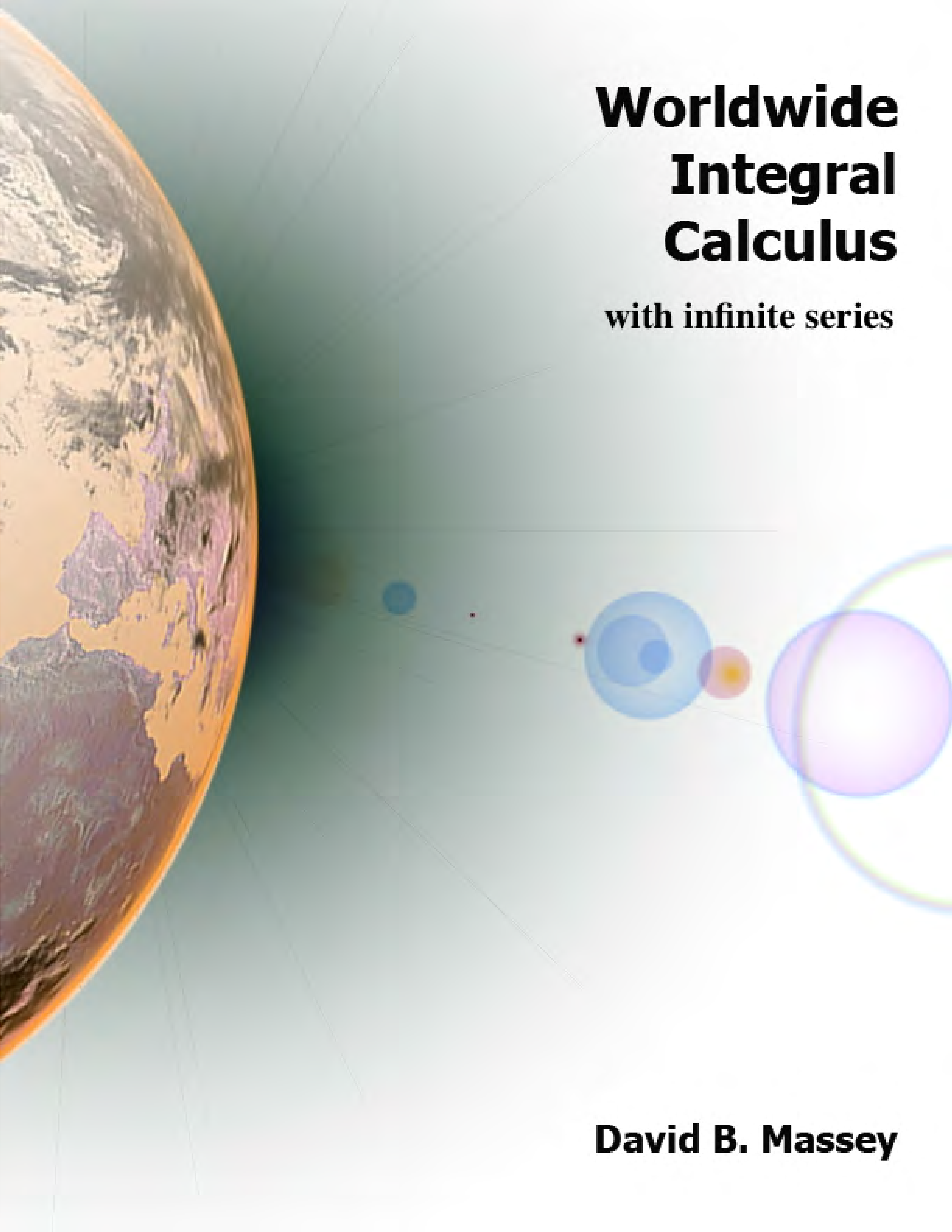

Worldwide Integral Calculus

with infinite series

David B. Massey

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0.1 Preface

Welcome to the Worldwide Integral Calculus textbook; the second textbook from the Worldwide Center of Mathematics.

Our goal with this textbook is, of course, to help you learn Integral Calculus (and power series methods) – the Calculus of integration. But why publish a new textbook for this purpose when so many already exist? There are several reasons why we believe that our textbook is a vast improvement over those already in existence.

• Even if this textbook is used as a classic printed text, we believe that the exposition, explanations, examples, and layout are superior to every other Calculus textbook. We have tried to write the text as we would speak the material in class; though, of course, the book contains far more details than we would normally present in class. In the book, we emphasize intuitive ideas in conjunction with rigorous statements of theorems, and provide a large number of illustrative examples. Where we think it will be helpful to you, we include proofs, or sketches of proofs, in the midst of the sections, but the extremely technical proofs are contained in the Technical Matters appendices to chapters, or are contained in referenced external sources. This greatly improves the overall readability of our textbook, while still allowing us to give mathematically precise definitions and statements of theorems.

• Our textbook is an Adobe pdf file, with linked/embedded/accompanying video content, annotations, and hyperlinks. With the videos contained in the supplementary files, you effectively possess not only a textbook, but also an online/electronic version of a course in Integral Calculus. Depending on the version of the files that you are using, clicking on the video frame to the right of each section title will either open an online, or an embedded, or a locally installed video lecture on that section. The annotations replace classic footnotes, without affecting the readability or formatting of the other text. The hyperlinks enable you to quickly jump to a reference elsewhere in the text, and then jump back to where you were.

• The pdf format of our textbook makes it incredibly portable. You can carry it on a laptop computer, on many handheld devices, e.g., an iPad, or can print any desired pages.

• Rather than force you to buy new editions of textbooks to obtain corrections and minor revisions, updates of this textbook are distributed free of cost.

• Because we have no print or dvd costs for the electronic version of this book and/or videos, we can make them available for download at an extremely low price. In addition, the printed, bound copies of this text and/or disks with the electronic files are priced as low as possible, to help reduce the burden of excessive textbook prices.

In this book, we assume you are already familiar with Differential Calculus. Specifically, we assume that you know the definition of the derivative of a function, that it represents the instantaneous rate of change, and that you know the "rules" for calculating derivatives. We will also need l'Hôpital's Rule and parameterized curves. Referring to the Worldwide Differential Calculus textbook [2], this means that you should know the contents of Chapters 1 and 2, Section 3.5, and Appendix A.

Our discussion of definite integrals, and their applications, is fairly traditional. However, our approach to infinite series is somewhat unusual. Our approach is motivated by two factors. First, we believe that the primary use that students will have for infinite series, outside of a Calculus class, is that many important functions have convergent power series representations, and these power series representations allow the student to mathematically manipulate and estimate the functions involved, in ways that would be difficult/impossible without power series. Second, statistical data that we collected over several years has made it clear that, in general, students do not grasp the basic idea that, when x is close to zero, smaller powers of x are more significant than larger powers of x in a power series or, even, in a polynomial function.

Consequently, we place emphasis on polynomial approximations and power series representations for functions, and, in a sense, view the classic convergence tests for sequences and series of constants as the "technical details" required to understand power series. We still include a chapter, Chapter 5, on sequences and series of constants, but that chapter comes **after** Chapter 4, which is on power series and approximating functions with polynomials. We firmly believe that this ordering of topics is better for the student and for applications, even though it may seem a bit awkward not to have the rigorous mathematical foundations of sequences and series come before their use in discussing power series.

This book is organized as follows:

Other than the Technical Matters sections, each section is accompanied by a video file, which is either a separate file, or an embedded video. Each video contains a classroom lecture of the essential contents of that section; if the student would prefer not to read the section, he or she can receive the same basic content from the video. Each non-technical section ends with exercises. The answers to all of the odd-numbered exercises are contained in Appendix C, at the end of the book.

Important definitions are boxed in green, important theorems are boxed in blue. Remarks, especially warnings of common misconceptions or mistakes, are shaded in red. Important conventions or fundamental principles, that will be used throughout the book, are boxed in black.

Very technical definitions and proofs from each section are contained in the Technical Matters appendices at the ends of some chapters, or in external sources. Our favorite external technical source is the excellent textbook by William F. Trench, *Introduction to Real Analysis*, [4], which is available, courtesy of the author, as a free pdf. For producing answers to various exercises or for help with examples or visualization, you may find the free web site wolframalpha.com very useful.

Internal references through the text are hyperlinked; simply click on the boxed-in link to go to the appropriate place in the textbook. If you have activated the “forward” and ”back” buttons in your pdf-viewer software, clicking on the “back” button will return you to where you started, before you clicked on the hyperlink.

Some terms or names are annotated; these are clearly marked in the margins by little blue “balloons”. Comments will pop up when you click on such annotated items.

Occasionally, when looking at approximations, we write an equals sign in quotes, as in “=”. We use this to denote “equal as far as a calculator is concerned”, i.e., equal to the precision of many/most/all calculators.

We sincerely hope that you find using our modern, multimedia textbook to be as enjoyable as using a mathematics textbook can be.

David B. Massey
August 2009

Chapter 1

Anti-differentiation: the Indefinite Integral

In this chapter, we discuss anti-differentiation, which is also called *indefinite integration*. This is the process for "undoing" differentiation. In the first section, we start with the basic techniques/results, and then in the remaining sections, we include some more-complicated methods.

The indefinite integral should not be confused with the definite integral, which is the topic of the next chapter. The definite integral is the mathematically precise notion of what it means to "take a continuous sum of infinitesimal contributions." The reason that both indefinite and definite integration are referred to as "integration" is because calculating continuous sums and finding anti-derivatives are related by the *Fundamental Theorem of Calculus*, Theorem 2.4.10.

1.1 Basic Anti-Differentiation

This section is about the process and formulas involved in un-doing differentiation, that is, in *anti-differentiating*. This means that you are given a function $f(x)$ and are asked to produce some/all functions $F(x)$ which have $f(x)$ as their derivative. This comes up often in applications, such as when you're given the acceleration $a(t)$ of an object and want the velocity $v(t)$, or when calculating *definite integrals* via the Fundamental Theorem of Calculus (see Section 2.3 and Theorem 2.4.10).

Since you know differential Calculus, you know what it means to have a function $F(x)$ and then be asked to calculate its derivative $F'(x)$. For instance, if $F(x) = x^3$, then $F'(x) = 3x^2$.

But what about the "reverse" question? What if you are given the function $f(x) = 3x^2$ and asked to produce an *anti-derivative* of $f(x)$, that is, if you are asked to find a function $F(x)$ whose derivative equals the given $f(x)$?

Certainly, $F(x) = x^3$ is **one** anti-derivative of $3x^2$. Are there any others? According to a corollary to the Mean Value Theorem, the only other anti-derivatives of $3x^2$ are functions that differ by a constant from the one anti-derivative that we produced, i.e., every other anti-derivative $F(x)$ of $f(x) = 3x^2$ is of the form $F(x) = x^3 + C$, for some constant C.

Definition 1.1.1. *Given a function $f(x)$, defined on an open interval I, a function $F(x)$, on I, such that $F'(x) = f(x)$ is called an* **anti-derivative of** *$f(x)$, with respect to x.*

Thus, an anti-derivative $y = F(x)$ of $f(x)$ is a solution to the differential equation $dy/dx = f(x)$.

If $F(x)$ is an anti-derivative of $f(x)$, on an open interval, then every anti-derivative of $f(x)$, on that interval, is given by $y = F(x) + C$, where C is a constant. The collection $y = F(x) + C$ is called the **(general) anti-derivative of** *$f(x)$, with respect to x; it is the general solution y to the differential equation $dy/dx = f(x)$.*

The notation for the general anti-derivative of $f(x)$, with respect to x, is

$$\int f(x)\, dx.$$

This is also called the **(indefinite) integral of** *$f(x)$, with respect to x.*

Remark 1.1.2. We have several important comments to make.

• First, it is important that $\int f(x)\,dx$ is not one particular function, but it **almost** is; $\int f(x)\,dx$ is actually a collection, or set, of functions, any two of which differ by a constant.

We write

$$\int 3x^2\,dx \;=\; x^3 \;+\; C,$$

where including the $+C$ is extremely important, for changing the value of C changes which element of the set of all anti-derivatives of $3x^2$ you are talking about. Technically, we ought to write

$$\int 3x^2\,dx \;=\; \{x^3 + C \mid C \in \mathbb{R}\},$$

but this is very cumbersome, and no one (well...no one that we know of) ever writes this.

• Second, you should notice that it follows from the definition that the units of $\int f(x)\,dx$ are the units of $f(x)$ times the units of x.

For instance, if $f(x)$ is in kilograms per cubic meter, and x is in cubic meters, then $\int f(x)\,dx$ is in kilograms.

• Third, you should think of the anti-differentiation, with respect to x, operator, $\int (\quad)\,dx$, as essentially being the inverse operator of $\frac{d}{dx}(\quad)$, differentiation with respect to x. That is, the anti-differentiation operator is a compound symbol; it starts with a $\int$, and ends with a differential, like dx, which, together, tell you to anti-differentiate whatever is in-between with respect to the variable which appears in the differential.

We wrote “essentially” above because, if you first differentiate and then anti-differentiate, you get what you started with, except that there is an additional $+C$; that is, you end up with a collection of functions that all differ by constants, instead of simply the one function that you started with.

• We should also comment on the term “indefinite integral.” There is another notion, called the *definite integral* of a function over a closed interval; see Section 2.3. The definite integral is defined in such a way that it agrees with one's intuitive idea of what a “continuous sum of infinitesimal contributions” should mean. This would seem to be unrelated to anti-differentiating. However, there is a theorem, the **Fundamental Theorem of Calculus**, which tells us: i) every continuous function possesses an anti-derivative (Theorem 2.4.7), and ii) the primary step used to obtain a nice formula for a definite integral is to produce an anti-derivative of the given function (Theorem 2.4.10).

Hence, anti-differentation is frequently referred to simply as “integration”, and definite integration is also simply referred to as “integration”; the context should always make it clear whether the meaning is anti-differentiation or definite integration. In addition, the symbols for anti-differentiating $\int (\ )\, dx$ are essentially the same as the symbols used for definite integration.

All of the differentiation formulas which you have learned yield corresponding anti-differentiation formulas; it's just a matter of reading things "in reverse", for, if $F'(x) = f(x)$, then the corresponding integration rule is $\int f(x)\, dx = F(x) + C$, where C denotes an arbitrary constant. In this context, $f(x)$ is frequently referred to as the *integrand*.

For instance, we have a **Power Rule for Integration**:

Theorem 1.1.3. *For all x in an open interval for which the functions involved are defined,*

1. $\int 0\, dx = C$;
2. $\int 1\, dx = x + C$;
3. *if* $p \neq -1$, $\displaystyle\int x^p\, dx = \frac{x^{p+1}}{p+1} + C$; *and*
4. $\displaystyle\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + C$.

Remark 1.1.4. In many books, only the third formula above is referred to as the Power Rule for Integration.

As is frequently the case, you should try to remember this rule not in symbols, but in words; it says that, as long as the exponent is not -1, you obtain the anti-derivative of x raised to a constant exponent by adding one to the exponent, and dividing by the new exponent (and then adding a C).

Remark 1.1.5. There are two fairly common, horrific mistakes associated with the integration rule $\displaystyle\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + C$.

The first big mistake is to treat the $p = -1$ case in the same manner as the cases where $p \neq -1$. If you were to do this, you would obtain

$$\int x^{-1}\, dx \;=\; \frac{x^{-1+1}}{-1+1} + C \;=\; \frac{x^0}{0} + C.$$

The undefined division by 0 should immediately tell you that you've done something wrong, and remind you that you must treat the $p = -1$ case differently.

The second big mistake may come later, when we have more integration rules. It will then be tempting to look at the formula $\int \frac{1}{x}\, dx = \ln|x| + C$ and think that it implies that

$$\int \frac{1}{\text{anything}}\, dx \;=\; \ln|\text{anything}| + C.$$

This is **completely wrong** (in general); it is not true that the derivative of the expression on the right would be the integrand. The problem is that, if you differentiate the expression on the right, you do, in fact, get a $1/\text{anything}$ factor, but then the Chain Rule tells you that that is multiplied by $d(\text{anything})/dx$.

Example 1.1.6. Find the function $P(r)$, with domain $r > 0$, such that

$$\frac{dP}{dr} \;=\; \sqrt{r} \qquad \text{and} \qquad P(9) = -7.$$

Solution:

We find that

$$P \;=\; \int r^{1/2}\, dr = \frac{r^{3/2}}{3/2} + C \;=\; \frac{2}{3}\, r^{3/2} + C.$$

We need to determine C. We have

$$-7 \;=\; P(9) \;=\; \frac{2}{3}\,(9)^{3/2} + C \;=\; 18 + C.$$

Therefore, $C = -25$, and so

$$P = \frac{2}{3}r^{3/2} - 25.$$

The linearity of the derivative gives us the linearity of the anti-derivative.

Theorem 1.1.7. **(Linearity of Anti-differentiation)** *If a and b are constants, not both zero, then*

$$\int af(x) + bg(x)\,dx = a \cdot \int f(x)\,dx + b \cdot \int g(x)\,dx.$$

Remark 1.1.8. The prohibition against $a = b = 0$ in Theorem 1.1.7 is there for just one reason: we do not want both of the arbitrary constants on the right to be eliminated by multiplying by zero. We will explain this more fully.

Since each indefinite integral actually yields a set, or collection, of functions, there may be some question in your mind about what it means to multiply a set of functions by a constant, like a or b, and what it means to add two such sets. In other words, you may wonder exactly what the right-hand side of the equality in Theorem 1.1.7 means.

For instance, what does it mean to write that

$$\int (5x^3 - 7\sqrt{x})\,dx = 5 \cdot \int x^3\,dx - 7 \cdot \int \sqrt{x}\,dx?$$

We know, from the Power Rule for Integration, that $\int x^3\,dx = x^4/4 + C_1$ and

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{3/2}}{3/2} + C_2 = 2x^{3/2}/3 + C_2,$$

where we have used C_1 and C_2, in place of using simply C twice, since we don't want to assume that we have to pick the two arbitrary constants to be the same thing.

So, what does $5 \cdot \int x^3\,dx$ mean? It means the collection of functions obtained by taking 5 times any function from the collection of functions $x^4/4 + C_1$; that is, the collection of functions

$5x^4/4 + 5C_1$, where C_1 could be any constant. But, if C_1 can be anything, then $5C_1$ can be anything, and we might as well just call it B_1, where B_1 can be any number. Thus, we can write the collection of functions $5 \cdot \int x^3\,dx$ as $5x^4/4 + B_1$.

However, here's the part that can be confusing; instead of using a new constant name, like B_1, it is fairly standard to just use the name C_1 again, i.e., to use C_1 to now denote 5 times the old value of C_1. Assuming that we had not determined some value for the old C_1, there is no harm in doing this, but it certainly can make things look confusing, for you frequently see calculations like

$$5 \cdot \int x^3\,dx \;=\; 5\big(x^4/4 + C_1\big) \;=\; 5x^4/4 + C_1,$$

where the C_1 on the far right above is actually 5 times the C_1 in the middle.

Similarly, we write

$$-7 \cdot \int \sqrt{x}\,dx \;=\; \int x^{1/2}\,dx \;=\; -7\big(2x^{3/2}/3 + C_2\big) \;=\; -14x^{3/2}/3 + C_2.$$

Therefore,

$$5 \cdot \int x^3\,dx - 7 \cdot \int \sqrt{x}\,dx \;=\; 5x^4/4 + C_1 - 14x^{3/2}/3 + C_2 \;=$$

$$5x^4/4 - 14x^{3/2}/3 + (C_1 + C_2) \;=\; 5x^4/4 - 14x^{3/2}/3 + C,$$

where $C = C_1 + C_2$ can be any real number.

Using this example as a guide, you can see what to do more generally: whenever you have a *linear combination* of indefinite integrals, i.e., a sum of constants multiplied times indefinite integrals, you do **not** include an arbitrary constant for each individual indefinite integral; instead, for each indefinite integral you write **one** particular anti-derivative, and then put in a single $+C$ at the end.

The fact that an indefinite integral is actually a collection of functions can lead to seemingly bizarre results. For instance, while it's true that $\int x\,dx = \int x\,dx$, it would nonetheless be bad to write that $\int x\,dx - \int x\,dx = 0$. Why? Because our operations on sets of functions tell us that the correct calculation is

$$\int x\,dx - \int x\,dx \;=\; x^2/2 + C_1 - \left(x^2/2 + C_2\right) \;=\; C_1 - C_2 \;=\; C,$$

which is the same as $\int 0\,dx$.

This agrees with what we wrote above: when you have a linear combination of indefinite integrals, you should use one particular anti-derivative for each integral, and then add a C at the end. Thus, in the above calculation, you should get $x^2/2 - x^2/2 + C$, which is just C.

Example 1.1.9. Calculate the indefinite integral

$$\int \left(\frac{5}{w} - 3 + 7w^3 + 5\sqrt[9]{w}\right)\,dw.$$

Solution:

We calculate

$$\int \left(\frac{5}{w} - 3 + 7w^3 + 5\sqrt[9]{w}\right)\,dw \;=\; \int \left(5\cdot\frac{1}{w} - 3 + 7w^3 + 5w^{1/9}\right)\,dw \;=$$

$$5\ln|w| - 3w + 7\cdot\frac{w^4}{4} + 5\cdot\frac{w^{(1/9)+1}}{(1/9)+1} + C \;=$$

$$5\ln|w| - 3w + \frac{7w^4}{4} + \frac{9w^{10/9}}{2} + C.$$

Example 1.1.10. Suppose that an object moves in a straight line with constant acceleration a meters per second per second. Show that the position of the object $p = p(t)$, in meters, is given by

$$p = at^2/2 + v_0 t + p_0,$$

where p_0 is the initial position of the object (i.e., the position at $t = 0$), in meters, v_0 is the initial velocity in m/s, and t is the time in seconds.

Solution:

Acceleration a is the derivative, with respect to t, of the velocity v, i.e., $a = dv/dt$. This is exactly the same as writing that v is an anti-derivative of a, with respect to t, i.e., $v = \int a\, dt$. Since a is a constant, we find

$$v \;=\; \int a\, dt \;=\; a \int 1\, dt \;=\; at + C \quad \text{m/s},$$

for some constant C. Therefore, $v = at + C$, but we would like to give some physical meaning to the constant C.

How do we do this? We are not given any other data. The answer is that we use *tautological initial data*; that is, we use initial data that is simply obviously true. We use that, when $t = 0$, the velocity v equals v_0. Why is this true? Because it says something that's clearly true: at time 0, the velocity equals the velocity at time 0. It may seem strange, but using this tautological initial data actually gets us somewhere.

We have $v = at + C$. Now, plug in that, when $t = 0$, $v = v_0$. You find $v_0 = a \cdot 0 + C$, i.e., $C = v_0$. Thus, we conclude that $v = at + v_0$. Notice that no one has to give you any extra data to conclude that $C = v_0$; it follows from the equation $v = at + C$.

Now, we have that $v = at + v_0$, and we know that the velocity v equals the rate of change of p, with respect to time, i.e., $v = dp/dt$. This is the same as writing that $p = \int v\, dt$. Therefore, we have

$$p \;=\; \int v\, dt \;=\; \int (at + v_0)\, dt \;=\; a\left(\int t\, dt\right) + v_0\left(\int 1\, dt\right) \;=\; at^2/2 + v_0 t + C \quad \text{meters},$$

where this C is definitely not the same C that we used in the equation for v.

How do we find this new C? We plug in more tautological initial data, namely that $p = p_0$ when $t = 0$. We find that $p_0 = a(0)^2/2 + v_0(0) + C$, and so $C = p_0$. Thus, we conclude

$$p = at^2/2 + v_0 t + p_0 \quad \text{meters}.$$

Other integration formulas obtained at once from differentiation formulas are:

Theorem 1.1.11. *As functions on the entire real line* $(-\infty, \infty)$, *we have*

1. $$\int \cos x\, dx \;=\; \sin x + C;$$

2. $$\int \sin x\, dx \;=\; -\cos x + C; \text{ and}$$

3. $$\int e^x\, dx \;=\; e^x + C.$$

Note that the integration formulas for sin and cos have the negative sign in the "opposite" place from the differentiation formulas. This frequently leads to confusion. It shouldn't. Remember: you are finding anti-derivatives. This means that the derivative of what you end up with should be what you started with (i.e., the integrand).

Let's look at another example in which you are given the acceleration of an object, and are asked to find the velocity and position, but, this time, we have an acceleration which is not constant.

Example 1.1.12. Suppose that the acceleration, in m/s^2, of an object moving in a straight line is $a = \sin t$, where t is the time in seconds. Find the velocity and position of the object, as functions of time, in terms of the initial velocity and initial position.

Solution:

We find that

$$v \;=\; \int a\, dt \;=\; \int \sin t\, dt \;=\; -\cos t + C,$$

and, plugging in the tautological initial data, we find $v_0 = -\cos(0) + C = -1 + C$. Thus, $C = v_0 + 1$, and

$$v \;=\; -\cos t + v_0 + 1 \;\; \text{m/s}.$$

Now, we integrate the velocity to find the position:

$$p \;=\; \int v\, dt \;=\; \int (-\cos t + v_0 + 1)\, dt \;=\; -\int \cos t\, dt + (v_0 + 1)\int 1\, dt \;=$$

$$-\sin t + (v_0 + 1)t + C.$$

Finally, we plug in the tautological initial data, in order to give physical meaning to this last C:

$$p_0 \;=\; -\sin(0) + (v_0+1)(0) + C.$$

Therefore, $C = p_0$, and we find

$$p \;=\; -\sin t + (v_0+1)t + p_0 \;\text{ meters.}$$

We shall not rewrite, as anti-differentiation formulas, every one of the differentiation formulas that you should know; the standard anti-differentiation formulas are contained in Appendix B. However, we'll go ahead and give two more, before looking at the Chain Rule and the Product Rule in their indefinite integral forms.

Theorem 1.1.13. *As functions on the open interval* $(-1,1)$*,*

$$\int \frac{1}{\sqrt{1-x^2}}\,dx \;=\; \sin^{-1}x + C.$$

As functions on the open interval $(-\infty,\infty)$*,*

$$\int \frac{1}{1+x^2}\,dx \;=\; \tan^{-1}x + C.$$

Example 1.1.14. Calculate

$$\int \frac{7z^2+5}{z^2+1}\,dz.$$

Solution:

We begin by "simplifying" via long division of polynomials, except the division is not so long in this case. We get clever and write $7z^2 = 7(z^2+1) - 7$, and find

$$\frac{7z^2+5}{z^2+1} \;=\; \frac{7(z^2+1)-7+5}{z^2+1} \;=\; 7 - 2\cdot\frac{1}{z^2+1}.$$

Thus,

$$\int \frac{7z^2+5}{z^2+1}\,dz \;=\; \int\left(7-2\cdot\frac{1}{z^2+1}\right)dz \;=\; 7\int 1\,dz \;-\; 2\cdot\int\frac{1}{z^2+1}\,dz \;=$$

$$7z - 2\tan^{-1} z + C.$$

Substitution: the Chain Rule in anti-derivative form:

The Chain Rule for differentiation tells you how to differentiate a composition of functions. If f and g are differentiable, then $\big(f(g(x))\big)' = f'(g(x))g'(x)$ or, letting $u = g(x)$, the Chain Rule can be written as

$$\frac{d}{dx}\big(f(u)\big) \;=\; f'(u)\,\frac{du}{dx}.$$

As an anti-derivative formula, this becomes

Theorem 1.1.15. (**Integration by Substitution**) *If f and g are differentiable functions, then*

$$\int f'(g(x))g'(x)\,dx \;=\; f(g(x)) + C,$$

or, letting $u = g(x)$,

$$\int f'(u)\,\frac{du}{dx}\,dx \;=\; \int f'(u)\,du \;=\; f(u) + C.$$

The second formula for substitution is particularly easy to use; it looks as though the dx's cancel, as in multiplying fractions. This is **not** what's happening, but it does make substitution easier to remember. It also means that the differential notation

$$du = \frac{du}{dx}\,dx,$$

which we introduced in [2], yields correct formulas in integrals, and, hence, we use it extensively.

Example 1.1.16. Calculate the integral

$$\int \cos(e^x + 7)\cdot e^x\,dx.$$

Solution:

How should you approach an integral like

$$\int \cos(e^x + 7)\cdot e^x\,dx?$$

First, you should realize that it's not just a linear combination of integrals of specific functions that you've memorized. You should then think "Well...$\cos(e^x+7)$ is a composition of functions. What if I make a substitution for the "inside" function? I'll let $u = e^x + 7$, so that $\cos(e^x + 7)$ becomes $\cos u$, and see if the remaining part of the integrand looks like du."

In fact, for easy cases, you can do this in your head. If $u = e^x + 7$, then

$$du \;=\; \frac{du}{dx}\,dx \;=\; e^x\,dx,$$

and you see that this is the remaining "factor" in the integral. Thus, by substitution, our original integral is transformed into an integral in terms of u that is very simple:

$$\int \cos(e^x + 7)\cdot e^x\,dx \;=\; \int \cos u\,du \;=\; \sin u + C \;=\; \sin(e^x+7) + C.$$

Nice.

Example 1.1.17. Calculate the integral

$$\int \frac{t}{1+t^2}\,dt.$$

Solution:

You once again realize that this integral is not some linear combination of basic integrals that have memorized, and there's no obvious composition of functions this time. You might think that $\tan^{-1} t$ is involved somehow, since $\int 1/(1+t^2)\,dt = \tan^{-1} t + C$, but the t in the numerator causes a problem.

You might think that you can factor the integrand and use that in some way:

$$\int \frac{t}{1+t^2}\,dt \;=\; \int t\cdot\frac{1}{1+t^2}\,dt \;\overset{?}{\underset{?}{=}}\; \int t\,dt\cdot\int\frac{1}{1+t^2}\,dt \;=\; \frac{t^2}{2}\cdot\tan^{-1} t + C.$$

THIS IS COMPLETELY WRONG. You can't just integrate each factor in a product and then multiply the results; dealing with products in an integral is more complicated than that.

We will get to the integral form of the Product Rule shortly, but, for now, you should differentiate $t^2\tan^{-1} t/2$ (using the Product Rule) and verify that you don't get anything close to our integrand $t/(1+t^2)$.

Great. Now we know one way **NOT** to find the integral $\int t/(1+t^2)\,dt$. We also know that we're discussing substitution here, and so you should suspect that a substitution is involved.

With practice, you should actually see relatively quickly that if you let $w = 1+t^2$, then $dw = 2t\,dt$, and we have a $t\,dt$ in the integral, so this substitution might be good. We can always "fix" multiplying by a constant, like the 2 in $dw = 2t\,dt$. We divide by 2 to get

$$\frac{1}{2}\,dw \;=\; t\,dt,$$

and

$$\int \frac{t}{1+t^2}\,dt \;=\; \int\frac{1}{1+t^2}\cdot t\,dt \;=\; \int\frac{1}{w}\cdot\frac{1}{2}\,dw \;=$$

$$\frac{1}{2}\int \frac{1}{w}\,dw \;=\; \frac{1}{2}\ln|w|+C \;=\; \frac{1}{2}\ln|1+t^2|+C \;=\; \frac{1}{2}\ln(1+t^2)+C,$$

where the last equality follows from the fact that $1+t^2 \geq 0$ (in fact, $1+t^2 \geq 1$).

It would be a good exercise for you to differentiate our final answer above, and see how the Chain Rule comes into play to produce our initial integrand.

By making the substitution $u = x/a$, so that $x = au$, we easily obtain:

Theorem 1.1.18. *Suppose that $a > 0$ is a constant. As functions on the open interval $(-a, a)$,*

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx \;=\; \sin^{-1}\left(\frac{x}{a}\right)+C.$$

As functions on the open interval $(-\infty, \infty)$,

$$\int \frac{1}{a^2+x^2}\,dx \;=\; \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C.$$

Integration by Parts: the Product Rule in anti-derivative form:

The Product Rule as an anti-derivative formula is

Theorem 1.1.19. (**Integration by Parts**) *If f and g are differentiable functions, then*

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx \;=\; f(x)g(x) + C,$$

or, letting $u = f(x)$ and $v = g(x)$,

$$\int u\,dv + \int v\,du \;=\; uv + C$$

or

$$\int u\,dv \;=\; uv - \int v\,du.$$

It is the last formula above that most people memorize as THE formula for integration by parts.

Here's how a basic integration by parts attack on a problem goes. You look at your integral, and make a choice that let's you write the integral in the form $\int u\,dv$. Then, you apply the integration by parts formula to obtain that your integral equals $uv - \int v\,du$. Then, you hope that the new integral $\int v\,du$ is easier (or, at least, no harder) to integrate than the integral that you started with.

Example 1.1.20. Calculate

$$\int ze^z\,dz \qquad \text{and} \qquad \int z^2e^z\,dz.$$

Solution:

You should look at $\int ze^z\,dz$, and realize quickly that ze^z does not result from one of the basic derivative formulas that you should have memorized, and so this integration is not a basic one that you should know immediately. In addition, if you think for a minute or so, you should be able to convince yourself that there's no substitution that will help. However, we do see that the integrand is the product of two very different-looking functions, z and e^z; this is a hint that integration by parts may be good to use.

Now what do you do? You identify some factor in the integrand that will be u; that factor should **not** contain the differential (here, dz). The remaining part of the integrand, together with the differential, should be dv. With u and dv determined, you calculate du and $v = \int dv$. You do not need to include an arbitrary constant in your calculation of v; we need **some** v, not all possible v's. Then, you write $uv - \int v\,du$, look at your new integral, and hope that it's easier.

Let's see how this works for $\int ze^z\,dz$. There are two obvious choices for u: either $u = z$ or $u = e^z$. Either one of these will lead to a formula that is **true**, but only one will lead to something **useful**.

Let's look at the **bad** choice first, so that you can see how you can tell when you've made a bad choice. Let's try $u = e^z$. That leaves $z\,dz$ to be dv. Now, if $u = e^z$ and $dv = z\,dz$, then $du = e^z\,dz$ and $v = \int dv = \int z\,dz = z^2/2$, where, as we discussed, we don't include a $+C$ in our calculation of v. Applying the integration by parts formula, we find

$$\int ze^z\,dz \;=\; \int e^z z\,dz \;=\; \int u\,dv \;=\; uv - \int v\,du \;=$$

$$(e^z)(z^2/2) - \int (z^2/2)e^z\,dz. \tag{1.1}$$

This is true, but not particularly helpful for calculating $\int ze^z\,dz$. The power of z went up, and, after moving the constant $1/2$ outside the integral, the rest of the integral, the e^z, is the same as what we started with. You should realize that the new integral is harder to deal with than the original. We could try to integrate by parts again, but, we leave it as an exercise for you to verify, depending on your new choice of u, that either the power of z goes up again (and the e^z remains), or the power of z goes back down to 1, but you end up with exactly the integral that we started with.

So, let's make the other choice for u in integrating by parts to calculate $\int ze^z\,dz$. Let $u = z$, which means that $dv = e^z\,dz$. Then, we find that $du = dz$ and $v = \int dv = \int e^z\,dz = e^z$. Thus, we obtain

$$\int ze^z\,dz \;=\; \int u\,dv \;=\; uv - \int v\,du \;=$$

$$ze^z - \int e^z\,dz \;=\; ze^z - e^z + C.$$

Why did this choice of u work better than our earlier one? Before, when we picked u to not include the power of z, the power of z was left in dv; when we integrated dv to get v, the power of z went up. Now, when we pick u to be the power of z, namely z^1, the power of z goes down in the calculation of du. For this reason, it is frequently (but not always – see the next example) a good idea in integration by parts problems which include powers of the variable to let the power of the variable be u.

With this in mind, how do you integrate $\int z^2e^z\,dz$? You integrate by parts, letting $u = z^2$, which means $dv = e^z\,dz$. What you should get is actually exactly what we got in Formula 1.1, except you need to multiply Formula 1.1 by 2, and rearrange, to obtain:

$$\int z^2e^z\,dz \;=\; z^2e^z - 2\int ze^z\,dz.$$

Now, even if we had not already calculated $\int ze^z\,dz$, you should realize that the new integral on the right above is easier than the one you started with, and you would calculate it by a second integration by parts (if we had not already done so). What we find is

$$\int z^2e^z\,dz \;=\; z^2e^z - 2\int ze^z\,dz \;=\; z^2e^z - 2\,(ze^z - e^z + C) \;=\; z^2e^z - 2ze^z + 2e^z + C.$$

It might be instructive to differentiate the final result above, on the right, and verify that you obtain z^2e^z.

Example 1.1.21. Calculate

$$\int t^5\ln t\,dt \qquad \text{and} \qquad \int \ln t\,dt.$$

Solution:

You look at $\int t^5\ln t\,dt$, and you realize immediately that this doesn't come from one of our basic derivative/integral formulas. You might think about a substitution, like $w = \ln t$, for a minute or so, but then realize that it doesn't get you anywhere. Then, you decide that, since the integrand is the product of two different kinds of functions, maybe integration by parts would be a good thing to try.

If you look at our previous example, you'd probably be tempted to let u be the power of t, i.e., let $u = t^5$, which would lead to $dv = \ln t\,dt$. However, this is one of those times when picking the power of the variable to be u is a bad idea. Perhaps the most obvious reason why this is bad is because we don't know how to calculate $v = \int dv = \int \ln t\,dt$. You may think that we've discussed this integral, and that it equals $1/t + C$. This is **very wrong**. We know that $(\ln t)' = 1/t$ or, what's the same thing, $\int 1/t\,dt = \ln t + C$ (for $t > 0$), but we don't know $\int \ln t\,dt$

(it's possible that you do, but it hasn't been discussed in the book up to this point). In fact, calculating $\int \ln t\, dt$ is the second part of this example.

So, to calculate

$$\int t^5 \ln t\, dt,$$

we'll try integration by parts with $u = \ln t$, which means that $dv = t^5\, dt$.

We find that $du = (1/t)\, dt$ and $v = \int dv = \int t^5\, dt = t^6/6$ (remember, we don't need a $+C$ here). Applying the integration by parts formula, we find

$$\int t^5 \ln t\, dt \;=\; \int u\, dv \;=\; uv - \int v\, du \;=\; (\ln t)(t^6/6) - \int (t^6/6)(1/t)\, dt \;=$$

$$\frac{t^6 \ln t}{6} - \frac{1}{6}\int t^5\, dt \;=\; \frac{t^6 \ln t}{6} - \frac{1}{6}\cdot\frac{t^6}{6} + C \;=\; \frac{t^6 \ln t}{6} - \frac{t^6}{36} + C.$$

Now, let's look at $\int \ln t\, dt$. How could this possibly be an integration by parts problem? There's no product in the integrand! Admittedly, this does not look like an integration by parts problem. Nonetheless, it is. Let $u = \ln t$, which means that $dv = dt$. Then, $du = (1/t)\, dt$, and $v = \int dv = \int dt = t$. Applying the integration by parts formula, we obtain

$$\int \ln t\, dt \;=\; \int u\, dv \;=\; uv - \int v\, du \;=\; (\ln t)(t) - \int t(1/t)\, dt \;=$$

$$t \ln t - \int 1\, dt \;=\; t\ln t - t + C.$$

We wish to look at one more integration by parts example, a complicated example.

Example 1.1.22. Calculate

$$\int e^x \cos x\, dx.$$

Solution:

We will calculate this integral via integration by parts. We could use either $u = e^x$ or $u = \cos x$; this time, it actually makes little difference. We will pick $u = e^x$, which means that $dv = \cos x\, dx$. (As an exercise, you should try starting with $u = \cos x$ instead.) We find, then, that $du = e^x\,dx$ and $v = \int dv = \int \cos x\,dx = \sin x$. The integration by parts formula tells us that we have

$$\int e^x \cos x\,dx \;=\; \int u\,dv \;=\; uv - \int v\,du \;=\; e^x \sin x - \int (\sin x)e^x\,dx. \tag{1.2}$$

But our new integral, $\int (\sin x)e^x\,dx$ is clearly just as difficult to integrate as the original integral. Are we getting anywhere? The answer is "yes", but it's not obvious yet. We will integrate $\int (\sin x)e^x\,dx$ by parts also.

You have to be careful. If you make the choice $u = \sin x$ here, you will obtain that $\int (\sin x)e^x\,dx$ equals $e^x \sin x - \int e^x \cos x\,dx$. If you substitute this into Formula 1.2, you will find that you have undone our original integration by parts, and you will come to the stunning conclusion that $\int e^x \cos x\,dx = \int e^x \cos x\,dx$. It's true, but not very helpful.

When integrating $\int (\sin x)e^x\,dx$ by parts, you need to make the choice of u that is analogous to your choice of u for the original integration by parts; you let $u = e^x$ again, and so $dv = \sin x\,dx$. Then, $du = e^x\,dx$ and $v = \int dv = \int \sin x\,dx = -\cos x$. Applying integration by parts, we find

$$\int (\sin x)e^x\,dx \;=\; \int u\,dv \;=\; uv - \int v\,du \;=\; e^x(-\cos x) - \int (-\cos x)e^x\,dx \;=$$

$$-e^x \cos x + \int e^x \cos x\,dx.$$

If you insert this result into Formula 1.2, you obtain

$$\int e^x \cos x\,dx \;=\; e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

At this point, you may be thinking to yourself "Aaaaagggghhhhhh! We spent all that time integrating by parts, only to end up with the same integral that we started with!" However, the fact that the new occurrence of the original integral has a negative sign in front of it saves us. If you simply add $\int e^x \cos x\,dx$ to each side of this last equation (and fix the loss of the arbitrary constant on the right), you obtain

$$2 \cdot \int e^x \cos x\,dx \;=\; e^x \sin x + e^x \cos x + C,$$

and so

$$\int e^x \cos x\, dx \;=\; \frac{e^x \sin x + e^x \cos x}{2} + C.$$

We end this section with a possibly surprising complication that exists for anti-differentiation; a type of complication which does not occur for differentiation.

Remark 1.1.23. From the derivative formulas in [2], we see that the derivative of any elementary functionis again an elementary function. You might hope that anti-derivatives/integrals would behave equally as well. **They do not.** It is easy to give elementary functions $f(x)$ for which it is possible to prove that there is no elementary function $F(x)$ such that $F'(x) = f(x)$, i.e., $f(x)$ has no elementary anti-derivative. Such functions $f(x)$ include e^{x^2}, e^{-x^2}, $\sin(x^2)$, and $\cos(x^2)$. This was first proved by Liouville in 1835.

The Fundamental Theorem of Calculus, Theorem 2.4.7, guarantees that the functions e^{x^2}, e^{-x^2}, $\sin(x^2)$, and $\cos(x^2)$, and, in fact, all continuous functions, have **some** anti-derivative, but those anti-derivatives need not be elementary functions.

1.1.1 Exercises

Calculate the general anti-derivatives in Exercises 1 through 21.

1. $\displaystyle\int (4x^2 + 4x + 9)\, dx$

2. $\displaystyle\int \left(\frac{5}{w} - 7e^w + 6\sqrt[3]{w}\right) dw$

3. $\displaystyle\int \left(5\sin t - \frac{3}{\sqrt{1-t^2}}\right) dt$

4. $\displaystyle\int \frac{1 + v + \sqrt{v}}{v^2}\, dv$

5. $\displaystyle\int \left(\frac{1}{y} + \frac{1}{y^2+1}\right) dy$

6. $\displaystyle\int \frac{5}{3z-7}\, dz$

7. $\displaystyle\int \cos(2\theta - 1)\, d\theta$

8. $\displaystyle\int e^{p+4}\,dp$

9. $\displaystyle\int \frac{r}{r^2-4}\,dr$

10. $\displaystyle\int \left(\frac{x}{\sqrt{x^2-1}}+\frac{1}{|x|\sqrt{x^2-1}}\right)dx$

11. $\displaystyle\int (5\omega-3)^{100}\,d\omega$

12. $\displaystyle\int \big(2\cos(2t+5)+3\sin(9t)\big)\,dt$

13. $\displaystyle\int \ln\left[(x+2)^{x+5}\right]\,dx$

14. $\displaystyle\int \frac{e^{1/x}}{x^3}\,dx$

15. $\displaystyle\int \frac{5}{x\ln x}\,dx$

16. $\displaystyle\int \frac{e^{1/x}}{6x^2}\,dx$

17. $\displaystyle\int \tan\theta\,d\theta$

18. $\displaystyle\int \frac{5}{4+x^2}\,dx$

19. $\displaystyle\int t^4\sqrt{t^5+6}\,dt$

20. $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$ Hint: $x^2+4x+5=(x+2)^2+1$.

21. $\displaystyle\int \frac{1}{\sqrt{-x^2+6x-8}}\,dx$

In each of Exercises 22 through 31, find the anti-derivative of the given function which satisfies the given initial condition. The anti-derivative of each function with a lower-case name is denoted by the upper-case version of the same letter.

22. $h(x)=4x^2+4x+9$, such that $H(-1)=2$.

23. $p(w)=\dfrac{5}{w}-7e^w+6\sqrt[3]{w}$, such that $P(-1)=0$.

24. $q(t)=5\sin t-\dfrac{3}{\sqrt{1-t^2}}$, such that $Q(0)=7$.

25. $k(v) = \dfrac{1+v+\sqrt{v}}{v^2}$, such that $K(1) = -2$.

26. $b(y) = \dfrac{1}{y} + \dfrac{1}{y^2+1}$, such that $B(1) = 0$.

27. $f(x) = x^2 + x\sin x$, such that $F(\pi) = 2\pi$.

28. $s(t) = \dfrac{2}{t(\ln t)^2}$, such that $S(e^2) = 5$.

29. $g(x) = x\sqrt{x+1}$, such that $G(0) = 1$.

30. $w(y) = \dfrac{\tan^{-1}\left(\frac{y}{2}\right)}{4+y^2}$, such that $W(2) = \pi$.

31. $r(t) = te^{1-t^2} - t$, such that $R(1) = \sqrt{2}$.

In each of Exercises 32 through 41, use integration by parts to find the indicated anti-derivative.

32. $\int xe^{3x}\,dx$

33. $\int (x-5)^2 e^x\,dx$

34. $\int t\sin(2t)\,dt$

35. $\int t^2\cos t\,dt$

36. $\int \sqrt{p}\,\ln p\,dp$

37. $\displaystyle\int \frac{\ln t}{t^2}\,dt$

38. $\int e^x \sin x\,dx$

39. $\int e^{2x}\sin(5x)\,dx$

40. $\int \tan^{-1} w\,dw$

41. $\int w\tan^{-1} w\,dw$ Hint: At some point, you may want to use that $w^2 = (1+w^2) - 1$.

42. Suppose that the net force F, acting on an object of mass m, pushes the object along the x-axis with an acceleration function, in m/s^2, of

$$a(t) = \sin(2t),$$

where $t \geq 0$ is measured in seconds.

a. Recall that $F = ma$ and that momentum $p = mv$. Find the momentum of the object, as a function of time, if the mass of the object is 10 kilograms, and momentum at time $t = 0$ is 20 kilogram meters per second.

b. What is the momentum of the object at time $t = 4$ seconds?

43. Repeat the preceding problem with the new acceleration function $a(t) = t \cdot \sin(2t)$.

44. For each positive integer n, define $f_n(\theta) = \sin^n \theta \cos \theta$ and $g_n(\theta) = \cos^n \theta \sin \theta$

a. Find $\displaystyle\int f_n(\theta)\, d\theta$ and $\displaystyle\int g_n(\theta)\, d\theta$.

b. Find the specific anti-derivatives $F_n(\theta)$ and $G_n(\theta)$ that satisfy the initial conditions $F_n(0) = 5$ and $G_n\left(\frac{\pi}{2}\right) = 4$.

In Exercises 45 through 48, you are given the velocity of a particle at time t, and the position $p(t_0)$ of the particle at a specific time t_0. Find the position function.

45. $v(t) = 3t^2 - 4t + 3$, $p(1) = 4$.

46. $v(t) = t + \cos(t)$, $p(0) = 0$.

47. $v(t) = 2t\sqrt{18 + 7t^2}$, $p(1) = 8$.

48. $v(t) = at + b$, $p(2) = 5$. Leave your answer in terms of a and b.

49. In the following steps, you will calculate the general anti-derivatives for $\sin^2 x$ and $\cos^2 x$.

a. Apply integration by parts to $\displaystyle\int \sin x \cdot \sin x\, dx$ (written suggestively).

b. Integration by parts yields a new anti-derivative. Use a trigonometric identity to write this new anti-derivative in terms of $\sin^2 x$, and solve your integration by parts equation for $\displaystyle\int \sin^2 x\, dx$.

c. What is $\displaystyle\int \cos^2 x\, dx$? Hint: Use your answer to part (b).

d. From the cosine double angle formula, $\cos(2x) = 2\cos^2 x - 1$. Use this to integrate $\cos^2 x$, and explain why the different-looking answer that you obtain is, in fact, the same as your answer from part (c).

Exercises 50 and 51 show that the argument in Exercise 49 can be generalized to calculate anti-derivatives of higher powers of $\sin$ and $\cos$.

50. a. Use integration by parts to prove that

$$\int \sin^n t\,dt = -\frac{1}{n}\cos t \sin^{n-1} t + \frac{n-1}{n}\int \sin^{n-2} t\,dt.$$

Assume $n \geq 2$.

b. Use this formula to calculate $\int \sin^2 t\,dt$. Check your answer by comparing to the previous problem.

51. a. Use integration by parts to prove that

$$\int \cos^n t\,dt = \frac{1}{n}\cos^{n-1} t \sin t + \frac{n-1}{n}\int \cos^{n-2} t\,dt.$$

b. Use the formula in part (a) to determine $\int \cos^2 t\,dt$.

52. Suppose that instantaneous rate of change, with respect to time, of a population of an island at time t, measured in years, where $t = 0$ corresponds to the year 2000, is given by

$$\frac{1}{2\sqrt{6,250,000 + t}} - \frac{500}{(t+1)^2}.$$

The population of the island in 2000 is 3000.

a. Find an explicit formula for the population at time t.

b. What is the predicted population in 2050?

53. Prove that the argument used to calculate $\int \ln t\,dt$ can be generalized. Assume that $f(t)$ is differentiable and prove that

$$\int f(t)\,dt = tf(t) - \int tf'(t)\,dt.$$

54. Calculate $\int e^x \sinh x\,dx$. Hint: do *not* use integration by parts.

55. Consider the following logic in calculating $\int e^x \sinh x \, dx$ using integration by parts.

$$\begin{aligned}
\int e^x \sinh x \, dx &= e^x \sinh x - \int e^x \cosh x \, dx \\
&= e^x \sinh x - \left(e^x \cosh x - \int e^x \sinh x \, dx \right) \Rightarrow \\
0 &= e^x \sinh x - e^x \cosh x \Rightarrow \\
e^x \sinh x &= e^x \cosh x.
\end{aligned}$$

Since $e^x > 0$, we can divide and conclude that $\sinh x = \cosh x$. What is the flaw in this argument?

56. Prove the formula
$$\int t^n e^t \, dt = t^n e^t - n \int t^{n-1} e^t \, dt.$$
Assume that $n \geq 1$.

57. Calculate $\displaystyle\int \frac{1}{x^2 + a^2} \, dx$, where $a \neq 0$ is a constant. Hint: use Theorem 4.2.14 and an appropriate substitution.

58. Prove that $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$.

59. Calculate $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$, where $a > 0$ is a constant.

60. Calculate $\displaystyle\int \frac{1}{|x|\sqrt{x^2 - 1}} \, dx$. Hint: consider $\sec^{-1} x$.

61. Calculate $\displaystyle\int e^x e^{e^x} \, dx$. Hint: use substitution.

62. Let $f_1(x) = e^x$ and, for all integers $n \geq 2$, let $f_n(x) = e^{f_{n-1}(x)}$. Prove that
$$\int f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x) \, dx = f_n(x) + C.$$

63. Calculate $\displaystyle\int 2^x \, dx$. Hint: rewrite 2^x as an exponential expression with base e.

64. Calculate $\displaystyle\int \frac{x + 4}{\sqrt{x + 2}} \, dx$.

65. Calculate $\displaystyle\int \ln(1 + x^2) \, dx$.

66. Calculate $\displaystyle\int \frac{e^{3x}-e^{2x}}{e^{2x}-1}\,dx$.

Consider a simple electric circuit with an inductor, but no resistor or capacitor. A battery supplies voltage $V(t)$. If inductance is constantly L, in henrys, then Kirchoff's Law from Example 2.7.18 of [2] tells us that the current i at time t satisfies the differential equation $L\dfrac{di}{dt} = V(t)$. In Exercises 67 through 70, find an explicit formula for $i(t)$, given the condition $i(t_0) = i_0$.

67. $L = 12$, $V(t) = \sin t$, $i(0) = 0$.

68. $L = 9$, $V(t) = 12$, $i(3) = 12$.

69. $L = 3$, $V(t) = \dfrac{t}{t^2+1}$, $i(0) = 0$.

70. $L = 13$, $V(t) = \dfrac{\ln(1/t)}{t^2}$, $i(1) = 2$.

For each of the functions in Exercises 71 through 74, verify that (a) $\dfrac{d}{dx}\left[\displaystyle\int f(x)\,dx\right] = f(x)$ and (b) $\displaystyle\int\left[\frac{d}{dx}f(x)\right]dx = f(x)+C$. Assume an appropriate domain for $f(x)$.

71. $f(x) = x^4$.

72. $f(x) = 3\cos(2x)$.

73. $f(x) = \ln x$.

74. $f(x) = \dfrac{1}{1+x^2}$.

75. In the following steps, you will find the general anti-derivatives for functions of the form

$$t^n \ln t.$$

   a. Suppose that $n \neq -1$. Apply integration by parts to find $\displaystyle\int t^n \ln t\,dt$.

   b. Find $\displaystyle\int t^{-1}\ln t\,dt$.

76. Suppose that water flows out of a hole 0.1 square meters in area from the bottom of a cylindrical tank with a base radius of 2 meters and an initial height of 10 meters at a rate

$$\frac{dV}{dt} = 0.007798t - 1.3999 \text{ cubic meters per second.}$$

a. If the tank starts out full, what is the function $V(t)$ for the volume in the tank at time t?

b. Calculate the amount of water remaining in the tank one minute after the leak starts.

c. Verify that $\frac{dV}{dt} = 0$ precisely when the tank is empty.

77. A particle is traveling along the curve $y = x^2$, so that its x-coordinate is a function of time t, measured in minutes. Suppose that the horizontal velocity (i.e., velocity in only the horizontal direction) is given by $dx/dt = 0.5\cos^3 t$ miles per minute.

a. What is the maximum horizontal *speed* (absolute value of velocity) on the time interval $0 \le t \le 20\pi$?

b. Find the function $x(t)$, the x-coordinate of the particle at time t, subject to the condition that the particle is at the origin at time $t = 0$.

c. Find the function $y(t)$, the y-coordinate of the particle at time t.

d. What is the vertical velocity of the particle at time $t = \pi$ minutes?

78. An enclosed room is built in order to experiment with the effects of pressure changes on objects. Suppose that the equipment is capable of decreasing the pressure in the room at a rate of $-\frac{2}{t+1} - 0.04t$ kilopascals, kPa, per second, and that it can run for up to one minute before overheating.

a. If the internal pressure in the room is 101.325 kPa (equal to the standard atmospheric pressure or atm), find an expression for $P(t)$.

b. How long will it take to reach 50 kPa (this is approximately the atmospheric pressure five and a half kilometers above the surface of the Earth)?

c. What is the lowest pressure that can be reached in the room?

In Exercises 79 through 82, you are given the acceleration, $a = a(t)$ in m/s^2, of an object moving in a straight line, where t is the time in seconds. Find the velocity v and position p of the object, as functions of time, in terms of the initial velocity v_0 and initial position p_0.

79. $a = t + 1$

80. $a = \sin t + \cos t$

81. $a = e^{-3t}$

82. $a = te^{-t}$

1.2 Special Trigonometric Integrals and Trigonometric Substitutions

In this section, we will discuss some special, important, trigonometric integrals, and then give three examples of *integration by trigonometric substitution*.

Proposition 1.2.1.

$$\int \tan\theta\, d\theta \;=\; -\ln|\cos\theta| + C \;=\; \ln|\sec\theta| + C.$$

Proof. We have
$$\int \tan\theta\, d\theta \;=\; \int \frac{\sin\theta}{\cos\theta}\, d\theta.$$
Let $u = \cos\theta$, so that $du = -\sin\theta\, d\theta$, i.e., $-du = \sin\theta\, d\theta$. We find

$$\int \frac{\sin\theta}{\cos\theta}\, d\theta \;=\; \int \frac{-du}{u} \;=\; -\ln|u| + C \;=\; -\ln|\cos\theta| + C.$$

□

Proposition 1.2.2.
$$\int \sec\theta\, d\theta \;=\; \ln|\sec\theta + \tan\theta| + C.$$

Proof. As $(\tan\theta)' = \sec^2\theta$ and $(\sec\theta)' = \sec\theta\tan\theta$, it follows that

$$(\tan\theta + \sec\theta)' = \sec^2\theta + \sec\theta\tan\theta = (\sec\theta)(\tan\theta + \sec\theta).$$

This means that, if $u = \tan\theta + \sec\theta$, then $du/d\theta = u\sec\theta$, i.e.,

$$\int \sec\theta\, d\theta \;=\; \int \frac{1}{u}\, du \;=\; \ln|u| + C \;=\; \ln|\sec\theta + \tan\theta| + C.$$

□

Proposition 1.2.3.

$$\int \sec^3\theta\, d\theta \;=\; \frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2} + C.$$

Proof. This is a "fun" integration by parts problem.

$$\int \sec^3\theta\, d\theta \;=\; \int \sec\theta(1+\tan^2\theta)\, d\theta \;=\; \int \sec\theta\, d\theta \;+\; \int \sec\theta\tan^2\theta\, d\theta \;=$$

$$\ln|\sec\theta + \tan\theta| \;+\; \int (\tan\theta)(\sec\theta\tan\theta)\, d\theta.$$

We approach this last integral by parts; let $u = \tan\theta$ and $dv = (\sec\theta\tan\theta)\, d\theta$. Then, $du = \sec^2\theta\, d\theta$ and $v = \int dv = \sec\theta$. We find

$$\int (\tan\theta)(\sec\theta\tan\theta)\, d\theta \;=\; \int u\, dv \;= uv - \int v\, du \;=\; \tan\theta\sec\theta - \int \sec^3\theta\, d\theta.$$

Combining this with our previous equality, we obtain

$$\int \sec^3\theta\, d\theta \;=\; \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int \sec^3\theta\, d\theta.$$

Adding $\int \sec^3\theta\, d\theta$ to each side, fixing the missing $+C$, and dividing by 2 yields the desired result. □

Remark 1.2.4. You may wonder why we didn't give integral formulas for the co-functions in the previous three propositions. The reason for this is that you should memorize fundamental integral formulas, and then use general techniques to quickly derive others, if possible.

Since replacing θ by $\pi/2-\theta$ changes any trig function into the corresponding co-function, the substitution $u = \pi/2-\theta$ tells us that the integrals of the co-functions of those in Proposition 1.2.1, Proposition 1.2.2, and Proposition 1.2.3 are obtained by negating what we obtained and replacing all of the trig functions by their co-functions, i.e.,

$$\int \cot\theta\,d\theta \;=\; \ln|\sin\theta| + C,$$

$$\int \csc\theta\,d\theta \;=\; -\ln|\csc\theta + \cot\theta| + C$$

and

$$\int \csc^3\theta\,d\theta \;=\; -\frac{\csc\theta\cot\theta + \ln|\csc\theta + \cot\theta|}{2} + C.$$

The following two *iteration formulas* for integrating powers of sine and cosine are frequently useful. They are proved by using integration by parts.

Proposition 1.2.5. *If $n \geq 2$ is an integer, then*

$$\int \sin^n\theta\,d\theta \;=\; -\frac{1}{n}\sin^{n-1}\theta\cos\theta \;+\; \frac{n-1}{n}\int \sin^{n-2}\theta\,d\theta,$$

and

$$\int \cos^n\theta\,d\theta \;=\; \frac{1}{n}\cos^{n-1}\theta\sin\theta \;+\; \frac{n-1}{n}\int \cos^{n-2}\theta\,d\theta.$$

Proof. These are obtained from integration by parts, and the two demonstrations are entirely similar. We will derive the $\int \cos^n\theta\,d\theta$ formula, and leave the other as an exercise.

We have

$$\int \cos^n\theta\,d\theta \;=\; \int \cos^{n-1}\theta\cdot\cos\theta\,d\theta.$$

Let $u = \cos^{n-1}\theta$ and $dv = \cos\theta\,d\theta$. Then, $du = -(n-1)\cos^{n-2}\theta\sin\theta\,d\theta$ and $v = \int dv = \sin\theta$. We find

$$\int \cos^n\theta\,d\theta \;=\; \int u\,dv \;=\; uv - \int v\,du \;=$$

$$\cos^{n-1}\theta\sin\theta - \int \sin\theta\cdot(-(n-1)\cos^{n-2}\theta\sin\theta)\,d\theta \;=$$

$$\cos^{n-1}\theta\sin\theta + (n-1)\int \cos^{n-2}\theta\sin^2\theta\,d\theta \;=$$

$$\cos^{n-1}\theta\sin\theta + (n-1)\int (\cos^{n-2}\theta)(1-\cos^2\theta)\,d\theta \;=$$

$$\cos^{n-1}\theta\sin\theta \;+\; (n-1)\int \cos^{n-2}\theta\,d\theta \;-\; (n-1)\int \cos^{n}\theta\,d\theta.$$

Therefore, we have concluded that

$$\int \cos^{n}\theta\,d\theta \;=\; \cos^{n-1}\theta\sin\theta \;+\; (n-1)\int \cos^{n-2}\theta\,d\theta \;-\; (n-1)\int \cos^{n}\theta\,d\theta.$$

Adding $(n-1)\int \cos^{n}\theta\,d\theta$ to each side of the equation, dividing by n, and replacing the missing $+C$ yields the desired result. □

Integration by trigonometric substitution:

Integration by trigonometric substitution (trig substitution) refers to having an integral involving some variable, say x, and "letting" x equal an expression involving a trig function, e.g., $x = a\sin\theta$ or $x = a\tan\theta$, and then using properties of trigonometric functions to produce a manageable integral in terms of θ. We placed the word "letting" in quotes in the previous sentence because we don't really get to "let" x be anything; it is what it is.

What we **can** do, however, is define a new variable θ in terms of x. Hence, what we really do are things like "let $\theta = \sin^{-1}(x/a)$" or "let $\theta = \tan^{-1}(x/a)$", which actually imply more than $x = a\sin\theta$ and $x = a\tan\theta$, respectively; they also imply that the values of θ are restricted to the intervals $[-\pi/2, \pi/2]$ and $(-\pi/2, \pi/2)$, respectively.

Let's look at three examples.

Example 1.2.6. Evaluate the integral $\displaystyle\int \sqrt{a^2 - x^2}\,dx$, where $a > 0$ is a constant.

Solution:

As the integrand is $\sqrt{a^2 - x^2}$, we must have that $-a \le x \le a$; it follows that $-1 \le x/a \le 1$. This is important since the closed interval $[-1, 1]$ is the domain of $\sin^{-1}$.

We now let $\theta = \sin^{-1}(x/a)$, so that $x = a\sin\theta$ and $-\pi/2 \le \theta \le \pi/2$. Then,

$$dx \;=\; a\cos\theta\,d\theta,$$

and

$$\sqrt{a^2 - x^2} \;=\; \sqrt{a^2 - a^2\sin^2\theta} \;=\; a\sqrt{1 - \sin^2\theta} \;=\; a\sqrt{\cos^2\theta} \;=\; a\cos\theta,$$

where we used that $\sqrt{a^2} = a$, since $a > 0$, and that $\sqrt{\cos^2\theta} = \cos\theta$, since $\cos\theta \geq 0$, because $-\pi/2 \leq \theta \leq \pi/2$.

Therefore, we obtain:

$$\int \sqrt{a^2 - x^2}\, dx \;=\; \int a\cos\theta \cdot a\cos\theta\, d\theta \;=\; a^2 \int \cos^2\theta\, d\theta.$$

Now, either by using the trig identity that $\cos^2\theta = (1 + \cos(2\theta))/2$, or by using the cosine iteration formula (which is proved using integration by parts) from Appendix B, we know that

$$\int \cos^2\theta\, d\theta \;=\; \frac{1}{2}\big[(\sin\theta\cos\theta) + \theta\big] + C.$$

Thus,

$$\int \sqrt{a^2 - x^2}\, dx \;=\; \frac{a^2}{2}\big[(\sin\theta\cos\theta) + \theta\big] + C \;=\; \frac{1}{2}\big[(a\sin\theta \cdot a\cos\theta) + a^2\theta\big] + C \;=$$

$$\frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right] + C.$$

Example 1.2.7. Evaluate the integral $\int \sqrt{a^2 + x^2}\, dx$, where $a > 0$ is a constant.

Solution: You might suspect that this integral would turn out to be something similar to the previous answer. After all, all that we changed was a minus sign to a plus sign. However, this seemingly simple change alters the problem dramatically.

Let $\theta = \tan^{-1}(x/a)$, so that $x = a\tan\theta$ and $-\pi/2 < \theta < \pi/2$. Then, $dx = a\sec^2\theta\, d\theta$, and

$$\sqrt{a^2 + x^2} \;=\; \sqrt{a^2 + a^2\tan^2\theta} \;=\; a\sqrt{1 + \tan^2\theta} \;=\; a\sqrt{\sec^2\theta} \;=\; a\sec\theta,$$

where $\sqrt{a^2} = a$, since $a > 0$, and $\sqrt{\sec^2\theta} = \sec\theta$, since $\sec\theta > 0$ because $-\pi/2 < \theta < \pi/2$. Thus, we find

$$\int \sqrt{a^2 + x^2}\, dx \;=\; \int a\sec\theta \cdot a\sec^2\theta\, d\theta \;=\; a^2 \int \sec^3\theta\, d\theta \;=$$

$$a^2 \cdot \frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2} + C \;=\;$$

$$\frac{x\sqrt{a^2+x^2} + a^2 \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|}{2} + C \;=\;$$

$$\frac{x\sqrt{a^2+x^2} + a^2 \ln\left|\sqrt{a^2+x^2} \;+\; x\right|}{2} + C,$$

where, in the last step, we used the property of the natural logarithm that $\ln(w/a) = \ln w - \ln a$, and absorbed the resulting constant $-a^2 \ln a/2$ into the constant C.

Example 1.2.8. Evaluate the integral

$$\int \frac{1}{(a^2+x^2)^n}\, dx,$$

where $a > 0$ is a constant and $n \geq 1$ is an integer.

Solution:

Again, we let $\theta = \tan^{-1}(x/a)$, so that $x = a\tan\theta$ and $-\pi/2 < \theta < \pi/2$. Then, $dx = a\sec^2\theta\, d\theta$, and

$$a^2 + x^2 \;=\; a^2 + a^2\tan^2 \;=\; a^2(1+\tan^2\theta) \;=\; a^2\sec^2\theta.$$

Thus,

$$\int \frac{1}{(a^2+x^2)^n}\, dx \;=\; \int \frac{1}{(a^2\sec^2\theta)^n}\, a\sec^2\theta\, d\theta \;=\;$$

$$a^{1-2n}\int \frac{1}{\sec^{2n-2}\theta}\, d\theta \;=\; a^{1-2n}\int \cos^{2n-2}\theta\, d\theta.$$

This final integral can be calculated using the cosine iteration formula in Proposition 1.2.5.

For instance, when $n = 1$, we recover a formula that we already knew:

$$\int \frac{1}{a^2+x^2}\, dx \;=\; a^{1-2}\int 1\, d\theta \;=\; \frac{1}{a}\,\theta + C \;=\; \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

When $n = 2$, we obtain

$$\int \frac{1}{(a^2+x^2)^2}\, dx \;=\; a^{1-4}\int \cos^2\theta\, d\theta \;=\; \frac{1}{a^3}\left(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\,\theta\right) + C \;=\;$$

$$\frac{1}{2a^3}\left(\frac{ax}{a^2+x^2}+\tan^{-1}\left(\frac{x}{a}\right)\right)+C.$$

1.2.1 Exercises

In each of Exercise 1 through 19, calculate the anti-derivative.

1. $\int \tan(3\theta)\,d\theta.$
2. $\int 2\sec(5t)\,dt.$
3. $\int \sec(4y)+\tan(3y)\,dy.$
4. $\int 2z\cot(z^2)\,dz.$
5. $\int (\cos x)\,(\cot(\sin x))\,\,dx.$
6. $\int \frac{\sec^4 u-1}{\sec u-1}\,du.$ Hint: factor before anti-differentiating.
7. $\int \csc t(1+\csc^2 t)\,dt.$
8. $\int \sqrt{25-\phi^2}\,d\phi.$
9. $\int \phi\sqrt{25-\phi^2}\,d\phi.$ Hint: do not use a trig substitution. Compare your answer to that of the previous problem.
10. $\int \sqrt{36+y^2}\,dy.$
11. $\int \frac{dk}{(121+k^2)^2}.$
12. $\int \frac{dx}{x^4+18a^2x^2+81}.$
13. $\int \sqrt{10-49v^2}\,dv.$
14. $\int \sqrt{x^2+6x+21}\,dx.$ Hint: complete the square.

15. $\displaystyle\int \frac{dz}{(1+z^2)^3}.$

16. $\displaystyle\int \tan^2 y\, dy.$

17. $\displaystyle\int \cot^2 v\, dv.$

18. $\displaystyle\int e^x \sqrt{16 - e^{2x}}\, dx.$

19. $\displaystyle\int \sqrt{e^{2x} - 16}\, dx.$

20. Prove the sine iteration formula:

$$\int \sin^n \theta\, d\theta = -\frac{1}{n} \sin^{n-1} \theta \cos \theta + \frac{n-1}{n} \int \sin^{n-2} \theta\, d\theta.$$

Assume $n \geq 2$.

21. The reduction formula below gives a method for calculating integrals of powers of the tangent function. Verify this formula in the case $n = 2$.

$$\int \tan^n \phi\, d\phi = \frac{1}{n-1} \tan^{n-1} \phi - \int \tan^{n-2} \phi\, d\phi.$$

Use the angle addition identities to calculate anti-derivatives in the next three exercises. Assume in each exercise that a and b are positive integers. These integrals are extremely important in Fourier analysis.

22. $\displaystyle\int \sin(ax) \sin(bx)\, dx.$

23. $\displaystyle\int \sin(ay) \cos(by)\, dy.$

24. $\displaystyle\int \cos(aw) \cos(bw)\, dw.$

25. Explain why the results in the last three problems do *not* hold in the case $a = b$.

26. Assume that $n \geq 1$ and show that:

$$\int z^n \sin z\, dz = -z^n \cos z + n \int z^{n-1} \cos z\, dz.$$

27. Assume that $n \geq 1$ and show that:

$$\int z^n \cos z \, dz = z^n \sin z - n \int z^{n-1} \sin z \, dz.$$

Use the integral formulas from the previous problems to calculate the anti-derivatives in Exercises 28 - 38.

28. $\displaystyle\int x^2 \sin x \, dx.$

29. $\displaystyle\int y^2 \cos y \, dy.$

30. $\displaystyle\int w^3 \sin w \, dw.$

31. $\displaystyle\int z^3 \cos z \, dz.$

32. $\displaystyle\int \sin^3 \phi \, d\phi.$

33. $\displaystyle\int \sin^4 \psi \, d\psi.$

34. $\displaystyle\int \tan^3 t \, dt.$

35. $\displaystyle\int \tan^4 u \, du.$

36. $\displaystyle\int \sin(9x) \sin(2x) \, dx.$

37. $\displaystyle\int \cos(4x) \cos x \, dx.$

38. $\displaystyle\int \sin(3t) \cos(5t) \, dt.$

Recall that the momentum, p, of a mass m moving with velocity v is given by $p = mv$. In Exercises 39 - 42 you are given the mass, acceleration, and the momentum at one specific time. Find the momentum function, $p(t)$. The units of acceleration are meters per second per second. The units of momentum are $\dfrac{\text{kg} \cdot \text{m}}{\text{sec}}$.

39. $m = 12$ kg, $a(t) = 2t^3\sqrt{16 - t^2}$, $p(0) = 8$.

40. $m = 9$ kg, $a(t) = \dfrac{1}{t^2\sqrt{t^2-15}}$, $p(4) = 6$.

41. $m = 5$ kg, $a(t) = \dfrac{6t^3}{\sqrt{t^2+9}}$, $p(0) = 0$.

42. $m = 20$ kg, $a(t) = \dfrac{t}{\sqrt{20-8t-t^2}}$, $p(1) = 1$.

Integrate the following products of trig. functions.

43. $\displaystyle\int \sin^3 x \cos^2 x \, dx$.

44. $\displaystyle\int \sin^3 w \cos^3 w \, dw$.

45. $\displaystyle\int \tan y \sec^2 y \, dy$.

46. $\displaystyle\int \tan^3 z \sec^2 z \, dz$.

Find the general solution to the separable differential equation.

47. $\dfrac{dx}{dt} = \dfrac{\sqrt{9-t^2}\sqrt{4+x^2}}{t^2 x}$.

48. $\dfrac{dy}{dx} = \dfrac{3\sqrt{y^2-16}}{2x^2\sqrt{x^2+4}}$.

49. $\dfrac{dx}{dz} = \dfrac{z^3}{x\sqrt{z^2+9-9x^2-x^2z^2}}$.

50. $\dfrac{dy}{dx} = \dfrac{\sqrt{2y-y^2-2x^2y+x^2y^2}}{y-1}$. Assume $|x| < 1$ and $|y| < \sqrt{2}$.

1.3 Integration by partial fractions

A *rational function* is one defined by the quotient of two polynomial functions, e.g.,

$$\frac{x^3 - 7x + \pi}{x + 5} \qquad \text{and} \qquad \frac{x^2 - 5x + 1}{x^3 + x^2},$$

where the domains exclude roots of the denominators. This section describes the fundamental techniques for integrating rational functions. *Integration by partial fractions* refers to an algebraic technique for obtaining the *partial fractions decomposition* of a rational function, and then integrating the resulting, easier summands in the decomposition. The partial fractions decomposition is essentially what you get by "undoing" the work you have to do to add rational functions and write the result as a single rational function, i.e., you have to "undo" finding a (least) common denominator and simplifying the numerator after writing all of the fractions over the common denominator.

Example 1.3.1. For instance, if you want to write

$$\frac{3}{x+7} + \frac{5}{x-2}$$

as a rational function, that is, as the quotient of two polynomials, you would use the common denominator $(x+7)(x-2)$ and obtain

$$\frac{3}{x+7} + \frac{5}{x-2} = \frac{3(x-2)}{(x+7)(x-2)} + \frac{5(x+7)}{(x+7)(x-2)} =$$

$$\frac{3x - 6 + 5x + 35}{(x+7)(x-2)} = \frac{8x + 29}{x^2 + 5x - 14}.$$

The corresponding partial fractions problem is to start with

$$\frac{8x + 29}{x^2 + 5x - 14}$$

and produce its partial fractions decomposition, i.e., to find that this rational function equals

$$\frac{3}{x+7} + \frac{5}{x-2}.$$

How does this help with integration? It means that, if we want to calculate the integral

$$\int \frac{8x+29}{x^2+5x-14}\, dx,$$

then, what we need to do is calculate

$$\int \left(\frac{3}{x+7} + \frac{5}{x-2}\right) dx \;=\; 3\cdot\int \frac{1}{x+7}\, dx \;+\; 5\cdot\int \frac{1}{x-2}\, dx \;=$$

$$3\ln|x+7| \;+\; 5\ln|x-2| \;+\; C,$$

where the final two integrals were accomplished via the easy substitutions $u = x+7$ and $w = x-2$, respectively.

So, how do you find partial fractions decompositions, and how do you integrate all of the resulting summands? Before we state the general process/technique, we shall first give a few examples of integration by partial fractions, and work through them slowly, in order to discuss most of the cases and issues that you will typically encounter.

Example 1.3.2. Let's look at the example above, in reverse. We know how to integrate the summands that we know will appear. The question is: how do you produce the partial fractions decomposition of

$$\frac{8x+29}{x^2+5x-14}?$$

(We are, of course, pretending that we don't already know the answer.)

First, you note that the numerator has smaller degree than denominator. If this were not the case, the first step would be to (long) divide the numerator by the denominator. We shall look at such a case in the next example.

The next step is to show either that the denominator factors into a product of two degree one (i.e., linear) real polynomials, or you *complete the square* to show that $x^2 + 5x - 14$ is an irreducible quadratic polynomial, that is, a quadratic polynomial that does **not** factor into two linear (real) polynomials. Hopefully, you see quickly (even if you have forgotten our work prior to this example) that $x^2 + 5x - 14$ factors as

$$x^2 + 5x - 14 \;=\; (x+7)(x-2).$$

However, if you did not notice this factorization, you would need to complete the square. Since the leading coefficient (the coefficient in front of x^2) is a 1, we complete the square by squaring half of the coefficient in front of the x term, and then adding and subtracting the resulting quantity; in this example, this means we add and subtract $(5/2)^2 = 25/4$. What do we mean by "adding and subtracting 25/4?" Exactly what we wrote; it's true that this adds zero in the end, but it adds zero in a very clever way. We obtain

$$x^2 + 5x - 14 \;=\; x^2 + 5x + \frac{25}{4} - \frac{25}{4} - 14.$$

The point is that $x^2 + 5x + \frac{25}{4}$ is a perfect square; it equals $\left(x + \frac{5}{2}\right)^2$. Thus, we find that

$$x^2 + 5x - 14 \;=\; \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} - 14 \;=\; \left(x + \frac{5}{2}\right)^2 - \frac{81}{4}. \tag{1.3}$$

The fact that we end up with something squared **minus** a positive quantity means that this result can be factored (over the real numbers) as the difference of squares; hence, as we already knew, $x^2 + 5x - 14$ factors as

$$x^2 + 5x - 14 \;=\; \left(x + \frac{5}{2}\right)^2 - \frac{81}{4} \;=\; \left(x + \frac{5}{2}\right)^2 - \left(\frac{9}{2}\right)^2 \;=$$

$$\left(x + \frac{5}{2} + \frac{9}{2}\right)\left(x + \frac{5}{2} - \frac{9}{2}\right) \;=\; (x+7)(x-2).$$

Had we completed the square and ended up with something squared **plus** a positive quantity in Formula 1.3, then the quadratic polynomial would have no real roots and, hence, would be an irreducible quadratic polynomial. We shall look at such an example in Example 1.3.4.

Discussing what to do in other cases is making this example much longer than it really is. All that we have really done to help us with our current problem is to factor the denominator, i.e., so far, what's relevant to this example is that

$$\frac{8x+29}{x^2+5x-14} = \frac{8x+29}{(x+7)(x-2)}.$$

We would like to "undo" the obtaining of the common denominator and write an equality of functions:

$$\frac{8x+29}{(x+7)(x-2)} = \frac{?}{x+7} + \frac{?}{x-2}, \tag{1.4}$$

but what goes in place of the question marks?

It can be shown that there are unique polynomials that go where the question marks are and, because the numerator on the lefthand side has smaller degree than the denominator, the degrees of the numerators on the righthand side of Formula 1.4 must be less than degrees of the denominators. However, the denominators have degree 1, and so the numerators must be polynomials of degree 0 (or the zero polynomial), i.e., the numerators must be constants.

Therefore, we know that there exist unique constants A and B such that, for all x unequal to -7 and 2, we have the equality

$$\frac{8x+29}{(x+7)(x-2)} = \frac{A}{x+7} + \frac{B}{x-2}, \tag{1.5}$$

but how do we figure out what A and B are?

The first step is to "clear the denominators" by multiplying both sides of the equation by the big denominator on the left, and canceling factors. We obtain the equality

$$8x+29 = A(x-2) + B(x+7), \tag{1.6}$$

where, initially, this equality is required to hold for all x, except $x=-7$ and $x=2$. However, the polynomial functions on each side of the equality are defined and continuous everywhere; hence, if they are equal for all x, other than -7 and 2, then they must, in fact, be equal when $x=-7$ and when $x=2$.

It is important to emphasize that the equality in Formula 1.6 must now hold at those x values which we couldn't use earlier, those which made the original denominators in Formula 1.5 equal zero. The reason that this is important is because the easiest way to determine A and B is to plug $x = -7$ and $x = 2$ into Formula 1.6. We find:

When $x = -7$:

$$8(-7) + 29 \;=\; A(-7-2) + B(0), \quad \text{and so} \quad -27 = -9A. \text{ Hence, } \; A = 3.$$

When $x = 2$:

$$8(2) + 29 \;=\; A(0) + B(2+7), \quad \text{and so} \;\; 45 = 9B. \text{ Hence, } \; B = 5.$$

Thus, plugging $A = 3$ and $B = 5$ back into Formula 1.5, we finally obtain the partial fractions decomposition

$$\frac{8x+29}{x^2+5x-14} \;=\; \frac{8x+29}{(x+7)(x-2)} \;=\; \frac{3}{x+7} \;+\; \frac{5}{x-2}.$$

If you now wish to integrate this, you proceed as we indicated before this example, by making easy substitutions, and you obtain

$$\int \frac{8x+29}{x^2+5x-14}\,dx \;=\; 3\ln|x+7| \;+\; 5\ln|x-2| \;+\; C.$$

We refer to our method above for determining A and B by plugging in exceptional values of x as the *exceptional values method*. There are at least two other methods for determining A and B, which are very similar to each other.

One method is simply to substitute **any** two, distinct, x values into Formula 1.6; this will yield two linear equations, involving A and B. You solve these equations simultaneously, and you will still find $A = 3$ and $B = 5$. (Try it.)

The other method is one that we refer to as *matching coefficients*. You "multiply out" Formula 1.6, then collect the powers of x, and match the coefficients of the corresponding powers of x on each side of the equation, since two polynomial functions (in the variable x) are equal if and only if the coefficients in front of each power of x agree.

Following this method, we obtain

$$8x + 29 \;=\; Ax - 2A + Bx + 7B \;=\; (A + B)x + (-2A + 7B).$$

The constant terms on each side of the equation must agree, and so must the coefficients in front of x. Therefore, we find that $29 = -2A + 7B$ and $8 = A + B$. Solving these equations, we once again find that $A = 3$ and $B = 5$.

Example 1.3.3. Consider the integral

$$\int \left(\frac{x^3 + 2x^2 - 21x + 71}{x^2 + 5x - 14} \right) dx.$$

True, it's disgusting-looking, but it's really not so bad. We want to produce the partial fractions decomposition of the integrand, and integrate the resulting "pieces."

This time, the degree of numerator is greater than or equal to the degree of the denominator. The first step is thus to long divide $x^3 + 2x^2 - 21x + 71$ by $x^2 + 5x - 14$.

$$\begin{array}{r|l} & \qquad\qquad x \quad - 3 \\ \hline x^2 + 5x - 14 & x^3 + 2x^2 - 21x + 71 \\ & -x^3 - 5x^2 + 14x \\ \hline & \qquad - 3x^2 \;\; - 7x + 71 \\ & \qquad\;\; 3x^2 + 15x - 42 \\ \hline & \qquad\qquad\qquad 8x + 29 \end{array}$$

What does this division tell us? It tells us that there is an equality

$$\frac{x^3 + 2x^2 - 21x + 71}{x^2 + 5x - 14} \;=\; x - 3 \;+\; \frac{8x + 29}{x^2 + 5x - 14}.$$

The remaining fractional part is precisely what we integrated in the last example. Hence, we conclude that

$$\int \left(\frac{x^3 + 2x^2 - 21x + 71}{x^2 + 5x - 14} \right) dx \;=\; \int (x - 3)\, dx \;+\; \int \left(\frac{8x + 29}{x^2 + 5x - 14} \right) dx \;=$$

$$\frac{x^2}{2} \;-\; 3x \;+\; 3\ln|x+7| \;+\; 5\ln|x-2| \;+\; C.$$

Example 1.3.4. Consider the integral

$$\int \frac{x+1}{(x-5)(x^2+6x+11)}\,dx.$$

The numerator has smaller degree than the denominator, so there's no need to long divide. We either need to factor $x^2+6x+11$, or complete the square. A few seconds of thought should convince you that it doesn't factor into linear terms with integer coefficients; still, it could factor with real, but irrational coefficients, or be irreducible over the real numbers. Completing the square will tell us, either way. We add and subtract $(6/2)^2 = 9$, and find

$$x^2+6x+11 \;=\; x^2+6x+9-9+11 \;=\; (x+3)^2+2.$$

This polynomial has no real roots, and so is irreducible (over the real numbers). What do we do?

Now that we know that $x^2+6x+11$ is irreducible, we temporarily put off using that it equals $(x+3)^2+2$. What we want to do now is write an equality of functions

$$\frac{x+1}{(x-5)(x^2+6x+11)} \;=\; \frac{?}{x-5} \;+\; \frac{?}{x^2+6x+11},$$

but what should the question marks be this time?

As before, it can be shown that there are unique polynomials that take the places of these question marks and, because the rational function on the left has a numerator of smaller degree than the denominator, the question marks are polynomials of smaller degree than the respective denominators. Thus, there exist unique constants A, B, and C such that

$$\frac{x+1}{(x-5)(x^2+6x+11)} \;=\; \frac{A}{x-5} \;+\; \frac{Bx+C}{x^2+6x+11}. \tag{1.7}$$

We wish to determine A, B, and C.

Again, we clear the denominators by multiplying both sides of the equation by the big denominator on the left, and canceling. We obtain

$$x + 1 \;=\; A(x^2 + 6x + 11) \;+\; (Bx + C)(x - 5),$$

which now must hold for all x. Plugging in $x = 5$ immediately tells us that $6 = A \cdot 66 + B \cdot 0$, so that $A = 1/11$, but there are no easy (real) values to plug in for x to immediately yield B and C. We will multiply things out and match coefficients.

We find

$$x + 1 \;=\; Ax^2 + 6Ax + 11A + Bx^2 - 5Bx + Cx - 5C \;=$$

$$(A + B)x^2 + (6A - 5B + C)x + (11A - 5C).$$

Therefore, noting that 0 is the coefficient of x^2 in the polynomial $x + 1$, we obtain the three simultaneous equations

$$A + B = 0, \qquad 6A - 5B + C = 1, \text{ and } \qquad 11A - 5C = 1.$$

As we already know that $A = 1/11$, we really only need two of these equations, say the first and the last, to find that $B = -1/11$ and $C = 0$. Substituting these values into Formula 1.7, we obtain

$$\frac{x + 1}{(x - 5)(x^2 + 6x + 11)} \;=\; \frac{1}{11} \cdot \left(\frac{1}{x - 5} \;-\; \frac{x}{x^2 + 6x + 11} \right).$$

Before integrating, we wish to rewrite $x/(x^2 + 6x + 11)$ in a different form. Recall that we completed the square to find that $x^2 + 6x + 11 = (x + 3)^2 + 2$. Thus,

$$\frac{x}{x^2 + 6x + 11} \;=\; \frac{x}{(x + 3)^2 + 2} \;=\; \frac{(x + 3) - 3}{(x + 3)^2 + 2},$$

where we added and subtracted a 3 in the numerator in order to have the same quantities in parentheses in the numerator and denominator. We shall see momentarily why this is useful.

Putting together all of our work above, we have

$$\frac{x + 1}{(x - 5)(x^2 + 6x + 11)} \;=\; \frac{1}{11} \left[\frac{1}{x - 5} \;-\; \frac{x + 3}{(x + 3)^2 + 2} \;+\; 3 \cdot \frac{1}{(x + 3)^2 + 2} \right].$$

This, finally, is the partial fractions decomposition that we need. We now discuss how to integrate the main "pieces" of the righthand side above, and will then put these pieces together to obtain the final answer.

Certainly, the integral of $1/(x-5)$ is easy. By making the substitution $u = x-5$, we find quickly that

$$\int \frac{1}{x-5}\,dx \;=\; \ln|x-5| + C_1.$$

By making the substitution $w = (x+3)^2+2$, we find that $dw = 2(x+3)\,dx$, and so $dw/2 = (x+3)\,dx$. Hence,

$$\int \frac{x+3}{(x+3)^2+2}\,dx \;=\; \int \frac{1}{2}\cdot\frac{dw}{w} \;=\; \frac{1}{2}\ln|w| + C_2 \;=\; \frac{1}{2}\ln\big((x+3)^2+2\big) + C_2,$$

where the absolute value signs disappeared since $(x+3)^2+2 \geq 0$.

If we let $v = x+3$, then $dv = dx$, and we find

$$\int \frac{1}{(x+3)^2+2}\,dx \;=\; \int \frac{1}{v^2+(\sqrt{2})^2}\,dv,$$

which, by the second formula in Theorem 1.1.18, is equal to

$$\frac{1}{\sqrt{2}}\cdot\tan^{-1}\left(\frac{v}{\sqrt{2}}\right) + C_3 \;=\; \frac{1}{\sqrt{2}}\cdot\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C_3.$$

Combining all of our work above, we, at long last, conclude that

$$\int \frac{x+1}{(x-5)(x^2+6x+11)}\,dx \;=$$

$$\frac{1}{11}\left[\ln|x-5| \;-\; \frac{1}{2}\ln\big((x+3)^2+2\big) \;+\; \frac{3}{\sqrt{2}}\cdot\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)\right] + C.$$

Example 1.3.5. As our final example of integration by partial fraction, we will calculate the integral

$$\int \frac{1}{(x+2)(x-1)^3(x^2+4)}\,dx.$$

Note that the denominator of the integrand has a *repeated linear factor*, $(x-1)^3$. This factor is, of course, a polynomial of degree 3, but it is important that it is a repeated linear term. As you would expect, we want to find the partial fractions decomposition:

$$\frac{1}{(x+2)(x-1)^3(x^2+4)} = \frac{A}{x+2} + \frac{?}{(x-1)^3} + \frac{Bx+C}{x^2+4},$$

where x^2+4 is clearly an irreducible quadratic polynomial, and the question mark should be a general polynomial of degree 2, one degree less than the denominator. The slightly tricky point is that, instead of writing the polynomial in the numerator of the $(x-1)^3$ term in powers of x, it is best to write it in powers on $(x-1)$. Thus, we replace the question mark with $D(x-1)^2 + E(x-1) + F$. This yields a summand

$$\frac{D(x-1)^2 + E(x-1) + F}{(x-1)^3} = \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{F}{(x-1)^3}.$$

This leads to the general rule that, when looking for the partial fractions decomposition of a rational function, when there are repeated linear factors in the denominator, you have summands of the form a constant over each repeated power of the repeated linear factor, up to the power that appears in the original denominator. In a similar fashion, if we have repeated powers of an irreducible quadratic factor in the denominator, we have summands of the form a general linear polynomial over each repeated power of the repeated irreducible quadratic factor, up to the power that appears in the original denominator.

Hence, we want to find constants A through F such that

$$\begin{aligned} \frac{1}{(x+2)(x-1)^3(x^2+4)} &= \\ \frac{A}{x+2} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{F}{(x-1)^3} + \frac{Bx+C}{x^2+4}&. \end{aligned} \tag{1.8}$$

However, even before finding the constants, we can easily write the integral of this rational function, with the unknown constants still to be determined. You should be able to show quickly that

$$\begin{aligned} \int \left(\frac{A}{x+2} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{F}{(x-1)^3} + \frac{Bx}{x^2+4} + \frac{C}{x^2+4} \right) dx &= \\ A\ln|x+2| \ + \ D\ln|x-1| \ - \ \frac{E}{x-1} \ - \ \frac{F}{2(x-1)^2} \ +& \\ \frac{B}{2}\ln(x^2+4) + \frac{C}{2}\tan^{-1}\left(\frac{x}{2}\right) + K,& \end{aligned} \tag{1.9}$$

where K is an arbitrary constant (which cannot be determined, unlike A through F).

We need to find the constants A through F to finish the problem. You clear all the denominators in Formula 1.8 by multiplying both sides by the big denominator on the left, and obtain

$$1 = A(x-1)^3(x^2+4) + D(x+2)(x-1)^2(x^2+4) +$$

$$E(x+2)(x-1)(x^2+4) + F(x+2)(x^2+4) + (Bx+C)(x+2)(x-1)^3.$$

What a mess! Still A and F are easy to find; plug in $x = -2$ and $x = 1$. We find $1 = A(-2-1)^3((-2)^2+4)$ so that $A = -1/216$, and $1 = F(1+2)(1^2+4)$, so that $F = 1/15$. To find the remaining constants in the partial fractions decomposition, we leave it as an exercise (a horrible exercise) for you to expand the terms, collect the powers of x, match the coefficients, and solve the resulting simultaneous equations. Alternatively, many calculators, computer programs, and the web site wolframalpha.com can produce partial fractions decompositions. What you should find is that $B = -9/1000$, $C = 13/500$, $D = 46/3375$, and $E = -11/225$. Now, you substitute the values of A through F into Formula 1.9, and you're finished.

As we saw in the examples above, there are always two major steps in using partial fractions to integrate a rational function $f(x)$:

- Find the partial fractions decomposition of $f(x)$.
- Integrate each summand in the partial fractions decomposition.

Of course, we have yet to state exactly what the partial fractions decomposition for an arbitrary rational function is. We shall do so now.

Suppose $n(x)$ and $q(x)$ are (real) polynomial functions, and that the leading coefficient of $q(x)$ is a 1, i.e., $q(x)$ is not the zero function, and the coefficient of the largest power of x appearing in $q(x)$, with a non-zero coefficient, is a 1. (If the leading coefficient of $q(x)$ were some other (non-zero) constant c, immediately factor out the c, follow the procedure given below, and then, in the end, multiply by $1/c$.)

We want to define the partial fractions decomposition of the rational function $f(x) = n(x)/q(x)$, whose domain is all x such that $q(x) \neq 0$.

1. If the degree of $n(x)$ is greater than, or equal to, the degree of $q(x)$, then you first (long) divide $q(x)$ into $n(x)$; if what you obtain from the division is a polynomial $p(x)$ and a remainder of $r(x)$, then exactly what that means is that

$$f(x) = \frac{n(x)}{q(x)} = p(x) + \frac{r(x)}{q(x)},$$

where $r(x) = 0$ or has degree less than the degree of $q(x)$. As it is easy to integrate the polynomial portion $p(x)$, we will concentrate on the portion $r(x)/q(x)$. In addition, if $r(x) = 0$, then there is nothing left to do, so we will assume that $r(x) \neq 0$.

So, at this point, we want to define and integrate the partial fractions decomposition of $r(x)/q(x)$, where $r(x)$ and $q(x)$ are polynomials, $r(x) \neq 0$, $q(x) \neq 0$, the degree of $r(x)$ is strictly less than the degree of $q(x)$, and the leading coefficient of $q(x)$ equals 1.

2. Factor $q(x)$ into, possibly repeated, linear terms, each of the form $(x-r)^m$, and, possibly repeated, irreducible quadratic terms, which, after completing the square, can each be written in the form $[(x-r)^2 + b^2]^m$.

3. $r(x)/q(x)$ is equal to a sum of contributions, where you have a contribution from each, possibly repeated, irreducible factor of $q(x)$. For a factor of the form $(x-r)^m$, the contribution to the partial fractions decomposition is a sum of the form

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m},$$

where all of the A_i's are constants (to be determined).

For a factor of the form $[(x-r)^2 + b^2]^m$, the contribution to the partial fractions decomposition is a sum of the form

$$\frac{B_1(x-r) + C_1}{(x-r)^2 + b^2} + \frac{B_2(x-r) + C_2}{[(x-r)^2 + b^2]^2} + \cdots + \frac{B_m(x-r) + C_m}{[(x-r)^2 + b^2]^m},$$

where all of the B_i's and C_i's are constants (to be determined).

4. Now, set $r(x)/q(x)$ equal to the sum of all of the contributions described above, and clear the denominators by multiplying both sides of the equation by $q(x)$, and canceling factors. The result is that the polynomial $r(x)$ equals a sum of polynomials with (possibly many) unknown constants; these polynomial functions must be equal for all x.

5. Determine the unknown constants above by plugging in exceptional values (roots of $q(x)$) for x, or by multiplying out the entire righthand side, gathering the powers of x together, and matching coefficients with the coefficients of $r(x)$. A combination of these two methods may be easiest.

6. After determining all of the constants, the sum of the contributions described above, with the all of the values of the A_i's, B_i's, and C_i's inserted is the *partial fractions decomposition* of $r(x)/q(x)$.

7. The individual summands in the partial fractions decomposition are integrated as follows. We will omit the arbitrary constants produced by the indefinite integral.

The integrals with powers of a linear factor are easy:

$$\int \frac{A}{x-r}\,dx = A\ln|x-r|,$$

and if $m \geq 2$,

$$\int \frac{A}{(x-r)^m}\,dx = \frac{A(x-r)^{-m+1}}{-m+1}$$

The integrals with powers of an irreducible quadratic term are harder:

$$\int \frac{B(x-r)+C}{[(x-r)^2+b^2]^m}\,dx = \int \frac{B(x-r)}{[(x-r)^2+b^2]^m}\,dx + \int \frac{C}{[(x-r)^2+b^2]^m}\,dx.$$

In the first integral, the substitution $u = (x-r)^2 + b^2$ yields $du = 2(x-r)\,dx$, so that we have

$$\int \frac{B(x-r)}{[(x-r)^2+b^2]^m}\,dx = \frac{B}{2}\int \frac{du}{u^m},$$

which, if $m = 1$, is

$$\frac{B}{2}\ln|u| = \frac{B}{2}\ln((x-r)^2+b^2)$$

and, if $m \geq 2$, is

$$\frac{B}{2}\cdot\frac{u^{-m+1}}{-m+1} = \frac{B}{2}\cdot\frac{((x-r)^2+b^2)^{-m+1}}{-m+1}.$$

Finally, we need to integrate terms of the form

$$\int \frac{C}{[(x-r)^2+b^2]^m}\,dx.$$

After pulling the constant C outside the integral, and making the trivial substitution $u = x - r$, we are reduced to an integral of the form in Example 1.2.8; see that example for the technique.

1.3.1 Exercises

Use partial fractions to calculate the anti-derivatives in Exercises 1-25.

1. $\displaystyle\int \frac{11x+26}{x^2+5x+6}\,dx$

2. $\displaystyle\int \frac{t+58}{t^2+6t-16}\,dt$

3. $\displaystyle\int \frac{8u-35}{u^2-7u}\,du$

4. $\displaystyle\int \frac{-42}{y^2-49}\,dy$

5. $\displaystyle\int \frac{11z-2}{2z^2-z-1}\,dz$

6. $\displaystyle\int \frac{2w^3+6w^2-4w+11}{w^2+3w-4}\,dw$

7. $\displaystyle\int \frac{p^2+p+3}{p+3}\,dp$

8. $\displaystyle\int \frac{-2\theta^3+4\theta^2+6\theta-2}{\theta^2-1}\,d\theta$

9. $\displaystyle\int \frac{x^3+12x^2+51x+74}{x^2+8x+15}\,dx$

10. $\displaystyle\int \frac{s^3+7s^2+s+10}{s+7}\,ds$

11. $\displaystyle\int \frac{7m^2+12m+8}{(m+3)(m^2+m+1)}\,dm$

12. $\displaystyle\int \frac{6n^2-4n+20}{(n-2)(n^2+5)}\,dn$

13. $\displaystyle\int \frac{-3v^2+4v-66}{(v-6)(v^2-4v+18)}\,dv$

14. $\displaystyle\int \frac{3r}{r^4+5r^2+4}\,dr$

15. $\displaystyle\int \frac{12t^2+52t+150}{(t^2+5t+30)(2t+3)}\,dt$

16. $\displaystyle\int \frac{3x^2+8x+5}{(x^2+2x+5)(x+3)}\,dx$

17. $\displaystyle\int \frac{-2y^2-60}{(y^2+4y+20)(y+12)}\,dy$

18. $\displaystyle\int \frac{1-2w}{(w-1)(w^2+w+1)}\,dw$

19. $\displaystyle\int \frac{-3\phi^2-30}{(\phi^2+\phi+5)(\phi+2)}\,d\phi$

20. $\displaystyle\int \frac{4s^2+4s+6}{(s^2-s+3)(s+3)}\,ds$

21. $\displaystyle\int \frac{5u^2-3u-29}{(u+3)(u-2)^2}\,du$

22. $\displaystyle\int \frac{-4t^2-10t-14}{(t+1)^2(t-1)}\,dt$

23. $\displaystyle\int \frac{-3x^2-17x-26}{(x+3)^2(x+2)}\,dx$

24. $\displaystyle\int \frac{2j^2+18j+43}{(j+5)^3}\,dj$

25. $\displaystyle\int \frac{7r^3-2r^2+9r-22}{(r+3)(r-1)^3}\,dr$

26. Prove that $\displaystyle\int \frac{(A+C)x+AD+BC}{x^2+(B+D)x+BD}\,dx = A\ln|x+B|+C\ln|x+D|+K$ where A, B, C, D, and K are constants and neither A nor C is zero.

27. Suppose A, B, C and D are non-zero constants and B and D are both positive. What is $\displaystyle\int \frac{(A+C)x^2+AD+BC}{x^4+(B+D)x^2+BD}\,dx$?

Solve the separable differential equations in Exercises 28 - 32. If an initial condition is given, solve for the integration constant.

28. $\dfrac{dy}{dx} = \dfrac{7x^2+44x+80}{(x+5)^2(x-2)}$, $y(3)=5/8$.

29. $\dfrac{dy}{dx} = \dfrac{2x^3-3x^2-32x-23}{x^2-2x-15}$, $y(6)=42$.

30. $\dfrac{dy}{dx} = \dfrac{(6x^3-17x^2-27x+16)(y^2+2y+10)(y+3)}{(x-3)^3(x+5)(5y^2+16y-10)}$.

31. $\dfrac{dy}{dx} = \dfrac{(-x-2)(y^3+9y)}{(3y^2+2y+27)(x^2+11x+30)}$.

32. $\dfrac{dy}{dx} = \dfrac{(3x^3-23x^2+53x-32)(y^2-3y+10)(y+4)}{(x^2-6x+8)(3y^2+8y+22)}$.

33. Recall that the logistic model for population growth is given by

$$\frac{dP}{dt} = kP(M-P)$$

where M and k are non-zero constants and $P=P(t)$ is the population at time t. Suppose $P_0=P(0)$ is the initial population. Use the fact that this differential equation is separable and integrate using partial fractions to arrive at the solution to the differential equation:

$$P = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0}.$$

Integrals involving square-roots can often be solved by a technique involving partial fractions and a *rationalizing substitution.* The idea is to make a substitution that allows the integrand to be written as a rational function, without a square-root symbol, and then to integrate using partial fractions. The method is outlined in Exercises 34. Use the method to calculate the anti-derivatives in Exercises 35 -40.

34. Consider $\int \frac{dx}{x\sqrt{x+1}}$.

    a. Let $u = \sqrt{x+1}$. Show that the integral can than be written as $\int \frac{2\,du}{u^2-1}$.

    b. Use partial fractions to conclude that the integral in part (a) is $\ln|u-1| - \ln|u+1| + C$.

    c. Rewrite the anti-derivative in terms of x, thus establishing: $\int \frac{dx}{x\sqrt{x+1}} = \ln|\sqrt{1+x} - 1| - \ln|\sqrt{1+x}+1| + C$.

35. $\int \frac{\sqrt{x}}{x-9}\,dx$.

36. $\int \frac{dx}{x - \sqrt{x+30}}$.

37. $\int \frac{dx}{\sqrt[3]{x} - 1}$.

38. $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$.

39. $\int \frac{x}{(x+2)(\sqrt{x+3})}\,dx$.

40. $\int \frac{dx}{a - \sqrt{x+b}}$ where a and b are positive constants.

A common application of partial fraction integration is the reaction rate of two chemicals that combine to form a third chemical. This is called a *bimolecular reaction* and is justified by the *law of mass action.*

Specifically, assume chemicals A and B have initial concentrations of a and b in mols per unit volume and react to form chemical C. Then under certain conditions, the rate of increase in the concentration, c, of chemical C is given by:

$$\frac{dc}{dt} = k(a-c)(b-c)$$

where k is some constant. Assume throughout the next four problems that $c < a < b$.

41. Find an explicit solution for the concentration of chemical C, $c(t)$, if $a = 3$, $b = 4$, $k = 2$ and $c(0) = 0$.

42. Find an explicit solution for the concentration of chemical C, $c(t)$, if $a = 8$, $b = 10$, $k = 1$ and $c(0) = 0$.

43. What is the concentration of chemical C after 15 seconds if $a = 2$, $b = 5$, $k = 4.5$ and $c(0) = 0$? Assume that t is measured in minutes.

44. Assume that $a \neq b$ and prove that a general solution to the differential equation is

$$c(t) = \frac{ab(1 - e^{(a-b)kt})}{b - ae^{(a-b)kt}}.$$

Assume that $c(0) = 0$.

The 18th century mathematician Leonhard Euler developed a quick way of determining the coefficients of partial fraction decompositions of rational functions. Let $P(x)/Q(x)$ is a rational function and assume $a+bx$ is a factor of $Q(x)$, but that $(a+bx)^2$ is not a factor of $Q(x)$. Then, $\dfrac{P(x)}{Q(x)} = \dfrac{k}{a+bx} + R(x)$, and we'd like to determine k.

Multiplying by $a + bx$ and using l'Hôpital's Rule, Euler proved that

$$k = \lim_{x \to -a/b} \frac{bP(x)}{Q'(x)}.$$

Use this method to answer Exercises 45-49.

45. We considered the partial fraction decomposition of $\dfrac{x+1}{(x-5)(x^2+6x+11)}$ in Example 1.3.4. The coefficient A of the term $\dfrac{A}{x-5}$ was found to be $1/11$. Let's use Euler's method to find this term.

    a. Show that $a = -5$, $b = 1$ and that $Q'(x) = 3x^2 + 2x - 19$.

b. Show that based on Euler's method, $A = \lim_{x\to 5} \dfrac{x+1}{3x^2+2x-10}$.

c. Show that this limit is indeed 1/11.

46. What is the coefficient of $x+1$ in the partial fraction decomposition of $\dfrac{x^{13}}{x^{23}+1}$?

47. Determine A, B and C in the equation

$$\frac{x^2+3x+5}{(x+2)(x+3)(x+5)} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{x+5}$$

using Euler's method. Note that $(x+2)(x+3)(x+5) = x^3+10x^2+31x+30$.

48. Use Euler's method to find the partial fraction decomposition of $\dfrac{3x^3+4x^2+4x+5}{x^4-1}$. Hint: use Euler to find the $A/(x-1)$ term. Subtract this term from the original function to find the remaining term.

49. Let m and n be positive integers. Let $f(x) = \dfrac{x^m+(2n-1)x}{1-x^{2n}}$. What is the coefficient A of the term $A/(x-1)$ in the partial fraction decomposition?

50. Consider $\displaystyle\int \frac{1}{x^4+1}\,dx$. According to Courant, "Even Leibnitz found this a troublesome integration."

a. Complete the square to show that $x^4+1 = (x^2+1)^2 - 2x^2$.

b. Notice that the expression in (a) is the difference of squares. Use this fact to prove $x^4+1 = (x^2+1+\sqrt{2}x)(x^2+1-\sqrt{2}x)$.

c. Use the methods in this chapter to prove that

$$\frac{1}{x^4+1} = K\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right),$$

where $K = \sqrt{2}/4$.

d. Finally, conclude that

$$\int \frac{dx}{x^4+1} =$$

$$K\Big(\tfrac{1}{2}\ln|x^2+\sqrt{2}x+1| - \tfrac{1}{2}\ln|x^2-\sqrt{2}x+1| + \tan^{-1}(\sqrt{2}x+1) + \tan^{-1}(\sqrt{2}x-1)\Big) + C.$$

1.4 Integration using Hyperbolic Sine and Cosine

Corresponding to sine and cosine, we will define the *hyperbolic sine* and *hyperbolic cosine* functions, $f(x) = \sinh x$ and $g(x) = \cosh x$, respectively; when speaking, hyperbolic cosine is usually referred to by pronouncing *cosh* phonetically, while hyperbolic sine is usually referred to by pronouncing *sinh* as though it were the word *cinch*.

Once we have hyperbolic sine and cosine, we could, but will not, define the remaining four hyperbolic trig. functions, in analogy with how the other trig. functions are defined in terms of sine and cosine, e.g., hyperbolic tangent is hyperbolic sine divided by hyperbolic cosine.

Instead, we will concentrate on the properties of sinh and cosh which make them so useful for anti-differentiating certain types of functions.

The definitions of hyperbolic sine and cosine do not look analogous in any way to the definitions of the usual sine and cosine functions. Later, in Example 4.6.21, we will see that *Euler's Formula* provides a close relationship between the hyperbolic and standard trig. functions, but you have to be willing to deal with complex numbers.

Definition 1.4.1. *The* **hyperbolic sine function**, $\sinh : \mathbb{R} \to \mathbb{R}$, *is defined by*

$$\sinh x \;=\; \frac{e^x - e^{-x}}{2},$$

and the **hyperbolic cosine function**, $\cosh : \mathbb{R} \to \mathbb{R}$, *is defined by*

$$\cosh x \;=\; \frac{e^x + e^{-x}}{2}.$$

The following algebraic/graphical/Calculus properties of sinh and cosh are easy to verify, and we leave them as exercises.

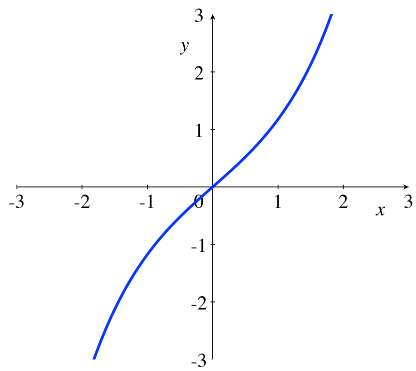


Figure 1.1: The graph of $y = \sinh x$.

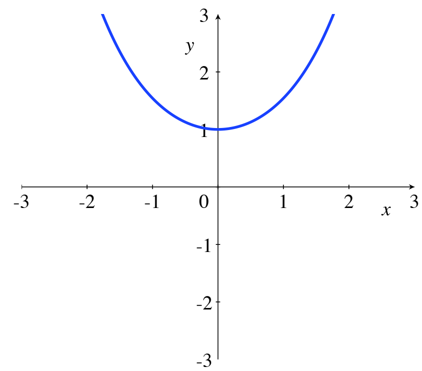


Figure 1.2: The graph of $y = \cosh x$.

Proposition 1.4.2. *Hyperbolic sine and cosine have the following properties:*

1. $\sinh x$ *is an odd function;*
2. $\cosh x$ *is an even function;*
3. $\sinh x$ *is one-to-one and its range is the entire real line;*
4. $\cosh x$ *is not one-to-one, but is one-to-one when restricted to the domain* $[0, \infty)$*, and its range is the interval* $[1, \infty)$*;*
5. $\sinh x$ *is strictly increasing, is negative when* x *is negative, and positive when* x *is positive;*
6. $\cosh x$ *is strictly decreasing on the interval* $(-\infty, 0]$*, strictly increasing on the interval* $[0, \infty)$*, and obtains its global minimum value of* 1 *when* $x = 0$*;*
7. $\sinh' x = \cosh x$*;*
8. $\cosh' x = \sinh x$*;*
9. $1 + \sinh^2 x = \cosh^2 x$*.*

The last property above is where the term *hyperbolic* comes from; a point $(x, y) = (\cosh t, \sinh t)$ lies on the hyperbola given by $x^2 - y^2 = 1$. You should think of this as being analogous to the fact that a point $(x, y) = (\cos t, \sin t)$ lies on the circle given by $x^2 + y^2 = 1$; hence, the "usual" trigonometric functions are sometimes referred to as the *circular* trigonometric functions.

As hyperbolic sine is one-to-one, onto, and possesses an everywhere non-zero derivative, the inverse function $\sinh^{-1}$ exists and is differentiable everywhere. However, as cosh is not one-to-one, we restrict its domain to $[0, \infty)$ before producing an inverse function. We can find explicit formulas for these inverse functions by writing $x = (e^y - e^{-y})/2$ and $x = (e^y + e^{-y})/2$, respectively, and solving for y in each case, while paying attention to the restrictions on the domain and codomain of $\cosh^{-1}$. The algebra involved is simply the quadratic formula, and applying the natural logarithm, but the "variable" to which you apply the quadratic formula is e^y; we leave this algebra as an exercise also.

You should find the following formulas for *inverse hyperbolic sine* and *inverse hyperbolic cosine.*

Proposition 1.4.3. *The function* $\sinh^{-1} : \mathbb{R} \to \mathbb{R}$ *is given by*

$$\sinh^{-1} x \;=\; \ln\left(x + \sqrt{x^2+1}\right),$$

and the function $\cosh^{-1} : [1, \infty) \to [0, \infty)$ *is given by*

$$\cosh^{-1} x \;=\; \ln\left(x + \sqrt{x^2-1}\right).$$

By using either the algebraic formulas above, or the general formula for the derivative of an inverse function, we find (via another exercise for you):

Proposition 1.4.4. *The function* $\sinh^{-1} : \mathbb{R} \to \mathbb{R}$ *is differentiable (everywhere), and*

$$(\sinh^{-1} x)' \;=\; \frac{1}{\sqrt{x^2+1}}.$$

The function $\cosh^{-1} : [1, \infty) \to [0, \infty)$ *is continuous; for all* $x > 1$*,* $\cosh^{-1}$ *is differentiable, and*

$$(\cosh^{-1} x)' \;=\; \frac{1}{\sqrt{x^2-1}}.$$

We **could** immediately rewrite the formulas above as anti-derivative formulas, but we want to emphasize that the real value of hyperbolic sine and cosine when anti-differentiating stems from three properties: that $1 + \sinh^2 x = \cosh^2 x$, $\sinh' x = \cosh x$, and $\cosh' x = \sinh x$.

Example 1.4.5. Suppose that you don't remember the derivatives of inverse hyperbolic sine and cosine that we gave above, but you need to find the anti-derivative

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx,$$

where $a > 0$ is a constant.

When you see $\sqrt{x^2+a^2}$, you should realize that there are (at least) two reasonable choices for methods of attack; these rely on your knowing that $1+\tan^2\theta = \sec^2\theta$ and that $1+\sinh^2 t = \cosh^2 t$.

We will first use $\tan\theta$ and see that the calculation takes a while. We will then look at how much easier the integral is using $\sinh t$.

Using circular trig. functions:

Using tangent, you should think "I want x to be $a\tan\theta$, so that $x^2+a^2 = a^2\tan^2\theta + a^2 = a^2(\tan^2\theta+1) = a^2\sec^2\theta$", and then $\sqrt{x^2+a^2} = a\sec\theta$. That's how you think about the problem, but, being more careful, what you really do is not "let $x = a\tan\theta$"; after all, x is already defined in the problem. You define the new variable θ by letting $\theta = \tan^{-1}(x/a)$, so that, yes, $x = a\tan\theta$, but now we also know that $-\pi/2 < \theta < \pi/2$. This restriction on θ is important, because, what we really get is that

$$\sqrt{x^2+a^2} \;=\; \sqrt{a^2\sec^2\theta} \;=\; |a\sec\theta|.$$

To know that this last quantity is equal to $a\sec\theta$, you have to use that $a > 0$ and that, since $-\pi/2 < \theta < \pi/2$, $\sec\theta > 0$.

Therefore, since $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$ and

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx \;=\; \int \frac{1}{a\sec\theta}\,a\sec^2\theta\,d\theta \;=\; \int \sec\theta\,d\theta \;=$$

$$\ln|\sec\theta+\tan\theta| + C \;=\; \ln(\sec\theta+\tan\theta) + C,$$

where, to eliminate the absolute value signs, we used that our restrictions on θ imply that $\sec\theta + \tan\theta \geq 0$. To finish the problem, you now use that $\tan\theta = x/a$ and that $\sec\theta = \sqrt{x^2+a^2}/a$.

Our final result:

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx \;=\; \ln\left(\frac{x}{a}+\frac{\sqrt{x^2+a^2}}{a}\right)+C \;=\; \ln\left(x+\sqrt{x^2+a^2}\right)+C,$$

where, in the last step, we used that $\ln\left[\left(x+\sqrt{x^2+a^2}\right)/a\right] = \ln\left(x+\sqrt{x^2+a^2}\right) - \ln a$, and then we "absorbed" the $-\ln a$ into the constant C.

Using hyperbolic trig. functions:

Using hyperbolic sine, you should think "I want x to be $a\sinh t$, so that $x^2+a^2 = a^2\sinh^2 t + a^2 = a^2(\sinh^2 t+1) = a^2\cosh^2 t$", and then $\sqrt{x^2+a^2} = a\cosh t$. Again, that's how you think about the problem, but what you really do is not "let $x = a\sinh t$"; you define the new variable t by letting $t = \sinh^{-1}(x/a)$. However, we don't need any special restrictions on t to know that $\sqrt{a^2\cosh^2 t} = a\cosh t$, because $\cosh t \geq 1 \geq 0$.

As $x = a\sinh t$, $dx = a\cosh t\,dt$, and we find

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx \;=\; \int \frac{1}{a\cosh t}\,a\cosh t\,dt \;=\; \int 1\,dt \;=\; t+C \;=\; \sinh^{-1}\left(\frac{x}{a}\right)+C,$$

which is the nicest form in which to leave the answer. On the other hand, if you wish to see that this answer agrees with the answer that we obtained above, using tangent, then you use the formula for $\sinh^{-1}$ in Proposition 1.4.3.

Example 1.4.6. Suppose that $a > 0$, and that we want the general anti-derivative of the function $f(x) = \dfrac{1}{\sqrt{x^2-a^2}}$ with domain $x > a$, i.e., we want to find

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx,$$

where $x > a > 0$.

We want $x = a\cosh t$, so that

$$\sqrt{x^2-a^2} \;=\; \sqrt{a^2\cosh^2 t - a^2} \;=\; a\sqrt{\sinh^2 t} \;=\; a|\sinh t|.$$

Thus, we let $t = \cosh^{-1}(x/a)$, which implies both that $x = a\cosh t$ and, as $x/a > 1$, that $t > 0$. Since $t > 0$, $|\sinh t| = \sinh t$.

Therefore, $dx = a \sinh t\, dt$ and

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx \;=\; \int \frac{1}{a\sinh t}\, a\sinh t\, dt \;=\; \int 1\, dt \;=\; t + C \;=\; \cosh^{-1}\left(\frac{x}{a}\right) + C.$$

The results from the last two examples are worth recording in a proposition.

Proposition 1.4.7. *Suppose that $a > 0$. Then,*

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx \;=\; \sinh^{-1}\left(\frac{x}{a}\right) + C,$$

and, for $x > a$,

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx \;=\; \cosh^{-1}\left(\frac{x}{a}\right) + C.$$

1.4.1 Exercises

Calculate the anti-derivatives of the functions.

1. $\displaystyle\int \frac{7}{\sqrt{x^2 + 9}}\, dx.$
2. $\displaystyle\int \frac{12}{\sqrt{z^2 - 625}}\, dz.$
3. $\displaystyle\int \frac{dy}{\sqrt{4y^2 + 49}}.$
4. $\displaystyle\int \frac{dt}{\sqrt{7t^2 - 81}}.$
5. $\displaystyle\int \cosh(3x)\, dx.$

6. $\displaystyle\int s \sinh s \, ds.$

7. $\displaystyle\int \frac{2x}{\sqrt{x^4+9}} \, dx.$

8. $\displaystyle\int \cosh^2 z - \sinh^2 z \, dz.$ Hint: why is this trivial?

9. $\displaystyle\int \frac{dx}{\sqrt{4x^2+4x+9}}.$

10. $\displaystyle\int \frac{dy}{\sqrt{y^2-10y+9}}.$

11. $\displaystyle\int \frac{1+\sqrt{t^3+3t}}{\sqrt{t^2+3}} \, dt,\ t > 0.$

12. $\displaystyle\int \frac{\sqrt{u^2-4}+\sqrt{u^2-4}}{\sqrt{u^4-16}} \, du.$

13. $\displaystyle\int \frac{\sqrt{x^2-4}+\sqrt{x^2+9}}{\sqrt{x^4+5x^2-36}} \, dx.$

14. $\displaystyle\int \frac{u^2}{(u^2+10)^{3/2}} \, du.$

15.

   a. Calculate $\displaystyle\int \sinh x \cosh x \, dx$ using integration by parts.

   b. Calculate $\displaystyle\int \sinh x \cosh x \, dx$ directly by writing $\sinh x$ and $\cosh x$ in terms of their definitions using the exponential function.

16. Calculate $\displaystyle\int \sqrt{x^2+1} \, dx.$

In each of Exercises 17 through 25, prove the given statements from Proposition 1.4.2.

17. $\sinh x$ is an odd function.

18. $\cosh x$ is an even function.

19. $\sinh x$ is one-to-one and its range is the entire real line.

20. $\cosh x$ is not one-to-one, but is one-to-one when restricted to the domain $[0, \infty)$, and its range is the interval $[1, \infty)$.

21. $\sinh x$ is strictly increasing, is negative when x is negative, and positive when x is positive.

22. $\cosh x$ is strictly decreasing on the interval $(-\infty, 0]$, strictly increasing on the interval $[0, \infty)$, and obtains its global minimum value of 1 when $x = 0$.

23. $\sinh' x = \cosh x$.

24. $\cosh' x = \sinh x$.

25. $1 + \sinh^2 x = \cosh^2 x$.

26. What is $\int \sqrt{x^2 - 1}\, dx$?

27. What is $\int x^2 \sqrt{x^2 - 1}\, dx$?

28. What is $\int x^2 \sqrt{x^2 + 1}\, dx$?

29. More generally, what is $\int x^2 \sqrt{x^2 + a^2}\, dx$?

Use the previous four problems to calculate the integrals in the next four problems.

30. $\int \sqrt{9x^2 - 16}\, dx$.

31. $\int 25x^2 \sqrt{25x^2 - 1}\, dx$.

32. $\int \sqrt{x^6 + 100x^4}\, dx$.

33. $\int cx^2 \sqrt{b^2x^2 + a^2}\, dx$, $b > 0$, $a > 0$.

In exercises 34 though 37, you are given the acceleration function of a particle and the velocity at one specific time. Find the velocity function.

34. $a(t) = \dfrac{1}{\sqrt{t^2 + 16t + 25}}$, $v(0) = 10$. Assume that $t > -8 + \sqrt{39}$.

35. $a(t) = \dfrac{1}{\sqrt{t^2 - 6t + 16}}$, $v(6) = 8$.

36. $a(t) = \dfrac{12}{\sqrt{6t^2 + 24}}$, $v(3) = 6$.

37. $a(t) = 2t \cosh(t^2 + 2)$, $v(0) = 4$.

38. What is $\int \sinh^{-1} x\,dx$? Hint: use a technique similar to the one used in calculating $\int \ln x\,dx$.

39. What is $\int \cosh^{-1} x\,dx$?

In the next four exercises you are given the acceleration function of a particle. Find a formula for the position vector. Make sure to retain any integration constants. Use the previous two problems.

40. $a(t) = \dfrac{1}{\sqrt{t^2+9}}$.

41. $a(t) = \dfrac{1}{\sqrt{t^2-16}}$.

42. $a(t) = \dfrac{5}{\sqrt{4t^2+25}}$, $v(0) = 0$.

43. $a(t) = \dfrac{6}{\sqrt{9t^2-11}}$, $v(2) = 8$.

Solve the following separable differential equations.

44. $\dfrac{dx}{dt} = \dfrac{\sqrt{12x^2-8}}{4+6t^2}$.

45. $\dfrac{dx}{dt} = \dfrac{(x^2+1)^{3/2}}{(\sqrt{t^2-3})}$.

46. $\dfrac{dx}{dt} = \dfrac{e^{t(x+1)} - e^{t(x-1)}}{e^{x(t+1)} + e^{x(t-1)}}$.

47. $\dfrac{dx}{dt} = \sqrt{\dfrac{x^4+x^2+t^2+t^2x^2}{t^4-t^2-x^2+t^2x^2}}$. Hint: look for a factor common to the numerator and denominator.

Calculate the integrals of the following products of hyperbolic and trigonometric functions. You may find it easier write the hyperbolic functions in terms of exponential functions.

48. $\int \sinh x \cos x\,dx$.

49. $\int \cosh x \cos x\,dx$.

50. $\int \cosh x \sinh x \sin x \cos x\,dx$.

Chapter 2

Continuous Sums: the Definite Integral

As we wrote at the beginning of the previous chapter, the definite integral is the mathematically precise notion of what it means to "take a continuous sum of infinitesimal contributions." It is the definition of integration as a continuous sum that yields all of its applications, many of which we will look at in Chapter 3. However, the basic method of calculating definite integrals is to use the *Fundamental Theorem of Calculus*, Theorem 2.4.10, to conclude that the problem of calculation of an integral (typically) boils down to producing an anti-derivative. Thus, as we wrote in the previous chapter, anti-differentiation is also referred to as calculating an *indefinite integral*.

We begin this chapter with a discussion of basic properties, notations, and techniques involving summations; not surprisingly, differences are involved in crucial ways. We then move on to the problem of how to approximate a "continuous sum"; of course, part of the problem is that we don't (yet) have a mathematically rigorous definition of what a continuous sum is. After we have our approximations, we then use limits of our approximations to define the definite integral.

2.1 Sums and Differences

In this section, we need to recall some notation and properties related to summations. We shall also define the *difference operator*, and use it to derive some useful summation formulas.

Recall the *sigma notation* for summations.

Definition 2.1.1. *Suppose that we have two integers m and n, where $m \leq n$, we let $[m..n]$ denote the set of integers between, or equal to, m and n. We call such a set an* **integer interval**.

Suppose we have a function f, whose domain includes $[m..n]$. Then, we write $\sum_{k=m}^{n} f(k)$ for the **summation, as k goes from m to n, of $f(k)$**. *This means that*

$$\sum_{k=m}^{n} f(k) = f(m) + f(m+1) + \cdots + f(n-1) + f(n).$$

In this context k is frequently referred to as the **index of summation**, *and $f(k)$ is frequently written as f_k. The integer interval $[m..n]$ is called the* **range of the index of summation**.

For example,

$$\sum_{k=-1}^{3} k^2 \;=\; (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 \;=\; 15,$$

or, as another example,

$$p(x) \;=\; a_0 + a_1 x + \cdots + a_n x^n \;=\; \sum_{k=0}^{n} a_k x^k,$$

where x^0 in the summation is to be interpreted as equaling 1, even if $x = 0$.

Note that the indexing variable k is a *dummy variable*; if we replaced it with an i or j, or any other variable (which is not already present), the summation would not change, e.g.,

$$\sum_{j=-1}^{3} j^2 \;=\; (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 = 15,$$

Summations can be "split". For instance, in the example above,

$$\sum_{j=-1}^{3} j^2 \;=\; (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 \;=$$

$$\left[(-1)^2 + 0^2\right] \;+\; \left[1^2 + 2^2 + 3^2\right] \;=\; \sum_{j=-1}^{0} j^2 \;+\; \sum_{j=1}^{3} j^2.$$

In general, we have

Proposition 2.1.2. (**splitting summations**) *If m, n, and p are integers, with $m \le n \le p$, and f is a function whose domain includes the integer interval $[m..p]$, then*

$$\sum_{k=m}^{p} f_k \;=\; \sum_{k=m}^{n} f_k \;+\; \sum_{k=n+1}^{p} f_k,$$

where, if $n = p$, the last summation should be interpreted as being 0.

Looking once again at

$$\sum_{j=-1}^{3} j^2 \;=\; (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2,$$

we note that we can replace each term j^2 by $\big((j+6)-6\big)^2$ (which may seem like a silly thing to do, but it leads us to an important property); this means that we have the easy equality

$$(-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 \;=\; (6-7)^2 + (7-7)^2 + (8-7)^2 + (9-7)^2 + (10-7)^2,$$

which equals $\sum_{k=6}^{10}(k-7)^2$.

In general,

Proposition 2.1.3. (**shifting indices**) *Suppose we have a function f, whose domain includes the integer interval $[m..n]$. Let p be an integer.*
Then, by making the substitution $k = j + p$, so that $j = k - p$, we obtain

$$\sum_{j=m}^{n} f_j \;=\; \sum_{k=m+p}^{n+p} f_{k-p}.$$

Thus, you get the same sum if you add p to each of the bounds of the summation, and simultaneously replace each occurrence of the index k by k **minus** p.

Consider now

$$\sum_{k=7}^{9}\big(1.7k^2 - \sqrt{3}\,\sin k\big) \;=$$

$$\big(1.7(7)^2 - \sqrt{3}\,\sin(7)\big) \;+\; \big(1.7(8)^2 - \sqrt{3}\,\sin(8)\big) \;+\; \big(1.7(9)^2 - \sqrt{3}\,\sin(9)\big) \;=$$

$$1.7\big(7^2+8^2+9^2\big) \;+\; (-\sqrt{3})\big(\sin 7 + \sin 8 + \sin 9\big) \;=$$

$$1.7\sum_{k=7}^{9} k^2 \;+\; (-\sqrt{3})\sum_{k=7}^{9}\sin k.$$

More generally, the algebraic properties of real numbers (i.e., associativity, commutativity, and distributivity) immediately imply:

Proposition 2.1.4. (**linearity of summation**) *If a and b are constants, and f and g are functions whose domain includes the integer interval $[m..n]$, then*

$$\sum_{k=m}^{n}\big(af_k + bg_k\big) \;=\; a\sum_{k=m}^{n} f_k \;+\; b\sum_{k=m}^{n} g_k.$$

Example 2.1.5. The properties of summations described above may seem simple, but they allow us to derive formulas involving sums that really don't look so obvious. Consider, for instance, the problem of "simplifying"

$$2\sum_{k=3}^{50} k(k-1) \;-\; \sum_{k=1}^{50}(k+1)(k+2). \tag{2.1}$$

We would like to combine the two summations, using linearity; however, the ranges of the indices of summation would need to be the same, and they are not. We will fix this "problem" in two different ways. Understand that the point of this example is not that you will necessarily agree that what we end up with is simpler than what we started with; the point is for you to understand the types of manipulations that we use.

First approach: We split the second summation, and have

$$2\sum_{k=3}^{50} k(k-1) \;-\; \sum_{k=1}^{50}(k+1)(k+2) \;=$$

$$2\sum_{k=3}^{50} k(k-1) \;-\; \sum_{k=3}^{50}(k+1)(k+2) \;-\; \sum_{k=1}^{2}(k+1)(k+2).$$

Now we can use linearity on the first two of the three summations above to obtain

$$\sum_{k=3}^{50}\bigl[2k(k-1)-(k+1)(k+2)\bigr] \;-\; \sum_{k=1}^{2}(k+1)(k+2) \;=$$

$$\left[\sum_{k=3}^{50}\bigl(2k^2-2k-k^2-3k-2\bigr)\right] \;-\; 6 \;-\; 12 \;=$$

$$-18 \;+\; \sum_{k=3}^{50}\bigl(k^2-5k-2\bigr)$$

Second approach: We first shift the index in the second summation in Formula 2.1; we add 2 to each of the bounds and replace k by $(k-2)$ in the summation. We obtain

$$2\sum_{k=3}^{50} k(k-1) \;-\; \sum_{k=1}^{50}(k+1)(k+2) \;=$$

$$2\sum_{k=3}^{50} k(k-1) \;-\; \sum_{k=3}^{52} \big((k-2)+1\big)\big((k-2)+2\big) \;=$$

$$2\sum_{k=3}^{50} k(k-1) \;-\; \sum_{k=3}^{52} (k-1)k.$$

The range of the index of summation on the right above still does not match that of the summation on the left, but we can split off part of the sum

$$2\sum_{k=3}^{50} k(k-1) \;-\; \sum_{k=3}^{52} (k-1)k.$$

$$2\sum_{k=3}^{50} k(k-1) \;-\; \left[\sum_{k=3}^{50} k(k-1)\right] \;-\; (50)(51) \;-\; (51)(52) \;=$$

$$-\;(50)(51) \;-\; (51)(52) \;+\; \sum_{k=3}^{50} \big[2k(k-1) - k(k-1)\big] \;=$$

$$-\;(50)(51) \;-\; (51)(52) \;+\; \sum_{k=3}^{50} \big[k(k-1)\big],$$

which does not look very similar to our answer from the first approach, but is nonetheless equally correct.

We now want to define notation related to differences.

Definition 2.1.6. *Suppose that m and n are integers, and $m < n$, and suppose that we have a real function f, whose domain is $[m..n]$. Then, we define the* **finite difference** *function Δf to be the function with domain $[(m+1)..n]$ given by*

$$(\Delta f)(k) \;=\; f(k) - f(k-1).$$

Remark 2.1.7. We have been very formal with our notation above. Typically, we write things like $\Delta k^2 = k^2 - (k-1)^2 = k^2 - (k^2 - 2k + 1) = 2k - 1$, in place of writing that, if $f(k) = k^2$, then $(\Delta f)(k) = 2k - 1$.

We should also remark that, in the notation $[m..n]$, we mean to allow the cases where $m = -\infty$ and/or $n = \infty$. To be precise, we mean that, if m and n are integers, then

$$[-\infty..n] \;=\; \{k \mid k \text{ is an integer, and } k \leq n\},$$

$$[m..\infty] \;=\; \{k \mid k \text{ is an integer, and } m \leq k\},$$

and $[-\infty..\infty]$ is the entire set of integers.

Finally, in Definition 2.1.6, if $m = -\infty$, then the $m + 1$ which appears in the domain of Δf should also be taken to equal $-\infty$.

Like summations, the finite difference operator is linear.

Proposition 2.1.8. (**linearity of differences**) *If a and b are constants, and f and g are functions whose domain is $[m..n]$, then*

$$\Delta\big(af(k) + bg(k)\big) \;=\; a\,\Delta f(k) \;+\; b\,\Delta g(k).$$

Despite the fact that it is trivial to prove, the following result turns out to be very useful.

Proposition 2.1.9. (**telescoping sums**) *Suppose that m and n are integers, and $m < n$. If f is a real function, whose domain is $[m..n]$, then*

$$\sum_{k=m+1}^{n} \Delta f(k) \;=\; f(n) - f(m).$$

Proof. A rigorous proof would use mathematical induction. However, it is easy to see why this is true.

$$\sum_{k=m+1}^{n} \Delta f(k) =$$

$$\big(f(m+1) - f(m)\big) + \big(f(m+2) - f(m+1)\big) + \big(f(m+3) - f(m+2)\big) + \ldots$$

$$\ldots + \big(f(n-2) - f(n-3)\big) + \big(f(n-1) - f(n-2)\big) + \big(f(n) - f(n-1)\big).$$

Note that every term, except $f(n)$ and $f(m)$, occurs once with a plus sign and once with a minus sign. Thus, all those intermediate terms "collapse" to 0. The result follows. □

Proposition 2.1.10. *Suppose that b is a constant. We have the following formulas for finite differences:*

1. $\Delta k = 1$;
2. $\Delta k^2 = 2k - 1$;
3. $\Delta k^3 = 3k^2 - 3k + 1$; *and*
4. $\Delta b^{k+1} = b^k(b-1)$.

Proof. These are all easy computations. For instance,

$$\Delta k^3 = k^3 - (k-1)^3 = k^3 - (k^3 - 3k^2 + 3k - 1) = 3k^2 - 3k + 1.$$

We leave the proofs of the remaining items as exercises. □

Corollary 2.1.11. *Suppose that b is a constant. We have the following formulas for finite differences:*

1. $$k = \Delta\left[\frac{k(k+1)}{2}\right];$$
2. $$k^2 = \Delta\left[\frac{k(k+1)(2k+1)}{6}\right]; \text{ and}$$
3. *if $b \neq 1$,*
$$b^k = \Delta\left[\frac{b^{k+1}}{b-1}\right].$$

Consequently,

a. if $n \geq 1$,
$$\sum_{k=1}^{n} k \;=\; \frac{n(n+1)}{2};$$

b. if $n \geq 1$,
$$\sum_{k=1}^{n} k^2 \;=\; \frac{n(n+1)(2n+1)}{6}; \text{ and}$$

c. if $n \geq 0$ and $b \neq 1$, then
$$\sum_{k=0}^{n} b^k \;=\; \frac{b^{n+1}-1}{b-1},$$
provided that b^0 is interpreted as equaling 1 when $b = 0$.

Proof. Items (a), (b), and (c) follow immediately from Items (1), (2), and (3), respectively, by applying Proposition 2.1.9. We shall prove Items (1), (2), and (3).

From Proposition 2.1.10, we have

$$\Delta k^2 \;=\; 2k - \Delta k,$$

and so, by the linearity of finite differences,

$$k \;=\; \frac{1}{2} \cdot \Delta(k^2 + k) \;=\; \Delta\left[\frac{k(k+1)}{2}\right].$$

From Proposition 2.1.10 and Item (1), we have

$$\Delta k^3 \;=\; 3k^2 \;-\; 3\,\Delta\left[\frac{k(k+1)}{2}\right] \;+\; \Delta k.$$

Therefore, linearity gives us

$$k^2 \;=\; \Delta\left[\frac{1}{3} \cdot \left(k^3 + \frac{3k(k+1)}{2} - k\right)\right] \;=\; \Delta\left[\frac{2k^3 + 3k(k+1) - 2k}{6}\right] \;=$$

$$\Delta \left[\frac{k(k+1)(2k+1)}{6} \right].$$

Finally, Proposition 2.1.10 and linearity immediately yield Item 3.

$\square$

We should remark that the summation $\sum_{k=0}^{n} b^k$ that appears in Item (c) of Corollary 2.1.11 is one which will be very important to us in Chapter 4 and Chapter 5; it is called a *geometric sum*.

2.1.1 Exercises

Calculate the sums.

1. $\displaystyle\sum_{k=3}^{7} (3k+2).$

2. $\displaystyle\sum_{j=-2}^{5} \left(j^2 - j\right).$

3. $\displaystyle\sum_{t=1}^{5} \ln t.$

4. $\displaystyle\sum_{m=4}^{7} \sin(m\pi).$

5. $\displaystyle\sum_{k=1}^{5} (k-1)(k-2)(k-3).$

6. $\displaystyle\sum_{t=-5}^{5} \cosh t.$

7. $\displaystyle\sum_{s=-5}^{5} \sinh s.$

8. $\displaystyle\sum_{k=0}^{6} (\sin(k\pi) + \cos(k\pi)).$

9. $\displaystyle\sum_{j=0}^{5} 2^{-j}.$

10. $\displaystyle\sum_{m=-2}^{3} \frac{1}{m}$.

Evaluate the sums by reindexing.

11. $\displaystyle\sum_{j=6}^{12} (3j - 18)$.

12. $\displaystyle\sum_{k=4}^{8} (k-3)^2 - \sum_{j=1}^{5} (9j + 20)$.

13. $\displaystyle\left(\sum_{k=6}^{9} \left(k^2 - 10k + 25\right)\right) - \left(\sum_{j=-3}^{0} \left(j^2 + 8j + 16\right)\right)$.

Prove the following statements of Proposition 2.1.10

14. $\Delta k = 1$.

15. $\Delta k^2 = 2k - 1$.

16. $\Delta b^{k+1} = b^k(b-1)$.

Calculate $\Delta f(k)$ for the following functions.

17. $f(k) = 5k$

18. $f(k) = C$, where C is a constant.

19. $f(k) = 3k^2 - 4k + 7$.

20. $f(k) = \ln k$.

21. $f(k) = \sin(2k\pi)$.

22. $f(k) = 3^k$.

23. Prove that if $f(x) = \sin x$, then $(\Delta f)(x) = (\sin x)(1 - \cos 1) + (\sin 1)\cos x$.

24. Prove that if $g(x) = \cos x$, then $(\Delta g)(x) = (\cos x)(1 - \cos 1) + (\sin 1)\sin x$.

25. a. What is $\displaystyle\sum_{n=0}^{U} (-1)^n$ when U is even?

    b. What is $\displaystyle\sum_{n=0}^{U} (-1)^n$ when U is odd?

26. Is the following statement true? Assume f and g are differentiable functions.

$$\left(\sum_{j=1}^{3} f(j)\right)\left(\sum_{j=1}^{3} g(j)\right) = \sum_{j=1}^{3} f(j)g(j)$$

Prove or give a counterexample.

27. What is $\displaystyle\sum_{k=1}^{100} \frac{1}{k(k+1)}$? Hint: use partial fractions and find a telescoping sum.

28. What is $\displaystyle\sum_{k=-J}^{J} f(k)$ if f is an odd function, $f(0) = y_0$, and J is some positive integer?

29. What is $\displaystyle\sum_{j=17}^{25} \left(\sqrt{j} - \sqrt{j-1}\right)$?

30. Suppose that f is a differentiable function. Prove that there exist numbers c_i in each open interval $(i-1, i)$, for $i = 1, 2, 3$, such that $f(3) - f(0) = f'(c_1) + f'(c_2) + f'(c_3)$.

31. Recall that the average of n numbers is given by the formula:

$$\bar{x} = \frac{\sum_{i=1}^{n} x}{n}.$$

Prove that $\displaystyle\sum_{i=1}^{n} (x - \bar{x}) = 0$.

The *sample standard deviation* of a data set, s, measures how spread out a data set is. For example, if $A = \{84, 84, 84\}$ and $B = \{80, 84, 88\}$, then the sample standard deviation of B will be larger than the sample standard deviation of A since there the points of B are more dispersed than those in set A. The sample standard deviation of a set with n data points is

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x - \bar{x})^2}.$$

The units of s are the same as those of the data.

32. A Calculus test is given to two sections of students. The grades of the students in section A are $\{85, 65, 73, 40, 64, 90\}$ and the grades of the students in section B are $\{84, 90, 96, 88, 100\}$. What are s_A and s_B?

33. Suppose that the week-to-week changes in a stock's price, in dollars per share, over a six week period are: $\{+2.5, +4, -7, 0, +3\}$. What is s for this set? The standard deviation of a stock's price measures the *volatility* of the stock.

34. The recorded annual rainfall, in inches, for a city over a five year span is $\{23, 47, 35, 42, 29\}$. What is the sample standard deviation?

35. We define the *sample variance* to be the square of the sample standard deviation:

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}.$$

Prove the following useful alternative formula:

$$s^2 = \frac{\left(\sum x_i^2\right) - (1/n)\left(\sum x_i\right)^2}{(n-1)}.$$

All sums are taken from 1 to n where n is the number of items in the data set.

Oftentimes statistics are used to measure the relationship between two variables. For example, researchers could be interested in the relationship between drug dosage and cancer cell counts. Basketball executives may be interested in the relationship between a team's free throw percentage, and the team's overall winning percentage. The *linear correlation coefficient*, r, quantifies the relationship. Specifically, r helps answer the question: to what extant are the two variables linearly related? r is always between -1 and $+1$. If r is close to $+1$ (resp. -1), then a strong positive (resp. negative) linear relationship exists between the variables in the sense that as one variable increases, the second variable tends to increase (resp. decrease) proportionally. If x_i and y_i are two data sets with means $\bar{x}$ and $\bar{y}$ and standard deviations s_x and s_y, then r is given by:

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{(n-1)s_x s_y}.$$

36. Suppose you work for a large retail store and your manager asks you study the rela-

tionship between revenue (sales) and the unemployment rate. Assume you assemble the unemployment rates and sales over the last six months in the table below.

Unemp (%)	Sales ($)
5	50350
7	37570
8	33140
9	25550
8	38750
5	55450

Let x be the unemployment rate and y the sales.

a. What is $\bar{x}$?

b. What is $\bar{y}$?

c. What is s_x?

d. What is s_y?

e. What is r?

37. Prove the following useful formula for the numerator of the formula for r:

$$\sum(x_i - \bar{x})(y_i - \bar{y}) = \left(\sum x_i y_i\right) - \frac{1}{n}\sum x_i \sum y_i.$$

38. Another common question in statistics is, What is the line that best fits the data? Assuming you believe that two variables have a linear relationship, you'd like to know the what line in the form $\hat{y} = mx + b$ appears to best describe the data. Here we use a 'hat' to emphasize that the output of the above formula is a predicted value, as opposed to an actual value. In classical linear regression, formulas for the parameters m and b are:

$$m = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x - \bar{x})^2}$$
$$b = \bar{y} - m\bar{x}.$$

a. Use these formulas to calculate the regression line of the above data.

b. What sales does the line predict for an unemployment rate of 6%?

Linear Regression FAQ

You probably have three good questions now:

Q In what sense is this the line of best fit?

A The line of best fit should be 'close' to the data. We quantify closeness by the square of the difference between the predicted and actual data. More specifically, the quantity $\sum(\hat{y} - y)^2$ is minimized.

Q Where do the formulas for m and b come from?

A We find m and b by minimizing the function:

$$f(m, b) = \sum(\hat{y} - y)^2 = \sum(mx + b - y)^2.$$

This is an elementary exercise in multivariable Calculus. In single variable Calculus, we find minima of a function by locating points where the derivative is zero. This procedure has a natural analog in higher dimensions.

Q When is it appropriate to use linear regression?

A There are several very important technical assumptions lurking behind linear regression that are seldom checked. One of these assumptions is that the data should be normally distributed about its mean at each y value. Many important economic models, including the Capital Asset Pricing Model and the Nobel Prize winning Black-Scholes option pricing method rely on the normality assumption. The extremely non-normal market fluctuations sparked by the burst of the housing market bubble in 2008-2009 has caused many people to question the validity of the normality assumption.

Exercises 39 - 43 are based on the lyrics of the song *The Twelve Days of Christmas.* You may find it helpful to have the lyrics in hand.

39. Let $f(m)$ be the total number of gifts received on the mth day. On the third day of Christmas, for example, a total of six gifts are given. Three french hens, two turtle doves, and a partridge in a pair tree. Write a formula for $f(m)$ using summation notation.

40. What is $f(12)$?

41. The cumulative number of gifts received after m days, call this function $F(m)$, is also growing. By the end of the third day, for example, 10 gifts have been received: 6 from the 3rd day, 3 from the 2nd day and 1 from the 1st day. Express $F(m)$ using summation notation.

42. What is $F(12)$?

43. Research the terms *triangular numbers* and *tetrahedral numbers* on the internet. How do these terms relate to these exercises?

44. By how much does $\ln 2$ differ from $\displaystyle\sum_{n=1}^{4} \frac{(-1)^{n-1}}{n}$? Does the summation get closer to $\ln 2$ if the upper limit is changed from 4 to 5?

45. Let $A_n = \displaystyle\sum_{u=1}^{n} \left(\frac{u^2+1}{2}\right)^2$ and $B_n = \displaystyle\sum_{u=1}^{n} \left(\frac{u^2-1}{2}\right)^2$. Find an expression for $A_n + B_n$. Hint: combine the sums immediately rather than dealing with each one individually.

The Fibonacci sequence is defined recursively. Set $F_0 = 0$, $F_1 = 1$ and, for $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$. This gives a sequence $\{0, 1, 1, 2, 3, 5, 8, ...\}$. Prove the following facts about the Fibonacci sequence.

46. $\Delta F_k = F_{k-2}$.

47. $\displaystyle\sum_{k=0}^{2n} F_k = F_{2n+2} - 1$.

48. $\displaystyle\sum_{k=1}^{n} F_{2k-1} = F_{2n}$.

49. $\displaystyle\sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1$.

50. $\displaystyle\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$.

2.2 Prelude to the Definite Integral: Riemann Sums

In differential Calculus, the instantaneous rate of change of a function is defined as follows: we have a notion of the average rate of change of the function, and we believe, for small changes in the independent variable, that the average rate of change approximates **something**. We then take limits of the average rates of change to define the instantaneous rate of change, the derivative.

In this section, we do something analogous: we will define *Riemann sums*, which are supposed to be approximations to "continuous sums of an infinite number of infinitesimal contributions". As with the passage from average rates of change to instantaneous ones, we pass from Riemann sums to continuous sums by taking limits. The continuous sum that we end up with is called the *definite integral*.

Let's begin with an extended example.

Example 2.2.1. Suppose that a car is traveling along a straight road, on which a coordinate axis has been laid out in meters. Let $p(t)$ denote the position of the car (on the axis), at time t seconds (after some initial starting time), and let $v(t)$ denote the velocity of the car, in meters per second, at time t seconds.

If the driver of the car were to brake hard, or step down hard on the accelerator, he or she could easily change the velocity of the car by a noticeable amount in 2 seconds. However, it would be difficult to change the velocity of the car in a significant way in a substantially smaller time interval, like 0.25 seconds.

We want to discuss how you would estimate the change in the position, the *displacement*, of the car, between times 0 and 2 seconds, if you were given **some** velocity measurements. First, we should clarify that, when we have two numbers a and b, where $a < b$, and we write that x is *between* a and b, we mean to allow for the possibility that x equals a or b, i.e., "x is between a and b" means $a \leq x \leq b$. If $a < b$, and we write that x is *strictly between* a and b, we mean to **disallow** the possibility that x equals a or b, i.e., "x is strictly between a and b" means $a < x < b$.

Now, suppose that we know that,

- at some time between 0 and 0.3 seconds, the car is moving with velocity 30 m/s;

• at some time between 0.3 and 0.8 seconds, the car is moving with velocity 20 m/s;

• at some time between 0.8 and 1.2 seconds, the car is moving with velocity 10 m/s; and,

• at some time between 1.2 and 2 seconds, the car is moving at -2 m/s (i.e., at 2 m/s in the negative direction).

How can we estimate the displacement of the car between times $t = 0$ and $t = 2$ seconds, i.e., how can we estimate $p(2) - p(0)$?

Note that we have the equality

$$p(2) - p(0) \;=\; \tag{2.2}$$

$$\big(p(2) - p(1.2)\big) \;+\; \big(p(1.2) - p(0.8)\big) \;+\; \big(p(0.8) - p(0.3)\big) \;+\; \big(p(0.3) - p(0)\big),$$

which results from the fact that all of the terms, other than $p(2)$ and $p(0)$, cancel out, that is, the sum telescopes, as in Proposition 2.1.9. Thus, to estimate $p(2) - p(0)$, we can estimate the four sub-displacements appearing in Formula 2.2, and add them together.

Okay. Fine. So how do we approximate each of the sub-displacements?

The answer is that, for each of the four intervals of time $[0, 0.3]$, $[0.3, 0.8]$, $[0.8, 1.2]$, and $[1.2, 2]$ (subintervals of $[0, 2]$), we use, as an approximation, that the velocity is constant on the given interval. Why do we do this? For two reasons. First, the subintervals of time are fairly small, small enough so that we believe that the velocity cannot change **too** much on each subinterval. Second, we don't really have much of a choice, considering that the only data that we are given is the velocity at **some** time in each of the given subintervals.

Thus, as an approximation, we assume/pretend that, on the subintervals $[0, 0.3]$, $[0.3, 0.8]$, $[0.8, 1.2]$, and $[1.2, 2]$, the velocity is constantly 30, 20, 10, and -2 m/s, respectively. Now, if the velocity is a constant during an interval of time, then that velocity is also the average velocity during the interval of time, and we know that the average velocity on an interval of time is the change in position on the interval divided by the change in time on the interval. Thus, we have the four approximations (in m/s):

$$\frac{p(2) - p(1.2)}{2 - 1.2} \approx -2, \qquad \frac{p(1.2) - p(0.8)}{1.2 - 0.8} \approx 10, \qquad \frac{p(0.8) - p(0.3)}{0.8 - 0.3} \approx 20, \text{ and}$$

$$\frac{p(0.3) - p(0)}{0.3 - 0} \approx 30.$$

Therefore, we approximate that the displacement of the car, between times $t = 0$ and $t = 2$ seconds:

$$p(2) - p(0) \;=$$

$$\big(p(2) - p(1.2)\big) \;+\; \big(p(1.2) - p(0.8)\big) \;+\; \big(p(0.8) - p(0.3)\big) \;+\; \big(p(0.3) - p(0)\big) \;\approx$$

$$-2(2 - 1.2) \;+\; 10(1.2 - 0.8) \;+\; 20(0.8 - 0.3) \;+\; 30(0.3 - 0) \;=$$

$$-1.6 \;+\; 4 \;+\; 10 \;+\; 9 \;=\; 21.4 \text{ meters.}$$

Therefore, we conclude that the displacement of the car, between times $t = 0$ and $t = 2$ seconds, is approximately 21.4 meters. Note that this is an estimate of the **change in** the position; we cannot estimate (in a reasonable manner) the actual position of the car at $t = 2$ seconds, unless we have $p(0)$ (or, at least, an estimate of $p(0)$).

Our data and approximations of the displacements on the subintervals look best in a table.

subinterval (sec.)	[0, 0.3]	[0.3, 0.8]	[0.8, 1.2]	[1.2, 2]
time (sec.)	t_1	t_2	t_3	t_4
velocity (m/s)	30	20	10	-2
Approx. Δp (meters)	9	10	4	-1.6
Approx. total $\Delta p = 9 + 10 + 4 + (-1.6) = 21.4$ meters.				

Note that we gave the names t_1, t_2, t_3, and t_4 to the times in the corresponding subintervals at which we were given the velocities that appear in the 3rd row. Our approximation of the total displacement is the sum of the entries in the 4th row of the table.

How could we obtain a better approximation of the displacement $p(2) - p(0)$ or, at least, an approximation that we'd expect to be better? We could subdivide the subintervals above into even smaller subintervals of time, and be given velocities at some time in each of the new subintervals.

For instance, if we thought that 0.8 seconds was too large of a change in time over which to approximate the velocity as being constant, we might subdivide the last subinterval $[1.2, 2]$ above into two subintervals, say $[1.2, 1.6]$ and $[1.6, 2]$. We would then need to know which of these new subintervals contains the time t_4 (both would, if $t_4 = 1.6$), and we'd need to be given the velocity at some time in the other new subinterval. For this example, let's assume that t_4 is in the subinterval $[1.2, 1.6]$, and that we know, at some time t_5 in the interval $[1.6, 2]$, that the velocity is -5 m/s. What approximation do we get for the total displacement now?

Our new table is:

subinterval (sec.)	[0, 0.3]	[0.3, 0.8]	[0.8, 1.2]	[1.2, 1.6]	[1.6, 2]
time (sec.)	t_1	t_2	t_3	t_4	t_5
velocity (m/s)	30	20	10	-2	-5
Approx. Δp (meters)	9	10	4	-0.8	-2
Approx. total $\Delta p = 9 + 10 + 4 + (-0.8) + (-2) = 20.2$ meters.					

Our new approximation for the displacement $p(2) - p(0)$ is no longer 21.4 meters, but is now 20.2 meters.

Do we really know that this new approximation is better than the first one? Not really. For all we know, in the last subinterval $[1.6, 2]$, the velocity is actually closer to -2 m/s most of time, and is only close to -5 m/s for a tiny subinterval around t_5. We can try to appeal to our intuition and/or physical experience to say that, in 0.4 seconds, the velocity cannot change a significant amount, but that's no proof, and it also leaves us with the question of: what's a "significant" amount? Nonetheless, we "suspect" that subdividing our subintervals into smaller subintervals leads to better approximations, provided we take our subdivisions small enough.

Despite the remark/warning above, it is nonetheless true that there is a strong sense in which using small enough subintervals guarantees a close approximation; see the remark below.

Remark 2.2.2. It is possible to make precise in what way the approximation "gets better" as the time intervals get smaller. First, we need to assume that the velocity function $v(t)$ is continuous on the interval $[0, 2]$. Now, call a subdivision of the interval $[0, 2]$ into a finite collection of subintervals a *partition* of $[0, 2]$ (technically, the partition is just the set of endpoints of the subintervals; see Definition 2.2.3). Call the length of the longest (any one of the longest) subintervals the *mesh* of the partition. Thus, if the mesh of the partition is less than some (small) positive constant δ, then every subinterval in the partition has length less than δ; informally, this means that saying that the mesh of a partition is small implies that every subinterval in the partition is small.

Now, we can state carefully in what sense picking small subintervals of time, and being given the velocity at one time per time interval, allows you to accurately approximate the actual displacement. Given any $\epsilon > 0$ (think: ϵ could be an arbitrarily small positive number), we can guarantee that our approximation for the displacement is within ϵ of the actual displacement by using partitions with a small enough mesh, i.e., there exists a number $\delta > 0$ such that, for every partition of $[0, 2]$, with mesh less than δ, for every choice of one time (a *sample point*) per subinterval at which to be given the velocity, the sum of the products of the lengths of each subinterval with the given velocities for the subintervals will be within ϵ of the displacement $p(2) - p(0)$.

The above statement is a result of the definition of the definite integral, Definition 2.3.1, Theorem 2.3.8, and the Fundamental Theorem of Calculus, Theorem 2.4.10.

We wish to discuss partitions, meshes, sample points, and the summing process above in a more general context.

Suppose that $a < b$. It turns out to be convenient to define a partition of the interval $[a, b]$ by giving the endpoints of the subintervals that we "chop" [a, b] into, rather than defining the partition to be the collection of subintervals themselves.

Definition 2.2.3. *A* **partition** $\mathcal{P}$ *of the interval* $[a, b]$*, into* n *subintervals, is an ordered set of numbers* $x_0, x_1, \ldots, x_n$ *such that* $x_0 = a$*,* $x_n = b$*, and* $x_0 < x_1 < \cdots < x_n$*.*

If $1 \leq i \leq n$*, then the closed interval* $[x_{i-1}, x_i]$ *is the* i**-th subinterval of the partition** $\mathcal{P}$*; note that the finite difference* Δx_i *equals* $x_i - x_{i-1}$*, which is the length of the* i*-th subinterval of the partition.*

The **mesh** *of a partition* $\mathcal{P}$*, denoted* $||\,\mathcal{P}\,||$*, is the maximum length of a subinterval of the partition, i.e.,* $||\,\mathcal{P}\,|| \;=\; \max\big\{\Delta x_i \mid 1 \leq i \leq n\big\}$*.*

A set $\mathcal{S}$ of **sample points** for a partition $\mathcal{P}$ is an ordered set of points, one for each subinterval of the partition, i.e., an ordered set of elements $s_1, s_2, \ldots, s_n$ such that, for all i, where $1 \leq i \leq n$, s_i is in the i-th subinterval of the partition, that is, $x_{i-1} \leq s_i \leq x_i$.

A **sampled partition** of the interval $[a, b]$ is an ordered pair $(\mathcal{P}, \mathcal{S})$, consisting of a partition $\mathcal{P}$ of the interval $[a, b]$ and a set $\mathcal{S}$ of sample points for $\mathcal{P}$.

For instance, in Example 2.2.1, we had the partition $\mathcal{P} = \{0, 0.3, 0.8, 1.2, 2\}$. The 1st subinterval is $[0, 0.3]$, the 2nd is $[0.3, 0.8]$, the 3rd subinterval is $[0.8, 1.2]$, and the 4th is $[1.2, 2]$. The mesh is $||\,\mathcal{P}\,|| = 2 - 1.2 = 0.8$. We were not given explicit sample points; we were simply told that, in each subinterval, there was a sample point (a time) at which we knew the velocity, and we were given those velocities.

Now, we need to define the general setup for the type of summation that we used in Example 2.2.1. We continue to assume that $a < b$.

Definition 2.2.4. *Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$, into n subintervals, let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a set of sample points for $\mathcal{P}$, and let f be a real-valued function whose domain includes the set of sample points $\mathcal{S}$.*

Then, the **Riemann sum**, *$\mathcal{R}^{\mathcal{S}}_{\mathcal{P}}(f)$, of f, with respect to $\mathcal{P}$ and $\mathcal{S}$, is defined to be*

$$\sum_{i=1}^{n} f(s_i)\Delta x_i \ =$$

$$f(s_1)(x_1 - x_0) + f(s_2)(x_2 - x_1) + \cdots + f(s_{n-1})(x_{n-1} - x_{n-2}) + f(s_n)(x_n - x_{n-1}).$$

For instance, in Example 2.2.1, for each of our sampled partitions, the associated Riemann sums of the velocity are precisely our approximations of the displacement $p(2) - p(0)$.

We need some terminology for when one partition has "smaller" subintervals than another, and for when we enlarge our set of sample points. It will be helpful to recall the notion of a *subset*; a set A is a subset of a set B, denoted $A \subseteq B$, if and only if every element of A is also an element of B.

Definition 2.2.5. *A partition $\mathcal{Q}$ of an interval $[a,b]$ is a* **refinement** *of a partition $\mathcal{P}$ of $[a,b]$ if and only if every point in $\mathcal{P}$ is in $\mathcal{Q}$, i.e., if and only if $\mathcal{P} \subseteq \mathcal{Q}$.*

If $(\mathcal{P}, \mathcal{S})$ is a sampled partition of $[a,b]$, then another sampled partition $(\mathcal{Q}, \mathcal{T})$ of $[a,b]$ is a **refinement** *of $(\mathcal{P}, \mathcal{S})$ if and only if $\mathcal{P} \subseteq \mathcal{Q}$ and $\mathcal{S} \subseteq \mathcal{T}$.*

Remark 2.2.6. As an example, our second sampled partition in Example 2.2.1, the one given in the second table, is a refinement of the original sampled partition in that example.

Understand the point of a refinement: the subintervals of a refinement are obtained from the subintervals of the original partition by subdividing some of the original subintervals into more, smaller, subintervals. Thus, if $\mathcal{P}$ and $\mathcal{Q}$ are partitions of an interval, and $\mathcal{P} \subseteq \mathcal{Q}$, then we have an inequality of meshes $m(\mathcal{Q}) \leq ||\,\mathcal{P}\,||$.

For a refinement of a sampled partition, we not only subdivide some of the original subintervals, but we keep the original sample points and throw in some new ones, to give us sample points in each new subinterval that does not contain an old sample point (or, if an old sample point is the endpoint of a subinterval in the refined partition, then we assign that point to one of the two adjacent subintervals, and must select a new sample point for the other adjacent subinterval).

Our interest in Riemann sums is not limited to velocity and displacement.

Example 2.2.7. Suppose that a balloon is being inflated between times $t = 0$ and $t = 120$ seconds, and the inflation rates, measured in in^3/s, are 8, 7, 5, and 2 at times 50, 70, 90, and 110 seconds, respectively.

Estimate the change in the volume $V = V(t)$ of air in the balloon between times $t = 40$ and $t = 120$ seconds.

Solution:

We will begin by selecting a partition of $[40, 120]$ that allows us to use 50, 70, 90, and 110 as sample points. An obvious choice is to subdivide $[40, 120]$ into 4 subintervals of equal length $\Delta t = (120 - 40)/4 = 20$. Then, the subintervals of the partition would be $[40, 60]$, $[60, 80]$, $[80, 100]$, and $[100, 120]$, and the points 50, 70, 90, and 110 would, in fact, be sample points for this partition (in fact, they are the midpoints of the subintervals).

subinterval (sec.)	[40,60]	[60,80]	[80,100]	[100, 120]
time (sec.)	50	70	90	110
inflation rate (in^3/s)	8	7	5	2
Approx. ΔV (in^3)	160	140	100	40
Approx. total $\Delta V = 160 + 140 + 100 + 40 = 440$ in^3.				

We should emphasize that 440 in^3 is the approximate **change in** the volume between times $t = 40$ and $t = 120$ seconds. With no knowledge of the volume of air in the balloon at time $t = 40$ seconds, we cannot reasonably approximate the actual volume $V(120)$ of air in the balloon at time $t = 120$ seconds.

Let's look at another example.

Example 2.2.8. Suppose that a circular rod, of length 1 meter, and cross-sectional area 0.01 m^2 (i.e., of radius $0.1/\sqrt{\pi}$ meters) is lying along the x-axis.

Figure 2.1: A rod of varying density.

Suppose that, for all x such that $0 \leq x \leq 1$, at each point in the cross section of the rod at x meters, the density of the rod is $\delta(x) = (1 + x)$ kg/m^3. We would like to estimate the total mass of the rod, by partitioning the interval $[0, 1]$ into 5 intervals of equal length.

Thus, each subinterval of our partition will have length $\Delta x = (1 - 0)/5 = 0.2$ meters, and so our partition is $\mathcal{P} = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. While we shall not use it, we remark that the mesh of $\mathcal{P}$ is $||\,\mathcal{P}\,|| = 0.2$.

Let's first use the left endpoint of each subinterval of our partition as the sample point for that subinterval. Thus, our left sample set is $L = \{0, 0.2, 0.4, 0.6, 0.8\}$.

This will yield the *left Riemann sum*, once we determine what function we're finding Riemann sums of. Average density is mass per volume. In an analogous way to how we dealt with velocity, position, and time in Example 2.2.1, we will assume that, for a small x-interval, the average density is approximated fairly well by the (instantaneous) density at any x-coordinate in the given interval. Therefore, if we take the density $\delta(x)$ at a point, and multiply times a little chunk of volume around that point, we will obtain an approximation of the mass of the chunk. A nice, manageable chunk of volume "surrounding" a given x-coordinate is given simply by taking an x-interval, containing the given x-coordinate, and multiplying by the cross-sectional area, 0.01 m^2. Thus, if we're going to multiply by lengths of subintervals in our Riemann sums, then the function that we want to find Riemann sums of, in order to approximate the mass, is cross-sectional area times density, i.e.,

$$f(x) = 0.01(1 + x) \ \ \text{kg/m}.$$

For $0 \leq x \leq 1$ meter, let $M(x)$ denote the mass of the rod between 0 and x, so that, if $0 \leq a < b \leq 1$, then $M(b) - M(a)$ is the mass of the rod between $x = a$ and $x = b$. We obtain a table of data that looks like:

$[x_{i-1}, x_i]$ (m)	[0, 0.2]	[0.2, 0.4]	[0.4,0.6]	[0.6, 0.8]	[0.8,1]
s_i (m)	0	0.2	0.4	0.6	0.8
$f(s_i)$ (kg/m)	0.01	0.01(1.2)	0.01(1.4)	0.01(1.6)	0.01(1.8)
ΔM (kg)	0.01(0.2)	0.01(1.2)(0.2)	0.01(1.4)(0.2)	0.01(1.6)(0.2)	0.01(1.8)(0.2)
Approx. total $M = \mathcal{R}^L_{\mathcal{P}}(f) = (0.01)(0.2)(1+1.2+1.4+1.6+1.8) = 0.014$ kg.					

Now let's look at the Riemann sum, where we use the same partition, but we use right endpoints of the subintervals as sample points. Thus, we let $R = \{0.2, 0.4, 0.6, 0.8, 1\}$, and will calculate $\mathcal{R}^R_{\mathcal{P}}(f)$, the *right Riemann sum* of the same function $f(x)$ that we used above.

$[x_{i-1}, x_i]$ (m)	[0, 0.2]	[0.2, 0.4]	[0.4,0.6]	[0.6, 0.8]	[0.8,1]
s_i (m)	0	0.2	0.4	0.6	0.8
$f(s_i)$ (kg/m)	0.01(1.2)	0.01(1.4)	0.01(1.6)	0.01(1.8)	0.01(2)
ΔM (kg)	0.01(1.2)(0.2)	0.01(1.4)(0.2)	0.01(1.6)(0.2)	0.01(1.8)(0.2)	0.01(2)(0.2)
Approx. total $M = \mathcal{R}^R_{\mathcal{P}}(f) = (0.01)(0.2)(1.2+1.4+1.6+1.8+2) = 0.016$ kg.					

There is a third common set of sample points that frequently get used: the midpoints of the intervals; we would normally denote this set by M, but, in our current example, M is already being used to denote the mass. So, we let $C = \{0.1, 0.3, 0.5, 0.7, 0.9\}$, and call $\mathcal{R}^C_{\mathcal{P}}(f)$ the *midpoint Riemann sum*. We calculate:

$[x_{i-1}, x_i]$ (m)	[0, 0.2]	[0.2, 0.4]	[0.4,0.6]	[0.6, 0.8]	[0.8,1]
s_i (m)	0	0.2	0.4	0.6	0.8
$f(s_i)$ (kg/m)	0.01(1.1)	0.01(1.3)	0.01(1.5)	0.01(1.7)	0.01(1.9)
ΔM (kg)	0.01(1.1)(0.2)	0.01(1.3)(0.2)	0.01(1.5)(0.2)	0.01(1.7)(0.2)	0.01(1.9)(0.2)
Approx. total $M = \mathcal{R}^C_{\mathcal{P}}(f) = (0.01)(0.2)(1.1+1.3+1.5+1.7+1.9) = 0.015$ kg.					

The fact that the midpoint Riemann sum ends up in-between the left and right Riemann sums, and is, in fact, the average of the left and right Riemann sums, is **not** generally true. These things are true in this example because our function f is linear, which also causes the midpoint Riemann sum to be exactly the value of what we will later call the *definite integral of f over* $[0, 1]$. In the completely general setting, the midpoint Riemann sums **can** yield worse approximations of the definite integral than the left and right Riemann sums. However, generally, we expect midpoint Riemann sums to yield better approximations than the left and right Riemann sums.

What we would like to see is that, if we use partitions of $[0,1]$ with arbitrarily small (positive) meshes, then the associated Riemann sums of f get arbitrarily close to some value; that value must then be the total mass of the rod.

Suppose that n is a natural number. Divide the interval $[0,1]$ into n subintervals of equal length, so that the associated partition is

$$\mathcal{P}_n \;=\; \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}.$$

We will calculate the left, right and midpoint Riemann sums of f with respect to $\mathcal{P}_n$, and see that all three of them approach the same limit as $n \to \infty$. (Here, we are using the limit of a *sequence*, which means that n takes on only integer values. We discussed this in [2]; also see Definition 4.5.1.)

The left sample set is $L_n = \{0, 1/n, 2/n, \ldots, (n-1)/n\}$, where $s_i = (i-1)/n$. The right sample set is $R_n = \{1/n, 2/n, \ldots, 1\}$, where $s_i = i/n$. The midpoint sample set is $C_n = \{1/(2n), 3/(2n), \ldots, (2n-1)/(2n)\}$, where $s_i = (i-1/2)/n = (2i-1)/(2n)$.

We have:

$$\mathcal{R}_{\mathcal{P}_n}^{L_n}(f) \;=\; \sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)\cdot\frac{1}{n} \;=\; \sum_{i=1}^{n} 0.01\left(1+\frac{i-1}{n}\right)\cdot\frac{1}{n} \;=$$

$$\frac{0.01}{n}\left[\left(\sum_{i=1}^{n} 1\right) + \frac{1}{n}\cdot\left(\sum_{i=1}^{n} i\right) - \frac{1}{n}\cdot\left(\sum_{i=1}^{n} 1\right)\right].$$

As $\displaystyle\sum_{i=1}^{n} 1 = n$, and $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, by Corollary 2.1.11, we find that

$$\mathcal{R}_{\mathcal{P}_n}^{L_n}(f) \;=\; \frac{0.01}{n}\left[n + \frac{1}{n}\cdot\frac{n(n+1)}{2} - \frac{1}{n}\cdot n\right] \;=\; \frac{0.01}{n}\left[\frac{2n+(n+1)-2}{2}\right] \;=$$

$$0.01\left[\frac{3-1/n}{2}\right].$$

Thus, as $n \to \infty$, we see that the left Riemann sums approach $0.01(3/2) = 0.015$ kg.

Similarly, we find

$$\begin{aligned}
\mathcal{R}^{R_n}_{\mathcal{P}_n}(f) &= \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} = \sum_{i=1}^{n} 0.01\left(1+\frac{i}{n}\right)\cdot\frac{1}{n} = \\
&\frac{0.01}{n}\left[\left(\sum_{i=1}^{n} 1\right) + \frac{1}{n}\cdot\left(\sum_{i=1}^{n} i\right)\right] = \\
&\frac{0.01}{n}\left[n + \frac{1}{n}\cdot\frac{n(n+1)}{2}\right] = \frac{0.01}{n}\left[\frac{2n+(n+1)}{2}\right] = \\
&0.01\left[\frac{3+1/n}{2}\right].
\end{aligned}$$

Thus, as $n \to \infty$, we see that the right Riemann sums also approach $0.01(3/2) = 0.015$ kg.

Finally, for the midpoint Riemann sums, we have

$$\begin{aligned}
\mathcal{R}^{C_n}_{\mathcal{P}_n}(f) &= \sum_{i=1}^{n} f\left(\frac{2i-1}{2n}\right)\cdot\frac{1}{n} = \sum_{i=1}^{n} 0.01\left(1+\frac{2i-1}{2n}\right)\cdot\frac{1}{n} = \\
&\frac{0.01}{n}\left[\left(\sum_{i=1}^{n} 1\right) + \frac{1}{n}\cdot\left(\sum_{i=1}^{n} i\right) - \frac{1}{2n}\left(\sum_{i=1}^{n} 1\right)\right] = \\
&\frac{0.01}{n}\left[n + \frac{1}{n}\cdot\frac{n(n+1)}{2} - \frac{1}{2n}\cdot n\right] = \frac{0.01}{n}\cdot\frac{3n}{2} = 0.015.
\end{aligned}$$

Thus, for each n, $\mathcal{R}^{C_n}_{\mathcal{P}_n}(f) = 0.015$ kg, and so, certainly, as $n \to \infty$, we see that the limit of the midpoint Riemann sums is also 0.015 kg.

In differential Calculus, it is extremely helpful to picture instantaneous rates of change graphically. This is accomplished by noting that the slope of a secant line yields the average rate of change, and then we take limits to arrive at the notion of a tangent line; we then visualize the instantaneous rate of change as the slope of the appropriate tangent line.

Our question now is: can we do something similar for Riemann sums and their limits, in order to picture these things geometrically?

The answer is (as you may have suspected): YES. We discuss this in the example below.

Example 2.2.9. How can we graphically represent Riemann sums? As a specific example, how can we graphically represent the left, right, and midpoint Riemann sums from Example 2.2.8?

Recall that the function we were finding Riemann sums of was

$$y = f(x) = 0.01(1+x).$$

(We shall omit the units throughout this example.) The graph is, of course, a straight line.

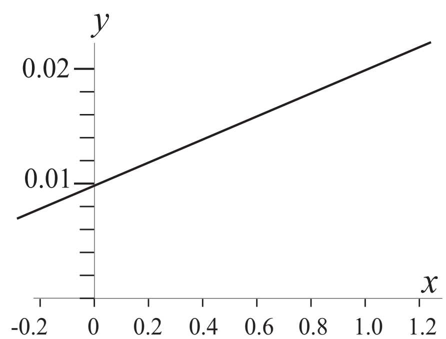


Figure 2.2: The graph of $y = 0.01(1+x)$.

Let's first look at the partition of $[0,1]$ given by $\mathcal{P} = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$, and the left sample set $L = \{0, 0.2, 0.4, 0.6, 0.8\}$. How can we visualize $\mathcal{R}_{\mathcal{P}}^{L}(f)$? We do it in terms of the areas of rectangles.

The Riemann sum is

$$\mathcal{R}_{\mathcal{P}}^{L}(f) = f(s_1)\Delta x_1 + f(s_2)\Delta x_2 + f(s_3)\Delta x_3 + f(s_4)\Delta x_4 + f(s_5)\Delta x_5 =$$

$$f(0)\cdot 0.2 + f(0.2)\cdot 0.2 + f(0.4)\cdot 0.2 + f(0.6)\cdot 0.2 + f(0.8)\cdot 0.2.$$

As $f(x)$ is non-negative on the interval $[0,1]$, we can interpret $f(s_i)\Delta x_i$ as the area of a rectangle of height $f(s_i)$ and width Δx_i; we draw this rectangle over the i-th subinterval on the x-axis. Thus, we represent $f(0)\Delta x_1 = f(0)\cdot 0.2$ by the rectangle in Figure 2.3, and the entire left Riemann sum is equal to the total area of all five of the inscribed rectangles in Figure 2.4

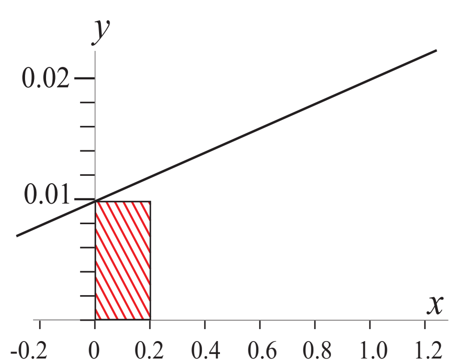


Figure 2.3: The first left summand.

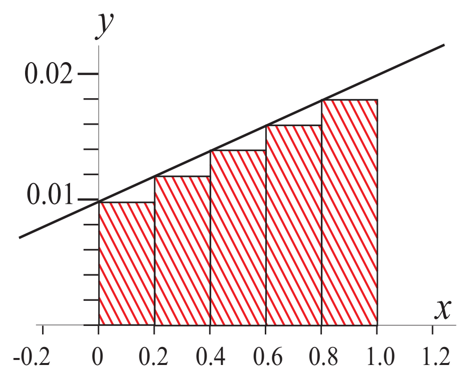


Figure 2.4: The left Riemann sum.

What about the right Riemann sum? We have

$$\mathcal{R}^R_{\mathcal{P}}(f) \ = \ f(s_1)\Delta x_1 + f(s_2)\Delta x_2 + f(s_3)\Delta x_3 + f(s_4)\Delta x_4 + f(s_5)\Delta x_5 \ =$$

$$f(0.2)\cdot 0.2 + f(0.4)\cdot 0.2 + f(0.6)\cdot 0.2 + f(0.8)\cdot 0.2 + f(1)\cdot 0.2.$$

Again, we interpret $f(s_i)\Delta x_i$ as the area of a rectangle of height $f(s_i)$ and width Δx_i. Now, however, the corresponding rectangles are **above** the line $y = 0.01(1+x)$. Thus, the entire right Riemann sum is equal to the total area of all five of the superscribed rectangles in Figure 2.5.

In order to compare the left and right Riemann sums visually, we have given both collections of relevant rectangles in Figure 2.6.

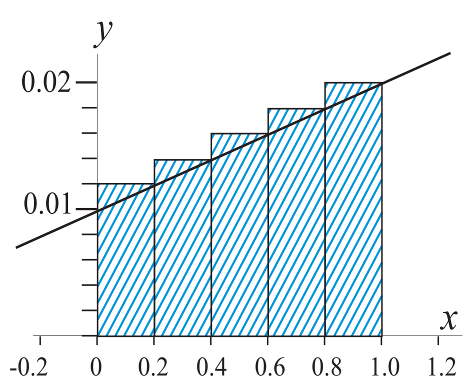


Figure 2.5: The right Riemann sum.

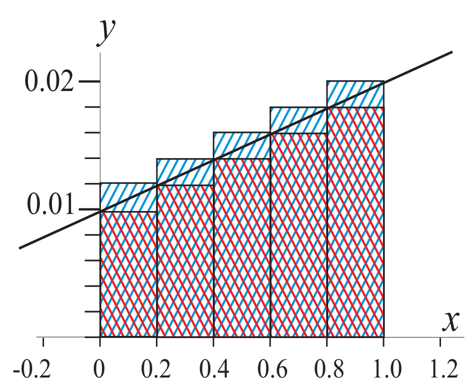


Figure 2.6: Combined Riemann sums.

What happens when we refine our partition into 10 subintervals of equal length, and look at both the left and right Riemann sums in terms of area of rectangles? We obtain the collections of inscribed and circumscribed rectangles in Figure 2.7.

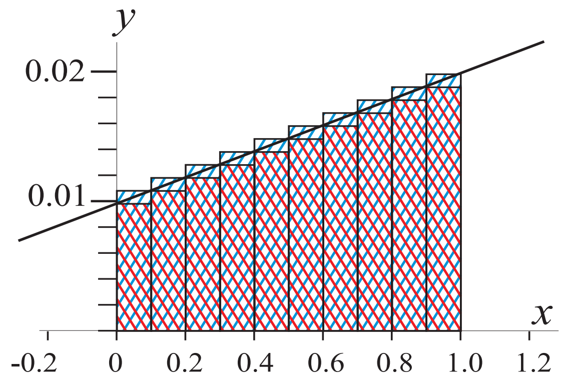


Figure 2.7: The rectangles for the refined Riemann sums.

As you can see, the total areas of the left and right rectangles have gotten closer together, that is, the **difference** between the area of the circumscribed rectangles and the area of the inscribed rectangles has gotten smaller; both areas have gotten closer to being the actual area of the trapezoid under the graph of $f(x) = 0.01(1 + x)$, and above the interval $[0, 1]$ in the x-axis.

Therefore, we see that, **as we refine our partition, the total area of the rectangles representing the Riemann sums approaches the actual area under the graph and above the closed interval on the x-axis.**

Of course, we know how to calculate the area of a trapezoid; it's one half the sum of the lengths of the bases times the height, where the bases are the two parallel sides. Thus, the area of our trapezoid is

$$\frac{1}{2}\big(f(0) + f(1)\big) \cdot 1 \;=\; \frac{1}{2}(0.01 + 0.02) \;=\; 0.015,$$

which agrees with what we found for the limit of our Riemann sums in Example 2.2.8

We have looked at the Riemann sums graphically only for the left and right Riemann sums. What about the midpoint (Figure 2.8) Riemann sums? What about Riemann sums with arbitrary sample sets?

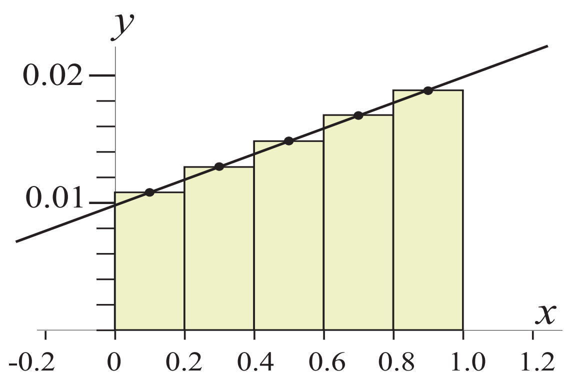


Figure 2.8: The midpoint Riemann sum.

Of course, we can't sketch rectangles corresponding to every choice that you might make for sample points. However, it is true that, **since our $f(x)$ is continuous, all limits of Riemann sums, with arbitrary sample sets approach the same value, so long as the meshes of the partitions approach zero**; that "same value", for a non-negative function, is the area under the graph and above the closed interval under consideration. This follows from Definition 2.3.1 and Theorem 2.3.8.

In fact, for $f(x) = 0.01(1 + x)$, it is easy to see that, since the left and right Riemann sums approach the area under the graph (as we take partitions with arbitrarily small mesh), so must Riemann sums using any sample sets whatsoever. Why is it easy to see this? Because $f(x) = 0.01(1 + x)$ is *monotonically increasing*, which implies that the smallest that f ever gets on a closed subinterval is at the left endpoint of the subinterval, and largest that f ever gets on a closed subinterval is at the right endpoint of the subinterval.

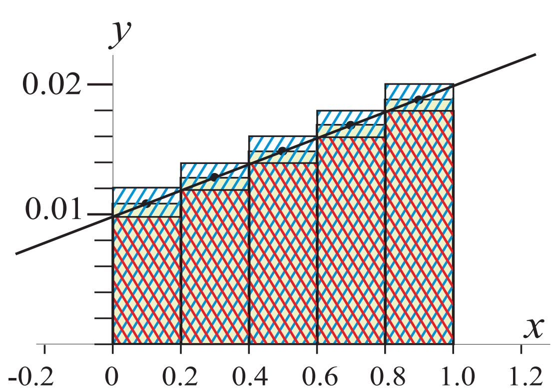


Figure 2.9: Left, right, and midpoint Riemann sums.

Thus, for all sets of sample points for a given partition, the rectangles representing the Riemann sums are "trapped" between the left and right Riemann sum rectangles. See Figure 2.9, where we have included the midpoint Riemann sum, with our previous partition of the interval $[0, 1]$ into 5 subintervals of equal length.

You may be thinking: "Ah, so, we can calculate limits of Riemann sums simply by calculating areas, like that of a trapezoid. This is easy." The problem is that we don't know the areas under the graphs of functions, at least, not by applying easy formulas from basic geometry.

Suppose, for instance, that our function f has been given, not by a function whose graph is a straight line, but rather by

$$f(x) = 0.01(1 + x^2).$$

Then, exactly how do we calculate the area under the graph of this f and over the interval $[0, 1]$?

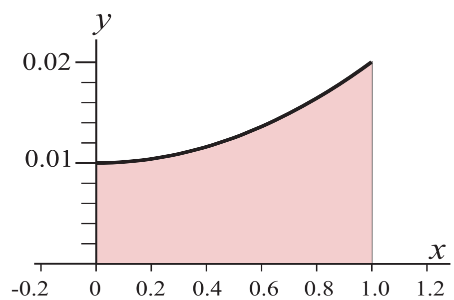


Figure 2.10: Area under a parabola.

The answer is that we calculate the limit of Riemann sums to find the area, not the other way around. Only in very few cases can we calculate the limit of Riemann sums from known formulas for the area.

Before we leave this example, we should comment on how we would have graphically represented the Riemann sums had $f(x)$ been negative, or sometimes positive and sometimes negative.

Suppose that we have a continuous function f such that $f(x) < 0$ for all x in the interval $[0, 1]$. Then, given a partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ of $[0, 1]$, and a sample set $\mathcal{S} = \{s_1, \ldots, s_n\}$ for $\mathcal{P}$, the summand $f(s_i)\Delta x_i$ in the Riemann sum $\mathcal{R}_{\mathcal{P}}^{\mathcal{S}}(f)$ **cannot** be represented by area, since it will be a negative quantity. However, we can consider a rectangle under the subinterval $[x_{i-1}, x_i]$ of height $-f(s_i)$, and then $f(s_i)\Delta x_i$ will be equal to **negative** the area of this rectangle beneath the x-axis. See Figure 2.11.

Thus, the limit of the Riemann sums of our negative function f, as the mesh of the partitions approaches zero, will be equal to **negative the area above the graph and under the interval** $[0,1]$.

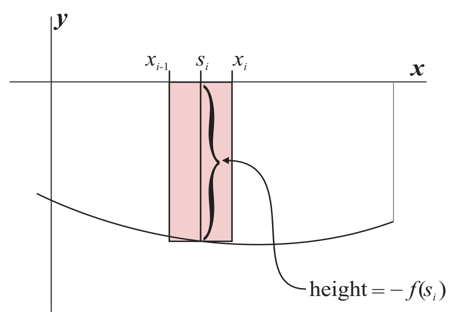


Figure 2.11: A graph below the x-axis.

Finally, what if f is continuous, but is negative sometimes and positive other times? As we shall see in Theorem 2.3.16, the limit of Riemann sums of such an f is a sum of contributions from where $f \geq 0$ and where $f \leq 0$ and, thus, in terms, of area, the limit of Riemann sums, as the mesh of the partitions approaches zero, is the equal to the area under the graph and above the x-axis **minus** the area above the graph and under the x-axis.

Thus, for a function f such as that in Figure 2.12, the limit of $\mathcal{R}_{\mathcal{P}}^{\mathcal{S}}(f)$, as the mesh of the partition $\mathcal{P}$ of $[0,1]$ approaches zero, will equal the difference of areas $A_2 - A_1$.

In no way are we claiming that the area under the x-axis and above the graph is negative; we are simply saying that the contribution to the limit of the Riemann sums from that portion of the graph is $-A_1$, where A_1 itself is positive.

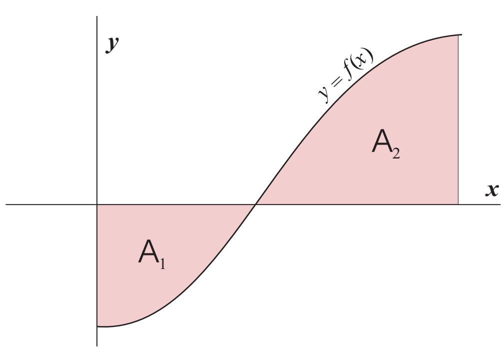


Figure 2.12: A graph below and above the x-axis.

2.2.1 Exercises

In each of Exercises 1 through 5, calculate the mesh of the given partition.

1. $\mathcal{P} = \{-2, 0, 3, 5, 7, 9\}$.

2. $\mathcal{P} = \{\ln 1, \ln 2, \ln 3, \ln 4, \ln 5\}$.

3. $\mathcal{P} = \{0, 1/4, 1/3, 1/2\}$.

4. $\mathcal{P} = \{0, 1/2, 2/3, 3/4, 4/5\}$.

5. $\mathcal{P} = \left\{0, \frac{1}{2^n}, \frac{2}{2^n}, ..., \frac{2^n-1}{2^n}, 1\right\}$, where n is a positive integer.

6. Explain why $x_0 = 0$, $x_1 = 2$, $x_2 = -1$, $x_3 = 3$, and $x_4 = 4$ cannot be used as a partition.

For the following true/false questions, in Exercises 7 through 13, assume all partitions mentioned are on the same interval, $[0, 1]$.

7. If $||Q|| \leq ||P||$, then $\mathcal{Q}$ is a refinement of $\mathcal{P}$.

8. If $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then $||Q|| \leq ||P||$.

9. Given a specific mesh $\epsilon \leq 1$, where ϵ is a rational number, there exists a partition $\mathcal{P}$ of $[0, 1]$, which has mesh ϵ, where every point of $\mathcal{P}$ is a rational number.

10. If $(\mathcal{Q}, \mathcal{T})$ is a refinement of $(\mathcal{P}, \mathcal{S})$, then $\mathcal{P}$ is a refinement of $\mathcal{Q}$.

11. If $\mathcal{P}$ is a refinement of $\mathcal{Q}$, then $(\mathcal{Q}, \mathcal{T})$ is a refinement of $(\mathcal{P}, \mathcal{S})$.

12. The sampled points of a partition must all be distinct.

13. If f is a continuous function and $\mathcal{P}$ is a partition of some interval, then $\mathcal{R}^{\mathcal{M}}_{\mathcal{P}}(f)$ is always between $\mathcal{R}^{\mathcal{R}}_{\mathcal{P}}(f)$ and $\mathcal{R}^{\mathcal{L}}_{\mathcal{P}}(f)$.

In each of Exercises 14 through 16, say (a) whether $\mathcal{Q}$ is a refinement of partition $\mathcal{P}$ and (b) whether $(\mathcal{Q}, \mathcal{T})$ is a refinement of $(\mathcal{P}, \mathcal{S})$.

14. $\mathcal{P} = \{0, 1/4, 1/2, 1\}$, $\mathcal{Q} = \{0, 1/2, 1\}$, $\mathcal{S} = \{1/8, 2/3, 3/4\}$, $\mathcal{T} = \{1/8, 3/4\}$.

15. $\mathcal{P} = \{0, 2/5, 1\}$, $\mathcal{Q} = \{0, 1/3, 2/5, 3/5, 1\}$, $\mathcal{S} = \{1/3, 2/3\}$, $\mathcal{T} = \{1/4, 1/3, 2/3, 3/4\}$.

16. $\mathcal{P} = \{0, 1/8, 2/5, 3/4, 1\}$, $\mathcal{Q} = \{0, 1/8, 3/13, 2/5, 5/9, 3/4, 4/5, 1\}$, $\mathcal{S} = \{1/16, 2/5, 5/9, 3/4\}$, $\mathcal{T} = \{1/7, 4/21, 5/18, 1/2, 8/13, 7/9, 9/10\}$.

17. Suppose a car is traveling with variable velocity for three minutes, and that

    - at some point between 0 and 40 seconds, the car is moving with velocity 15 m/s;
    - at some point between 40 and 90 seconds, the car is moving with velocity 20 m/s;
    - at some point between 90 and 120 seconds, the car is moving with velocity 25 m/s;
    - at some point between 120 and 180 seconds, the car is moving with velocity 15 m/s.

    Use Riemann sums to approximate the total displacement of the car from its initial position.

18. Suppose a car is traveling with variable velocity for one hour, and that

    - at some point between 0 and 12 minutes, the car is moving with velocity 40 miles/hour;
    - at some point between 12 and 30 minutes, the car is moving with velocity -10 miles/hour;
    - at some point between 30 and 40 minutes, the car is moving with velocity 20 miles/hour;
    - at some point between 40 and 60 minutes, the car is moving with velocity 0 miles/hour.

    Use Riemann sums to approximate the total displacement of the car from its initial position.

19. Suppose a car is traveling with variable *acceleration* for 30 minutes, and that

    - at some point between 0 and 10 minutes, the car is accelerating at a rate of 5 miles per hour per hour.
    - at some point between 10 and 15 minutes, the car is accelerating at a rate of -3 miles per hour per hour.
    - at some point between 15 and 25 minutes, the car is accelerating at a rate of 1 mile per hour per hour.
    - at some point between 25 and 30 minutes, the car is accelerating at a rate of 4 miles per hour per hour.

    Use Riemann sums to approximate the total change in velocity of the car from its initial velocity.

20. Suppose that a balloon starts with no air in it, and is inflated between $t = 0$ and $t = 90$ seconds. Suppose further that,

- at some time between 0 and 20 seconds, the balloon is inflating at a rate of 2 in^3/s.
- at some time between 20 and 50 seconds, the balloon is inflating at a rate of 5 in^3/s.
- at some time between 50 and 75 seconds, the balloon is inflating at a rate of -3 in^3/s. Perhaps someone lost their grip on the balloon momentarily.
- at some time between 75 and 90 seconds, the balloon is inflating at a rate of 4 in^3/s.

Use Riemann sums to approximate the total volume of air in the balloon at $t = 90$ seconds.

21. If $f(t) = c$ on the interval $[a, b]$, prove that the Riemann sum of f on $[a, b]$ using any sample set and any partition is $c(b - a)$. Give a physical interpretation of this fact if f is a velocity function.

22. Let $f(x) = mx$ be a line through the origin defined on $[0, k]$. Assume $m > 0$. What is the limiting Riemann sum of this function over this interval? Use the fact that Riemann sums approximate the area under a curve. Your answer may be non-rigorous.

23. Suppose f is a continuous function on $[a, b]$. Then f achieves its minimum m and maximum M. Let $\mathcal{P}$ be a partition of $[a, b]$. Prove that $m(b - a) \leq \mathcal{R}_{\mathcal{P}}(f) \leq M(b - a)$.

24. Consider the following function with domain $[0, 1]$.

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational;} \\ 0,, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that no matter what partition is chosen, and no matter how small its mesh, we can always find two sets of sample points $\mathcal{S} = \{s_i\}$ and $\mathcal{S}' = \{s_i'\}$ for that partition such that the Riemann sum is 1 on $\mathcal{S}$ and 0 on $\mathcal{S}'$.

In each of Exercises 25 through 27, calculate the left Riemann sum for the given function and partition.

25. $h(x) = -x$, $\mathcal{P} = \{-3, -2, -1, 0, 1, 2, 3\}$.
26. $g(x) = \sin x$, $\mathcal{P} = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\right\}$.
27. $f(x) = x^2$, $\mathcal{P} = \{0, 0.1, 0.2, ..., 0.9, 1\}$.

In each of Exercises 28 through 30, calculate the right Riemann sum for the given function and partition.

28. $k(x) = e^x/2$, $\mathcal{P} = \{0, \ln 2, \ln 3, \ln 4\}$.

29. $l(x) = e^{-x}/2$, $\mathcal{P} = \{0, \ln 2, \ln 3, \ln 4\}$.

30. $j(x) = \cosh x$, $\mathcal{P} = \{0, \ln 2, \ln 3, \ln 4\}$. Hint: Use the previous two problems and the fact that Riemann sums are linear, a fact you will prove in a later exercise.

In each of Exercises 31 through 34, calculate the midpoint Riemann sums for the following functions and partitions.

31. $y(x) = \cos x$, $\mathcal{P} = \{-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\}$

32. $r(x) = 1/x^2$, $\mathcal{P} = \{1/10, 1/8, 1/5, 1/2, 1\}$.

33. $g(x) = \tan^{-1} x$, $\mathcal{P} = \{0, 1, 2, 3, 4\}$.

34. $h(x) = \sin x \cos x$, $\mathcal{P} = \{0, \pi/2, \pi, 3\pi/2, 2\pi\}$.

35. Let $f(x) = \sqrt{1 - x^2}$. Calculate the left Riemann sums for the following partitions.
    a. $\mathcal{P}_1 = \{-1, 0, 1\}$.
    b. $\mathcal{P}_2 = \{-1, -0.5, 0, 0.5, 1\}$.
    c. $\mathcal{P}_3 = \{-1, -2/3, -1/3, 0, 1/3, 2/3, 1\}$.
    d. Based on geometric intuition, what is the limiting Riemann sum, as the mesh approaches 0, using partitions of the interval $[-1, 1]$?

36. Suppose $f(x)$ is a continuous odd function. What is the midpoint Riemann sum using the partition $\mathcal{P} = \{-a_3, -a_2, -a_1, 0, a_1, a_2, a_3\}$, where $0 < a_1 < a_2 < a_3$?

The work done on an object experiencing a constant force F, as the object travels along the x-axis, from point a to b, is $W = F(b - a)$ (provided that the force acts parallel to the x-axis).

The total work done on an object experiencing a variable force $F = F(x)$ can be approximated by Riemann sums. Namely, suppose $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ and that $\mathcal{S} = \{s_1, \ldots, s_n\}$ is a sample set for the partition. Then the approximate work done is $\mathcal{W}_{\mathcal{P}}^{\mathcal{S}}(F) = \sum_{i=1}^{n} F(s_i)(x_i - x_{i-1})$. The metric units of work are Newton-meters, or *joules*.

In each of Exercises 37 through 39, assume an object moves along the interval $[0, 1]$, that the partition is $\mathcal{P} = \{0, 1/4, 1/2, 3/4, 1\}$, and that the sample set is $\mathcal{S} = \{1/4, 1/2, 3/4, 1\}$; calculate $\mathcal{W}_{\mathcal{P}}^{\mathcal{S}}(F)$.

37. $F(x) = x^2$.

38. $F(x) = \sin(\pi x)$.

39. $F(x) = 3x$.

40. Let $f(x) = 1/x$ on the interval $I_n = [\frac{1}{n}, 1]$, where $n = 2, 3, \ldots$. Let $\mathcal{P} = \left\{\frac{1}{n}, \frac{1}{n-1}, , ..., \frac{1}{2}, 1\right\}$. Calculate the left Riemann sum of f over this partition using the left endpoints for $n = 2, 3$ and 4.

41. Suppose that f and g are two continuous functions on the interval $[a, b]$. Let $(\mathcal{P}, \mathcal{S})$ be a partition and sample set for this interval. Prove that the Riemann sum is linear in the sense that

$$\mathcal{R}_{\mathcal{P}}^{\mathcal{S}}(cf + g) = c \cdot \mathcal{R}_{\mathcal{P}}^{\mathcal{S}}(f) + \mathcal{R}_{\mathcal{P}}^{\mathcal{S}}(g)$$

where c is any real number.

42. Consider the function,

$$h(x) = \begin{cases} \frac{1}{x}\sin\left(\frac{\pi}{x}\right), & \text{if } 0 < x \leq 1; \\ 0, & \text{if } x = 0, \end{cases}$$

and suppose that $\mathcal{P}$ is a partition of the interval $[0, 1]$.

a. Show that the Riemann sums can be made arbitrarily large, i.e., for all $N > 0$, there exists a sample set $\mathcal{S}$ such that $\mathcal{R}_{\mathcal{P}}^{\mathcal{S}} > N$.

b. Similarly, show that the Riemann sums can be made arbitrarily small, i.e., for all $N > 0$, there exists a sample set $\mathcal{S}$ such that $\mathcal{R}_{\mathcal{P}}^{\mathcal{S}} < -N$.

43. Consider the function

$$f(x) = \begin{cases} 3, & \text{if } 0 \leq x < 1 \text{ or } 1 < x \leq 2; \\ 100 & x = 1. \end{cases}$$

a. Let $\mathcal{P}_n = \{0, \frac{1}{2^n}, \frac{2}{2^n}, ..., 2\}$ be the uniform mesh where the distance between consecutive points is $1/2^n$. Calculate $\mathcal{R}_{\mathcal{P}_n}^{R}(f)$,

b. Show that $\lim_{n\to\infty} \mathcal{R}_{\mathcal{P}_n}^{R}(f) = 6$.

44. Let $f(x) = c$ be a constant function on the interval $[0, 1]$ with $c > 0$. Let

$$g(x) = \begin{cases} c, & \text{if } x \neq 1/2; \\ 0, & \text{if } x = 1/2. \end{cases}$$

Let $\mathcal{P}_n$ be the partition on $[0,1]$ where the distance between each sample point is $1/n$. Show that $\lim_{n\to\infty} \mathcal{R}^R_{\mathcal{P}}(f) = \lim_{n\to\infty} \mathcal{R}^R_{\mathcal{P}}(g) = c$.

The idea is that the Riemann sum of a function with a single discontinuity approaches the Riemann sum of the same function with the discontinuity removed.

Suppose that a partition $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ of the interval $[a,b]$ has been given, and that f is continuous on $[a,b]$. The Extreme Value Theorem tells us that f attains maximum and minimum values on any closed subinterval of $[a,b]$.

Let $U(\mathcal{P})$ be a sample set chosen in such a way that, for each s_i in the sample set $U(\mathcal{P})$, $f(s_i)$ equals the maximum value of f on $[x_i, x_{i+1}]$. Similarly, let $L(\mathcal{P})$ be a sample set chosen in such a way that, for each s_i in the sample set $L(\mathcal{P})$, $f(s_i)$ equals the minimum value of f on $[x_i, x_{i+1}]$.

45. Show that $\mathcal{R}^{U(\mathcal{P})}_{\mathcal{P}} \geq \mathcal{R}^{L(\mathcal{P})}_{\mathcal{P}}$.

46. Show that if $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then $\mathcal{R}^{U(\mathcal{Q})}_{\mathcal{Q}} \leq \mathcal{R}^{U(\mathcal{P})}_{\mathcal{P}}$ and $\mathcal{R}^{L(\mathcal{Q})}_{\mathcal{Q}} \geq \mathcal{R}^{L(\mathcal{P})}_{\mathcal{P}}$.

47. Show that if $\mathcal{P}'$ is another partition, not necessarily a refinement of $\mathcal{P}$ or vice versa, then $\mathcal{R}^{L(\mathcal{P})}_{\mathcal{P}} \leq \mathcal{R}^{U(\mathcal{P}')}_{\mathcal{P}'}$.

We can use Riemann sums to approximate lengths as well as areas. Suppose we'd like to approximate the length of the graph of a function $f(x)$ on the interval $[a,b]$. If $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition of the interval $[a,b]$, then we can measure the lengths of the line segments connecting the points $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$. Recall that this length is given by $\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$.

48. Let $f(x) = \sqrt{1-x^2}$.

    a. Consider the partition $\mathcal{P}_1 = \{-1, 0, 1\}$ of $[-1,1]$. Approximate the length of the graph by calculating

    $$\sum_{i=1}^{2} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.$$

    b. Approximate the length of the graph, via the same technique, but using the refined partition $\mathcal{P}_2 = \{-1, -1/2, 0, 1/2, 1\}$.

    c. Based on classical geometry, what is the length of this graph? Are your approximations close?

In each of Exercises 49 through 51, use the technique in the previous problem to approximate the lengths of the graphs over the specified interval using the given partition.

49. $g(x) = \sin x$, $[0, \pi]$, $\mathcal{P} = \{0, \pi/4, \pi/2, 3\pi/4, \pi\}$.

50. $h(x) = x^2$, $[0, 1]$, $\mathcal{P} = \{0, 1/4, 1/2, 3/4, 1\}$.

51. $j(x) = 1/x$, $[1, 5]$, $\mathcal{P} = \{1, 2, 3, 4, 5\}$.

52. Given a partition $\{x_0, ..., x_n\}$ of $[a, b]$, we've been using

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

to approximate the length of the graph of f on the interval $[a, b]$. This doesn't look like a Riemann sum, but we can "fix" it.

Prove that, if f is differentiable on an open interval containing $[a, b]$, and no two x_i's are equal, then there exists s_i in the open interval (x_{i-1}, x_i) such that

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \;=\; \sum_{i=1}^{n} \left(\sqrt{1 + [f'(s_i)]^2}\right) \Delta x_i.$$

2.3 The Definite Integral

As we saw in the previous section, the process of taking Riemann sums, and using partitions with meshes that get arbitrarily small, arises in a number of different contexts. We might be given the velocity of an object over an interval of time, and want to find the displacement. We might be given the rate at which a balloon is being inflated over an interval of time, and want to know how much the volume of air in the balloon changes. We might be given a rod of variable density, and want to know its total mass. We might want to know how much area is "trapped" between the graph of a function and the x-axis.

In this section, we define the *definite integral* as the limit of Riemann sums, and discuss some of the basic properties of definite integrals. We do not give many applications of definite integrals here; such applications will be the sole topic of the entire next chapter, Chapter 3. It is also somewhat difficult to look at serious applications before we have the basic tool for calculating definite integrals, the second part of the Fundamental Theorem of Calculus, Theorem 2.4.10.

Many of the proofs in this section are extremely technical and we have deferred them to the Technical Matters section, Section 2.A.

In Example 2.2.8 and Example 2.2.9 in the previous section, we saw that our Riemann sums $\sum_{i=1}^{n} f(s_i)\Delta x_i$ approached a limit as we let the mesh of the partition approach zero, but we used special partitions and sample sets; we used subintervals which all had the same length, and we used the left and right Riemann sums. What we would like to know is that other manners of choosing partitions and sample sets would have approximated the same quantities, that is, we would like to know that we could have used arbitrary partitions and arbitrary sample sets, and still obtained the same limit, so long as the meshes of the partitions approach zero. Technically, this means that we would like for there to exist a limit L that our Riemann sums get arbitrarily close to (within ϵ, for arbitrary $\epsilon > 0$) if we make the mesh of our partitions small enough (less than some $\delta > 0$, where δ would typically depend on how ϵ was chosen).

Thus, we make the definition below, though it should **not** be clear at this point that many/any functions satisfy the strong requirements.

Definition 2.3.1. (**The Definite Integral**) *Suppose that f is defined on the closed interval $[a, b]$, where $a < b$, and that there exists a real number L such that, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all partitions $\mathcal{P} = \{x_0, \dots, x_n\}$ of $[a, b]$, with mesh less than δ, and for all sample sets $\mathcal{S} = \{s_1, \dots, s_n\}$ for $\mathcal{P}$,*

$$\left| \sum_{i=1}^{n} f(s_i)\Delta x_i \; - \; L \right| < \epsilon.$$

Then, we write that

$$\lim_{||\,\mathcal{P}\,|| \to 0} \mathcal{R}_{\mathcal{P}}^{\mathcal{S}}(f) \;=\; \lim_{||\,\mathcal{P}\,|| \to 0} \sum_{i=1}^{n} f(s_i)\Delta x_i \;=\; L,$$

and we say that f is **Riemann integrable on** $[a, b]$.

When f is Riemann integrable, the limit above is usually written

$$\int_a^b f(x)\, dx,$$

and is called the **(definite) integral of f on $[a, b]$, or the integral of $f(x)$, with respect to x, as x goes from a to b.**

In this context, f is referred to as the **integrand**, and a and b are the **limits of integration**.

Remark 2.3.2. There are several points that we need to make now.

First, you should understand the point of the definition of the integral. A function is Riemann integrable if and only if the Riemann sums $\sum_{i=1}^{n} f(s_i)\Delta x_i$ approach a specific limit, **regardless of how you choose the partitions and/or sample points**, as long as you take small enough subintervals in your partitions. You are allowed to use partitions in which the subintervals have different lengths and choose sample points randomly. For f to be Riemann integrable means that none of this matters, as long as the mesh of the partitions approaches 0.

It's the strongest property that you ask for, given our discussion in the previous section, and it's a little difficult to believe that many interesting functions satisfy such a strong condition, but we shall see that most of the functions with which you're familiar are, in fact, Riemann integrable. (We are deliberately leaving you in suspense for the moment about which functions those are.)

We should also mention that it is customary in Calculus textbooks to drop the modifier "Riemann" from the term "Riemann integrable", and simply say that a function is "integrable", whenever they mean "Riemann integrable." The justification given for this is always that "Riemann integrability is the only type of integrability that will be used in this textbook." Strangely, this is not usually the case; every, or almost every, Calculus textbook includes a discussion of *improper integrals* (see Section 2.5).

Improper integrals specifically involve defining an integral $\int_a^b f(x)\,dx$ in certain cases where f is not Riemann integrable on $[a, b]$.

For this reason, we shall **not** drop the modifier "Riemann" from "Riemann integrable." We shall reserve the simpler term "integrable" for functions on intervals for which the Riemann integral exists or for which the improper integral exists.

It is important to note that the units on the integral are the units of the Riemann sums, i.e., the units of f times the units of x.

Finally, you should realize that the variable x, which appears in $\displaystyle\int_a^b f(x)\,dx$, is a dummy variable, which means that this variable name is irrelevant in the determining the actual value of the integral. For instance, we have the following equalities:

$$\int_a^b f(x)\,dx \;=\; \int_a^b f(t)\,dt \;=\; \int_a^b f(u)\,du.$$

All of these integrals mean the same thing: you partition the interval $[a, b]$ into arbitrarily small subintervals, you evaluate f at a sample point in each subinterval, you multiply the values of f at the sample points times the lengths of the respective subintervals, you add, and you take the limit of totals that you get, as the mesh of the partitions approaches zero. The point of writing all of that, without referring to a variable, was to make it clear that it is irrelevant what the variable is.

However, it is true that, in applications, there will typically be variables of physical significance, like t for time or x for position, and it would, in fact, be a little odd to change the variable name when writing integrals related to these quantities.

The fact that Definition 2.3.1 defines the Riemann integral as one single thing follows from the standard type of proof (see Theorem 2.A.1) that limits are unique; thus, there aren't two (or more) different limits L_1 and L_2 that satisfy the ϵ-δ condition that we required. We state this as a theorem.

Theorem 2.3.3. *The limit of Riemann sums, the definite integral, given in Definition 2.3.1 is unique, if it exists.*

In addition, if f is Riemann integrable on the interval $[a,b]$, $\int_a^b f(x)\,dx = L$, and $(\mathcal{P}_n, \mathcal{S}_n)$ is a sequence of sampled partitions of $[a,b]$ such that $\lim_{n\to\infty} ||\,\mathcal{P}_n\,|| = 0$, then

$$\lim_{n\to\infty} \mathcal{R}_{\mathcal{P}_n}^{\mathcal{S}_n}(f) = L.$$

Well, this is all great, but **do we know any functions which are Riemann integrable**? For that matter, do we know any functions which **aren't** Riemann integrable? How you calculate definite integrals, when they exist, is a separate question, and our best answer to that will have to wait until Section 2.4, but, for now, we'd at least like to know some results on when integrals exist and when they don't.

There is an easy, fundamental, criterion which implies that a function is **not** Riemann integrable. If a function is arbitrarily large in absolute value on an interval, then the function will **not** be Riemann integrable. We give a careful definition before stating this result as theorem.

Definition 2.3.4. *A set E of real numbers is* **bounded** *if and only if there exist real numbers p and q such that, for all x in E, $p \le x \le q$. Equivalently, E is bounded if and only if there exists a real number $M \ge 0$ such that, for all x in E, $-M \le x \le M$, i.e., $|x| \le M$.*

Suppose that f is a real function and that a set E is contained in the domain of f. We say that f **is bounded on** *E if and only if the set of values of f on E is bounded, i.e., there exists $M \ge 0$ such that, for all x in E, $|f(x)| \le M$.*

If f is not bounded on E, then we say that f **is unbounded on** *E.*

Example 2.3.5. As a particular case of a bounded set, note that closed, bounded intervals are intervals of the form $[a, b]$.

As an example of an unbounded function on a closed, bounded interval, consider

$$f(x) = \begin{cases} 0\ , & \text{if } x = 0; \\ \dfrac{1}{x}\ , & \text{if } 0 < x \leq 1. \end{cases}$$

This function is unbounded on $[0, 1]$, since f becomes arbitrarily large as x approaches 0 from the right.

On the other hand, the Extreme Value Theorem (see [2], or [4]) tells us that **every continuous function on a closed, bounded interval is bounded**.

The most basic way in which a function can fail to be Riemann integrable is given by:

Theorem 2.3.6. *If f is unbounded on the interval $[a, b]$, then f is* **not** *Riemann integrable on $[a, b]$.*

Proof. Suppose that f is unbounded on $[a, b]$, and let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Then, f must be unbounded on at least one of the subintervals $[x_{i-1}, x_i]$. Therefore, by changing the sample point s_i in this subinterval, you can make $|f(s_i)\Delta x_i|$ arbitrarily large. This means that, as the mesh of the partitions approaches 0, the Riemann sums do not approach a limit that is independent of how the sample points are chosen, i.e., the definite integral does not exist. $\square$

Example 2.3.7. Theorem 2.3.6 tells us immediately that the function $f(x)$ from Example 2.3.5 is **not** Riemann integrable.

Alright. Now we have a basic way in which functions can fail to be Riemann integrable. But, we want a theorem that tells us that lots of functions **are** Riemann integrable. There is such a theorem. Informally, what it says is that a function on a closed, bounded interval is Riemann integrable if and only if the function is bounded and the set of points where the function is discontinuous is "small". The technical requirement for "small" is that the set of points where f is discontinuous has to have *measure zero*. A discussion of this result, in full generality, is beyond the scope of the textbook. See [4] and [3].

However, all of the functions that **we** shall want to integrate in this book will either be continuous, or have a finite number of discontinuities; the good news is that the empty set and finite sets of points definitely have measure zero.

Thus, we have:

Theorem 2.3.8. *Bounded functions on closed bounded intervals, which have, at most, a finite number of discontinuities, are Riemann integrable.*

In particular, as continuous functions on closed, bounded intervals are bounded, all continuous functions on closed, bounded intervals are Riemann integrable.

Proof. We give the proof in Theorem 2.A.5. □

It is convenient to give a name to functions which may have, at most, a finite number of discontinuities.

Definition 2.3.9. *A real function f is* **piecewise-continuous** *on an interval I provided that f is defined on I and is continuous at all, except (possibly) a finite number of, points in I. In particular, a continuous function is also piecewise-continuous.*

Example 2.3.10. The function

$$g(x) = \begin{cases} 0\,, & \text{if } 0 \leq x \leq 0.1; \\ \dfrac{1}{x}\,, & \text{if } 0.1 < x \leq 1 \end{cases}$$

is piecewise-continuous and, hence, Theorem 2.3.8 tells that $g(x)$ is Riemann integrable.

It is interesting to compare this with the function $f(x)$ from Example 2.3.5:

$$f(x) = \begin{cases} 0\ , & \text{if } x = 0; \\ \dfrac{1}{x}\ , & \text{if } 0 < x \leq 1. \end{cases}$$

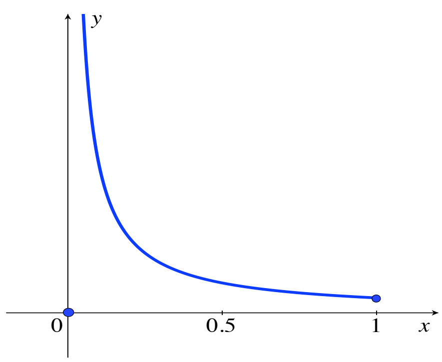


Figure 2.13: Graph of $y = f(x)$.

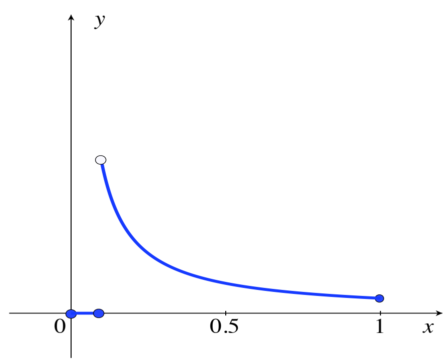


Figure 2.14: Graph of $y = g(x)$.

While the graphs of f and g may appear to be similar (or maybe they don't to you), it is important to keep in mind that unbounded functions are **very** different from bounded piecewise-continuous functions.

Before stating more theorems, we should translate our discussion about area in Example 2.2.9 into a proposition about definite integrals, a proposition which tells us how to visualize integrals in terms of area. This proposition could, instead, be used as a **definition** for area below or above graphs of certain types of functions; we choose to believe that you have a preconceived notion of area, and that the following proposition tells you rigorously what that area equals in terms of limits of Riemann sums, i.e., in terms of definite integrals.

Proposition 2.3.11. *Suppose that $a < b$, and f is Riemann integrable on $[a, b]$.*

1. *If $f \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx$ is equal to the area under the graph of $y = f(x)$ and above the interval $[a, b]$.*
2. *If $f \leq 0$ on $[a, b]$, then $-\int_a^b f(x)\,dx = \int_a^b [-f(x)]\,dx$ is equal to the area under the interval $[a, b]$ and above the graph of $y = f(x)$; thus, $\int_a^b f(x)\,dx$ is equal to negative the area under the interval $[a, b]$ and above the graph of $y = f(x)$.*

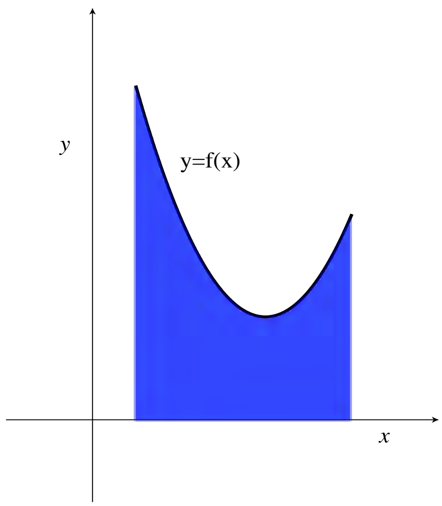


Figure 2.15: Integral of a positive function is area under the graph.

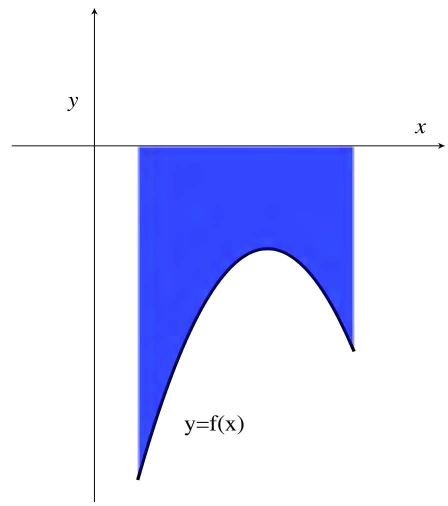


Figure 2.16: Integral of a negative function is negative the area above the graph.

Be careful: the area under the x-axis and above the graph is NOT, itself, negative; it's positive, as area always is. The point is that, to interpret the integral of a negative function in terms of area, you negate the positive area above the graph, which, of course, yields a negative number.

A special case of Proposition 2.3.11 is when $y = f(x) = k$, where k is a constant.

Proposition 2.3.12. *Suppose that $a < b$, and k is a constant. Then, the function $f(x) = k$ is Riemann integrable and*

$$\int_a^b k\,dx \;=\; k(b-a).$$

Proof. Of course, constant functions are continuous and, hence, Riemann integrable by Theorem 2.3.8. However, our proof of the formula will also prove that constant functions are Riemann integrable.

We shall simply show that **every** Riemann sum of $f(x) = k$, regardless of the partition $\mathcal{P}$ and the sample set $\mathcal{S}$, has $\mathcal{R}_{\mathcal{P}}^{\mathcal{S}}(f)$ equal to precisely $k(b-a)$. Thus, $L = k(b-a)$ clearly satisfies the conditions for L in Definition 2.3.1.

Let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$, and let $\mathcal{S} = \{s_1, \dots, s_n\}$ be a sample set for $\mathcal{P}$. Then,

$$\mathcal{R}_{\mathcal{P}}^{\mathcal{S}}(f) \;=\; \sum_{i=1}^{n} f(s_i)\Delta x_i \;=\; \sum_{i=1}^{n} k\Delta x_i \;=\; k\sum_{i=1}^{n} \Delta x_i,$$

and this last summation telescopes to yield $x_n - x_0 = b - a$. □

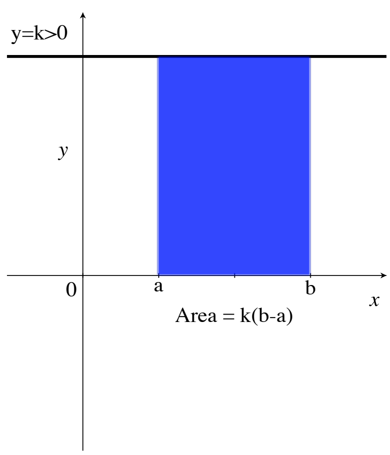


Figure 2.17: Area under $y = k > 0$.

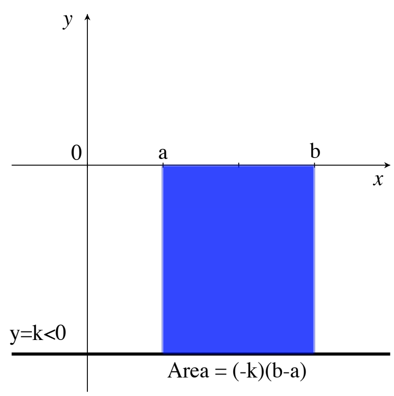


Figure 2.18: Area above $y = k < 0$.

Remark 2.3.13. The definition of the definite integral looks very technical. Thinking in terms of area can help you to see things in some cases, but many physical problems, which don't involve areas, deal with subdividing, taking Riemann sums, and taking limits. So, more generally, you may be wondering how you should think about

$$\int_a^b f(x)\,dx \;=\; \lim_{||\mathcal{P}||\to 0} \mathcal{R}_{\mathcal{P}}^{\mathcal{S}}(f) \;=\; \lim_{||\mathcal{P}||\to 0} \sum_{i=1}^{n} f(s_i)\Delta x_i\ .$$

One possible way to think about definite integrals – a way that we shall use extensively throughout this textbook – is informal, but is, nonetheless, extremely useful. We discuss the problems *infinitesimally*.

If we take a partition with mesh less than some positive δ, then all of the Δx_i's are less than δ, and any x-coordinate in the interval $[a,b]$ is within δ of one of the sample points. Thus, as the meshes approach zero, the sample points get arbitrarily close to each point in the interval $[a,b]$, and the subinterval(s) containing a given point become(s) arbitrarily small.

Thus, we think, intuitively, that, if we select an x in $[a,b]$, as the mesh of the partitions approaches 0, and we look at $\sum_{i=1}^{n} f(s_i)\Delta x_i$, in the limit, one of the summands "becomes" $f(x)$ times an infinitesimal change in x, represented by dx. That is, in the limit, we think that, for each x in the interval $[a,b]$, one of the $f(s_i)\Delta x_i$ approaches the infinitesimal summand $f(x)\,dx$, and the summation $\sum_{i=1}^{n}$ becomes the "continuous summation" $\int_a^b$.

While we always keep in the backs of our minds that we are really using partitions, sample points, Riemann sums, and taking limits, in many physical applications it is very intuitive, and time-saving, to **think of the definite integral as the continuous sum of infinitesimal contributions**, and to analyze problems using this informal terminology.

For instance, we arrived at Proposition 2.3.11 by looking back at our discussion in Example 2.2.9 and using the definition of the definite integral in Definition 2.3.1. Thus, we consider area as a limit of Riemann sums. But now, let's discuss area in terms of continuous sums of infinitesimal contributions. Hopefully, this will seem more intuitive to you, but, keep in mind, that the real **definition** of the integral is the limit of Riemann sums.

Consider a function $f(x)$, defined on the closed interval $[a,b]$, where $a < b$, and, for now, assume that $f(x) \geq 0$, for all x in $[a,b]$. If we assume that f is Riemann integrable on $[a,b]$, how do we picture $\int_a^b f(x)\,dx$?

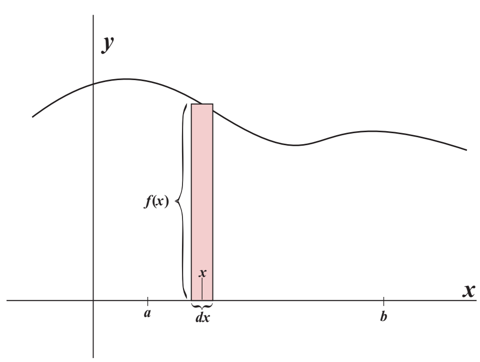


Figure 2.19: A rectangle with infinitesimal width and area.

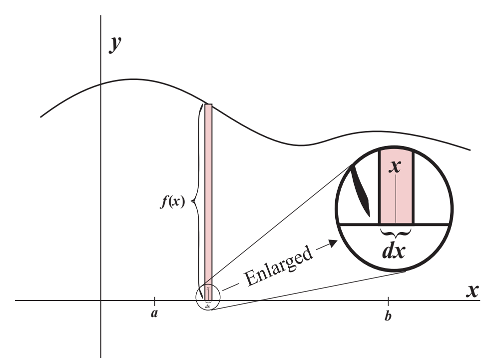


Figure 2.20: A magnified infinitesimal rectangle.

Pick some x-coordinate between a and b, and draw a rectangle of height $f(x)$ and a very small width, which we think of as the infinitesimal dx; the infinitesimal subinterval of width dx should include x. See Figure 2.19 and Figure 2.20, which show the type of thin rectangle that you might draw for yourself, in order to represent a rectangle of infinitesimal width and area, together with a "fancier" illustration of a really (think: "infinitesimally") thin rectangle being magnified. Intuitively, we think "the **only** x-coordinate in the infinitesimal subinterval is x." Why? Because if there were some other x-coordinate, say $\hat{x}$, in the subinterval, then the subinterval would have to have length equal to at least $|x-\hat{x}|$, and so would not be infinitesimal. Our infinitesimally wide rectangle has infinitesimal area dA, which equals the height times the width, i.e., $dA = f(x)dx$.

Now, to obtain the total area under the graph of f and above the interval $[a, b]$ on the x-axis, we simply take the continuous sum of all of the infinitesimal areas as x goes from a to b. Thus, the total area is

$$\int_{x=a}^{x=b} dA \;=\; \int_a^b f(x)\,dx.$$

Note that, on the first integral above, we had to include "$x =$" in the limits of integration; by default, the limits of integration refer to the variable that you have d of, that is, the variable in the differential. So, if we had not included "$x =$" in the limits of integration in $\int_{x=a}^{x=b} dA$, then it would have meant that A was going from a to b, not x. As we have dx in the righthand integral above, we do not need to explicitly state in that integral that the limits of integration refer to x.

Understand – we are not claiming that we have given actual definitions that make what we have written mean anything rigorous; we are merely trying to get you used to this manner of intuitive thinking, in terms of continuous sums of infinitesimal contributions.

Of course, this is precisely what we concluded in Example 2.2.9, except there we didn't call it the integral; we simply said the area under the graph was the limit of the Riemann sums. That's what we are saying here too, it's just that we have new terminology, the definite integral, together with our new notation for the integral, and instead of explicitly adding up Riemann sums, we talk about taking a continuous sum of infinitesimal contributions.

What do we do when $f(x) \leq 0$ (and is still Riemann integrable)? We draw a picture like that in Figure 2.19, except that now the infinitesimal rectangle lies under the x-axis and above the graph; thus, the rectangle looks like that in Figure 2.11, except that now we think of the width as the infinitesimal dx and the height is $-f(x)$. Therefore, the total area A between the x-axis and the graph of $y = f(x)$ is

$$A = \int_{x=a}^{x=b} dA \;=\; \int_a^b -f(x)\,dx \;=\; -\int_a^b f(x)\,dx,$$

and, hence, $\int_a^b f(x)\,dx = -A$. (Note that we "cheated" a bit here, and used that we could "pull out" the negative sign in the integral; we won't really know that this is "legal" until we have Theorem 2.3.19.)

Finally, what do we do when f is Riemann integrable, and $f(x) \geq 0$ on some (finite number of) closed subintervals in $[a,b]$, and $f(x) \leq 0$ on (a finite number of) other closed subintervals in $[a,b]$? In this case, as we discussed at the end of Example 2.2.9, $\int_a^b f(x)\,dx$ is a sum of the contributions from where $f \geq 0$ and where $f \leq 0$, which means that $\int_a^b f(x)\,dx$ can be interpreted as the area under the graph of $y = f(x)$ and above the interval $[a,b]$ minus the area above the graph and under the interval $[a,b]$. (Again, we "cheated" a bit here, and used that we could split the interval over $[a,b]$ into the sum of integrals over subintervals whose union is $[a,b]$; we won't really know that this is "legal" until we have Theorem 2.3.16 and/or Theorem 2.3.18.)

We now need to state a number of theorem about definite integrals. Despite the fact that all of the examples of Riemann integrable functions that we shall look at will be bounded, piecewise-continuous functions, the remaining theorems of this section are true for completely general Riemann integrable functions, and it's just as easy to state and prove them in that generality; so we will.

The following theorem says that, as far as integration is concerned, you can "ignore" what happens at a finite set of points in the domain. We give the proof in the Technical Matters section; see Theorem 2.A.8.

Theorem 2.3.14. *Suppose that f and g are defined on a closed interval $[a,b]$, and that, except possibly for a finite set points in $[a,b]$, f and g are equal at each point in $[a,b]$.*

Then, f is Riemann integrable on $[a,b]$ if and only if g is, and when f and g are Riemann integrable,

$$\int_a^b f(x)\,dx \;=\; \int_a^b g(x)\,dx.$$

Example 2.3.15. The function given by

$$f(x) = \begin{cases} 7\,, & \text{if } 1 \leq x < 3 \text{ or } 3 < x \leq 5; \\ 4\,, & \text{if } x = 3 \end{cases}$$

is equal to the function that is constantly 7 on the interval $[1,5]$, except at the point $x=3$.

Thus, by Theorem 2.3.14

$$\int_1^5 f(x)\,dx \;=\; \int_1^5 7\,dx \;=\; 7(5-1) \;=\; 28,$$

where the next-to-last equality follows from Proposition 2.3.12.

In terms of area, it seems reasonable that what happens at a finite number of points shouldn't affect the integral, since the area under (or over) a single point and above (or below, respectively) the x-axis should be zero, and so does not change the area being considered. See Figure 2.21.

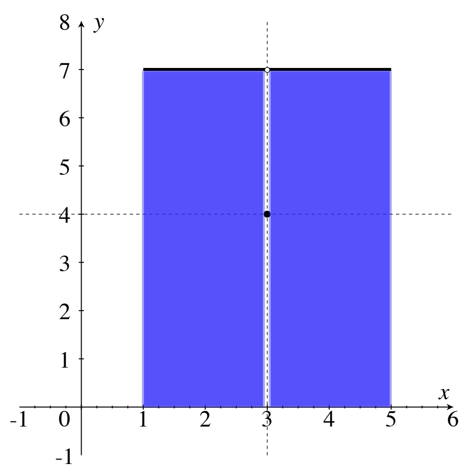


Figure 2.21: Area under a point is zero.

The following theorem is about subdividing, or splitting, the interval over which you're integrating. Thinking in terms of continuous sums, Item 1 of the theorem says that, if the continuous sum on a bigger interval is defined, then so is the continuous sum on any smaller subinterval; Item 2 simply says that the sum of the infinitesimal contributions over all x in the interval $[a,b]$ is equal to sum of those contributions as x goes from a to some intermediate x-coordinate c plus the sum of the contributions as x goes on from c to b.

Theorem 2.3.16. *Suppose that $a < b$.*

1. If f is Riemann integrable on $[a, b]$ and $a \leq c < d \leq b$, then f is Riemann integrable on $[c, d]$, i.e., if f is Riemann integrable on a given closed interval, then f is Riemann integrable on any closed subinterval of the given interval.

2. Suppose that $a < c < b$. Then, f is Riemann integrable on $[a, b]$ if and only if f is Riemann integrable on $[a, c]$ and $[c, b]$ and, when these equivalent conditions hold,

$$\int_a^b f(x)\,dx \;=\; \int_a^c f(x)\,dx \;+\; \int_c^b f(x)\,dx.$$

Proof. See Theorem 2.A.6 in the Technical Matters section, Section 2.A. $\square$

For a non-negative, Riemann integrable function f, Item 2 of Theorem 2.3.16 is easy to picture in terms of area; see Figure 2.22.

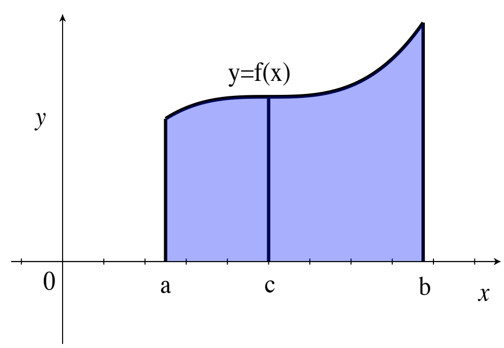


Figure 2.22: Area over $[a, b]$ equals area over $[a, c]$ plus area over $[c, b]$.

You may wonder about counting the line segment at $x = c$ once in $\int_a^b f(x)\,dx$ versus twice in the sum of integrals from a to c and from c to b, but, remember: the area of a line segment is 0, and adding it or subtracting it has no affect on the area calculation.

The formula in Item 2 of Theorem 2.3.16 is so useful that we would like for it to be true regardless of what inequalities (or equalities) are satisfied by a, b, and c. For instance, we would like for the formula to hold if $a = c$. This means that we must have

$$\int_a^b f(x)\,dx \;=\; \int_a^a f(x)\,dx \;+\; \int_a^b f(x)\,dx,$$

and so, we would need for $\int_a^a f(x)\,dx$ to equal 0.

Also, if $a = b$, we should have

$$\int_a^a f(x)\,dx \;=\; \int_a^c f(x)\,dx \;+\; \int_c^a f(x)\,dx.$$

If $\int_a^a f(x)\,dx = 0$, then we need for $\int_c^a f(x)\,dx$ to equal $-\int_a^c f(x)\,dx$.

Therefore, we make the following definitions:

Definition 2.3.17. *If a is in the domain of f, we say that f is Riemann integrable on the interval $[a, a]$ and define*

$$\int_a^a f(x)\,dx \;=\; 0.$$

If $a < b$, and f is Riemann integrable on $[a, b]$, then we define

$$\int_b^a f(x)\,dx \;=\; -\int_a^b f(x)\,dx.$$

With the definitions above, and given Theorem 2.3.16, it is easy to check that

Theorem 2.3.18. *Suppose that f is Riemann integrable on a closed interval containing a, b, and c, then*

$$\int_a^b f(x)\,dx \;=\; \int_a^c f(x)\,dx \;+\; \int_c^b f(x)\,dx,$$

regardless of the order of a, b, and c.

The following two theorems describe fundamental properties of integration, and are true essentially because the corresponding results for Riemann sums are true.

Theorem 2.3.19. (**Linearity of Integration**) *Definite integration over a closed interval is a linear operation, i.e., if f and g are Riemann integrable on $[a, b]$, then, for all constants r and s, the function $rf + sg$ is Riemann integrable on $[a, b]$, and*

$$\int_a^b \big(rf(x) + sg(x)\big)\,dx \;=\; r\int_a^b f(x)\,dx \;+\; s\int_a^b g(x)\,dx.$$

Proof. This proof actually splits into two distinct parts; that the integral of a sum is the sum of the integrals, and that you can move constants, multiplied in the integrand, outside of the integral.

The statement about constant multiples is easy, and follows from the fact that constants distribute over Riemann sums; we leave this part of the proof as an exercise, Exercise 52. The fact that the integral of a sum is the sum of the integrals is harder; we prove this in Theorem 2.A.7 in the Technical Matters section, Section 2.A. □

If you think of integrals in terms of areas, the following theorem seems fairly obvious; see Figure 2.23.

Theorem 2.3.20. (**Monotonicity of Integration**) *If f and g are Riemann integrable on the interval $[a, b]$ and, for all x in $[a, b]$, $f(x) \le g(x)$, then*

$$\int_a^b f(x)\,dx \;\le\; \int_a^b g(x)\,dx.$$

Proof. The inequality $f(x) \le g(x)$ implies that the corresponding inequality holds on the Riemann sums for each sampled partition. As we are assuming that f and g are Riemann integrable, it is easy to conclude that the integral of f on $[a, b]$ is less than or equal to the integral of g. We give the details in Theorem 2.A.9. □

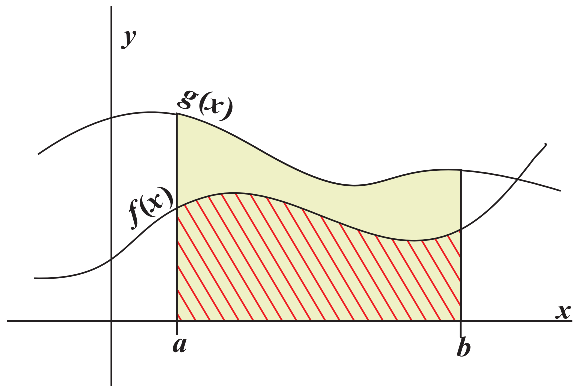


Figure 2.23: Comparing areas under the two graphs.

Theorem 2.3.21. *If f is Riemann integrable on the interval $[a, b]$, then so is $|f|$, and,*

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Proof. The proof that f being Riemann integrable implies that $|f|$ is Riemann integrable is very technical; we refer you to Theorem 3.3.5 of [4]. Assuming this fact, the given inequality follows as a corollary to Theorem 2.3.20 and Theorem 2.3.19.

To see this, note that $-|f(x)| \leq f(x) \leq |f(x)|$. Applying Theorem 2.3.20 and Theorem 2.3.19, we obtain that $-|f|$ is Riemann integrable on $[a, b]$ and

$$-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx.$$

The inequality in the theorem is now immediate. □

Now, consider the continuous function $f(x) = x$ on a closed interval $[a, b]$. Theorem 2.3.8 tells us that

$$\int_a^b x\, dx$$

exists. Given that the integral exists, Theorem 2.3.3 tells us that we may calculate the value of the integral by using the limit of Riemann sums over any sequence of sampled partitions, as long as the meshes of the partitions approach zero.

So, we will let $\mathcal{P}_n$ be the partition of $[a, b]$ into n subintervals of equal length, i.e., letting $\Delta x = (b-a)/n$,

$$\mathcal{P}_n \;=\; \{a, a+\Delta x, a+2\Delta x, \ldots, a+(n-1)\Delta x, a+n\Delta x\},$$

where $x_i = a + i\Delta x$ and, of course, $b = a + n\Delta x$.

We will use the right Riemann sums, so that our sample points are also given by $s_i = a+i\Delta x$. Let's denote the corresponding Riemann sum by simply $\mathcal{R}_n$. For $f(x) = x$, we obtain

$$\mathcal{R}_n \;=\; \sum_{i=1}^{n} f(s_i)\Delta x_i \;=\; \sum_{i=1}^{n} \left[(a+i\Delta x)\cdot \Delta x\right].$$

Using linearity of summations, Proposition 2.1.4, and Item a from Corollary 2.1.11, we find

$$\mathcal{R}_n \;=\; a\Delta x \sum_{i=1}^{n} 1 \;+\; (\Delta x)^2 \sum_{i=1}^{n} i \;=\; an\Delta x \;+\; (\Delta x)^2 \cdot \frac{n(n+1)}{2} \;=$$

$$a(b-a) \;+\; \frac{(b-a)^2}{2}\cdot\frac{n^2+n}{n^2} \;=\; a(b-a) \;+\; \frac{(b-a)^2}{2}\cdot\left(1+\frac{1}{n}\right).$$

Therefore, we find

$$\int_a^b x\,dx \;=\; \lim_{n\to\infty} \mathcal{R}_n \;=\; \lim_{n\to\infty}\left[a(b-a) \;+\; \frac{(b-a)^2}{2}\cdot\left(1+\frac{1}{n}\right)\right] \;=$$

$$a(b-a) + \frac{(b-a)^2}{2} \;=\; (b-a)\left[a + \frac{b-a}{2}\right] \;=\; (b-a)\cdot\frac{b+a}{2} \;=\; \frac{b^2-a^2}{2}.$$

Hence, we have shown

Proposition 2.3.22.

$$\int_a^b x\,dx \;=\; \frac{b^2}{2} - \frac{a^2}{2}.$$

Note that, even though we assumed $a < b$ in our proof of Proposition 2.3.22, it follows from Definition 2.3.17 that the proposition above remains true when $b \leq a$.

Example 2.3.23. Using the area interpretation of the integral, it is easy to derive Proposition 2.3.22 geometrically.

First, suppose that $b > 0$. Then, the integral $\int_0^b x\,dx$ equals the area under the graph of $y = x$ and above the interval $[0, b]$ on the x-axis. This is the area of a triangle of width b and height b.

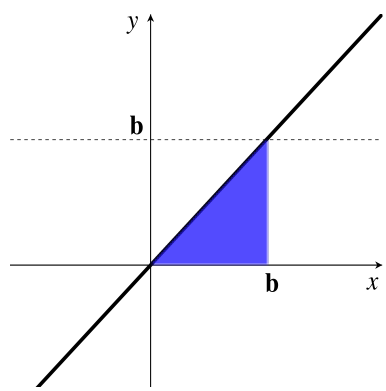


Figure 2.24: Area under $y = x$.

Thus,

$$\int_0^b x\,dx \;=\; \frac{b^2}{2}.$$

This formula holds even if $b < 0$ for, in that case,

$$\int_0^b x\,dx \;=\; -\int_b^0 x\,dx,$$

and $\int_b^0 x\,dx$ is equal to negative the area below the interval $[b, 0]$ and above the graph of $y = x$; this area is again $b^2/2$.

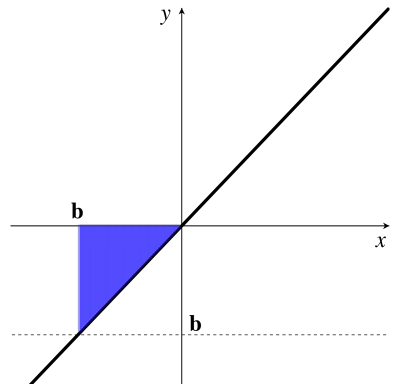


Figure 2.25: Area above $y = x$.

Hence,

$$\int_0^b x\,dx \;=\; -\int_b^0 x\,dx \;=\; -\left(-\frac{b^2}{2}\right) \;=\; \frac{b^2}{2}.$$

Of course, it follows that

$$\int_0^a x\,dx \;=\; \frac{a^2}{2},$$

and so, using Theorem 2.3.18, we find

$$\int_a^b x\,dx \;=\; \int_a^0 x\,dx \;+\; \int_0^b x\,dx \;=\; -\int_0^a x\,dx \;+\; \int_0^b x\,dx \;=\; -\frac{a^2}{2} + \frac{b^2}{2},$$

which agrees with what we obtained in Proposition 2.3.22.

Example 2.3.24. Now that we have Proposition 2.3.22 and Proposition 2.3.12, we can combine them with linearity to calculate

$$\int_a^b (mx + c)\,dx,$$

where m and c are constants.

We find

$$\int_a^b (mx + c)\,dx \;=\; m\int_a^b x\,dx \;+\; \int_a^b c\,dx \;=\; m\left(\frac{b^2 - a^2}{2}\right) \;+\; c(b - a).$$

Example 2.3.25. Consider the integral $\int_{-3}^{6} |x|\, dx$.

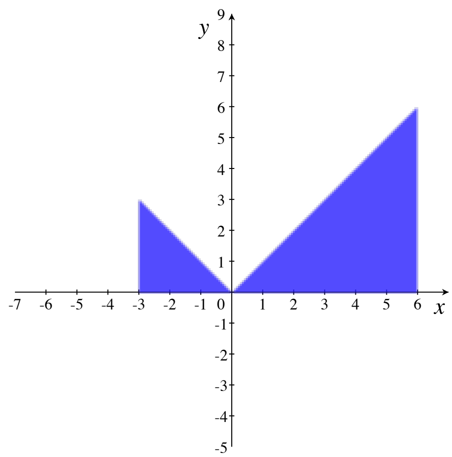


Figure 2.26: Area under $y = |x|$.

By Theorem 2.3.16, we have

$$\int_{-3}^{6} |x|\, dx \;=\; \int_{-3}^{0} |x|\, dx \;+\; \int_{0}^{6} |x|\, dx.$$

Why split up the integral like this? Because $|x| = x$, if $x \geq 0$, and $|x| = -x$, if $x \leq 0$. Thus, $|x| = -x$ on the interval $[-3, 0]$, and $|x| = x$ on the interval $[0, 6]$. Therefore, we obtain

$$\int_{-3}^{6} |x|\, dx \;=\; \int_{-3}^{0} |x|\, dx \;+\; \int_{0}^{6} |x|\, dx \;=\; \int_{-3}^{0} -x\, dx \;+\; \int_{0}^{6} x\, dx \;=$$

$$-\int_{-3}^{0} x\, dx \;+\; \int_{0}^{6} x\, dx \;=\; -\left(\frac{0^2 - (-3)^2}{2}\right) \;+\; \left(\frac{6^2 - 0^2}{2}\right) \;=\; \frac{9}{2} + \frac{36}{2} \;=\; \frac{45}{2}.$$

Note that this agrees with what you would get from interpreting the integral in terms of area; see Figure 2.26.

Example 2.3.26. The cosine function is continuous and, for all x, $-1 \leq \cos x \leq 1$. If we want bounds on the integral of $\cos x$, we can apply Theorem 2.3.20 twice.

For instance, even though we do not yet know how to calculate $\int_0^{\pi/2} \cos x\, dx$, we can conclude that

$$-\frac{\pi}{2} \;=\; \int_0^{\pi/2} -1\, dx \;\leq\; \int_0^{\pi/2} \cos x\, dx \;\leq\; \int_0^{\pi/2} 1\, dx \;=\; \frac{\pi}{2}.$$

In fact, as we shall see later, $\int_0^{\pi/2} \cos x\, dx = 1$, which is, indeed, between $-\pi/2$ and $\pi/2$.

Proposition 2.3.27.

$$\int_a^b x^2\, dx \;=\; \frac{b^3}{3} - \frac{a^3}{3}.$$

Proof. We will prove that, if $b > 0$, then $\int_0^b x^2\, dx = b^3/3$. The case where $b < 0$ is similar, and we leave it as an exercise. Once you know that $\int_0^b x^2\, dx = b^3/3$, the proposition follows from applying Theorem 2.3.18:

$$\int_a^b x^2\, dx \;=\; \int_a^0 x^2\, dx \;+\; \int_0^b x^2\, dx \;=\; -\int_0^a x^2\, dx \;+\; \int_0^b x^2\, dx \;=$$

$$-\frac{a^3}{3} + \frac{b^3}{3}.$$

So, assume that $b > 0$. How do we show that $\int_0^b x^2\, dx = b^3/3$? As x^2 is continuous, we know that the integral exists, and so we may calculate it by taking a limit of Riemann sums, in which the meshes of our partitions approach 0. This was how we derived Proposition 2.3.22.

Let $\mathcal{P}_n$ be the partition of $[0, b]$ into n subintervals of equal length, i.e., the partition in which $x_i = ib/n$, and $\Delta x_i = b/n$. If we once again use right Riemann sums, our sample set R_n is the one in which s_i also equals ib/n.

Our corresponding Riemann sum for x^2 is

$$\mathcal{R}_n \;=\; \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^2 \cdot \frac{b}{n} \;=\; \frac{b^3}{n^3} \sum_{i=1}^{n} i^2.$$

By Item (b) of Corollary 2.1.11, we know that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

and so

$$\mathcal{R}_n = \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{b^3}{6} \cdot \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

Therefore, as $n \to \infty$, $\mathcal{R}_n \to (b^3/6)(2) = b^3/3$ and, hence, $\int_0^b x^2\,dx = b^3/3$, as we wanted to show. □

Example 2.3.28. The area interpretation of the definite integral lets us calculate the definite integral of $f(x)$ in cases where the region between the x-axis and the graph of $y = f(x)$ consists of rectangles, triangles, and trapezoids.

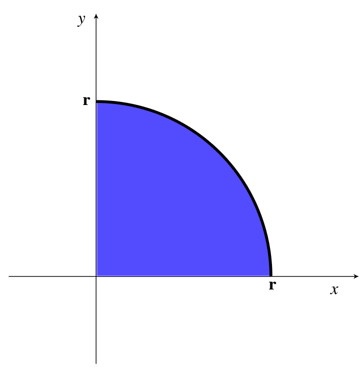


Figure 2.27: Area under $y = \sqrt{r^2 - x^2}$.

Of course, we also know that the area inside a circle of radius r is πr^2, which enables us to calculate integrals like:

$$\int_0^r \sqrt{r^2 - x^2}\,dx,$$

where $r > 0$ is a constant. This integral represents the area inside the first quadrant quarter of a circle of radius r, centered at the origin; see Figure 2.27.

Thus,

$$\int_0^r \sqrt{r^2 - x^2}\,dx = \frac{\pi r^2}{4}.$$

Example 2.3.29. Let's return to the rod of varying density in Example 2.2.8, and discuss the problem of calculating its total mass, but now we shall use our "continuous sum of infinitesimal contributions" language.

A circular rod, of length 1 meter, and cross-sectional area 0.01 m^2 (i.e., of radius $0.1/\sqrt{\pi}$ meters) is lying along the x-axis.

Figure 2.28: A rod of varying density.

For all x such that $0 \leq x \leq 1$, at each point in the cross section of the rod at x meters, the density of the rod is $\delta(x) = (1 + x)$ kg/m^3. What is the total mass of the rod?

Solution:

We shall present the solution using the language of infinitesimals. Consider the infinitesimal amount of volume dV lying along the infinitesimal subinterval at x of infinitesimal length dx.

The infinitesimal volume dV is just equal to the cross-sectional area times the length dx, i.e.,

$$dV \;=\; 0.01\,dx \quad \mathrm{m}^3,$$

and the infinitesimal mass dM of this infinitesimal volume is simply the density at x times the (infinitesimal) volume dV, i.e.,

$$dM \;=\; \delta(x)\,dV \;=\; (1+x)0.01\,dx \quad \mathrm{kg}.$$

Finally, the total mass M of the rod is the continuous sum of all of the infinitesimal masses as x goes from 0 to 1 meter, i.e.,

$$M \;=\; \int_{x=0}^{x=1} dM \;=\; \int_0^1 (1+x)0.01\,dx \quad \mathrm{kg}.$$

Suppose that $a < b$. We now want to develop a reasonable notion of "the mean, or average, value of f on the interval $[a, b]$."

What **should** the average value of f mean? Well...if there were a finite number of f-values, we would simply add up all of the values and then divided by the number of values; this is what is normally meant by the "average" or "mean". The problem, of course, is that we have an infinite number of values $f(x)$, one for each x in the interval $[a, b]$. How do we deal with this?

As you may have guessed, we take the average of a finite number of f-values, and then take a limit. Let $n \geq 1$ be an integer. Subdivide the interval $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$. As usual, in the i-th subinterval, we pick a sample point s_i. We may now take the average value of f at the n sample points; we obtain

$$\frac{f(s_1) + f(s_2) + \cdots + f(s_n)}{n} = \frac{\sum_{i=1}^{n} f(s_i)}{n}.$$

So, what would be a reasonable notion of the average value of f on the entire interval? We can try taking the limit of the average above, as n approaches ∞, but why would this limit exist, and what does it have to do with integration?

Things become more clear if we multiply the numerator and denominator of the above fractions by Δx. Then, we obtain

$$\frac{\sum_{i=1}^{n} f(s_i)\Delta x}{n\Delta x} = \frac{\sum_{i=1}^{n} f(s_i)\Delta x}{b - a}.$$

If f is Riemann integrable on $[a, b]$, then, as $n \to \infty$, this last numerator approaches $\int_a^b f(x)\,dx$, and so, the limit of the average value of f at the sample points approaches

$$\frac{1}{b - a}\int_a^b f(x)\,dx.$$

Therefore, we make the following definition:

Definition 2.3.30. *Suppose that $a < b$ and that f is Riemann integrable on $[a, b]$. Then, we define the* **mean value, or average value, of f on** $[a, b]$ *to be*

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

For continuous functions, we have the following theorem about the mean value of f.

Theorem 2.3.31. (**Mean Value Theorem for Integration**) *Suppose that f is continuous on the closed interval $[a, b]$. Then, there exists c in $[a, b]$ such that*

$$\int_a^b f(x)\,dx \;=\; (b-a)f(c).$$

Thus, if $a < b$, there exists c in the interval $[a, b]$ such that $f(c)$ equals the mean value of f on $[a, b]$, i.e., f attains its mean value on $[a, b]$ at some point in $[a, b]$.

Proof. If $a = b$, then the theorem is obviously true. So, assume that $a < b$.

As f is continuous on $[a, b]$, the Extreme Value Theorem (see [2]), tells us that f attains a global minimum value m on $[a, b]$ and a global maximum value M on $[a, b]$.

Thus, for all x in $[a, b]$, $m \leq f(x) \leq M$. By Theorem 2.3.20, this implies that

$$m(b-a) \;=\; \int_a^b m\,dx \;\leq\; \int_a^b f(x)\,dx \;\leq\; \int_a^b M\,dx \;=\; M(b-a).$$

Since $a < b$, $b - a \neq 0$, and so we may divide to obtain

$$m \;\leq\; \frac{1}{b-a}\int_a^b f(x)\,dx \;\leq\; M.$$

Now, the function f is continuous on $[a, b]$, attains the values m and M, and $\frac{1}{b-a}\int_a^b f(x)\,dx$ is a value between m and M. The Intermediate Value Theorem (see [2]), implies that there exists c in $[a, b]$ such that

$$f(c) \;=\; \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Multiply each side of the above equality by $b - a$ to obtain the theorem. $\square$

Remark 2.3.32. The area interpretation of integration helps us visualize the mean value of a function on an interval. For instance, if $f \geq 0$ and continuous on the interval $[a, b]$, then the mean value of f on $[a, b]$ is the height of a rectangle with base $[a, b]$ such that the rectangle has the same area as that below the graph of f and above the interval $[a, b]$.

For instance, as we shall see in Example 2.4.13, the mean value of $y = f(x) = 1/(1 + x^2)$ on the interval $[-1, 1]$ is $\pi/4$.

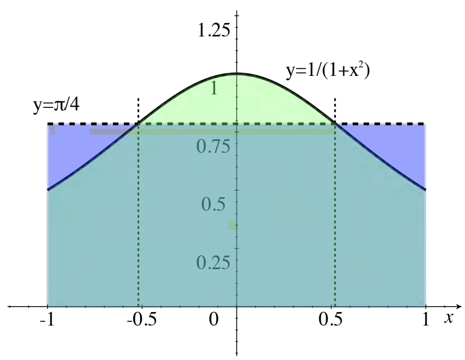


Figure 2.29: Equal areas under $y = 1/(1 + x^2)$ and $y = \pi/4$.

The Mean Value Theorem for Integration guarantees that, since $f(x) = 1/(1 + x^2)$ is continuous, there exists at least one x value in $[-1, 1]$ at which f attains its mean value. In fact, we see in Figure 2.31 that there are two such x values, approximately ± 0.5. We calculate these values precisely in Example 2.4.13.

2.3.1 Exercises

Calculate the definite integrals.

1. $\displaystyle\int_3^7 5\,dx.$

2. $\displaystyle\int_{-3}^6 5t - 7\,dt.$

3. $\displaystyle\int_{-4}^8 |u-4| + 2\,du.$

4. $\displaystyle\int_{-3}^7 2|4-8z|\,dz.$

5. $\displaystyle\int_0^{12} 5w^2\,dw.$

6. $\displaystyle\int_{-3}^7 (v+3)\,dv.$

7. $\displaystyle\int_2^{-3} x^2 + x\,dx.$

8. $\displaystyle\int_{-4}^6 f(x)\,dx$, where $f(x) = \begin{cases} 3 & x < -1 \\ -5 & x \geq -1. \end{cases}$

9. $\displaystyle\int_4^4 e^{-x^2}\,dx.$

Answer the following true/false questions. If the statement is true, why is it true? If the statement is false, provide a counterexample.

10. If f is differentiable on an open interval containing $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx$ exists.

11. If $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, then $f(x) \leq g(x)$ for all $x \in [a,b]$.

12. If f is integrable, then f is continuous.

13. $\displaystyle\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$ for all integrable functions f.

In each of exercises 14 through 18, calculate the average value of the function over the interval.

14. $c(x) = -10$, $[-11, 9]$.

15. $g(x) = 12x - 10$, $[4, 9]$.

16. $h(x) = \dfrac{x^2}{6}$, $[0, 12]$.

17. $m(t) = |t| + 3$, $[-3, 6]$.

18. $p(x) = \begin{cases} 2x & x \neq 1 \\ 0 & x = 1 \end{cases}$, $[-2, 4]$.

19. Recall that the height of a projectile with initial velocity v_0 and h_0 is

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0.$$

Calculate the average height between times $t = 0$ and $t = t_1$.

20. Prove that the function $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational;} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$ is not integrable.

21. Construct an example of a function that is integrable but that does not satisfy the mean value property. That is, construct a function that does not achieve its average value over its domain.

In each of Exercises 22 through 25, you are given the velocity function of a particle, in meters per second. Calculate the average velocity of the function over the specified interval.

22. $v(t) = 3t + 2$, $[4, 8]$.

23. $v(t) = t^2 - 5t + 3$, $[2, 6]$.

24. $v(t) = t^3 - 2t^2 + 3t + 2$, $[-1, 3]$.

25. $v(t) = |t - 4|$, $[2, 6]$.

In Exercises 26 - 29, you are given the same velocity function and interval as in the previous four problems. For each problem, calculate (a) the acceleration function, and (b) the average acceleration over the interval.

26. $v(t) = 3t + 2$, $[4, 8]$.

27. $v(t) = t^2 - 5t + 3$, $[2, 6]$.

28. $v(t) = t^3 - 2t^2 + 3t + 2$, $[-1, 3]$.

29. $v(t) = |t - 4|$, $[2, 6]$.

30. Suppose $p_1(t)$ and $p_2(t)$ are the position functions of two particles and that $p_2(t) = p_1(t) + C$ where C is some constant. The position functions have common domain $[a, b]$.

a. What is the relationship between the average positions of the two particles?

b. What is the relationship between the average velocities of the two particles?

c. What is the relationship between the average accelerations of the two particles?

31. Suppose that $v_1(t)$ and $v_2(t)$ are the velocity functions of two particles and that $v_2(t) = v_1(t) + C$ where C is some constant. The velocity functions have common domain $[a, b]$.

    a. What is the relationship between the average velocities of the two particles?

    b. What is the relationship between the average accelerations of the two particles?

32. Is the function below integrable on $[-\pi, \pi]$?

$$f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

33. Is the function below integrable on $[-\pi, \pi]$?

$$g(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Estimate the definite integrals using the given partitions using the left endpoint sample set.

34. $\displaystyle\int_0^1 \sin(\pi x)\,dx$, $\mathcal{P} = \{0, 1/4, 1/2, 3/4, 1\}$.

35. $\displaystyle\int_0^1 \sqrt{x}\,dx$, $\mathcal{P} = \{0, 1/4, 1/3, 3/4, 1\}$.

36. $\displaystyle\int_{-1}^1 \frac{1}{1+x^2}\,dx$, $\mathcal{P} = \{-1, -1/2, 0, 1/2, 1\}$.

37. $\displaystyle\int_{-1}^1 \cosh x\,dx$, $\mathcal{P} = \{-1, -1/2, 0, 1/2, 1\}$.

Calculate the definite integrals below by determining the area enclosed between their graphs and the x-axis. Remember that area below the x-axis is counted with a minus sign in the integral.

38. $\displaystyle\int_{-4}^7 2x - 3\,dx$.

39. $\int_{-5}^{5} \sqrt{25-x^2}+4\,dx.$

40. $\int_{2}^{6} |x-3|+5\,dx.$

The idea of using integrals to calculate the area between a curve and the x-axis can be generalized. To calculate the area of the region between the graphs of $f(x)$ and $g(x)$, we just need to notice that if $f > g \geq 0$, then the area between the graphs of f and g is the area bounded by the graph of f and the x-axis minus the area bounded by the graph of g and the x-axis. By the linearity of the integral, this area is given by $\int_a^b \big(f(x) - g(x)\big)\,dx$. In each of Exercises 41 through 44, calculate the area bounded by the two graphs along the interval.

41. $f(x) = 4x$, $g(x) = 2$, $[8, 12]$.

42. $f(x) = 5x$, $g(x) = 3x$, $[0, 10]$.

43. $f(x) = x$, $g(x) = x^2$, $[0, 1]$.

44. $f(x) = \sqrt{25-x^2}$, $g(x) = 3$, $[-4, 4]$.

45. Consider a triangle contained entirely in the first quadrant with vertices $P = (x_1, y_1)$, $Q = (x_2, y_2)$ and $R = (x_2, y_3)$. Assume further that $x_1 < x_2$ and that $y_3 > y_2$.

    a. Show that the equations of the lines between P and R, and P and Q are

$$
\begin{aligned}
f(x) &= \frac{y_3 - y_1}{x_2 - x_1}\,x - \frac{x_2(y_3 - y_1)}{x_2 - x_1} + y_3 \\
g(x) &= \frac{y_2 - y_1}{x_2 - x_1}\,x - \frac{x_2(y_2 - y_1)}{x_2 - x_1} + y_2
\end{aligned}
$$

    respectively.

    b. Calculate $\int_{x_1}^{x_2} f(x) - g(x)\,dx$ and conclude that the area of the triangle is

$$\frac{1}{2}\left[x_2y_3 - x_2y_2 + x_1y_2 - x_1y_3\right] = \frac{1}{2}(x_1 - x_2)(y_2 - y_3).$$

46. Suppose a 10 meter long rod lies along the x-axis The rod is a rectangular prism with cross-sectional area 2 m^2. The density of the rod is given by $\delta x = (x^2+1)$ kg / m^3. Express the total mass of the rod in terms of a definite integral. You need not calculate the integral.

47. Suppose a 10 meter metallic cone lies along the x-axis with its vertex at the origin. More specifically, the projection of the cone onto the xy plane is defined by the line segments $y = x$ and $y = -x$ for $x \in [0, 10]$. The density at each point in the cross section of the rod is given by $\delta(x) = (12-x)$ kg/m^3. Express the total mass of the cone in terms of a definite integral. You need not calculate the integral. Hint: in this case, dA, the infinitesimal area, is no longer constant.

Recall that a parameterized planar curve with domain $[a, b]$ is a map $\vec{\alpha}(t)$ into the xy-plane. That is, $\vec{\alpha}(t) = (x(t), y(t))$. Definite integrals can be used to calculate the length of curves. Specifically, if $\vec{\alpha}(t)$ is a differentiable curve on $[a, b]$, the length of the curve is $\int_a^b |\vec{\alpha}'(t)|\, dt$. Calculate the lengths of the curves in Exercises 48 - 51.

48. $\vec{\alpha}(t) = (t, 3)$, $[-2, 4]$.

49. $\vec{\alpha}(t) = (t, 4t)$, $[0, 5]$.

50. $\vec{\alpha}(t) = (t, mt)$, $[a, b]$.

51. $\vec{\alpha}(t) = (R\cos t, R\sin t)$, $[0, \pi]$, where $R \geq 0$.

52. Prove the constant multiple portion of Theorem 2.3.19, i.e., prove that if f is Riemann integrable on the closed interval $[a, b]$ and c is a constant, then cf is Riemann integrable on $[a, b]$, and
$$\int_a^b cf(x)\, dx \;=\; c\int_a^b f(x)\, dx.$$

2.4 The Fundamental Theorem of Calculus

The definite integral has many applications, all of which stem from the fact that the definition of the definite integral as a limit of Riemann sums makes the definite integral a "continuous sum of infinitesimal contributions". However, actually **calculating** definite integrals by taking limits of Riemann sums is extraordinarily painful, and, for definite integrals to really be manageable, we need an easy way to calculate these limits of Riemann sums that come up in so many physical situations.

The second part of the *Fundamental Theorem of Calculus*, Theorem 2.4.10, provides this "easy" method; it tells us that we may calculate definite integrals by anti-differentiating. This theorem is truly "fundamental", for it tells us that the two basics ideas of Calculus, differentiating and integrating, are inextricably related.

In Proposition 2.3.12, Proposition 2.3.22, and Proposition 2.3.27, we gave formulas for the definite integrals of $f(x) = k$, $g(x) = x$, and $h(x) = x^2$ over the interval $[a, b]$. Those formulas may not have had an obvious pattern, but it may become clear what's going on after we introduce some new notation.

Definition 2.4.1. *If $F(x)$ is a real function, defined at $x = a$ and $x = b$, then we define the* **evaluation notation***, $\big|_a^b$ by*

$$F(x)\Big|_a^b \;=\; F(b) - F(a).$$

We also write

$$F(x)\Big|_{x=a}^{x=b} \;=\; F(b) - F(a),$$

if we want to emphasize the name of the variable that takes on the values a and b.

We read this as: F evaluated from a to b, or $F(x)$ evaluated from $x = a$ to $x = b$.

Now that we have this new notation, let's rewrite our formulas from Proposition 2.3.12, Proposition 2.3.22, and Proposition 2.3.27. We have

$$\int_a^b k\,dx \;=\; k(b-a) \;=\; kx\Big|_a^b\,,$$

$$\int_a^b x\,dx \;=\; \frac{b^2}{2}-\frac{a^2}{2} \;=\; \frac{x^2}{2}\Big|_a^b ,$$

and

$$\int_a^b x^2\,dx \;=\; \frac{b^3}{3}-\frac{a^3}{3} \;=\; \frac{x^3}{3}\Big|_a^b .$$

Do you see the pattern here? It may not be clear from just three examples, but you may be able to guess what happens more generally.

In each of the three formulas above, the derivative of the function being evaluated from a to b on the right is precisely the function being integrated from a to b on the left. In other words, the function being evaluated from a to b on the right is an **anti-derivative** of the integrand.

Is this true more generally? **YES**. This result is part of the *Fundamental Theorem of Calculus*, and it is the reason that the notation $\int f(x)\,dx$ and terminology "indefinite integral" for anti-derivatives are so similar to the notation and terminology for definite integrals.

Once we have the Fundamental Theorem, we will no longer need to compute definite integrals by the cumbersome process of taking limits of Riemann sums. All of our anti-derivative formulas from Chapter 1 will become formulas for computing definite integrals.

Understand: **ALL** of the applications of the definite integral are due to the fact that it represents a continuous sum of infinitesimal contributions, i.e., that the definite integral is a limit of Riemann sums. The Fundamental Theorem of Calculus is **NOT** the definition of the definite integral; the value of the Fundamental Theorem is that it allows us to calculate these continuous sums much more easily.

We need a few results before we can prove the Fundamental Theorem of Calculus.

Suppose that f is a Riemann integrable function on the interval $[a,b]$. By Theorem 2.3.16 (and Definition 2.3.17), for all x in $[a,b]$, the function f is also integrable on the interval $[a,x]$. Thus, we may make the following definition.

Definition 2.4.2. *Suppose that f is a Riemann integrable function on the interval $[a,b]$. Then, the* **integral function of** f **on** $[a,b]$ *is the function $I_f^{[a,b]}$, with domain $[a,b]$ and codomain $(-\infty,\infty)$, given by*

$$I_f^{[a,b]}(x) \;=\; \int_a^x f(t)\,dt.$$

Note the use of the dummy variable t in the integrand; it would be confusing to use x in the integrand, as x is the upper-limit of integration. We did not have to use t as our dummy variable (though it is a standard choice); we could have used any variable name, other than x, a, b, and f. The dummy variable d would also be a bit confusing.

It will be important to us that the following theorem is true, and its proof is instructive, since it uses a number of basic properties of integration.

Theorem 2.4.3. *Suppose that f is Riemann integrable on the interval $[a,b]$. Then, the integral function, $I_f^{[a,b]}$, of f on $[a,b]$ is continuous.*

Proof. Let c be in $[a,b]$. We need to show that the limit, as x approaches c, of $I_f^{[a,b]}(x)$ is equal to $I_f^{[a,b]}(c)$, where, if c equals a or b, we mean that we take the corresponding one-sided limit, so that we are always looking at x's in the interval $[a,b]$. For simplicity, we will use the two-sided limit notation throughout the proof.

We will show that

$$\lim_{x\to c} I_f^{[a,b]}(x) \;=\; I_f^{[a,b]}(c)$$

by showing that

$$\lim_{x\to c} \left| I_f^{[a,b]}(x) \;-\; I_f^{[a,b]}(c) \right| \;=\; 0.$$

Note that $\left| I_f^{[a,b]}(x) \;-\; I_f^{[a,b]}(c) \right| = \left| I_f^{[a,b]}(c) \;-\; I_f^{[a,b]}(x) \right|$.

Now,

$$I_f^{[a,b]}(x) \;-\; I_f^{[a,b]}(c) \;=\; \int_a^x f(x)\,dx \;-\; \int_a^c f(x)\,dx \;=\; \int_c^x f(x)\,dx$$

and, hence, we wish to show that

$$\lim_{x\to c} \left| \int_c^x f(x)\,dx \right| \;=\; 0.$$

We accomplish this by using that, since f is Riemann integrable on $[a,b]$, f is bounded on $[a,b]$, by Theorem 2.3.6. Thus, there exists $M\geq 0$ such that, for all x in $[a,b]$, $-M\leq f(x)\leq M$. Now, if $x\leq c$, the monotonicity of integration, Theorem 2.3.20, implies that

$$-M(c-x) \;\leq\; \int_x^c -M\,dt \;\leq\; \int_x^c f(t)\,dt \;\leq\; \int_x^c M\,dt \;=\; M(c-x).$$

Similarly, if $c \le x$, then

$$-M(x-c) \ \le\ \int_c^x -M\,dt \ \le\ \int_c^x f(t)\,dt \ \le\ \int_c^x M\,dt \ =\ M(x-c).$$

Hence, in either case, we obtain

$$\left|\int_c^x f(x)\,dx\right| \ \le\ M|c-x|.$$

As x approaches c, $|c-x|$ approaches 0, and, thus, so does $\left|\int_c^x f(x)\,dx\right|$. □

Remark 2.4.4. The conclusion of Theorem 2.4.3 is that the function on $[a,b]$ that sends x to $\int_a^x f(t)\,dt$ is continuous.

An essentially identical argument shows that the function on $[a,b]$ that sends x to $\int_x^b f(t)\,dt$ is continuous.

In Definition 1.1.1, we defined what is meant by an anti-derivative of a function f on an **open** interval I; it's a function F on I such that $F' = f$ (at all points in I). We also reminded you that the Mean Value Theorem for Derivatives implies that any two anti-derivatives of the same f must differ by a constant. That is, if F_1 and F_2 are both anti-derivatives of f on I, then there exists a constant C such that, for all x in I, $F_1(x) = F_2(x) + C$.

It will be useful for us to define the notion of an anti-derivative of a function on a **closed** interval, or on any interval whatsoever, and to prove that any two such anti-derivatives differ by a constant.

Definition 2.4.5. *Let f be a function on the interval J. An* **anti-derivative of f on J** *is a continuous function on J, which, on the interior of J (the open interval of points other than possible endpoints of the original interval), is an anti-derivative of f, i.e., a continuous function F on J such that F differentiable on the interior of J, and such that, for all x in the interior of J, $F'(x) = f(x)$.*

Theorem 2.4.6. *Suppose that f is a function on the interval J. Then, any two anti-derivatives of f on J differ by a constant.*

Proof. This follows easily from the result on open intervals. If J consists of a single point, then the result is trivially true. So, assume that is not the case, i.e., assume that the interior of J is a non-empty open interval.

Suppose that both F_1 and F_2 are anti-derivatives of f on J. Then, we know that there exists a constant C such that, for all x in the interior of J, $F_1(x) = F_2(x) + C$.

Suppose that a is a left endpoint of the interval J, and that a is contained in J. As F_1 and F_2 are continuous at a, we have

$$F_1(a) \;=\; \lim_{x\to a^+} F_1(x) \;=\; \lim_{x\to a^+} (F_2(x) + C) \;=\; F_2(a) + C.$$

In the exact same fashion, we see that, if b is a right endpoint of J, which is contained in J, then $F_1(b) = F_2(b) + C$. □

We are now going to prove the Fundamental Theorem of Calculus. The theorem is usually broken into two parts. The first part tells us that the integral function $I_f^{[a,b]}$ of a continuous function f on the interval $[a, b]$ is an anti-derivative of f on $[a, b]$. The second part tells us that, if we already know an anti-derivative of f on $[a, b]$, then we can use that to evaluate $\int_a^b f(x)\,dx$. It is this second part that is used most often in applications.

Theorem 2.4.7. (**Fundamental Theorem of Calculus, Part 1**) *Suppose that f is Riemann integrable on $[a, b]$ and is continuous at a point x_0 in (a, b). Then, the integral function $I_f^{[a,b]}$ of f on $[a, b]$ is differentiable at x_0 and*

$$\big(I_f^{[a,b]}\big)'(x_0) = f(x_0).$$

Thus, if f is continuous on $[a, b]$, then $I_f^{[a,b]}$ is an anti-derivative of f on $[a, b]$.

Proof. See Theorem 2.A.10. □

If f is continuous on an interval J, then, for all a and x in J, f is continuous on the interval $[a, x]$ or $[x, a]$ (depending on which interval is defined), and so, in either case, $\int_a^x f(t)\,dt$ exists. It is easy to conclude the following corollary from Theorem 2.4.7.

Corollary 2.4.8. *Suppose that f is continuous on an interval J, and that a is a point in J. Then, the function of x, with domain J, given by*

$$\int_a^x f(t)\,dt$$

is an anti-derivative of f on J.

Remark 2.4.9. The first part of the Fundamental Theorem, and its corollary, are of great theoretical importance; they tell us that all continuous functions have anti-derivatives.

For instance, quick...what's an anti-derivative of the continuous function $f(x) = e^{-x^2}$? That's easy: $F(x) = \int_0^x e^{-t^2}\,dt$ is an anti-derivative of e^{-x^2}.

Understand: this means that, if you want to calculate

$$\frac{d}{dx}\left(\int_0^x e^{-t^2}\,dt\right),$$

you don't first calculate $\int_0^x e^{-t^2}\,dt$, and then take the derivative. You simply apply the Fundamental Theorem to conclude immediately that

$$\frac{d}{dx}\left(\int_0^x e^{-t^2}\,dt\right) = e^{-x^2}.$$

What's another anti-derivative of e^{-x^2}? Easy again: $\int_{37}^x e^{-t^2}\,dt$.

How can both $\int_0^x e^{-t^2}\,dt$ and $\int_{37}^x e^{-t^2}\,dt$ be anti-derivatives of e^{-x^2}? Because they differ by a constant:

$$\int_0^x e^{-t^2}\,dt - \int_{37}^x e^{-t^2}\,dt = \int_0^{37} e^{-t^2}\,dt,$$

which is just a fixed constant.

We should make a final comment. Recall Remark 1.1.23, where we stated that the function $f(x) = e^{-x^2}$ has no elementary anti-derivative. This is true. The Fundamental Theorem guarantees that continuous functions possess anti-derivatives; it does **not** tell us that those anti-derivatives have to be easily expressible in terms of familiar functions.

Anti-derivatives of elementary functions need not be elementary functions.

We now come to the second part of the Fundamental Theorem of Calculus; the part that we discussed at the beginning of the section, the part that is most often used in applications. In a way, the result that we will state, and prove, should seem kind of obvious.

Suppose that $F(x)$ is an anti-derivative of $f(x)$ on $[a, b]$. Then, for all x in (a, b), $f(x) = F'(x)$, and the derivative $F'(x)$ is the limit of the change in F divided by the change in x, i.e., the limit, as Δx approaches 0, of $\Delta F/\Delta x$. But what's the definite integral, $\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx$? You take limits of Riemann sums; you take the limit of what you get when you multiply values of F' times small changes in x, and add these together. In other words, as x goes from a to b, we add up a bunch of quantities of the form

$$F'(x)\Delta x \;\approx\; \frac{\Delta F}{\Delta x}\cdot \Delta x \;=\; \Delta F,$$

and we take the limit of this as Δx approaches 0. Now we use that the sum $\sum(\Delta F)$ telescopes, just as in Proposition 2.1.9, to give us $F(b) - F(a)$. Therefore, the Riemann sums approximately equal $F(b) - F(a)$ and, as Δx approaches 0, the approximation "should" become an equality. Technically, we need that $f(x)$ is continuous, but our informal reasoning should make the following theorem easy to believe.

Theorem 2.4.10. (**Fundamental Theorem of Calculus, Part 2**) *Suppose that f is continuous on an interval $[a, b]$, and that F is an anti-derivative of f on $[a, b]$.*

Then,

$$\int_a^b f(x)\,dx \;=\; F(x)\Big|_a^b \;=\; F(b) - F(a).$$

Proof. This actually follows very quickly from our previous results.

By Theorem 2.4.7, $I_f^{[a,b]}$ is an anti-derivative of f on $[a, b]$. As F is also an anti-derivative of f on $[a, b]$, Theorem 2.4.6 implies that there exists a constant C such that, for all x in $[a, b]$,

$$\int_a^x f(t)\,dt \;=\; F(x) + C.$$

When $x = a$, the left side of the above equality is 0, and we obtain that $0 = F(a) + C$, i.e., $C = -F(a)$. Therefore, for all x in $[a, b]$,

$$\int_a^x f(t)\,dt \;=\; F(x) - F(a).$$

When $x = b$, we obtain

$$\int_a^b f(t)\,dt \;=\; F(b) - F(a),$$

which, noting that we may now use x as our dummy variable of integration, is what we wanted to show. $\square$

It is now easy for us to calculate definite integrals by using our anti-derivative formulas from Chapter 1.

Example 2.4.11. Calculate $\int_1^4 \left(7x^3 + 3\sqrt{x} + \frac{5}{x}\right)\,dx$.

Solution:

This calculation would be ridiculously complicated if we had to explicitly use limits of Riemann sums. But, we don't have to use Riemann sums; we now have the Fundamental Theorem of Calculus.

From our formulas in Section 1.1, we quickly find that

$$7\cdot\frac{x^4}{4} + 3\cdot\frac{x^{3/2}}{3/2} + 5\ln x \;=\; \frac{7x^4}{4} + 2x^{3/2} + 5\ln x$$

is an anti-derivative of $7x^3 + 3\sqrt{x} + \frac{5}{x}$ on $[1, 4]$ (actually, on all of $(0, \infty)$).

Therefore, the Fundamental Theorem tells us that

$$\int_1^4 \left(7x^3 + 3\sqrt{x} + \frac{5}{x}\right) dx \;=\; \left(\frac{7x^4}{4} + 2x^{3/2} + 5\ln x\right)\Bigg|_1^4 \;=$$

$$\left(\frac{7\cdot 4^4}{4} + 2\cdot 4^{3/2} + 5\ln 4\right) - \left(\frac{7\cdot 1^4}{4} + 2\cdot 1^{3/2} + 5\ln 1\right) \;=$$

$$448 + 16 + 5\ln 4 - \frac{7}{4} - 2 - 0 \;=\; \frac{1841}{4} + 5\ln 4 \;\approx\; 467.18147.$$

You may be thinking "Wait – wouldn't I get a different answer if I used a different anti-derivative of $7x^3 + 3\sqrt{x} + \frac{5}{x}$?" The answer had better be "no", and it is.

Any anti-derivative of $7x^3+3\sqrt{x}+\frac{5}{x}$ differs from the one we used by some constant C. Thus, the only other possibilities for us to use for the anti-derivative are all of the form

$$\frac{7x^4}{4} + 2x^{3/2} + 5\ln x + C.$$

Does the C cause us to get a different answer? No, because when we evaluate from 1 to 4, the C gets cancelled out:

$$\left(\frac{7x^4}{4} + 2x^{3/2} + 5\ln x + C\right)\Bigg|_1^4 \;=$$

$$\left(\frac{7\cdot 4^4}{4} + 2\cdot 4^{3/2} + 5\ln 4 + C\right) - \left(\frac{7\cdot 1^4}{4} + 2\cdot 1^{3/2} + 5\ln 1 + C\right) \;=$$

$$\left(\frac{7\cdot 4^4}{4} + 2\cdot 4^{3/2} + 5\ln 4\right) - \left(\frac{7\cdot 1^4}{4} + 2\cdot 1^{3/2} + 5\ln 1\right),$$

which is what we had before.

The moral of the story is that, when evaluating definite integrals by use of the Fundamental Theorem, you just use SOME anti-derivative; you don't need to ever put in a general $+C$.

Example 2.4.12. Find the area under the graph of $y = \sin x$ and above the interval $[0, \pi]$.

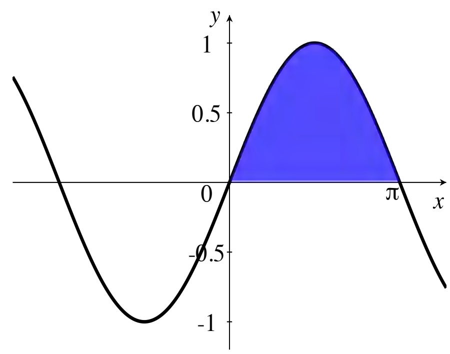


Figure 2.30: The area under part of the graph of $y = \sin x$.

Solution:

This area is simply

$$\int_0^\pi \sin x\,dx \;=\; -\cos x\,\Big|_0^\pi \;=\; -\cos\pi - (-\cos 0) \;=\; 1+1 \;=\; 2.$$

Example 2.4.13. Find the average value of

$$f(x) = \frac{1}{1+x^2}$$

on the interval $[-1,1]$, and find all values of c in $[-1,1]$ that are guaranteed to exist by the Mean Value Theorem for Integration, Theorem 2.3.31, i.e., all values of c in $[-1,1]$ such that $f(c)$ equals the average value of f on $[-1,1]$.

Solution:

By definition (see Definition 2.3.30), the average value of f on $[-1,1]$ is equal to

$$\frac{1}{1-(-1)}\int_{-1}^{1} \frac{1}{1+x^2}\,dx.$$

By the Fundamental Theorem and Theorem 1.1.13, we find that this equals

$$\frac{1}{2}\big(\tan^{-1}(1) - \tan^{-1}(-1)\big) \;=\; \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) \;=\; \frac{\pi}{4}.$$

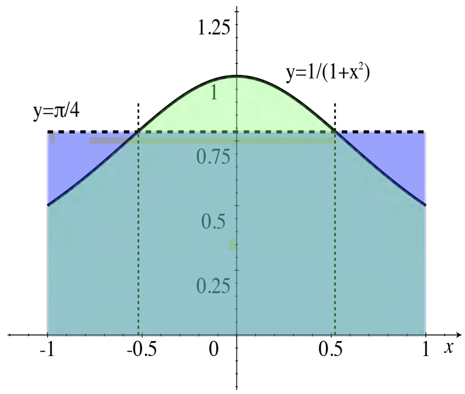


Figure 2.31: Equal areas under $y = 1/(1+x^2)$ and $y = \pi/4$.

For what c in $[-1, 1]$ does $f(c) = \pi/4$? We solve

$$\frac{1}{1+c^2} \;=\; \frac{\pi}{4}$$

and find that we must have

$$c^2 \;=\; \frac{4-\pi}{\pi},$$

i.e.,

$$c \;=\; \pm\sqrt{\frac{4-\pi}{\pi}} \;\approx\; \pm 0.5227232;$$

both of which are in the interval $[-1, 1]$.

Example 2.4.14. In Example 2.3.28, we discussed the fact that the definite integral

$$\int_0^r \sqrt{r^2 - x^2}\; dx$$

yields one quarter of the area inside a circle of radius r.

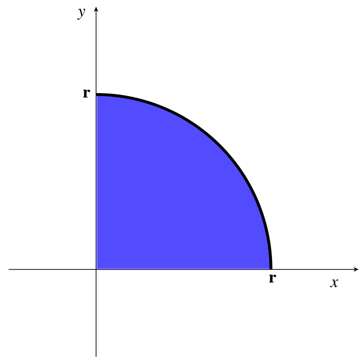


Figure 2.32: Area under $y = \sqrt{r^2 - x^2}$.

Then, we used our knowledge from high school geometry about the area inside a circle of radius r to conclude that

$$\int_0^r \sqrt{r^2 - x^2}\, dx \;=\; \frac{\pi r^2}{4}.$$

However, now that we have the Fundamental Theorem, we can use integration/anti-differentiation to verify that the area inside one quarter of a circle of radius r is $\pi r^2/4$, i.e., that the area inside a circle of radius r is πr^2. In other words, we can show that the formula from high school geometry is correct.

In Example 1.2.6, we found (replacing a with r) that

$$\int \sqrt{r^2 - x^2}\, dx \;=\; \frac{1}{2}\left[x\sqrt{r^2 - x^2} + r^2 \sin^{-1}\left(\frac{x}{r}\right)\right] + C.$$

Thus, the Fundamental Theorem tells us

$$\int_0^r \sqrt{r^2 - x^2}\, dx \;=\; \frac{1}{2}\left[x\sqrt{r^2 - x^2} + r^2 \sin^{-1}\left(\frac{x}{r}\right)\right]\Bigg|_0^r \;=$$

$$\frac{1}{2}\Big((0 + r^2 \sin^{-1}(1)) - (0 + r^2 \sin^{-1}(0))\Big) \;=\; \frac{1}{2}\cdot r^2 \cdot \frac{\pi}{2} \;=\; \frac{\pi r^2}{4}.$$

Substitution in Definite Integrals:

We discussed the use of substitutions in finding anti-derivatives in Theorem 1.1.15. Now that we have definite integration and the Fundamental Theorem, we can state a similar substitution for definite integrals.

Theorem 2.4.15. (**Substitution in Definite Integrals**) *Suppose that g is continuously differentiable on an open interval which contains the closed interval $[a, b]$, and that f is continuous on the closed interval between the minimum and maximum values of g on $[a, b]$. Then,*

$$\int_a^b f(g(x))\, g'(x)\, dx \;=\; \int_{g(a)}^{g(b)} f(u)\, du.$$

Proof. We will assume that $g(a) \leq g(b)$. The case where $g(b) \leq g(a)$ is essentially identical.

As f is continuous on $[g(a), g(b)]$, f possesses an anti-derivative F on $[g(a), g(b)]$ by Theorem 2.4.7, i.e., there is a function F, which is continuous on $[g(a), g(b)]$, and such that, for all x in $(g(a), g(b))$, $F'(x) = f(x)$.

Since the composition of continuous functions is continuous, and, for all x in (a, b)

$$(F \circ g)'(x) \;=\; F'(g(x)) \cdot g'(x),$$

it follows that $F \circ g$ is an anti-derivative of $(f \circ g) \cdot g'$ on $[a, b]$. The hypotheses guarantee that $f(g(x))\, g'(x)$ is continuous on $[a, b]$.

Therefore, we may apply the Fundamental Theorem, Theorem 2.4.10, to both sides of the desired equality, and conclude that they are both equal to $F(g(b)) \;-\; F(g(a))$. □

Example 2.4.16. Let's calculate $\int_0^2 xe^{-x^2}\, dx$ in two slightly different-looking ways.

Our first method is to note that xe^{-x^2} is continuous everywhere, and use the Fundamental Theorem, Theorem 2.4.10, to calculate the value of integral. We proceed as we did for indefinite integrals in Section 1.1, except that we carry along the limits of integration throughout the process.

Let $u = -x^2$, so that $du = -2x\, dx$. Then, $x\, dx = -du/2$, and obtain

$$\int_0^2 xe^{-x^2}\, dx \;=\; \int_{x=0}^{x=2} e^u(-du/2) \;=\; \frac{-1}{2}\int_{x=0}^{x=2} e^u\, du,$$

where we need to explicitly write $x = 0$ and $x = 2$ for the limits of integration, not just 0 and 2.

This is important: if we wrote simply $\int_0^2 e^u\,du$ for the integral, then the limits of integration would be telling us values of u, and then the Fundamental Theorem would give us $-(e^2-1)/2 = (1-e^2)/2$ for the value of the integral. **This is wrong**.

Proceeding correctly, we find

$$\frac{-1}{2}\int_{x=0}^{x=2} e^u\,du \;=\; \frac{-1}{2}e^u\Big|_{x=0}^{x=2} \;=\; \frac{-1}{2}e^{-x^2}\Big|_0^2 \;=\; \frac{-e^{-4}}{2}-\frac{-1}{2} \;=\; \frac{1-e^{-4}}{2}.$$

Notice that, in the above process, all that we were really doing was calculating an anti-derivative, as we did in Section 1.1, and carrying the limits of integration $x = 0$ and $x = 2$ throughout the calculation.

What's the other method? To apply Theorem 2.4.15, and actually change the limits of integration to describe what u does. This saves us cumbersome notation, and means that we don't need to reinsert what u was, in terms of x, at the end of the calculation. Aside from that, the process looks the same.

So, as before, let $u = -x^2$, so that $du = -2x\,dx$. Then, of course, $x\,dx = -du/2$, but, now, we also note the values of u that we should use for the limits of integration in the u integral; when $x = 0$, $u = -0^2 = 0$ and, when $x = 2$, $u = -2^2 = -4$. Therefore, we obtain

$$\int_0^2 xe^{-x^2}\,dx \;=\; \int_0^{-4} e^u(-du/2) \;==\; \frac{-1}{2}e^u\Big|_0^{-4} \;=\; \frac{-e^{-4}}{2}-\frac{-1}{2} \;=\; \frac{1-e^{-4}}{2}.$$

Thus, we see that, when substituting into a definite integral, changing the limits of integration to describe what your substitution variable is doing saves some time – maybe not a lot of time, but some.

However, there are some times when changing your limits of integration saves you a **LOT** of time. Consider the integral

$$\int_0^1 e^{-[x(x-1)]^2}(2x-1)\,dx.$$

Make the substitution $u = x(x-1) = x^2 - x$, so that $du = (2x-1)\,dx$. Then, the integral becomes

$$\int_{x=0}^{x=1} e^{-u^2}\,du.$$

We cannot produce a "nice" anti-derivative of e^{-u^2}; this function has no elementary anti-derivative. See Remark 1.1.23. Nonetheless, had we changed our limits of integration to describe what u does, we would have found that, when $x = 0$, $u = 0$, and when $x = 1$, $u = 0$; thus, our integral becomes

$$\int_0^0 e^{-u^2}\, du \;=\; 0.$$

Therefore, in this example, switching to the u limits of integration enables us to calculate an integral that we would not be able to calculate otherwise.

2.4.1 Exercises

In each of Exercises 1 through 15, calculate the definite integrals using the Fundamental Theorem of Calculus.

1. $\displaystyle\int_3^5 x^3 + x + 2\, dx.$

2. $\displaystyle\int_4^6 \frac{y}{y^2-3}\, dy.$

3. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 3t\, dt.$

4. $\displaystyle\int_2^{200} \frac{5}{z}\, dz.$

5. $\displaystyle\int_5^6 \sqrt{100-u^2}\, du.$

6. $\displaystyle\int_5^{10} \frac{dw}{\sqrt{w^2+25}}.$

7. $\displaystyle\int_{-3}^3 \frac{dv}{(9+v^2)^2}\, dv.$

8. $\displaystyle\int_0^4 \cosh(5y)\, dy.$

9. $\displaystyle\int_1^3 \frac{8u-35}{u^2-7u}\, du.$

10. $\displaystyle\int_0^{\pi} \sin t \cos t\, dt.$

11. $\displaystyle\int_9^{15} \frac{dx}{\sqrt{x^2-49}}$.

12. $\displaystyle\int_0^{\pi} \cos^2\theta\, d\theta$.

13. $\displaystyle\int_0^1 (2x+6)e^{(x+3)^2}\, dx$.

14. $\displaystyle\int_0^4 \sqrt{16+y^2}\, dy$.

15. $\displaystyle\int_0^2 \frac{z^2+z+3}{z+3}\, dz$.

16. Suppose we want to calculate $\displaystyle\int_{-1}^{1} \frac{dx}{x}$. Since $\ln|x|$ is an anti-derivative of the integrand, the Fundamental Theorem of Calculus tells us that

$$\int_{-1}^{1} \frac{dx}{x} = \ln|x|\Big|_{-1}^{1} = \ln 1 - \ln 1 = 0.$$

What is wrong with this argument?

17. Suppose that f is continuous on $[a,b]$ and that $G(x) = \displaystyle\int_a^{x^2} f(t)\, dt$. What is dG/dx? Hint: Use the Chain Rule.

In each of Exercises 18 through 22, find the average value of the function on the closed interval if it is defined. If it is undefined, explain why.

18. $k(x) = \sqrt{9-x^2}$, $[2,5]$.

19. $j(y) = \sqrt{1+y^2}$, $[0,1]$.

20. $h(z) = \dfrac{1}{z^2}$, $[-2,2]$.

21. $g(u) = \cosh u$, $[-k,k]$, $k>0$.

22. $f(v) = \sec v$, $[0,\pi]$.

23. Find the area under the graph of $y = 3e^x + 2$ and above the interval $[0, \ln 7]$ on the x-axis. Note that $3e^x + 2 \geq 0$.

24. Find the area trapped between the graph $y = y(x) = -2 + \dfrac{3}{1+x^2}$ and the interval $[-1, 1]$ on the x-axis. Note that, for x in $[-1, 1]$, $y(x)$ is sometimes positive and sometimes negative. You need to calculate the total area, both above and below the given interval.

Given a probability density function, $f(x)$, for a continuous random variable, X, the probability that $a \le X \le b$ is $\int_a^b f(x)\,dx$.

25. A continuous random variable is *uniformly distributed* if $f(x) = C$ for some (necessarily positive) constant. This means that any all possible outcomes are equally likely to appear. Suppose the life-span of a certain organism is uniformly distributed between 4 and 9 years and the probability density function is $f(x) = 1/5$. What is the probability a given organism will live for 6 to 8 years?

26. If $I = [a, b]$ is the set of all possible outcomes of a random variable with density function $f(x)$, then it must be true that $\int_a^b f(x)\,dx = 1$. This means the probability the random variable falls within the set of all its possible outcomes is 100%. Prove that a uniformly distributed random variable on the interval $[a, b]$ has density function $f(x) = 1/(b - a)$.

27. A random variable has a range of possible outcomes between 1 and 4.

    a. The density function is $f(x) = cx^2$. What is c?

    b. What is the probability of an outcome between 2 and 3?

The *expected value* or *expectation* of random variable X is a weighted average of the possible values that the random variable can achieve and is denoted $E(X)$. For a discrete variable, $E(X) = \sum_{i=1}^{n} x_i p(x_i)$ where each x_i is a possible outcome that will occur with probability $p(x_i)$. For a continuous random variable that can achieve a value on the interval $[a, b]$, $E(X) = \int_a^b x f(x)\,dx$ where $f(x)$ is the density function.

28. Suppose two standard six-sided dice are rolled and we're interested in the average, or expected sum on the two dice. Thus, we let X can be any integer between two and 12.

    a. Complete the table below by determining the probability of each possible outcome and entering it in the second column. Note that there are 36 possible outcomes–$(1, 1), (1, 2), (1, 3), ..$ etc. For example, there are four ways to achieve a sum of five: $(1, 4), (4, 1), (2, 3), (3, 2)$. The probability of a sum of five is therefore $4/36 = 1/9$.

x_i	$p(x_i)$	$x_i \cdot (x_i)$
1		
2		
3		
4		
5	$4/36$	$20/36$
6		
7		
8		
9		
10		
11		
12		

b. What is the most likely sum? That is, what sum has the highest probability of occurring?

c. Fill in the third column by multiplying the first two columns together. In row 5, for example, $x_i p(x_i) = 20/36 = 5/9$.

d. What is the sum of all the entries in the third column? This is the expected, or average sum.

29. What is the expected value of a continuous random variable uniformly distributed over the interval $[a, b]$?

30. Let $f(x) = \frac{3}{4}(1 - x^2)$.

    a. Verify that this is a legitimate density function of a random variable with possible outcomes on the interval $[-1, 1]$ by verifying that $\displaystyle\int_{-1}^{1} f(x)\,dx = 1$.

    b. What is $E(X)$?

More generally, we can define the expectation of a function of the random variable. For example, in the dice example above, we may have been interested in the square of the sum of the dice, rather than just the sum. The expectation of the square of the sum is $\sum (x_i)^2 \cdot p(x_i)$. In general, if $g(X)$ is a function of a discrete variable, we write: $E(g(X)) = \sum g(x_i) \cdot p(x_i)$. If $g(X)$ is a function of a continuous variable, we write: $E(g(X)) = \displaystyle\int_a^b g(x) \cdot p(x)\,dx$.

31. What is $E(X^2)$ in the dice example?

32. What is $E(X^2)$ for a uniformly distributed random variable on the interval $[a, b]$?

33. What is $E(X^2)$ for a random variable with domain $[-1, 1]$ and density function $f(x) = \frac{3}{4}(1 - x^2)$?

Variance **measures the dispersion of a variable, or how spread out it is. Let** $\mu = E(X)$**. Then, the variance of a discrete variable is** $\mathbf{Var}(X) = \sum (x-\mu)^2 f(x)$**. The variance of a continuous random variable is** $\mathbf{Var}(X) = \int_a^b (x-\mu)^2 f(x)\,dx$**.**

34. Show that for a discrete variable, $\text{Var}(X) = E(X^2) - E(X)^2$.

35. Show that for a continuous variable, $\text{Var}(X) = E(X^2) - E(X)^2$. Hint: the proof is nearly identical to the previous problem. This shows that the rules governing finite sums are in some sense not that different from the rules governing definite integrals.

36. What is the variance of the sum of two dice?

37. What is the variance of a uniformly distributed random variable on the interval $[a, b]$?

38. What is the variance of a random variable with density function $f(x) = \frac{3}{4}(1 - x^2)$?

39. Let a and b be positive integers. Prove that $\int_{-\pi}^{\pi} \sin ax \cos bx\,dx = 0$. Hint: prove that $\sin ax \cos bx = \dfrac{1}{2}(\sin(ax + bx) + \sin(ax - bx))$ and use the fact that $\cos x$ is an even function.

40. Let a and b be positive integers. Evaluate $\int_{-\pi}^{\pi} \sin ax \sin bx\,dx$, when

    a. $a \neq b$;

    b. $a = b$.

Use the Fundamental Theorem of Calculus and the Chain Rule to solve Exercises 41 through 44.

41. What is $\dfrac{d}{dx}\displaystyle\int_0^{3x} 3t\,dt$?

42. What is $\dfrac{d}{dx}\displaystyle\int_{2x}^{5x} t^2\,dt$?

43. What is $\dfrac{d}{dx}\displaystyle\int_x^{x^2} 1\,du$?

44. What is $\dfrac{d}{dx}\displaystyle\int_0^{x^2} xt\,dt$?

45. Let $g(x) = \int_0^{x^2} e^{x+t}\,dt$. What is $g'(2)$? Hint: move e^x outside the integral.

46. Let a and b be positive integers. Evaluate $\int_{-\pi}^{\pi} \cos(ax)\cos(bx)\,dx$ when

a. $a \neq b$;

b. $a = b$.

47. Let $A = \displaystyle\int_0^{\pi/2} \frac{\sin^k x}{\sin^k x + \cos^k x}\, dx$.

a. Show that $A = \displaystyle\int_0^{\pi/2} \frac{\cos^k\left(\frac{\pi}{2} - x\right)}{\cos^k\left(\frac{\pi}{2} - x\right) + \sin^k\left(\frac{\pi}{2} - x\right)}\, dx$.

b. Make the substitution $u = \dfrac{\pi}{2} - x$ and show that $A = \displaystyle\int_0^{\pi/2} \frac{\cos^k u}{\sin^k u + \cos^k u}\, du$. Let the right-hand side of this equation be B.

c. Prove that $A + B = \pi/2$ and conclude that $A = \pi/4$.

48. Consider the expression $\displaystyle\int_2^4 \int_1^5 x^2y+2y\, dx\, dy$. We evaluate by first viewing y as a constant and integrating with respect to x, and then integrating with respect to y.

a. Evaluate $\int_1^5 x^2y + 2y\, dx$. Assume y is a constant.

b. Take the definite integral of your answer to part (a) with respect to y.

c. Do you get the same answer if you switch the order of integration? That is, is your answer to part (b) the same as $\displaystyle\int_1^5 \int_2^4 x^2y + 2y\, dy\, dx$?

Use the method in the previous problem to evaluate the integrals in the next two exercises.

49. a. $\displaystyle\int_0^{2\pi} \int_0^1 r\, dr\, d\theta$.

b. $\displaystyle\int_0^1 \int_0^{2\pi} r\, d\theta\, dr$.

50. a. $\displaystyle\int_0^{\pi} \int_0^{\pi} \sin x \cos y\, dx\, dy$.

b. $\displaystyle\int_0^{\pi} \int_0^{\pi} \sin x \cos y\, dy\, dx$.

It's not always true that switching the order of integration results in the same value. However, if a function is reasonably well-behaved, then switching the order of integration has no effect on the final outcome. This is part of the content of *Fubini's Theorem.*

2.5 Improper Integrals

In this section, we will define the integral of a function over a set of real numbers, where the set of real numbers need not be a closed, bounded, interval; we also define integrals when the value of the integrand is unbounded, even if the interval of integration is itself bounded. This really requires a new definition, for Theorem 2.3.6 tells us that the Riemann integral of an unbounded function does not exist.

The actual calculation of our new type of integral will involve calculating our usual integrals on intervals of the form $[a, b]$, where now either a or b varies, and we will then take a limit as a or b approaches some "problematic" value. Of course, the calculation of the integrals on the intervals $[a, b]$ can/will still use the Fundamental Theorem of Calculus, and, hence, we may apply all of our techniques from earlier sections to find anti-derivatives.

As we wish to be able to discuss integrals over intervals $[a, b]$ and over intervals $(a, b]$ (and over other sets), the notation $\int_a^b$ does not suffice to distinguish between the types of intervals that we care about. Thus, we adopt some new notation; if E is a subset of the real numbers, we will write

$$\int_E f(x)\,dx$$

for the integral of f over the set E. Of course, right now, this has no meaning for us, unless E is a closed interval $[a, b]$, in which case $\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx$. Our goal in this section is to define $\int_E f(x)\,dx$ for sets E that need not be closed intervals.

Before we really start looking at new types of integrals, it will be helpful to have a new piece of terminology.

Definition 2.5.1. *If f is a real function, with domain D, an* **extension** *of f is a function $\hat{f}$ whose domain is larger (or equal to) D, and such that f and $\hat{f}$ agree at all points of D, i.e., for all x in D, $\hat{f}(x) = f(x)$.*

In other words, an extension $\hat{f}$, of f, is a function whose domain includes the domain, D, of f, and such that the restriction of $\hat{f}$ to D is equal to f.

Example 2.5.2. For example, the function $\hat{f}$, with domain $[0,1]$ given by

$$\hat{f} \;=\; \begin{cases} \dfrac{\sin x}{x} & \text{if } 0 < x \le 1; \\ 1 & \text{if } x = 0 \end{cases}$$

is an extension of the function $f(x) = (\sin x)/x$, with domain $(0,1]$. In fact, $\hat{f}$ is a **continuous** extension of f to $[0,1]$, since

$$\lim_{x \to 0^+} \frac{\sin x}{x} \;=\; 1.$$

Now, let's start our discussion of more general notions of integration by looking at the easiest case, one where we already know the answer: the case of a function which is Riemann integrable.

Example 2.5.3. Consider $f(x) = x^2$ on the interval $[0,1]$. Certainly, f is Riemann integrable on $[0,1]$, since f is continuous on $[0,1]$. Thus, we know what $\int_{[0,1]} x^2\,dx$ means; it means $\int_0^1 x^2\,dx$, which, by the Fundamental Theorem of Calculus, is equal to $(x^3/3)\big|_0^1 = 1/3$.

What if we omit the 0 from the interval of integration, i.e., what should $\int_{(0,1]} x^2\,dx$ mean? Theorem 2.3.14 tells us that, for Riemann integrals on closed intervals, altering the function at a finite number of points does not change the integrability of the function or the value of the integral. Thus, intuitively, it seems reasonable that omitting a single point of integration, like 0, should not affect the integral. Therefore, in this case, it seems reasonable to make the definition that

$$\int_{(0,1]} x^2\,dx \;=\; \int_{[0,1]} x^2\,dx \;=\; \frac{1}{3}.$$

While this seems reasonable, it should, in a way, seem like "cheating"; we wanted to integrate $f(x) = x^2$ on the interval $(0,1]$, and yet we used information about f on the larger interval $[0,1]$, i.e., we used that there was a continuous extension of f from the interval $(0,1]$ to the interval $[0,1]$. Could we have defined $\int_{(0,1]} x^2\,dx$ without using the extension to $[0,1]$? Yes.

For all a such that $0 < a \le 1$, $f(x) = x^2$ is Riemann integrable on the interval $[a,1]$, and

$$\int_a^1 x^2\,dx \;=\; \frac{x^3}{3}\bigg|_a^1 \;=\; \frac{1}{3} - \frac{a^3}{3},$$

and this integral uses only that $f(x) = x^2$ is defined on $(0,1]$.

Now, we can take the limit as a approached 0 from the right to obtain our previous answer. That is, we could have defined $\int_{(0,1]} x^2\,dx$ by

$$\int_{(0,1]} x^2\,dx \;=\; \lim_{a\to 0^+}\int_a^1 x^2\,dx \;=\; \lim_{a\to 0^+}\left(\frac{1}{3}-\frac{a^3}{3}\right) \;=\; \frac{1}{3}.$$

The point is that, in this example, we obtain the same value for $\int_{(0,1]} x^2\,dx$, regardless of whether we use the fact that x^2 extends to $[0,1]$ or whether we instead use the limits of integrals, as our lower-limit of integration approaches 0.

Example 2.5.4. Let's consider a more-complicated example. Let f be the function, with domain $(0,1]$, given by

$$f(x) \;=\; \frac{\sin x}{x}.$$

How many choices do we have for reasonable ways to define the integral $\int_{(0,1]} f(x)\,dx$? At least two.

First, we can define an extension $\hat{f}$ to the closed interval $[0,1]$, and then define

$$\int_{(0,1]} \frac{\sin x}{x}\,dx \;=\; \int_{[0,1]} \hat{f}(x)\,dx.$$

Of course, we either need to pick a particular extension, or show that the value of $\int_{[0,1]} \hat{f}(x)\,dx$ is always the same, regardless of what extension we select. We could, in fact, take $\hat{f}$ to be the continuous extension of f given in Example 2.5.2; then, $\hat{f}$ would certainly be Riemann integrable by Theorem 2.3.8.

However, **we don't have to use continuous extension of** f. Since $(\sin x)/x$ **does** possess a continuous extension to the interval $[0,1]$, it follows that $(\sin x)/x$ is bounded on $(0,1]$. Thus, Theorem 2.3.8 and Theorem 2.3.14 imply that it doesn't matter how we define $\hat{f}$ at $x=0$; no matter what value we chose for $\hat{f}(0)$, $\hat{f}$ will be bounded and, at least, piecewise-continuous – hence, Riemann integrable – and changing the value at one point will not affect the value of the integral. Therefore, we could define

$$\int_{(0,1]} f(x)\,dx \;=\; \int_{[0,1]} \hat{f}\,dx,$$

for **any** extension of f to a function $\hat{f}$ on $[0,1]$.

However, we could take the second approach, as we did in Example 2.5.3; without using an extension at all, we could define

$$\int_{(0,1]} \frac{\sin x}{x}\,dx \;=\; \lim_{a\to 0^+} \int_a^1 \frac{\sin x}{x}\,dx.$$

This leads to a new question or two. If $0 < a \leq 1$, we know that $(\sin x)/x$ is continuous on $[a,1]$ and, hence, $\int_a^1 \frac{\sin x}{x}\,dx$ exists. But, how do we know that the limit as $a \to 0^+$ exists and, even if the limit does exist, how do we know that that limit is equal to what we'd get by extending $(\sin x)/x$ to $[0,1]$ and then taking the Riemann integral of our extension over $[0,1]$?

Actually, the answers to these questions are easy, given our earlier results. Suppose that $\hat{f}$ is an extension of f to $[0,1]$. Then, as we discussed above, $\hat{f}$ is Riemann integrable on $[0,1]$. By Remark 2.4.4, the function on $[0,1]$ that sends a to $\int_a^1 \hat{f}(x)\,dx$ is continuous; in particular,

$$\int_0^1 \hat{f}(x)\,dx \;=\; \lim_{a\to 0^+} \int_a^1 \hat{f}(x)\,dx \;=\; \lim_{a\to 0^+} \int_a^1 f(x)\,dx,$$

where the second equality above follows from the fact that, if $0 < x \leq 1$, then $\hat{f}(x) = f(x)$, since $\hat{f}$ is an extension of f.

Thus, $\lim_{a\to 0^+} \int_a^1 f(x)\,dx$ exists and equals what we would obtain by calculating the Riemann integral of any extension of f to $[0,1]$. Therefore, it seems reasonable to define the integral of f over the half-open interval $(0,1]$ by

$$\int_{(0,1]} \frac{\sin x}{x}\,dx \;=\; \lim_{a\to 0^+} \int_a^1 \frac{\sin x}{x}\,dx.$$

What we have seen in the previous two examples is that, in those cases, it made sense to define $\int_{(0,1]} f(x)\,dx$ by

$$\int_{(0,1]} f(x)\,dx \;=\; \lim_{a\to 0^+} \int_a^1 f(x)\,dx,$$

where each $\int_a^1 f(x)\,dx$ is a Riemann integral. However, in those cases, we could also have defined $\int_{(0,1]} f(x)\,dx$ by extending f to the closed interval $[0,1]$ and then using the Riemann integral of the extended function. Now we will look at an example where the approach via extensions does not work, but the limit idea still yields a meaningful result.

Example 2.5.5. Consider the function

$$f(x) \;=\; \frac{1}{\sqrt{x}} \;=\; x^{-1/2}$$

on the half-open interval $(0,1]$. As x approaches 0 from the right, $1/\sqrt{x}$ approaches ∞, and so f is unbounded on $(0,1]$. Hence, any extension of f to the closed interval $[0,1]$ will also be unbounded and, therefore, will **not** be Riemann integrable (Theorem 2.3.6).

On the other hand, using the Fundamental Theorem, and the Power Rule for Integration, we find

$$\lim_{a\to 0^+}\int_a^1 x^{-1/2}\,dx \;=\; \lim_{a\to 0^+}\left(\left.\frac{x^{1/2}}{1/2}\right|_a^1\right) \;=\; \lim_{a\to 0^+}(2-2\sqrt{a}) \;=\; 2.$$

This means that the area under the graph of $y = 1/\sqrt{x}$ and over the interval $[a,1]$ approaches 2 as $a \to 0^+$. We say, simply, that the area under the graph of $y = 1/\sqrt{x}$ and over the interval $(0,1]$ equals 2.

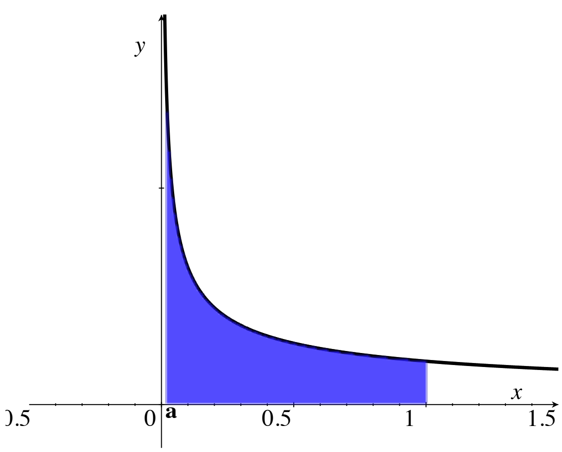


Figure 2.33: Area under the graph of $y = 1/\sqrt{x}$ over $[a,1]$.

What we see in this example is that defining the integral over a half-open interval $(a,b]$ in terms of limits of integrals over closed intervals gives us a well-defined number, while the Riemann integral of an extension to the closed interval $[a,b]$ does not exist.

Before making a definition, we wish to look at one more example.

Example 2.5.6. Suppose that $f(x) = e^{-x}$ and we wish to integrate f over the interval $[0, \infty)$. Note that $-e^{-x}$ is an anti-derivative of e^{-x}.

In this example, the function f itself is bounded on the given interval; e^{-x} is between 0 and 1 for x in $[0, \infty)$. On the other hand, the interval that we want to integrate over is unbounded, since it "goes out to infinity". How should we define

$$\int_{[0,\infty)} e^{-x}\, dx?$$

Here, there is no closed interval to which we can possibly extend f. However, the limit approach still yields an answer:

$$\lim_{b\to\infty} \int_0^b e^{-x}\, dx \;=\; \lim_{b\to\infty} \left(-e^{-x} \Big|_0^b \right) \;=\; \lim_{b\to\infty} \big(-e^{-b} - (-e^0) \big) \;=\; 0+1 \;=\; 1.$$

This means that the area under the graph of $y = e^{-x}$ and over the interval $[0, b]$ approaches 1 as $b \to \infty$. We say, simply, that the area under the graph of $y = e^{-x}$ and over the interval $[0, \infty)$ equals 1.

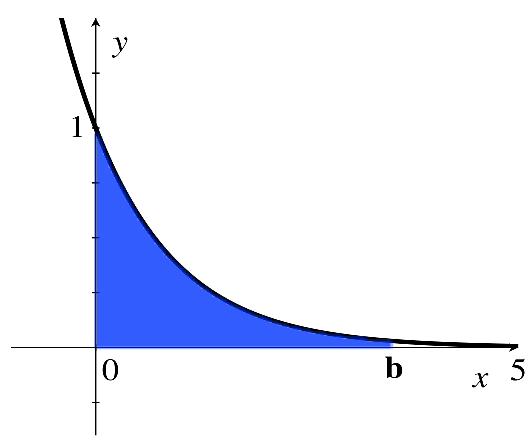


Figure 2.34: Area under the graph of $y = e^{-x}$ over $[0, b]$.

In light of the above discussion and examples, we make the following definition:

Definition 2.5.7. *Suppose that f is defined on the half-open interval $[a, b)$, where b may be ∞, and suppose that, for all c such that $a \leq c < b$, f is Riemann integrable on the closed interval $[a, c]$.*

Then, we let

$$\int_a^b f(x)\, dx \;=\; \int_{[a,b)} f(x)\, dx \;=\; \lim_{c \to b^-} \int_a^c f(x)\, dx,$$

provided that the limit exists, in which case we say that f **is integrable on** *$[a, b)$, or that the integral $\int_a^b f(x)\, dx$* **converges**. *Otherwise, we say that $\int_a^b f(x)\, dx$* **diverges**.

Similarly, suppose that f is defined on the half-open interval $(a, b]$, where a may be $-\infty$, and suppose that, for all c such that $a < c \leq b$, f is Riemann integrable on the closed interval $[c, b]$.

Then, we let

$$\int_a^b f(x)\, dx \;=\; \int_{(a,b]} f(x)\, dx \;=\; \lim_{c \to a^+} \int_c^b f(x)\, dx,$$

provided that the limit exists, in which case we say that f **is integrable on** *$(a, b]$, or that the integral $\int_a^b f(x)\, dx$* **converges**. *Otherwise, we say that $\int_a^b f(x)\, dx$* **diverges**.

Naturally, if $\int_a^b f(x)\, dx$ converges, we define $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$.

Note that we do **not** have a notational conflict; if f is, in fact, Riemann integrable on $[a, b]$, then, by Theorem 2.4.3 and Remark 2.4.4, the Riemann integral $\int_a^b f(x)\, dx$ equals both of the one-sided limits of integrals given above.

The way that we usually conclude that f is Riemann integrable on all of the closed intervals $[a, c]$ (or $[c, b]$) contained in $[a, b)$ (or $(a, b]$) is that f is continuous on the entire half-open interval. Assuming this is the case, the only way that the integral $\int_a^b f(x)\, dx$ can possibly fail to converge is for either the interval of integration to be unbounded, or for the function f to be unbounded on the interval of integration. We give these two types of integrals which involve unbounded activity a name:

Definition 2.5.8. *An integral $\int_a^b f(x)\, dx$, in which a or b is $\pm\infty$, or such that f is unbounded on the interval (a, b) is called an* **improper integral**.

The integrals in Example 2.5.5 and Example 2.5.6 are improper integrals, and yet the integrals converge. The importance of improper integrals is that, for continuous functions, they're the only types of integrals which **might** diverge.

While the integrals in Definition 2.5.7 are the basic new types of integrals that we are defining in this section, we are also interested in more-complicated integrals, ones which break up into a finite number of pieces which are of the types found in Definition 2.5.7 .

Example 2.5.9. Consider the integral

$$\int_{-1}^{8} \frac{1}{x^{2/3}}\, dx.$$

The point $x = 0$ is in the interval $[-1, 8]$, the interval over which we're supposed to integrate. However, as x approaches 0 from the left or right, the integrand goes to ∞. This is a type of improper integral. The question is: how should we define what such an integral means?

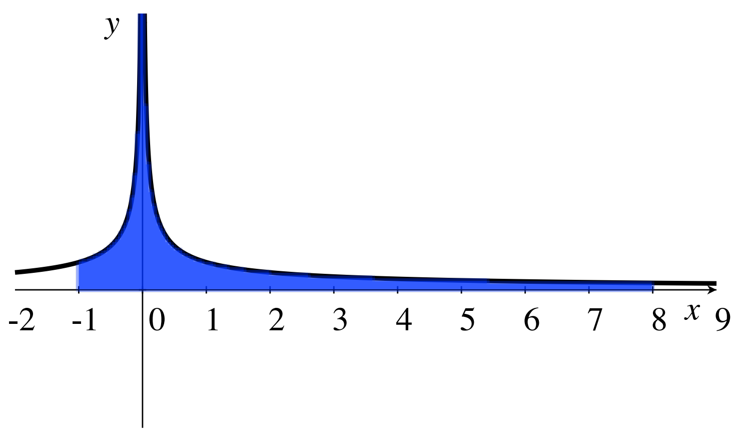


Figure 2.35: The $y = x^{-2/3}$ becomes unbounded from either side of $x = 0$.

The answer is: we want Theorem 2.3.16, on splitting up integrals, to remain true. This means that we want it to be true that

$$\int_{-1}^{8} \frac{1}{x^{2/3}}\, dx \;=\; \int_{-1}^{0} \frac{1}{x^{2/3}}\, dx \;+\; \int_{0}^{8} \frac{1}{x^{2/3}}\, dx,$$

where each of the summands on the right is an integral of the type we defined in Definition 2.5.7. We now calculate by taking limits:

$$\lim_{b\to 0^-}\int_{-1}^{b} x^{-2/3}\,dx \;+\; \lim_{a\to 0^+}\int_{a}^{8} x^{-2/3}\,dx \;=\; \lim_{b\to 0^-}\left(\,3x^{1/3}\Big|_{-1}^{b}\,\right) \;+\; \lim_{a\to 0^+}\left(\,3x^{1/3}\Big|_{a}^{8}\,\right) \;=$$

$$\lim_{b\to 0^-}\left(\,3b^{1/3}-3(-1)^{1/3}\,\right) \;+\; \lim_{a\to 0^+}\left(\,3(8)^{1/3}-3a^{1/3}\,\right) \;=\; 3 \;+\; 6 \;=\; 9.$$

There can also be multiple "problem points".

Example 2.5.10. Consider the integral

$$\int_{1}^{\infty}\frac{1}{(x-2)(x-4)}\,dx.$$

After possibly removing a finite number of points (problem points, where unboundedness comes into play), we want to split the interval $[1,\infty)$ into a finite number of closed or half-open intervals on which the integrand $1/[(x-2)(x-4)]$ is continuous; in this splitting, we allow a pair of closed or half-open intervals to intersect each other in, at most, one point. We then add together the resulting integrals, provided **all of them** exist; otherwise, we say that the original integral diverges.

Thus, we start with the interval $[1,\infty)$. We remove the two points $x=2$ and $x=4$, where $1/[(x-2)(x-4)]$ is undefined. For now, this gives us intervals $[1,2)$, $(2,4)$, and $(4,\infty)$. The intervals $(2,4)$ and $(4,\infty)$ are **not** closed or half-open. Why do we care? We do not want to have to deal with some sort of simultaneous limits at the two endpoints of the intervals.

Therefore, we further split the intervals $(2,4)$, and $(4,\infty)$. Where do we split them? At any point in-between the endpoints. It is a theorem, which we incorporated into the statement of Definition 2.5.11, that is doesn't matter where the splitting occurs. So, we split the interval $(2,4)$ into $(2,2.5]$ and $[2.5,4)$ (pairs of half-open intervals may intersect at a point), and we split $(4,\infty)$ into $(4,7]$ and $[7,\infty)$.

We end up with five half-open intervals: $I_1=[1,2)$, $I_2=(2,2.5]$, $I_3=[2.5,4)$, $I_4=(4,7]$, and $I_5=[7,\infty)$, whose union is equal to the original interval $[1,\infty)$, minus a finite number of points, and pairs of the half-open intervals intersect each other in, at most, one point.

Now, we define

$$\int_1^\infty \frac{1}{(x-2)(x-4)}\,dx$$

to equal the sum

$$\int_{I_1} \frac{1}{(x-2)(x-4)}\,dx \;+\; \int_{I_2} \frac{1}{(x-2)(x-4)}\,dx \;+\; \int_{I_3} \frac{1}{(x-2)(x-4)}\,dx \;+$$

$$\int_{I_4} \frac{1}{(x-2)(x-4)}\,dx \;+\; \int_{I_5} \frac{1}{(x-2)(x-4)}\,dx,$$

provided that **all** of these integrals exist, in which case we say that $\int_1^\infty \frac{1}{(x-2)(x-4)}\,dx$ converges. If one (or more) of the five separate integrals, appearing in the summation, diverges, then we say that $\int_1^\infty \frac{1}{(x-2)(x-4)}\,dx$ diverges.

In the remainder of this example, we will show that

$$\int_1^\infty \frac{1}{(x-2)(x-4)}\,dx$$

diverges, by showing that the first improper integral in the summation above diverges, i.e., we will show that

$$\int_1^2 \frac{1}{(x-2)(x-4)}\,dx \;=\; \lim_{b\to 2^-}\int_1^b \frac{1}{(x-2)(x-4)}\,dx$$

diverges to ∞.

We find an anti-derivative of $\frac{1}{(x-2)(x-4)}$ via partial fractions, as in Section 1.3. So, we first determine constants A and B such that

$$\frac{1}{(x-2)(x-4)} \;=\; \frac{A}{x-2} \;+\; \frac{B}{x-4},$$

for all x, other than $x=2$ and $x=4$. Clearing the denominators, by multiplying each side of the equality by the big denominator on the left, i.e., by $(x-2)(x-4)$, we obtain

$$1 \;=\; A(x-4) \;+\; B(x-2),$$

which needs to hold for **all** x. Plugging in $x=2$, we find that $1 = A\cdot(-2) + B\cdot 0$, so that

$A = -1/2$. Plugging in $x = 4$, we find that $1 = A \cdot 0 + B \cdot 2$, and so $B = 1/2$. Thus,

$$\frac{1}{(x-2)(x-4)} \;=\; \frac{-1/2}{x-2} \;+\; \frac{1/2}{x-4},$$

and we want to calculate

$$\lim_{b\to 2^-} \int_1^b \frac{1}{(x-2)(x-4)}\,dx \;=\; \lim_{b\to 2^-} \int_1^b \left(\frac{-1/2}{x-2} \;+\; \frac{1/2}{x-4}\right) dx, \tag{2.3}$$

or show that it doesn't exist. We need to find an anti-derivative of $1/(x-a)$, where a is 2 or 4. We accomplish this by making the substitution $u = x - a$, so that $du = dx$, and

$$\int \frac{1}{x-a}\,dx \;=\; \int \frac{1}{u}\,du \;=\; \ln|u| \;+\; C \;=\; \ln|x-a| \;+\; C.$$

Thus, Formula 2.3 becomes

$$\lim_{b\to 2^-} \int_1^b \frac{1}{(x-2)(x-4)}\,dx \;=\; \lim_{b\to 2^-} \left(\frac{-1}{2}\ln|x-2| + \frac{1}{2}\ln|x-4|\right)\Bigg|_1^b.$$

As b is approaching 2 from the left, we know that all of the x's that we are considering are less than 2 and, hence, $|x-2| = 2-x$ and $|x-4| = 4-x$. Now, evaluating at $x = b$ and subtracting the value at $x = 1$, we find

$$\lim_{b\to 2^-} \int_1^b \frac{1}{(x-2)(x-4)}\,dx \;=\; \lim_{b\to 2^-} \left(\frac{-1}{2}\ln(2-b) + \frac{1}{2}\ln(4-b) - \frac{1}{2}\ln 3\right),$$

where we used that $\ln 1 = 0$. As b approaches 2 from the left, $\frac{1}{2}\ln(4-b) - \frac{1}{2}\ln 3$ approaches $\frac{1}{2}\ln 2 - \frac{1}{2}\ln 3$, but $\frac{-1}{2}\ln(2-b)$ approaches $(-1/2)(-\infty) = \infty$.

Therefore, $\int_1^2 \frac{1}{(x-2)(x-4)}\,dx$ diverges, and so does the integral over the bigger interval $[1,\infty)$. We should mention that, just because $\int_1^2 \frac{1}{(x-2)(x-4)}\,dx$ diverges to ∞, that does **not** imply that $\int_1^\infty \frac{1}{(x-2)(x-4)}\,dx$ also diverges to ∞; other parts of the summand that we split the integral into may (and do) diverge to $-\infty$. Thus, we simply say that $\int_1^\infty \frac{1}{(x-2)(x-4)}\,dx$ diverges, without trying to specify the manner in which it diverges.

We summarize our discussion and example above into a definition. In this definition, we will split a set into a union of closed or half-open intervals (as we did above); the point is that each of the intervals that we split things into has a “problem”, an unboundedness issue, at, at most, one endpoint. It is important that the definition gives you the same result, regardless of how you pick the half-open intervals. The proof of this is in Theorem 2.A.11.

Definition 2.5.11. *Let E be a subset of the real numbers which is the union of a finite number of intervals. Let f be a real function, which is defined and continuous on the set E except, perhaps, at a finite set of points P. Let $E - P$ denote the set of points in E which are not in the set P.*

Then, $E - P$ is a union of a finite number of closed or half-open intervals $I_1, I_2, \ldots, I_n$ on which f is defined and continuous. Given any such decomposition of $E - P$ into intervals, we define the **integral of f on E** *by*

$$\int_E f(x)\,dx \;=\; \int_{I_1} f(x)\,dx \;+\; \int_{I_2} f(x)\,dx \;+\; \ldots \;+\; \int_{I_n} f(x)\,dx,$$

provided that each integral in the summation on the right converges; in this case, we say that $\int_E f(x)\,dx$ **converges** *or that f is* **integrable on** *E. If any one of the integrals in the summation above diverges, then we say that $\int_E f(x)\,dx$* **diverges**.

These definitions of converges, diverges, and the value of the integral are independent of the choice of the intervals $I_1, I_2, \ldots, I_n$, as long as the given conditions are satisfied.

If E is an interval $[a, b]$, $[a, b)$, $(a, b]$, or (a, b), then we may also use other notation; we set

$$\int_a^b f(x)\,dx \;=\; \int_E f(x)\,dx, \qquad \text{and} \qquad \int_b^a f(x)\,dx \;=\; -\int_E f(x)\,dx.$$

Note that, with our terminology, we may say that a function f is integrable on a set E, **even if f is not defined at a finite number of the points in E.** For instance, in Example 2.5.9, we would say that $1/x^{2/3}$ is integrable on $(-1, 8)$, even though $1/x^{2/3}$ is not defined at $x = 0$. Be aware that other books might use different terminology, and might not say that f is integrable on a set on which f is not defined.

Just as we have linearity for Riemann integrals, Theorem 2.3.19, we also have linearity for our more general integrals, provided the individual integrals converge. The proof is essentially identical, except that, to deal with the improper integrals, you must use that limits are "linear", i.e., you can pull constants out of limits and split up sums.

Theorem 2.5.12. (**Linearity of Improper Integrals**) *Let E be a subset of the real numbers which is the union of a finite number of intervals. Suppose that f and g are integrable on E, and that a and b are any real numbers. Then, $af + bg$ is integrable on E and*

$$\int_E af(x) + bg(x)\,dx \;=\; a\int_E f(x)\,dx \;+\; b\int_E g(x)\,dx.$$

Example 2.5.13. As we saw in Example 2.5.5 and Example 2.5.9 (with different upper-limits of integration), the integrals

$$\int_0^1 \frac{1}{x^{1/2}}\,dx \qquad \text{and} \qquad \int_0^1 \frac{1}{x^{2/3}}\,dx$$

both converge; we leave it as an exercise for you to show that they converge to 2 and 3, respectively.

Therefore,

$$\int_0^1 \left(\frac{5}{x^{1/2}} - \frac{7}{x^{2/3}}\right)dx$$

converges, and equals $5\cdot 2 - 7\cdot 3 = -11$.

It is sometimes possible to tell that an improper integral converges, **without** being able to determine what it converges to. This seemingly bizarre fact stems from a defining property of the real numbers: every non-empty set of real numbers, which has an upper bound, has a LEAST upper bound. See Definition 5.1.15 and Theorem 5.1.18. This *least upper bound property* is the main reason that we can sometimes tell that limits, as in improper integrals, exist without knowing the value.

We first give a theorem which says that there's only one way for some types of improper integrals to diverge.

Theorem 2.5.14. *Let I be an interval of the form $[a, b)$ or $(a, b]$, where we allow the intervals $[a, \infty)$ or $(-\infty, b]$. Suppose that, for all x in I, $f(x) \geq 0$, and that, for all closed intervals $[c, d]$ contained in I, f is Riemann integrable on $[c, d]$.*

Then, if there exists a real number M (an upper bound) such that, for all $[c, d]$ contained in I, $\int_c^d f(x)\,dx \leq M$, then $\int_a^b f(x)\,dx$ converges, and what it converges to is less than, or equal to, M; in other words, if there is an upper bound M on all of the $\int_c^d f(x)\,dx$, then $\int_a^b f(x)\,dx$ converges to the least such upper bound.

In particular, if $\int_a^b f(x)\,dx$ diverges, what it diverges to is ∞.

Proof. See Theorem 2.A.12. $\square$

Remark 2.5.15. The analogous statement for $f(x) \leq 0$ is true, and is obtained by applying Theorem 2.5.14 to $-f(x)$, which would be non-negative.

What you find for non-positive f is: if there exists a real number M (a lower-bound) such that, for all $[c, d]$ contained in I, $M \leq \int_c^d f(x)\,dx$, then $\int_a^b f(x)\,dx$ converges to the greatest such lower bound. In particular, if $\int_a^b f(x)\,dx$ diverges, what it diverges to is $-\infty$.

Corollary 2.5.16. *Let I be an interval of the form $[a, b)$ or $(a, b]$, where we allow the intervals $[a, \infty)$ or $(-\infty, b]$. Suppose that, for all x in I, $0 \leq f(x) \leq g(x)$, and that, for all closed intervals $[c, d]$ contained in I, f and g are Riemann integrable on $[c, d]$.*

Then, if $\int_a^b g(x)\,dx$ converges, then so does $\int_a^b f(x)\,dx$, and what it converges to is something less than, or equal to, $\int_a^b g(x)\,dx$. This implies that, if $\int_a^b f(x)\,dx$ diverges, then so does $\int_a^b g(x)\,dx$.

Proof. This is easy now. Since $f \leq g$ on I, Theorem 2.3.20 tells us that, for all $[c, d]$ contained in I,

$$\int_c^d f(x)\,dx \;\leq\; \int_c^d g(x)\,dx.$$

If $\int_a^b g(x)\,dx$ converges, then Theorem 2.5.14 tells us that it converges to the least upper bound M on the integrals $\int_c^d g(x)\,dx$, but, by the inequality above, this M is an upper bound on all of the integrals $\int_c^d f(x)\,dx$. The corollary now follows by applying Theorem 2.5.14 again. $\square$

Example 2.5.17. Consider the integrals

$$\int_0^1 \frac{1+\sin x}{\sqrt{x}}\,dx \qquad \text{and} \qquad \int_1^\infty \frac{2+\sin x}{x}\,dx.$$

Producing manageable anti-derivatives of these integrands is problematic/impossible. Nonetheless, we can use Corollary 2.5.16 to determine quickly whether they converge or not.

First, note that, for all x in $(0,1]$,

$$0 \;\leq\; \frac{1+\sin x}{\sqrt{x}} \;\leq\; \frac{2}{\sqrt{x}},$$

and that, for all x in $[1,\infty)$,

$$0 \;\leq\; \frac{1}{x} \;\leq\; \frac{2+\sin x}{x}.$$

Now, we find

$$\int_0^1 \frac{2}{\sqrt{x}}\,dx \;=\; \lim_{a\to 0^+}\int_a^1 \frac{2}{\sqrt{x}}\,dx \;=\; \lim_{a\to 0^+} 4x^{1/2}\Big|_a^1 \;=\; 4\cdot \lim_{a\to 0^+}(1-a^{1/2}) \;=\; 4,$$

and

$$\int_1^\infty \frac{1}{x}\,dx \;=\; \lim_{b\to\infty}(\ln b - \ln 1) \;=\; \infty.$$

Combining all of the above with Corollary 2.5.16, we conclude that $\displaystyle\int_0^1 \frac{1+\sin x}{\sqrt{x}}\,dx$ converges (to something less than or equal to 4, but we don't know what), while $\displaystyle\int_1^\infty \frac{2+\sin x}{x}\,dx$ diverges to ∞.

2.5.1 Exercises

In Exercises 1 through 17, evaluate the given integral if it converges; otherwise, show that the integral diverges.

1. $\displaystyle\int_0^\infty \frac{dx}{(2x+5)^2}$.

2. $\displaystyle\int_1^\infty \frac{dt}{(4+5t)^{3/2}}$.

3. $\displaystyle\int_0^\infty \frac{dx}{7x-5}$.

4. $\displaystyle\int_1^\infty \frac{dy}{(4y-3)^{1/2}}$.

5. $\displaystyle\int_0^\infty \sin z\,dz$.

6. $\displaystyle\int_0^\infty \frac{du}{(u+3)(u+5)}$.

7. $\displaystyle\int_{-4}^\infty \frac{du}{(u+3)(u+5)}$.

8. $\displaystyle\int_0^\pi \frac{dv}{1-\cos v}$. Hint: use a half-angle identity to evaluate the integral.

9. $\displaystyle\int_0^1 \frac{dt}{\sqrt{1-t^2}}$.

10. $\displaystyle\int_0^\infty \frac{dw}{1+w^2}$.

11. $\displaystyle\int_0^1 x^{-1/2}\ln x\,dx$.

12. $\displaystyle\int_0^1 x^{-2}\ln x\,dx$.

13. $\displaystyle\int_{-k}^k \frac{dt}{\sqrt[3]{t}}\,dt,\ k>0$.

14. $\displaystyle\int_0^\infty \frac{\tan^{-1} z}{1+z^2}\,dz$.

15. Let $n>1$. Does $\displaystyle\int_0^1 \frac{dx}{x^n}$ converge or diverge? If it converges, what does it converge to?

16. Let $n=1$. Does $\displaystyle\int_0^1 \frac{dx}{x^n}$ converge or diverge? If it converges, what does it converge to?

17. Let $n<1$. Does $\displaystyle\int_0^1 \frac{dx}{x^n}$ converge or diverge? If it converges, what does it converge to?

Use Corollary 2.5.16 to determine whether the integrals in Exercises 18 through 24 converge or diverge. If they converge, you need not calculate the integrals.

18. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx.$

19. $\displaystyle\int_{2}^{\infty} \frac{|\sin x|}{1+x^2}\,dx.$

20. $\displaystyle\int_{1}^{\infty} \frac{e^t}{t}\,dt.$

21. $\displaystyle\int_{0}^{1} \frac{e^{-y}}{\sqrt{y}}\,dy.$

22. $\displaystyle\int_{1}^{\infty} \frac{e^{-y}}{\sqrt{y}}\,dy.$

23. $\displaystyle\int_{0}^{\infty} \frac{\tan^{-1} x}{x^3+5x+1}\,dx.$ Hint: See Exercise 14 above.

24. $\displaystyle\int_{1}^{\infty} \frac{s}{e^s-1}\,ds.$ Hint: Justify and use the fact that $e^s > 1 + \dfrac{s^3}{6}$, for all $s > 0$.

25. Prove that the elliptic integral $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ converges. Assume $k^2 < 1$.

26. Formulate a similar comparison theorem for functions that grow without bound at some real number a. You may limit your statement to either a right or left-hand limit.

27. Find all real numbers p, if any exist, such that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^p}\,dx$ converges.

28. Find all real numbers p, if any exist, such that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^p}\,dx$ converges.

Let $\Gamma(n) = \displaystyle\int_{0}^{\infty} e^{-x}x^{n-1}\,dx$**. This function is called the** ***gamma function*****.**

29. Calculate $\Gamma(1)$, and $\Gamma(2)$.

30. Prove inductively that $\Gamma(n) = (n-1)!$ for $n = 1, 2, 3, \ldots$..

A random variable is said to be an exponentially distributed if its density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0 \end{cases}$$

for some $\lambda > 0$**.**

31. What is $E(X)$, the expected value of an exponential random variable?

32. What is $E(X^2)$, if X is an exponential random variable?

33. What is the variance, $Var(X)$, of an exponential random variable?

A data set (in statistics) or a random variable (in probability) is said to be *normally distributed* if its density function is

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}.$$

The graph of this density function is the familiar *bell curve***.**

34. Prove that the expected value of a normally distributed random variable is μ.

35. What is $E(X^2)$ for a normally distributed random variable?

36. Prove that the variance of a normally distributed random variable is σ^2.

37. Recall that the cumulative distribution function of a continuous random variable is $F(a) = \int_{-\infty}^{a} f(x)\,dx$. $F(a)$ is the probability that the random variable X is less than or equal to a. What is $F(a)$ for an exponentially distributed random variable?

38. Using the notation of the previous problem, prove that $\lim_{a\to\infty} F(a) = 1$ if F is the cumulative distribution function of an exponentially distributed random variable.

39. The *survival function* of a a random variable is the complement of the cumulative distribution function. That is, if $F(x)$ is the cumulative distribution function, then $S(x) = 1 - F(x)$ is the survival function. The name of this function can be understood from an actuarial perspective where the random variable is a life. Then $S(x)$ is the probability that the life will survive longer than x years. What is $S(x)$ for an exponential function?

40. The *moment generating function* of a distribution provides an alternative method of calculating the mean and variance of the distribution, and is defined as follows:

$$\phi(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

where $f(x)$ is the density function. Prove that if X is exponentially distributed with parameter λ, then $\phi(t) = \dfrac{\lambda}{\lambda - t}$.

41. Show that $\phi'(0) = E(X)$ if X is exponentially distributed.

42. Show that $\phi''(0) = E(X^2)$ if X is exponentially distributed. These two results are not specific to the exponential distribution; they hold for arbitrary continuous distributions (Bonus exercise: try to prove this!)

43. The amount of time it takes to wait in line to register for classes is exponentially distributed with mean 12 minutes (so $\lambda = 1/12$).

    a. What's the probability a student will have to wait for less than 12 minutes?

    b. What's the probability a student's waiting time will be between 10 and 20 minutes?

    c. What's the probability a student's waiting time exceeds 25 minutes?

44. Recall that the mass $m(t)$ of a decaying compound is given by $m(t) = m_0 e^{-\lambda t}$ where $t > 0$. However, this apparently exact formula is merely a model for a complicated physical process. Radioactive decay can be be viewed in probabilistic terms using an exponential distribution. First, note that $e^{-\lambda t}$ is the survival function of an exponential random variable. Assuming the life of the atom is exponentially distributed, it makes sense to define the *mean life* of an atom as $ML = \int_0^\infty t\lambda e^{-\lambda t}\,dt$. Calculate the mean lifetime of the following atoms (λ is given so that t is in years).

    a. uranium-239, $\lambda = 1.55 \times 10^{-10}$.

    b. carbon-14, $\lambda = 1.15 \times 10^{-5}$.

    c. tritium, $\lambda = 0.056$.

45. Suppose we have 100 tritium atoms. How many atoms are anticipated to decay between 8 and 12 years based on the model in the previous problem? Note that this is the same as asking for the probability that a single tritium atom will decay between 8 and 12 years from now.

46. Is $\displaystyle\int_0^\infty x\sin(x^2)\,dx$ convergent? Justify your answer.

If $f(t)$ is a continuous function on $[0,\infty)$, then the *Laplace transform* of f is the function $F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\,dt$. Laplace transforms provide a powerful technique for solving differential equations. Calculate the Laplace transforms of the functions in Exercises 47 through 52. Note that s values for which the Laplace transform integral does not exist are not in the domain of the transformed function $F(s)$.

47. $f(t) = 1$.

48. $f(t) = kt$, where k is a constant.

49. $f(t) = e^{at}$, where a is a constant.

50. $f(t) = \sin(ct)$, where c is a constant.

51. $f(t) = \cosh t$.

52. $f(t) = \sinh t$.

53. Suppose $p > -1$. Recall the definition of the gamma function from above. Show that the Laplace transform of t^p is $F(s) = \Gamma(p+1)/s^{p+1}$. In particular, if p is a positive integer, then $F(s) = p!/s^{p+1}$.

2.6 Numerical Techniques for Approximating Integrals

In this section, it may seem like we're backing up. In Section 2.2, we looked at Riemann sums, then, in Section 2.3, we took limits of Riemann sums to define the definite integral, the continuous sum of infinitesimal contributions. Then, after seeing how tedious it is to calculate integrals as limits of Riemann sums, in Section 2.4, we presented the Fundamental Theorem of Calculus, which tells us that, if we have an anti-derivative F of a continuous function f, then it's easy to calculate the values of definite integrals of f, in terms of F. Great. So, what's our problem?

Our problem is that there are continuous functions f for which we cannot produce manageable anti-derivatives, and so we cannot use the Fundamental Theorem to calculate definite integrals for such f. The classic example is $f(x) = e^{-x^2}$. Definite integrals of this function are of fundamental importance in probability and statistics, and yet, as we mentioned in Remark 1.1.23, e^{-x^2} has no elementary anti-derivative.

Thus, while the first part of the Fundamental Theorem, Theorem 2.4.7, tells us that definite integrals of continuous functions on closed intervals exist, our question, for functions f without "nice" anti-derivatives is: can we approximate $\int_a^b f(x)\,dx$ in a better way than by using simply Riemann sums?

The answer to this question is: "yes". We will look at two "rules" for approximating the values of definite integrals: the *Trapezoidal* (or *Trapezoid*) *Rule* and *Simpson's Rule*. Both of these rules use summations that look similar to Riemann sums, but the summations are, in fact, **not** Riemann sums.

The Trapezoidal Rule is very easy to derive. We approximate the graph of f, the function we want to integrate, by using line segments between points on the graph, and (assuming for the convenience of this discussion that $f \geq 0$) we then find the area of the region, a trapezoid, below each line segment and above the relevant interval. Adding these areas gives an approximation of the corresponding definite integral.

It turns out that, for many functions, the approximation of the definite integral using the Trapezoidal Rule is **worse** than the approximation using midpoint Riemann sums (Riemann sums in which all subintervals have the same length, and the sample points are the midpoints of the subintervals; see Example 2.2.8). Hence, in some sense, the Trapezoidal Rule is useless.

However, Simpson's Rule is just as easy to use as the Trapezoidal Rule, and yet, approximating an integral using Simpson's Rule is usually **stunningly** more accurate than using the Trapezoidal Rule or using midpoint Riemann sums. Simpson's Rule is based on approximating the graph of f with a collection of parabola segments, instead of with the line segments used in the Trapezoidal Rule. Since graphs of typical functions curve, and parabolas curve, it should seem reasonable that Simpson's Rule is typically more accurate than the Trapezoidal Rule.

Below, we first adopt some notation and remind you what a midpoint Riemann sum is. After that, we discuss and derive the Trapezoidal Rule, and then discuss and state Simpson's Rule, leaving the actual derivation of Simpson's Rule for the Technical Matters section, Section 2.A. We also give statements regarding bounds on the errors when you use the three approximations; the proofs of these bounds are also in Section 2.A. We first give examples in which we do not consider the error bounds. Finally, we give an extended example where we discuss the error, and bounds on the error, in more detail.

Suppose that we have a function f on a closed interval $[a, b]$.

We subdivide the interval $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$. For $0 \leq k \leq n$, the corresponding partition of $[a, b]$, i.e., the collection of endpoints of the subintervals, is given by $x_k = a + k\Delta x$, that is,

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ \dots \ , x_{n-1} = a + (n-1)\Delta x, \quad x_n = a + n\Delta x = b.$$

For $1 \leq k \leq n$, the k-th subinterval is the closed interval $[x_{k-1}, x_k]$; its midpoint is $s_k = (x_{k-1} + x_k)/2$.

If A is an approximation of $\int_a^b f(x)\, dx$, then we refer to the difference $E = \int_a^b f(x)\, dx - A$ as the *error* in the approximation.. As we usually care about whether A is within plus or minus some amount of the actual value of the integral, it is usually the absolute value of E that we are interested in. We refer to the absolute value of the error as the *absolute error*.

We should remark that we will give all our calculations below to 12 decimal places. This may seem like ridiculous precision; however, when you're trying to compare various approximations to each other, you want to go out to enough decimal places to actually see where the approximations differ.

We first use midpoint Riemann sums to approximate the definite integral. Graphically, we view the midpoint Riemann sum (for $f \geq 0$) as shown in Figure 2.36.

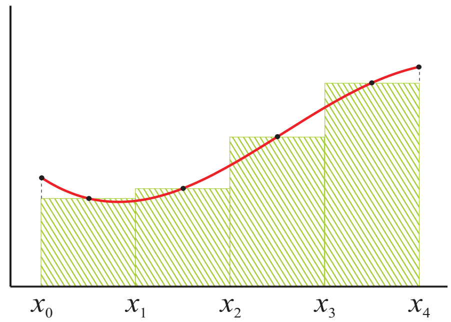


Figure 2.36: A midpoint Riemann sum.

Definition 2.6.1. (**Midpoint Approximation for Integrals**) *Using the notation above, the* **midpoint approximation of** $\int_a^b f(x)\,dx$*, using* n *subintervals, is*

$$\int_a^b f(x)\,dx \;\approx\; \Delta x\left[f\left(\frac{x_0+x_1}{2}\right) \;+\; f\left(\frac{x_1+x_2}{2}\right) \;+\; \cdots \;+\; f\left(\frac{x_{n-1}+x_n}{2}\right)\right].$$

Example 2.6.2. Consider the integral $\int_0^1 e^{-x^2}\,dx$. A calculator will tell you, correctly, that, to 12 decimal places, the value of this definite integral is 0.746824132812. What approximations do we obtain from the Midpoint Approximation using $n = 2$ and $n = 4$?

When $n = 2$, we have $\Delta x = (1-0)/2 = 1/2$, $x_0 = 0$, $x_1 = 1/2$, and $x_2 = 1$ and, of course, $f(x) = e^{-x^2}$.

The Midpoint Approximation is

$$\int_0^1 e^{-x^2}\,dx \;\approx\; \frac{1}{2}\left[e^{-(1/4)^2} + e^{-(3/4)^2}\right],$$

which, to 12 decimal places is 0.754597943772. Thus, the absolute error here is

$$\left|0.746824132812 \;-\; 0.754597943772\right| \;=\; 0.00777381096;$$

not great, but pretty good, considering that we used only 2 subintervals.

Let's try $n = 4$. Now, $\Delta x = (1-0)/4 = 1/4$, $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, and $x_4 = 1$.

Our new Midpoint Approximation is

$$\int_0^1 e^{-x^2}\, dx \;\approx\; \frac{1}{4}\left[e^{-(1/8)^2} + e^{-(3/8)^2} + e^{-(5/8)^2} + e^{-(7/8)^2}\right],$$

which, to 12 decimal places is 0.748747131891. Thus, the absolute error here is

$$\left|0.746824132812 \;-\; 0.748747131891\right| \;=\; 0.001922999079;$$

this is roughly $1/4$ of the absolute error that we had when we used $n = 2$.

Now let's look at the Trapezoidal Rule. As we stated at the beginning of the section, the Trapezoidal Rule does not involve Riemann sums, but rather something that looks similar to Riemann sums. The idea of the Trapezoidal Rule is simple; over each of our subintervals $[x_{k-1}, x_k]$, you approximate the function f by the unique linear function $L_k(x) = m_k x + b_k$ whose graph passes through the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$, i.e., on the graph of f, you "connect the dots" (with line segments) between the points of the graph of f which correspond to the ends of the subinterval.

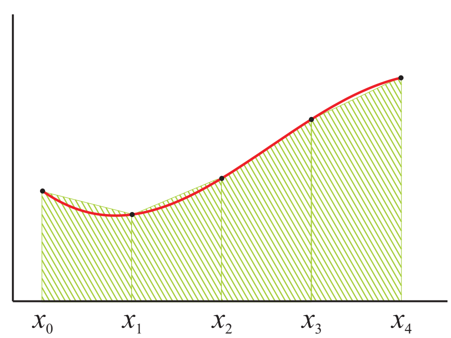


Figure 2.37: Typical areas involved in the Trapezoidal Rule.

At this point, instead of integrating f, you calculate each of the integrals I_k of the linear functions $L_k(x) = m_k x + b_k$ over the interval $[x_{k-1}, x_k]$, and then you add these integrals together. What you obtain for I_k is

$$I_k = \frac{\Delta x}{2}\Big[f(x_{k-1}) + f(x_k)\Big].$$

If f is positive, you should recognize this as the area of a trapezoid: one half the "height" (here, it's the width) times the sum of the lengths of the bases. Adding these together, we find the approximation

$$\int_a^b f(x)\,dx \approx I_1 + I_2 + I_3 + \cdots + I_{n-1} + I_n =$$

$$\frac{\Delta x}{2}\Big[f(x_0) + f(x_1)\Big] + \frac{\Delta x}{2}\Big[f(x_1) + f(x_2)\Big] + \frac{\Delta x}{2}\Big[f(x_2) + f(x_3)\Big] + \cdots +$$
$$\frac{\Delta x}{2}\Big[f(x_{n-2}) + f(x_{n-1})\Big] + \frac{\Delta x}{2}\Big[f(x_{n-1}) + f(x_n)\Big].$$

Factoring out the $\frac{\Delta x}{2}$, and combining the pairs of overlapping terms everywhere, except for at the $f(x_0)$ and $f(x_n)$ terms, we obtain the following approximation.

Definition 2.6.3. (**Trapezoidal Rule**) *Using the notation above, the* **Trapezoidal Rule Approximation of** $\int_a^b f(x)\,dx$*, using* n *subintervals, is*

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big].$$

Note that the pattern of the coefficients for $f(x_k)$ in the Trapezoidal Rule is that the first and last coefficients are 1's and, aside from that, the coefficients are always 2's.

Example 2.6.4. As in Example 2.6.2, consider the integral $\int_0^1 e^{-x^2}\,dx$. Recall that, to 12 decimal places, the value of this definite integral is 0.746824132812. What approximations do we obtain from the Trapezoidal Rule using $n = 2$ and $n = 4$?

When $n = 2$, as before, we have $\Delta x = (1-0)/2 = 1/2$, $x_0 = 0$, $x_1 = 1/2$, and $x_2 = 1$.

The Trapezoidal Rule Approximation is

$$\int_0^1 e^{-x^2}\,dx \;\approx\; \frac{1}{4}\left[e^{-(0)^2} + 2e^{-(1/2)^2} + e^{-(1)^2}\right],$$

which, to 12 decimal places is 0.731370251829. Thus, the absolute error here is

$$\left|0.746824132812 \;-\; 0.731370251829\right| \;=\; 0.015453880983.$$

Note that this is roughly double the absolute error that we had when we used the midpoint Riemann sum with $n = 2$.

What about when $n = 4$? As before, $\Delta x = (1-0)/4 = 1/4$, $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, and $x_4 = 1$.

Our new Trapezoidal Rule Approximation is

$$\int_0^1 e^{-x^2}\,dx \;\approx\; \frac{1}{8}\left[e^{-(0)^2} + 2e^{-(1/4)^2} + 2e^{-(1/2)^2} + 2e^{-(3/4)^2} + e^{-(1)^2}\right],$$

which, to 12 decimal places is 0.742984097800. Thus, the absolute error here is

$$\left|0.746824132812 \;-\; 0.742984097800\right| \;=\; 0.003840035012.$$

As with the Midpoint Approximation, this new approximation, using the Trapezoidal Rule with $n = 4$ has roughly $1/4$ of the absolute error that we had when we used the Trapezoidal Rule with $n = 2$. However, you should also notice that it is still true that the absolute error, with $n = 4$, using the Trapezoidal Rule, is roughly double the error, with $n = 4$, using the midpoint Riemann sum.

Example 2.6.5. When would you really want to use the Trapezoidal Rule?

One situation where trapezoidal approximation is the best way to go is when you're given a table of data and, perhaps, the subintervals do not even all have the same length.

Consider, for instance, approximating $\int_0^7 f(x)\,dx$, when you're given the following table of values.

x	0	1	3	7
$f(x)$	2	3.5	4	3

How can you give a reasonable approximation of $\int_0^7 f(x)\,dx$? It's simple; you use what we derived earlier for the Trapezoidal Rule: that the integral of f on the subinterval $[x_{k-1}, x_k]$ is approximated by the area of the trapezoid

$$\frac{\Delta x}{2}\Big[f(x_{k-1}) + f(x_k)\Big],$$

where Δx may change with each subinterval. So, you have to use $\Delta x = x_k - x_{k-1}$ on the k-th subinterval, and then you add the contributions from all of the subintervals.

Thus, in our current example, we find

$$\int_0^7 f(x)\,dx \;\approx\; \frac{1-0}{2}\big(2+3.5\big) + \frac{3-1}{2}\big(3.5+4\big) + \frac{7-3}{2}\big(4+3\big) \;=\; 24.25.$$

Now we want to look at Simpson's Rule. What is Simpson's Rule? In a sense, Simpson's Rule is the next step after the Trapezoidal Rule. The Trapezoidal Rule takes pairs of successive points on the graph of f, as determined by the endpoints of the subintervals. Two points determine a line, and so determine a linear function $L = mx + b$. On the k-th subinterval, we approximate f by the linear function $L_k = m_k x + b_k$ and integrate this linear function over the integral $[x_{k-1}, x_k]$, instead of integrating f. Then, we add together the approximations over all of the subintervals.

Simpson's Rule uses the fact that three points determine a parabola (or a line, if the points are collinear), and so determine a unique polynomial $q = q(x) = ax^2 + bx + c$ of degree less than or equal to 2. Simpson's Rule takes three successive points on the graph of f, as determined by the endpoints of the subintervals, and approximates f by the corresponding polynomial $q = q(x) = ax^2 + bx + c$ over the **two** subintervals whose endpoints correspond to the three points we took on the graph. For instance, over the two successive subintervals $[x_0, x_1]$ and $[x_1, x_2]$, i.e., over the interval $[x_0, x_2]$, Simpson's Rule approximates f by the unique function $q(x) = ax^2 + bx + c$ whose graph passes through the three points $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$.

When you determine the function $q(x)$ and integrate it over $[x_0, x_2]$, what you find is relatively simple. You get $\frac{\Delta x}{3}\Big[f(x_0)+4f(x_1)+f(x_2)\Big]$. (See Proposition 2.A.14.) We can do this for pairs of subintervals, and approximate the entire integral $\int_a^b f(x)\,dx$, **provided that n, the number of subintervals, is even**.

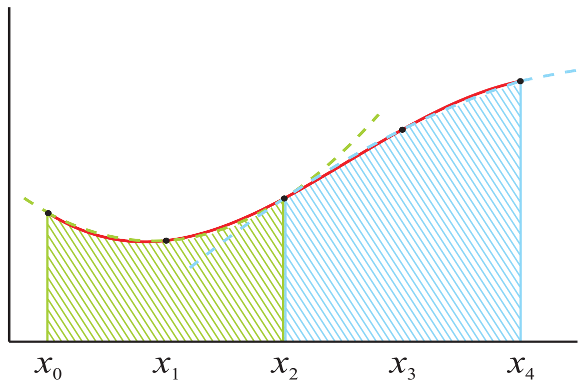


Figure 2.38: Typical areas involved in Simpson's Rule.

Note that we had to make Figure 2.38 unusually large for you to have any hope of seeing that the dotted parabolas do not **exactly** fit the graph.

Assuming that n is even, if we add together the contributions over pairs of intervals, the approximation that we obtain is

$$\int_a^b f(x)\,dx \;\approx$$
$$\frac{\Delta x}{3}\Big[f(x_0)+4f(x_1)+f(x_2)\Big] \;+\; \frac{\Delta x}{3}\Big[f(x_2)+4f(x_3)+f(x_4)\Big] \;+\; \cdots \;+$$
$$\frac{\Delta x}{3}\Big[f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})\Big]+\frac{\Delta x}{3}\Big[f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big].$$

Factoring out the $\frac{\Delta x}{3}$, and combining the pairs of overlapping terms everywhere, except for at the $f(x_0)$ and $f(x_n)$ terms, we obtain the following approximation.

Definition 2.6.6. (**Simpson's Rule**) *Using the notation above, the* **Simpson's Rule approximation of** $\int_a^b f(x)\,dx$*, using n subintervals, where n is even, is*

$$\int_a^b f(x)\,dx \;\approx\; \frac{\Delta x}{3}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+4f(x_{n-1})+f(x_n)\Big].$$

Note that the pattern of the coefficients for $f(x_k)$ in Simpson's Rule is that the first and last coefficients are 1's and, aside from that, the coefficients alternate 4, 2, 4, 2, etc. The next-to-last coefficient will always be a 4, due to the fact that n is even.

Example 2.6.7. As in our previous two examples, consider the integral $\int_0^1 e^{-x^2}\,dx$. Recall that, to 12 decimal places, the value of this definite integral is 0.746824132812. What approximations do we obtain from Simpson's Rule using $n = 2$ and $n = 4$?

When $n = 2$, we still have $\Delta x = (1-0)/2 = 1/2$, $x_0 = 0$, $x_1 = 1/2$, and $x_2 = 1$.

The Simpson's Rule Approximation is

$$\int_0^1 e^{-x^2}\,dx \;\approx\; \frac{1}{6}\left[e^{-(0)^2} + 4e^{-(1/2)^2} + e^{-(1)^2}\right],$$

which, to 12 decimal places is 0.747180428910. Thus, the absolute error here is

$$\left|0.746824132812 \;-\; 0.747180428910\right| \;=\; 0.000356296098.$$

Wow! This absolute error is roughly 1/20 of the absolute error using $n = 2$ for the midpoint Riemann sum, and is even roughly 1/5 of the absolute error using $n = 4$ for the midpoint Riemann sum (and, remember, the midpoint Riemann sums gave better approximations than the Trapezoidal Rule).

Thus, what we see is that the approximation via Simpson's Rule, even with a smaller n value, is significantly more accurate than those obtained from midpoint Riemann sums or the Trapezoidal Rule. This is typical.

Let's see what happens when $n = 4$. As before, $\Delta x = (1-0)/4 = 1/4$, $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, and $x_4 = 1$.

Our new Simpson's Rule Approximation is

$$\int_0^1 e^{-x^2}\,dx \;\approx\; \frac{1}{12}\left[e^{-(0)^2} + 4e^{-(1/4)^2} + 2e^{-(1/2)^2} + 4e^{-(3/4)^2} + e^{-(1)^2}\right],$$

which, to 12 decimal places is 0.746855379791. Thus, the absolute error here is

$$\left|0.746824132812 \;-\; 0.746855379791\right| \;=\; 0.000031246979.$$

This new approximation, using Simpson's Rule with $n = 4$ has roughly $1/10$ of the absolute error that we had when we used Simpson's Rule with $n = 2$.

It is reasonable to ask if it is possible to make precise in what sense midpoint Riemann sums, the Trapezoidal Rule, and Simpson's Rule are "reasonable" approximation methods. Basically, the question is: what can an instructor say to a student who wants credit for claiming "$\int_0^1 e^{-x^2}\,dx$ is **approximately** $1,000,000$, it's just that the error is really large"?

A possible answer is provided by the following three theorems, which we prove in the Technical Matters section, Section 2.A. These theorems give upper bounds on the absolute error when approximating an integral by using midpoint Riemann sums, the Trapezoidal Rule, and/or Simpson's Rule, provided that the integrand possesses enough derivatives. In particular, these theorems imply that, for all three approximation techniques, the error approaches zero as n approaches ∞.

Theorem 2.6.8. *Suppose that $a < b$ and the midpoint Riemann sum with n subintervals of equal length Δx is used to approximate $\int_a^b f(x)\,dx$.*

If $f''(x)$ exists for all x in some open interval containing the closed interval $[a, b]$ and, if there exists a number $M \geq 0$ such that, for all x in $[a, b]$, $|f''(x)| \leq M$, then the absolute value of the error, E_{mdpt}, satisfies the inequality

$$|E_{\mathrm{mdpt}}| \;\leq\; \frac{M(\Delta x)^3 n}{24} \;=\; \frac{M(b-a)^3}{24n^2}.$$

Proof. See Theorem 2.A.15. $\square$

If, in fact, f'' above is continuous on the interval $[a, b]$, then the best, i.e., smallest, value that you can use for M above is the maximum value of f'' on $[a, b]$.

Theorem 2.6.9. *Suppose that $a < b$ and the Trapezoidal Rule with n subintervals of equal length Δx is used to approximate $\int_a^b f(x)\,dx$.*

If $f''(x)$ exists for all x in some open interval containing the closed interval $[a, b]$ and, if there exists a number $M \geq 0$ such that, for all x in $[a, b]$, $|f''(x)| \leq M$, then the absolute value of the error, E_{trap}, satisfies the inequality

$$|E_{\text{trap}}| \;\leq\; \frac{M(\Delta x)^3 n}{12} \;=\; \frac{M(b-a)^3}{12n^2}$$

Proof. See Theorem 2.A.16. ☐

Remark 2.6.10. Note that our upper bound for the absolute value of the error in approximating with the Trapezoidal Rule is precisely twice the upper bound that we had for approximating with midpoint Riemann sums. This is why references frequently say that midpoint Riemann sums generally provide a better approximation to definite integrals than does the Trapezoidal Rule. Understand, however, that the general upper bounds on the absolute values of the errors that appear in the theorems don't have to be very close to the true absolute values of the errors; they are simply gross upper bounds.

In particular, the absolute value of the error using the Trapezoidal Rule **can** be strictly smaller than what you obtain from midpoint Riemann sums. That's just not what we expect, in general.

Theorem 2.6.11. *Suppose that $a < b$, n is even, and Simpson's Rule with n subintervals of equal length Δx is used to approximate $\int_a^b f(x)\,dx$.*

If $f^{(4)}(x)$ exists for all x in some open interval containing the closed interval $[a, b]$ and, if there exists a number $M \geq 0$ such that, for all x in $[a, b]$, $|f^{(4)}(x)| \leq M$, then the absolute value of the error, E_{Simp}, satisfies the inequality

$$|E_{\text{Simp}}| \;\leq\; \frac{M(\Delta x)^5 n}{180} \;=\; \frac{M(b-a)^5}{180n^4}.$$

Proof. See Theorem 2.A.17. □

Remark 2.6.12. If $f(x)$ is a quadratic polynomial, then it should not be surprising that Simpson's Rule would give the exact value of the integral; after all, Simpson's Rule is obtained by approximating the given function with quadratic functions.

What is somewhat surprising is that, even if $f(x)$ is a cubic polynomial, the error bound above tells us that Simpson's Rule still yields the exact value of the integral, for the fourth derivative of a cubic polynomial is 0, and so the M in the error statement could be taken to be 0.

Let's look at an example for which we **can** produce an easy anti-derivative, and can easily find bounds on the second and fourth derivatives, so that we can see how two different approximations compare with the exact answer. Understand, however, that the true value of these approximations is in cases like $\int_0^1 e^{-x^2}\,dx$, where we **can't** produce a convenient anti-derivative.

Example 2.6.13. Consider

$$\int_1^2 \frac{1}{x^2}\,dx \;=\; \int_1^2 x^{-2}\,dx \;=\; \frac{x^{-1}}{-1}\bigg|_1^2 \;=\; -\frac{1}{2}+1 \;=\; \frac{1}{2}.$$

Let's approximate $\int_1^2 \frac{1}{x^2}\,dx$ using midpoint Riemann sums, the Trapezoidal Rule, and Simpson's Rule with both $n=2$ and $n=4$. First, though, let's go ahead and calculate the theoretical error bounds, so that we can see how our actual errors compare with the bounds. We will also calculate how big we would need to pick n to have an error bound in each of the three approximations that would guarantee an error of less than 0.00001.

So, we have $f(x) = 1/x^2 = x^{-2}$, $a=1$, and $b=2$. We calculate the derivatives that we need for the error bounds: $f'(x) = -2x^{-3}$, $f''(x) = 6x^{-4}$, $f'''(x) = -24x^{-5}$, and $f^{(4)}(x) = 120x^{-6}$. Thus, the maximum value of $|f''(x)|$ on the interval $[1,2]$ is 6, and the maximum value of $|f^{(4)}(x)|$ on $[1,2]$ is 120.

Therefore, when $n=2$, we know that

$$|E_{\mathrm{mdpt}}| \;\leq\; \frac{6(2-1)^3}{24\cdot 2^2} \;=\; \frac{1}{16} \;=\; 0.0625;$$

$$|E_{\text{trap}}| \leq \frac{6(2-1)^3}{12 \cdot 2^2} = \frac{1}{8} = 0.125;$$

and

$$|E_{\text{Simp}}| \leq \frac{120(2-1)^5}{180 \cdot 2^4} = \frac{1}{24} = 0.041\overline{6}.$$

When $n = 4$, we find

$$|E_{\text{mdpt}}| \leq \frac{6(2-1)^3}{24 \cdot 4^2} = \frac{1}{64} = 0.015625;$$

$$|E_{\text{trap}}| \leq \frac{6(2-1)^3}{12 \cdot 4^2} = \frac{1}{32} = 0.03125;$$

and

$$|E_{\text{Simp}}| \leq \frac{120(2-1)^5}{180 \cdot 4^4} = \frac{1}{384} = 0.0026041\overline{6}.$$

You need to realize that these numbers are **bounds** on the possible amounts of error, **not** the actual errors. In fact, there's a very important point here; if we know precisely what the error in using an approximation for an integral is, then we know the actual value of the integral; take the approximation and add/subtract the error. Of course, in this example, we **do** know the actual value of the integral, but – we'll write it again – the value of these approximation techniques is that we can use them in cases where we do not have a way to calculate the exact integral. Thus, the most that we can hope for, typically, is to calculate a nice bound on the error, not to calculate the error itself.

How many subintervals would we need to use with each of the approximations to produce bounds on the absolute values of the errors that would guarantee our approximation is within ± 0.00001?

With midpoint Riemann sums, we would need

$$|E_{\text{mdpt}}| \leq \frac{6(2-1)^3}{24 \cdot n^2} \leq \frac{1}{100,000},$$

that is, we must require $4n^2 \geq 100,000$. Thus, we would need $n \geq \sqrt{25,000} \approx 158.11$, i.e., the smallest n that makes the upper bound on the absolute value of the error less than or equal to 0.00001 is $n = 159$.

It is possible that a smaller n actually yields an approximation within the desired amount, we just can't determine such an n from our theoretical bound.

With the Trapezoidal Rule, we would need

$$|E_{\text{trap}}| \;\leq\; \frac{6(2-1)^3}{12\cdot n^2} \;\leq \frac{1}{100,000},$$

that is, we must require $2n^2 \geq 100,000$. Thus, we would need $n \geq \sqrt{50,000} \approx 223.61$, i.e., the smallest n that makes the upper bound on the absolute value of the error less than or equal to 0.00001 is $n = 224$.

Again, it is possible that a smaller n actually yields an approximation within the desired amount, we just can't determine such an n from our theoretical bound.

With Simpson's Rule, we would need

$$|E_{\text{Simp}}| \;\leq\; \frac{120(2-1)^5}{180\cdot n^4} \;\leq \frac{1}{100,000},$$

that is, we must require $3n^4/2 \geq 100,000$. Thus, we would need $n \geq 16.07$, i.e., the smallest n that makes the upper bound on the absolute value of the error less than or equal to 0.00001 would be $n = 17$ **if** we didn't need an even n for Simpson's Rule, but we do. So, the smallest n that we can actually use for Simpson's Rule is $n = 18$.

We'll write it for a final time: it is possible that a smaller n actually yields an approximation within the desired amount, we just can't determine such an n from our theoretical bound.

You can see from the numbers above why Simpson's Rule is the preferred approximation method, and why, in general, we prefer midpoint Riemann sums to the Trapezoidal Rule. All three approximations are roughly equally as complicated to calculate for a fixed n, and yet, we can know that Simpson's Rule with $n = 18$ gives us a bound on the error that requires $n = 159$ for midpoint Riemann sums, and $n = 224$ for the Trapezoidal Rule. This is typical.

Okay - it's time to actually calculate the approximations of $\int_1^2 \frac{1}{x^2}\,dx$ when $n = 2$ and $n = 4$.

- $n = 2$:

  $\Delta x = (b-a)/2 = 1/2$, $x_0 = 1$, $x_1 = 3/2$, $x_2 = 2$.

  midpoint Riemann sum:

$$\frac{1}{2}\left(\frac{1}{(5/4)^2} + \frac{1}{(7/4)^2}\right) \;=\; \frac{1}{2}\left(\frac{16}{25} + \frac{16}{49}\right) \;\approx\; 0.483265306122.$$

Trapezoidal Rule:

$$\frac{1}{4}\left(\frac{1}{1^2} + 2 \cdot \frac{1}{(3/2)^2} + \frac{1}{2^2}\right) \; = \; 0.5347\overline{2}.$$

Simpson's Rule:

$$\frac{1}{6}\left(\frac{1}{1^2} + 4 \cdot \frac{1}{(3/2)^2} + \frac{1}{2^2}\right) \; = \; 0.504\overline{629}.$$

As the actual value of the integral is 0.5, we see that the absolute values of the errors in the three cases are (to within 12 decimal places), respectively, 0.016734693878, $0.0347\overline{2}$, and $0.004\overline{629}$. Recall that the upper bounds on these, which we calculated earlier, were, respectively, 0.0625, 0.125, and $0.041\overline{6}$. What we see is that the upper bounds are certainly upper bounds, but that these upper bounds are not particularly close to the actual absolute values of the errors. This, too, is typical.

• $n = 4$:

$\Delta x = (b-a)/4 = 1/4$, $x_0 = 1$, $x_1 = 5/4$, $x_2 = 3/2$, $x_3 = 7/4$, $x_4 = 2$.

midpoint Riemann sum:

$$\frac{1}{4}\left(\frac{1}{(9/8)^2} + \frac{1}{(11/8)^2} + \frac{1}{(13/8)^2} + \frac{1}{(15/8)^2}\right) \; \approx \; 0.495547936480.$$

Trapezoidal Rule:

$$\frac{1}{8}\left(\frac{1}{1^2} + 2 \cdot \frac{1}{(5/4)^2} + 2 \cdot \frac{1}{(3/2)^2} + 2 \cdot \frac{1}{(7/4)^2} + \frac{1}{2^2}\right) \; = \; 0.508993764172.$$

Simpson's Rule:

$$\frac{1}{12}\left(\frac{1}{1^2} + 4 \cdot \frac{1}{(5/4)^2} + 2 \cdot \frac{1}{(3/2)^2} + 4 \cdot \frac{1}{(7/4)^2} + \frac{1}{2^2}\right) \; = \; 0.500417611489.$$

Here, we see that the absolute values of the errors in the three cases are (to within 12 decimal places), respectively, 0.00445206352, 0.0089937641720, and 0.000417611489. The upper bounds on these, which we calculated earlier, were, respectively, 0.015625, 0.03125, and $0.00260041\overline{6}$.

Again, our upper bounds are clearly upper bounds, but they are not very close to the actual absolute values of the errors.

Remark 2.6.14. Before leaving this section, we should make a final remark. These days, all scientific calculators can calculate definite integrals. How do they do this? They use a form of Simpson's Rule, with many, very small, subintervals, so that the result is typically accurate to within the number of significant digits that the display can handle.

2.6.1 Exercises

For the definite integrals in Exercises 1 - 15, calculate (a) the exact value of the definite integral, (b) a midpoint approximation, (c) a trapezoidal approximation, and (d) an approximation using Simpson's Rule. For each of the approximations, use $n = 4$. View (a) as an opportunity to practice integration techniques from the previous chapters.

1. $\displaystyle\int_0^4 \frac{5y}{y-7}\,dy.$

2. $\displaystyle\int_0^4 \sqrt{x^2+16}\,dx.$

3. $\displaystyle\int_6^{10} \frac{9x-10}{x^2-3x-10}\,dx.$

4. $\displaystyle\int_{-4}^4 \cosh(4s)\,ds.$

5. $\displaystyle\int_2^6 \frac{4t^2+6t-2}{t^2-1}\,dt.$

6. $\displaystyle\int_0^{12} \frac{3}{\sqrt{x^2+4}}\,dx.$

7. $\displaystyle\int_0^{\pi/6} \tan(2\theta)\,d\theta.$

8. $\displaystyle\int_1^5 \ln h\,dh.$

9. $\displaystyle\int_2^6 \frac{-3z^3+5z^2+5z-5}{z^3-z^2}\,dz.$

10. $\displaystyle\int_0^6 \sqrt{36-t^2}\,dt.$

11. $\displaystyle\int_{12}^{16} \frac{24}{\sqrt{m^2-9}}\,dm.$

12. $\displaystyle\int_0^1 2\frac{du}{(144+u^2)^2}\,du.$

13. $\displaystyle\int_3^7 \frac{2p^2+1}{p^3+p}\,dp.$

14. $\displaystyle\int_0^{2\pi} \sin^2\phi\,d\phi.$

15. $\displaystyle\int_{-6}^{0} \sqrt{x^2+12x+45}\,dx.$

16. Prove that the midpoint approximation is in fact equal to $\displaystyle\int_a^b mx\,dx$ for arbitrary n. Hint: show $\int_{x_i}^{x_{i+1}} mx+b\,dx$ is equal to the area of the rectangle approximating it.

Determine n, the number of intervals "necessary" to give an approximation of the definite integral using the specified method with absolute error less than that given.

17. $\displaystyle\int_0^{\pi/4} \sin(4\theta)\,d\theta$, Midpoint, Error$<0.001$.

18. $\displaystyle\int_0^{\pi/4} \sin(4\theta)\,d\theta$, Trapezoid, Error$<0.001$.

19. $\displaystyle\int_{5134}^{14238} 97x^3-56\pi x^2+42ex-500\,dx$, Simpson's Rule, Error$<0.000001$.

Estimate each of the following unpleasant definite integrals using (a) the Midpoint Rule, (b) the Trapezoidal Rule and (c) Simpson's Rule.

20. $\displaystyle\int_1^5 \frac{e^x}{x}\,dx$, $n=4$.

21. $\displaystyle\int_0^{\pi} \sin(\sin t)\,dt$. $n=6$.

22. $\displaystyle\int_0^{\pi/2} \sin(y^2)\,dy$. $n=8$. Note: this is one of the Fresnel integrals, introduced in the exercises of the previous chapter.

23. $\displaystyle\int_0^1 \frac{dz}{\sqrt{z^4+1}}$. $n = 10$.

24. $\displaystyle\int_0^{1/2} \frac{\ln(1+x)}{1+x}\,dx$. $n = 6$.

25. Use the fact that $\ln x = \displaystyle\int_1^x \frac{dt}{t}$ and the Trapezoidal Rule with $n = 10$ to approximate $\ln 2$.

26. Redo the previous problem using $n = 10$ and Simpson's rule.

27. Use the formula $\displaystyle\frac{\pi}{4} = \int_0^1 \frac{dy}{1+y^2}$ to estimate π. Use the trapezoid formula with $n = 10$.

28. Redo the previous problem using $n = 10$ and Simpson's rule.

In Exercises 29 - 32, calculate an upper bound for the absolute error of the definite integral in terms of n when approximated with (a) the Midpoint Method and (b) the Trapezoidal Rule.

29. $\displaystyle\int_1^9 \frac{dy}{y}$.

30. $\displaystyle\int_0^{2\pi} \sin^2 t\,dt$.

31. $\displaystyle\int_0^{12} e^z\,dz$.

32. $\displaystyle\int_{-1}^{1} e^{-x^2}\,dx$. Hint: To find the maximum absolute value of the derivatives, note that all derivatives are of the form $p(x)e^{-x^2}$ where $p(x)$ is a polynomial. It takes some fortitude to factor these polynomials and correctly apply the first or second derivative tests, but some of the roots are obvious.

33. What is the upper bound on the absolute error of $\displaystyle\int_0^3 e^{x^2}\,dx$ when estimated with Simpson's Rule? Leave your answer in terms of n. Hint: even though Simpson's Rule requires more derivative taking than the Midpoint Method, this problem is easier than the previous problem since the polynomial appearing in the 4th derivative is easier to analyze.

34. Generalize the previous problem. Let $E(n, b)$ be the upper bound of Simpson's approximation of $\displaystyle\int_0^b e^{x^2}\,dx$ when $[0, b]$ is subdivided into n equally spaced intervals. What is $E(n, b)$?

35. Prove that if $n = 4$, the Simpson's approximation of $\displaystyle\int_a^b x^2\,dx$ is exact.

A random variable is said to follow the *standard normal distribution* if it is normally distributed with mean zero and standard deviation one ($\mu = 0$, $\sigma = \sigma^2 = 1$). Use this information to solve Exercises 36 - 40.

36. Show that the density function of a standard normal distribution is $\dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

37. a. Suppose a data set follows a standard normal distribution. Use the Midpoint Rule with $n = 4$ to approximate the proportion of data between 0 and 1.

    b. Based on your result in (a), approximately how much of the data falls between -1 and 1? Hint: the density function is symmetric about the y-axis.

38. a. Suppose a data set follows a standard normal distribution. Use the Trapezoidal Rule with $n = 4$ to approximate the proportion of data between 0 and 2.

    b. Based on your result in (a), approximately how much of the data falls between -2 and 2?

39. Suppose that the mature height of a certain tree species is normally distributed with mean 16 feet and standard deviation of 2 feet. What proportion of trees are between 14 and 18 feet tall? Hint: given a normal distribution with parameters μ and σ, make the substitution $z = \dfrac{x - \mu}{\sigma}$ before evaluating. You should be able to use one of the prior two problems to help answer this question.

40. Suppose a test is administered to gauge the effect of alcohol on driving ability. A large sample of individuals is given one alcoholic beverage, and their response time to a certain stimulus is measured. Suppose the response time is normally distributed with mean three seconds and standard deviation 0.6 seconds. What proportion of the individuals' response times falls between 1.8 and 4.2 seconds?

Up to this point, we've used the various methods in this chapter to approximate definite integrals of a given continuous, or at least integrable, function. Since these approximations depend only on knowledge of the function at finitely many points, they may be adapted to applications where a definite integral is required, but the integrand is known only at finitely many points.

41. Recall that if the force exerted on an object x meters from the origin is given by $f(x)$, then the work done in moving the object from a to b meters from the origin is $W = \displaystyle\int_a^b f(x)\,dx$. Suppose a 3 kilogram mass is being moved from 9 to 17 meters from the origin and the acceleration of the particle is measured every 2 meters in meters per second per second as shown in the table below. Use the Midpoint Method to approximate the total work done.

x	Acceleration
10	2.5
12	5
14	2
16	-1

42. Suppose that a device is placed in a race car that measure the instantaneous velocity at two second intervals. The data is shown below. Use Simpson's Rule to estimate the total distance traveled by the car between $t = 0$ and $t = 14$ seconds.

Time (t)	Velocity (miles/hour)
0	0
2	15
4	40
6	66
8	82
10	92
12	108
14	116

43. Redo the previous problem using the Trapezoidal Rule.

Estimate each of the following definite integrals using the specified method and number of partitions. Then make a conjecture about the value of the (possibly improper) integral. As a side note, each of these integrals have the property that they can be calculated directly using a powerful tool from complex analysis called ***residues*****.**

44. $\displaystyle\int_0^{100} \frac{1}{1+x^4}\,dx$, Midpoint Method, $n = 10$. Make a conjecture regarding the value of $\displaystyle\int_0^{\infty} \frac{1}{1+x^4}\,dx$.

45. $\displaystyle\int_{-10}^{10} \frac{\cos z}{1+z^2}\,dz$, Trapezoidal Rule, $n = 10$. Make a conjecture regarding the value of $\displaystyle\int_{-\infty}^{\infty} \frac{\cos z}{1+z^2}\,dz$.

46. $\displaystyle\int_{-\pi}^{\pi} \frac{dt}{5+3\cos t}$, Simpson's Rule, $n = 8$. Make a conjecture regarding the exact value of $\displaystyle\int_{-\pi}^{\pi} \frac{dt}{5+3\cos t}$.

An ordinary or simple pendulum is one in which the path of the pendulum is assumed to trace out a portion of a circle. This contrasts with a cycloidal pendulum

where the path is a cycloid. Assuming no friction, the period T of an ordinary pendulum can be stated in terms of the elliptic integral

$$T = 2\sqrt{\frac{L}{g}} \cdot \int_{-1}^{1} \frac{dw}{\sqrt{(1-w^2)(1-w^2\sin^2(A/2))}}$$

where L is the radius of the path, g is the acceleration due to gravity, and A is the amplitude of oscillation. That is A measures the angular position of the pendulum when it is first dropped.

47. Suppose $A = \pi/2$, $g = 9.8$ m/sec^2 and $L = 1$ meter. Use Simpson's Rule with $n = 4$ to estimate the period. As the integral is improper, make your estimation of the interval $[-.95, .95]$.

48. Redo the previous problem with $A = \pi/4$.

49. Argue that if the period is small, the period can be approximated by

$$2\sqrt{\frac{L}{g}} \cdot \int_{-1}^{1} \frac{dw}{\sqrt{1-w^2}}.$$

This expression is therefore independent of the amplitude. Evaluate this integral directly using $L = 1$ meter.

50. Is your answer to the previous problem closer to the Simpson's approximation with $A = \pi/2$ or $A = \pi/4$? Why does this make sense?

Appendix 2.A Technical Matters

In this section, we want to describe the technical framework for proving many of the theorems on definite integrals. Still, there are many details beyond the scope of what we want to present even in this technical appendix; our primary reference for these details is [4].

Theorem 2.A.1. *The limit of Riemann sums, the definite integral, given in Definition 2.3.1 is unique, if it exists.*

In addition, if f is Riemann integrable on the interval $[a,b]$, $\int_a^b f(x)\,dx = L$, and $(\mathcal{P}_n, \mathcal{S}_n)$ is a sequence of sampled partitions of $[a,b]$ such that $\lim_{n\to\infty} ||\,\mathcal{P}_n\,|| = 0$, then

$$\lim_{n\to\infty} \mathcal{R}_{\mathcal{P}_n}^{\mathcal{S}_n}(f) \;=\; L.$$

Proof. Clearly, proving the sequence statement proves the entire theorem.

Suppose that $\int_a^b f(x)\,dx$ exists and equals L, and suppose that $(\mathcal{P}_n, \mathcal{S}_n)$ is a sequence of sampled partitions of $[a,b]$ such that $\lim_{n\to\infty} ||\,\mathcal{P}_n\,|| = 0$.

Let $\epsilon > 0$. Then, by definition of the Riemann integral, there exists $\delta > 0$ such that, for all partitions $\mathcal{P} = \{x_0, \ldots, x_n\}$ of $[a,b]$, such that $||\mathcal{P}|| < \delta$, for all sample sets $\{s_1, \ldots, s_n\}$ for $\mathcal{P}$,

$$\left|\sum_{i=1}^{n} f(s_i)\Delta x_i \;-\; L\right| < \epsilon.$$

As $\lim_{n\to\infty} ||\,\mathcal{P}_n\,|| = 0$, there exists a positive integer N such that, for all $n \geq N$, $||\,\mathcal{P}_n\,|| < \delta$. Hence, for all $n \geq N$, if $\mathcal{P}_n = \{x_0, \ldots, x_n\}$ and $\mathcal{S}_n = \{s_1, \ldots, s_n\}$,

$$\left|\sum_{i=1}^{n} f(s_i)\Delta x_i \;-\; L\right| < \epsilon,$$

that is,

$$\lim_{n\to\infty} \mathcal{R}_{\mathcal{P}_n}^{\mathcal{S}_n}(f) \;=\; L.$$

□

Before stating any further results, we remark that Theorem 2.3.6 tells us that unbounded functions (see Definition 2.3.4) are **not** Riemann integrable. Consequently, below, we will be concerned with the case where f is a bounded function, until we get to the results on improper integrals.

It is a defining property of the real numbers that every non-empty set of real numbers, which is bounded above, has a *least upper bound* or *supremum*. It follows, by negating, that every non-empty set of real numbers, which is bounded below, has a *greatest lower bound* or *infimum*. For a bounded set V of real numbers, you should think of the supremum and infimum as the numbers that "want" to be the maximum and minimum values, respectively, in V, though neither the supremum or infimum need actually be contained in V.

Now suppose that f is a bounded function (see Definition 2.3.4) on a set E of real numbers. This means that the set $V = f(E) = \{f(x) \mid x \in E\}$ is bounded, and so possesses a supremum and infimum; we denote these, respectively, by $M_f(E)$ and $m_f(E)$.

The Extreme Value Theorem, (see [2] or [4]), tells us that a continuous function f on a closed interval $[a, b]$ is bounded and that, in fact, $M_f([a, b])$ and $m_f([a, b])$ are in the set $f([a, b])$, so that they are, respectively, the maximum and minimum values of f on $[a, b]$.

Now, suppose that f is bounded on the closed interval $[a, b]$, and that $\mathcal{P} = (x_0, \dots, x_n)$ is a partition of $[a, b]$. We would like to pick sample points which maximize or minimize the Riemann sums, but such sample points need not exist (though they would if f were continuous). However, we can "fake" it by using the supremum and infimum of f on each subinterval of the partition. Thus, we define the *upper* and *lower sums* $U_f(\mathcal{P})$ and $L_f(\mathcal{P})$, respectively, of f with respect to the partition $\mathcal{P}$ to be

$$U_f(\mathcal{P}) \;=\; \sum_{i=1}^{n} M_f([x_{i-1}, x_i])\, \Delta x_i,$$

and

$$L_f(\mathcal{P}) \;=\; \sum_{i=1}^{n} m_f([x_{i-1}, x_i])\, \Delta x_i.$$

It follows immediately that $U_f(\mathcal{P})$ is the least upper bound of all of the Riemann sums of f, using the partition $\mathcal{P}$, and that $L_f(\mathcal{P})$ is the greatest lower bound of all of the Riemann sums of f, using the partition $\mathcal{P}$.

Note that it is immediate that, if a partition $\mathcal{P}^*$ is a refinement of a partition $\mathcal{P}$, then

$$U_f(\mathcal{P}^*) - L_f(\mathcal{P}^*) \;\leq\; U_f(\mathcal{P}) - L_f(\mathcal{P}).$$

The fundamental integrability result that we need, for bounded functions, is the following, which we state without proof; see Theorem 3.2.7 of [4].

Theorem 2.A.2. *Suppose that f is bounded on the interval $[a, b]$. Then, f is Riemann integrable on $[a, b]$ if and only if, for all $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a, b]$ such that*

$$U_f(\mathcal{P}) - L_f(\mathcal{P}) < \epsilon.$$

Given Theorem 2.A.2, the integrability of continuous functions follows almost immediately from a basic fact: continuous functions on closed bounded intervals $[a, b]$ are *uniformly continuous.* That is:

Theorem 2.A.3. *If f is continuous on the closed interval $[a, b]$, then, for all $\epsilon > 0$, there exists $\delta > 0$ such that, if x_1 and x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$, then $f(x_1) - f(x_2) < \epsilon$.*

Proof. See Theorem 2.2.12 of [4]. □

The "uniform" in "uniformly continuous" refers to the fact that, for a given $\epsilon > 0$, there is a uniform $\delta > 0$ that works everywhere in the interval; normal continuity allows δ to vary as you change one of the points.

It is now easy to prove:

Theorem 2.A.4. *Suppose that f is continuous on the interval $[a, b]$, where $a < b$. Then, f is Riemann integrable on $[a, b]$.*

Proof. Let $\epsilon > 0$. Then, $\epsilon/(b - a) > 0$, and so, by Theorem 2.A.3, there exists $\delta > 0$ such that, if x_1 and x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon/(b - a)$.

Let $\mathcal{P} = (x_0, \dots, x_n)$ be a partition of $[a, b]$ with mesh $< \delta$. We claim that $U_f(\mathcal{P}) - L_f(\mathcal{P}) < \epsilon$, which would prove that f is Riemann integrable by Theorem 2.A.2.

As $||\,\mathcal{P}\,|| < \delta$, for any two points a_i and b_i in the subinterval $[x_{i-1}, x_i]$, $|a_i - b_i| < \delta$. By considering the points in the subinterval where f attains its maximum and minimum values, we conclude that $M_f([x_{i-1}, x_i]) - m_f([x_{i-1}, x_i]) < \epsilon/(b - a)$ and, hence, that

$$\sum_{i=1}^{n} \Big(M_f([x_{i-1}, x_i]) - m_f([x_{i-1}, x_i])\Big)\Delta x_i \; < \; \sum_{i=1}^{n} \frac{\epsilon}{b-a}\,\Delta x_i \; = \; \frac{\epsilon}{b-a}\sum_{i=1}^{n} \Delta x_i \; = \; \epsilon,$$

i.e., that

$$U_f(\mathcal{P}) - L_f(\mathcal{P}) < \epsilon.$$

□

We can now prove Theorem 2.3.8.

Theorem 2.A.5. *Bounded, piecewise-continuous functions on closed intervals are Riemann integrable.*

Proof. By dividing the interval $[a,b]$ into subintervals such that each subinterval has a discontinuity at, at most, one endpoint. By Theorem 2.3.16, it is enough for us to prove that f is Riemann integrable on such subintervals. Thus, we will assume that f is bounded on $[a,b]$, with $|f(x)| \leq B$, for x in $[a,b]$, and has a single discontinuity at a. Note that, if $B = 0$, f is identically 0 on $[a,b]$, and we're finished; so assume that $B > 0$.

Now, let $\epsilon > 0$. Consider first the partition of $[a,b]$ given by $\mathcal{P} = (a, a + \epsilon/(4B), b)$. As f is continuous on $[a + \epsilon/(4B), b]$, f is integrable on this subinterval, and so, by Theorem 2.A.2, there exists a partition $\widehat{\mathcal{P}} = (x_0, \dots, x_n)$ of $[a + \epsilon/(4B), b]$ such that $U_f(\widehat{\mathcal{P}}) - L_f(\widehat{\mathcal{P}}) < \epsilon/2$. Let $\mathcal{P}^*$ be the partition of $[a,b]$ given by $(a, x_0, \dots, x_n)$.

Then, from the definitions, it follows at once that

$$U_f(\mathcal{P}^*) \;=\; M_f([a,x_0]) \cdot \epsilon/(4B) \;+\; U_f(\widehat{\mathcal{P}})$$

and

$$L_f(\mathcal{P}^*) \;=\; m_f([a,x_0]) \cdot \epsilon/(4B) \;+\; L_f(\widehat{\mathcal{P}}).$$

Also, note that $M_f([a,x_0]) \leq B$, while $m_f([a,x_0]) \geq -B$, so that $-m_f([a,x_0]) \leq B$, and $M_f([a,x_0]) - m_f([a,x_0]) \leq 2B$.

Therefore,

$$U_f(\mathcal{P}^*) \;-\; L_f(\mathcal{P}^*) \;=\; M_f([a,x_0]) \cdot \epsilon/(4B) \;+\; U_f(\widehat{\mathcal{P}}) \;-\; m_f([a,x_0]) \cdot \epsilon/(4B) \;-\; L_f(\widehat{\mathcal{P}}) \;=$$

$$\Big(M_f([a,x_0]) \;-\; m_f([a,x_0])\Big) \cdot \frac{\epsilon}{4B} \;+\; \Big[U_f(\widehat{\mathcal{P}}) \;-\; L_f(\widehat{\mathcal{P}})\Big] \;<\; \frac{\epsilon}{2} \;+\; \frac{\epsilon}{2} \;=\; \epsilon,$$

and we are finished by Theorem 2.A.2. □

Theorem 2.A.6. *Suppose that $a < b$.*

1. *If f is Riemann integrable on $[a, b]$ and $a \leq c < d \leq b$, then f is Riemann integrable on $[c, d]$, i.e., if f is Riemann integrable on a given closed interval, then f is Riemann integrable on any closed subinterval of the given interval.*

2. *Suppose that $a < c < b$. Then, f is Riemann integrable on $[a, b]$ if and only if f is Riemann integrable on $[a, c]$ and $[c, b]$ and, when these equivalent conditions hold,*

$$\int_a^b f(x)\,dx \;=\; \int_a^c f(x)\,dx \;+\; \int_c^b f(x)\,dx.$$

Proof. Suppose that f is Riemann integrable on $[a, b]$ and $a \leq c < d \leq b$. By Theorem 2.3.6, f must be bounded on $[a, b]$ and, hence, is bounded on $[c, d]$.

Let $\epsilon > 0$. By Theorem 2.A.2, there exists a partition $\mathcal{P}$ of $[a, b]$ such that $U_f(\mathcal{P}) - L_f(\mathcal{P}) < \epsilon$. Now, let $\mathcal{P}^*$ be the partition of $[c, d]$ formed from the points in $[c, d] \cap \mathcal{P}$, together with c and d. Then,

$$U_f(\mathcal{P}^*) - L_f(\mathcal{P}^*) \;\leq\; U_f(\mathcal{P}) - L_f(\mathcal{P}) < \epsilon,$$

and so, by Theorem 2.A.2, f is integrable on $[c, d]$.

Suppose now that $a < c < b$.

If f is Riemann integrable on $[a, b]$, then part (1) tells us that f is Riemann integrable on $[a, c]$ and $[c, b]$.

Suppose that f is Riemann integrable on $[a, c]$ and $[c, b]$. Then, f must be bounded on $[a, c]$ and $[c, b]$ and, hence, on $[a, b]$. Let $\epsilon > 0$. Then, $\epsilon/2 > 0$. By Theorem 2.A.2, there exist partitions $\mathcal{P}^1$ and $\mathcal{P}^2$ of $[a, c]$ and $[c, b]$, respectively, such that $U_f(\mathcal{P}^1) - L_f(\mathcal{P}^1) < \epsilon/2$ and $U_f(\mathcal{P}_2) - L_f(\mathcal{P}_2) < \epsilon/2$. Then, the partition $\mathcal{P}^* = \mathcal{P}^1 \cup \mathcal{P}^2$ of $[a, b]$ is such that

$$U_f(\mathcal{P}^*) - L_f(\mathcal{P}^*) \;<\; \frac{\epsilon}{2} \;+\; \frac{\epsilon}{2} \;=\; \epsilon.$$

Therefore, by Theorem 2.A.2, f is Riemann integrable on $[a, b]$.

Now, suppose that f is Riemann integrable on $[a, c]$ and $[c, b]$ and, hence, by the above, Riemann integrable on $[a, b]$. Let $L_1 = \int_a^c f(x)\,dx$, $L_2 = \int_c^b f(x)\,dx$, and $L = \int_a^b f(x)\,dx$. For each integer $n \geq 1$, let $\mathcal{P}_n^1$ and $\mathcal{P}_n^2$ be partitions of $[a, c]$ and $[c, b]$, respectively, with mesh $< 1/n$. Let $\mathcal{S}_n^1$ and $\mathcal{S}_n^2$ be sample sets for the partitions $\mathcal{P}_n^1$ and $\mathcal{P}_n^2$, respectively. Let $\mathcal{P}_n = \mathcal{P}_n^1 \cup \mathcal{P}_n^2$ and $\mathcal{S}_n = \mathcal{S}_n^1 \cup \mathcal{S}_n^2$. Then, $\mathcal{P}_n$ is a partition of $[a, b]$ of mesh $< 1/n$, and $\mathcal{S}_n$ is a sample set for $\mathcal{P}_n$ (possibly using the sample point c as a sample point for two different subintervals).

Using Theorem 2.3.3 three times, we find

$$L_1 \;+\; L_2 \;=\; \lim_{n\to\infty} \mathcal{R}^{\mathcal{S}^1_n}_{\mathcal{P}^1_n}(f) \;+\; \lim_{n\to\infty} \mathcal{R}^{\mathcal{S}^2_n}_{\mathcal{P}^2_n}(f) \;=\; \lim_{n\to\infty} \mathcal{R}^{\mathcal{S}_n}_{\mathcal{P}_n}(f) \;=\; L.$$

□

Theorem 2.A.7. (**Linearity of Integration**) *Definite integration over a closed interval is a linear operation, i.e., if f and g are Riemann integrable on $[a,b]$, then, for all constants r and s, the function $rf+sg$ is Riemann integrable on $[a,b]$, and*

$$\int_a^b \big(rf(x)+sg(x)\big)\,dx \;=\; r\int_a^b f(x)\,dx \;+\; s\int_a^b g(x)\,dx.$$

Proof. Suppose that f and g are Riemann integrable on $[a,b]$. We will prove that $f+g$ is Riemann integrable on $[a,b]$ and that

$$\int_a^b \big(f(x)+g(x)\big)\,dx \;=\; \int_a^b f(x)\,dx \;+\; \int_a^b g(x)\,dx.$$

That constants can be pulled out of integrals is left to you in Exercise 52.

Let $[c,d]$ be any closed subinterval of $[a,b]$. It is trivial to see that

$$M_{f+g}([c,d]) \;\leq\; M_f([c,d]) + M_g([c,d])$$

and

$$m_f([c,d]) + m_g([c,d]) \;\leq\; m_{f+g}([c,d]).$$

Therefore, for any partition $\mathcal{P}$ of $[a,b]$, $U_{f+g}(\mathcal{P}) \leq U_f(\mathcal{P}) + U_g(\mathcal{P})$ and $L_f(\mathcal{P}) + L_g(\mathcal{P}) \leq L_{f+g}(\mathcal{P})$; hence,

$$U_{f+g}(\mathcal{P}) \;-\; L_{f+g}(\mathcal{P}) \;\leq\; [U_f(\mathcal{P}) \;-\; L_f(\mathcal{P})] \;+\; [U_g(\mathcal{P}) \;-\; L_g(\mathcal{P})]\,.$$

It follows immediately from Theorem 2.A.2 that $f+g$ is integrable on $[a,b]$.

Now, let $\mathcal{P}_n$ be a partition of $[a,b]$ of mesh $< 1/n$, and $\mathcal{S}_n$ be a sample set for $\mathcal{P}_n$. Using

Theorem 2.3.3 three times, we find

$$\int_a^b \big(f(x)+g(x)\big)\,dx \;=\; \lim_{n\to\infty} \mathcal{R}_{\mathcal{P}_n}^{\mathcal{S}_n}(f+g) \;=\; \lim_{n\to\infty} \mathcal{R}_{\mathcal{P}_n}^{\mathcal{S}_n}(f) \;+\; \lim_{n\to\infty} \mathcal{R}_{\mathcal{P}_n}^{\mathcal{S}_n}(g) \;=$$

$$\int_a^b f(x)\,dx \;+\; \int_a^b g(x)\,dx.$$

□

Using Theorem 2.3.19 (see Theorem 3.3.1 of [4]), it is easy to conclude Theorem 2.3.14.

Theorem 2.A.8. *Suppose that f and g are defined on a closed interval $[a,b]$, and that, except possibly for a finite set points in $[a,b]$, f and g are equal at each point in $[a,b]$.*

Then, f is Riemann integrable on $[a,b]$ if and only if g is, and when f and g are Riemann integrable,

$$\int_a^b f(x)\,dx \;=\; \int_a^b g(x)\,dx.$$

Proof. As f and g differ on at most a finite number of points, f is bounded if and only if g is bounded. If both functions are unbounded, then Theorem 2.3.6 tells us that both functions are not Riemann integrable.

Now, suppose that f and g are both bounded, and that f is Riemann integrable. Then, $g-f$ is bounded, and is equal to 0 except at, possibly, a finite number of points, $r_1,\dots,r_p$. We claim that $\int_a^b (g-f)(x)\,dx = 0$; if we show this, we are finished by Theorem 2.3.19, since we would then have

$$\int_a^b f(x)\,dx \;=\; \int_a^b f(x)\,dx \;+\; \int_a^b (g-f)(x)\,dx \;=\; \int_a^b g(x)\,dx.$$

Let M be the maximum of $|(g-f)(r_i)|$ for $1\le i\le p$. Let $\mathcal{P}=(x_0,\dots,x_n)$ be a partition of the interval $[a,b]$, and let $\mathcal{S}=(s_1,\dots,s_n)$ be a set of sample points for $\mathcal{P}$.

Then,

$$\left|\sum_{i=1}^n (g-f)(s_i)\Delta x_i\right| \;\le\; \sum_{i=1}^n |(g-f)(s_i)|\cdot|\Delta x_i| \;\le\; M\cdot||\,\mathcal{P}\,||,$$

and, hence,

$$\lim_{||\,\mathcal{P}\,||\to 0} \mathcal{R}_{\mathcal{P}}^{\mathcal{S}}(g-f) \;=\; 0.$$

□

Theorem 2.A.9.

Theorem 2.A.10. (**Fundamental Theorem of Calculus, Part 1**) *Suppose that f is Riemann integrable on $[a,b]$ and is continuous at a point x_0 in (a,b). Then, the integral function $I_f^{[a,b]}$ of f on $[a,b]$ is differentiable at x_0 and*

$$\left(I_f^{[a,b]}\right)'(x_0) = f(x_0).$$

Thus, if f is continuous on $[a,b]$, then $I_f^{[a,b]}$ is an anti-derivative of f on $[a,b]$.

Proof. From the definition of the derivative, we have

$$\left(I_f^{[a,b]}\right)'(x_0) \;=\; \lim_{x\to x_0} \frac{I_f^{[a,b]}(x) - I_f^{[a,b]}(x_0)}{x-x_0} \;=\; \lim_{x\to x_0} \frac{\int_a^x f(t)\,dt \;-\; \int_a^{x_0} f(t)\,dt}{x-x_0} \;=$$

$$\lim_{x\to x_0} \frac{\int_{x_0}^x f(t)\,dt}{x-x_0}.$$

We must show that this limit equals $f(x_0)$. We shall show that the limit from the right equals $f(x_0)$, and leave the nearly identical argument from the left as an exercise.

Let $\epsilon > 0$. By the definition of f being continuous at x_0, there exists $\delta > 0$ such that, for all x such that $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$. We claim that, if $0 < x - x_0 < \delta$, then

$$\left|\frac{\int_{x_0}^x f(t)\,dt}{x-x_0} \;-\; f(x_0)\right| < \epsilon, \tag{2.4}$$

which is what it means that $\lim_{x\to x_0^+} \frac{\int_{x_0}^x f(t)\,dt}{x-x_0} = f(x_0)$.

To prove the claim in Formula 2.4, fix an x such that $0 < x - x_0 < \delta$. Then, for all t in $[x_0, x]$, we have $0 \leq t - x_0 \leq x - x_0 < \delta$, and so, $|t - x_0| < \delta$. Using the defining property of δ again, we conclude that, for all t in $[x_0, x]$, $|f(t) - f(x_0)| < \epsilon$.

Therefore, for all t in $[x_0, x]$,

$$f(x_0) - \epsilon \;\leq\; f(t) \;\leq f(x_0) + \epsilon.$$

By the monotonicity of integration, Theorem 2.3.20, and Proposition 2.3.12, we conclude that

$$(f(x_0)-\epsilon)(x-x_0) = \int_{x_0}^{x} (f(x_0)-\epsilon)\,dt \le \int_{x_0}^{x} f(t)\,dt \le \int_{x_0}^{x} (f(x_0)+\epsilon)\,dt = (f(x_0)+\epsilon)(x-x_0).$$

Hence,

$$f(x_0)-\epsilon \;\le\; \frac{\int_{x_0}^{x} f(t)\,dt}{x-x_0} \;\le\; f(x_0)+\epsilon,$$

i.e.,

$$\left| \frac{\int_{x_0}^{x} f(t)\,dt}{x-x_0} \;-\; f(x_0) \right| < \epsilon,$$

as we claimed in Formula 2.4.

The final conclusion of the theorem follows from the first one, together with the continuity of $I_f^{[a,b]}$, which follows from Theorem 2.4.3. □

Theorem 2.A.11. *The integral defined in Definition 2.5.11 is independent of the splitting into subintervals (of the type described in the definition).*

Proof. We will prove a lemma, from which the theorem follows immediately.

Suppose that f is continuous on the interval $I = [a, b)$. Let c be such that $a < c < b$. We claim that $\int_a^b f(x)\,dx$ exists if and only if $\int_c^b f(x)\,dx$ exists and, in this case,

$$\int_a^b f(x)\,dx \;=\; \int_a^c f(x)\,dx \;+\; \int_c^b f(x)\,dx.$$

This is easy:

$$\int_a^b f(x)\,dx \;=\; \lim_{d\to b^-} \int_a^d f(x)\,dx \;=\; \lim_{d\to b^-} \left[\int_a^c f(x)\,dx \;+\; \int_c^d f(x)\,dx \right] \;=$$

$$\int_a^c f(x)\,dx \;+\; \lim_{d\to b^-} \left[\int_c^d f(x)\,dx \right] \;=\; \int_a^c f(x)\,dx \;+\; \int_c^b f(x)\,dx.$$

□

Theorem 2.A.12. *Let I be an interval of the form $[a, b)$ or $(a, b]$, where we allow the intervals $[a, \infty)$ or $(-\infty, b]$. Suppose that, for all x in I, $f(x) \geq 0$, and that, for all closed intervals $[c, d]$ contained in I, f is Riemann integrable on $[c, d]$.*

Then, if there exists a real number M (an upper bound) such that, for all $[c, d]$ contained in I, $\int_c^d f(x)\,dx \leq M$, then $\int_a^b f(x)\,dx$ converges, and what it converges to is less than, or equal to, M; in other words, if there is an upper bound M on all of the $\int_c^d f(x)\,dx$, then $\int_a^b f(x)\,dx$ converges to the least such upper bound.

In particular, if $\int_a^b f(x)\,dx$ diverges, what it diverges to is ∞.

Proof. We will deal with the case where $I = [a, b)$; the other cases are entirely analogous.

For $a \leq d < b$, define the function $F(d) = \int_a^d f(x)\,dx$, which exists by assumption. As $f \geq 0$, F is an increasing function on the interval I.

If F gets unboundedly large, then, since F is increasing, $\lim_{d \to b^-} F(d) = \infty$. If F is bounded above by M, then, as F is increasing, F converges to the least upper bound of the set $F(I)$, which is $\leq M$. □

Lemma 2.A.13. *Let $q(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, where c_i denotes a constant. Then,*

$$\int_{-h}^{h} q(x)\,dx \;=\; \frac{h}{3}\Big(6c_0 + 2c_2 h^2\Big) \;=\; \frac{h}{3}\Big(q(-h) + 4q(0) + q(h)\Big).$$

Proof. This is a trivial calculation, which we leave to the reader. □

Proposition 2.A.14. *Let $q(x) = ax^2 + bx + c$. Fix x_0. Let $h > 0$. Let $x_1 = x_0 + h$, $x_2 = x_1 + h$, and $y_i = q(x_i)$, for $i = 1, 2, 3$. Then,*

$$\int_{x_0}^{x_2} q(x)\,dx \;=\; \frac{h}{3}\Big(y_0 + 4y_1 + y_2\Big).$$

Proof. One can make the substitution $u = x - (x_0 + h)$ into the integral and reduce oneself to the case in Lemma 2.A.13, or do a messy, but simple, algebra problem. One has to show that

$$\left[\frac{a(x_0 + 2h)^3}{3} + \frac{b(x_0 + 2h)^2}{2} + c(x_0 + 2h)\right] \;-\; \left[\frac{ax_0^3}{3} + \frac{bx_0^2}{2} + cx_0\right] \;=$$

$$\frac{h}{3}\Big\{\big[ax_0^2+bx_0+c\big] \;+\; 4\big[a(x_0+h)^2+b(x_0+h)+c\big] \;+\; \big[a(x_0+2h)^2+b(x_0+2h)+c\big]\Big\}.$$

We leave either verification as an exercise. □

In the proofs below, for notational convenience, we apply the inequalities above in the case where $a=0$; the general case follows at once by making the substitution $u=x-a$.

Theorem 2.A.15. *Suppose that $a<b$ and the midpoint Riemann sum with n subintervals of equal length Δx is used to approximate $\int_a^b f(x)\,dx$.*

If $f''(x)$ exists for all x in some open interval containing the closed interval $[a,b]$ and, if there exists a number $M\geq 0$ such that, for all x in $[a,b]$, $|f''(x)|\leq M$, then the absolute value of the error, E_{mdpt}, satisfies the inequality

$$|E_{\mathrm{mdpt}}| \;\leq\; \frac{M(\Delta x)^3 n}{24} \;=\; \frac{M(b-a)^3}{24n^2}$$

Proof. We shall prove this for one subinterval, centered at 0. The general result follows by re-centering the intervals, via substitution, and then adding, which leads to the multiplication by n in the inequality. Let $h=\Delta x/2$.

We will show that

$$\left|\int_{-h}^{h} f(x)\,dx \;-\; 2hf(0)\right| \;\leq\; \frac{Mh^3}{3}.$$

We must look ahead to the Taylor-Lagrange Theorem, Theorem 4.3.3,, which immediately implies that: if f is twice-differentiable on an open interval I around a point a, and $M>0$ is a constant such that, for all x in I, $|f(x)|\leq M$, then, for all x in I,

$$\big|f(x) \;-\; \big(f(a)+f'(a)(x-a)\big)\big| \;\leq\; \frac{M}{2}(x-a)^2; \tag{2.5}$$

Now the proof is easy.

$$\left|\int_{-h}^{h} f(x)\,dx \;-\; 2hf(0)\right| \;=\; \left|\int_{-h}^{h} f(x)\,dx \;-\; \int_{-h}^{h}\big(f(0)+f'(0)x\big)\,dx\right| \;\leq$$

$$\int_{-h}^{h}\big|f(x) \;-\; f(0) \;-\; f'(0)x\big|\,dx \;\leq\; \int_{-h}^{h}\frac{M}{2}x^2\,dx \;=\; \frac{Mh^3}{3}.$$

□

Theorem 2.A.16. *Suppose that $a < b$ and the Trapezoidal Rule with n subintervals of equal length Δx is used to approximate $\int_a^b f(x)\,dx$.*

If $f''(x)$ exists for all x in some open interval containing the closed interval $[a,b]$ and, if there exists a number $M \geq 0$ such that, for all x in $[a,b]$, $|f''(x)| \leq M$, then the absolute value of the error, E_{trap}, satisfies the inequality

$$|E_{\text{trap}}| \;\leq\; \frac{M(\Delta x)^3 n}{12} \;=\; \frac{M(b-a)^3}{12n^2}$$

Proof. We shall prove this for one subinterval, centered at 0. The general result follows by re-centering the intervals, via substitution, and then adding, which leads to the multiplication by n in the inequality. Let $h = \Delta x/2$.

We will show that

$$\left| \int_{-h}^{h} f(x)\,dx \;-\; 2h \cdot \frac{f(h)+f(-h)}{2} \right| \;\leq\; \frac{2Mh^3}{3}.$$

We integrate by parts twice.

$$\int_{-h}^{h} f(x)\,dx \;=\; xf(x)\Big|_{-h}^{h} \;-\; \int_{-h}^{h} xf'(x)\,dx \;=\; hf(h) + hf(-h) \;-\; \int_{-h}^{h} xf'(x)\,dx \;=$$

$$h\big(f(h)+f(-h)\big) \;-\; \left[\frac{1}{2}x^2 f'(x)\Big|_{-h}^{h} \;-\; \frac{1}{2}\int_{-h}^{h} x^2 f''(x)\,dx\right].$$

Therefore,

$$\left| \int_{-h}^{h} f(x)\,dx \;-\; 2h \cdot \frac{f(h)+f(-h)}{2} \right| \;=\; \frac{1}{2}\left| h^2 f'(h) - h^2 f'(-h) - \int_{-h}^{h} x^2 f''(x)\,dx \right| \;=$$

$$\frac{1}{2}\left| \int_{-h}^{h} (h^2 - x^2) f''(x)\,dx \right| \;\leq\; \frac{1}{2}\int_{-h}^{h} (h^2 - x^2)M\,dx \;=\; \frac{M}{2}\cdot\frac{4h^3}{3} \;=\; \frac{2Mh^3}{3}.$$

□

Theorem 2.A.17. *Suppose that $a < b$, n is even, and Simpson's Rule with n subintervals of equal length Δx is used to approximate $\int_a^b f(x)\,dx$.*

If $f^{(4)}(x)$ exists for all x in some open interval containing the closed interval $[a, b]$ and, if there exists a number $M \geq 0$ such that, for all x in $[a, b]$, $|f^{(4)}(x)| \leq M$, then the absolute value of the error, E_{Simp}, satisfies the inequality

$$|E_{\text{Simp}}| \;\leq\; \frac{M(\Delta x)^5 n}{180} \;=\; \frac{M(b-a)^5}{180n^4}.$$

Proof. We shall prove this for two subintervals, centered at 0. The general result follows by re-centering the intervals, via substitution, and then adding, which leads to the multiplication by $n/2$ in the inequality. Let $h = \Delta x$.

Let

$$\psi(t) \;=\; \int_{-t}^{t} f(x)\,dx \;-\; \frac{t}{3}\Big[f(-t) \;+\; 4f(0) \;+\; f(t)\Big].$$

We will show that there exists a c such that $-h < c < h$ and

$$\psi(h) \;=\; -\frac{h^5 f^{(4)}(c)}{90}.$$

Let

$$\phi(t) \;=\; \psi(t) \;-\; \left(\frac{t}{h}\right)^5 \psi(t).$$

One easily calculates that $\phi(0) = \phi'(0) = \phi''(0) = 0$ and, for $t \neq 0$,

$$\phi'''(t) \;=\; -\frac{2t^2}{3}\left(\frac{f'''(t) - f'''(-t)}{2t} \;+\; \frac{90}{h^5}\,\psi(h)\right).$$

Now, $\phi(h) = 0$, and so, by Rolle's Theorem, there exists c_1 such that $0 < c_1 < h$ and $\phi'(c_1) = 0$. Applying Rolle's Theorem again, we find that there exists c_2 such that $0 < c_2 < c_1 < h$ and $\phi''(c_2) = 0$. Applying Rolle's Theorem yet again, we find that there exists c_3 such that $0 < c_3 < c_2 < c_1 < h$ and

$$\phi'''(c_3) = 0.$$

Applying the Mean Value Theorem, we conclude that there exists c such that $-c_3 < c < c_3$ and

$$f'''(c_3) \;-\; f'''(-c_3) \;=\; 2c_3 f^{(4)}(c).$$

Combining the previous 3 displayed formulas, we conclude that

$$f^{(4)}(c) \;+\; \frac{90}{h^5}\,\psi(h) \;=\; 0,$$

i.e.,

$$\psi(h) \;=\; -\frac{h^5 f^{(4)}(c)}{90},$$

which is what we wanted to show. □

Chapter 3

Applications of Integration

In this chapter, we will apply our results on anti-differentiation, definite integrals, and the Fundamental Theorem of Calculus to a wide variety of problems involving displacement, distance traveled, area in the plane, volume, surface areas, mass, centers of mass, rotational inertia, work, and hydrostatic pressure.

Throughout this chapter, we state many of our general results as Propositions, though, in many cases, these Propositions could be used as Definitions. For instance, we will see the definite integral of speed, with respect to time, yields the total distance traveled. We assume that you have a preconceived notion of the distance traveled, and then conclude that that distance can be calculated by integrating speed. However, we could, instead, **define** the distance traveled by the integral of the speed. Mathematically, this latter approach is more rigorous; it is, however, somewhat intuitively unsatisfying.

Thus, throughout this chapter, we shall usually assume that we have predefined physical terms, such as distance traveled, volume, mass, etc., and state our integration formulas for these quantities as Propositions.

We should remark that many of the "applications" in this chapter are actually "pre-applications", or what some people refer to as "toy problems". A pre-application or toy problem is a problem that is stripped of many physical complications or is not inherently of interest in and of itself, but rather is designed mainly to provide some basic example of a fundamental idea that will serve as a building block to tackling more-difficult actual applications.

3.1 Displacement and Distance Traveled in a Straight Line

In this section, we will apply the definite integral and the Fundamental Theorem of Calculus to problems involving the net change in position, the *displacement*, of an object, and the *total distance traveled* by the object. If the object changes its direction, these two quantities will **not** be the same; the total distance traveled would be greater, while the displacement could, in fact, turn out to be zero if the object ends up back where it started.

The second part of the Fundamental Theorem of Calculus, Theorem 2.4.10, tells us that, if F is an anti-derivative of a continuous function f on the interval $[a, b]$, then

$$\int_a^b f(t)\,dt \;=\; F(t)\Big|_a^b \;=\; F(b) - F(a).$$

This means that, if we start with a function $g(t)$, which is differentiable on an open interval containing $[a, b]$, and $g'(t)$ is continuous, then

$$\int_a^b g'(t)\,dt \;=\; \int_a^b \frac{dg}{dt}\,dt \;=\; g(b) - g(a).$$

Suppose now that we have an object moving in a straight line, on which we've chosen positive and negative directions, i.e., suppose that we have an object moving along a coordinate. Let $p(t)$ denote the position (i.e., the coordinate value) of the object at time t.

Assuming that p is continuously differentiable, we know that the velocity $v = v(t)$ of the object is the rate of change of the position, with respect to time, dp/dt, and the Fundamental Theorem tells us that

$$\int_a^b v(t)\,dt \;=\; p(b) - p(a).$$

The quantity on the right above is the change in the position of the object between times $t = a$ and $t = b$; recall that this is called the *displacement* of the object between times a and b.

Of course, when we refer to the velocity function $v = v(t)$, we are implicitly assuming that the position function is differentiable, for, otherwise, "the velocity" has no good meaning. Thus, we can summarize our discussion above by:

Proposition 3.1.1. *If the velocity $v = v(t)$, as a function of time t, of an object on a coordinate axis is continuous on the interval $[a, b]$, then the* **displacement** *of the object between times $t = a$ and $t = b$ is given by*

$$\int_a^b v(t)\,dt.$$

Example 3.1.2. A particle is moving in a straight line in such a way that its velocity $v = v(t)$, in m/s, at time t seconds, is given by

$$v \;=\; 3t^2 - 12t + 8.$$

What is the displacement of the particle between times 1 and 3 seconds? Between times 2 and 4 seconds? What is the displacement between times 1 and t seconds, for arbitrary t?

Solution:

To answer the first two questions, we need to calculate

$$\int_1^3 (3t^2 - 12t + 8)\,dt \qquad \text{and} \qquad \int_2^4 (3t^2 - 12t + 8)\,dt.$$

We want to apply the Fundamental Theorem, and so we need to find the (or, an) anti-derivative

$$\int (3t^2 - 12t + 8)\,dt \;=\; 3 \cdot \frac{t^3}{3} - 12 \cdot \frac{t^2}{2} + 8t + C \;=\; t^3 - 6t^2 + 8t + C \;=\; t(t-2)(t-4) + C.$$

Therefore, the displacement between times 1 and 3 seconds is

$$\int_1^3 (3t^2 - 12t + 8)\,dt \;=\; t(t-2)(t-4)\Big|_1^3 \;=\; -3 - 3 \;=\; -6 \text{ meters;}$$

this means that the particle ended up, at time 3 seconds, 6 meters in the negative direction from where it started at time 1 second.

Note that we do **not** know where the particle is at times 1 and 3 seconds, i.e., we don't know the position function $p(t)$. However, **except for the arbitrary constant** $+C$ **above**, we know $p(t)$, and that's enough to determine the **change in** the position.

The displacement of the particle between times 2 and 4 seconds is

$$\int_2^4 (3t^2 - 12t + 8)\, dt \;=\; t(t-2)(t-4)\Big|_2^4 \;=\; 0 - 0 \;=\; 0 \text{ meters.}$$

This does **not** mean that the particle did not move between times 2 and 4 seconds; it merely means that the particle, at time 4 seconds, ended up at the same place where the particle was at time 2 seconds. Of course, this means that the particle had to turn around at some point. Do not confuse **displacement**, the net change in position, with the **total distance traveled**. The displacement here, between times 2 and 4 seconds is 0. We shall calculate the total distance traveled by this particle, between times 2 and 4 seconds, in the next example.

Finally, in this example, we were asked to find the displacement of the particle between times 1 and t. This is easy, except that it would now be bad form to use t for the integration variable. You should use essentially any other variable for the dummy variable of integration; the only variables you should avoid are t and v, and maybe d would look confusing too. We'll use z for the integration variable. Then, we quickly find that the displacement between times 1 and t seconds is

$$\int_1^t (3z^2 - 12z + 8)\, dz = z(z-2)(z-4)\Big|_1^t = t(t-2)(t-4) - (1)(-1)(-3) = t^3 - 6t^2 + 8t - 3 \text{ meters.}$$

In the example above, we ran into the issue of the distinction between displacement and total distance traveled. This distinction arises because, if the velocity v of an object is continuous and always positive on an interval of time $[a, b]$, then the distance traveled by the object between times a and b is the same as the displacement, namely $\int_a^b v\, dt$. On the other hand, if the velocity v of an object is continuous and always negative on an interval of time $[a, b]$, then the displacement will be negative; it will be the negation of the distance traveled by the object between times a and b, i.e., the distance traveled will be $-\int_a^b v\, dt = \int_a^b -v\, dt$.

Thus, to find the distance traveled, we want to integrate v on intervals where v is positive, and we want to integrate $-v$ on intervals where v is negative. The way to write this in one formula is to say that we want to integrate the absolute value $|v|$ in all cases.

Proposition 3.1.3. *If the velocity $v = v(t)$, as a function of time t, of an object on a coordinate axis is continuous on the interval $[a, b]$, then the (total)* **distance traveled** *by the object between times $t = a$ and $t = b$ is given by*

$$\int_a^b |v(t)|\, dt.$$

Remark 3.1.4. Proposition 3.1.3 tells us, in a new way, something that we already knew. It tells us that the total distance traveled, $r = r(t)$, by an object, between some initial time t_0 and some arbitrary time t, is given by

$$r(t) \;=\; \int_{t_0}^t |v(z)|\, dz,$$

where we once again have used z as a new dummy variable, and we're assuming that v is continuous.

Now, the first part of the Fundamental Theorem, Theorem 2.4.7, tells us that

$$\frac{dr}{dt} \;=\; |v(t)|,$$

i.e., that the (instantaneous) *speed* of an object can either be defined as the absolute value of the velocity or as the instantaneous rate of change of the distance traveled, with respect to time.

However, it is important to remember that **average** speed need **not** be the absolute value of the average velocity. For instance, an object that returns to where it started, after some amount of time, will have zero displacement and, hence, zero average velocity, but not zero average speed. Average speed is the average rate of change of the distance traveled, with respect to time. Thus, it is probably best, in the average and/or instantaneous setting, to define speed as the rate of change of the distance traveled, with respect to time, and then take it as a theorem that the instantaneous speed is equal to the absolute value of the instantaneous velocity.

Let's look at an example of calculating the (total) distance traveled.

Example 3.1.5. In Example 3.1.2, we had a particle moving in a straight line in such a way that its velocity $v = v(t)$, in m/s, at time t seconds, was given by

$$v \;=\; 3t^2 - 12t + 8.$$

We found that the displacement of the particle between times 2 and 4 seconds is

$$\int_2^4 (3t^2 - 12t + 8)\,dt \;=\; t(t-2)(t-4)\Big|_2^4 \;=\; 0 - 0 \;=\; 0 \text{ meters.}$$

Now we'll ask a different question.

What was the **distance traveled** by the particle between 2 and 4 seconds?

Solution:

It's certainly easy to write the appropriate integral. The distance traveled by the particle between 2 and 4 seconds was

$$\int_2^4 \left|3t^2 - 12t + 8\right|\,dt \qquad \text{meters,}$$

but how do we calculate the integral of the absolute value?

One thing that you definitely **don't** do is find an anti-derivative of $3t^2 - 12t + 8$, and then take its absolute value, in hope of producing an anti-derivative of $\left|3t^2 - 12t + 8\right|$. This would be **completely wrong**.

There's really only one thing to do; split up the integral into integrals over subintervals on which $3t^2 - 12t + 8$ is always ≥ 0 or is always ≤ 0. Then use that, if $3t^2 - 12t + 8 \geq 0$, then $\big|3t^2 - 12t + 8\big| = 3t^2 - 12t + 8$ and, if $3t^2 - 12t + 8 \leq 0$, then $\big|3t^2 - 12t + 8\big| = -(3t^2 - 12t + 8)$.

As $v(t) = 3t^2 - 12t + 8$ is a continuous function, it can switch signs only at points where it hits 0. We find these points, then check the sign of $v(t)$ in-between the zeroes. Setting $3t^2 - 12t + 8 = 0$, and using the quadratic formula, we find that the zeroes of $v(t)$ occur where

$$t \;=\; \frac{12 \pm \sqrt{144 - 96}}{6} \;=\; 2 \;\pm\; \frac{2\sqrt{3}}{3}.$$

Obviously, $2 - 2\sqrt{3}/3 < 2$. However, $2 + 2\sqrt{3}/3$ is in the interval $[2, 4]$. Thus, the sign of $v(t)$ does not switch from positive to negative, or vice-versa, on the interval $[2, 2 + 2\sqrt{3}/3]$ or on the interval $[2 + 2\sqrt{3}/3, 4]$. As $v(2) = -4$ and $v(4) = 8$, we find that $v(t) \leq 0$ on the interval $[2, 2 + 2\sqrt{3}/3]$, and that $v(t) \geq 0$ on the interval $[2 + 2\sqrt{3}/3, 4]$.

Therefore, the distance traveled, in meters, by the particle between times 2 and 4 seconds was

$$\int_2^4 \big|3t^2 - 12t + 8\big|\, dt \;=\; \int_2^{2+2\sqrt{3}/3} \big|3t^2 - 12t + 8\big|\, dt \;+\; \int_{2+2\sqrt{3}/3}^4 \big|3t^2 - 12t + 8\big|\, dt \;=$$

$$\int_2^{2+2\sqrt{3}/3} -(3t^2 - 12t + 8)\, dt \;+\; \int_{2+2\sqrt{3}/3}^4 (3t^2 - 12t + 8)\, dt.$$

Using plus or minus our anti-derivative from Example 3.1.2, we find that the sum above equals

$$-t(t-2)(t-4)\Big|_2^{2+2\sqrt{3}/3} \;+\; t(t-2)(t-4)\Big|_{2+2\sqrt{3}/3}^4 \;=$$

$$\Big[-\big(2 + 2\sqrt{3}/3\big)\big(2\sqrt{3}/3\big)\big(-2 + 2\sqrt{3}/3\big) - 0\Big] \;+\; \Big[0 - \big(2 + 2\sqrt{3}/3\big)\big(2\sqrt{3}/3\big)\big(-2 + 2\sqrt{3}/3\big)\Big] \;=$$

$$\frac{16\sqrt{3}}{9} \;+\; \frac{16\sqrt{3}}{9} \;=\; \frac{32\sqrt{3}}{9} \;\approx\; 6.1584.$$

It is somewhat interesting to note that, since the displacement between times 2 and 4 seconds was 0, the average velocity between times 2 and 4 seconds was 0, but the average **speed** was approximately $6.1584/(4-2) = 3.0792$ meters per second.

As you can see, the calculation of the distance traveled and/or average speed can be, and usually is, significantly more difficult than the calculation of the displacement and/or average velocity.

We should make a final remark in this section.

Remark 3.1.6. It may seem as though we have competing notions of what **average velocity** and **average speed** mean.

Suppose that, at time t, we let $p(t)$ denote the position of an object which is moving along line, let $r(t)$ denote the total distance traveled by the object, and let $v(t)$ denote the velocity, which we will assume is a continuous function of t.

Then, the definition of the average velocity and average speed of the object, between times $t = a$ and $t = b$ (or, on the interval $[a, b]$), where $a \neq b$, are

$$\text{average velocity} \;=\; \frac{p(b)-p(a)}{b-a} \qquad \text{and} \qquad \text{average speed} \;=\; \frac{r(b)-r(a)}{b-a}.$$

However, in Definition 2.3.30, the average value of **any** Riemann integrable function $f(t)$ on the interval $[a, b]$ was defined to be

$$\frac{1}{b-a}\int_a^b f(t)\,dt,$$

and we could consider the average value of the (instantaneous) velocity and speed functions, $v(t)$ and $|v(t)|$, respectively.

Do we need to worry that "average velocity" and "average speed" might refer to the average values of the velocity and speed functions, instead of referring to our original definitions? **No**.

As we discussed in this section, $\int_a^b v(t)\,dt = p(b) - p(a)$ and $\int_a^b |v(t)|\,dt = r(b) - r(a)$. Thus, the average value of the velocity function $v(t)$ equals the average velocity from our original definition, and the average value of the speed function $|v(t)|$ equals the average speed from our original definition.

3.1.1 Exercises

Throughout the exercises, assume that units of length are meters and that units of time are given in seconds and, hence, that velocities are given in m/s and accelerations in m/s^2.

In Exercises 1 - 10, the velocity function of particle is given. Find the total displacement of the particle between the times t_0 and t_1.

1. $v(t) = t^3 - 3t^2 + 1$, $t_0 = 0$, $t_1 = 10$.
2. $v(t) = 1/(t-3) + \cos 4t$, $t_0 = 2\pi$, $t_1 = 4\pi$.
3. $v(t) = \cosh 3t$, $t_0 = -5$, $t_1 = 5$.
4. $v(t) = \sinh 3t$, $t_0 = -5$, $t_1 = 5$.
5. $v(t) = \dfrac{1}{\sqrt{4t^2 - 64}}$, $t_0 = 5$, $t_1 = 8$.
6. $v(t) = t^2/(t^2 - 1)$, $t_0 = 3$, $t_1 = 6$.
7. $v(t) = \tan^2 t$, $t_0 = 0$, $t_1 = \pi/4$.
8. $v(t) = \dfrac{2t}{\sqrt{12 - t^2}}$, $t_0 = 0$, $t_1 = 3$.
9. $v(t) = \dfrac{4}{25 + 36t^2}$, $t_0 = 0$, $t_1 = 1$.
10. $v(t) = (\cos t)e^{\sin t}$, $t_0 = 0$, $t_1 = \pi$.

Calculate the total distance traveled by the particle over the given time interval with the given velocity function.

11. $v(t) = t^2 - 5t + 6$, $[0, 5]$.
12. $v(t) = 2t^2 + 12t + 1$, $[-4, 4]$.
13. $v(t) = \sin 2t$, $[0, \pi/2]$.
14. $v(t) = e^t - 1$, $[0, 2]$.
15. $v(t) = e^{-3t} + t^2 + \cos^2 t + \cosh t + 1$, $[-4, 4]$.

16. $v(t) = \dfrac{t^2 - 9t + 18}{t^2 + 3t + 2}$, $[0, 8]$.

17. $v(t) = \dfrac{6t}{\sqrt{24 - 3t^2}}$, $[-2, 2]$.

18. $v(t) = \sqrt{4t^2 + 4t + 10}$, $[-3, 5]$.

19. $v(t) = \ln(t/2)$, $[1, 4]$.

20. $v(t) = \sin t + \cos t$, $[0, 2\pi]$.

In Exercises 21 - 25, calculate (a) the average velocity and (b) the average speed of the particle traveling with velocity function $v(t)$ over the given time interval.

21. $v(t) = \sin 3t$, $[-\pi/6, \pi/3]$.

22. $v(t) = t^2 + 5t - 14$, $[0, 12]$.

23. $v(t) = \dfrac{6t + 8}{t^2 + t - 6}$, $[-2, 1]$.

24. $v(t) = \sec^3 4t$, $[-\pi/10, \pi/12]$.

25. $v(t) = t \cosh t^2$, $[-2, 4]$.

In Exercises 26 - 30, a particle is moving with velocity $v(t)$. Calculate the position function in terms of the time t from the given initial time t_0 where the particle's position is $p(t_0)$. Assume that $t > t_0$.

26. $v(t) = t^2 + 5t - 24$, $p(0) = 12$.

27. $v(t) = t \ln t$, $p(1) = 0$.

28. $v(t) = \cot^2 t$, $p(\pi/4) = 3$. Assume $t < \pi$.

29. $v(t) = \dfrac{t - 1}{t^2 + 1}$, $p(1) = 5$.

30. $v(t) = \sec t \tan t$, $p(0) = -3$. Assume $t < \pi/2$.

In Exercises 31 - 34, a particle is moving with velocity $v(t)$. Calculate the total distance $D(t)$ traveled by the particle between the specified starting time t_0 and an arbitrary time t, where $t \geq t_0$.

31. $v(t) = e^t + \cos^2 t + t^4$, $t_0 = 0$.

32. $v(t) = t^2 + 7t - 18$, $t_0 = 3$.

33. $v(t) = t^2 + 7t - 18$, $t_0 = 1$.

34. $v(t) = \sinh t$, $t_0 = 0$.

35. Suppose $v(t) = (t+2)(t-3)$ and we're interested in calculating the total distance traveled by the particle between $t = 0$ and $t = t_1$ where $t_1 > 0$. Then the integral $D(t_1) = \int_0^{t_1} |(t+2)(t-3)|\, dt$ will need to be evaluated in two pieces because of the sign change at $t = 3$. Is the function $D(t_1)$ continuous at $t_1 = 3$? Is it differentiable?

36. Redo the previous problem if $v(t) = (t-3)^2$. Why is this function easier to work with when calculating the distance function than the one in the previous problem?

37. Suppose a paratrooper jumps from an airplane with initial velocity $v(0) = 0$ and accelerates downward at $g = 9.8$ m/s^2 for three seconds. She opens her parachute and then accelerates upward at a rate of 2 m/s^2 for 5 more seconds.

    a. Write an expression for $v(t)$, where "up" is used as the positive direction.

    b. What is the paratrooper's displacement over the eight seconds?

    c. What is the paratrooper's average velocity over the eight seconds?

38. Suppose the A train is traveling at 60 mph. The conductor sees the B train in front of him on the same track moving in the same direction with speed 30 mph. The conductor puts on the breaks causing constant deceleration of a mph when the trains are exactly 1/4 of a mile apart.

    a. How far will the A train travel, in terms of a, before it comes to rest?

    b. How far will the B train travel, in terms of a, before the A train comes to rest?

    c. What is the minimum a must be to prevent a collision?

39. Suppose $v(t)$ is continuous. Prove, or provide a counterexample to, the statement

$$\int_a^b |v(t)|\, dt = \left| \int_a^b v(t)\, dt \right| .$$

40. Consider the position function

$$p(t) = \begin{cases} t^2 \sin(1/t) & \text{if } t \neq 0; \\ 0 & \text{if } t = 0. \end{cases}$$

a. Prove that p is differentiable for all t.

b. Prove that $p'(t)$ is not continuous at $t = 0$.

c. What are the ramifications of (a) and (b) on the application of the fundamental theorem of calculus? Specifically, is it true that $\int_0^b p'(t)\,dt = p(b) - p(0)$?

In each of Exercises 41 through 43, you are given the acceleration function $a(t)$ for a particle and the velocity of the particle at time $t = 0$. Calculate the average velocity of the particle between times $t = 0$ and $t = t_1$.

41. $a(t) = 2\sin 3t - 3\cos 2t$. $v(0) = 4$, $t_1 = \pi/6$.

42. $a(t) = \dfrac{t^2}{t+1} + 4$. $v(0) = 6$, $t_1 = 9$.

43. $a(t) = \dfrac{3}{25 - t^2}$. $v(0) = 4$, $t_1 = 3$.

44. Calculate the average acceleration of the functions in the three previous problems.

In some applications, it is reasonable to consider the limit of the displacement or speed of a moving particle as $t \to \infty$. These limits may or may not exist and should be evaluated using improper integrals.

45. Suppose a force is acting on a particle in such a way that the velocity of the particle is $v(t) = e^{-t}$. What is the limit as $t_1 \to \infty$ of the distance the particle has traveled between times $t = 0$ and $t = t_1$?

46. Using the velocity function in the previous problem,

a. What is the average velocity between times $t = 0$ and $t = t_1$?

b. What is the limit of the average velocity between times $t = 0$ and $t = t_1$ as $t_1 \to \infty$?

47. Consider a particle traveling in such a way that its velocity function is modeled by a damped oscillation. Specifically, assume $v(t) = e^{-t}\sin t$.

a. Show that $|v(t)| \to 0$ as $t \to \infty$.

b. Show that $\int_{(k-1)\pi}^{k\pi} |v(t)|\,dt = \frac{1}{2}e^{-k\pi}(1 + e^{\pi})$ for $k = 1, 2, 3, \ldots.$

c. What is $\int_0^{n\pi} |v(t)|\,dt$, where n is a positive integer? In other words, what's the total distance traveled by the particle during this time interval?

d. What is the limit, if it exists, of $\int_0^{n\pi} |v(t)|\,dt$ as $n \to \infty$?

As we shall discuss at length in Section 3.3, the formula for the distance traveled by an object traveling along a curved path in two or three-dimensional Euclidean space is a natural generalization of our current formula for distance traveled along the x-axis. Suppose that $\vec{p}$ is a differentiable curve which gives the position of an object in three-space. Let $\vec{p}(t) = (x(t), y(t), z(t))$, where $t \in [a, b]$. Then the velocity vector and instantaneous speed of the particle at $t = t_0$ are, respectively:

$$\vec{v}(t_0) = \vec{p}\,'(t) = (x'(t_0), y'(t_0), z'(t_0) \qquad \text{and} \qquad |\vec{v}(t_0)| = \sqrt{[x'(t_0)]^2 + [y'(t_0)]^2 + [z'(t_0)]^2}.$$

This notation is convenient because the formula for the total distance traveled between times t_0 and t is exactly the same as the one-dimensional formula:

$$r(t) = \int_{t_0}^{t} |v(z)|\, dz.$$

48. Suppose an object is traveling in a circular orbit with radius R. The position function is $\vec{p}(t) = (R\cos t, R\sin t)$. What is the total distance traveled by the particle between times t_1 and t_2? Show that this is equal to the arc length of a sector of a circle with central angle $t_2 - t_1$.

49. The previous problem confirms our intuition that calculating the distance traveled by a dynamic particle is equivalent to measuring the length of a path. A DNA molecule is shaped like a helix- a spiral around a cylinder. The approximate radius of the molecule is 10 angstroms, and the helix rises by about 34 angstroms per revolution.

    a. Show that this path can be parameterized as $\vec{p}(t) = (10\cos t, 10\sin t, 17t/\pi)$.

    b. Calculate $\vec{p}\,'(t)$. If we think of $\vec{p}$ as the position of a particle traversing the helix, then $\vec{p}\,'(t)$ is the velocity vector.

    c. Calculate $|\vec{p}\,'(t)|$. One can interpret this as either the instantaneous speed of a particle traversing the helix, or as the length of the tangent vector.

    d. What is the length, or distance traveled by a particle traversing the helix, in one revolution?

    e. A DNA molecule has approximately 285 million turns. What is the approximate length of a DNA molecule?

3.2 Area in the Plane

Given a continuous or, at least, integrable (Riemann integrable, or in the extended manner defined for improper integrals), we have looked at examples of calculating the area under the graph and above intervals on the x-axis or, for negative functions, areas above the graph and under intervals on the x-axis.

In this section, we will look at the more general problem of calculating the area "trapped" between the graphs of two different functions. This is a relatively easy application of integration. However, as we shall see, the main new difficulty is reminiscent of the difficulty we saw in the last section, where we wanted to calculate distance traveled, instead of displacement: we need to integrate the absolute value of a function.

Back in Proposition 2.3.11, we saw that if $f \geq 0$ and integrable on an interval $[a, b]$, then $\int_a^b f(x)\,dx$ is equal to the area under the graph and above the interval $[a, b]$. We also saw that if $f \leq 0$ and integrable on an interval $[a, b]$, then $-\int_a^b f(x)\,dx = \int_a^b [-f(x)]\,dx$ is equal to the area above the graph and under the interval $[a, b]$. One formula that unites these two results is given in:

Proposition 3.2.1. *Suppose that we have a function $y = f(x)$ defined and integrable on an interval, I, from a to b. Then, the* **area between the graph** *of $y = f(x)$ and the x-axis, for x in the interval I (or, on I), is*

$$\int_a^b |f(x)|\,dx.$$

Of course, the absolute value signs in Proposition 3.2.1 give a very succinct way of expressing the result, but they hide the difficulty: to actually evaluate $\int_a^b |f(x)|\,dx$, we have to do what we did for distance traveled problems in the previous section, namely, split the integral up into pieces over various subintervals on which $f(x)$ is ≥ 0 and on which $f(x)$ is ≤ 0. We then use that, if $f(x) \geq 0$, then $|f(x)| = f(x)$ and, if $f(x) \leq 0$, then $|f(x)| = -f(x)$.

Example 3.2.2. Calculate the area between the x-axis and the graph of $y = \ln x$, for $\frac{1}{2} \le x \le 2$. See Figure 3.1.

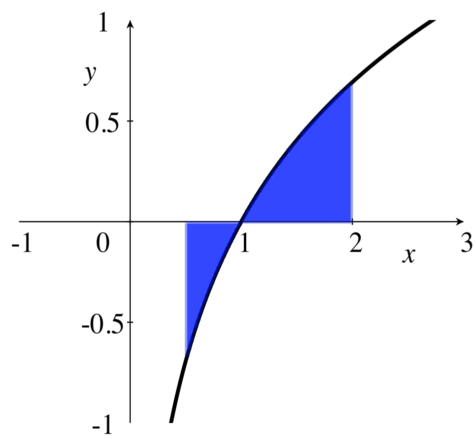


Figure 3.1: Area between the x-axis and $y = \ln x$, $\frac{1}{2} \le x \le 2$.

Solution:

The area of the region is given by

$$\int_{1/2}^{2} |\ln x|\, dx.$$

But how do we calculate this?

We use that $\ln x \le 0$ if $0 < x \le 1$, $\ln x \ge 0$ if $x \ge 1$ and, as we saw in Example 1.1.21, integration by parts tells us that $\int \ln x\, dx = x \ln x - x + C$. (Don't get confused: the **derivative** of $\ln x$ is $1/x$, but we're integrating here.)

Therefore,

$$\int_{1/2}^{2} |\ln x|\, dx \;=\; \int_{1/2}^{1} |\ln x|\, dx \;+\; \int_{1}^{2} |\ln x|\, dx \;=\; \int_{1/2}^{1} -\ln x\, dx \;+\; \int_{1}^{2} \ln x\, dx \;=$$

$$-(x \ln x - x)\Big|_{1/2}^{1} \;+\; (x \ln x - x)\Big|_{1}^{2} \;=$$

$$\left[-(0-1)+\left(\frac{1}{2}\ln\left(\frac{1}{2}\right)-\frac{1}{2}\right)\right]+[(2\ln 2-2)-(0-1)] \;=\; \frac{1}{2}-\frac{1}{2}\ln 2+2\ln 2-1 \;=$$

$$\frac{3\ln 2-1}{2}.$$

If we think of the x-axis as the graph of $y = g(x) = 0$, then finding the area between the graph of $y = f(x)$ and the x-axis is the same as finding the area between the graphs of $y = f(x)$ and $y = g(x)$. Our question now is: can we use integration to find the area trapped between two graphs in the more general case in which we don't assume that one graph is the x-axis?

Let's think about this. Consider the functions $y = f(x) = x^2$ and $y = g(x) = 4$. Suppose we want to find the area between the graphs of f and g for $1 \leq x \leq 3$.

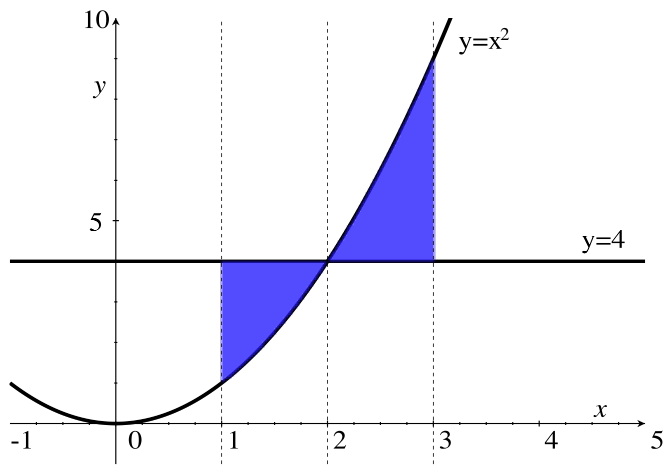


Figure 3.2: The area between the graphs of $y = x^2$ and $y = 4$.

What we do is split the problem into two pieces, pieces which correspond to which function is bigger, f or g. Hence, we find the area between the graphs for $1 \leq x \leq 2$ and add to that the area between the graphs for $2 \leq x \leq 3$.

How do we find the area between the graphs for $1 \leq x \leq 2$? What we **could** do is find the area under the graph of $y = 4$, namely $\int_1^2 4\,dx$ (which, even without integrating, we know is $4 \cdot 1$, since the region is a rectangle of height 4 and width 1), and then subtract the missing area, $\int_1^2 x^2\,dx$, under the graph of $y = x^2$. We would find that the area between the graphs for $1 \leq x \leq 2$ is

$$\int_1^2 4\,dx \;-\; \int_1^2 x^2\,dx \;=\; \int_1^2 (4 - x^2)\,dx \;=\; \int_1^2 (g(x) - f(x))\,dx.$$

Similarly, we **could** calculate the area between the graphs of f and g for $2 \leq x \leq 3$ by taking the area, $\int_2^3 x^2\,dx$, under the graph of $y = x^2$, and above the interval $[2, 3]$ and then subtracting the missing area $\int_2^3 4\,dx$. We would find that the area between the graphs for $1 \leq x \leq 2$ is

$$\int_2^3 x^2\,dx \;-\; \int_2^3 4\,dx \;=\; \int_2^3 (x^2 - 4)\,dx \;=\; \int_2^3 (f(x) - g(x))\,dx.$$

However, there's one psychological reason and one practical reason why, on the different subintervals, you don't want to think of taking the entire area under one graph and subtracting the entire "missing" area under the other graph; you want to think of integrating the differences, i.e., you want to think of the problem in terms of $\int_1^2 (g(x) - f(x))\,dx$ and $\int_2^3 (f(x) - g(x))\,dx$ in the first place.

Psychologically, you shouldn't think of involving any regions other than those that are trapped between the two graphs. Instead of subtracting areas of entire regions, you should think of taking the continuous sum of areas of infinitesimal rectangles that lie between the two graphs, rectangles of infinitesimal width dx, and height $(g(x) - f(x))$, if $1 \leq x \leq 2$, or height $(f(x) - g(x))$, if $2 \leq x \leq 3$. See Figure 3.3. Thus, we have rectangles of infinitesimal area $dA = (g(x) - f(x))\,dx$ or $dA = (f(x) - g(x))\,dx$, and we should add up these infinitesimal areas, by taking the integral, to find the total area.

In a practical sense, for other functions and intervals, it could be significantly easier to calculate $\int_a^b (f(x) - g(x))\,dx$ than to separately calculate $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ and then subtract. How is this possible? Nasty terms might cancel out in $f(x) - g(x)$. For instance, if $f(x) = x^2 + e^{-x^2}$ and $g(x) = 4 + e^{-x^2}$, then $\int_2^3 (f(x) - g(x))\,dx = \int_2^3 (x^2 - 4)\,dx$, which is easy to calculate, but you would not succeed in calculating either $\int_2^3 f(x)\,dx$ or $\int_2^3 g(x)\,dx$.

So, what we have seen is that, over subintervals $[a, b]$ where $g(x) \geq f(x)$, i.e., where $g(x) - f(x) \geq 0$, we find the area between the graphs of $y = f(x)$ and $y = g(x)$ by calculating $\int_a^b (g(x) - f(x))\,dx$. Over subintervals $[a, b]$ where $f(x) \geq g(x)$, i.e., where $f(x) - g(x) \geq 0$, we find the area between the graphs of $y = f(x)$ and $y = g(x)$ by calculating $\int_a^b (f(x) - g(x))\,dx$. Therefore, regardless of which function is greater, our integrand is always $|f(x) - g(x)|$.

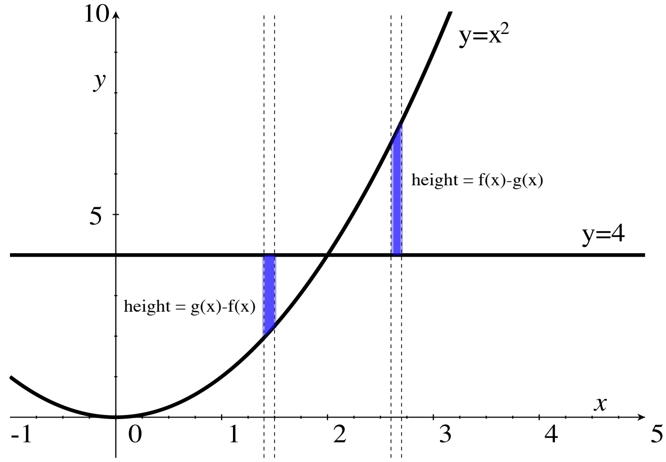


Figure 3.3: Infinitesimally wide rectangles between the graphs of $y = x^2$ and $y = 4$.

Proposition 3.2.3. *Suppose that we have functions $y = f(x)$ and $y = g(x)$ defined on an interval, I, from a to b, and that $f(x) - g(x)$ is integrable on I. Then, the* **area between the graphs** *of $y = f(x)$ and $y = g(x)$, for x in the interval I (or, on I), is*

$$\int_a^b |f(x) - g(x)|\, dx.$$

Note that, if $g(x) = 0$, so that the graph of $y = g(x)$ is just the x-axis, then Proposition 3.2.3 reduces to Proposition 3.2.1.

Example 3.2.4. Let's finish our problem from the discussion above. We have the functions $y = f(x) = x^2$ and $y = g(x) = 4$, and we want to find the area between the graphs of f and g for $1 \leq x \leq 3$.

Solution:

We need to calculate

$$\int_1^3 \left|x^2 - 4\right|\, dx.$$

In order to deal with the absolute value, we need to know where $x^2 - 4 \geq 0$ and where $x^2 - 4 \leq 0$. As $x^2 - 4$ is continuous, we first find where $x^2 - 4 = 0$. This is easy; it happens when $x = \pm 2$. Deleting these two zeroes divides the real line into three subintervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, and the sign, $\pm$, of $x^2 - 4$ cannot change on a given one of these subintervals, for the Intermediate Value Theorem tells us that continuous functions must pass through zero to switch signs.

Thus, to find the sign of $x^2 - 4$ on each of these subintervals, we may simply pick **any** x value in the subinterval, evaluate $x^2 - 4$ there, and see whether it's positive or negative. In fact, as we're integrating from 1 to 3, we don't actually care about what happens on the interval $(-\infty, -2)$. From the interval $(-2, 2)$, we'll pick $x = 0$, and find then that $x^2 - 4 = -4 < 0$; thus, $x^2 - 4 < 0$ for all x in the interval $(-2, 2)$, and so $x^2 - 4 \leq 0$ for all x in $[-2, 2]$. In particular, $x^2 - 4 \leq 0$ for each x in the interval $[1, 2]$. When $x = 3$, $x^2 - 4 = 5 > 0$, and so $x^2 - 4 \geq 0$ for all x in the interval $[2, \infty)$. Of course, it follows that $x^2 - 4 \geq 0$ for all x in the interval $[2, 3]$.

Hence, we find

$$\int_1^3 |x^2 - 4|\; dx \;=\; \int_1^2 -\left(x^2 - 4\right)\, dx \;+\; \int_2^3 \left(x^2 - 4\right)\, dx \;=$$

$$\left(-\frac{x^3}{3} + 4x\right)\Big|_1^2 \;+\; \left(\frac{x^3}{3} - 4x\right)\Big|_2^3 \;=$$

$$\left[\left(-\frac{8}{3} + 8\right) - \left(-\frac{1}{3} + 4\right)\right] + \left[(9 - 12) - \left(\frac{8}{3} - 8\right)\right] \;=\; 4.$$

Example 3.2.5. Of course, not all problems about area between graphs involve a change in signs of $f(x) - g(x)$. Consider the problem of finding the area between the graphs of $y = \sin x$ and $y = \cos x$, for $0 \leq x \leq \pi/6$.

All that you need to know is that, for $0 \leq x \leq \pi/6$, $\cos x \geq \sin x$, so that $\cos x - \sin x \geq 0$, and, hence, $|\cos x - \sin x| = \cos x - \sin x$. Thus, the area between the graphs is

$$\text{Area} \;=\; \int_0^{\pi/6} (\cos x - \sin x)\, dx \;=\; (\sin x + \cos x)\Big|_0^{\pi/6} \;=\; \frac{1}{2} + \frac{\sqrt{3}}{2} - 0 - 1 \;=\; \frac{\sqrt{3} - 1}{2}.$$

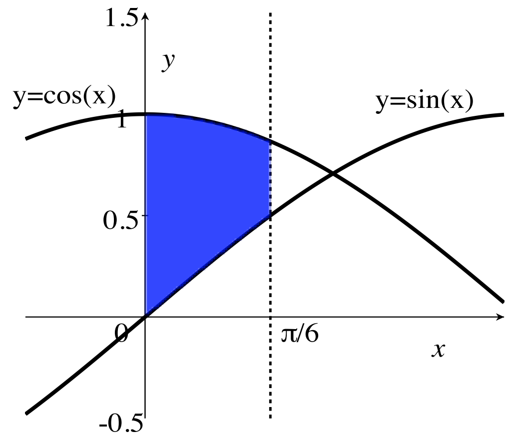


Figure 3.4: Area between $y = \sin x$ and $y = \cos x$, $0 \le x \le \pi/6$.

Really, in a problem where one function is always bigger than (or equal to) the other function, you shouldn't think in terms of absolute values at all; just take the bigger function, subtract the smaller function, and integrate.

There are times when you are not explicitly given the limits of integration. Let's look at a typical such problem.

Example 3.2.6. Find the area of the bounded region between the graphs of $y = f(x) = x^3$ and $y = g(x) = x^2$.

Solution: If you look at the graphs in Figure 3.5, you can see the region between the graphs of $y = x^3$ and $y = x^2$ naturally breaks up into three pieces: one piece where $x \le 0$, one piece where $0 \le x \le 1$, and one piece where $x \ge 1$. But the pieces where $x \le 0$ and $x \ge 1$ are unbounded, i.e., go out infinitely far. The **bounded** region between the graphs is the portion where $0 \le x \le 1$; the part that's really trapped, or completely bordered, by the graphs.

After we know this, the problem is easy, except for one small issue. You probably normally think that x^3 is bigger than x^2. Right? Doesn't cubing a number give you something bigger than squaring it? **Not for numbers between** 0 **and** 1**!** (And certainly not for negative numbers, either.) If $0 \le x \le 1$, then $x^2 \ge x^3$. Therefore, for $0 \le x \le 1$, $x^2 - x^3 \ge 0$, and we don't need to use absolute values to find the area. The area of the bounded region is simply

$$\text{Area} \;=\; \int_0^1 (x^2 - x^3)\,dx \;=\; \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 \;=\; \frac{1}{12}.$$

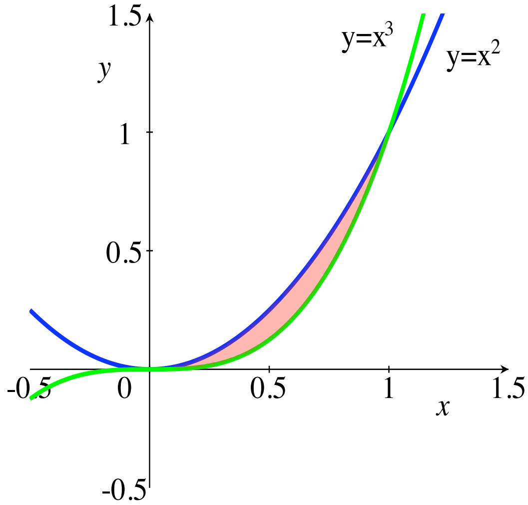


Figure 3.5: Area of bounded region between $y = x^3$ and $y = x^2$.

In the following example, we look at a problem in which nasty pieces of the functions involved cancel out.

Example 3.2.7. Let $f(x) = e^{-x} + e^{x^2}$ and $g(x) = e^{-3x+4} + e^{x^2}$. Find the area between the graphs of f and g for $0 \leq x \leq 3$.

Solution:

You might be inclined to graph f and g by hand, or by using a calculator or computer. The problem is that the e^{x^2} terms get so big so fast that you will have difficulty seeing the relatively small difference between the values of the functions. The good news is that we don't **need** to see a picture in order to solve the problem.

We need to calculate

$$\int_0^3 |f(x) - g(x)|\, dx \;=\; \int_0^3 \left|e^{-x} - e^{-3x+4}\right|\, dx.$$

To deal with the absolute value, we need to determine where $e^{-x} - e^{-3x+4}$ is positive and where it's negative. So, we find where $e^{-x} - e^{-3x+4} = 0$, i.e., where $e^{-x} = e^{-3x+4}$. Taking natural logs of both sides of the equation (or using that the exponential function is one-to-one), we find that $-x = -3x + 4$, and so $x = 2$.

Therefore, since $e^{-x} - e^{-3x+4}$ is continuous and is zero only at $x = 2$, we must have that $e^{-x} - e^{-3x+4}$ is either always positive or always negative on each of the intervals $(-\infty, 2)$ and $(2, \infty)$. We simply check the sign of the function at some point in each interval. When $x = 0$, $e^{-x} - e^{-3x+4} = e^0 - e^4 < 0$; hence, $e^{-x} - e^{-3x+4} < 0$ for all x in $(-\infty, 2)$, and so $e^{-x} - e^{-3x+4} \leq 0$ for all x in $(-\infty, 2]$. Similarly, when $x = 3$, $e^{-x} - e^{-3x+4} = e^{-3} - e^{-5} > 0$, and we conclude that $e^{-x} - e^{-3x+4} \geq 0$ for all x in $[2, \infty)$.

Thus, we find

$$\int_0^3 \left|e^{-x} - e^{-3x+4}\right| dx \;=\; \int_0^2 \left|e^{-x} - e^{-3x+4}\right| dx \;+\; \int_2^3 \left|e^{-x} - e^{-3x+4}\right| dx \;=$$

$$\int_0^2 -\left(e^{-x} - e^{-3x+4}\right) dx \;+\; \int_2^3 \left(e^{-x} - e^{-3x+4}\right) dx.$$

Let's produce an anti-derivative for all functions of the form e^{ax+b}, so that we can use this result to integrate each piece in the integrals above. Consider $\int e^{ax+b}\, dx$, where a and b are constants, and $a \neq 0$. Make the substitution $u = ax + b$, so that $du = a\, dx$, i.e., $dx = du/a$. Then, we find

$$\int e^{ax+b}\, dx \;=\; \int e^u \cdot \frac{du}{a} \;=\; \frac{1}{a}\int e^u\, du \;=\; \frac{1}{a}\, e^u + C \;=\; \frac{e^{ax+b}}{a} + C.$$

Now, we apply this formula to e^{-x} and e^{-3x+4}, and obtain

$$\int_0^2 -\left(e^{-x} - e^{-3x+4}\right) dx \;=\; e^{-x} - \frac{e^{-3x+4}}{3}\,\Bigg|_0^2 \;=\; e^{-2} - \frac{e^{-2}}{3} - \left(1 - \frac{e^4}{3}\right) \;=$$

$$\frac{2e^{-2}}{3} - 1 + \frac{e^4}{3}.$$

We also find that

$$\int_2^3 \left(e^{-x} - e^{-3x+4}\right) dx \;=\; -e^{-x} + \frac{e^{-3x+4}}{3}\,\Bigg|_2^3 \;=$$

$$-e^{-3}+\frac{e^{-5}}{3}-\left(-e^{-2}+\frac{e^{-2}}{3}\right) \;=\; -e^{-3}+\frac{e^{-5}}{3}+\frac{2e^{-2}}{3}.$$

Putting together all of our above work, we find:

$$\text{Area} \;=\; \int_0^3 \left|e^{-x}-e^{-3x+4}\right|dx \;=$$

$$\frac{2e^{-2}}{3}-1+\frac{e^4}{3}-e^{-3}+\frac{e^{-5}}{3}+\frac{2e^{-2}}{3} \;=\; \frac{e^4+4e^{-2}+e^{-5}}{3}-e^{-3}-1.$$

In the next example, we consider the same two functions that we did above, but now we look at the area of an unbounded region, a region that extends out infinitely far. This, of course, leads to an improper integral.

Example 3.2.8. Let $f(x)=e^{-x}+e^{x^2}$ and $g(x)=e^{-3x+4}+e^{x^2}$. Find the area between the graphs of f and g for $0\le x<\infty$.

Solution:

These are the same functions that we used in the previous example, and so we can use much of our work from that problem.

We need to calculate

$$\int_0^\infty \left|e^{-x}-e^{-3x+4}\right|\,dx.$$

We **could** do this either one of two ways.

First, we could proceed exactly as we did in the previous example, and write

$$\int_0^\infty \left|e^{-x}-e^{-3x+4}\right|\,dx \;=\; \int_0^2 \left|e^{-x}-e^{-3x+4}\right|\,dx \;+\; \int_2^\infty \left|e^{-x}-e^{-3x+4}\right|dx \;=$$

$$\int_0^2 -\left(e^{-x}-e^{-3x+4}\right)\,dx \;+\; \int_2^\infty \left(e^{-x}-e^{-3x+4}\right)dx.$$

We already calculated, in the previous example, the integral on the left; we would still need to calculate the improper integral on the right.

However, given that we already calculated $\int_0^3 \left|e^{-x} - e^{-3x+4}\right| \, dx$ is the previous example, it would also be reasonable to split the integral up in the following manner:

$$\int_0^\infty \left|e^{-x} - e^{-3x+4}\right| \, dx \;=\; \int_0^3 \left|e^{-x} - e^{-3x+4}\right| \, dx \;+\; \int_3^\infty \left|e^{-x} - e^{-3x+4}\right| dx \;=$$

$$\frac{e^4 + 4e^{-2} + e^{-5}}{3} - e^{-3} - 1 \;+\; \int_3^\infty \left(e^{-x} - e^{-3x+4}\right) dx,$$

where, again, we still need to calculate the improper integral on the right.

We'll use the second splitting of the integral, though it makes little difference. We find

$$\int_3^\infty \left(e^{-x} - e^{-3x+4}\right) dx \;=\; \lim_{p\to\infty} \left\{ -e^{-x} + \frac{e^{-3x+4}}{3} \Big|_3^p \right\}.$$

$$\lim_{p\to\infty} \left(-e^{-p} + \frac{e^{-3p+4}}{3} + e^{-3} - \frac{e^{-5}}{3} \right) \;=\; e^{-3} - \frac{e^{-5}}{3}.$$

Therefore, our final answer is

$$\text{Area} \;=\; \int_0^\infty \left|e^{-x} - e^{-3x+4}\right| \, dx \;=$$

$$\frac{e^4 + 4e^{-2} + e^{-5}}{3} - e^{-3} - 1 \;+\; e^{-3} - \frac{e^{-5}}{3} \;=\; \frac{e^4 + 4e^{-2}}{3} - 1.$$

What if you want to find the area between curves that aren't given to you by having y in terms of x, but instead are given to you by x being specified in terms of y? For instance, how do you find the area between the curves/lines $x = 4 - y^2$ and $x = 2 - y$ for $0 \leq y \leq 2$?

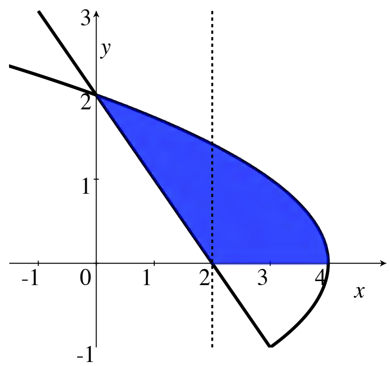


Figure 3.6: Area between $x = 4 - y^2$ and $x = 2 - y$, $0 \leq y \leq 2$.

There are several ways to approach this problem, some good, some not so good. Let's start with the not so good.

• Probably the worst way to approach this problem is to think "I know how to do area problems only when I have y in terms of x, so I'll rewrite both equations, solving for y". You get $y = \sqrt{4-x}$ (there's no plus or minus sign because we're looking where $y \geq 0$) and $y = 2 - x$. Understand that in other problems, this step, of solving algebraically for y in terms of x, might be impossible (in any reasonable way). That's a big downside to this method in general, but we can try to proceed anyway with our specific problem.

So, now you need to find the area between the two graphs of $y = \sqrt{4-x}$ and $y = 2 - x$, for $0 \leq y \leq 2$. If you look at Figure 3.6, it's easy to see another difficulty: in terms of x, the region that we're trying to find the area of is between $y = 2 - x$ and $y = \sqrt{4-x}$ when $0 \leq x \leq 2$, but is between $y = 0$ and $y = \sqrt{4-x}$ when $2 \leq x \leq 4$. This means that we need to split the problem up as the sum of two integrals, i.e., we need to calculate

$$\int_0^2 \left(\sqrt{4-x} \ - \ (2-x)\right) dx \ + \ \int_2^4 \left(\sqrt{4-x} - 0\right) dx.$$

Yuck. We would do this if we had to, and find the answer of 10/3, but surely there's an easier way.

• Even if you want to find areas by always integrating with respect to x, you don't have to do what we did in the previous method. You could just tell yourself that "flipping" the graph, interchanging the x- and y-axes, doesn't change the area of the region. In fact, instead of actually flipping the graph – to save time, effort, and space – you can simply relabel the old

x-axis as now being the y-axis, and the old y-axis as now being the x-axis, and swap all of the x and y's in the equations and in the bounds. This way, we are now looking for the area between the curves/lines $y = 4 - x^2$ and $y = 2 - x$ for $0 \le x \le 2$, but the region looks the same as it did before. Note the new labels on the axes in Figure 3.7.

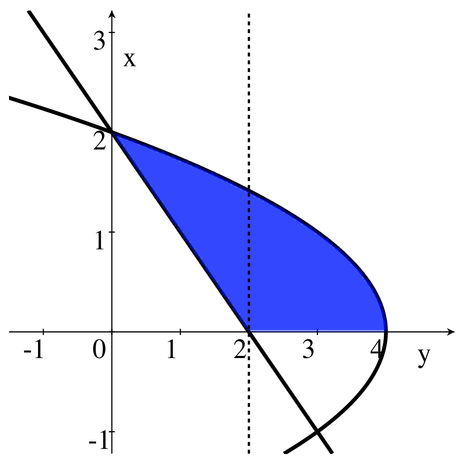


Figure 3.7: Make the x-axis the vertical axis, and the y-axis the horizontal.

Now we know what to do. To calculate the area, we evaluate

$$\int_0^2 \big((4 - x^2) - (2 - x)\big)\, dx \;=\; \int_0^2 \left(2 + x - x^2\right) dx \;=\; 2x + \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^2 \;=\; \frac{10}{3}. \tag{3.1}$$

• But the best method for finding the area of the given region is to realize that, while you **could** switch the x's and y's, as we did above, we don't need to; just don't panic about using y for your integration variable in the first place. Had you not exchanged the x's and y's in the first place, but instead used y for the integration variable, you would have had y's throughout Formula 3.1, in place of x's, but of course you'd get the same answer:

$$\int_0^2 \big((4 - y^2) - (2 - y)\big)\, dy \;=\; \int_0^2 \left(2 + y - y^2\right) dy \;=\; 2y + \frac{y^2}{2} - \frac{y^3}{3}\bigg|_0^2 \;=\; \frac{10}{3}.$$

Geometrically, with the x-axis and y-axis in their usual positions, this means we are taking the continuous sum of areas of infinitesimally high rectangles, of height dy, whose length is given by the difference of the x-coordinates on the two curves, as in Figure 3.8

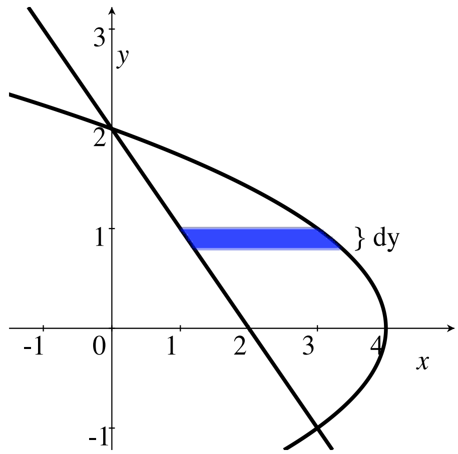


Figure 3.8: An infinitesimal rectangle between $x = 4 - y^2$ and $x = 2 - y$.

Even though it's really nothing new, for the sake of completeness, and for reference, we give the general proposition which relates to the discussion above.

Proposition 3.2.9. *Suppose that we have functions $x = f(y)$ and $x = g(y)$ defined on an interval, I, from a to b, and that $|f(y) - g(y)|$ is integrable on I. Then, the* **area between the graphs** *of $x = f(y)$ and $x = g(y)$, for y in the interval I, is*

$$\int_a^b |f(y) - g(y)|\, dy.$$

A particular case of Proposition 3.2.9 is when one of the curves is the graph of $x = 0$, i.e., the y-axis. Then, the integral for the area collapses to just the integral of the absolute value of the other function.

Example 3.2.10. Suppose, for instance, that you want to find the area between the graph of $x = \sin y$ and the y-axis, for $-\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$.

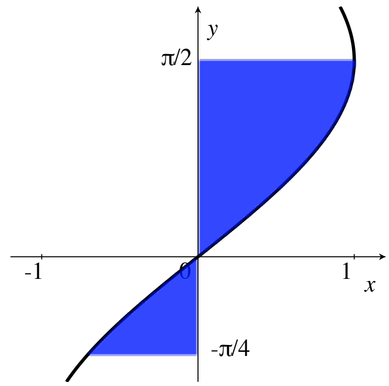


Figure 3.9: Area between $x = \sin y$ and the y-axis, $-\pi/4 \leq y \leq \pi/2$.

We need to calculate $\int_{-\pi/4}^{\pi/2} |\sin y|\, dy$. We find

$$\int_{-\pi/4}^{\pi/2} |\sin y|\, dy \;=\; \int_{-\pi/4}^{0} |\sin y|\, dy \;+\; \int_{0}^{\pi/2} |\sin y|\, dy \;=$$

$$\int_{-\pi/4}^{0} -\sin y\, dy \;+\; \int_{0}^{\pi/2} \sin y\, dy \;=\; \cos y\Big|_{-\pi/4}^{0} \;+\; (-\cos y)\Big|_{0}^{\pi/2} \;=$$

$$1 - \frac{1}{\sqrt{2}} \;+\; 0 - (-1) \;=\; 2 - \frac{1}{\sqrt{2}} \;=\; \frac{4 - \sqrt{2}}{2}.$$

3.2.1 Exercises

In each of Exercises 1 through 5, find the total area between the x-axis and the graph of $y = f(x)$, for the indicated values of x. Also, possibly using a calculator or graphing software, sketch the region whose area you are calculating, and include in your sketch some typical "infinitesimal" rectangles with "infinitesimal" width dx.

1. $y = \sin x$, $0 \leq x \leq 2\pi$

2. $y = \cos x$, $0 \leq x \leq 2\pi$

3. $y = 9 - x^2$, $-4 \leq x \leq 2$

4. $y = x^3 + x^2 - 2x$, $-3 \leq x \leq 2$

5. $y = xe^{(x^2)}$, $-1 \leq x \leq 1$

In each of Exercises 6 through 10, find the area between the y-axis and the graph of $x = f(y)$, for the indicated values of y. Also, possibly using a calculator or graphing software, sketch the region whose area you are calculating, and include in your sketch some typical "infinitesimal" rectangles with "infinitesimal" height dy.

6. $x = \sin y$, $0 \leq y \leq 2\pi$

7. $x = \cos y$, $0 \leq y \leq 2\pi$

8. $x = 9 - y^2$, $-4 \leq y \leq 2$

9. $x = y^3 + y^2 - 2y$, $-3 \leq y \leq 2$

10. $x = ye^{(y^2)}$, $-1 \leq y \leq 1$

In each of Exercises 11 through 20, find the area of the bounded region between the graphs of the two given functions. Use integration with respect to whichever variable seems most convenient.

11. $y = 3 - x^2, \ \ y = -6$

12. $y = 3 - x^2, \ \ y = 3x - 1$

13. $x = 3 - y^2, \ \ x = 3y - 1$

14. $x = y^2, \ \ y = x^2$

15. $y = x(e^x - e)$, $y = 0$

16. $x = y(e^y - e)$, $x = 0$

17. $x = \dfrac{4}{1+y^2}, \ \ x = 2$

18. $y = \dfrac{10}{1+x^2}, \ \ y = 2$

19. $y = \dfrac{10}{1+x^2}, \ \ y = 1 - \dfrac{7}{1+x^2}$

20. $x = \dfrac{10}{1+y^2}$, $x = 1 - \dfrac{7}{1+y^2}$

In each of Exercises 21 through 36, calculate the total area bounded by the graphs of $y = f(x)$ and $y = g(x)$, or $x = f(y)$ and $x = g(y)$, for values of the independent variable in the given interval.

21. $f(x) = x^2 - 16$, $g(x) = 9$, $[-10, 10]$.
22. $f(x) = \sin x$, $g(x) = \cos x$, $[0, 2\pi]$.
23. $f(y) = \sin y$, $g(y) = \sin 2y$, $[0, 2\pi]$.
24. $f(y) = e^{y^2} + e^{3y+1}$, $g(y) = e^{2y+2} + e^{y^2}$, $[-4, 3]$.
25. $f(x) = x^4 + x^3 - 3x^2 + 5x + 5$, $g(x) = x^4 + 2x^3 - 3x^2 + 5x + 6$, $[-4, 6]$.
26. $f(x) = \ln(x^x \tan x) - \ln(\sin x)$, $g(x) = \ln(e^x \sec x)$, $[1, e/2]$.
27. $f(y) = 3/(y+2)$, $g(y) = 2/(y-4)$, $[-1, 3]$.
28. $f(y) = \dfrac{\tan y}{1 - \tan y \tan 1}$, $g(y) = \dfrac{-\tan 1}{1 - \tan y \tan 1}$, $[0, \pi/4]$.
29. $f(x) = x^4 + x^2 + 2$, $g(x) = x^3 + 3x$, $[-3, 3]$.
30. $f(x) = \tan^{-1} x + x^2 + 3$, $g(x) = 2x^2 + x + \tan^{-1} x - 4$, $[-3, 5]$.
31. $f(y) = e^{2y} + 4$, $g(y) = e^{-2y} + 2$, $[-5, 5]$.
32. $f(y) = \dfrac{y^4 + y^2}{\sqrt{y^6 + y^4}}$, $g(y) = \dfrac{y^2}{\sqrt{y^2 + 1}}$, $[1, 4]$.
33. $f(x) = \sinh^{-1} x$, $g(x) = \cosh^{-1} x$, $[1, 10]$.
34. $f(x) = \dfrac{x^2}{(x-2)(x+2)}$, $g(x) = \dfrac{4}{(x-2)(x+2)}$, $[-1, 1]$.
35. $f(y) = |y|$, $g(y) = -|y| + 1$, $[-1/2, 1/2]$.
36. $f(y) = 2^y$, $g(y) = 3^y$, $[-1, 2]$.
37. Prove that if f, g and h are continuous functions on $[a, b]$ then the area between the graphs of f and g is equal to the area between the graphs of $f + h$ and $g + h$.
38. Suppose the area enclosed by two continuous functions f and g on the interval $[a, b]$ is $A = \int_a^b |f(x) - g(x)|\, dx$. Suppose c is an arbitrary real number. Is it true that the area between the graphs of $cf(x)$ and $cg(x)$ is cA? If not, how can you correct this statement?

39. Calculate the limiting area between the curves e^{-at} and e^{-bt} between $t = 0$ and $t = U$ where $U \to \infty$. Assume a and b are positive distinct real numbers.

40. Calculate the area between the curves $h(y) = \ln(ay)$ and $g(y) = \ln(by)$ on the interval $[c, d]$ where a, b and c are positive and $a \neq b$.

41. Calculate the area between the curves $a(t) = 1/t$ and $b(t) = 1/t^2$ on the interval $[1, U]$. Does the limiting area converge as $U \to \infty$?

42. Redo the previous problem with $a(t) = 1/t^2$ and $b(t) = 1/t^3$.

43. What is the area enclosed by the curves $f(x) = \sin nx$ and $g(x) = \cos nx$ on the interval $[0, 2\pi]$ where n is a positive integer?

In each of Exercises 44 through 47, use the Midpoint Rule to approximate the areas bound by the two curves on the interval $[0, 1]$. Use a partition of $n = 4$ evenly spaced points.

44. $f(x) = e^{x^2}$, $g(x) = e^x$.

45. $f(x) = e^{x^k}$, $g(x) = e^x$, $k > 0$.

46. $f(x) = e^{x^k}$, $g(x) = e^{x^m}$, $k, m > 0$.

47. $f(x) = \sqrt{1 + x^3}$, $g(x) = x + 1$.

In Exercises 48 through 51, you are given the area A between the graphs of $f(x)$ and $g(x)$ over an interval I. You are also given the two functions with one unknown parameter c. Solve for the parameter.

48. $A = 24$, $f(x) = 2x^2 - 3x + 4$, $g(x) = x^2 + cx - 6$, $I = [0, 6]$.

49. $A = -2 + 2\sqrt{3} - \pi/6$, $f(x) = \sin x$, $g(x) = c$, $I = [0, \pi]$, $c > 0$.

50. $A = 16$, $f(x) = c - |x|$, $g(x) = |x|$, $I = [-c, c]$.

51. $A = 9\pi - 18$, $f(x) = \sqrt{c^2 - x^2}$, $g(x) = c - x$, $I = [0, c]$.

52. Consider the region R in the first quadrant, under the graph of $y = 1 - x^2$. Find the value of b so that the horizontal line given by $y = b$ splits R into two sections with equal areas.

53. Suppose that f is continuous on $[a, b]$.

    a. Prove that $|f|$ is integrable on $[a, b]$.

b. Argue that one can choose $k = \pm 1$ such that $k\int_a^b f(x)\,dx \geq 0$.

c. Conclude that $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

d. Use part (c) to show that a lower bound for the area enclosed between the graphs of $f(x)$ and $g(x)$ on the interval $[a,b]$ is $\left|\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right|$.

54. Argue that if f and h are continuous on $[a,b]$, then the area enclosed by f and the x-axis is less than or equal to the sum of the area enclosed by f and h, and the area enclosed by h and the x-axis.

55. Suppose f and g are continuous on $[a,b]$. Let $h(x) = |f(x)|$ and $j(x) = |g(x)|$. Prove that the area enclosed by j and h on the interval $[a,b]$ is less than or equal to the area enclosed by f and g on the interval $[a,b]$.

It's often more convenient to describe a function f using polar coordinates. If we assume that the distance from the origin to a point on the graph, r, depends only on the angle between the x-axis and ray connecting the origin to the point, θ then we can write $r = f(\theta)$. Under reasonable assumptions, the area of the polar region defined for $\theta \in [a,b]$ is $A = \dfrac{1}{2}\displaystyle\int_a^b f^2(\theta)\,d\theta$. Here, we assume that $b - a \leq 2\pi$ to prevent double counting. Calculate the areas of the polar regions in Exercises 36 - 40.

56. $r = \theta$, $\theta \in [0,\pi]$.

57. $r = \sin 3\theta$, $\theta \in [0,\pi/3]$.

58. $4 = \sin n\theta$, $\theta \in [0,\pi/n]$, n a positive integer.

59. $r = r_0$, $\theta \in [0,2\pi]$. What familiar shape is this?

60. $r = 1 + \sin\theta$, $\theta \in [0,2\pi]$.

The area between two polar regions defined by the functions $r_1(\theta)$ and $r_2(\theta)$ is given by $A = \frac{1}{2}\int_a^b |r_1(\theta) - r_2(\theta)|\,d\theta$. Calculate the areas between the polar curves over the given angular interval in Exercises 41 - 45

61. $r_1(\theta) = 1/2$, $r_2(\theta) = \cos 3\theta$, $I = [-\pi/9,\pi/9]$.

62. $r_1(\theta) = 3\cos\theta$, $r_2(\theta) = 1 + \cos\theta$, $\theta \in [-\pi/3,\pi/3]$.

63. $r_1 = \sin\theta$, $r_2 = \cos\theta$, $I = [0,\pi/4]$.

64. Recalculate the previous problem by writing the equations of the graphs of the two functions in cartesian coordinates.

65. Use the polar area formula to find the area of the annular region enclosed by circles of radii R_1 and R_2, $0 < R_1 < R_2$.

Recall that the area below the graph of a probability density function (pdf) is interpreted as a probability. Specifically, if $f(x)$ is a pdf of a random variable X, then the probability that X is between a and b is $\int_a^b f(x)\,dx$. In finance and economics, the area between two curves can be used to assess the materiality in choosing one model over another. We explore this idea in Exercises 46 - 48.

66. Suppose that the XYZ corporation is considering a \$200,000 investment. It is known with certainty that the most XYZ can lose is its initial investment of \$200,000 and the most it can profit is \$700,000. Analyst A believes the profit is uniformly distributed on the interval $[-200000, 700000]$. Analyst B believes the profit has density function $f_2(x) = \dfrac{x^2}{117,000,000}$.

    a. What is $f_1(x)$, the pdf based on Analyst A's assumptions?

    b. What is the area between $f_1(x)$ and $f_2(x)$ on the interval $[500, 700]$? What is the probabilistic interpretation of this area?

67. Under which analyst's assumptions is it more likely for XYZ to experience a positive profit from this investment?

68. Calculate the area between the graphs of the functions $xf_1(x)$ and $xf_2(x)$ on the interval $[-200, 700]$. What does this area represent?

69. Suppose $f(x)$ is a strictly increasing continuous function defined on all reals with the property that $f(0) = 0$. Then f possesses an inverse function g. Just as $\int_0^a f(x)\,dx$ is the area below the graph of f on the interval $[0, a]$, $\int_0^b g(y)\,dy$ is the area to the left of the graph of g along the (vertical) interval $[0, b]$.

    a. Assume a and b are positive. Draw a picture to prove that

    $$ab \le \int_0^a f(x)\,dx + \int_0^b g(y)\,dy.$$

    This is *Young's Inequality.*

    b. Based on the picture, formulate a conjecture about when the inequality is an equality.

70. Use the previous problem and the function $f(x) = x^k$, $k > 0$ to prove that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

if $a, b \geq 0$, $1 < p$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

71. a. Suppose f and g are two continuous functions, both of which are positive for all real x. Suppose $p > 1$ is a real number, and q is the unique real such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Suppose f and g have the additional properties:

$$\left(\int_a^b [f(x)]^p \, dx\right)^{1/p} = 1 \qquad \text{and} \qquad \left(\int_a^b [g(x)]^q \, dx\right)^{1/q} = 1.$$

Integrate both sides of the inequality in the previous problem to prove that

$$\int_a^b f(x)g(x)\, dx \leq 1.$$

b. Now suppose f and g satisfy all the conditions in part (a) except the last two. Prove that

$$\int_a^b f(x)g(x)\, dx \;\leq\; \left(\int_a^b [f(x)]^p \, dx\right)^{1/p} \cdot \left(\int_a^b [g(x)]^q \, dx\right)^{1/q}.$$

This result is known as *Hölder's Inequality*.

3.3 Distance Traveled in Space and Arc Length

In Section 3.1, we looked at an object which was moving in a straight line, and found that its displacement and distance traveled could be calculated by integrating its velocity $v(t)$ and the speed $|v(t)|$, respectively.

Suppose, however, that an object is moving along some curved path in a plane or in space. How do we use the velocity and/or speed of the object to calculate the displacement and the distance traveled? Amazingly, the answer is that, aside from writing things in terms of *vectors*, the formulas look exactly the same.

It's also true that the total distance traveled by an object is equal to the length of the curve (or line) that it travels along, provided that the object doesn't move back over points that it's already hit. This means that we can use the same techniques to calculate the *arc length* of a curve that we use in calculating the distance traveled.

In this section, we will use a small amount of material on vectors and vector-valued functions, such as the material in Appendix A. While this discussion could be put off until you take multi-variable Calculus, it is really not much more difficult than analyzing motion in a straight line.

Displacement and distance traveled:

Suppose that an object is moving in xyz-space, $\mathbb{R}^3$, in such a way that, at time t, its x-, y-, and z-coordinates are given by continuously differentiable functions $x(t)$, $y(t)$, $z(t)$. Then, the *position function* or *position vector* or, simply, *position* of the object at time t is

$$\vec{p}(t) \;=\; (x(t), y(t), z(t)).$$

When we say that $\vec{p} : [a, b] \to \mathbb{R}^3$ is continuously differentiable, you may wonder what is meant at the endpoints a and b of the closed interval, since you can't take two-sided limits there. We shall always mean that $\vec{p}$ is defined and differentiable on an open interval which contains $[a, b]$, and that $\vec{p}\,'$ is continuous when restricted to the closed interval $[a, b]$.

For a given t value, we sometimes think of $\vec{p}(t)$ as simply the point $(x(t), y(t), z(t))$, and, at other times, think of $\vec{p}(t)$ as the vector which is represented by the arrow from the origin to the point $\vec{p}(t)$; the context should always make it clear how we are thinking of $\vec{p}(t)$.

If the object is moving in the xy-plane, $\mathbb{R}^2$, we can still think of it as moving in xyz-space, but always with z-coordinate equal to 0. Thus, we will discuss motion in xyz-space, and that will also tell you what happens for objects moving in a plane; you take our discussion for motion in space and either set $z(t) = 0$ for all t, or just omit the z-coordinate. On the other hand, diagrams are much easier to produce and understand in the xy-plane; so our diagrams will typically show the 2-dimensional situation, and we leave to you to picture the 3-dimensional case.

The *velocity* (function or vector) of the object is

$$\vec{v}(t) \;=\; \vec{p}\,'(t) \;=\; (x'(t), y'(t), z'(t)).$$

While we shall not derive it here, we will mention that, if the velocity vector $\vec{v}(t)$ is drawn as initiating from the point $\vec{p}(t)$, and $\vec{v}(t) \neq \vec{0}$, then $\vec{v}(t)$ will be tangent to the curve that the object is moving along. See Figure 3.10.

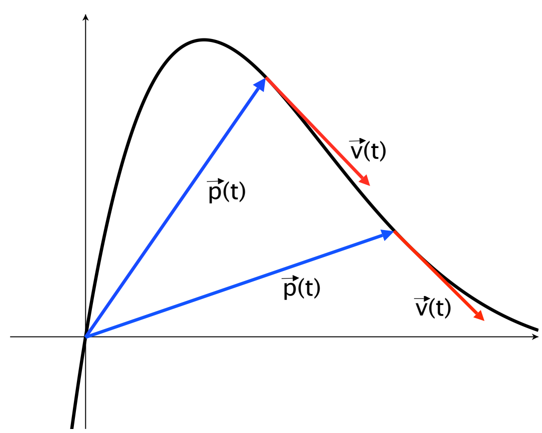


Figure 3.10: Velocity vectors are tangent to the curve defined by $\vec{p}(t)$.

As the velocity is the derivative of the position, if we are given $\vec{v}(t)$, we can anti-differentiate each component function, and obtain

$$\int \vec{v}(t)\, dt \;=\; (x(t) + C_x,\; y(t) + C_y,\; z(t) + C_z) \;=$$

$$(x(t), y(t), z(t)) + (C_x, C_y, C_z) \;=\; \vec{p}(t) + \vec{C},$$

where C_x, C_y, and C_z are constants.

Now, suppose that we have two times a and b, where $a \leq b$. Then, by definition, the definite integral from a to b of a vector-valued function is obtained by integrating each component separately; so,

$$\int_a^b \vec{v}(t)\,dt \;=\; \left(\int_a^b x'(t)\,dt,\ \int_a^b y'(t)\,dt,\ \int_a^b z'(t)\,dt\right).$$

By applying the Fundamental Theorem of Calculus to each component, we find that this last vector of integrals equals

$$\big(x(b)-x(a),\ y(b)-y(a),\ z(b)-z(a)\big) \;=\; \vec{p}(b)-\vec{p}(a).$$

The vector $\vec{p}(b)-\vec{p}(a)$ is the change in the position of the object between times $t=a$ and $t=b$; as in the case of motion in a line, this change in position is the *displacement* of the object. The magnitude of the displacement is the (straight line) distance between the points $\vec{p}(a)$ and $\vec{p}(b)$.

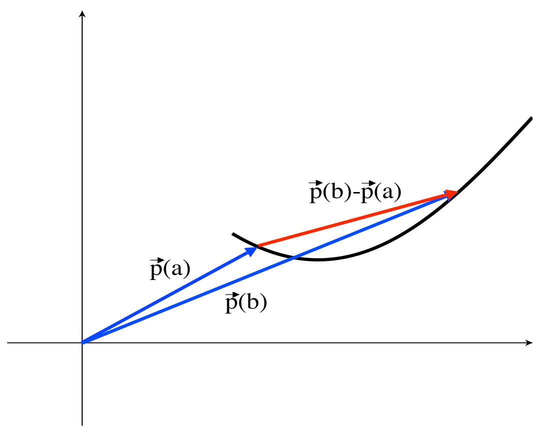


Figure 3.11: A typical displacement vector.

From our discussion above, we arrive at the more general vector version of Proposition 3.1.1:

Proposition 3.3.1. *If the velocity $\vec{v}=\vec{v}(t)$, as a function of time t, of an object is continuous on the interval $[a,b]$, then the* **displacement** *of the object between times $t=a$ and $t=b$ is given by*

$$\int_a^b \vec{v}(t)\,dt.$$

Example 3.3.2. A particle's velocity is given by $\vec{v}(t) = (e^t, \sqrt{2}, -e^{-t})$ meters per second, where t is in seconds. What is the displacement of the particle between times $t = 0$ and $t = 1$ seconds? What is the magnitude of the displacement?

Solution:

This is easy. We find the displacement is

$$\vec{p}(1) - \vec{p}(0) \;=\; \int_0^1 \vec{v}(t)\,dt \;=\; \int_0^1 (e^t, \sqrt{2}, -e^{-t})\,dt \;=\; (e^t, \sqrt{2}\,t, e^{-t})\Big|_0^1 \;=$$

$$(e, \sqrt{2}, e^{-1}) - (1, 0, 1) \;=\; (e-1, \sqrt{2}, e^{-1} - 1) \;\text{ meters.}$$

The magnitude $|\vec{p}(1) - \vec{p}(0)|$ of the displacement vector is

$$|\vec{p}(1) - \vec{p}(0)| \;=\; \left|(e, \sqrt{2}, e^{-1}) - (1, 0, 1)\right| \;=\; \sqrt{(e-1)^2 + (\sqrt{2})^2 + (e^{-1} - 1)^2} \;=$$

$$\sqrt{e^2 - 2e + 1 + 2 + e^{-2} - 2e^{-1} + 1} \;=\; \sqrt{(e - e^{-1})^2 - 2(e + e^{-1} - 3)} \;\text{ meters,}$$

where we have left the answer in this form for a reason that will be apparent in the next example.

If we look at the example above, and anti-differentiate $\vec{v}(t)$, we find that the position $\vec{p}(t)$ is given by

$$\vec{p}(t) \;=\; (e^t, \sqrt{2}\,t, e^{-t}) \;+\; \vec{C},$$

where $\vec{C}$ is a constant vector. This means that, between times 0 and 1 second, the particle moves from the point $(1, 0, 1) + \vec{C}$ to the point $(e, \sqrt{2}, e^{-1}) + \vec{C}$, not along a straight line, but rather along the curved path $\vec{p}(t) = (e^t, \sqrt{2}\,t, e^{-t}) + \vec{C}$.

The (straight line) distance between the initial and final points of the particle is the distance between $(e, \sqrt{2}, e^{-1}) + \vec{C}$ and $(1, 0, 1) + \vec{C}$; this is equal to the magnitude of the displacement $|\vec{p}(1) - \vec{p}(0)|$.

But, what if we want to know the total distance traveled by the particle **along the curved path** given by $\vec{p}(t) = (e^t, \sqrt{2}\,t, e^{-t}) + \vec{C}$? This distance traveled along the curved path **should** (and will) turn out to be bigger than $\sqrt{(e + e^{-1})^2 - 2(e + e^{-1} - 3)}$.

To determine how to calculate the distance traveled by an object, first think about approximating the distance traveled during a small interval of time, and then we add together the resulting small distances, and take the limit as the size of the time interval approaches zero. Not surprisingly, this results in an integral.

Suppose that we have two times $t_0 < t_1$, and let's think about what happens when $\Delta t = t_1 - t_0$ is small (close to 0). If Δt is close to 0, then the distance Δs that the object travels (along its possibly curved path) should approximately equal the distance along a straight line between the starting position $\vec{p}(t_0)$ and the ending position $\vec{p}(t_1)$; this straight line distance is $|\vec{p}(t_1) - \vec{p}(t_0)|$. Thus, we have

$$\Delta s \;\approx\; |\vec{p}(t_1) - \vec{p}(t_0)|.$$

We can rewrite this distance as

$$\Delta s \;\approx\; \left|\frac{\vec{p}(t_0 + \Delta t) - \vec{p}(t_0)}{\Delta t}\right| \Delta t,$$

where we used that Δt is positive. As $\Delta t \to 0$, $\big(\vec{p}(t_0 + \Delta t) - \vec{p}(t_0)\big)/\Delta t$ approaches $\vec{v}(t) = \vec{p}\,'(t_0)$.

Using differential notation and infinitesimal terminology, this means that the infinitesimal distance ds traveled by the object in the infinitesimal time interval dt is

$$ds = |\vec{v}(t)|\, dt.$$

The infinitesimal value ds is frequently referred to an *element of arc length*. Of course, to obtain the total distance traveled, we take a continuous sum of the infinitesimal distances traveled, i.e., we take an integral.

Proposition 3.3.3. *If the velocity $\vec{v} = \vec{v}(t)$, as a function of time t, of an object is continuous on the interval $[a, b]$, then the* **distance traveled** *by the object between times $t = a$ and $t = b$ is given by*

$$\int_{t=a}^{t=b} ds \;=\; \int_a^b |\vec{v}(t)|\, dt,$$

where $ds = |\vec{v}(t)|\, dt$.

Example 3.3.4. Let's return to the situation in Example 3.3.2. A particle's velocity is given by $\vec{v}(t) = (e^t, \sqrt{2}, -e^{-t})$ meters per second, where t is in seconds. What is the distance traveled by the particle between times $t = 0$ and $t = 1$ seconds?

Solution:

We calculate that the distance traveled:

$$\int_0^1 |\vec{v}(t)|\,dt \;=\; \int_0^1 \sqrt{(e^t)^2 + (\sqrt{2})^2 + (-e^{-t})^2}\;\,dt \;=\; \int_0^1 \sqrt{e^{2t} + 2 + e^{-2t}}\;\,dt \;=$$

$$\int_0^1 \sqrt{(e^t + e^{-t})^2}\;dt \;=\; \int_0^1 (e^t + e^{-t})\,dt \;=\; \left(e^t - e^{-t}\right)\Big|_0^1 \;=\; (e - e^{-1}) - (1 - 1) \;=$$

$$e - e^{-1}\;\text{ meters.}$$

Note that, since $e + e^{-1} - 3 > 0$, the distance traveled is greater than the magnitude of the displacement,

$$\sqrt{(e - e^{-1})^2 - 2(e + e^{-1} - 3)}\;\text{ meters,}$$

which we found in Example 3.3.2. In other words, the distance traveled along a curved path between two points is greater than the straight line distance between the points. Good!

If the velocity $\vec{v}(t)$ of an object is continuous on the interval $[a, b]$, then we may define the *distance traveled function* between times a and t, where $a \le t \le b$, to be

$$s(t) \;=\; \int_a^t |\vec{v}(z)|\,dz,$$

where z is just a dummy variable, which we introduce since t is now a limit of integration.

Then, the first part of the Fundamental Theorem of Calculus, Theorem 2.4.7, tells us that

$$\frac{ds}{dt} \;=\; |\vec{v}(t)|.$$

Definition 3.3.5. *The* **speed** *of an object, whose velocity function is continuous, is the magnitude of the velocity or, equivalently, the rate of change of the distance traveled, with respect to time.*

Remark 3.3.6. It is possible to weaken the assumptions that we made about our position and velocity functions, and still calculate the distance traveled via integration.

Assuming that you don't believe that objects can teleport, then any moving object should have a continuous position function. If $\vec{p}(t)$ is continuous, the distance traveled by an object during times t such that $a < t < b$ is the same as the distance traveled during times t such that $a \leq t \leq b$.

Thus, to calculate the distance traveled, it is enough for $\vec{p}(t)$ to be continuous on the closed interval $[a, b]$, continuously differentiable on the open interval (a, b), and for the integral of $|\vec{v}(t)| = |\vec{p}\,'(t)|$ on the open interval (a, b), as defined in Definition 2.5.11, to exist. This distance traveled is thus, again, simply

$$\int_a^b |\vec{v}(t)|\, dt,$$

provided that this integral exists.

We should mention that, while we assume, for physical reasons (non-teleportation), that the position function is continuous, it is fairly common in physics and engineering to allow the velocity of an object to be **discontinuous**.

For instance, a pitched baseball may be moving at 90 mph in one direction and then, when it's hit by a bat, moves in the other direction at around 110 mph. The change in velocity of the baseball is 200 mph, and for most practical purposes, that changes occurs instantly at the time when the ball is struck by the bat. Thus, we think of the velocity function as being discontinuous at the time at which the bat strikes the ball.

Of course, what really happens is that the bat strikes the ball, and the ball deforms while making contact with the bat for a very small amount of time. Hence, the velocity function of the ball is not truly discontinuous. However, it is so difficult to analyze exactly what happens to the ball's velocity (or, more precisely, the velocity of the center of mass of the ball; see Section 3.8) during the tiny time interval of contact with the bat that the situation is usually idealized and described as an instantaneous change in velocity of 200 mph.

This discussion does not merely apply to baseballs; it applies often when one object strikes another. If you jump into the air 1 foot, and land on the ground, then, just before you strike the ground, your speed downward is approximately 8 ft/s. After you hit the ground, your speed is zero. Once again, for most practical purposes, it is reasonable to say that you stop instantly.

If you think about it, if an object's velocity changes by some finite amount in zero time, then the acceleration of the object must be infinite, and so, by Newton's 2nd Law of Motion, the force acting on the object must be infinite. Thus, we typically say things like "the bat strikes the baseball with an instantaneous infinite force" or "the ground exerts an instantaneous infinite force on you when you land on it after jumping". Such instantaneous infinite forces are usually described via the *Dirac delta* "function".

What's really happening in these cases is that an extremely large force is acting in a complicated manner over an extremely small period of time. We just idealize the situation to the instantaneous, discontinuous, setting.

Example 3.3.7. Suppose that a particle moves in the xy-plane in such a way that its position $\vec{p}(t)$, in feet, at time t seconds, where $-1 \leq t \leq 1$, is given by

$$\vec{p}(t) \;=\; \left(t, \sqrt{1-t^2}\right) \;=\; \left(t, (1-t^2)^{1/2}\right).$$

Describe geometrically the path that the particle takes, and find the distance that the particle travels between $t=-1$ and $t=1$ seconds.

Solution:

At time t seconds, x-coordinate of the particle is $x=t$, and the y-coordinate is $y=\sqrt{1-t^2}$. Thus, (x,y) is a point on the path that the particle moves along if and only if $-1 \leq x \leq 1$, $y \geq 0$, and $y=\sqrt{1-x^2}$. Squaring both sides of this last equation yields $y^2 = 1-x^2$, or $x^2+y^2=1$. Therefore, the path of the particle lies on the circle of radius 1 foot, centered at the origin, but only the part where $y \geq 0$. Hence, the particle moves along the top half of the unit circle, centered at the origin.

We find the velocity

$$\vec{v}(t) \;=\; \vec{p}\,'(t) \;=\; \left(1,\; \frac{1}{2}\left(1-t^2\right)^{-1/2}(-2t)\right) \;=\; \left(1,\; \frac{-t}{\sqrt{1-t^2}}\right) \;\; \text{ft/s},$$

for $-1 < t < 1$. Note that the velocity vector does not exist when $t = \pm 1$.

In Figure 3.12, we have drawn the semicircle and indicated, in red, some velocity vectors. Note that the lengths of the velocity vectors do **not** match the speed; we had to scale the magnitude in order to fit things in a diagram of reasonable size. What is important for you to get from the diagram is the direction of the velocity, and that the magnitude of the velocity, the speed, gets larger as the particle gets closer to the points $(-1,0)$ and $(1,0)$, "becoming infinite" exactly at these endpoints of the semicircle.

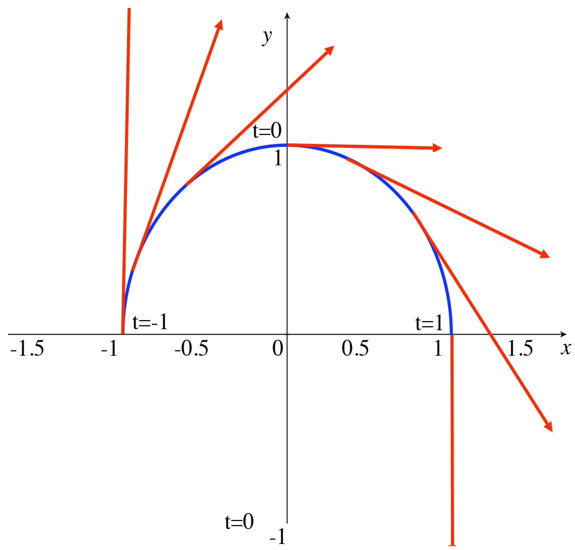


Figure 3.12: Velocity vectors have infinite magnitude when $t = x = \pm 1$.

We know from high school geometry how far the particle travels; the length/circumference of the top half of the unit circle is one half of circumference of the entire circle: $(2\pi \cdot 1)/2 = \pi$ feet. Let's make certain that the integral $\int_{-1}^{1} |\vec{v}(t)|\, dt$ gives us the same answer.

First, we find the speed, in ft/s:

$$|\vec{v}(t)| \;=\; \sqrt{(1)^2 \;+\; \left(\frac{-t}{\sqrt{1-t^2}}\right)^2} \;=\; \sqrt{\frac{1-t^2}{1-t^2} \;+\; \frac{t^2}{1-t^2}} \;=\; \frac{1}{\sqrt{1-t^2}}.$$

Thus, the distance traveled by the particle, between times $t = -1$ and $t = 1$ second, is given by

$$\int_{-1}^{1} |\vec{v}(t)|\, dt \;=\; \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, dt.$$

This is an improper integral (see Section 2.5) with two problem points: $t = -1$ and $t = 1$. Hence, we split the integral, and calculate

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, dt \;=\; \int_{-1}^{0} \frac{1}{\sqrt{1-t^2}}\, dt \;+\; \int_{0}^{1} \frac{1}{\sqrt{1-t^2}}\, dt \;=$$

$$\lim_{a\to -1^+} \int_{a}^{0} \frac{1}{\sqrt{1-t^2}}\, dt \;+\; \lim_{b\to 1^-} \int_{0}^{b} \frac{1}{\sqrt{1-t^2}}\, dt \;=$$

$$\lim_{a\to -1^+}\left(\sin^{-1} t\,\Big|_a^0\right) + \lim_{b\to 1^-}\left(\sin^{-1} t\,\Big|_0^b\right) = \big(0 - \lim_{a\to -1^+}\sin^{-1} a\big) + \big(\lim_{b\to 1^-}\sin^{-1} b - 0\big) =$$

$$-\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi \text{ feet.}$$

Whew! It sure took a while to verify that the length of a semicircle of radius 1 is π.

Arc length:

The term *length* in this context is usually augmented and is referred to as **arc length**, in order to emphasize that we mean the length along something which curves (i.e., along something which is composed of "arcs").

Example 3.3.8. If all we want is to use integration to verify that the arc length of a semicircle of radius 1 is equal to π, perhaps it would be better to think of a particle which moves around the semicircle at a different speed from that of the particle in Example 3.3.7.

Suppose that a particle moves in the xy-plane in such a way that its position $\vec{p}(t)$, in feet, at time t seconds, where $0 \leq t \leq \pi$, is given by

$$\vec{p}(t) = (-\cos t, \sin t).$$

Then, we see that the x- and y-coordinates of the particle are given by $x = -\cos t$ and $y = \sin t$, so that $x^2+y^2 = (-\cos t)^2+\sin^2 t = 1$. Hence, once again, the particle is always on the circle of radius 1, centered at the origin. We also see that $y = \sin t \geq 0$ for $0 \leq t \leq \pi$, that $\vec{p}(0) = (-1, 0)$, and that $\vec{p}(\pi) = (1, 0)$. Finally, it is important to note that $\vec{p}(t)$ is one-to-one, i.e., the particle is never at the same point at two (or more) different times.

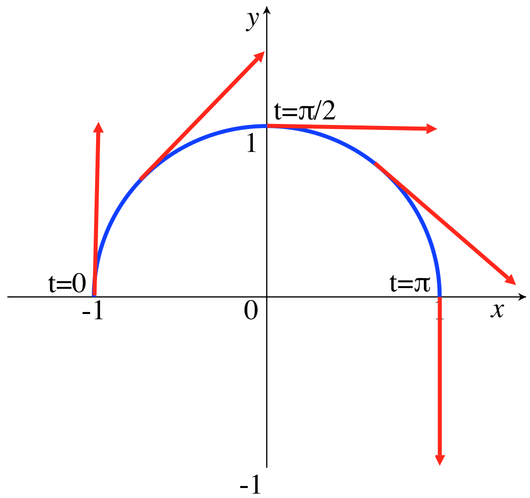


Figure 3.13: Moving around the semicircle with constant speed.

Thus, as in Example 3.3.7, the particle starts at $(-1, 0)$ and moves clockwise around the semicircle of radius 1, centered at the origin, ending at $(1, 0)$. However, the **speed** at which this particle is moving is very different from the speed of the particle in Example 3.3.7.

We find $\vec{v}(t) = \vec{p}\,'(t) = (\sin t, \cos t)$, and so the speed is

$$|\vec{v}(t)| \;=\; \sqrt{\sin^2 t + \cos^2 t} \;=\; 1 \;\; \text{ft/s}.$$

Hence, we find that our current particle moves with a constant speed of 1 ft/s.

Our integral for the distance that the particle travels, i.e., for the arc length of a semicircle of radius 1 is now:

$$\int_0^{\pi} |\vec{v}(t)|\, dt \;=\; \int_0^{\pi} 1\, dt \;=\; t\,\Big|_0^{\pi} \;=\; \pi \;\; \text{feet}.$$

Comparing this example with Example 3.3.7, we see that selecting/imagining a particle that's moving in the "right" way can make the calculation of the arc length of a curve **much** easier.

One of the important things that you should have gotten out of the last two examples is that, if you want the arc length of a curve, you can think of an object moving along the curve and calculate the distance traveled by the object, provided that the object does not travel along any portions of the curve more than once; this means that the position function should be *one-to-one*. We also saw that the calculation of the distance traveled can be made much easier by making a nice choice of a position function.

In fact, it doesn't matter whether or not we actually think of an object moving along a given curve. What's important is that we have a one-to-one, continuously differentiable function $\vec{p}(t)$ from a closed interval $[a,b]$, where $a<b$, into the xy-plane or into xyz-space, that plays the role of a position function, in that the points on the curve that we're interested in are precisely the points that you get from $\vec{p}(t)$, i.e., the curve whose arc length we want is the *range* of the function $\vec{p}$. The term *simple* is frequently used in this context to indicate that $\vec{p}$ is one-to-one.

We also require, for now, a condition, *regularity*, which, in terms of motion, would say that the object never stops at times in the open interval (a,b); this technical condition is very useful mathematically.

Definition 3.3.9. *A* **simple regular parameterization of a curve in** $\mathbb{R}^n$ *(e.g., in $\mathbb{R}^2$, the xy-plane, or in $\mathbb{R}^3$, xyz-space) is a one-to-one, continuously differentiable function $\vec{p}$ from a closed interval $[a,b]$, where $a<b$, into $\mathbb{R}^n$, such that, for all t in $[a,b]$, $\vec{p}\,'(t)\neq\vec{0}$.*

The range of a simple regular parameterization, that is, the set of points that $\vec{p}(t)$ "passes through", is called a **simple regular curve***.*

We say that a simple regular parameterization $\vec{p}$, with domain $[a,b]$, **parameterizes** *the simple regular curve which is its range, and that the parameterization* **starts at the point $\vec{p}(a)$ and ends at the point $\vec{p}(b)$***.*

Remark 3.3.10. We have defined a simple regular curve C as a set of points in $\mathbb{R}^2$ or $\mathbb{R}^3$, and you should have an intuitive idea of what a curve looks like, but how can we rigorously, mathematically, say when a set of points matches your intuition for what a curve is?

What we have said in Definition 3.3.9 is that a rigorous definition of a simple regular curve is that it is the range of a simple regular parametrization; but keep in mind that, as we saw in the previous two examples, a simple regular curve, like a semicircle, can have more than one simple regular parameterization. In fact, it is not difficult to show that any simple regular curve has an infinite number of different simple regular parameterizations. Nonetheless, as we saw in the previous two examples, the curve itself has a length which is independent of the simple regular parameterization, i.e., $\int_a^b |\vec{p}\,'(t)|\,dt$ will give the same number, regardless of what simple regular parameterization you use for a given simple regular curve.

Finally, we should mention that, in many references, a *curve* is defined to be the parameterization $\vec{p}$, not the range of this function. This is mathematically convenient, but does not agree with most people's intuition for what a "curve" is, and so we will not use the terminology.

What we have seen in our discussion and examples is that an infinitesimal piece of arc length, ds is equal to $|\vec{p}\,'(t)|\,dt$, and so we have:

Proposition 3.3.11. *Suppose that C is a simple regular curve in $\mathbb{R}^2$ or in $\mathbb{R}^3$. Then, for any simple regular parameterization $\vec{p}(t)$, with domain $[a,b]$, of C, the arc length of C is*

$$\int_{t=a}^{t=b} ds \;=\; \int_a^b |\vec{p}\,'(t)|\,dt.$$

Curves in the plane:

In our next examples of arc length calculations, we are going to look at simple regular curves in $\mathbb{R}^2$, the xy-plane. In the plane, there are three common ways to specify simple regular parameterizations.

One way is to specify appropriate coordinate (or component) functions $x = x(t)$ and $y = y(t)$, and then the parameterization is, of course, $\vec{p}(t) = (x(t), y(t))$. Then, Proposition 3.3.11 tells us:

Proposition 3.3.12. *If $\vec{p}(t) = (x(t), y(t))$, for $a \leq t \leq b$, is a simple regular parameterization of a curve C in $\mathbb{R}^2$, then*

$$ds \;=\; \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt,$$

and

$$\text{arc length of } C \;=\; \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt.$$

However, you are probably most familiar with curves described as the graph of $y = f(x)$, where $a \leq x \leq b$, and where f is continuously differentiable. There is an obvious parameterization here: $x = t$ and $y = f(t)$, i.e., $\vec{p}(t) = (t, f(t))$. This $\vec{p}$ is clearly one-to-one, since $(t_1, f(t_1)) = (t_2, f(t_2))$ immediately implies that $t_1 = t_2$. In addition, using that $t = x$, we have

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \;=\; \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{df}{dx}\right)^2} \;=\; \sqrt{1 + \left(\frac{df}{dx}\right)^2} \;>\; 0.$$

Therefore, if f is continuously differentiable on an open interval containing $[a, b]$, then $\vec{p}(t) = (t, f(t))$ is a simple regular parameterization.

Proposition 3.3.13. *If $y = f(x)$ is continuously differentiable on an open interval containing $[a, b]$, then*

$$ds \;=\; \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

and the arc length of the graph of f, for $a \leq x \leq b$, is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx.$$

While it's less common, a curve may also be described by specifying x as a function of y, instead of giving y as a function of x. Of course, all this does is interchange the roles of x and y.

Proposition 3.3.14. *If $x = f(y)$ is continuously differentiable on an open interval containing $[a, b]$, then*

$$ds \;=\; \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\; dy,$$

and the arc length of the graph of f, for $a \leq y \leq b$, is

$$\int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\; dy.$$

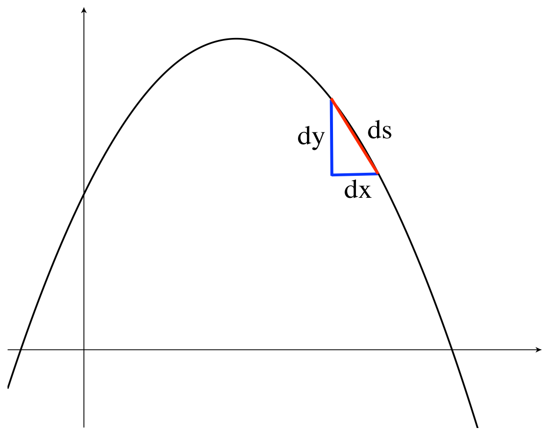


Figure 3.14: Infinitesimally, arc length and straight line distance are "equal".

All three of the formulas for arc length above are frequently thought of as stemming from an infinitesimal version of the Pythagorean Theorem.

That is, if we perform algebra with the differentials, assuming that dt, dx, and dy are positive, then what we are integrating in Proposition 3.3.12, Proposition 3.3.13, and Proposition 3.3.14 is always

$$ds \;=\; \sqrt{(dx)^2 \;+\; (dy)^2},$$

which is frequently written as

$$(ds)^2 \;=\; (dx)^2 \;+\; (dy)^2.$$

Before we give a couple of examples, we should mention that the anti-derivatives that arise in calculating arc length are usually very difficult, or impossible, to find (as elementary functions). Such definite integrals can be approximated using methods such as Simpson's Rule (Definition 2.6.6). The examples below are **very** special.

Example 3.3.15. Find the arc length of the graph of $y = \dfrac{2}{3}\,x^{3/2}$, for $0 \leq x \leq 8$.

Solution:

We use Proposition 3.3.13 and find

$$\text{arc length} \;=\; \int_0^8 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\;dx \;=\; \int_0^8 \sqrt{1+(x^{1/2})^2}\;dx \;=\; \int_0^8 \sqrt{1+x}\;dx.$$

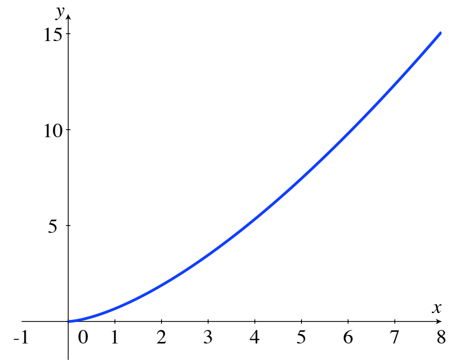


Figure 3.15: The graph of $y = 2x^{3/2}/3$.

We make the substitution $u = 1 + x$, which tells us that $du = dx$, and that, as x goes from 0 to 8, u goes $1 + 0 = 1$ to $1 + 8 = 9$. Thus, we have that

$$\text{arc length} \;=\; \int_1^9 u^{1/2}\,du \;=\; \frac{u^{3/2}}{3/2}\Big|_1^9 \;=\; \frac{2}{3}\left(9^{3/2} - 1^{3/2}\right) \;=\; \frac{52}{3}.$$

Example 3.3.16. Find the arc length of the graph of $x = \dfrac{e^y + e^{-y}}{2} = \cosh(y)$, for $0 \leq y \leq \ln 3$.

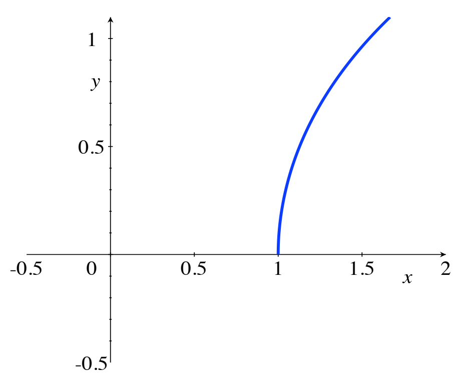


Figure 3.16: The graph of $x = (e^y + e^{-y})/2$.

Solution:

We use Proposition 3.3.14 and find

$$\text{arc length} \;=\; \int_0^{\ln 3}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \;=\; \int_0^{\ln 3}\sqrt{1+\left(\frac{e^y-e^{-y}}{2}\right)^2}\,dy \;=$$

$$\int_0^{\ln 3}\sqrt{\frac{4}{4}+\frac{(e^y)^2-2+(e^{-y})^2}{4}}\,dy \;=\; \int_0^{\ln 3}\sqrt{\frac{(e^y)^2+2+(e^{-y})^2}{4}}\,dy \;=$$

$$\int_0^{\ln 3}\sqrt{\left(\frac{e^y+e^{-y}}{2}\right)^2}\,dy \;=\; \int_0^{\ln 3}\frac{e^y+e^{-y}}{2}\,dy \;=\; \frac{1}{2}\left(e^y-e^{-y}\right)\Big|_0^{\ln 3} \;=$$

$$\frac{1}{2}\left[(3-3^{-1})-(1-1)\right] \;=\; \frac{4}{3}.$$

If some finite collection of simple regular curves intersect each other in a finite number of points, then the arc length of all of the curves combined is simply the sum of the arc lengths of the individual curves; this is because the overlap at a finite number of points does not have any effect on the total arc length. We call the union of a finite number of simple regular curves, which may intersect each other in a finite number of points, a **piecewise-simple regular curve**.

We can find arc lengths of some piecewise-simple regular curves by using a parameterization $\vec{p}(t)$ that is not one-to-one, but for which a finite number of points might be repeated for a finite number of t values.

Example 3.3.17. Verify, using integration, that the arc length of a circle of radius $r>0$ is $2\pi r$.

Solution:

Consider the parameterization of the circle of radius r, centered at the origin, given by

$$\vec{p}(t) \;=\; (r\cos t, r\sin t) \;=\; r(\cos t,\sin t), \quad \text{for} \quad 0\le t\le 2\pi.$$

This is **not** a simple regular parameterization, since the function is not one-to-one: $\vec{p}(0) = \vec{p}(2\pi) = (1,0)$. However, this overlap at $(1,0)$ is the only problem; $\vec{p}(t)$ is continuously differentiable, its range is the circle in question, the only "repeated point" occurs when $t = 0$ and $t = 2\pi$, and

$$|\vec{p}\,'(t)| \;=\; |r(-\sin t, \cos t)| \;=\; r\,|(-\sin t, \cos t)| \;=\; r\sqrt{(-\sin t)^2 + (\cos t)^2} \;=\; r \neq 0.$$

Thus, we claim that the arc length is given by

$$\int_0^{2\pi} |\vec{p}\,'(t)|\,dt \;=\; \int_0^{2\pi} r\,dt \;=\; rt\Big|_0^{2\pi} \;=\; 2\pi r,$$

even though the parameterization is not one-to-one.

Why is this calculation correct? Because we can look at the circle C in question as the union of the top semicircle C_1 (including the endpoints at $(-r, 0)$ and $(r, 0)$) and the bottom semicircle C_2 (including the endpoints). Both C_1 and C_2 are simple regular curves, parameterized by the restrictions of $\vec{p}$ to the intervals $[0, \pi]$ and $[\pi, 2\pi]$, respectively. Thus, C is piecewise-simple regular curve and its arc length is the sum of the arc lengths of C_1 and C_2, i.e.,

$$\int_0^{\pi} |\vec{p}\,'(t)|\,dt \;+\; \int_{\pi}^{2\pi} |\vec{p}\,'(t)|\,dt \;=\; \int_0^{2\pi} |\vec{p}\,'(t)|\,dt,$$

which is what we found to equal $2\pi r$.

Parameterizing by arc length:

Suppose that $\vec{p} : [a, b] \to \mathbb{R}^n$ is a simple regular parameterization of a curve C, which has length L. Just as we defined the distance traveled function (immediately before Definition 3.3.5), we can define the *arc length function* $l : [a, b] \to [0, L]$ by

$$s = l(t) \;=\; \int_a^t |\vec{p}\,'(u)|\,du;$$

thus, $s = l(t)$ gives the arc length along C from the point $\vec{p}(a)$ to the point $\vec{p}(t)$.

As before, the first part of the Fundamental Theorem of Calculus, Theorem 2.4.7, tells us that

$$l'(t) \;=\; \frac{ds}{dt} \;=\; |\vec{p}\,'(t)|.$$

As $\vec{p}$ is a regular parameterization, $|\vec{p}\,'(t)| > 0$, and so the Inverse Function Theorem implies that l has a monotonically increasing, differentiable inverse function, i.e., there exists a differentiable function $t = m(s)$ such that $m(0) = a$, $m(L) = b$, and $l(m(s)) = s$, for all s in the interval $[0, L]$, and $m(l(t)) = t$, for all t in the interval $[a, b]$. From the Chain Rule (or the formula for the derivative of an inverse function), we have that

$$m'(s) \;=\; \frac{1}{l'(m(s))} \;=\; \frac{1}{|\vec{p}\,'(m(s))|}$$

It is easy to check that the composed function $\vec{q} = \vec{p} \circ m : [0, L] \to \mathbb{R}^n$ is again a simple regular parameterization of C, and has the property that $\vec{q}(s)$ is the unique point on the curve C which is distance (along C) s away from $\vec{q}(0) = \vec{p}(a)$.

Definition 3.3.18. *A parameterization such as $\vec{q}$, where the parameter measures arc length along C, is said to be a* **parameterization by arc length**.

Note that if $\vec{q} : [0, L] \to \mathbb{R}^n$ is a simple regular parameterization, parameterized by arc length, then, applying the Chain Rule (to each component of the vector), we find

$$|\vec{q}\,'(s)| \;=\; |(\vec{p} \circ m)'(s)| \;=\; |m'(s)\vec{p}\,'(m(s))| \;=\; |m'(s)|\,|\vec{p}\,'(m(s))| \;=$$

$$\frac{1}{|\vec{p}\,'(m(s))|} \cdot |\vec{p}\,'(m(s))| \;=\; 1.$$

This, of course, gives us what it had better give us: if we use $\vec{q}$ to define the arc length function, we get

$$\int_0^s |\vec{q}\,'(u)|\, du \;=\; \int_0^s 1\, du \;=\; u\Big|_0^s \;=\; s,$$

which, in words, says that if you calculate the arc length between the starting point of a (parameterized) simple regular curve and the point that's the distance s away (along C) from the starting point, then what you get is s. If that sounds stupidly obvious, GOOD.

One final remark on parameterizing by arc length: if we return to the situation of an object moving along a curve, with a simple regular position function $\vec{p} = \vec{p}(t)$, where t is time, then the parameter t is also the arc length (so that $\vec{p}(t)$ is a parameterization by arc length) if and only if, for all t, $|\vec{p}\,'(t)| = 1$. Thus, a parameterization by arc length is also referred to as a *parameterization with unit speed, or speed* 1.

More general parameterized curves:

If we combine parameterizations like those in Example 3.3.7 and in Example 3.3.17, we find that we can deal with curves that are the ranges of parameterizations which may fail to satisfy various conditions at a finite number of points. We don't need for our parameterizations to have **closed** intervals as their domains. We also don't require that $\vec{p}$ is continuously differentiable everywhere, and so $\vec{p}\,'$ need not be defined at some points, and even the improper integral (recall Section 2.5) of $|\vec{p}\,'(t)|$ is not automatically guaranteed to exist.

All of these things make the definition below very technical. You need to recall our most general notion of integrating over fairly general sets of points from Definition 2.5.11. The term "rectifiable", used below, is a technical term which implies that a reasonable notion of arc length exists for the curve in question.

Definition 3.3.19. *A* **rectifiable parameterization of a curve in** $\mathbb{R}^n$ *is a continuous function $\vec{p}$ from an interval I, whose interior is non-empty, into $\mathbb{R}^n$, such that there exists a finite set of points F (which could be empty) in I such that, the restriction of $\vec{p}$ to the points of I which are not in F (i.e., to the set $I - F$) is one-to-one, continuously differentiable, has an everywhere non-zero derivative, and such that the (possibly) improper integral $\int_I |\vec{p}\,'(t)|\,dt$ exists.*

The range of a rectifiable parameterization is a **rectifiable, piecewise-regular, curve**.

Of course, we have:

Proposition 3.3.20. *Suppose that C is a rectifiable, piecewise-regular, curve. Then, for any rectifiable parameterization $\vec{p}(t)$, with domain I, of C, the arc length of C is*

$$\int_I |\vec{p}\,'(t)|\,dt.$$

3.3.1 Exercises

You are given the velocity function $\vec{v}(t)$ of a object in Exercises 1 - 6. Calculate the displacement of the object over the given time interval.

1. $\vec{v}(t) = (3\sin 2\pi t, 3\cos 2\pi t, 4t),\ [0, 2]$.

2. $\vec{v}(t) = (t^2 + t, 4t^3, 0),\ [0, 3]$.

3. $\vec{v}(t) = (3 + 4t, 5 - 7t, 11t),\ [-3, 7]$.

4. $\vec{v}(t) = (-15, t + 4, \sinh t),\ [-5, 8]$.

5. $\vec{v}(t) = (|t|, |t|, |t|),\ [-3, 3]$.

6. $\vec{v}(t) = (t\ln t, 1/t, e^{-t}),\ [1, 5]$.

Say whether the given map is a simple regular parameterization of a curve. If it's not, say why it is not.

7. $\vec{\lambda}(t) = (2t^3, 16, \cos t),\ t \in [-10, 10]$.

8. $\vec{\alpha}(t) = (\cos t, |t + 2|, \sin t),\ t \in [-\pi, 0]$.

9. $\vec{u}(t) = (\cos t, |t + 2|, \sin t),\ t \in [0, \pi]$.

10. $\vec{w}(t) = (1, t, t^2),\ t \in [-12, 20]$.

11. $\vec{h}(t) = (t^3, t^3, t^3),\ t \in [-5, 5]$.

In each of Exercises 12 through 14, give an arc length parameterization of the given curve.

12. $\vec{\alpha}(t) = (\cos 5t, \sin 5t),\ t \in [0, 2\pi/5]$.

13. $\vec{w}(t) = (3 + 6t, 2 - 5t, 7 + t),\ t \in [0, 4]$.

14. $\vec{z}(t) = \big(t^3/3, 0, 3 + (t^2/2)\big),\ t \in [1, 3]$.

15. Suppose $\phi : [a, b] \to \mathbb{R}^n$ is a regular curve parameterized by arc length. Show that the length of the curve is $b - a$.

16. A regular curve $\alpha : [a, b] \to \mathbb{R}^n$ is *closed* if $\alpha(a) = \alpha(b)$. Note that α must still be one-to-one everywhere else.

a. If α is a closed regular position function, show that the total displacement over the interval is $\vec{0}$.

b. Give a simple example that shows that while the displacement is zero, the total distance traveled may be non-zero.

You are given the position function of a particle. Calculate the total distance traveled by the particle over the given time interval.

17. $\vec{p}(t) = (-t, t, 3)$, $t \in [0, \pi/4]$.

18. $\vec{p}(t) = (t, 12, 0.5e^{2t})$, $t \in [0, 1]$.

19. $\vec{p}(t) = (\sin 3t, \cos 3t, 2t^{3/2})$, $t \in [0, \pi]$.

Calculate the arc length of the graph of the function over the interval I.

20. $f(x) = ax + b$, $a \neq 0$, $I = [c, d]$.

21. $g(x) = \ln(\sin x)$, $I = [\pi/4, \pi/2]$.

22. $f(y) = e^{ay}$, $a > 0$, $I = [0, 1]$.

23. $h(y) = 5 \ln y$, $I = [1, 5]$.

24. $f(x) = \dfrac{x^3}{12} + \dfrac{1}{x}$. $I = [1, 2]$.

25. Generalize the previous problem. Let $a > 0$ and $f(x) = \dfrac{x^3}{6\sqrt{a}} + \dfrac{\sqrt{a}}{2x}$, $I = [1, 2]$ and calculate the arc length of the graph.

26. Suppose $\vec{p}(t) = (a \cos t, a \sin t, b \cos t, b \sin t)$, $t \in [0, 2\pi]$ is the position function of a particle traveling in $\mathbb{R}^4$. What is the total distance traveled?

27. Parameterize the curve in the previous example by arc length.

28. What is the range of the curve (t^3, t^3), $t \in \mathbb{R}$? Is this curve regular?

29. Is the path $(t, |t|)$ regular? Is it rectifiable on the interval $[-1, 1]$?

30. The intersection of a sphere centered at the origin with radius R and the plane $z = z_0$, $-R < z_0 < R$ is a circle. Give a parameterization of this circle and calculate its circumference in terms of z_0 and R.

31. Suppose you make a journey (on the Earth) consisting of three legs. Assume the Earth is a sphere centered at the origin with radius R. You start at the North Pole and travel to a point on the equator along a line of longitude. You then walk along the equator through 90 degrees of longitude. Finally, you travel back along the equator.

a. Give a piecewise differentiable parameterization for your journey.

b. What is the total distance traveled?

32. In *Differential Calculus*, we defined the *curvature*, κ, function. If α is a regular parameterization, then we set $\kappa(t) = \left|\dfrac{T'(t)}{\alpha'(t)}\right|$ where $T(t)$ is the unit tangent vector. Prove that if α is parameterized by an arc length parameter s, then $\kappa(s) = \left|\dfrac{dT}{ds}\right|$.

33. Suppose C is the range of some simple regular curve $\phi : [a, b] \to \mathbb{R}^3$. Suppose $\psi : [c, d] \to \mathbb{R}^3$ is another simple regular parameterization of C. We'd like to make sure that the arc length of C is the same whether we use ϕ or ψ.

    a. Assume without loss of generality that $\phi(a) = \psi(c)$ and $\phi(b) = \psi(d)$. Let $f : [a, b] \to [c, d]$ be the function $f = \psi^{-1} \circ \phi$. Let $u = f(t)$ and show that $\dfrac{d\phi}{dt} = \dfrac{d\psi}{du}\dfrac{du}{dt}$.

    b. Carefully justify the equality:

$$\int_a^b |\phi'(t)|\,dt = \int_c^d |\psi'(u)|\,du.$$

34. Assume the bottom of a tire with radius r is initially positioned at the origin. The bottom of the tire is marked with a red piece of tape. As the tire rolls clockwise, the tape traces out a *cycloid*. The figure below shows a cycloid with $r = 2$.

    a. Let θ be the angle between the angle the tape makes with the vertical and let $\vec{a}(\theta)$ be the vector from the center of the tire to the piece of tape. What is $\vec{a}(\theta)$?

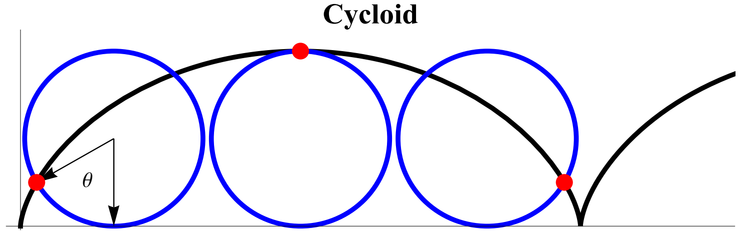
    b. Let $\vec{b}(\theta)$ be the vector from the origin to the center of the tire. What is $\vec{b}(\theta)$?

    c. Using vector addition, determine the position function, $\vec{p}(\theta)$ of the piece of tape.

    d. What is the distance traveled by the piece of tape from $0 \le \theta \le 2\pi$?

    e. Show that the velocity vector vanishes when $\theta = 2k\pi$, k an integer.

Cycloid



35. What is the arc length of the cycloid between two consecutive cusps?

36. Suppose that an outer circle orbits clockwise along the outside of an inner circle. This is a common gear configuration. As in the previous problem, suppose a point on the outer circle is marked in red. The path traced about by the mark is called an *epicycloid*. Suppose the inner and outer circles have radii r_1 and r_2, respectively and that the outer circle is initially positioned on top of the inner circle with the marker touching the inner circe.

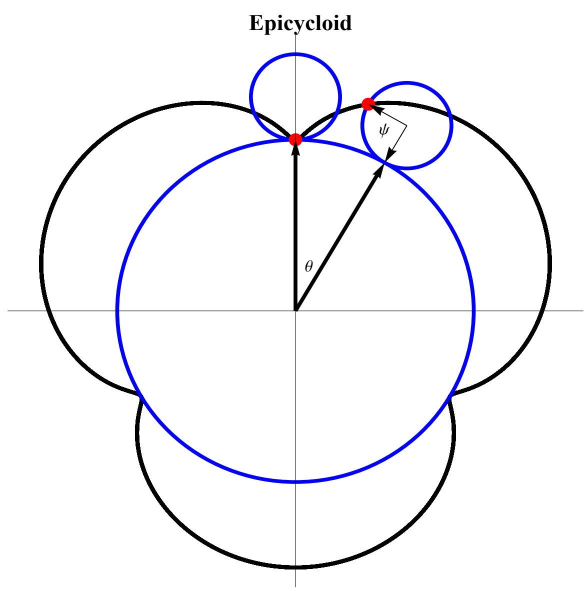
    a. Let ψ be the angle through which the outer circle rotates, and θ the angle between the y-axis and the vector connecting the centers of the two circles, measured clockwise. Show that the position vector of the center of the outer circle is given by $\vec{a}(\theta) = ((r_1 + r_2)\sin\theta, (r_1 + r_2)\cos\theta))$.

    b. Show that the vector from the center of the outer circle to the red marker is $\vec{b}(\psi) = (-r_2 \sin\frac{(r_1 + r_2)\psi}{r_1}, -r_2 \cos\frac{(r_1 + r_2)\psi}{r_1})$.

    c. Justify the equation $r_1\theta = r_2\psi$.

    d. Conclude that the position function of the red mark is given by

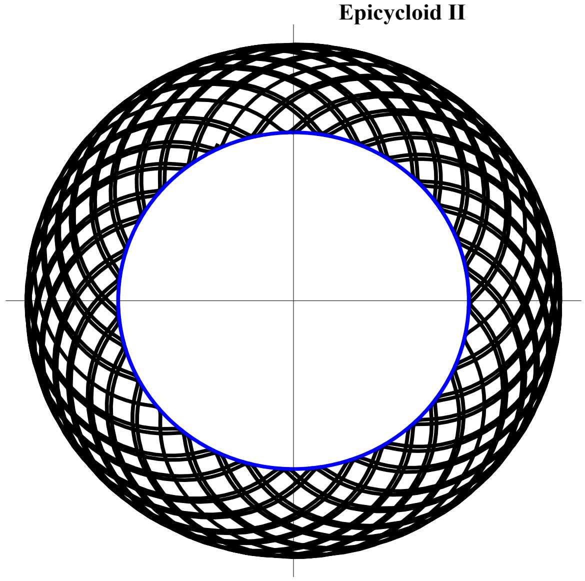
$$\vec{p}(\psi) = \left((r_1 + r_2)\sin(\frac{r_2\psi}{r_1}) - r_2 \sin\frac{(r_1 + r_2)\psi}{r_1}, (r_1 + r_2)\cos(\frac{r_2\psi}{r_1}) - r_2 \cos\frac{(r_1 + r_2)\psi}{r_1}\right).$$



37. Calculate the velocity vector of the epicycloid in the previous problem.

38. A curve $c(t)$ with domain $(-\infty, \infty)$ is periodic if there exists a constant t_0 such that $c(t) = c(t - t_0)$. An epicycloid may fail to be periodic, as suggested by the picture below.

Determine a necessary and sufficient condition for an epicycloid to be periodic. Hint: consider the relationship between r_1 and r_2.



39. Suppose $c(\psi)$ is a parameterization of an epicycloid with infinitely many cusps and that $r_1 = 1$. Prove that given $\epsilon > 0$, there exists ψ_0 such that $c(\psi_0)$ is a cusp and the arc length between the points $p = (1, 0)$ and $c(\psi_0)$ is less than ϵ. Note: there's nothing special about the point $(1, 0)$. Given an arbitrary point p on the inner circle, the above argument holds. We say the cusps are *dense* on the inner circle.

40. Let $c(t) = (t, t\sin(1/t))$ for $t \neq 0$ and let $c(0) = 0$.

    a. Show that c is everywhere continuous.

    b. Show that c is not a rectifiable parameterization.

41. If a small circle of radius r_2 revolves around the inside of a larger circle with radius r_1, the curve defined by the path of a fix point on the inner cycloid is a *hypocycloid*. A parameterization of a hypocycloid is given below.

$$x(\theta) = (r_1 - r_2)\cos\theta + r_2\cos\left(\frac{(r_1 - r_2)\theta}{r_2}\right)$$
$$y(\theta) = (r_1 - r_2)\sin\theta - r_2\sin\left(\frac{(r_1 - r_2)\theta}{r_2}\right)$$

Calculate $x'(\theta)$ and $y'(\theta)$.

42. Show that in a hypocycloid where the ratio of r_1 to r_2 is 4 to 1, the coordinates satisfy $x^{2/3} + y^{2/3} = C$, where C is some constant.

43. **Students not familiar with the dot product should skip this problem.** Suppose p and q are two distinct points in $\mathbb{R}^3$. We'd like to confirm our intuition that the straight line segment connecting these two points is the shortest curve among those with endpoints p and q. More precisely, we'll show that any curve between these two points must be at least as long as the straight line segment between them.

    a. Let $\vec{c} : [a, b] \to \mathbb{R}^3$ be any simple regular curve such that $c(a) = p$ and $c(b) = q$. Let $\vec{v}$ be any unit vector (so, $|\vec{v}| = 1$). Prove that $(q - p) \cdot \vec{v} = \int_a^b \vec{c}\,'(t) \cdot \vec{v}\, dt$.

    b. Prove that $\int_a^b \vec{c}\,'(t) \cdot \vec{v}\, dt \leq \int_a^b |\vec{c}\,\prime(t)|\, dt$. Hint: one way to do this is to use the fact that $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos\theta$.

    c. Let $v = \dfrac{p - q}{|p - q|}$ and conclude that

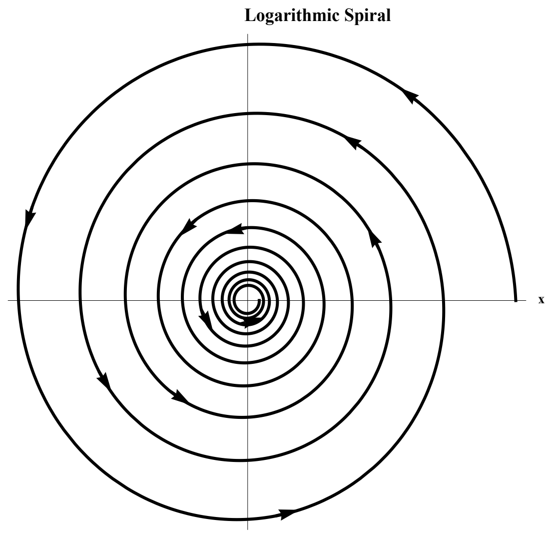
    $$|p - q| \leq \int_a^b |\vec{c}\,'(t)|\, dt.$$

    In other words, the length of an arbitrary curve from p to q is at least as long as the straight line segment between p and q.

44. A curve may be parametrized for unbounded time, and yet still have finite length. Let $\vec{p}(t) = (ae^{-bt} \cos t, ae^{-bt} \sin t)$ where $a, b > 0$ and $t \in \mathbb{R}$. The range of this parameterization is called the *logarithmic spiral*.

    a. Calculate $|\vec{p}\,'(t)|$.

    b. What is $\displaystyle\lim_{t\to\infty} \int_0^t |\vec{p}\,'(u)|\, du$?



45. It's often more convenient to parameterize a curve in polar coordinates rather than Cartesian coordinates. Recall that the relationship between the two coordinate systems is given by the equations:

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

   a. Suppose $c(\theta) = (x(\theta), y(\theta)) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$. This means the distance from a point on the curve to the origin, r, is a function of θ. Calculate $dx/d\theta$ and $dy/d\theta$ via the chain and product rules.

   b. Show that

$$\text{arc length} = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Calculate the arc length of the polar parameterized curves in Exercises 46 through 49 over the given interval.

46. $r = 1 + \sin\theta$, $\theta \in [0, 2\pi]$.

47. $r = C$, $C > 0$ a constant, $\theta \in [0, 2\pi]$. What familiar curve is this?

48. $r = 3\cos\theta$, $\theta \in [0, \pi/4]$.

49. $r = \theta$, $\theta \in [0, 2\pi]$. This curve is called the *Archimedean spiral.*

3.4 Area Swept Out and Polar Coordinates

In this section, we discuss how a parameterized curve in the plane leads to a notion of cumulative area "swept out" as points move along the curve. This notion is somewhat analogous to "total distance traveled", in that moving back over the same points gets counted each time you hit the points, not counted just once and/or not subtracted when moving in the opposite direction.

We can, however, use the swept out area to calculate the area of a region, provided that we make certain that we don't hit the same points more than once during our sweep or, really, that any overlap occurs for a finite number of parameter values.

We then apply this to the specific case of curves and regions that are described in terms of polar coordinates.

As in the previous section, we will assume that we have a curve, now in the xy-plane, that is described parametrically; thus, we have a function $\vec{p} : [t_0, t_1] \to \mathbb{R}^2$, where $[t_0, t_1]$ is a closed interval in $\mathbb{R}$. This means that we have functions $x = x(t)$ and $y = y(t)$ such that $\vec{p}(t) = (x(t), y(t))$. In our current setting, we do not need to be as restrictive as we were when discussing arc length. We don't need a simple, or piecewise-simple, regular parameterization; it is enough for us to assume that $x(t)$ and $y(t)$ extend to continuously differentiable functions on an open interval containing $[t_0, t_1]$. It is frequently helpful to imagine a particle moving in the xy-plane and to think that $x(t)$ and $y(t)$ give the x- and y-coordinates of the particle at time t.

We think of $\vec{p}(t)$ in two different ways: as a point, with coordinates $x(t)$ and $y(t)$, and as a vector, represented by the arrow from the origin to the point $(x(t), y(t))$. The context will always make it clear how we are thinking of $\vec{p}(t)$. Whether you think of $\vec{p}(t)$ as the position vector of a particle or as a purely mathematical point, for each t value in the interval $[a, b]$, we wish to consider the line segment from the origin to the point $\vec{p}(t) = (x(t), y(t))$; we will denote this line segment, including the endpoints, by $\ell_{\vec{p}(t)}$. Note that, if $\vec{p}(t) = \vec{0}$, then the "line segment" $\ell_{\vec{p}(t)}$ is really just a point (the origin).

As t changes, as the "particle" moves, the line segment $\ell_{\vec{p}(t)}$ "sweeps out" area in the xy-plane; see Figure 3.17. If we assume that all line segments $\ell_{\vec{p}(t)}$ intersect each other only at the origin, then this swept out area would simply be the area of the region which is the union of all of the $\ell_{\vec{p}(t)}$.

How do you calculate the area swept out? As you no doubt noticed in Figure 3.17, we labeled the points on our parameterized curve as $(x(t), y(t))$ and $(x(t+h), y(t+h))$. You may have guessed, correctly, that we are going to estimate the area of the "sector", the area swept out along the curve, between "times" t and $t+h$, and then pass to the integral by seeing what happens as h approaches 0, i.e., as we change t by an infinitesimal amount.

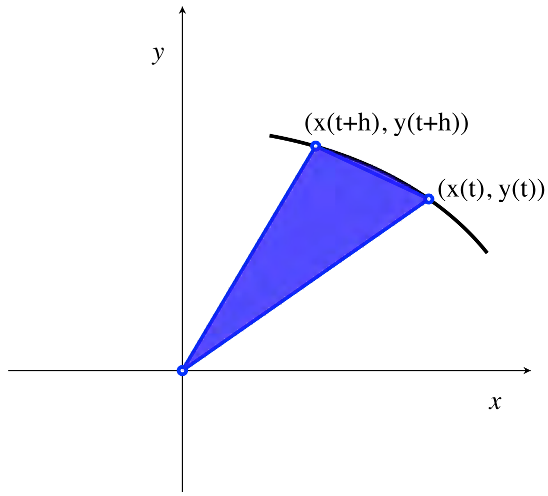


Figure 3.17: Area swept out as particle moves.

However, if we allow the line segments $\ell_{\vec{p}(t)}$ to overlap each other in more than the origin, for more than a finite number of t values, then, generally, the area swept out would be more than the area of the region which is the union of all of the $\ell_{\vec{p}(t)}$, for the area swept out counts the area of a region (always with a plus sign) each time that region is passed over.

The area of the blue sector in Figure 3.17 is approximated well, for values of h close to 0, by the area of the triangle with vertices $(0,0)$, $(x(t), y(t))$ and $(x(t+h), y(t+h))$. But how do you calculate the area of a triangle given the coordinates of the vertices?

Consider a triangle, $\triangle$, with vertices $(0,0)$, (a,b), and (c,d), as in Figure 3.18. The area of $\triangle$ is the area of the triangle with vertices $(0,0)$, $(a,0)$, (a,b), plus the area of the parallelogram with vertices $(a,0)$, $(c,0)$, (a,b), (c,d), minus the area of the triangle with vertices $(0,0)$, $(c,0)$, (c,d). Thus, we find

$$\text{area of } \triangle \;=\; \frac{1}{2}\,ab \;+\; \frac{1}{2}\,(b+d)(c-a) \;-\; \frac{1}{2}\,cd \;=\; \frac{1}{2}\,(bc-ad).$$

If we switch the positions of (a,b) and (c,d), we would, of course obtain that the area of the triangle is $(ad-bc)/2$. Hence, a formula that gives us the correct answer, without having to know the relative positions of (a, b) and (c, d) is

$$\text{area of } \triangle \;=\; \frac{1}{2}\,\big|ad - bc\big|,$$

and it can be verified (you can do it!) that this formula is correct regardless of where the points (a, b) and (c, d) are located, even if they're in different quadrants.

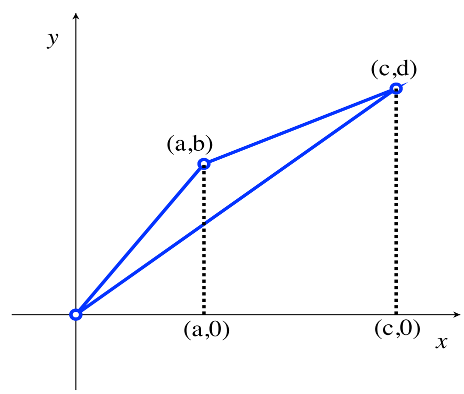


Figure 3.18: Finding the area of a triangle, given vertices.

Returning to the sector in Figure 3.17, we find that its area is approximately equal to the area of the triangle with vertices $(0, 0)$, $(x(t), y(t))$, $(x(t+h), y(t+h))$, which we now know is equal to

$$\text{area of sector} \;=\; \text{area of } \triangle \;=\; \frac{1}{2}\,\big|x(t+h)y(t) \;-\; y(t+h)x(t)\big|.$$

When h is close to 0, the definition of the derivative gives us the approximations $x(t+h) \approx x(t) + h\,x'(t)$ and $y(t+h) \approx y(t) + h\,y'(t)$. Inserting these into the previous formula, we find that

$$\text{area of sector} \;\approx\; \frac{1}{2}\,\Big|\big(x(t) + h\,x'(t)\big)y(t) \;-\; \big(y(t) + h\,y'(t)\big)x(t)\Big| \;=\; \frac{1}{2}\,\big|x'(t)y(t) - y'(t)x(t)\big|\,\big|h\big|.$$

In infinitesimal terms, this means that the infinitesimal amount of area, dA, swept out, along the parameterized curve $\vec{p}(t)$ in an infinitesimal positive amount of time dt is

$$dA \;=\; \frac{1}{2}\,\big|x'(t)y(t) - y'(t)x(t)\big|\,dt.$$

As is frequently the case in this chapter, we **could** take the following proposition as a definition, but we prefer to use our discussion above as the basis for a proof.

Proposition 3.4.1. *The* **area swept out** *by the parameterized curve* $\vec{p} : [t_0, t_1] \to \mathbb{R}^2$, *where* $\vec{p}(t) = (x(t), y(t))$ *is continuously differentiable on an open interval containing* $[t_0, t_1]$, *is*

$$\int_{t_0}^{t_1} \frac{1}{2}\left|x'(t)y(t) - y'(t)x(t)\right| dt.$$

This is the area of the region consisting of all of the line segments $\ell_{\vec{p}(t)}$, *for* $t_0 \leq t \leq t_f$, *provided that, for all* $t_0 < t < t_f$ *(note the strict inequalities), the* $\ell_{\vec{p}(t)}$ *intersect each other only at the origin, that is, provided that the only overlap (other than the origin) during the sweeping is, possibly, at the initial and final* t *values.*

Note that the line segment from the origin to $\vec{p}(t) \neq \vec{0}$ overlaps the line segment from the origin to $\vec{p}(r) \neq \vec{0}$, at a point other than origin, if and only if there exists a real number (a *scalar*) $\lambda > 0$ such that $\vec{p}(t) = \lambda\, \vec{p}(r)$, i.e., such that $x(t) = \lambda\, x(r)$ and, simultaneously, $y(t) = \lambda\, y(r)$.

Example 3.4.2. Let's look at the parameterized curve $\vec{p}(t)$ given by $x = x(t) = t^2$, $y = y(t) = t^3$, and $-1 \leq t \leq 1$. We see that $x^3 = t^6 = y^2$, so that the points in the range of this parameterized curve lie on the graph of $y^2 = x^3$.

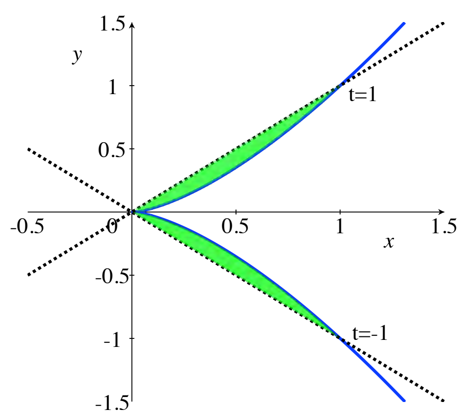


Figure 3.19: Area swept out is in green.

Looking at the curve $y^2 = x^3$ and noting that $y(t)$ increases as t increases, it may be intuitively clear that there is no overlapping (outside the origin) of line segments out to $\vec{p}(t)$ at various t values. To really show this carefully, suppose that (t^2, t^3) and (r^2, r^3) are not equal to $\vec{0}$, i.e., that $t \neq 0$ and $r \neq 0$, and that there is a $\lambda > 0$ such that $t^2 = \lambda r^2$ and $t^3 = \lambda r^3$. We want to show that either this implies that $t = r$, so that "two" line segments intersecting outside the origin implies that the parameter values were the same (and so, the "two" lines segments were actually the same one), or that t and r are the endpoints of the interval; in this example, those endpoints are -1 and 1.

However, this is easy; just divide the two sides of $t^3 = \lambda r^3$ by the corresponding sides of $t^2 = \lambda r^2$ (which we can do, since t, r, and λ are not 0), and you immediately obtain that $t = r$.

Thus, the area swept out by the parameterized curve is equal to the area of the region consisting of the union of the line segments out to points on the curve. In Figure 3.19, the swept out area is in green.

The area swept out is

$$\text{area} \;=\; \int_{-1}^{1} \big|x'(t)y(t) - y'(t)x(t)\big|\, dt \;=\; \int_{-1}^{1} \frac{1}{2}\big|2t \cdot t^3 - 3t^2 \cdot t^2\big|\, dt \;=$$

$$\frac{1}{2}\int_{-1}^{1} t^4\, dt \;=\; \frac{1}{2} \cdot \frac{t^5}{5}\bigg|_{-1}^{1} \;=\; \frac{1}{2}\left(\frac{1}{5} - \frac{-1}{5}\right) \;=\; \frac{1}{5}.$$

Example 3.4.3. Let's look at a variant of the previous example. Consider the parameterized curve $\vec{p}(t)$ given by $x = x(t) = t^4$, $y = y(t) = t^6$, and $-1 \leq t \leq 1$. We see that $x^3 = t^{12} = y^2$, so that the points in the range of this parameterized curve still lie on the graph of $y^2 = x^3$. Note, however, that this time the y-coordinate is always ≥ 0, so that the range of $\vec{p}(t)$ is only the top half of the graph of $y^2 = x^3$.

When $t = -1$, the point/particle is at $(1, 1)$, and moves down the curve as t increases to 0, at which time, the point/particle is at the origin. Then, as t increases to 1, the point/particle moves back up to where it started at $(1, 1)$.

It is "obvious" that, as the point/particle moves, it sweeps through the same region twice, the region in green in Figure 3.20. To show this, without appealing to the picture, we look at points (t^4, t^6) and (r^4, r^6) that are not equal to $\vec{0}$, such that there is a $\lambda > 0$ such that $t^4 = \lambda r^4$ and $t^6 = \lambda r^6$. We want to show that this implies that $t = \pm r$, so that, aside from $t = r = 0$, the line segments from the origin out to the curve not only overlap for pairs of values, but that, at those pairs, the corresponding points on the curve are actually the same.

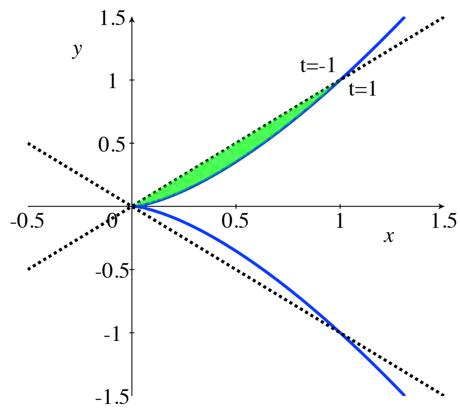


Figure 3.20: Area swept out is in green.

This is easy; once again, we just "divide the equations": $t^6 = \lambda r^6$ and $t^4 = \lambda r^4$ (which we can do, since t, r, and λ are not 0), and we obtain that $t^2 = r^2$, and so $t = \pm r$.

Thus, the area swept out by the parameterized curve is equal to twice the area of the green region in Figure 3.20, which, by symmetry, should be equal to exactly what we calculated in Example 3.4.2. Let's check.

The area swept out is

$$\int_{-1}^{1} \left| x'(t)y(t) - y'(t)x(t) \right| dt \;=\; \int_{-1}^{1} \frac{1}{2}\left|4t^3 \cdot t^6 - 6t^5 \cdot t^4\right| dt \;=$$

$$\int_{-1}^{1} |t|^9 \, dt \;=\; \int_{-1}^{0} -t^9 \, dt \;+\; \int_{0}^{1} t^9 \, dt \;=\; -\left.\frac{t^{10}}{10}\right|_{-1}^{0} \;+\; \left.\frac{t^{10}}{10}\right|_{0}^{1} \;=\; \frac{1}{10} \;+\; \frac{1}{10} \;=\; \frac{1}{5},$$

as we expected.

Area in polar coordinates:

A classic example of parameterized curves and the area swept out by them is provided by curves described using *polar coordinates*.

A point with Cartesian coordinates (x, y) is said to have *polar coordinates* (r, θ) provided that

$$x \;=\; r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

You should realize immediately that this definition, and the periodicity of sine and cosine, imply that every point (x, y) possesses an **infinite** number of polar coordinates! For instance, the point $(x, y) = (1, 1)$ has polar coordinates $(r, \theta) = (\sqrt{2}, \pi/4)$, $(r, \theta) = (-\sqrt{2}, 3\pi/4)$, and, more generally, polar coordinates

$$(r, \theta) \;=\; \left((-1)^n\sqrt{2},\; \frac{\pi}{4} + n\pi\right),$$

where n is any integer. Even worse, the origin, $(x, y) = (0, 0)$ has polar coordinates $r = 0$ and $\theta =$ **any** real number!

Be aware that some sources require that $r \geq 0$ in polar coordinates. While this convention makes polar coordinates slightly easier to deal with, it destroys our ability to easily describe some beautiful curves in terms of polar coordinates. We shall **not** assume that r must be non-negative.

Given r and θ for a point P, it is easy to produce **the** Cartesian coordinates of P; you simply use that $x = r\cos\theta$ and $y = r\sin\theta$. The question is: given x and y, how do you produce **one** corresponding set of polar coordinates, i.e., given x and y, how do you produce **a** simultaneous solution to $x = r\cos\theta$ and $y = r\sin\theta$?

Squaring and adding yields

$$x^2 \;+\; y^2 \;=\; r^2\cos^2\theta \;+\; r^2\sin^2\theta \;=\; r^2.$$

Thus, $r = \pm\sqrt{x^2 + y^2}$.

We see, of course, that $r = 0$ if and only if the point is the origin. If we are considering a point other than the origin, and decide to use a positive r, then we must have $r = \sqrt{x^2 + y^2}$, and then a θ, where $0 \leq \theta < 2\pi$, is uniquely determined by dividing the equations $x = r\cos\theta$ and $y = r\sin\theta$ by r to obtain:

$$\cos\theta \;=\; \frac{x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad \sin\theta \;=\; \frac{y}{\sqrt{x^2 + y^2}}.$$

At this point, you can use inverse trig functions to obtain a θ, as long as you're careful about what quadrant the point is in. Once you have one pair of polar coordinates (r_0, θ_0) for a point (other than the origin), every other pair that represents the same point is of the form $\big((-1)^n r_0, \theta_0 + n\pi\big)$ for integer values of n.

For points other than the origin, the most standard choice of polar coordinates has $r > 0$ and $0 \leq \theta < 2\pi$; with these choices, r is the distance of the point from the origin and θ is the counter-clockwise angle between the positive x-axis and the line segment from the origin to the point.

Negative values of θ correspond to line segments which make **clockwise** angles of $|\theta|$ with the positive x-axis. The relation between a point with polar coordinates (r, θ) and a point with polar coordinates $(-r, \theta)$ is that they lie on the same line through the origin, the same distance from the origin, but on opposite sides.

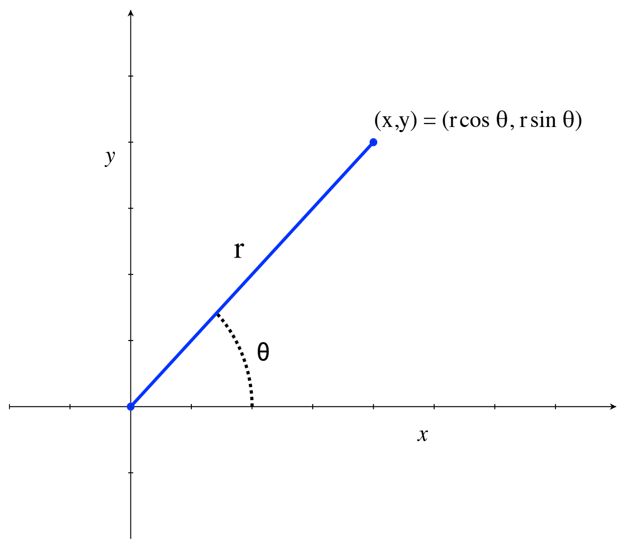


Figure 3.21: The polar coordinates r and θ.

Polar coordinates are useful for dealing with problems – Calculus and non-Calculus problems – in which data is given in terms of distances and angles, instead of in terms of x- and y-coordinates. But what does any of this have to do with parameterized curves and area swept out???

Well...many cool-looking curves can be described very easily by specifying equations that involve the polar coordinates of the points on the curve. For instance, the equation

$$r = 1 + \cos\theta$$

describes a curve called a *cardioid*, as shown in Figure 3.22. The name is derived from the fact that the shape is vaguely heart-like.

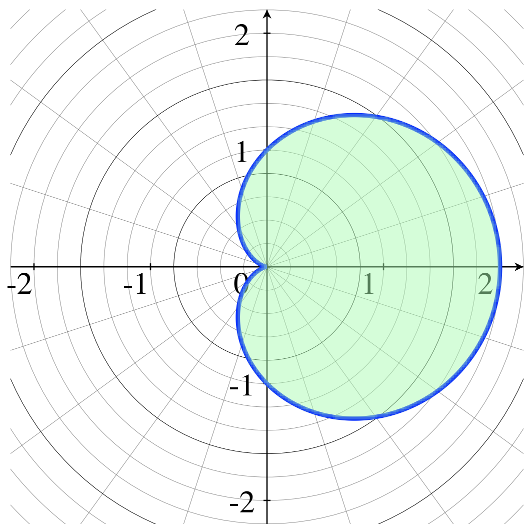


Figure 3.22: The cardioid given by $r = 1 + \cos\theta$.

We had a computer graph this for us, but you can do it by hand. How? It's not too hard. Note that, since $-1 \leq \cos\theta \leq 1$, we have that $0 \leq 1 + \cos\theta \leq 2$, so that $r \geq 0$ and, hence, represents distance from the origin. Now, just think about what happens.

As θ increases from 0 to π, that is, as you're drawing and you increase your counter-clockwise angle with the positive x-axis, $\cos\theta$ starts at 1 and decreases to -1; hence, the distance that your pen or pencil is from the origin, $r = 1 + \cos\theta$, should start at 2 and decrease to 0 as the angle increases to π. As the angle θ increases from π to 2π, r steadily increases from 0 back to 2. As $\cos\theta$ is 2π-periodic, you continue to get the same points as θ increases beyond 2π or if you take θ to be negative.

We would like to apply our discussion from earlier in this section to calculate the area of the region inside the cardioid, the green region in Figure 3.22. In fact, we would like to calculate the area swept out by curves given in polar coordinates in the form $r = f(\theta)$, for $\theta_0 \leq \theta \leq \theta_1$, where f is continuously differentiable on an open interval containing the interval $[\theta_0, \theta_1]$.

This is actually easy for us; given $r = f(\theta)$, $x = r\cos\theta$, and $y = r\sin\theta$, we obtain a parameterization of the curve

$$\vec{p}(\theta) \;=\; (x(\theta),\; y(\theta)) \;=\; \big(f(\theta)\cos\theta,\; f(\theta)\sin\theta\big).$$

The area swept out by the curve as θ goes from θ_0 to θ_1, where $\theta_0 \leq \theta_1$ is thus, by Proposition 3.4.1, equal to

$$\int_{\theta_0}^{\theta_1} \frac{1}{2}\left|\big[f(\theta)\cos\theta\big]' f(\theta)\sin\theta \;-\; \big[f(\theta)\sin\theta\big]' f(\theta)\cos\theta\right| d\theta.$$

We leave it as an exercise for you to verify that this integrand simplifies greatly and yields the expression in the proposition below.

Proposition 3.4.4. *Suppose that f is continuously differentiable on an open interval containing the interval $[\theta_0, \theta_1]$. Then, the area swept out by the parameterized curve given in polar coordinates by $r = f(\theta)$, where $\theta_0 \leq \theta \leq \theta_1$, is*

$$\int_{\theta_0}^{\theta_1} \frac{1}{2} r^2\, d\theta \;=\; \int_{\theta_0}^{\theta_1} \frac{1}{2}\,[f(\theta)]^2\, d\theta.$$

Example 3.4.5. Let's apply Proposition 3.4.4 to calculate the area inside the curve given by $r = 1 + \cos\theta$. Note that, when we discussed drawing this curve, it was clear that there were no overlapping line segments from the origin out to the curve for $0 \leq \theta < 2\pi$ (at points other than the origin). The line segment when $\theta = 0$ and when $\theta = 2\pi$ are/is the same, but that's allowed in Proposition 3.4.1 and doesn't affect the area calculation.

Therefore, the area inside the cardioid given by $r = 1 + \cos\theta$ is

$$\frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2\, d\theta \;=\; \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta.$$

The hard part of this integral is integrating $\cos^2\theta$. You either use Proposition 1.2.5, with $n = 2$, or use the trig identity $\cos^2\theta = (1 + \cos(2\theta))/2$ and an easy substitution of $u = 2\theta$. One way or the other, you need to find that $\int \cos^2\theta\, d\theta = (\theta + \cos\theta\sin\theta)/2 + C$, so that our area integral above becomes

$$\frac{1}{2}\left(\theta + 2\sin\theta + \frac{1}{2}(\theta + \cos\theta\sin\theta)\right)\Bigg|_0^{2\pi} \;=\; \frac{3\pi}{2}.$$

Example 3.4.6. We looked at the cardioid described in polar coordinates by $r = 1 + \cos\theta$. Perhaps surprisingly, the graph of $r = 1 + \cos(3\theta)$ looks dramatically different. This graph is usually referred to as a 3-leaved rose.

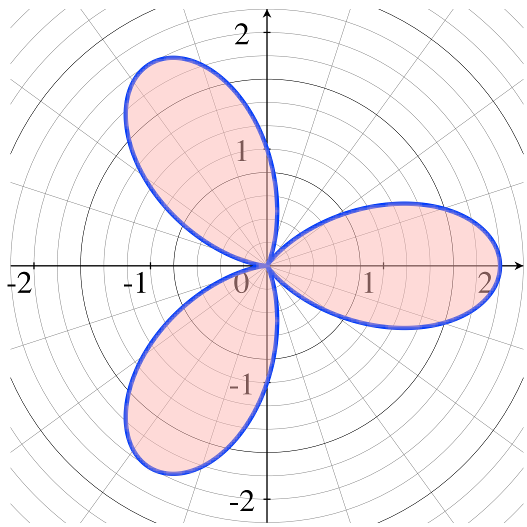


Figure 3.23: The 3-leaved rose given by $r = 1 + \cos(3\theta)$.

How could you sketch this by hand, and how do you show that the apparent symmetries under rotations by $2\pi/3$ radians and by reflection about the x-axis, actually **are** symmetries?

Once again, $0 \leq r \leq 2$, and it helps to determine the values of θ that make r equal 0 and those that make r equal 2.

We have $r = 0$ precisely when $\cos(3\theta) = -1$, which occurs exactly when $3\theta = \pi + 2\pi n$ for some integer n. Thus, $r = 0$ when $\theta = \pi/3 + n(2\pi/3)$; for $0 \leq \theta \leq 2\pi$, this occurs when θ equals $\pi/3$ ($n = 0$), π ($n = 1$), and $5\pi/3$ ($n = 2$).

To have $r = 2$, we must have $\cos(3\theta) = 1$, which occurs exactly when $3\theta = 2\pi n$ for some integer n. Thus, $r = 2$ when $\theta = n(2\pi/3)$; for $0 \leq \theta \leq 2\pi$, this occurs when θ equals 0 ($n = 0$), $2\pi/3$ ($n = 1$), and $4\pi/3$ ($n = 2$).

Hence, if you had to draw this 3-leaved rose, you could start at $\theta = 0$ and $r = 2$. As you increase θ from 0 to $\pi/3$, i.e., as you draw points at an increasing counter-clockwise angle from the positive x-axis, from an angle of 0 to an angle of $\pi/3$ (60°), the value of r drops from 2 down to 0, i.e., you decrease the distance from the point you're drawing to the origin from 2 down to

0. This is what yields the top half of the right-hand leaf.

Now, as θ increases from $\pi/3$ to $2\pi/3$, you keep increasing the angle at which you're drawing points, and let the distance from the origin increase from 0 back out to 2. This yields upper right-hand portion of the leaf on the upper-left of the rose. Now continue in this fashion, until you come all the way around to $\theta = 2\pi$, at which point, you will have closed the graph, and the points would start repeating if you let θ get bigger.

How do you verify the apparent symmetries? Notice that

$$r \;=\; 1+\cos(3\theta) \;=\; 1+\cos(3\theta+2\pi) \;=\; 1+\cos\big(3(\theta+2\pi/3)\big)$$

This means that a pair (r_0, θ_0) satisfies $r = 1+\cos(3\theta)$ if and only if $(r_0, \theta_0 + 2\pi/3)$ satisfies the same equation; in other words, when you rotate the graph by $2\pi/3$, one third of the way around a circle, you get the same graph. Thus, we've verified the rotational symmetry.

Now notice that, because cosine is an even function,

$$r \;=\; 1+\cos(3\theta) \;=\; 1+\cos(-3\theta) \;=\; 1+\cos\big(3(-\theta)\big).$$

This means that a pair (r_0, θ_0) satisfies $r = 1+\cos(3\theta)$ if and only if $(r_0, -\theta_0)$ satisfies the same equation. This proves that the graph is symmetric under reflection about the x-axis.

What is the area enclosed by the entire 3-leaved rose? Using the two symmetries that we verified above, it is enough for us to find the area of half of one "leaf", and then multiply by 6. Recall that we saw that the top half of the right-hand leaf is produced as θ goes from 0 to $\pi/3$. Therefore, we find that the area enclosed is

$$6\cdot\frac{1}{2}\int_0^{\pi/3}(1+\cos(3\theta))^2\,d\theta \;=\; 3\int_0^{\pi/3}(1+2\cos(3\theta)+\cos^2(3\theta))\,d\theta.$$

We leave it as an exercise for you to show that this yields

$$3\cdot\left(\theta+\frac{2\sin(3\theta)}{3}+\frac{3\theta+\cos(3\theta)\sin(3\theta)}{6}\right)\Bigg|_0^{\pi/3} \;=\; \frac{3\pi}{2}.$$

We should make one final remark here. We have used the term "3-leaved rose" as a generic, qualitative description of the graph of $r = 1 + \cos(3\theta)$. A very similar-looking 3-leaved rose, a *trifolium*, has an equation $r = 2\cos(3\theta)$; this is a more-standard "3-leaved rose". We compare the two graphs in Figure 3.24, where $r = 2\cos(3\theta)$ describes the inside rose (in black).

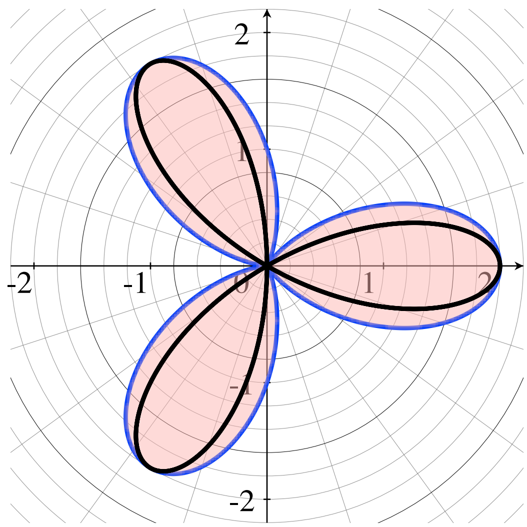


Figure 3.24: $r = 1 + \cos(3\theta)$ in blue. $r = 2\cos(3\theta)$ in black.

We leave it to you as an exercise to show that the area inside this trifolium is π, which is significantly less than $3\pi/2$, which agrees with what we see in Figure 3.24.

Example 3.4.7. As we saw earlier, $r = 1 + \cos\theta$ describes a cardioid. You might think that

$$r = \cos\theta$$

would also describe some interesting polar curve, a curve that you never saw in terms of Cartesian coordinates. However, you'd be wrong if you thought this.

If you multiply the equation by r, you don't affect the set of points described by the equation; for multiplying by r adds the solution $r = 0$, but this just describes the origin, which is already a point on the curve, since $0 = \cos(\pi/2)$. Thus, the curve may be described by

$$r^2 \;=\; r\cos\theta.$$

But this is easy to change into Cartesian coordinates; the equation becomes

$$x^2 + y^2 \;=\; x.$$

Subtracting x from each side, and completing the square yields

$$x^2 - x + y^2 \;=\; 0 \qquad \text{and so} \qquad \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4},$$

which, hopefully, you recognize as an equation for a circle of radius $1/2$, centered at $(1/2, 0)$.

You should convince yourself that, in polar coordinates, this circle is swept out as θ goes from 0 to π; if you went to 2π, you'd sweep out the circle twice.

Of course, we know that the area inside a circle of radius $1/2$ is $\pi(1/2)^2 = \pi/4$, but we can "check" this by calculating the integral

$$\int_0^{\pi} \frac{1}{2}\, r^2\, d\theta \;=\; \int_0^{\pi} \frac{1}{2}\cos^2\theta\, d\theta.$$

As we saw in Proposition 1.2.5, $\int \cos^2\theta\, d\theta = (\theta + \cos\theta\sin\theta)/2 + C$, and so, we conclude that

$$\text{area} \;=\; \left.\frac{\theta + \cos\theta\sin\theta}{4}\right|_0^{\pi} \;=\; \frac{\pi}{4}.$$

3.4.1 Exercises

In Exercises 1 through 5, you are given the component functions $x = x(t)$ and $y = y(t)$ for a parameterized curve. a) Find the area swept out by the parameterized curve

over the given interval. b) Think of the parameterized curve as giving the position, at time t, of a particle in the xy-plane. Eliminate the parameter t, to arrive at an equation containing only x and y; sketch the graph of this curve, and decide how the particle moves along the curve during the given times. Shade in the area swept out.

1. $x = t$, $y = t^2$; $0 \leq t \leq 2$.

2. $x = \sin t$, $y = \cos t$; $\frac{\pi}{2} \leq t \leq \pi$.

3. $x = t^3$, $y = t^2$; $-1 \leq t \leq 1$.

4. $x = e^{2t}$, $y = e^{3t}$; $-1 \leq t \leq 1$.

5. $x = \ln t$, $y = t$; $1 \leq t \leq 3$

6. Suppose that $a > b > 0$. Find the area swept out by the curve $\vec{p}(t) = (t^a, t^b)$, for t in the interval $[0, 1]$.

In Exercises 7 through 12, you are given a curve $r = f(\theta)$, described in polar coordinates. Sketch the curve and find the area it encloses.

7. $r = 1 + \sin\theta$.

8. $r = 1 + \sin(4\theta)$.

9. $r = 5 + 4\cos\theta$.

10. $r^2 = 9\sin(2\theta)$.

11. $r = 2(1 + \sin\theta)$.

12. $r = 3\cos(5\theta)$.

13. Find the area inside the big loop and outside the small loop of the graph of $r = \frac{1}{2} + \sin\theta$; this figure is known as a *limaçon*. See Figure 3.25.

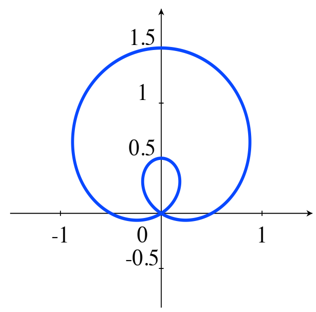


Figure 3.25: The limaçon given by $r = 0.5 + \sin\theta$.

14. Find the area swept out by the spiral $x = (10 - t)\cos t$, $y = (10 - t)\sin t$, $0 \le t \le 10$. See Figure 3.26.

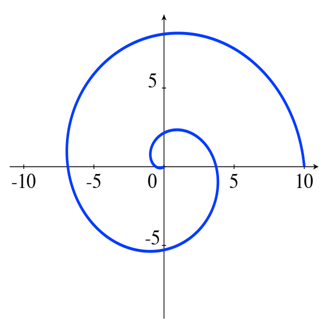


Figure 3.26: The spiral $x = (10 - t)\cos t$, $y = (10 - t)\sin t$, $0 \le t \le 10$.

15. Find the area swept out by the exponential spiral $x = e^{-t}\cos(10t)$, $y = e^{-t}\sin(10t)$, $0 \le t < \infty$. See Figure 3.27.

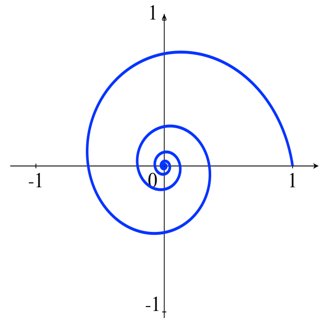


Figure 3.27: An exponential spiral.

16. The most famous example of area swept out appears in *Kepler's 2nd Law of Planetary Motion*.

    When a planet moves around the sun, Kepler determined that the planet moves in an ellipse, contained in a plane (which we take as the xy-plane), with the star at one of the foci. Placing the sun at the origin in the xy-plane (*heliocentric coordinates*), a planet moves along an ellipse given in polar coordinates by

$$r = \frac{p}{1 + \epsilon \cos\theta},$$

    where $p > 0$ and $0 < \epsilon < 1$. The constant ϵ is the *eccentricity* of the ellipse.

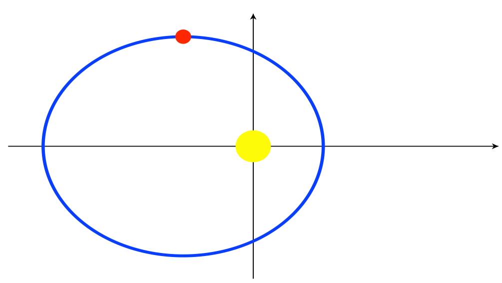


Figure 3.28: A planet, moving in an ellipse around the sun.

    Applying Newton's Law of Universal Gravitation and vector Calculus, it is not terribly difficult to show that, as functions of time, the position of a planet given by $\theta = \theta(t)$ and

$$r = r(t) = \frac{p}{1 + \epsilon \cos\big(\theta(t)\big)},$$

satisfies the differential equation

$$r\,\frac{d^2\theta}{dt^2} + 2\,\frac{dr}{dt} \cdot \frac{d\theta}{dt} = 0.$$

a. Show that
$$\frac{d}{dt}\left(r^2\,\frac{d\theta}{dt}\right) = 0.$$

b. Derive Kepler's 2nd Law of Planetary Motion: a planet sweeps an equal area in an equal amount of time, i.e., given an amount of time $T > 0$, the amount of area that the planet sweeps out between times t and $t + T$ is independent of t. (Hint: Do **not** explicitly use that $r = p/(1 + \epsilon \cos\theta)$.)

3.5 Volume

In this section, we will discuss how integration allows us to calculate the volume of solid regions for which we're given the area of cross sections which are perpendicular to some axis. We will look at many examples in which the solids are *solids of revolution*, solids obtained by revolving a plane region around an axis and looking at the solid region which is "swept out". In the context of solids of revolution, integrating the cross-sectional area is referred to as the *disk method* or *washer method*, due to the shapes of the cross sections.

We will also look at a second method for finding the volumes of solids of revolution; this method has some aspects that are similar to the disk and washer methods, but it does not use planar cross sections of the solid. This second method for finding volumes of solids of revolution is the *cylindrical shell method*.

Suppose that we have a solid region, S, in space, and that, after fixing a coordinate axis, say the x-axis, we know the area of every cross-sectional slice of S which is perpendicular to the x-axis, and that the solid region lies between the x cross sections where $x = a$ and $x = b$, where $a < b$.

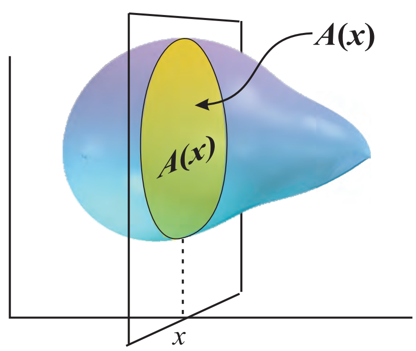


Figure 3.29: A typical x cross section of a solid.

In order to determine the volume of S, given the cross-sectional area function $A(x)$, we take partitions of the interval $[a, b]$, and look at Riemann sums of estimates of the volume over the i-th subinterval, of width Δx_i, using a sample point s_i in the subinterval. Our estimate for the volume over the i-th subinterval is the area $A(s_i)$ times a small thickness Δx_i. Thus, we obtain Riemann sums of the form $\sum_i A(s_i)\Delta x_i$.

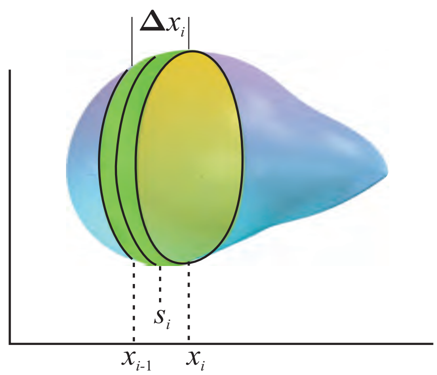


Figure 3.30: A typical summand in a Riemann sum for volume.

Taking the limit, we obtain:

Proposition 3.5.1. *If a solid region lies between the $x = a$ and $x = b$ cross sections, and the cross-sectional area function $A : [a, b] \to \mathbb{R}$ of the solid is continuous, then the volume of the solid region S is*

$$\int_a^b A(x)\,dx.$$

Remark 3.5.2. Of course, we generally think of the situation infinitesimally. At each x-coordinate between a and b, the infinitesimal volume dV of the solid S above the infinitesimal interval from x to $x + dx$, of infinitesimal width dx, is given by $dV = A(x)\,dx$, and the total volume is the continuous sum of the infinitesimal chunks of volume dV as x goes from a to b, i.e.,

$$\text{volume of } S \;=\; \int_{x=a}^{x=b} dV \;=\; \int_a^b A(x)\,dx.$$

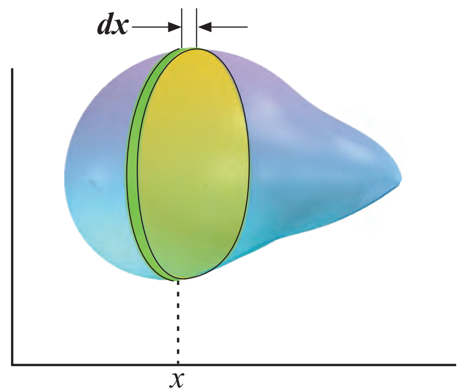


Figure 3.31: An infinitesimal piece of volume in a solid.

Example 3.5.3. Suppose that we have a rectangle of area B (for "base") which lies in the plane at a fixed positive z-coordinate H (for "height"). If we connect all of the points of the rectangle to the origin via straight lines, then the set of the points on all the line segments form a solid, an upside-down pyramid. We would like to determine the volume of this pyramid in terms of B and H.

If we had a formula for the z cross-sectional area, $A(z)$, for $0 \leq z \leq h$, then we would simply calculate

$$\int_0^h A(z)\, dz,$$

but how do we obtain a formula for $A(z)$?

Call the length of each of one pair of parallel sides of the base rectangle L and let W be the length of each of the other sides. Each z cross section of the pyramid is a rectangle, of a length and width that varies as z varies. Let $l = l(z)$ and $w = w(z)$ denote the length and width of the cross-sectional rectangle at z, with l and w chosen to correspond to the "same" sides as L and W, respectively.

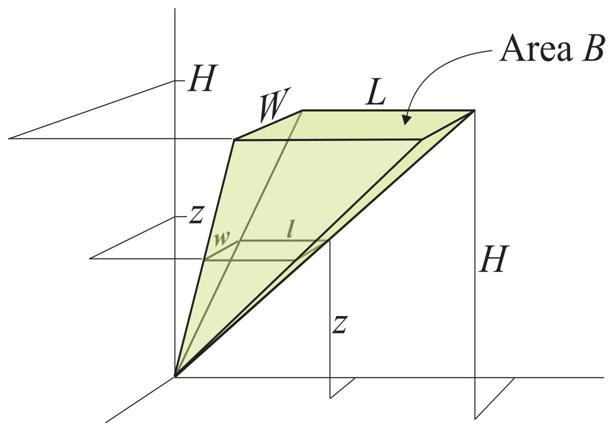


Figure 3.32: An upside-down slanted pyramid.

By using similar triangles in the faces of the pyramid, we find that the length $l(z)$ and width $w(z)$ satisfy:

$$\frac{l(z)}{L} = \frac{z}{H} \qquad \text{and} \qquad \frac{w(z)}{W} = \frac{z}{H}.$$

Therefore, the cross-sectional area $A(z)$ is given by

$$A(z) = l(z)\,w(z) = \frac{LW}{H^2}\,z^2 = \frac{B}{H^2}\,z^2.$$

Now it's easy to find the volume. We calculate

$$\text{volume} = \int_0^H A(z)\,dz = \int_0^H \frac{B}{H^2}\,z^2\,dz = \frac{B}{H^2}\cdot\frac{z^3}{3}\bigg|_0^H = \frac{B}{H^2}\,\frac{H^3}{3} = \frac{1}{3}BH.$$

The formula that we just derived, that the volume was one third of the area of the base times the height, may seem familiar to you; it's the formula for the volume of a cone. You may be thinking to yourself "wait, we didn't have a cone; we had some upside-down slanty pyramid thing".

It's true that, when most people picture a cone, they picture a right circular cone, i.e., a disk (a filled-in circle) connected by line segments to a vertex which lies on the line through the center of the circle, perpendicular to the plane containing the circle. However, the object that we looked at in Example 3.5.3 is also a type of *cone*; it, and right circular cones, are particular cases of the following general definition.

Definition 3.5.4. *Suppose that we have a region R, contained in a plane P in $\mathbb{R}^3$, and have a point v in $\mathbb{R}^3$, but not in the plane P.*

Then, the set of points on all line segments connecting points of R to the point v is called the **cone**, *with* **base** *R and* **vertex** *v.*

If R contains at least 3 non-collinear points, so that there is only one plane P containing R. then the (perpendicular) distance from vertex v to the plane P is called the **height of the cone**.

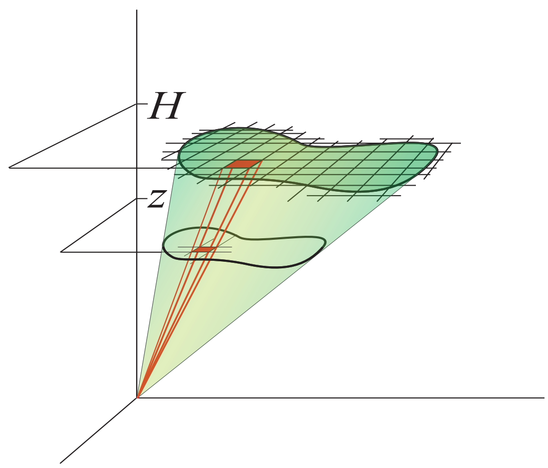


Figure 3.33: A cone whose base is a general plane region.

Thus, a right circular cone is a (general) cone, and so is the upside-down slanty pyramid thing from Example 3.5.3. However, the really cool thing about Example 3.5.3 is that it gives us a formula for the volume of **any** cone, at least, any cone that has a base which has some area. Why? Because to approximate the area of a non-rectangular base, you partition the region into little rectangles (with some rectangles possibly sticking out of the region, or not covering part of the region near the boundary), and add together all of the areas of the little rectangles; then you perform a limiting operation, as the rectangles get arbitrarily small.

But, each partition of the plane region into little rectangles partitions the cone into a collection of narrow cones with rectangular bases, i.e., narrow cones whose volume we calculated in Example 3.5.3.

Thus, we find

Proposition 3.5.5. *The volume V of a general cone, of height H, whose base has area B, is*

$$V \;=\; \frac{1}{3}BH.$$

Now let's turn to another classic geometry formula: the formula for the volume inside a sphere.

Example 3.5.6. We wish to derive the well-known formula for the volume V inside a sphere of radius R:

$$V = \frac{4}{3}\pi R^3.$$

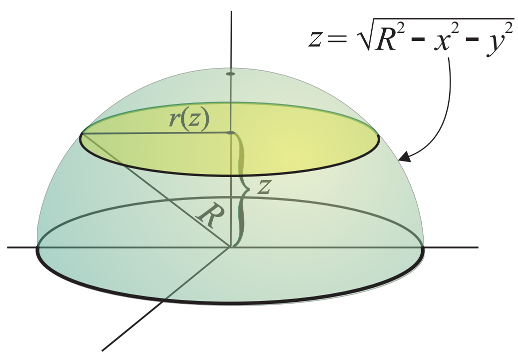


Figure 3.34: A hemisphere of radius R.

For convenience, we will use the sphere of radius R which is centered at the origin. This sphere is the set of points where $x^2 + y^2 + z^2 = R^2$. We will determine the volume inside the top hemisphere, where $z = \sqrt{R^2 - x^2 - y^2}$, and then double the result.

Each z cross section, for $0 \leq z \leq R$, of the hemisphere yields a circle of some radius $r(z)$, and so the cross-sectional area of the solid is $A(z) = \pi\big(r(z)\big)^2$. We need to find the function $r(z)$, or $\big(r(z)\big)^2$, calculate the integral $\int_0^R \pi\big(r(z)\big)^2\,dz$, and then double the result.

To find a formula for $\big(r(z)\big)^2$, you need to sketch a picture, draw the "correct" auxiliary line segment, and use the Pythagorean Theorem. You should quickly find that $z^2 + \big(r(z)\big)^2 = R^2$, and we therefore obtain the very simple formula $\big(r(z)\big)^2 = R^2 - z^2$. Thus, $A(z) = \pi(R^2 - z^2)$ and

$$\frac{V}{2} \;=\; \int_0^R A(z)\,dz \;=\; \int_0^R \pi(R^2 - z^2)\,dz \;=\; \pi\left(R^2 z - \frac{z^3}{3}\right)\Big|_0^R \;=$$

$$\pi\left(R^3 - \frac{R^3}{3}\right) \;=\; \frac{2}{3}\,\pi R^3.$$

We conclude what we wanted, namely that $V = \dfrac{4}{3}\,\pi R^3$.

Solids of revolution:

The sphere, or hemisphere, that we looked at, above, is a particular example of a **solid of revolution**; the hemisphere is "swept out" by revolving, around the z-axis, a quarter of a disk, specifically, the quarter disk, in the 1st-quadrant of the yz-plane, that's inside the circle of radius R, centered at the origin.

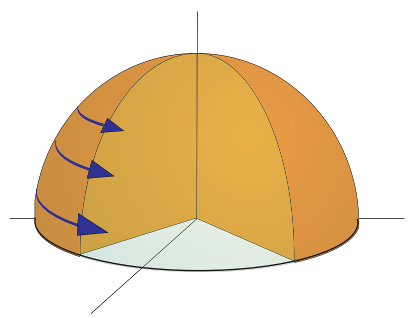


Figure 3.35: The solid hemisphere as a solid of revolution.

We can revolve other plane regions about other axes. We assume that our region B and our *axis of revolution* ℓ are in the same plane, and that B does not contain points on each side of ℓ, i.e., B is completely on one side of ℓ or the other, possibly having an edge along ℓ. We also assume that each cross section of B itself by a line (in the plane) perpendicular to ℓ is a line segment, a point, or is empty.

With these assumptions, when we revolve B in space, about ℓ, each of the cross-sectional line segments will sweep out either a **disk** (a filled-in circle), if one end of the line segment is on ℓ, or will sweep out what's known as a **washer**, a big disk minus a smaller disk with the same center; you get a washer when the revolved cross-sectional line segment in B does not have an endpoint on ℓ. (The technical term for a washer is an *annulus*.)

For instance, we saw a typical swept out disk in Figure 3.34, when we were calculating the volume of the hemisphere in Example 3.5.6. An example that yields washers is given by revolving the bounded region between the graphs of $y = x^2$ and $y = x$ around the y-axis (we return to this example in Example 3.5.10); see Figure 3.36.

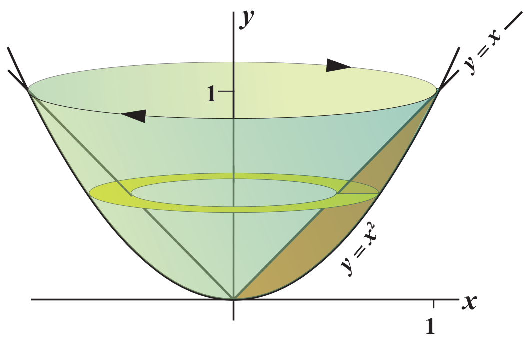


Figure 3.36: An example in which the cross sections are washers.

The area of a disk is, of course, πR^2, where R is the radius of the disk. Thus, if our axis of revolution is the x-axis, then the x cross-sectional area $A(x)$ of our solid of revolution will be

$$A(x) \;=\; \pi(R(x))^2 \;=\; \pi R^2,$$

where $R = R(x)$ is the radius of the disk, i.e., the length of the x cross-sectional line segment in the plane region B.

If the cross sections are washers, then there's the outside radius $R = R(x)$ of the washer, and the inside radius $r = r(x)$, the radius of the "missing" disk. In this case, we have that the x cross-sectional area $A(x)$ of our solid of revolution will be

$$A(x) \;=\; \pi(R(x))^2 - \pi(r(x))^2 \;=\; \pi(R^2 - r^2).$$

In terms of the x cross-sectional line segment in the plane region B, $r(x)$ is the distance from the x-axis to the closest end of the line segment, and $R(x)$ is the distance from the x-axis to the farthest end of the line segment.

It is a common, but terrible, mistake to write that the area of a washer is $\pi(R-r)^2$, instead of $\pi(R^2-r^2)$. Just keep in mind that it's the differences between the areas inside two circles, and you should have no trouble getting it right.

Example 3.5.7. Consider the region B in the xy-plane which is below the graph of $y = x^2$, and above the interval $[0, 2]$ on the x-axis. Revolve the region B around the x-axis. Find the volume of the resulting solid of revolution S.

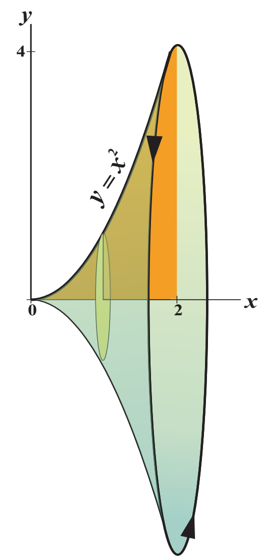


Figure 3.37: The cross sections perpendicular to the x-axis are disks.

Solution:

The cross section of S at any x in the interval $[0, 2]$ is a disk, whose radius is the y-coordinate on the curve $y = x^2$, i.e., whose radius is x^2. So, the radius function is $R = R(x) = x^2$,

$$A(x) = \pi R^2 = \pi(x^2)^2 = \pi x^4,$$

$$dV = A(x)\, dx = \pi x^4\, dx.$$

Therefore, we find that the volume is

$$V = \int_{x=0}^{x=2} dV = \int_0^2 \pi x^4 \, dx = \frac{\pi x^5}{5}\bigg|_0^2 = \frac{32\pi}{5}.$$

Example 3.5.8. Consider the bounded region B in the xy-plane which is "trapped" between the graphs of $y = x^2$ and $y = x$, i.e., the region between $x = 0$ and $x = 1$ that's above the graph of $y = x^2$ and below the graph of $y = x$. Revolve the region B around the x-axis. Find the volume of the resulting solid of revolution S.

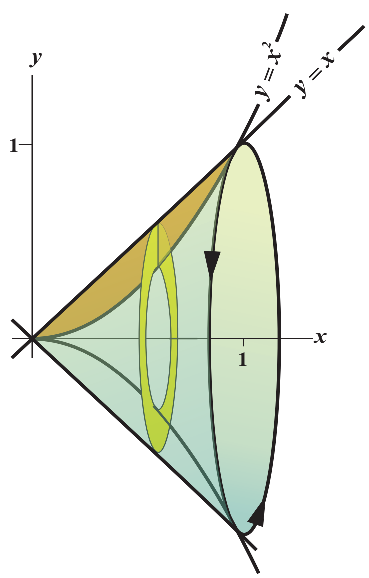


Figure 3.38: The cross sections are now washers.

Solution:

Unlike the plane region in Example 3.5.7, the current region B does not abut the x-axis; there is a gap between B and the axis of revolution. Thus, our cross sections of S, perpendicular to the x-axis are not disks, but are, instead, washers.

The inside radius, the radius of the hole in the washer, $r(x)$, at the x cross section is the distance from the axis to the closest part of B, i.e., to the corresponding point on the curve given by $y = x^2$. Therefore, $r = r(x) = x^2$.

The outside radius, the big radius of the entire washer, $R(x)$, at the x cross section is the distance from the axis to the farthest part of B, i.e., to the corresponding point on the curve given by $y = x$. Therefore, $R = R(x) = x$.

Hence, our cross-sectional area is

$$A(x) \;=\; \pi\big(R^2 - r^2\big) \;=\; \pi\big(x^2 - (x^2)^2\big) \;=\; \pi(x^2 - x^4),$$

an infinitesimal piece of volume is

$$dV \;=\; A(x)\,dx \;=\; \pi(x^2 - x^4)\,dx,$$

and the volume of the solid of revolution is

$$V \;=\; \int_{x=0}^{x=1} dV \;=\; \int_0^1 \pi(x^2 - x^4)\,dx \;=\; \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\bigg|_0^1 \;=\; \pi\left(\frac{1}{3} - \frac{1}{5}\right) \;=\; \frac{2\pi}{15}.$$

Example 3.5.9. What if we consider the same region as in Example 3.5.8, but revolve the region about the line given by $y = 2$? Then, what volume do we obtain for the solid of revolution S?

Solution: The line ℓ given by $y = 2$ is parallel to the x-axis, so that cross sections of the region B, perpendicular to ℓ, are the same as the cross sections perpendicular to the x-axis, and x still goes from 0 to 1. We obtain washers as the cross sections of the solid S, with an inside radius $r = r(x)$ and an outside radius $R = R(x)$.

It takes some thought to come up with the functions $r(x)$ and $R(x)$. Certainly, looking at a diagram helps.

The inside radius $r(x)$ is the distance from ℓ to the closest curve; that closest curve is given by $y = x$. The question is: at a fixed x-coordinate, what is the distance from $y = 2$ to $y = x$?

If you think about it, and look at the diagram, hopefully you come up with the answer. The distance $r(x)$ is the distance from $y = 2$ to the x-axis, minus the distance from the x-axis to the corresponding point on $y = x$. The distance from $y = 2$ to the x-axis is obviously 2. The distance from the x-axis to the corresponding point on the graph of $y = x$ is simply the y-coordinate on the graph, namely x. Therefore, $r = r(x) = 2 - x$.

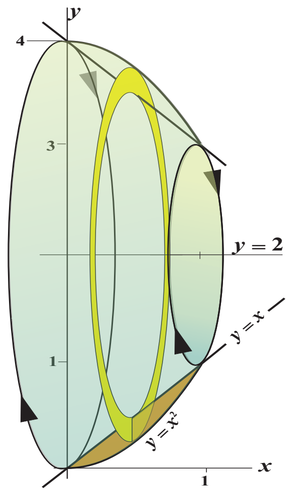


Figure 3.39: The cross sections are washers, but our axis of revolution has changed.

Similarly, the outer radius of the washers, the distance $R(x)$, is the distance from $y = 2$ to the x-axis, minus the distance from the x-axis to the corresponding point on $y = x^2$. We find $R = R(x) = 2 - x^2$.

Hence, our cross-sectional area is

$$A(x) \;=\; \pi\big(R^2 - r^2\big) \;=\; \pi\Big((2-x^2)^2 - (2-x)^2\Big) \;=\; \pi\big((4-4x^2+x^4) - (4-4x+x^2)\big) \;=$$

$$\pi\big(4x - 5x^2 + x^4\big),$$

an infinitesimal piece of volume is

$$dV \;=\; A(x)\,dx \;=\; \pi\big(4x - 5x^2 + x^4\big)\,dx,$$

and the volume of the solid of revolution is

$$V \;=\; \int_{x=0}^{x=1} dV \;=\; \int_0^1 \pi\big(4x - 5x^2 + x^4\big)\,dx \;=$$

$$\pi\left(2x^2 - \frac{5x^3}{3} + \frac{x^5}{5}\right)\Bigg|_0^1 \;=\; \frac{8\pi}{15}.$$

Example 3.5.10. What if we consider the same region as in Example 3.5.8, but revolve the region about the y-axis? Then, what volume do we obtain for the solid of revolution S?

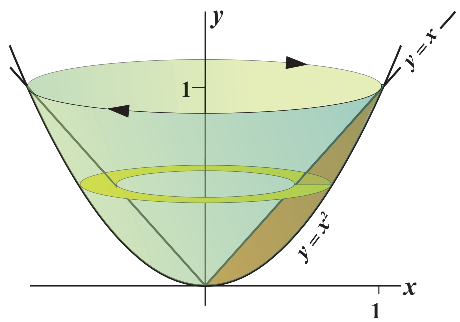


Figure 3.40: Washers around the y-axis.

Solution: Now, cross sections perpendicular to our axis of revolution are y cross sections, which are still washers. However, as we are using y cross sections, we need to write everything in terms of y.

The region B lies between the lines $y = 0$ and $y = 1$ (it is a special property of the region B that makes the y "limits" the same as the x limits; these would not be the same, in general).

The inside radius $r(y)$ of the cross-sectional washer is the x-coordinate on the closest curve to the axis of revolution, i.e., the x-coordinate on the graph of $y = x$. Thus, $r = r(y) = y$. The outside radius $R(y)$ is the x-coordinate on the farthest curve, i.e., on the graph of $y = x^2$. In terms of y, this x-coordinate is $\sqrt{y}$. Therefore, $R = R(y) = \sqrt{y}$.

Hence, our cross-sectional area is

$$A(y) \;=\; \pi\big(R^2 - r^2\big) \;=\; \pi\big((\sqrt{y})^2 - y^2\big) \;=\; \pi(y - y^2),$$

an infinitesimal piece of volume is

$$dV \;=\; A(y)\,dy \;=\; \pi(y - y^2)\,dy,$$

and the volume of the solid of revolution is

$$V = \int_{y=0}^{y=1} dV = \int_0^1 \pi(y - y^2)\, dy = \pi \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\bigg|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}.$$

For solids of revolution, there's a minor change in how we want to think about obtaining the infinitesimal pieces of volume that we get by taking cross-sectional area and multiplying by an infinitesimal thickness.

Suppose that we let our axis of revolution be the x-axis, that we have a region B in the xy-plane, and that we generate a solid of revolution by revolving B in space around the x-axis.

Then, what we've been doing is taking cross sections of our solid S, perpendicular to the x-axis, and producing an infinitesimal chunk of volume $dV = A(x)dx$, where $A(x)$ is the area of the x cross section of S. Instead of revolving and then taking cross sections, we've seen that we can first take cross-sectional line segments in the plane region B, perpendicular to the x-axis, and revolve those around the x-axis to obtain disks or washers with area $A(x)$. We then multiply by an infinitesimal thickness dx.

The slight change in thinking that we wish to bring up now is that we could have first multiplied the cross-sectional line segment in B by dx, to obtain a rectangle of infinitesimal width, and **then** revolved this rectangle about the x-axis to generate our thickened disks or washers.

Why look at things in terms of revolving infinitesimally thin rectangles, perpendicular to the axis of revolution? Because it leads us to consider calculating infinitesimal volumes, and so total volumes, in a different way.

Cylindrical Shells

We now wish to take cross-sectional line segments in B which, instead of being perpendicular to the x-axis, are **parallel** to the x-axis. This means that we look at cross sections perpendicular to the y-axis. For each y-coordinate for which we have points in the region B, let $h(y)$ be the length of the cross-sectional line segment in B at the given y-coordinate. We thicken these line segments to obtain infinitesimally wide/thick rectangles by multiplying by an infinitesimal thickness, which is now dy (**not** dx). We then revolve these rectangles around the x-axis, generating infinitesimally thin (right circular) cylinders (think of aluminum cans, with no tops or bottoms); these are called **cylindrical shells**.

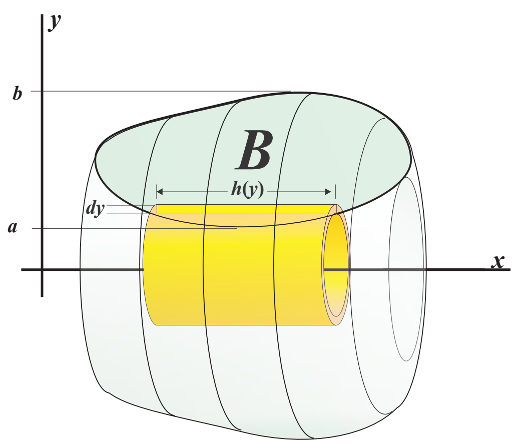


Figure 3.41: A typical cylindrical shell.

The surface area of a cylinder (still with no top or bottom) is the circumference times the height, so $A(y)$ equals $2\pi rh$, where r is the radius of the cylinder and h is its height. Since we're rotating around the x-axis, the radius, in terms of y is simply y, while the height h is what we wrote above for the length of the cross-sectional line segment, namely $h(y)$. We multiply this area by the infinitesimal thickness dy to obtain the infinitesimal piece of volume

$$dV = 2\pi y\, h(y)\, dy.$$

Therefore, if the region B lies between $y = a$ and $y = b$, where $a < b$, and we generate a solid of revolution by revolving B around the x-axis, we find, using cylindrical shells, that the volume is

$$V \;=\; \int_{y=a}^{y=b} 2\pi rh\, dy \;=\; \int_a^b 2\pi y\, h(y)\, dy.$$

Of course, if you revolve a region around a different axis, then the formula above needs to be changed accordingly.

Example 3.5.11. Let's return to the solid of revolution from Example 3.5.7. We had the region B in the xy-plane which is below the graph of $y = x^2$, and above the interval $[0, 2]$ on the x-axis. We revolved the region B around the x-axis, and found the volume V of the resulting solid of revolution S by using disks. We calculated that $V = 32\pi/5$.

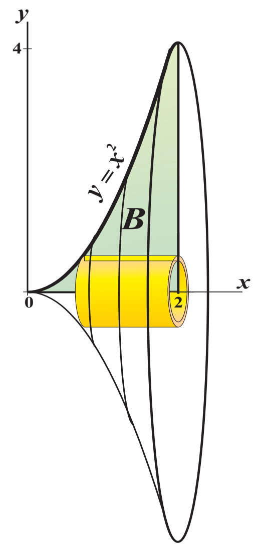


Figure 3.42: Volume by cylindrical shells.

We will now redo this problem, using cylindrical shells. As you will see, the integral that we obtain will not look very much like the disk-method integral; nonetheless, we'll get the same answer. We'd better!

As we discussed above, when using shells, we see that the y-coordinates of the region B are between 0 and 4. For each y-coordinate between 0 and 4, we find that the length $h(y)$ of the cross-sectional line segment is 2 minus the distance from the y-axis to the curve $y = x^2$; this distance is the x-coordinate on the curve, which is $\sqrt{y}$. Thus, $h(y) = 2 - \sqrt{y}$, and the volume of the solid of revolution is

$$V = \int_{y=0}^{y=4} dV = \int_0^4 2\pi r h \, dy = \int_0^4 2\pi y(2 - y^{1/2}) \, dy = 2\pi \int_0^4 \left(2y - y^{3/2}\right) dy =$$

$$2\pi \left(y^2 - \frac{y^{5/2}}{5/2} \right) \Bigg|_0^4 = 2\pi \left(16 - \frac{2}{5} \cdot 32 \right) = \frac{32\pi}{5}.$$

Math works again!!!

Example 3.5.12. You may wonder why you would ever prefer using cylindrical shells over using disks or washers to find the volume of a solid of revolution.

Consider the curved triangular region B in the xy-plane, which has part of the y-axis as its left edge, its top edge is given by $y = 2 - x$, and its bottom curved edge is given by $y = \sqrt{x}$. This region lies between $x = 0$ and $x = 1$, and between $y = 0$ and $y = 2$.

Let's revolve B around the y-axis and calculate the volume of S, the resulting solid of revolution, both by using disks and by using cylindrical shells.

By disks:

If we use disks, then we take cross sections of B that are perpendicular to the axis of revolution – here, the y-axis. You may immediately see the issue that arises; the formula for the radius of the disks changes as the y-coordinate passes through $y = 1$, the y-coordinate where the graphs of $y = 2 - x$ and $y = \sqrt{x}$ intersect.

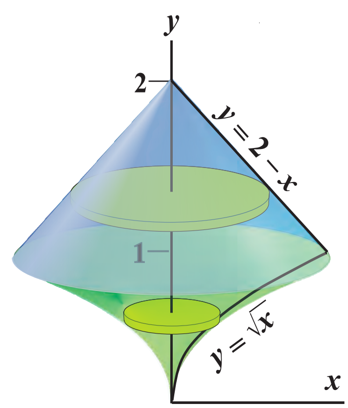


Figure 3.43: Volume by disks.

For $0 \leq y \leq 1$, the radius of each cross-sectional disk is determined solely by the graph of $y = \sqrt{x}$ (and the y-axis). The radius, $r = r(y)$ is the x-coordinate of the corresponding point on the graph of $y = \sqrt{x}$; thus, $r(y) = y^2$ for $0 \leq y \leq 1$.

However, for $1 \leq y \leq 2$, the radius of each cross-sectional disk is determined solely by the graph of $y = 2 - x$. Again, the radius, $r = r(y)$ is the x-coordinate of the corresponding point on the graph, but now it's the graph of $y = 2 - x$; solving for x in terms of y, we find that $r(y) = 2 - y$ for $1 \leq y \leq 2$.

How do we deal with this changing formula for the radius of the disks? By splitting the integral.

$$V = \int_0^2 \pi r^2 \, dy = \int_0^1 \pi r^2 \, dy + \int_1^2 \pi r^2 \, dy = \int_0^1 \pi (y^2)^2 \, dy + \int_1^2 \pi (2-y)^2 \, dy =$$

$$\int_0^1 \pi y^4\,dy \;+\; \int_1^2 \pi(4-4y+y^2)\,dy \;=\; \pi\left[\; \frac{y^5}{5}\bigg|_0^1 \;+\; \left(4y-2y^2+\frac{y^3}{3}\right)\bigg|_1^2\;\right] \;=$$

$$\pi\left[\;\frac{1}{5} \;+\; \left(8-8+\frac{8}{3}\right)-\left(4-2+\frac{1}{3}\right)\;\right] \;=\; \frac{8\pi}{15}.$$

By cylindrical shells:

However, even though the integrals that appear when using the disk method are easy, many people find it aesthetically unpleasing to have to split the integral up into two pieces; if you use cylindrical shells, there is no need to split things up.

When using cylindrical shells, we take cross sections parallel to the y-axis, i.e., perpendicular to the x-axis, for $0 \le x \le 1$. For a given x-coordinate between 0 and 1, the height $h(x)$ of the cylindrical shell, which is the length of the x cross-sectional line segment in B, is the difference between the corresponding y-coordinates on the two graphs. Thus, $h(x) = (2-x) - \sqrt{x}$. The radius $r(x)$ of the cylindrical shell is the distance from the x cross-sectional line segment in B to the y-axis; this distance is simply x.

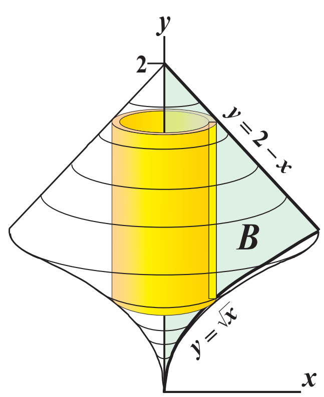


Figure 3.44: Volume by cylindrical shells.

Therefore, the infinitesimal volume of each cylindrical shell is given by

$$dV \;=\; 2\pi r h\,dx \;=\; 2\pi x(2-x-x^{1/2})\,dx,$$

and so the volume is

$$V = \int_{x=0}^{x=1} dV = \int_0^1 2\pi x(2 - x - x^{1/2})\,dx = \int_0^1 2\pi(2x - x^2 - x^{3/2})\,dx =$$

$$2\pi\left(x^2 - \frac{x^3}{3} - \frac{x^{5/2}}{5/2}\right)\Bigg|_0^1 = 2\pi\left(1 - \frac{1}{3} - \frac{2}{5}\right) = \frac{8\pi}{15}.$$

A final note on this example:

You should convince yourself that, had we revolved the region B around the x-axis, instead of around the y-axis, then using washers would **not** require us to split our integral into two pieces, but using cylindrical shells would have.

Example 3.5.13. In the previous example, we saw that using cylindrical shells might sometimes be easier for calculating volumes of solids of revolution. However, which method to use is not always a question of mild convenience or aesthetics.

Consider the region B under the graph of $y = f(x) = x - x^5$ and above the interval $[0, 1]$ on the x-axis; see Figure 3.45.

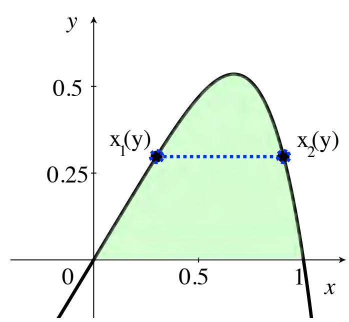


Figure 3.45: The region B under $y = x - x^5$.

If we revolve B about the x-axis, the volume V of the resulting solid of revolution is easy to calculate via disks; we obtain

$$V = \int_0^1 \pi(x - x^5)^2\,dx = \pi\int_0^1 \left(x^2 - 2x^6 + x^{10}\right)\,dx =$$

$$\pi\left(\frac{x^3}{3}-\frac{2x^7}{7}+\frac{x^{11}}{11}\right)\Bigg|_0^1 \;=\; \frac{32\pi}{231}.$$

However, attempting to calculate the volume V using cylindrical shells leads to insurmountable difficulties. We can set $f'(x)$ equal to 0 and find that the maximum y value for $0 \leq x \leq 1$ is $4/5^{5/4} \approx 0.53499$, but we have a bigger problem. Looking at Figure 3.45, you can see that, for each y value between 0 and $4/5^{5/4}$, there are two corresponding x values on the graph of $y = x - x^5$; call the smaller one $x_1(y)$ and the larger one $x_2(y)$. Using cylindrical shells, we would obtain

$$V \;=\; \int_0^{4/5^{5/4}} 2\pi y(x_2(y) - x_1(y))\, dy,$$

but we need formulas for x_1 and x_2 in terms of y. This means we need to solve $y = x - x^5$ for x; however, there is no "nice" formula for the solutions to general quintic equations, and so we are completely stuck.

You may be thinking "ah - while the shell method sometimes gives nicer solutions, it can run into insurmountable problems that the disk/washer method does not have". This is not the case; you should revolve the region B about the y-axis, instead of about the x-axis, and convince yourself that using cylindrical shells is now the manageable method, while, this time, using washers leads to the problem of solving $y = x - x^5$ for x.

The moral of the story should be an obvious, general principle: when you have multiple techniques for obtaining solutions, and one of them doesn't work, try another one!

Example 3.5.14. (**Gabriel's horn**) In the final example of this section, we will calculate the volume of a solid of revolution, even though the solid "goes out to infinity".

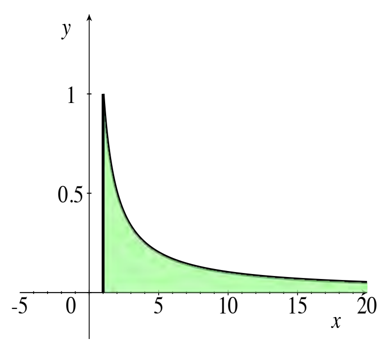


Figure 3.46: The region B under $y = 1/x$ and above $[1, \infty)$.

Consider the region B under the graph of $y = 1/x$ and above the interval $[1, \infty)$ on the x-axis. When you revolve B around the x-axis, the solid of revolution S that is generated looks like some sort of long, narrow, trumpet, a "horn", where the mouthpiece has been stretched out to infinity. Of course, the solid region should be filled in, but we've drawn just the surface in order to make it look more horn-like.

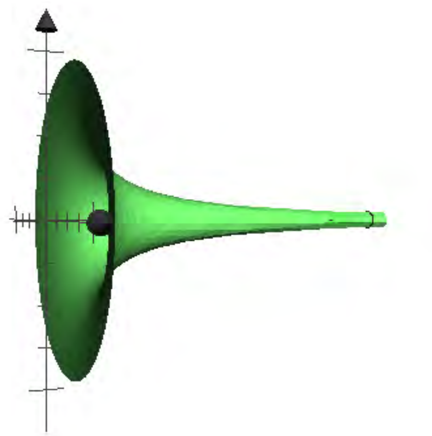


Figure 3.47: Gabriel's Horn.

We'll use disks to calculate the volume of S. Our disks are cross sections perpendicular to the x-axis; the radii are given by $r(x) = 1/x$, and so $dV = \pi(1/x)^2\, dx$. The total volume is

$$V = \int_{x=1}^{x=\infty} dV = \int_1^{\infty} \pi x^{-2}\, dx = \lim_{b\to\infty} \int_1^b \pi x^{-2}\, dx = \pi \left[\lim_{b\to\infty} \left. \frac{x^{-1}}{-1} \right|_1^b \right] =$$

$$\pi \left[\lim_{b\to\infty} \left(-\frac{1}{b} + 1 \right) \right] = \pi.$$

3.5.1 Exercises

Suppose that the solids in Exercises 1 - 4 have cross-sectional area $A(x)$ for x in the given range. Calculate the volume of the solids.

1. $A(x) = 3x^2 + 1$, $2 \le x \le 4$.

2. $A(x) = \cos x$, $-\pi/2 \leq x \leq \pi/2$.

3. $A(x) = x - x^3$, $0 \leq x \leq 1$.

4. $A(x) = 2\pi x$, $3 \leq x \leq 6$.

5. Suppose S is a general cone with a square base lying in the plane $z = 10$. Each side of the base has length 10 and the vertex of the cone is the point $(0, 0, 2)$. What is the volume of the general cone?

6. A *frustum* of a pyramid is constructed by "chopping off" the top of the pyramid. More specifically, let R_1 and R_2 be squares in the planes $z = 0$ and $z = 10$, respectively. R_1 has sides of lengths 12 and R_2 has sides of lengths 4. Then the frustum S is the of points on all line segments connecting R_1 and R_2. What is the volume of S?

7. Redo the previous problem where the side lengths of R_1 and R_2 are a and b, respectively, and where R_2 lies in the plane $z = h$. Assume R_1 still lies in the plane $z = 0$.

8. Suppose that a solid has its base in the xy-plane, and that each cross section, perpendicular to the x-axis, for $0 \leq x \leq 1$, is a (filled-in) square, one of whose sides goes from the x-axis out to the curve $y = x^2$. See Figure 3.48. Find the volume of the solid.

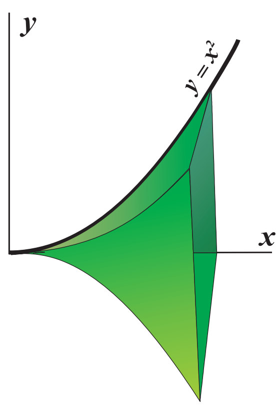


Figure 3.48:

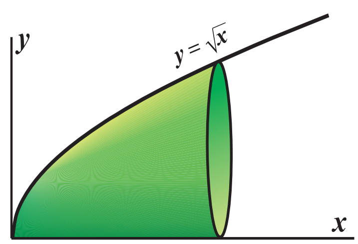


Figure 3.49:

9. Suppose that a solid is formed in such a way that each cross section perpendicular to the x-axis, for $0 \leq x \leq 1$, is a disk, a diameter of which goes from the x-axis out to the curve $y = \sqrt{x}$. See Figure 3.49. Find the volume of the solid.

Determine the volume of the region obtained by revolving the region lying below the graph of the given function and above the x-axis about the specified axis.

10. $y = \sin x$, x-axis, x in $[0, \pi]$.

11. $y = x^2 + x + 1$, $y = 10$, x in $[0, 2]$.

12. $y = \ln x$, x-axis, x in $[1, b]$.

13. $y = x^2$, x-axis, x in $[0, 1]$.

14. $y = x^2 + k$, x-axis, x in $[0, 1]$.

15. $y = x^n$, x-axis, x in $[0, 1]$ and $n > 0$.

16. $x + y = c$, $c > 0$, x-axis, x in $[0, c]$.

17. $y = \sin x$, about the line $y = 2$ for x in $[0, \pi]$.

18. $y = 2 + \sin x$, about the line $y = 2$ for x in $[0, \pi]$.

19. Revolve the region in the first quadrant bounded by the lines $f(x) = x$, and $g(x) = 5(x-3)$, about the x-axis. What is the volume of the solid of revolution?

20. Revolve the region bounded by the graphs of $h(x) = e^{-x}$, $j(x) = 2$ and the line $x = 1$ about the x-axis. What is the volume of the resulting solid?

21. Let a and b be the two solutions in $[0, \pi)$ to the equation $g(x) = k(x)$ where $g(x) = \sin x$ and $k(x) = \sin 2x$. What is the volume of the solid obtained by revolving the region defined by the graphs of g and k and the lines $x = a$ and $x = b$ about the x-axis?

22. Let $h(x) = x^2 + 3x - 1$, $k(x) = 7x - 4$. What is the volume obtained by revolving the region bounded by the graphs of these two functions about

    a. the x-axis;

    b. the y-axis?

In each of Exercises 23 through 27, approximate the volume obtained by revolving, about the y-axis, the region lying below the graph of the given function and above the given interval on the x-axis. Use the midpoint method of approximation along with the shell method of calculating volume. Partition the interval of integration into $n = 4$ evenly spaced subintervals.

23. $f(x) = \sqrt{\sin x}$, x in $[0, \pi]$.

24. $h(x) = \ln(\sin x) + 3$, x in $[\pi/4, 3\pi/4]$.

25. $g(x) = \tan^{-1} e^x$, x in $[-4, 0]$.

26. $j(x) = \cosh(1/x)$, x in $[1, 5]$.

27. $k(x) = e^{\cos x}$, x in $[0, \pi]$.

28. Let $u(x) = x^m$ and $v(x) = x^n$ where $m > n \geq 1$. What is the volume obtained by revolving the region bounded by the graphs of these two functions for x in $[0, 1]$ about the x-axis?

29. What is the volume obtained by revolving the region described in the previous problem about the y-axis?

30. Suppose C is a right circular cone with its vertex at the origin, and with its circular base in the plane $z = h$. Assume the radius of the base is r. What is the average area of a cross section parallel to the xy plane? Express your answer in terms of the volume of the cone and its height.

31. Suppose G is a general cone with vertex at the origin and with its base in the plane $z = h$. What is the average area of a cross section parallel to the xy plane?

32. Suppose that $f \geq 0$, that f is continuous, and that F is an anti-derivative of f. Prove that the volume obtained by revolving, about the y-axis, the region below the graph of $f(x)$ and above the interval $[0, b]$ (on the x-axis) is $V = 2\pi bF(b) - 2\pi \int_0^b F(x)\,dx$.

33. Let $f(x) = \dfrac{|\sin x|}{x}$ for $x > 1$. Consider the solid obtained by revolving the region lying below the graph of f and above the x-axis about the x-axis. Does this region have finite volume? Hint: compare the volume to another region with a known volume.

34. Recall that the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ defines an ellipse. Let R be the portion of the ellipse lying in the upper-half plane, $y > 0$.

    a. What is the volume obtained by revolving the region about the x-axis?

    b. Set $a = b$ in your result from part (a) and verify that this is the correct volume formula for a sphere.

35. Let $f(x) = \cosh x$ and $g(x) = x + 1$. Let a and b be the two x values where the graphs of the functions intersect. Calculate the volume obtained by revolving the area between these two curves on the interval $[a, b]$ about the x-axis.

36. Calculate the volume obtained by revolving the region in the previous problem about the y-axis.

37. A *torus* in three dimensions can be obtained as follows. Let D be a disc in the xy plane with center $(R, 0)$ and with radius r where $r < R$. Rotate the disc about the y-axis to realize a torus. For the next few problems, we refer to r as the *inner radius* and R as the *outer radius*. Use the disc method to calculate the volume of this torus.

38. Another way to construct a torus is to bend a cylinder until the top and bottom meet. The volume of the resultant torus is identical to that of the original cylinder. Use the setup in the previous problem and calculate the volume of the torus by calculating the volume of a cylinder.

39. Calculate the volume obtained by revolving the region below the graph of $y = \sin x$ and above the x-axis on the interval x in $[0, \pi]$ about the y-axis using:

    a. The shell method;

    b. The disk/washer method.

40. Suppose a torus shaped inner-tube is floating in a swimming pool. The inner radius is 4 inches, and the outer radius is 18 inches. A portion of the inner tube is submerged below the water level. If the water level is half an inch, what is the volume of the submerged portion of the inner tube?

41. Suppose a spherical volleyball with radius 65 cm is floating in a pool so that the ball is submerged up to 5 cm. What is the volume of the submerged portion of the ball?

42. Suppose an object with volume V is partially submerged in some fluid. Let V_s be the volume of the submerged portion of the object. Suppose the density of the object is ρ_o and the density of the fluid is ρ_f. A basic result of fluid statics states that

$$\frac{V_s}{V} = \frac{\rho_o}{\rho_f}.$$

    Calculate the densities of the inner tube and volleyball mentioned in the previous two problems if the density of water 1 gm / cm^3.

43. Suppose that the volleyball of radius 65 cm is again floating in water. The ratio of the volumes of the submerged portion of the ball to the unsubmerged portion is 1 : 4. To what depth is the volleyball submerged?

44. Suppose we try to generalize Gabriel's Horn.

    a. Let $f(x) = 1/x^n$ and consider the solid formed by revolving the region lying below the graph of f and above the x-axis about the x-axis on the interval x in $[1, \infty)$. Does the volume exist when $n \geq 1$? If so, what is the volume?

b. Does the volume exist when $0 < n < 1$? If so, what is the volume?

45. Calculate the volume of the solid obtained by revolving the region bounded by the graph of $y = 1/x$, the line $x = 1$ and the line $x = a$ where $0 < a < 1$, about the y-axis. Does this integral converge as $a \to 0$?

46. Let R be the upper half of the disc centered at the origin with radius r. What is the volume of the region obtained by revolving R about the line $y = b$ where $b \geq r$ and $b \leq 0$.

47. Let $f(x) = cx^2$ and $g(x) = c^2\sqrt{x}$ where $c > 0$. Let V_c be the volume obtained by revolving the region bounded by the graphs of f and g about the x-axis. What is V_c?

48. Calculate the volume of the region defined by revolving the region lying below the graph of $y = \sin(x^2)$ and above the x-axis about the y-axis. Assume x in $[0, \sqrt{\pi}]$.

49. Suppose an annulus with inner radius r and outer radius R is situated in the plane $z = \pi$. The center of the annulus is the point $(2, 700, \pi)$. What is the volume of the general cone with base the annulus and with a vertex at the origin?

50. Calculate the volume, if it exists, of the solid obtained by revolving the region lying below the graph of $y = \sec x$ and above the x-axis about the x-axis on the interval $[0, \pi/2]$.

51. Let $g(x) = e^{-x^2}$, $x \in [0, \infty)$. Calculate the volume, if it exists, of the region obtained by revolving the region lying below the graph of $g(x)$ and above the x-axis about the y-axis.

52. What is the volume obtained by revolving the region lying below the graph of $y = e^{-x}$ and above the x-axis about the x-axis on the interval $[0, \infty)$?

53. Consider the function
$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ e^{x} & x < 0 \end{cases}.$$

a. What is the volume obtained by revolving the region lying below the graph of $f(x)$ and above the x-axis about the x-axis on the interval $(-\infty, \infty)$?

b. Use the disc method to try and calculate the volume obtained by revolving the region below the graph of $f(x)$ and above the x-axis around the y-axis for $a \leq y \leq 1$. Does this integral converge as $a \to 0$?

c. Try calculating the volume in part (b) using the shell method. Does your answer concur with that in part (b)?

54. A right circular cone is obtained by revolving the region bounded by the curves $y = mx$, $x = 0$, and $y = h$, where $m \neq 0$, about the y-axis. What is the volume of the cone? Leave your answer in terms of m and h. Make sure to consider the case that m is negative.

55. Archimedes noted that if a right circular cone, a hemisphere, and a cylinder all have the same heights and radii, then the ratios of their volumes is $1:2:3$ in the order specified above. Verify this claim. Observe that if we append a complete sphere to the tail of this sequence, the ratio pattern extends to $1:2:3:4$.

56. Suppose a regular n-gon (equilateral, equiangular and convex) is situated in the plane $z = h$ with its center on the z-axis. Recall that the *apothem* is the distance from the center of the polygon to the midpoint of one the sides.

    a. Prove that the area of the n-gon with apothem a is given by $B_n = a^2 n \tan \dfrac{\pi}{n}$.

    b. What is the volume, V_n, of the solid defined by the line segments connecting the origin to the polygon?

    c. Show that $B_n \to \pi a^2$ and $V_n \to \dfrac{\pi a^2 h}{3}$ as $n \to \infty$. Geometrically, why does this make sense?

3.6 Surface Area

In this section, we will look at *surfaces of revolution*; these are the outer surfaces (or, parts of the surfaces) of solids of revolution, which we discussed in the previous section. We will use the definitions and results on arc lengths of curves from Section 3.3, and will briefly recall the needed material here.

Suppose that C is a simple regular curve (Definition 3.3.9) in the xy-plane. We want to revolve C around some line ℓ, the axis of revolution, look at the surface of revolution that is swept out, and find its area, i.e., calculate the surface area of a surface of revolution.

Consider an infinitesimal piece of arc length ds on C, which is a distance r from ℓ. Then, ds "looks" infinitesimally like the length of part of a straight line which is parallel to ℓ, and the infinitesimal area dA swept out by ds is that of a right, circular cylinder of radius r and height ds. (It is not trivial to show that the tilt and straightening of the arc length to make it "parallel" to the axis is negligible, compared to the "infinitesimal" arc length; it is nonetheless true.) Thus,

$$dA \;=\; 2\pi r\, ds.$$

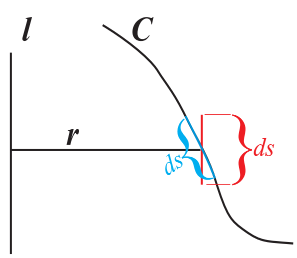


Figure 3.50: A negligible tilt of the arc length.

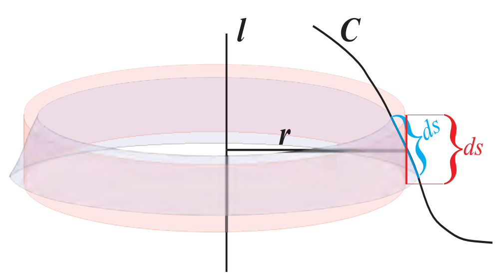


Figure 3.51: Revolving around the axis.

Recall from Proposition 3.3.12 that, for a simple regular parameterization $\vec{p}(t) = (x(t), y(t))$,

$$ds \;=\; \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt.$$

Therefore, we have:

Proposition 3.6.1. *If $\vec{p}(t) = (x(t), y(t))$, for $a \leq t \leq b$, is a simple regular parameterization of C, and $r(t)$ equals the distance from $\vec{p}(t)$ to ℓ, then the area of the surface of revolution generated by revolving C around ℓ is*

$$\text{surface area} \;=\; \int_{t=a}^{t=b} dA \;=\; \int_{t=a}^{t=b} 2\pi r\, ds \;=\; \int_a^b 2\pi r(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

As in Section 3.3, if we are given the curve C as part of the graph of a function $y = f(x)$ or $x = f(y)$, then ds can be written as

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{or} \qquad \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy,$$

respectively; if you were using one of these forms for ds, then you would need to write the distance to ℓ, the "radius" r, in terms of x or y, respectively. Rather than rewrite Proposition 3.6.1 in every conceivable form, we shall, instead, look at four specific examples.

We should remark at this point that, as with arc length integrals, the integrals/anti-derivatives that arise in trying to calculate the areas of surfaces of revolution are usually ridiculously difficult or impossible (to obtain as elementary functions). Of course, you can approximate the integrals very closely by using numerical techniques from Section 2.6. The examples that we are about to give are **very** special ones, for which we will be able to obtain exact areas.

Example 3.6.2. You may already know a/the formula for the surface area of a sphere of radius R, but let's calculate it as the area of a surface of revolution, and make sure that we get the well-known result.

Consider the upper semi-circle C of radius R, centered at the origin in the xy-plane. This semi-circle has a simple regular parameterization given by

$$\vec{p}(t) \;=\; (x(t), y(t)) \;=\; (R\cos t, R\sin t), \quad \text{for } 0 \leq t \leq \pi,$$

and the sphere of radius R, centered at the origin, is the surface of revolution obtained by revolving C around the x-axis. As we are revolving around the x-axis, $r = r(t)$ is the y-coordinate on C, i.e., $r = R\sin t$.

We calculate

$$ds \;=\; \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \;=\; \sqrt{(-R\sin t)^2 + (R\cos t)^2}\,dt \;=\; R\,dt,$$

and

$$\text{surface area of a sphere of radius } R \;=\; \int_{t=0}^{t=\pi} 2\pi r\,ds \;=\; \int_0^{\pi} 2\pi(R\sin t)\,R\,dt \;=$$

$$2\pi R^2(-\cos t)\Big|_0^{\pi} \;=\; 2\pi R^2\big(-(-1)-(-1)\big) \;=\; 4\pi R^2,$$

which is what you may have learned in high school.

Remark 3.6.3. You may look at the formula $V = \frac{4}{3}\pi R^3$ for the volume inside a sphere of radius R and the formula $A = 4\pi R^2$ for the surface area of the sphere, and notice that $dV/dR = A$. Is this just a coincidence? No.

Just as we used integrals involving cylindrical shells, thickened cylinders, to calculate the volumes of solids of revolution in Section 3.5, we **could** have used integrals involving *spherical shells*, infinitesimally thickened spheres (remember: the sphere is just the surface, not the inside) to calculate the volume inside a sphere.

Let $A(r)$ denote the surface area of a sphere of radius r. Then, consider the volume inside a sphere of radius R. For each r, where $0 \leq r \leq R$, we can consider the infinitesimal volume of the infinitesimally thickened sphere of radius r (with the same center as the big sphere); this infinitesimal volume is

$$dV \;=\; A(r)\,dr,$$

and the total volume inside the sphere of radius R is

$$V(R) \;=\; \int_0^R A(r)\,dr.$$

assuming that we already know that $V(R) = \frac{4}{3}\pi R^3$, we obtain that

$$\frac{4}{3}\pi R^3 \;=\; \int_0^R A(r)\,dr.$$

Now, the first part of the Fundamental Theorem of Calculus, Theorem 2.4.7, tells us that the derivative, with respect to R of the right-hand side above is simply $A(R)$, which means that, if we differentiate both sides of the equality, we obtain

$$4\pi R^2 \;=\; A(R),$$

which is what we found in Example 3.6.2.

Example 3.6.4. Let C be the portion of the graph of $y = x^2$ between $x = 0$ and $x = 2$. Revolve C around the y-axis and find the area of the resulting surface of revolution (which looks like some sort of rounded cup). See Figure 3.52.

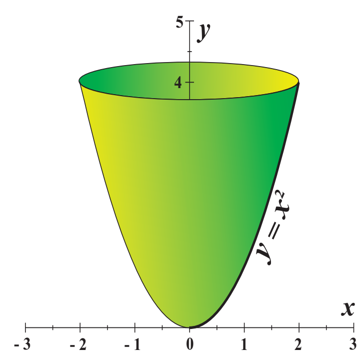


Figure 3.52: Graph of $y = x^2$, $0 \leq x \leq 2$, revolved around the y-axis.

Solution:

We will calculate

$$\text{surface area} \;=\; \int_{x=0}^{x=2} 2\pi r\,ds.$$

As we are given y as a function of x, we want to use that

$$ds \;=\; \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \;=\; \sqrt{1+(2x)^2}\,dx \;=\; \sqrt{1+4x^2}\,dx.$$

We also need to write r as a function of x. But, as we are revolving C around the y-axis, the distance to the axis of revolution, r, is simply the x-coordinate on C, i.e., in terms of x, $r = x$.

Thus, the integral that we obtain is:

$$\text{surface area} \;=\; \int_0^2 2\pi x\sqrt{1+4x^2}\,dx \;=\; \int_0^2 2\pi x(1+4x^2)^{1/2}\,dx.$$

Hopefully, you see fairly quickly that the substitution $u = 1+4x^2$ will enable us to evaluate the integral. We find that $du = 8x\,dx$, or $du/8 = x\,dx$. We also see that, when $x = 0$, $u = 1 + 4\cdot 0^2 = 1$ and, when $x = 2$, $u = 1 + 4 \cdot 2^2 = 17$.

Hence, we have:

$$\text{surface area} \;=\; 2\pi\int_1^{17} u^{1/2}\cdot\frac{1}{8}\,du \;=\; \frac{\pi}{4}\cdot\left.\frac{u^{3/2}}{3/2}\right|_1^{17} \;=\; \frac{\pi}{6}\big[(17)^{3/2}-1\big].$$

Example 3.6.5. Let C be the portion of the graph of $y = x^3$ between $x = 0$ and $x = 2$. Revolve C around the x-axis and find the area of the resulting surface of revolution. See Figure 3.53.

Solution:

Once again, we will calculate

$$\text{surface area} \;=\; \int_{x=0}^{x=2} 2\pi r\,ds,$$

and, as we are again given y as a function of x, we want to use that

$$ds \;=\; \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \;=\; \sqrt{1+(3x^2)^2}\,dx \;=\; \sqrt{1+9x^4}\,dx.$$

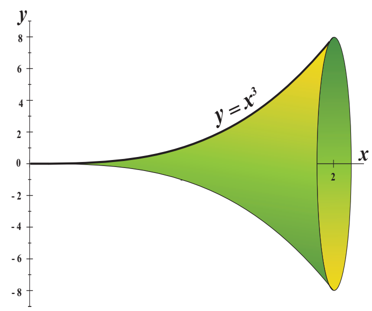


Figure 3.53: Graph of $y = x^3$, $0 \leq x \leq 2$, revolved around the x-axis.

We again need to write r as a function of x. But, this time, as we are revolving C around the x-axis, the distance to the axis of revolution, r, is the y-coordinate on C, i.e., in terms of x, $r = y = x^3$.

Thus, the integral that we obtain is:

$$\text{surface area} \;=\; \int_0^2 2\pi x^3 \sqrt{1+9x^4}\,dx \;=\; \int_0^2 2\pi x^3 (1+9x^4)^{1/2}\,dx.$$

The substitution $u = 1 + 9x^4$ will enable us to evaluate the integral. We find that $du = 36x^3\,dx$, or $du/36 = x^3\,dx$. We also see that, when $x = 0$, $u = 1 + 9 \cdot 0^4 = 1$ and, when $x = 2$, $u = 1 + 9 \cdot 2^4 = 145$.

Hence, we have:

$$\text{surface area} \;=\; 2\pi \int_1^{145} u^{1/2} \cdot \frac{1}{36}\,du \;=\; \frac{\pi}{18} \cdot \frac{u^{3/2}}{3/2}\bigg|_1^{145} \;=\; \frac{\pi}{27}\big[(145)^{3/2} - 1\big].$$

Example 3.6.6. (**Gabriel's horn, revisited**) In Example 3.5.14, we looked at the region B under the graph of $y = 1/x$ and above the interval $[1, \infty)$ on the x-axis, revolved B around the x-axis, and found that the resulting solid of revolution, Gabriel's horn, had volume π.

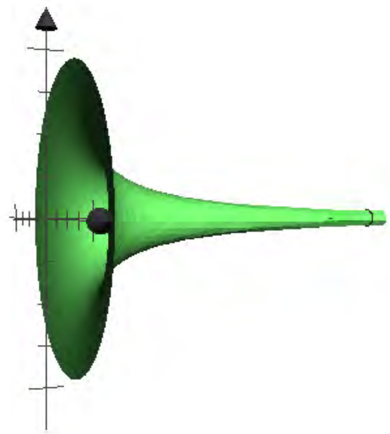


Figure 3.54: Gabriel's Horn.

In this example, we wish to show that the surface area of Gabriel's horn is **infinite**. (When we write "surface area" here, we mean the area of the "sides", i.e., we are excluding the disk that could fill the flared end of the horn at $x = 1$. However, as the surface area is infinite without the disk at $x = 1$, the surface area would certainly still be infinite if we included the disk.)

The curve that we are revolving around the x-axis is the graph of $y = 1/x = x^{-1}$ for $x \geq 1$. We find

$$ds \;=\; \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \;=\; \sqrt{1+(-x^{-2})^2}\,dx \;=\; \sqrt{1+x^{-4}}\,dx,$$

and r, the distance from a point on the graph to the x-axis, in terms of x, is given by the y-coordinate of the graph of $y = 1/x$, i.e., $r = 1/x$.

Therefore,

$$\text{surface area} \;=\; \int_1^\infty 2\pi\cdot\frac{1}{x}\cdot\sqrt{1+x^{-4}}\,dx \;=\; 2\pi\cdot\lim_{b\to\infty}\int_1^b \frac{1}{x}\cdot\sqrt{1+x^{-4}}\,dx.$$

An easy substitution will **not** let us find an anti-derivative of $\frac{1}{x}\cdot\sqrt{1+x^{-4}}$. However, note that, for $x \geq 1$,

$$\frac{1}{x}\cdot\sqrt{1+x^{-4}} \;\geq\; \frac{1}{x}.$$

Thus, Theorem 2.3.20 tells us that, if $b \geq 1$, then

$$\int_1^b \frac{1}{x}\cdot\sqrt{1+x^{-4}}\,dx \;\geq\; \int_1^b \frac{1}{x}\,dx \;=\; \ln x\,\Big|_1^b \;=\; \ln b.$$

As b goes to infinity, so does $\ln b$, and this forces the larger quantity $\int_1^b \frac{1}{x} \cdot \sqrt{1+x^{-4}}\, dx$ to go to infinity also. We conclude:

$$\text{surface area of Gabriel's horn} \;=\; 2\pi \cdot \infty \;=\; \infty.$$

Remark 3.6.7. The results of Example 3.5.14 and Example 3.6.6 tell us that Gabriel's horn has finite volume, but infinite surface area. These results are sometimes described as "you can fill Gabriel's horn, but you can't paint it". The clever student then asks "What if you filled the horn with paint? Wouldn't that paint the surface?".

This seeming contradiction is caused by a lack of precision in claiming that having infinite surface area means that a surface can't be painted. What **is** true is that having infinite surface area implies that the surface cannot be painted with a finite amount of paint, **if** we are required to have a **uniformly thick** layer of paint everywhere. However, if it were possible to have arbitrarily thin layers of paint, then the surface of Gabriel's horn could be painted.

We'll try to describe a similar problem, in which it's hopefully easier to see what's going on.

Suppose you took a cube that's 1 foot long on each side, and you fill it with paint. Then, the volume of paint is finite; it's 1 ft^3. The surface area of the cube is the combined area of the 6 sides, namely, 6 ft^2.

Now, imagine chopping the cube in half by a cut which is parallel to two of the faces, while simultaneously sealing the two new exposed sides (or, you could think of inserting dividers into the cube first, then chopping the cube in half). The total volume of paint in the two half-cubes is still 1 ft^3, but now the surface area has gone up, because we created two new faces; the surface area is now $6+2=8$ ft^2.

Now, by making a cut parallel to the original cut, divide (and seal) one of the two half-cubes from above; the volume of paint remains 1 ft^3, but we added two more faces, for a new surface area of 10 ft^2. Imagine continuing this process indefinitely, each time, taking one of your smallest two pieces, and dividing it into two pieces by making a cut parallel to all of the other cuts. The volume of paint is always 1 ft^3, but the surface area gets arbitrarily large or, in the limit, is infinite.

Is there a contradiction here? No, but note that the layer of paint on the sides of the smaller and smaller pieces can't be any thicker than the width of each piece, which is getting arbitrary small (close to zero).

3.6.1 Exercises

Calculate the area of the surface obtained by revolving the graph of the function about the x-axis. Recall that the notation $x \in [a, b]$ means x takes on values in this interval. In the first problem, for example, the portion of the graph of $y = 3x + 4$ between $x = 3$ and $x = 7$ is being revolved.

1. $y = 3x + 4$, $x \in [3, 7]$.
2. $y = \sin x$, $x \in [0, \pi]$.
3. $y = e^x$, $x \in [0, \pi]$.
4. $y = \cosh x$, $x \in [0, 1]$.
5. $y = \sqrt{x}$, $x \in [0, 9]$.
6. $y = \sinh x$, $x \in [0, 10]$.

Calculate the area of the surface obtained by revolving the graph of the function about the y-axis.

7. $y = 2x - 3$, $x \in [4, 7]$.
8. $y = 4x^2$, $x \in [0, 6]$.
9. $y = x^{1/3}$, $x \in [1, 8]$.
10. $y = x^3$, $x \in [0, 2]$.
11. $y = \cosh x$, $x \in [0, 4]$.

Calculate the area of the surface obtained by revolving the graph of the function about the given axis.

12. $y = 3x + 13$, $x = 0$, $x \in [-4, 0]$.
13. $y = 3x + 1$, $x = 4$, $x \in [0, 4]$.
14. $y = x^2$, $y = 1$, $x \in [0, 1]$.
15. $y = x^2$, $x = 1$, $x \in [0, 1]$.

16. $y = x - 5$, $y = 3$, $x \in [5, 8]$.

17. $y = mx + b$, $m \neq 0$, $y = b$, $x \in [0, 1]$.

Approximate the areas of the surfaces obtained by revolving the graph of the function about the y-axis. Use the Midpoint Rule with $n = 4$ evenly spaced partitions.

18. $y = \ln x$, $x \in [1, 9]$.

19. $y = x^2 + 3x + 5$, $x \in [0, 12]$.

20. $y = x^4$, $x \in [3.7]$.

21. $y = x^{-2}$, $x \in [4, 8]$.

22. $y = \tanh x$, $x \in [0, 4]$.

23. $y = \dfrac{1}{1+x}$, $x \in [0, 1]$.

24. Suppose that $f(x)$ is a positive differentiable function. Argue that the surface area obtained by revolving the graph of $f(x)$ about the x-axis on the interval $[a, b]$ is

$$A = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

25. Given the setup in the previous problem, what is the surface area obtained by revolving $f(x)$ about the line $y = y_0$? Assume $f(x) > y_0$ for all $x \in [a, b]$.

26. Suppose again that $f(x)$ is a positive differentiable function. Argue that the surface area obtained by revolving the graph of $f(x)$ about the y-axis over the interval $[a, b]$ is

$$A = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\, dx.$$

Assume $a > 0$.

27. Given the setup in the previous problem, what is the surface area obtained by revolving the graph of $f(x)$ about the line $x = x_0$? Assume $x_0 < a$.

28. a. What is the surface area obtained by revolving the line $y = mx$ about the x-axis for $x \in [0, a]$?

b. If the surface area function you obtained in part (a) is $S(m)$, is S a continuous function of m? Is it differentiable?

29. Let $a, b > 0$. What is the area obtained by revolving the curve $y = a\sqrt{x+b}$ about the x-axis for $x \in [0,1]$?

30. Consider the torus obtained by revolving the circle $(x-2)^2 + y^2 = 1$ about the y axis. Calculate the area of the torus by computing the area of the upper half of the torus and multiplying by two.

31. a. What is the surface area obtained by revolving the line $y = mx$ about the y-axis for $x \in [0, a]$?

    b. If the surface area function you obtained in part (a) is $S(m)$, is S a continuous function of m? Is it differentiable?

32. Consider the horizontal line segment in the xy-plane with end points $(1,3)$ and $(2,3)$.

    a. Find a parameterization of the line segment.

    b. Find the area of the surface obtained by revolving the segment about the y-axis.

    c. What familiar shape is this? Check that your answer to part (b) is correct by calculating the area using classical geometry.

33. Consider again the horizontal line segment in the xy-plane with end points $(1,3)$ and $(2,3)$.

    a. Find the area of the surface obtained by revolving the segment about the y-axis.

    b. What familiar shape is this? Check that your answer to part (a) is correct by calculating the area using classical geometry.

34. Let C be a right circular cone obtained by revolving the line $y = x$ about the y-axis. Suppose that sand is poured into the cone.

    a. Calculate the surface area of the sand in the cone as a function of y, the height of the sand in the cone. Include the circular base of the cone in your analysis.

    b. If $S(y)$ is the surface area of the sand in the cone, what is dS/dy?

35. Prove that the distance between a point (x_1, y_1) and the line $y = mx$ is given by the equation
$$d = \frac{|y_1 - mx_1|}{\sqrt{m^2+1}}.$$

36. Assume $f(x)$ is a continuous function with $f(x) > mx$. Use the previous problem to derive an integral equation for the area obtained by revolving the graph of the function $f(x)$ about the line $y = mx$ for $x \in [a, c]$.

37. Generalize the result in the previous problem. How does the equation for the surface area change if the axis of rotation has the more general equation $y = mx + b$?

38. Let θ be the angle the line $y = mx$ makes with the x-axis. What is the surface area of the figure obtained by revolving the graph of $f(x)$ about the line $y = mx$ in terms of θ? Assume $x \in [a, c]$. Hint: no need to start from scratch. Use the previous problems and find an equation relating m and θ.

39. With the notation of the previous problem, derive formulas for the surface area in the special cases. Assume that in all cases, the graph of $f(x)$ lies on just one side of the axis of rotation. Assume $x \in [a, c]$.

    a. $\theta = 0$.

    b. $\theta = \pi/4$.

    c. $\theta = \pi/2$.

40. In light of these results, justify the statement "If a graph of a function lies in the first quadrant and on one side of the line $y = mx$, then calculating the surface area obtained by revolving the graph about $y = mx$ is easy if we can calculate the area obtained by revolving about the x-axis and y-axis.

41. What is the surface area of the region obtained by revolving the line segment $y = 2x + 4$, $x \in [3, 5]$ about the line $y = 2x$?

42. Let D be a the disk with radius 3 centered at the point $(4, 2)$. Then the line $y = \dfrac{1}{2}x$ cuts the disc into two equal hemispheres. Calculate the surface area of the figure formed by revolving the hemisphere about the line $y = \dfrac{1}{2}x$.

43. Let C be the circle defined by the equation $(x - 3)^2 + (y - 3)^2 = 4$. What is the surface area obtained by revolving the portion of C above the line $y = x$ about the line $y = x$?

44. What is the surface area obtained by revolving the graph of the function $h(x) = 4x - 2$ about the line $y = 2x$ for $x \in [1, 5]$? What shape is this?

In Exercises 45-49, setup an integral to calculate the surface area of the region obtained by rotating the graph of $f(x)$ about the given line. Do not evaluate the integral.

45. $f(x) = e^x$, $x \in [2, 5]$, $y = 3x$.

46. $f(x) = \cos^2 x$, $x \in [-1, 0]$, $y = x$.

47. $f(x) = \dfrac{x + 2}{x + 1}$, $x \in [2, 4]$, $y = 2x$.

48. $f(x) = x^2 + 5x + 1$, $x \in [0, 3]$, $y = \dfrac{1}{3}x$.

49. $f(x) = \sqrt{x+1}$, $x \in [3, 8]$, $y = 4x$.

3.7 Mass and Density

You are probably familiar with the concept of *density*, mass per volume. However, you may have never thought about "instantaneous density". The issue is that objects may have different densities at different points. The density $\delta(P)$ **at** a particular point P is defined by taking "small" chunks (rectangular solids) of the object around P, looking at the mass of such a chunk divided by its volume, and taking the limit of this process as the volume of the chunk approaches zero. A serious discussion of this limit belongs in multivariable Calculus, but this informal description should suffice for the applications in this section.

In terms of derivatives, we usually write that $\delta(P) = dm/dV$, i.e., that the density is the instantaneous rate of change of the mass, with respect to volume. Or, in terms of infinitesimal quantities and differentials, the infinitesimal amount of mass dm at P is given by $dm = \delta(P)\,dV$, where dV represents an infinitesimal volume in the object at P. Because it will help with the later discussion, we will borrow notation and terminology from multivariable Calculus, and use undefined terms like "manageable type", a "continuous function of three variables", and an "integral over a solid region", so that we can state:

Proposition 3.7.1. *Suppose that a density function $\delta(P)$ is continuous throughout a manageable solid region S. Then, an infinitesimal piece of mass at a point P in S is given by $dm = \delta(P)\,dV$, where dV is an infinitesimal volume in S around P, and the total mass of S is the integral of δ, with respect to the volume, over the solid region S, i.e.,*

$$\text{mass of } S \;=\; \int_S dm \;=\; \int_S \delta(P)\,dV.$$

Okay. Great. But now your question should be: How can we have a density which varies in a solid object, and still deal with the situation using single-variable Calculus? Actually, the problem is not so bad – we just need to have situations in which the density varies only in a single dimension. Okay. Fine. What does this mean?

One type of problem that we can handle is essentially what we discussed at the beginning of Section 3.5. We suppose that we have a solid region, S, in space, and that we know the area $A(x)$ of every cross-sectional slice of S which is perpendicular to the x-axis, and that the solid region lies between the x cross sections where $x = a$ and $x = b$, where $a < b$.

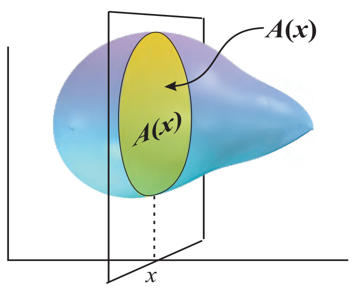


Figure 3.55: An x cross section with a constant density $\delta(x)$.

Now, however, we also assume that the density of S at each point in a given x cross section is a constant $\delta(x)$; that is, we allow the density to vary as the x cross section varies, but we do not allow the density to vary inside the cross sections.

Proposition 3.7.2. *Suppose that $a < b$, and we have a solid region, S, in space, which lies between the x cross sections where $x = a$ and $x = b$. Further, suppose we have a continuous function $A(x)$, which gives x the cross-sectional area of S, and a continuous density function $\delta(x)$ which gives the density of S at each point in the x cross section.*
Then, the infinitesimal mass dm of an infinitesimally thickened x cross section is given by

$$dm = \delta(x)\, dV \;\; = \;\; \delta(x)A(x)\, dx,$$

and

$$\text{mass of } S \;\; = \;\; \int_{x=a}^{x=b} dm \;\; = \;\; \int_a^b \delta(x)A(x)\, dx.$$

Before we begin with examples, we should discuss the units that are used for mass and weight in the Metric and English (a.k.a. FPS) Systems.

The standard unit of mass in the Metric System is a *kilogram*, abbreviated as kg. The weight of an object which has a mass of 1 kg is the force that gravity exerts on the mass, which is the mass times the acceleration produced by gravity. Hence, the weight, at sea level on the Earth, of a 1 kg object is (approximately) (1 kg)(9.8 m/s^2) = 9.8 kg-m/s^2; this is 9.8 Newtons, where a Newton, abbreviated by N, is 1 kg-m/s^2 and is the standard unit of force in the Metric System. If we assume that we first have Newtons, meters, and seconds, we could have defined 1 kilogram as 1 N/(m/s^2). The distinction between mass and weight in the Metric System is frequently blurred in common speech. People often say something like "I weigh 70 kilos (kilograms)". They mean, of course, that their mass is 70 kg; there simply is no good verb form of the word "mass".

You should be familiar with the standard unit of force in the English System; it's the *pound*, abbreviated lb. Just as one 1 N/(m/s^2) is called a kilogram, we give a name to 1 lb/(ft/s^2); it is called a *slug*, which is not abbreviated. The acceleration produced by gravity, using English units, is (approximately) 32 ft/s^2, and so, 1 slug, at sea level on the Earth, weighs (approximately) 32 lb.

Example 3.7.3. Consider the sphere of radius R, centered at the origin, and let S be the solid region contained inside the upper hemisphere, as in Example 3.5.6. Assume that all distances are measured in meters, and that the solid region S is composed of some highly compressible material, so that the lower portion of S is more dense than the upper part. Assume that the density of S at each point in a given z cross section is $\delta(z) = 1000(2R - z)$ kg/m^3. Find the mass m of S.

Solution:

As we saw in Example 3.5.6, the infinitesimal volume of a thickened z cross section of S is $dV = \pi(R^2 - z^2)\,dz$ cubic meters. The infinitesimal mass of this thickened slice of S is

$$dm \;=\; \delta(z)\,dV \;=\; 1000\pi(2R - z)(R^2 - z^2)\,dz \;\;\text{kg},$$

and the total mass of S is

$$m \;=\; \int_{z=0}^{z=R} dm \;=\; \int_0^R 1000\pi(2R - z)(R^2 - z^2)\,dz \;=$$

$$1000\pi \int_0^R (2R^3 - R^2 z - 2Rz^2 + z^3)\,dz \;=\; 1000\pi\left(2R^3 z \;-\; R^2\cdot\frac{z^2}{2} \;-\; 2R\cdot\frac{z^3}{3} \;+\; \frac{z^4}{4}\right)\Bigg|_0^R \;=$$

$$1000\pi\left(\frac{13R^4}{12} \;-\; 0\right) \;=\; \frac{13,000\pi R^4}{12} \;\;\text{kilograms.}$$

There is another situation in which we can find the mass of a solid by integrating a density function which depends on only one variable. Suppose that we have a solid of revolution, S, as we discussed in Section 3.5, and that the density of the solid at any point P depends solely on the distance r from the point P to the axis of revolution. Then, corresponding to finding volumes of solids of revolutions by using cylindrical shells, we can integrate to add up the infinitesimal amounts of mass of each cylindrical shell to produce the total mass

For instance, suppose that B is a region located in the first and/or fourth quadrant of the xy-plane, and we create a solid of revolution S by revolving B around the y-axis. A given x-coordinate determines an x cross section of B, which will (typically) be a line segment. When that line segment is revolved around the y-axis, all of the points on the corresponding cylinder are at distance x from the y-axis, and the infinitesimal volume of the corresponding cylindrical shell is

$$dV \;=\; 2\pi x h(x)\,dx,$$

where the height function $h(x)$ is determined by the shape of the region B. Now, if the density at each point P in S depends only on the distance to the y-axis, then S has the same density at each point in the cylindrical shell of radius x; this means that $\delta = \delta(x)$, and the infinitesimal mass dm of the cylindrical shell of radius x is

$$dm \;=\; \delta(x)\,dV \;=\; 2\pi x h(x)\delta(x)\,dx.$$

Now, of course, we integrate to add up all of these infinitesimal blobs of mass.

Example 3.7.4. Recall Example 3.5.12, in which we looked at the curved triangular region B in the xy-plane, which has part of the y-axis as its left edge, its top edge is given by $y = 2 - x$, and its bottom curved edge is given by $y = \sqrt{x}$. This region lies between $x = 0$ and $x = 1$. We revolved B around the y-axis, generating a solid of revolution S.

Assume that all distances are measured in meters, and that, if P is a point in S and the distance from P to the y-axis is r, then the density of S at P is $200(1 + r)$ kg/m^3. Find the total mass of S.

Solution:

As we saw in Example 3.5.12, the infinitesimal volume of a cylindrical shell with radius $r = x$ is given by

$$dV \;=\; 2\pi r h\,dx \;=\; 2\pi x(2 - x - x^{1/2})\,dx \;\text{ cubic meters.}$$

As the radius of the shell is $r = x$, the density of S at each point in this shell is $\delta(x) = 200(1+x)$

kg/m^3, and the infinitesimal mass of this shell is

$$dm \;=\; \delta(x)\,dV \;=\; 200(1+x)\cdot 2\pi x(2-x-x^{1/2})\,dx \;\; \text{kg}.$$

Therefore, the total mass of S, in kilograms, is given by

$$m \;=\; \int_{x=0}^{x=1} dm \;=\; \int_0^1 400\pi(1+x)\,x\,(2-x-x^{1/2})\,dx.$$

To calculate this integral, you "pull out" the 400π, multiply out (i.e., expand) the remaining terms, and use the Power Rule multiple times. We leave it as an exercise for you to verify that the result is

$$m \;=\; \frac{3340\pi}{21} \;\; \text{kilograms.}$$

There are two other "types" of densities that are commonly used, when the object in question is essentially 1-dimensional, like a wire, or essentially 2-dimensional, like a thin metal plate (a *lamina*); the corresponding densities are referred to as *length-density*, and *area-density*.

What is length-density?

Consider a thin straight wire. For many purposes (like ours), the wire is considered as a 1-dimensional object, a line segment. Suppose that we lay out the wire along the x-axis, between 0 and L, where L is the length of the wire. Then, for each x between 0 and L, we want the length-density, $\delta_\ell(x)$, to be the limit of the quotient obtained by taking the mass of a small length of the wire around x, divided by that small length. Thus, by definition of $\delta_\ell(x)$, we want the infinitesimal amount of mass dm, at x, on the wire to be given by

$$dm = \delta_\ell(x)\,dx,$$

i.e., $\delta_\ell(x)$ is defined to be dm/dx.

While it is standard to deal with length-densities as we did in the previous paragraph, it is, of course, true that a wire is actually a 3-dimensional solid object; it's just that two of the wire's dimensions are very small. If we think of the wire as having 3 dimensions, and think of usual density, then can we make sense of what length-density means? Yes.

At each x-coordinate between 0 and L, the wire has a tiny x cross-sectional area $A(x)$. We suppose, as we did earlier, that, in each x cross section, the density has a constant value $\delta(x)$. An infinitesimal piece of volume dV on the wire at x is given by $dV = A(x)\,dx$ and, as each point on the wire in the x cross section has density $\delta(x)$, the infinitesimal mass dm of this piece is

$$dm \;=\; \delta(x)\,dV \;=\; \delta(x)A(x)\,dx.$$

Thus, if you wish to think in terms of the usual density of a 3-dimensional solid object, you use the setup that we just discussed, and define length-density by $\delta_\ell(x) = \delta(x)A(x)$. Then, you once again obtain that

$$dm \;=\; \delta_\ell(x)\,dx.$$

Proposition 3.7.5. *Suppose that $a < b$, and we have an idealized 1-dimensional object (think of a thin wire), laid out along the x-axis, lying between $x = a$ and $x = b$. Further, suppose we have a continuous length-density function $\delta_\ell(x)$ for the object. Then, the infinitesimal mass dm of an infinitesimal portion of the object is given by $dm = \delta_\ell(x)\,dx$, and*

$$\text{mass of the object} \;=\; \int_{x=a}^{x=b} dm \;=\; \int_a^b \delta_\ell(x)\,dx.$$

Example 3.7.6. A wire has been stretched out along the x-axis. Its left end is at $x = 0$ and its right end is at $x = 4$ feet. Suppose that the length-density of the wire is given by $\delta_\ell(x) = e^{-x}$ slugs/ft.

What is the mass of the "left half" of the wire between $x = 0$ and $x = 2$ feet? What is the mass of the "right half" of the wire between $x = 2$ and $x = 4$ feet? What is the x-coordinate x_m of the point (the *mass midpoint*) such that half of the mass of the wire lies to left of x_m and half of the mass lies to the right?

Solution:

The infinitesimal mass dm is simply

$$dm \;=\; \delta_\ell(x)\,dx \;=\; e^{-x}\,dx \;\;\text{slugs}.$$

Using the substitution $u = -x$, it is easy to show that $\int e^{-x}\,dx \;=\; -e^{-x} + C$.

Thus, the mass M_L of the left half of the wire is

$$\int_{x=0}^{x=2} dm \;=\; \int_0^2 e^{-x}\,dx \;=\; -e^{-x}\Big|_0^2 \;=\; -e^{-2}-(-e^0) \;=\; 1-e^{-2} \;\approx\; 0.8646647 \;\text{ slugs.}$$

The mass M_R of the right half of the wire is

$$\int_{x=2}^{x=4} dm \;=\; \int_2^4 e^{-x}\,dx \;=\; -e^{-x}\Big|_2^4 \;=\; -e^{-4}-(-e^{-2}) \;=\; e^{-2}-e^{-4} \;\approx\; 0.1170196 \;\text{ slugs.}$$

The mass midpoint, x_m, is the point such that

$$\int_{x=0}^{x=x_m} dm \;=\; \int_{x=x_m}^{x=4} dm,$$

that is

$$\int_0^{x_m} e^{-x}\,dx \;=\; \int_{x_m}^4 e^{-x}\,dx, \qquad \text{which yields} \qquad -e^{-x_m}-(-1) \;=\; -e^{-4}-(-e^{-x_m}).$$

Thus, we need

$$1+e^{-4} \;=\; 2e^{-x_m}, \qquad \text{which gives us} \qquad x_m \;=\; \ln\left(\frac{2}{1+e^{-4}}\right) \;\approx\; 0.6749973 \;\text{ feet.}$$

Now we want to look at area-density. What is area-density?

In a similar way to how we dealt with length-density, if we have a lamina (a plane region which has been thickened slightly, e.g., a thin sheet of metal), then we **could** think of the usual 3-dimensional density, where one of the dimensions is very small. However, in analogy with what we did for thin wires, we typically just think of the *area-density*, $\delta_{\mathrm{ar}}(P)$, at each point P on our idealized 2-dimensional plate. Thus, an infinitesimal chunk of mass dm on the plate will be given by $dm = \delta_{\mathrm{ar}}(P)\,dA$, where dA is an infinitesimal area. Of course, we still need to be able to reduce this to a one-variable problem.

Therefore, if our idealized 2-dimensional region B is in the xy-plane, we will assume that the density is constant in each cross-sectional line segment perpendicular the x-axis (or the y-axis), i.e., we assume that, for each fixed x, $\delta_{\text{ar}} = \delta_{\text{ar}}(x)$ and that we know the function $h(x)$ which is the height (length, width) of the x cross-sectional line segment in B. Then, the infinitesimal area of a thickened x cross section, an infinitesimally thin rectangle, is $dA = h(x)\,dx$ and its infinitesimal mass is

$$dm = \delta_{\text{ar}}(x)\,dA = \delta_{\text{ar}}(x)h(x)\,dx.$$

Proposition 3.7.7. *Suppose that $a < b$, and we have an idealized 2-dimensional object S (think of a thin metal plate), in the xy-plane, lying between $x = a$ and $x = b$. Further, suppose that we have a continuous height function $h(x)$ which gives us the height of the x cross section of S, and a continuous area-density function $\delta_{\text{ar}}(x)$, which gives the area-density of x at each point in the x cross section.*

Then, the infinitesimal mass dm of an infinitesimally thickened x cross section of the object is given by

$$dm = \delta_{\text{ar}}(x)\,dA = \delta_{\text{ar}}(x)h(x)\,dx,$$

and

$$\text{mass of the object} = \int_{x=a}^{x=b} dm = \int_a^b \delta_{\text{ar}}(x)h(x)\,dx.$$

Example 3.7.8. Suppose that a thin metal plate occupies the triangular region in the first quadrant of the xy-plane below the line $y = 2 - x$, where all distances are measured in meters.

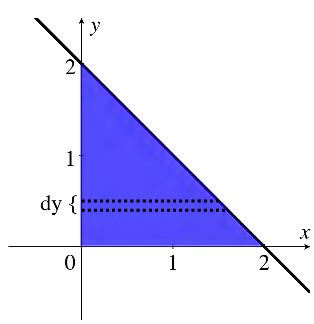


Figure 3.56: A triangular metal plate of varying density.

Suppose that the area-density at each point in any given y-coordinate is constant, equal to $\delta_{\text{ar}}(y) = e^{-y}$ kg/m^2. Find the mass of the plate.

Solution:

The length $h(y)$ of an infinitesimally thin rectangle in the plate, at a given y-coordinate, is simply the corresponding x-coordinate on the line given by $y = 2 - x$, i.e., $h(y) = 2 - y$ meters. Hence, the infinitesimal area of the infinitesimally thin rectangle at each y-coordinate is $dA = h(y)\,dy = (2-y)\,dy$. The infinitesimal mass of this rectangle is

$$dm \;=\; \delta_{\mathrm{ar}}(y)\,dA \;=\; e^{-y}(2-y)\,dy$$

and, thus, the total mass of the plate is

$$\text{mass of plate} \;=\; \int_{y=0}^{y=2} dm \;=\; \int_0^2 e^{-y}(2-y)\,dy \;=\; \int_0^2 2e^{-y}\,dy \;-\; \int_0^2 ye^{-y}\,dy \;\;\text{kg.}$$

This first integral is easy. We find

$$\int_0^2 2e^{-y}\,dy \;=\; -2e^{-y}\Big|_0^2 \;=\; -2e^{-2} - (-2e^0) \;=\; 2 - 2e^{-2}.$$

We need to calculate the second integral and subtract. How do you calculate $\int_0^2 ye^{-y}\,dy$? By parts (Theorem 1.1.19).

Let's calculate the indefinite integral, and then we'll plug in the limits of integration. Let $u = y$ and that leaves us with $dv = e^{-y}\,dy$. Then, $du = dy$, $v = \int dv = \int e^{-y}\,dy = -e^{-y}$, and Integration by Parts tells us that

$$\int ye^{-y}\,dy \;=\; \int u\,dv \;=\; uv - \int v\,du \;=\; y(-e^{-y}) - \int -e^{-y}\,dy \;=\; -ye^{-y} - e^{-y} + C.$$

Therefore, we obtain

$$\int_0^2 ye^{-y}\,dy \;=\; -ye^{-y} - e^{-y}\Big|_0^2 \;=\; -e^{-y}(y+1)\Big|_0^2 \;=\; -e^{-2}(3) + 1 \;=\; 1 - 3e^{-2},$$

and so,

$$\text{mass of plate} \;=\; 2 - 2e^{-2} - (1 - 3e^{-2}) \;=\; 1 + e^{-2} \;\;\text{kg.}$$

3.7.1 Exercises

Throughout the exercises, assume all information is given in standard metric units. That is, mass is given in kilograms, time in seconds and lengths in meters.

1. Suppose that S is a rectangular solid in the first octant with $0 \leq x \leq 3$, $0 \leq y \leq 6$, and $0 \leq z \leq 5$, and that the density is constant along each cross section parallel to the xy-plane, with $\delta(z) = \cosh z$. Calculate the mass of S.

2. Suppose that S is a rectangular solid with sides parallel to the coordinate axes, where each cross section parallel to the xy plane has uniform density $\delta(z)$, and where $a \leq z \leq b$. Prove that the mass is given by $m = A(z)\int_a^b \delta(z)\,dz$. That is, show that area may be factored out of the usual equation.

3. Suppose that S is a solid right circular cone, centered along the y-axis, with vertex at the origin, and that the base lies in the plane $y = 6$ and has radius 4. Suppose that each cross section parallel to the xz-plane has density $\delta(y) = 5y$. Calculate the mass of S.

4. Generalize the result in the previous problem. Suppose S is a solid right circular cone, centered along the y axis, with height h, base radius r, vertex at the origin and density function $\delta = \delta(y)$. Give an integral for the mass of S.

5. Suppose that S is the solid ball centered at the origin with radius 9, and that the cross sections parallel to the yz-plane have density $\delta(x) = x^2 + 1$. Calculate the mass of S.

6. Suppose that S is the solid ellipsoid defined by the equation

$$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{9} = 1,$$

and that the cross sections parallel to the xz-plane have density $\delta(y) = 2y^4 + y^2 + 1$. Calculate the mass of S.

In each of Exercises 7 through 14, calculate the mass of the solid of revolution S. For each problem, assume the density function, δ, depends only on the distance r from the point to the axis of revolution (as in Example 3.7.4).

7. Let S be the solid obtained by revolving the first quadrant region below the curve $y = \sqrt{x}$ and above the interval $[0, 4]$ about the y-axis, if $\delta(r) = r + 1$.

8. S is the solid cone obtained by revolving the plane region bounded by the lines $y = x$, $x = 0$, $y = 8$ about the y-axis, with density function $\delta(r) = 1 + r^3$.

9. More generally, assume S is the solid cone obtained by revolving the plane region bounded by the lines $y = mx$, $y = y_0 > 0$ and $x = 0$ with $m > 0$, about the y-axis, given a density function $\delta(r)$. (Here, just give an appropriate integral for the mass.)

10. Consider the plane region B in the first quadrant bounded by the lines $y = 5$ and $x = 4$. Let S be the solid obtained by revolving B about the y-axis, with density function $\delta(r) = 2 + \cos(r^2)$.

11. More generally, assume that B is the plane region in the first quadrant bounded by the lines $y = h > 0$, $x = b > 0$ and S is obtained by revolving B about the y-axis, given a density function $\delta(r)$. (Here, just give an appropriate integral for the mass.)

12. Using the notation in the previous problem, what is the mass if $h = 5$, $b = 7$, but $\delta(r)$ is changed to e^{-r}?

13. Suppose B is the plane region in the first quadrant below the line $y = 4$, with $5 \leq x \leq 7$. Let S be the solid obtained by revolving B about the y-axis, with $\delta(r) = \ln r$.

14. Consider the region in the first quadrant bounded by the curves $y = 1.5x$ and $y = \cos \pi x$, with x in $[0, 1/3]$. Let S be the region obtained by revolving this region about the y-axis, with $\delta(r) = r + 1$.

In each of Exercises 15 through 19, you are given the length-density function, $\delta_\ell(x)$, of an infinitesimally thin wire lying on the x-axis over a given interval. For each problem, calculate (a) the total mass of the wire and (b) the mass midpoint (recall Example 3.7.6). You may need to use technology to numerically approximate the mass midpoint.

15. $\delta_\ell(x) = x + 7, \ \ 0 \leq x \leq 4.$

16. $\delta_\ell(x) = \dfrac{4}{9 - x^2}, \ \ 1 \leq x \leq 2.$

17. $\delta_\ell(x) = \sin x, \ \ 0 \leq x \leq \pi/2.$

18. $\delta_\ell(x) = x \ln x, \ \ 3 \leq x \leq 9.$

19. $\delta_\ell(x) = \cosh x, \ \ -1 \leq x \leq 1.$

20. Consider the following two quantities. C is the point in $[a, b]$ such that $\int_a^C \delta_\ell(x)\,dx = \int_C^b \delta_\ell(x)\,dx$, and $K = \dfrac{1}{b-a}\int_a^b \delta_\ell(x)\,dx$. Briefly describe what C and K measure in physical terms.

21. Suppose the length-density of a wire is $\delta_\ell(x) = ce^x$ for x in $[0, 10]$. What is the mass midpoint?

22. Suppose the length-density of a wire is $\delta_\ell(x) = \ln(cx)$ for x in $[1, c]$, where $c > 1$. Calculate the mass of the wire in terms of c.

In each of Exercises 23 through 28, you are given a region B in the xy-plane, occupied by a thin metal plate, and an area-density function for the plate. If δ_{ar} is given as a function of x (resp. y), then δ_{ar} gives the area-density of the plate at each point in the x (resp. y) cross section. Calculate the mass of the metal plate.

23. Let B be the intersection of the disc centered at the origin with radius 3 and the first quadrant, $\delta_{\text{ar}}(x) = 3x + 1$.

24. B is the intersection of the disc centered at the origin with radius 5 and the first quadrant, $\delta_{\text{ar}}(y) = \sqrt{25 - y^2} + 1$.

25. Let B be the triangular region bounded by the coordinate axes and the line $y = 2x - 6$, $\delta_{\text{ar}}(x) = x^2 + 1$.

26. B is the region in the first quadrant bounded by the curves $y = \cos x$ and $y = \sin x$, with $0 \le x \le \pi/4$, and $\delta_{\text{ar}}(x) = x + 1$.

27. B is the region in the first quadrant bounded by $y = \sqrt{x}$ and $x = 4$, $\delta_{\text{ar}}(y) = y^{3/2} + 2$.

28. B is a triangle in the first quadrant with vertices $(0, 0)$, $(0, a)$, $(b, 0)$. $\delta_{\text{ar}}(y) = 1/(b + 1)$.

29. Suppose B is triangle in the first quadrant with vertices $(0, 0)$, $(0, 2)$, $(5, 0)$ with density function $\delta_{\text{ar}}(x) = cx^2$. What is c if the total mass is 25 kg?

The average density of a solid region with varying density is the total mass divided by the total volume. Use this idea to answer each of Exercises 30 through 35.

30. Suppose S is a solid region where the density function of any plane parallel to the yz plane is given by $\delta(x)$. Suppose further that the area of each cross section parallel to the yz

plane is $A(x)$ and that $a \le x \le b$. Show that the average density of S is given by

$$\text{average density} = \frac{\int_a^b \delta(x)A(x)\,dx}{\int_a^b A(x)\,dx}.$$

31. Calculate the average density of the solid in Exercise 3.

32. Calculate the average density of the solid in Exercise 5.

33. Suppose that $0 \le a < b$, that $h(x)$ is continuous and positive, and that B is the plane region under the graph of $h(x)$ and above the interval $[a, b]$ on the x-axis. Let S be the solid obtained by revolving B about the y-axis. Suppose further that the density of the object, $\delta(r)$, depends only on the distance from the y-axis. Show that the average density of the solid is
$$\text{average density} = \frac{\int_a^b 2\pi x h(x)\delta(x)\,dx}{\int_a^b 2\pi x h(x)\,dx}.$$

34. Calculate the average density of the solid in Exercise 7.

35. Calculate the average density of the solid in Exercise 8.

It's common in physical applications for the density and mass of a region to evolve over time. For example, we could be considering a rectangular solid initially containing water, but where oil is added and mixed with the water. Over time, the density and mass of the solid will evolve. In each of Exercises 36 through 39, the density function is a function of time as well as position.

36. S is a rectangular solid with sides parallel to the coordinate planes, with $-2 \le x \le 4$, $3 \le y \le 6$, and $7 \le z \le 9$. The density function $\delta(x, t)$ of a cross section parallel to the yz plane at time t is $\delta(x, t) = (t - 3)^2 e^x$.

    a. Calculate $m(t)$, the mass of the solid at time t.

    b. Calculate dm/dt.

    c. Assume $t > 0$. Is there a finite time when the mass is minimal?

37. Suppose S is a spherical ball centered at the origin with radius 3 and that the density function is $\delta(r, t) = \dfrac{r + 1}{t}$, where r is the distance between a point in the interior of the ball and the origin. Assume $t > 0$.

    a. Calculate $m(t)$, the mass of the ball at time t.

b. Calculate dm/dt.

38. In the previous problem, what is the average mass of the ball between times $t = 2$ and $t = 5$?

39. Suppose B is the region in the first quadrant bounded by the y-axis, the graph of e^x and $y = 9$. Let S be the solid region obtained by revolving B about the y-axis. Suppose that, at time t, the density of S at each point that's a distance r from the y-axis is given by $\delta(r, t) = tr + t^2 + 1$. What is $m(t)$?

In the chapter, we saw that there was a special formula for calculating the mass of a 3-dimensional object when the density at a point depends on the distance between the point and an axis. A similar formula exists in 2 dimensions. Suppose B is a annulus or a disc in the xy-plane bounded by $r = r_0$ and $r = r_1$ and that the area-density function, $\delta(r)$, depends only on the distance between a point on the surface and the origin. Then the total mass is

$$\textbf{mass} = 2\pi \int_{r_0}^{r_1} r\delta(r)\, dr.$$

Use this formula to calculate the mass of the surface in each of Exercises 40 through 43.

40. B is a disk with radius 3 centered at the origin and with density function $\delta(r) = 3r + 1$.

41. B is a disk with radius 5 centered at the origin with density function $\delta(r) = \dfrac{r+2}{r+1}$.

42. B is an annulus centered at the origin bounded by circles of radii 4 and 7. $\delta(r) = 1/r$.

43. B is an annulus centered at the origin bounded by circles of radii 2 and 6. $\delta(r) = 4e^{r^2}$.

3.8 Centers of Mass and Moments

We frequently discuss objects as though they are located at specific points. When we state that an object of mass m is located at a point P, we are usually thinking of an idealized "point-mass": an imaginary object which occupies a single point in space at any given time. Of course, objects in real life exist at an infinite number of points in space, but we can think of a given solid object as consisting of an infinite number of infinitesimal point-masses.

Now, suppose that you have a solid object, occupying an infinite number of points in space, and a force is acting on the object. We would like to apply Newton's 2nd Law of Motion to determine the acceleration of the object. But, if you're thinking of the object as being composed of an infinite number of point-masses, which are possibly moving separately in a very complicated manner, what does "the acceleration of the object" mean?

The answer is that we can define a point, called the *center of mass* of an object, whose acceleration is given by Newton's 2nd Law. Given an object, or collection of objects, of (total) mass M, the center of mass is a point P in space such that, in many physical problems, the object(s) can be treated as a point-mass, with mass M, located at the point P. If you were trying to balance a rigid wire or metal plate on your finger, the center of mass is where you would place your finger. If two children want to balance on a see-saw, the center of mass is the point where the base (fulcrum) of the see-saw needs to be.

Note that the center of mass of an object need **not** actually be located at a point on the object. Perhaps the easiest example of this is a uniformly dense annulus: think of a thin metal disk with a smaller disk removed from its center; by symmetry, the center of mass is located at the center of the annulus, but that part has been removed. What this means is that there is no place where you could place your finger in order to balance an annulus. Hopefully, this seems intuitively clear.

For a solid object, the center of mass can be determined, in principle, via integration, from the shape and density function (Section 3.7) of the object. For an object with constant density, the center of mass is also known as the *centroid* of the object.

In order to discuss how integration can be used to find centers of mass of objects, we first need to understand what happens for a finite number of point-masses. We shall describe this, and then define the center of mass.

The discussion of the center of mass is most natural using the language of vectors, which we looked at in Section 3.3, and for which there's a quick summary in Appendix A. We shall use vectors now, but we shall also describe the situation later for wires and thin plates, and there we shall not refer to vectors. Our treatment follows that in the excellent volumes of The Feyman Lectures on Physics, [1].

Suppose that we have n point-masses, whose masses are constants $m_1, \ldots, m_n$ (not all zero), and the masses are moving in space (or in a plane, or in a line). Let $\vec{r}_i = \vec{r}_i(t)$ denote the position of the mass m_i, for $i = 1, \ldots, n$, at time t. (In other places, we have used $\vec{p}$ to denote position; it is standard in our current situation to use $\vec{r}$, for *radial* vector.)

Suppose now that we have a force $\vec{F}_i = \vec{F}_i(t)$ acting on each mass m_i, and that $\vec{F}_i$ is the only force acting on m_i. Then, the vector form of Newton's 2nd Law of Motion tells us that the net force acting on an object (of constant mass) is the mass times the acceleration of the object, i.e.,

$$\vec{F}_i \;=\; m_i \frac{d^2\vec{r}_i}{dt^2}.$$

Now suppose that we think of all of the masses together as one object, possibly because they're very close together, but possibly not. Then, the total mass of the new "object" is $M = \sum_{i=1}^n m_i$, and the net force acting on the object is $\vec{F} = \sum_{i=1}^n \vec{F}_i$. We want to be able to apply Newton's 2nd Law to the collective mass M; that is, we want that $\vec{F}$ is equal to the mass M times the acceleration. But what is this "acceleration" the acceleration of? Acceleration is, of course, the second derivative of position, with respect to time, but what position do we take the second derivative of?

Using that the masses m_i are constant, so that M is also constant, we have

$$\vec{F} \;=\; \sum_{i=1}^{n} \vec{F}_i \;=\; \sum_{i=1}^{n} \frac{d^2(m_i\vec{r}_i)}{dt^2} \;=\; M\,\frac{d^2\left[\left(\sum_{i=1}^n m_i\vec{r}_i\right)/\,M\right]}{dt^2}.$$

Therefore, if we define a position vector

$$\vec{r}_{\rm cm} \;=\; \vec{r}_{\rm cm}(t) \;=\; \frac{\displaystyle\sum_{i=1}^{n} m_i\vec{r}_i}{M} \;=\; \frac{\displaystyle\sum_{i=1}^{n} m_i\vec{r}_i}{\displaystyle\sum_{i=1}^{n} m_i},$$

then we obtain that

$$\vec{F} = M\frac{d^2\vec{r}_{\mathrm{cm}}}{dt^2}. \tag{3.2}$$

Hence, we make the following definition:

Definition 3.8.1. *The* **center of mass** *of a collection/system of objects with masses $m_1, \ldots, m_n$ at positions $\vec{r}_1, \ldots, \vec{r}_n$, respectively, is the point*

$$\vec{r}_{\mathrm{cm}} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i}.$$

If we let $\vec{r}_i = (x_i, y_i, z_i)$, then we find that

$$\overline{x} = x\text{-coordinate of the center of mass} = \left(\sum_{i=1}^{n} m_i x_i\right) \Big/ \left(\sum_{i=1}^{n} m_i\right);$$

$$\overline{y} = y\text{-coordinate of the center of mass} = \left(\sum_{i=1}^{n} m_i y_i\right) \Big/ \left(\sum_{i=1}^{n} m_i\right);$$

$$\overline{z} = z\text{-coordinate of the center of mass} = \left(\sum_{i=1}^{n} m_i z_i\right) \Big/ \left(\sum_{i=1}^{n} m_i\right).$$

The quantity $M = \sum_{i=1}^{n} m_i$ is the total mass of the system. The quantities $\sum_{i=1}^{n} m_i x_i$, $\sum_{i=1}^{n} m_i y_i$, and $\sum_{i=1}^{n} m_i z_i$ are referred to by various terms involving the word **moment***; we shall refer to them as the* **x-, y-, and z-components, respectively, of the mass-moment about the origin***. In fact, as we shall consider only mass-moment about only the origin, we will usually write simply the* **x-, y-, and z-components of the moment***.*

Different sources also use different notations for the components of the moment. We shall try to avoid confusion; if M is the total mass, then we write $M\overline{x}$, $M\overline{y}$, and $M\overline{z}$ for the x-, y-, and z-components of the moment, respectively.

Remark 3.8.2. Note that if we replaced $\vec{r}_{\text{cm}}$ in Formula 3.2 with $\vec{r} = \vec{r}_{\text{cm}} + t\vec{v}_0 + \vec{r}_0$, where $\vec{v}_0$ and $\vec{r}_0$ are constant vectors, then the equation would still hold since the 2nd derivative of $t\vec{v}_0 + \vec{r}_0$, with respect to t, is zero. However, when there is only one non-zero mass, i.e., when $m_{i_0} \neq 0$ for exactly one index i_0, we, of course, want the center of mass to be where that non-zero mass is, namely, at $\vec{r}_{i_0}$. Since we want this to be true at all times, and regardless of which mass is non-zero, we are required to choose $\vec{v}_0 = \vec{0}$ and $\vec{r}_0 = \vec{0}$, i.e., to define the center of mass as we did.

Suppose now that we have a thin wire, of (possibly) variable length-density $\delta_\ell(x)$, laid out along the x-axis; see Section 3.7. Then, its center of mass has only an x-coordinate. How do we find it?

It's easy, now that we know what to do for a finite number of point-masses. We think of chopping the wire up into a large (but finite) number of small pieces, taking approximating Riemann sums, and taking a limit. Of course, we describe the result infinitesimally instead. Around the point at a given x value, an infinitesimal chunk of mass dm contributes dm to the total mass and $x\,dm = x\delta_\ell(x)\,dx$ to the x-component of the moment. What we conclude is:

Proposition 3.8.3. *Suppose that we have an idealized 1-dimensional object, laid out along the x-axis, between $x = a$ and $x = b$, with a continuous length-density function $\delta_\ell(x)$.*

Then, the x-component of the moment is

$$\int_{x=a}^{x=b} x\,dm \;=\; \int_a^b x\delta_\ell(x)\,dx,$$

the total mass is

$$M \;=\; \int_{x=a}^{x=b} dm \;=\; \int_a^b \delta_\ell(x)\,dx,$$

and the center of mass is located at the x-coordinate given by

$$\overline{x} \;=\; \frac{\int_{x=a}^{x=b} x\,dm}{M} \;=\; \frac{\int_a^b x\delta_\ell(x)\,dx}{\int_a^b \delta_\ell(x)\,dx}.$$

Remark 3.8.4. Suppose you have a thin wire with a constant (positive) length-density. Where do you intuitively believe that the center of mass is located? Hopefully, it seems clear that it should be the midpoint of the wire. Let's make certain that that's what we get from the formula in Proposition 3.8.3.

Suppose that $\delta_\ell(x) = \delta_\ell > 0$ is a constant, and that the wire lies between $x = a$ and $x = b$, where $a < b$. Then,

$$\frac{\int_{x=a}^{x=b} x\,dm}{M} = \frac{\int_a^b x\delta_\ell\,dx}{\int_a^b \delta_\ell\,dx} = \frac{\delta_\ell \int_a^b x\,dx}{\delta_\ell \int_a^b dx} = \frac{\frac{x^2}{2}\Big|_a^b}{x\Big|_a^b} = \frac{\frac{b^2}{2} - \frac{a^2}{2}}{b-a} = \frac{b+a}{2},$$

which is the midpoint of the wire, as we expected.

Recall that the center of mass of an object with constant density, as above, is called the centroid. What we have just seen, in the case of a wire, is that the constant density cancels out in the calculation; so that the location of the centroid is independent of what the constant density actually is. Hence, the centroid depends only on the shape of the object. As we shall see, this is also true for thin metal plates, and solid objects.

It is important that, in general, you **cannot** take the fraction

$$\frac{\int_a^b x\delta_\ell(x)\,dx}{\int_a^b \delta_\ell(x)\,dx}$$

and somehow cancel the $\delta_\ell(x)$ in the integral in the numerator with the $\delta_\ell(x)$ in the integral in the denominator. This is true for constant density because, then, we may factor the density out of the integrals.

Example 3.8.5. Let's look again at the wire from Example 3.7.6. The wire was stretched out along the x-axis. Its left end was at $x = 0$ and its right end was at $x = 4$ feet. The length-density of the wire was given by $\delta_\ell(x) = e^{-x}$ slugs/ft.

We found that the x-coordinate x_m of the point such that half of the mass of the wire lies to the left of x_m and half of the mass lies to the right was $x_m = \ln\left(\frac{2}{1+e^{-4}}\right) \approx 0.6749973$ feet. Is this the same as the center of mass? No. The calculation of x_m did not take into account the actual x-coordinates of the infinitesimal chunks of mass. Let's calculate the center of mass and see what we actually get.

Back in Example 3.7.6, we already calculated the masses of the left and right halves of the wire; they were/are $M_L = 1 - e^{-2}$ slugs and $M_R = e^{-2} - e^{-4}$ slugs. Thus, the total mass of the wire is

$$M \;=\; M_L \;+\; M_R \;=\; 1 - e^{-4} \;\text{ slugs.}$$

The x-component of the moment is

$$M\overline{x} \;=\; \int_{x=0}^{x=4} x\,dm \;=\; \int_0^4 x\delta_\ell(x)\,dx \;=\; \int_0^4 xe^{-x}\,dx.$$

This integral is evaluated using integration by parts. We actually already calculated this indefinite integral (with y's in place of x's) back in Example 3.7.8, where we found that

$$\int ye^{-y}\,dy \;=\; \int u\,dv \;=\; uv - \int v\,du \;=\; y(-e^{-y}) - \int -e^{-y}\,dy \;=\; -ye^{-y} - e^{-y} + C.$$

Therefore,

$$M\overline{x} \;=\; \int_{x=0}^{x=4} x\,dm \;=\; \int_0^4 xe^{-x}\,dx \;=\; \left(-xe^{-x} - e^{-x}\right)\Big|_0^4 \;=$$

$$-4e^{-4} - e^{-4} - (0-1) \;=\; 1 - 5e^{-4} \;\text{ ft-slugs.}$$

Dividing this by the total mass, we find that

$$\overline{x} \;=\; \frac{1-5e^{-4}}{1-e^{-4}} \;\approx\; 0.925370558545 \;\text{ feet.}$$

Notice that this is almost 50% larger than x_m, the "mid-mass" point.

Now, consider a thin metal plate, lying in the xy-plane. As in Section 3.7, we will assume that we have an area-density function for the plate which depends on a single variable, say x. That is, we assume that our plate lies between the vertical lines $x = a$ and $x = b$, and that we have a continuous area-density function $\delta_{\text{ar}}(x)$, for $a \leq x \leq b$, which gives the area-density of our object at each point in the x cross section. We assume that we have a continuous function $h(x)$ which gives the height/length of the x cross section of the object.

Then, we know that an infinitesimally wide vertical slab of mass is given by $dm = \delta_{\rm ar}(x)\,dA = \delta_{\rm ar}(x)h(x)\,dx$. In addition, the contribution of this slab of mass to the x-component of the moment is easy; it's $x\,dm = x\delta_{\rm ar}(x)h(x)\,dx$.

The slightly difficult part is the contribution of this vertical slab of mass to the y-component of the moment. We would like to say that it's $y\,dm$, but there are an infinite number of y coordinates in each vertical slab. So what y-coordinate do we use? Actually, that's easy; we use the y-coordinate $\overline{y}_x$ of the center of mass of the infinitesimal slab given by thickening the x cross section. **Since the density is constant in a vertical slab**, $\overline{y}_x$ is simply the y-coordinate of the midpoint of the x cross section, which is one half the sum of the lower and upper y-coordinates of the cross section.

Therefore, we obtain:

Proposition 3.8.6. *Suppose that $a < b$, and we have an idealized 2-dimensional object S (think of a thin metal plate), in the xy-plane, lying between the lines $x = a$ and $x = b$, under the graph of $y = f(x)$ and above the graph of $y = g(x)$, where $f(x)$ and $g(x)$ are continuous functions and $f(x) \geq g(x)$ on the interval $[a, b]$. Further, suppose that we have a continuous area-density function $\delta_{\rm ar}(x)$, which gives the area-density of x at each point in the x cross section.*

Then, the total mass of the object is

$$M \;=\; \int_{x=a}^{x=b} dm \;=\; \int_a^b \delta_{\rm ar}(x)\,dA \;=\; \int_a^b \delta_{\rm ar}(x)\big(f(x) - g(x)\big)\,dx.$$

The x-component of the moment is

$$M\overline{x} \;=\; \int_{x=a}^{x=b} x\,dm \;=\; \int_a^b x\delta_{\rm ar}(x)\,dA \;=\; \int_a^b x\delta_{\rm ar}(x)\big(f(x) - g(x)\big)\,dx.$$

The y-component of the moment is

$$M\overline{y} \;=\; \int_{x=a}^{x=b} \overline{y}_x\, dm \;=\; \int_a^b \frac{f(x)+g(x)}{2}\, \delta_{\rm ar}(x)\, dA \;=$$

$$\int_a^b \frac{f(x)+g(x)}{2}\, \delta_{\rm ar}(x)\big(f(x)-g(x)\big)\, dx \;=\; \int_a^b \delta_{\rm ar}(x) \left[\frac{f^2(x)-g^2(x)}{2}\right] dx.$$

The x- and y-components of the center of mass, $\overline{x}$ and $\overline{y}$, are are the corresponding components of the moment divided by the mass M.

Example 3.8.7. Suppose that we have a thin metal plate, occupying the region in the xy-plane which is bounded on the top and left by the graph of $y = x^2$, on the right by the line given by $x = 2$, and on the bottom by the line given by $y = 1$. Suppose that the area-density function is $\delta_{\rm ar}(x) = x$ kg/m^2.

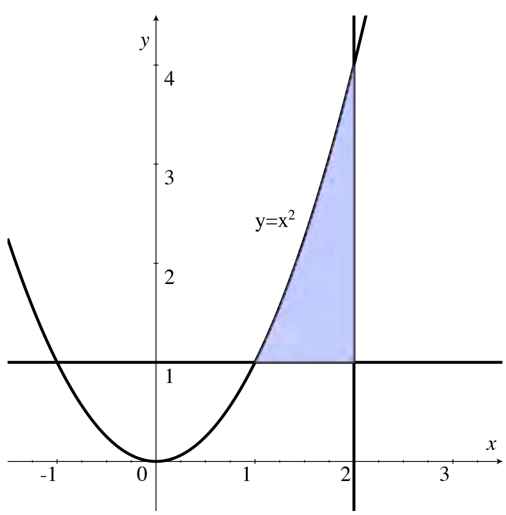


Figure 3.57: A thin metal plate of variable density.

Find the center of mass of the plate.

Solution:

First, let's find an expression for the infinitesimal mass dm of an infinitesimally thin rectangle at a given x value. We have

$$dm \;=\; \delta_{\text{ar}}(x)\,dA \;=\; \delta_{\text{ar}}(x)(x^2-1)\,dx \;=\; x(x^2-1)\,dx \;=\; (x^3-x)\,dx.$$

Now, the contribution to the x-component of the moment from this chunk of mass is

$$x\,dm \;=\; x\cdot(x^3-x)\,dx \;=\; (x^4-x^2)\,dx,$$

and the contribution to the y-component of the moment is

$$\overline{y}_x\,dm \;=\; \frac{x^2+1}{2}\,x(x^2-1)\,dx \;=\; \frac{1}{2}\,x(x^4-1)\,dx \;=\; \frac{1}{2}(x^5-x)\,dx.$$

Thus, we find that the mass is

$$M \;=\; \int_{x=1}^{x=2} dm \;=\; \int_1^2 (x^3-x)\,dx \;=\; \left(\frac{x^4}{4}-\frac{x^2}{2}\right)\Big|_1^2 \;=\; (4-2)-\left(\frac{1}{4}-\frac{1}{2}\right) \;=\; \frac{9}{4}.$$

We also easily calculate

$$\overline{x} \;=\; \frac{\int_{x=1}^{x=2} x\,dm}{M} \;=\; \frac{\int_1^2 (x^4-x^2)\,dx}{9/4} \;=\; \frac{\left(\frac{x^5}{5}-\frac{x^3}{3}\right)\Big|_1^2}{9/4} \;=\; \frac{\left(\frac{32}{5}-\frac{8}{3}\right)-\left(\frac{1}{5}-\frac{1}{3}\right)}{9/4} \;=$$

$$\frac{58/15}{9/4} \;=\; \frac{232}{135} \;\approx\; 1.7185,$$

and

$$\overline{y} \;=\; \frac{\int_{x=1}^{x=2} \overline{y}_x\,dm}{M} \;=\; \frac{\int_1^2 \frac{1}{2}(x^5-x)\,dx}{9/4} \;=\; \frac{\frac{1}{2}\left(\frac{x^6}{6}-\frac{x^2}{2}\right)\Big|_1^2}{9/4} \;=\; \frac{\frac{1}{2}\left[\left(\frac{64}{6}-2\right)-\left(\frac{1}{6}-\frac{1}{2}\right)\right]}{9/4} \;=\; 2.$$

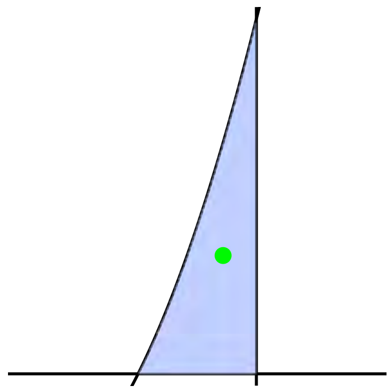


Figure 3.58: Center of mass indicated in green.

In Figure 3.58, we have indicated the center of mass in green.

Example 3.8.8. Consider the triangular region in the first quadrant, under the line given by $y = kx$, where $k > 0$, and above the interval $[0, b]$, where $b > 0$. Find the centroid of this region.

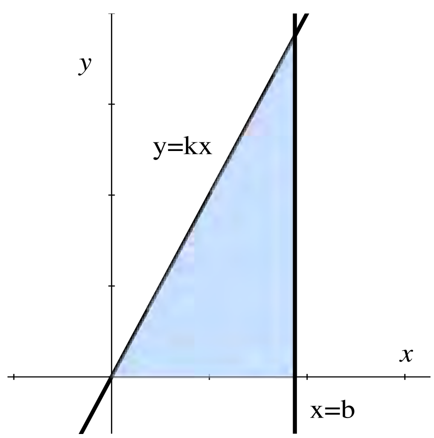


Figure 3.59: A triangular metal plate of constant area-density.

Solution:

As we are finding a centroid, we assume that the area-density $\delta_{\mathrm{ar}} > 0$ is a constant. The infinitesimal mass dm of the infinitesimally thin rectangle at x is given by

$$dm \;=\; \delta_{\mathrm{ar}}\, dA \;=\; \delta_{\mathrm{ar}} kx\, dx.$$

The infinitesimal contribution from this rectangle to the x-component of the moment is

$$x\, dm \;=\; x\delta_{\mathrm{ar}} kx\, dx \;=\; \delta_{\mathrm{ar}} kx^2\, dx,$$

and the infinitesimal contribution from this rectangle to the y-component of the moment is

$$\overline{y}_x\, dm \;=\; \frac{kx+0}{2}\,\delta_{\mathrm{ar}} kx\, dx \;=\; \frac{\delta_{\mathrm{ar}} k^2}{2}\, x^2\, dx.$$

The mass of the plate is

$$\text{mass} \;=\; \int_{x=0}^{x=b} \delta_{\mathrm{ar}}\, dA \;=\; \delta_{\mathrm{ar}} \int_{x=0}^{x=b} dA \;=\; \delta_{\mathrm{ar}} \cdot (\text{area of the plate}).$$

Of course, when the area-density is constant, it's always true that the mass of the plate is simply δ_{ar} times A, the area of the plate. For a more-complicated region, we would have to integrate to find the area and, even here, we **could** integrate kx to find the area of the triangle, but it's a **triangle**; it's area is $1/2$ the base times the height, i.e., $A = (1/2)b(kb) = kb^2/2$. Hence, we have

$$M \;=\; \frac{\delta_{\mathrm{ar}} kb^2}{2}.$$

The x-component of the moment is

$$M\overline{x} \;=\; \int_{x=0}^{x=b} x\, dm \;=\; \int_0^b \delta_{\mathrm{ar}} kx^2\, dx \;=\; \frac{\delta_{\mathrm{ar}} kb^3}{3},$$

and the y-component of the moment is

$$M\overline{y} \;=\; \int_{x=0}^{x=b} \overline{y}_x\, dm \;=\; \int_0^b \frac{\delta_{\mathrm{ar}} k^2}{2}\, x^2\, dx \;=\; \frac{\delta_{\mathrm{ar}} k^2 b^3}{6}.$$

Thus, we find

$$\overline{x} = \frac{\delta_{\mathrm{ar}} k b^3/3}{\delta_{\mathrm{ar}} k b^2/2} = \frac{2b}{3}$$

and

$$\overline{y} = \frac{\delta_{\mathrm{ar}} k^2 b^3/6}{\delta_{\mathrm{ar}} k b^2/2} = \frac{kb}{3}.$$

If we let P denote the vertex at the right angle, i.e., the point $(b, 0)$, then the centroid is located at the point with x-coordinate which is $1/3$ of the way along the base from P, and with y-coordinate which is $1/3$ of the height of the triangle away from P.

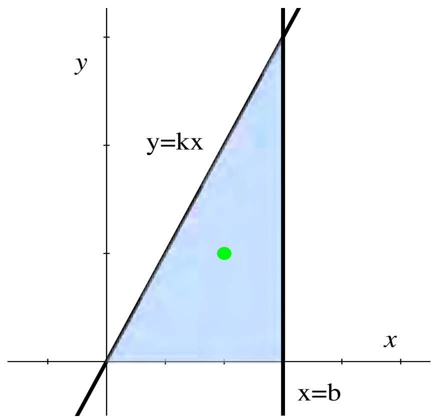


Figure 3.60: The centroid is indicated in green.

Now we would like to look at one example involving the center of mass (actually, here, the centroid) of a solid region.

Example 3.8.9. Recall the general definition of a cone from Definition 3.5.4. Consider, as we did just before Proposition 3.5.5, the cone whose base is a region contained in the plane at the fixed z-coordinate $z = H > 0$, with vertex at the origin.

By looking at the case where the plane region was a rectangle, as in Example 3.5.3, we can/did conclude that the cross-sectional area of the cone at a given z-coordinate is proportional to z^2, i.e., there is a constant k such that $A(z) = kz^2$.

Without knowing more about the base of the cone, it is not possible to find the x- and y-coordinates of the centroid. However, we can easily find the z-coordinate.

We assume that the density is a constant $\delta > 0$. An infinitesimal slab of mass dm is obtained by taking the area of the z cross section and multiplying it by an infinitesimal thickness dz. Hence,

$$dm \;=\; \delta\, dV \;=\; \delta A(z)\, dz \;=\; \delta k z^2\, dz.$$

The contribution to the z-component of the moment of this slab is

$$z\, dm \;=\; z \cdot \delta k z^2\, dz \;=\; \delta k z^3\, dz.$$

Therefore, we find

$$\overline{z} \;=\; \frac{\int_{z=0}^{z=H} z\, dm}{\int_{z=0}^{z=H} dm} \;=\; \frac{\int_0^H \delta k z^3\, dz}{\int_0^H \delta k z^2\, dz} \;=\; \frac{\delta k H^4/4}{\delta k H^3/3} \;=\; \frac{3}{4}H.$$

Note that this is independent of the area or shape of the base of the cone and, of course, independent of δ.

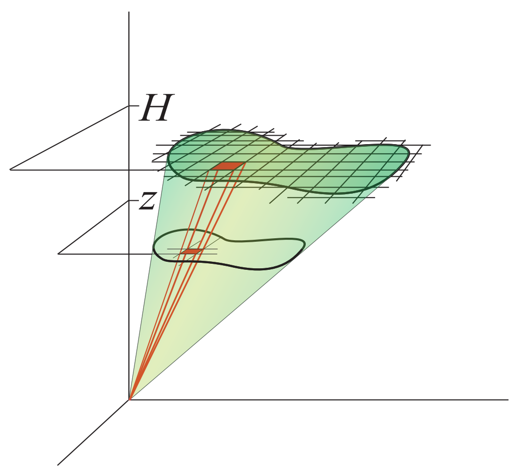


Figure 3.61: A cone whose base is a general plane region.

What would we need to know in order to find the x- and y-coordinates of the centroid? We would need for each z cross section to be very symmetric, let's say they're circular disks, and would need to know the x- and y-coordinates, $a(z)$ and $b(z)$, respectively, of the center of each z cross-sectional disk, for these coordinates are clearly the coordinates of the centroid of each disk itself. In analogy with what we wrote for thin metal plates, we write $\overline{x}_z$ and $\overline{y}_z$ for $a(z)$

and $b(z)$, respectively.

We would then have

$$\overline{x} = \frac{\int_{z=0}^{z=H} \overline{x}_z \, dm}{\int_{z=0}^{z=H} \, dm} = \frac{\int_0^H \overline{x}_z \delta k z^2 \, dz}{\int_0^H \delta k z^2 \, dz} = \frac{\int_0^H \overline{x}_z \, z^2 \, dz}{\int_0^H z^2 \, dz} = \frac{\int_0^H \overline{x}_z \, z^2 \, dz}{H^3/3}$$

and

$$\overline{y} = \frac{\int_{z=0}^{z=H} \overline{y}_z \, dm}{\int_{z=0}^{z=H} \, dm} = \frac{\int_0^H \overline{y}_z \delta k z^2 \, dz}{\int_0^H \delta k z^2 \, dz} = \frac{\int_0^H \overline{y}_z \, z^2 \, dz}{\int_0^H z^2 \, dz} = \frac{\int_0^H \overline{y}_z \, z^2 \, dz}{H^3/3}.$$

The most simple, standard case would be that of a right circular cone, centered along the z-axis. Then, $\overline{x}_z = 0$ and $\overline{y}_z = 0$, and the above formulas yield that $\overline{x}$ and $\overline{y}$ are both zero. Good – that's what we would automatically conclude by appealing to the symmetry of the right circular cone.

3.8.1 Exercises

In Exercises 1 - 5 you are given the position vectors of a collection of objects in two or three dimensions along with the mass of each object. Calculate the center of mass of each system. For each of the two dimensional systems, make a plot showing the position of the objects and the center of mass.

1.

$\vec{r_i}$	m_i
$(0,0)$	5
$(0,1)$	10
$(1,0)$	4

2.

$\vec{r_i}$	m_i
$(-1,0)$	3
$(0,1)$	6
$(1,0)$	4
$(0,-1)$	9

3.

$\vec{r_i}$	m_i
$(1,1)$	1
$(2,2)$	2
$(3,3)$	3
$(4,4)$	4

4.

$\vec{r_i}$	m_i
$(0,0,0)$	9
$(0,0,1)$	15
$(0,1,0)$	12
$(1,0,0)$	8

5.

$\vec{r_i}$	m_i
$(-1,0,0)$	2
$(1,0,0)$	4
$(0,1,0)$	3
$(0,-1,0)$	6
$(0,0,1)$	5
$(0,0,-1)$	7

6. Consider a collection of five objects. Four of the objects lie at the corners of a square and have mass m. The fifth object lies at the geometric center of the square and has mass k. What is the center of mass of the system?

In Exercises 7 - 12, you are given the length-density function $\delta_\ell(x)$, of an idealized 1-dimensional object that lies along the x-axis. For each problem, calculate (a) the x-component of the moment and (b), the x-coordinate of the center of mass.

7. $\delta_\ell(x) = 3x^2 + 2x + 4$, x in $[2, 7]$.

8. $\delta_\ell(x) = \cos x$, x in $[\frac{-\pi}{4}, \frac{\pi}{4}]$.

9. $\delta_\ell(x) = \dfrac{2x+1}{3x-2}$, x in $[4, 12]$.

10. $\delta_\ell(x) = x \ln x$, x in $[3, 6]$.

11. $\delta_\ell(x) = 5^x$, x in $[2, 3]$.

12. $\delta_\ell(x) = \dfrac{1}{\sqrt{1+x^2}}$, x in $[0, 10]$.

13. Explain why an area-density function of $\delta_a(x) = x^2 - 3$ is valid on the x-interval $[2, 5]$ but not on the interval $[1, 4]$.

14. Consider an annulus centered at the origin with outer radius 2 and inner radius 1. Suppose the top half of the annulus is made of a material with uniform density δ and that the lower half of the annulus is made of a second material with uniform density $k\delta$. What is the center of mass?

In Exercises 15 - 20, you are given an idealized 2-dimensional object S in the xy-plane and the area-density function $\delta_a(x)$ of the object which gives the area-density of x at each point in the x cross section. Find (a) M, (b) $M\bar{x}$, (c) $M\bar{y}$, (d) $\bar{x}$, and (e) $\bar{y}$.

15. S is bounded by the curves $f(x) = x$ and $g(x) = x^2$ between $x = 0$ and $x = 1$. $\delta_a(x) = 2x + 1$.

16. S is a rectangle with vertices $(1, 1)$, $(1, 5)$, $(6, 5)$, and $(6, 1)$, $\delta_a(x) = e^x$.

17. S is the region bounded by the curves $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{1 - x^2}$, $x = -1/2$, $x = 1/2$ and $\delta_a(x) = x^2 + 1$.

18. S is a triangular region with vertices $(0, 0)$, $(1, 1)$ and $(1, -1)$. $\delta_a(x) = 2 - x^2$.

19. S is bounded by the lines $f(x) = 3x + 5$, $g(x) = x - 3$, $x = -4$ and $x = 3$, $\delta_a(x) = x^2 + 5$.

20. S is the region below the curve $f(x) = \cos x$ and above the line $g(x) = 1/2$ and x is between $-\pi/3$ and $\pi/3$. $\delta_a(x) = |x| + 1$.

21. Find the centroid of the region occupied by the metal plate in Example 3.8.7.

22. What is the centroid of a right circular cone where the base lies in the plane $z = 9$ and the vertex is the origin?

23. Consider a cone of uniform density which has a circular disc as its base. The base lies in the plane $z = 1$, has center $(1, 1, 1)$ and radius 1. The cone is the union of all line segment between the origin and points on the disc. What is the centroid of the cone?

24. Must there be any mass at the center of mass of a system? If not, give an example.

25. Suppose $f(x)$ is a continuous function on the interval $[a, b]$ and that $f(x) > 0$. Let S be the region lying below the graph of $f(x)$, above the x-axis, and between $x = a$ and $x = b$ and suppose S has constant density δ. Show that the center of mass, or centroid, has coordinates $\bar{x} = \dfrac{1}{A}\displaystyle\int_a^b x f(x)\,dx$ and $\bar{y} = \dfrac{1}{2A}\displaystyle\int_a^b f(x)^2\,dx$ where A is the area of S.

Use the previous problem to find the centroid of the regions in Exercises 26-31. Assume the density is constant and that each region lies below the graph of $f(x)$, above the x-axis and between $x = a$ and $x = b$.

26. $f(x) = 2x + 4$, $x = 3$, $x = 7$.

27. $f(x) = \sin x$, $x = 0$, $x = \pi$.

28. $f(x) = \cosh x$, $x = -1$, $x = 1$.

29. $f(x) = mx$ where $m > 0$, $x = 0$, $x = b$.

30. $f(x) = \sqrt{16 - x^2}$, $x = -4$, $x = 4$.

31. $f(x) = e^x$, $x = 0$, $x = 1$.

32. Suppose f and g are two continuous functions on the interval $[a, b]$ and that $f > g$. Let S be the region lying between the graphs of the two functions and bounded by $x = a$ and $x = b$. Assume S has constant density δ and prove the centroid of S has coordinates $\bar{x} = \dfrac{1}{A}\displaystyle\int_a^b x(f(x) - g(x))\,dx$ and $\bar{y} = \dfrac{1}{2A}\displaystyle\int_a^b \left(f(x)^2 - g(x)^2\right)\,dx$.

Use the previous problem to find the centroid of the regions in Exercises 33 - 38 bounded by the graphs of $f(x)$ and $g(x)$, and by $x = a$ and $x = b$. Assume in each case the density is constant.

33. $f(x) = 2x + 9$, $g(x) = 2x + 5$, $x = 4$, $x = 8$.

34. $f(x) = 9$, $g(x) = x^2$, $x = -3$, $x = 3$.

35. $f(x) = \sqrt{64 - x^2}$, $g(x) = \sqrt{9 - x^2}$, $x = -3$, $x = 3$.

36. $f(x) = \sqrt{x}$, $g(x) = x$, $x = 0$, $x = 1$.

37. $f(x) = 2e^x$, $g(x) = 2$, $x = 0$, $x = 3$.

38. $f(x) = 5x^2 + 3$, $g(x) = 2x^2$, $x = -2$, $x = 2$.

In Exercises 39 - 44, you are given a dynamic system where a group of particles of varying masses are moving. The path of the i-th object, which has mass m_i, has position $\vec{p}_i(t)$ at time t. Calculate the location of the center of mass as a function of t.

39. $\vec{p}_1(t) = (0, 0)$, $\vec{p}_2(t) = (t, t)$, $m_1 = 8$, $m_2 = 4$.

40. $\vec{p}_1(t) = (t, 0)$, $\vec{p}_2(t) = (-t, 0)$, $m_1 = 5$, $m_2 = 10$.

41. $\vec{p}_1(t) = (t, 0)$, $\vec{p}_2(t) = (-t, 0)$, $\vec{p}_3(t) = (0, t)$, $m_1 = m_2 = 6$, $m_3 = 5$.

42. $\vec{p}_1(t) = (\cos 2t, \sin 2t)$, $\vec{p}_2(t) = (2\cos t, 2\sin t)$, $m_1 = 100$, $m_2 = 300$. This example models two objects orbiting a central mass-less point. The first "planet" completes a revolution in half the time as the outer planet.

43. $\vec{p}_1(t) = (\cos 2t, \sin 2t)$, $\vec{p}_2(t) = (2\cos t, 2\sin t)$, $\vec{p}_3 = (0, 0)$, $m_1 = 100$, $m_2 = 300$, $m_3 = 1000$. This system is similar to the previous one, but now includes a massive "sun".

44. $\vec{p}_1(t) = (-2 + \cos t, \sin t)$, $\vec{p}_2(t) = (2 + \cos t, \sin t)$, $m_1 = 30$, $m_2 = 20$.

45. Suppose three particles are initially at rest, and have initial positions $\vec{p}_1(0) = (-2, 2)$, $\vec{p}_2(0) = (1, -3)$, $\vec{p}_3(0) = (4, 1)$. At $t = 0$, a constant external force is applied to each particle. The first particle experiences a force of 6 N in the negative x direction, the

second a force of 14 N in the positive x direction and the third, a force of 16 N in the positive y direction. The masses of the three particles are 4 kg, 4 kg and 8 kg respectively.

a. Calculate the position function, $\vec{p}_i(t)$, for each of the three particles.

b. Calculate the position function of the center of mass at time t. Call this function $\vec{c}(t)$.

c. Calculate the magnitude of the acceleration of the center of mass using your answer from part (b).

d. What is the total external force acting on the system? Hint: resolve the force into x and y components first.

e. Calculate the magnitude of the acceleration of the center of mass by dividing the total external force acting on the system by the total mass of the system. Your answer should agree with the answer to part (c).

In Exercises 46 - 50 you are given a system similar to that in the previous problem. Assume each particle starts at rest and that a constant force is applied to each particle starting at $t = 0$. For each problem, calculate the magnitude of the acceleration of the center of mass in two ways: first, by explicitly calculating the position function of the center of mass, and second, by calculating the total force acting on the system. The force acting on particle i is given in x and y components. For example, $F_1 = (3, 0)$ indicates a 3 N force in the positive x direction.

46. $\vec{p}_1(0) = (1, 0)$, $\vec{p}_2(0) = (0, 1)$, $F_1 = (3, 0)$, $F_2 = (0, 3)$, $m_1 = 5$, $m_2 = 5$.

47. $\vec{p}_1(0) = (0, 0)$, $\vec{p}_2(0) = (2, 3)$, $\vec{p}_3(0) = (-1, -3)$, $F_1 = (2, 0)$, $F_2 = (0, -4)$, $F_3 = (-3, 0)$, $m_1 = 3$, $m_2 = 6$, $m_3 = 12$.

48. $\vec{p}_1(0) = (1, 0)$, $\vec{p}_2(0) = (0, 1)$, $\vec{p}_3(0) = (0, 0)$, $F_1 = (2, 0)$, $F_2 = (0, 2)$, $F_3 = (-2, 0)$, $m_1 = 4$, $m_2 = 8$, $m_3 = 12$.

49. $\vec{p}_1(0) = (1, 0)$, $\vec{p}_2(0) = (0, 1)$, $\vec{p}_3(0) = (-1, 0)$, $\vec{p}_4(0) = (0, -1)$, $F_1 = (3, 0)$, $F_2 = (0, 3)$, $F_3 = (-1, 0)$, $F_4(0) = (0, -1)$, $m_1 = m_2 = 1 = m_3 = m_4 = 5$.

50. $\vec{p}_1(0) = (1, 0, 0)$, $\vec{p}_2(0) = (0, 1, 0)$, $\vec{p}_3(0) = (0, 0, 1)$, $\vec{p}_4(0) = (0, 0, 0)$, $F_1 = (2, 0, 0)$, $F_2 = (0, 3, 0)$, $F_3 = (0, 0, 4)$, $F_4(0) = (0, -3, 0)$, $m_1 = m_2 = 6$, $m_3 = m_4 = 12$.

3.9 Work and Energy

The technical notion of *work* does not coincide precisely with what is frequently referred to as work in day-to-day speech. *Work* is defined to be force applied over a displacement. We shall clarify this below, but, for now, we want to emphasize that, if an object does not move, then no work is done on the object. This conflicts with what most people would refer to as the work required to hold some heavy weight motionless in their hands. We have a tendency to think of work as corresponding to applying force over a period of time, not over a displacement; when you're in a physics or mathematics class, you need to stop thinking this way.

Work is, in a technical sense, equivalent to energy, though we usually use the terms slightly differently when speaking and writing. We typically say things like "it takes energy to produce work", but it's also true that the work done on a object can get "converted" into kinetic energy, potential energy, heat, etc. All of this falls under the heading of the principle of *Conservation of Energy*, which states that energy cannot be created or destroyed, merely converted from one form to another. Since the work of Einstein showed that mass can be converted into energy, according to the famous equation $E = mc^2$, which was first put into practice with the development of the atomic bomb, the principles of Conservation of Energy and Conservation of Mass are frequently now combined, and referred to as Conservation of Mass-Energy.

Assume that we have an object whose motion is constrained to a straight line, which we take to be the x-axis. We suppose that we have a constant force F which acts parallel to the x-axis, with the sign of F indicating the direction of the force. If the object begins at x-coordinate x_0 and ends up at x-coordinate x_1, then the displacement of the object is $\Delta x = x_1 - x_0$.

Definition 3.9.1. *The* **work** *done by the constant force F on the object, in displacing the object Δx, is $F\Delta x$.*

In the metric system, the standard unit of work/energy is 1 **joule***, which is equal to 1 Newton-meter. In the English system, you simply use the* **foot-pound***.*

Remark 3.9.2. Note that work can be positive or negative. If the force is in the direction of the displacement, i.e., if the force and displacement are both positive, or the force and displacement are both negative, then the work is positive. However, if the force and displacement are in opposite directions, then the work is negative. For example, suppose that we exert a force upward, raising an object against the force of gravity. Then, we do (or, our force does) positive work on the object. However, gravity does negative work on the object during that displacement.

Instead of saying that a particular force does work, it is common to say that whatever is producing the force does the work. For instance, if Sally pushes an object along the ground, then we would usually say that Sally does work on the object, rather than refer to the force that Sally exerts.

So, what do we do when the force is not constant? We do what we always do: chop things up into small pieces, over which we can assume that the force is approximately constant. Then, we take Riemann sums, and limits, to arrive at the definite integral. Infinitesimally, we write simply that the infinitesimal work dW done by F in displacing an object by the infinitesimal amount dx is $dW = F\,dx$, and so:

Proposition 3.9.3. *If a continuous force $F = F(x)$ (parallel to the displacement) acts on an object, as the object is displaced from x_0 to x_1, then the work W done by F on the object is given by*

$$W \;=\; \int_{x_0}^{x_1} F\,dx.$$

Remark 3.9.4. In some of our later problems, in which we look at lifting liquid out of a tank, it will be more natural to consider the amount of displacement q as a function of the force, i.e., we will use that, essentially, q is a function of F. This means that, instead of using that $dW = F\,dx$, we will use that $dW = q\,dF$.

Example 3.9.5. Suppose, when an object is located at position x meters, that the object experiences a force of $F = -k/x^2$ Newtons, where k is a positive constant. Find, in terms of k, how much work the force does on the object, as the object moves from $x = 2$ to $x = 1$ meter.

Solution: We simply calculate

$$W \;=\; \int_2^1 F\,dx \;=\; \int_2^1 -\frac{k}{x^2}\,dx \;=\; -k\int_2^1 x^{-2}\,dx \;=\; -k\cdot\frac{x^{-1}}{-1}\Big|_2^1 \;=$$

$$k\left(1-\frac{1}{2}\right) \;=\; \frac{k}{2}\ \text{joules.}$$

Before we look at more examples, it will be useful to first discuss the work/energy equivalence and, in particular, discuss the relationship between work, kinetic energy, and (gravitational) potential energy.

Definition 3.9.6. *The* **kinetic energy** *of an object of mass m, moving with speed v, is*

$$E_K \;=\; \frac{1}{2}mv^2.$$

Note that it is classical to use v in the formula for kinetic energy, even though the v is *speed*, not velocity. For motion in a straight line, where the direction of the velocity is given by a plus or minus sign, the distinction between speed and velocity in the calculation of the kinetic energy is irrelevant, for v is squared. However, for the more general situation of arbitrary motion is space, it is important that you use speed in calculating the kinetic energy.

The relationship between work and kinetic energy is given by:

Theorem 3.9.7. *Suppose that F is the sum of all forces acting (along the x-axis) on an object of constant mass m, at all times between t_0 and t_1, where $t_0 < t_1$. Suppose that the position $x = x(t)$ of the object is continuously differentiable, that the velocity v is a continuously differentiable function of x.*

Let x_0 and x_1 denote the positions of the object, at times t_0 and t_1, respectively, and let v_0 and v_1 denote the velocities of the object, at positions x_0 and x_1, respectively.

Then, F is a continuous function of x, and the work done by F in displacing the object from x_0 to x_1 is given by

$$W \;=\; \int_{x_0}^{x_1} F\,dx \;=\; \frac{1}{2}mv_1^2 \;-\; \frac{1}{2}mv_0^2,$$

i.e., the work done by the net force is equal to the change in kinetic energy.

Proof. Let $v = dx/dt$ denote the velocity. By Newton's 2nd Law of Motion,

$$F \;=\; ma \;=\; m\frac{dv}{dt} \;=\; m\,\frac{dv}{dx}\cdot\frac{dx}{dt} \;=\; m\frac{dv}{dx}\,v,$$

where the third equality follows from the Chain Rule.

Thus,

$$W = \int_{x_0}^{x_1} F\,dx = \int_{x_0}^{x_1} mv\,\frac{dv}{dx}\,dx = \int_{v_0}^{v_1} mv\,dv = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2.$$

□

Example 3.9.8. Suppose that $x(t) = 4 - t^2$ is the position function, in meters, of a 10 kilogram object, which is moving along the x-axis between times $t = 0$ and $t = 3$ seconds. Calculate the work done in displacing the object from $x(0) = 4$ meters to $x(3) = -5$ meters.

Solution:

By Theorem 3.9.7, all that matters is the change in kinetic energy, so we need the velocities at times $t = 0$ and $t = 3$. We find

$$v(t) = x'(t) = -2t \quad \text{m/s}.$$

Thus the work done is

$$\frac{1}{2}(10)(v^2(3) - v^2(0)) = 5\big((-6)^2 - 0^2\big) = 180 \quad \text{joules}.$$

Remark 3.9.9. A special application of Theorem 3.9.7 comes up fairly often.

Suppose that some force F_{ext}, which we're thinking of as external, acts on an object with constant mass, and that the only other force is F_{you}, exerted by you (or anything/anyone else). Then, the total force F acting on the object is $F = F_{\text{ext}} + F_{\text{you}}$. Suppose that the initial speed of the object is equal to the final speed of the object; frequently, we consider the case where the object starts at rest, i.e., with zero velocity, and ends up at rest.

Then, Theorem 3.9.7 tells us that

$$\int_{x_0}^{x_1} F\,dx = \int_{x_0}^{x_1} (F_{\text{ext}} + F_{\text{you}})\,dx \;=\; 0,$$

i.e.,

$$\int_{x_0}^{x_1} F_{\text{you}}\,dx \;=\; \int_{x_0}^{x_1} -F_{\text{ext}}\,dx. \tag{3.3}$$

In words, if an object starts and ends with the same kinetic energy, then the work that you do on the object is the negative of the work done by the external force; this is frequently referred to as the **work done against** F_{ext}.

Note that Formula 3.3 is correct even though, usually, it would **not** be the case, at every time or position, that F_{you} is equal to $-F_{\text{ext}}$. For instance, if you begin with an object at rest, and apply a force to lift it, acting against the force of gravity, $-mg$, and end with the object higher, at rest, then the force that you exert cannot **always** be equal to mg; for you must apply a force greater than mg to make the mass accelerate, i.e., to make it start moving upward, and you must apply a force less than mg later in order for the mass to decelerate back down to zero velocity.

It is crucial in Theorem 3.9.7 and in Formula 3.3 that the mass of the object is constant. In our derivations, we used Newton's 2nd Law of Motion that the net force, F, acting on an object is equal to the mass times the acceleration, ma. This may **not** be correct if the mass is changing. See Remark 3.9.17. Even if $F = ma$, we used that m was constant when we found that $\int mv\,dv = mv^2/2 + C$.

Definition 3.9.10. *The (gravitational)* **potential energy** *of an object of mass m, at height h above the surface of the Earth, is*

$$E_P \;=\; mgh,$$

where g is the acceleration due to gravity on Earth.

We have assumed in the above formula that the heights involved are small enough that the force of gravity on the object does not vary significantly. We have also taken zero potential energy to correspond to zero height; however, in fact, it is really only the change in potential energy that occurs in calculations, and so, if it's more convenient to take the zero potential energy location elsewhere, you can do so.

Remark 3.9.9 applies to the case of (gravitational) potential energy.

Theorem 3.9.11. *Suppose that a force F_{you} is used to raise an object of mass m from rest at height $h = h_0$ (above the surface of the Earth) to rest at height $h = h_1$, and that the only other force acting on the object is the force of gravity. Make the same assumptions on the continuity and differentiability of the velocity and position functions as in Theorem 3.9.7 (using, here, h in place of x).*

Then, the work done by the force F_{you} is

$$W = \int_{h_0}^{h_1} F_{\text{you}}\, dh = mgh_1 - mgh_0,$$

i.e., the work done by the force F_{you} is equal to the change in potential energy.

Proof. There are two forces acting on the object: F_{you} and the force of gravity, $-mg$. By Remark 3.9.9, we find

$$\int_{h_0}^{h_1} F_{\text{you}}\, dh = \int_{h_0}^{h_1} -(-mg)\, dh = mg(h_1 - h_0).$$

□

Remark 3.9.12. In what sense is potential energy actually energy? The potential energy of an object is actually the potential energy of the object relative to an ambient (surrounding or external) force. What we mean is that, in the presence of the gravitational field of the Earth, if you release an object of mass m from rest, at a height h above the ground, the object will fall, increasing its speed. In fact, ignoring air resistance, Theorem 3.9.7 tells us that the work done by the force of gravity will convert the object's potential energy into kinetic energy; you can calculate the speed v at which the object will strike the ground simply by solving for v in the equation $mgh = mv^2/2$. The fact that the object will spontaneously start moving, and acquire kinetic energy, is why we want to say the object had an initial energy to begin with.

However, it is important to realize that an object, in and of itself, without the presence of some external force, does **not** possess "potential energy". If the Earth suddenly vanished, an object with positive gravitational potential energy would immediately lose this potential energy.

There is potential energy associated with force fields other than gravitational force. All that is really required to have a reasonable notion of potential energy is that you have a *conservative* force field. While this is a topic for multivariable Calculus, physically, a conservative force field is one in which the work done by the force field in moving an object from one point to another is independent of the path along which the object moves; all that matters is the beginning and end points of the path.

Now we're ready to handle a classic example, after we make the following comment/warning:

In the following example, we use Newton's 2nd Law of Motion and conclude that the net force acting on an object is equal to the mass of the object times its acceleration, i.e., we conclude that $F = ma$, **even though the mass of the object is changing**. This is okay in this example since we assume that the mass which is leaving the object (a leaky bucket of sand) has the same velocity as the bucket as the sand leaks out. See Remark 3.9.17 for a detailed discussion of the issue.

Example 3.9.13. Suppose that you are lifting up a bucket of sand 40 feet, at a constant velocity of 2 ft/s. Suppose that the bucket initially contains 50 lb of sand, but is leaking sand at a rate of 1 pound of sand per second. How much work do you do in lifting the bucket of sand?

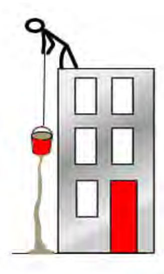


Figure 3.62: Lifting a leaking bucket of sand.

Assume that the rope/wire used to lift the bucket has negligible weight, that by "the velocity of the bucket" we mean the velocity of the center of mass of the bucket and the sand that's in it, and that the instantaneous velocity of the sand as it leaves the bucket is equal to the velocity of the bucket, i.e., assume that the escaping sand leaves the bucket with velocity 2 ft/s (and then starts decelerating once gravity is the only force acting on it).

Note that gravity acts downward, while the displacement is upward. Thus, the work done by gravity will be negative, while the work that you do will be positive. We take the acceleration of gravity to be $g = 32$ ft/s^2.

Set up a y-axis vertically, with 0 at the initial position of the bucket, and the positive direction being upward. Let t be the amount of time elapsed, in seconds, from the time at which you start raising the bucket.

The force of gravity F_{grav} on the bucket is its weight, with a minus sign, to indicate that gravity acts downward. As the bucket initially weighs 50 lb, and loses 1 pound per second, we have

$$F_{\text{grav}} \;=\; -(50-t) \;\; \text{lb}.$$

But we need F_{grav} as a function of y, so that we can calculate the work done by gravity $W_{\text{grav}} = \int_0^{40} F_{\text{grav}}\, dy$.

Since the bucket is moving with a constant velocity of 2 ft/s, the position of the bucket at time t seconds is $y = 2t$ feet. Hence, $t = y/2$, and so

$$F_{\text{grav}} \;=\; -\left(50 - \frac{y}{2}\right) \;\; \text{lb}.$$

As we mentioned before starting this problem, since the sand is leaving the bucket at the same velocity as the bucket, the total/net force $F = F_{\text{you}} + F_{\text{grav}}$ acting on the bucket is ma, where m is the mass of the bucket of sand and a is its acceleration. But, the bucket of sand is moving with constant velocity; so $a = 0$, and we conclude that

$$F \;=\; F_{\text{you}} + F_{\text{grav}} \;=\; 0,$$

i.e., that $F_{\text{you}} = -F_{\text{grav}} = \left(50 - \frac{y}{2}\right)$ ft-lb.

Now, we calculate easily

$$W_{\text{you}} \;=\; \int_0^{40} F_{\text{you}}\, dy \;=\; \int_0^{40} \left(50 - \frac{y}{2}\right) dy \;=\; \left(50y - \frac{y^2}{4}\right)\Bigg|_0^{40} \;=$$

$$(2000 - 400) \; = \; 1600 \; \text{ ft-lb.}$$

A standard example of a force which changes when the position changes is provided by a spring, which has a block attached to it. In a sense, we don't really need to have a block attached to the spring for a discussion about the work involved in compressing or stretching the spring. However, it is convenient to assume that the spring itself has negligible mass, and the only relevant mass in the problem is that of the block.

Thus, let's suppose that the left end of a horizontal spring is attached to a wall, and a block is attached to the right end of the spring. Suppose that the center of the block is at $x = 0$ meters, when the spring is at its natural length. This is called the *equilibrium position* for the block.

When the block is to the left of the equilibrium position, so that the spring is compressed, the force exerted by the spring pushes the block to the right. When the block is to the right of the equilibrium position, so that the spring is stretched, the spring force pulls the block to the left. See Figures 3.63 and 3.64.

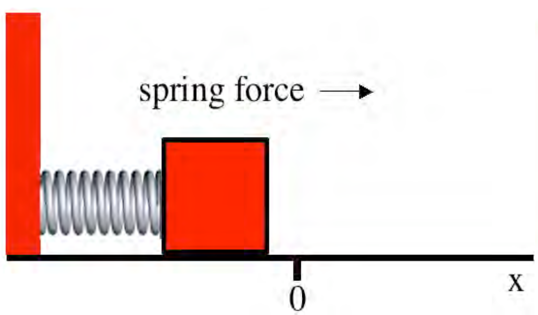


Figure 3.63: A compressed spring.

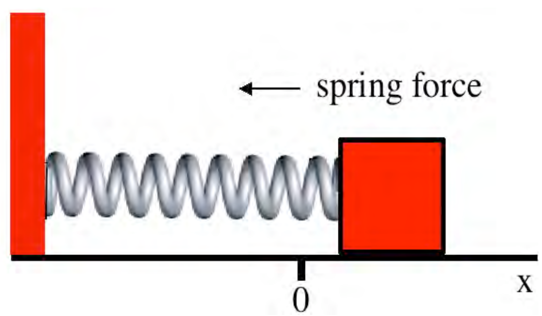


Figure 3.64: A stretched spring.

Springs, if they are not compressed or stretched too far, are typically assumed to obey *Hooke's Law*: the force that the spring exerts is proportional to the displacement from the equilibrium position, and acts in the direction opposite the displacement.

Let $F = F(x)$ denote the force exerted by the spring on the block, when the center of the block is at position x, as measured from its equilibrium position. Note that the displacement of the center of the block from its equilibrium position and the displacement of the right end of the spring from its equilibrium position are the same; thus, we may use the block's displacement and equilibrium position in applying Hooke's Law. Hence, what Hooke's Law tells us is that there exists a positive constant k such that

$$F = -kx.$$

The constant k depends on the spring in question, and is called the *spring constant*.

Example 3.9.14. Suppose that a spring has spring constant $k = 50$ N/m. How much work does it take to compress the spring 1 meter from the equilibrium position?

Solution:

While it is not stated explicitly in the problem, it is implicit that the spring/mass starts at rest and ends at rest. In addition, unlike the leaky bucket problem in Example 3.9.13, the mass is not changing. Hence, we may apply Formula 3.3 to conclude that the work done by the compressing force is negative the work done by the spring, i.e.,

$$\int_0^{-1} F_{\text{comp}}\,dx = \int_0^{-1} -F_{\text{spring}}\,dx = \int_0^{-1} -(-kx)\,dx = \left.\frac{kx^2}{2}\right|_0^{-1} = 25 \text{ joules.}$$

Now we come to two examples where, instead of using that an infinitesimal amount of work dW is $F\,dx$, we use that $dW = q\,dF$, where dF is an infinitesimal amount of force and q is the displacement of the mass on which dF acts.

Example 3.9.15. Suppose that we have a tank, which is a right circular cylinder of radius 8 feet and height 30 feet. The tank contains water up to a height of 10 feet (and the rest of the tank is empty). Assume that the weight-density of water (weight per volume, a.k.a., the *specific weight*) is $\delta_w = 62.4$ lb/ft^3.

We want to find the amount of work required to pump all of the water to the top of the tank. We are not discussing any extra work involved in then moving the water off to a pipe at some location at the top of the tank; in particular, we mean the work involved in moving the water to the top of the tank **at rest**. As the water is not initially moving, and we are calculating the work involved in leaving the water at the top of the tank at rest, the work involved in pumping the water is negative the work that gravity does on the water as we move the water to the top, i.e., the work done against gravity.

Okay. So how do we find the work done against gravity? Set up a z-axis, with $z = 0$ at the bottom of the tank, and with the positive direction being up. For each z-coordinate where we have water, that is, for z between 0 and 10, we calculate the work involved in lifting the infinitesimally thickened z cross section of water. We then add up, i.e., take the integral of, all of these infinitesimal pieces of work as z goes from 0 to 10.

Consider the infinitesimally thickened z cross section of water. As in Section 3.5, the infinitesimal volume of this piece is $dV = A(z)\,dz$, where $A(z)$ is the cross-sectional area. In our current setting, $A(z)$ is constant; it's the area inside a circle of radius 8 feet. Thus, $A(z) = 64\pi$ ft^2, and $dV = 64\pi\,dz$. The (negation of the) infinitesimal force that gravity exerts of this chunk of volume is

$$dF \;=\; \delta_w\,dV \;=\; (62.4)(64\pi)\,dz \;\;\text{lb.}$$

How much do we have to displace the slab of volume at coordinate z to get it to the top of the tank? As the tank is 30 feet high, the needed displacement is $30-z$ feet. Thus, the infinitesimal work done against gravity in order to lift the infinitesimally thick piece of volume at a given z-coordinate between 0 and 10 is

$$dW \;=\; (30-z)\,dF \;=\; (30-z)(62.4)(64\pi)\,dz \;\;\text{ft-lb.}$$

Therefore, the total work required to move all of the water to the top of the tank is

$$W \;=\; \int_{z=0}^{z=10} dW \;=\; \int_0^{10} (30-z)(62.4)(64\pi)\,dz \;=\; (62.4)(64\pi)\left(30z - \frac{z^2}{2}\right)\Big|_0^{10} \;=$$

$$(62.4)(64\pi)\,(300-50) \;\approx\; 3,136,566.1 \;\;\text{ft-lb.}$$

Note that the limits of integration refer to the z-coordinates at which there is water which needs to be lifted. It is a common mistake to let z go from 0 to the height of the tank. Keep in mind what the integral is doing: it's adding up all of the infinitesimal pieces of work from each of the z cross sections at which there's water that's being lifted.

We **could** have obtained this same answer in a different way. In Exercise 32, we ask you to show something that you may have guessed: that you can calculate the work in this problem (or any problem like it) by taking the total weight of the water and then multiplying by the "right" displacement. What is this right displacement? The distance between z-coordinate of the center of mass of the water and the top of the tank.

In more-complicated problems, calculating the work by finding the total weight and z-coordinate of the center of mass would not be significantly easier than calculating a corresponding work integral. However, in our current problem, the calculation via the center of mass is trivial. By symmetry, the z-coordinate of the center of mass is $z = 5$ feet. The total weight of the water is $\delta_w = 62.4$ lb/ft^3 times the volume of a right circular cylinder of radius 8 feet and height 10 feet. Therefore, we quickly find that work required to move all of the water to the top of the tank is

$$(62.4)(64\pi)(10)(30-5) \ \ \text{ft-lb},$$

which, of course, agrees with what we calculated above.

In this example, we change the tank from the previous problem; instead of using a right circular cylinder, we use an upside-down cone.

Example 3.9.16. Suppose that we have a tank, which is a right circular **cone**, which is (mysteriously) balancing on its point. Suppose that the top radius of the cone is 8 feet and height is 30 feet. The tank contains water up to a height of 10 feet (and the rest of the tank is empty). Assume that the weight-density of water (weight per volume) is $\delta_w = 62.4$ lb/ft^3. We once again want to find the amount of work required to pump all of the water to the top of the tank.

What changes from our previous example? In a sense, very little. The only thing that changes is our formula for the cross-sectional area of the tank at a given z value between 0 and 10 feet. Our z cross sections are still disks, but the radius of the disks gets smaller as we get closer to the bottom of the tank.

By using similar triangles, we quickly conclude that the radius r of the cross section at z satisfies

$$\frac{z}{30} \;=\; \frac{r}{8}, \qquad \text{and so} \qquad r \;=\; \frac{4z}{15}.$$

Therefore, the cross-sectional area is given by

$$A(z) \;=\; \pi\left(\frac{4z}{15}\right)^2 \;=\; \frac{16\pi z^2}{225} \ \ \text{ft}^2.$$

Hence, instead of calculating

$$W \;=\; \int_{z=0}^{z=10} dW \;=\; \int_0^{10} (30-z)(62.4)(64\pi)\,dz,$$

which is what we did in Example 3.9.15, we need to calculate

$$W = \int_{z=0}^{z=10} dW = \int_0^{10} (30-z)(62.4)\left(\frac{16\pi z^2}{225}\right) dz.$$

We find

$$W = \frac{(62.4)(16\pi)}{225}\int_0^{10}(30z^2 - z^3)\,dz = \frac{(62.4)(16\pi)}{225}\left(10z^3 - \frac{z^4}{4}\right)\Big|_0^{10} =$$

$$\frac{(62.4)(16\pi)}{225}(10,000 - 2500) \approx 104,552.2 \text{ ft-lb.}$$

We end this section with a serious discussion of Newton's 2nd Law of Motion for objects with changing mass. We should explicitly mention that this is part of Newtonian Mechanics, and does not include a discussion that takes Einstein's theories of relativity into account; this means that we are dealing with velocities which are small compared to the velocity of light, so that the relativistic effects are insignificant.

Remark 3.9.17. When an object changes mass, it either gains or loses mass, and the appropriate form of Newton's 2nd Law depends on the relative velocity of the center of mass of the object and the mass which is gained or lost. Understand that part of the issue with changing mass is that what we are calling "the object" changes; we suddenly start calling the old object, together with the new particles that joined it, the "new object", or we stop referring to particles that used to be part of the object as part of the "new object".

Let m denote the mass of the object at time t, and let F denote the net force acting on the object. Then, dm/dt will be positive if the object is gaining mass and negative if the object is losing mass. Let v_{rel} denote the velocity, relative to the velocity of the center of mass of the object, of the gained or lost mass instantaneously as it joins or departs the object; that is, v_{rel} is the velocity of the center of mass of the gained or lost mass according to someone is moving along with the center of mass of the object, and who considers their own velocity to be zero. This means that if an outside observer says that the velocity of the center of mass of the object is v and the velocity of the center of mass of the gained or lost mass is v_{gl}, then

$$v_{\mathrm{rel}} = v_{\mathrm{gl}} - v.$$

Then, the correct form of Newton's 2nd Law is

$$F = m\frac{dv}{dt} - v_{\text{rel}}\frac{dm}{dt} = ma + (v - v_{\text{gl}})\frac{dm}{dt}. \tag{3.4}$$

Therefore, if the center of mass of the mass that is gained or lost is moving at the same velocity as the center of mass of the object, such as in the leaky bucket problem in Example 3.9.13, then $v - v_{\text{gl}} = 0$ and Formula 3.4 collapses to the familiar

$$F = ma.$$

On the other hand, if $v_{\text{gl}} = 0$, so that $v_{\text{rel}} = -v$, then Formula 3.4 becomes

$$F = m\frac{dv}{dt} + v\frac{dm}{dt} = \frac{d(mv)}{dt},$$

i.e., the net force equals the rate of change of the momentum, with respect to time. This situation is a reasonable approximation of what happens as hail gains mass as it falls through the atmosphere and smashes into essentially motionless water molecules, which become part of the hail. This case may also apply to a rocket, if the rocket is expelling burned fuel at velocity 0, i.e., at velocity $-v$, relative to the rocket.

We should remark that you need to be careful with Conservation of Energy calculations when objects collide or separate. When objects collide, momentum is conserved, but kinetic energy need **not** be conserved; some of the kinetic energy present before the collision may be "used up" in deforming the objects which are colliding. These deformations involve the objects heating up, so that some of the kinetic energy is converted into heat. Such a collision, in which kinetic energy is not conserved, is called an *inelastic collision*. Thus, in an inelastic collision, the total kinetic energy drops. A collision in which the objects actually merge to form one object, such as when motionless water molecules attach to hail, is referred to as *totally inelastic*.

When objects separate, there is the analogous concept of a *totally inelastic separation*. In a totally inelastic separation, the kinetic energy goes up. This implies that extra energy had to come from somewhere, such as a chemical reaction which releases heat. A typical example of this is when a rocket expels burned/burning fuel.

3.9.1 Exercises

In Exercises 1 through 5, calculate the work done in moving a particle subject to the force $F(x)$ as it moves between the endpoints of the interval. Assume that x is position of the particle along the x-axis, measured in meters, and F is measured in Newtons.

1. $F(x) = 3x^2 - 2x + 1$, $-3 \leq x \leq 6$.
2. $F(x) = 4\sin(2x) + \ln x$, $1 \leq x \leq \pi$.
3. $F(x) = \dfrac{3}{x^2}$, $5 \leq x \leq 9$.
4. $F(x) = \dfrac{x^2 + x + 1}{x + 1}$, $0 \leq x \leq 4$.
5. $F(x) = 5e^{-4x}$, $1 \leq x \leq 3$.

6. You exert a force F_{you} on an object. The only other force acting on the object is an external force, F_{ext}. The object is moving along the x-axis. If the initial and final *positions* of the object are the same, is it true that the total work experienced by the object is zero? Explain your answer.
7. You exert a force F_{you} on an object. The only other force acting on the object is an external force, F_{ext}. The object is moving along the x-axis. If the initial and final *speeds* of the object are the same, is it true that the total work experienced by the object is zero? Explain your answer.

In Exercises 8 through 12, you are given the position function $x(t)$ of an object with constant mass m kilograms moving along the x-axis between times t_0 and t_1. Calculate the work done in displacing the object from $x(t_0)$ to $x(t_1)$. The displacement is measured in meters and time is measured in seconds.

8. $x(t) = \cos t$, $t_0 = 0$, $t_1 = \pi/2$.
9. $x(t) = \cos t$, $t_0 = 0$, $t_1 = \pi$.
10. $x(t) = -4.9t^2 + 2t + 100$, $t_0 = 0$, $t_1 = 3$.
11. $x(t) = 3\ln(t + 1)$, $t_0 = 0$, $t_1 = 9$.
12. $x(t) = \cosh t$, $t = 0$, $t = 4$.

In Exercises 13 through 17, an object experiences a force depending on its position x, and a constant k. The total work, W, done by the force on the object as it moves along the x-axis from positions x_0 to x_1 is also given. Solve for k. W is measured in joules, F is measured in Newtons, and displacement is measured in meters.

13. $F = -k/x^2$, $x_0 = 2$, $x_1 = 1$, $W = 112$.

14. $F = x + k$, $x_0 = 3$, $x_1 = 12$, $W = 96$.

15. $F = -k/x$, $x_0 = 4$, $x_1 = 8$, $W = 80$.

16. $F = ke^x$, $x_0 = 0$, $x_1 = 3$, $W = 72$.

17. $F = kx^2$, $x_0 = 4$, $x_1 = 5$, $W = 3$.

In Exercises 18 through 22, you are given the mass and velocity, $v(x)$, of an object moving along the x-axis as a function of its position. Calculate the total work done as the object moves from x_0 to x_1. Mass is measured in kilograms, velocity in meters per second, and displacement in meters.

18. $v(x) = 3x^2 + 2$, $m = 12$, $x_0 = 2$, $x_1 = 5$.

19. $v(x) = 9$, $m = 15$, $x_0 = 2$, $x_1 = 29$.

20. $v(x) = \sin \pi x$, $m = 20$, $x_0 = 0$, $x_1 = 1$.

21. $v(x) = |9x - 2|$, $m = 9$, $x_0 = 1$, $x_1 = 3$.

22. $v(x) = e^{-x^2}$, $m = 11$, $x_0 = 5$, $x_1 = 5$.

23. A worker is cleaning windows on a tall building. The combined mass of the worker and the lift used to transport him is 150 kg. If the distance between each floor is 3 meters, how much work is required to elevate the worker and the lift from the 12th to the 22nd floor?

24. A ball is dropped from a height of 50 meters. What is the speed of the ball when it hits the ground? Solve by equating formulas for potential and kinetic energy.

25. Redo Example 3.9.13 under the following assumptions: the bucket of sand is lifted at constant velocity 4 ft/s. The bucket initially contains 80 lb of sand and the sand is leaking at a rate of 0.5 lb per second. How much work is done lifting the sand 60 feet?

26. Consider a bucket-lifting problem where all the initial data is the same, as in Example 3.9.13, with the following exceptions: The bucket is to be lifted to a height of 120 feet instead of 40 feet. The bucket leaks sand until only 1 lb of residual sand remains. How much work is done in lifting the bucket to 120 feet?

27. Generalize Example 3.9.13: Suppose that you are a lifting a bucket of sand h feet at a constant velocity v ft / s. The bucket initially contains P_0 lb of sand, but is leaking sand at a rate of r pounds of sand per second. Assume $P_0 > \dfrac{rh}{v}$. How much work is done in lifting the bucket?

In Exercises 28 through 30, calculate the work required to compress the spring with the given constant by d meters from the equilibrium position. Use Hooke's Law.

28. $k = 10$ N/m, $d = 2$ meters.

29. $k = 12$ N/m, $d = 3$ meters.

30. $k = 6$ N/m, $d = 5$ meters.

31. Generalize Theorem 3.9.11 by removing the assumptions that the initial and final velocities of the object are zero, and calculating $\int_{h_0}^{h_1} F_{\text{you}}\, dh$ in terms of changes in the potential energy and the kinetic energy of the object.

32. Suppose that a tank has height H and has z cross-sectional area given by the continuous function $A(z)$, where $z = 0$ corresponds to the bottom of the tank, and up is the positive z direction. Suppose also that the tank contains a liquid, which fills the portion of the tank between $z = 0$ and $z = b$, where $0 \leq b \leq H$, and that, at each fixed z-coordinate between 0 and b, every point in the z cross section has the same weight-density $\delta_w(z)$. Finally, suppose that $\delta_w(z)$ is continuous.

    Show that the work required to pump all of the liquid to the top of the tank is equal to the total weight of the liquid times the distance between the top of the tank and the z-coordinate of the center of mass of the liquid.

We now consider a particle which is subjected to a force as the particle travels along a path in three dimensions. We use the vector material in Appendix A.

Let $\vec{\alpha}(t) = (x(t), y(t), z(t))$ be a parameterization of the path where $a < t < b$. Suppose that $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector force field. Then, we define the work done by the force to the particle along the path using integration and the dot product:

$$\textbf{Work} = \int_a^b \vec{F}(\vec{\alpha}(t)) \cdot \vec{\alpha}'(t)\, dt.$$

Here, we are assuming that $\vec{F}$ is continuous and $\vec{\alpha}$ is differentiable. Note that even though the particle is traveling in three dimensions, the calculation of work involves a one-dimensional integral.

33. Suppose $\vec{F}(x, y, z) = (2x, xyz, 5 - z)$, is a vector force field acting on a particle whose position is given by $\vec{\alpha}(t) = (t^2, t, 5)$, $1 < t < 2$.

    a. Show that $\vec{F}(\vec{\alpha}(t)) = (2t^2, 5t^3, 0)$.

    b. Show that $\vec{F}(\vec{\alpha}(t)) \cdot \vec{\alpha}\,'(t) = 9t^3$.

    c. Calculate the work done by $\vec{F}$ on the particle on the path $\alpha(t)$ by calculating $\displaystyle\int_1^2 9t^3\,dt$.

In Exercises 34 through 38, calculate the work done on a particle by the force field $\vec{F}$ as the particle moves along the parameterized path. $\vec{F}$ is in Newtons, distances are in meters, and time is in seconds.

34. $\vec{F}(x, y, z) = (y, x, z^2)$, $\vec{\alpha}(t) = (\cos t, \sin t, t)$, $0 < t < 2\pi$.

35. $\vec{F}(x, y, z) = (yz, xz, xy)$, $\vec{\alpha}(t) = (2t, t, -3t)$, $-1 < t < 1$.

36. $\vec{F}(x, y, z) = (3x^2 - 2x + 1, 0, 0)$, $\vec{\alpha}(t) = (t, 0, 0)$, $-3 < t < 6$. Note that this is just another way of presenting Exercise 1 of this section.

37. $\vec{F}(x, y, z) = (\sinh x, y, z)$, $\vec{\alpha}(t) = (0, 5\sin t, 5\cos t)$, $0 < t < 2\pi$.

38. $\vec{F}(x, y) = \left(\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2}\right)$, $\vec{\alpha}(t) = (\cos t, \sin t)$, $0 < t < 2\pi$. In this case, the particle's path and the force field are both two dimensional. We can fit this exercise into our definitions above by declaring the third components of both the particle's path and the force field to be constantly zero.

In Exercises 39 through 42, it will be helpful to recall Example 3.9.15 and Example 3.9.16. In particular, you should use that the weight-density of water is $\delta_w = 62.4$ lb/ft^3.

39. A rectangular tank has length 20 feet, width 8 feet, and height 30 feet. The tank contains water up to a height of 10 feet (and the rest of the tank is empty). Find the amount of work required to pump all of the water to the top of the tank.

40. Suppose that we have a tank, which is a right circular cylinder of radius 8 feet and height 30 feet, and the tank is initially full. Find the amount of work required to pump all of the water to the top of the tank.

41. Suppose that we have a tank, which is a right circular cone, which is balancing on its point. Suppose that the tank is full, and that the top radius of the cone is 8 feet and height is 30 feet. Find the amount of work required to pump all of the water to the top of the tank.

42. Suppose that we have a tank, which is a right circular cone, which has its circular base at the bottom, and its vertex at the top. Suppose that the radius of the base is 8 feet and height is 30 feet. The tank contains water up to a height of 10 feet (and the rest of the tank is empty). Find the amount of work required to pump all of the water to the top of the tank.

3.10 Hydrostatic Pressure

The term “hydrostatic pressure” refers to pressure that acts on a surface which is submerged in motionless water (“static” for motionless, “hydro” for water), and while we shall, in fact, discuss mainly the case of water on Earth, our discussion applies to essentially any body of liquid anywhere.

When an object is partially, or totally, submerged in a liquid, such as water, the surfaces of the object which are exposed to the liquid experience pressure, force per area, which depends solely on the depth of the liquid at each point on the exposed surface. Assuming that the body of liquid is open above the submerged surface, this pressure can be thought a result of the weight of the liquid above each point.

Imagine a plate of area A submerged in a liquid which has a weight-density (weight per volume) of δ_w, and assume that the plate lies at a constant depth D beneath the (open) surface of the liquid. We assume that the air pressure on the surface of the liquid is negligible. Then the weight, i.e., the force F due to gravity, of the liquid above the plate is the weight-density δ_w of the liquid times the volume, AD, of liquid above the plate. Thus, $F = \delta_w AD$, and so the pressure, the force per area, is

$$P \;=\; F/A \;=\; \delta_w D. \tag{3.5}$$

While a complete discussion of the physics and mathematics is beyond the scope of this book, amazingly, the pressure P on a submerged surface does **not** depend on the orientation of the surface inside the liquid; the pressure P at any point at depth D on the submerged surface produces a *pressure vector*, which has magnitude $P = \delta_w D$ and points perpendicularly (a.k.a., normally) into the surface. In fact, the liquid does not even have to be open to the air (or vacuum) directly above the submerged surface; all that matters is the weight-density of the liquid, and the depth of each point on the submerged surface below the line determined by the open surface of the liquid **even if that open surface of the liquid does not lie directly above the submerged surface**.

When a solid object is submerged (partially or totally) in a liquid, the fact that lower parts of the surface of the object experience more pressure than higher parts explains the existence of *buoyancy force*, the force with which the liquid pushes up on the object. Buoyancy force is easy to describe using *Archimedes Principle*: the buoyancy force exerted on a submerged object is equal to the weight of the amount of the liquid displaced, i.e., if the volume of the submerged portion of the object is V, then the buoyancy force pushing upward on the object is equal to $\delta_w V$. This assumes that the entire surface of the object which lies below the level of the surface of the liquid is exposed to the liquid (for instance, the bottom of the object should not be flush with the base of a water tank).

The serious mathematical treatment of many of the general results pertaining to hydrostatic pressure, in particular, Archimedes Principle, requires the use of multivariable Calculus and integrating vector fields. However, we shall restrict ourselves to a type of problem which we can handle with single variable Calculus; we will look at the total force on a flat surface which is vertical inside our liquid, i.e., a submerged surface that is part of a plane which is perpendicular to the surface of the liquid.

Example 3.10.1. Suppose that one end on a swimming pool is parabolic (and vertical). Set up a vertical y-axis, with $y = 0$ at the bottom of the pool, and where the end of the pool is the region above $y = x^2$ and below $y = 9$, where all distances are measured in feet. Assume the pool is full of water, and that the weight-density of water is $\delta_w = 62.4$ lb/ft^3.

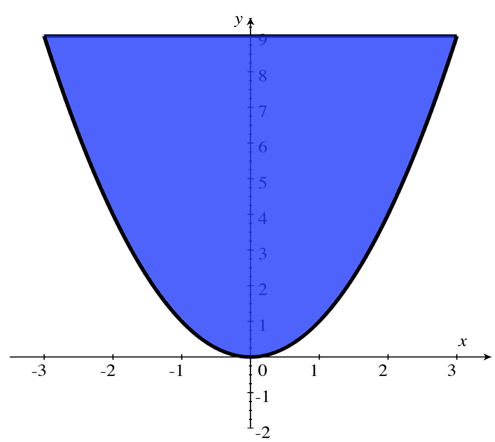


Figure 3.65: The parabolic end of a swimming pool.

We want to determine how much force the water exerts on this parabolic end.

How do you approach such a problem? Well, for $0 \leq y \leq 9$, all of the points at that given y-coordinate have depth $D = 9-y$, and so experience hydrostatic pressure of $P = \delta_w D = \delta_w(9-y)$. We need to multiply this pressure by the infinitesimal area dA at that y-coordinate, to obtain the infinitesimal force dF.

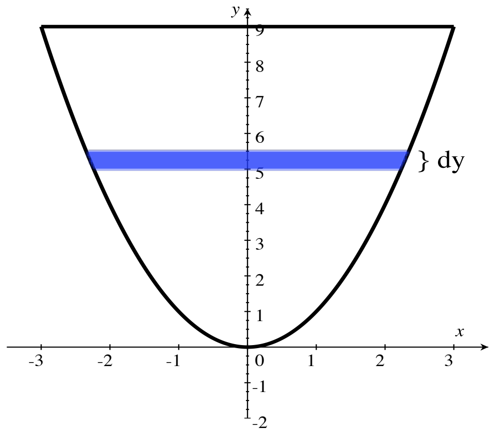


Figure 3.66: A strip of infinitesimal area at a given y-coordinate.

The infinitesimal area at coordinate y is the infinitesimal height dy times the length L of the infinitesimally high rectangle inside our region at the given y-coordinate. That length is the distance between the left x-coordinate on the graph of $y = x^2$, at the given y value, and the right x-coordinate on the graph. Solving for x in terms of y, we find that $x = \pm\sqrt{y}$, and so the two x-coordinates in question are $-\sqrt{y}$ and $\sqrt{y}$. Hence, $L = 2\sqrt{y}$, and

$$dF \;=\; P\,dA \;=\; \delta_w(9-y)L\,dy \;=\; 62.4(9-y)2\sqrt{y}\,dy \text{ lb.}$$

Therefore, the total hydrostatic force F on the parabolic end of the full pool is the continuous sum of all of the infinitesimal contributions:

$$F \;=\; \int_{y=0}^{y=9} dF \;=\; \int_0^9 62.4(9-y)2\sqrt{y}\,dy \;=\; 124.8\int_0^9 (9y^{1/2} - y^{3/2})\,dy \;=$$

$$124.8\left(9\cdot\frac{y^{3/2}}{3/2} \;-\; \frac{y^{5/2}}{5/2}\right)\Bigg|_0^9 \;=\; 124.8\left(6\cdot 27 - \frac{2}{5}\cdot 243\right) \;=\; 8087.04 \;\text{ lb.}$$

Let's record the result of our discussion above in a proposition.

Proposition 3.10.2. *Suppose you have a vertical surface, where y is the vertical coordinate, and that the portion of (one side of) the surface between $y = a$ and $y = b$ ($a \leq b$) is submerged in water, where the top level of the water is at $y = T$. If $L(y)$ is the cross-sectional length of surface at y, and δ_w is the weight-density of water, then the total force, pushing into the given portion of the surface, due to hydrostatic pressure, is given by*

$$\int_a^b \delta_w (T - y) L(y)\, dy.$$

Example 3.10.3. Consider the same swimming pool as in the previous example, but now assume that the water in the pool is only 4 feet deep, i.e., there is water against the parabolic wall between $y = 0$ and $y = 4$ feet. Now how much force does the water exert against the wall?

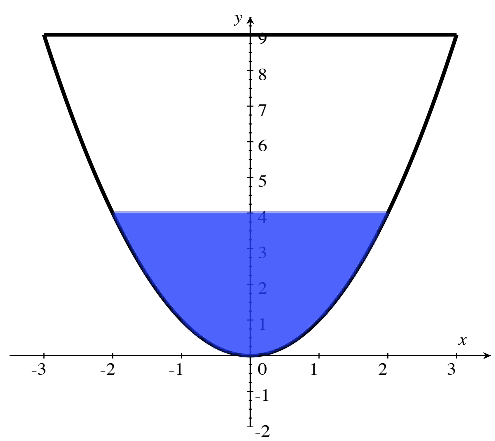


Figure 3.67: Now the pool is not full.

Solution: What changes and what stays the same from the previous example? In our current situation, for $0 \leq y \leq 4$, all of the points at a given y value have **depth** $D = 4 - y$, because the depth (as far as hydrostatic pressure is concerned) is measured from the waterline, not from the top of the pool. Thus, the hydrostatic pressure at each point at a given y-coordinate is $P = \delta_w D = \delta_w (4 - y)$.

The infinitesimal area at coordinate y is still the infinitesimal height dy times the length L of the infinitesimally high rectangle inside our region at the given y-coordinate. That length is still $L = 2\sqrt{y}$, and so

$$dF \;=\; P\,dA \;=\; \delta_w(4-y)L\,dy \;=\; 62.4(4-y)2\sqrt{y}\,dy \text{ lb.}$$

The total hydrostatic force F on the parabolic end of the full pool is the continuous sum of all of the infinitesimal contributions, but now there are contributions only for $0 \leq y \leq 4$. Therefore,

$$F \;=\; \int_{y=0}^{y=4} dF \;=\; \int_0^4 62.4(4-y)2\sqrt{y}\,dy \;=\; 124.8\int_0^4 (4y^{1/2} - y^{3/2})\,dy \;=$$

$$124.8\left(4\cdot\frac{y^{3/2}}{3/2} \;-\; \frac{y^{5/2}}{5/2}\right)\Bigg|_0^4 \;=\; 124.8\left(\frac{8}{3}\cdot 8 - \frac{2}{5}\cdot 32\right) \;=\; 1064.96 \;\text{ lb,}$$

which is far less force than we found in the previous example in which the pool was full.

3.10.1 Exercises

Throughout these exercises assume that the weight-density of water is 62.4 lb/ft^3 or 9800 N/m^3.

1. Suppose that one side of a swimming pool is a flat, vertical rectangular wall of length 12 feet and height 10 feet, and the pool is full, i.e., the water is 10 feet deep. Find the total hydrostatic force on the given wall.

2. Consider the previous exercise.

   a. How much force is exerted on the lower half (half, by height) of the wall?

   b. How much force is exerted on the upper half of the wall?

   c. Explain why one of these values should obviously be larger than the other.

3. Suppose that one side of a swimming pool is a wall which is a flat, vertical isosceles trapezoid, with height 10 meters, and that the upper parallel side has length 12 meters, and the lower parallel side has length 8 meters. Suppose that the pool is full, i.e., the water is 10 meters deep. Find the total hydrostatic force on the given wall.

4. Consider the previous exercise.

   a. How much force is exerted on the lower half (half, by height) of the wall?

   b. How much is exerted on the upper half of the wall?

   c. Explain why, this time, unlike in Exercise 2, it is not so obvious that one of these values should obviously be larger than the other.

5. Suppose that one side of a tank is a flat, vertical semi-circular wall of radius 6 feet, with the "big end" up, i.e., the diameter of the semi-circle is at the top of the tank. Suppose that the water is 4 feet deep (that is, the top of the water is 4 feet above the bottom of the semicircle).

   a. Set this up with $y = 0$ at the top of the wall. Show that the total hydrostatic force is given by
   $$\int_{-6}^{-2} 62.4(-y)2\sqrt{36 - y^2}\, dy \ \ \text{lb.}$$

   b. Evaluate the integral from part a.

6. Suppose now that the semi-circular wall, of radius 6 feet, from the previous problem is upside-down, i.e., the diameter of the semi-circle is at the bottom of the tank. If the water in the tank is still 4 feet deep, determine the hydrostatic force on the wall.

7. Suppose that one side of a tank is a wall which is a flat, vertical isosceles triangle, with the equal angles at the top, and a vertex at the bottom. Let H be the height of the triangle, and let W be the width at the top of the triangle. Suppose that the tank is full, i.e., the water is H meters deep. Find the total hydrostatic force on the given wall.

8. Consider the previous exercise.

   a. How much force is exerted on the lower half (half, by height) of the wall?

   b. How much is exerted on the upper half of the wall?

   c. What do you observe about your answers in parts (b) and (c)?

9. Let's generalize what you should have observed in the previous exercise. Suppose that one side of a tank is a wall which is flat and vertical. Let $y = 0$ correspond to the bottom of the wall, and $y = b$ correspond to the top. Suppose that the cross-sectional length of the wall is given by $L(y) = ky$, for some constant $k > 0$. Suppose the tank is full, so that the top of the water is also at $y = b$.

   Show that the hydrostatic force on the lower half (half, by height) of the wall is equal to the hydrostatic force on the upper half.

10. Consider a vertical end of a swimming pool, as in Example 3.10.1 and Example 3.10.3. Let A be the area of the region R of the vertical end that's covered with water. Set up a vertical y-axis. Then, show that the total hydrostatic force acting on the swimming pool end is equal to $\delta_w AD$, where δ_w is the weight-density of water and D is the depth (below the waterline) of the y-coordinate of the centroid of R (see Section 3.8).

Chapter 4

Understanding Functions via Polynomials and Power Series

Fundamentally, the question that we address in this chapter is: what does it mean to "know what a function is"?

Do you really know what a function $f(x)$ is if you can't calculate the value of f, to some desired accuracy, for each x in the domain of f? Or, another way to look at this is: does your calculator know what functions are better than you do? It can calculate values, to the accuracy of the display, of functions like e^x, $\ln x$, $\sin x$, and $\tan^{-1} x$. Can we do the same thing?

However, knowing what a function **is** goes beyond what your calculator can easily handle. We'd also like to know when one interesting combination of functions is equal to another interesting combination of functions, **as functions**, not just for specific values of the independent variable; in other words, we'd like to produce identities between functions, and/or recognize when we have them.

As we shall see, you can often approximate functions to any desired accuracy by using polynomials, and many functions that we use often are equal to *power series*, basically polynomials with an infinite number of terms.

Because polynomial functions are so easy to evaluate and manipulate, approximating via polynomials, and/or representing a function as some sort of never-ending polynomial, is extremely valuable. As you may recall, in *Worldwide Differential Calculus*, [2], we actually defined the exponential function $\exp(x) = e^x$ by using a power series, which made it very easy to derive the important properties of $\exp(x)$, and allowed us to approximate the value of $e = \exp(1)$ to any desired accuracy.

Most of the technical details of this chapter are theorems that will appear in the next chapter. Most textbooks place the study of these theorems before the results on polynomial approximation and power series. Sadly, this seems to lead to many students missing the most important and useful aspects of the theory.

4.1 Approximating Polynomials

In this section, we will discuss approximating polynomials by using just some of their terms/summands. This will also require us to rewrite polynomials, where, instead of using powers of x, we use powers of $(x-a)$, where a is some fixed value, e.g., powers of $(x-2)$. As we shall see, this rewriting is important for approximating well for values of x near a.

Throughout this section, m and n will both denote integers which are greater than, or equal to, zero.

Of course, you know that a *polynomial function* is a function $p(x)$ defined by a polynomial, e.g.,

$$p(x) \ = \ 4 - 2x + 5x^2 + 9x^5.$$

Note that the term *polynomial* actually refers to the algebraic expression, while *polynomial function* means the function defined by the algebraic expression; this distinction will never cause us a problem, for we will always be very explicit about the algebraic ways in which we wish to rewrite polynomials, even though the original and rewritten polynomials will, of course, both define the same function.

Using our sigma/summation notation (from Definition 2.1.1), we can write a general polynomial, in powers of x, as

$$p(x) \ = \ c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_dx^d \ = \ \sum_{k=0}^{d} c_kx^k,$$

where d is the degree of the polynomial, provided that $c_d \neq 0$. Note that, in the above, formula, we have assumed that $x^0 = 1$, **even if** $x = 0$. This is very notational convenient.

We shall adopt the convention that the expression x^0 is equal to 1, **even when** $x = 0$.

We do this despite the fact that 0^0 is, in general, undefined. This convention simply saves us from having to write a lot of extra words about what we mean in various formulas, in the special case where $x = 0$.

Much of this section is based on the simple observation:

Proposition 4.1.1. *When x is close to 0, higher powers of x are closer to 0. More precisely, if $1 \leq m < n$, and $0 < |x| < 1$, then $|x^n| < |x^m|$, and $\lim_{p\to\infty} |x^p| = 0$.*

So? What does this do for us? It gives us the (imprecise) general principle:

When x is close to 0, a polynomial function $p(x)$ can be approximated "well" by leaving off terms of the form x^n, where n is "large".

We shall make the above general principle more precise in Section 4.3.

Let's look at an example.

Example 4.1.2. Consider the polynomial function

$$p(x) = 4 - 2x + 5x^2 + 9x^5.$$

You may be used to thinking of the highest-power term $9x^5$ as the most important term of the polynomial and, if x is large in absolute value, then that's certainly correct. But, **when x is close to** 0, x^5 will be **very** small, even after multiplying by 9. The point is: when x is close to 0, the 0-th degree term, 4, is the most important, $-2x$ is the second most important term, and $5x^2$ is third most important; the $9x^5$ term is the **least** important if x is close enough to 0.

How close does x need to be to 0 to make the lower-degree terms larger in absolute value than the higher-degree terms? That depends on the precise coefficients in $p(x)$, and we will address such questions in a more general context in Section 4.3. For now, we're going to plug in some numbers and look at the approximations that we obtain when we omit the terms of "large" degree.

However, before we start plugging in numbers, let's adopt the notation that $p^n(x)$ means the sum of all of the terms of $p(x)$, from the constant term up to the x^n term (even if that term is missing, i.e., has a zero coefficient).

Hence,

$$p^0(x) = 4, \qquad p^1(x) = 4 - 2x, \qquad p^2(x) = 4 - 2x + 5x^2, \qquad p^3(x) = 4 - 2x + 5x^2,$$

$$p^4(x) = 4 - 2x + 5x^2, \qquad \text{and} \qquad p^5(x) = p(x) = 4 - 2x + 5x^2 + 9x^5.$$

We won't need beyond p^5, but it is still worth noting that $p^n(x) = 4 - 2x + 5x^2 + 9x^5$, for all $n \geq 5$.

When x is close to 0, we want to approximate $p(x)$ by leaving off its terms of "large degree", i.e., we want to look at the approximation $p(x) \approx p^n(x)$, for various n, when x is close to 0.

Let's fix $x = 0.1$, and look at the approximations $p(x) \approx p^0(x)$, $p(x) \approx p^1(x)$, and $p(x) \approx p^2(x)$. First, we find

$$p(0.1) \;=\; 4 - 2(0.1) + 5(0.1)^2 + 9(0.1)^5 \;=\; 4 - 0.2 + 0.05 + 0.00009 \;=\; 3.85009.$$

Now,

$$p^0(0.1) = 4, \qquad p^1(0.1) = 4 - 2(0.1) = 3.8, \qquad \text{and} \qquad p^2(0.1) = 4 - 2(0.1) + 5(0.1)^2 = 3.85.$$

As you can see, the partial polynomials, or *partial sums*, above give "reasonable" approximations to the actual value of $p(0.1)$, that are accurate to within (plus or minus) 0.14991, 0.05009, and 0.00009, respectively.

What happens if we pick an x which is even closer to 0? Let's look at $x = 0.01$. Then, we find

$$p(0.01) \;=\; 4-2(0.01)+5(0.01)^2+9(0.01)^5 \;=\; 4-0.02+0.0005+0.0000000009 \;=\; 3.9805000009,$$

and

$$p^0(0.01) = 4, \qquad p^1(0.1) = 4-2(0.01) = 3.98, \qquad \text{and} \qquad p^2(0.01) = 4-2(0.01)+5(0.01)^2 = 3.9805.$$

We see that the partial sums now give even better approximations to the actual value of $p(0.01)$; the approximations by the partial sums now are accurate to within (plus or minus) 0.0194999991, 0.0005000009, and 0.0000000009, respectively.

Let's look at another sort of approximation problem.

Example 4.1.3. Suppose again that $p(x) = 4 - 2x + 5x^2 + 9x^5$. If x is close to 0, then what power of x best approximates $\big(p(x) - 4 + 2x\big)/5$?

Solution:

We find

$$\big(p(x) - 4 + 2x\big)/5 \;=\; \big(4 - 2x + 5x^2 + 9x^5 - 4 + 2x\big)/5 \;=\; x^2 + 9x^5/5.$$

Thus, the question is: when x is close to 0, what power of x best approximates $x^2 + 9x^5/5$?

The answer is: the 2nd power, since the smallest degree term, the x^2 term, is most relevant when x is close to 0.

Okay. Great. Now we know that, when x is close to 0, we can approximate polynomials, that are written in terms of powers of x, by taking the first "few" lowest degree terms, i.e., by using the partial sums. We expect the approximation to be better when x is closer to 0, or when we take a partial sum with more terms.

But what if we want to approximate a polynomial when x is close to some x value other than 0? For instance, can we easily approximate the value of a polynomial when x is close to -2? Or when x is close to an arbitrary x value a?

The answer is : yes, and we can do it in exactly the same way that we approximated when x was close to 0, **provided that the polynomial is written in terms of powers NOT of x, but of** $(x - a)$.

For example, let's look at $q(x) = 7 + 3(x+2) - 5(x+2)^4$. Just as with powers of x, when the quantity $(x + 2)$ is close to 0, higher powers of $(x + 2)$ are closer to 0, and are less significant. Hence, when $(x + 2)$ is close to 0, we might use either approximation

$$q(x) \;\approx\; 7, \qquad \text{or} \qquad q(x) \;\approx\; 7 + 3(x + 2).$$

Note that saying that $(x + 2)$ is close to 0 is the same as saying that x is close to -2 and, more generally, saying that x is close to a is the same as saying that $(x - a)$ is close to 0. This is why, if we want to approximate a polynomial well for values of x near some fixed value a, we want our original polynomial to be written in powers of $(x - a)$.

We want to adopt terminology for polynomials written in terms of powers of $(x-a)$, and also for their *partial sums*.

Definition 4.1.4. *A* **polynomial centered at** a *is a polynomial written in terms of powers of* $(x-a)$ *(or with some other variable in place of* x*), i.e., a polynomial of the form*

$$q(x) \;=\; b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \cdots + b_d(x-a)^d \;=\; \sum_{k=0}^{d} b_k(x-a)^k,$$

where, if $b_d \neq 0$*,* d *denotes the degree of* $q(x)$*. Hence, a polynomial written in terms of powers of* x *is centered at* 0*.*

The n**-th order partial sum**, $q^n(x)$*, of such a* $q(x)$ *is the sum of the first* n *terms of* $q(x)$*, i.e., all of the terms of* $q(x)$ *from* b_0 *to* $b_n(x-a)^n$*. Hence,*

$$q^n(x) \;=\; \sum_{k=0}^{n} b_k(x-a)^k.$$

In particular, if $n \geq d$*, then* $q^n(x) = q(x)$

Note that the polynomial function defined by a polynomial centered at a is the same function that you have if you expand the polynomial and write it in terms of powers of x. The point is that the polynomial **function** does not "care" where it's centered. This is the main reason that we sometimes distinguish, in this book, between a polynomial, as an algebraic expression, and the associated polynomial function. However, you needn't worry about being confused; when we write something about a polynomial centered at a, it will be clear what we mean.

The point (for us) of polynomials centered at a is:

If $q(x)$ is a polynomial centered at a, and x is close to a, then the polynomial function $q(x)$ can be approximated "well" by using the partial sums $q^n(x)$.

Before we give an example of approximating a polynomial that's centered somewhere other than 0, we will make a couple of definitions, so that we can more easily discuss the error in approximating by partial sums.

Definition 4.1.5. *Suppose that* $q(x) = \sum_{k=0}^{d} b_k(x-a)^k$ *is a polynomial centered at* a*. Then, the difference between* $q(x)$ *and its* n*-th order partial sum* $q^n(x)$ *is called the* n**-th remainder**, *and is denoted* $R^n(x)$*. Thus,*

$$R^n(x) \;=\; q(x) - q^n(x), \qquad \text{or, equivalently} \qquad q(x) = q^n(x) + R^n(x).$$

The **error in approximating** $q(x)$ **by** $q^n(x)$ *is* $E^n(x) = |R^n(x)|$*, the absolute value of the remainder. Thus,* $q^n(x)$ *approximates* $q(x)$ *to within plus or minus the error, i.e.,*

$$q^n(x) - E^n(x) \;\leq\; q(x) \;\leq\; q^n(x) + E^n(x).$$

Example 4.1.6. Let's return to our example, above, of a polynomial centered at -2:

$$q(x) \;=\; 7 + 3(x+2) - 5(x+2)^4.$$

Suppose that x is within 0.1 of -2. What's that maximum amount of error that we could have in approximating $q(x)$ by $q^1(x)$?

Solution:

To say that "x is within 0.1 of -2" is the same as saying "x is between $-2-0.1$ and $-2+0.1$", i.e., that $-2-0.1 \leq x \leq -2+0.1$. Adding 2 everywhere, we see that this is equivalent to $-0.1 \leq x+2 \leq 0.1$, which, in turn, is equivalent to

$$|x+2| \;\leq\; 0.1.$$

The error in approximating $q(x)$ by $q^1(x) = 7 + 3(x+2)$ is

$$E^1(x) \;=\; |R^1(x)| \;=\; |q(x) - q^1(x)| \;=\; |-5(x+2)^4| \;=\; 5|x+2|^4.$$

Therefore, if $|x+2| \;\leq\; 0.1$, then we have the following upper bound on the error in approximating $q(x)$ by $q^1(x)$:

$$E^1(x) \;=\; 5|x+2|^4 \;\leq\; 5(0.1)^4 \;=\; 0.0005.$$

It is important to note that this is an **upper bound** on the error in the approximation, given that all that we know about x is that $|x+2| \;\leq\; 0.1$. Of course, if we know that some particular x is even closer to -2, then the error $E^1(x)$ will be smaller than 0.0005.

At this point, we know that, when x is close to a, we can approximate a polynomial $q(x)$, which is in terms of powers of $(x-a)$, by using the partial sums $q^n(x)$. But what do we do if we are given a polynomial centered at one point, say, centered at 0, and we want to approximate the values of the polynomial function when x is near some other point $a \neq 0$?

What we need to be able to do is start with a polynomial function $p(x)$, of degree d, and figure out how to rewrite $p(x)$ as a polynomial $q(x)$ centered at any point a that we desire. That is, given $p(x)$ and a, we need to be able to produce the coefficients b_k so that

$$p(x) \;=\; b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \cdots + b_d(x-a)^d \;=\; \sum_{k=0}^{d} b_k(x-a)^k.$$

Before we describe what happens generally, let's look at an example.

Example 4.1.7. Suppose that, once again, $p(x) = 4 - 2x + 5x^2 + 9x^5$, and we want to approximate $p(x)$ by some lower-degree polynomial, so that the approximation is good when x is close to 1.

What we need to do is rewrite $p(x)$ as a polynomial $q(x)$, which is centered at 1, and then use the partial sums of $q(x)$ for our approximations. Thus, we want to find b_0, b_1, b_2, b_3, b_4, and b_5 so that

$$b_0 + b_1(x-1) + b_2(x-1)^2 + b_3(x-1)^3 + b_4(x-1)^4 + b_5(x-1)^5 \;=\; p(x). \tag{4.1}$$

We **could** (but won't) approach this in a purely algebraic fashion: you expand all of the terms on the left, collect the terms with the same powers of x, and match the coefficient of each x^k on the left with the coefficient of x^k in $p(x)$. The coefficients on the left will involve all of the b_k, and you will end up with 6 (linear) equations and 6 unknowns. These equations will actually be easy to solve: you first find b_5, then b_4 is easy to determine after that, then b_3, and so on.

However, Calculus gives us a much easier, faster way to determine the b_k. First, plug in $x = 1$ on both sides of Formula 4.1, and note that all of the powers $(x-1)^k$, where $k \geq 1$, become 0. Therefore, we obtain quickly that

$$b_0 \;=\; p(1).$$

Now, differentiate each side of Formula 4.1 to obtain

$$b_1 + 2b_2(x-1) + 3b_3(x-1)^2 + 4b_4(x-1)^3 + 5b_5(x-1)^4 \;=\; p'(x) \tag{4.2}$$

and, once again, plug $x = 1$ into both sides of the equation; all of the powers of $(x-1)$ are 0, and we find

$$b_1 \;=\; p'(1).$$

You may guess at this point that b_k always equals $p^{(k)}(1)$, the k-th derivative of p at 1. This is **not** correct.

Differentiate again (differentiate Formula 4.2), to obtain

$$2b_2 + 2\cdot 3b_3(x-1) + 3\cdot 4b_4(x-1)^2 + 4\cdot 5b_5(x-1)^3 \;=\; p''(x) \tag{4.3}$$

and plug in $x = 1$ to find

$$b_2 \;=\; \frac{p''(1)}{2}.$$

Hmmmm...what is the pattern here? Let's differentiate more times, and always plug in $x = 1$. You'll see what happens. We find:

$$2\cdot 3b_3 + 2\cdot 3\cdot 4b_4(x-1) + 3\cdot 4\cdot 5b_5(x-1)^2 \;=\; p'''(x),$$

$$2\cdot 3\cdot 4b_4 + 2\cdot 3\cdot 4\cdot 5b_5(x-1) \;=\; p^{(4)}(x),$$

$$2 \cdot 3 \cdot 4 \cdot 5 b_5 \;=\; p^{(5)}(x),$$

and so

$$b_3 = \frac{p'''(1)}{3!}, \qquad b_4 = \frac{p^{(4)}(1)}{4!}, \qquad \text{and} \qquad b_5 = \frac{p^{(5)}(1)}{5!},$$

where, for $n \geq 1$, $p^{(n)}$ denotes the n-th derivative of p, and $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$ is n factorial. For $n = 0$, there are special definitions; $p^{(0)}$ is the 0-th derivative of p, and so is just p itself (p differentiated zero times), while $0!$ is defined to be 1.

With these notations, what we have seen is that, for all k, we have the general formula

$$b_k \;=\; \frac{p^{(k)}(1)}{k!}.$$

For our particular $p(x) = 4 - 2x + 5x^2 + 9x^5$, we find

$$p'(x) \;=\; -2 + 10x + 5 \cdot 9x^4, \qquad p''(x) \;=\; 10 + 4 \cdot 5 \cdot 9x^3, \qquad p^{(3)}(x) \;=\; 3 \cdot 4 \cdot 5 \cdot 9x^2,$$

$$p^{(4)}(x) \;=\; 2 \cdot 3 \cdot 4 \cdot 5 \cdot 9x, \qquad \text{and} \qquad p^{(5)}(x) \;=\; 2 \cdot 3 \cdot 4 \cdot 5 \cdot 9.$$

Therefore,

$$b_0 = p(1) = 16, \qquad b_1 = p'(1) = 53, \qquad b_2 = \frac{p''(1)}{2} = 95, \qquad b_3 = \frac{p'''(1)}{2 \cdot 3} = 90,$$

$$b_4 = \frac{p^{(4)}(1)}{2 \cdot 3 \cdot 4} = 45, \qquad \text{and} \qquad b_5 = \frac{p^{(5)}(1)}{2 \cdot 3 \cdot 4 \cdot 5} = 9.$$

Finally, we find

$$p(x) \;=\; 4 - 2x + 5x^2 + 9x^5 \;=\;$$
$$16 + 53(x-1) + 95(x-1)^2 + 90(x-1)^3 + 45(x-1)^4 + 9(x-1)^5.$$

So, how do you produce a reasonable approximation of $p(x)$, when x is close to 1? Use the partial sums of the polynomial centered at 1, e.g.,

$$16 + 53(x-1) \qquad \text{or} \qquad 16 + 53(x-1) + 95(x-1)^2$$

would provide reasonable lower-degree approximations of $p(x)$, when x is close to 1.

The process that we went through in the above example works in general. We conclude:

Proposition 4.1.8. *Suppose that $p(x)$ is a polynomial function of degree d, and that a is a fixed number. Then, there is a unique polynomial, of degree d,*

$$q(x) \;=\; \sum_{k=0}^{d} b_k (x-a)^k,$$

centered at a, such that there is an equality of functions $p(x) = q(x)$; that polynomial is the one which has coefficients given by

$$b_k \;=\; \frac{p^{(k)}(a)}{k!}.$$

Example 4.1.9. Rewrite the polynomial $p(x) = (x-2)^3$ as a polynomial centered at -1, and use the 2nd partial sum of this new polynomial to approximate $p(-0.9)$.

Solution:

In order to use Proposition 4.1.8, we need to calculate the derivatives of p at -1. We find

$$p'(x) = 3(x-2)^2, \qquad p''(x) = 6(x-2), \qquad \text{and} \qquad p'''(x) = 6,$$

and so

$$p(-1) = -27, \qquad p'(-1) = 27, \qquad p''(-1) = -18, \qquad \text{and} \qquad p'''(-1) = 6.$$

Therefore, Proposition 4.1.8 tells us that

$$(x-2)^3 \;=\; -27 + 27(x+1) + \frac{-18}{2}(x+1)^2 + \frac{6}{6}(x+1)^3 \;=$$

$$-27 + 27(x+1) - 9(x+1)^2 + (x+1)^3.$$

The approximation

$$(x-2)^3 \;\approx\; -27+27(x+1)-9(x+1)^2,$$

applied to $x=-0.9$ yields

$$p(-0.9) \;=\; (-2.9)^3 \;\approx\; -27+27(0.1)-9(0.1)^2 \;=\; -24.39.$$

The actual value of $(-2.9)^3$ is -24.389; so our approximation is pretty good.

4.1.1 Exercises

In Exercises 1 through 6, you are given a polynomial $p(x)$, centered at some a, and values x_0 near, or equal to, the center. For each x_0, determine $p(x_0)$, and the values of the 1st and 2nd order partial sums of $p(x)$, evaluated at x_0. In each case, find the error in approximating $p(x_0)$ by the 1st and 2nd partial sums.

1. $p(x)=4-3x+5x^2-9x^3+100x^4$; $x_0=0,\ 0.1,\ 0.01,\ -0.001$.
2. $p(x)=4-3(x+2)+5(x+2)^2-9(x+2)^3+100(x+2)^4$; $x_0=-2,\ -1.9,\ -1.99,\ -2.001$.
3. $p(x)=7+(x-1)+12(x-1)^2-15(x-1)^6$; $x_0=1,\ 1.1,\ 1.01,\ 0.999$.
4. $p(x)=7+x+12x^2-15x^6$; $x_0=0,\ 0.1,\ 0.01,\ -0.001$.
5. $p(x)=1+x+x^2+x^3+x^4+x^5$; $x_0=0,\ -0.01,\ 0.001$.
6. $p(x)=1+(x-\pi)+(x-\pi)^2+(x-\pi)^3+(x-\pi)^4+(x-\pi)^5$; $x_0=\pi,\ \pi-0.01,\ \pi+0.001$.

7. Let $p(x)=4-3x+5x^2-9x^3+100x^4$. When x is close to 0, what power of x best approximates
$$\frac{1}{5}\big(p(x)-4+3x\big)?$$
8. Let $p(x)=4-3(x+2)+5(x+2)^2-9(x+2)^3+100(x+2)^4$. When x is close to -2, what power of $(x+2)$ best approximates
$$\frac{1}{5}\big(p(x)-4+3(x+2)\big)?$$

9. Let $p(x) = 7 + (x-1) + 12(x-1)^2 - 15(x-1)^6$. When x is close to 1, what polynomial of the form $c(x-1)^n$ best approximates $p(x) - 7 - (x-1)$?

10. Let $p(x) = 7 + (x-1) + 12(x-1)^2 - 15(x-1)^6$. When x is close to 1, what polynomial of the form $c(x-1)^n$ best approximates the derivative $\big[p(x) - 7 - (x-1)\big]'$?

In each of Exercises 11 through 15, rewrite the polynomial $q(x)$ as a polynomial centered at the given a.

11. $q(x) = x^2 + 5x + 6$, $a = -2$.

12. $q(x) = (x+2)^3 - (x+2) + 1$, $a = 2$.

13. $q(x) = x^3 + 15x^2 + 75x + 125$, $a = 5$.

14. $q(x) = 1 + x + 2x^2 + 3x^3 + 4x^4$, $a = 1$.

15. $q(x) = 1 - x + 2x^2 - 3x^3 + 4x^4$, $a = 1$.

In each of Exercises 16 through 20, use the (re-centered) 1st order partial sum from your answers to Exercises 11 through 15 to approximate $q(x)$ at the given value.

16. Approximate $q(x) = x^2 + 5x + 6$ at $x = -2.1$.

17. Approximate $q(x) = (x+2)^3 - (x+2) + 1$ at $x = 2.05$.

18. Approximate $q(x) = x^3 + 15x^2 + 75x + 125$ at $x = 4.99$.

19. Approximate $q(x) = 1 + x + 2x^2 + 3x^3 + 4x^4$ at $x = 1.1$.

20. Approximate $q(x) = 1 - x + 2x^2 - 3x^3 + 4x^4$ at $x = 1.1$.

In Exercises 21 through 25, use the method from Example 4.1.6 to calculate an upper bound on the error in approximating $q(x)$ by $q^{d-1}(x)$, where d is the degree of q, for x values within the given amount of the center.

21. $q(x) = 3 + 4(x+2) - 6(x+2)^2$, x is within 0.2 of -2.

22. $q(x) = 4x^4 + 3x^3 + 2x^2 + x + 1$, x is within 0.1 of 0.

23. $q(x) = 5(x-3)^5 - 7(x-3)$, x is within 0.01 of 3.

24. $q(x) = 6(x-1)^3 - 5(x-1) + 7$, x is within 0.05 0f 1.

25. $q(x) = 12(x+4)^4 + (x+4)^3 - (x+4)^2 + (x+4)$, x is within 0.3 of -4.

26. a. Let $p(x)$ be a polynomial (centered at 0) which contains only even powers of x. Suppose $b_0, ..., b_d$ are the coefficients of $p(x)$ centered at a and that $c_0, ..., c_d$ are the coefficients of $p(x)$ centered at $-a$. Prove that $b_i = c_i$ when i is even, and $b_i = -c_i$ when i is odd.

    b. State and prove a similar conclusion when $p(x)$ contains only odd powers of x.

4.2 Approximation of Functions by Polynomials

In the previous section, we discussed approximating polynomials $p(x)$, centered at some point a, for values of x close to a, by using the first few lowest-degree terms, i.e., by using partial sums.

In this section, our goal is to investigate approximating a **non-polynomial** function $f(x)$, by using a polynomial function $p(x)$ and, if we want the approximation to be good for x near a, then we will use a polynomial which is centered at a. Why do this? Because **polynomials are our friends**; we can easily evaluate polynomials by hand (if we have to), because all that's involved is raising to powers, multiplying by constants, and adding (or subtracting). We understand polynomials, and if we can use them to accurately approximate other functions, then we can say, in some serious sense, that we understand the functions being approximated.

In this section, we will give the intuitive idea, and look at examples. In the next section, we will investigate how you can place some rigorous bounds on how good/bad the approximation may be.

Suppose that we have two functions $f(x)$ and $p(x)$, which are both n times differentiable, for some integer $n \geq 0$. We're thinking of $f(x)$ as a non-polynomial function, and of $p(x)$ as a polynomial centered at a, but, for now, they could be any two functions which are differentiable enough.

If we want $p(x)$ to provide a good approximation of $f(x)$, when x is close to a, what should we want to be true? Well, first of all, we can require that **at** a, f and g have the same value, i.e., require that $f(a) = p(a)$; to make this fit in with what we will see later, we also say that f and p have the same 0-th derivative at a. This is equivalent to saying that the graphs of f and p intersect at the point $(a, f(a)) = (a, p(a))$. As f and g are differentiable, they're continuous, and so, requiring them to be equal at a forces them to not be **too** far apart if x is close to a. But can we do better?

How about if we continue to require $f(a) = p(a)$, and also require that f and p are changing at the same rate at $x = a$? This certainly seems like it should force f and p to be close to each other for x near a. This new condition is simply that $f'(a) = p'(a)$, and, combined with requiring that $f(a) = p(a)$, means that the graphs of $y = f(x)$ and $y = p(x)$ have the same tangent lines at the point $(a, f(a)) = (a, p(a))$.

But two graphs which intersect at a point, and have the same tangent line there, may not have the same concavity. Wouldn't requiring that the two graphs have the same tangent line and concavity at the point $(a, f(a)) = (a, p(a))$ force the values of the two functions to be even closer together when x is close to a? This would mean that we would require $f''(a) = p''(a)$.

Hopefully, by now, you can see where we're heading. The general principle (which doesn't **always** work) is that, if we want for $f(x)$ and $p(x)$ to be really close to each other, when x is near a, then we could/should force f and p, and as many of their derivatives as we wish, to be equal at a.

In Figure 4.1, the function whose graph is in black has the same 0-th and 1st derivatives at $x = 1$ as the other three functions whose graphs appear in red, green, and blue, but only the functions whose graphs are black and blue also have equal 2nd derivatives at 1. As you can see the black and blue graphs are very close together near $x = 1$.

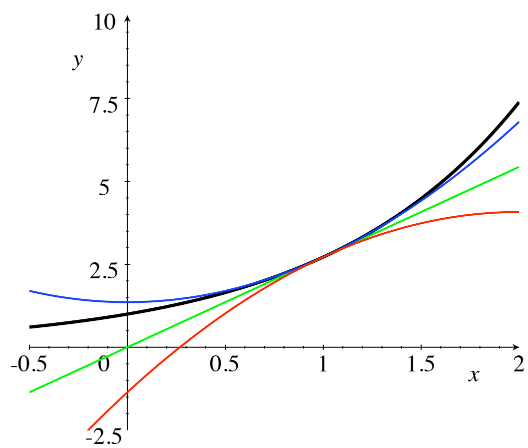


Figure 4.1: Graphs of four functions with some matching derivatives at $x = 1$.

Now we want to assume that $f(x)$ is n times differentiable at a, and we want to determine a polynomial $p(x) = \sum_{k=0}^{n} c_k(x-a)^k$, centered at a, which approximates $f(x)$ well, when x is close to a. From our discussion above, we see that we want $f^{(k)}(a) = p^{(k)}(a)$, for all k such that $0 \leq k \leq n$. The good news is that this completely determines the coefficients c_k of $p(x)$ in terms of the derivatives of f at a.

We will now derive a formula for the c_k, and the discussion will look just like that in Example 4.1.7, and our conclusion will look very similar to Proposition 4.1.8, though we cannot possibly have an equality of $f(x)$ and a polynomial approximation $p(x)$, if f itself is not a polynomial function.

Theorem 4.2.1. *Suppose that $f(x)$ is a function which is n times differentiable at a, and that $p(x) = \sum_{k=0}^{n} c_k(x-a)^k$ is a polynomial, centered at a, such that, for all k such that $0 \leq k \leq n$, there is an equality of derivatives $f^{(k)}(a) = p^{(k)}(a)$. Then,*

$$c_k = \frac{f^{(k)}(a)}{k!}.$$

Proof. The real **proof** of the formula is by *mathematical induction*, but the argument that we give should be completely convincing.

We want to see that:

$$\text{if} \quad p(x) = \sum_{k=0}^{n} c_k(x-a)^k, \quad \text{then} \quad p^{(k)}(a) = (k!)c_k. \tag{4.4}$$

For, then, requiring $f^{(k)}(a) = p^{(k)}(a)$ forces us to have the equality $f^{(k)}(a) = (k!)c_k$, and so $c_k = f^{(k)}(a)/(k!)$.

So, we need to see why Formula 4.4 is true. We have

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots + c_n(x-a)^n.$$

Hence,

$$p'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1},$$

$$p''(x) = 2c_2 + (2\cdot 3)c_3(x-a) + (3\cdot 4)c_4(x-a)^2 + \cdots + [(n-1)n]c_n(x-a)^{n-2},$$

$$p'''(x) = (2\cdot 3)c_3 + (2\cdot 3\cdot 4)c_4(x-a) + \cdots + [(n-2)(n-1)n]c_n(x-a)^{n-3},$$

and so on. What you can see is that the constant term of $p^{(k)}(x)$ is $(k!)c_k$, and all of the other terms, all of the positive powers of $(x-a)$, are zero when $x = a$. Formula 4.4 follows, and we are done. $\square$

We make the following definition.

Definition 4.2.2. *Suppose that $f(x)$ is a function which is n times differentiable at a. Then, the polynomial*

$$T_f^n(x;a) \;=\; \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \;=$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the **n-th order Taylor polynomial of f, centered at** a.

When $a = 0$, we write simply $T_f^n(x)$ in place of $T_f^n(x;0)$, and refer to the polynomial

$$T_f^n(x) \;=\; \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k \;=$$

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n$$

as the **n-th order Maclaurin polynomial of f**.

Remark 4.2.3. We want to emphasize that, for all n, $T_f^n(a;a) = f(a)$; that is, the Taylor polynomials always give the actual value of f at $x = a$.

Note that the n-th order Taylor polynomial will have degree n, provided that $f^{(n)}(a) \neq 0$; for this reason, the n-th order Taylor polynomial is sometimes referred to as the **degree n Taylor polynomial**, without worrying about whether or not $f^{(n)}(a)$ is zero.

The point of our discussion up to this point is that:

We expect that, for n big enough, $T_f^n(x;a)$ approximates $f(x)$ well, when x is near a.

Note that our previous discussion does not **prove** that for n big enough, $T_f^n(x;a)$ approximates $f(x)$ well, for x near a; we will discuss bounds on the error in the approximation in the next section. In the remainder of this section, we will calculate some specific Taylor/Maclaurin polynomials, and compare the approximations that we get from them with the "actual values" of $f(x)$ that we get from calculators. This is somewhat circular, since modern computers and scientific calculators, with floating point units, use polynomial approximation to give you highly accurate results.

Remark 4.2.4. Note that the 1st order Taylor polynomial $T_f^1(x;a) = f(a) + f'(a)(x-a)$, centered at a, is precisely the *linearization* of f at a, $L_f(x;a)$, which we discussed in [2]. Hence, the approximation

$$f(x) \;\approx\; T_f^1(x;a)$$

is the *linear approximation* of $f(x)$, when x is close to a.

As we shall see, using Taylor polynomials of order higher than 1 typically yields a better approximation than you obtain by using merely the linearization.

Example 4.2.5. Suppose that $f(x) = e^x$. Let's find $T_f^n(x)$, the n-th order Maclaurin polynomial of f, and use the 1st and 3rd order Maclaurin polynomials to approximate $e^{0.1}$. we'll compare these approximations with the calculator value of $e^{0.1}$.

The coefficient c_k of the x^k term in $T_f^n(x)$ is $c_k = f^{(k)}(0)/(k!)$. So, we need all of the derivatives of $f(x)$ at 0. But all of the derivatives of e^x are e^x, which has value 1 when $x = 0$. Therefore, $c_k = 1/(k!)$, and the n-th order Maclaurin polynomial of e^x is

$$T_f^n(x) \;=\; \sum_{k=0}^{n} \frac{x^k}{k!} \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

In a way, this should be very **unsurprising**; in [2], we defined $\exp(x) = e^x$ to be the limit of the polynomials

$$T_f^n(x) \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

This leads to a general question which we will address in later sections: when is a function $f(x)$ the limit, as $n \to \infty$, of its Taylor polynomials $T_f^n(x)$?

For now, however, we just want to investigate how well $T_f^1(0.1)$ and $T_f^3(0.1)$ approximate $e^{0.1}$ (where, in this section, we are going to rely on our calculators for highly accurate values).

We shall use an equality sign in quotes to denote "equal, according to the full precision of (our) calculators".

Your calculator should tell you that

$$e^{0.1} \text{ ``}=\text{''} \ 1.10517091808,$$

and we find that

$$T_f^1(0.1) \ = \ 1 + 0.1 \ = \ 1.1,$$

while

$$T_f^3(0.1) \ = \ 1 + 0.1 + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} \ = \ 1.105166\overline{6}.$$

As you can see, the approximation by T_f^1 is fairly good, but the approximation by T_f^3 is much better.

Before we go on, it will be convenient to adopt some new notation. Rather than write the n-th order Taylor polynomial for arbitrary n, it's nice to use some notation which implies that there is no particular n at which we need to stop. Thus, for example, when $f(x) = e^x$, rather than writing that, for all n,

$$T_f^n(x) \ = \ \sum_{k=0}^{n} \frac{x^k}{k!} \ = \ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!},$$

we instead write

$$T_f(x) \ = \ T_f^\infty(x) \ = \ \sum_{k=0}^{\infty} \frac{x^k}{k!} \ = \ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

We use the notations $T_f(x)$ and $T_f^\infty(x)$ interchangeably. We are **not** claiming (yet!) that this infinite sum defines a function. For now, it is formal notation that simply indicates, for arbitrarily large n, what $T_f^n(x)$ is.

More generally, we make the definition:

Definition 4.2.6. *A* **power series, centered at** a*, is a formal algebraic expression*

$$\sum_{k=0}^{\infty} c_k(x-a)^k \;=\; c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots ,$$

where the c_k *are constants.*

Suppose that $f(x)$ *is a function which is* **infinitely differentiable** *at* a*, i.e., such that the* n*-th derivative* $f^{(n)}(a)$ *exists for all* n*. Then, the power series* $T_f(x;a) = T_f^{\infty}(x;a)$*, centered at* a*, given by*

$$T_f(x;a) \;=\; T_f^{\infty}(x;a) \;=\; \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \;=$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

is the **Taylor series of** f**, centered at** a*; when* $a=0$*, this is also referred to as the* **Maclaurin series of** f.

Remark 4.2.7. Once we have the notion of a Taylor series, it is natural to **think of the** n**-th order Taylor polynomial as the** n**-th order partial sum of the Taylor series**.

We also want to state again that, at this point, we have not defined power series **functions**. Just as there is a difference between a polynomial as an algebraic object and as a polynomial function, there is a difference between a power series as an algebraic object and as a function (assuming that we can even make sense of the infinite sum as a function).

Example 4.2.8. Find the Taylor series for $f(x) = \sin x$, centered at $\pi/2$, and use the first two non-zero terms to estimate $\sin(1.5)$. Compare your estimate with the calculator value of $\sin(1.5)$.

Solution: First, it is important that all angles are in **radians**, as is always the case in a Calculus course, unless you are explicitly told otherwise.

The Taylor series for $f(x)$, centered at $\pi/2$ is the power series

$$T_f\left(x;\frac{\pi}{2}\right) \;=\; \sum_{k=0}^{\infty} c_k\left(x-\frac{\pi}{2}\right)^k \;=\; c_0+c_1\left(x-\frac{\pi}{2}\right)+c_2\left(x-\frac{\pi}{2}\right)^2+c_3\left(x-\frac{\pi}{2}\right)^3+\cdots,$$

where $c_k = \dfrac{f^{(k)}(\frac{\pi}{2})}{k!}$.

Thus, as is always the case when finding Taylor series, the problem boils down to calculating $f^{(k)}(a)$, which here is $f^{(k)}(\frac{\pi}{2})$, for all k.

The derivatives of sine repeat every four times you take a derivative:

$$f(x)=\sin x, \qquad f'(x)=\cos x, \qquad f''(x)=-\sin x, \qquad f'''(x)=-\cos x, \qquad f^{(4)}(x)=\sin x,$$

and, now that we're back at $\sin x$, the derivatives just keep cycling. Therefore, the derivatives (including the 0-th), evaluated at $a=\pi/2$, are

$$1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0,\ldots$$

and so, the Taylor series, centered at $\pi/2$ is

$$T_f\left(x;\frac{\pi}{2}\right) \;=$$

$$1+0\left(x-\frac{\pi}{2}\right)-\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2+0\left(x-\frac{\pi}{2}\right)^3+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4+0\left(x-\frac{\pi}{2}\right)^5-\frac{1}{6!}\left(x-\frac{\pi}{2}\right)^6+\cdots \;=$$

$$1-\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4-\frac{1}{6!}\left(x-\frac{\pi}{2}\right)^6+\frac{1}{8!}\left(x-\frac{\pi}{2}\right)^8-\frac{1}{10!}\left(x-\frac{\pi}{2}\right)^{10}+\cdots.$$

Hopefully, you can see the pattern here. We have only the even powers of $\left(x-\dfrac{\pi}{2}\right)$ and, for each term, we divide by a factorial that matches the exponent, and the sign alternates between $+$ and $-$.

For many purposes, the formula above for $T_f\left(x;\frac{\pi}{2}\right)$, in which the pattern is clear, is good enough. However, frequently, we want to write a formula for c_k, regardless of what k is, so that we can write one "nice-looking" summation formula for the power series. But, in some cases, like our current one, we don't need to produce a formula for **every** c_k; we really only need a formula for the coefficients of the non-zero terms. In this example, the non-zero terms are the even-powered terms, and we would like a formula for c_{2k}, where k can be any integer ≥ 0, and then we will use a summation that runs through only the even powers of $\left(x - \frac{\pi}{2}\right)$ in the first place:

$$T_f\left(x; \frac{\pi}{2}\right) \;=\; \sum_{k=0}^{\infty} c_{2k}\left(x - \frac{\pi}{2}\right)^{2k} \;=$$
$$c_0 + c_2\left(x - \frac{\pi}{2}\right)^2 + c_4\left(x - \frac{\pi}{2}\right)^4 + c_6\left(x - \frac{\pi}{2}\right)^6 + c_8\left(x - \frac{\pi}{2}\right)^8 + c_{10}\left(x - \frac{\pi}{2}\right)^{10} + \cdots .$$

So, now the question is: can we write a nice formula for c_{2k}? Let's see, we have

$$c_{2\cdot 0} = c_0 = 1, \qquad c_{2\cdot 1} = c_2 = -\frac{1}{2!}, \qquad c_{2\cdot 2} = c_4 = \frac{1}{4!} \qquad c_{2\cdot 3} = c_6 = -\frac{1}{6!}, \qquad c_{2\cdot 4} = c_8 = \frac{1}{8!}, \ldots .$$

We can see that the pattern is that

$$c_{2k} \;=\; \pm\frac{1}{(2k)!},$$

where the sign alternates from one even-subscripted coefficient to the next, but can we write an algebraic formula for this alternating sign? Yes. The standard "trick" for this is to look at $(-1)^k$ or $(-1)^{k+1} = (-1)^{k-1}$; for $(-1)^k$ equals 1 when k is even, and -1 when k is odd, while, on the other hand, $(-1)^{k+1}$ equals 1 when k is odd, and -1 when k is even.

Therefore, looking at our coefficients, we see that we want

$$c_{2k} \;=\; (-1)^k\frac{1}{(2k)!},$$

so that

$$T_f\left(x; \frac{\pi}{2}\right) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k} \;=$$
$$1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \frac{1}{8!}\left(x - \frac{\pi}{2}\right)^8 - \frac{1}{10!}\left(x - \frac{\pi}{2}\right)^{10} + \cdots .$$

We were told to use the first two non-zero terms of the Taylor series to approximate $\sin(1.5)$. Before we do this, we should answer the question: why would we expect this approximation to be any good? The answer: because 3 is close to π, so that $3/2 = 1.5$ is close to $\pi/2$. As our x value, 1.5, is close to the center, we expect that the Taylor polynomials will provide a good approximation.

The polynomial that we get from the first two non-zero terms of the Taylor series is

$$T_f^2\left(x; \frac{\pi}{2}\right) \;=\; 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2,$$

which is the same as $T_f^3\left(x; \frac{\pi}{2}\right)$, since the cubed term is missing (i.e., has a coefficient of 0).

A calculator tells us that

$$T_f^2\left(1.5; \frac{\pi}{2}\right) \;=\; 1 - \frac{1}{2!}\left(1.5 - \frac{\pi}{2}\right)^2 \;\text{“}=\text{”}\; 0.997493940056.$$

Comparing this with the calculator value of

$$\sin(1.5) \;\text{“}=\text{”}\; 0.997494986604,$$

we see that the approximation by the 2nd order Taylor polynomial, centered at $\pi/2$ is accurate (according to our calculators) to within 0.000001046548. Pretty good!

Let's look at $\sin x$ again, but center now at 0.

Example 4.2.9. Find the Maclaurin series for $f(x) = \sin x$, and use the first two non-zero terms to estimate $\sin(0.1)$ and $\sin(1.5)$. Compare your estimates with the calculator values.

Solution:

There is no question what we need to do; if $f(x) = \sin x$, then

$$T_f(x) \;=\; \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k,$$

and our problem is to find $f^{(k)}(0)$ and, ideally, to find a nice formula for at least the non-zero derivative at 0.

We saw in the previous example that the derivatives of sine repeat:

$$f(x) = \sin x, \qquad f'(x) = \cos x, \qquad f''(x) = -\sin x, \qquad f'''(x) = -\cos x, \qquad f^{(4)}(x) = \sin x,$$

and so, the derivatives (including the 0-th), evaluated at $a = 0$, are

$$0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1, \ldots$$

and so, the Maclaurin series for $\sin x$ is

$$T_f(x) \ = \ 0 \ + \ 1 \cdot x \ + \ 0 \cdot x^2 \ - \ \frac{1}{3!}\, x^3 \ + 0 \cdot x^4 \ + \ \frac{1}{5!}\, x^5 \ + \ 0 \cdot x^6 \ - \ \frac{1}{7!}\, x^7 + \cdots \ =$$

$$x \ - \ \frac{x^3}{3!} \ + \ \frac{x^5}{5!} \ - \ \frac{x^7}{7!} \ + \ \frac{x^9}{9!} \ - \ \frac{x^{11}}{11!} \ + \ \cdots .$$

We should remark that the fact that sine is an odd function and that the Maclaurin series for $\sin x$ has only odd-powered (non-zero) terms is **not** a coincidence; it is an exercise for you to show that, evaluated at 0, the even derivatives of odd functions are zero, and the odd derivatives of even functions are zero.

Though we shall not need it, we leave it to you to verify that

$$T_f(x) \ = \ \sum_{k=0}^{\infty} (-1)^k \, \frac{x^{2k+1}}{(2k+1)!}.$$

The first two non-zero terms of $T_f(x)$ correspond to the 3rd order Maclaurin polynomial, $T_f^3(x)$, which, in fact, is the same as $T_f^4(x)$.

Hence, we were told to use the approximation

$$\sin x \ \approx \ x - \frac{x^3}{3!},$$

when $x = 0.1$ and when $x = 1.5$.

When $x = 0.1$, we find that

$$T_f^3(0.1) \;=\; 0.1 - \frac{(0.1)^3}{6} \;=\; 0.0998333\overline{3}, \qquad \text{and} \qquad \sin(0.1) \text{ “ = ” } 0.099833416647,$$

which is a very close approximation.

On the other hand, when $x = 1.5$, we have

$$T_f^3(1.5) \;=\; 1.5 - \frac{(1.5)^3}{6} \;=\; 0.9375, \qquad \text{and} \qquad \sin(1.5) \text{ “ = ” } 0.997494986604,$$

which is **not** a particularly close approximation.

It should not be surprising that the approximation when $x = 1.5$ is not as good here, or that the approximation when $x = 1.5$ in the previous example, where we were centered at $\pi/2$, was **much** better. The value $x = 0.1$ is relatively close to 0.1, while 1.5 is not. On the other hand, 1.5 is very close to $\pi/2$.

Example 4.2.10. If x is close to 0, then $6(x - \sin x)$ is best approximated by what power of x?

Solution:

In fact, we cannot **really** justify our answer to this until we have some bound on the error in the Taylor approximation, which we won't look at until the next section.

However, we expect that, when x is close to 0, $f(x) = \sin x$ is close to its Maclaurin polynomials, which we can read off of the Maclaurin series from the previous example. Hence, we expect that

$$\sin x \;\approx\; x \;-\; \frac{x^3}{3!} \;+\; \frac{x^5}{5!} \;-\; \frac{x^7}{7!} \;+\dots\;+\; (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}.$$

Then, performing some easy algebra, we expect that

$$x - \sin x \;\approx\; \frac{x^3}{3!} \;-\; \frac{x^5}{5!} \;+\; \frac{x^7}{7!} \;-\dots\;+\; (-1)^n\frac{x^{2n-1}}{(2n-1)!}$$

and

$$6(x - \sin x) \;\approx\; x^3 \;-\; \frac{6x^5}{5!} \;+\; \frac{6x^7}{7!} \;-\dots\;+\; (-1)^n\frac{6x^{2n-1}}{(2n-1)!}.$$

When x is close to 0, we know that the x^3 term on the right is the most important; the other terms will be extremely close to 0.

Therefore, when x is close to 0, we expect that $6(x - \sin x)$ is best approximated by x^3.

We conclude this section by listing a few important Maclaurin series, some of which we've derived, and some of which we leave as exercises.

Theorem 4.2.11. *If $f(x) = e^x$, then*

$$T_f(x) \;=\; \sum_{k=0}^{\infty} \frac{x^k}{k!} \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

If $f(x) = \sin x$, then

$$T_f(x) \;=\; \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \;=\; x \;-\; \frac{x^3}{3!} \;+\; \frac{x^5}{5!} \;-\; \frac{x^7}{7!} \;+\; \frac{x^9}{9!} \;-\; \frac{x^{11}}{11!} \;+\; \cdots .$$

If $f(x) = \cos x$, then

$$T_f(x) \;=\; \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \;=\; 1 \;-\; \frac{x^2}{2!} \;+\; \frac{x^4}{4!} \;-\; \frac{x^6}{6!} \;+\; \frac{x^8}{8!} \;-\; \frac{x^{10}}{10!} \;+\; \cdots .$$

Remark 4.2.12. Note that $\sin x$ is an odd function, and that its Maclaurin series contains only odd-powered terms, while $\cos x$ is an even function, and that its Maclaurin series contains only even-powered terms. These are **not** coincidences.

In Exercise 37, we outline how you show that the Maclaurin series of an odd function contains only odd powers of x, and that the Maclaurin series of an even function contains only even powers of x.

If $f(x) = \ln(1+x)$, *then*

$$T_f(x) \;=\; \sum_{k=0}^{\infty}(-1)^k\,\frac{x^{k+1}}{k+1} \;=\; x \;-\; \frac{x^2}{2} \;+\; \frac{x^3}{3} \;-\; \frac{x^4}{4} \;+\; \frac{x^5}{5} \;-\; \frac{x^6}{6} \;+\; \cdots .$$

If $f(x) = \dfrac{1}{1-x}$, *then*

$$T_f(x) \;=\; \sum_{k=0}^{\infty} x^k \;=\; 1 \;+\; x \;+\; x^2 \;+\; x^3 \;+\; x^4 \;+\; \cdots .$$

Remark 4.2.13. Because we ended with a list of Maclaurin series, you may have gotten the impression that Taylor/Maclaurin **series** were the most important topic in this section. This is **not** the case. We introduced Taylor and Maclaurin series here simply as a convenience; referring to the series enables us to omit any reference to the final term in a Taylor polynomial, and to leave out the phrase “for all n”.

The main point of this section is simple: **the n-th order Taylor polynomial, centered at a, of $f(x)$ should approximate $f(x)$ very well, when x is close to a and n is big**.

4.2.1 Exercises

In Exercises 1 through 5, (a) use the five Maclaurin series given in Theorem 4.2.11 to determine the 1st, 2nd, and 3rd order Maclaurin polynomials of the given function f, (b) using a calculator, compare the values of $f(2)$ with the values of $T_f^1(2)$, $T_f^2(2)$, and $T_f^3(2)$, and explain why your results are not surprising, and (c) using a calculator, compare the values of $f(0.001)$ with the values of $T_f^1(0.001)$, $T_f^2(0.001)$, and $T_f^3(0.001)$, and explain why your results are not surprising.

1. $f(x) = e^x$.

2. $f(x) = \sin x$.

3. $f(x) = \cos x$.

4. $f(x) = \ln(1 + x)$.

5. $f(x) = \dfrac{1}{1 - x}$.

In each of Exercises 6 through 10, (a) determine the 2nd order Taylor polynomial, centered at the given a, for the given function, (b) using a graphing calculator or computer graphing program, graph, in a single window, the original function and $T_f^2(x; a)$, for values of x "close" to a.

6. $p(x) = (x - 1)^3 + (x - 1)^2 + (x - 1)$, $a = 0$.

7. $g(x) = \sinh x$, $a = 0$.

8. $h(x) = \cosh x$, $a = 0$.

9. $f(x) = e^{-x}$, $a = 1$.

10. $r(x) = (1 + x)^4$, $a = 3$.

In each of Exercises 11 through 13, find the 3rd order Taylor polynomial, $T_f^3(x; a)$, for the given f and the given a.

11. $f(x) = \sqrt[3]{x}, a = 1$.

12. $f(x) = \sqrt{1 + x^2}, a = 0$.

13. $f(x) = \ln(x^2), a = 1.$

In Exercises 14 through 18, (a) calculate the specified $T_f^n(x;a)$ and (b) the error between the Taylor approximation and the value given by your calculator.

14. $f(x) = \sin x$.

    a. $T_f^3\left(4.7; \dfrac{3\pi}{2}\right)$

    b. $\left|\sin(4.7) - T_f^3\left(4.7; \dfrac{3\pi}{2}\right)\right|$

15. $g(x) = \sinh x$.

    a. $T_g^3(0.1; 0)$

    b. $|\sinh(0.1) - T_g^3(0.1; 0)|$

16. $h(x) = \ln x$.

    a. $T_h^2(0.9; 1)$

    b. $|\ln(0.9) - T_h^2(0.9; 1)|$

17. $v(x) = \sec x$.

    a. $T_v^2(-0.1; 0)$

    b. $|\sec(-0.1) - T_v^2(-0.1; 0)|$

18. $w(x) = \cos x$.

    a. $T_w^2\left(1.6; \dfrac{\pi}{2}\right)$

    b. $\left|\cos(1.6) - T_w^2\left(1.6; \dfrac{\pi}{2}\right)\right|$

19. What power of x do you expect to approximate $\sin x$ well, when x is near 0? That is, if you want to approximate $\sin x$ by a function x^n, for x values close to 0, what should n be?

20. What power of x do you expect to approximate $6(x - \sin x)$ well, when x is near 0? That is, if you want to approximate $6(x - \sin x)$ by a function x^n, for x values close to 0, what should n be?

21. If you want to approximate $1 - \cos x$ well, by a function of the form cx^n, when x is close to 0, what should you pick for the constant c and the positive integer n?

22. Using a graphing calculator or graphing software, graph $f(x) = 1 - \cos x$ and $g(x) = cx^n$, using your c and n values from the previous exercise. Graph the functions in the same window, with $-1 \leq x \leq 1$ and $-0.5 \leq y \leq 0.5$.

In each of Exercises 23 through 33, determine the Taylor series for the given function, centered at the given a. Give the Taylor series using summation notation, or by giving the first 5 non-zero terms, followed by dots.

23. $s(x) = e^{3x}$, $a = -1$.

24. $u(t) = \sin t + \cos t$, $a = \pi/2$.

25. $k(t) = \cos(2t)$, $a = \pi/4$.

26. $s(t) = 5^t$, $a = 0$.

27. $k(x) = \cosh(2x)$, $a = 0$.

28. $r(t) = \sinh(2t)$, $a = 0$.

29. $f(t) = \ln(1 + 3t)$, $a = 3$.

30. $m(x) = x$, $a = 1$.

31. $n(x) = x^2$, $a = -1$.

32. $j(x) = \ln x$, $a = 1$.

33. $s(x) = \dfrac{1}{1+x}$, $a = 1$.

34. Describe in a paragraph or two the idea behind why the Taylor polynomials of f, centered at a, "should" approximate $f(x)$ well for values of x close to a.

35. What is $T_f^n(x;a) - T_f^{(n-1)}(x;a)$? Assume f is infinitely differentiable in a neighborhood of $x = a$.

36. The approximation $\sin x \approx x$, when x is close 0, is used often in physics and engineering. Where does this come from?

    If you wanted a better approximation of $\sin x$, when x is close to 0, and were willing to use a polynomial with two non-zero terms (instead of one), what polynomial would you use? Why?

37. In this exercise, we give the steps which show that the Maclaurin series of an even or odd function contain only even or, respectively, odd powers of x.

a. Suppose that f is an odd function, which is defined at 0. Show that $f(0) = 0$.

b. Show that the derivative of an even function is odd, and that the derivative of an odd function is even. Conclude that, for infinitely differentiable functions, even-numbered derivatives of odd functions are odd, and odd-numbered derivatives of even functions are odd.

c. Show that, if f is infinitely differentiable and defined at 0, then the only (possible) non-zero terms in the Maclaurin series of f are the even-powered terms if f is even, and the odd-powered terms if f is odd.

38. Derive the Maclaurin series for $\cos x$ given in Theorem 4.2.11.

39. Derive the Maclaurin series for $\ln(1 + x)$ given in Theorem 4.2.11.

40. Derive the Maclaurin series for $1/(1 - x)$ given in Theorem 4.2.11.

4.3 Error in Approximation by Polynomials

In the previous section, we introduced Taylor polynomials, centered at a, of a function $f(x)$. We tried to convince you, on an intuitive level, that the n-th order Taylor polynomial **should** approximate the original function well for values of x near the center when n is large, because we forced n derivatives of f and the Taylor polynomial to match at a. In this section, we will actually give a result on the error involved, and show how to apply it.

As in Section 4.1, we wish to give names and notations for the *remainder* and *error* when approximating a function by its Taylor polynomials.

Definition 4.3.1. *Suppose that f is n times differentiable at a.*

Then, we let

$$R_f^n(x;a) \;=\; f(x) \;-\; T_f^n(x;a)$$

and refer to it as the **n-th order Taylor remainder for f, centered at a.**

We let $E_f^n(x;a) = |R_f^n(x;a)|$, and refer to it as the **n-th order Taylor error for f, centered at a.**

If $a = 0$, we naturally refer to these as the **n-th order Maclaurin remainder and Maclaurin error for f.**

Remark 4.3.2. As $T_f^n(a;a) = f(a)$ for all n, the n-th order Taylor remainder and error are both zero at $x = a$, i.e., $E_f^n(a;a) = |R_f^n(a;a)| = 0$.

We now state and prove the one result that we shall use to establish upper bounds on the error in approximating a function by its Taylor polynomials: the *Taylor-Lagrange Theorem.* The theorem is a generalization of the Mean Value Theorem for derivatives. Various forms of this result are known simply as *Taylor's Theorem.* The result that we give is actually Taylor's Theorem with Lagrange's form of the remainder, and so we refer to the theorem as the Taylor-Lagrange Theorem.

The condition in the Taylor-Lagrange Theorem that $x \neq a$ is present because the theorem refers to a number c that's in the **open** interval between x and a. This does **not** mean that we don't know what happens when $x = a$; remember: when $x = a$, we have already seen that all of the Taylor remainders are 0.

Theorem 4.3.3. (**Taylor-Lagrange Theorem**) *Suppose that n is a non-negative integer, and that the $(n+1)$-st derivative of f exists on an open interval I around a point a.*

Then, for all $x \neq a$ in I, there exists a c in the open interval between x and a (i.e., in (x, a) if $x < a$, or in (a, x) if $a < x$) such that

$$R_f^n(x; a) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}.$$

In particular, for all x in I (even if $x = a$), there exists c in the **closed** *interval between x and a such that*

$$R_f^n(x; a) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}.$$

Proof. This proof is technical. However, as this result is the entire point of this section, we will give the proof here, rather than relegating it to an appendix.

Assume that $x \neq a$. Define the function $g(t)$, with domain I, by

$$g(t) = f(x) - R_f^n(x; a)\,\frac{(x-t)^{n+1}}{(x-a)^{n+1}} - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k.$$

Since f is $(n+1)$-differentiable on I, g is differentiable on I, and so is certainly continuous on the closed interval between x and a. One easily sees that $g(x) = g(a) = 0$. By Rolle's Theorem (see [2]), there exists c in the open interval between x and a such that $g'(c) = 0$.

We claim that this shows that $R_f^n(x; a) = \dfrac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}$, as desired, but we need to calculate $g'(t)$ in order to see this.

We find, using the Product Rule inside the summation,

$$g'(t) \ =$$

$$-R_f^n(x;a)\,\frac{(n+1)(x-t)^n(-1)}{(x-a)^{n+1}} - \sum_{k=0}^{n}\frac{1}{k!}\cdot\left[f^{(k+1)}(t)(x-t)^k + f^{(k)}(t)k(x-t)^{k-1}(-1)\right] =$$

$$(n+1)R_f^n(x;a)\,\frac{(x-t)^n}{(x-a)^{n+1}} - \sum_{k=0}^{n}\frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n}\frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}.$$

Let $j = k-1$, so that $k = j+1$, and use j to re-index the last summation above. We obtain

$$g'(t) \;=\; (n+1)R_f^n(x;a)\,\frac{(x-t)^n}{(x-a)^{n+1}} - \sum_{k=0}^{n}\frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{j=1}^{n-1}\frac{f^{(j+1)}(t)}{j!}(x-t)^j.$$

Now, in these two summations above, one with a plus sign, one with a minus sign, all of the terms cancel out, except for one: $-f^{(n+1)}(t)(x-t)^n/(n!)$.

Therefore, we find that

$$g'(t) \;=\; \frac{(n+1)(x-t)^n}{(x-a)^{n+1}}\cdot\left[R_f^n(x;a) - \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}\right].$$

Since $g'(c) = 0$ and c is in the **open** interval between x and a, we know that $x - c \neq 0$ and so, we conclude that

$$R_f^n(x;a) - \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \;=\; 0,$$

which is what we needed to show. □

Remark 4.3.4. It shouldn't be too difficult to remember the conclusion of the Taylor-Lagrange Theorem; the n-th remainder

$$R_f^n(x;a) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}$$

looks exactly like the $(n+1)$-st order term in the Taylor series for f, **except** that there is this unknown c stuck into the $(n+1)$-st derivative instead of the center a.

As we shall see, to use this form of the remainder, we always need to determine an upper bound on the absolute value of $f^{(n+1)}(c)$, an upper bound that does not contain a reference to the mysterious c.

You should note that the value of c can and, probably, will change when n changes and, for this reason, we could denote c by the more careful notation c_n. However, as our goal is always to place an upper bound on $f^{(n+1)}(c)$ which doesn't have the c in it, there is no practical need to write c_n in place of c.

Example 4.3.5. In Example 4.2.5, we looked at Maclaurin polynomials for $f(x) = e^x$. We saw that $T_f^1(0.1) = 1.1$ and $T_f^3(0.1) = 1.105166\overline{6}$, and we decided to trust our calculators, which told us that $e^{0.1}$ "=" 1.10517091808.

What if we don't want to rely on our calculator? Can we use the Taylor-Lagrange Theorem to obtain upper bounds on the error if we approximate $e^{0.1}$ by the 1st and 3rd order Maclaurin polynomials? We could also ask what order Maclaurin polynomial we would need to use, according to the Taylor-Lagrange Theorem, to obtain the 11 decimal place accuracy of our calculators.

Certainly e^x is infinitely differentiable on the entire real line, so the hypothesis of the Taylor-Lagrange Theorem about being differentiable $(n+1)$ times on an open interval I around a holds for all a and for $I = (-\infty, \infty)$.

Therefore, the error in approximating $e^{0.1}$ by the n-th order Maclaurin polynomial $T_f^n(x)$ is given by

$$E_f^n(0.1) \;=\; \left|R_f^n(0.1)\right| \;=\; \left|\frac{f^{(n+1)}(c)}{(n+1)!}\,(0.1)^{n+1}\right| \;=\; \frac{(0.1)^{n+1}}{(n+1)!}\left|f^{(n+1)}(c)\right|,$$

for some c in the closed interval $[0, 0.1]$. Of course, all of the derivatives of e^x are just e^x. Now, $f^{(n+1)}(c) = e^c$, which is always positive, and so we have

$$E_f^n(0.1) \;=\; \frac{(0.1)^{n+1}\,e^c}{(n+1)!},$$

for some c in the closed interval $[0, 0.1]$. But what do we do with this e^c factor, when we don't know what c is?

The quick answer: we say that the e^c factor is less than or equal to something which has no c in it. This means that we won't know the **exact** error, but will be able to say that the error is no more than a certain amount, which is what we really need in most applications.

Since c is in the closed interval $[0, 0.1]$, and $f(x) = e^x$ is an increasing function (since it's derivative e^x is always positive), regardless of what c is, we know that $e^c \leq e^{0.1}$. Thus, we know

$$E_f^n(0.1) \;\leq\; \frac{(0.1)^{n+1}\, e^{0.1}}{(n+1)!}.$$

At this point, your reaction may be something like "But...but...our whole goal was to find a good approximation of $e^{0.1}$, and now our upper bound on the error contains $e^{0.1}$. Isn't this circular?"

No – because we can now use some fairly awful upper bound, that doesn't have to be very accurate, for $e^{0.1}$, and then conclude how big n needs to be so that our gross upper bound guarantees that $E_f^n(0.1)$ is really small. So, no, our argument will not be "circular", but will seem close to it; we will use a gross upper bound for $e^{0.1}$ to obtain arbitrarily good approximations of $e^{0.1}$. Cool, huh?

How does this work? We will use that $e^{0.1} \leq 2$. Why is this true? Because certainly $e \leq 2^{10}$! So, now we have that

$$E_f^n(0.1) \;\leq\; \frac{(0.1)^{n+1}\, e^{0.1}}{(n+1)!} \;\leq\; \frac{(0.1)^{n+1}\cdot 2}{(n+1)!}. \tag{4.5}$$

Note that our upper bound on the error decreases as n increases, so that using higher-order Maclaurin polynomials gives a smaller upper bound on the error in the approximation. This is what we want and expect.

Now we're ready to answer the questions that we asked at the beginning of this section.

When $n = 1$, Formula 4.5 tells us that

$$E_f^1(0.1) \;\leq\; \frac{(0.1)^{1+1}\cdot 2}{(1+1)!} \;=\; 0.01.$$

In fact, what we really found before, trusting our calculators, is that

$$E_f^1(0.1) \;=\; \left|e^{0.1} - T_f^1(0.1)\right| \;\text{“ = ”}\; |1.10517091808 - 1.1| \;=\; 0.00517091808.$$

Thus, the actual error is roughly half of what we produced for an upper bound, but what's important is that the error **is less than or equal to our upper bound** of 0.01, as it had to be. It is not terribly surprising that our upper bound is "far" from the actual error; we used a rough upper bound for e^c, and then we used that $e \leq 2^{10}$.

The point is that it is more important to end up with a manageable upper bound

than to end up with a more accurate, but unmanageable, upper bound.

When $n = 3$, Formula 4.5 tells us that

$$E_f^3(0.1) \;\leq\; \frac{(0.1)^{3+1} \cdot 2}{(3+1)!} \;=\; \frac{0.0002}{24} \;=\; 0.000008333\overline{3}.$$

Of course, once again trusting our calculators, you should check that the "actual" error is roughly half of this.

Finally, if we were the ones setting up your calculator with a Maclaurin polynomial to approximate $e^{0.1}$ to within 11 decimal places, what order Maclaurin polynomial would Formula 4.5 tell us is good enough? That is, what's the smallest n such that our upper bound on the error $E_f^n(0.1)$ is less than $5(10)^{-12}$, so that the digit in the 12th decimal place is less than 5, and rounding will give the "correct" answer?

We want the smallest n such that

$$\frac{(0.1)^{n+1} \cdot 2}{(n+1)!} \;\leq\; 5(10)^{-12}.$$

The bad news is that there are no algebraic rules that will let you solve for such an n. The good news is that if n is 11, $(0.1)^{n+1} = (10)^{-12}$ and $2/(12!)$ is much less than 1, so $n = 11$ is definitely big enough, but how do we find the smallest such n? The easy way – just check the other n values less than 11, either by hand or with a calculator. You can start at $n = 0$ and go up, or start at $n = 11$ and go down.

We leave it as an exercise for you to show that the answer is $n = 7$. Hence, the 7th order Maclaurin polynomial for e^x can be used to approximate $e^{0.1}$ to 11 decimal place accuracy.

It is worth noting that $n = 7$ is **good enough**, and it's the smallest n that makes our upper bound less than $5(10)^{-12}$, but, since our upper bound is not, itself, the actual error (probably), it is **possible** that some more difficult, or different, argument would tell us a smaller n so that $E_f^n(0.1)$ is less than $5(10)^{-12}$. But we're happy with knowing that $n = 7$ forces our relatively easy upper bound on the error to be less than $5(10)^{-12}$, which, in turn, forces the error itself to be less than $5(10)^{-12}$.

Example 4.3.6. Suppose that x is within ± 0.2 of $\pi/2$. If we use the 4th order Taylor polynomial of $f(x) = \sin x$, centered at $\pi/2$, to approximate $\sin x$, how accurate can we be certain that our approximation is?

Solution:

As in the last example, we will find a gross upper bound on the error $E^n_f(x; \pi/2)$, and then see how big this upper bound is. We shall first look at the situation for general n, and then discuss what happens when $n = 4$; there is something special which happens in this example, something which did not occur in the previous example.

The Taylor-Lagrange Theorem tells us that the error $E^n_f(x)$ in approximating $\sin x$ by $T^n_f(x; \pi/2)$ is given by

$$E^n_f(x; \pi/2) \;=\; \left|R^n_f\left(x; \frac{\pi}{2}\right)\right| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}\left(x - \frac{\pi}{2}\right)^{n+1}\right| \;=\; \frac{\left|x - \dfrac{\pi}{2}\right|^{n+1}}{(n+1)!} \cdot \left|f^{(n+1)}(c)\right|,$$

where c is between x and $\pi/2$, and so is also within ± 0.2 of $\pi/2$.

We were told that x is within ± 0.2 of $\pi/2$; this means precisely that $|x - \pi/2| \leq 0.2$. Thus, we know that

$$E^n_f(x; \pi/2) \;\leq\; \frac{(0.2)^{n+1}}{(n+1)!} \cdot \left|f^{(n+1)}(c)\right|.$$

As in all such problems, we need to find an upper bound on $\left|f^{(n+1)}(c)\right|$. How do we do this? The manner by which you produce such an upper bound varies from function to function, but, this time, is particularly easy.

All of the derivatives of sine are equal to plus or minus sine or cosine, i.e., $f^{(n+1)}(c)$ is one of $\pm \sin c$ or $\pm \cos c$. As the values of sine and cosine are always between -1 and 1, the absolute values of plus or minus sine or cosine are all less than or equal to 1. Hence, $\left|f^{(n+1)}(c)\right| \leq 1$, and

$$E^n_f(x; \pi/2) \;\leq\; \frac{(0.2)^{n+1}}{(n+1)!} \cdot \left|f^{(n+1)}(c)\right| \;\leq\; \frac{(0.2)^{n+1}}{(n+1)!} \cdot 1.$$

When $n = 4$, we find

$$E^4_f(x; \pi/2) \;\leq\; \frac{(0.2)^{4+1}}{(4+1)!} \;=\; 0.00000266\overline{6}.$$

Recall that we found the Taylor series for $\sin x$, centered at $\pi/2$, in Example 4.2.8; we found

$$T_f^\infty\left(x; \frac{\pi}{2}\right) = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \cdots . \tag{4.6}$$

It's 4th order partial sum, the 4th order Taylor polynomial is

$$T_f^4\left(x; \frac{\pi}{2}\right) = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4,$$

which has only three non-zero terms. It's pretty amazing that these three non-zero polynomial terms let us approximate $\sin x$ to within $\pm 0.000002\overline{6}$, provided that we know that x is within ± 0.2 of $\pi/2$.

However, by being slightly tricky, we can actually conclude that the error is substantially smaller. Since all of the odd-powered terms are missing/zero in Formula 4.6, $T_f^4(x; \pi/2) = T_f^5(x; \pi/2)$, which implies that $E_f^4(x; \pi/2) = E_f^5(x; \pi/2)$. Now, our upper bound on $E_f^n(x; \pi/2)$ says that

$$E_f^4(x; \pi/2) = E_f^5(x; \pi/2) \leq \frac{(0.2)^{5+1}}{(5+1)!} = 0.0000000088\overline{8}.$$

So, $T_f^4(x; \pi/2)$ yields a stunningly accurate approximation for x in the given range. For instance, when $x = 0.1 + \pi/2$, a calculator tells us that

$$\sin\left(0.1 + \frac{\pi}{2}\right) \text{ “ = ” } 0.995004165278,$$

while

$$T_f^4\left(0.1 + \frac{\pi}{2};\ \frac{\pi}{2}\right) = 1 - \frac{1}{2!}(0.1)^2 + \frac{1}{4!}(0.1)^4 = 0.99500416666\overline{6}.$$

Example 4.3.7. Let $f(x) = \cos(3x)$.

a. Suppose that $|x| \leq 0.1$. Find a “reasonable” upper-bound on the error in approximating $f(x)$ by $T_f^3(x; 0)$.

b. Suppose that $|x| \leq 0.1$. By using “reasonable” bounds, find the smallest n such that you can prove that the error in approximating $f(x)$ by $T_f^n(x; 0)$ is less than or equal to 0.0001.

c. Suppose that we want $T_f^3(x;0)$ to approximate $f(x)$ to within 0.0001. By using "reasonable" bounds, find the largest $\delta > 0$ such that this accuracy will be achieved provided that $|x| \leq \delta$.

Solution:

All of these involve finding an expression for the Lagrange form of the remainder for $T_f^n(x;0)$, and then producing a "reasonable" upper-bound for its absolute value.

We calculate the derivatives:

$$f'(x) = -3\sin(3x), \quad f''(x) = -3^2\cos(3x), \quad f'''(x) = 3^3\sin(3x), \quad f^{(4)}(x) = 3^4\cos(3x), \ldots .$$

It is easy to see that $f^{(n+1)}(x)$ equals $\pm 3^{n+1}\cos(3x)$ or $\pm 3^{n+1}\sin(3x)$.

Therefore, if c is between x and 0 (actually, regardless of what c is), $|f^{(n+1)}(c)| \leq 3^{n+1}$, and so the error $E_f^n(x;0)$ in approximating $f(x)$ by $T_f^n(x;0)$ satisfies the inequality

$$E_f^n(x;0) \;=\; \left|R_f^n(x;0)\right| \;=\; \left|\frac{f^{(n+1)}(c)}{(n+1)!}\,x^{n+1}\right| \;\leq\; \frac{3^{n+1}}{(n+1)!}\,|x|^{n+1}. \tag{4.7}$$

a. Suppose that $|x| \leq 0.1$. Then, the inequality in Formula 4.7 yields

$$E_f^n(x;0) \;\leq\; \frac{3^{n+1}}{(n+1)!}\,|x|^{n+1} \;\leq\; \frac{3^{n+1}}{(n+1)!}\,(0.1)^{n+1}$$

Hence, $E_f^3(x;0)$, the error in approximating $f(x)$ by $T_f^3(x;0)$, is less than or equal to

$$\frac{3^4}{4!}\,(0.1)^4 \;=\; 0.0003375.$$

b. We suppose again that $|x| \leq 0.1$ and, hence, as above, we know that

$$E_f^n(x;0) \;\leq\; \frac{3^{n+1}}{(n+1)!}\,(0.1)^{n+1}.$$

We want to find the smallest n so that this is less than or equal to 0.0001.

Notice that, from part (a), we know that, when $n = 3$, $\dfrac{3^{n+1}}{(n+1)!}\,(0.1)^{n+1} = 0.0003375$, which is bigger than we want. So, we just start checking n's, starting at $n = 4$, and calculate until we find one so that

$$\frac{3^{n+1}}{(n+1)!}\,(0.1)^{n+1} \leq 0.0001.$$

We were close at $n = 3$, so it sure seems like $n = 4$ or $n = 5$ will work.

When $n = 4$, we find

$$\frac{3^5}{5!}\,(0.1)^5 \;=\; 0.00002025 \;\leq 0.0001.$$

Thus, $n = 4$ is good enough.

c. We use that

$$E_f^3(x;0) \;\leq\; \frac{3^4}{4!}\,|x|^4,$$

and require this to be less than or equal to 0.0001.

We want

$$\frac{3^4}{4!}\,|x|^4 \;\leq\; 0.0001,$$

and so, we need

$$|x| \;\leq\; \left(\frac{4!}{3^4}(0.0001)\right)^{1/4} \quad \text{“}=\text{”}\ 0.073778794646688.$$

Let's try another one.

Example 4.3.8. Let $f(x) = e^{-x}$.

a. Suppose that $|x-1| \leq 0.1$. Find a “reasonable” upper-bound on the error in approximating $f(x)$ by $T_f^3(x;1)$.

b. Suppose that $|x-1| \leq 0.1$. By using “reasonable” bounds, find the smallest n such that you can prove that the error in approximating $f(x)$ by $T_f^n(x;1)$ is less than or equal to 0.0001.

c. Suppose that we want $T_f^3(x;1)$ to approximate $f(x)$ to within 0.0001. By using “reasonable” bounds, find the largest $\delta > 0$ such that this accuracy will be achieved provided that $|x-1| \leq \delta$.

Solution:

As in the previous example, all of these involve finding an expression for the Lagrange form of the remainder for $T^n_f(x;1)$, and then producing a "reasonable" upper-bound for its absolute value.

You should be able to find quickly that $f^{(n+1)}(x) = (-1)^{n+1}e^{-x}$. Therefore, $|f^{(n+1)}(c)| = e^{-c}$, and so the error $E^n_f(x;1)$ in approximating $f(x)$ by $T^n_f(x;1)$ satisfies the inequality

$$E^n_f(x;1) \;=\; \left|R^n_f(x;1)\right| \;=\; \left|\frac{f^{(n+1)}(c)}{(n+1)!}\,(x-1)^{n+1}\right| \;=\; \frac{e^{-c}}{(n+1)!}\,|x-1|^{n+1}, \tag{4.8}$$

for some c between x and 1, inclusive.

a. Suppose that $|x-1| \le 0.1$. This means that $-0.1 \le x-1 \le 0.1$, and so, $0.9 \le x \le 1.1$. As c is between x and 1, it follows that c itself is trapped in the same interval, i.e., that $0.9 \le c \le 1.1$. We need an upper bound on e^{-c}. Be careful! As a function of c, e^{-c} is **decreasing**, because of the negative sign. This means that smaller values of c yield larger values of e^{-c}. Hence, we use that $e^{-c} \le e^{-0.9}$.

   Then, Formula 4.8 yields

$$E^n_f(x;1) \;\le\; \frac{e^{-0.9}}{(n+1)!}\,|x-1|^{n+1} \;\le\; \frac{e^{-0.9}}{(n+1)!}\,(0.1)^{n+1}.$$

   If the $e^{-0.9}$ in this upper-bound seems too ugly, you can use another "reasonable" upper-bound, and write that, since $e^{-0.9} \le 1$,

$$E^n_f(x;1) \;\le\; \frac{e^{-0.9}}{(n+1)!}\,(0.1)^{n+1} \;\le\; \frac{1}{(n+1)!}\,(0.1)^{n+1}.$$

   Using this nicer upper-bound, we find that $E^3_f(x;1)$, the error in approximating $f(x)$ by $T^3_f(x;1)$, is less than or equal to

$$\frac{1}{4!}\,(0.1)^4 \;=\; 0.0000041\overline{6}.$$

b. We suppose again that $|x-1| \le 0.1$ and, hence, as above, we know that

$$E_f^n(x;1) \;\le\; \frac{1}{(n+1)!}\,(0.1)^{n+1}.$$

We want to find the smallest n so that this is less than or equal to 0.0001. You can easily check that $n = 3$ is the smallest n that works.

c. This is more difficult than part (c) in the previous example. Suppose that $|x-1| \le \delta$. Then, as in part (a), where δ was 0.1, we find that $1-\delta \le x \le 1+\delta$, and so the c in the Lagrange form of the remainder also satisfies $1-\delta \le c \le 1+\delta$.

Now, Formula 4.8 tells us that

$$E_f^n(x;1) \;=\; \frac{e^{-c}}{(n+1)!}\,|x-1|^{n+1},$$

where $|x-1| \le \delta$ and, as before, $e^{-c} \le e^{-(1-\delta)}$. We conclude that

$$E_f^n(x;1) \;\le\; \frac{e^{-(1-\delta)}}{(n+1)!}\,\delta^{n+1}.$$

What we want is, when $n = 3$, to find a δ that's as big as we can "reasonably" produce, in order to make the last quantity above less than or equal to 0.0001. That is, we want to know how big we can choose $\delta > 0$ so that

$$\frac{e^{-(1-\delta)}}{24}\,\delta^4 \;\le\; 0.0001. \tag{4.9}$$

Unfortunately, there is no nice algebra that we can do to "solve" the above inequality for an optimally large δ.

However, we just want some "reasonably large" δ that makes the inequality true. And so, we can just treat different factors separately; for instance, we can look for a $\delta > 0$ such that

$$e^{-(1-\delta)} \le 1 \qquad \text{and} \qquad \frac{\delta^4}{24} \le 0.0001.$$

These, together, would certainly imply the desired inequality in Formula 4.9.

The good news is that we can "solve" the two inequalities above. We find that we would like a $\delta > 0$ such that

$$\delta \leq 1 \qquad \text{and} \qquad \delta \leq \sqrt[4]{(24)(0.0001)} = 0.1\sqrt[4]{24} \leq 0.222.$$

Of course, requiring $\delta \leq 0.1\sqrt[4]{24}$ already implies that $\delta \leq 1$. Thus, choosing $\delta = 0.1\sqrt[4]{24}$ would be "reasonable".

4.3.1 Exercises

1. Let $f(x) = \sin\left(\dfrac{x}{2}\right)$.

   a. Suppose that $|x| \leq 0.1$. Find a "reasonable" upper-bound on the error in approximating $f(x)$ by $T_f^3(x;0)$.

   b. Suppose that $|x| \leq 0.1$. By using "reasonable" bounds, find the smallest n such that you can prove that the error in approximating $f(x)$ by $T_f^n(x;0)$ is less than or equal to 0.0001.

   c. Suppose that we want $T_f^3(x;0)$ to approximate $f(x)$ to within 0.0001. By using "reasonable" bounds, find the largest $\delta > 0$ such that this accuracy will be achieved provided that $|x| \leq \delta$.

2. Let $f(x) = e^{3x}$.

   a. Suppose that $|x| \leq 0.1$. Find a "reasonable" upper-bound on the error in approximating $f(x)$ by $T_f^3(x;0)$.

   b. Suppose that $|x| \leq 0.1$. By using "reasonable" bounds, find the smallest n such that you can prove that the error in approximating $f(x)$ by $T_f^n(x;0)$ is less than or equal to 0.0001.

   c. Suppose that we want $T_f^3(x;0)$ to approximate $f(x)$ to within 0.0001. By using "reasonable" bounds, find the largest $\delta > 0$ such that this accuracy will be achieved provided that $|x| \leq \delta$.

3. Let $f(x) = \sin(5x)$.

   a. Suppose that $|x - 1| \leq 0.1$. Find a "reasonable" upper-bound on the error in approximating $f(x)$ by $T_f^3(x;1)$.

b. Suppose that $|x-1| \leq 0.1$. By using "reasonable" bounds, find the smallest n such that you can prove that the error in approximating $f(x)$ by $T_f^n(x;1)$ is less than or equal to 0.0001.

c. Suppose that we want $T_f^3(x;1)$ to approximate $f(x)$ to within 0.0001. By using "reasonable" bounds, find the largest $\delta > 0$ such that this accuracy will be achieved provided that $|x-1| \leq \delta$.

4. Let $f(x) = e^{0.7x}$.

a. Suppose that $|x-1| \leq 0.1$. Find a "reasonable" upper-bound on the error in approximating $f(x)$ by $T_f^3(x;1)$.

b. Suppose that $|x-1| \leq 0.1$. By using "reasonable" bounds, find the smallest n such that you can prove that the error in approximating $f(x)$ by $T_f^n(x;1)$ is less than or equal to 0.0001.

c. Suppose that we want $T_f^3(x;1)$ to approximate $f(x)$ to within 0.0001. By using "reasonable" bounds, find the largest $\delta > 0$ such that this accuracy will be achieved provided that $|x-1| \leq \delta$.

5. Let $f(x) = \ln x$.

a. Suppose that $|x-1| \leq 0.1$. Find a "reasonable" upper-bound on the error in approximating $f(x)$ by $T_f^3(x;1)$.

b. Suppose that $|x-1| \leq 0.1$. By using "reasonable" bounds, find the smallest n such that you can prove that the error in approximating $f(x)$ by $T_f^n(x;1)$ is less than or equal to 0.0001.

c. Suppose that we want $T_f^3(x;1)$ to approximate $f(x)$ to within 0.0001. By using "reasonable" bounds, find the largest $\delta > 0$ such that this accuracy will be achieved provided that $|x-1| \leq \delta$.

6. Let $f(x) = \cos\left(\frac{x}{3}\right)$.

a. Suppose that $|x-1| \leq 0.1$. Find a "reasonable" upper-bound on the error in approximating $f(x)$ by $T_f^3(x;1)$.

b. Suppose that $|x-1| \leq 0.1$. By using "reasonable" bounds, find the smallest n such that you can prove that the error in approximating $f(x)$ by $T_f^n(x;1)$ is less than or equal to 0.0001.

c. Suppose that we want $T_f^3(x;1)$ to approximate $f(x)$ to within 0.0001. By using "reasonable" bounds, find the largest $\delta > 0$ such that this accuracy will be achieved provided that $|x-1| \leq \delta$.

7. Let $f(x) = x^5$.

   a. Suppose that $|x+1| \leq 0.1$. Find a "reasonable" upper-bound on the error in approximating $f(x)$ by $T_f^3(x; -1)$.

   b. Suppose that $|x+1| \leq 0.1$. By using "reasonable" bounds, find the smallest n such that you can prove that the error in approximating $f(x)$ by $T_f^n(x; -1)$ is less than or equal to 0.0001.

   c. Suppose that we want $T_f^3(x; -1)$ to approximate $f(x)$ to within 0.0001. By using "reasonable" bounds, find the largest $\delta > 0$ such that this accuracy will be achieved provided that $|x| \leq \delta$.

8. Let $f(x) = x^7$.

   a. Suppose that $|x+1| \leq 0.1$. Find a "reasonable" upper-bound on the error in approximating $f(x)$ by $T_f^3(x; -1)$.

   b. Suppose that $|x+1| \leq 0.1$. By using "reasonable" bounds, find the smallest n such that you can prove that the error in approximating $f(x)$ by $T_f^n(x; -1)$ is less than or equal to 0.0001.

   c. Suppose that we want $T_f^3(x; -1)$ to approximate $f(x)$ to within 0.0001. By using "reasonable" bounds, find the largest $\delta > 0$ such that this accuracy will be achieved provided that $|x| \leq \delta$.

9. Let $f(x) = x - \sin x$.

   a. What is $T_f^5(x; 0)$?

   b. Suppose that $|x| \leq 1$. Find a "reasonable" upper-bound on the error in approximating $f(x)$ by $T_f^5(x; 0)$.

10. Let $f(x) = 1 - \cos x$.

   a. What is $T_f^5(x; 0)$?

   b. Suppose that $|x| \leq 1$. Find a "reasonable" upper-bound on the error in approximating $f(x)$ by $T_f^5(x; 0)$.

Recall that the Extreme Value Theorem (see [2]) tells us that a continuous function defined on a closed interval is bounded. Combining this with the Taylor-Lagrange Theorem, it follows that, if $f^{(n+1)}$ is continuous on the closed interval $[a-\delta, a+\delta]$, then there exists a constant M such that $|R_f^n(x; a)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$.

Use this result in Exercises 11 through 16 to find an upper bound on $E_f^n(x;0) = |R_f^n(x;0)|$ where x is within $\pm\delta$ of zero; this amounts to finding the maximum of $|f^{(n+1)}(c)|$ for c in the interval $[-\delta, \delta]$.

11. $f(x) = e^x$, $\delta = 1$.

12. $f(x) = \cos x$, $\delta = 0.3$.

13. $f(x) = \sin x$, $\delta = 0.3$.

14. $f(x) = \ln(1+x)$, $\delta = 0.1$.

15. $f(x) = \cosh x$, $\delta = 1$.

16. $f(x) = \sinh x$, $\delta = 1$.

17. If f is an infinitely differentiable function, what is $E_f^n(a;a)$?

18. Explain why the Mean Value Theorem is a special case of the Taylor-Lagrange Theorem.

19. Prove that if $f^{(k+1)}$ is continuous at $x = a$, then $\lim\limits_{x\to a} \left| \dfrac{R_f^k(x;a)}{(x-a)^k} \right| = 0$.

20. a. Suppose that $k \geq 0$, and that $f^{(k+1)}(a)$ exists. Use the actual **definition** of the remainder (and the derivative), and iterate l'Hôpital's Rule (see [2]) to prove that

$$\lim_{x\to a} \frac{R_f^k(x;a)}{(x-a)^{k+1}} = \frac{1}{(k+1)!} f^{(k+1)}(a).$$

(Hint: To apply l'Hôpital's Rule, you will need to use that the existence of $f^{(k+1)}(a)$ implies that all lower-order derivatives of f exist for all x in an open interval around a.)

b. Prove a stronger (and harder) result than that in the previous exercise; prove that, if $f^{(k+1)}(a)$ exists, then $\lim\limits_{x\to a} \left| \dfrac{R_f^k(x;a)}{(x-a)^k} \right| = 0$.

21. In this exercise, you will reprove the Second Derivative Test (see [2]). Suppose that $f'(a) = 0$ and that $f''(a)$ exists.

a. Use the results of the previous exercise to show that $\lim_{x\to a} \dfrac{f(x)-f(a)}{(x-a)^2} = \dfrac{1}{2} f''(a)$.

b. Suppose that $f''(a) > 0$. Conclude that f attains a local minimum value at $x = a$.

c. Suppose that $f''(a) < 0$. Conclude that f attains a local maximum value at $x = a$.

22. Use the approach in the previous problem to prove the following Third Derivative Test. If $f^{(3)}(a)$ exists and is non-zero, $f'(a) = 0$, and $f''(a) = 0$, then f attains neither a local maximum nor a local minimum value at $x = a$.

23. Prove that, if k is even and $f'(a) = f''(a) = ... = f^{(k-1)}(a) = 0$, but $f^{(k)}(a) \neq 0$, then f attains a local minimum (respectively, maximum) value at $x = a$ if $f^{(k)}(a) > 0$ (respectively, $f^{(k)}(a) < 0$).

24. Prove that, given the same hypotheses as the previous problem except that k is odd, f attains neither a local maximum nor a local minimum value at $x = a$.

25. Let $f(x) = e^x$. For part (a), use the fact, proved in [2], that $e < 3$.

    a. Show that $0 < |R_f^k(1;0)| < \dfrac{3}{(k+1)!}$.

    b. Estimate the value of e to within 0.001 of the correct value.

26. In this problem, we will prove that e is irrational. We will do this by contradiction; so, assume that e is rational and is equal to p/q, where p and q are both positive integers. We wish to derive a contradiction from this assumption. Let $f(x) = e^x$.

    a. Use part (b) of the previous problem to argue that $q \geq 2$.

    b. Show that, since we're assuming that $e = p/q$, it follows that

    $$\frac{p}{q} \;=\; 1 + 1 + \frac{1}{2!} + ... + \frac{1}{q!} + R_f^q(1;0),$$

    where $0 < R_f^q(1;0) < \dfrac{3}{(q+1)!}$ by the previous exercise.

    c. Multiply both sides of the equation in part (b) by $q!$. Reach a contradiction by rearranging the resulting equation and arguing that one side is an integer and the other side is not.

In the next exercise, you'll prove a different version of Taylor's Theorem; you will prove ***Taylor's Theorem with Integral Remainder*****.**

27. Suppose that f is continuously differentiable. Then, the Fundamental Theorem of Calculus, Theorem 2.4.10, tells us that $\displaystyle\int_a^x f'(t)\,dt = f(x) - f(a)$. Also, recall the formula for

integration by parts, Theorem 1.1.19, in the context of definite integration :

$$\int_a^b h(y)g'(y)\,dy = h(b)g(b) - h(a)g(a) - \int_a^b h'(y)g'(y)\,dy.$$

a. Apply integration by parts to $\int_a^x f'(t)\,dt$ with $h(t) = f'(t)$ and $g(t) = t - x$ to obtain the formula

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt,$$

provided that f possesses a continuous second derivative.

b. Apply integration by parts a second time to obtain the formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}(x-a)^2 + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,dt,$$

provided that f possesses a continuous third derivative.

c. Argue inductively that

$$f(x) = f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt,$$

provided that f possesses a continuous $(n+1)$-th derivative. Thus, if f possesses a continuous $(n+1)$-th derivative,

$$R_f^n(x;a) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt.$$

This is *Taylor's Theorem with Integral Remainder.*

28. Suppose that $f^{(n+1)}$ exists and is continuous. Fix an a and an x.

a. Explain why there exists $M \geq 0$ such that $|f^{(n+1)}(t)| \leq M$ for all t in the closed interval between a and x.

b. Use the M from part (a) and Taylor's Theorem with Integral Remainder to show that $|R_f^n(x;a)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$. Note that we obtained this same bound prior to Exercise 11, by using Lagrange's form of the remainder.

4.4 Functions as Power Series

In the previous two sections, we first looked informally at approximating functions by their Taylor polynomials, and then we got formal and used the Taylor-Lagrange Theorem to put bounds on the error in the Taylor approximations. Along the way, as a convenient notational device, we introduced Taylor **series**; these are formal algebraic objects, which are like polynomials, centered at a, except they don't (necessarily) stop. We did **not** say, in those earlier sections, that Taylor series define **functions**.

In this section, we will discuss when a Taylor series $T_f^\infty(x;a)$ defines a function which is equal to the original function $f(x)$, for, at least, some values of x close to the center a. It is common to say that the original function can be *represented* by its Taylor series. We will then give examples/theorems involving familiar functions.

Note that this section deals with when the Taylor series $T_f^\infty(x;a)$ of a given function f defines a function in its own right, and when the function $T_f^\infty(x;a)$ equals the original function f. This section can be viewed as an introduction to using power series as functions. Results on functions that are initially **defined** as power series, or manipulations of power series functions, are dealt with in Section 4.5 and Section 4.6.

Suppose that $f(x)$ is infinitely differentiable at a point a. Then, for all n, we have defined the n-th order Taylor polynomial

$$T_f^n(x;a) \;=\; \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(x-a)^k.$$

Of course, the Taylor polynomials also give us Taylor polynomial *functions* and, in the previous sections, we looked at the error $E_f^n(x;a)$ in approximating $f(x)$ by using the functions $T_f^n(x;a)$.

We also defined the formal algebraic Taylor series:

$$T_f^\infty(x;a) \;=\; \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}\,(x-a)^k \;=$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots\,,$$

which was really just a way of indicating that we weren't thinking of looking at just one Taylor polynomial, but rather were interested in looking at what happens as we take arbitrarily high-order Taylor polynomials.

Our question now is: **can we somehow think of the infinite sum in the Taylor series as defining a function, and can we say when such a Taylor series function is equal to the original function** $f(x)$?

Since this is a Calculus book, limits may/should come to mind; we can define the infinite Taylor sum to be the limit, as n approaches infinity, of the n-th order partial sums, i.e., of the n-th order Taylor polynomials. (Note that, here, the limit uses only integer values for n; thus, it is the limit of a *sequence*, as defined in [2], and as we shall see again in Definition 4.5.1.)

Of course, we **can** make this definition; the question is: does such a definition give us anything useful? As we shall see, the answer is "yes", and so we make a definition.

Definition 4.4.1. *Suppose that f is infinitely differentiable at a.*

Then, the **value, at a point** x_0**, of the Taylor series of** f**, centered at** a*, is*

$$T_f^{\infty}(x_0; a) \;=\; \lim_{n\to\infty} T_f^n(x_0; a) \;=$$

$$\lim_{n\to\infty}\left[f(a) + f'(a)(x_0 - a) + \frac{f''(a)}{2!}(x_0-a)^2 + \frac{f'''(a)}{3!}(x_0-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x_0-a)^n\right],$$

provided that this limit exists; in this case, we say that the Taylor series $T_f^{\infty}(x;a)$ of f, centered at a, **converges at** x_0.

If the limit fails to exist, we say that $T_f^{\infty}(x;a)$ **diverges at** x_0.

The **Taylor series function of** f**, centered at** a*, is the function whose domain is the set of x_0 at which $T_f^{\infty}(x;a)$ converges, and its value at any x_0 in its domain is $T_f^{\infty}(x_0;a)$, as defined above. The codomain is taken to be all real numbers. The notation for this function is the same as that of the algebraic Taylor series: $T_f(x;a) = T_f^{\infty}(x;a)$.*

Usually, the Taylor series **function** is referred to simply as the Taylor series, and the context makes it clear whether we mean as a formal algebraic object or as a function; this is why there is not separate notation for the algebraic Taylor series and the Taylor series function.

Since we expect the Taylor polynomials of f to approximate f well, near the center, and for the approximation to get better as we take higher-order Taylor polynomials, what we hope/expect is that, if a Taylor series converges at some x_0, then what it converges to is $f(x_0)$, i.e., we expect that $f(x_0) = T_f^{\infty}(x_0; a)$. In other words, we expect, as functions of x, that $f(x) = T_f^{\infty}(x; a)$, at least for x values near a. Surprisingly, this is not **always** true (see Example 4.4.11).

However, something that's not surprising is the following theorem, which tells us that it's all a question of whether the remainder approaches 0 or not.

Theorem 4.4.2. *Suppose that f is infinitely differentiable at a. Then, for a given x, $\lim_{n\to\infty} E_f^n(x; a) = 0$ if and only if $T_f^{\infty}(x; a)$ converges to $f(x)$, i.e., if and only if*

$$f(x) = T_f^{\infty}(x; a).$$

Proof. As $E_f^n(x; a) = \left|R_f^n(x; a)\right|$, the equality $\lim_{n\to\infty} E_f^n(x; a) = 0$ is equivalent to $\lim_{n\to\infty} R_f^n(x; a) = 0$, which means precisely that

$$\lim_{n\to\infty} \left[f(x) - T_f^n(x; a)\right] = 0,$$

i.e.,

$$f(x) \;=\; \lim_{n\to\infty} T_f^n(x; a) \;=\; T_f^{\infty}(x; a).$$

□

Remark 4.4.3. We should make three comments.

First, we usually omit the explicit assumption in Theorem 4.4.2 that f is infinitely differentiable at a; this omission causes no confusion since, when we give a statement involving $T_f^{\infty}(x; a)$, it is clear that we are, in fact, assuming that the formal algebraic series $T_f^{\infty}(x; a)$ exists, i.e., that f is infinitely differentiable at a.

Second, the way in which we usually show that $E_f^n(x;a) \to 0$ is to find some upper bound $U_f^n(x;a)$ on $E_f^n(x;a)$, and then show that $U_f^n(x;a) \to 0$ as $n \to \infty$. This implies that $E_f^n(x;a) \to 0$, since

$$0 \;\leq\; E_f^n(x;a) \;\leq\; U_f^n(x;a),$$

and $E_f^n(x;a)$ gets pinched to 0 as $U_f^n(x;a) \to 0$.

Finally, it is, of course, always true that $T_f^\infty(x;a)$ converges to $f(x)$ at the single point $x = a$, for then all of the $(x-a)$'s are 0. What's interesting is not when $T_f^\infty(x;a)$ converges to $f(x)$ at just $x = a$, but rather when $T_f^\infty(x;a)$ converges to $f(x)$ **for all** x **near** a.

Before we look at some examples, we will state an easy lemma, which is useful, in various problems, in showing that $\lim_{n\to\infty} E_f^n(x;a) = 0$.

Lemma 4.4.4. *Suppose that x is a fixed real number.*

1. *Suppose that $|x| < 1$. Then, $\lim_{n\to\infty} |x|^n = 0$.*
2. *Regardless of the size of $|x|$,*

$$\lim_{n\to\infty} \frac{|x|^n}{n!} \;=\; 0.$$

Proof. Item 1 is easy. If $|x| = 0$, there is nothing to show. So, suppose $0 < |x| < 1$. Let $\epsilon > 0$. If $\epsilon \geq 1$, then, for all n, $|x|^n < \epsilon$. If $0 < \epsilon < 1$, then $\ln|x|$ and $\ln\epsilon$ are both negative, and selecting $n > (\ln\epsilon)/(\ln|x|)$ implie that $|x|^n < \epsilon$. This proves the limit statement in Item 1.

To show Item 2, suppose that n_0 is the first natural number greater than $|x|$. Then, for all $n > n_0$,

$$0 \;\leq\; \frac{|x|^n}{n!} \;=\; \frac{|x|^{n_0-1}}{(n_0-1)!}\cdot\frac{|x|}{n_0}\cdot\frac{|x|}{n_0+1}\cdot\frac{|x|}{n_0+2}\cdots\frac{|x|}{n} \;<\; \frac{|x|^{n_0-1}}{(n_0-1)!}\cdot\left(\frac{|x|}{n_0}\right)^{n-n_0+1},$$

where $|x|/n_0 < 1$. By Item 1, this last factor on the right approaches 0 as n approaches infinity and, hence, so does $|x|^n/(n!)$, since it gets pinched between 0 and 0. $\square$

Now we are ready to look at our first example of using Theorem 4.4.2. All of our examples in this section are important enough to state as theorems.

The following result is actually circular; if you look back in [2], you'll see that we defined $e^x = \exp(x)$ using what we now call its Maclaurin series. Nonetheless, it is a good warm-up exercise for us.

Theorem 4.4.5. *For all x, the function e^x is equal to its Maclaurin series function, i.e., there is an equality of functions*

$$e^x \;=\; T_f^{\infty}(x) \;=\; \sum_{k=0}^{\infty} \frac{x^k}{k!} \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

Proof. Let $f(x) = e^x$. We will use the Lagrange form of the remainder, together with Theorem 4.4.2.

Fix a particular value x_0 of x. We have that

$$0 \;\leq\; E_f^n(x_0) \;=\; \left|\frac{f^{(n+1)}(c)\, x_0^{n+1}}{(n+1)!}\right| \;=\; \left|f^{(n+1)}(c)\right| \cdot \frac{|x_0|^{n+1}}{(n+1)!}$$

for some c between 0 and x_0. As n goes to ∞, $n+1$ goes to ∞, and so Lemma 4.4.4 tells us that the factor on the right, above, $|x_0|^{n+1}/(n+1)!$, approaches 0 as n approaches ∞. We would like to know that the entire term

$$\left|f^{(n+1)}(c)\right| \cdot \frac{|x_0|^{n+1}}{(n+1)!}$$

approaches 0, for then the non-negative quantity $E_f^n(x_0)$ would be pinched to 0. The question is: what do we do about the $\left|f^{(n+1)}(c)\right|$ factor?

Since $f(x) = e^x$, $f^{(n+1)}(x) = e^x$, and so $f^{(n+1)}(c) = e^c$. As e^x is an increasing function, if $x_0 \leq 0$, so that $x_0 \leq c \leq 0$, then $e^c \leq e^0 = 1$, and if $x_0 \geq 0$, so that $0 \leq c \leq x_0$, then $e^c \leq e^{x_0}$.

In either case, e^c is less than or equal to a fixed value, 1 or e^{x_0}, and so, since $|x_0|^{n+1}/(n+1)! \to 0$, the non-negative quantity

$$\left|f^{(n+1)}(c)\right| \cdot \frac{|x_0|^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$, and we are finished. □

Remark 4.4.6. Understand the difference between the result of Theorem 4.4.5 and the result of Theorem 4.2.11: if $f(x) = e^x$, then, in Theorem 4.2.11, we stated that the formal Maclaurin series was

$$T_f(x) \;=\; \sum_{k=0}^{\infty} \frac{x^k}{k!} \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

There was no claim, at that time, that the Maclaurin series defined a function, or that, even if it did define a function, that that function had to be e^x; these two conclusions are what we proved in Theorem 4.4.5 .

We also discussed, back in [2], that sine and cosine are equal to their Maclaurin series (though we didn't phrase it that way). It is important here that $\sin x$ and $\cos x$ mean that x is interpreted as being in **radians**. In a way, these series help explain why radians are "more natural", mathematically, than degrees; these series are not correct if the x to which you apply sine and cosine is measured in degrees.

Theorem 4.4.7. *For all x, the functions $\sin x$ and $\cos x$ are equal to their Maclaurin series, i.e., there are equalities of functions*

$$\sin x \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

and

$$\cos x \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Proof. We can actually prove both of these at once. Suppose that $f(x)$ is equal to $\sin x$ or $\cos x$. Then, for all n, $f^{(n+1)}(x)$ is equal to $\pm\sin x$ or $\pm\cos x$. As $\sin x$ and $\cos x$ are always between -1 and 1, in any case, for all c, $|f^{(n+1)}(c)| \leq 1$.

Therefore, for each fixed value of x,

$$0 \;\leq\; E_f^n(x) \;=\; |f^{(n+1)}(c)| \cdot \left|\frac{x^{n+1}}{(n+1)!}\right| \;\leq\; 1 \cdot \frac{|x|^{n+1}}{(n+1)!},$$

and, by Lemma 4.4.4, the last term on the right above approaches 0 as n approaches ∞. $\square$

As we pointed out when we gave the Maclaurin series for $\sin x$ and $\cos x$ back in Theorem 4.2.11, it is helpful to remember that $\sin x$ is an odd function, and its Maclaurin series therefore contains only odd-powered terms, and that $\cos x$ is even, and so its Maclaurin series contains only even-powered terms, including a constant x^0 term.

Remark 4.4.8. You may be wondering why we use the Maclaurin series for e^x, $\sin x$, and $\cos x$ in the theorems above, instead of using Taylor series centered somewhere other than at 0. Was it really important for us to center our Taylor series at 0? The answer is: it wasn't **really** important, but we can only remember/memorize so many things.

It was unimportant for us to use Taylor series centered at 0 because, in fact, if $f(x)$ equals e^x, $\sin x$, or $\cos x$, and a is any center whatsoever, then the proofs that we gave when $a = 0$ would also show that $f(x)$ is equal, as a function, to its Taylor series function $T_f^\infty(x; a)$.

It's true that our Taylor error would now have a factor of $|x - a|^{n+1}/(n + 1)!$ in place of $|x|^{n+1}/(n + 1)!$, and that the c in the $f^{(n+1)}(c)$ could now be between a and x, instead of between 0 and x, but these things don't matter; it's still true that, for a given x and a, $\lim_{n\to\infty} |x - a|^{n+1}/(n + 1)! = 0$, and that we have fixed upper bounds on $|f^{(n+1)}(c)|$.

So, for instance, we conclude from our discussion and Example 4.2.8 that there is an equality of functions

$$\sin x \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k} \;=$$

$$1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \frac{1}{8!}\left(x - \frac{\pi}{2}\right)^8 - \frac{1}{10!}\left(x - \frac{\pi}{2}\right)^{10} + \cdots .$$

Should you memorize this??? Absolutely not. Memorize the Maclaurin series; they are simpler, and they're what get used in applications 99% of the time. If you need other Taylor series, you calculate them on-the-fly.

Example 4.4.9. In fact, by manipulating known Maclaurin series, you can find new power series representations of functions. For instance, we know that, for all x,

$$\cos x \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots . \tag{4.10}$$

But, if x is a number, then $\frac{\pi}{2} - x$ is just some number, and we can replace every x in Formula 4.10 with $\frac{\pi}{2} - x$. Thus, we conclude that, for all x,

$$\cos\left(\frac{\pi}{2} - x\right) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{\pi}{2} - x\right)^{2k}}{(2k)!} \;=\; 1 - \frac{\left(\frac{\pi}{2} - x\right)^2}{2!} + \frac{\left(\frac{\pi}{2} - x\right)^4}{4!} - \frac{\left(\frac{\pi}{2} - x\right)^6}{6!} + \cdots .$$

Now, as there is an identity $\sin x = \cos\left(\frac{\pi}{2} - x\right)$, and as raising $\left(\frac{\pi}{2} - x\right)$ to an even power is the same as raising $\left(x - \frac{\pi}{2}\right)$ to that even power, we conclude that

$$\sin x \;=\; 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \cdots , \tag{4.11}$$

which, of course, is exactly what we concluded in the previous remark. The point here is that the previous remark used our earlier calculation of the Taylor series of $\sin x$, centered at $\pi/2$, and considered bounds on the error; in this example, we simply messed with the Maclaurin series of $\cos x$ to obtain the result very quickly.

There is, however, a subtle point here. While our work in this example really does prove the equality in Formula 4.11, it requires a new result, which we won't get to until Corollary 4.6.8, to know that the equality in Formula 4.11 actually tells us that the given power series is, in fact, the **Taylor series** for $\sin x$, centered at $\pi/2$. Right now, we don't know that, if a function f equals a powers series centered at a (on some open interval around a), then that power series **has to be** the Taylor series of f centered at a. It's true; we just haven't proved it yet.

Theorem 4.4.10. (**Geometric Series Theorem**) *Suppose that a is a constant. Then, for all x such that $|x| < 1$, the function $a/(1-x)$ is equal to its Maclaurin series, i.e., there is an equality of functions*

$$\frac{a}{1-x} \;=\; \sum_{k=0}^{\infty} ax^k \;=\; a \;+\; ax \;+\; ax^2 \;+\; ax^3 \;+\; ax^4 \;+\; \cdots .$$

In addition, if $a \neq 0$, then the infinite series $\sum_{k=0}^{\infty} ax^k$ diverges if $|x| \geq 1$.

This series is called a (or sometimes **the***)* **geometric series***.*

Proof. Note that $a/(1-x)$ exists as long as $x \neq 1$, but that we are saying that, if $a \neq 0$, the infinite sum does **not** exist if $x \geq 1$ or $x \leq -1$. You may have guessed, correctly, that this means that our proof does not look like our proofs that e^x, $\sin x$, and $\cos x$ equal their Maclaurin series.

Let $f(x) = a/(1-x)$. Then, it is easy to see (or prove by induction) that $f^{(k)}(x) = k!a(1-x)^{-(k+1)}$. Hence, $f^{(k)}(0) = k!a$, and so the n-th order Maclaurin polynomial of f is

$$T_f^n(x) \;=\; a \,+\, ax \,+\, ax^2 \,+\, ax^3 \,+\, ax^4 \,+\, \cdots \,+\, ax^n \;=\; \sum_{k=0}^{n} ax^k.$$

Recall Corollary 2.1.11, Item c. Replacing b with x in that formula, we have that, if $x \neq 1$, then

$$T_f^n(x) \;=\; \sum_{k=0}^{n} ax^k \;=\; a \cdot \frac{x^{n+1}-1}{x-1} \;=\; \frac{a - ax^{n+1}}{1-x},$$

and so

$$E_f^n(x) \;=\; \left|f(x) - T_f^n(x)\right| \;=\; \left|\frac{a}{1-x} - \frac{a-ax^{n+1}}{1-x}\right| \;=\; |a| \cdot \frac{|x|^{n+1}}{|1-x|}.$$

When $|x| < 1$, this last quantity approaches 0, as n approaches ∞, by Lemma 4.4.4.

We leave the divergence claim as an exercise. $\square$

As we mentioned earlier, it is possible for a Taylor series of f to converge and, yet, not converge to the value of f.

Example 4.4.11. Consider the function given by

$$f(x) \;=\; \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly, f is differentiable for $x \neq 0$; we claim that f is also differentiable **at** $x = 0$. To see this, consider

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} \;=\; \lim_{h \to 0} \frac{e^{-1/h^2} - 0}{h}.$$

We consider each of the one-sided limits, make the substitution $u = 1/h$, and use l'Hôpital's Rule (see[2]). For instance,

$$\lim_{h\to 0^+} \frac{e^{-1/h^2}}{h} = \lim_{u\to\infty} \frac{e^{-u^2}}{1/u} = \lim_{u\to\infty} \frac{u}{e^{u^2}} = \lim_{u\to\infty} \frac{1}{2ue^{u^2}} = 0.$$

The limit as $h \to 0^-$ is just as easily shown to be 0.

More generally, it is not difficult to show that, for $x \neq 0$, the k-th derivative of f, $f^{(k)}(x)$, is of the form $e^{-1/x^2} \cdot r(x)$, where $r(x)$ is a rational function, i.e., a quotient of polynomials. Substitution and l'Hôpital's Rule (see[2]) can then be used to show that $f(x)$ is infinitely differentiable everywhere, **including** at $x = 0$, and that f and all of its derivatives are 0 at $x = 0$. Hence, the Maclaurin series for f has every term equal to 0, and so, converges to 0 at all x values.

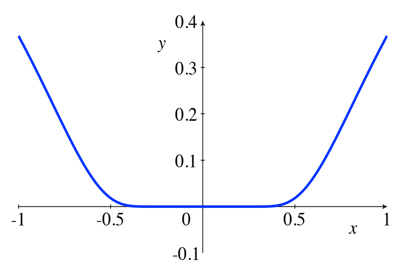


Figure 4.2: The graph of f is very flat near $x = 0$.

Therefore, the function $T_f^{\infty}(x)$ is the zero function, the function that is always 0, while $f(x)$ certainly is not.

From a graphical point-of-view, what we see is that the graph of f is extremely flat near $x = 0$, in other words, extremely close to being $y = 0$.

You can also tinker with the function f above to produce functions with other strange properties. For instance, consider

$$g(x) = \begin{cases} e^{-1/x^2}, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

Then, our discussion about $f(x)$ allows us to quickly see that $g(x)$ is also infinitely differentiable everywhere, including at $x = 0$, is zero for $x \leq 0$, and suddenly is non-zero for $x > 0$. As we shall see in Theorem 4.5.26, this sort of "weird" activity cannot happen for a function which equals a power series.

In the next theorem, we state a result about $\ln(1+x)$ equaling its Maclaurin series, for $-1 < x \leq 1$. In fact, we shall give, in this section, the proof only for $0 \leq x \leq 1$; the rest of the proof will be much easier after we have some results from the next section.

Theorem 4.4.12. *For $-1 < x \leq 1$, the function $\ln(1+x)$ is equal to its Maclaurin series, i.e., there is an equality of functions*

$$\ln(1+x) \;=\; \sum_{k=0}^{\infty}(-1)^k\,\frac{x^{k+1}}{k+1} \;=\; x \;-\; \frac{x^2}{2} \;+\; \frac{x^3}{3} \;-\; \frac{x^4}{4} \;+\; \frac{x^5}{5} \;-\; \frac{x^6}{6} \;+\; \cdots.$$

Proof. We give the proof for $0 \leq x \leq 1$; the remainder of the proof is in Example 4.6.15.

If $x = 0$, then we know that the convergence and equality hold. So, suppose that $0 < x \leq 1$.

If you haven't done so already, you should do the exercise from Section 4.2 to show that the Maclaurin series for $f(x) = \ln(1+x)$ is the given series. The key steps are, for $k \geq 1$, to recall that $k! = k \cdot ((k-1)!)$ and to check that

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!(1+x)^{-k}.$$

Therefore, for $n \geq 1$, there exists a c, where $0 < c < x \leq 1$, such that

$$E_f^n(x) \;=\; \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \;=\; \frac{n!}{(n+1)!}\cdot\left(\frac{x}{1+c}\right)^{n+1} \;=\; \frac{1}{n+1}\cdot\left(\frac{x}{1+c}\right)^{n+1} \;<\; \frac{x^{n+1}}{n+1},$$

which, since $0 < x \leq 1$, approaches 0 as n approaches ∞. Thus, once again, $E_f^n(x)$ gets pinched and must approach 0. $\square$

Remark 4.4.13. We have one cool remark about Theorem 4.4.12 and one warning.

Let's start with the warning. It is easy to get used to seeing factorials in Taylor and Maclaurin series and to mistakenly put factorials on the denominators of the Maclaurin series for $\ln(1+x)$. Don't do this! Make a special mental note: no factorials appear in the Maclaurin series for $\ln(1+x)$, at least, not after it's simplified.

The cool remark is that, when $x = 1$, we obtain that

$$\ln 2 \;=\; 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

The infinite sum/difference on the right is known as the *alternating harmonic series*, in contrast with the *harmonic series*, which has all plus signs. As we shall see in Proposition 5.2.16, the harmonic series diverges to ∞.

A weird thing happens when a sum/series converges, but diverges when you add up the absolute values of all of the summands: it is possible to rearrange the terms (including the negative signs) that you're adding and change the sum. By taking lots of positive terms earlier, you can make the sum diverge to ∞. By taking lots of negative terms earlier, you can make the sum diverge to $-\infty$. And, by rearranging in the right ways, you can make the infinite sum converge to any real number that you want. Thus, the order in which you add the terms changes the value of the summation, something which is not the case for finite sums.

We shall discuss such *conditionally convergent* series more carefully in Section 5.4.

The fact that the next function, $(1+x)^p$, equals its Maclaurin series was discovered by Sir Isaac Newton. The Maclaurin series which appears is called the *binomial series*, and the fact that it equals its Maclaurin series, for the given values of x, is called the *Binomial Theorem*.

A direct proof of the Binomial Theorem, using Lagrange's form of the remainder, is problematic, and so we outline and have you produce an alternative proof in Exercise 43 in Section 4.6. We include the statement in this section, since it is one of the fundamental Maclaurin series that is worth memorizing.

Theorem 4.4.14. (**Binomial Theorem**) *Let p be a real number. Then, for $-1 < x < 1$, the function $(1+x)^p$ is equal to its Maclaurin series, i.e., there is an equality of functions*

$$(1+x)^p \;=\; 1 \;+\; px \;+\; \frac{p(p-1)}{2!}\,x^2 \;+\; \frac{p(p-1)(p-2)}{3!}\,x^3 \;+$$

$$\frac{p(p-1)(p-2)(p-3)}{4!}\,x^4 \;+\; \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\,x^5 \;+\; \cdots .$$

If we let $\binom{p}{0} = 1$ and, for $k \geq 1$, let $\binom{p}{k} = p(p-1)(p-2)\cdots(p-k+1)/(k!)$, then the Binomial Theorem tells us that, for $|x| < 1$,

$$(1+x)^p \;=\; \sum_{k=0}^{\infty} \binom{p}{k} x^k.$$

Remark 4.4.15. The expression $\binom{p}{k}$ is read as "p choose k". When p is a positive integer $\geq k$, p choose k is used often in probability and combinatorics; it represents the number of possible (unordered) ways of choosing k objects from among p objects.

When p is a positive integer, all of the summands in the Binomial Series for $(1+x)^p$ will be zero after the x^p term for, after that, the factor $(p-p)$ will appear in the numerator of the coefficient.

For instance, if $p = 3$, Theorem 4.4.14 yields the familiar binomial expansion

$$(1+x)^3 \;=\; 1 \;+\; 3x \;+\; \frac{3(3-1)}{2}\,x^2 \;+\; \frac{3(3-1)(3-2)}{6}\,x^3 \;+$$

$$\frac{3(3-1)(3-2)(3-3)}{24}\,x^4 \;+\; \frac{3(3-1)(3-2)(3-3)(3-4)}{120}\,x^5 \;+\; \cdots \;=$$

$$1 \;+\; 3x \;+\; 3x^2 \;+\; x^3.$$

The value of the Binomial Theorem is that it holds when p is negative, a rational number, or even an irrational number.

Example 4.4.16. Use the Binomial Theorem to approximate $\sqrt{3.8}$.

Solution:

The instruction "to approximate" is pretty vague. We'll give one easy approximation.

To use the Binomial Theorem to get a "good" approximation, we need to apply it to something of the form $(1+x)^{1/2}$, where x is "close to" the center of the Binomial Series, i.e., where x is close to 0. But $\sqrt{3.8} = (1+2.8)^{1/2}$, and 2.8 doesn't seem very close to 0. So, what do we do?

Well...we think harder. We think:

$$\sqrt{3.8} \;=\; \sqrt{4\cdot\frac{3.8}{4}} \;=\; \sqrt{4}\cdot\sqrt{\frac{4-0.2}{4}} \;=\; 2\cdot(1-0.05)^{1/2}.$$

Now, we can apply the Binomial Theorem with $x = -0.05$, which is substantially closer to 0 than 2.8 was.

What is the easiest non-trivial approximation that we can obtain for $(1-0.05)^{1/2}$ from the Binomial Series? The one we get from the linear approximation, that

$$(1+x)^p \;\approx\; 1+px,$$

when x is close to 0. Thus, $(1-0.05)^{1/2} \approx 1 - 0.025 = 0.975$, and so

$$\sqrt{3.8} \;=\; 2\cdot(1-0.05)^{1/2} \;\approx\; 2\cdot 0.975 \;=\; 1.950.$$

As you can check on your calculator, $\sqrt{3.8}$ " = " 1.949358868961793. So, our quick approximation is pretty good.

4.4.1 Exercises

1. Recall that

$$\ln 2 \;=\; 1 \;-\; \frac{1}{2} \;+\; \frac{1}{3} \;-\; \frac{1}{4} \;+\; \frac{1}{5} \;-\; \frac{1}{6} \;+\; \frac{1}{7} \;-\; \frac{1}{8} \;+\; \frac{1}{9} \;-\; \frac{1}{10} \;+\; \cdots .$$

   Approximate $\ln 2$ by using the partial sum containing the first 4 terms in the infinite series. Compare this with the calculator value of 0.693147180560 for $\ln 2$. Would you say that your approximation is "good"? Does the approximation get better if you use the first 10 terms?

2. a. Using that

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ,$$

   we see that $6\left(e^x - 1 - x - \dfrac{x^2}{2}\right)$ is best approximated by what power of x when x is close to 0?

   b. Use your approximation from part (a) to estimate the value of

$$v = 6\left(e^{0.1} - 1 - 0.1 - \frac{(0.1)^2}{2}\right).$$

   Compare your estimate with the value of v from your calculator.

3. a. Using that

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,$$

   we see that $x - \sin x$ is best approximated by what function of the form cx^n when x is close to 0?

   b. Use your approximation from part (a) to estimate the value of $v = 0.1 - \sin(0.1)$. Compare your estimate with the value of v from your calculator. (Make certain that your calculator is set to use radians in trig functions.)

4. a. Using that

$$\cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ,$$

   we see that $2(1 - \cos x)$ is best approximated by what power of x when x is close to 0?

b. Use your approximation from part (a) to estimate the value of $v = 2(1 - \cos(0.01))$. Compare your estimate with the value of v from your calculator. (Make certain that your calculator is set to use radians in trig functions.)

5. a. Using that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots .$$

we see that $\dfrac{-x+\ln(1+x)}{x^2}$ is best approximated by what constant when x is close to 0?

b. Without using l'Hôpital's Rule, calculate

$$\lim_{x\to 0} \frac{-x+\ln(1+x)}{x^2}.$$

6. a. We know that, for $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

So, when x is close to zero, $1/(1-x)$ is best approximated by what quadratic polynomial?

b. Use your approximation in part (a) to estimate $\dfrac{1}{1.01}$. Compare this with the calculator value of $1/1.01$.

7. The Binomial Theorem, Theorem 4.4.14, with $p = \dfrac{1}{3}$, tells us that

$$(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^2 + \cdots .$$

Use the first three terms of this series to estimate $(1.09)^{1/3}$. Compare this with the calculator value of $(1.09)^{1/3}$.

8. Use the first two terms of the binomial series to approximate $\sqrt[5]{31}$, and compare this with the value from your calculator.

9. Explain how the Binomial Theorem, Theorem 4.4.14, yields the Geometric Series Theorem, Theorem 4.4.10, in the case where $a = 1$.

10. a. Find the Taylor series for e^x, centered at 5. Either use summation notation, or write out the first 5 non-zero terms, followed by dots.

    b. Determine the x values for which your Taylor series in part (a) equals the function e^x

11. a. Find the Taylor series for $\cos x$, centered at $\pi/4$. Either use summation notation, or write out the first 5 non-zero terms, followed by dots.

    b. Determine the x values for which your Taylor series in part (a) equals the function $\cos x$

12. a. Find the Taylor series for $\sin x$, centered at $\pi/4$. Either use summation notation, or write out the first 5 non-zero terms, followed by dots.

    b. Determine the x values for which your Taylor series in part (a) equals the function $\sin x$.

13. a. Find the Taylor series for $\ln(1 + x)$, centered at 2. Either use summation notation, or write out the first 5 non-zero terms, followed by dots. Hint: You may wish to use the formula for the k-th derivative from the proof of Theorem 4.4.12.

    b. Can you determine any x values, other than 2, for which your Taylor series in part (a) equals the function $\ln(1 + x)$?

14. a. Find the Taylor series for $1/(1 - x)$, centered at 2. Either use summation notation, or write out the first 5 non-zero terms, followed by dots.

    b. Can you determine any x values, other than 2, for which your Taylor series in part (a) equals the function $1/(1 - x)$?

15. Let $f(x) = xe^x$.

    a. Show that the k-th derivative of f, $f^{(k)}(x)$, is equal to $(x + k)e^x$.

    b. Find the Maclaurin series for $f(x)$, using the definition of Maclaurin series.

    c. Why is your answer to part (b) unsurprising?

    d. Show that $f(x)$ is equal to its Maclaurin series for all x.

16. Suppose that $f(x)$ is infinitely differentiable at 0. Then, $g(x) = xf(x)$ is also infinitely differentiable at 0.

a. Show that, for $k \geq 1$, $g^{(k)}(x) = xf^{(k)}(x) + kf^{(k-1)}(x)$.

b. Show that, if the Maclaurin series for $f(x)$ is $\sum_{i=0}^{\infty} c_i x^i$, then the Maclaurin series for $g(x)$ is $\sum_{i=0}^{\infty} c_i x^{i+1}$.

c. Suppose that $f(x)$ equals its Maclaurin series for all x in some set I. Show that $g(x)$ equals its Maclaurin series for all x in I. (Hint: Do **not** use the Lagrange form of the remainder; use the definition of the Taylor remainder and the properties of limits.)

4.5 Power Series as Functions I: Definitions & the Ratio Test

In the previous section, we began with a function $f(x)$ and discussed when it was equal to its Taylor or Maclaurin series or, as people frequently say, when a given function can be *represented* by its Taylor series. We saw that $f(x) = T_f^\infty(x; a)$ if and only if the error $E_f^n(x; a)$ approaches 0 as $n \to \infty$.

Showing that the error approaches 0 actually implies two essentially separate things at once; it implies that $T_f^\infty(x, a)$ converges to **something**, and that the something that it converges to is precisely $f(x)$.

But, what if you start with just a power series $\sum_{k=0}^{\infty} c_k(x-a)^k$, instead of some pre-defined function $f(x)$? Can we say when the power series, as a formal algebraic object, can be used to **define** a function? And, even if we can use a power series to define a function, is there any point to doing so???

The answer to both of these questions is: YES. Power series define functions wherever they converge and, to a large extent, as far as algebraic operations, differentiation, and integration are concerned, functions which are defined by power series can be treated just like polynomials, polynomials which never end.

In this section, we will begin our general investigation of using power series to define functions; though, actually, we introduced you to this idea in [2], when we defined the exponential function $\exp(x)$ using limits of polynomials.

We shall give fundamental definitions and results on convergence, and the domain of a power series function. In particular, we will state and use the Ratio Test to find the *radius of convergence*, a fundamental notion related to the domains of power series functions. We shall also define and discuss *real analytic functions*; these are functions which are locally equal to power series, but the center of the power series is allowed to move.

We will not prove the Ratio Test, or even state other convergence tests, until Chapter 5. However, before going into such an extensive discussion of convergence, we will, in the next section, Section 4.6, look at manipulations of convergent power series, in order to show **why** it's so nice to have functions defined, or represented by, power series.

We discussed partial sums and convergence of Taylor series earlier (recall that the n-th order partial sum of a Taylor series is just the n-th order Taylor polynomial). These are special cases of the general notions of partial sums, infinite series, and convergence of series, which we will look at in detail in Chapter 5. However, we need to briefly define and discuss these terms here, so that we can look at them in the specific case of power series.

We defined sequences and their convergence back in [2], and have referred to them a number of times throughout this textbook; they will now be so crucial to us that we want to give the definition again.

Definition 4.5.1. *Suppose that m is an integer, i.e., is in the set $\mathbb{Z}$. Denote by $\mathbb{Z}_{\geq m}$ the set of integers which are greater than or equal to m.*

A function $b : \mathbb{Z}_{\geq m} \to \mathbb{R}$ is called a **sequence (of real numbers)***. In place of $b(n)$, it is standard to write b_n.*

We say that **the sequence b_n converges to (a real number)** *L, and write $\lim_{n\to\infty} b_n = L$ if and only if, for all $\epsilon > 0$, there exists an integer $N \geq m$ such that, for all integers $n \geq N$, $|b_n - L| < \epsilon$.*

If a sequence does not converge to some L, then we say that the sequence **diverges***.*

When discussing the limit of a sequence, the initial value, m above, of the index of the sequence, n above, is frequently omitted, since all you typically care about is what happens when n is big.

Example 4.5.2. Consider the sequence $b_n = 2n + 1$, for $n \geq 0$. This is the sequence of odd natural numbers:

$$b_0 = 1, \quad b_1 = 3, \quad b_2 = 5, \quad b_3 = 7, \quad \dots .$$

This sequence clearly diverges to ∞.

The sequence $a_n = (-1)^n$, for $n \geq 0$, consists of alternating 1's and -1's:

$$a_0 = 1, \quad a_1 = -1, \quad a_2 = 1, \quad a_3 = -1, \quad \dots,$$

and, therefore, diverges, since there is not **one** limit L that it gets arbitrarily close to.

On the other hand, the sequence

$$c_n = \frac{1}{b_n} = \frac{1}{2n+1}$$

converges to 0, while the sequence

$$d_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

converges to 1.

It will be convenient to use k now, instead of n, for the indexing variable in our sequence; of course, this doesn't affect what the sequence is; the sequence $b_n = 2n + 1$, for $n \geq 0$, is the same as the sequence $b_k = 2k + 1$, for $k \geq 0$.

Given a sequence of real numbers, b_k, where $k \geq m$, we are interested in defining the infinite sum $\sum_{k=m}^{\infty} b_k$. The infinite summation is usually referred to as an *infinite series*.

Definition 4.5.3. *Given a sequence b_k, for $k \geq m$, we define, for each $n \geq m$, the* **partial sum** *to be $\sum_{k=m}^{n} b_k$, and the* **infinite sum** *or* **infinite series** *or, simply,* **series** *to be the infinite summation $\sum_{k=m}^{\infty} b_k$, which, technically, consists of the sequence b_k together with the summation instruction/symbol, telling you to add the sequence.*

The **sum of the series** *or* **value of the series** *$\sum_{k=m}^{\infty} b_k$ is the limit as $n \to \infty$ of the partial sums, and we write*

$$\sum_{k=m}^{\infty} b_k = \lim_{n\to\infty} \sum_{k=m}^{n} b_k,$$

provided the limit exists, in which case we say that the partial sums converge, that the series converges, or simply that the infinite sum exists and is equal to the limit of the partial sums.

If the limit of the partial sums does not exist, then we say that the series **diverges** *or, simply, that the infinite sum does not exist.*

Note that there are actually two sequences associated with an infinite sum/series: the sequence of things you're adding, the *terms*, and the *sequence of partial sums* $s_n = \sum_{k=m}^{n} b_k$. What we're interested in is what happens to the sequence of partial sums as $n \to \infty$.

There should be no confusion here. We are trying to define what an infinite sum should mean. Intuitively, it means the limit as you add more and more terms; this means that an infinite sum is the limit of the partial sums, **NOT** the limit of the terms. The terms are just the individual summands; they are not sums themselves.

What does any of this have to do with power series? Consider a power series, centered at a,

$p(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$. This means that we have specified the infinite sequence of coefficients c_0, c_1, c_2, ...

Now, when we plug in a specific x value, $x = x_0$, the power series gives a series of real numbers; letting $b_k = c_k(x_0 - a)^k$, we have the series

$$\sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} c_k(x_0 - a)^k,$$

and we just defined such an infinite sum in Definition 4.5.3. It's defined to be the limit of the partial sums

$$\lim_{n\to\infty} \sum_{k=0}^{n} c_k(x_0 - a)^k,$$

provided that this limit exists.

Thus, as with polynomials and Taylor series, we define the *partial sums* of a power series.

Definition 4.5.4. *For all integers $n \geq 0$, we define the* **n-th order partial sum** *of $p(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ to be the polynomial, or polynomial function,*

$$p^n(x) = \sum_{k=0}^{n} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n.$$

And now we can define a power series **function**. This is just a mild generalization of our definition in Definition 4.4.1 of a function defined by a Taylor series; however, now, we allow arbitrary power series, instead of restricting ourselves to Taylor series of given functions.

Definition 4.5.5. *Suppose we have the power series, centered at a, $p(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$.*

Then, the **value of $p(x)$ at a point** *x_0 is $p(x_0) = \lim_{n\to\infty} p^n(x_0) =$*

$$\lim_{n\to\infty} \left[c_0 + c_1(x_0-a) + c_2(x_0-a)^2 + c_3(x_0-a)^3 + \cdots + c_n(x_0-a)^n\right],$$

provided that this limit exists; in this case, we say that the power series $p(x)$ **converges at** *x_0.*

If the limit fails to exist, we say that $p(x)$ **diverges at** *x_0.*

The **power series function** *$p(x)$ is the function whose domain is the set of x_0 at which $p(x)$ converges, and its value at any x_0 in its domain is $p(x_0)$, as defined above. The codomain of a power series function is taken to be all real numbers. The notation for this function is the same as that of the algebraic power series, $p(x)$, and we frequently use the term "power series" to mean either the formal algebraic series or the function, letting the context make the distinction clear.*

Remark 4.5.6. It is trivial, but important, that every power series, centered at a, converges at $x = a$; for there, the value of every term, other than the constant term, is zero, and so every partial sum is simply the constant term c_0. Thus,

$$p(a) \;=\; \lim_{n\to\infty} p^n(a) \;=\; \lim_{n\to\infty} c_0 \;=\; c_0.$$

The real question is: where else, besides the center, does the power series converge?

Example 4.5.7. Recall from Section 4.4 that the Maclaurin series for e^x, $\sin x$, and $\cos x$ are, respectively,

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}, \qquad \text{and} \qquad \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}.$$

In fact, in Section 4.4, we did not consider the separate question of which x values make these series converge; we looked only at the question of which x values make the series converge **and** make what they converge to equal to the value of the original function.

However, the three power series above converge and equal the original function for **all** x and so, in particular the domain of each of the three power series functions above is the entire real line $(-\infty, \infty)$.

In Section 4.4, we also saw that the domain of the Maclaurin series (function) of $a/(1-x)$, for $a \neq 0$, is precisely $(-1, 1)$, for, in Theorem 4.4.10, we included the statement that the Maclaurin series diverges outside of the interval $(-1, 1)$.

On the other hand, you have to be somewhat careful when looking back at Section 4.4 for the domains of the Maclaurin series of $\ln(1 + x)$ and $(1 + x)^p$. In Section 4.4, we gave intervals on which the Maclaurin series of these two functions are equal to the functions themselves, but there was no claim that the Maclaurin series diverged outside of those intervals; in theory, the sets of points on which the Maclaurin series converge could be larger than the intervals on which the Maclaurin series agree with the original functions.

As we shall see later, the domain of the Maclaurin series function of $\ln(1+x)$ is precisely the interval on which the series converges to $\ln(1+x)$, namely, the interval $(-1,1]$.

However, the domain of the Maclaurin series function of $(1+x)^p$, the binomial series, depends on p, and can be any one of three possible intervals: $(-1,1)$, $(-1,1]$, or $[-1,1]$. While it is beyond the scope of this textbook, it can be shown that the binomial series for $(1+x)^p$ converges at $x=-1$ if and only if $p \geq 0$, and converges at $x=1$ if and only if $p > -1$. Thus, for instance, when $p=-1/2$, the interval on which the Maclaurin series of $1/\sqrt{1+x}$ converges is $(-1,1]$.

Of course, we haven't given an example of a power series that's not the Maclaurin series of some "nice" function. What can we say about the domain of the power series function

$$\sum_{k=1}^{\infty} \frac{(x-5)^k}{3^k k^2} = \frac{1}{3}(x-5) + \frac{1}{3^2 \cdot 2^2}(x-5)^2 + \frac{1}{3^3 \cdot 3^2}(x-5)^3 + \frac{1}{3^4 \cdot 4^2}(x-5)^4 + \dots ?$$

Below, we give a general result that tells us that the domain of this, or any, power series function must be an interval, and, in fact, that interval must have, as its center, the center of the power series, which is 5 for the series above. We will then state the Ratio Test, which tells you, in many cases, how to find the radius (half the length) of the interval on which the series converges.

Recall now that an *extended real number* is a real number or $\pm\infty$. We need one more definition before we can begin with the results and examples.

Definition 4.5.8. *Suppose that $R \geq 0$ is an extended real number, i.e., a non-negative real number or ∞.*

Then, an **interval centered at** *a is an interval which contains a and which has endpoints $a-R$ and $a+R$, where the interval may or may not include one or both of the endpoints.*

Note that, if $R=\infty$, then the interval is $(-\infty,\infty)$, and if $R=0$, then the interval (which was required to contain a) must be the closed interval which contains only a, i.e., must be $[a,a]$.

The extended real number R is called the **radius of the interval**.

As we mentioned, back in Example 4.5.7, there is the following result.

Theorem 4.5.9. *The set of points at which a power series,* $\sum_{k=0}^{\infty} c_k(x-a)^k$*, centered at* a*, converges is an interval centered at* a*.*

In addition, at points x *in the interior of this interval (if there are any), the series* $\sum_{k=0}^{\infty} |c_k| \cdot |x-a|^k$*, whose terms are the absolute values of the original series, converges.*

Proof. This result is beyond the scope of this textbook. See Theorem 4.5.2 of [4]. □

In light of this, we make another definition.

Definition 4.5.10. *The interval of points at which a power series converges, that is, the domain of the power series function, is called the* **interval of convergence** *of the power series, and the radius of that interval is called the* **radius of convergence** *of the power series.*

Thus, for a power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ *centered at* a*, the radius of convergence is the unique extended real number* $R \geq 0$ *such that, if* $|x-a| < R$*, then the series converges, and if* $|x-a| > R$*, then the series diverges.*

Remark 4.5.11. The second paragraph of Theorem 4.5.9 means that, when $R > 0$, at points where $|x-a| < R$, i.e., at points x in the open interval $(a-R, a+R)$ (meaning the open interval $(-\infty, \infty)$, when $R = \infty$), the power series converges in a fairly strong way; it converges *absolutely*, which we won't discuss any further until Section 5.4.

Example 4.5.12. In Example 4.5.7, using our current terminology, what we saw is that the intervals of convergence of the Maclaurin series for e^x, $\sin x$, and $\cos x$ are all $(-\infty, \infty)$ and, hence, the radii of convergence for all of those series are ∞.

We also discussed in Example 4.5.7 that, for $a \neq 0$, the Maclaurin series for $a/(1-x)$, i.e., the geometric series, has $(-1, 1)$ as its interval of convergence, and so, has radius of convergence 1. Furthermore, the Maclaurin series for $\ln(1+x)$ and $(1+x)^p$ also have radii of convergence equal to 1, with corresponding intervals of convergence $(-1, 1]$ and, depending on p, one of $(-1, 1)$, $(-1, 1]$, and $[-1, 1]$.

However, we have yet to explain any method whatsoever for determining the radius or interval of convergence of essentially arbitrary series, like $\sum_{k=1}^{\infty} \frac{(x-5)^k}{3^k k^2}$ from Example 4.5.7. For this, we use the Ratio Test, below.

There are many theorems related to the convergence of power series; by far the most important of these is the *Ratio Test*. We shall discuss the Ratio Test in this section, and defer the other convergence tests until Chapter 5. The Ratio Test is a theorem about series of constants, but which, when applied to many power series, yields the radii of convergence. However, the Ratio Test, in its usual form, can never tell you about convergence/divergence at the endpoints of the interval of convergence.

As we shall see in Theorem 5.3.23, the proof of the Ratio Test just amounts to comparing with what happens for geometric series (recall Theorem 4.4.10).

Theorem 4.5.13. **(The Ratio Test)** *Consider the series* $\sum_{k=m}^{\infty} b_k$.

1. *If there exists $r < 1$ (and, necessarily, > 0) and an integer $M \geq m$ such that, for all $k \geq M$,*

$$b_k \neq 0 \quad \text{and} \quad \left|\frac{b_{k+1}}{b_k}\right| \leq r,$$

then the given series converges.

2. *If there exists an integer $M \geq m$ such that, for all $k \geq M$,*

$$b_k \neq 0 \quad \text{and} \quad \left|\frac{b_{k+1}}{b_k}\right| \geq 1,$$

then the given series diverges.

In particular, suppose that $\lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right|$ *exists, as an extended real number; call its value L. Then,*

a. *if $L < 1$, the given series converges;*

b. *if $L > 1$, including $L = \infty$, the given series diverges;*

c. *if $L = 1$, the given series may converge or diverge.*

Remark 4.5.14. You usually try to calculate the limit L in the Ratio Test first, and hope that it exists and is unequal to 1. If L fails to exist or equals 1, then you consider cases 1) and 2).

Be aware, however, that, if L exists and $L = 1$, then there cannot possibly exist an r as in case 1) of the Ratio Test, and so you will not be able to use the Ratio Test to conclude that the series converges. On the other hand, even if $L = 1$, you may be able to use case 2) to conclude that the series diverges.

Also, if the limit L fails to exist, cases 1) and 2) may enable you to conclude convergence or divergence of a given infinite series.

Remark 4.5.15. Due to the nature of the Ratio Test and because factorials frequently appear in series, it is worth recalling how factorials work, before we look at examples.

From the definition, for $k \geq 0$,

$$(k+1)! \; = (k+1)k(k-1)(k-2)\cdots(3)(2)(1)$$

and

$$(2k+2)! \;=\; (2k+2)(2k+1)(2k)(2k-1)(2k-2)\cdots(3)(2)(1).$$

Hence,

$$(k+1)! \;=\; (k+1)(k!) \quad \text{and} \quad [2(k+1)]! \;=\; (2k+2)(2k+1)[(2k)!],$$

and so

$$\frac{(k+1)!}{k!} \;=\; k+1 \quad \text{and} \quad \frac{[2(k+1)]!}{(2k)!} \;=\; (2k+2)(2k+1).$$

The point of this is **not** that you should memorize these specific formulas for quotients involving factorials, but rather that, if you just think about what factorial means, such formulas become quick and easy to derive.

Example 4.5.16. Let's consider six different series of constants, series whose terms, for $k \geq 1$, are given by

$$a_k \;=\; \frac{1}{3^k}, \quad b_k \;=\; 3^k, \quad c_k \;=\; \frac{3^k}{k!}, \quad d_k \;=\; (-1)^k k, \quad e_k \;=\; \frac{1}{k}, \quad \text{and} \quad f_k \;=\; \frac{(-1)^{k+1}}{k}.$$

What does the Ratio Test say about the six corresponding infinite series/summations?

- $\sum_{k=1}^{\infty} a_k \ = \ \sum_{k=1}^{\infty} \frac{1}{3^k}$:

  Since
$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| \ = \ \lim_{k\to\infty}\frac{1/3^{k+1}}{1/3^k} \ = \ \lim_{k\to\infty}\frac{1}{3} \ = \ \frac{1}{3} \ < \ 1,$$

the Ratio Test tells us that

$$\sum_{k=1}^{\infty} a_k \ = \ \sum_{k=1}^{\infty}\frac{1}{3^k} \ = \ \frac{1}{3} \ + \ \frac{1}{3^2} \ + \ \frac{1}{3^3} \ + \ \frac{1}{3^4} \ + \ \cdots$$

converges.

In fact, $\sum_{k=1}^{\infty} \frac{1}{3^k}$ is equal to $\sum_{k=0}^{\infty} \frac{1}{3}\cdot\left(\frac{1}{3}\right)^k$ (if this isn't clear, write out the first few terms), and so this series is a geometric series; hence, we can apply Theorem 4.4.10, with $a = 1/3$ and $x = 1/3$ to conclude that, not only does $\sum_{k=1}^{\infty} a_k$ converge, but it converges to $\frac{1/3}{1-(1/3)} = 1/2$.

- $\sum_{k=1}^{\infty} b_k \ = \ \sum_{k=1}^{\infty} 3^k$:

  Since
$$\lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right| \ = \ \lim_{k\to\infty}\frac{3^{k+1}}{3^k} \ = \ \lim_{k\to\infty} 3 \ = \ 3 \ > \ 1,$$

the Ratio Test tells us that

$$\sum_{k=1}^{\infty} b_k \ = \ \sum_{k=1}^{\infty} 3^k \ = \ 3 \ + \ 3^2 \ + \ 3^3 \ + \ \cdots$$

diverges. Of course, it should be obvious that the partial sums are approaching ∞.

As with the $\sum a_k$, the current series is geometric, but, this time, with $a = 3$ and $x = 3$, and so Theorem 4.4.10 already told us that the series $\sum_{k=1}^{\infty} b_k$ diverges.

- $\sum_{k=1}^{\infty} c_k \ = \ \sum_{k=1}^{\infty} \frac{3^k}{k!}$:

  Since

$$\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| \ = \ \lim_{k\to\infty}\frac{3^{k+1}/(k+1)!}{3^k/k!} \ = \ \lim_{k\to\infty}\frac{3^{k+1}}{3^k}\cdot\frac{k!}{(k+1)!} \ = \lim_{k\to\infty}\frac{3}{k+1} \ = \ 0 \ < \ 1$$

the Ratio Test tells us that

$$\sum_{k=1}^{\infty} c_k \;=\; \sum_{k=1}^{\infty} \frac{3^k}{k!} \;=\; \frac{3}{1!} \;+\; \frac{3^2}{2!} \;+\; \frac{3^3}{3!} \;+\; \frac{3^4}{4!} \;+\; \cdots$$

converges.

This series may look familiar. In fact, Theorem 4.4.5 tells us that

$$e^3 \;=\; 1 \;+\; \frac{3}{1!} \;+\; \frac{3^2}{2!} \;+\; \frac{3^3}{3!} \;+\; \frac{3^4}{4!} \;+\; \cdots \;=\; 1 \;+\; \sum_{k=1}^{\infty} \frac{3^k}{k!}$$

and, thus, not only does $\sum_{k=1}^{\infty} \frac{3^k}{k!}$ converge, but it converges to $e^3 - 1$.

- $\sum_{k=1}^{\infty} d_k \;=\; \sum_{k=1}^{\infty} (-1)^k k$:

  Since
  $$\lim_{k\to\infty} \left|\frac{d_{k+1}}{d_k}\right| \;=\; \lim_{k\to\infty} \left|\frac{(-1)^{k+1}(k+1)}{(-1)^k k}\right| \;=\; \lim_{k\to\infty} \left(1+\frac{1}{k}\right) \;=\; 1,$$

  case c) of the Ratio Test, Theorem 4.5.13, tells us that, at this point, we cannot conclude whether $\sum_{k=1}^{\infty} d_k$ converges or diverges.

  On the other hand, case 2) of the Ratio Test applies, since, for all $k \geq 1$,

  $$\left|\frac{d_{k+1}}{d_k}\right| \;=\; 1 \;+\; \frac{1}{k} \;>\; 1.$$

  Therefore,
  $$\sum_{k=1}^{\infty} d_k \;=\; \sum_{k=1}^{\infty} (-1)^k k \;=\; -1 \;+\; 2 \;-\; 3 \;+\; 4 \;-\; 5 \;+ \cdots$$
  diverges.

- $\sum_{k=1}^{\infty} e_k \;=\; \sum_{k=1}^{\infty} \frac{1}{k}$ and $\sum_{k=1}^{\infty} f_k \;=\; \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$:

  The terms of these two series have the same absolute value and, hence, both of these series lead to the same ratio in the Ratio Test:

  $$\lim_{k\to\infty} \frac{1/(k+1)}{1/k} \;=\; \lim_{k\to\infty} \frac{k}{k+1} \;=\; \lim_{k\to\infty} \frac{(k+1)-1}{k+1} \;=\; \lim_{k\to\infty} \left(1-\frac{1}{k+1}\right) \;=\; 1.$$

Once again, case c) of the Ratio Test, Theorem 4.5.13, tells us that, at this point, we cannot conclude whether the two series converge or diverge.

This time, case 2) of the Ratio Test does **not** apply, since, for all $k \geq 1$,

$$\frac{1/(k+1)}{1/k} \;=\; 1 \;-\; \frac{1}{k+1} \;<\; 1.$$

You might hope that case 1) of the Ratio Test would allow us to conclude the convergence of both of the series, but, in fact, there is no $r < 1$ so that, for arbitrarily large k, $1 - 1/(k+1)$ is always less than or equal to r; as we mentioned in Remark 4.5.14, no such r can exist precisely because $\lim_{k\to\infty}(1 - 1/(k+1)) = 1$.

Actually, as we discussed in Remark 4.4.13, the series

$$\sum_{k=1}^{\infty} f_k \;=\; \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \;=\; 1 \;-\; \frac{1}{2} \;+\; \frac{1}{3} \;-\; \frac{1}{4} \;+\; \cdots$$

is called the *alternating harmonic series*, and converges to $\ln 2$.

However, the *harmonic series*

$$\sum_{k=1}^{\infty} e_k \;=\; \sum_{k=1}^{\infty} \frac{1}{k} \;=\; 1 \;+\; \frac{1}{2} \;+\; \frac{1}{3} \;+\; \frac{1}{4} \;+\; \cdots$$

diverges to ∞, as we shall see in Proposition 5.2.16.

Great. So now we have the Ratio Test, and we've looked at a few examples of how to use it. But, what does this have to do with power series? Another extended example should make it very clear how the Ratio Test allows you find the radius of convergence of many power series.

Example 4.5.17. Let's see what the Ratio Test tells us about the radius of convergence of each of the following power series:

$$\sum_{k=0}^{\infty}\frac{x^k}{k!},\qquad \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!},\qquad \sum_{k=0}^{\infty}\frac{(x+3)^{2k}}{7^k},\qquad \sum_{k=0}^{\infty}k!x^k,\qquad \text{and}\qquad \sum_{k=1}^{\infty}\frac{(x-5)^k}{3^k k^2}.$$

When we apply the Ratio Test to each series, we do so with the variable x in the test, and we obtain restrictions on x which guarantee convergence. We always explicitly, or implicitly, assume that x is **not** the center of the power series while applying the Ratio Test; we know that all power series converge at their centers, for every term becomes zero, except for the constant term. Of course, every term, other than the constant term, being zero causes a problem in the Ratio Test, since then the quotients in the Ratio Test are not defined for large values of k. Thus, we assume we're not taking x to be the center.

- $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k!}$:

We apply the Ratio Test with $b_k = \dfrac{x^k}{k!}$ (assuming that $x \neq 0$), and find

$$L = \lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}/(k+1)!}{x^k/k!}\right| = \lim_{k\to\infty}\left|x\cdot\frac{k!}{(k+1)!}\right| = \lim_{k\to\infty}\frac{|x|}{k+1} = 0 < 1,$$

regardless of the value of x. That is, no matter what number we put in for x, the Ratio Test limit comes out less than 1; this means that the power series converges **for all** x. Therefore, the interval of convergence is $(-\infty,\infty)$ and the radius of convergence is $R=\infty$.

Of course, we knew this already; this power series is the Maclaurin series of e^x, and converges to e^x, for all x. But, it's nice to know that we can use the Ratio Test to conclude that the series converges for all x, **without** having to know ahead of time that the series equals e^x.

- $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$:

Hopefully, you recognize this as the Maclaurin series for $\cos x$; hence, we know that power series converges to $\cos x$, for all x, but let's see if the Ratio Test will allow us to conclude that the interval of convergence is $(-\infty,\infty)$.

Note that it's extremely convenient that the indexing of the terms of this power series are designed to hit only the non-zero terms, i.e., even though the coefficients of all of the odd powers of x are zero in this series, our indexing skips those zero terms, and only adds the even powers of

x. Missing the non-zero terms makes the ratios in the Ratio Test defined, since we don't divide by zero.

We apply the Ratio Test and calculate

$$L = \lim_{k\to\infty}\left|\frac{x^{2(k+1)}/[2(k+1)]!}{x^{2k}/(2k)!}\right| = \lim_{k\to\infty}\left(x^2\cdot\frac{(2k)!}{(2k+2)!}\right) = \lim_{k\to\infty}\frac{x^2}{(2k+2)(2k+1)} = 0 < 1,$$

regardless of the value of x. So, yes, once again the Ratio Test allows us to conclude that the interval of convergence is the entire real line $(-\infty,\infty)$, and so the radius of convergence is ∞.

- $\displaystyle\sum_{k=0}^{\infty}\left(\frac{x+3}{7}\right)^k = \sum_{k=0}^{\infty}\frac{1}{7^k}(x+3)^k$:

We apply the Ratio Test and calculate

$$L = \lim_{k\to\infty}\left|\frac{(x+3)^{k+1}/7^{k+1}}{(x+3)^k/7^k}\right| = \lim_{k\to\infty}\frac{|x+3|}{7} = \frac{|x+3|}{7}.$$

Now, the Ratio Test tells us that the series converges if $|x+3|/7 < 1$ and diverges if $|x+3|/7 > 1$. Multiplying by 7, we find that the series, which is centered at -3, converges if $|x-(-3)| < 7$ and diverges if $|x-(-3)| > 7$; looking back at Definition 4.5.10, you can see that this tells us that the radius of convergence of the series is 7.

Thus, the largest **open** interval on which the series converges is the interval $(-3-7, -3+7) = (-10, 4)$, but this easy use of the Ratio Test does **not** tell you what happens at the endpoints of this interval, for the endpoints are exactly where the limit in the Ratio Test comes out to equal 1. This specific example shows you what happens in general: when you apply the Ratio Test, Theorem 4.5.13, simply calculating the limit L in cases a), b), and c) can **never** tell you about convergence/divergence at the endpoints of the interval of convergence.

However, in this particular example, case 2) of Ratio Test allows us to conclude that the series diverges at both endpoints of the interval of convergence. This isn't terribly surprising; the proof of the Ratio Test uses that we know exactly what happens for geometric series and, while it's slightly disguised, the series $\sum_{k=0}^{\infty}\left(\frac{x+3}{7}\right)^k$ is, in fact, geometric.

How do you see this? Well...the most basic geometric series, when $a = 1$ in Theorem 4.4.10, is

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k,$$

where the equality holds for $|x| < 1$, and the series diverges for $|x| \geq 1$. Replacing the x

everywhere with the quantity $(x+3)/7$, we find that

$$\frac{1}{1-\frac{x+3}{7}} \;=\; \sum_{k=0}^{\infty}\left(\frac{x+3}{7}\right)^k,$$

where the equality holds for $\left|\frac{x+3}{7}\right| < 1$, and the series diverges for $\left|\frac{x+3}{7}\right| \geq 1$. Note that the convergence/divergence is exactly what we found by using the Ratio Test.

- $\displaystyle\sum_{k=0}^{\infty} k!x^k$:

  We calculate the limit L from the Ratio Test:

$$L \;=\; \lim_{k\to\infty}\left|\frac{(k+1)!x^{k+1}}{k!x^k}\right| \;=\; \lim_{k\to\infty}(k+1)|x|.$$

If $x \neq 0$, this limit is ∞, which is greater than 1, regardless of the value of x. Thus, the Ratio Test tells us that this series converges only at the center $x=0$ or, equivalently, that the radius of convergence is 0.

- $\displaystyle\sum_{k=1}^{\infty}\frac{(x-5)^k}{3^k k^2}$:

  We calculate the limit L from the Ratio Test:

$$L \;=\; \lim_{k\to\infty}\left|\frac{\frac{(x-5)^{k+1}}{3^{k+1}(k+1)^2}}{\frac{(x-5)^k}{3^k k^2}}\right| \;=\; \lim_{k\to\infty}\left|\frac{(x-5)^{k+1}}{(x-5)^k}\cdot\frac{3^k}{3^{k+1}}\cdot\frac{k^2}{(k+1)^2}\right| \;=\; \frac{|x-5|}{3}\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^2.$$

Dividing by k in the numerator and denominator inside the parentheses, we find

$$L \;=\; \frac{|x-5|}{3}\lim_{k\to\infty}\left(\frac{1}{1+\frac{1}{k}}\right)^2 \;=\; \frac{|x-5|}{3}.$$

Therefore, the series converges when $|x-5|/3 < 1$ and diverges when $|x-5|/3 > 1$, i.e., converges when $|x-5| < 3$ and diverges when $|x-5| > 3$. Hence, the radius of convergence is 3, and the largest open interval on which the series converges is the interval $(5-3, 5+3) = (2, 8)$.

We leave it as an exercise for you to show that even the more-complicated parts of the Ratio Test do not allow you to conclude anything about convergence/divergence when $x = 2$ and $x = 8$.

One final note on applying the Ratio Test to this last series: as we saw, it can be very helpful to split up the quotient that appears in the Ratio Test into a product of quotients of similar-looking terms. The calculation of the limits frequently works out nicely when you do this.

Remark 4.5.18. You may have figured out from the above examples how to calculate the radius of convergence the "easy way"; the Ratio Test implies that, if you have the infinite series $\sum_{k=0}^{\infty} c_k(x-a)^k$, and

$$\lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right| = R,$$

where R is an extended real number, then R is the radius of convergence of the power series.

Example 4.5.19. Consider the last series from Example 4.5.17: $\displaystyle\sum_{k=1}^{\infty}\frac{(x-5)^k}{3^k k^2}$.

We calculate

$$R \;=\; \lim_{k\to\infty}\left|\frac{1/\left(3^k k^2\right)}{1/\left(3^{k+1}(k+1)^2\right)}\right| \;=\; \lim_{k\to\infty}\frac{3^{k+1}}{3^k}\cdot\frac{(k+1)^2}{k^2} \;=\; 3,$$

as we found in Example 4.5.17.

It's important when using the method of Remark 4.5.18 that the limit $\displaystyle\lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$ exists, at least as an extended real number, and that c_k is the coefficient of $(x-a)^k$, i.e., that you're looking at **all** of the powers of $(x-a)$.

Consider, for instance, the series

$$\sum_{k=0}^{\infty}\frac{(x+7)^{2k}}{9^k} \;=\; 1 \;+\; \frac{(x+7)^2}{9} \;+\; \frac{(x+7)^4}{9^2} \;+\; \frac{(x+7)^6}{9^3} \;+\; \cdots.$$

Note that the odd terms are missing. It might be tempting to think that the radius of convergence can be calculated from the coefficients of the **non-zero** terms, and look at

$$\lim_{k\to\infty}\left|\frac{1/9^k}{1/9^{k+1}}\right| \;=\; 9.$$

However, the radius of convergence is **not** 9. Why not?

You can apply the Ratio Test to the series and determine that the series converges when $\dfrac{|x+7|^2}{9} < 1$ and diverges when $\dfrac{|x+7|^2}{9} > 1$. Therefore, the radius of convergence is $\sqrt{9} = 3$.

In general, it is not particularly interesting to have functions defined by power series that converge only at their centers; such a function has a single point in its domain and, at that single point, the function equals the constant term in the series. Not too useful.

Therefore, our interest is really in functions which equal power series on a non-empty open interval around the center a of the series. As we shall see in the next section, this necessarily means that the function equals its Taylor series, centered at a. We give a name to such functions.

Definition 4.5.20. *Suppose that f is an infinitely differentiable function on an open subset $\mathcal{U}$ of $\mathbb{R}$.Let $\mathcal{V}$ be an open subset of $\mathcal{U}$, and let a be a point in $\mathcal{U}$.*

Then, $f(x)$ is **real analytic at** *a provided that, for all x in some open interval around a, $T_f^{\infty}(x;a)$ converges to $f(x)$. This is equivalent to saying that $f(x)$ is equal to* **some** *(convergent) power series, centered at a, on an open interval containing a.*

The function f is said to be **real analytic on** *$\mathcal{V}$ provided that f is real analytic at x, for all x in $\mathcal{V}$. A function which is real analytic on its whole domain is simply called a* **real analytic function**.

Example 4.5.21. Suppose that we have a power series $p(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$, centered at a, with a positive radius of convergence $R > 0$. Then, the function defined by $p(x)$, on the interval of convergence of $p(x)$, is, by definition, real analytic at a, since $p(x)$ equals the convergent power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ on (at least) the non-empty open interval $(a-R, a+R)$.

Remark 4.5.22. There is a subtle, but extremely important, point in the definition of a function being real analytic on an open subset or, more specifically, on an open interval.

Suppose that f is real analytic at a. This means that there exists an open interval I, around a, such that, for all x in I, $f(x) = T_f^\infty(x;a)$ or, equivalently, that there is some power series $\sum_{k=0}^\infty c_k(x-a)^k$ such that $f(x) = \sum_{k=0}^\infty c_k(x-a)^k$ on a non-empty open interval I around a.

The question is: does this mean that f is real analytic on I, i.e., if b is in I, must it be true that f is real analytic at b?

Your tendency might be to say "Sure – f equals its Taylor series for all x in I, and b is in I, so f equals its Taylor series at b".

The problem here is that what we mean by "its Taylor series" should not be the same both times in the sentence above. The first time that we wrote "its Taylor series", we meant "its Taylor series **centered at** a". However, f being real analytic **at** b means that the Taylor series of f, **centered at** b, converges to $f(x)$ for all x in an open interval around b. In other words, being real analytic at different points requires you to switch the center of your Taylor/power series to match the point under consideration, and then you need that the Taylor series $T_f^\infty(x;b)$ around this (possibly) new center b also converges to $f(x)$ on some open interval around b; this interval around b need **not** be the same open interval I on which $T_f^\infty(x;a)$ converged.

Fortunately, there is a theorem, Theorem 8.4 of [3], which tells us that, in fact, we don't really have to worry about the subtle point discussed above.

Theorem 4.5.23. *Let R be a positive extended real number. Suppose that $T_f^\infty(x;a)$ converges to $f(x)$ for all x in the open interval I defined by $|x-a| < R$. Then, for all b in I, $T_f^\infty(x;b)$ converges to $f(x)$ for all x such that $|x-b| < R - |b-a|$, that is, for all x in the largest open interval, centered at b, which is contained in I.*

As a special case, if $T_f^\infty(x;a)$ converges to $f(x)$ for all real numbers x, then so does $T_f^\infty(x;b)$, for every real number b.

In particular, if $T_f^\infty(x;a)$, or any power series $\sum_{k=0}^\infty c_k(x-a)^k$, converges to $f(x)$ for all x in an open interval I, centered at a, then f is real analytic on I.

Example 4.5.24. We know that e^x, $\sin x$, and $\cos x$ are equal to their Maclaurin series, for all x. Theorem 4.5.23 tells us that being equal to their Maclaurin series on the interval $(-\infty, \infty)$ in fact implies that, on the entire real line, these functions are equal to their Taylor series **centered anywhere**. Of course, we could have determined this by showing that the Taylor error, with an arbitrary center, approaches zero, but Theorem 4.5.23 gives it to us without any additional work.

Example 4.5.25. Let's look at a function which has a Maclaurin series with a positive, finite radius of convergence. Consider the function $f(x) = \ln(1+x)$, for all $x > -1$. Is this function real analytic on the entire interval $(-1, \infty)$?

Your immediate reaction may be to say "no; $\ln(1+x)$ equals its Maclaurin series only on the interval $(-1, 1]$ ". This is true, but, somewhat surprisingly, irrelevant. The question of being real analytic on the interval $(-1, \infty)$ is: for all $a > -1$, is it true that $\ln(1+x)$ equals its Taylor series **centered at** a on some open interval around a? That is, we allow the center of the series under consideration to change. Being real analytic does **not** require being equal to a **single** power series everywhere. Yes - being equal to a single power series on an open interval implies being real analytic; this follows from Theorem 4.5.23. But a function can be real analytic on an interval and "require" you to use Taylor series with different centers to represent the function and see the real analyticity.

If we take as given (even though we haven't proved it yet) that $\ln(1+x)$ is real analytic on the open interval $(-1, 1)$, then, if we show that, for all $a > 0$, $\ln(1+x)$ is real analytic at a, we will have shown that $\ln(1+x)$ is real analytic on the interval $(-1, \infty)$. Let's do this.

Suppose that $a > 0$, and let $f(x) = \ln(1+x)$. As we saw earlier, in the proof of Theorem 4.4.12, for $k \geq 1$, $f^{(k)}(x) = (-1)^{k-1}(k-1)!(1+x)^{-k}$.

Therefore, if $x \neq a$, for $n \geq 1$, there exists a c, strictly between x and a, such that the Taylor error satisfies the following equality:

$$E_f^n(x) \;=\; \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| \;=\; \frac{n!}{(n+1)!}\cdot\left|\frac{x-a}{1+c}\right|^{n+1} \;=\; \frac{1}{n+1}\cdot\left|\frac{x-a}{1+c}\right|^{n+1}.$$

What we want to do is produce an $R > 0$ (not necessarily the radius of convergence) such that, if $|x-a| < R$, then $|x-a|/|1+c| < 1$, for then the limit $\lim_{n\to\infty} E_f^n(x)$ will be 0, which would show that $\ln(1+x)$ equals its Taylor series at a on the non-empty open interval $(a-R, a+R)$.

But this is easy; let R be the minimum of the two values 1 and a, i.e., $R = \min\{a, 1\}$. Then $R \leq a$ and $R \leq 1$. Suppose now that $|x-a| < R$, i.e., suppose that $a - R < x < a + R$. Then, certainly, $|x-a| < 1$. In addition, since $R \leq a$, $a - R \geq 0$ and we must have $x > 0$, which, together with $a > 0$, implies that $c > 0$, so that $0 < 1/(1+c) < 1$. Therefore, $|x-a|/|1+c| < 1$, and so the Taylor error approaches 0.

Thus, we have shown that $\ln(1+x)$ is real analytic on the interval $(0, \infty)$, which, when combined with the real analyticity on $(-1, 1)$, shows that $\ln(1+x)$ is real analytic on the entire interval $(-1, \infty)$.

The following theorem tells us that, if a real analytic function on an open interval can be extended, a.k.a., *continued*, to a real analytic function on a larger open interval, then there's only one way to do it. The analog of this theorem in the setting of complex numbers is well-known, but it is difficult to find many references for this fact in the real numbers; we prove the result in Theorem 4.A.1.

Theorem 4.5.26. (**Real Analytic Continuation**) *Suppose that f and g are real analytic functions on an open interval I, and that $f(x) = g(x)$ for all x in an open interval J, which is contained in I. Then, f and g are equal on I.*

Remark 4.5.27. You may think that Theorem 4.5.26 is obvious, and must surely be true for any "nice" functions, like infinitely differentiable functions; surely we don't really need real analytic functions. Actually...we do.

Recall, from Example 4.4.11, that the following function is infinitely differentiable on the entire real line:

$$g(x) \;=\; \begin{cases} e^{-1/x^2}, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

Certainly the function $f(x) = 0$ is infinitely differentiable on the entire real line, and $f(x) = g(x)$ on the open interval $(-\infty, 0)$. Nonetheless, $f(x)$ and $g(x)$ are not equal at all points in the interval $(-\infty, \infty)$.

This does not contradict Theorem 4.5.26, since, as we saw in Example 4.4.11, the function $g(x)$ is not real analytic; more specifically, $g(x)$ is not real analytic at $x = 0$.

4.5.1 Exercises

In each of Exercises 1 through 6, determine whether the given sequence converges or diverges. Note that, as we are interested in what happens as $n \to \infty$, we don't bother specifying a starting value of the index. Assume that the indexing begins at a high enough value so that all of the terms are defined.

1. $b_n = \dfrac{5n^2 + 3n}{7n^2}$.
2. $a_n = \dfrac{5n^2 + 3(-1)^n}{7n^2}$.
3. $c_n = \dfrac{3(-1)^n n^2 + 4}{7n^2}$.
4. $b_n = \dfrac{n^2}{n!}$.
5. $a_n = \cos(n\pi/4)$.
6. $c_n = \dfrac{n^2}{1000n + 1}$.

In each of Exercises 7 through 12, determine the convergence or divergence of the given series.

7. $\displaystyle\sum_{k=1}^{\infty} \frac{1000}{5^k}$
8. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{5^k}$
9. $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^{100}}$
10. $\displaystyle\sum_{k=1}^{\infty} (-1)^k \, \frac{k^2 + 1}{k!}$
11. $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{(2k)!}$
12. $\displaystyle\sum_{k=1}^{\infty} (-1)^k$ (Hint: Do not use the Ratio Test.)

13. a. Find the radius of convergence of the power series

$$p(x) = \sum_{k=1}^{\infty} \frac{5^k(x-1)^k}{k^2} = 5(x-1) + \frac{5^2}{2^2}(x-1)^2 + \frac{5^3}{3^2}(x-1)^3 + \frac{5^4}{4^2}(x-1)^4 + \cdots .$$

b. Given the center of the series and the radius of convergence from part (a), what are the possibilities for the interval of convergence of $p(x)$?

c. Use the first 3 terms of $p(x)$ to estimate the values of $p(1.1)$, $p(1.01)$, and $p(2)$.

d. Explain why, in this exercise, unlike in the exercises from the previous section, it is not possible (in any easy way) to compare your estimates in part (c) with the actual values of $p(x)$ from your calculator.

e. Which of your estimates in part (c) do you expect to be good estimates of the actual values of $p(x)$? Why?

14. a. Find the radius of convergence of the power series

$$p(x) = \sum_{k=1}^{\infty} \sqrt{k}\,(x+3)^k = (x+3) + \sqrt{2}\,(x+3)^2 + \sqrt{3}\,(x+3)^3 + \sqrt{4}\,(x+3)^4 + \cdots .$$

b. Given the center of the series and the radius of convergence from part (a), what are the possibilities for the interval of convergence of $p(x)$?

c. Use the first 3 terms of $p(x)$ to estimate the values of $p(-2.9)$, $p(-2.99)$, and $p(-2)$.

d. Explain why, in this exercise, unlike in the exercises from the previous section, it is not possible (in any easy way) to compare your estimates in part (c) with the actual values of $p(x)$ from your calculator.

e. Which of your estimates in part (c) do you expect to be good estimates of the actual values of $p(x)$? Why?

15. a. Find the radius of convergence of the power series

$$p(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k^3+2} = \frac{1}{2} - \frac{x}{1^3+2} + \frac{x^2}{2^3+2} - \frac{x^3}{3^3+2} + \frac{x^4}{4^3+2} - \cdots .$$

b. Given the center of the series and the radius of convergence from part (a), what are the possibilities for the interval of convergence of $p(x)$?

c. Use the first 3 terms of $p(x)$ to estimate the values of $p(0.1)$, $p(0.01)$, and $p(1)$.

d. Explain why, in this exercise, unlike in the exercises from the previous section, it is not possible (in any easy way) to compare your estimates in part (c) with the actual values of $p(x)$ from your calculator.

e. Which of your estimates in part (c) do you expect to be good estimates of the actual values of $p(x)$? Why?

16. a. Find the radius of convergence of the power series

$$p(x) = \sum_{k=0}^{\infty} \frac{k!(x-4)^k}{7^{k+1}}.$$

b. Given the center of the series and the radius of convergence from part (a), what are the possibilities for the interval of convergence of $p(x)$?

c. Use the first 3 terms of $p(x)$ to estimate the values of $p(4.1)$, $p(4.01)$, and $p(5)$.

d. Explain why, in this exercise, unlike in the exercises from the previous section, it is not possible (in any easy way) to compare your estimates in part (c) with the actual values of $p(x)$ from your calculator.

e. Which of your estimates in part (c) do you expect to be good estimates of the actual values of $p(x)$? Why?

17. a. Find the radius of convergence of the power series

$$p(x) = \sum_{k=0}^{\infty} \frac{(x-4)^k}{7^{k+1}}.$$

b. Given the center of the series and the radius of convergence from part (a), what are the possibilities for the interval of convergence of $p(x)$?

c. Use the first 3 terms of $p(x)$ to estimate the values of $p(4.1)$, $p(4.01)$, and $p(5)$.

d. Explain why, in this exercise, unlike in the exercises from the previous section, it is not possible (in any easy way) to compare your estimates in part (c) with the actual values of $p(x)$ from your calculator.

e. Which of your estimates in part (c) do you expect to be good estimates of the actual values of $p(x)$? Why?

18. a. Find the radius of convergence of the power series

$$p(x) = \sum_{k=1}^{\infty} \frac{(x-4)^k}{7^{k+1}k}.$$

b. Given the center of the series and the radius of convergence from part (a), what are the possibilities for the interval of convergence of $p(x)$?

c. Use the first 3 terms of $p(x)$ to estimate the values of $p(4.1)$, $p(4.01)$, and $p(5)$.

d. Explain why, in this exercise, unlike in the exercises from the previous section, it is not possible (in any easy way) to compare your estimates in part (c) with the actual values of $p(x)$ from your calculator.

e. Which of your estimates in part (c) do you expect to be good estimates of the actual values of $p(x)$? Why?

19. a. Find the radius of convergence of the power series

$$p(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x+7)^k}{k^k}.$$

b. Given the center of the series and the radius of convergence from part (a), what are the possibilities for the interval of convergence of $p(x)$?

c. Use the first 3 terms of $p(x)$ to estimate the values of $p(-6.9)$, $p(-6.99)$, and $p(-6)$.

d. Explain why, in this exercise, unlike in the exercises from the previous section, it is not possible (in any easy way) to compare your estimates in part (c) with the actual values of $p(x)$ from your calculator.

e. Which of your estimates in part (c) do you expect to be good estimates of the actual values of $p(x)$? Why?

20. a. Find the radius of convergence of the power series

$$p(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k!(x+7)^k}{k^k}.$$

b. Given the center of the series and the radius of convergence from part (a), what are the possibilities for the interval of convergence of $p(x)$?

c. Use the first 3 terms of $p(x)$ to estimate the values of $p(-6.9)$, $p(-6.99)$, and $p(-6)$.

d. Explain why, in this exercise, unlike in the exercises from the previous section, it is not possible (in any easy way) to compare your estimates in part (c) with the actual values of $p(x)$ from your calculator.

e. Which of your estimates in part (c) do you expect to be good estimates of the actual values of $p(x)$? Why?

21. a. Find the radius of convergence of the power series

$$p(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{\sqrt[3]{k}}.$$

b. Given the center of the series and the radius of convergence from part (a), what are the possibilities for the interval of convergence of $p(x)$?

c. Use the first 3 terms of $p(x)$ to estimate the values of $p(0.1)$, $p(0.01)$, and $p(1)$.

d. Explain why, in this exercise, unlike in the exercises from the previous section, it is not possible (in any easy way) to compare your estimates in part (c) with the actual values of $p(x)$ from your calculator.

e. Which of your estimates in part (c) do you expect to be good estimates of the actual values of $p(x)$? Why?

22. Suppose that a power series $p(x)$ converges at $x = 1$ and at $x = 3$, but diverges at $x = -2$ and $x = 5$. What can you say about the center of the power series and the radius of convergence?

23. Suppose that someone tells you that they have a power series which converges at $x = 1$ and at $x = 3$, but diverges at $x = 2$. What can you conclude?

24. In this exercise, you will show that $f(x) = \dfrac{1}{1-x}$ is real analytic at all points a, except for $a = 1$, i.e., f is real analytic on the union of open intervals $(-\infty, 1) \cup (1, \infty)$.

a. Find a formula for the k-th derivative, $f^{(k)}(x)$.

b. Give Lagrange's form of the error for $E_f^k(x; a)$, where $a \neq 1$.

c. For each $a \neq 1$, find an $r > 0$ such that, for all x in the open interval $(a - r, a + r)$, the limit of your error from part (b) is zero, i.e., such that $1/(1 - x) = T_f^{\infty}(x; a)$ for all x in $(a - r, a + r)$. (Hint: Choose $r < |1 - a|/2$.)

d. Explain why $1/(1-x)$ being real analytic everywhere, except at 1, doesn't contradict the fact that the Maclaurin series for $1/(1 - x)$ converges if and only if $|x| < 1$.

4.6 Power Series as Functions II: Operations on Power Series

In the previous section, we discussed what it means for a power series to define a function. We looked at the *interval* and *radius of convergence*, we stated the *Ratio Test*, and looked at examples of using the Ratio Test to determine radii of convergence or, in some cases, intervals of convergence. We also discuss *real analytic functions*, functions which are locally equal, on open intervals, to their Taylor series at each point.

In this section, we will look at **why** power series functions are so nice to deal with. We will see that, as far as differentiation, integration, and algebraic operations are concerned, power series can be treated like polynomials, polynomials that just don't ever end.

For a number of technical proofs in this section, we refer you to the excellent real analysis book of Trench, [4].

We will begin with the following basic theorem on continuity, due to Abel.

Theorem 4.6.1. *Power series functions are continuous, i.e., continuous on their intervals of convergence, including at possible endpoints that are included in the intervals of convergence.*

Proof. See [4], Theorem 4.5.12. □

Remark 4.6.2. As we shall see below, power series functions are differentiable on the interior of the interval of convergence, i.e., on the open interval $(a-R, a+R)$, where a is the center and R is the radius of convergence. As differentiable functions are continuous, the main point of Theorem 4.6.1, for us, is that, if a power series $p(x)$ converges at one or both of the endpoints of the interval of convergence, then the one-sided limits of $p(x)$ as you approach the endpoint equal the value of $p(x)$ **at** the endpoint.

Why is this important? Suppose, for instance, that we have a function $f(x)$ which we know is continuous on the interval $(-1, 1]$ and we have a power series $p(x)$ which we know converges on the same half-open interval $(-1, 1]$. Suppose, further, that we know that, on the open interval

$(-1,1)$, $f(x) = p(x)$. Then, we can immediately conclude that, in fact, $f(x) = p(x)$ on the entire half-open interval $(-1,1]$. Why? Because Abel's theorem, Theorem 4.6.1, tells us that $p(x)$ is also continuous on the interval $(-1,1]$, and so

$$f(1) \;=\; \lim_{x\to 1^-} f(x) \;=\; \lim_{x\to 1^-} p(x) \;=\; p(1),$$

where the middle inequality follows from the fact that $f(x) = p(x)$ on the interval $(-1,1)$.

For example, below, we shall use integration of power series to show that, for all x in the open interval $(-1,1)$,

$$\ln(1+x) \;=\; \sum_{k=0}^{\infty}(-1)^k\,\frac{x^{k+1}}{k+1} \;=\; x \;-\; \frac{x^2}{2} \;+\; \frac{x^3}{3} \;-\; \frac{x^4}{4} \;+\; \frac{x^5}{5} \;-\; \frac{x^6}{6} \;+\; \cdots. \tag{4.12}$$

Looking ahead to results in Chapter 5, we find that the series on the right, above, converges when $x = 1$ and diverges when $x = -1$. As $\ln(1+x)$ is continuous, and Abel's theorem tells us that the power series is also continuous, we conclude that

$$\ln 2 \;=\; \lim_{x\to 1^-} \ln(1+x) \;=\; \lim_{x\to 1^-}\left[x \;-\; \frac{x^2}{2} \;+\; \frac{x^3}{3} \;-\; \frac{x^4}{4} \;+\; \frac{x^5}{5} \;-\; \frac{x^6}{6} \;+\; \cdots\right] \;=$$

$$1 \;-\; \frac{1}{2} \;+\; \frac{1}{3} \;-\; \frac{1}{4} \;+\; \frac{1}{5} \;-\; \frac{1}{6} \;+\; \cdots.$$

In other words, the equality given in Formula 4.12, which was for $-1 < x < 1$, also holds when $x = 1$. This provides an alternative way of showing that the alternating harmonic series converges to $\ln 2$; something we saw back in Theorem 4.4.12 and Remark 4.4.13, using Taylor errors.

In the remainder of this section, we will sometimes assume that we have an interval I on which a power series converges. This does not necessarily mean that I is the **entire** interval of convergence; I could, in fact, be a smaller interval inside the interval of convergence.

We want to look now at substituting into power series. Not all substitutions lead to new **power series**, but they give some infinite summation whose convergence or divergence can be discussed.

Example 4.6.3. Recall from Theorem 4.4.5 that we have the following equality

$$e^x \;=\; \sum_{k=0}^{\infty}\frac{x^k}{k!} \;=\; 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots, \tag{4.13}$$

for all x. Now, if x is some number, then $5(x-1)^3$ is just some number, and we can stick it into Formula 4.13 and obtain

$$e^{(5(x-1)^3)} \;=\; 1+5(x-1)^3+\frac{\left(5(x-1)^3\right)^2}{2!}+\frac{\left(5(x-1)^3\right)^3}{3!}+\frac{\left(5(x-1)^3\right)^4}{4!}+\cdots \;=$$

$$1+5(x-1)^3+\frac{25(x-1)^6}{2!}+\frac{125(x-1)^9}{3!}+\frac{625(x-1)^{12}}{4!}+\cdots.$$

Hence, we obtain a power series, centered at 1, which converges to $e^{(5(x-1)^3)}$, for all x.

Is this series necessarily equal to the Taylor series, centered at 1, for $e^{(5(x-1)^3)}$? Yes, though we won't really "know" this until we have Corollary 4.6.8.

Generalizing the above example, we have:

Theorem 4.6.4. *Suppose that $p(x)=\sum_{k=0}^{\infty}c_k(x-a)^k$ converges on an interval I, and we have a second function $f(x)$.*

Then, for all x such that $f(x)$ is in I, we have the following equality

$$p(f(x)) \;=\; \sum_{k=0}^{\infty}c_k(f(x)-a)^k,$$

in which the infinite sum converges.

In particular, if $f(x) = r(x-b)^m + a$, where r and b are constants and $m \geq 0$ is an integer, then, for all x such that $r(x-b)^m + a$ is in I, we have the following equality

$$p\big(r(x-b)^m + a\big) \;=\; \sum_{k=0}^{\infty}(c_k r^k)(x-b)^{mk},$$

in which the sum on the right is a convergent power series. In addition, the new interval of convergence consists precisely of those x values such that $r(x-b)^m + a$ is in I.

Example 4.6.5. Consider the geometric series from Theorem 4.4.10, with $a = 1$:

$$\frac{1}{1-x} \;=\; \sum_{k=0}^{\infty} x^k \;=\; 1 \;+\; x \;+\; x^2 \;+\; x^3 \;+\; x^4 \;+\; \cdots,$$

where the interval of convergence of the series, and the interval on which the equality holds, is the open interval $(-1, 1)$.

By substituting $-3(x+2)$ in for x, we immediately obtain that

$$\frac{1}{1+3(x+2)} \;=\; \sum_{k=0}^{\infty}\big(-3(x+2)\big)^k \;=\; \sum_{k=0}^{\infty}(-1)^k 3^k (x+2)^k \;=$$

$$1 \;-3(x+2) \;+\; 9(x+2)^2 \;-\; 27(x+2)^3 \;+\; 81(x+2)^4 \;+\; \cdots,$$

where the equality holds, and the series converges, on precisely the interval of those x's such that $-3(x+2)$ is in the interval $(-1, 1)$. To determine this interval in a more standard form, you need to manipulate the inequalities

$$-1 \;<\; -3(x+2) \;<\; 1.$$

Divide through by -3, reversing the inequalities, to obtain

$$-\frac{1}{3} \;<\; x+2 \;<\; \frac{1}{3}.$$

Now, subtract 2 everywhere,

$$-2 \;-\; \frac{1}{3} \;<\; x \;<\; -2 \;+\; \frac{1}{3},$$

to find that the new interval of convergence is $(-7/3, -5/3)$, which is centered at -2 and has radius $1/3$.

Example 4.6.6. Recall from Theorem 4.4.7 that, for all x,

$$\cos x \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} \;=\; 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots.$$

Replacing the x by $\dfrac{\pi}{2}-x$, we obtain

$$\cos\left(\frac{\pi}{2}-x\right) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{\pi}{2}-x\right)^{2k}}{(2k)!} \;=\; 1-\frac{\left(\frac{\pi}{2}-x\right)^2}{2!}+\frac{\left(\frac{\pi}{2}-x\right)^4}{4!}-\frac{\left(\frac{\pi}{2}-x\right)^6}{6!}+\cdots.$$

Now, it is a standard trig identity that $\sin x = \cos\left(\frac{\pi}{2}-x\right)$, and $\frac{\pi}{2}-x = -\left(x-\frac{\pi}{2}\right)$ is the same when raised to an even power as $x-\frac{\pi}{2}$ to the same power. Therefore, we obtain

$$\sin x \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{\left(x-\frac{\pi}{2}\right)^{2k}}{(2k)!} \;=\; 1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}-\frac{\left(x-\frac{\pi}{2}\right)^6}{6!}+\cdots.$$

Note that we obtained this same equality in Remark 4.4.8, using our calculation of the Taylor series for $\sin x$, centered at $\pi/2$, from Example 4.2.8.

You should find the last line of the example above to be interesting. In Remark 4.4.8, using Example 4.2.8, we found a power series, centered at $\pi/2$, which equals $\sin x$. In Example 4.6.6, we produced a power series, centered at $\pi/2$, which equals $\sin x$, by substituting $\frac{\pi}{2}-x$ into the

Maclaurin series for $\cos x$. What we found is that, despite the fact that we obtained the power series in completely different ways, we arrived at the same power series.

This may leave you wondering: "given a particular center a, can you have two **different** power series centered at a which converge on some open interval around a and which define the **same** function?" Or, putting it another way: "if $p(x)$ equals a power series, centered at a, on some open interval around a, is that power series centered at a unique?"

As we shall see in Corollary 4.6.8, for a fixed center, if a function equals a power series on an open interval around the center, then that power series is unique, for the given center. Of course, changing the center will change the power series.

The uniqueness of the power series representation of a function follows as a corollary to the theorem below, which tells us that you differentiate a power series as though it's a polynomial, i.e., by using linearity and the Power Rule on each term.

Theorem 4.6.7. (**Differentiating Power Series**) *Let R be the radius of convergence of $p(x)=\sum_{k=0}^{\infty} c_k(x-a)^k$ and let I denote the interval of convergence.*

Then, the radius of convergence of the series $\sum_{k=1}^{\infty} kc_k(x-a)^{k-1}$, obtained from $p(x)$ from term by term differentiation, is also R and, for all x in the open interval $(a-R, a+R)$,

$$p'(x) \;=\; \sum_{k=1}^{\infty} kc_k(x-a)^{k-1}.$$

If the series for $p(x)$ diverges at one or both of the endpoints of I, then so does the series for $p'(x)$. However, convergence of the series for $p(x)$ at an endpoint of I does **not** *necessarily imply convergence of the series for $p'(x)$ at that endpoint.*

Proof. See Theorem 4.5.5 of [4]. □

Note that the new summation in Theorem 4.6.7 begins at $k=1$, since the $k=0$ term c_0 in $p(x)$ has 0 for its derivative.

After you apply Theorem 4.6.7 to a series $p(x)$, you have a new power series for $p'(x)$, which again converges for all x in the same open interval I. Thus, you can differentiate $p'(x)$ and its power series, term by term, to produce a new power series for $p''(x)$. You can iterate this process any number of times, n, in order to produce a power series for $p^{(n)}(x)$; this power series will still be centered at a, converge to $p^{(n)}(x)$ on the open interval I, and have as its constant term $n!c_n$. The real proof of this last fact is by induction, but it is easy to see by writing out the first few derivatives.

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \cdots .$$

$$p'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \cdots .$$

$$p''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + 5\cdot 4c_5(x-a)^3 + \cdots .$$

$$p'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots .$$

$$p^{(4)}(x) = 4\cdot 3\cdot 2c_4 + 5\cdot 4\cdot 3\cdot 2c_5(x-a) + \cdots .$$

Now, the constant term of a power series is exactly what you get when you evaluate the series at the center, $x = a$, since then all of the $(x-a)$'s are 0. Thus, for all $n \geq 0$, $p^{(n)}(a) = n!c_n$. At this point, we can write k in place of n, if we want.

Thus, we obtain:

Corollary 4.6.8. *Suppose that $p(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ converges on an open interval I, which contains a.*

Then, $p(x)$ is infinitely differentiable on I and, for all k, $c_k = \dfrac{p^{(k)}(a)}{k!}$, and, hence, the original power series must, in fact, be the Taylor series for $p(x)$, centered at a.

In particular, if a function equals a power series, centered at a, on an open interval around a, then that power series, centered at a, is unique.

Remark 4.6.9. You should have learned in high school that, if two polynomials are equal, then all of the coefficients of the various powers of x must be the same for the two polynomials. It may not have been clear to you at the time, but what they were telling you was that, if two polynomials define the same **function**, then the coefficients of each power of x must be the same, i.e., the abstract algebraic polynomials must be the same.

The uniqueness statement in Corollary 4.6.8 is a generalization of this. It tells you that, if you have an equality of power series **functions**

$$\sum_{k=0}^{\infty} b_k(x-a)^k \;=\; \sum_{k=0}^{\infty} c_k(x-a)^k,$$

for all x in an open interval around a, then, in fact, the coefficients in front of the various powers of $(x-a)$ must be the same, i.e., for all k, $b_k = c_k$.

Example 4.6.10. Theorem 4.6.7 and Corollary 4.6.8 tell us that we could have obtained the Maclaurin series for $\cos x$ by differentiating, term by term, the Maclaurin series for $\sin x$.

We have

$$\sin x \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

and so

$$\cos x \;=\; \sum_{k=0}^{\infty}(-1)^k(2k+1)\frac{x^{2k}}{(2k+1)!} \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} \;=$$

$$1 - 3\cdot\frac{x^2}{3!} + 5\cdot\frac{x^4}{5!} - 7\cdot\frac{x^6}{7!} + \cdots \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

which, of course, is what we knew we'd get.

However, we can also use differentiation to obtain equalities that we didn't know before. For instance, we know that, for $-1 < x < 1$,

$$\frac{1}{1-x} \;=\; \sum_{k=0}^{\infty} x^k \;=\; 1 + x + x^2 + x^3 + \cdots.$$

Taking derivatives, we obtain that, for $-1 < x < 1$,

$$\frac{1}{(1-x)^2} \;=\; \sum_{k=1}^{\infty} kx^{k-1} \;=\; 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

Remark 4.6.11. This is a good time to remind you of Remark 4.2.12. Suppose that f is infinitely differentiable and defined at 0. If f is an even function, then only even-powered terms appear in the Maclaurin series of f. If f is an odd function, then only odd-powered terms appear in the Maclaurin series of f. This is true whether or not f is actually equal to its Maclaurin series on an open interval containing 0. The proof of this was outlined for you in Exercise 37 of Section 4.2.

You can make a stronger statement for power series functions. A power series function $p(x)$, centered at 0, with a non-zero radius of convergence, is even (respectively, odd) if and only if the only terms which appear (with non-zero coefficients) in $p(x)$ are the terms whose power is even (respectively, odd). You are asked to show this in Exercise 39.

One slightly subtle point is that, for a function to be even or odd, for every x in the domain, it is required that $-x$ is also in the domain. For a power series function, centered at 0, this means that if one endpoint of the interval of convergence I is included in I, then the other endpoint must also be included in I. That is, for power series, centered at 0, containing only even, or only odd terms, the interval of convergence is either of the form $(-R, R)$ or $[-R, R]$.

Example 4.6.12. Given a fixed center, the fact that power series representations of functions are unique, and must be the Taylor series, can save you a **lot** of work.

Consider the function $f(x) = e^{(x^2)} = e^{x^2}$.

Suppose that you want to calculate the Maclaurin series of f barehandedly, i.e., from the definition of Maclaurin/Taylor series. To do this, you need $f^{(k)}(0)$, for all k, but the higher-order derivatives of f get pretty ugly pretty fast.

$$
\begin{aligned}
f'(x) &= 2xe^{x^2}, \\
f''(x) &= 2(x \cdot 2xe^{x^2} + e^{x^2}) = 2(2x^2+1)e^{x^2}, \\
f'''(x) &= 2\big[(2x^2+1)2xe^{x^2} + 4xe^{x^2}\big] = 4x(2x^2+3)e^{x^2},
\end{aligned}
$$

and so on.

However, Corollary 4.6.8 tells us that, if we can produce **by any means** an equality between e^{x^2} and a power series, centered at 0, on an open interval, then that series **must** be the Maclaurin series of e^{x^2}.

That certainly makes things easier. By substituting x^2 in for x in the Maclaurin series equality for e^x, Theorem 4.4.5, we obtain, for all x,

$$e^{x^2} \;=\; \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} \;=\; 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots,$$

for all x. Hence, this series must be the Maclaurin series for e^{x^2}. Nice, huh?

Using Theorem 4.6.7 in reverse tells us how to anti-differentiate a power series function $p(x)$, **provided** that we know that the series obtained from term-by-term anti-differentiation of the series $P(x)$ actually converges. Once we know how to anti-differentiate, then, since power series functions are continuous by Theorem 4.6.1, we can apply the Fundamental Theorem, Theorem 2.4.10, and calculate the definite integral between two points x_0 and x_1 in the interval of convergence of $p(x)$ by evaluating $P(x_1) - P(x_0)$.

Theorem 4.6.13. (**Integrating Power Series**) *Let R be the radius of convergence of $p(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ and let I denote the interval of convergence.*

Then, for every constant C, the radius of convergence of the power series $P(x) = C + \sum_{k=0}^{\infty} \frac{c_k}{k+1}(x-a)^{k+1}$, obtained from $p(x)$ from term-by-term anti-differentiation, is also R, and, for all x in the open interval $(a-R, a+R)$,

$$\int p(x)\,dx \;=\; C + \sum_{k=0}^{\infty} \frac{c_k}{k+1}(x-a)^{k+1}.$$

If $p(x)$ converges at one or both of the endpoints of I, then so does $P(x)$; hence, $P(x)$ exists and is continuous on I, making $P(x)$ an anti-derivative of $p(x)$ on I (Definition 2.4.5). However, divergence of $p(x)$ at an endpoint of I does **not** *necessarily imply divergence of $P(x)$ at that endpoint.*

Proof. See Theorem 4.5.8 of [4]. □

Remark 4.6.14. Note that, while differentiation may destroy convergence at the endpoints of the interval of convergence, integration can lead to convergence at the endpoints, even when the original series diverged at the endpoints of the interval of convergence.

Example 4.6.15. Let's look again at our old friend, the basic geometric series: For $-1 < x < 1$,

$$\frac{1}{1-x} \;=\; \sum_{k=0}^{\infty} x^k \;=\; 1 + x + x^2 + x^3 + \cdots.$$

We'll first make a substitution; we'll replace all of the x's with $-x$'s, including the x in the inequality. We obtain:

for $-1 < -x < 1$,

$$\frac{1}{1+x} \;=\; \sum_{k=0}^{\infty} (-x)^k \;=\; \sum_{k=0}^{\infty} (-1)^k x^k \;=\; 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots.$$

Multiplying the inequality throughout by -1, and reversing the inequalities, we see that the inequality still amounts to $-1 < x < 1$, and so the equalities above hold on this interval.

Now, we anti-differentiate both sides of the equality to obtain that, for $-1 < x < 1$,

$$\ln(1+x) \;=\; C + \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} \;=\; C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots,$$

for some constant C.

How do you determine C? Plug in the only x value where the series is easy to evaluate: the center, $x = 0$. We find

$$\ln(1+0) \;=\; C + 0 + 0 + 0 + \cdots.$$

Therefore, $C = 0$, and we have, for $-1 < x < 1$,

$$\ln(1+x) \;=\; \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots.$$

This is the proof of this equality for $-1 < x < 0$, which we stated, but didn't prove, back in Theorem 4.4.12. Note that this proof does not instantly give you that the equality also holds when $x = 1$, though our Taylor error proof in Theorem 4.4.12 **did** give us the equality at $x = 1$. If we ignore our earlier proof of the convergence of the alternating harmonic series and, instead, use another result (like the Alternating Series Test, Theorem 5.4.17 to show that the alternating harmonic series converges, we can obtain the $x = 1$ **equality** by using Abel's theorem on continuity, as we discussed in Remark 4.6.2.

Note that integration changed divergence at $x = 1$ in the original series into convergence at $x = 1$ for the integrated series. Of course, this means that term-by-term differentiation of the Maclaurin series for $\ln(1 + x)$ destroys convergence at $x = 1$.

Example 4.6.16. Let's start from one of our series equalities from the last example:

for $-1 < x < 1$,

$$\frac{1}{1+x} \;=\; \sum_{k=0}^{\infty}(-1)^k x^k \;=\; 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots .$$

This time, we substitute x^2 in for x to obtain:

for $-1 < x^2 < 1$,

$$\frac{1}{1+x^2} \;=\; \sum_{k=0}^{\infty}(-1)^k x^{2k} \;=\; 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots . \tag{4.14}$$

The condition that $-1 < x^2 < 1$ once again is equivalent to $-1 < x < 1$, and so we have the above equalities for all x in the open interval $(-1, 1)$.

Now, recalling that $(\tan^{-1} x)' = 1/(1 + x^2)$, let's anti-differentiate the equality in Formula 4.14 to obtain

$$\tan^{-1} x \;=\; C \;+\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} \;=\; C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots .$$

Once again, plug in the x value of the center of the series to determine C. We find that

$C = \tan^{-1}(0) = 0$, and so, we arrive at

$$\tan^{-1} x \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots, \tag{4.15}$$

for $-1 < x < 1$.

Of course, $\tan^{-1} x$ is continuous, and so, using Abel's theorem again, we can conclude that this last equality holds at $x = \pm 1$, provided that we know that the series actually converges at $x = \pm 1$. In fact, the same *Alternating Series Test*, Theorem 5.4.17, that tells us that the alternating harmonic series converges also tells us quickly that the series in Formula 4.15 converges at both $x = \pm 1$, and so the equality in Formula 4.15 holds for all x in the closed interval $[-1, 1]$.

One cool "application" of this is that it implies that

$$\frac{\pi}{4} \;=\; \tan^{-1}(1) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1} \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.$$

Hence,

$$\pi \;=\; 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right),$$

and we could use this to approximate π by using partial sums to estimate the infinite sum. This is actually not a very efficient method for approximating π. For instance, we find

$$\pi \;\approx\; 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}\right) \;\text{“}=\text{”}\; 3.041839618929,$$

which is not very close to 3.141592653590. To get reasonable accuracy, you would have to use a partial sum with many, many terms.

We now need to discuss algebraic operations with power series. What we shall see is that you add, subtract, multiply, and divide power series as though they are never-ending polynomials. However, the precise manner in which you deal with multiplication and division requires some discussion.

Theorem 4.6.17. *Suppose that* $p(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ *and* $q(x) = \sum_{k=0}^{\infty} d_k(x-a)^k$ *converge on an interval* I.

Then, for all x *in* I, *we have the following equality of functions:*

$$p(x) \; + \; q(x) \;=\; \sum_{k=0}^{\infty}(c_k+d_k)(x-a)^k,$$

where, in particular, we mean that the series converges for all x *in* I.

In words, we say that summations of two power series with the same center is performed "term-wise".

Proof. This follows at once from the analogous theorem for series of constants, Theorem 5.2.17, which is left to you as an easy exercise in Section 5.2. □

The following theorem basically says that multiplication distributes over addition, even if the addition is of an infinite number of things. It follows at once from distributing in the partial sums, and using basic limit properties.

Theorem 4.6.18. *Suppose that* $p(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ *converges on an interval* I, *and that* $f(x)$ *is a function defined on* I.

Then, for all x *in* I, $\sum_{k=0}^{\infty} f(x)c_k(x-a)^k$ *converges, and*

$$f(x)p(x) \;=\; \sum_{k=0}^{\infty} f(x)c_k(x-a)^k.$$

In particular, if $f(x) = r(x-a)^m$, *where* r *is a constant and* $m \geq 0$ *is an integer, then we obtain that* $r(x-a)^m p(x)$ *equals the power series* $\displaystyle\sum_{k=0}^{\infty} rc_k(x-a)^{k+m}$.

Example 4.6.19. Of course, you can combine Theorem 4.6.17 and Theorem 4.6.18 to find power series representations of various combinations of known series. For instance, for all x,

$$5\sin x + 3x\cos x \;=\; 5\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!} \;+\; 3x\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} \;=\; \sum_{k=0}^{\infty}\left[\frac{5}{(2k+1)!}+\frac{3}{(2k)!}\right]x^{2k+1} \;=$$

$$8x \;+\; \left(\frac{5}{3!}+\frac{3}{2!}\right)x^3 \;+\; \left(\frac{5}{5!}+\frac{3}{4!}\right)x^5 \;+\; \left(\frac{5}{7!}+\frac{3}{6!}\right)x^7 \;+\; \cdots.$$

Note that, as we discussed in Remark 4.6.11, the odd function $5\sin x+3x\cos x$ has a Maclaurin series contained only odd-powered terms.

Example 4.6.20. Power series manipulations can help you calculate limits.

For instance, for all $x \neq 0$,

$$\frac{e^x-1-x}{x^2} \;=\; \frac{1}{x^2}\cdot\left[(-1-x)+1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right] \;=$$

$$\frac{1}{x^2}\cdot\left[\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\cdots\right] \;=\; \frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots.$$

Therefore, as power series are continuous, we find that

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} \;=\; \lim_{x\to 0}\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\right) \;=\; \frac{1}{2!}+0+0+\cdots \;=\; \frac{1}{2}.$$

Example 4.6.21. In this example, we “cheat” and must use that Calculus and the theory of infinite series work perfectly well when you use complex numbers in place of real numbers. This means that, in this example, we want to allow numbers of the form $a+bi$, where a and b are real numbers, and i is a square root of -1, i.e., $i^2=-1$.

Note that, when you look at positive integral powers of i, what you get repeats every four steps:

$$i^1=i,\qquad i^2=-1,\qquad i^3=i^2\cdot i=-i,\qquad i^4=(-1)i^2=(-1)(-1)=1,$$

$$i^5 = i \cdot i^4 = i \cdot 1 = i, \qquad i^6 = i^2 \cdot i^4 = i^2 \cdot 1 = i^2 = -1, \qquad \text{etc.}$$

Now, let's replace the x in

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

by $i\theta$, where we're thinking of θ being a real number (but it doesn't have to be). In fact, we're thinking of θ as an angle (hence, the use of variable θ); we shall see why in a minute. Using what we wrote about how the powers of i work, we obtain

$$e^{i\theta} \;=\; 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \frac{i^7\theta^7}{7!} + \cdots \;=$$

$$1 + i\theta - \frac{\theta^2}{2!} - i\,\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\,\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\,\frac{\theta^7}{7!} + \cdots \;=$$

$$\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) \;+\; i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \;=$$

$$\cos\theta \;+\; i\,\sin\theta.$$

This fundamental result, that

$$\boxed{e^{i\theta} \;=\; \cos\theta \;+\; i\,\sin\theta \qquad (4.16)}$$

is of great importance in many areas of mathematics, physics, and electrical engineering; this equality is known as *Euler's Formula*.

As a quick example of the usefulness of Euler's Formula, let's see how easily it lets you derive the angle addition formulas for sine and cosine. We will use that, if a, b, c and d are real numbers, then $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Suppose that α and β are real numbers. Euler's Formula tells us that

$$e^{i\alpha} \;=\; \cos\alpha \;+\; i\,\sin\alpha \qquad \text{and} \qquad e^{i\beta} \;=\; \cos\beta \;+\; i\,\sin\beta.$$

Multiplying, we obtain

$$e^{i\alpha} \cdot e^{i\beta} \;=\; \cos\alpha\cos\beta + i^2\sin\alpha\sin\beta + i\cos\beta\sin\alpha + i\cos\alpha\sin\beta,$$

and so

$$e^{i(\alpha+\beta)} \;=\; \big(\cos\alpha\cos\beta - \sin\alpha\sin\beta\big) + i\big(\cos\beta\sin\alpha + \cos\alpha\sin\beta\big).$$

Applying Euler's Formula one more time to $e^{i(\alpha+\beta)}$, we find that

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) \;=\; \big(\cos\alpha\cos\beta - \sin\alpha\sin\beta\big) + i\big(\cos\beta\sin\alpha + \cos\alpha\sin\beta\big)$$

and, therefore, matching the parts without the i's (the *real parts*) and the parts with the i's (the *imaginary parts*), we obtain the well-known angle addition formulas:

$$\cos(\alpha+\beta) \;=\; \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and

$$\sin(\alpha+\beta) \;=\; \cos\beta\sin\alpha + \cos\alpha\sin\beta.$$

So cool.

Example 4.6.22. Now that we have Euler's Formula, we can also give the close relationship between hyperbolic sine and cosine, from Definition 1.4.1, and the usual, circular, sine and cosine.

Euler's Formula tells us that $e^{i\theta} = \cos\theta + i\sin\theta$. Replacing θ with $-\theta$, and using that cosine is an even function and sine is an odd function, we find that $e^{-i\theta} = \cos\theta - i\sin\theta$.

Therefore, we obtain

$$\cosh(i\theta) \;=\; \frac{e^{i\theta} + e^{-i\theta}}{2} \;=\; \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} \;=\; \cos\theta$$

and

$$\sinh(i\theta) \;=\; \frac{e^{i\theta} - e^{-i\theta}}{2} \;=\; \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2} \;=\; i\sin\theta.$$

Or, replacing θ with $-i\theta$, and using $i^2 = -1$ and the even- and odd-ness of cosine and sine, we have

$$\cosh\theta \;=\; \cos(i\theta) \quad \text{and} \quad \sinh\theta \;=\; -i\sin(i\theta).$$

There are still three important operations with power series that we want to look at: multiplying two power series with the same center, dividing two power series with the same center, and composing power series, which, in fact, is best discussed in terms of real analytic functions (Definition 4.5.20).

Example 4.6.23. Consider the two polynomials, centered at a:

$$p(x) = b_0+b_1(x-a)+b_2(x-a)^2+b_3(x-a)^3 \;\;\text{and}\;\; q(x) = c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3.$$

Multiplying $p(x)$ times $q(x)$, and collecting powers of $(x-a)$, you find

$$p(x)\cdot q(x) \;=\; b_0c_0 \;+\; (b_0c_1+b_1c_0)(x-a) \;+\; (b_0c_2+b_1c_1+b_2c_0)(x-a)^2 \;+$$

$$(b_0c_3+b_1c_2+b_2c_1+b_3c_0)(x-a)^3 \;+\; \text{higher-degree terms}$$

You shouldn't memorize this, but hopefully you see the pattern: the coefficient in front of the $(x-a)^n$ term in the product is a sum of the products of coefficients of the two individual power series, and the indices of the coefficients from $p(x)$ and $q(x)$ always add up to n, the exponent of $(x-a)^n$. It is not hard to understand why this is true: to obtain an $(x-a)^n$ term in the product, you can take any term from $p(x)$ of degree k, where $0 \leq k \leq n$, and multiply it by a term from $q(x)$ of degree $n-k$, i.e., take $b_k(x-a)^k \cdot c_{n-k}(x-a)^{n-k} = b_kc_{n-k}(x-a)^n$. Of course, you get a summand of this form for each k such that $0 \leq k \leq n$.

This pattern actually continues where we have written "higher-degree terms"; it's just that many of the coefficients involved in the terms of degree ≥ 4 are zero, since those terms are zero (i.e., don't appear) in $p(x)$ and $q(x)$.

The same process that we used in the above example applies to infinite series.

Theorem 4.6.24. *Suppose that* $p(x) = \sum_{k=0}^{\infty} b_k(x-a)^k$ *and* $q(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ *converge on an open interval* I*.*

Then, for all x *in* I*, we have the following equality of functions:*

$$p(x) \cdot q(x) \;=\; \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_k c_{n-k} \right) (x-a)^n,$$

where, in particular, we mean that the series converges for all x *in* I*.*

Proof. This follows from Theorem 5.4.11, combined with the fact that power series converge absolutely on the interior of the interval of convergence; see Theorem 4.5.9. $\square$

Example 4.6.25. Find the 5th order Maclaurin polynomial of $e^x \sin x$, and use it to approximate $e^{0.1}\sin(0.1)$.

Solution:

We could do this "simply" by calculating the 5th order Maclaurin polynomial from the definition, Definition 4.2.2. However, instead, we will use Theorem 4.6.24 and just do the multiplication out to the degree 5 term.

We have

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

and

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots .$$

Don't get confused by the missing terms in the $\sin x$ series; if it helps, explicitly insert zero times each of the even powers of x. However, you shouldn't really have to write those terms in – just keep in mind that they're there or, equivalently, that when a term is missing, you don't get the corresponding product. Think: How do you get a degree 0 term in the product? You have to multiply a degree 0 term from e^x times a degree 0 term from $\sin x$; but there is no degree 0 term in the $\sin x$ series, so you don't get a degree 0 term in the product. How do you get a degree 1 term in the product? By multiplying a degree 0 term from e^x times a degree 1 term from $\sin x$ – yielding $1 \cdot x$ – and/or by multiplying a degree 1 term from e^x times a degree 0 term from $\sin x$ – but this portion doesn't appear since there is no degree 0 term from $\sin x$. Just continue in this manner.

What you should find is (we've included the 0's, in case you thought of it like that):

$$e^x \sin x \;=\; 0 + (1+0)x + (0+1+0)x^2 + \left(-\frac{1}{3!} + 0 + \frac{1}{2!} + 0\right)x^3 + \left(0 - \frac{1}{3!} + 0 + \frac{1}{3!} + 0\right)x^4 +$$

$$\left(\frac{1}{5!} + 0 - \frac{1}{2!}\cdot\frac{1}{3!} + 0 + \frac{1}{4!} + 0\right)x^5 + \cdots .$$

Simplifying, we find that the 5th order Maclaurin polynomial of $e^x \sin x$ is

$$x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5.$$

As 0.1 is reasonably close to the center 0, we expect that a decent approximation of $e^{0.1}\sin(0.1)$ is given by

$$0.1 + (0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{30}(0.1)^5 \;=\; 0.110333.$$

The calculator value of $e^{0.1}\sin(0.1)$ is 0.110332988730, so our approximation looks very good.

Now let's look at division.

Example 4.6.26. Knowing the Maclaurin series for e^x, $\sin x$, and $\cos x$, can we determine Maclaurin series or, at least, the 4th order Maclaurin polynomials, for $e^x/\sin x$ and $e^x/\cos x$ in an algebraic manner, rather than by taking lots of fairly ugly derivatives?

For $e^x/\sin x$, the answer is certainly **no**; you can't have a Maclaurin series for this function because the function itself is undefined at $x = 0$. More generally, if you have two power series centered at a, $p(x)$ and $q(x)$, and you want a power series centered at a which equals $p(x)/q(x)$, you at least need $q(a) \neq 0$, i.e., if $q(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$, you need $c_0 \neq 0$.

So, what about $e^x/\cos x$? Can we long divide one series into another to find

$$\frac{1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots} \;=\; d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4 + \cdots ?$$

The answer is that, yes, we could describe long division, but it's actually nicer to work backwards from the multiplication. That is, we're going to solve for the coefficients d_k that make the following equality hold

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots \;=\; \left(d_0+d_1x+d_2x^2+d_3x^3+d_4x^4+\cdots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right). \tag{4.17}$$

Calculating the first few terms in the product, we find

$$\left(d_0+d_1x+d_2x^2+d_3x^3+d_4x^4+\cdots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right) \;=$$

$$d_0+d_1x+\left(-\frac{1}{2!}+d_2\right)x^2+\left(-\frac{d_1}{2!}+d_3\right)x^3+\left(\frac{1}{4!}-\frac{d_2}{2!}+d_4\right)x^4+\cdots.$$

Now, we set this equal to the Maclaurin series for e^x, as in Formula 4.17, match the coefficients, and get 5 equations and 5 unknowns, but the equations are easy to solve:

$$d_0 \;=\; 1, \qquad d_1 \;=\; 1, \qquad -\frac{1}{2!}+d_2 \;=\; \frac{1}{2!}, \qquad -\frac{d_1}{2!}+d_3 \;=\; \frac{1}{3!},$$

and

$$\frac{1}{4!}-\frac{d_2}{2!}+d_4 \;=\; \frac{1}{4!}.$$

Solving the equations, in order, each new equation gives you a single new unknown, because you've already solved for the others. Therefore, it's easy to obtain

$$d_0 \;=\; 1, \qquad d_1 \;=\; 1, \qquad d_2=1, \qquad d_3=\frac{2}{3}, \qquad \text{and} \qquad d_4=\frac{1}{2}.$$

Thus, if $f(x)=\dfrac{e^x}{\cos x}$, then

$$T_f^4(x) \;=\; 1 \;+\; x \;+x^2 \;+\; \frac{2}{3}x^3 \;+\; \frac{1}{2}x^4.$$

We should remark that the Maclaurin/Taylor series obtained by division has a non-zero radius of convergence, provided that the two series being divided do also; however, the radius of convergence of the quotient series may be strictly smaller than those of the series in the numerator and denominator. In particular, there is a problem if the denominator hits 0. But, assuming that the power series $q(x)$, centered at a, has a non-zero radius of convergence and that $q(a) \neq 0$, then the continuity of power series functions implies that there is an open interval I, containing a, such that, for all x in I, $q(x) \neq 0$.

We now state the theorem which summarizes what we saw in the example above; see [4], pages 269-270, for a discussion.

Theorem 4.6.27. *Suppose that* $p(x) = \sum_{k=0}^{\infty} b_k(x-a)^k$, $q(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$, *that both series have a positive radius of convergence, and that* $c_0 \neq 0$.

Then, there is a unique power series, $r(x) = \sum_{k=0}^{\infty} d_k(x-a)^k$, *centered at* a, *which converges on an open interval* I, *containing* a, *such that, for all* x *in* I, $q(x) \neq 0$ *and*

$$r(x) \;=\; \frac{p(x)}{q(x)}.$$

The series $r(x)$ *is determined by the equality* $p(x) = r(x)q(x)$.

As the final topic for this section, we want to discuss composition of power series functions. In many ways, this topic is awful to think about.

Suppose you have two power series, $p(x) = \sum_{k=0}^{\infty} b_k(x-a)^k$ and $q(x) = \sum_{j=0}^{\infty} c_j(x-b)^j$, and that they converge on open intervals I and J, respectively, containing their respective centers.

Can you obtain a new convergent powers series for $p(q(x))$ by algebraically manipulating the given two series? Well...consider what you get when you substitute the series for $q(x)$ in for the x in the series for $p(x)$; you get

$$\sum_{k=0}^{\infty} b_k \left(-a + \sum_{j=0}^{\infty} c_j(x-b)^j\right)^k.$$

So, we have an infinite sum of arbitrarily large powers of a series. Now, we could raise a series to any power by iterating how we multiplied two series; this would be truly horrible.

Think about calculating

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)^4$$

in this manner. But, even if we were willing to do that for each integral power, then we'd need to multiply by constants and add together an infinite number of infinite series.

Even if we were willing and able to do this, we have ignored the issue of where the new series should be centered. Of course, we have indicated in the notation that we're allowing different centers for $p(x)$ and $q(x)$, but, even if we took them both to be centered at a, it's unclear that a is the best place to center the composition. In fact, discussing the centers of the series should make you think about possible problems with the domains and codomains involved in the composition.

We specified the domains of p and q as I and J, respectively, and power series functions have the entire real line, $\mathbb{R}$, as their codomains by default. Thus, we are discussing composing $p : I \to \mathbb{R}$ and $q : J \to \mathbb{R}$, but this may not be possible – for $p(q(x))$ to be defined, we would need for the value $q(x)$ to be in I. Thus, we need to restrict the codomain of q to I, which requires us to restrict the domain of q to the set of those x's such that $q(x)$ is in I; this set is denoted $q^{-1}(I)$. However, the set $q^{-1}(I)$ need not be an interval; it may consist of a union of non-intersecting intervals, and there's no reason to expect that the original center of the power series $q(x)$ will be in the restricted domain given by $q^{-1}(I)$. Therefore, there is no natural center for the composition.

Ugh – composition of power series functions is a mess! The calculations that we'd have to perform are practically impossible and, in the general case, it's not even theoretically possible to make sense of the thing that we're trying to calculate.

However, recall our definition of real analytic functions from Definition 4.5.20. A large part of the point of a real analytic function is that, while it can locally be written as a power series, the center of the power series that it equals is allowed to move. This makes a statement about compositions of analytic functions easy, but you should understand that this easy statement, while being theoretically nice, does **not**, in any way, make it easy to actually **calculate** power series representations of compositions of real analytic, or power series, functions.

Theorem 4.6.28. *Compositions of real analytic functions are real analytic.*

More precisely, if $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ are open subsets of real numbers, and $g : \mathcal{U} \to \mathcal{V}$ and $f : \mathcal{V} \to \mathcal{W}$ are real analytic functions, then $f \circ g : \mathcal{U} \to \mathcal{W}$ is a real analytic function.

Example 4.6.29. So, for instance, Theorem 4.6.28 tells us that $\sin(e^x)$ and $e^{\cos x}$ are real analytic on the entire real line, i.e., for all real numbers a, these functions are equal to their Taylor series, centered at a in some open interval around a.

However, Theorem 4.6.28 gives you no simple way of finding power series representations of these functions. If you want the first few terms of the Maclaurin series of these functions, denote either by $f(x)$, it's probably easiest simply to use the definition of the Maclaurin series and calculate the coefficients $f^{(k)}(0)/k!$.

4.6.1 Exercises

In each of Exercises 1 through 12, (a) manipulate the known Maclaurin series representations/equalities of e^x, $\sin x$, $\cos x$, $1/(1-x)$, $\ln(1+x)$, and $(1+x)^p$ to find Taylor series, centered at the given a, which equal the given function $f(x)$ on some open interval containing a, and give the largest open interval on which you're certain that the equality holds, and (b) use the first three non-zero terms of your Taylor series from part (a) to estimate the value of $f(x)$ at the given x, and compare with the answer from a calculator. In part (a), "give" the series either by writing the general series using summation notation, or by simply writing out the first 5 non-zero terms and then $+\cdots$.

1. $f(x) = 3\cos x - 5\sin x$, $a = 0$, $x = 0.1$
2. $f(x) = e^{x-2} + \sin(x-2)$, $a = 2$, $x = 2.1$
3. $f(x) = \dfrac{1}{1-4(x-2)^3}$, $a = 2$, $x = 1.9$
4. $f(x) = e^{x+1} + (2+x)^{1/2}$, $a = -1$, $x = -1.2$
5. $f(x) = x\ln(1+x^2)$, $a = 0$, $x = 0.2$
6. $f(x) = (x-1)\ln x$, $a = 1$, $x = 1.1$
7. $f(x) = \tan^{-1}(x+2)$, $a = -2$, $x = -2.1$
8. $f(x) = x\tan^{-1}(x+2) = (x+2)\tan^{-1}(x+2) - 2\tan^{-1}(x+2)$, $a = -2$, $x = -2.1$
9. $f(x) = 1/(1-x)^3$, $a = 0$, $x = -0.1$ (Hint: Let $f(x) = 1/(1-x)$ and consider $f''(x)$.)

10. $f(x) = \sqrt[3]{8+x^3}$, $a=0$, $x=0.1$

11. $f(x) = \cos(\sqrt{x-4})$, $a=4$, $x=4.01$

12. $f(x) = \displaystyle\int_0^x \frac{t^2}{\sqrt{1+t^2}}\,dt$, $a=0$, $x=0.1$

In each of Exercises 13 through 21, differentiate and anti-differentiate the given series, giving the result in summation notation (preceded by a big $C+$ for the anti-derivative) and writing out the first 5 (not counting the $C+$) non-zero terms of the summations. Give the center and radius of convergence of the original series, and of the derivative and anti-derivative series; note that the given series are those from Exercises 13 through 21 of Section 4.5.

13. $p(x) \;=\; \displaystyle\sum_{k=1}^{\infty} \frac{5^k(x-1)^k}{k^2} \;=\; 5(x-1) \;+\; \frac{5^2}{2^2}(x-1)^2 \;+\; \frac{5^3}{3^2}(x-1)^3 \;+\; \frac{5^4}{4^2}(x-1)^4 \;+\; \cdots.$

14. $p(x) \;=\; \displaystyle\sum_{k=1}^{\infty} \sqrt{k}\,(x+3)^k \;=\; \big(x+3\big) \;+\; \sqrt{2}\,\big(x+3\big)^2 \;+\; \sqrt{3}\,\big(x+3\big)^3 \;+\; \sqrt{4}\,\big(x+3\big)^4 \;+\; \cdots.$

15. $p(x) \;=\; \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k^3+2} \;=\; \frac{1}{2} \;-\; \frac{x}{1^3+2} \;+\; \frac{x^2}{2^3+2} \;-\; \frac{x^3}{3^3+2} \;+\; \frac{x^4}{4^3+2} \;-\; \cdots.$

16. $p(x) \;=\; \displaystyle\sum_{k=0}^{\infty} \frac{k!(x-4)^k}{7^{k+1}}.$

17. $p(x) \;=\; \displaystyle\sum_{k=0}^{\infty} \frac{(x-4)^k}{7^{k+1}}.$

18. $p(x) \;=\; \displaystyle\sum_{k=1}^{\infty} \frac{(x-4)^k}{7^{k+1}k}.$

19. $p(x) \;=\; \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x+7)^k}{k^k}.$

20. $p(x) \;=\; \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k!(x+7)^k}{k^k}.$

21. $p(x) \;=\; \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{\sqrt[3]{k}}.$

In each of Exercises 22 through 27, write out the first 5 non-zero terms of the product of the two given series, using the same (common) center as the given series.

22. $p(x) = \sum_{k=0}^{\infty} \frac{2^k(x-1)^k}{k!}$ and $q(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(x-1)^k}{k!}$

23. $p(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$ and $q(x) = \sum_{k=0}^{\infty} x^k$

24. $p(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ and $q(x) = \sum_{k=0}^{\infty} x^k$

25. $p(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ and $q(x) = \sum_{k=0}^{\infty} x^k$

26. $p(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-3)^k}{\sqrt{k}}$ and $q(x) = p(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-3)^k}{\sqrt{k}}$

27. $p(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-3)^k}{\sqrt{k}}$ and $q(x) = \sum_{k=0}^{\infty} \frac{(x-3)^k}{k!}$

28. Multiply the Maclaurin series for $\sin x$ and $\cos x$ together, and write out the first 5 non-zero terms. You should get the Maclaurin series for $a\sin(bx)$, for some constants a and b. What are a and b?

In each of Exercises 29 through 34, write out the first 4 non-zero terms of $p(x)/q(x)$, using the same (common) center as the given series.

29. $p(x) = \sum_{k=0}^{\infty} \frac{2^k(x-1)^k}{k!}$ and $q(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(x-1)^k}{k!}$

30. $p(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$ and $q(x) = \sum_{k=0}^{\infty} x^k$

31. $p(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ and $q(x) = \sum_{k=0}^{\infty} x^k$

32. $p(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ and $q(x) = \sum_{k=0}^{\infty} x^k$

33. $p(x) = 1 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-3)^k}{\sqrt{k}}$ and $q(x) = p(x) = 1 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-3)^k}{\sqrt{k}}$

34. $p(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-3)^k}{\sqrt{k}}$ and $q(x) = \sum_{k=0}^{\infty} \frac{(x-3)^k}{k!}$

35. What quadratic polynomial best approximates $\cos(\sqrt{x})$, when x is close to 0?

36. What cubic polynomial best approximates $\tan^{-1}(x+5)$, when x is close to -5?

37. What polynomial of degree 5 best approximates $x\ln(1+x^2)$, when x is close to 0?

38. What polynomial of degree 4 best approximates $e^{(x-1)^2}$, when x is close to 1?

39. In this exercise, you are asked to prove the statements made in Remark 4.6.11

    a. Suppose that a power series, centered at 0, with radius of convergence R, has only even-powered terms or only odd-powered terms. Prove that the interval of convergence is either $(-R, R)$ or $[-R, R]$.

    b. Prove that a power series function $p(x)$, centered at 0, with a non-zero radius of convergence, is even (respectively, odd) if and only if the only terms which appear (with non-zero coefficients) in $p(x)$ are the terms whose power is even (respectively, odd).

40. Recall Remark 1.1.23: the function $f(x) = e^{-x^2}$ has no elementary anti-derivative. However, elementary functions involve **finite** combinations of certain basic functions, and power series are **infinite** sums.

    a. Find a power series anti-derivative of e^{-x^2}.

    b. Use the 3rd order Maclaurin polynomial from your answer in part a) to approximate $\int_0^1 e^{-x^2}\,dx$.

41. The function $f(x) = \cos(x^2)$ has no elementary anti-derivative, but we can find a power series anti-derivative.

    a. Find a power series anti-derivative of $\cos(x^2)$.

    b. Use the 3rd order Maclaurin polynomial from your answer in part a) to approximate $\int_0^1 \cos(x^2)\,dx$.

42. The function $f(x) = \sin(x^2)$ has no elementary anti-derivative, but we can find a power series anti-derivative.

    a. Find a power series anti-derivative of $\sin(x^2)$.

    b. Use the 3rd order Maclaurin polynomial from your answer in part a) to approximate $\int_0^1 \sin(x^2)\,dx$.

43. In this exercise, we lead you through the steps of proving the Binomial Theorem, Theorem 4.4.14. Recall that $\binom{p}{0} = 1$ and, for $k \geq 1$, $\binom{p}{k} = p(p-1)(p-2)\cdots(p-k+1)/(k!)$.

a. Let
$$f(x) \;=\; \sum_{k=0}^{\infty}\binom{p}{k}x^k.$$
Use the Ratio Test to show that $f(x)$ converges (absolutely) for $|x| < 1$.

b. Calculate the Maclaurin series for $f'(x)$.

c. Show that, for $|x| < 1$,
$$(1+x)f'(x) \;=\; p\,f(x).$$

d. Solve the separable differential equation in the last part, and use the initial data that $f(0) = 1$, to show that, for $|x| < 1$, $f(x) \;=\; (1+x)^p$.

44. Square both sides of Euler's Formula, Formula 4.16, $e^{i\theta} \;=\; \cos\theta \;+\; i\,\sin\theta$. What well-known identities, using only real numbers, do you obtain for $\sin(2\theta)$ and $\cos(2\theta)$?

45. Cube both sides of Euler's Formula, Formula 4.16, $e^{i\theta} \;=\; \cos\theta \;+\; i\,\sin\theta$. What not-so-well-known identities, using only real numbers, do you obtain for $\sin(3\theta)$ and $\cos(3\theta)$?

4.7 Power Series Solutions of Differential Equations

One important application of power series functions is using them to find solutions to differential equations and/or initial value problems (IVP's). We discussed differential equations and IVP's at length in the final chapter of [2], before we had power series at our disposal.

How do power series help you solve differential equations and/or IVP's? You assume that there is a power series solution to the differential equation, centered at the x-coordinate of the initial data in the case of an IVP. You then plug in a power series with unknown coefficients, and this frequently leads to a manageable, but infinite, set of simultaneous equations to solve, or from which to derive a general formula for the coefficients. Your solution is then the power series that you end up with, or an approximate solution is given by using the first few non-zero terms.

Sometimes, you may recognize your power series solution as being equal to some combination of standard functions, i.e., you may recognize it as an elementary function when you're finished, but this is not something to be expected in general.

Example 4.7.1. Let's begin with a separable differential equation, one that we can solve explicitly, so that we can compare with the power series solution.

Consider the IVP:

$$\frac{dy}{dx} \;=\; xy; \qquad y(0) = 5.$$

The differential equation is separable. We find

$$\int \frac{1}{y}\,dy \;=\; \int x\,dx.$$

Thus, $\ln|y| = x^2/2 + C$ and, after some algebra, we find $y = Ae^{x^2/2}$, for some constant A. Plugging in the initial data, we find that $A = 5$, so that the unique solution to the IVP is

$$y = 5e^{x^2/2}.$$

Okay. So, how does the power series approach go?

We look for a power series solution to the IVP. Since the x value of the initial data is 0, we look for a power series centered at 0. Thus, we want to find the coefficients b_0, b_1, b_2, ... so that

$$y \;=\; y(x) \;=\; \sum_{k=0}^{\infty} b_k x^k \;=\; b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$

is a solution to the IVP.

First of all, we immediately know that we need $b_0 = 5$, because we have

$$5 \;=\; y(0) \;=\; b_0 + 0 + 0 + 0 + \cdots \;=\; b_0.$$

To find the rest of the b_k, we take our power series expression for y, and the differential equation

$$y' \;=\; xy$$

becomes an equality of power series functions, centered at 0. As power series functions are unique, by Corollary 4.6.8, and as we discussed in Remark 4.6.9, we can match coefficients on both sides of the equation, and solve for the b_k's. The only slight difficulty is in re-indexing the summations, but we'll get to that in a minute.

Since

$$y(x) \;=\; \sum_{k=0}^{\infty} b_k x^k,$$

we use the Power Rule, term-by term, and find

$$y' \;=\; \sum_{k=1}^{\infty} k b_k x^{k-1}.$$

If we want our index to be the power of x (as usual), then, as an intermediate step, let $j = k-1$, so that $k = j+1$. As k goes from 1 to ∞, j will go from 0 to ∞, and so

$$y' \;=\; \sum_{k=1}^{\infty} k b_k x^{k-1} \;=\; \sum_{j=0}^{\infty} (j+1) b_{j+1} x^j. \tag{4.18}$$

Now,

$$xy \;=\; x \cdot \sum_{k=0}^{\infty} b_k x^k \;=\; \sum_{k=0}^{\infty} b_k x^{k+1},$$

and, once again, if we want to index by the power of x, we let $j = k+1$, so that $k = j-1$, and this time we find

$$xy \;=\; \sum_{j=1}^{\infty} b_{j-1}x^j.$$

Combining this last equation with Formula 4.18, we find that our differential equation gives us

$$\sum_{j=0}^{\infty}(j+1)b_{j+1}x^j \;=\; \sum_{j=1}^{\infty} b_{j-1}x^j.$$

We would like to say, at this point, that we now match the coefficients of the powers of x on both sides of the equation. However, one summation starts at $j = 0$ and the other starts at $j = 1$. But this is no real problem; we simply split off the $j = 0$ term, and write

$$b_1 \;+\; \sum_{j=1}^{\infty}(j+1)b_{j+1}x^j \;=\; \sum_{j=1}^{\infty} b_{j-1}x^j.$$

Now we can match coefficients. The constant term on the left is b_1 and there is no constant term on the right; hence, $b_1 = 0$. For $j \geq 1$, the coefficient of x^j in the summation on the left must equal the coefficient of the x^j term of the summation on the right; thus, for $j \geq 1$, we have

$$(j+1)b_{j+1} \;=\; b_{j-1} \quad \text{or, equivalently,} \quad b_{j+1} \;=\; \frac{b_{j-1}}{j+1}, \tag{4.19}$$

and we already know that $b_0 = 5$ and $b_1 = 0$.

Equations like that in Formula 4.19 are frequently referred to as *iteration equations* and, when you have some of the initial b_k values, hopefully, the iteration equation is enough to give you the rest.

How does this work? Well...a real proof would be by using mathematical induction once or twice, but it's really not hard to see what's going on by looking at the first few coefficients.

Let's look at the odd-numbered coefficients first. We have $b_1 = 0$. Now, Formula 4.19, with $j = 2$, tells us that

$$b_3 \;=\; b_{2+1} \;=\; \frac{b_{2-1}}{2+1} \;=\; \frac{b_1}{3} \;=\; 0.$$

Putting $j = 4$ into the iteration formula gives us that $b_5 = b_3/5$, but we know that $b_3 = 0$. So, $b_5 = 0$. And so on. It's easy to see that the odd-numbered coefficients are all 0.

What about the even-numbered ones? We find, from the iteration formula,

$$b_2 \;=\; \frac{b_0}{2} \;=\; \frac{5}{2},$$

$$b_4 \;=\; \frac{b_2}{4} \;=\; \frac{5}{2\cdot 4} \;=\; \frac{5}{2^2\cdot 1\cdot 2},$$

$$b_6 \;=\; \frac{b_4}{6} \;=\; \frac{5}{2\cdot 4\cdot 6} \;=\; \frac{5}{2^3\cdot 1\cdot 2\cdot 3},$$

and

$$b_8 \;=\; \frac{b_6}{8} \;=\; \frac{5}{2\cdot 4\cdot 6\cdot 8} \;=\; \frac{5}{2^4\cdot 1\cdot 2\cdot 3\cdot 4}.$$

Hopefully, you see the pattern here; letting n equal half of the even j value, we have, for $n \geq 0$,

$$b_{2n} \;=\; \frac{5}{2^n n!}.$$

Thus, our power series solution to the IVP is

$$y \;=\; 5 \;+\; \frac{5}{2^1 1!}\,x^2 \;+\; \frac{5}{2^2 2!}\,x^4 \;+\; \frac{5}{2^3 3!}\,x^6 \;+\; \frac{5}{2^4 4!}\,x^8 \;+\; \cdots \;=\; \sum_{n=0}^{\infty} \frac{5}{2^n n!}\,x^{2n}.$$

If we had to leave our answer like this we would, but it's true that, even without looking back at our separable solution, you might notice that

$$y \;=\; 5\cdot \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^n}{n!} \;=\; 5e^{x^2/2},$$

as we found by separating the variables. When finding power series solutions to arbitrary differential equations, you should **not** expect to be able to easily (or, even with difficulty) recognize that your solution has a "closed form", i.e., has a form which clearly shows that the solution is an elementary function.

As you can see in this example, had we started with more general initial data $y(0) = y_0$, then y_0 would appear every place that we have a 5, i.e., y_0 would equal b_0, and we would have found that $y \;=\; y_0 e^{x^2/2}$.

But what if we had started with an initial x value other than 0? Suppose we had started with arbitrary initial data $y(x_0) = y_0$? In that case, you want to look for a power series solution which is now centered at x_0. Why? For two reasons. First, you want your power series solution to be easy to evaluate at x_0, and your typical power series is easy to evaluate only at the center. Second, in general, we only expect to solve initial value problems, and find a unique $y = y(x)$, for x values close to the initial x_0; for a power series solution, this means that we want to center at x_0 and find some non-zero, but possibly small, radius of convergence.

What happens if we have the IVP $y' = xy$, $y(x_0) = y_0$, and we take

$$y \;=\; \sum_{k=0}^{\infty} b_k (x - x_0)^k?$$

We once again find immediately that $b_0 = y_0$. That was easy. It's also easy to find

$$y' \;=\; \sum_{k=1}^{\infty} k b_k (x - x_0)^{k-1} \;=\; \sum_{j=0}^{\infty} (j+1) b_{j+1} (x - x_0)^j.$$

However, it's slightly more difficult to write xy as a power series centered at x_0, since, now, the x that we're multiplying by is itself not centered at x_0. We fix this by writing $x = (x - x_0) + x_0$.

Now,

$$xy \;=\; [(x - x_0) + x_0] \sum_{k=0}^{\infty} b_k (x - x_0)^k \;=$$

$$\sum_{k=0}^{\infty} b_k (x - x_0)^{k+1} \;+\; \sum_{k=0}^{\infty} x_0 b_k (x - x_0)^k \;=\; \sum_{j=1}^{\infty} b_{j-1} (x - x_0)^j \;+\; \sum_{j=0}^{\infty} x_0 b_j (x - x_0)^j.$$

Thus, our differential equation becomes

$$\sum_{j=0}^{\infty} (j+1) b_{j+1} (x - x_0)^j \;=\; \sum_{j=1}^{\infty} b_{j-1} (x - x_0)^j \;+\; \sum_{j=0}^{\infty} x_0 b_j (x - x_0)^j.$$

If we separate the two $j = 0$ terms, we obtain

$$b_1 \;+\; \sum_{j=1}^{\infty} (j+1) b_{j+1} (x - x_0)^j \;=\; x_0 b_0 \;+\; \sum_{j=1}^{\infty} (b_{j-1} + x_0 b_j)(x - x_0)^j.$$

Therefore, we obtain $b_0 = y_0$ (as we said before), $b_1 = x_0 b_0 = x_0 y_0$, and, for all $j \geq 1$,

$$(j+1)b_{j+1} \;=\; b_{j-1} \;+\; x_0 b_j,$$

that is,

$$b_{j+1} \;=\; \frac{b_{j-1} \;+\; x_0 b_j}{j+1}.$$

You may now use this iteration equation to generate as many of the b_k's as you wish. It is unlikely that you'll see a pattern; what you end up with should equal

$$y \;=\; y_0 e^{\frac{-x_0^2}{2}} e^{\frac{x^2}{2}},$$

which, when written in terms of $(x - x_0)$, is

$$y \;=\; y_0 e^{x_0(x-x_0)} \cdot e^{\frac{(x-x_0)^2}{2}}.$$

It would be difficult, indeed, to look at the power series solution, centered at x_0, and recognize the coefficients as those coming from this product.

Example 4.7.2. Let's look at a non-separable differential equation and IVP; one that we considered in Section 4.5 of [2].

Consider

$$y' \;=\; x - y; \qquad y(1) = -1.$$

We want to look for a power series solution, centered at the initial x value, 1.

So, suppose that

$$y \;=\; \sum_{k=0}^{\infty} b_k (x-1)^k.$$

Then, we immediately find that

$$-1 \;=\; y(1) \;=\; b_0 + 0 + 0 + 0 + \cdot,$$

so that

$$b_0 = -1.$$

Differentiating, we obtain

$$y' = \sum_{k=1}^{\infty} k b_k (x-1)^{k-1} = \sum_{j=0}^{\infty} (j+1) b_{j+1} (x-1)^j$$

and the differential equation becomes

$$\sum_{j=0}^{\infty} (j+1) b_{j+1} (x-1)^j = x - \sum_{j=0}^{\infty} b_j (x-1)^j.$$

We now rewrite x as $(x-1)+1$, in order to center it 1, and separate the degree 0 and 1 terms from the summations, to obtain

$$b_1 + 2b_2(x-1) + \sum_{j=2}^{\infty} (j+1) b_{j+1} (x-1)^j = 1 + (x-1) - b_0 - b_1(x-1) + \sum_{j=2}^{\infty} -b_j (x-1)^j.$$

Therefore,

$$b_1 = 1 - b_0 = 2,$$

$$2b_2 = 1 - b_1, \qquad \text{and so,} \qquad b_2 = -\frac{1}{2},$$

and, for all $j \geq 2$,

$$(j+1) b_{j+1} = -b_j, \qquad \text{or, equivalently,} \qquad b_{j+1} = -\frac{b_j}{j+1}.$$

Let's put off inserting that we know that $b_2 = -1/2$ for a moment, and look at the other b_k's.

When $j \geq 2$, we find

$$b_3 = -\frac{b_2}{3},$$

$$b_4 = -\frac{b_3}{4} = (-1)^2 \frac{b_2}{3 \cdot 4},$$

$$b_5 = -\frac{b_4}{5} = (-1)^3 \frac{b_2}{3 \cdot 4 \cdot 5},$$

and so on. The denominators look like factorials, but they're missing the multiplication by 2. However, that's easy to fix; multiply each numerator and denominator by 2. Then, inserting that $b_2 = -1/2$, you should see (but the real "proof" requires induction) that, for $k \geq 2$,

$$b_k \;=\; (-1)^{k-2}\,\frac{2b_2}{k!} \;=\; \frac{(-1)^{k-1}}{k!} \;=\; -\frac{(-1)^k}{k!}.$$

Therefore, our solution is

$$y \;=\; -1 \;+\; 2(x-1) \;+\; \sum_{k=2}^{\infty} -\frac{(-1)^k}{k!}\,(x-1)^k. \tag{4.20}$$

We can write this in a closed form if we really want, because the summation looks like something closely related to the Maclaurin series for e^x, and, in fact, it is.

Replacing the x with $-(x-1)$ in the equality between e^x and its Maclaurin series gives us

$$e^{-x+1} \;=\; e^{-(x-1)} \;=\; \sum_{k=0}^{\infty} \frac{[-(x-1)]^k}{k!} \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(x-1)^k.$$

Hence,

$$-e^{-x+1} \;=\; \sum_{k=0}^{\infty} -\frac{(-1)^k}{k!}(x-1)^k \;=\; -1 \;+\; (x-1) \;+\; \sum_{k=2}^{\infty} -\frac{(-1)^k}{k!}(x-1)^k.$$

Thus,

$$1 \;-\; (x-1) \;-e^{-x+1} \;=\; \sum_{k=2}^{\infty} -\frac{(-1)^k}{k!}(x-1)^k,$$

and we conclude, from Formula 4.20, that our solution to the IVP can be written as

$$y \;=\; y(x) \;=\; -1 \;+\; 2(x-1) \;+\; 1 \;-\; (x-1) \;-e^{-x+1} \;=\; x \;-\; 1 \;-\; e^{-x+1}.$$

We've done enough work at this point that there's been lots of room for small mistakes. Since it's relatively easy, it might be a good idea to check that our solution to the IVP really is a solution.

We find

$$y(1) \ = \ 1 - 1 - e^{-1+1} \ = \ 0 - 1 \ = \ -1,$$

which means that the initial data condition is satisfied. Also, if $y = x - 1 - e^{-x+1}$, then $y' = 1 + e^{-x+1}$, and it is easy to verify that $y' = x - y$.

Thus, we have indeed found a solution to the given initial value problem. In fact, we found the **unique** solution and, in this particular example, uniqueness is not difficult to establish. However, in more general IVP's, showing that the solution is unique is not so easy, and we will not go into the matter here.

What we **have** shown is that the unique **real analytic** solution at, or near, $x = 1$, is

$$y = x - 1 - e^{-x+1}.$$

Example 4.7.3. As our final example of this section, let's look at a simple 2nd order differential equation, with no initial conditions.

Consider

$$y'' \ = \ -4y,$$

and we'll look for a power series solution centered at 0.

So, suppose that $y = \sum_{k=0}^{\infty} b_k x^k$ is a solution to our differential equation. We quickly find

$$y' \ = \ \sum_{k=1}^{\infty} k b_k x^{k-1},$$

and

$$y'' \ = \ \sum_{k=2}^{\infty} k(k-1) b_k x^{k-2} \ = \ \sum_{j=0}^{\infty} (j+2)(j+1) b_{j+2} x^j.$$

Therefore, we need

$$\sum_{j=0}^{\infty} (j+2)(j+1) b_{j+2} x^j \ = \ \sum_{j=0}^{\infty} -4 b_j x^j.$$

It follows that $y = \sum_{k=0}^{\infty} b_k x^k$ will be a solution to $y'' = -4y$ if and only if, for all $j \geq 0$,

$$(j+2)(j+1)b_{j+2} \;=\; -4b_j, \qquad \text{or, equivalently,} \qquad b_{j+2} \;=\; -\frac{4b_j}{(j+2)(j+1)}.$$

As the iteration formula above refers to an index which is 2 bigger on the left than the only b term on the right, there will be one formula for the odd-indexed coefficients and another formula for the even-indexed coefficients. In addition, b_0 and b_1 can each be anything at all.

Looking at the even coefficients first, we find

$$b_2 \;=\; -\frac{4b_0}{2\cdot 1},$$

$$b_4 \;=\; -\frac{4b_2}{4\cdot 3} \;=\; (-1)^2\,\frac{4^2 b_0}{4\cdot 3\cdot 2\cdot 1},$$

$$b_6 \;=\; -\frac{b_4}{6\cdot 5} \;=\; (-1)^3\,\frac{4^3 b_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1},$$

and, generally, for $n \geq 0$,

$$b_{2n} \;=\; (-1)^n\,\frac{4^n b_0}{(2n)!} \;=\; b_0\cdot(-1)^n\,\frac{2^{2n}}{(2n)!}$$

A similar calculation shows that, for $n \geq 0$,

$$b_{2n+1} \;=\; (-1)^n\,\frac{4^n b_1}{(2n+1)!} \;=\; b_1\cdot(-1)^n\,\frac{2^{2n}}{(2n+1)!} \;=\; \frac{b_1}{2}\cdot(-1)^n\,\frac{2^{2n+1}}{(2n+1)!}$$

Therefore,

$$y \;=\; \sum_{k=0}^{\infty} b_k x^k \;=\; \sum_{k\text{ even}} b_k x^k \;+\; \sum_{k\text{ odd}} b_k x^k \;=\; \sum_{n=0}^{\infty} b_{2n} x^{2n} \;+\; \sum_{n=0}^{\infty} b_{2n+1} x^{2n+1} \;=$$

$$\sum_{n=0}^{\infty} b_0\cdot(-1)^n\,\frac{2^{2n}}{(2n)!}\,x^{2n} \;+\; \sum_{n=0}^{\infty} \frac{b_1}{2}\cdot(-1)^n\,\frac{2^{2n+1}}{(2n+1)!}\,x^{2n+1} \;=$$

$$b_0 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} \quad + \quad \frac{b_1}{2} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}.$$

If you look back at the equalities between $\cos x$, $\sin x$, and their Maclaurin series, you should see that we just found that

$$y \;=\; b_0 \cos(2x) \;+\; \frac{b_1}{2} \sin(2x),$$

which means that $y(0) = b_0$ and $y'(0) = b_1$, which would be determined by initial data.

We should remark that, above, when we split our summation into the even degree parts and the odd degree parts, we implicitly assumed that rearranging the order of the summation did not affect the sum; in general, this requires that the series *converges absolutely*, as we shall discuss in Section 5.4.

Before we leave this section, we want to point out that the examples that we gave were actually simple ones, though they may not have seemed so simple.

Consider, for instance, the problem of finding a power series solution to the differential equation

$$y' \;=\; y^2 \;+\; x.$$

You can look for a power series solution, centered at 0, $y = \sum_{k=0}^{\infty} b_k x^k$, but what do you do with the y^2 part??? Well...you square your series, that is, you multiply it times itself, and use the product series. Yes, it's ugly, but it's not ridiculously difficult to find at least the first few terms of the solution.

What about solving

$$y' \;=\; (\sin x)y \;+\; x?$$

Again, what you have to do is fairly unpleasant; you let $y = \sum_{k=0}^{\infty} b_k x^k$ and multiply this by the Maclaurin series for $\sin x$ to deal with the $(\sin x)y$ part of the problem.

The point is: while solving differential equations via power series may be time-consuming and tedious, what you need to do, and actually doing it, is frequently/usually very straightforward.

4.7.1 Exercises

In each of Exercises 1 through 10, you are given an initial value problem, in which the initial data is specified at a particular value $x = x_0$. Assume that the initial value problem has a unique power series solution, $y = \sum_{k=0}^{\infty} b_k(x - x_0)^k$ (at least, for x values near x_0). (a) Determine iterative formulas for the b_k. (b) Calculate b_0 through b_8 explicitly. (c) Approximate the value of your solution at $x = x_0 + 0.1$ by using the first 4 non-zero terms in your power series.

1. $y' = 5y + x;\;\; y(0) = 2$
2. $y' = 5y + x;\;\; y(1) = 2$
3. $y' = 5y + e^x;\;\; y(0) = 2$
4. $y' = 5y + xe^x;\;\; y(0) = 2$
5. $y' = 5y + x^2e^x + 1;\;\; y(0) = 2$
6. $y' = 5y + \sin x;\;\; y(0) = 2$
7. $y' = (x - 3)^2y + 1;\;\; y(3) = -1$
8. $y'' = 4y;\;\; y(0) = 1,\; y'(0) = 0$
9. $y'' = 4y + 2e^x;\;\; y(0) = 1,\; y'(0) = 0$
10. $y'' = (x - 2)y + 1;\;\; y(2) = 1,\; y'(2) = 0$

In each of Exercises 11 through 18, you are given an initial value problem, with initial data specified at $x = x_0$. Find a power series solution, centered at x_0, and, by manipulating known Maclaurin series, rewrite your solution in a "closed" form.

11. $y' = 5y + x;\;\; y(0) = 2$
12. $y' = 5y + x;\;\; y(1) = 2$
13. $y' = (x + 2)^2y;\;\; y(-2) = 7$
14. $y' = 5y + e^{x-1};\;\; y(1) = 2$
15. $y' = ay + bx$; $y(0) = 0$, where a and b are constants, $a \neq 0$.
16. $y'' = 5y';\;\; y(-2) = 7,\; y'(-2) = 0$
17. $y'' = 4y;\;\; y(0) = 1,\; y'(0) = 0$

18. $y'' = -k^2 y;\ \ y(0) = 1,\ y'(0) = 0$, where k is a constant.

19. Find the 3rd order Maclaurin polynomial of the solution $y = y(x)$ to the initial value problem
$$y' = y^2 + x; \qquad y(0) = 1.$$

20. Find the 3rd order Maclaurin polynomial of the solution $y = y(x)$ to the initial value problem
$$y' = y \sin x + x; \qquad y(0) = 1.$$

Appendix 4.A Technical Matters

Theorem 4.A.1. *Suppose that f and g are real analytic functions on an open interval I, and that $f(x) = g(x)$ for all x in an open interval J, which is contained in I. Then, f and g are equal on I.*

Proof. By considering $f - g$, we are reduced to showing:

Suppose f is real analytic on an open interval I, and is 0 on an open subinterval $J \subseteq I$. Then, f is 0 on all of I.

We will use here that intervals are *connected*, i.e., that an interval is not the disjoint union of two non-empty open sets.

We say that the *germ of f is 0 at x* if and only if there exists an open interval around x on which f is everywhere equal to 0; we write $[f]_x = 0$.

By definition, $A = \{x \in I \mid [f]_x = 0\}$ is an open subset of I. By our assumption that f is 0 on J, A is non-empty. We claim that the set $B = \{x \in I \mid [f]_x \neq 0\}$, which is the complement of A in I, is also open. As I is connected, it will follow that B is empty, i.e., that f is zero on all of I.

Suppose that $b \in B$. We wish to show that there is an open interval K around b such that $K \subseteq B$. Suppose not. Then, by picking smaller and smaller open intervals around b, we can produce a sequence of points $a_i \in A$ such that $\lim_{i\to\infty} a_i = b$. As f is real analytic at each a_i and is zero on an open interval around a_i, all of the coefficients of the Taylor series of f at each a_i must be 0, i.e., for all i, for all k, $f^{(k)}(a_i) = 0$. As the derivatives are continuous, $f^{(k)}(b) = \lim_{i\to\infty} f^{(k)}(a_i) = 0$. As all of the derivatives $f^{(k)}(b)$ are 0, and f is real analytic at b, it follows that f is 0 in an open interval around b, i.e., $b \notin B$. This contradiction shows that B would have to be open and, hence, empty. $\square$

Chapter 5

Theorems on Sequences and Series

As we have discussed in prior chapters, with various levels of formalism, a *sequence* of real numbers is essentially just an infinite list of real numbers, that is, for all integers n, greater than or equal to some initial integer m, we have a real number b_n. A sequence *converges* if and only if $\lim_{n\to\infty} b_n$ exists, where n is allowed to take on only integer values $\geq m$. Convergence of certain kinds of sequences is a fundamental property of the real numbers, which we shall not attempt to derive.

An infinite *series* is what you get when you add an infinite sequence of *terms*; it's a limit of *partial sums*. We will discuss many tests for deciding when such limits of partial sums exist.

Unfortunately, while it is extremely important to distinguish between sequences and series, confusion between these two terms is common – perhaps stemming from the fact that both words start with "se" or, perhaps, because the partial sums themselves form a second sequence, in addition to the sequence of terms. Throughout this chapter, you need to keep in mind that, in an infinite series, you are trying to make sense of an infinite sum; this is the limit of the partial sums, **not** the limit of the individual summands (the terms).

5.1 Theorems on Sequences

In our discussion and use of power series, and several other times throughout this textbook, we have used *infinite sequences*, "lists" of real numbers that never stop. In this section, we will state, prove, and look at examples of some fundamental definitions and results on sequences.

Informally, a *sequence* of real numbers is a list of real numbers, the *terms*, such that there's a first number, a second number, a third number, and so on – in general, there's an n-th number in the list, for every natural number n.

For instance, we sometimes list a few numbers in the list, with some obvious pattern, and then, once the pattern is clear, we indicate the remainder of the list with dots, e.g.,

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \dots,$$

where the n-th term is $b_n = 1/n$.

It is convenient in some cases to start the subscript, the *index*, at some number other than 1; for instance, we might consider the sequence given by

$$a_n \;=\; \frac{1}{(n-1)(n-2)}, \qquad \text{for } n \geq 3.$$

This causes some mild confusion when speaking of the terms of the sequence, because it's then true that the first term of this sequence is a_3, the second term is a_4, and so on. The problem is: if you said "the n-th term", when using n for the indexing variable, it might be unclear whether you meant a_n or a_{n+2}. For this reason, it's best to talk about a_n, which is very clear, rather than to use common English phrasing. Typically, we care about the limit of a sequence, as the index approaches ∞, so confusing a_n and a_{n+2} is unlikely to cause a problem.

The notion of a sequence as a list is informal. Really, a sequence is a function; you pick an integer (greater than or equal to some starting integer m), and you get back a real number. We gave the rigorous definitions of a sequence and its convergence/divergence in [2], and in Definition 4.5.1; however, we will give those definitions again here.

Definition 5.1.1. *Suppose that m is an integer, i.e., is in the set $\mathbb{Z}$. Denote by $\mathbb{Z}_{\geq m}$ the set of integers which are greater than or equal to m.*

A function $b : \mathbb{Z}_{\geq m} \to \mathbb{R}$ is called a **sequence (of real numbers)**. *In place of $b(n)$, it is standard to write b_n.*

We say that **the sequence b_n converges to (a real number)** *L, and write $\lim_{n\to\infty} b_n = L$ if and only if, for all $\epsilon > 0$, there exists an integer $N \geq m$ such that, for all integers $n \geq N$, $|b_n - L| < \epsilon$.*

If a sequence does not converge to some real number L, then we say that the sequence **diverges**.

The sequence b_n **diverges to ∞**, *and we write $\lim_{n\to\infty} b_n = \infty$, if and only if, for all real numbers A, there exists $N \geq m$ such that, for all $n \geq N$, $b_n > A$.*

The sequence b_n **diverges to $-\infty$**, *and we write $\lim_{n\to\infty} b_n = -\infty$, if and only if, for all real numbers A, there exists $N \geq m$ such that, for all $n \geq N$, $b_n < A$.*

It is important that a sequence can have only one limit.

Theorem 5.1.2. *If a sequence b_n converges to L_1 and converges to L_2, then $L_1 = L_2$, i.e., if a sequence converges to some limit, then that limit is the unique limit.*

Proof. This proof is identical to the proof of uniqueness of limits of functions of a real variable, except that you allow only integers for the value of the variable. We refer you to the proof in [2]. $\square$

Remark 5.1.3. Because the same notation is used for limits over the integers and limits over the real numbers, some confusion can arise.

Think about $\lim_{n\to\infty} \sin(n\pi)$. Do we mean the limit of the sequence $a_n = \sin(n\pi)$, so that n takes on only integer values (with an unspecified starting value)? Or do we mean that n can be any real number? It makes a difference.

If we mean that n varies over the real numbers, then $\lim_{n\to\infty} \sin(n\pi)$ does not exist, because the sine function continues to oscillate between -1 and 1 as n increases through the real numbers.

On the other hand, if n is an integer, then $\sin(n\pi) = 0$, and so the limit $\lim_{n\to\infty} \sin(n\pi)$ **of the sequence** $\sin(n\pi)$ is just $\lim_{n\to\infty} 0 = 0$.

How do we avoid this confusion? There are several ways. One is to explicitly state that we are taking the limit of a sequence. Another is to refer to n's that take on only integer values as *discrete variables*, and n's that can take on arbitrary real values as *continuous variables*. Finally, it is common to use the letters i, j, k, l, m, and n as discrete variables, and the letters r, s t, u, v, w, x, y, and z as continuous variables. One of these ways should always make it clear which type of limit is intended.

For instance, many people would write simply

$$\lim_{n\to\infty} \sin(n\pi) \;=\; \lim_{n\to\infty} 0 \;=\; 0,$$

while

$$\lim_{x\to\infty} \sin(x\pi) \text{ does not exist,}$$

automatically assuming that n denotes a discrete variable and x denotes a continuous variable.

In the above remark, we saw that, if we take the function $f(x) = \sin(x\pi)$, whose domain is the set of real numbers, then $\lim_{x\to\infty} f(x)$ does not exist, but if we restrict the domain of f to the integers, then $\lim_{n\to\infty} f(n)$ **does** exist. However, the theorem below tells us that, if $\lim_{x\to\infty} f(x)$ exists, and equals L, then $\lim_{n\to\infty} f(n)$ also exists and equals L.

Theorem 5.1.4. *Suppose that M is a real number and that $f(x)$ is defined for all real $x \geq M$.*

If $\lim_{x\to\infty} f(x)$ exists as an extended real number, and equals L, then the limit $\lim_{n\to\infty} f(n)$ of the restriction of f to the integers $\geq M$ exists as an extended real number and equals L.

Proof. We shall prove the case in which L is a real number, and leave the cases where $L = \pm\infty$ as an exercise.

Suppose that $\lim_{x\to\infty} f(x)$ equals a real number L, and let $\epsilon > 0$. Then, there exists a real number $N \geq M$ such that, if x is a real number and $x > N$, then $|f(x) - L| < \epsilon$. However, this is true for all real numbers $> N$; in particular, it's true for the integers $> N$.

This is precisely what it means for the sequence $f(n)$ to converge to L. $\square$

Example 5.1.5. Theorem 5.1.4 is very useful when trying to calculate limits of sequences which arise from restricting functions of a continuous variable.

An easy example is to consider $g(x) = 1/x$. Then, we know that $\lim_{x\to\infty} g(x) = 0$. It follows that

$$\lim_{n\to\infty} \frac{1}{n} \;=\; 0,$$

where, in this latter limit, we mean the limit of the sequence.

Example 5.1.6. Let's look at something a bit more complicated. Consider the limit of the sequence n/e^n.

We'd like to use l'Hôpital's Rule (see [2]), but l'Hôpital's Rule requires us to take derivatives of the numerator and denominator. If n really takes on only integer values, then we can't take derivatives.

However, consider, instead, the function of a continuous variable $f(x) = x/e^x$. Then, l'Hôpital's Rule tells us

$$\lim_{x\to\infty} \frac{x}{e^x} \;=\; \lim_{x\to\infty} \frac{x'}{(e^x)'} \;=\; \lim_{x\to\infty} \frac{1}{e^x} \;=\; 0,$$

and now Theorem 5.1.4 tells us that, because the "continuous limit" above is 0, we also have the discrete limit equaling the same thing, i.e.,

$$\lim_{n\to\infty} \frac{n}{e^n} \;=\; 0.$$

Understanding that the existence of the continuous variable limit implies the existence (and equality) of the discrete variable limit, it is fairly standard to be a bit sloppy, and never write anything explicit about the continuous variable function, but rather simply apply l'Hôpital's Rule to the function containing the discrete variable n. That is, it's pretty standard to write

$$\lim_{n\to\infty} \frac{n}{e^n} \;=\; \lim_{n\to\infty} \frac{n'}{(e^n)'} \;=\; \lim_{n\to\infty} \frac{1}{e^n} \;=\; 0.$$

Just remember: there's really a theorem at work here.

Many of the results that we proved for limits $\lim_{x\to\infty} f(x)$ have analogs for sequences, which can be proved simply by replacing the continuous variable x by the discrete variable n; such results include arithmetic with limits, the Pinching Theorem, and composition with continuous functions, all of which we state below.

Theorem 5.1.7. (**Arithmetic of Convergent Sequences**) *Suppose that the sequences a_n and b_n converge, and that c is a constant. Then,*

1. $\lim_{n\to\infty} c = c$;
2. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$;
3. $\lim_{n\to\infty} ca_n = c \cdot \lim_{n\to\infty} a_n$;
4. $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$;
5. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, *provided that* $\lim_{n\to\infty} b_n \neq 0$.

In particular, the sequences $a_n + b_n$, ca_n, $a_n \cdot b_n$, and a_n/b_n converge.

Example 5.1.8. Since we know that

$$\lim_{n\to\infty} \frac{1}{n} = \mathbf{0},$$

multiplying this limit times itself two and three times gives us that

$$\lim_{n\to\infty} \frac{1}{n^2} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{n^3} = 0.$$

Combining these with Properties 1-4 in Theorem 5.1.7 tells us that

$$\lim_{n\to\infty} \left[\left(5 - \frac{7}{n^2}\right)\left(3 + \frac{4}{n^3}\right)\right] = (5 - 7\cdot 0)(3 + 4\cdot 0) = 15.$$

Just like functions of a continuous variable, sequences satisfy a *Pinching Theorem.*

Theorem 5.1.9. (**Pinching Theorem for Sequences**) *Suppose that, for $n \geq m$, we have the inequality of sequences $a_n \leq b_n \leq c_n$, and that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, where L is an extended real number. Then, $\lim_{n\to\infty} b_n = L$.*

Example 5.1.10. Consider

$$\lim_{n\to\infty} \frac{\sin^2 n}{n^3}.$$

Because $0 \leq \sin^2 n \leq 1$, for $n \geq 1$, we know that

$$0 \;\leq\; \frac{\sin^2 n}{n^3} \;\leq\; \frac{1}{n^3}.$$

Since $1/n^3 \to 0$ as $n \to \infty$ (by Theorem 5.1.7), the Pinching Theorem tells us that

$$\lim_{n\to\infty} \frac{\sin^2 n}{n^3} \;=\; 0.$$

As $-|a_n| \leq a_n \leq |a_n|$, the following is an immediate corollary of the Pinching Theorem, which is useful in special cases.

Corollary 5.1.11. $\lim_{n\to\infty} |a_n| = 0$ *if and only if* $\lim_{n\to\infty} a_n = 0$.

Example 5.1.12. Since we know that $1/n \to 0$ as $n \to \infty$, it follows from the corollary above that

$$\lim_{n\to\infty} \frac{(-1)^n}{n} \;=\; 0.$$

You can compose continuous functions with convergent sequences (remember: sequences are technically functions) to produce new convergent sequences.

Theorem 5.1.13. *Suppose that* $\lim_{n\to\infty} a_n = L$*, where* L *is a real number. Suppose also that* $f(x)$ *is a function whose domain contains all of the values* a_n*, and that* f *is continuous at* L*. Then, the new sequence* $f(a_n)$ *converges to* $f(L)$*, i.e.,*

$$\lim_{n\to\infty} f(a_n) \;=\; f\left(\lim_{n\to\infty} a_n\right).$$

Example 5.1.14. For instance, since $\lim_{n\to\infty}(1/n) = 0$,

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) \;=\; \cos(0) \;=\; 1.$$

As another example of Theorem 5.1.13, consider the limit from Example 5.1.8:

$$\lim_{n\to\infty}\left[\left(5 \;-\; \frac{7}{n^2}\right)\left(3 \;+\; \frac{4}{n^3}\right)\right] \;=\; (5-7\cdot 0)(3+4\cdot 0).$$

Instead of appealing to the arithmetic properties of limits of sequences, as we did in Example 5.1.8, we could let $f(x)$ be the continuous function $(5-7x^2)(3+4x^3)$, and then use Theorem 5.1.13 to obtain

$$\lim_{n\to\infty}\left[\left(5 \;-\; \frac{7}{n^2}\right)\left(3 \;+\; \frac{4}{n^3}\right)\right] \;=\; \lim_{n\to\infty} f\left(\frac{1}{n}\right) \;=\; f\left(\lim_{n\to\infty}\frac{1}{n}\right) \;=\; f(0) = 15.$$

Before we can give any useful ways of determining the convergence/divergence of a sequence, we must discuss two fundamental properties which essentially define the real numbers. We refer you to [3] for more details.

First, we need two definitions.

Definition 5.1.15. *Suppose that E is a set of real numbers, i.e., a subset of $\mathbb{R}$. Suppose that M is a real number (which may or may not be in E).*

Then, M is an **upper bound** *of E if and only if, for all x in E, $x \leq M$. If an upper bound exists for E, then we say that E is* **bounded above**.

An upper bound M of E is the **least upper bound** *or* **supremum** *if and only if M is the smallest upper bound of E, i.e., if $N < M$, then N is not an upper bound of E, i.e., there exists x in E such that $N < x$.*

M is a **lower bound** *of E if and only if, for all x in E, $M \leq x$. If a lower bound exists for E, then we say that E is* **bounded below**.

A lower bound M of E is the **greatest lower bound** *or* **infimum** *if and only if M is the biggest lower bound of E, i.e., if $M < N$, then N is not a lower bound of E, i.e., there exists x in E such that $x < N$.*

As we defined earlier in Definition 2.3.4, the set E is **bounded** *if and only if E is bounded above and below.*

Remark 5.1.16. When we apply any of the terms in Definition 5.1.15 to a sequence, such as saying "the sequence a_n is bounded above", we mean that the set E under consideration is the set $\{a_n \mid n \geq m\}$ of values of the sequence, i.e., the range of the sequence function.

We also make the following, seemingly unrelated, definition.

Definition 5.1.17. *Suppose that b_n, for $n \geq m$, is a sequence of real numbers. Then, the sequence b_n is a* **Cauchy sequence** *if and only if, for all $\epsilon > 0$, there exists an integer $M \geq m$ such that, for all $j, k \geq M$, $|b_j - b_k| < \epsilon$.*

Cauchy sequences, least upper bounds, and greatest lower bounds are related in that they all provide ways of defining what's known as the *completeness of the real numbers*:

Theorem 5.1.18. Least Upper Bound Property & Completeness of the Real Numbers:

1. *Every non-empty set of real numbers, which is bounded above, possesses a least upper bound.*

2. *Every non-empty set of real numbers, which is bounded below, possesses a greatest lower bound.*

3. *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Remark 5.1.19. The greatest lower bound property follows very easily from the least upper bound property by negating everything, which interchanges upper bounds and lower bounds.

All of the completeness properties in Theorem 5.1.18 fail if you look solely at rational numbers. There are non-empty sets of rational numbers, which are bounded above/below by rational numbers, which have no rational supremum/infimum, and there are Cauchy sequences of rational numbers which do not converge to a rational number. Thus, any of the properties in Theorem 5.1.18 can be used to show why the rational numbers are "incomplete".

In fact, the completeness properties in Theorem 5.1.18 actually tell us what "extra stuff" we want to combine with the rational numbers in order to obtain the real numbers. The real numbers are constructed from the rational numbers by, basically, letting real numbers be suprema/infima of bounded-above/below sets of rational numbers, or by letting real numbers be limits of Cauchy sequences of rational numbers.

We should remark that the nature of our completeness properties is why we can frequently tell that sequence converges to **something**, without being able to say what the something is. The limit of the sequence may be a little mysterious when it's described as the least upper bound, or greatest lower bound, of some set of real numbers, or as being the thing that a Cauchy sequence is converging to.

Monotonic sequences will be especially important to us. These sequences are just monotonic functions which have their domains restricted to being sets of integers.

Definition 5.1.20. *Suppose that b_n, for $n \geq m$, is a sequence of real numbers.*

- *b_n is* **increasing** *if and only if, for all integers k and l such that $k \geq l \geq m$, $b_k \geq b_l$.*
- *b_n is* **decreasing** *if and only if, for all integers k and l such that $k \geq l \geq m$, $b_k \leq b_l$.*
- *b_n is* **monotonic** *if and only if it is increasing or decreasing.*

Remark 5.1.21. Note that increasing sequences are automatically bounded below by the initial term of the sequence, and decreasing sequences are automatically bounded above by the initial term of the sequence.

Therefore, the assumption that an increasing sequence is bounded is equivalent to the assumption that it's bounded above, and the assumption that a decreasing sequence is bounded is equivalent to the assumption that it's bounded below.

When you're interested in whether or not a sequence converges, you can ignore any finite number of initial terms in the sequence. Hence, the following terminology is frequently convenient to use.

Definition 5.1.22. *We say that a sequence a_n, for $n \geq m$,* **eventually** *has a certain property if there exists an integer $M \geq m$ such that the new sequence a_n, for $n \geq M$, has the property.*

Thus, for instance, a sequence is eventually increasing if it is always increasing after some point.

The following theorem gives one of the most basic ways in which one concludes that a series converges.

Theorem 5.1.23. *Suppose that a sequence b_n, for $n \geq m$, is eventually monotone. Then, b_n converges if and only if it is bounded.*

More specifically,

1. *if b_n is increasing for $n \geq m$,*
   a. *if $\lim_{n\to\infty} b_n = L$, then, for all $n \geq m$, $b_m \leq b_n \leq L$, and*
   b. *if, for all $n \geq m$, $b_n \leq B$, then $\lim_{n\to\infty} b_n$ exists and is $\leq B$.*
2. *if b_n is decreasing for $n \geq m$,*
   a. *if $\lim_{n\to\infty} b_n = L$, then, for all $n \geq m$, $L \leq b_n \leq b_m$, and*
   b. *if, for all $n \geq m$, $B \leq b_n$, then $\lim_{n\to\infty} b_n$ exists and is $\geq B$.*

In particular, the only way for an eventually increasing sequence to diverge is to diverge to ∞, and the only way for an eventually decreasing sequence to diverge is to diverge to $-\infty$. This means that, for every eventually monotone sequence b_n, we can write $\lim_{n\to\infty} b_n = L$, where L is an extended real number, i.e., a real number or $\pm\infty$.

Proof. Assume that the sequence is increasing for all $n \geq M$. The decreasing case is completely analogous.

Suppose that b_n converges to L. We will show that, for all $n \geq M$, $b_n \leq L$. Since the sequence is increasing for $n \geq M$, this will show that the sequence b_n is bounded below by the minimum value in the set $\{b_n \mid m \leq n \leq M\}$ and above by the maximum of L and the maximum value in the set $\{b_n \mid m \leq n \leq M-1\}$.

Suppose, to the contrary, that there exists $n_0 \geq M$ such that $b_{n_0} > L$. Let $\epsilon = b_{n_0} - L > 0$. As b_n is increasing for $n \geq M$, for all $n \geq n_0$,

$$b_n - L \;\geq\; b_{n_0} - L \;\geq\; \epsilon.$$

This contradicts that L is the limit of the sequence, and this contradiction establishes that sequence b_n is bounded.

We need now to show the other direction in the proof. So, suppose that the sequence b_n, $n \geq m$ is bounded. Then, the set $E = \{b_n \mid n \geq M\}$ is bounded; let B be an upper bound of E. By Theorem 5.1.18, E has a least upper bound; call it L, which is necessarily $\leq B$. We claim that $\lim_{n\to\infty} b_n = L$.

Note, first, that, since L is **an** upper bound for E, for all $n \geq M$, $b_n \leq L$, i.e., $L - b_n \geq 0$. Now, let $\epsilon > 0$. Then, $L - \epsilon$, which is less than L is **not** an upper bound of E, i.e., there exists

$n_0 \geq M$ such that $b_{n_0} > L - \epsilon$. This means that $L - b_{n_0} < \epsilon$. As b_n is increasing for $n \geq M$, it follows that, for all $n \geq n_0$, $L - b_n < \epsilon$. As $L - b_n \geq 0$, this establishes what we wanted: for all $n \geq n_0$, $|L - b_n| < \epsilon$.

We still need to make an argument to justify the last statement in the theorem. Suppose now only that b_n is increasing for all $n \geq M$. Then, as we saw above, b_n is bounded below by the minimum value in the set $\{b_n \mid m \leq n \leq M\}$. As we also saw above, if the sequence b_n diverges, then it's unbounded; thus, it must not be bounded above. Pick some real number A, and let A_0 be the maximum of A and 1 plus the maximum value in the set $\{b_n \mid m \leq n \leq M-1\}$. As b_n is not bounded above, there exists $n_0 \geq M$ such that $b_{n_0} \geq A_0$. As b_n, for $n \geq M$, is increasing, it follows that, for all $n \geq n_0$, $b_n \geq b_{n_0} \geq A_0 \geq A$, which shows that b_n diverges to ∞. □

Corollary 5.1.24. (**Comparison of Monotonic Sequences**)

1. *Suppose that, for $n \geq m$, a_n and b_n are increasing, and $a_n \leq b_n$. Then, if b_n converges, so does a_n and, in this case, $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$; in particular, if a_n diverges, then so does b_n.*

2. *Suppose that, for $n \geq m$, a_n and b_n are decreasing, and $a_n \leq b_n$. Then, if a_n converges, so does b_n and, in this case, $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$; in particular, if b_n diverges, then so does a_n.*

Proof. This is immediate from Theorem 5.1.23. □

Example 5.1.25. Let's look at an example which is relevant to the next section, where we look at infinite series.

For $n \geq 0$, define sequences

$$a_n = \sum_{k=0}^{n} \frac{1}{2^k+1} \qquad \text{and} \qquad b_n = \sum_{k=0}^{n} \frac{1}{2^k}.$$

As we are adding something non-negative to get from one term of a_n to the next, and something non-negative to get from one term of b_n to the next, the sequences a_n and b_n are both increasing. Moreover, clearly, we have $a_n \leq b_n$ (in fact, $a_n < b_n$). We also know from Theorem 4.4.10 that the sequence b_n converges to 2.

It follows from Corollary 5.1.24 that the sequence a_n converges to something ≤ 2, but we have no idea what that something is – though the something must also be greater than or equal to every a_n. In fact, the limit of the sequence a_n is precisely the smallest number that is greater than or equal to all of the a_n's, i.e., the least upper bound of the sequence.

Consider now the sequence $1/n$, for $n \geq 1$; this is the *harmonic sequence*:

$$\frac{1}{1}, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \frac{1}{5}, \ \frac{1}{6}, \ \frac{1}{7}, \ \frac{1}{8} \ \ldots.$$

Suppose that we look at just the terms in the harmonic sequence that have even denominators:

$$\frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{6}, \ \frac{1}{8}, \ \frac{1}{10}, \ \frac{1}{12}, \ \frac{1}{14}, \ \frac{1}{16} \ \ldots.$$

This is the sequence $1/(2n)$, for $n \geq 1$. Notice that each of the terms in this *even harmonic sequence* appear in the harmonic sequence, and they appear in the same order, i.e., the even harmonic sequence is obtained from deleting terms from the harmonic series, without rearranging the remaining terms.

Sequences obtained in this manner are call *subsequences*.

Definition 5.1.26. *Suppose that we have a sequence a_n, for $n \geq m$. A* **subsequence** *of the sequence a_n is a sequence formed by taking some (or all) of the terms of a_n in order. Rigorously, this means that a subsequence of a_n, for $n \geq m$, is a sequence a_{n_k}, where n_k, for $k \geq 1$, is a sequence of integers such that $n_1 \geq m$ and, for $i < j$, $n_i < n_j$, i.e., the indices in the subsequence are a strictly increasing sequence of some of the indices from the sequence a_n.*

There is a theorem and a corollary about subsequences that will be of interest to us.

Theorem 5.1.27. *If $\lim_{n\to\infty} a_n = L$, where L is an extended real number, then the limit of every subsequence of a_n is also L.*

In particular, if a sequence a_n has two (or more) subsequences which have different limits, then a_n diverges.

Proof. While we could write this in full mathematical detail, that would obscure what's really going on, which is very elementary.

Suppose that $\lim_{n\to\infty} a_n = L$. This means that you can make a_n as close to L as you want (or, in the cases where $L = \pm\infty$, you can make L as big or as negative as you want) by picking the index n to be big enough. But, this is true for **every** a_n as long as the index is big enough; in particular, it would be true for the indices (which are big enough) of the a's appearing in any subsequence. □

Recall from Theorem 5.1.23 that a monotone sequence always approaches a limit L, if we allow extended real numbers L. Combining this with the theorem above, we immediately obtain:

Corollary 5.1.28. *If a sequence a_n is monotonic, and a subsequence of a_n has an extended real number L as its limit, then $\lim_{n\to\infty} a_n = L$.*

Example 5.1.29. Consider the sequence a_n, for $n \geq 1$, $a_n = \dfrac{1}{n} + (-1)^n$.

If we look at the subsequence where n is even, i.e., where $n = 2k$, $k \geq 1$, we find that

$$a_{2k} = \frac{1}{2k} + (-1)^{2k} = \frac{1}{2k} + 1,$$

which clearly approaches 1 as $k \to \infty$.

On the other hand, if we look at the subsequence where n is odd, i.e., where $n = 2k - 1$, $k \geq 1$, we find that

$$a_{2k-1} = \frac{1}{2k-1} + (-1)^{2k-1} = \frac{1}{2k-1} - 1,$$

which clearly approaches -1 as $k \to \infty$.

As there are two sequences of a_n which approach different limits, the (entire) sequence a_n diverges, by Theorem 5.1.27.

5.1.1 Exercises

In each of Exercises 1 through 15, determine whether the given sequence converges or diverges. If the sequence converges, find its limit. As we are interested in what happens as $n \to \infty$, we do not specify an initial n value; assume that n is big enough so that all of the given terms are defined.

1. $1 + \dfrac{5}{n}$
2. $\dfrac{5+3n}{n}$
3. $\dfrac{n^2}{n^2+1}$
4. $\dfrac{2^n + n^2}{2^n}$
5. $\dfrac{2n\cos\left(\frac{n\pi}{4}\right)}{n^2+1}$
6. $\dfrac{n}{10\ln n}$
7. $\ln(2n) - \ln(n+1)$
8. $\dfrac{2n}{\sqrt{n^2+3}}$
9. $\left(7 + \dfrac{1}{n}\right)^{1/n}$
10. $\left(1 + \dfrac{3}{n}\right)^{n}$
11. $\dfrac{(5+n)^n}{n^n}$
12. $\dfrac{e^n}{n!}$
13. $\dfrac{100^n}{n!}$
14. $\tan^{-1} n$
15. $n - \sqrt{n^2+7}$

In each of Exercises 16 through 19, use the Pinching Theorem, Theorem 5.1.9, to show that the given sequence converges.

16. $\dfrac{2+(-1)^n}{n}$

17. $\dfrac{5\sin n+7}{n^2}$

18. $\dfrac{3\sin n-2\cos n}{\sqrt{n+1}}$

19. $\dfrac{4n+3(-1)^n}{n}$

In each of Exercises 20-24, find two subsequences of the given sequence which converge to different limits. Conclude that each sequence diverges.

20. $2+(-1)^n$

21. $\sin\left(\frac{n\pi}{4}\right)$

22. $\dfrac{2n\cos\left(\frac{n\pi}{4}\right)}{n+1}$

23. $\frac{1}{2},\ -\frac{1}{2},\ \frac{2}{3},\ -\frac{2}{3},\ \frac{3}{4},\ -\frac{3}{4},\ \frac{4}{5},\ -\frac{4}{5},\ \ldots$

24. $\frac{2}{1},\ 0,\ 0,\ \frac{3}{2},\ 0,\ 0,\ 0,\ \frac{4}{3},\ 0,\ 0,\ 0,\ 0,\ \frac{5}{4},\ 0,\ 0,\ 0,\ 0,\ 0,\ \frac{6}{5},\ \ldots$

25. Suppose that a sequence a_n, $n \geq m$, has the property that, for all $n \geq m$, $a_n \leq 7$.

    a. Is it true that the sequence must converge to a number that's ≤ 7? Prove it or give a counterexample.

    b. Is it true that, if the sequence converges to L, then $L \leq 7$? Prove it or give a counterexample.

    c. Is it true that, if a_n is increasing, then it converges to a number that's ≤ 7? Prove it or give a counterexample.

26. Suppose that a sequence a_n, $n \geq m$, has the property that, for all $n \geq m$, $3 \leq a_n \leq 7$.

    a. Is it true that the sequence must converge to a number L such that $3 \leq L \leq 7$? Prove it or give a counterexample.

    b. Is it true that, if the sequence converges to L, then $3 \leq L \leq 7$? Prove it or give a counterexample.

    c. Is it true that, if a_n is monotonic, then it converges to a number L such that $3 \leq L \leq 7$? Prove it or give a counterexample.

27. Suppose that the sequence a_n converges, and that the sequence b_n diverges. Prove that the sequence $a_n + b_n$ diverges.

28. Give an example of a sequence a_n which diverges, and a sequence b_n which diverges such that the sequence $a_n + b_n$ converges.

29. Suppose that the sequence a_n converges to a non-zero limit, and that the sequence b_n diverges. Prove that the sequence b_n/a_n diverges.

30. Perhaps the most famous sequence of all time is the *Fibonacci sequence*, F_n, $n \geq 0$, given by:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots,$$

where the sequence is given iteratively by specifying that $F_0 = 0$, $F_1 = 1$, and, for all $n \geq 2$,

$$F_n \ = \ F_{n-1} + F_{n-2}.$$

That is, after the first two terms, each term is the sum of the two terms before it. In this exercise, we will look at a closed (non-iterative) formula for F_n.

a. Suppose that a and b are constants. Consider the sequence c_n, $n \geq 0$, given by

$$c_n \ = \ a\left(\frac{1+\sqrt{5}}{2}\right)^n \ + \ b\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Determine a and b so that $c_0 = 0$ and $c_1 = 1$.

b. Show that

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 \ = \ \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \left(\frac{1-\sqrt{5}}{2}\right)^2 \ = \ \frac{3-\sqrt{5}}{2}.$$

c. Using the constants a and b that you found in part (a), show that that the sequence c_n in part (a) is, in fact, the Fibonacci sequence F_n. Since you know that $c_0 = 0$, and $c_1 = 1$, to show that $c_n = F_n$, you need to show that c_n satisfies the Fibonacci iteration formula, i.e., you need to show that,for $n \geq 2$,

$$c_n \ = \ c_{n-1} \ + \ c_{n-2}.$$

d. Calculate
$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}.$$

The value that you obtain for this limit is known as the *Golden Ratio*.

While they may appear to be simply interesting mathematical curiosities, the Fibonacci sequence and the Golden Ratio surprisingly appear frequently in nature, for example, in the arrangements of leaves on a plant and in the spiral shell of a nautilus.

5.2 Theorems on Series I: Basic Properties

In the last few sections of the previous chapter, we looked at infinite sums in the guise of power series. For a power series function $p(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$, we are interested in convergence of the series for different x values. But, for any fixed value of x, a power series simply becomes a summation $\sum b_k$ of a sequence of numbers b_k, and we can forget that the sequence of numbers that we're adding are of the form $b_k = c_k(x-a)^k$; that is, we can forget that the sequence comes from the terms of a power series. Such a summation of fixed numbers is simply called a *series* or an *infinite series*; to emphasize that the terms are not changing as a function of some variable x, the terminology *series of constants* is often used.

When we discussed series earlier, we put off any discussion of general convergence theorems, other than the Ratio Test, Theorem 4.5.13, though we did also look at the convergence of a few specific series; in these latter cases, we established convergence by showing that certain functions equal their Taylors series in Section 4.4.

In this section, we will state, but (usually) not prove a large number of results on convergence and divergence of series of constants. Starting with an infinite sequence b_k, for $k \geq m$, we add more and more terms of the sequence together in the *partial sums*, and see if these partial sums approach anything; if they do, then that thing is, by definition, the sum of the series $\sum_{k=m}^{\infty} b_k$, and we say that the series converges to, or equals, this limit. This means that a series is, in fact, the limit of a sequence, but it's the limit of the sequence of *partial sums*, not the sequence of terms.

The theorems that we shall state are usually referred to as *tests* for convergence or divergence.

Even though we gave the following definition earlier in Definition 4.5.3, it is appropriate to give it again here. After the definition, we also repeat some of the comments that we made earlier, because these comments are vitally important to understanding infinite series.

Definition 5.2.1. *Given a sequence b_k, for $k \geq m$, we define, for each $n \geq m$, the* **partial sum** *to be $\sum_{k=m}^{n} b_k$, and the* **infinite sum** *or* **infinite series** *or, simply,* **series** *to be the infinite summation $\sum_{k=m}^{\infty} b_k$, which, technically, consists of the sequence b_k together with the summation instruction/symbol, telling you to add the sequence. The terms of the sequence being added are also referred to as the* **terms of the series**.

The **sum of the series** *or* **value of the series** $\sum_{k=m}^{\infty} b_k$ *is the limit as* $n \to \infty$ *of the partial sums, and we write*

$$\sum_{k=m}^{\infty} b_k = \lim_{n\to\infty} \sum_{k=m}^{n} b_k,$$

provided the limit exists, in which case we say that the partial sums **converge**, *that the series* **converges**, *or simply that the infinite sum exists and is equal to the limit of the partial sums.*

If the limit of the partial sums does not exist, then we say that the series **diverges** *or, simply, that the infinite sum does not exist.*

Note that there are actually two sequences associated with an infinite sum/series: the sequence of things you're adding, the *terms*, and the *sequence of partial sums* $s_n = \sum_{k=m}^{n} b_k$. What we're interested in is what happens to the sequence of partial sums as $n \to \infty$.

As with approximating Taylor series by Taylor polynomials, the partial sums of any infinite series are frequently used as approximations of the infinite summation, assuming that the infinite sum exists, i.e., assuming that the series converges.

We are trying to define what an infinite sum should mean. Intuitively, it means the limit as you add more and more terms; this means that an infinite sum is the limit of the partial sums, **NOT** the limit of the terms. The terms are just the individual summands; they are not sums themselves.

Remark 5.2.2. You may wonder about the difference between a series and its value or sum. Technically, a series is actually the sequence, together with the summation sign(s), indicating that you want to add the given sequence, and the value or sum is what you get when you actually do the summation, i.e., the limit of the partial sums.

What's the point? Well...even though $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ (look back at Theorem 4.4.10 and use $a = 1/2$ and $x = 1/2$), you shouldn't talk about the series that consists of just the number 1. The series means the sum written out as a sum of the terms; its value is just a number (provided the series converges).

Fortunately, this distinction seems to cause no problem; when people discuss infinite series, there is always a clear sequence being summed.

One last point, which may be obvious. When we write something like $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, there is no need to assert separately that the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges; we've already stated what it converges to.

Okay. So, given an infinite series of constants, how do you find the sum? You might say "that's easy – just take the limit of the partial sums". The problem, and it's a big one, is that there are very few series, or types of series, for which we can write nice, manageable expressions for the partial sums. In fact, basically, there are only two kinds of series for which formulas for the partial sums are easy to obtain: *geometric series* and *telescoping series.*

Using different terminology, we derived the formula for the partial sums of geometric series way back in Corollary 2.1.11, and we looked at geometric series themselves in Theorem 4.4.10. However, we wish to restate these results now, using a different letter r for the base.

Theorem 5.2.3. *Consider the geometric series* $\sum_{k=0}^{\infty} ar^k$ *and the sequence of partial sums* $s_n = \sum_{k=0}^{n} ar^k$, *for* $n \geq 0$.

1. *If* $r \neq 1$, *then,*
$$s_n = a \cdot \frac{1-r^{n+1}}{1-r}.$$
2. *If* $|r| < 1$, *then the series converges to* $a/(1-r)$, *that is*
$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$
3. *If* $|r| \geq 1$ *and* $a \neq 0$, *then the series diverges.*

Example 5.2.4. A classic social problem arises when a number of people are at dinner, and there's one piece of bread left. No one wants to take the entire remaining piece, and so each person rips off part of the bread and takes that piece, leaving some behind.

Let's suppose that we start with one piece of bread, and that each person who takes some bread takes $1/3$ of the remaining bread. What do we get if we keep track of the total bread taken (as fractions of the initial 1 piece) and the total bread remaining, assuming that this thirding-process goes on forever (ignoring that this is physically impossible and/or that we would at some point split atoms and blow up everything).

The first person takes $1/3$, leaving behind $2/3$. The second person takes $1/3$ of the remaining $2/3$, so $(1/3)(2/3)$ of the original one piece, leaving behind $(2/3)(2/3) = (2/3)^2$. The third person takes $(1/3)(2/3)^2$, leaving behind $(2/3)(2/3)^2 = (2/3)^3$, and so on.

After n iterations of bread-dividing, we find that the total bread taken is

$$\frac{1}{3} + \frac{1}{3}\left(\frac{2}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right)^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \cdots + \frac{1}{3}\left(\frac{2}{3}\right)^{n-1},$$

while the bread remaining is simply $\left(\frac{2}{3}\right)^n$.

Hopefully, you see that the sum for the amount of bread taken is a geometric sum. Using the notation from Theorem 5.2.3, we have $a = 1/3$, $r = 2/3$, and the summation above is s_{n-1}, which by Theorem 5.2.3, is given by

$$s_{n-1} = \sum_{k=0}^{n-1} ar^k = \frac{1}{3} \cdot \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} = 1 - \left(\frac{2}{3}\right)^n.$$

This, of course, is what we'd **better** get; the total bread taken should equal the initial 1 minus the bread remaining $(2/3)^n$.

Thus, not surprisingly, the infinite sum of all the bread taken (ignoring explosions) is 1, the entire piece of bread, and you can reach this conclusion by summing the series or by seeing that the limit of remaining bread $\lim_{n\to\infty}(2/3)^n$ is zero.

Example 5.2.5. Theorem 5.2.3 tells us instantly that $\sum_{k=0}^{\infty}(1.1)^k$ diverges, while $\sum_{k=0}^{\infty} 100(0.9)^k$ converges to $100/(1-0.9) = 1000$.

What if these series hadn't been written with the general summations formulas? For instance, what if you were given

$$100 + 100(0.9) + 100(0.9)^2 + 100(0.9)^3 + \cdots ?$$

Technically, the first four terms don't determine the series, but sometimes we rely on your being able to see "clear" patterns. You are supposed to see that you go from each term to the next by multiplying by the same thing each time – here, 0.9. Once you see that each term is obtained from the previous one by multiplying by the same thing each time, the series is a geometric series, and the thing that you're multiplying by each time is the r in Theorem 5.2.3. The initial term – here, 100 – is the a in Theorem 5.2.3. Now that you've identified the series as being geometric, and have determined the r and the a, you apply Theorem 5.2.3.

Suppose the indexing variable doesn't start at 0. What can we say about the convergence/divergence of

$$\sum_{k=2}^{\infty}(1.1)^k \qquad \text{and} \qquad \sum_{k=2}^{\infty}100(0.9)^k \text{ ?}$$

There are two options here. One is to reindex, and the other is to realize that leaving off the first two terms from our original series (the series that started at $k=0$) changes each partial sum by the same fixed amount. Let's look at both approaches.

- Suppose we let $j = k-2$, so that, when k starts at 2, j starts at 0. Then, $k = j+2$, and our two sums become

$$\sum_{k=2}^{\infty}(1.1)^k \;=\; \sum_{j=0}^{\infty}(1.1)^{j+2} \;=\; \sum_{j=0}^{\infty}(1.1)^2(1.1)^j$$

and

$$\sum_{k=2}^{\infty}100(0.9)^k \;=\; \sum_{j=0}^{\infty}100(0.9)^{j+2} \;=\; \sum_{j=0}^{\infty}100(0.9)^2(0.9)^j.$$

Even though the indexing variable is now j, instead of k, that doesn't change anything. In fact, at this point, if you really wanted to, you could replace the j's with k's. Either way, you should see that both series are geometric, with r still equaling 1.1 and 0.9, respectively. Hence, the first series diverges, while the second series converges to

$$\frac{100(0.9)^2}{1-0.9} \;=\; 810.$$

If you're worried that you would never think to switch indices by letting $j = k-2$, don't let that bother you; if you just wrote out a few terms, you'd be able to rewrite the summation with a different index, or you'd recognize the series as a new geometric series, with the same r, but a different a. For instance, you'd get

$$\sum_{k=2}^{\infty}(1.1)^k \;=\; (1.1)^2 \;+\; (1.1)^3 \;+\; (1.1)^4 \;+\; (1.1)^5 \;+\; \cdots\,,$$

and should immediately see that you have a geometric series with $r = 1.1$, since that's what you multiply by to get from one term to the next, and the initial term is $a = (1.1)^2$. This would be enough data to answer any geometric series question, but, if you really want to write this geometric series, using an index which starts at 0, you would get

$$\sum_{j=0}^{\infty}(1.1)^2(1.1)^j \qquad \text{or} \qquad \sum_{k=0}^{\infty}(1.1)^2(1.1)^k,$$

or you could use some other indexing variable.

- Another method for dealing with

$$\sum_{k=2}^{\infty}(1.1)^k \qquad \text{and} \qquad \sum_{k=2}^{\infty}100(0.9)^k$$

is to realize that these are the same series as those starting at $k = 0$, except that the first two terms are missing from the sum, i.e.,

$$\sum_{k=2}^{\infty}(1.1)^k = \left(\sum_{k=0}^{\infty}(1.1)^k\right) - \left(\sum_{k=0}^{1}(1.1)^k\right)$$

and

$$\sum_{k=2}^{\infty}100(0.9)^k = \left(\sum_{k=0}^{\infty}100(0.9)^k\right) - \left(\sum_{k=0}^{1}100(0.9)^k\right).$$

In fact, these subtractions exist on the level of the partial sums, i.e.,

$$\sum_{k=2}^{n}(1.1)^k = \left(\sum_{k=0}^{n}(1.1)^k\right) - \left(\sum_{k=0}^{1}(1.1)^k\right)$$

and

$$\sum_{k=2}^{n}100(0.9)^k = \left(\sum_{k=0}^{n}100(0.9)^k\right) - \left(\sum_{k=0}^{1}100(0.9)^k\right).$$

It follows that the divergence of $\sum_{k=0}^{\infty}(1.1)^k$ implies the divergence of $\sum_{k=2}^{\infty}(1.1)^k$, and the convergence of $\sum_{k=0}^{\infty}100(0.9)^k$ to $100/(1-0.9) = 1000$ implies that $\sum_{k=2}^{\infty}100(0.9)^k$ converges to

$$1000 - \left(\sum_{k=0}^{1}100(0.9)^k\right) = 1000 - \big(100 + 100(0.9)\big) = 810,$$

as we found before.

What we observed in the example above about leaving off a finite number of terms in a series is worth stating as a proposition.

Proposition 5.2.6. *Suppose that b_k, for $k \geq m$, is a sequence of real numbers and $n \geq m$. Then, $\sum_{k=m}^{\infty} b_k$ converges if and only if $\sum_{k=n+1}^{\infty} b_k$ converges, and when the series converge,*

$$\sum_{k=m}^{\infty} b_k \;=\; \sum_{k=m}^{n} b_k \;+\; \sum_{k=n+1}^{\infty} b_k.$$

In words, omitting or including a finite number of terms in a series does not affect whether or not the series converges or diverges; in the case of convergence, the sum of the series is changed by the sum of terms which are omitted/included.

Remark 5.2.7. Because of Proposition 5.2.6, when we are discussing whether or not a series converges, but not precisely what it might converge to, we sometimes write merely "the series $\sum b_k$", knowing that the sum goes out to ∞ and that, as far as convergence/divergence is concerned, it doesn't matter where we start the summation (though it is tacitly assumed that the series starts at some m such that b_k is defined for all $k \geq m$.)

Example 5.2.8. Before we leave the topic of geometric series, we should mention their relationship with *repeating decimal expansions* of real numbers.

It is beyond the scope of the subject matter in this book to prove that every rational number, i.e., every quotient of two integers (where the denominator is not 0), has a terminating or repeating decimal expansion, but, given a repeating decimal, our knowledge of geometric series does tell us how to write the repeating decimal as a quotient of integers.

Consider the real number given by the infinite repeating decimal

$$123.45678678678\overline{678}.$$

We would like to know what this is as a fraction. Let's call the number x. Then, $x - 123.45 = 0.00678678\overline{678}$, and so

$$100(x - 123.45) \;=\; 0.678678\overline{678}.$$

Therefore, if we knew $0.678678\overline{678}$ as a fraction, we'd be able to write x as a fraction.

However, $0.678678\overline{678}$ **means** the sum of the geometric series

$$0.678678\overline{678} \;=\; \frac{678}{1000} \;+\; \frac{678}{(1000)^2} \;+\; \frac{678}{(1000)^3} \;+\; \frac{678}{(1000)^4} \;+\; \cdots,$$

and we know how to sum this. It's a geometric series, with $a = 678/1000$ and $r = 1/1000$. Thus,

$$0.678678\overline{678} \;=\; \frac{a}{1-r} \;=\; \frac{\frac{678}{1000}}{1-\frac{1}{1000}} \;=\; \frac{678}{999}.$$

This same argument, applied to the general case of a repeating block, which starts in the 1/10's place, shows that such a repeating decimal is equal to the fraction with the repeating block in the numerator and a denominator with a matching number of 9's as digits.

As for our original x, we see that it's given by

$$x \;=\; \frac{12345}{100} \;+\; \frac{1}{100}\cdot\frac{678}{999},$$

where we'll let you finish by finding a common denominator and adding the numerators.

Of course, we should mention the result that perplexes so many people:

$$0.999999\overline{9} \;=\; \frac{9}{10} \;+\; \frac{9}{(10)^2} \;+\; \frac{9}{(10)^3} \;+\; \frac{9}{(10)^3} \;+\; \cdots \;=\; \frac{\frac{9}{10}}{1-\frac{1}{10}} \;=\; 1.$$

This bothers people because the thinking is that the repeating 9's never "jump the decimal point", and that 1 should be the "next" real number after $0.999999\overline{9}$. We'll make two comments about this.

First, given two real numbers $a < b$, there are always an infinite number of real numbers in-between a and b. For instance, $(a+b)/2$ would be halfway between a and b; so, there is never a "next real number". Second, infinite decimal expansions are **defined** to be the sum of the infinite series formed by placing each digit over the appropriate power of 10. This means that, by definition, an infinite decimal expansion is a **limit**; the limit of the partial sums. If you think of $0.999999\overline{9}$ this way, there shouldn't be any confusion. What real number is being approached by the partial sums

$$0.9,\ 0.99,\ 0.999,\ 0.9999,\ 0.99999,\ 0.999999,\ \dots\ ?$$

Uhhhhh...1, obviously.

We wrote earlier that, aside from geometric series, there's only one other type of series for which we can find nice formulas for the partial sums: *telescoping series*.

Recall, back in Definition 2.1.6, that, starting with a function f, the finite difference function Δf is defined by $(\Delta f)(k) = f(k) - f(k-1)$, where, if $f(k)$ is defined for integers k such that $m \leq k \leq n$, then $(\Delta f)(k)$ is defined for $m+1 \leq k \leq n$. For instance, $\Delta k^2 = k^2 - (k-1)^2 = 2k-1$ (as we saw in Proposition 2.1.10).

The definition that we give below contains the ambiguous term "clear"; we will try to explain the point, and the problem with giving a precise definition, in the example that comes after the definition.

Definition 5.2.9. *A series $\sum_{k=m}^{\infty} b_k$ is* **telescoping** *provided that there is a clear, elementary function f such that, for all $k \geq m$, $b_k = (\Delta f)(k)$.*

Our interest in telescoping series stems from Proposition 2.1.9, which, in our current context, tells us precisely:

Proposition 5.2.10. *Suppose that we have a (telescoping) series $\sum_{k=m}^{\infty} b_k$ in which $b_k = (\Delta f)(k)$, for some f which is defined for all integers greater than or equal to $m-1$.*

Then, for all $n \geq m$, the partial sum $s_n = \sum_{k=m}^{n} b_k$ satisfies the equality

$$s_n \;=\; f(n) - f(m-1).$$

Therefore, the series $\sum_{k=m}^{\infty} b_k$ *converges if and only if the sequence* $f(n)$ *converges, i.e., if and only if* $\lim_{n\to\infty} f(n)$ *exists, and when the series converges, it converges to*

$$\left(\lim_{n\to\infty} f(n)\right) \; - \; f(m-1).$$

Recall that the proof for the above statement about the partial sums is not complicated; it's just that things cancel in pairs in the summation:

$$\sum_{k=m}^{n} b_k \;=\; b_n \;+\; b_{n-1} \;+\; \cdots \;+\; b_{m+1} \;+\; b_m \;=$$

$$\big(f(n)-f(n-1)\big) \;+\; \big(f(n-1)-f(n-2)\big) \;+\; \cdots\; \big(f(m+1)-f(m)\big) \;+\; \big(f(m)-f(m-1)\big) \;=$$

$$f(n) \;-\; f(m-1).$$

It's true that the rigorous **proof** requires mathematical induction, but the above observation is the heart of the matter.

Example 5.2.11. To produce a telescoping series, you can simply pick an f and then take its finite difference. For instance, we'll take $f(k) = -1/k$. Then,

$$(\Delta f)(k) \;=\; \frac{-1}{k} \;-\; \frac{-1}{k-1} \;=\; \frac{1}{k-1} \;-\; \frac{1}{k}$$

and

$$\sum_{k=2}^{\infty}\left[\frac{1}{k-1} \;-\; \frac{1}{k}\right]$$

is a basic example of a telescoping series. We can now apply Proposition 5.2.10 to conclude that $\sum_{k=2}^{\infty}\left[\frac{1}{k-1} \;-\; \frac{1}{k}\right]$ converges, and

$$\sum_{k=2}^{\infty}\left[\frac{1}{k-1} \;-\; \frac{1}{k}\right] \;=\; \left(\lim_{n\to\infty} f(n)\right) \;-\; f(1) \;=\; \lim_{n\to\infty}\frac{-1}{n} \;-\; (-1) \;=\; 0 \;+\; 1 \;=\; 1.$$

Of course, we could "simplify" the terms of this series by using that

$$\frac{1}{k-1} - \frac{1}{k} = \frac{k-(k-1)}{k(k-1)} = \frac{1}{k(k-1)}.$$

So, now we see that

$$\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = \sum_{k=2}^{\infty} \left[\frac{1}{k-1} - \frac{1}{k}\right] = 1.$$

The question is: do we want to say that the series $\sum_{k=2}^{\infty} \frac{1}{k(k-1)}$ telescopes? No, because the terms are not explicitly written as the difference function of some f. Of course, what's not clear to one person may be clear to another – still, not many people could look at $\frac{1}{k(k-1)}$ and think "ah - that's clearly $(\Delta f)(k)$, where $f(k) = 1/k$".

You should note how well-disguised the telescoping nature of the series is now; we've shown that

$$\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \frac{1}{5 \cdot 4} + \cdots = 1,$$

and there is no cancellation of pairs staring us in the face that makes this summation obvious.

You might think that it shouldn't matter if the terms of the series are **clearly** a finite difference. In the definition of a telescoping series $\sum_{k=m}^{\infty} b_k$, why don't we just go ahead and write that there just has to **exist** an elementary function f, or maybe just an arbitrary function f, such that $b_k = (\Delta f)(k)$?

The problem is that, for every series $\sum_{k=m}^{\infty} b_k$, there **always** exists a function f such that, for all $k \geq m$, $b_k = (\Delta f)(k)$, and, in general, it is difficult/impossible to know, without a good bit of work, whether such a function f is elementary.

Consider any series $\sum_{k=m}^{\infty} b_k = \sum_{j=m}^{\infty} b_j$. Define $s(m-1) = 0$, and, for $k \geq m$, define the k-th partial sum $s(k) = \sum_{j=m}^{k} b_j$. Then, $(\Delta s)(m) = s(m) - 0 = b_m$, and, for $k \geq m$,

$$(\Delta s)(k) = s(k) - s(k-1) = \sum_{j=m}^{k} b_j - \sum_{j=m}^{k-1} b_j = b_k.$$

Understand the situation we're in. We want telescoping series so that we can obtain nice formulas for the partial sums via Proposition 5.2.10. What we've just seen is that the partial sums yield a function which tells us that any series telescopes, but, if we use the partial sum function for the telescoping, we arrive at the useless result that knowing the partial sums would enable us to determine the partial sums. Hence, we need to put some restrictions on the type, or clarity, of the function f that we're allowed to choose such that $b_k = (\Delta f)(k)$.

The point is, in order for Proposition 5.2.10 to be useful, the function f needs to be manageable, i.e., elementary, and it needs to be easily discernible.

One of the things that we just saw in the previous example leads us to an interesting result, a result that can tell us that there's an easy way to see that some series **diverge**.

Suppose that we have a series $\sum_{k=m}^{\infty} b_k$, and it converges to L. If we let s_n be the partial sum $s_n = \sum_{k=m}^{n} b_k$, then, as we saw in the example above, $b_n = s_n - s_{n-1}$. However, as the series converges to L, the partial sums converge to L, and so $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} = L$. Therefore,

$$\lim_{n\to\infty} b_n \;=\; \lim_{n\to\infty} (s_n - s_{n-1}) \;=\; \lim_{n\to\infty} s_n \;-\; \lim_{n\to\infty} s_{n-1} \;=\; L - L \;=\; 0.$$

Thus, if a series converges, then the terms **must** approach 0. This is usually used in the logically equivalent, contrapositive, form, which we state first.

Theorem 5.2.12. (Term Test for Divergence)

1. *If the terms of a series do not approach* 0*, then the series diverges.*
2. *Equivalently, if a series converges, then the terms* **must** *approach* 0*.*
3. *If the terms of a series approach* 0*, the series may either converge or diverge, i.e., you can conclude nothing merely from the fact that the terms approach* 0*.*

Item 2 of the Term Test tells us that any convergent series must have its terms approach 0; so that every convergent series that we've discussed so far, like the alternating harmonic series or geometric series with $|r| < 1$, are examples of convergent series for which the terms approach 0. However, we have mentioned before the *harmonic series*,

$$\sum_{k=1}^{\infty} \frac{1}{k} \;=\; 1 \;+\; \frac{1}{2} \;+\; \frac{1}{3} \;+\; \frac{1}{4} \;+\; \cdots,$$

which is a fundamental example of a series in which the terms approach 0, and yet the series still diverges. We will show that the harmonic series diverges in Proposition 5.2.16, after we give some easy examples of using the Term Test for Divergence. But what's going on? How can a series diverge if the terms approach 0?

The point of the harmonic series example is to show that **it's not enough for the terms of a series to approach 0 in order to conclude convergence of the series; the terms need to approach 0 fast enough to make up for the fact that we're adding more and more of them together**.

The reason that the Term Test is called the Term Test **for Divergence** is that the test tells you that some series diverge, but it can **NEVER** tell you, by itself, that a series converges.

Example 5.2.13. Consider the series

$$\sum_{k=1}^{\infty} \left(\frac{1}{4} + \frac{7}{k} \right).$$

As $k \to \infty$, the terms $\frac{1}{4} + \frac{7}{k}$ approach $\frac{1}{4}$, not 0. Therefore, the series diverges by the Term Test for Divergence, Theorem 5.2.12.

Understand what's going on here: the sequence of **terms** converges, and what the terms converge to is $1/4$, but the value of the series is what you get when you add together an infinite number of terms, so, basically, in the series, it's as though you're **adding together an infinite number of $1/4$'s**; not surprisingly, this adds up to ∞, and so the series diverges to ∞.

It is a fundamental error to confuse the limit of the terms with the sum of the series. The sum of the series is absolutely **not** the limit of the terms. The terms are what are being added in the series. The sum of the series is the limit of the partial sums, which is thus what is approached as you add more and more terms.

Example 5.2.14. What does the Term Test for Divergence tell us about the convergence or divergence of

$$\sum_{k=1}^{\infty} \frac{7}{k} \, ?$$

It is tempting to say "ah – now the terms approach 0, so the series must converge, by the Term Test for Divergence".

But remember Item 3 of the Term Test of Divergence; you can conclude **nothing** about the convergence or divergence of the series when the terms approach 0.

In fact, the series $\sum_{k=1}^{\infty} \frac{7}{k}$ is, term-wise, 7 times the harmonic series, and so, like the harmonic series, it diverges.

Example 5.2.15. What about the series $\displaystyle\sum_{k=0}^{\infty}(-1)^k$?

Here, the limit of the terms does not exist, since, as k gets bigger, $(-1)^k$ continues to alternate between -1 and 1. Does the Term Test for Divergence tell us anything about this series?

Yes - if the limit of the terms fails to exist, then it is true that "the terms of the series do not approach 0"; in fact, the terms don't approach anything, so, in particular, they don't approach 0. Hence, the series $\sum_{k=0}^{\infty}(-1)^k$ diverges by the Term Test for Divergence.

We have put it off long enough. We are going to show that the harmonic series diverges. In fact, this will follow, as a special case of our p-series Test, Corollary 5.3.15, but we give a more basic argument here because it is instructive.

Proposition 5.2.16. *The harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{5.1}$$

diverges to ∞.

Proof. We want to show that, as $n \to \infty$, the sequence of partial sums

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

goes to ∞.

Note that, as each partial sum is obtained from earlier partial sums by adding **positive** terms, the sequence of partial sums S_n is increasing. Therefore, by Theorem 5.1.23, the sequence S_n diverges if and only if the sequence is unbounded.

We shall find a lower bound on **some** of the partial sums, the partial sums of the form S_{2^p}, and show that this lower bound approaches ∞, which forces the harmonic series to diverge. Technically, we should use mathematical induction, but we'll make the idea clear.

The "trick" is to group together collections of terms in such a way that each group clearly adds up to something greater than 1/2. We will indicate the groups with parentheses:

$$(1)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)+\cdots.$$

You always end a group at 1 over a power of 2, and start the group at 1 over the next number after the last power of 2. Notice that, in each group, all of the summands are greater than or equal to the last one in that group. So that, what we've written above is clearly greater than or equal to

$$1 \;+\; \frac{1}{2} \;+\; 2\cdot\frac{1}{4} \;+\; 4\cdot\frac{1}{8} \;+\; 8\cdot\frac{1}{16} \;+\; \cdots \;=\; 1 \;+\; \frac{1}{2} \;+\; \frac{1}{2} \;+\; \frac{1}{2} \;+\; \frac{1}{2} \;+\; \cdots.$$

Thus, what we find is that, for each $p \geq 1$,

$$S_{2^p} \geq \; 1+\frac{p}{2}.$$

As $p \to \infty$, $1+\frac{p}{2} \to \infty$, and thus $S_{2^p} \to \infty$. Therefore, the increasing sequence of partial sums is unbounded and approaches ∞. □

The harmonic series should be an example that you always keep in mind. It's the most basic example of a series in which the terms approach 0, and yet the series nonetheless **diverges**. This shows that you **cannot** conclude convergence of a series simply because the terms approach 0.

Multiplying a series by a constant, and adding two series works as you'd expect. The proofs using limits of partial sums are straightforward; we leave them for you as an exercise.

Theorem 5.2.17. *Suppose that we have infinite series* $\sum_{k=m}^{\infty} a_k$ *and* $\sum_{k=m}^{\infty} b_k$.

1. *Assume that* $c \neq 0$. *Then, the series* $\sum_{k=m}^{\infty} a_k$ *converges if and only if the series* $\sum_{k=m}^{\infty} ca_k$ *converges and, when the series converge,*

$$\sum_{k=m}^{\infty} ca_k \;=\; c \cdot \sum_{k=m}^{\infty} a_k.$$

2. *Suppose that* $\sum_{k=m}^{\infty} a_k$ *converges. Then,* $\sum_{k=m}^{\infty} b_k$ *converges if and only if* $\sum_{k=m}^{\infty}(a_k + b_k)$ *converges and, when the series converge,*

$$\sum_{k=m}^{\infty} (a_k + b_k) \;=\; \sum_{k=m}^{\infty} a_k \;+\; \sum_{k=m}^{\infty} b_k.$$

By letting $c = -1$ in Item 1 above, and combining with Item 2, you can, naturally, obtain an analogous result about subtracting one series from another.

Example 5.2.18. By Theorem 5.2.3, the series $\sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^k$ converges and, by Remark 4.4.13 and Proposition 5.2.6, the series $\sum_{k=2}^{\infty}(-1)^{k-1}\left(\frac{1}{k}\right)$ also converges (it's the alternating harmonic series, except that the first term is missing). In fact, we know what these series converge to:

$$\sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^k \;=\; \left[\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k\right] - 1 - \frac{1}{3} \;=\; \frac{1}{1-\frac{1}{3}} - \frac{4}{3} \;=\; \frac{1}{6},$$

and

$$\sum_{k=2}^{\infty}(-1)^{k-1}\left(\frac{1}{k}\right) \;=\; \left[\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{k}\right)\right] - 1 \;=\; (\ln 2) - 1.$$

Thus, Theorem 5.2.17 tells us that

$$\sum_{k=2}^{\infty}\left[6\left(\frac{1}{3}\right)^{k} + (-1)^{k}\left(\frac{1}{k}\right)\right] = \sum_{k=2}^{\infty}\left[6\left(\frac{1}{3}\right)^{k} - (-1)^{k-1}\left(\frac{1}{k}\right)\right] =$$

$$6\cdot\sum_{k=2}^{\infty}\left(\frac{1}{3}\right)^{k} - \sum_{k=2}^{\infty}(-1)^{k-1}\left(\frac{1}{k}\right) = 6\cdot\frac{1}{6} - \big((\ln 2)-1\big) = 2-\ln 2.$$

Example 5.2.19. You need to be careful not to try to apply Theorem 5.2.17 to sums of two **divergent** series. The term-by-term sum of two divergent series may diverge or converge.

For instance, the harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges. If you take the harmonic series, and add it to itself term-wise, you get

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}+\frac{1}{k}\right) = \sum_{k=1}^{\infty}\frac{2}{k},$$

which diverges by Item 1 of Theorem 5.2.17, since it's 2 times the harmonic series (in the term-wise sense). What this shows is that the term-by-term sum of two divergent series may again be divergent.

On the other hand, using Item 1 of Theorem 5.2.17 with $c=-1$, we conclude that $\sum_{k=1}^{\infty}\left(-\frac{1}{k}\right)$ diverges. But the term-wise sum of $\sum_{k=1}^{\infty}\frac{1}{k}$ and $\sum_{k=1}^{\infty}\left(-\frac{1}{k}\right)$ is

$$\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k}\right] = \sum_{k=1}^{\infty}0 = 0,$$

so that the term-wise sum of these two divergent series converges to 0.

The point is that the term-by-term sum of two divergent series may converge or may diverge. There is no theorem that quickly tells you what happens.

However, you may correctly suspect that the above example of two divergent series being added, term-wise, and producing a convergent series can happen only because of cancellation. If all of the terms of both series had been ≥ 0, or all of the terms of both series had been ≤ 0, then the term-by-term sum of two divergent series would again be divergent. We will see this in the next section.

In the example above, we saw that standard algebraic operations may not work so well when dealing with divergent series. Actually, things may be worse than you suspect.

Suppose that we have a series

$$\sum_{k=m}^{\infty} a_k \;=\; a_m \;+\; a_{m+1} \;+\; a_{m+2} \;+\; a_{m+3} \;+\; a_{m+4} \;+\; a_{m+5} \;+\; \cdots .$$

Since addition is associative, you might suspect that we can throw in parentheses anywhere that we want and not change the sum. However, if the series diverges, it is not true that grouping the terms necessarily leads to a series that continues to diverge.

Example 5.2.20. Consider the series

$$1 \;-\; 1 \;+\; 1 \;-\; 1 \;+\; 1 \;-\; 1 \;+\; \cdots .$$

We write this summation without any parentheses, because we assume that it's obvious that we mean the series $\sum_{k=1}^{\infty}(-1)^{k-1}$. This series diverges by the Term Test for Divergence, Theorem 5.2.12; the terms do not approach 0. However, you're so used to the associativity of addition that you might believe that this series is the same as

$$(1 \;-\; 1) \;+\; (1 \;-\; 1) \;+\; (1 \;-\; 1) \;+\; \cdots$$

or

$$1 \;+\; (-\,1 \;+\; 1) \;+\; (-\,1 \;+\; 1) \;+\; (-\,1 \;+\; 1) \;+\cdots ,$$

but these latter two series converge to 0 and 1, respectively, assuming that the groupings indicate the terms of the series. Thus, it is possible to start with a divergent series and, by grouping some terms together to form a new series, arrive at convergent series, and it's possible to get convergent series which converge to different numbers.

However, for convergent series, "associativity" works as you would expect.

Theorem 5.2.21. *Suppose that the terms of a series* $\sum_{j=n}^{\infty} b_j$ *are formed from a convergent series* $\sum_{k=m}^{\infty} a_k$ *by grouping disjoint finite sums of the* a_k *without changing the order, i.e., suppose that the series* $\sum_{j=n}^{\infty} b_j$ *is formed by "adding some parentheses" to the convergent series* $\sum_{k=m}^{\infty} a_k$*.*

Then, $\sum_{j=n}^{\infty} b_j$ *converges, and* $\displaystyle\sum_{j=n}^{\infty} b_j = \sum_{k=m}^{\infty} a_k$*.*

Proof. The partial sums of $\sum_{j=n}^{\infty} b_j$ are sums of the a_k's, in order, and so form a subsequence of the sequence of partial sums of $\sum_{k=m}^{\infty} a_k$, which we are assuming is a convergent sequence. Theorem 5.1.27 immediately yields the theorem. □

As we shall see in the next section, if all of the terms in a series are non-negative, it is **not** possible to group a divergent series in such a way that you get a convergent series; that is, the weird thing that happened in Example 5.2.20 can't happen if all of the terms in the series are non-negative.

5.2.1 Exercises

In each of Exercises 1 through 10, list the first 5 terms of the series, and the first 5 partial sums. Give the partial sums written out as summations, and (possibly using a calculator), also give each partial sum as a single number.

1. $$\frac{1}{1} + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \frac{1}{13} + \frac{1}{16} + \frac{1}{19} + \cdots$$

2. $$\frac{1}{1} - \frac{1}{5} + \frac{1}{10} - \frac{1}{15} + \frac{1}{20} - \frac{1}{25} + \frac{1}{30} + \cdots$$

3. $\displaystyle\sum_{k=0}^{\infty} (-2)^k$

4. $\displaystyle\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k$

5. $\displaystyle\sum_{k=1}^{\infty} \left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right]$

6. $\displaystyle\sum_{j=1}^{\infty} \frac{j}{j^3+1}$

7. $\displaystyle\sum_{p=-3}^{\infty} p^2$

8. $\displaystyle\sum_{p=-3}^{\infty} (-1)^p p^2$

9. $\displaystyle\sum_{n=4}^{\infty} \frac{2}{n-3}$

10. $\displaystyle\sum_{k=-2}^{\infty} \frac{2}{k+3}$

In each of Exercises 11 through 20, you are given m and a formula for the partial sum $s_n = \sum_{i=m}^{n} b_i$ of an infinite series $\sum_{i=m}^{\infty} b_i$. (a) Give the terms $b_m, \ldots, b_{m+4}$, (b) give a formula for the general term b_i, and (c) determine whether the series $\sum_{i=m}^{\infty} b_i$ converges or diverges; if it converges, determine what it converges to.

11. $s_n = \dfrac{1}{n}, \;\; m = 1$

12. $s_n = 1 - \dfrac{1}{n}, \;\; m = 1$

13. $s_n = (-1)^n, \;\; m = 0$

14. $s_n = 1, \;\; m = 0$

15. $s_n = 2 + \dfrac{n}{e^n}, \;\; m = 0$

16. $s_n = \dfrac{5}{\sqrt{n}}, \;\; m = 1$

17. $s_n = \dfrac{1}{(n-2)(n-1)}, \;\; m = 3$

18. $s_n = \ln(5n^2) - \ln(n^2+1), \;\; m = 1$

19. $s_n = \dfrac{2n+3}{n}, \;\; m = 1$

20. $s_n = \dfrac{2^n}{n!}, \;\; m = 1$

In each of Exercises 21 through 30, you are given a series which is geometric, telescopic, or diverges by the Term Test for Divergence, Theorem 5.2.12. Determine if the series converges or diverges; if it converges, determine what it converges to.

21. $\displaystyle\sum_{k=1}^{\infty}\left[1+\frac{1}{k}\right]$

22. $\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{k+1}-\frac{1}{k}\right]$

23. $\displaystyle\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{k}$

24. $\displaystyle\sum_{j=0}^{\infty}\frac{7}{5^{j}}$

25. $\displaystyle\sum_{j=0}^{\infty}\left(\frac{7}{5}\right)^{j}$

26. $\displaystyle\sum_{j=0}^{\infty}\left(\frac{5}{7}\right)^{j}$

27. $\displaystyle\sum_{j=0}^{\infty}\left(-\frac{5}{7}\right)^{j}$

28. $\displaystyle\sum_{p=1}^{\infty}\left[(p+1)^{2}-p^{2}\right]$

29. $\displaystyle\sum_{q=1}^{\infty}\left[\tan^{-1}(q)-\tan^{-1}(q+1)\right]$

30. $\displaystyle\sum_{q=1}^{\infty}\tan^{-1}(q)$

In each of Exercise 31 through 40, determine if the given series converges or diverges. If it converges, determine what it converges to.

31. $\displaystyle\sum_{k=1}^{\infty}\left[\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{5(-1)^{k}}{3^{k}}\right]$

32. $\displaystyle\sum_{p=1}^{\infty}\frac{1}{10p}$

33. $\displaystyle\sum_{p=1}^{\infty} \frac{(-1)^p}{10p}$

34. $\displaystyle\sum_{j=1}^{\infty} \left[\frac{1}{j} + \frac{5(-1)^j}{3^j}\right]$

35. $\displaystyle\sum_{r=0}^{\infty} \left[\frac{2}{7^r} + \frac{5(-1)^r}{3^r}\right]$

36. $\displaystyle\sum_{r=0}^{\infty} \left[\frac{r}{7} + \frac{5(-1)^r}{3^r}\right]$

37. $\displaystyle\sum_{k=1}^{\infty} \left[\frac{k-1}{k} + \frac{3}{2^k}\right]$

38. $\displaystyle\sum_{k=1}^{\infty} \left[\frac{3}{2^k} + 5\left(\sin\left(\frac{1}{k}\right) - \sin\left(\frac{1}{k+1}\right)\right)\right]$

39. $\displaystyle\sum_{k=1}^{\infty} \left[\frac{-7}{4^k} + 5\left((k+1)^2 - k^2\right)\right]$

40. $\displaystyle\sum_{k=0}^{\infty} \left[\frac{3}{2^k 7} + \frac{1}{3^k 100}\right]$

In each of Exercise 41 through 44, write the repeating decimal as a quotient of positive integers.

41. $0.230123012301\overline{2301}$

42. $0.1415727272\overline{72}$

43. $2.7182818281\overline{8281}$

44. $123.123123\overline{123}$

45. Suppose that $\sum_{k=1}^{\infty} a_k = 7$ and $\sum_{k=1}^{\infty} b_k = -2$. What can you conclude about the series $\sum_{k=1}^{\infty}(2a_k - 3b_k)$?

46. Suppose that $\sum_{k=1}^{\infty} a_k = 3$ and $\sum_{k=1}^{\infty} b_k = -3$. What can you conclude about the series $\sum_{k=1}^{\infty}(2a_k - 3b_k)$?

47. Suppose that $\sum_{k=1}^{\infty} a_k = 7$ and $\sum_{k=1}^{\infty} b_k$ diverges. What can you conclude about the series $\sum_{k=1}^{\infty}(2a_k - 3b_k)$?

48. Suppose that $\sum_{k=1}^{\infty} a_k$ diverges and $\sum_{k=1}^{\infty} b_k$ diverges. What can you conclude about the series $\sum_{k=1}^{\infty}(2a_k - 3b_k)$?

49. Consider the series

$$\sum_{k=0}^{\infty} a_k \;=\; \frac{3}{2} \;-\; 1 \;+\; \frac{5}{4} \;-\; 1 \;+\; \frac{9}{8} \;-\; 1 \;+\; \frac{17}{16} \;-\; 1 \;+\; \cdots,$$

in which, for all $n \geq 0$, $a_{2n} = (2^{n+1}+1)/2^{n+1}$ and $a_{2n+1} = -1$.

   a. Show that the series diverges.

   b. By grouping terms together, i.e., by inserting parentheses into the summation, show that you can obtain a convergent series.

50. Consider the series $\displaystyle\sum_{k=1}^{\infty}\left[(-1)^k + \frac{1}{k^2}\right]$.

   a. Show that the series diverges.

   b. By grouping terms together, i.e., by inserting parentheses into the summation, show that you can obtain a convergent series.

51. Explain why the harmonic series is an important example, i.e., what it is an important example of.

52. Prove that the "even harmonic series" $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2k}$ diverges.

53. Prove that the "odd harmonic series" $\displaystyle\sum_{k=0}^{\infty}\frac{1}{2k+1}$ diverges.

54. Prove Theorem 5.2.17, on multiplying series by constants and adding two series.

55. A man is jogging home, accompanied by his dog. The man is a mile from home, and jogs at a constant speed of 5 miles per hour. His dog runs in front of him at a constant speed of 10 miles per hour and, when the dog gets home, he turns around and runs, at 10 mph, back to the man. When the dog reaches the man, the dog turns around and runs home again, at 10 mph. The dog continues running home and returning the man, at 10 mph, for the entire time it takes for the man to jog home. Assume that the time it takes the dog to turn around is negligible, so that the dog is really running at 10 mph for the whole trip.

   a. Write an infinite series for the total distance traveled by the dog. Your series should have as terms the distance traveled by the dog during each leg of his trip.

b. Sum the series from part (a).

c. Use a simple distance/speed/time argument to obtain the same answer that you obtained in part (b).

5.3 Theorems on Series II: Non-negative Series

Series in which all of the terms are greater than or equal to zero, *non-negative series*, or in which all of the terms are less than or equal to zero, *non-positive series*, are easier to deal with than series that contain both positive and negative terms. However, all of the results on convergence/divergence for non-positive series can be obtained by negating, term-by-term, to obtain non-negative series, and then looking at the results for non-negative series. Consequently, it is customary to state the results, the convergence and divergence tests, for non-negative series only, leaving the results for non-positive series as quick corollaries.

Non-negative series are more manageable than series with both positive and negative terms because, for a non-negative series, the sequence of partial sums is increasing, and Theorem 5.1.23 then tells us that the sequence of partial sums converges, i.e., the series converges, if and only if the sequence of partial sums is bounded above.

This will allow us to derive three useful tests for convergence of non-negative series: the Integral Test, the Comparison test, and the Limit Comparison Test.

Definition 5.3.1. *A series $\sum_{k=m}^{\infty} b_k$ is* **non-negative** *provided that, for all $k \geq m$, $b_k \geq 0$.*

Example 5.3.2. A geometric series $\sum_{k=0}^{\infty} ar^k$ is non-negative if and only if $a \geq 0$ and $r \geq 0$.

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ is non-negative, while the alternating harmonic series $\sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{k}$ is **not** non-negative (nor is it non-positive).

The series

$$\sum_{k=1}^{\infty} \left(1+\sin\left(\frac{k\pi}{2}\right)\right)\left(\frac{1}{k}\right) \;=\; 2\cdot 1 \;+\; 1\cdot\frac{1}{2} \;+\; 0\cdot\frac{1}{3} \;+\; 1\cdot\frac{1}{4} \;+\; 2\cdot\frac{1}{5} \;+\; 1\cdot\frac{1}{6} \;+\; 0\cdot\frac{1}{7} \;+\; \cdots$$

is non-negative, even though there are an infinite number of terms which equal 0.

From the definition of a non-negative series, the sequence of partial sums of a non-negative series is increasing and so, from Theorem 5.1.23, we immediately conclude:

Theorem 5.3.3. *The partial sums of a non-negative series are an increasing sequence and, consequently, a non-negative series converges if and only if the partial sums are bounded above; if the partial sums are bounded above by M, then the sum of the convergent series is $\leq M$. A non-negative series diverges if and only if it diverges to ∞.*

Recall the definition of a subsequence in Definition 5.1.26. If we have a series $\sum_{n=m}^{\infty} a_n$ (non-negative or not), and form a new series by using as terms a subsequence of the a_n, we refer to this new series as a *subseries*.

Now, if we have a **non-negative** series $\sum_{n=m}^{\infty} a_n$, then the partial sums of any subseries are less than or equal to partial sums from $\sum_{n=m}^{\infty} a_n$ itself. Therefore, Theorem 5.3.3 immediately yields the following corollary.

Corollary 5.3.4. *Suppose that $\sum_{n=m}^{\infty} a_n$ is a non-negative series which converges. Then, every subseries of $\sum_{n=m}^{\infty} a_n$ converges.*

In other words, if a subseries of a non-negative series diverges (necessarily to ∞), then the original series also diverges to ∞.

Example 5.3.5. Consider the last non-negative series from Example 5.3.2:

$$\sum_{k=1}^{\infty}\left(1+\sin\left(\frac{k\pi}{2}\right)\right)\left(\frac{1}{k}\right) = 2\cdot 1 + 1\cdot\frac{1}{2} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{4} + 2\cdot\frac{1}{5} + 1\cdot\frac{1}{6} + 0\cdot\frac{1}{7} + \cdots .$$

If we consider the subseries using those terms where k is even, we obtain

$$1\cdot\frac{1}{2} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{6} + \ + 1\cdot\frac{1}{8} + \cdots = \sum_{j=1}^{\infty}\frac{1}{2}\cdot\frac{1}{j},$$

which is, term-wise, $1/2$ of the harmonic series. Thus, by Theorem 5.2.17, $\sum_{j=1}^{\infty}\frac{1}{2}\cdot\frac{1}{j}$ diverges and by Corollary 5.3.4, so does the original series $\sum_{k=1}^{\infty}\left(1+\sin\left(\frac{k\pi}{2}\right)\right)\left(\frac{1}{k}\right)$.

Example 5.3.6. The conclusion of Corollary 5.3.4 is not true (in general) if we omit the condition that the series is non-negative. Consider the alternating harmonic series,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots,$$

which, as we discussed in Remark 4.4.13, converges to $\ln 2$. However, if you just add up all of the negative terms, the terms with even denominators, you get, term-wise, $-1/2$ times the harmonic series, which diverges.

Thus, series that contain both positive and negative terms may converge, even though sub-series may diverge.

Recall that Theorem 5.2.21 told us that grouping together sums in a convergent series has no effect on the convergence. In Example 5.2.20, we saw a divergent series that contains both positive and negative terms, for which it is possible to group the terms to form new series that converge. As another corollary to Theorem 5.3.3, we will now show that you cannot group the summations in a divergent **non-negative** series in order to produce a convergent series.

Corollary 5.3.7. *Suppose that the terms of a series $\sum_{j=n}^{\infty} b_j$ are formed from a divergent non-negative series $\sum_{k=m}^{\infty} a_k$ by grouping disjoint finite sums of the a_k without changing the order, i.e., by "adding some parentheses" to the divergent series $\sum_{k=m}^{\infty} a_k$.*

Then, $\sum_{j=n}^{\infty} b_j$ also diverges.

Proof. Each partial sum of $\sum_{j=n}^{\infty} b_j$ is also a partial sum of $\sum_{k=m}^{\infty} a_k$, and the partial sums of $\sum_{k=m}^{\infty} a_k$ diverge to ∞. Therefore, the partial sums of $\sum_{j=n}^{\infty} b_j$ also diverge to ∞. □

Now that we know, from Theorem 5.2.21 and Corollary 5.3.7, that grouping terms (i.e., inserting parentheses around finite numbers of terms) does not affect the convergence or divergence of a non-negative series, we would like to know that changing the order of the summation, i.e., rearranging the order in which the terms are added, also does not affect non-negative series.

For instance, we know that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

converges and, in fact, we stated that it converges to $\pi^2/6$. Does it follow that the series formed by switching pairs of terms,

$$\frac{1}{2^2} + \frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{5^2} + \cdots, \tag{5.2}$$

converges, and converges to $\pi^2/6$? You might think "sure - a sum doesn't care what order you add things in", and that's true for finite sums, but, as we shall see in the next section, it is false, in general, for series containing both positive and negative terms. However, non-negative series are well-behaved with respect to such *rearrangements*.

First, we need to give the technical definition of a rearrangement. Note that the only thing that we really need to rearrange is the order of the indices of the sequence of terms. For instance, if we let $a_k = 1/k^2$, then the series in Formula 5.2 is

$$a_2 + a_1 + a_4 + a_3 + a_6 + a_5 + \cdots.$$

Suppose that, for all integers $j \geq 1$, we define $r(j) = j+1$ if j is odd, and $r(j) = j-1$ if j is even. Then,

$$r(1) = 2,\ r(2) = 1,\ r(3) = 4,\ r(4) = 3,\ r(5) = 6,\ r(6) = 5,\ \cdots.$$

Then, the series in Formula 5.2 is simply $\sum_{k=1}^{\infty} b_{r(k)}$. All that we used here to reorder, or rearrange, our series was the one-to-one correspondence (the *bijection*) r between the original indices and the newly ordered indices.

Definition 5.3.8. *Consider a (not necessarily non-negative) series* $\sum_{k=m}^{\infty} b_k$.

A **permutation of the indices** *of the series is a one-to-one correspondence, a bijection,* $r : \mathbb{Z}_{\geq m} \to \mathbb{Z}_{\geq m}$.

A **rearrangement of the series** $\sum_{k=m}^{\infty} b_k$ *is a series* $\sum_{k=m}^{\infty} b_{r(k)}$, *where* r *is a permutation of the indices.*

Theorem 5.3.9. *Consider a* **non-negative** *series* $\sum_{k=m}^{\infty} b_k$.

1. *If* $\sum_{k=m}^{\infty} b_k$ *converges to* L*, then every rearrangement of* $\sum_{k=m}^{\infty} b_k$ *converges to* L*.*
2. *If* $\sum_{k=m}^{\infty} b_k$ *diverges, then every rearrangement of* $\sum_{k=m}^{\infty} b_k$ *diverges.*

Proof. First, note that Item 2 follows from Item 1, since the original series is a rearrangement of any rearrangement of itself; so, if a rearrangement converges, Item 1 implies that the original series converges.

Hence, we need to prove Item 1. Suppose that $\sum_{k=m}^{\infty} b_k$ converges to L and that we have a rearrangement $\sum_{k=m}^{\infty} b_{r(k)}$. Let $S_n = \sum_{k=m}^{n} b_k$ denote the partial sums of $\sum_{k=m}^{\infty} b_k$, and let $R_n = \sum_{k=m}^{n} b_{r(k)}$ denote the partial sums of $\sum_{k=m}^{\infty} b_{r(k)}$.

For each n, all of the terms in S_n eventually appear in some partial sum of $\sum_{k=m}^{\infty} b_{r(k)}$; let $R_{\alpha(n)}$ be such a partial sum. Note that $\alpha(n) \geq n$ (since the R partial sum has to contain at least each term of the S partial sum). Note also that, since $\sum b_k$ is non-negative, any extra terms in the R partial sum are non-negative and, hence, $S_n \leq R_{\alpha(n)}$.

Similarly, for each p, all of the terms in R_p eventually appear in some partial sum of $\sum_{k=m}^{\infty} b_k$; let $S_{\beta(p)}$ be such a partial sum. Then, $\beta(p) \geq p$, and $R_p \leq S_{\beta(p)}$.

Combining our inequalities, and letting $p = \alpha(n)$, we conclude that, for all $n \geq m$, $n \leq \alpha(n) \leq \beta(\alpha(n))$ and

$$S_n \;\leq\; R_{\alpha(n)} \;\leq\; S_{\beta(\alpha(n))}.$$

As $n \to \infty$, both $\alpha(n)$ and $\beta(\alpha(n))$ approach ∞, and so, by the Pinching Theorem for Sequences, Theorem 5.1.9, the sequence of partial sums $R_{\alpha(n)}$ converges to L, i.e., $\sum_{k=m}^{\infty} b_{r(k)}$ converges to L. □

Remark 5.3.10. You should understand the point of Theorem 5.2.21, Corollary 5.3.7, and Theorem 5.3.9. Together, they tells us that, **for non-negative series**, the sum of a series, which is the limit of its partial sums, behaves like a true sum of the collection of terms: the order in which we take the terms and/or how we group them does not affect the sum.

We want to emphasize again that these things are **not** true, in general, for series with both positive and negative terms. Such a series can diverge, and yet, some groupings of the terms can lead to convergent series (Example 5.2.20). Moreover, rearrangements of general convergent series can lead to series that converge to different values or even diverge to $\pm\infty$; see Theorem 5.4.20.

We would like a test for convergence of series that allows us to use our knowledge of integrals, since we have many techniques for integrating. Moreover, it's not **too** hard to believe that integrals could be useful when looking at series; infinite series involve sums and integrals are limits of Riemann sums. In fact, we have the very useful:

Theorem 5.3.11. (**The Integral Test**) *Suppose that m is an integer, and that the function f is defined and Riemann integrable on every closed, bounded subinterval $[a, b]$ of $[m, \infty)$, for instance, f could be continuous on $[m, \infty)$. Suppose further that, for $x \geq m$, $f(x) \geq 0$ and is decreasing.*

Then, the series $\sum_{k=m}^{\infty} f(k)$ converges if and only if the improper integral $\int_m^\infty f(x)\,dx$ converges. In addition, in the case where the integral and the series converge,

$$\int_m^\infty f(x)\,dx \;\leq\; \sum_{k=m}^{\infty} f(k) \;\leq\; f(m) \;+\; \int_m^\infty f(x)\,dx.$$

Proof. This proof is very intuitive; we will show that inequalities exist between the partial sums of the series and integrals in which the upper limits of integration are finite values of x.

Suppose that k is an integer $\geq m$. As f is decreasing, for all x in the interval $[k, k+1]$, $f(k+1) \leq f(x) \leq f(k)$ and so, by the monotonicity of integration, Theorem 2.3.20,

$$\int_k^{k+1} f(k+1)\,dx \;\leq\; \int_k^{k+1} f(x)\,dx \;\leq\; \int_k^{k+1} f(k)\,dx.$$

Thus,

$$f(k+1) \;\leq\; \int_k^{k+1} f(x)\,dx \;\leq\; f(k),$$

and, hence, for all $n \geq m$,

$$\sum_{k=m}^{n} f(k+1) \;\leq\; \sum_{k=m}^{n} \left[\int_k^{k+1} f(x)\,dx\right] \;\leq\; \sum_{k=m}^{n} f(k).$$

Reindexing on the left, and adding together the integrals, we find

$$\sum_{k=m+1}^{n+1} f(k) \;\leq\; \int_m^{n+1} f(x)\,dx \;\leq\; \sum_{k=m}^{n} f(k).$$

If the integral $\int_m^\infty f(x)\,dx$ converges, then the left-hand inequality tells us that the partial sums of the infinite series are bounded above and, therefore, the series converges by Theorem 5.3.3. If the infinite series converges, then the right-hand inequality above tells us that the integrals $\int_m^{n+1} f(x)\,dx$ are bounded above. Then, by Theorem 2.5.14, the integral $\int_m^\infty f(x)\,dx$ converges.

In the case where the integral and series converge, by taking limits, we immediately obtain

$$\sum_{k=m+1}^{\infty} f(k) \;\leq\; \int_m^{\infty} f(x)\,dx \;\leq\; \sum_{k=m}^{\infty} f(k).$$

The right-hand inequality above yields the left-hand inequality in the statement of the theorem. The right-hand inequality in the statement of the theorem is obtained by adding $f(m)$ to each side of the left-hand inequality above. $\square$

Example 5.3.12. Let's look at $f(x) = 1/x$, for $x \geq 1$, This function is certainly positive and decreasing. Therefore, the Integral test tells us that $\sum_{k=1}^\infty \frac{1}{k}$ converges if and only if $\int_1^\infty \frac{1}{x}\,dx$ converges. Of course, the series here is the harmonic series, which we proved diverges in Proposition 5.2.16. So we can conclude that the integral diverges. However, we can see easily that the integral diverges,

$$\int_1^\infty \frac{1}{x}\,dx \;=\; \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \;=\; \lim_{b\to\infty}\,(\ln b - \ln 1) \;=\; \infty,$$

and so the Integral test gives us another proof that the harmonic series diverges.

Example 5.3.13. What does the Integral Test tells us about $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$?

We look at the function $f(x) = 1/x^2$; this function is non-negative and decreasing for $x \geq 1$, and

$$\int_1^{\infty} \frac{1}{x^2}\, dx \;=\; \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big|_1^b \;=\; \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) \;=\; 1.$$

Hence, the Integral Test tells us that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, and

$$1 \;\leq\; \sum_{k=1}^{\infty} \frac{1}{k^2} \;\leq\; \frac{1}{1^2} \;+\; 1 \;=\; 2.$$

In fact, it can be shown, using *Fourier series* techniques (see [3]) that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges to $\pi^2/6$, which is, in fact, between 1 and 2.

Example 5.3.14. Use the Integral Test to decide whether or not $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges.

Solution:

The function $f(x) = \dfrac{1}{x(\ln x)^2}$ is certainly non-negative and decreasing for $x \geq 2$. We find the indefinite integral by substitution of $u = \ln x$, so that $du = (1/x)dx$, and

$$\int \frac{1}{x(\ln x)^2}\, dx \;=\; \int \frac{1}{u^2}\, du \;=\; -\frac{1}{u} + C \;=\; -\frac{1}{\ln x} + C.$$

And now we calculate the definite integral:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx \;=\; \lim_{b\to\infty}\left(-\frac{1}{\ln b} + \frac{1}{\ln 2}\right) \;=\; \frac{1}{\ln 2}.$$

Therefore, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges, and

$$\frac{1}{\ln 2} \le \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} \le \frac{1}{2(\ln 2)^2} + \frac{1}{\ln 2}.$$

As we saw in Example 5.3.12 and Example 5.3.13, the Integral Test works very quickly to let us decide about the convergence or divergence of a series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p},$$

where p was, respectively, 1 and 2 in the two examples. More generally, such a series is called a *p-series* and, as a corollary to the Integral Test, we find that the harmonic series, the $p = 1$ case, is the dividing case between p-series converging and diverging.

Corollary 5.3.15. (**The p-Series Test**) *The p-series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if $p > 1$ and diverges if $p \le 1$.

In addition, in the convergent case, where $p > 1$,

$$\frac{1}{p-1} \le \sum_{k=1}^{\infty} \frac{1}{k^p} \le 1 + \frac{1}{p-1}.$$

Proof. We leave this proof as an exercise. □

Even if the indexing does not begin at 1, we still call a series $\displaystyle\sum_{k=m}^{\infty} \frac{1}{k^p}$ a p-series and, since omitting a finite number of terms does not affect convergence/divergence, such a p-series still converges if and only if $p > 1$.

Example 5.3.16. The p-Series Test tells us immediately that $\sum_{k=1}^{\infty} \frac{1}{k^{0.999}}$ diverges, while $\sum_{k=1}^{\infty} \frac{1}{k^{1.001}}$ converges to some value v such that $1000 \leq v \leq 1001$.

It is important not to confuse p-series with geometric series, even though they look very similar.

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \text{ is geometric, while } \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ is a } p\text{-series.}$$

Geometric series have a fixed base and variable exponent, while p-series have a variable base and a fixed exponent.

Our next convergence test is easy to see. Suppose that we have two sequences of terms, where one is always less than or equal to the other, i.e., for $k \geq m$, we have $0 \leq a_k \leq b_k$. Then, all of the partial sums satisfy the analogous inequality

$$\sum_{k=m}^{n} a_k \leq \sum_{k=m}^{n} b_k.$$

Recalling from Theorem 5.3.3 that the only way for a non-negative series to diverge is for it to diverge to ∞, we immediately conclude:

Theorem 5.3.17. (**The Comparison Test**) *Suppose that, for all $k \geq m$, we have the inequalities $0 \leq a_k \leq b_k$.*

1. *If $\sum_{k=m}^{\infty} b_k$ converges, then so does $\sum_{k=m}^{\infty} a_k$, and*

$$\sum_{k=m}^{\infty} a_k \;\leq\; \sum_{k=m}^{\infty} b_k.$$

2. *If $\sum_{k=m}^{\infty} a_k$ diverges (necessarily to ∞), then so does $\sum_{k=m}^{\infty} b_k$.*

Remark 5.3.18. The conclusions of the Comparison Test should not be hard to remember, provided that you keep in mind that the only way for a non-negative series to diverge is for it to diverge to ∞. If you remember this, the Comparison Test basically says that a series being less than or equal to than something that's $< \infty$ implies the series is $< \infty$, and a series being greater than or equal to something that's ∞ implies that the series is ∞.

If you think about it, it's also not hard to see what the Comparison Test does **not** say. If the bigger series diverges, it diverges to ∞, and all you know about the smaller series is that it adds up to something $\leq \infty$; but that means that the smaller series could converge to something finite or diverge to ∞, so you conclude nothing. If the smaller series converges, then the bigger series adds up to something larger, but that something could be finite or ∞, so once again you conclude nothing.

You need for the bigger non-negative series to converge, or the smaller non-negative series to diverge in order to conclude anything from the Comparison Test.

Example 5.3.19. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+3}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k-0.5}$.

As $\dfrac{1}{k^2+3} \leq \dfrac{1}{k^2}$, and we know from the p-Series Test that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the Comparison Test tells us that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+3}$ also converges.

As $\dfrac{1}{k-0.5} \geq \dfrac{1}{k}$, for $k \geq 1$, and we know that the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the Comparison Test tells us that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k-0.5}$ also diverges.

Note that the facts that $\dfrac{1}{k^2} \leq \dfrac{1}{k^2-0.5}$ and that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges do **not** allow us to conclude from the Comparison Test anything about the convergence or divergence of $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2-0.5}$.

What we saw at the end of the example above, that comparing $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2-0.5}$ tells us nothing about the convergence/divergence of $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2-0.5}$, is somewhat frustrating. Yeah - it's true that $\dfrac{1}{k^2} \leq \dfrac{1}{k^2-0.5}$, and yet you may have the feeling that, when k is really big, the -0.5 just shouldn't matter – it's negligible – and the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2-0.5}$ should do whatever the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ does.

The following theorem says that this line of thinking is valid.

Theorem 5.3.20. (**The Limit Comparison Test**) *Suppose that, for $k \geq m$, $a_k \geq 0$ and $b_k \geq 0$. Suppose also that*

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L,$$

where L may be infinity (so, the limit exists as an extended real number).

1. *If $L \neq 0$ and $L \neq \infty$, then the two series $\sum_{k=m}^{\infty} a_k$ and $\sum_{k=m}^{\infty} b_k$ do the same thing, i.e., either both converge or both diverge.*
2. *If $L = 0$, and $\sum_{k=m}^{\infty} b_k$ converges, then $\sum_{k=m}^{\infty} a_k$ converges.*
3. *If $L = \infty$, and $\sum_{k=m}^{\infty} b_k$ diverges, then $\sum_{k=m}^{\infty} a_k$ diverges.*

Proof. We give the proof in Case 1, and leave the other two cases as an exercise.

So, suppose that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L,$$

where $L > 0$ is a real number. Note, first, that this means that there has to exist m_0 such that, for all $k \geq m_0$, $b_k \neq 0$, for, otherwise, the fraction doesn't even exist for arbitrarily large k.

Let $\epsilon = L/2 > 0$. Then, by definition of the limit of a sequence, there exists an integer $N \geq m_0$ such that, for all integers $n \geq N$,

$$\left|\frac{a_n}{b_n} - L\right| \leq \epsilon = \frac{L}{2}.$$

Thus, for $n \geq N$,

$$-\frac{L}{2} \leq \frac{a_n}{b_n} - L \leq \frac{L}{2},$$

and, hence,

$$\left(\frac{L}{2}\right) b_n \leq a_n \leq \left(\frac{3L}{2}\right) b_n, \tag{5.3}$$

where we used that $b_n > 0$ for $n \geq N$.

Now, suppose that $\sum_{n=m}^{\infty} b_n$ converges. Then, by Proposition 5.2.6, $\sum_{n=N}^{\infty} b_n$ converges, and by Theorem 5.2.17, $\sum_{n=N}^{\infty} \left(\frac{3L}{2}\right) b_n$ converges. Hence, using the inequality on the right-hand side of Formula 5.3 and the Comparison Test, Theorem 5.3.17, we conclude that $\sum_{n=N}^{\infty} a_n$ converges. Applying Proposition 5.2.6 again allows us to conclude that $\sum_{n=m}^{\infty} a_n$ converges.

The other conclusion is entirely similar. Suppose that $\sum_{n=m}^{\infty} a_n$ converges. Then, by Proposition 5.2.6, $\sum_{n=N}^{\infty} a_n$ converges. Using the inequality on the left-hand side of Formula 5.3 and the Comparison Test, we conclude that $\sum_{n=N}^{\infty} \left(\frac{L}{2}\right) b_n$ converges. By Theorem 5.2.17, $\sum_{n=N}^{\infty} b_n$ converges. Applying Proposition 5.2.6 again allows us to conclude that $\sum_{n=m}^{\infty} b_n$ converges. □

Example 5.3.21. So, suppose you want to decide whether or not

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - 0.5}$$

converges. How do you think about such a problem?

You think the way we discussed before proving the Limit Comparison Test. You think: "when k is big, the 0.5 (added or subtracted) is negligible, and this series should do whatever

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ does, which is converge, since it's a p-series with $p = 2 > 1$." Thus, just by looking at the series (and knowing the appropriate convergence tests!), you immediately conclude that $\sum_{k=1}^{\infty} \frac{1}{k^2 - 0.5}$ converges.

Of course, we still need to show that the 0.5 was indeed "negligible" by calculating the required limit in the Limit Comparison Test. This means that we let one of the series be the one we don't know about, $\sum_{k=1}^{\infty} \frac{1}{k^2 - 0.5}$, and we let the other series (that we limit compare with) be the one that we know about, whose terms look like those of the original series without the "negligible" parts, i.e., we limit compare with $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

We calculate

$$\lim_{k\to\infty} \frac{\frac{1}{k^2 - 0.5}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k^2 - 0.5} = \lim_{k\to\infty} \frac{1}{1 - \frac{0.5}{k^2}} = \frac{1}{1-0} = 1,$$

and see that, yes, we may use the Limit Comparison Test between $\sum_{k=1}^{\infty} \frac{1}{k^2 - 0.5}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$, and conclude that $\sum_{k=1}^{\infty} \frac{1}{k^2 - 0.5}$ converges because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ does.

Remark 5.3.22. It was not important in the example above, or in the Limit Comparison Test in general, that $\lim_{k\to\infty} (a_k/b_k)$ equals 1; if the limit had turned out to be 37, we could still have used the Limit Comparison Test in exactly the way that we did.

However, what you typically get for the limit in the Limit Comparison Test is 1. Why? Because of the way you arrive at the series that you limit compare with.

You start with some non-negative series $\sum a_k$. You look at it and think "ah, when k is big, these parts of the terms are insignificant/negligible and, if I ignore the negligible parts, I get a non-negative series $\sum b_k$, and I know what the series $\sum b_k$ does, so I'll limit compare with that."

But, if you've correctly decided that a_k and b_k look alike, except for parts that are negligible when k is big, then that precisely means that $\lim_{k\to\infty} (a_k/b_k) = 1$.

As our last two results on the convergence/divergence of non-negative series, we will state and prove the *Ratio Test* and the *Root Test* for non-negative series. We stated the Ratio Test for arbitrary series back in Theorem 4.5.13, and we shall look at both of these tests again in the next section on series which contain both positive and negative terms. However, in a strong sense, the Ratio and Root Tests are really theorems about non-negative series; it's just that these results can be extended to arbitrary series by taking absolute values of the terms (see Theorem 5.4.13 and Theorem 5.4.14).

Theorem 5.3.23. (**The Ratio Test for Non-Negative Series**) *Consider the series* $\sum_{k=m}^{\infty} b_k$, *where* $b_k \geq 0$.

1. *If there exists* $r < 1$ *(and, necessarily,* > 0*) and an integer* $M \geq m$ *such that, for all* $k \geq M$,
$$b_k \neq 0 \quad \text{and} \quad \frac{b_{k+1}}{b_k} \leq r,$$
then the given series converges.
2. *If there exists an integer* $M \geq m$ *such that, for all* $k \geq M$,
$$b_k \neq 0 \quad \text{and} \quad \frac{b_{k+1}}{b_k} \geq 1,$$
then the given series diverges.

In particular, suppose that $\lim_{k\to\infty} \frac{b_{k+1}}{b_k}$ *exists, as an extended real number; call its value* L. *Then,*

a. *if* $L < 1$, *the given series converges;*

b. *if* $L > 1$, *including* $L = \infty$, *the given series diverges;*

c. *if* $L = 1$, *the given series may converge or diverge.*

Proof. Let's dispose of Case 2 first. Suppose that there exists an integer $M \geq m$ such that, for all $k \geq M$,
$$b_k \neq 0 \quad \text{and} \quad \frac{b_{k+1}}{b_k} \geq 1 \quad \text{i.e.,} \quad b_{k+1} \geq b_k.$$

Then, for all $k \geq M$, $b_k \geq b_M > 0$. Therefore, $\lim_{k\to\infty} b_k$ is not zero, and so the series $\sum b_k$ diverges by the Term Test for Divergence, Theorem 5.2.12.

Now, let's look at Case 1. Suppose that there exists $r < 1$ and an integer $M \geq m$ such that, for all $k \geq M$,

$$b_k \neq 0 \quad \text{and} \quad \frac{b_{k+1}}{b_k} \leq r \quad \text{i.e.,} \quad b_{k+1} \leq b_k r.$$

Then, for all $k \geq M$,

$$b_k \leq b_M r^k.$$

Thus, the series $\sum_{k=M}^{\infty} b_k$ is, term-by-term, less than or equal to $\sum_{k=M}^{\infty} b_M r^k$. As $0 < r < 1$, $\sum_{k=M}^{\infty} b_M r^k$ converges, as it's a geometric series with $|r| < 1$. Hence, by the Comparison Test, $\sum_{k=M}^{\infty} b_k$ converges, and so $\sum_{k=m}^{\infty} b_k$ converges. □

Do not confuse the limits in the Limit Comparison Test, Theorem 5.3.20, and in the Ratio Test, Theorem 5.3.23. In the Limit Comparison Test, you expect, and want, to get 1 for the limit of the ratio of the terms of the two series. In the Ratio Test, if you get a 1 for the limit, that's **bad**, because it means that you can't conclude anything from the test.

Example 5.3.24. Consider the series

$$\sum_{k=1}^{\infty} \frac{k!}{5^k}, \qquad \sum_{k=1}^{\infty} \frac{k^{1000}}{1.01^k}, \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{7^k(k+1)}{k^k}.$$

Let's see what the Ratio Test tells us about the convergence or divergence of these series.

For the first series, we calculate the limit of ratios

$$\lim_{k\to\infty} \frac{(k+1)!/5^{k+1}}{k!/5^k} = \lim_{k\to\infty} \left[\frac{(k+1)!}{k!} \cdot \frac{5^k}{5^{k+1}}\right] = \lim_{k\to\infty} \left[\frac{k+1}{5}\right] = \infty,$$

and so the Ratio Test tells us that $\sum_{k=1}^{\infty} \frac{k!}{5^k}$ diverges.

For the second series, we calculate the limit of ratios

$$\lim_{k\to\infty} \frac{(k+1)^{1000}/1.01^{k+1}}{k^{1000}/1.01^k} \;=\; \lim_{k\to\infty}\left[\frac{(k+1)^{1000}}{k^{1000}}\cdot\frac{1.01^k}{1.01^{k+1}}\right] \;=$$

$$\lim_{k\to\infty}\left[\left(1+\frac{1}{k}\right)^{1000}\cdot\frac{1}{1.01}\right] \;=\; 1\cdot\frac{1}{1.01} \;<1,$$

and so the Ratio Test tells us that $\displaystyle\sum_{k=1}^{\infty}\frac{k^{1000}}{1.01^k}$ converges.

Note that, even though the exponent on the base k is large, 1000, and the base 1.01 is barely bigger than 1, the Ratio Test and the convergence of the series is telling us that, when k is large, 1.01^k is significantly larger than k^{1000}.

Let's look at the third and final series. We calculate the limit of ratios

$$\lim_{k\to\infty}\frac{7^{k+1}(k+2)/(k+1)^{k+1}}{7^k(k+1)/k^k} \;=\; \lim_{k\to\infty}\left[\frac{7^{k+1}}{7^k}\cdot\frac{k+2}{k+1}\cdot\frac{k^k}{(k+1)^{k+1}}\right] \;=$$

$$\lim_{k\to\infty}\left[7\;\cdot\;\frac{1+\frac{2}{k}}{1+\frac{1}{k}}\;\cdot\;\left(\frac{k}{k+1}\right)^k\;\cdot\;\frac{1}{k+1}\right] \;=\; \lim_{k\to\infty}\left[7\;\cdot\;\frac{1+\frac{2}{k}}{1+\frac{1}{k}}\;\cdot\;\frac{1}{\left(1+\frac{1}{k}\right)^k}\;\cdot\;\frac{1}{k+1}\right] \;=$$

$$7\cdot 1\cdot\frac{1}{e}\cdot 0 \;=\; 0<1,$$

and so the series $\displaystyle\sum_{k=1}^{\infty}\frac{7^k(k+1)}{k^k}$ converges.

Remark 5.3.25. You might think that the p-Series Test, Corollary 5.3.15, would follow easily from the Ratio Test; however, that is not the case. If $b_k = 1/k^p$, then

$$\lim_{k\to\infty}\frac{b_{k+1}}{b_k} \;=\; \lim_{k\to\infty}\frac{1/(k+1)^p}{1/k^p} \;=\; \lim_{k\to\infty}\frac{k^p}{(k+1)^p} \;=\; \lim_{k\to\infty}\frac{1}{\left(1+\frac{1}{k}\right)^p} \;=\; 1,$$

so the Ratio Test is useless here.

There is a test that looks very similar to the Ratio Test; it's called the *Root Test*. Like the proof of the Ratio Test, the proof of the Root Test involves a comparison with a geometric series in which $0 \le r < 1$. We leave this proof as an exercise.

Theorem 5.3.26. (**The Root Test for Non-Negative Series**) *Consider the series* $\sum_{k=m}^{\infty} b_k$*, where* $b_k \ge 0$.

1. *If there exists* $r < 1$ *(and, necessarily,* > 0*) and an integer* $M \ge m$ *such that, for all* $k \ge M$*,*

$$\sqrt[k]{b_k} \;\le\; r,$$

   then the given series converges.

2. *If there exists an integer* $M \ge m$ *such that, for all* $k \ge M$*,*

$$\sqrt[k]{b_k} \;\ge\; 1,$$

   then the given series diverges.

In particular, suppose that $\lim_{k\to\infty} \sqrt[k]{b_k}$ *exists, as an extended real number; call its value* L*. Then,*

a. *if* $L < 1$*, the given series converges;*

b. *if* $L > 1$*, including* $L = \infty$*, the given series diverges;*

c. *if* $L = 1$*, the given series may converge or diverge.*

Example 5.3.27. The most obvious time to use the Root Test is when the terms are explicitly k-th powers. For instance,

$$\sum_{k=1}^{\infty} \left(\frac{1}{2} + \frac{1}{k}\right)^k$$

certainly converges by the Root Test, since

$$\lim_{k\to\infty} \sqrt[k]{\left(\frac{1}{2} + \frac{1}{k}\right)^k} \;=\; \lim_{k\to\infty} \left(\frac{1}{2} + \frac{1}{k}\right) \;=\; \frac{1}{2} \;<\; 1.$$

Remark 5.3.28. The Ratio Test and the Root Test look very similar, and which one is easier to use depends on the series, though roots are frequently harder to manage than ratios. For instance, if the terms of the series involve factorials, like $k!$, then the Root Test will involve k-th roots of factorials, which are certainly not pleasant to deal with (see, however, Corollary 5.3.30). On the other hand, there are times when the Ratio Test fails (i.e., doesn't tell us about convergence or divergence), and yet the Root Test easily yields a conclusion.

Consider the series

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^4} + \cdots,$$

that is, a geometric series, with $r = 1/2$, **except** that we have each term twice. It may seem obvious that, since $\sum_{k=0}^{\infty} 1/2^k$ converges to 1, that the given series converges to twice that; in fact, this is true and follows from the term-wise addition theorem, Theorem 5.2.17.

However, the Ratio Test is useless here for concluding convergence, since every other ratio of successive terms is precisely 1.

On the other hand, the Root Test works fine for this series. The terms are $b_k = \dfrac{1}{2^{k/2}}$, if k is even, and $b_k = \dfrac{1}{2^{(k+1)/2}}$, if k is odd. So that

$$\sqrt[k]{b_k} = \begin{cases} \dfrac{1}{2^{1/2}} & \text{if } k \text{ is even;} \\[2ex] \dfrac{1}{2^{\left(1+\frac{1}{k}\right)/2}} & \text{if } k \text{ is odd.} \end{cases}$$

Therefore, $\lim_{k\to\infty} \sqrt[k]{b_k} = 1/\sqrt{2} < 1$, and so the Root Test tells us that the series converges.

As a final advantage of the Root Test, notice that the Root Test has no problem if an infinite number of terms in a series are 0, unlike the Ratio Test.

The following result tells us that the Root Test "works" any time that the Ratio Test "works", but the converse is not true, as we showed in the remark above.

Proposition 5.3.29. *Suppose that $b_k \geq 0$ and $\lim_{k\to\infty}(b_{k+1}/b_k) = L$, where L is an extended real number. Then,*

$$\lim_{k\to\infty} \sqrt[k]{b_k} = L.$$

In particular, if the limit in the Ratio Test, Theorem 4.5.13, tells you whether a series converges or diverges, then the limit in the Root Test, Theorem 5.3.26, tells you the same thing.

Proof. See Remark 3.36 and Theorem 3.37 of [3]. □

The following corollary can be helpful in cases where you decide to use the Root Test.

Corollary 5.3.30.

$$\lim_{k\to\infty} \sqrt[k]{k} = 1 \qquad \text{and} \qquad \lim_{k\to\infty} \frac{k}{\sqrt[k]{k!}} = e.$$

In particular, $\lim_{k\to\infty} \sqrt[k]{k!} = \infty$.

Proof. For the first equality, you can take natural logarithms, using that the natural logarithm is continuous, and reduce the problem to showing that $\lim_{k\to\infty}(\ln k)/k = 0$, which follows easily by applying l'Hôpital's Rule (see [2]) to $\lim_{x\to\infty}(\ln x)/x$. Alternatively, you may apply Theorem 5.3.29 to the sequence $a_k = k$.

To show the second equality, consider the sequence $b_k = k^k/k!$. Then, we find

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \lim_{k\to\infty} \frac{(k+1)^{k+1}/(k+1)!}{k^k/k!} = \lim_{k\to\infty}\left[\frac{k!}{(k+1)!}\cdot\frac{(k+1)^k}{k^k}\cdot(k+1)\right] =$$

$$\lim_{k\to\infty}\left[\frac{1}{k+1}\cdot\left(1+\frac{1}{k}\right)^k\cdot(k+1)\right] = e.$$

Now, by the proposition, $\lim_{k\to\infty} \sqrt[k]{b_k} = e$, i.e.,

$$\lim_{k\to\infty} \frac{k}{\sqrt[k]{k!}} = \lim_{k\to\infty} \sqrt[k]{\frac{k^k}{k!}} = e.$$

$\square$

5.3.1 Exercises

In each of Exercises 1 through 10, use the Integral Test, Theorem 5.3.11, or the p-Series Test, Corollary 5.3.15, possibly combined with other results, to (a) decide whether the given series converges or diverges, and (b) for the convergent series, give upper and lower bounds for the sum.

1. $\displaystyle\sum_{k=5}^{\infty} \frac{2}{k^{0.1}}$

2. $\displaystyle\sum_{j=0}^{\infty} \frac{1}{j^2+1}$

3. $\displaystyle\sum_{p=4}^{\infty} \frac{7}{p^{8/7}}$

4. $\displaystyle\sum_{m=1}^{\infty} \frac{1+\sqrt{m}}{m^{5/4}}$

5. $\displaystyle\sum_{p=4}^{\infty} \frac{7+p^{1/7}}{p^{8/7}}$

6. $\displaystyle\sum_{j=2}^{\infty} \frac{5}{j(\ln j)^3}$

7. $\displaystyle\sum_{k=1}^{\infty} \frac{3k+5}{k^3}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2+5}{k^3}$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{3k^3+5}{k^3}$

10. $\displaystyle\sum_{q=3}^{\infty} \frac{e^q}{(e^q+1)^2}$

In each of Exercises 11 through 20, you are given a series $\sum_{k=m}^{\infty} a_k$. Find a series $\sum_{k=m}^{\infty} b_k$, which is a p-series, constant multiple of a p-series, or a geometric series, to compare with $\sum_{k=m}^{\infty} a_k$ to determine its convergence or divergence via the Comparison Test, Theorem 5.3.17.

11. $\displaystyle\sum_{k=1}^{\infty} \frac{1+\sin k}{k^2}$

12. $\displaystyle\sum_{j=0}^{\infty} \frac{1}{2^j+1}$

13. $\displaystyle\sum_{p=4}^{\infty} \frac{p+1}{0.9^p}.$

14. $\displaystyle\sum_{m=1}^{\infty} \frac{4}{2^m+m}$

15. $\displaystyle\sum_{m=1}^{\infty} \frac{4}{(0.9)^m+m^2}$

16. $\displaystyle\sum_{j=3}^{\infty} \frac{5}{j^2 \ln j}$

17. $\displaystyle\sum_{k=1}^{\infty} \frac{3+(-1)^k}{\sqrt{k}}$

18. $\displaystyle\sum_{k=1}^{\infty} \frac{k+5}{k^2}$

19. $\displaystyle\sum_{k=1}^{\infty} \frac{2+(-1)^k}{k^{1.1}}$

20. $\displaystyle\sum_{q=3}^{\infty} \frac{e^q}{(e^q+1)^2}$

In each of Exercises 21 through 30, you are given a series $\sum_{k=m}^{\infty} a_k$. Find a p-series, constant multiple of a p-series, or a geometric series $\sum_{k=m}^{\infty} b_k$ to limit compare with $\sum_{k=m}^{\infty} a_k$ to determine its convergence or divergence via the Limit Comparison Test, Theorem 5.3.20.

21. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k+1000}$

22. $\displaystyle\sum_{j=3}^{\infty} \frac{1}{2^j-7}$

23. $\displaystyle\sum_{p=4}^{\infty} \frac{p+1}{0.9^p}.$

24. $\displaystyle\sum_{m=1}^{\infty} \frac{4}{2^m + m}$

25. $\displaystyle\sum_{m=1}^{\infty} \frac{4}{(0.9)^m + m^2}$

26. $\displaystyle\sum_{j=3}^{\infty} \frac{5}{j^2 \ln j}$

27. $\displaystyle\sum_{k=1}^{\infty} \frac{3 + \sqrt[3]{k}}{7 + \sqrt{k}}$

28. $\displaystyle\sum_{k=5}^{\infty} \frac{k-5}{k^2}$

29. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{4^k + 1}}{4^k}$

30. $\displaystyle\sum_{q=3}^{\infty} \frac{e^q}{(e^q - 1)^2}$

In each of Exercises 31 through 37, use the Ratio Test or Root Test to determine whether the given series converges or diverges.

31. $\displaystyle\sum_{k=1}^{\infty} \frac{5k}{k! + 1}$

32. $\displaystyle\sum_{j=3}^{\infty} \frac{1}{2^j - 7}$

33. $\displaystyle\sum_{p=4}^{\infty} \frac{p+1}{0.9^p}.$

34. $\displaystyle\sum_{m=1}^{\infty} \frac{4}{2^m + m}$

35. $\displaystyle\sum_{m=1}^{\infty} \frac{4}{(0.9)^m m^2}$

36. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{4^k + 1}}{4^k}$

37. $\displaystyle\sum_{q=3}^{\infty} \frac{e^q}{(e^q-1)^2}$

In each of Exercises 38 through 43, combine any/all of your convergence and divergence tests and theorems to decide whether the given series converges or diverges.

38. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k} + \frac{1}{2^k+1}\right)$

39. $\displaystyle\sum_{j=3}^{\infty} \frac{1}{\sqrt{2^j+7}}$

40. $\displaystyle\sum_{p=4}^{\infty} \frac{p!\,p^2}{p^p}.$

41. $\displaystyle\sum_{m=2}^{\infty} \frac{5-2\sin m}{m^2-m}$

42. $\displaystyle\sum_{m=1}^{\infty} \frac{7+(-1)^m}{m+m^{1/2}}$

43. $\displaystyle\sum_{k=3}^{\infty} \frac{k-2}{k^2-2}$

44. Prove the p-Series Test, Corollary 5.3.15, as a corollary of the Integral Test, Theorem 5.3.11.

45. Prove Cases 2 and 3 of the Limit Comparison Test, Theorem 5.3.20.

46. Prove the Root Test, Theorem 5.3.26.

47. Prove that $\lim_{k\to\infty} \left(\ln k - \dfrac{1}{k}\displaystyle\sum_{j=1}^{k} \ln j\right) = 1.$

5.4 Theorems on Series III: Series with Positive and Negative Terms

Non-negative series are relatively "easy" to deal with, for the partial sums always form an increasing sequence, which converges if and only if it's bounded above. Series with both positive and negative terms are not so well-behaved.

We end up classifying convergent series into two kinds: series that converge *absolutely* and series that converge *conditionally*. A series $\sum b_k$ converges absolutely if the non-negative series $\sum |b_k|$ converges. A series converges conditionally if it converges, but does not converge absolutely.

As we shall see, series which converge absolutely have very nice properties. On the other hand, series which converge conditionally are very strange; they do not really correspond to what we'd like to think of as the sum of a collection of numbers. For instance, if a series converges conditionally, then, by rearranging the terms of the series, i.e., by changing the order of the summation, we can make the series converge to anything that we want, or diverge to $\pm\infty$.

While we may use any of our tests on non-negative series to investigate absolute convergence, we will have only one general test for concluding conditional convergence: the *Alternating Series Test*, Theorem 5.4.17.

Given an infinite series $\sum_{k=m}^{\infty} b_k$, we know that, by definition, Definition 5.2.1, the series converges if and only if the sequence of partial sums $s_n = \sum_{k=m}^{n} b_k$ converges. However, the completeness of the real numbers, Theorem 5.1.18, tells us that a sequence of real numbers (and now we're thinking of the sequence of partial sums) converges if and only if it's a Cauchy sequence.

Thus, we conclude

Theorem 5.4.1. (**Cauchy Criterion for Series Convergence**) *A series $\sum_{k=m}^{\infty} b_k$ converges if and only if, for all $\epsilon > 0$, there exists an integer $M \geq m$ such that, for all integers p and q such that $M \leq p \leq q$,*

$$\left|\sum_{k=p}^{q} b_k\right| < \epsilon.$$

From the Cauchy Criterion, we quickly conclude an important corollary:

Corollary 5.4.2. *Consider the series* $\sum_{k=m}^{\infty} |b_k|$ *formed by taking the absolute value of each term of the series* $\sum_{k=m}^{\infty} b_k$*.*

If $\sum_{k=m}^{\infty} |b_k|$ *converges, then* $\sum_{k=m}^{\infty} b_k$ *converges and*

$$-\sum_{k=m}^{\infty} |b_k| \;\leq\; \sum_{k=m}^{\infty} b_k \;\leq\; \sum_{k=m}^{\infty} |b_k|.$$

Proof. Suppose that $\sum_{k=m}^{\infty} |b_k|$ converges. We will use the Cauchy Criterion to conclude that $\sum_{k=m}^{\infty} b_k$ converges. The stated inequality then follows easily from the analogous inequalities on the partial sums.

Let $\epsilon > 0$. As $\sum_{k=m}^{\infty} |b_k|$ converges, the Cauchy Criterion tells us that there exists an integer $M \geq m$ such that, for all integers p and q such that $M \leq p \leq q$,

$$\left| \sum_{k=p}^{q} |b_k| \right| \;<\; \epsilon, \tag{5.4}$$

where the outer absolute value signs are now unnecessary (since the sum is non-negative).

However, it is an easy exercise to show that

$$-\sum_{k=p}^{q} |b_k| \;\leq\; \sum_{k=p}^{q} b_k \;\leq \sum_{k=p}^{q} |b_k|, \tag{5.5}$$

(which, in words, says something obvious: the sum of some positive and negative terms is less than or equal to the sum in which you make everything positive, and greater than or equal to the sum in which you make everything negative).

Now, Formula 5.4 tells us that $\sum_{k=p}^{q} |b_k| < \epsilon$ and, negating, that $-\epsilon < -\sum_{k=p}^{q} |b_k|$. Combining this with Formula 5.5, we find that, for $M \leq p \leq q$,

$$-\epsilon < \sum_{k=p}^{q} b_k \;<\; \epsilon,$$

i.e., $\left|\sum_{k=p}^{q} b_k\right| < \epsilon$. Therefore, the Cauchy Criterion tells us that $\sum_{k=m}^{\infty} b_k$ converges. $\square$

Example 5.4.3. So, if we take the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, and negate every term whose denominator is a prime number squared, we obtain the series

$$\sum_{k=1}^{\infty} b_k \ = \ \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} - \frac{1}{11^2} + \cdots .$$

The series $\sum_{k=1}^{\infty} b_k$ might look impossible to deal with, but it's easy for us now; if we take the series whose terms are the absolute values of the b_k's, we get back $\sum_{k=1}^{\infty} \frac{1}{k^2}$ which converges (it's a p-series with $p = 2 > 1$). Thus, Corollary 5.4.2 tells us that $\sum_{k=1}^{\infty} b_k$ converges.

Corollary 5.4.2 tells us that the convergence of $\sum_{m}^{\infty} |b_k|$ implies the convergence of $\sum_{m}^{\infty} b_k$; we give this type of convergence of $\sum_{m}^{\infty} b_k$ a name.

Definition 5.4.4. *If $\sum_{k=m}^{\infty} |b_k|$ converges, then we say that $\sum_{k=m}^{\infty} b_k$* **converges absolutely**.

If $\sum_{k=m}^{\infty} b_k$ converges, but does not converge absolutely (i.e., $\sum_{k=m}^{\infty} |b_k|$ diverges), then we say that $\sum_{k=m}^{\infty} b_k$ **converges conditionally**.

Remark 5.4.5. Thus, there are three things that a series can do: converge absolutely, converge conditionally, or diverge. For a non-negative series (or a non-positive series), there are only two possibilities: absolute convergence or divergence.

Note that, to show absolute convergence of $\sum_{k=m}^{\infty} b_k$, you have to show one thing: that $\sum_{k=m}^{\infty} |b_k|$ converges. However, to show conditional convergence, you have to show two things: that $\sum_{k=m}^{\infty} |b_k|$ diverges and, yet, $\sum_{k=m}^{\infty} b_k$ nonetheless converges.

Example 5.4.6. Using our new terminology, what we saw in Example 5.4.3 is that the series converges absolutely.

On the other hand, the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges conditionally. Why? Because, when we take the series with absolute values around each term, we get the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$, which we know diverges (it's a p-series, with $p \leq 1$), and yet we know that the alternating harmonic series converges (to $\ln 2$) from Remark 4.4.13.

Example 5.4.7. What about the series

$$\frac{1}{1} - \frac{1}{1^2} + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{3^2} + \frac{1}{4} - \frac{1}{4^2} + \cdots ? \tag{5.6}$$

We claim that this series diverges. How do you see this? There are several ways, but we'll look at a way that leads to useful general theorem.

It is easy to spot a convergent subseries hiding inside the series in Formula 5.6; we find the convergent subseries

$$0 - \frac{1}{1^2} + 0 - \frac{1}{2^2} + 0 - \frac{1}{3^2} + 0 - \frac{1}{4^2} + \cdots ,$$

where we have filled in zeroes for the missing terms; this series converges, since it's a negated p-series with $p = 2 > 1$ (the extra zeroes change nothing). Of course, the original series is formed from this "zeroed out" subseries by adding

$$\frac{1}{1} + 0 + \frac{1}{2} + 0 + \frac{1}{3} + 0 + \frac{1}{4} + 0 \cdots ,$$

which is the harmonic series (with extra zeroes), and so diverges. Now, it follows from Theorem 5.2.17 that the sum of these last two series, which equals the original series, diverges.

The argument that we gave in the example above is completely general and allows us to conclude:

Theorem 5.4.8. *Suppose that a series $\sum a_k$ has a convergent subseries, where the subseries converges to L. Delete the terms of the convergent subseries from $\sum a_k$, forming a new subseries $\sum b_j$ from the remaining terms.*

Then, $\sum b_j$ converges if and only if $\sum a_k$ converges and, when they both converge,

$$\sum a_k \; = \; L \; + \; \sum b_j.$$

In words, removing a convergent subseries from a series does not change whether or not the series converges or diverges, but, in the convergent case, it does change the sum of the series by exactly the sum of what's removed.

Recall the definition of a permutation of indices and rearrangement of a series from Definition 5.3.8. The following theorem tells us that rearrangement has no effect on absolutely convergent series.

Theorem 5.4.9. *Suppose that we have infinite series $\sum_{k=m}^{\infty} a_k$.*

1. *If $\sum_{k=m}^{\infty} a_k$ converges absolutely, then every rearrangement of $\sum_{k=m}^{\infty} a_k$ converges absolutely to the same value as $\sum_{k=m}^{\infty} a_k$.*
2. *If $\sum_{k=m}^{\infty} a_k$ does* **not** *converge absolutely, then no rearrangement of $\sum_{k=m}^{\infty} a_k$ converges absolutely.*

Proof. The "...to the same value" statement is very technical to prove; we refer you to Theorem 4.3.24 of [4], but we'll prove the rest.

First, Item 2 follows from Item 1, for if some rearrangement of $\sum_{k=m}^{\infty} a_k$ were to converge absolutely, then, by Item 1, any rearrangement of that rearrangement would converge absolutely. However, $\sum_{k=m}^{\infty} a_k$ itself is a rearrangement of any rearrangement of $\sum_{k=m}^{\infty} a_k$, and so $\sum_{k=m}^{\infty} a_k$ would have to converge absolutely.

Let's prove what's left of Item 1. Assume that $\sum_{k=m}^{\infty} a_k$ converges absolutely, i.e., that $\sum_{k=m}^{\infty} |a_k|$ converges. Then, Theorem 5.3.9 tells us that, for every permutation r of the indices, $\sum_{k=m}^{\infty} |a_{r(k)}|$ converges, i.e., $\sum_{k=m}^{\infty} a_{r(k)}$ converges absolutely. $\square$

Absolute convergence also behaves well with respect to multiplying by a constant or adding two series.

Theorem 5.4.10. *Suppose that we have infinite series* $\sum_{k=m}^{\infty} a_k$ *and* $\sum_{k=m}^{\infty} b_k$*, which converge absolutely. Let* c *be a constant.*

Then, $\sum_{k=m}^{\infty} ca_k$ *and* $\sum_{k=m}^{\infty}(a_k + b_k)$ *converge absolutely.*

Proof. The result follows quickly from looking at the partial sums. We find

$$\sum_{k=m}^{n} |ca_k| \;=\; |c| \sum_{k=m}^{n} |a_k|$$

and, using the triangle inequality, $|a+b| \leq |a| + |b|$; hence,

$$\sum_{k=m}^{n} |a_k + b_k| \;\leq\; \sum_{k=m}^{n} (|a_k| + |b_k|) \;=\; \sum_{k=m}^{n} |a_k| \;+\; \sum_{k=m}^{n} |b_k|.$$

The conclusions follow quickly. We leave the remaining details as an exercise. □

In the following theorem, we describe a product of two series, in a way that leads to the power series product that we looked at in Theorem 4.6.24; this product, $\sum c_n$ below, is known as the *Cauchy product* of the series.

Theorem 5.4.11. *Suppose that we have two series* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{j=0}^{\infty} b_j$*, and that* $\sum_{k=0}^{\infty} a_k$ *converges absolutely.*

Consider the Cauchy product series $\sum_{n=0}^{\infty} c_n$*, where*

$$c_n \;=\; \sum_{p=0}^{n} a_p b_{n-p} \;=\; a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0.$$

a. If $\sum_{j=0}^{\infty} b_j$ *converges, then* $\sum_{n=0}^{\infty} c_n$ *converges.*

b. If $\sum_{j=0}^{\infty} b_j$ *converges absolutely, then* $\sum_{n=0}^{\infty} c_n$ *converges absolutely.*

Proof. See Theorem 4.3.27 and Exercise 4.3.40 of [4]. □

Example 5.4.12. Of course, it's easy to find examples in which taking a Cauchy product of a convergent series with a divergent series yields a divergent series; you are asked to produce such an example in Exercise 30.

However, something that you may think would be true is not; in Exercise 31, you are asked to find two conditionally convergent series whose Cauchy product diverges.

The Ratio and Root Tests for Non-Negative Series, Theorem 5.3.23 and Theorem 5.3.26, immediately yield tests for absolute convergence, simply by inserting absolute value signs. Typically, you **cannot** conclude that series a $\sum b_k$ diverges by showing that $\sum |b_k|$ diverges; the series could still converge conditionally. However, recall that the divergence conclusions of the Ratio and Root Tests for Non-Negative Series are arrived at by showing that the terms do not approach 0, i.e., that the series diverges by the Term Test for Divergence, Theorem 5.2.12. But, if $|b_k|$ does not approach 0, then neither does b_k and so, if $\sum |b_k|$ diverges by the Term Test, then so does $\sum b_k$.

Thus, we immediately conclude the general Ratio and Root Tests.

Theorem 5.4.13. (**The Ratio Test**) *Consider the series* $\displaystyle\sum_{k=m}^{\infty} b_k$.

1. *If there exists $r < 1$ (and, necessarily, > 0) and an integer $M \geq m$ such that, for all $k \geq M$,*
$$b_k \neq 0 \quad \text{and} \quad \left|\frac{b_{k+1}}{b_k}\right| \leq r,$$
then the given series converges absolutely.

2. *If there exists an integer $M \geq m$ such that, for all $k \geq M$,*
$$b_k \neq 0 \quad \text{and} \quad \left|\frac{b_{k+1}}{b_k}\right| \geq 1,$$
then the given series diverges.

In particular, suppose that $\lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right|$ *exists, as an extended real number; call its value* L. *Then,*

a. *if* $L < 1$, *the given series converges absolutely;*

b. *if* $L > 1$, *including* $L = \infty$, *the given series diverges;*

c. *if* $L = 1$, *the given series may converge or diverge.*

Theorem 5.4.14. (*The Root Test*) *Consider the series* $\sum_{k=m}^{\infty} b_k$.

1. *If there exists* $r < 1$ *(and, necessarily,* > 0*) and an integer* $M \geq m$ *such that, for all* $k \geq M$,
$$\sqrt[k]{|b_k|} \ \leq \ r,$$
then the given series converges absolutely.

2. *If there exists an integer* $M \geq m$ *such that, for all* $k \geq M$,

$$\sqrt[k]{|b_k|} \ \geq \ 1,$$

then the given series diverges.

In particular, suppose that $\lim_{k\to\infty}\sqrt[k]{|b_k|}$ *exists, as an extended real number; call its value* L. *Then,*

a. *if* $L < 1$, *the given series converges absolutely;*

b. *if* $L > 1$, *including* $L = \infty$, *the given series diverges;*

c. *if* $L = 1$, *the given series may converge or diverge.*

Example 5.4.15. Consider the series

$$\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}.$$

To use the Ratio Test, we calculate the limit

$$L = \lim_{k\to\infty} \left| \frac{\frac{(-2)^{k+1}}{(k+1)!}}{\frac{(-2)^k}{k!}} \right| = \lim_{k\to\infty} \left[\frac{2^{k+1}}{2^k} \cdot \frac{k!}{(k+1)!} \right] = \lim_{k\to\infty} \frac{2}{k+1} = 0 < 1.$$

Therefore, the Ratio Test tells us that the series converges absolutely.

We can show the same thing with the Root Test. We calculate

$$L = \lim_{k\to\infty} \sqrt[k]{\left| \frac{(-2)^k}{k!} \right|} = \lim_{k\to\infty} \frac{2}{\sqrt[k]{k!}} = 0,$$

where, in the last step, we used that $\lim_{k\to\infty} \sqrt[k]{k!} = \infty$, from Corollary 5.3.30. Thus, the Root Test also tells us that the series converges absolutely. Of course, after we proved absolute convergence by the Ratio Test, we could have simply used Proposition 5.3.29 to conclude that the Root Test would yield the same limit.

We know that the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges, because we showed in Theorem 4.4.12 that the Maclaurin series for $\ln(1+x)$ converges and is equal to $\ln(1+x)$ at $x = 1$, and this yields the convergence of the alternating harmonic series to $\ln 2$.

We have also discussed the fact that alternating harmonic series converges **conditionally**, because it converges, but does not do so absolutely, since the harmonic series itself – what you get when you take the absolute values of the terms – diverges.

In fact, up to now, we have had no good tests for concluding the convergence of a series which does not converge absolutely. But we shall give one now. It is the *Alternating Series Test*, and it applies to series, like the harmonic series, in which the signs of the terms alternate, i.e., go $+, -, +, -, +, -, \cdots$ or $-, +, -, +, -, +, \cdots$.

Definition 5.4.16. *A series $\sum_{k=m}^{\infty} a_k$ is* **alternating** *if and only if either $a_k = (-1)^k |a_k|$ for all $k \geq m$, or $a_k = (-1)^{k-1}|a_k|$ for all $k \geq m$, i.e., if and only if the terms of the series alternate between being non-negative and being non-positive.*

Our primary interest in alternating series stems from:

Theorem 5.4.17. (**Alternating Series Test**) *Suppose that a series $\sum_{k=m}^{\infty} a_k$ satisfies the following three conditions:*

1. *$\sum_{k=m}^{\infty} a_k$ is an alternating series;*
2. *the terms are decreasing in absolute value, i.e., for all $k \geq m$, $|a_{k+1}| \leq |a_k|$;*
3. *the terms approach zero, i.e., $\lim_{k\to\infty} a_k = 0$ or, equivalently, $\lim_{k\to\infty} |a_k| = 0$.*

Then, the series $\sum_{k=m}^{\infty} a_k$ converges.

Furthermore, if we let s_∞ denote the sum of the series, and let $s_n = \sum_{k=m}^{n} a_k$, then s_n approximates s to within $|a_{n+1}|$; more precisely,

$$|s_\infty \; - \; s_n| \; \leq \; |a_{n+1}|.$$

In words, this says that the partial sum approximates the infinite sum to within the absolute value of the next term that's not in the partial sum.

Proof. This proof is not difficult. We give the main ideas and leave the details as an exercise for you.

First note, since the terms are decreasing in absolute value, that if a term is zero, then all of the subsequent terms must be zero, which would make the infinite sum actually a finite sum; convergence would be immediate, and the inequality would not be difficult to establish. So, in what follows, we will assume that all of the terms are non-zero.

We will assume, for convenience, that $m = 1$; this is clearly an unimportant assumption. We will also assume that the first term, a_1, is positive. The case where a_1 is negative will follow by negating the terms in the proof that we give for the case where a_1 is positive.

It is easy to show that the odd partial sums s_{2n+1}, for $n \geq 0$, form a decreasing sequence, which is bounded below by $a_1 + a_2$; thus, by Theorem 5.1.23, the sequence s_{2n+1} converges to some limit L_1, and $a_1 + a_2 \leq L_1$.

Similarly, it is easy to show that the even partial sums s_{2n}, for $n \geq 1$, form an increasing sequence, which is bounded above by a_1; thus, by Theorem 5.1.23, the sequence s_{2n} converges to some limit L_2, and $L_2 \leq a_1$.

We need to show that $L_1 = L_2$, which would show that all of the partial sums converge to this common limit. But this is easy, because

$$|L_1 - L_2| \;=\; \lim_{n\to\infty} |s_{2n+1} - s_{2n}| \;=\; \lim_{n\to\infty} |a_{2n+1}| \;=\; 0.$$

Therefore, we have shown that $\sum_{k=m}^{\infty} a_k$ converges to $L_1 = L_2$, which is $\leq a_1$. In the case where a_1 may be positive or negative, what this shows is that $\sum_{k=m}^{\infty} a_k$ converges and $|\sum_{k=m}^{\infty} a_k| \leq |a_1|$.

To show the inequality in the theorem, note that

$$s_\infty \;-\; s_n \;=\; \sum_{k=n+1}^{\infty} a_k,$$

and this new series, which starts at a_{n+1} is itself an alternating series which satisfies the requirements of the Alternating Series Test. Therefore, from our inequality above, which uses the first term of the alternating series, we immediately conclude that

$$|s_\infty \;-\; s_n| \;=\; \left|\sum_{k=n+1}^{\infty} a_k\right| \;\leq\; |a_{n+1}|.$$

□

Example 5.4.18. Consider the *alternating p-series*,

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^p} \;=\; 1 \;-\; \frac{1}{2^p} \;+\; \frac{1}{3^p} \;-\; \frac{1}{4^p}.$$

- If $p \leq 0$, the Term Test for Divergence, Theorem 5.2.12, tells us that the series diverges.

- If $p > 0$, then the Alternating Series Test tells us that the series converges.

- The p-Series Test, Corollary 5.3.15, tells us that

$$\sum_{k=1}^{\infty} \left| (-1)^{k-1} \frac{1}{k^p} \right| \;=\; \sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Therefore, the alternating p-series, $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^p}$, converges absolutely if $p > 1$, converges conditionally if $0 < p \leq 1$, and diverges if $p \leq 0$.

For $p > 0$, we can easily estimate the entire infinite summation by the partial sums. For instance, the last part of Theorem 5.4.17 tells us that

$$\left| \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^3} \;-\; \sum_{k=1}^{4} (-1)^{k-1} \frac{1}{k^3} \right| \;\leq\; \frac{1}{5^3},$$

i.e.,

$$-\frac{1}{5^3} + \sum_{k=1}^{4} (-1)^{k-1} \frac{1}{k^3} \;\leq\; \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^3} \;\leq\; \frac{1}{5^3} + \sum_{k=1}^{4} (-1)^{k-1} \frac{1}{k^3}.$$

Recall now our discussion of power series as functions from Section 4.5. In that section, we gave you one convergence/divergence test: the Ratio Test. This enabled us to determine the radius of convergence of power series, but could never tell us what happened at the endpoints of the intervals of convergence.

At long last, we are in a position to consider examples in which we determine the entire intervals of convergence of some power series, **including** what happens at the endpoints. Recall that power series converge absolutely on the interior of the interval of convergence, but, at an endpoint, it is possible to have divergence, conditional convergence, or absolute convergence.

Example 5.4.19. Determine the intervals of convergence of the series

$$p(x) \;=\; \sum_{k=1}^{\infty} (-1)^k \frac{(x-2)^k}{3^k \sqrt{k}} \qquad \text{and} \qquad q(x) \;=\; \sum_{k=1}^{\infty} (-1)^k \frac{7^k (x+1)^k}{k^2},$$

and determine whether any convergence at an endpoint of the intervals of convergence is absolute or conditional.

Solution:

As in Section 4.5, we use the Ratio Test, Theorem 5.4.13, to quickly determine the radii of convergence. Because of the absolute values, we can drop the minus signs and find

$$L_p \;=\; \lim_{k\to\infty}\left|\frac{\dfrac{(x-2)^{k+1}}{3^{k+1}\sqrt{k+1}}}{\dfrac{(x-2)^k}{3^k\sqrt{k}}}\right| \;=\; \lim_{k\to\infty}\left[\frac{3^k}{3^{k+1}}\cdot\frac{\sqrt{k}}{\sqrt{k+1}}\cdot\frac{|x-2|^{k+1}}{|x-2|^k}\right] \;=\; \frac{1}{3}\,|x-2|$$

and

$$L_q \;=\; \lim_{k\to\infty}\left|\frac{\dfrac{7^{k+1}(x+1)^{k+1}}{(k+1)^2}}{\dfrac{7^k(x+1)^k}{k^2}}\right| \;=\; \lim_{k\to\infty}\left[\frac{7^{k+1}}{7^k}\cdot\frac{k^2}{(k+1)^2}\cdot\frac{|x+1|^{k+1}}{|x+1|^k}\right] \;=\; 7\,|x+1|.$$

Therefore, $p(x)$ converges absolutely if $|x-2|/3 < 1$, and diverges if $|x-2|/3 > 1$, i.e., converges absolutely if $|x-2| < 3$, and diverges if $|x-2| > 3$. Similarly, $q(x)$ converges absolutely if $7|x+1| < 1$, and diverges if $7|x+1| > 1$, i.e., converges absolutely if $|x+1| < 1/7$, and diverges if $|x+1| > 1/7$.

Thus, the interiors of the intervals of convergence of $p(x)$ and $q(x)$ are, respectively, the open intervals $(2-3, 2+3) = (-1, 5)$ and $(-1-\frac{1}{7}, -1+\frac{1}{7}) = (-8/7, -6/7)$. The question is: what happens at the endpoints of these intervals?

The beginning of the answer is: you plug the endpoint x values into the power series, look at the resulting series of constants, and use tests **other** than the Ratio Test.

• So, let's consider the series

$$p(x) \;=\; \sum_{k=1}^{\infty}(-1)^k\,\frac{(x-2)^k}{3^k\sqrt{k}},$$

when we plug in the endpoints of $(-1, 5)$, i.e., when $x = -1$ and when $x = 5$.

When $x = -1$, we find the series of constants

$$\sum_{k=1}^{\infty}(-1)^k\,\frac{(-1-2)^k}{3^k\sqrt{k}} \;=\; \sum_{k=1}^{\infty}(-1)^k\,\frac{(-1)^k}{\sqrt{k}} \;=\; \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}.$$

This is a p-series, with $p = 1/2$, and so it diverges by the p-series Test, Corollary 5.3.15.

When $x = 5$, we find the series of constants

$$\sum_{k=1}^{\infty}(-1)^k \frac{3^k}{3^k\sqrt{k}} \;=\; \sum_{k=1}^{\infty}(-1)^k \frac{1}{\sqrt{k}}.$$

This is an alternating series, which converges by Theorem 5.4.17; note that the series does not converge absolutely, as taking the absolute values of the terms would yield the previous divergent p-series with $p = 1/2$.

Hence, we find that the interval of convergence of $p(x)$ is the interval $(-1, 5]$, and the convergence is conditional at $x = 5$.

- Now, let's consider the series

$$q(x) \;=\; \sum_{k=1}^{\infty}(-1)^k \frac{7^k(x+1)^k}{k^2},$$

when we plug in the endpoints of $(-8/7, -6/7)$, i.e., when $x = -8/7$ and when $x = -6/7$.

You can quickly see that, when we plug in these endpoints, we get $\sum_{k=1}^{\infty}\frac{1}{k^2}$ and $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}$, respectively. These both converge absolutely, by the p-series Test, Corollary 5.3.15, with $p = 2$.

Hence, we find that the interval of convergence is the closed interval $[-8/7, -6/7]$, and that $q(x)$ converges absolutely everywhere in the interval.

As the final result of this section, we wish to state, and give an idea of the proof of, a result that tells us that the summation in a conditionally convergent series does **not** correspond to what we normally think of as adding together a collection of numbers; the theorem tells us that the order in which we add the terms in a conditionally convergent series can have a **dramatic** effect on the summation. This is **not** the case for absolutely convergent series, as we saw in Theorem 5.4.9.

Theorem 5.4.20. *Suppose that we have infinite series $\sum_{k=m}^{\infty} a_k$, which converges conditionally. Let L be any extended real number.*

Then, there exists a rearrangement of the series $\sum_{k=m}^{\infty} a_{r(k)}$ such that

$$\sum_{k=m}^{\infty} a_{r(k)} \;=\; L.$$

In words, a conditionally convergent series can be rearranged to converge to anything whatsoever, or diverge to $\pm\infty$.

Proof. The details of the proof can be found in Theorem 4.3.26 of [4], but we will try here to indicate how such a thing is possible.

Let P denote the sum of the positive terms of $\sum_{k=m}^{\infty} a_k$, taken in order, and let N denote the sum of the negative terms of $\sum_{k=m}^{\infty} a_k$, taken in order.

P and N are defined by non-negative and non-positive subseries, respectively, and we may assume that we have inserted zeroes for each "missing" term of each subseries. This enables us to write $P = \sum_{k=m}^{\infty} b_k$ and $N = \sum_{k=m}^{\infty} c_k$, where $b_k \geq 0$, $c_k \leq 0$, and $a_k = b_k + c_k$.

We claim that $P = \infty$ and $N = -\infty$. Why is this true? As $\sum b_k$ and $\sum c_k$ are non-negative and non-positive, respectively, the only choice other than equaling $\pm\infty$ is that the series converge. But, if either one of these series converges, then, since $\sum_{k=m}^{\infty} a_k$ converges, Theorem 5.4.8 tells us that the other series must also converge. However, as $\sum b_k$ and $\sum c_k$ are non-negative and non-positive, the only way for them to converge is absolutely, which would imply, by Theorem 5.4.10, that their sum $\sum a_k$ converges absolutely. This would contradict that $\sum a_k$ converges conditionally.

Therefore, the sum of the positive terms of $\sum a_k$ is ∞, and the sum of the negative terms is $-\infty$.

What does this have to do with anything? Well...intuitively, things are fairly easy now. If you want to produce a rearrangement of $\sum a_k$ that diverges to ∞, then you simply put lots of (sizable) positive terms first, then a negative term, then lots of positive terms, then a negative term; the point being that by adding together lots of positive terms early, you can overwhelm the sporadic negative terms. You have to have every negative term in the series **somewhere**, but they can come after big chunks of positive terms that add up to arbitrarily large positive numbers.

Similarly, to produce a rearrangement that diverges to $-\infty$ just take lots of (sizable) negative terms early, then a positive term, and keep going.

Once you believe that there are rearrangements that yield ∞ and $-\infty$, it's easy to believe, if you were very careful with picking your rearrangement, that you could make it add up to anything between $-\infty$ and ∞, i.e., make the rearrangement add up to any real number whatsoever. $\square$

5.4.1 Exercises

In each of Exercises 1 through 15, determine whether the given series converges absolutely, converges conditionally, or diverges. We omit the starting and final values for the index on the summation; assume the starting index is big enough so that all of the terms are defined and the final value is ∞.

1. $\sum \dfrac{\sin k}{k^2}$
2. $\sum \dfrac{-2+\sin k}{k}$
3. $\sum \dfrac{-2+\sin k}{k^3}$
4. $\sum (-1)^k \dfrac{1}{\sqrt{k+4}}$
5. $\sum (-1)^k \sqrt{1+\dfrac{4}{k}}$
6. $\sum \left(\dfrac{5}{k^{3/2}} + (-1)^{k-1}\dfrac{1}{3k+4}\right)$
7. $\sum \left((-1)^{k-1}\dfrac{5}{k^{3/2}} + \dfrac{1}{3k+4}\right)$
8. $\sum \dfrac{(-1)^k}{3(k!)+4}$
9. $\sum \dfrac{(-1)^k}{\sqrt{3(k!)+4}}$
10. $\sum \dfrac{(-1)^k}{k\ln k}$
11. $\sum \left(3 + \dfrac{(-1)^k}{k\ln k}\right)$

12. $\sum \dfrac{\cos(k\pi/2)}{k^5 \ln k}$

13. $\sum \dfrac{\cos(k\pi/2)}{\sqrt{k}}$

14. $\sum (-1)^{k-1} \dfrac{\sqrt{k^3+1}}{\sqrt{k^5+1}}$

15. $\sum (-1)^{k-1} \dfrac{\sqrt{k^3+1}}{\sqrt{k^6+1}}$

In each of Exercises 16 through 19, find the first 5 terms of the Cauchy product of the two given series; see Theorem 5.4.11. Also prove that each Cauchy product converges.

16. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)^2}$ and $\displaystyle\sum_{j=0}^{\infty} (-1)^j \frac{1}{j+1}$

17. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+1)^2}$ and $\displaystyle\sum_{j=0}^{\infty} (-1)^j \frac{1}{j+1}$

18. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$ and $\displaystyle\sum_{j=0}^{\infty} (-1)^j \frac{1}{\sqrt{j+1}}$

19. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{1}{k!}$ and $\displaystyle\sum_{j=0}^{\infty} \frac{1}{j!}$

In each of Exercises 20 through 23, show that the alternating series converges, and give a partial sum which estimates the entire sum of the infinite series to within 0.01.

20. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k}$

21. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{\sqrt{k}}$

22. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^2}$

23. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{\sqrt{k^2+1}}$

24. If $\sum a_n$ converges absolutely and $\sum b_n$ converges absolutely, what can you say about $\sum(a_n + b_n)$?

25. If $\sum a_n$ converges absolutely and $\sum b_n$ converges conditionally, what can you say about $\sum(a_n + b_n)$?

26. If $\sum a_n$ converges conditionally and $\sum b_n$ converges conditionally, what can you say about $\sum(a_n + b_n)$?

27. If $\sum a_n$ converges absolutely and $\sum b_n$ diverges, what can you say about $\sum(a_n + b_n)$?

28. If $\sum a_n$ converges conditionally and $\sum b_n$ diverges, what can you say about $\sum(a_n + b_n)$?

29. Complete the proof of Theorem 5.4.10.

30. Give an example of an absolutely convergent series and a divergent series such that the Cauchy product (see Theorem 5.4.11) of the two series diverges.

31. Find two conditionally convergent series whose Cauchy product diverges.

32. Fill in the details of the proof of the Alternating Series Test, Theorem 5.4.17.

33. Write a brief essay, in which you discuss the ways in which conditionally convergent series do not behave like "normal" sums of numbers.

34. Suppose you have a power series $p(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$, with radius of convergence R. Prove that, if $p(x)$ converges absolutely at one end of the interval of convergence, then it converges absolutely at the other end. Is the same statement true if "absolutely" is replaced both times with "conditionally"?

In each of Exercises 35 through 42, you are given a power series from the Exercises in Section 4.5, for which you may have already found the radius of convergence. Finish finding the precise interval of convergence by determining what happens at the endpoints of the interval, i.e., if the power series has radius of convergence R, where $0 < R \leq \infty$, and is centered at a, determine whether the series converges absolutely, converges conditionally, or diverges at $x = a - R$ and $x = a + R$.

35.

$$p(x) \;=\; \sum_{k=1}^{\infty} \frac{5^k(x-1)^k}{k^2} \;=\; 5(x-1) \;+\; \frac{5^2}{2^2}(x-1)^2 \;+\; \frac{5^3}{3^2}(x-1)^3 \;+\; \frac{5^4}{4^2}(x-1)^4 \;+\; \cdots .$$

36.

$$p(x) \;=\; \sum_{k=1}^{\infty} \sqrt{k}\,(x+3)^k \;=\; \big(x+3\big) \;+\; \sqrt{2}\,\big(x+3\big)^2 \;+\; \sqrt{3}\,\big(x+3\big)^3 \;+\; \sqrt{4}\,\big(x+3\big)^4 \;+\; \cdots\,.$$

37.

$$p(x) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{x^k}{k^3+2} \;=\; \frac{1}{2} \;-\; \frac{x}{1^3+2} \;+\; \frac{x^2}{2^3+2} \;-\; \frac{x^3}{3^3+2} \;+\; \frac{x^4}{4^3+2} \;-\; \cdots\,.$$

38. $p(x) \;=\; \displaystyle\sum_{k=0}^{\infty} \frac{(x-4)^k}{7^{k+1}}.$

39. $p(x) \;=\; \displaystyle\sum_{k=1}^{\infty} \frac{(x-4)^k}{7^{k+1}k}.$

40. $p(x) \;=\; \displaystyle\sum_{k=1}^{\infty}(-1)^{k+1} \frac{(x+7)^k}{k^k}.$

41. $p(x) \;=\; \displaystyle\sum_{k=1}^{\infty}(-1)^{k+1} \frac{k!(x+7)^k}{k^k}.$

42. $p(x) \;=\; \displaystyle\sum_{k=1}^{\infty}(-1)^{k+1} \frac{x^k}{\sqrt[3]{k}}.$

Appendix A

An Introduction to Vectors and Motion

A brief introduction to vectors in 2 and 3 dimensions

A *vector* is frequently described in physics and engineering classes as something which has a magnitude and a direction. Typical physical vector quantities in the plane are things such as "move north 3 miles", "travel west at 5 meters per second", and "a force of 10 pounds pushing south".

A vector in the real line, $\mathbb{R}$, the xy-plane, $\mathbb{R}^2$, or xyz-space, $\mathbb{R}^3$, is usually represented by an arrow, which points in the direction of the vector and has length equal to the magnitude of the vector (even if vector itself does not have length units). However, the arrow representing a given vector has no fixed starting and ending point; parallel arrows which have the same length and which have their tails and heads at corresponding ends (i.e., point in the same direction) represent the same vector.

A vector in the real line is simply a real number r. The magnitude of r is the absolute value $|r|$. There are two possible "directions" for a vector in the real line; if $r > 0$, then r has the positive direction and, if $r < 0$, then r has the negative direction. The *zero vector* $r = 0$ is said to have every direction (in this case, **both** directions); this is convenient in statements of some results, and makes sense, in that, moving 0 feet in the positive direction is the same as moving 0 feet in the negative direction.

We shall now discuss vectors in $\mathbb{R}^2$. The analogous statements and definitions for vectors in $\mathbb{R}^3$ should be clear, and we will explicitly state many of them after our discussion about vectors in the plane.

We denote the arrow from the point (a, b) to the point (c, d) by $\overrightarrow{(a,b)(c,d)}$. Two arrows

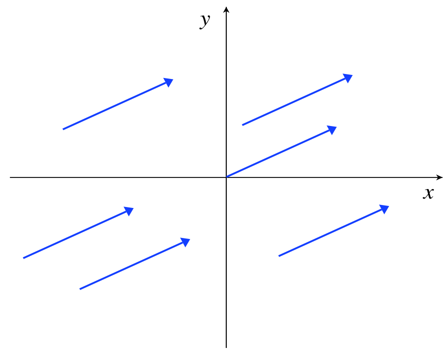


Figure A.1: Arrows which all represent the same vector.

$\overrightarrow{(a_1, b_1)(c_1, d_1)}$ and $\overrightarrow{(a_2, b_2)(c_2, d_2)}$ have the same direction and magnitude, i.e., represent the same vector, if and only if, when we "drag them to where they both start at the origin", we get the same arrow. This means that $\overrightarrow{(a_1, b_1)(c_1, d_1)}$ and $\overrightarrow{(a_2, b_2)(c_2, d_2)}$ represent the same vector if and only if

$$\overrightarrow{(0,0)(c_1 - a_1, d_1 - b_1)} \;=\; \overrightarrow{(0,0)(c_2 - a_2, d_2 - b_2)},$$

i.e., if and only if $c_1 - a_1 = c_2 - a_2$ and $d_1 - b_1 = d_2 - b_2$. This means that, if we set our default starting point for arrows as the origin, we can specify vectors simply by specifying the ending points of the arrows. Thus, given an ordered pair of real numbers, like $(2, -5)$, we may refer to $(2, -5)$ as a point, and picture it as a point, or we may refer to it as a vector, and picture it as the vector represented by an arrow from the origin to the point $(2, -5)$.

Vectors are frequently denoted by including little arrows over variable names, e.g., the vector $\vec{v} = (2, -5)$. Note that it is not common to put an arrow over $(2, -5)$ itself; we either explicitly state that $(2, -5)$ is a vector, or let the context determine whether $(2, -5)$ is being used as a point or a vector.

If we have a vector $\vec{v} = (a, b)$, we refer to a as the *x-component* of $\vec{v}$ and to b as the *y-component* of $\vec{v}$.

The *magnitude*, $|(a, b)|$, of a vector (a, b) is the Euclidean length of the arrow from the origin to the point (a, b), i.e., $|(a, b)| = \sqrt{a^2 + b^2}$. Note that the use of "absolute value" signs for the magnitude does **not** lead to confusion, for, as we discussed above, a vector in the real line is simply a single real number, and the analogous notion of magnitude would be that the magnitude of a real number a is $\sqrt{a^2}$, but this is the same as the absolute value of a.

There is only one vector (in the plane) whose magnitude is zero: the *zero vector* $\vec{0} = (0, 0)$. A *unit vector* means a vector whose magnitude is 1.

We define two other operations on vectors, *vector addition* and *scalar multiplication*. The sum of two vectors (a, b) and (c, d) is defined by adding the corresponding components:

$$(a, b) + (c, d) \ = \ (a + c, b + d).$$

The fact that we are adding the ordered pairs means that we are using them as vectors, not points. The scalar multiplication $r(a, b)$ of the vector (a, b) by a real number (a scalar) r is defined by

$$r(a, b) \ = \ (ra, rb).$$

If $r \neq 0$, then $\left(\frac{1}{r}\right)\vec{v}$ is frequently written as $\frac{\vec{v}}{r}$.

From the definition of scalar multiplication, it follows quickly that $|r(a, b)| = |r| \cdot |(a, b)|$. Multiplying by a positive scalar leaves the direction of the vector unchanged, but "scales" the magnitude because, if $r > 0$, then $|r(a, b)| = |r| \cdot |(a, b)| = r|(a, b)|$. Multiplying by a negative scalar produces a vector which points in the opposite direction from the original vector, but the magnitude is scaled by the absolute value of the scalar; thus, $-2\vec{v}$ has twice the magnitude of $\vec{v}$, but points in the opposite direction.

The negation, $-\vec{v}$, of a vector $\vec{v}$ is equal to $-1\vec{v}$. Subtraction of vectors $\vec{v} = (a, b)$ and $\vec{w} = (c, d)$ is defined component-wise, so that $\vec{v} - \vec{w} = (a - c, b - d)$. This is written in terms of vector addition and scalar multiplication as

$$\vec{v} - \vec{w} \ = \ \vec{v} + (-1\vec{w}).$$

Note that the distance between two points (a, b) and (c, d) is simply the magnitude of the vector $(a, b) - (c, d)$ (or $(c, d) - (a, b)$).

If $\vec{v} \neq \vec{0}$, then there is a unique unit vector in the direction of $\vec{v}$, namely, $\frac{\vec{v}}{|\vec{v}|}$. In fact, since most directions in the plane, using your intuitive notion of "direction", do not have some predetermined name, the *direction* of a non-zero vector $\vec{v}$ is defined to be the unit vector $\frac{\vec{v}}{|\vec{v}|}$. As in the case of "vectors" in the real line, we say that the zero vector $\vec{0}$ has every direction.

The *dot product* of two vectors $\vec{v} = (a, b)$ and $\vec{w} = (c, d)$ is defined by

$$(a, b) \cdot (c, d) \ = \ ac + bd,$$

i.e., you multiply corresponding components and add. Note that the dot product of two vectors is **not** a vector; it is a real number (a scalar). The dot product has a number of nice properties: for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, and scalars a and b,

1. (commutativity) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$;
2. (distributivity/linearity) $\vec{u} \cdot (a\vec{v} + b\vec{w}) = a(\vec{u} \cdot \vec{v}) + b(\vec{u} \cdot \vec{w})$; and
3. $\vec{v} \cdot \vec{v} = |\vec{v}|^2$.

However, the reason that the dot product is of great interest is because it has the following geometric interpretation

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\theta,$$

where θ is the angle between the vectors $\vec{v}$ and $\vec{w}$. In particular this means that $\vec{v}$ and $\vec{w}$ are perpendicular if and only if $\vec{v} \cdot \vec{w} = 0$; note that our convention that the zero vector has **every** direction makes this statement true even if $\vec{v}$ or $\vec{w}$ is the zero vector.

Vectors in $\mathbb{R}^3$ work just like vectors in $\mathbb{R}^2$, except that you now have three components, instead of two. We also frequently refer to $\mathbb{R}^2$ as being contained in $\mathbb{R}^3$ by thinking of the point or vector $(a, b, 0)$ as being the same as the point or vector (a, b), i.e., it is standard to think of the xy-plane as being the same as the plane where $z = 0$ inside xyz-space.

Given an ordered triple of real numbers, like $(2, -5, 3)$, we may refer to $(2, -5, 3)$ as a point, and picture it as a point, or we may refer to it as a vector, and picture it as the vector represented by an arrow from the origin to the point $(2, -5, 3)$.

If we have a vector $\vec{v} = (a, b, c)$, we refer to a as the *x-component* of $\vec{v}$, to b as the *y-component* of $\vec{v}$, and to c as the *z-component*.

The *magnitude*, $|(a, b, c)|$, of a vector (a, b, c) is the Euclidean length of the arrow from the origin to the point (a, b, c), i.e., $|(a, b)| = \sqrt{a^2 + b^2 + c^2}$.

There is only one vector in $\mathbb{R}^3$ whose magnitude is zero: the *zero vector* $\vec{0} = (0, 0, 0)$. A *unit vector* means a vector whose magnitude is 1.

We define *vector addition* and *scalar multiplication* as you would expect: the sum of two vectors (a, b, c) and (d, e, f) is defined by adding the corresponding components:

$$(a, b, c) + (d, e, f) = (a + d, b + e, c + f).$$

The scalar multiplication $r(a, b, c)$ of the vector (a, b, c) by a real number (a scalar) r is defined by

$$r(a, b, c) = (ra, rb, rc).$$

If $\vec{v} \neq \vec{0}$, then there is a unique unit vector in the direction of $\vec{v}$, namely, $\dfrac{\vec{v}}{|\vec{v}|}$. As in the case of "vectors" in $\mathbb{R}$ and $\mathbb{R}^2$, we say that the zero vector $\vec{0}$ has every direction.

The *dot product* of two vectors $\vec{v} = (a, b, c)$ and $\vec{w} = (d, e, f)$ is defined by

$$(a, b, c) \cdot (d, e, f) \;=\; ad + be + cf,$$

i.e., you multiply corresponding components and add. The dot product in $\mathbb{R}^3$ has the same nice properties as it did in $\mathbb{R}^2$: for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, and scalars a and b,

1. (commutativity) $\vec{v} \cdot \vec{w} \;=\; \vec{w} \cdot \vec{v}$;
2. (distributivity/linearity) $\vec{u} \cdot (a\vec{v} + b\vec{w}) = a(\vec{u} \cdot \vec{v}) + b(\vec{u} \cdot \vec{w})$; and
3. $\vec{v} \cdot \vec{v} \;=\; |\vec{v}|^2$.

In addition, it is still true in $\mathbb{R}^3$ that

$$\vec{v} \cdot \vec{w} \;=\; |\vec{v}|\,|\vec{w}| \cos\theta,$$

where θ is the angle between the vectors $\vec{v}$ and $\vec{w}$.

Vector-valued functions

Suppose we have a function $\vec{p}(t) = (x(t), y(t))$ or $\vec{p}(t) = (x(t), y(t), z(t))$, whose domain is a subset of the real numbers and whose codomain is a subset of either $\mathbb{R}^2$ or $\mathbb{R}^3$; this is a *vector-valued function*. The functions $x(t)$, $y(t)$, and $z(t)$ are the *component functions* of $\vec{p}(t)$. Such a function $\vec{p}(t)$ is *continuous* if and only if each of its component functions is continuous; such a function is *differentiable* if and only if each of its component functions is differentiable and, when this is the case, the derivative $\dfrac{d\vec{p}}{dt} = \vec{p}\,'(t)$ is defined by

$$\vec{p}\,'(t) \;=\; \lim_{h \to 0} \frac{\vec{p}(t+h) - \vec{p}(t)}{h} \;=\; (x'(t), y'(t)) \;\text{ or }\; (x'(t), y'(t), z'(t)).$$

Differentiation of vector-valued functions is linear: if $\vec{p}(t)$ and $\vec{q}(t)$ are differentiable vector-valued functions, both in $\mathbb{R}^n$ for the same n, and a and b are constants (scalars), then $a\,\vec{p}(t) + b\,\vec{q}(t)$ is differentiable and

$$\big(a\,\vec{p}(t) + b\,\vec{q}(t)\big)' \;=\; a\,\vec{p}\,'(t) \;+\; b\,\vec{q}\,'(t).$$

More surprising is that there is a *product rule* for the dot product: if $\vec{p}(t)$ and $\vec{q}(t)$ are differentiable vector-valued functions, both in $\mathbb{R}^n$ for the same n, then $\vec{p}(t) \cdot \vec{q}(t)$ is differentiable and

$$\big(\vec{p}(t) \cdot \vec{q}(t)\big)' \;=\; \vec{p}(t) \cdot \vec{q}\,'(t) \;+\; \vec{p}\,'(t) \cdot \vec{q}(t).$$

An immediate consequence of this is that, if $\vec{p}(t)$ is differentiable and always has magnitude 1 (i.e., is always a unit vector), then $1 = \vec{p}(t) \cdot \vec{p}(t)$ for all t; therefore, differentiating with respect to t and applying the Product Rule, we find

$$0 = 2\,\vec{p}(t) \cdot \vec{p}\,'(t),$$

so that $\vec{p}(t)$ is always perpendicular to $\vec{p}\,'(t)$.

Motion in the plane and in space

We will now discuss motion in the plane and in space. Throughout our discussion, we shall use notation for the case of motion in $\mathbb{R}^3$. The case of motion in $\mathbb{R}^2$ is analogous; simply ignore the z-component in what we write, or take the z-component to be zero.

Suppose that an object is moving in $\mathbb{R}^3$. Then, at each time t, the object has an x-coordinate, $x(t)$, a y-coordinate, $y(t)$, and a z-coordinate, $z(t)$. The functions $x(t)$, $y(t)$, and $z(t)$ are said to describe a *parameterized curve* in space. The *position vector* (or, simply position), $\vec{p}(t)$, of the object, at time t, is simply the vector $(x(t), y(t), z(t))$. In place of $x(t)$, $y(t)$, and $z(t)$, it is common to write $p_x(t)$, $p_y(t)$, and $p_z(t)$, respectively, for the x-, y-, and z-components of $\vec{p}(t)$.

If the position function $\vec{p}(t)$ is differentiable, the *velocity vector* (or, simply, velocity), $\vec{v}(t)$, of the object, at time t, is the derivative of the position vector, i.e.,

$$\vec{v}(t) \;=\; \frac{d}{dt}\vec{p}(t) \;=\; \vec{p}\,'(t).$$

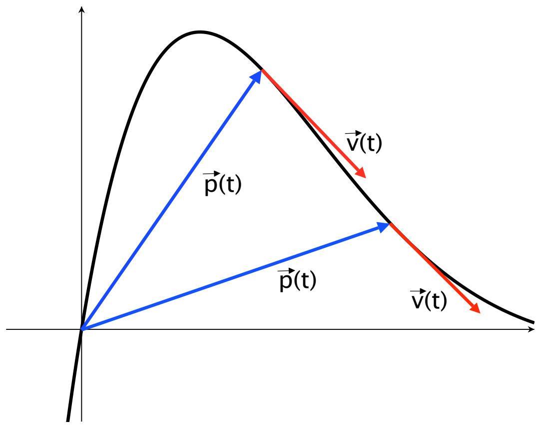


Figure A.2: Velocity vectors are tangent to the curve defined by $\vec{p}(t)$.

It is standard to indicate the velocity vector $\vec{v}(t)$ as having its initial point at the point $\vec{p}(t)$; if the velocity vector $\vec{v}(t_0)$ is non-zero, it will be tangent to the curve defined by $\vec{p}(t)$ at the point $\vec{p}(t_0)$.

The magnitude $|\vec{v}(t)|$ of the velocity is the *speed* of the object at time t. See Section 3.3 for why this notion of speed is equivalent to the instantaneous rate of change of the distance traveled, with respect to time.

The *acceleration vector* (or, simply, acceleration), $\vec{a}(t)$, of the object, at time t, is the derivative of the velocity vector, i.e.,

$$\vec{a}(t) \;=\; \frac{d}{dt}\vec{v}(t) \;=\; \vec{v}\,'(t) \;=\; \vec{p}\,''(t),$$

provided that $\vec{v}(t)$ is differentiable.

Appendix B

Tables of Integration Formulas

In the tables below a, b, p, and C denote arbitrary real constants, except that, in the formulas involving b^x and $\log_b$, we assume that $b > 0$, and in the formulas involving the inverse trig functions, we assume that $a > 0$. We use f and g to denote differentiable real functions, and u, v, and x to denote variable names, either independent variables, or dependent variables, i.e., the values of functions. In any formulas involving divisions, we assume that we are in a situation that does **not** lead to division by zero. More generally, we assume that we are in a situation where all of the functions involved in the formulas are defined.

Formulas for Reducing Complicated Integrals to Easier Ones	
Linearity:	$(af \pm bg)'(x) = af'(x) \pm bg'(x)$.
Integration by Parts:	$(f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x)$.
Integration by Substitution:	$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ or $\dfrac{dv}{dx} = \dfrac{dv}{du} \cdot \dfrac{du}{dx}$.

Algebraic Integrals	
$\int 0\,dx = C.$	$\displaystyle\int b^x\,dx = \frac{b^x}{\ln b} + C.$
$\displaystyle\int x^p\,dx = \frac{x^{p+1}}{p+1} + C, \text{ if } p \neq -1$	$\displaystyle\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C.$
$\int e^x\,dx = e^x + C.$	

Trigonometric Integrals	
$\int \cos x\,dx = \sin x + C.$	$\int \sin x\,dx = -\cos x + C.$
$\int \sec^2 x\,dx = \tan x + C.$	$\int \csc^2 x\,dx = -\cot x + C.$
$\int \sec x \tan x\,dx = \sec x + C.$	$\int \csc x \cot x\,dx = -\csc x + C.$

Inverse Circular and Hyperbolic Trigonometric Integrals
$\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \text{ if } a > 0.$
$\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C, \text{ if } a \neq 0.$
$\displaystyle\int \frac{1}{\lvert x\rvert\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C, \text{ if } a \neq 0.$
$\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C, \text{ if } a > 0.$
$\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C, \text{ if } x > a > 0.$

Iteration Formulas for an integer $n \geq 2$
$$\int \sin^n \theta \, d\theta \;=\; -\frac{1}{n} \sin^{n-1} \theta \cos \theta \;+\; \frac{n-1}{n} \int \sin^{n-2} \theta \, d\theta.$$
$$\int \cos^n \theta \, d\theta \;=\; \frac{1}{n} \cos^{n-1} \theta \sin \theta \;+\; \frac{n-1}{n} \int \cos^{n-2} \theta \, d\theta.$$

Appendix C

Answers to Odd-Numbered Exercises

For producing answers to various exercises or for help with examples or visualization, you may also find the free web site wolframalpha.com very useful.

Chapter 1

Section 1.1

1. $\frac{4}{3}x^3 + 2x^2 + 9x + C$

3. $-5\cos t - 3\sin^{-1} t + C$

5. $\ln|y| + \tan^{-1} y + C$

7. $1/2\sin(2\theta - 1) + C$

9. $1/2\ln|r^2 - 4| + C$

11. $(5w - 3)^{101}/505 + C$

13. $-(x+2)(x+5) + (x+7)(x+2)\ln|x+2| + C$

15. $5\ln|\ln x| + C$

17. $-\ln|\cos\theta| + C$

19. $\frac{2}{15}(t^5 + 6)^{3/2} + C$

21. $\sin^{-1}(x-3) + C$

23. $P(w) = 5\ln|w| - 7e^w + (9/2)w^{4/3} + 7e^{-1} - 9/2$

25. $K(v) = -v^{-1} + \ln|v| - 2v^{-1/2} + 1$

27. $F(x) = 1/3x^3 - x\cos x + \sin x + \pi - 1/3\pi^3$

29. $G(x) = 2/3(x+1)^{3/2} - 2/5(x+1)^{5/2} + 17/6$

31. $R(t) = -1/2e^{1-t^2} - 1/2t^2 + \sqrt{2} + 1$

33. $(x-5)^2e^x - 2[(x-5)e^x - e^x] + C =$ $(x^2 - 12x + 37)e^x + C$

35. $t^2\sin t + 2t\cos t - 2\sin t + C$

37. $-\dfrac{\ln t}{t} - \dfrac{1}{t} + C$

39. $\dfrac{1}{29}e^{2x}[2\sin(5x) - 5\cos(5x)] + C$

41. $\dfrac{1}{2}(w^2\tan^{-1} w + \tan^{-1} w - w) + C$

43.
   a. $p(t) = -5t\cos(2t) + 2.5\sin(2t) + 20$
   b. $p(4) \approx 25.3833963$

45. $p(t) = t^3 - 2t^2 + 3t + 2$

47. $p(t) = 2/21(18 + 7t^2)^{3/2} - 82/21$

49.
   a. $-\sin x\cos x + \int \cos x\cos x dx$
   b. $-\sin x\cos x/2 + x/2 + C.$
   c. $\sin x\cos x/2 + x/2 + C$
   d. $(\sin(2x))/4 + x/2 + C$

57. $(\tan^{-1}(x/a))/a + C.$

59. $\sin^{-1}(x/a) + C$

61. $e^{e^x} + C$

63. $\frac{2^x}{\ln 2} + C$

65. $x\ln(1+x^2) - 2x + 2\tan^{-1}(x) + C$

67. $(1-\cos t)/12$

69. $(\ln(t^2+1))/6$

75.

a. $\int t^n \ln t dt = \frac{t^{n+1}}{n+1}\ln t - \frac{t^{n+1}}{(n+1)^2} + C$

b. $(\ln t)^2/2 + C$

77.

a. maximum speed is 0.5 mi/h

b. $x(t) = 1/2\sin t - 1/6\sin^3 t.$

c. $y(t) = (x(t))^2 = \left(\frac{1}{2}\sin t - \frac{1}{6}\sin^3 t\right)^2$

d. 0 mi/h

79. $v = t^2/2 + t + v_0$, $p = t^3/6 + t^2/2 + v_0 t + p_0$

81. $v = -e^{-3t}/3 + v_0 + 1/3$, $p = e^{-3t}/9 + (v_0 + 1/3)t + p_0 - 1/9$

Section 1.2

1. $\dfrac{1}{3}\ln|\sec(3\theta)| + C.$

3. $\dfrac{1}{4}\ln|\sec(4y) + \tan(4y)| + \dfrac{1}{3}|\sec(3y)| + C.$

5. $\ln|\sin(\sin x)| + C.$

7. $-\dfrac{3}{2}\ln|\csc t + \cot t| - \dfrac{1}{2}\csc t\cot t + C.$

9. $-\dfrac{1}{3}\left(25-\phi^2\right)^{3/2} + C.$

11. $\dfrac{k}{242(121+k^2)} + \dfrac{1}{2662}\cdot\tan^{-1}(k/11) + C.$

13. $\dfrac{1}{2}v\sqrt{10-49v^2} + \dfrac{5}{7}\sin^{-1}(7v/\sqrt{10}) + C.$

15. $\dfrac{z(5+5z^2)}{8(1+z^2)^2} + \dfrac{3}{8}\tan^{-1} z + C.$

17. $-v - \cot v + C.$

19. $\sqrt{e^{2x}-16} - 4\tan^{-1}\dfrac{\sqrt{e^{2x}-16}}{4} + C.$

21. Argument is similar to the iterated $\cos\theta$ formula, which is derived in the text.

23. $-\dfrac{1}{2}\left[\dfrac{\cos(ay-by)}{a-b} + \dfrac{\cos(ay+by)}{a+b}\right] + C.$

25. 22 and 24 become integrals of $\sin^2(ax)$ and $\cos^2(bx)$, resp. 23 is just the integral of $\cos(ay)\sin(ay)$.

27. Idea: use integration by parts.

29. $2y\cos y + (y^2-2)\sin y + C.$

31. $(3z^2-6)\cos z + (z^3-6z)\sin z + C.$

33. $\dfrac{3\psi}{8} - \dfrac{1}{4}\sin^3\psi\cos\psi - \dfrac{3}{16}\sin 2\psi + C.$

35. $u - \dfrac{4\tan u}{3} + \dfrac{\sec^2 u\tan u}{3} + C$. Answers may vary based on how $1+\tan^2 u = \sec^2 u$ is applied.

37. $\dfrac{\sin 3x}{6} + \dfrac{\sin 5x}{10} + C.$

39. $p(t) = -\dfrac{8}{5}(16-t^2)^{3/2}(32+3t^2) + 3372.8.$

41. $p(t) = (10t^2 - 180)\sqrt{t^2+9} + 540.$

43. $\dfrac{\cos 5x}{80} - \dfrac{\cos 3x}{48} - \dfrac{\cos x}{8} + C.$

45. $\dfrac{1}{2}\tan^2 y + C$ or $\dfrac{1}{2}\sec^2 y + C.$

47. $\sqrt{4+x^2} = -\dfrac{\sqrt{9-t^2}}{t} - \sin^{-1}(t/3) + C.$

49. $\sqrt{(1-x^2)^{3/2}} = (z^2-18)\sqrt{z^2+9} + C.$

Section 1.3

1. $4\ln|x+2| + 7\ln|x+3| + C.$

3. $5\ln|u| + 3\ln|u-7| + C.$

5. $\dfrac{5}{2}\ln|2z+1| + 3\ln|z-1| + C.$

7. $\dfrac{1}{2}p^2 - 2p + 9\ln|p+3| + C.$

9. $\dfrac{1}{2}x^2 + 4x + \ln|x+3| + 3\ln|x+5| + C.$

11. $5\ln|m+3| + \ln|m^2+m+1| + C.$

13. $\ln|v^2-4v+18| - 5\ln|v-6| + C.$

15. $2\left(\ln|2t+3| + \ln|t^2+5t+30|\right) + C.$

17. $-3\ln|y+12| - \dfrac{1}{2}\tan^{-1}\left(\dfrac{y+2}{4}\right) + \dfrac{1}{2}\ln|y^2 + 4y + 20| + C.$

19. $\dfrac{3}{2}\ln|\phi^2+\phi+5| - \dfrac{3}{\sqrt{19}}\tan^{-1}\left(\dfrac{2\phi+1}{\sqrt{19}}\right) - 6\ln|\phi+2| + C.$

21. $\ln|u+3| + 4\ln|u-2| + \dfrac{3}{u-2} + C$.

23. $\ln|x+3| - \dfrac{2}{x+3} - 4\ln|x+2| + C$.

25. $3\ln|r-1| - \dfrac{7}{r-1} + \dfrac{1}{(r-1)^2} + 4\ln|r+3| + C$.

27. $\dfrac{A}{\sqrt{B}}\tan^{-1}\left(\dfrac{x}{\sqrt{B}}\right) + \dfrac{C}{D}\tan^{-1}\left(\dfrac{x}{\sqrt{D}}\right) + K$.

29. $y(x) = x^2 + x - \ln|x-5| + \ln|x+3| - \ln 9$.

31. $\dfrac{2}{3}\tan^{-1}\left(\dfrac{y}{3}\right) + 3\ln|y| = 3\ln|x+5| - 4\ln|x+6| + C$.

33. See the "Separable Differential Equations" chapter of *Worldwide Differential Calculus* for a complete proof.

35. $2\sqrt{x} + 3\ln|\sqrt{x}-3| - 3\ln|\sqrt{x}+3| + C$.

37. $3(\sqrt[3]{x} + \frac{1}{2}x^{2/3} + \ln|\sqrt[3]{x}-1|) + C$.

39. $2\sqrt{3+x} + 4\tanh^{-1}\sqrt{x+3} + C$.

41. $c(t) = \dfrac{12(1-e^{-2t})}{4-3e^{-2t}}$.

43. Approximately 1.552 mols per unit volume.

45.

a. $a=-5$, $b=1$ is trivial. Use the product rule to calculate $Q'(x)$.

b. Substitute the variables into the formula in the explanatory paragraph.

c. Evaluate the limit via direct substitution.

47. $A=1$, $B=-2.5$, $C=2.5$.

49. -1.

Section 1.4

Many of the integrals have equivalent but different looking answers depending on whether the are expressed in terms of hyperbolic trig functions or logarithms. We indicate both answers for some but not all of the problems.

1. $7\sinh^{-1}(x/3) + C$ or $7\ln(x+\sqrt{x^2+9}) + C$

3. $\dfrac{1}{2}\sinh^{-1}(2y/7) + C$ or $\dfrac{1}{2}\ln(2y+\sqrt{4y^2+49}) + C$.

5. $\dfrac{1}{3}\sinh(3x) + C$.

7. $\sinh^{-1}(x^2/3) + C$ or $\ln(x^2+\sqrt{x^4+9}) + C$.

9. $\dfrac{1}{2}\ln(2x+1+\sqrt{4x^2+4x+9}) + C$.

11. $\dfrac{2}{3}t^{3/2} + \ln(t+\sqrt{t^2+3}) + C$.

13. $\sinh^{-1}(x/3) + \cosh^{-1}(x/2) + C$ or $\ln(x+\sqrt{x^2+9}) + \ln(x+\sqrt{x^2-4}) + C$.

15.

a. $(\cosh^2 x)/2 + C_1 = (\sinh^2 x)/2 + C_2$.

b. Same as (a).

The two answers coincide.

17. Prove $\sinh(-x) = -\sinh(x)$ by using the definition of $\sinh x$ in terms of the exponential function.

19. $\sinh(x)$ is one-to-one since its derivative, $\cosh x$, is strictly positive. The formula for $\sinh^{-1} x$ gives an explicit inverse formula with domain equal to all real numbers.

21. $\sinh x$ is strictly increasing since its derivative, $\cosh x$, is strictly positive. The second part follows immediately from the fact that $\sinh 0 = 0$.

23. Prove this by differentiating the definition of $\sinh x$ using what you know about the derivative of the exponential function.

25. Prove this by writing $\sinh x$ and $\cosh x$ in terms of the exponential function.

27. $\dfrac{x}{8}(2x^2-1)\sqrt{x^2-1} - \dfrac{1}{8}\cosh^{-1}x + C$.

29. $\dfrac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \dfrac{a^4}{8}\sinh^{-1}(x/a) + C$.

31. $-\dfrac{1}{40}\cosh^{-1}(5x) + \dfrac{25}{4}x^3\sqrt{25x^2-1} - \dfrac{1}{8}x\sqrt{25x^2-1} + C$.

33. $\dfrac{c}{8b^3}[bx(2b^2x^2+a^2)\sqrt{b^2x^2+a^2}$
$-a^4\ln(bx+\sqrt{b^2x^2+a^2})] + K$.

35. $v(t) = \sinh^{-1}((t-3)/\sqrt{7}) + 8 - \sinh^{-1}(3/\sqrt{7})$.

37. $v(t) = \sinh(t^2+2) + 4 - \sinh(2)$.

39. $x\cosh^{-1}x - \sqrt{x^2-1} + C$.

41. $p(t) = t\cosh^{-1}(t/4) - \sqrt{t^2-16} + C_0 t + C_1$.

43. $p(t) = 2t\cosh^{-1}\left(\dfrac{3t}{\sqrt{11}}\right) - \dfrac{2}{3}\sqrt{9t^2-11} + (8-\ln 11)t + C$.

45. $\dfrac{x}{\sqrt{1+x^2}} = \ln(t + \sqrt{t^2 - 3}) + C.$

47. $\sinh^{-1} x - \cosh^{-1} t = C.$

49. $\dfrac{1}{2}(\sin x \cosh x + \cos x \sinh x) + C.$

Section 2.1

1. 85.

3. $\ln 120 \approx 4.787.$

5. 30.

7. 0.

9. 63/32.

11. 63.

13. 0.

15. $\Delta k^2 = k^2 - (k-1)^2 = 2k - 1.$

17. 5.

19. $6k - 7.$

21. 0.

23.

$$\begin{aligned}(\Delta f)(x) &= \sin x - \sin(x-1)\\ &= \sin x - (\sin x \cos 1 - \sin 1 \cos x)\\ &= (\sin x)(1 - \cos 1) + \sin 1 \cos x.\end{aligned}$$

25.

a. 1.

b. 0.

27. 100/101.

29. 1.

31.

$$\begin{aligned}\sum(x - \bar{x}) &= \sum x - \sum \bar{x}\\ &= \sum x - n\bar{x}\\ &= \sum x - \sum x = 0.\end{aligned}$$

33. $s \approx 4.444.$

35. $\sum(x_i - \bar{x})^2 = \sum x_i^2 - 2\bar{x}\sum x_i + \sum \bar{x}^2 = \sum x_i^2 - (2/n)\sum x_i \sum x_i + (n/n^2)(\sum x)^2 = \sum x_i^2 - (1/n)(\sum x_i)^2$. Divide by $n-1$ to get the final result.

37. $\sum(x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \bar{x}\sum y_i = \bar{y}\sum x_i + \sum \bar{x}\bar{y} = \sum x_i y_i - (2/n)\sum x_i \sum y_i + (n/n^2)\sum x_i \sum y_i = (\sum x_i y_i) - (1/n)\sum x_i \sum y_i.$

39. $f(m) = \sum_{i=1}^{m} i.$

41. $F(m) = \displaystyle\sum_{i=1}^{m} \frac{i(i+1)}{2}.$

43. The number of gifts received on the m-th day is the m-th triangular number. The cumulative number of gifts received by the end of the m-th day is the m-th tetrahedral number.

45. $\dfrac{1}{2}\displaystyle\sum_{k=1}^{n}(u^4 + 1).$

47. Prove by induction on n.

49. Prove by induction on n.

Section 2.2

1. $m(\mathcal{P}) = 3.$

3. $m(\mathcal{P}) = 1/4.$

5. $m(\mathcal{P}) = 1/2^n.$

7. False.

9. True.

11. False.

13. False.

15.

a. Yes.

b. Yes.

17. 3250 meters.

19. 13/12 miles per hour.

21. Outline: the Riemann sum is telescoping. The total displacement of a particle moving with constant velocity over an interval is equal to the velocity times the width of the interval (= the total time traveled).

23. Follows from the fact that if s_i is a sample point, then $m \le f(s_i) \le M$.

25. 3.

27. 0.285.

29. $\approx 0.27683.$

31. $-\pi.$

33. ≈ 3.92923.

35.

a. 1.
b. ≈ 1.366.
c. ≈ 1.459.
d. $\pi/2 \approx 1.571$.

37. $15/32 = 0.46875$.

39. $15/8 = 1.875$.

41. Idea: Riemann sum linearity follows from the linearity of finite summations.

43.

a. $\mathcal{R}^{R}_{\mathcal{P}_n}(f) = \dfrac{100 + 3(2^{n+1} - 1)}{2^n}$.
b. Rewrite the answer to (a) as $\dfrac{100}{2^n} + 6 \cdot \dfrac{2^n - 0.5}{2^n}$. The first term vanishes as $n \to \infty$. The second term approaches 6.

45. Follows from the fact that f evaluated at the representative from $U(\mathcal{P})$ is greater than f evaluated at the representative from $L(\mathcal{P})$ on each interval of the partition.

47. Let $\mathcal{Q}$ be a common refinement of $\mathcal{P}$ and $\mathcal{P}'$. Then by parts (a) and (b), $\mathcal{R}^{L(\mathcal{P})}_{\mathcal{P}} \leq \mathcal{R}^{L(\mathcal{Q})}_{\mathcal{Q}} \leq \mathcal{R}^{U(\mathcal{Q})}_{\mathcal{Q}} \leq \mathcal{R}^{U(\mathcal{P}')}_{\mathcal{P}'}$.

49. ≈ 3.79.

51. ≈ 4.1365.

Section 2.3

1. 20.

3. 64.

5. 2880.

7. $-55/6$.

9. 0.

11. False. E.g.: $f(x) = x$, $g(x) = 1$, $a = 0$, $b = 1.2$.

13. False.

15. 68.

17. 5.5.

19. $-\dfrac{1}{6}gt_1^2 + \dfrac{1}{2}v_0t_1 + h_0$.

21. Answers may vary. For example, on the interval $[0, 1]$, take $f(x) = x$ if $x \neq 0.5$ and $f(1/2) = 10$.

23. 1/3 meters per second.

25. 1 meter per second.

27.

a. $a(t) = 2t - 5$.
b. 3 meters per second per second.

29.

a. $a(t) = \begin{cases} -1 & t \in (2,4) \\ 1 & t \in (4,6) \end{cases}$

   $a(t)$ is undefined at $t = 4$.
b. The average value of the acceleration function is undefined; however, average acceleration, as the change in velocity divided by the change in time, is 0.

31.

a. The avg. velocity of the second particle equals the avg. velocity of the first particle plus C.
b. The avg. accelerations are equal.

33. Yes, since the function is continuous.

35. ≈ 0.4987.

37. ≈ 2.40.

39. $40 + \frac{25\pi}{2}$.

41. 152.

43. 1/6.

45.

a. Use the point-slope equation for a line.
b. Integrate carefully. Looks complicated but many items are just constants.

47. $\displaystyle\int_0^{10} (12 - x)\pi x^2\, dx$.

49. $5\sqrt{17}$.

51. πR.

Section 2.4

1. 148.

3. 0.

5. $24 - \dfrac{25\sqrt{3}}{2} - \dfrac{25\pi}{3} + 50\sin^{-1}\dfrac{3}{5}$.

7. $(2+\pi)/108$.

9. $\ln 72$.

11. $\cosh^{-1}\left(\frac{15}{7}\right) - \cosh^{-1}\left(\frac{9}{7}\right) = \ln\left(\frac{15+4\sqrt{11}}{9+4\sqrt{2}}\right)$.

13. $e^9(e^7-1)$.

15. $9\ln\left(\dfrac{5}{3}\right) - 2$.

17. $2xf(x^2)$.

19. $\dfrac{1}{2}\left(\sqrt{2} + \sinh^{-1} 1\right)$.

21. $\dfrac{\sinh(k)}{k}$.

23. $18 + 2\ln 7$.

25. 0.4.

27.

   a. $c = 1/21$.
   b. $19/63$.

29. $(a+b)/2$.

31. $329/6$.

33. 0.2.

35. Multiply out the $(x-\mu)^2$ factor and simplify.

37. $\dfrac{(b-a)^2}{12}$.

39. Follow the hint.

41. $27x$.

43. $2x-1$.

45. $5e^6 - e^2$.

47.

   a. Make the substitution $y = (\pi/2) - x$.
   b. Use the fact that cos and sin are co-functions.
   c. The integrand of $A+B$ is 1 so the integral of $A+B$ is $\pi/2$.

49.

   a. π.
   b. π.

Section 2.5

1. $1/10$.

3. Diverges.

5. Diverges.

7. Diverges.

9. $\pi/2$.

11. -4.

13. Converges to 0.

15. Diverges.

17. Converges to $1/(1-n)$.

19. Converges.

21. Converges.

23. Converges.

27. Diverges for all p.

29. Both are equal to 1.

31. $1/\lambda$.

33. $1/\lambda^2$.

35. $\mu^2 + \sigma^2$.

37. $1 - e^{-\lambda x}$.

39. $e^{-\lambda x}$.

41. Idea: take the derivative of ϕ given in the previous problem.

43.

   a. $\approx 63.2\%$.
   b. $\approx 24.6\%$.
   c. $\approx 12.4\%$.

45. ≈ 12.8 atoms.

47. $1/s$

49. $1/(s-a)$

51. $s/(s^2-1)$

Section 2.6

1.

   a. $20 + 35\ln\dfrac{3}{7}$ or $20 - 70\tanh\dfrac{2}{5} \approx -9.65543$.
   b. ≈ -9.52603.
   c. ≈ -9.91667.

d. ≈ -9.66667.

3.

a. $\ln \frac{253125}{16} \approx 9.66904$.

b. ≈ 9.49344.

c. ≈ 10.04141.

d. ≈ 9.73300.

5.

a. $16 + \ln \frac{30625}{9} \approx 24.13235$.

b. ≈ 23.98561.

c. ≈ 24.44286.

d. ≈ 24.18413.

7.

a. $\ln 2/2 \approx 0.34657$.

b. ≈ 0.34248.

c. ≈ 0.35491.

d. ≈ 0.34726.

9.

a. $-\frac{31}{3} + \ln 25 \approx -7.11446$.

b. ≈ -7.23043.

c. ≈ -6.87083.

d. ≈ -7.07685.

11.

a. $24 \ln \left(\frac{16+\sqrt{247}}{12+3\sqrt{15}}\right) \approx 7.07470$.

b. ≈ 7.07118.

c. ≈ 7.08175.

d. ≈ 7.07474.

13.

a. $\ln \frac{7\sqrt{5}}{3} \approx 1.65202$.

b. ≈ 1.64581.

c. ≈ 1.66453.

d. ≈ 1.65243.

15.

a. $4.5\left(2\sqrt{5} + \ln(2+\sqrt{5})\right) \approx 26.62097$.

b. ≈ 26.53705.

c. ≈ 26.78876.

d. ≈ 26.62160.

17. 18 intervals.

19. 2 intervals.

21.

a. ≈ 1.80972.

b. ≈ 1.74036.

c. ≈ 1.78867.

23.

a. ≈ 0.92733.

b. ≈ 0.92645.

c. ≈ 0.92704.

25. ≈ 0.69377.

27. $\pi \approx 3.139926$.

29.

a. $\frac{128}{3n^2}$.

b. $\frac{256}{3n^2}$.

31.

a. $\frac{72e^{12}}{n^2}$.

b. $\frac{144e^{12}}{n^2}$.

33. $\frac{5913e^9}{5n^4}$.

35. Outline: $\Delta x = (b-a)/4$, let $f(x) = x^2$. The Simpson's approximation is $\frac{\Delta x}{3}\left(f(a) + 4f(a+\Delta x) + ... + f(b)\right)$. Write everything out in terms of a and b.

37.

a. ≈ 0.34198.

b. ≈ 0.68397.

39. The question is equivalent to the problem: What proportion of the data is between -1 and 1 standard deviations from the mean? We approximated this in problem 37b): ≈ 0.68397.

41. 51 Joules.

43. Since t is given in seconds and the velocity in miles per hour, one of the two must be converted. Answer: ≈ 0.25611 miles.

45. ≈ 1.591517. Turns out the limiting integral is equal to the interesting number π/e.

47. ≈ 4.20156 seconds.

49. If A is small, then $\sin^2(A/2) \approx 0$. The integral is exactly $2\pi\sqrt{\dfrac{1}{9.8}}$.

Section 3.1

1. 1510 m.

3. $\dfrac{2\sinh 15}{3} \approx 1,089,672$ m.

5. $\dfrac{1}{2}\ln\left(\dfrac{1}{2}(2+\sqrt{3})\right) \approx 0.3119$ m.

7. $1 - \dfrac{\pi}{4} \approx 0.2146$ m.

9. $\dfrac{2}{15}\tan^{-1}\dfrac{6}{5} \approx 0.1168$ m.

11. 9.5 m.

13. 1.0 m.

15. $\cos 4\sin 4 + \dfrac{2}{3}(82 + 3\sinh 4 + \sinh 12) \approx 54,361$ m. Note that there are equivalent algebraic expressions for this integral.

17. $8\sqrt{3}(-1+\sqrt{2}) \approx 5.74$ m.

19. $-1 + \ln 8 \approx 1.08$ m.

21.

a. $\pi/6$ m/sec.
b. $\pi/2$ m/sec.

23.

a. $(-2/3)\ln 4$ m/sec.
b. $4\ln 5 - 4\ln 3 - (4/3)\ln 2$ m/sec.

25.

a. $\dfrac{1}{12}(\sinh 16 - \sinh 4)$ m/sec. Answers may also be written in terms of the exponential function.
b. $\dfrac{1}{12}(\sinh 16 + \sinh 4)$ m/sec.

27. $p(t) = \dfrac{1}{4} - \dfrac{t^2}{4} + \dfrac{t^2}{2}\ln t$.

29. $p(t) = -\tan^{-1} t + \dfrac{1}{2}\ln(1+t^2) + 5 + \dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$.

31. $d(t) = -1 + e^t + \dfrac{t}{2} + \dfrac{t^5}{5} + \dfrac{1}{4}\sin 2t$.

33. $d(t) = 18t - \dfrac{7t^2}{2} - \dfrac{t^3}{3}$.

35. The distance function is continuous and differentiable at $t = 3$.

37.

a. $v(t) = \begin{cases} -9.8t & t \in [0,3] \\ -29.4 + 2(t-3) & t \in [3,8]. \end{cases}$
b. -166.1 m.
c. -20.7625 m/sec.

39. Statement is false. Answers may vary, but one easy counterexample is $v(t) = t$ on the interval $[-1, 1]$.

41. $\dfrac{14}{3} - \dfrac{258}{37\pi}$ m/sec.

43. $4 + \dfrac{39}{15}\ln 2 - \ln 5$ m/sec.

45. 1 meter.

47.

a. $|v(t)| = |e^{-t}||\sin t| \le |e^{-t}| \to 0$.
b. Outline: first, note that $v(t)$ doesn't change sign on such an interval. Use repeated integration by parts to calculate the integral.
c. This integral is the sum of finitely many terms of the form given in part (b). Thus,
$$\int_0^{n\pi} |v(t)|\,dt = \frac{1+e^{\pi}}{2}\sum_{k=1}^{n} e^{-k\pi}.$$
d. Since $e^{-\pi} < 1$, the series above is geometric. It converges to $\dfrac{1+e^{-\pi}}{2(1-e^{-\pi})}$.

49.

a. Only thing to check is that the helix rises by 34 angstroms per revolution. Note that an increase in t by 2π leads to an increase in the z coordinate of 34 angstroms.

b. $\alpha'(t) = (-10\sin t, 10\cos t, 17/\pi)$.

c. $|\alpha'(t)| = \sqrt{10 + \dfrac{289}{\pi^2}} \approx 6.2675$.

d. $2\pi\sqrt{10 + \dfrac{289}{\pi^2}} \approx 39.38$ angstroms.

e. ≈ 1.122 meters.

Section 3.2

1. 4

3. 110/3

5. $e - 1$

7. 4

9. 133/12

11. 36

13. 125/6

15. $(e-2)/2$

17. $2(\pi - 2)$

19. $34\tan^{-1} 4 - 8 \approx 37$.

21. 500.

23. 5.

25. 783/2.

27. $5\ln 5$.

29. 636/5.

31. $2\left(\cosh 10 + \sinh^{-1} 1 - \sqrt{2}\right)$.

33. $10\left(\sinh^{-1} 10 - \cosh^{-1} 10\right) - \sinh^{-1} 1 + \sqrt{2} + 3\sqrt{11} - \sqrt{101}$.

35. 1/2.

37. Follows from the fact that $|(f+h)-(g+h)| = |f-g|$.

39. $\left|\dfrac{1}{a} - \dfrac{1}{b}\right|$.

41. Area $= \frac{1}{U}\left(-(U-1) + U\ln U\right)$. Area $\to \infty$ as $U \to \infty$.

43. $4\sqrt{2}$.

45. $\dfrac{1}{4}\left[(e^{1/8} - e^{(1/8)^k} + \ldots + e^{7/8} - e^{(7/8)^k}\right]$.

47. ≈ 0.39133.

49. $c = 1/2$.

51. $c = 6$.

53.

a. f continuous implies $|f|$ is continues implies $|f|$ is integrable.

b. If the integral is positive, take $k = 1$. If the integral is not positive, take $k = -1$.

c. $\left|\int_a^b f(x)\,dx\right| = k\int_a^b f(x)\,dx = \int_a^b kf(x)\,dx \le \int_a^b |f(x)|\,dx$. The last inequality follows from the fact that $kf(x) \le |f(x)|$.

d. By the argument in (c), $\int_a^b |f(x) - g(x)|\,dx \ge \left|\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right|$.

55. Follows from the inequality $||x| - |y|| \le |x-y|$. Specifically, $|j - h| \le ||f| - |g|| \le |f-g|$.

57. $\pi/12$.

59. πr_0^2. This is a circle.

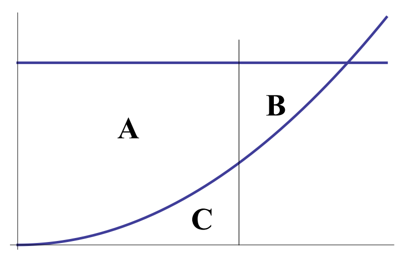
61. $\dfrac{1}{18}\left(3\sqrt{3} - \pi\right)$.

63. $\dfrac{\sqrt{2}-1}{2}$.

65. $\pi(R_2^2 - R_1^2)$.

67. Analyst B.

69.

a. 

In the picture, the horizontal line is $y = b$, the vertical line is $x = a$. So ab is the area $A + C$ whereas the right-hand side of the equation has area $A + B + C$. This proves the inequality.

b. The inequality is an equality if $f(a) = b$.

71.

a. $f(x)g(x) \le \dfrac{f(x)^p}{p} + \dfrac{g(x)^q}{q}$ holds point-wise.

Integrate point-wise and use the "additional properties" to arrive at the result.

b. Set $h(x) = f(x) / \left(\int_a^b f(x)^p \, dx \right)^{1/p}$. Define $j(x)$ similarly using $g(x)$ and q. Then the "additional properties" hold for j and h. By part (a), $\int_a^b j(x)h(x)\, dx \leq 1$. Multiply both sides of the equation by the denominators of h and j to arrive at the inequality.

Section 3.3

1. $(0, 0, 8)$.

3. $(110, -90, 220)$.

5. $(9, 9, 9)$.

7. No, the derivative is the zero vector when $t = 0$.

9. Yes. The map avoids the cusp at $t = -2$.

11. No, the derivative is the zero vector when $t = 0$.

13. $\vec{w}(s) = (3 + \frac{6s}{\sqrt{62}}, 2 - \frac{5s}{\sqrt{62}}, 7 + \frac{s}{\sqrt{62}})$.

15. Follows from the fact that $\left| \frac{d\vec{\phi}}{ds} \right| = 1$ for all $s \in [a, b]$.

17. $\frac{\pi\sqrt{2}}{4}$.

19. $2(-1 + (1 + \pi)^{3/2})$.

21. $\ln(1 + \sqrt{2})$.

23. $5\sqrt{2} - \sqrt{26} + 5 \ln \frac{5 + \sqrt{26}}{1 + \sqrt{2}}$.

25. $\frac{7}{6\sqrt{a}} + \frac{\sqrt{a}}{4}$.

27. $\vec{p}(s) = (a \cos \frac{s}{a^2+b^2}, a \sin \frac{s}{a^2+b^2}, ,b \cos \frac{s}{a^2+b^2}, b \sin \frac{s}{a^2+b^2})$.

29. It is not regular as it fails to be differentiable at $t = 0$. It is not rectifiable since it is not one-to-one on any open interval containing zero.

31.

a. Let $\vec{p}(t) = R(\sin t, 0, \cos t)$ for $t \in [0, \pi/2]$, $R(\cos(t - \frac{\pi}{2}), \sin(t - \frac{\pi}{2}, 0)$ for $t \in [\pi/2, \pi]$ and $R(0, -\cos t, -\sin t)$ for $t \in [\pi, \frac{3\pi}{2}]$.

b. $3\pi R/2$.

33.

a. Apply the chain rule to $\phi(t) = \psi(f(t))$.

b. Use part (a) and the Substitution Rule. Also, since $\phi(a) = \psi(c)$, and all the functions are one-to-one, it must be that $|\frac{du}{dt}| > 0$.

35. $8r$.

37. $\vec{p}'(\psi) = \frac{r_2(r_1+r_2)}{r_1} \cdot$
$\left(\cos \frac{r_2\psi}{r_1} - \cos \frac{(r_1+r_2)\psi}{r_1}, -\sin \frac{r_2\psi}{r_1} + \sin \frac{(r_1+r_2)\psi}{r_1}\right)$.

39. Idea: cusps occur whenever $\psi = 2n\pi$, n an integer. The cusps therefore occur, in terms of θ, at $\theta = (r_2/r_1)(2n\pi)$. When r_2/r_1 is irrational, cusps will start repeating themselves at a frequency controlled by r_1. However, if r_2/r_1 is irrational, then for an arbitrary θ we can find an n such $(r_2/r_1)2n\pi$ is arbitrarily close to θ.

41. $x'(\theta) = (r_2 - r_1) \sin \theta + (r_2 - r_1) \sin \frac{(r_1 - r_2)\theta}{r_2}$.
$y'(\theta) = (r_1 - r_2) \cos \theta + (r_2 - r_1) \cos \frac{(r_1 - r_2)\theta}{r_2}$.

43.

a. Let $f(t) = \vec{c}(t) \cdot \vec{v}$. Then $f'(t) = \vec{c}' \cdot \vec{v}$. Use this fact and Fundamental Theorem of Calculus to obtain the result.

b. Use the hint and the fact that $\vec{v}$ is a unit vector. I.e., $|\vec{v}| = 1$.

c. $\int_a^b |\vec{c}'(t)| \, dt \geq \int_a^b \vec{c}' \cdot \vec{v} \, dt = \frac{(p - q) \cdot (p - q)}{|p - q|} = |p - q|$.

45.

a. $\frac{dx}{d\theta} = r'(\theta) \cos \theta - r(\theta) \sin \theta$, $\frac{dy}{d\theta} = r'(\theta) \sin \theta + r(\theta) \cos \theta$.

b. Follows from the fact that $x'(\theta)^2 + y'(\theta)^2 = r(\theta)^2 + r'(\theta)^2$.

47. $2\pi C$. This is a circle.

49. $\pi\sqrt{1 + 4\pi^2} + \frac{1}{2} \sinh^{-1} 2\pi$.

Section 3.4

1.

a. $4/3$.

b. $y = x^2$.

3.

a. $1/5$.
b. $x^2 = y^3$.

5.

a. $(1/2)(-8 + 23 + \ln 27)$.
b. $x = \ln y$.

7. $3\pi/2$.

9. 33π.

11. 6π.

13. $\frac{3\sqrt{3}}{4} + \frac{\pi}{4}$.

15. $5/2$.

Section 3.5

1. 58.

3. $1/4$.

5. $800/3$.

7. $\frac{1}{3}\left(\frac{a^3}{h} - \frac{b^4 a}{h}\right)$.

9. $\pi/8$.

11. $1574\pi/15$.

13. $\pi/5$.

15. $\frac{\pi}{2n+1}$.

17. $\frac{\pi(16-\pi)}{2}$.

19. $6929\pi/16$.

21. $\frac{3\sqrt{3}\pi}{16}$.

23. ≈ 24.4919.

25. ≈ 5.73224.

27. ≈ 25.3993.

29. $2\pi\left(\frac{1}{n+2} - \frac{1}{m+2}\right)$.

31. $B/3$.

33. The integral is finite by comparison with Gabriel's Horn.

35. ≈ 5.23016.

37. $2\pi^2 R r^2$.

39. $2\pi^2$.

41. $4750\pi/3$ cm^3.

43. ≈ 27.672 cm.

45. $2\pi(1-a)$. The integral converges to 2π.

47. $V_c = \frac{3c^{(16/3)}}{10}$.

49. $\pi^2(R^2 - r^2)/3$.

51. π.

53.

a. π.
b. The integral converges to 2π.
c. The integral converges to 2π.

55. The volumes of a right circular cone, hemisphere, cylinder and sphere are $\pi r^2 h/3$, $2\pi r^2 h/3$, $\pi r^2 h$, and $4\pi r^2 h/3$ respectively.

Section 3.6

1. $152\pi\sqrt{10}$.

3. $\pi(e^\pi\sqrt{1+e^{2\pi}} - \sqrt{2} + \sinh^{-1} e^\pi - \sinh^{-1} 1)$.

5. $\frac{\pi}{6}\left(37\sqrt{37} - 1\right)$.

7. $33\pi\sqrt{5}$.

9. $\frac{5\pi}{27}\left(29\sqrt{145} - 2\sqrt{10}\right)$.

11. $2\pi\left(4\sinh 4 - \cosh 4 + 1\right)$.

13. $16\pi\sqrt{10}$.

15. $\frac{\pi}{6}\left(1 + \sqrt{5} + 3\sinh^{-1} 2\right)$.

17. $\pi|m|\sqrt{1+m^2}$.

19. ≈ 8496.

21. ≈ 150.8.

23. ≈ 3.395.

25. $A = \int_a^b 2\pi\left(f(x) - y_0\right)\sqrt{1 + [f'(x)]^2}\, dx$.

27. $A = \int_a^b 2\pi\left(x - x_0\right)\sqrt{1 + [f'(x)]^2}\, dx$.

29. $\frac{\pi a}{6}\left[(a^2 + 4b + 4)^{3/2} - (a^2 + 4b)^{3/2}\right]$.

31.

a. $S(m) = \begin{cases} \pi a\sqrt{1+m^2} & m \neq 0 \\ 0 & m = 0 \end{cases}.$

b. S is neither continuous nor differentiable at $m = 0$.

33.

a. 3π.

b. Annulus.

35. One way to do this is to write the equation perpendicular to $y = mx$ passing through (x_1, y_1). Find where the two lines intersect and use the distance formula.

37. $A = \int_a^c 2\pi \frac{|f(x) - mx - b|}{\sqrt{1+m^2}} \sqrt{1 + f'(x)^2}\, dx$.

39.

a. $A = \int_a^c 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx$. Here we assume $f(x) > 0$.

b. $A = \frac{\sqrt{2}}{2} \int_a^c 2\pi(f(x) - x)\sqrt{1 + f'(x)^2}\, dx$. We assume $f(x) > x$, or that f lies about the line $y = x$.

c. $A = \int_a^c 2\pi x\sqrt{1 + f'(x)^2}\, dx$. We assume $x > 0$.

41. 16π.

43. 16π.

45. $\int_2^5 2\pi \frac{e^x - 3x}{\sqrt{10}} \cdot \sqrt{1 + e^{2x}}\, dx$.

47. $\frac{2\pi}{\sqrt{5}} \int_2^4 \left(2x - \frac{x+2}{x+1}\right) \sqrt{1 + \frac{1}{(1+x)^4}}\, dx$.

49. $\frac{2\pi}{17} \int_3^8 (4x - \sqrt{x+1}) \sqrt{1 + \frac{1}{4+4x}}\, dx$.

Section 3.7

1. $18 \sinh 5$.

3. 720π.

5. $83,592\pi/5$.

7. $3456\pi/35$.

9. $\int_0^{y_0/m} \delta(x) 2\pi x(y_0 - mx)\, dx$.

11. $2\pi h \int_0^b x\delta(x)\, dx$.

13. $2\pi(49 \ln 49 - 25 \ln 25 - 24)$.

15.

a. 36.

b. $x = -7 + \sqrt{85}$.

17.

a. 1.

b. $x = \pi/3$.

19.

a. $2 \sinh 1$.

b. $x = 0$.

21. $x = \ln\left(\frac{1}{2}(e^{10} - 1) + 1\right)$.

23. $\frac{108 + 9\pi}{4}$.

25. $45/2$.

27. ≈ 14.689 kg.

29. $c = 6/5$.

31. 22.5 kg/m^3.

33. Numerator follows from comments in this chapter. Denominator follows from the shell volume formula.

35. 103.4 kg/m^3.

37.

a. $117\pi/t$.

b. $-117\pi/t^2$.

39. $\pi(t - 1)^2 (16 + 9 \ln 9(\ln 9 - 2)) + 3\pi t(\ln 9)^2 \ln 81$.

41. $\pi (35 - 2\ln(6))$.

43. $4\pi e^4(e^{32} - 1)$.

Section 3.8

1. $(4/19, 10/19)$.

3. $(3, 3)$.

5. $(2/27, -1/9, -2/9)$.

7.

a. $25225/12$.

b. $1009/192$.

9.

a. $\dfrac{2}{27}(660 + 7\ln(17/5))$.

b. $\dfrac{2(660 + 7\ln(17/5))}{3(48 + 7\ln(17/5))}$.

11.

a. $\dfrac{25(-4 + 13\ln 5)}{(\ln 5)^2}$.

b. $\dfrac{-4 + 13\ln 5}{4\ln 5}$.

13. Density functions should not take on negative values.

15.

a. $1/3$.

b. $11/60$.

c. $3/20$.

d. $11/20$.

e. $9/20$.

17.

a. $\dfrac{1}{96}\left(-21\sqrt{3} + 3\sqrt{15} - 20\pi + 768\sin^{-1}(1/4)\right)$.

b. 0.

c. $13/8$.

d. 0.

e. $\dfrac{156}{-20\pi - 21\sqrt{3} + 3\sqrt{15} + 768}$.

19.

a. $2401/6$.

b. $4802/15$.

c. $31213/30$.

d. $4/5$.

e. $13/5$.

21. $(27/16, 39/20)$.

23. $(0, 0, 3/4)$.

25. Substitute $g(x) = 0$ and cancel out $\delta(x)$ in the formulas for $\bar{x}$ and $\bar{y}$.

27. $\left(\dfrac{\pi}{2}, \dfrac{\pi}{8}\right)$.

29. $\left(\dfrac{2c}{3}, \dfrac{cm}{3}\right)$.

31. $\left(\dfrac{1}{e-1}, \dfrac{1+e}{4}\right)$.

33. $(6, 19)$.

35. $\left(0, \dfrac{330}{6\sqrt{55} - 9\pi + 128\sin^{-1}(3/8)}\right) \approx (0, 5.044)$.

37. $\left(\dfrac{7-4e^3}{8-2e^3}, \dfrac{e^6-7}{2e^3-8}\right) \approx (2.27975, 12.3225)$.

39. $(t/3, t/3)$.

41. $(0, 5t/17)$.

43. $\left(\dfrac{\cos 2t + 6\cos t}{14}, \dfrac{\sin 2t + 6\sin t}{14}\right)$.

45.

a. $\vec{p}_1(t) = (-2 - (3/4)t^2, 2)$, $\vec{p}_2(t) = (1 + (7/4)t^2, -3)$, $\vec{p}_3(t) = (4, 1 + t^2)$.

b. $\left(\dfrac{t^2+7}{4}, \dfrac{1+2t^2}{4}\right)$.

c. $\dfrac{\sqrt{5}}{2} \approx 1.1$.

d. $8\sqrt{5}$ N.

e. Same as (c).

47. $\dfrac{3\sqrt{2}}{10}$.

49. $\dfrac{\sqrt{2}}{10}$.

Section 3.9

1. 225 joules.

3. $4/15$ joules.

5. $\dfrac{5(e^8-1)}{4e^{12}}$ joules.

7. Yes.

9. 0 joules.

11. $-4.21875m$ joules.

13. $k = 224$.

15. $k = -80/\ln 2$.

17. $k = 9/61$.

19. 0 joules.

21. 2592 joules.

23. $44,100$ joules.

25. 4575 ft-lb.

27. $hP_0 - \frac{h^2 r}{2v}$ ft-lb.

29. 54 joules.

31. $W = \dfrac{1}{2}(mv_1^2 + mv_0^2) + mg(h_1 - h_0)$.
This is the sum of the changes in the kinetic and potential energies.

33. $63/4$ joules.

35. -12 joules.

37. 0 joules.

39. $\int_0^{10} (30-z)(62.4)(160)\,dz$ ft-lb.

41. $\int_0^{30} (30-z)(62.4)\left(\frac{16\pi z^2}{225}\right)\,dz$ ft-lb.

Section 3.10

1. 37440 lb.

3. $13,720,000/3$ N.

5. 7530.4 lb.

7. $\dfrac{\delta_w W H^2}{6}$.

9. You can either show that the force on one half of the wall is half the total force, or show directly that the forces on the two halves are equal.

Section 4.1

1.

a. $x_0 = 0$: $p(x_0) = 4$; $p^1(x_0) = 4$; $p^2(x_0) = 4$; errors are all 0.

b. $x_0 = 0.1$: $p(x_0) = 3.751$; $p^1(x_0) = 3.7$; $p^2(x_0) = 3.75$; errors: 0.051 and 0.001.

c. $x_0 = 0.01$: $p(x_0) = 3.970492$; $p^1(x_0) = 3.97$; $p^2(x_0) = 3.9705$; errors: 0.000492 and 0.000008.

d. $x_0 = -0.001$: $p(x_0) = 4.0030050091$; $p^1(x_0) = 4.003$; $p^2(x_0) = 4.003005$; errors: 0.0000050091 and 0.0000000091.

3.

a. $x_0 = 1$: $p(x_0) = 7$; $p^1(x_0) = 7$; $p^2(x_0) = 7$; errors are all 0.

b. $x_0 = 1.1$: $p(x_0) = 7.219985$; $p^1(x_0) = 7.1$; $p^2(x_0) = 7.22$; errors: 0.119985 and 0.000015.

c. $x_0 = 1.01$: $p(x_0) = 7.0112$; $p^1(x_0) = 7.01$; $p^2(x_0) = 7.0112$; errors: 0.0012 and $1.5(10^{-11})$.

d. $x_0 = 0.999$: $p(x_0) = 6.999012$; $p^1(x_0) = 6.999$; $p^2(x_0) = 6.999012$; errors: 0.000012 and $1.5(10^{-17})$.

5.

a. $x_0 = 0$: $p(x_0) = 1$; $p^1(x_0) = 1$; $p^2(x_0) = 1$; errors are all 0.

b. $x_0 = -0.01$: $p(x_0) = 0.9900990099$; $p^1(x_0) = 0.99$; $p^2(x_0) = 0.9901$; errors: 0.0000990099 and 0.0000009901.

c. $x_0 = 0.001$: $p(x_0) = 1.001001001001001$; $p^1(x_0) = 1.001$; $p^2(x_0) = 1.001001$; errors: 0.000001001001001 and 0.000000001001001.

7. x^2.

9. $12(x-1)^2$.

11. $q(x) = (x+2) + (x+2)^2$.

13. $q(x) = 1000 + 300(x-5) + 30(x-5)^2 + (x-5)^3$.

15. $q(x) = 3 + 10(x-1) + 17(x-1)^2 + 13(x-1)^3 + 4(x-1)^4$.

17. 63.35.

19. 14.

21. $E^1(x) = 0.24$.

23. $E^4(x) = 0.07$.

25. $E^3(x) = 0.0972$.

Section 4.2

1.

a. $T_f^1(x) = 1 + x$; $T_f^2(x) = 1 + x + \frac{x^2}{2}$; $T_f^3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

b. $f(2) = 7.389$; $T_f^1(2) = 3$; $T_f^2(2) = 5$; $T_f^3(2) = 6.33\overline{3}$.

c. $f(0.001) = 1.0010005$; $T_f^1(0.001) = 1.001$; $T_f^2(0.001) = 1.0010005$; $T_f^3(0.001) = 1.0010005$.

3.

a. $T_f^1(x) = 1$; $T_f^2(x) = 1 - \frac{x^2}{2}$; $T_f^3(x) = 1 - \frac{x^2}{2}$.

b. $f(2) = -0.4161468$; $T_f^1(2) = 1$; $T_f^2(2) = -1$; $T_f^3(2) = -1$.

c. $f(0.001) = 0.9999995$; $T_f^1(0.001) = 1$; $T_f^2(0.001) = 0.9999995$; $T_f^3(0.001) = 0.9999995$.

5.

a. $T_f^1(x) = 1 + x$; $T_f^2(x) = 1 + x + x^2$; $T_f^3(x) = 1 + x + x^2 + x^3$.

b. $f(2) = -1$; $T_f^1(2) = 3$; $T_f^2(2) = 7$; $T_f^3(2) = 15$.

c. $f(0.001) = 1.001001001$; $T_f^1(0.001) = 1.001$; $T_f^2(0.001) = 1.001001$; $T_f^3(0.001) = 1.001001001$.

7. $T_g^2(x;0) = x$.

9. $T_f^2(x;1) = \frac{1}{e} - \frac{x-1}{e} + \frac{(x-1)^2}{2e}$.

11. $T_f^3(x;1) = 1+\frac{1}{3}(x-1)-\frac{1}{9}(x-1)^2+\frac{5}{81}(x-1)^3$.

13. $T_f^3(x;1) = 2(x-1) - (x-1)^2 + \frac{2}{3}(x-1)^3$.

15.

a. $T_g^3(0.1;0) = 0.100166\overline{6}$.

b. $|\sinh(0.1) - T_g^3(0.1;0)| = 0.00000008$.

17.

a. $T_v^2(-0.1;0) = 1.005$.

b. $|\sec(-0.1) - T_v^2(-0.1;0)| = 0.0000209$.

19. $n = 1$.

21. $c = \frac{1}{2}; n = 2$.

23. $T_s^\infty(x;-1) = \frac{1}{e^3} + \frac{3(x+1)}{e^3} + \frac{9(x+1)^2}{2e^3} + \frac{27(x+1)^3}{3!e^3} + \frac{81(x+1)^4}{4!e^3} + \cdots$

25. $T_k^\infty(t;\frac{\pi}{4}) = -2(x-\frac{\pi}{4}) + \frac{8}{3!}(x-\frac{\pi}{4})^3 - \frac{32}{5!}(x-\frac{\pi}{4})^5 + \cdots$

27. $T_k^\infty(x;0) = \sum_{j=0}^\infty \frac{4^j x^{2j}}{(2j)!}$.

29. $T_f^\infty(t;3) = \ln(10) - \sum_{k=1}^\infty \frac{(-3)^k(x-3)^k}{10^k k}$.

31. $T_n^\infty(x;-1) = 1 - 2(x+1) + (x+1)^2$.

33. $T_s^\infty(x;1) = \sum_{k=0}^\infty \frac{(-1)^k(x-1)^k}{2^{k+1}}$.

35. $\frac{f^{(n)}(a)}{n!}(x-a)^n$.

Section 4.3

1. a. $E_f^3(x;0) \le (0.1)^4/(2^4 4!)$; b. $n = 2$ (actually, since $T_f^1(x;0) = T_f^2(x;0)$, $n = 1$ is a better answer); c. $|x| \le (2^4 4!(0.0001))^{1/4}$.

3. a. $E_f^3(x;1) \le (5^4(0.1)^4)/(4!)$; b. $n = 5$; c. $|x-1| \le (4!(0.0001)/5^4)^{1/4}$.

5. a. $E_f^3(x;1) \le (0.1)^4/((0.9)^4 4)$; b. $n = 3$; c. $|x-1| \le 0.1(4^{1/4})/[1+0.1(4^{1/4})]$.

7. a. $E_f^3(x;-1) \le 11(0.1)^4/2$; b. $n = 4$; c. need $5(1+\delta)\delta^4 \le 0.0001$; certainly $\delta = (0.00001)^{1/4}$ works.

9. a. $T_f^5(x;0) = \frac{x^3}{3!} - \frac{x^5}{5!}$; b. $E_f^5(x;0) \le 1/6!$.

11. $E_f^n(x;0) \le e/(n+1)!$.

13. $E_f^n(x;0) \le (0.3)^{n+1}/(n+1)!$.

15. $E_f^n(x;0) \le (e+e^{-1})/(n+1)!$.

17. 0.

25. b. Use $k = 6$: $e \approx 1+1+1/2!+1/3!+1/4!+1/5!+1/6! = 2.718055\overline{5}$.

Section 4.4

1. 4th partial sum $= 0.5833\overline{3}$. 10th partial sum "=" 0.6456349206. Yes - the 10th partial sum is a better estimate of $\ln 2$.

3. a. $(1/6)x^3$; b. Calculator: $0.1 - \sin(0.1) = 0.00016658335317$; Approximation by $(1/6)(0.1)^3 = 0.000166\overline{6}$.

5. a. $-1/2$; b. $-1/2$.

7. Calculator: $(1.09)^{1/3} = 1.0291424665715$; Approximation by $1 + (1/3)(0.09) - (1/9)(0.09)^2 = 1.0291$.

9. In the Binomial Theorem, replace x by $-x$ and let $p = -1$ to obtain the Geometric Series Theorem with $a = 1$.

11.

a. $T_f^\infty(x;\pi/4) = (1/\sqrt{2}) - (1/\sqrt{2})(x-\pi/4) - (1/\sqrt{2})(x-\pi/4)^2/2! + (1/\sqrt{2})(x-\pi/4)^3/3! + (1/\sqrt{2})(x-\pi/4)^4/4! - \cdots$.

b. All x.

13.

a. $T_f^\infty(x;2) = \ln 3 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{3^k k}(x-2)^k$.

b. $\ln(1+x)$ equals its Taylor series, centered at

2, for $-1 < x \leq 5$, but a smaller interval, such as $1 < x < 3$, is what you're expected to find.

15. b. and c. $T_f^\infty(x;0)$ equals the power series obtained by multiplying each term of the Maclaurin Series of e^x by x.

Section 4.5

1. Converges.

3. Diverges.

5. Diverges.

7. Converges.

9. Converges.

11. Converges.

13.

a. 1/5.

b. $\left[\frac{4}{5},\frac{6}{5}\right], \left(\frac{4}{5},\frac{6}{5}\right], \left[\frac{4}{5},\frac{6}{5}\right), \left(\frac{4}{5},\frac{6}{5}\right)$

c.

$$p(1.1) \approx 83/144$$
$$p(1.01) \approx 0.050614$$
$$p(2) \approx 224/9$$

d. We do not have an explicit algebraic function that we can type in on the calculator.

e. We expect good estimates for $x \in (\frac{4}{5}, \frac{6}{5})$ since the remainder term converges to zero within the radius of convergence.

15.

a. 1.

b. $(-1,1), [-1,1), (-1,1], [-1,-1]$.

c.

$$p(0.1) \approx 1403/3000$$
$$p(0.01) \approx 0.49667$$
$$p(1) \approx 4.15$$

d. We do not have an explicit algebraic function that we can type in on the calculator.

e. We expect good estimates for $x \in (-1,1)$ since the remainder term converges to zero within the radius of convergence.

17.

a. 7.

b. $(-3,11), [-3,11), (-3,11], [-3,11]$.

c.

$$p(4.1) \approx 0.144927$$
$$p(4.01) \approx 0.1430615$$
$$p(5) \approx 0.16618$$

d. We do not have an explicit algebraic function that we can type in on the calculator.

e. We expect good estimates for $x \in (-3,11)$ since the remainder term converges to zero within the radius of convergence.

19.

a. 1/5.

b. $(-\infty,\infty)$.

c.

$$p(-6.9) \approx 0.097537$$
$$p(-6.99) \approx 0.009975$$
$$p(-6) \approx 85/108$$

d. We do not have an explicit algebraic function that we can type in on the calculator.

e. We expect good estimates for all real x.

21.

a. 1.

b. $(-1,1)$, $[-1,1]$, $[-1,1)$, $(-1,1]$.

c.

$$p(0.1) \approx 0.092756$$
$$p(0.01) \approx 0.0099213$$
$$p(1) \approx 0.899661.$$

d. We do not have an explicit algebraic function that we can type in on the calculator.

e. We expect good estimates for $x \in (-1,1)$ since the remainder term converges to zero within the radius of convergence.

23. That they are mistaken.

Section 4.6

1.

a. $3-5x-\frac{3x^2}{2}+\frac{5x^3}{6}+\frac{x^4}{8}+\dots$ Converges on $(-\infty,\infty)$.

b. Taylor approx: 2.485, Calculator: 2.485845413.

3.

a. $\sum_{k=0}^{\infty}4^k\cdot(x-2)^{3k}$. Converges on $\left(2-\frac{1}{\sqrt[3]{4}},2+\frac{1}{\sqrt[3]{4}}\right)$.

b. Taylor approx: 0.996016, Calculator: 0.9960159.

5.

a. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k+1}}{k}$. Converges on $(-1,1)$.

b. Taylor approx: 0.00784427, Calculator: 0.0078441426.

7.

a. $\sum_{k=0}^{\infty}(-1)^k\frac{(x+2)^{2k+1}}{2k+1}$. Converges on $(-3,-1)$.

b. Taylor approx: -0.0996687, Calculator: -0.0996687.

9.

a. $\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}x^k$. Converges on $(-1,1)$.

b. Taylor approx: 0.76, Calculator: 0.7513148.

11.

a. $1-\frac{x-4}{2}+\frac{(x-4)^2}{24}-\frac{(x-4)^3}{720}+\frac{(x-4)^4}{40320}+\dots$ Converges on $[4,\infty)$.

b. Taylor approx: 0.995004 , Calculator: 0.995004.

The radius of convergence and center of the series in Exercises 13-21 of this chapter can be found in the solutions to Exercise 13-21 of Chapter 4.5. By the theorems in this chapter, the radius of convergence is the same for the differentiated and integrated series.

13.

$$\int p(x)\,dx = C+\sum_{k=1}^{\infty}\frac{5^k(x-1)^{k+1}}{(k+1)k^2}$$

$$p'(x)=\sum_{k=1}^{\infty}\frac{5^k(x-1)^{k-1}}{k}.$$

15.

$$\int p(x)\,dx = C+\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{(k+1)(k^3+2)}$$

$$p'(x)=\sum_{k=1}^{\infty}k(-1)^k\frac{x^{k-1}}{k^3+2}.$$

17.

$$\int p(x)\,dx = C+\sum_{k=0}^{\infty}\frac{(x-4)^{k+1}}{(k+1)7^{k+1}}$$

$$p'(x)=\sum_{k=1}^{\infty}\frac{k(x-4)^{k-1}}{7^{k+1}}.$$

19.

$$\int p(x)\,dx = C+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x+7)^{k+1}}{(k+1)k^k}$$

$$p'(x)=\sum_{k=2}^{\infty}(-1)^{k+1}\frac{(x+7)^{k-1}}{k^{k-1}}$$

21.

$$\int p(x)\,dx = C+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{k+1}}{(k+1)\sqrt[3]{k}}$$

$$p'(x)=\sum_{k=2}^{\infty}(-1)^{k+1}k^{2/3}x^{k-1}.$$

23. $1+\frac{x}{2}+\frac{5x^2}{6}+\frac{7x^3}{12}+\frac{47x^4}{60}+\dots$

25. $1+x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{13x^4}{24}+\dots$

27. $(x-3)+\left(\frac{-1}{\sqrt{2}}+1\right)(x-3)^2+\left(\frac{1}{2}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}\right)(x-3)^3+\left(\frac{1}{\sqrt{3}}-\frac{1}{2\sqrt{2}}-\frac{1}{3}\right)(x-3)^4+\left(\frac{1}{\sqrt{5}}-\frac{1}{2}+\frac{1}{2\sqrt{3}}-\frac{1}{6\sqrt{2}}+\frac{1}{24}\right)(x-3)^5+\dots$

29. $1+3(x-1)+4.5(x-1)^2+4.5(x-1)^3$.

31. $x-x^2-\dfrac{x^3}{6}+\dfrac{x^4}{6}$.

33.

35. $1-\dfrac{x}{2}+\dfrac{x^2}{24}$.

37. $x^3-\dfrac{x^5}{2}$.

39.

a. Idea: an even series converges at a iff the series converges at $-a$.
b. Straightforward, but keep in mind there are four things to prove.

41.

a. $C+\sum_{k=0}^{\infty}(-1)^k\dfrac{x^{4k+1}}{(4k+1)(2k!)}$
b. 1

43.

a. Follow the hint.
b. $\sum_{k=1}^{\infty}\dfrac{p!}{(k-1)!(p-k)!}x^{k-1}$.
c. Multiply the result of (b) by $1+x$.
d. Separate the variables and integrate.

45. $\cos(3\theta)=\cos^3\theta-3\cos\theta\sin^2\theta$ $\sin(3\theta)=3\cos^2\theta\sin\theta-\sin^3\theta$.

Section 4.7

1.

a. $b_{k+1}=\dfrac{5b_k}{k+1}$, $k\geq 2$.
b.

$$\begin{aligned} b_0 &= 2\\ b_1 &= 10\\ b_2 &= 51/2\\ b_3 &= 85/2\\ b_4 &= 425/8\\ b_5 &= 425/8\\ b_6 &= 2125/48\\ b_7 &= 10625/336\\ b_8 &= 53125/2688 \end{aligned}$$

c. ≈ 3.2975.

3.

a. $b_{k+1}=\frac{5b_k+\frac{1}{k!}}{k+1}$, $k\geq 1$,
b.

$$\begin{aligned} b_0 &= 1\\ b_1 &= 6\\ b_2 &= 31/2\\ b_3 &= 26\\ b_4 &= 781/24\\ b_5 &= 651/20\\ b_6 &= 19531/720\\ b_7 &= 4069/210\\ b_8 &= 488281/40320. \end{aligned}$$

c. ≈ 1.7576.

5.

a. $b_{k+1}=\dfrac{5b_k+\frac{1}{(k-2)!}}{k+1}$, $k\geq 3$,
b.

$$\begin{aligned} b_0 &= 2\\ b_1 &= 11\\ b_2 &= 55/2\\ b_3 &= 277/6\\ b_4 &= 1391/24\\ b_5 &= 6967/120\\ b_6 &= 6971/144\\ b_7 &= 34861/1008\\ b_8 &= 871567/40320. \end{aligned}$$

c. $17227/6000$.

7.

a. $b_j=\dfrac{b_{j-3}}{j}$, $j\geq 3$.

b.

$$
\begin{aligned}
b_0 &= -1 \\
b_1 &= 1 \\
b_2 &= 0 \\
b_3 &= -1/3 \\
b_4 &= 1/4 \\
b_5 &= 0 \\
b_6 &= -1/18 \\
b_7 &= 1/28 \\
b_8 &= 0.
\end{aligned}
$$

c. ≈ -0.900308.

9.

a. $b_{k+2} = \dfrac{4b_k}{(k+2)(k+1)} + \dfrac{2}{(k+2)!}$

b.

$$
\begin{aligned}
b_0 &= 1 \\
b_1 &= 0 \\
b_2 &= 3 \\
b_3 &= 1/3 \\
b_4 &= 13/12 \\
b_5 &= 1/12 \\
b_6 &= 53/360 \\
b_7 &= 1/120 \\
b_8 &= 71/6720.
\end{aligned}
$$

c. ≈ 1.030442.

11. $y(x) = 2 + 10x + \frac{51}{25}\left(e^{5x} - 1 - 5x\right)$.

13. $y(x) = 7e^{p(x)}$ where $p(x) = (x^3/3) + 2x^3 + 4x + 8/3$.

15. $y(x) = -\dfrac{b(1 - e^{ax} + ax)}{a^2}$.

17. $y = \dfrac{1}{2}(e^{2x} + e^{-2x})$.

19. $p(x) = 1 + x + \dfrac{3}{2}x^2 + \dfrac{4}{3}x^3$.

Section 5.1

1. Converges to 1.

3. Converges to 1.

5. Converges to 0.

7. Converges to $\ln 2$.

9. Converges to 1.

11. Converges to e^5.

13. Converges to 0.

15. Converges to 0.

17. Use $2/n^2$ and $12/n^2$ as pinching sequences.

19. Use $\dfrac{4n-3}{n}$ and $\dfrac{4n+3}{n}$ as pinching sequences.

21. Let $a_n = \sin\left(\frac{4n\pi}{4}\right)$ and $b_n = \sin\left(\frac{(8n+2)\pi}{4}\right)$.

23. Use the subsequences $1/2, 2/3, 3/4, \ldots$ and $-1/2, -2/3, -3/4, \ldots$.

25.

a. No, the sequence need not converge. Let $a_n = 5 + (-1)^n$.

b. Yes.

c. Yes, since the sequence is monotonic and bounded.

27. Idea: If $a_n + b_n$ converges and a_n converges, then $b_n = (a_n + b_n) - a_n$ converges.

29. Idea: If b_n/a_n converges and a_n converges, then $b_n = \dfrac{b_n}{a_n} \cdot a_n$ converges.

Section 5.2

1.

$$
\begin{aligned}
s_1 &= 1/1 = 1 \\
s_2 &= 1 + 1/4 = 5/4 \\
s_3 &= 1 + 1/4 + 1/7 = 39/28 \\
s_4 &= 1 + 1/4 + 1/7 + 1/10 = 209/140 \\
s_5 &= 1 + 1/4 + 1/7 + 1/10 + 1/13 = 2857/1820.
\end{aligned}
$$

3.

$$
\begin{aligned}
s_0 &= 1 \\
s_1 &= 1 - 2 = -1 \\
s_2 &= 1 - 2 + 4 = 3 \\
s_3 &= 1 - 2 + 4 - 8 = -5 \\
s_4 &= 1 - 2 + 4 - 8 + 16 = 11
\end{aligned}
$$

5.

$s_1 = \frac{1}{1} - \frac{1}{4} = \frac{3}{4}$
$s_2 = (1-1/4) + (1/4-1/9) = 8/9$
$s_3 = (1-1/4) + (1/4-1/9) + (1/9-1/16) = 15/16$

Similarly, $s_4 = 24/25$ and $s_5 = 35/36$.

7.

$$\begin{aligned} s_{-3} &= (-3)^2 = 9 \\ s_{-2} &= (-3)^2 + (-2)^2 = 13 \\ s_{-1} &= (-3)^2 + (-2)^2 + (-1)^2 = 14 \\ s_0 &= s_{-1} + 0^2 = 14 \\ s_1 &= s_0 + 1^2 = 15. \end{aligned}$$

9.

$$\begin{aligned} s_4 &= 2/1 = 2 \\ s_5 &= \frac{2}{1} + \frac{2}{2} = 3 \\ s_6 &= \frac{2}{1} + \frac{2}{2} + \frac{2}{3} = 11/3 \\ s_7 &= \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} = 25/6 \\ s_8 &= \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5} = 137/30. \end{aligned}$$

11.

a. $b_1 = 1, b_2 = -1/2, b_3 = -1/6, b_4 = -1/12, b_5 = -1/20$.

b. $b_i = \dfrac{-1}{i(i-1)}$ if $i > 1$. $b_1 = 1$.

c. Converges to 0.

13.

a. $b_0 = 1, b_1 = -2, b_2 = 2, b_3 = -2, b_4 = 2, b_5 = -2$.

b. $b_0 = 1$, $b_i = (-1)^i \cdot 2$ for $i > 0$.

c. Diverges.

15.

a. $b_0 = 2$, $b_1 = 1/e$, $b_2 = \dfrac{2-e}{e^2}$, $b_3 = \dfrac{3-2e}{e^3}$, $b_4 = \dfrac{4-3e}{e^4}$.

b. $b_i = \frac{i-(i-1)e}{e^i}$ if $i > 0$. $b_0 = 2$.

c. Converges to 2.

17.

a. $b_3 = 1/2$, $b_4 = -1/3$, $b_5 = -1/12$, $b_6 = -1/30$, $b_7 = -1/60$

b. $b_i = \dfrac{-2}{(i-1)(i-2)(i-3)}$ if $i > 3$. $b_3 = 1/2$.

c. Converges to 0.

19.

a. $b_1 = 5, b_2 = -3/2, b_3 = -1/2, b_4 = -1/4, b_5 = -3/20$

b. $b_i = \dfrac{-3}{i(i-1)}$ if $i > 1$. $b_1 = 5$.

c. Converges to 2.

21. Diverges.

23. Diverges.

25. Diverges.

27. Converges to $7/12$.

29. Converges to $-\pi/4$.

31. Converges to $-1/4$.

33. Converges to $-\dfrac{\ln 2}{10}$.

35. Converges to $73/12$.

37. Diverges.

39. Diverges.

41. $767/3333$.

43. $271801/99990$.

45. Converges to 20.

47. Diverges.

49.

a. The subsequences don't converge to the same value.

b. After grouping pairs, we have:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots.$$

which converges.

51. Answers may vary. It shows the terms of a series may converge to zero while the series diverges.

53. Compare with $1/(3k)$, for example.

55.

a. $1+\sum_{k=1}^{\infty}\frac{2}{3^k}$.

b. 2.

c. The dog runs twice as fast as the man, and so covers twice the distance.

Section 5.3

1. Diverges by the p-Series Test.

3.

a. Converges by the p-Series Test.

b. Using inequalities from p-Series Test, between $49-7\left(1+2^{-8/7}+3^{-8/7}\right)$ and $56-7\left(1+2^{-8/7}+3^{-8/7}\right)$.

5. Diverges.

7.

a. Converges by the p-Series Test.

b. Between 5.5 and 13.5.

9. Diverges.

11. Compare with the convergent series $\sum\frac{2}{k^2}$.

13. Compare with the divergent series $\sum\frac{p+1}{1^p}$.

15. Compare with the convergent series $\sum\frac{4}{m^2}$.

17. Compare with the divergent series $\sum\frac{2}{k^{1/2}}$.

19. Compare with the convergent series $\sum\frac{3}{k^{1.1}}$.

21. Diverges. Limit compare with $\sum\frac{1}{k}$.

23. Diverges. Limit compare with $\sum(p+1)$.

25. Converges. Limit compare with $\sum\frac{1}{m^2}$.

27. Diverges. Limit compare with $\sum\frac{1}{k^{1/6}}=\sum\frac{k^{1/3}}{k^{1/2}}$.

29. Converges. Limit compare with $\sum\frac{1}{2^k}=\sum\frac{\sqrt{4^k}}{4^k}$.

31. Converges.

33. Diverges.

35. Diverges.

37. Converges.

39. Converges.

41. Converges.

43. Diverges.

45. Hint: one can use an argument similar to the proof of the ratio test.

47. Apply the natural logarithm to the second formula in Corollary 5.3.30.

Section 5.4

1. Converges absolutely.

3. Converges absolutely.

5. Diverges.

7. Diverges.

9. Converges absolutely.

11. Diverges.

13. Converges conditionally.

15. Converges absolutely.

17. Converges by Theorem 5.4.11. $c_0=1$, $c_1=-1/4$, $c_2=23/72$, $c_4\approx 0.183287$, $c_5=139/2000$.

19. Converges by Theorem 5.4.11. $c_0=1$, $c_1=0$, $c_2=0$, $c_3=0$, $c_4=0$, $c_5=0$.

21. $\sum_{k=1}^{9999}(-1)^{k-1}\frac{1}{\sqrt{k}}$.

23. $\sum_{k=1}^{99}(-1)^{k-1}\frac{1}{\sqrt{k^2+1}}$.

25. Converges conditionally.

27. Diverges.

29. The final step of both parts of the theorem follow by the Cauchy criterion and the fact that the series a_n and b_n converge absolutely.

31. Let $a_n=b_n=\frac{(-1)^n}{\sqrt{n+1}}$.

33. Unlike finite sums, infinite sums are not, in general, commutative. The order in which terms are summed matters.

35. Converges on $[0.8, 1.2]$ and absolutely at both endpoints.

37. Converges on $[-1, 1]$ and absolutely at both endpoints.

39. Converges on $[-3, 11)$. Converges conditionally at $x = -3$.

41. Converges on $(-7 - e, -7 + e)$. Diverges at both endpoints by the Term Test for Divergence.

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David B. Massey was born in Jacksonville, Florida in 1959. He attended Duke University as an undergraduate mathematics major from 1977 to 1981, graduating *summa cum laude*. He remained at Duke as a graduate student from 1981 to 1986. He received his Ph.D. in mathematics in 1986 for his results in the area of complex analytic singularities.

Professor Massey taught for two years at Duke as a graduate student, and then for two years, 1986-1988, as a Visiting Assistant Professor at the University of Notre Dame. In 1988, he was awarded a National Science Foundation Postdoctoral Research Fellowship, and went to conduct research on singularities at Northeastern University. In 1991, he assumed a regular faculty position in the Mathematics Department at Northeastern. He has remained at Northeastern University ever since, where he is now a Full Professor.

Professor Massey has won awards for his teaching, both as a graduate student and as a faculty member at Northeastern. He has published 32 research papers, and two research-level books. In addition, he was a chapter author of the national award-winning book on teaching: "Dear Jonas: What can I say?, Chalk Talk: E-advice from Jonas Chalk, Legendary College Teacher", edited by D. Qualters and M. Diamond, New Forums Press, (2004).

Professor Massey founded the Worldwide Center of Mathematics, LLC, in the fall of 2008, in order to give back to the mathematical community, by providing free or very low-cost materials and resources for students and researchers.