INTEGRAL CALCULUS

Made Easy

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Fundamental Integration Formulae

1.1 INTRODUCTION

Integration is the inverse process of differentiation. In differentiation, we are given a function and we are required to find its derivative or differential co-efficient. In integration, the derivative of some function is given and we are required to find that function.

1.2 ANTIDERIVATIVE OR PRIMITIVE

A function $F(x)$ is said to be the antiderivative of the function $f(x)$ on the interval $[a, b]$ if

$$\frac{d}{dx}[F(x)] = f(x), \quad \forall\, x \in [a, b].$$

And we write

$$\frac{d}{dx}[F(x)] = f(x)$$

$$\Rightarrow \qquad \int f(x)\, dx = F(x).$$

SOME SOLVED EXAMPLES

Example 1. *Find the antiderivative of the function $f(x) = x^3$.*

Solution. From the definition of antiderivative it follows that the function $F(x) = \frac{x^4}{4}$ is an antiderivative because

$$\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3.$$

Example 2. $\frac{d}{dx}(\sin x) = \cos x$

Solution. $\int \cos x\, dx = \sin x$.

Also, if c is any constant,

Then $\quad \frac{d}{dx}(\sin x + c) = \cos x.$

Therefore ; In general $\int \cos x\, dx = \sin x + c.$

It shows that different values of c will give different integrals and hence a given function may have an indefinite number of integrals.

Remark. The presence of indefinite constant c justifies the name Indefinite Integral.

Thus, we may conclude that

$$\frac{d}{dx}[F(x)] = f(x)$$

$$\Rightarrow \qquad \int f(x)\,dx = F(x) + c$$

where $c \to$ constant and known as constant of integration.

Note (*i*) The symbol $\int$ is elongated S, and represents the summation and stands for integral of the given function.

(*ii*) $f(x)$, the function which is to be integrated, is called the integrand.

(*iii*) $\int dx$ is called integral w.r.t. x.

(*iv*) c is called constant of integration and it means any real number.

(*v*) x in $\int f(x)\,dx$ stands for variable of integration.

(*vi*) The integral of a function may or may not exist. In other words, it may not be possible to find a function whose derivative is equal to the given function.

(*vii*) The integral of a function if it exists is not unique and any two integrals of a function differ by constant.

(*viii*) The geometrical meaning assigned to the integral is area of some region.

(*ix*) Integral is used to find physical quantities like centre of mass, momentum etc.

Result 1. *Prove that* $\int x^n\,dx = \frac{x^{n+1}}{(n+1)} + c$, *when* $n \neq -1$.

Sol. We have $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{(n+1)} = x^n \quad \therefore \quad \int x^n\,dx = \frac{x^{n+1}}{n+1} + c.$

Thus, we may write

(*i*) $\int x^4\,dx = \frac{x^{4+1}}{4+1} + c = \frac{x^5}{5} + c$

(*ii*) $\int x^{3/2}\,dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c = \frac{x^{5/2}}{\frac{5}{2}} + c = \frac{2}{5}x^{5/2} + c$

(*iii*) $\int \frac{1}{x^{2/3}}\,dx = \int x^{-2/3}\,dx = \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + c = \frac{x^{1/3}}{\frac{1}{3}} + c = 3x^{1/3} + c.$

The above Result is also known as **Power Formula.**

Note. The Power Formula, could be memorise as increase the power of x by one and divide by the increased power.

Result 2. *Prove that* $\int \frac{1}{x}\,dx = \log|x| + c$, *where* $x \neq 0$.

Sol. Clearly either $x > 0$ or $x < 0$.

Therefore, there may be two cases :

Case I. When $x > 0$, then $|x| = x$

$$\therefore \qquad \frac{d}{dx}[\log |x|] = \frac{d}{dx}(\log x) = \frac{1}{x}.$$

Therefore, in this case, we get

$$\int \frac{1}{x}\,dx = \log |x| + c.$$

Case II. When $x < 0$, then $|x| = -x$

$$\therefore \qquad \frac{d}{dx}[\log |x|] = \frac{d}{dx}[\log(-x)] = \frac{1}{(-x)}(-1) = +\frac{1}{x}$$

Therefore, in this case, we get

$$\int \frac{1}{x}\,dx = \log |x| + c$$

Thus, from both the cases, we have

$$\int \frac{1}{x}\,dx = \log |x| + c.$$

1.3 SOME STANDARD ELEMENTARY INTEGRALS

Based on differentiation and definition of integration, we have the following standard results. The student is strongly advised to commit these results to memory, because no further progress is otherwise possible.

(*i*) $\frac{d}{dx}(c) = 0 \quad \Rightarrow \quad \int 0 \,.\, dx = c$

(*ii*) $\frac{d}{dx}(x) = 1 \quad \Rightarrow \quad \int 1 \,.\, dx = x + c$

(*iii*) $\frac{d}{dx}(kx) = k \quad \Rightarrow \quad \int k\,dx = kx + c$

(*iv*) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{(n+1)} \,.\, (n+1)x^n = x^n \; ; n \neq -1$

$$\Rightarrow \qquad \int x^n \,.\, dx = \frac{x^{n+1}}{n+1} + c \; ; n \neq -1$$

(*v*) $\frac{d}{dx}[\log |x|] = \frac{1}{x} \quad \Rightarrow \quad \int \frac{1}{x}\,dx = \log |x| + c$

(*vi*) $\frac{d}{dx}(e^x) = e^x \quad \Rightarrow \quad \int e^x\,dx = e^x + c.$

(*vii*) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = \frac{a^x(\log a)}{(\log a)} = a^x \; ; a > 0 \; ; a \neq 1$

$$\Rightarrow \qquad \int a^x\,dx = \frac{a^x}{\log a} + c \; ; a > 0, a \neq 1$$

(viii) $\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x \, dx = \sin x + c$

(ix) $\frac{d}{dx}(-\cos x) = \sin x \quad \Rightarrow \quad \int \sin x \, dx = -\cos x + c$

(x) $\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x \, dx = \tan x + c$

(xi) $\frac{d}{dx}(-\cot x) = \text{cosec}^2 x \quad \Rightarrow \quad \int \text{cosec}^2 x \, dx = -\cot x + c$

(xii) $\frac{d}{dx}(\sec x) = \sec x \tan x \quad \Rightarrow \quad \int \sec x \tan x \, dx = \sec x + c$

(xiii) $\frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x \quad \Rightarrow \quad \int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + c$

(xiv) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

(xv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad \Rightarrow \quad \int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

(xvi) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \quad \Rightarrow \quad \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

Note. We know that $\frac{d}{dx}(-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

From this result, it should not be concluded that $\sin^{-1} x = -\cos^{-1} x$. Rather, $\sin^{-1} x$ and $\cos^{-1} x$ differ by a constant.

(xvii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \quad \Rightarrow \quad \int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + c$

(xviii) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2} \quad \Rightarrow \quad \int \frac{-1}{1+x^2} dx = \cot^{-1} x + c$

(xix) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}} \quad \Rightarrow \quad \int \frac{-1}{x\sqrt{x^2-1}} dx = \text{cosec}^{-1} x + c.$

1.4 INTEGRATION OF ALGEBRAIC, LOGARITHMIC, EXPONENTIAL AND RATIONAL FUNCTIONS

Example. *Evaluate the following integrals :*

(i) $\int \frac{1}{x^{1/3}} dx$ (ii) $\int 5^x \, dx$ (iii) $\int \frac{1}{x^{3/2}} dx$

(iv) $\int x^{-6} \, dx$ (v) $\int \frac{1}{\sqrt[n]{x}} dx$ (vi) $\int e^{3 \log x} \, dx$

(vii) $\int e^{1x} \, .dx$ (viii) $\int a^{3\log_a x} \, .dx$ (ix) $\int \frac{a^x}{b^x} dx$

(x) $\int \sqrt[3]{p^2} \, dp.$

Solution. (i) $$\int \frac{1}{x^{1/3}} \, .dx = \int x^{-1/3} \, .dx$$

$$= \frac{x^{\left(-\frac{1}{3}+1\right)}}{\left(-\frac{1}{3}+1\right)} + c \qquad \left[\because \int x^n \, dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \frac{x^{2/3}}{\frac{2}{3}} + c = \frac{3}{2} x^{2/3} + c.$$

(ii) $$\int 5^x \, .dx = \frac{5^x}{\log 5} + c. \qquad \left[\because \int a^x \, dx = \frac{a^x}{\log a} + c\right]$$

(iii) $$\int \frac{1}{x^{3/2}} \, dx = \int x^{-3/2} \, dx$$

$$= \frac{x^{\left(-\frac{3}{2}+1\right)}}{\left(-\frac{3}{2}+1\right)} + c \qquad \left[\because \int x^n \, dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \frac{x^{-1/2}}{-\frac{1}{2}} + c = -2x^{-1/2} + c.$$

(iv) $$\int x^{-6} \, dx = \frac{x^{(-6+1)}}{(-6+1)} + c \qquad \left[\because \int x^n \, dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= -\frac{x^{-5}}{5} + c.$$

(v) $$\int \frac{1}{\sqrt[n]{x}} \, dx = \int \frac{1}{x^{1/n}} \, dx = \int x^{-1/n} \, dx \qquad \left[\because \int x^n \, dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \frac{x^{\left(-\frac{1}{n}+1\right)}}{\left(-\frac{1}{n}+1\right)} + c = \frac{x^{\left(\frac{n-1}{n}\right)}}{\left(\frac{n-1}{n}\right)} + c = \frac{n}{n-1} x^{\frac{n-1}{n}} + c.$$

(vi) $$\int e^{3\log x} \, .dx = \int e^{\log x^3} dx \qquad [\because \; m \log n = \log n^m]$$

$$= \int x^3 \, dx \qquad [\because \; e^{\log f(x)} = f(x)]$$

$$= \frac{x^{3+1}}{3+1} + c = \frac{x^4}{4} + c.$$

(vii) $\int e^{1x} . dx = e^x + c.$

(viii) $\int a^{3 \log_a x} . dx = \int a^{\log_a x^3} dx$ $[\because \quad a^{\log_a x} = x]$

$$= \int x^3 . dx = \frac{x^{3+1}}{3+1} + c = \frac{x^4}{4} + c.$$

(ix) $\int \frac{a^x}{b^x} . dx = \int \left(\frac{a}{b}\right)^x dx$ $\left[\because \int a^x dx = \frac{a^x}{\log a} + c\right]$

$$= \frac{\left(\frac{a}{b}\right)^x}{\log\left(\frac{a}{b}\right)} + c.$$

(x) $\int \sqrt[3]{p^2} . dp = \int (p^2)^{1/3} dp = \int p^{2/3} dp = \frac{p^{\frac{2}{3}+1}}{\frac{2}{3}+1} + c$ $\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$

$$= \frac{p^{5/3}}{\frac{5}{3}} + c = \frac{3}{5} p^{5/3} + c.$$

1.5 IMPORTANT EXTENSIONS OF ELEMENTARY FORMS

(a) All the results mentioned in the standard elementary integrals holds good when x is replaced by $(x + a)$ in any formula.
where $a \to$ any constant.

For example :

(i) $\int (x+a)^n . dx = \frac{(x+a)^{n+1}}{n+1} + c ; \quad (n \neq -1)$

$\because \quad \frac{d}{dx}\left[\frac{(x+a)^{n+1}}{(n+1)}\right] = \frac{1}{(n+1)} . (n + 1) (x + a)^n = (x + a)^n.$

(ii) $\int \frac{1}{x+a} . dx = \log | x + a | + c ; x \neq -a.$

(iii) $\int \sec^2 (x+a) . dx = \tan (x + a) + c$

(iv) $\int e^{x+a} . dx = e^{x+a} + c.$

(b) If x be replaced by $(ax + b)$ on both sides of any standard result mentioned in the list, the standard form remains true, provided the result on R.H.S. is divided by 'a' *i.e.*, the coefficient of x.
where a and b being constants.

For example :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1) . a} + c ; \quad (n \neq -1)$

$\because \quad \dfrac{d}{dx}\left[\dfrac{(ax+b)^{n+1}}{(n+1).a}\right] = \dfrac{1}{(n+1).a} \cdot (n+1)(ax+b)^n \cdot a = (ax+b)^n.$

(ii) $\quad \displaystyle\int \frac{1}{ax+b} \cdot dx = \frac{\log|ax+b|}{a} + c.$

(iii) $\quad \displaystyle\int \sec^2(ax+b)\, dx = \frac{\tan(ax+b)}{a} + c$

(iv) $\quad \displaystyle\int e^{ax+b} \cdot dx = \frac{e^{ax+b}}{a} + c.$

1.6 THEOREMS ON INTEGRATION

Theorem 1. *If $f(x)$ be an integrable function on x, then* $\dfrac{d}{dx}\left[\int f(x)\, dx\right] = f(x)$.

Proof. Let us consider that

$$\int f(x) \cdot dx = g(x) + c \quad ...(1)$$

$\therefore$ By definition

$$\frac{d}{dx}[g(x)] = f(x) \quad ...(2)$$

$\Rightarrow \quad \dfrac{d}{dx}\left[\int f(x)\, dx\right] = \dfrac{d}{dx}[g(x) + c]$ [$\because$ By using equation (1)]

$= \dfrac{d}{dx}[g(x)] + \dfrac{d}{dx}(c)$ [$\because$ By using equation (2)]

$= f(x) + 0 = f(x)$

$\therefore \quad \dfrac{d}{dx}\left[\int f(x)\, dx\right] = f(x).$

Note. This result shows that integration and differentiation are inverse processes and they neutralize the effect of each other.

Theorem 2. $\int kf(x)\, dx = k\int f(x)\, dx$

where $k \to$ constant.

i.e., the integral of the product of a constant and a function is equal to the product of constant and the integral of the function.

Proof. Let us consider that

$$\int f(x)\, dx = g(x) \quad ...(1)$$

Then, by the definition of an integral, we have

$$\frac{d}{dx}[g(x)] = f(x) \quad ...(2)$$

$\Rightarrow \quad \dfrac{d}{dx}[kg(x)] = k \cdot \dfrac{d}{dx}[g(x)]$

$= k \cdot f(x)$ [$\because$ By using equation (1)]

$\Rightarrow \quad \int k \cdot f(x)\, dx = k \cdot g(x)$ [$\because$ By definition of an integral]

$\Rightarrow \quad \int k \cdot f(x)\, dx = k \cdot \int f(x)\, dx$ $\left[\because \int f(x)\, dx = g(x)\right]$

Remark. If $f(x) = 1$, then

$$\int k \,.\, dx = k \,.\, \int 1 \,.\, dx = k \int x^0 \,.\, dx = kx + c.$$

This implies that, the integration of a constant k with respect to x is kx.

Theorem 3. *Prove that the indefinite integral of algebraic sum of two or more functions is equal to the algebraic sum of their integrals.*

i.e., $$\int [f(x) + g(x)] \,.\, dx = \int f(x) \,.\, dx + \int g(x) \,.\, dx.$$

Proof. Let us consider that

$$\int f(x) \,.\, dx = f_1(x) \quad \ldots(1)$$

and $$\int g(x) \,.\, dx = g_1(x) \quad \ldots(2)$$

$\therefore$ $$\frac{d}{dx} [f_1(x)] = f(x) \quad \ldots(3)$$

and $$\frac{d}{dx} [g_1(x)] = g(x) \quad \ldots(4)$$

$$\Rightarrow \quad \frac{d}{dx} [f_1(x) + g_1(x)] = \frac{d}{dx} [f_1(x)] + \frac{d}{dx} [g_1(x)]$$

$$= f(x) + g(x) \qquad [\because \text{ By using equations (3) and (4)}]$$

$$\therefore \quad \int [f(x) + g(x)] \,.\, dx = f_1(x) + g_1(x) \qquad [\because \text{ By definition of an integral}]$$

$$= \int f(x)\, dx + \int g(x)\, dx \qquad [\because \text{ By using equations (1) and (2)}]$$

Remark. (*i*) The generalization of the above result is

$$\int [f_1(x) + f_2(x) + \ldots\ldots + f_n(x)] \,.\, dx$$

$$= \int f_1(x)\, dx + \int f_2(x)\, dx + \ldots\ldots + \int f_n(x)\, dx.$$

(*ii*) It can be easily derived that the indefinite integral of the difference of two functions is equal to the difference of their integrals.

(*iii*) The results of theorems 2 and 3 can be generalized to the form

$$\int [k_1 f_1(x) \pm k_2 f_2(x) \ldots\ldots \pm k_n f_n(x)]\, dx$$

$$= k_1 \int f_1(x)\, dx \pm k_2 \int f_2(x)\, dx \ldots\ldots \pm k_n \int f_n(x)\, dx$$

i.e., The integration of the linear combination of a finite number of functions is equal to the linear combination of their integrals.

1.7 IMPORTANT EXTENSION FORMULAE OF STANDARD INTEGRAL FORMS

(*i*) $$\int x^n \,.\, dx = \frac{x^{n+1}}{n+1} + c\,; \quad (n \neq -1)$$

$$\Rightarrow \quad \int (ax + b)^n \,.\, dx = \frac{(ax+b)^{n+1}}{(n+1)\,.\,a} + c\,; \; (n \neq -1)$$

(*ii*) $$\int \frac{1}{x} \,.\, dx = \log |x| + c \quad \Rightarrow \quad \int \frac{1}{(ax+b)} \,.\, dx = \frac{\log |ax+b|}{a} + c$$

(iii) $\int a^x \, . \, dx = \dfrac{a^x}{\log a} + c \, ; (a > 0, a \neq 1)$

$\Rightarrow \quad \int a^{mx+b} \, . \, dx = \dfrac{a^{mx+b}}{m \, . \log a} + c \, ; \qquad (a > 0 \, ; a \neq 1)$

(iv) $\int e^x \, . \, dx = e^x + c \quad \Rightarrow \quad \int e^{mx+b} \, . \, dx = \dfrac{e^{mx+b}}{m} + c$

(v) $\int \cos x \, . \, dx = \sin x + c \quad \Rightarrow \quad \int \cos (ax+b) \, dx = \dfrac{\sin (ax+b)}{a} + c$

(vi) $\int \sin x \, dx = -\cos x + c \quad \Rightarrow \quad \int \sin (ax+b) \, dx = -\dfrac{\cos (ax+b)}{a} + c$

(vii) $\int \sec^2 x \, dx = \tan x + c \quad \Rightarrow \quad \int \sec^2 (ax+b) \, dx = \dfrac{\tan (ax+b)}{a} + c$

(viii) $\int \operatorname{cosec}^2 x \, dx = -\cot x + c \quad \Rightarrow \quad \int \operatorname{cosec}^2 (ax+b) \, dx = \dfrac{-\cot (ax+b)}{a} + c$

(ix) $\int \sec x \tan x \, dx = \sec x + c$

$\Rightarrow \quad \int \sec (ax+b) \tan (ax+b) \, dx = \dfrac{\sec (ax+b)}{a} + c$

(x) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

$\Rightarrow \quad \int \operatorname{cosec} (ax+b) \cot (ax+b) \, dx = -\dfrac{\operatorname{cosec} (ax+b)}{a} + c.$

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following integrals :*

(i) $\int (1-x) \sqrt{x} \, dx$ (ii) $\int \sqrt[3]{x} \, dx$

(iii) $\int (x^2 - 3x + 4) \, dx$ (iv) $\int a^x \, . \, e^x \, dx$

(v) $\int 2^{2x} \, . \, 3^x \, dx$ (vi) $\int (x^2 - 2x + 4)^2 \, dx$

(vii) $\int x^{5/4} \, dx$ (viii) $\int 9^{x+2} \, dx$

(ix) $\int 3^{2 \log_3 x} \, dx$

Solution. (i) $\int (1-x) \sqrt{x} \, dx = \int (\sqrt{x} - x\sqrt{x}) \, dx = \int x^{1/2} \, dx - \int x^{3/2} \, dx$

$$\left[\because \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \right]$$

$$= \frac{x^{1/2+1}}{\frac{1}{2}+1} - \frac{x^{3/2+1}}{\frac{3}{2}+1} + c = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + c.$$

(ii) $$\int \sqrt[3]{x}\, dx = \int x^{1/3}\, dx = \frac{x^{1/3+1}}{\frac{1}{3}+1} + c \qquad \left[\because \int x^n\, dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \frac{3}{4}x^{4/3} + c.$$

(iii) $$\int (x^2 - 3x + 4)\, dx = \int x^2\, dx - 3\int x\, dx + 4\int dx \qquad \left[\because \int x^n\, dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \frac{x^3}{3} - \frac{3x^2}{2} + 4x + c.$$

(iv) $$\int a^x . e^x\, dx = \int (ae)^x\, dx = \frac{(ae)^x}{\log (ae)} + c \qquad \left[\because \int a^x\, dx = \frac{a^x}{\log a} + c\right]$$

$$= \frac{a^x\, e^x}{\log (ae)} + c.$$

(v) $$\int 2^{2x} . 3^x dx = \int 4^x . 3^x dx = \int 12^x\, dx \qquad \left[\because \int a^x\, dx = \frac{a^x}{\log a} + c\right]$$

$$= \frac{12^x}{\log 12} + c.$$

(vi) $$\int (x^2 - 2x + 4)^2\, dx \qquad [\because \; (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca]$$

$$= \int (x^4 + 4x^2 + 16 - 4x^3 - 16x + 8x^2)\, dx$$

$$= \int (x^4 - 4x^3 + 12x^2 - 16x + 16)\, dx$$

$$= \int x^4\, dx - 4\int x^3\, dx + 12\int x^2\, dx - 16\int x\, dx + 16\int dx$$

$$\left[\because \int x^n\, dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \frac{x^5}{5} - \frac{4x^4}{4} + 12\frac{x^3}{3} - 16\frac{x^2}{2} + 16x + c$$

$$= \frac{1}{5}x^5 - x^4 + 4x^3 - 8x^2 + 16x + c.$$

(*vii*) $\int x^{5/4}\,dx = \dfrac{x^{5/4+1}}{\frac{5}{4}+1} + c$ $\left[\because \int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c\right]$

$= \dfrac{4}{9}x^{9/4} + c.$

(*viii*) $\int 9^{x+2}\,dx = \int 9^x . 9^2\,dx$

$= \int 81 . 9^x\,dx = 81\int 9^x\,dx$

$= 81\left(\dfrac{9^x}{\log 9}\right) + c.$ $\left[\because \int a^x\,dx = \dfrac{a^x}{\log a} + c\right]$

(*ix*) $\int 3^{2\log_3 x}\,dx = \int 3^{\log_3 x^2}\,dx$ $[\because m\log n = \log n^m]$

$= \int x^2\,dx = \dfrac{x^{2+1}}{2+1} + c$ $[\because a^{\log_a x} = x]$

$= \dfrac{1}{3}x^3 + c.$

Example 2. *Evaluate the following integrals :*

(*i*) $\int 3^{x+2}\,dx$ (*ii*) $\int \dfrac{1}{2}\sec^2 x\,dx$

(*iii*) $\int \sqrt{x}(x^3 + 2x^2 - x + 3)\,dx$ (*iv*) $\int \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^3 dx$

(*v*) $\int \dfrac{x^3 - 1}{x^2}\,dx.$

Solution. (*i*) $\int 3^{x+2}\,dx = \int 3^x\, 3^2 . dx = 9\int 3^x\,dx$

$= 9\left(\dfrac{3^x}{\log 3}\right) + c.$ $\left[\because \int a^x\,dx = \dfrac{a^x}{\log a} + c\right]$

(*ii*) $\int \dfrac{1}{2}\sec^2 x\,dx = \dfrac{1}{2}\int \sec^2 x\,dx = \dfrac{1}{2}\tan x + c.$

(*iii*) $\int \sqrt{x}(x^3 + 2x^2 - x + 3)\,dx$

$= \int x^{1/2}(x^3 + 2x^2 - x + 3)\,dx$

$= \int (x^{7/2} + 2x^{5/2} - x^{3/2} + 3x^{1/2})\,dx$

$= \int x^{7/2} . dx + 2\int x^{5/2}\,dx - \int x^{3/2}\,dx + 3\int x^{1/2}\,dx$

$$= \frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1} + 2.\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} - \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{x^{9/2}}{\frac{9}{2}} + 2.\frac{x^{7/2}}{\frac{7}{2}} - \frac{x^{5/2}}{\frac{5}{2}} + \frac{3x^{3/2}}{\frac{3}{2}} + c$$

$$= \frac{2}{9}x^{9/2} + \frac{4}{7}x^{7/2} - \frac{2}{5}x^{5/2} + 2x^{3/2} + c.$$

(iv) $\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^3 dx = \int\left(x^{1/2} - x^{-1/2}\right)^3 dx$ $\quad [\because \ (a-b)^3 = a^3 - b^3 - 3ab(a-b)]$

$$= \int\left[(x^{1/2})^3 - (x^{-1/2})^3 - 3x^{1/2}.x^{-1/2}(x^{1/2} - x^{-1/2})\right]dx$$

$$= \int[x^{3/2} - x^{-3/2} - 3(x^{1/2} - x^{-1/2})]\,dx \qquad [\because \ x^{1/2}.x^{-1/2} = x^0 = 1]$$

$$= \int(x^{3/2} - x^{-3/2} - 3x^{1/2} + 3x^{-1/2}).\,dx$$

$$= \int x^{3/2}\,dx - \int x^{-3/2}.\,dx - 3\int x^{1/2}\,dx + 3\int x^{-1/2}\,dx$$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} - \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{3x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{x^{5/2}}{\frac{5}{2}} - \frac{x^{-1/2}}{-\frac{1}{2}} - \frac{3x^{3/2}}{\frac{3}{2}} + \frac{3x^{1/2}}{\frac{1}{2}} + c$$

$$= \frac{2}{5}x^{5/2} + 2x^{-1/2} - 2x^{3/2} + 6x^{1/2} + c.$$

(v) $\int\frac{x^3-1}{x^2}dx = \int\left(\frac{x^3}{x^2} - \frac{1}{x^2}\right)dx = \int(x - x^{-2})\,dx$

$$= \int x.\,dx - \int x^{-2}.\,dx = \frac{x^{1+1}}{1+1} - \frac{x^{-2+1}}{-2+1} + c = \frac{x^2}{2} - \frac{x^{-1}}{-1} + c = \frac{x^2}{2} + \frac{1}{x} + c.$$

Example 3. *Evaluate the following integrals :*

(i) $\int\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)dx$ $\qquad$ *(ii)* $\int\frac{ax^2+bx+c}{x^2}dx$

(iii) $\int\left(\frac{3}{\sqrt{x}} + 5x^4\right)dx$ $\qquad$ *(iv)* $\int\left(5x^3 + 2x^{-5} - 7x + \frac{1}{\sqrt{x}} + \frac{5}{x}\right)dx$

(v) $\int\frac{1+x+x^2}{x^2(1+x)}dx.$

Solution. (*i*) $\int\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)dx$

$$= \int\left[x\left(x^2+\frac{1}{x^2}\right)+\frac{1}{x}\left(x^2+\frac{1}{x^2}\right)\right]dx = \int\left(x^3+\frac{1}{x}+x+\frac{1}{x^3}\right)dx$$

$$= \int\left(x^3+x+\frac{1}{x}+\frac{1}{x^3}\right)dx = \int x^3\,dx+\int x\,.dx+\int\frac{1}{x}dx+\int\frac{1}{x^3}dx$$

$$= \frac{x^{3+1}}{3+1}+\frac{x^{1+1}}{1+1}+\log|x|+\frac{x^{-3+1}}{-3+1}+c$$

$$= \frac{x^4}{4}+\frac{x^2}{2}+\log|x|+\frac{x^{-2}}{-2}+c$$

$$= \frac{x^4}{4}+\frac{x^2}{2}+\log|x|-\frac{1}{2x^2}+c.$$

(*ii*) $\int\frac{ax^2+bx+c}{x^2}dx = \int\left(\frac{ax^2}{x^2}+\frac{bx}{x^2}+\frac{c}{x^2}\right)dx = \int\left(a+\frac{b}{x}+\frac{c}{x^2}\right)dx$

$$= a\int 1\,.dx+b\int\frac{1}{x}dx+c\int x^{-2}\,.dx$$

$$= ax+b\log|x|+\frac{cx^{-2+1}}{-2+1}+C$$

$$= ax+b\log|x|+\frac{cx^{-1}}{-1}+C$$

$$= ax+b\log|x|-\frac{c}{x}+C.$$

(*iii*) $\int\left(\frac{3}{\sqrt{x}}+5x^4\right)dx = \int 3x^{-1/2}\,dx+\int 5x^4\,dx = 3\int x^{-1/2}\,dx+5\int x^4\,dx$

$$= \frac{3x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+\frac{5x^{4+1}}{4+1}+c = 3\,.\,\frac{2}{1}\,.\,x^{1/2}+\frac{5\,.\,x^5}{5}+c$$

$$= 6\sqrt{x}+x^5+c.$$

(*iv*) $\int\left(5x^3+2x^{-5}-7x+\frac{1}{\sqrt{x}}+\frac{5}{x}\right)dx$

$$= 5\int x^3\,dx+2\int x^{-5}\,dx-7\int x\,.dx+\int\frac{1}{\sqrt{x}}dx+5\int\frac{1}{x}dx$$

$$= \frac{5.x^{3+1}}{3+1}+\frac{2\,.\,x^{-5+1}}{-5+1}-\frac{7\,.\,x^{1+1}}{1+1}+\frac{x^{-1/2+1}}{-\frac{1}{2}+1}+5\log|x|+c$$

$$= \frac{5x^4}{4}+\frac{2x^{-4}}{-4}-\frac{7\,.\,x^2}{2}+\frac{x^{1/2}}{\frac{1}{2}}+5\log|x|+c$$

$$= \frac{5x^4}{4} - \frac{1}{2x^4} - \frac{7x^2}{2} + 2\sqrt{x} + 5 \log |x| + c.$$

(v) $\int \frac{1+x+x^2}{x^2(1+x)} dx = \int \frac{(1+x)+x^2}{x^2(1+x)} dx = \int \left[\frac{(1+x)}{x^2(1+x)} + \frac{x^2}{x^2(1+x)}\right] dx$

$$= \int \frac{1}{x^2} dx + \int \frac{1}{1+x} dx = \int x^{-2} dx + \int \frac{1}{1+x} dx$$

$$= \frac{x^{-2+1}}{-2+1} + \log |1+x| + c = \frac{x^{-1}}{-1} + \log |1+x| + c$$

$$= -\frac{1}{x} + \log |1+x| + c.$$

Example 4. *Evaluate the following integrals :*

(i) $\int \sqrt{ax+b}\, dx$ (ii) $\int (4x^3 - 4x^{-5})\, dx$

(iii) $\int \frac{x^3 - x^2 + x - 1}{x-1} dx$ (iv) $\int \frac{x^2}{1+x^2} dx$

(v) $\int (x-3)^2 . \sqrt{x}\, dx$.

Solution. (i) $\int \sqrt{ax+b}\, dx = \int (ax+b)^{1/2} dx = \frac{(ax+b)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right).a} + c$

$$= \frac{(ax+b)^{3/2}}{\left(\frac{3}{2}.a\right)} + c = \frac{2}{3a}(ax+b)^{3/2} + c.$$

(ii) $\int (4x^3 - 4x^{-5})\, dx = 4\int x^3 . dx - 4\int x^{-5} dx$

$$= \frac{4x^{3+1}}{3+1} - \frac{4x^{-5+1}}{-5+1} + c = \frac{4x^4}{4} - \frac{4x^{-4}}{-4} + c$$

$$= x^4 + \frac{1}{x^4} + c.$$

(iii) $\int \frac{x^3 - x^2 + x - 1}{x-1} dx = \int \frac{(x-1)(x^2+1)}{(x-1)} dx$ $\left[\because \begin{array}{l} x^3 - x^2 + x - 1 \\ = x^2(x-1) + 1(x-1) \\ = (x-1)(x^2+1) \end{array}\right]$

$$= \int (x^2+1)\, dx = \int x^2 . dx + 1\int dx = \frac{x^3}{3} + x + c.$$

(iv) $\int \frac{x^2}{1+x^2} dx = \int \frac{1+x^2-1}{1+x^2} dx$ **[Add and subtract 1 to the numerator]**

$$= \int \left[\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right] dx = \int 1 . dx - \int \frac{1}{1+x^2} dx$$

$= x - \tan^{-1} x + c$ $\left[\because \int \frac{1}{1+x^2} dx = \tan^{-1} x + c\right]$

(v) $\int (x-3)^2 \cdot \sqrt{x}\, dx = \int (x^2 - 6x + 9) \cdot \sqrt{x}\, dx$

$$= \int (x^2 - 6x + 9) \cdot x^{1/2}\, dx = \int (x^{5/2} - 6x^{3/2} + 9x^{1/2})\, dx$$

$$= \int x^{5/2}\, dx - 6\int x^{3/2}\, dx + 9\int x^{1/2}\, dx.$$

$$= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} - \frac{6x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{9x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{7/2}}{7/2} - \frac{6x^{5/2}}{5/2} + \frac{9x^{3/2}}{3/2} + c$$

$$= \frac{2}{7}x^{7/2} - \frac{12}{5}x^{5/2} + 6x^{3/2} + c.$$

Example 5. *Evaluate the following integrals :*

(i) $\int (e^x + 3\cos x - 4x^3 + 2)\, dx$ (ii) $\int \left(e^{3x} - 2e^x + \frac{1}{x}\right) dx$

(iii) $\int \frac{x^2}{1+x}\, dx$ (iv) $\int \frac{1-x}{1+x}\, dx$

(v) $\int \frac{x^2-4}{x+1}\, dx$ (vi) $\int \frac{x+2}{(x+1)^2}\, dx$

(vii) $\int \frac{2x}{(2x+1)^2}\, dx$ (viii) $\int \frac{(x+1)^2}{x\sqrt{x}}\, dx.$

Solution. (i) $\int (e^x + 3\cos x - 4x^3 + 2)\, dx$

$$= \int e^x\, dx + 3\int \cos x\, dx - 4\int x^3\, dx + 2\int dx$$

$$= e^x + 3\sin x - \frac{4x^4}{4} + 2x + c$$

$$= e^x + 3\sin x - x^4 + 2x + c.$$

(ii) $\int \left(e^{3x} - 2e^x + \frac{1}{x}\right) dx = \int e^{3x}\, dx - 2\int e^x\, dx + \int \frac{1}{x}\, dx$

$$= \frac{e^{3x}}{3} - 2e^x + \log |x| + c \qquad \left[\because \int e^{ax}\, dx = \frac{e^{ax}}{a} + c, \quad \int \frac{1}{x}\, dx = \log|x| + c.\right]$$

$$= \frac{1}{3}e^{3x} - 2e^x + \log |x| + c.$$

(iii) $\int \frac{x^2}{1+x}\, dx$

Since the degree of numerator is greater than the degree of denominator, therefore by actual division, we have

$$\int \frac{x^2}{1+x}\,dx = \int \left(x - \frac{x}{1+x}\right)dx = \int x\,dx - \int \frac{x}{1+x}\,dx$$

$$= \int x\,dx - \int \frac{1+x-1}{1+x}\,dx$$

$$\begin{array}{l} x+1\,\overline{)\,x^2\,(}\,x \\ \quad\;\; +x^2 + x \\ \quad\;\;\; - \;\; - \\ \hline \qquad\quad -x \\ \hline \end{array}$$

[Add and subtract 1 to the numerator of second intergral]

$$= \int x\,dx - \int 1.\,dx + \int \frac{1}{1+x}\,dx$$

$$= \frac{x^2}{2} - x + \log |\, 1 + x \,| + c. \qquad \left[\begin{array}{l} \because \displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c \\ \displaystyle\int \frac{1}{x}\,dx = \log|x| + c \end{array}\right]$$

(*iv*) $$\int \frac{1-x}{1+x}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{x}{1+x}\,dx$$

$$= \int \frac{1}{x+1}\,dx - \int \frac{1+x-1}{1+x}\,dx$$

[Add and subtract 1 to the numerator of the second integral]

$$= \int \frac{1}{x+1}\,dx - \int 1\,dx + \int \frac{1}{1+x}\,dx$$

$$= \log |\, 1 + x \,| - x + \log |\, 1 + x \,| + c$$

$$= 2 \log |\, 1 + x \,| - x + c.$$

(*v*) $$\int \frac{x^2-4}{x+1}\,dx$$

Since the degree of numerator is greater than the degree of denominator, therefore by actual division, we have :

$$\int \frac{x^2-4}{x+1}\,dx = \int \left[(x-1) - \frac{3}{x+1}\right]dx$$

$$\begin{array}{l} x+1\,\overline{)\,x^2 - 4\,(}\; x - 1 \\ \quad\;\; \underline{\;\; x^2 \quad \pm x} \\ \qquad\quad -x - 4 \\ \qquad\quad -x - 1 \\ \qquad\quad\; + \;\; + \\ \hline \qquad\qquad\;\; -3 \\ \hline \end{array}$$

$$= \int x\,dx - \int dx - 3\int \frac{1}{1+x}\,dx$$

$$= \frac{x^2}{2} - x - 3 \log |\, x + 1 \,| + c.$$

(*vi*) $$\int \frac{x+2}{(x+1)^2}\,dx = \int \frac{x+1+1}{(x+1)^2}\,dx$$ [Note this step]

$$= \int \left[\frac{x+1}{(x+1)^2} + \frac{1}{(x+1)^2}\right]dx$$

$$= \int \frac{1}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx$$

$$= \int \frac{1}{x+1}\,dx + \int (x+1)^{-2}\,dx$$

$$= \log|x+1| + \frac{(x+1)^{-2+1}}{-2+1} + c$$

$$= \log|x+1| - \frac{1}{1+x} + c.$$

(*vii*) $\int \frac{2x}{(2x+1)^2}\,dx = \int \frac{2x+1-1}{(2x+1)^2}\,dx$ [Add and subtract 1 to the numerator.]

$$= \int \left[\frac{2x+1}{(2x+1)^2} - \frac{1}{(2x+1)^2}\right]dx$$

$$= \int \frac{1}{(2x+1)}\,dx - \int (2x+1)^{-2}\,dx$$

$$\left[\because \int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c\right]$$

$$= \frac{\log|2x+1|}{2} - \frac{(2x+1)^{-2+1}}{2(-2+1)} + c$$

$$= \frac{1}{2}\log|2x+1| + \frac{1}{2(2x+1)} + c.$$

(*viii*) $\int \frac{(x+1)^2}{x\sqrt{x}}\,dx$ $[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$$= \int \frac{x^2+2x+1}{x^{3/2}}\,dx = \int \left[\frac{x^2}{x^{3/2}} + \frac{2x}{x^{3/2}} + \frac{1}{x^{3/2}}\right]dx$$

$$= \int x^{1/2}\,dx + 2\int x^{-1/2}\,dx + \int x^{-3/2}\,dx$$

$$= \frac{x^{1/2+1}}{\left(\frac{1}{2}+1\right)} + \frac{2x^{-1/2+1}}{\left(-\frac{1}{2}+1\right)} + \frac{x^{-\frac{3}{2}+1}}{\left(-\frac{3}{2}+1\right)} + c$$

$$= \frac{2}{3}x^{3/2} + 4x^{1/2} - 2x^{1/2} + c$$

$$= \frac{2}{3}x\sqrt{x} + 4\sqrt{x} - \frac{2}{\sqrt{x}} + c.$$

Example 6. *Evaluate the following integrals :*

(i) $\int \left(\frac{x}{a}+\frac{a}{x}+x^a+a^x+ax\right).dx \; ; a>0$

(ii) $\int (5^{3x+1}).dx$ *(iii)* $\int (e^{x\log a}+e^{a\log x}+e^{a\log a}).dx$

(iv) $\int \left(1+\frac{1}{1+x^2}-\frac{2}{\sqrt{1-x^2}}-x\right).dx$ *(v)* $\int (x^a+a^x+e^x.a^x+\sin\alpha).dx$.

Solution. *(i)* $\int \left(\frac{x}{a}+\frac{a}{x}+x^a+a^x+ax\right).dx$

$$= \frac{1}{a}\int x.dx + a\int \frac{1}{x}dx + \int x^a.dx + \int a^x.dx + a\int x.dx$$

$$= \frac{1}{a}.\frac{x^2}{2} + a\log|x| + \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + a.\frac{x^2}{2} + c$$

$$= \frac{x^2}{2a} + a\log|x| + \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + \frac{ax^2}{2} + c.$$

(ii) $\int 5^{3x+1}.dx = \frac{5^{3x+1}}{(\log 5).3}+c$

$$= \frac{5^{3x+1}}{3\log 5}+c. \qquad \left[\because \int a^{mx+b}.dx = \frac{a^{mx+b}}{m.\log a}; a>0; a\neq 1\right]$$

(iii) $\int (e^{x\log a}+e^{a\log x}+e^{a\log a}).dx$

$$= \int (e^{\log a^x}+e^{\log x^a}+e^{\log a^a}).dx \qquad [\because m\log n = \log n^m]$$

$$= \int (a^x+x^a+a^a).dx \qquad [\because e^{\log f(x)} = f(x)]$$

$$= \int a^x.dx + \int x^a.dx + \int a^a.dx = \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + a^a.x + c.$$

(iv) $\int \left(1+\frac{1}{1+x^2}-\frac{2}{\sqrt{1-x^2}}-x\right).dx$

$$= \int 1.dx + \int \frac{1}{1+x^2}dx - 2\int \frac{1}{\sqrt{1-x^2}}dx - \int x.dx$$

$$= x + \tan^{-1}x - 2\sin^{-1}x - \frac{x^2}{2} + c.$$

$$\left[\because \int \frac{1}{1+x^2}.dx = \tan^{-1}x + c; \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + c\right]$$

(v) $\int (x^a + a^x + e^x . a^x + \sin\alpha) . dx$

$$= \int x^a . dx + \int a^x \, dx + \int (ea)^x . dx + \int \sin\alpha . dx$$

$$= \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + \frac{(ea)^x}{\log ea} + (\sin\alpha) . x + c.$$

Example 7. *Evaluate the following integrals :*

(i) $\int \frac{x^4}{x^2+1} . dx$ (ii) $\int \frac{(a^x+b^x)^2}{a^x b^x} . dx$

(iii) $\int \frac{(x^4+x^2+1)}{(x^2-x+1)} dx$ (iv) $\int \frac{x}{\sqrt{1+x}} dx.$

Solution. (i) $\int \frac{x^4}{x^2+1} . dx$

$$= \int \frac{(x^4-1+1)}{(x^2+1)} . dx \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \int \left[\frac{x^4-1}{x^2+1} + \frac{1}{x^2+1}\right] . dx \qquad \left[\begin{array}{l} \because \ (x^4-1) = (x^2)^2 - 1^2 = (x^2-1)(x^2+1) \\ \because \ a^2 - b^2 = (a+b)(a-b) \end{array}\right]$$

$$= \int \frac{(x^2-1)(x^2+1)}{x^2+1} . dx + \int \frac{1}{x^2+1} . dx$$

$$= \int (x^2-1) . dx + \int \frac{1}{1+x^2} dx = \int x^2 . dx - \int 1 . dx + \int \frac{1}{1+x^2} dx$$

$$= \frac{x^3}{3} - x + \tan^{-1} x + c \qquad \left[\because \int \frac{1}{1+x^2} dx = \tan^{-1} x + c\right]$$

(ii) $\int \frac{(a^x+b^x)^2}{a^x b^x} . dx$

$$= \int \frac{(a^x)^2 + (b^x)^2 + 2a^x . b^x}{a^x b^x} . dx \qquad [\because \ (a+b)^2 = a^2 + 2ab + b^2]$$

$$= \int \left[\frac{(a^x)^2}{a^x b^x} + \frac{(b^x)^2}{a^x b^x} + \frac{2a^x b^x}{a^x b^x}\right] . dx$$

$$= \int \left(\frac{a^x}{b^x} + \frac{b^x}{a^x} + 2\right) . dx$$

$$= \int \left(\frac{a}{b}\right)^x . dx + \int \left(\frac{b}{a}\right)^x . dx + \int 2 . dx \qquad \left[\because \int a^x . dx = \frac{a^x}{\log a}\right]$$

$$= \frac{\left(\frac{a}{b}\right)^x}{\log\left(\frac{a}{b}\right)} + \frac{\left(\frac{b}{a}\right)^x}{\log\left(\frac{b}{a}\right)} + 2x + c.$$

(iii) $\int \frac{x^4+x^2+1}{x^2-x+1} \cdot dx$

$$= \int \frac{(x^4+2x^2+1)-x^2}{(x^2-x+1)} \cdot dx \qquad \text{[Add and subtract } x^2 \text{ to the numerator]}$$

$$= \int \frac{(x^2+1)^2-x^2}{(x^2-x+1)} \cdot dx$$

$$= \int \frac{(x^2+1+x)(x^2+1-x)}{(x^2-x+1)} \cdot dx = \int (x^2+x+1) \cdot dx$$

$$[\because \; (a^2-b^2) = (a-b)(a+b)]$$

$$= \int x^2 \cdot dx + \int x \cdot dx + \int 1 \cdot dx = \frac{x^3}{3} + \frac{x^2}{2} + x + c.$$

(iv) $\int \frac{x}{\sqrt{1+x}} dx = \int \frac{(1+x)-1}{\sqrt{1+x}} \cdot dx$ [Add and subtract 1 to the numerator]

$$= \int \left(\frac{1+x}{\sqrt{1+x}} - \frac{1}{\sqrt{1+x}} \right) \cdot dx = \int (1+x)^{1/2} \cdot dx - \int (1+x)^{-1/2} \cdot dx$$

$$= \frac{(1+x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right) \cdot 1} - \frac{(1+x)^{\frac{-1}{2}+1}}{\left(\frac{-1}{2}+1\right) \cdot 1} + c = \frac{(1+x)^{3/2}}{3/2} - \frac{(1+x)^{1/2}}{1/2} + c$$

$$= \frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + c.$$

Example 8. *Evaluate the following integrals :*

(i) $\int \frac{1}{\sqrt{2x+1}+\sqrt{2x+2}} dx$ (ii) $\int \frac{2x}{\sqrt{a+x}+\sqrt{a-x}} dx$

(iii) $\int x\sqrt{5x-2}\, dx$ (iv) $\int \left(a^{5x-3} + e^{2x-3}\right) dx$.

Solution. (i) Let $I = \int \frac{1}{\sqrt{2x+1}+\sqrt{2x+2}} dx$

$$= \int \frac{1}{\sqrt{2x+1}+\sqrt{2x+2}} \times \frac{\sqrt{2x+1}-\sqrt{2x+2}}{\sqrt{2x+1}-\sqrt{2x+2}} dx \qquad \text{[On rationalization]}$$

$$= \int \frac{\sqrt{2x+1}-\sqrt{2x+2}}{(2x+1)-(2x+2)} dx \qquad [\because \; (a+b)(a-b) = a^2-b^2]$$

$$= \int \frac{\sqrt{2x+1}-\sqrt{2x+2}}{-1} dx = \int \sqrt{2x+2}\, dx - \int \sqrt{2x+1}\, dx$$

$$= \int (2x+2)^{1/2} dx - \int (2x+1)^{1/2} dx$$

$$\left[\because \; \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a \cdot (n+1)} + c\right]$$

$$= \frac{(2x+2)^{\frac{1}{2}+1}}{2.\left(\frac{1}{2}+1\right)} - \frac{(2x+1)^{\frac{1}{2}+1}}{2.\left(\frac{1}{2}+1\right)} + c$$

$$= \frac{1}{3}(2x+2)^{3/2} - \frac{1}{3}(2x+1)^{3/2} + c$$

$$= \frac{1}{3}[(2x+2)^{3/2} - (2x+1)^{3/2}] + c.$$

(*ii*) $\int \frac{2x}{\sqrt{a+x}+\sqrt{a-x}}\,dx$

$$= \int \frac{2x}{\sqrt{a+x}+\sqrt{a-x}} \times \frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}\,dx \qquad \text{[On rationalization]}$$

$$= \int \frac{2x(\sqrt{a+x}-\sqrt{a-x})}{(a+x)-(a-x)}\,dx \qquad [\because\ (a+b)(a-b) = a^2 - b^2]$$

$$= \int \frac{2x(\sqrt{a+x}-\sqrt{a-x})}{2x}\,dx = \int \sqrt{a+x}\,dx - \int \sqrt{a-x}\,dx$$

$$= \int (a+x)^{1/2}\,dx - \int (a-x)^{1/2}\,dx$$

$$= \frac{(a+x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)} - \frac{(a-x)^{\frac{1}{2}+1}}{(-1)\left(\frac{1}{2}+1\right)} + c \qquad \left[\because\ \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c\right]$$

$$= \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(a-x)^{3/2} + c$$

$$= \frac{2}{3}[(a+x)^{3/2} + (a-x)^{3/2}] + c.$$

(*iii*) Let $\quad I = \int x\sqrt{5x-2}\,dx$

$$= \frac{1}{5}\int 5x\sqrt{5x-2}\,dx \qquad \text{[Multiply and divided by 5]}$$

$$= \frac{1}{5}\int (5x-2+2)\sqrt{5x-2}\,dx \qquad \text{[Add and subtract 2]}$$

$$= \frac{1}{5}\int (5x-2)\sqrt{5x-2}\,dx + \frac{2}{5}\int \sqrt{5x-2}\,dx$$

$$= \frac{1}{5}\int (5x-2)^{3/2}\,dx + \frac{2}{5}\int (5x-2)^{1/2}\,dx$$

$$= \frac{1}{5}\frac{(5x-2)^{\frac{3}{2}+1}}{5.\left(\frac{3}{2}+1\right)} + \frac{2}{5}\frac{(5x-2)^{\frac{1}{2}+1}}{5.\left(\frac{1}{2}+1\right)}$$

$$\left[\because \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a.(n+1)} + c\right]$$

$$= \frac{2}{125}(5x-2)^{5/2} + \frac{4}{75}(5x-2)^{3/2} + c.$$

(iv) $\int (a^{5x-3} + e^{2x-3})\,dx$

$$= \int a^{5x-3}\,dx + \int e^{2x-3}\,dx$$

$$= \frac{a^{5x-3}}{5\log a} + \frac{e^{2x-3}}{2} + c.$$

$$\left[\begin{array}{l}\because \int a^{bx+c}\,dx = \dfrac{a^{bx+c}}{b\log a} + c \\ \int e^{ax+b}\,dx = \dfrac{e^{ax+b}}{a} + c\end{array}\right]$$

Example 9. *Evaluate the following integrals :*

(i) $\int log_x x . dx$ (ii) $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} . dx$

(iii) $\int \frac{(2x+1)^2}{x-2} . dx$ (iv) $\int \frac{1}{\sqrt{5x+3}+\sqrt{5x-2}} . dx$

(v) $\int \frac{e^{5\log_e x} - e^{4\log_e x}}{e^{3\log_e x} - e^{2\log_e x}} . dx$ (vi) $\int \frac{1}{\sqrt{x+1}-\sqrt{x-2}} . dx.$

Solution. (i) $\int \log_x x . dx = \int 1 . dx = x + c.$ $[\because \log_a . a = 1]$

(ii) $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} . dx$

$$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{(\sqrt{x+a}-\sqrt{x+b})}{(\sqrt{x+a}-\sqrt{x+b})} . dx \quad \text{[On rationalization]}$$

$$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})}{(x+a)-(x+b)} . dx \quad [\because (a^2-b^2) = (a+b)(a-b)]$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b} . dx$$

$$= \frac{1}{a-b}\int (x+a)^{1/2} . dx - \frac{1}{a-b}\int (x+b)^{1/2} . dx$$

$$= \frac{1}{a-b} . \frac{(x+a)^{3/2}}{3/2} - \frac{1}{a-b}\frac{(x+b)^{3/2}}{3/2} + c$$

$$= \frac{1}{a-b} \cdot \frac{2}{3} (x+a)^{3/2} - \frac{1}{a-b} \cdot \frac{2}{3} (x+b)^{3/2} + c$$

$$= \frac{2}{3} \frac{(x+a)^{3/2}}{a-b} - \frac{2}{3} \cdot \frac{(x+b)^{3/2}}{a-b} + c.$$

(*iii*) $\int \frac{(2x+1)^2}{x-2}\, dx$.

Note. If the integrand is a rational function whose numerator and denominator are polynomials and degree of numerator is greater than or equal to that of the denominator, then, first divide the numerator by the denominator and then use the result :

$$\left[\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}\right]$$

$$\int \frac{(2x+1)^2}{x-2} . dx = \int \frac{4x^2+4x+1}{x-2} . dx$$

$$= \int \left[(4x+12) + \frac{25}{x-2}\right] . dx$$

$$= 4 \int x . dx + 12 \int 1 . dx + 25 \int \frac{1}{x-2} . dx$$

$$\begin{array}{r|l} x-2 & 4x^2+4x+1 \;(\; 4x+12 \\ & \underline{+4x^2-8x} \\ & \quad - \quad + \\ & \qquad 12x+1 \\ & \qquad \underline{+12x-24} \\ & \qquad\quad - \quad + \\ & \qquad\qquad \underline{25} \end{array}$$

$$= \frac{4x^2}{2} + 12x + 25 \log | x-2 | + c = 2x^2 + 12x + 25 \log | x-2 | + c.$$

(*iv*) $\int \frac{1}{\sqrt{5x+3}+\sqrt{5x-2}} . dx$

$$= \int \left[\frac{1}{\sqrt{5x+3}+\sqrt{5x-2}} \times \frac{\sqrt{5x+3}-\sqrt{5x-2}}{\sqrt{5x+3}-\sqrt{5x-2}}\right] . dx \quad \text{[On rationalization]}$$

$$= \int \frac{\sqrt{5x+3}-\sqrt{5x-2}}{(5x+3)-(5x-2)} . dx \qquad [\because \; (a^2-b^2) = (a+b)(a-b)]$$

$$= \int \frac{\sqrt{5x+3}-\sqrt{5x-2}}{5}\, dx = \frac{1}{5}\left[\int \sqrt{5x+3} . dx - \int \sqrt{5x-2} . dx\right]$$

$$= \frac{1}{5}\left[\int (5x+3)^{1/2} . dx - \int (5x-2)^{1/2} . dx\right]$$

$$= \frac{1}{5}\left[\frac{(5x+3)^{3/2}}{\frac{3}{2}.5} - \frac{(5x-2)^{3/2}}{\frac{3}{2}.5}\right] + c$$

$$= \frac{1}{5}\left[\frac{2}{15}(5x+3)^{3/2} - \frac{2}{15}(5x-2)^{3/2}\right] + c$$

$$= \frac{2}{75}\left[(5x+3)^{3/2} - (5x-2)^{3/2}\right] + c.$$

(v) $\int \frac{e^{5\log_e x} - e^{4\log_e x}}{e^{3\log_e x} - e^{2\log_e x}} . dx$

$$= \int \frac{e^{\log_e x^5} - e^{\log_e x^4}}{e^{\log_e x^3} - e^{\log_e x^2}} . dx \qquad [\because \; m \log_e n = \log_e n^m]$$

$$= \int \frac{x^5 - x^4}{x^3 - x^2} . dx \qquad [\because \; e^{\log_e f(x)} = f(x)]$$

$$= \int \frac{x^4(x-1)}{x^2(x-1)} . dx = \int x^2 . dx = \frac{x^3}{3} + c.$$

(vi) $\int \frac{1}{\sqrt{x+1} - \sqrt{x-2}} . dx$

$$= \int \frac{1}{\sqrt{x+1} - \sqrt{x-2}} \times \frac{\sqrt{x+1} + \sqrt{x-2}}{\sqrt{x+1} + \sqrt{x-2}} . dx \qquad \text{[On rationalization]}$$

$$= \int \frac{\sqrt{x+1} + \sqrt{x-2}}{(x+1) - (x-2)} . dx \qquad [\because \; (a^2 - b^2) = (a+b)(a-b)]$$

$$= \int \frac{\sqrt{x+1} + \sqrt{x-2}}{3} . dx = \frac{1}{3}\left[\int (x+1)^{1/2} . dx + \int (x-2)^{1/2} . dx\right]$$

$$= \frac{1}{3}\left[\frac{(x+1)^{3/2}}{\frac{3}{2}.1} + \frac{(x-2)^{3/2}}{\frac{3}{2}.1}\right] + c = \frac{1}{3}\left[\frac{2}{3}(x+1)^{3/2} + \frac{2}{3}(x-2)^{3/2}\right] + c$$

$$= \frac{2}{9}\left[(x+1)^{3/2} + (x-2)^{3/2}\right] + c.$$

Example 10. *Evaluate the following integrals :*

(i) $\int \frac{x^6 + 1}{x^2 + 1} . dx$ (ii) $\int \frac{x^4}{x+1} . dx.$

Solution. First, see note given in the part (*iii*) of above example.

(i) $\int \frac{x^6 + 1}{x^2 + 1} . dx = \int \frac{x^6 + 1}{x^2 + 1} . dx$

$$= \int \left(x^4 - x^2 + 1 + \frac{0}{x^2 + 1}\right) . dx$$

$$= \int (x^4 - x^2 + 1) . dx$$

$$= \int x^4\, dx - \int x^2 \,.\, dx + \int 1 \,.\, dx$$

$$= \frac{x^5}{5} - \frac{x^3}{3} + x + c.$$

$$\begin{array}{r l}
x^2+1 \,\big)\, x^6+1 \,\big(\, & x^4 - x^2 + 1 \\
x^6 + x^4 & \\
- \quad - & \\
\hline
-x^4 + 1 & \\
-x^4 - x^2 & \\
+ \quad + & \\
\hline
x^2 + 1 & \\
x^2 + 1 & \\
- \quad - & \\
\hline
0 &
\end{array}$$

(*ii*) $\displaystyle\int \frac{x^4}{x+1} \,.\, dx = \int \frac{x^4}{x+1} \,.\, dx$

$$= \int \left(x^3 - x^2 + x - 1 + \frac{1}{x+1} \right) . \, dx$$

$$= \int x^3 \,.\, dx - \int x^2 \,.\, dx + \int x \,.\, dx - \int 1 \,.\, dx + \int \frac{1}{x+1} \,.\, dx$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + \log | x + 1 | + c.$$

$$\begin{array}{r l}
x+1 \,\big)\, x^4 \,\big(\, & x^3 - x^2 + x - 1 \\
x^4 + x^3 & \\
- \quad - & \\
\hline
-x^3 & \\
-x^3 - x^2 & \\
+ \quad + & \\
\hline
x^2 & \\
x^2 + x & \\
- \quad - & \\
\hline
-x & \\
-x - 1 & \\
+ \quad + & \\
\hline
1 &
\end{array}$$

Example 11. *If* $f'(x) = \dfrac{1}{x} + \dfrac{1}{1+x^2}$ *and* $f(1) = \dfrac{\pi}{4}$, *find* $f(x)$.

Solution. We have $\quad f(x) = \displaystyle\int f'(x) \,.\, dx = \int \left(\frac{1}{x} + \frac{1}{1+x^2} \right) . \, dx$

$$= \int \frac{1}{x} \,.\, dx + \int \frac{1}{1+x^2} \,.\, dx$$

$\Rightarrow \qquad f(x) = \log | x | + \tan^{-1} x + c \qquad$ ...(1)

when $x = 1$, $f(1) = \dfrac{\pi}{4}$

$\Rightarrow \qquad \dfrac{\pi}{4} = \log 1 + \tan^{-1} 1 + c$

$\Rightarrow \qquad \dfrac{\pi}{4} = 0 + \dfrac{\pi}{4} + c \Rightarrow c = 0$

$\therefore \qquad f(x) = \log | x | + \tan^{-1} x.$

Example 12. *If $f''(x) = a \sin x + b \cos x$ and $f'(0) = 4$, $f(0) = 3$, $f\left(\frac{\pi}{2}\right) = 5$, find $f(x)$.*

Solution. We have $f(x) = \int f'(x) \, . \, dx$

$= \int (a \sin x + b \cos x) \, . \, dx$

$= a \int \sin x \, dx + b \int \cos x \, . \, dx$

$\Rightarrow \quad f(x) = -a \cos x + b \sin x + c$ ...(1)

when $x = 0, f'(0) = 4$

$\Rightarrow \quad 4 = a \sin 0 + b \cos 0$

$\Rightarrow \quad 4 = 0 + b \Rightarrow b = 4$ ...(2)

when $x = 0, f(0) = 3$

By using equation (1)

$\Rightarrow \quad 3 = -a \cos 0 + b \sin 0 + c$

$\Rightarrow \quad 3 = -a + c$

$\Rightarrow \quad c - a = 3$ ...(3)

when $x = \frac{\pi}{2}, f\left(\frac{\pi}{2}\right) = 5$

$\therefore$ By using equation (1)

$\Rightarrow \quad 5 = -a \cos \frac{\pi}{2} + b \sin \frac{\pi}{2} + c$

$\Rightarrow \quad 5 = -a(0) + b(1) + c$

$\Rightarrow \quad 5 = b + c$

$\Rightarrow \quad 5 = 4 + c$ [By using equation (2)]

$\Rightarrow \quad c = 1$ ...(4)

Substituting the vlaue of c in equation (3), we have

$\Rightarrow \quad 1 - a = 3 \quad \Rightarrow \quad a = -2$

Now, substituting the values of a, b and c in equation (1), we have

$f(x) = 2 \cos x + 4 \sin x + 1.$

1.8 INTEGRATION OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

Example 13. *Evaluate the following integrals :*

(i) $\int \tan^2 x \, . \, dx$ *(ii)* $\int \sqrt{1 - \sin 2x} \, . \, dx$

(iii) $\int (\sin x + \cos x) \, . \, dx$ *(iv)* $\int \operatorname{cosec} x (\operatorname{cosec} x + \cot x) \, . \, dx$

(v) $\int \frac{1 - \sin x}{\cos^2 x} \, . \, dx.$

Solution. (*i*) $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$ $[\because \ \sec^2 A - \tan^2 A = 1]$

$= \int \sec^2 x . dx - \int 1 . dx = \tan x - x + c.$

(*ii*) $\int \sqrt{1-\sin 2x}\, .\, dx = \int [(\cos^2 x + \sin^2 x) - 2 \sin x \cos x]^{1/2} . dx$

$[\because \ \cos^2 A + \sin^2 A = 1, \sin 2A = 2 \sin A \cos A]$

$= \int [(\cos x - \sin x)^2]^{1/2} . dx = \int (\cos x - \sin x) . dx$

$= \int \cos x\, dx - \int \sin x\, dx = \sin x - (-\cos x) + c$

$= \sin x + \cos x + c.$

(*iii*) $\int (\sin x + \cos x) . dx = \int \sin x\, dx + \int \cos x\, dx$

$= -\cos x + \sin x + c.$

(*iv*) $\int \operatorname{cosec} x (\operatorname{cosec} x + \cot x) . dx = \int \operatorname{cosec}^2 x\, dx + \int \operatorname{cosec} x \cot x . dx$

$= -\cot x - \operatorname{cosec} x + c.$

(*v*) $\int \frac{1-\sin x}{\cos^2 x} . dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) . dx$

$= \int (\sec^2 x - \sec x \tan x) . dx$

$= \int \sec^2 x . dx - \int \sec x \tan x . dx$

$= \tan x - \sec x + c.$

Example 14. *Evaluate the following integrals :*

(*i*) $\int (6 \sec^2 x - 7 \operatorname{cosec}^2 x + 3e^x)\, dx$ (*ii*) $\int \tan(3z - 5) \sec(3z - 5)\, dz$

(*iii*) $\int \sin^2 \frac{x}{2} dx$ (*iv*) $\int \sin^3 (2x + 1)\, dx$

(*v*) $\int \cos^4 2x\, dx$.

Solution. (*i*) $\int (6 \sec^2 x - 7 \operatorname{cosec}^2 x + 3e^x)\, dx$

$= 6 \int \sec^2 x\, dx - 7 \int \operatorname{cosec}^2 x\, dx + 3 \int e^x\, dx$

$= 6 \tan x - 7(-\cot x) + 3e^x + c = 6 \tan x + 7 \cot x + 3e^x + c.$

(*ii*) $\int \tan (3z - 5) \sec (3z - 5)\, dz$

$= \frac{\sec (3 - 2z)}{-2} + c.$ $\left[\because \ \int \sec x \tan x\, dx = \sec x + c\right]$

(*iii*) $\int \sin^2 \frac{x}{2} dx = \frac{1}{2} \int 2 \sin^2 \frac{x}{2}\, dx$ [Multiply and divided by 2]

$= \frac{1}{2} \int (1 - \cos x)\, dx$ $\left[\begin{array}{l} \because \ 1 - \cos 2A = 2 \sin^2 A \\ \Rightarrow \ 1 - \cos A = 2 \sin^2 \frac{A}{2} \end{array}\right]$

$$= \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos x\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\sin x + c = \frac{1}{2}(x - \sin x) + c.$$

(*iv*) $\int \sin^3 (2x+1)\,dx$

$$= \int \left[\frac{3}{4}\sin(2x+1) - \frac{1}{4}\sin 3(2x+1)\right]dx$$

$$\left[\begin{array}{l} \because\ \sin 3A = 3\sin A - 4\sin^3 A \\ \Rightarrow\ 4\sin^3 A = 3\sin A - \sin 3A \\ \Rightarrow\ \sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A \end{array}\right]$$

$$= \frac{3}{4}\int \sin(2x+1)\,dx - \frac{1}{4}\int \sin(6x+3)\,dx$$

$$= \frac{3}{4}\left[\frac{-\cos(2x+1)}{2}\right] - \frac{1}{4}\left[\frac{-\cos(6x+3)}{6}\right] + c$$

$$= -\frac{3}{8}\cos(2x+1) + \frac{1}{24}\cos(6x+3) + c.$$

(*v*) $\int \cos^4 2x\,dx = \int (\cos^2 2x)^2\,dx$

$$= \int \left(\frac{1+\cos 4x}{2}\right)^2 dx$$

$$\left[\begin{array}{l} \because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ 1+\cos 4A = 2\cos^2 2A \\ \Rightarrow\ \left(\frac{1+\cos 4A}{2}\right) = \cos^2 2A \end{array}\right]$$

$$= \frac{1}{4}\int (1 + 2\cos 4x + \cos^2 4x)\,dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 4x + \frac{1+\cos 8x}{2}\right)dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 4x + \frac{1}{2} + \frac{1}{2}\cos 8x\right)dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 4x + \frac{1}{2}\cos 8x\right)dx$$

$$= \frac{3}{8}\int dx + \frac{1}{2}\int \cos 4x\,dx + \frac{1}{8}\int \cos 8x\,dx$$

$$= \frac{3}{8}x + \frac{1}{2}\left(\frac{\sin 4x}{4}\right) + \frac{1}{8}\left(\frac{\sin 8x}{8}\right) + c$$

$$= \frac{3}{8}x + \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x + c.$$

Example 15. *Evaluate the following integrals :*

(i) $\int \frac{\sin^2 x}{1+\cos x}\,dx$ (ii) $\int \frac{1}{1+\sin x}\,.\,dx$

(iii) $\int \frac{1}{\sin^2 x \cos^2 x}\,.\,dx$ (iv) $\int \sin x\,\sqrt{1+\cos 2x}\,.\,dx$

(v) $\int \frac{2+3\cos x}{\sin^2 x}\,dx.$

Solution. (i) $\int \frac{\sin^2 x}{1+\cos x}\,.\,dx = \int \frac{1-\cos^2 x}{1+\cos x}\,dx$ $[\because \sin^2 A + \cos^2 A = 1]$

$$= \int \frac{(1+\cos x)(1-\cos x)}{1+\cos x}\,dx = \int (1-\cos x)\,.\,dx$$

$$= \int 1\,.\,dx - \int \cos x\,dx = x - \sin x + c.$$

(ii) $\int \frac{1}{1+\sin x}\,dx = \int \left[\frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\right].\,dx$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\,dx \qquad [\because a^2 - b^2 = (a+b)(a-b)]$$

$$= \int \frac{1-\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx - \int \frac{\sin x}{\cos^2 x}\,dx$$

$[\because \sin^2 A + \cos^2 A = 1]$

$$= \int \sec^2 x\,dx - \int \sec x \tan x\,dx = \tan x - \sec x + c.$$

(iii) $\int \frac{1}{\sin^2 x \cos^2 x}\,.\,dx = 4\int \frac{1}{(2\sin x\cos x)^2}\,.\,dx$ [Multiply and divided by 4]

$$= 4\int \frac{1}{(\sin 2x)^2}\,dx = 4\int \text{cosec}^2\, 2x\,dx$$

$[\because \sin 2A = 2\sin A\cos A]$

$$= \frac{4(-\cot 2x)}{2} + c = -2\cot 2x + c.$$

(iv) $\int \sin x\,.\,\sqrt{1+\cos 2x}\,.\,dx$

$$= \int \sin x\,.\,\sqrt{2\cos^2 x}\,dx \qquad [\because \cos 2A = 2\cos^2 A - 1]$$

$$= \sqrt{2}\int \sin x\,.\cos x\,dx$$

$$= \frac{\sqrt{2}}{2}\int 2\sin x\cos x\,dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{\sqrt{2}}\int \sin 2x\,dx \qquad [\because \sin 2A = 2\sin A\cos A]$$

$$= \frac{1}{\sqrt{2}}\left(\frac{-\cos 2x}{2}\right) + c = \frac{-1}{2\sqrt{2}} \cos 2x + c.$$

(v) $\int \frac{2 + 3\cos x}{\sin^2 x}\, dx = \int \left(\frac{2}{\sin^2 x} + \frac{3\cos x}{\sin^2 x}\right) . dx$

$$= \int \frac{2}{\sin^2 x} dx + 3 \int \frac{\cos x}{\sin^2 x} dx$$

$$= 2 \int \operatorname{cosec}^2 x\, dx + 3 \int \cot x \operatorname{cosec} x\, dx$$

$$= -2 \cot x - 3 \operatorname{cosec} x + c.$$

Example 16. *Integrate the following functions :*

(i) $(\tan x + \cot x)^2$ (ii) $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

(iii) $\frac{\sec^2 x}{\operatorname{cosec}^2 x}$ (iv) $(2 \tan x - 3 \cot x)^2$

(v) $\frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x}$.

Solution. (i) $\int (\tan x + \cot x)^2 . dx$

$$= \int (\tan^2 x + \cot^2 x + 2 \tan x \cot x) . dx$$

$[\because \ \sec^2 A - \tan^2 A = 1, \operatorname{cosec}^2 A - \cot^2 A = 1]$

$$= \int \left(\sec^2 x - 1 + \operatorname{cosec}^2 x - 1 + 2 . \tan x . \frac{1}{\tan x}\right) . dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x - 2 + 2) . dx = \int (\sec^2 x + \operatorname{cosec}^2 x) dx$$

$$= \int \sec^2 x . dx + \int \operatorname{cosec}^2 x . dx = \tan x - \cot x + c.$$

(ii) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} . dx$ $[\because \ \cos 2A = 2\cos^2 A - 1]$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{(\cos x - \cos \alpha)} . dx = \int \frac{2\cos^2 x - 2\cos^2 \alpha}{\cos x - \cos \alpha} . dx$$

$$= 2 \int \frac{(\cos^2 x - \cos^2 \alpha)}{\cos x - \cos \alpha} . dx = 2 \int \frac{(\cos x + \cos \alpha)(\cos x - \cos \alpha)}{(\cos x - \cos \alpha)} . dx$$

$$= 2 \int (\cos x + \cos \alpha) . dx = 2 \int \cos x . dx + 2 \int \cos \alpha . dx$$

$$= 2 \int \cos x\, dx + 2 \cos \alpha \int 1 . dx = 2 \sin x + 2x \cos \alpha + c.$$

(iii) $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} . dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \tan^2 x\, dx$ $[\because \ \sec^2 A - \tan^2 A = 1]$

$$= \int (\sec^2 x - 1)\, dx = \int \sec^2 x . dx - \int 1 . dx = \tan x - x + c.$$

(iv) $\int (2\tan x - 3\cot x)^2 \, . \, dx$

$$= \int [(2\tan x)^2 + (3\cot x)^2 - 2(2\tan x)(3\cot x)] \, . \, dx$$

$$= \int (4\tan^2 x + 9\cot^2 x - 12\tan x\cot x) \, . \, dx$$

$$= \int [4(\sec^2 x - 1) + 9(\text{cosec}^2 x - 1) - 12] \, . \, dx$$

$$\left[\because \tan x \cot x = \tan x \, . \, \frac{1}{\tan x} = 1\right]$$

$$= \int (4\sec^2 x - 4 + 9\,\text{cosec}^2 x - 9 - 12) \, . \, dx$$

$$= \int (4\sec^2 x + 9\,\text{cosec}^2 x - 25) \, . \, dx$$

$$= 4\int \sec^2 x \, dx + 9\int \text{cosec}^2 x \, . \, dx - 25\int 1 \, . \, dx$$

$$= 4\tan x - 9\cot x - 25x + c.$$

(v) $\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x} \, . \, dx$

$$= \int \left(\frac{\sin^3 x}{\sin^2 x\cos^2 x} - \frac{\cos^3 x}{\sin^2 x \cos^2 x}\right) . \, dx = \int \left(\frac{\sin x}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right) . \, dx$$

$$= \int \frac{1}{\cos x} \, . \, \frac{\sin x}{\cos x} \, dx - \int \frac{1}{\sin x} \, . \, \frac{\cos x}{\sin x} \, dx$$

$$= \int \sec x \tan x \, dx - \int \text{cosec}\, x \cot x \, . \, dx = \sec x + \text{cosec}\, x + c.$$

Example 17. *Evaluate the following integrals :*

(i) $\int \frac{\cos^2 x - \sin^2 x}{\sqrt{1 + \cos 4x}} \, dx$ *(ii)* $\int \frac{3\cos x + 4}{\sin^2 x} \, . \, dx$

(iii) $\int \sqrt{1 + \sin \frac{x}{2}} \, . \, dx$ *(iv)* $\int \frac{4 + 5\sin x}{\cos^2 x} \, . \, dx$

(v) $\int \sqrt{1 - \sin x} \, . \, dx$ *(vi)* $\int \frac{\sin^2 x}{(1 + \cos x)^2} \, . \, dx$

(vii) $\int \frac{\sec x}{\sec x + \tan x} \, . \, dx.$

Solution. *(i)* $\int \frac{\cos^2 x - \sin^2 x}{\sqrt{1 + \cos 4x}} \, . \, dx = \int \frac{\cos 2x}{\sqrt{2\cos^2 2x}} \, . \, dx = \frac{1}{\sqrt{2}} \int \frac{\cos 2x}{\cos 2x} \, . \, dx$

$$= \frac{1}{\sqrt{2}} \int 1 \, . \, dx = \frac{x}{\sqrt{2}} + c.$$

(ii) $\int \frac{3\cos x + 4}{\sin^2 x} \, . \, dx = \int \left[\frac{3\cos x}{\sin^2 x} + \frac{4}{\sin^2 x}\right] . \, dx$

$$= 3\int \frac{\cos x}{\sin^2 x}\,.\,dx + 4\int \frac{1}{\sin^2 x}\,.\,dx$$

$$= 3\int \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\,dx + 4\int \operatorname{cosec}^2 x\,dx$$

$$= 3\int \operatorname{cosec} x \cot x\,.\,dx + 4\int \operatorname{cosec}^2 x\,dx$$

$$= -\,3\operatorname{cosec} x - 4\cot x + c.$$

(*iii*) $\int \sqrt{1+\sin\frac{x}{2}}\,.\,dx$

$$= \int \sqrt{\cos^2\frac{x}{4} + \sin^2\frac{x}{4} + 2\sin\frac{x}{4}\cos\frac{x}{4}}\,.\,dx \qquad \left[\begin{array}{l} \because\ \cos^2 A + \sin^2 A = 1 \\ \quad\ \sin 2A = 2\sin A\cos A \\ \Rightarrow\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \end{array}\right]$$

$$= \int \sqrt{\left(\cos\frac{x}{4} + \sin\frac{x}{4}\right)^2}\,.\,dx = \int \left(\cos\frac{x}{4} + \sin\frac{x}{4}\right).\,dx$$

$$= \int \cos\frac{x}{4}\,.\,dx + \int \sin\frac{x}{4}\,.\,dx = \frac{(\sin x/4)}{1/4} + \frac{(-\cos x/4)}{1/4} + c$$

$$= 4\sin\frac{x}{4} - 4\cos\frac{x}{4} + c = 4\left(\sin\frac{x}{4} - \cos\frac{x}{4}\right) + c.$$

(*iv*) $\int \frac{4+5\sin x}{\cos^2 x}\,.\,dx$

$$= \int \left(\frac{4}{\cos^2 x} + \frac{5\sin x}{\cos^2 x}\right).\,dx = \int \frac{4}{\cos^2 x}\,.\,dx + \int 5\,\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\,.dx$$

$$= 4\int \sec^2 x\,.\,dx + 5\int \sec x\tan x\,.\,dx = 4\tan x + 5\sec x + c.$$

(*v*) $\int \sqrt{1-\sin x}\,.\,dx$

$$= \int \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\cos\frac{x}{2}\sin\frac{x}{2}}\,.\,dx \qquad \left[\begin{array}{l} \because\ \cos^2 A + \sin^2 A = 1 \\ \quad\ \sin 2A = 2\sin A\cos A \\ \Rightarrow\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \end{array}\right]$$

$$= \int \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}\,.\,dx = \int \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right).\,dx$$

$$= \int \cos\frac{x}{2}\,.\,dx - \int \sin\frac{x}{2}\,.\,dx = \frac{(\sin x/2)}{1/2} - \frac{(-\cos x/2)}{1/2} + c$$

$$= 2\sin\frac{x}{2} + 2\cos\frac{x}{2} + c = 2\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + c.$$

(*vi*) $\int \frac{\sin^2 x}{(1+\cos x)^2} \, . \, dx$

$= \int \left(\frac{\sin x}{1+\cos x}\right)^2 . \, dx$ $\left[\begin{array}{l} \because \ \sin 2A = 2 \sin A \cos A \\ \Rightarrow \ \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \end{array}\right]$

$= \int \left(\frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}}\right)^2 . \, dx$ $\left[\begin{array}{l} \because \ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow \ 1+\cos A = 2 \cos^2 \frac{A}{2} \end{array}\right]$

$= \int \left(\frac{\sin x/2}{\cos x/2}\right)^2 . \, dx = \int \tan^2 \frac{x}{2} . \, dx$ $[\because \ \sec^2 A - \tan^2 A = 1]$

$= \int \left(\sec^2 \frac{x}{2} - 1\right) . \, dx = \int \sec^2 \frac{x}{2} . \, dx - \int 1 . \, dx$

$= \frac{\tan x/2}{1/2} - x + c = 2 \tan \frac{x}{2} - x + c.$

(*vii*) $\int \frac{\sec x}{\sec x + \tan x} . \, dx = \int \frac{\sec x}{(\sec x + \tan x)} \times \frac{(\sec x - \tan x)}{(\sec x - \tan x)} . \, dx$ [On rationalization]

$= \int \frac{\sec^2 x - \sec x \tan x}{\sec^2 x - \tan^2 x} . \, dx$ $[\because \ a^2 - b^2 = (a-b)(a+b)]$

$= \int (\sec^2 x - \sec x \tan x) . \, dx$ $[\because \ \sec^2 A - \tan^2 A = 1]$

$= \int \sec^2 x . \, dx - \int \sec x . \tan x . \, dx$

$= \tan x - \sec x + c.$

Example 18. *Evaluate the following :*

(*i*) $\int \frac{\cos x - \cos 2x}{1 - \cos x} . \, dx$ (*ii*) $\int \sin x \cos x (\sin 2x + \cos 2x) . \, dx$

(*iii*) $\int \frac{\tan x}{(\sec x + \tan x)} . \, dx$ (*iv*) $\int \left(\frac{1-\cos 2x}{1+\cos 2x}\right) . \, dx.$

Solution. (*i*) $\int \frac{\cos x - \cos 2x}{1-\cos x} . \, dx = \int \frac{\cos x - (2\cos^2 x - 1)}{1 - \cos x} . \, dx$

$= \int \frac{-2\cos^2 x + 1 + \cos x}{1 - \cos x} dx = \int \frac{-(2\cos^2 x - \cos x - 1)}{-(\cos x - 1)} . \, dx$

$= \int \frac{(2\cos x + 1)(\cos x - 1)}{(\cos x - 1)} . \, dx = \int (2 \cos x + 1) . \, dx$

$= 2 \int \cos x . \, dx + \int 1 . \, dx = 2 \sin x + x + c.$

(*ii*) $\int \sin x \cos x (\sin 2x + \cos 2x) \, . \, dx$

$$= \frac{1}{2}\int (2 \sin x \cos x) \, . \, (\sin 2x + \cos 2x) \, . \, dx \quad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int \sin 2x (\sin 2x + \cos 2x) \, . \, dx \quad [\because \ \sin 2A = 2 \sin A \cos A]$$

$$= \frac{1}{2}\int (\sin^2 2x + \sin 2x \cos 2x) \, . \, dx$$

$$= \frac{1}{4}\int (2 \sin^2 2x + 2 \sin 2x \cos 2x) \, . \, dx$$

[Multiply and divided by 2 again]

$$= \frac{1}{4}\int (1 - \cos 4x + \sin 4x) \, . \, dx \quad [\because \ 1 - \cos 2A = 2 \sin^2 A]$$

$$= \frac{1}{4}\left[\int 1 \, . \, dx - \int \cos 4x \, . \, dx + \int \sin 4x \, . \, dx\right]$$

$$= \frac{1}{4}\left[x - \frac{\sin 4x}{4} - \frac{\cos 4x}{4}\right] + c.$$

(*iii*) Please try yourself. [**Ans.** $\sec x - \tan x + x + c$]

(*iv*) $\int \frac{1 - \cos 2x}{1 + \cos 2x} \, . \, dx$

$$= \int \frac{2 \sin^2 x}{2 \cos^2 x} \, . \, dx = \int \tan^2 x \, dx \quad \left[\begin{array}{l} \because \ 1 - \cos 2A = 2 \sin^2 A \\ 1 + \cos 2A = 2 \cos^2 A \end{array}\right]$$

$$= \int (\sec^2 x - 1) \, . \, dx = \int \sec^2 x \, . \, dx - \int 1 \, . \, dx$$

$$= \tan x - x + c.$$

Example 19. *Evaluate the following integrals :*

(*i*) $\int \left(\frac{x^2 + \sin^2 x}{1 + x^2}\right) . \sec^2 x \, . \, dx$ (*ii*) $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} \, . \, dx$

(*iii*) $\int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x} \, . \, dx$ (*iv*) $\int \frac{1}{1 + \sec x} \, dx$

(*v*) $\int \left(\sqrt{x} - \cos^2 \frac{x}{2}\right) dx.$

Solution. (*i*) $\int \left(\frac{x^2 + \sin^2 x}{1 + x^2}\right) . \sec^2 x \, . \, dx$

$$= \int \left[\frac{x^2 + (1 - \cos^2 x)}{1 + x^2}\right] . \sec^2 x \, . \, dx \quad [\because \ \sin^2 A + \cos^2 A = 1]$$

$$= \int \left(\frac{1 + x^2 - \cos^2 x}{1 + x^2}\right) . \sec^2 x \, . \, dx$$

$$= \int \left[\frac{1+x^2}{1+x^2} \cdot \sec^2 x - \frac{\cos^2 x}{1+x^2} \cdot \sec^2 x\right] . dx$$

$$= \int \sec^2 x \, . dx - \int \frac{1}{1+x^2} . dx = \tan x - \tan^{-1} x + c.$$

(ii) $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} . dx$ $\quad [\because \; (a^3 + b^3) = (a+b)^3 - 3ab(a+b)]$

$$= \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)}{\sin^2 x \, . \cos^2 x} . dx$$

$[\because \; \sin^2 A + \cos^2 A = 1]$

$$= \int \frac{1 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} . dx$$

$$= \int \left[\frac{1}{\sin^2 x \cos^2 x} - \frac{3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x}\right] . dx$$

$$= \int \left(\frac{1}{\sin^2 x \cos^2 x} - 3\right) . dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} . dx - 3 \int 1 . dx \qquad [\because \; \sin^2 A + \cos^2 A = 1]$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} . dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} . dx - 3 \int 1 . dx$$

$$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx - 3 \int 1 . dx = \tan x - \cot x - 3x + c.$$

(iii) $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} . dx$

$$= \int \frac{(\sin^4 x)^2 - (\cos^4 x)^2}{1 - 2\sin^2 x \cos^2 x} . dx$$

$$= \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{1 - 2\sin^2 x \cos^2 x} \qquad [\because \; (a^2 - b^2) = (a+b)(a-b)]$$

$$= \int \frac{(\sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x - 2\sin^2 x \cos^2 x)(\sin^4 x - \cos^4 x)}{1 - 2\sin^2 x \cos^2 x} . dx$$

[Add and subtract $2 \sin^2 x \cos^2 x$]

$$= \int \frac{[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x] . [(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)]}{1 - 2\sin^2 x \cos^2 x} . dx$$

$$= \int \frac{(1 - 2\sin^2 x \cos^2 x)(\sin^2 x - \cos^2 x)}{(1 - 2\sin^2 x \cos^2 x)} . dx \qquad [\because \; \sin^2 A + \cos^2 A = 1]$$

$$= \int (\sin^2 x - \cos^2 x) \,.\, dx$$

$$= \int - \cos 2x \, dx = - \frac{\sin 2x}{2} + c. \qquad [\because \cos 2A = \cos^2 A - \sin^2 A]$$

(*iv*) $$\int \frac{1}{1 + \sec x} \,.\, dx = \int \frac{1}{1 + \frac{1}{\cos x}} dx$$

$$= \int \frac{1}{\frac{\cos x + 1}{\cos x}} \,.\, dx = \int \frac{\cos x}{1 + \cos x} dx$$

$$= \int \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{1 + \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} \,.\, dx \qquad [\because \cos 2A = \cos^2 A - \sin^2 A]$$

$$= \int \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\left(1 - \sin^2 \frac{x}{2}\right) + \cos^2 \frac{x}{2}} \,.\, dx \qquad [\because \cos^2 A + \sin^2 A = 1]$$

$$= \int \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \,.\, dx = \int \frac{\cos^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \,.\, dx - \int \frac{\sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \,.\, dx$$

$$= \frac{1}{2} \int 1 \,.\, dx - \frac{1}{2} \int \tan^2 \frac{x}{2} \,.\, dx = \frac{1}{2} \int 1 \,.\, dx - \frac{1}{2} \int \left(\sec^2 \frac{x}{2} - 1\right) . \, dx$$

$$= \frac{1}{2} \int 1 \,.\, dx - \frac{1}{2} \int \sec^2 \frac{x}{2} \,.\, dx + \frac{1}{2} \int 1 \,.\, dx$$

$$= \frac{1}{2} x - \frac{1}{2} \frac{\tan x/2}{1/2} + \frac{1}{2} \,.\, x + c = x - \tan \frac{x}{2} + c.$$

(*v*) $$\int \left(\sqrt{x} - \cos^2 \frac{x}{2}\right) . \, dx$$

$$= \int \sqrt{x} \,.\, dx - \int \cos^2 x/2 \,.\, dx \qquad \left[\begin{array}{l} \because \cos 2A = 2 \cos^2 A - 1 \\ \Rightarrow \cos A = 2 \cos^2 \frac{A}{2} - 1 \end{array}\right]$$

$$= \int x^{1/2} \, dx - \int \frac{1 + \cos x}{2} \,.\, dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{1}{2} [x + \sin x] + c$$

$$= \frac{x^{3/2}}{3/2} - \frac{1}{2} x - \frac{1}{2} \sin x + c = \frac{2}{3} x^{3/2} - \frac{1}{2} x - \frac{1}{2} \sin x + c.$$

1.9 EVALUATION OF $\int \sin^n x\, dx$ AND $\int \cos^n x\, dx$

In order to evaluate $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$ for $n = 2, 3, 4$ the integrands $\sin^n x$ and $\cos^n x$ are expressed in terms of sines and cosines of multiples of x by using the following formulae :

(i) $\cos 2A = 1 - 2\sin^2 A$ (ii) $\cos 2A = 2\cos^2 A - 1$

(iii) $\sin 3A = 3\sin A - 4\sin^3 A$ (iv) $\cos 3A = 4\cos^3 A - 3\cos A$.

Example 20. *Evaluate the following integrals :*

(i) $\int \cos^2 nx\, dx$ (ii) $\int \cos^4 x\, dx$

(iii) $\int \sin^4 x\, dx$ (iv) $\int \sin^3 x\, dx$.

Solution. (i) $\int \cos^2 nx\, dx = \frac{1}{2}\int 2\cos^2 nx\, dx$ [Multiply and divided by 2]

$$= \frac{1}{2}\int (1 + \cos 2nx)\, dx \qquad [\because \cos 2A = 2\cos^2 A - 1]$$

$$= \frac{1}{2}\int 1 . dx + \frac{1}{2}\int \cos 2nx . dx = \frac{x}{2} + \frac{1}{2} . \frac{\sin 2nx}{2n} + c$$

$$= \frac{x}{2} + \frac{1}{4n} . \sin 2nx + c$$

(ii) $\int \cos^4 x . dx = \int (\cos^2 x)^2 . dx$

$$= \int \left(\frac{1 + \cos 2x}{2}\right)^2 . dx = \frac{1}{4}\int (1 + \cos^2 2x + 2\cos 2x) . dx$$

$$= \frac{1}{4}\int \left(1 + \frac{1 + \cos 4x}{2} + 2\cos 2x\right) . dx$$

$$= \frac{1}{4}\int \left(1 + \frac{1}{2} + \frac{1}{2}\cos 4x + 2\cos 2x\right) . dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + \frac{1}{2}\cos 4x + 2\cos 2x\right) . dx$$

$$= \int \left(\frac{3}{8} + \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x\right) . dx$$

$$= \frac{3}{8}\int 1 . dx + \frac{1}{8}\int \cos 4x . dx + \frac{1}{2}\int \cos 2x . dx$$

$$= \frac{3x}{8} + \frac{1}{8} \frac{\sin 4x}{4} + \frac{1}{2} . \frac{\sin 2x}{2} + c$$

$$= \frac{3x}{8} + \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + c.$$

(iii) $\int \sin^4 x . dx = \int (\sin^2 x)^2 . dx$ $[\because \cos 2A = 1 - 2\sin^2 A]$

$$= \int \left(\frac{1 - \cos 2x}{2}\right)^2 . dx = \frac{1}{4}\int (1 + \cos^2 2x - 2\cos 2x) dx$$

$$= \frac{1}{4}\int \left(1 + \frac{1 + \cos 4x}{2} - 2\cos 2x\right) . dx$$

$$= \frac{1}{4}\int \left(1+\frac{1}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right).dx$$
$$= \frac{1}{4}\int \left(\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right).dx$$
$$= \int \left(\frac{3}{8}+\frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x\right).dx$$
$$= \frac{3}{8}\int 1.dx + \frac{1}{8}\int \cos 4x.dx - \frac{1}{2}\int \cos 2x\, dx$$
$$= \frac{3x}{8} + \frac{1}{8}\frac{\sin 4x}{4} - \frac{1}{2}\frac{\sin 2x}{2} + c$$
$$= \frac{3x}{8} + \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + c.$$

(iv) $\int \sin^3 x.dx = \int \left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right).dx$

$$\left[\begin{array}{l} \because\ \sin 3A = 3\sin A - 4\sin^3 A \\ \Rightarrow 4\sin^3 A = 3\sin A - \sin 3A \\ \Rightarrow \sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A \end{array}\right]$$

$$= \frac{3}{4}\int \sin x.dx - \frac{1}{4}\int \sin 3x\, dx$$
$$= \frac{3}{4}(-\cos x) - \frac{1}{4}\left(\frac{-\cos 3x}{3}\right) + c = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + c.$$

Example 21. *Evaluate the following integrals :*

(i) $\int \sin^2 (2x+5).dx$ (ii) $\int \sin^4 x \cos^4 x.dx$

(iii) $\int \cos^3 x.dx.$

Solution. (i) $\int \sin^2 (2x+5).dx$

$$= \int \frac{1-\cos 2(2x+5)}{2}dx$$

$$\left[\begin{array}{l} \because\ \cos 2A = 1 - 2\sin^2 A \\ \Rightarrow 2\sin^2 A = 1 - \cos 2A \\ \Rightarrow \sin^2 A = \frac{1-\cos 2A}{2} \end{array}\right.$$

$$= \frac{1}{2}\int 1.dx - \frac{1}{2}\int \cos(4x+10).dx$$
$$= \frac{x}{2} - \frac{1}{2}\frac{\sin(4x+10)}{4} + c$$
$$= \frac{x}{2} - \frac{1}{8}\sin(4x+10) + c.$$

(ii) $\int \sin^4 x.\cos^4 x.dx = \int (\sin x \cos x)^4.dx$

$$= \frac{1}{16}\int (2\sin x\cos x)^4.dx = \frac{1}{16}\int (\sin 2x)^4.dx$$

[Multiply and divided by 16]

$$= \frac{1}{16}\int (\sin^2 2x)^2\, dx$$

$$= \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2 . dx \qquad \left[\begin{array}{l} \because \cos 2A = 1 - 2\sin^2 A \\ \Rightarrow 2\sin^2 A = 1 - \cos 2A \\ \Rightarrow \sin^2 A = \dfrac{1-\cos 2A}{2} \\ \Rightarrow \sin^2 2A = \dfrac{1-\cos 4A}{2} \end{array}\right]$$

$$= \frac{1}{64}\int (1 + \cos^2 4x - 2\cos 4x) . dx$$

$$= \frac{1}{64}\int \left(1 + \frac{1+\cos 8x}{2} - 2\cos 4x\right) . dx$$

$$= \frac{1}{64}\int \left(1 + \frac{1}{2} + \frac{\cos 8x}{2} - 2\cos 4x\right) . dx$$

$$= \frac{1}{64}\int \left(\frac{3}{2} + \frac{\cos 8x}{2} - 2\cos 4x\right) . dx$$

$$= \frac{1}{128}\int (3 + \cos 8x - 4\cos 4x) . dx$$

$$= \frac{1}{128}\left[3\int 1 . dx + \int \cos 8x . dx - 4\int \cos 4x . dx\right]$$

$$= \frac{1}{128}\left[3x + \frac{1}{8}\sin 8x - \frac{4\sin 4x}{4}\right] + c$$

$$= \frac{1}{128}\left[3x + \frac{1}{8}\sin 8x - \sin 4x\right] + c.$$

(*iii*) $\int \cos^3 x . dx = \int \left(\frac{\cos 3x + 3\cos x}{4}\right) . dx$

$$= \frac{1}{4}\int \cos 3x . dx + \int 3\cos x . dx \qquad \left[\begin{array}{l} \because \cos 3A = 4\cos^3 A - 3\cos A \\ \Rightarrow 4\cos^3 A = \cos 3A + 3\cos A \\ \Rightarrow \cos^3 A = \dfrac{\cos 3A + 3\cos A}{4} \end{array}\right]$$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right] + c$$

1.10 EVALUATION OF $\int \sin mx \cos nx \, dx$, $\int \cos mx \cos nx \, dx$ AND $\int \sin mx \sin nx \, dx$

In order to evaluate $\int \sin mx \cos nx \, dx$, $\int \cos mx \cos nx \, dx$, $\int \sin mx \sin nx \, dx$, the integrands $\sin mx \cos nx$, $\cos mx \cos nx$, $\sin mx \sin nx$ are expressed in terms of the sum (or difference) of sines and cosines of multiples of x by using the formulae :

(*i*) $2\sin A \cos B = \sin(A+B) + \sin(A-B)$
(*ii*) $2\cos A \sin B = \sin(A+B) - \sin(A-B)$
(*iii*) $2\cos A \cos B = \cos(A+B) + \cos(A-B)$
(*iv*) $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

In applying these formulae, it is advisable to have A > B, because A > B would imply A – B > 0.

Example 22. *Evaluate the following integrals :*

(*i*) $\int \cos mx \cos nx\, dx$; *when (a)* $m \neq n$ *(b)* $m = n$.

(*ii*) $\int \sin mx \sin nx\, dx$; $m \neq n \Rightarrow m - n \neq 0$.

Solution. (*i*) (*a*) It is given that, when $m \neq n$,

$$\int \cos mx \cos nx\, dx = \frac{1}{2}\int 2\cos mx \cos nx\, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int [\cos(m+n)x + \cos(m-n)x]\, dx$$

$$[\because\ 2\cos A \cos B = \cos(A+B) + \cos(A-B)]$$

$$= \frac{1}{2}\int \cos(m+n)x\, dx + \frac{1}{2}\int \cos(m-n)\,x\,.\,dx$$

$$= \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + c.$$

(*b*) When $m = n$;

$$\int \cos mx \cos nx\, dx = \int \cos^2 nx\, dx$$

$$= \frac{1}{2}\int 2\cos^2 nx = \frac{1}{2}\int (1+\cos 2nx)\,.\,dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int 1\,.\,dx + \frac{1}{2}\int \cos 2nx\,.\,dx$$

$$= \frac{x}{2} + \frac{1}{2}\cdot\frac{\sin 2nx}{2n} + c = \frac{x}{2} + \frac{1}{4n}\,.\,\sin 2nx + c.$$

(*ii*) $\int \sin mx \sin nx\, dx = \frac{1}{2}\int 2\sin mx \sin nx\, dx$ [Multiply and divided by 2]

$$= \frac{1}{2}\int [\cos(m-n)x - \cos(m+n)x]\, dx$$

$$[\because\ 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2}\int \cos(m-n)\,x\,.\,dx - \frac{1}{2}\int \cos(m+n)\,x\,.\,dx$$

$$= \frac{1}{2}\,\frac{\sin(m-n)x}{(m-n)} - \frac{1}{2}\,\frac{\sin(m+n)x}{m+n} + c$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{(m-n)} - \frac{\sin(m+n)x}{m+n}\right] + c.$$

Example 23. *Evaluate the following integrals :*

(*i*) $\int \sin 3x \sin 2x\, dx$ (*ii*) $\int \cos 4x \cos x\,.\,dx$

(*iii*) $\int \sin 4x \cos 3x\, dx$ (*iv*) $\int \sin 3x \cos 4x\, dx$.

Solution. (*i*) $\int \sin 3x \sin 2x\, dx$

$$= \frac{1}{2}\int 2\sin 3x \sin 2x\, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int [\cos(3x - 2x) - \cos(3x + 2x)]\,dx$$

$$[\because\ 2\sin A \sin B = \cos(A - B) - \cos(A + B)]$$

$$= \frac{1}{2}\int (\cos x - \cos 5x)\,dx = \frac{1}{2}\int \cos x\,.\,dx - \frac{1}{2}\int \cos 5x\,dx$$

$$= \frac{1}{2}\sin x - \frac{1}{2}\frac{\sin 5x}{5} + c = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + c.$$

(*ii*) $\int \cos 4x \cos x\,dx = \frac{1}{2}\int 2\cos 4x \cos x\,dx$ [Multiply and divided by 2]

$$= \frac{1}{2}\int [\cos(4x + x) + \cos(4x - x)]\,dx$$

$$[\because\ 2\cos A \cos B = \cos(A + B) + \cos(A - B)]$$

$$= \frac{1}{2}\int (\cos 5x + \cos 3x)\,dx = \frac{1}{2}\int \cos 5x\,dx + \frac{1}{2}\int \cos 3x\,dx$$

$$= \frac{1}{2}\frac{\sin 5x}{5} + \frac{1}{2}\cdot\frac{\sin 3x}{3} + c = \frac{\sin 5x}{10} + \frac{\sin 3x}{6} + c.$$

(*iii*) $\int \sin 4x \cos 3x\,dx$

$$= \frac{1}{2}\int 2\sin 4x \cos 3x\,dx$$ [Multiply and divided by 2]

$$= \frac{1}{2}\int [\sin(4x + 3x) + \sin(4x - 3x)]\,dx$$

$$[\because\ 2\sin A \cos B = \sin(A + B) + \sin(A - B)]$$

$$= \frac{1}{2}\int (\sin 7x + \sin x)\,dx = \frac{1}{2}\int \sin 7x\,dx + \frac{1}{2}\int \sin x\,dx$$

$$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) + \frac{1}{2}[-\cos x] + c$$

$$= \frac{-\cos 7x}{14} - \frac{\cos x}{2} + c.$$

(*iv*) $\int \sin 3x \cos 4x\,dx$

$$= \frac{1}{2}\int 2\sin 3x \cos 4x\,dx$$ [Multiply and divided by 2]

$$= \frac{1}{2}\int [\sin(3x + 4x) + \sin(3x - 4x)]\,dx$$

$$[\because\ 2\sin A \cos B = \sin(A + B) + \sin(A - B)]$$

$$= \frac{1}{2}\int [\sin 7x + \sin(-x)]\,dx$$

$$= \frac{1}{2}\int \sin 7x\,dx - \frac{1}{2}\int \sin x\,dx \qquad [\because\ \sin(-\theta) = -\sin\theta]$$

$$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}(-\cos x) + c$$

$$= -\frac{1}{14}\cos 7x + \frac{1}{2}\cos x + c.$$

Example 24. *Evaluate the following integrals :*

(*i*) $\int \sin^3 x \cos^3 x \, dx$ (*ii*) $\int \cos x \cos 2x \cos 3x \, dx$

(*iii*) $\int \sin x \sin 2x \sin 3x \, dx$ (*iv*) $\int \frac{\sin 4x}{\cos 2x} dx$

(*v*) $\int \frac{\sin 4x}{\sin x} dx$.

Solution. (*i*) $\int \sin^3 x \cos^3 x \, dx = \int (\sin x \cos x)^3 \, dx$

$= \frac{1}{8}\int (2 \sin x \cos x)^3 \, dx$ [Multiply and divided by 8]

$= \frac{1}{8}\int (\sin 2x)^3 \, dx$

$= \frac{1}{8}\int \sin^3 2x \, dx$

$= \frac{1}{8}\int \left(\frac{3 \sin 2x - \sin 6x}{4}\right) dx$ $\left[\begin{array}{ll} \because & \sin 3A = 3 \sin A - 4 \sin^3 A \\ \Rightarrow & 4 \sin^3 A = 3 \sin A - \sin 3A \\ \Rightarrow & \sin^3 A = \dfrac{3 \sin A - \sin 3A}{4} \\ \Rightarrow & \sin^3 2A = \dfrac{3 \sin 2A - \sin 6A}{4} \end{array}\right]$

$= \frac{1}{32}\left[\int 3 \sin 2x \, dx - \int \sin 6x \, dx\right]$

$= \frac{1}{32}\left[\frac{-3 \cos 2x}{2} - \left(-\frac{\cos 6x}{6}\right)\right] + c$

$= \frac{1}{32}\left[-\frac{3}{2} \cos 2x + \frac{1}{6} \cos 6x\right] + c.$

(*ii*) $\int \cos x \cos 2x \cos 3x \, dx$

$= \frac{1}{2}\int (2 \cos x \cos 2x) . \cos 3x \, dx$ [Multiply and divided by 2]

$= \frac{1}{2}\int (2 \cos 2x \cos x) \cos 3x \, dx$

$= \frac{1}{2}\int [\cos (2x + x) + \cos (2x - x)] \cos 3x \, dx$

[$\because$ $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$]

$= \frac{1}{2}\int (\cos 3x + \cos x) \cos 3x \, dx$

$= \frac{1}{2}\int (\cos^2 3x + \cos 3x \cos x) \, dx$

$= \frac{1}{4}\int (2 \cos^2 3x + 2 \cos 3x \cos x) \, dx$

[Multiply and divided by 2 again]

$$= \frac{1}{4}\int (1+\cos 6x)\,dx + \frac{1}{4}\int [\cos(3x+x) + \cos(3x-x)]\,dx$$

$$\begin{bmatrix} \because & \cos 2A = 2\cos^2 A - 1 \\ \Rightarrow & 2\cos^2 A = 1 + \cos 2A \\ \Rightarrow & 2\cos^2 3A = 1 + \cos 6A \end{bmatrix}$$

$[\because\ 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$

$$= \frac{1}{4}\int 1\,.dx + \frac{1}{4}\int \cos 6x\,.dx + \frac{1}{4}\int \cos 4x\,dx + \frac{1}{4}\int \cos 2x\,dx$$

$$= \frac{1}{4}\left[x + \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right] + c.$$

(*iii*) $\int \sin x\ \sin 2x \sin 3x\,dx$

$$= \frac{1}{2}\int (2\sin 2x \sin x)\sin 3x\,dx$$ [Multiply and divided by 2]

$$= \frac{1}{2}\int [\cos(2x-x) - \cos(2x+x)]\,.\sin 3x\,dx$$

$[\because\ 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$

$$= \frac{1}{2}\int (\cos x - \cos 3x)\,.\sin 3x\,dx$$

$$= \frac{1}{2}\int (\sin 3x\cos x - \sin 3x\cos 3x)\,dx$$

$$= \frac{1}{4}\int 2\sin 3x\cos x\,dx - \frac{1}{4}\int 2\sin 3x\cos 3x\,dx$$

[Multiply and divided by 2 again]

$$= \frac{1}{4}\int [\sin(3x+x) + \sin(3x-x)]\,dx - \frac{1}{4}\int \sin 6x\,dx$$

$$\begin{bmatrix} \because\ 2\sin A\cos B = \sin(A+B) + \sin(A-B) \\ 2\sin A\cos A = \sin 2A \end{bmatrix}$$

$$= \frac{1}{4}\int (\sin 4x + \sin 2x)\,dx - \frac{1}{4}\int \sin 6x\,dx$$

$$= \frac{1}{4}\left[\frac{-\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6}\right] + c.$$

(*iv*) $\int \frac{\sin 4x}{\cos 2x}\,dx = \int \frac{2\sin 2x\cos 2x}{\cos 2x}\,dx$ $[\because\ \sin 2A = 2\sin A\cos A]$

$$= 2\int \sin 2x\,dx = \frac{-2\cos 2x}{2} + c = -\cos 2x + c.$$

(*v*) $\int \frac{\sin 4x}{\sin x}\,dx = \int \frac{2\sin 2x\cos 2x}{\sin x}\,dx$ $[\because\ \sin 2A = 2\sin A\cos A]$

$$= \int \frac{4\sin x\cos x\cos 2x}{\sin x}\,dx = 4\int \cos 2x\cos x\,dx$$

$$= 2\int 2\cos 2x\cos x\,dx$$ $[\because\ 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$

$$= 2\int [\cos(2x+x) + \cos(2x-x)]\,dx$$

$$= 2\int \cos 3x \, dx + 2\int \cos x \, dx = 2\frac{\sin 3x}{3} + 2\sin x + c$$

$$= \frac{2}{3}\sin 3x + 2\sin x + c.$$

Example 25. *Evaluate the following integrals :*

(i) $\int \sin 4x \sin 8x \, dx$ (ii) $\int \cos 2x \cos 4x \cos 6x \, dx$.

Solution. (i) $\int \sin 4x \sin 8x \, dx$

$$= \frac{1}{2}\int 2\sin 8x \sin 4x \, dx \quad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int [\cos(8x - 4x) - \cos(8x + 4x)] \, dx$$

$$[\because \; 2\sin A \sin B = \cos(A - B) - \cos(A + B)]$$

$$= \frac{1}{2}\int (\cos 4x - \cos 12x) \, dx = \frac{1}{2}\int \cos 4x \, dx - \frac{1}{2}\int \cos 12x \, dx$$

$$= \frac{1}{2}\frac{\sin 4x}{4} - \frac{1}{2}\cdot\frac{\sin 12x}{12} + c = \frac{1}{8}\sin 4x - \frac{1}{24}\sin 12x + c.$$

(ii) $\int \cos 2x \cos 4x \cos 6x \, . \, dx$

$$= \frac{1}{2}\int (2\cos 4x \cos 2x) \, . \cos 6x \, dx \quad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int [\cos(4x + 2x) + \cos(4x - 2x)] \, . \cos 6x \, dx$$

$$[\because \; 2\cos A \cos B = \cos(A + B) + \cos(A - B)]$$

$$= \frac{1}{2}\int (\cos 6x + \cos 2x) \, . \cos 6x \, dx$$

$$= \frac{1}{2}\int (\cos^2 6x + \cos 6x \cos 2x) \, dx$$

$$= \frac{1}{4}\int (2\cos^2 6x + 2\cos 6x \cos 2x) \, dx$$

[Multiply and divided by 2 again]

$$= \frac{1}{4}\int [(1 + \cos 12x) + \cos(6x + 2x) + \cos(6x - 2x)] \, dx$$

$$[\because \; \cos 2A = 2\cos^2 A - 1 \Rightarrow 2\cos^2 A = 1 + \cos 2A]$$

$$= \frac{1}{4}\left[\int 1 . dx + \int \cos 12x \, dx + \int \cos 8x \, dx + \int \cos 4x \, dx\right]$$

$$= \frac{1}{4}\left[x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right] + c.$$

Example 26. *Evaluate the following integrals :*

(i) $\int \frac{\cos x}{1 + \cos x} dx$ (ii) $\int \frac{1}{1 + \cos x} dx$

(iii) $\int \sin x\sqrt{1-\cos 2x}\,dx$ (iv) $\int \sin mx \cos nx\,dx$.

Solution. (i) $\int \frac{\cos x}{1+\cos x}dx$

$$= \int \frac{1+\cos x - 1}{1+\cos x}dx \quad \text{[Add and subtract 1 to the numerator]}$$

$$= \int \left(1-\frac{1}{1+\cos x}\right)dx$$

$$= \int 1.dx - \int \frac{1}{1+\cos x}dx$$

$$= \int 1\,dx - \int \frac{1}{2\cos^2\frac{x}{2}}dx \quad \left[\begin{array}{l}\because\ (1+\cos 2A) = 2\cos^2 A \\ \Rightarrow\ 1+\cos A = 2\cos^2\frac{A}{2}\end{array}\right]$$

$$= \int 1\,dx - \frac{1}{2}\int \sec^2\frac{x}{2}dx = x - \frac{1}{2}\left(\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right) + c$$

$$= x - \tan\frac{x}{2} + c.$$

(ii) $\int \frac{1}{1+\cos x}dx = \int \frac{1}{2\cos^2\frac{x}{2}}dx \quad \left[\begin{array}{l}\because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ 1+\cos A = 2\cos^2\frac{A}{2}\end{array}\right]$

$$= \frac{1}{2}\int \sec^2\frac{x}{2}dx = \frac{1}{2}\left[\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right] + c$$

$$= \tan\frac{x}{2} + c.$$

(iii) $\int \sin x\sqrt{1-\cos 2x}\,dx$

$$= \int \sin x\sqrt{2\sin^2 x}\,dx \quad [\because\ 1-\cos 2A = 2\sin^2 A]$$

$$= \sqrt{2}\int \sin x \sin x\,dx = \sqrt{2}\int \sin^2 x\,dx$$

$$= \frac{\sqrt{2}}{2}\int 2\sin^2 x\,dx \quad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{\sqrt{2}}\int (1-\cos 2x)\,dx = \frac{1}{\sqrt{2}}\left[\int 1.dx - \int \cos 2x\,dx\right]$$

$$= \frac{1}{\sqrt{2}}\left(x - \frac{\sin 2x}{2}\right) + c.$$

(*iv*) $\int \sin mx \cos nx\, dx$

$= \frac{1}{2}\int 2\sin mx \cos nx\, dx$ [Multiply and divided by 2]

$= \frac{1}{2}\int [\sin(m+n)x + \sin(m-n)x]\, dx$

[$\because$ $2\sin A\cos B = \sin(A+B) + \sin(A-B)$]

$= \frac{1}{2}\int \sin(m+n)x\, dx + \frac{1}{2}\int \sin(m-n)x\, dx$

$= \frac{1}{2}\left[-\frac{\cos(m+n)x}{(m+n)}\right] + \frac{1}{2}\left[-\frac{\cos(m-n)x}{(m-n)}\right] + c$

$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{(m+n)} + \frac{\cos(m-n)x}{(m-n)}\right] + c.$

Example 27. *Evaluate the following integrals :*

(*i*) $\int \frac{7\cos^3 x + 9\sin^3 x}{3\sin^2 x\cos^2 x}\, dx$ (*ii*) $\int \cos x\sqrt{1-\cos 2x}\, dx$

(*iii*) $\int \sin^{-1}(\cos x)\, dx$ (*iv*) $\int \frac{\sin x}{1+\sin x}\, dx$.

Solution. (*i*) $\int \frac{7\cos^3 x + 9\sin^3 x}{3\sin^2 x\cos^2 x}\, dx$

$= \frac{1}{3}\int \left(\frac{7\cos^3 x}{\sin^2 x\cos^2 x} + \frac{9\sin^3 x}{\sin^2 x\cos^2 x}\right) dx$

$= \frac{1}{3}\int \left(\frac{7\cos x}{\sin x \,.\, \sin x} + \frac{9\sin x}{\cos x\cos x}\right) dx$

$= \frac{7}{3}\int \operatorname{cosec} x\cot x\, dx + \frac{9}{3}\int \sec x\tan x\, dx$

$= \frac{-7}{3}\operatorname{cosec} x + 3\sec x + c.$

(*ii*) $\int \cos x\sqrt{1-\cos 2x}\, dx$

$= \int \cos x\sqrt{2\sin^2 x}\, dx$ [$\because$ $(1-\cos 2A) = 2\sin^2 A$]

$= \sqrt{2}\int \sin x\cos x\, dx$

$= \frac{\sqrt{2}}{2}\int 2\sin x\cos x\, dx$ [Multiply and divided by 2]

$= \frac{1}{\sqrt{2}}\int \sin 2x\, dx$ [$\because$ $2\sin A\cos A = \sin 2A$]

$$= \frac{1}{\sqrt{2}} \cdot \frac{-\cos 2x}{2} + c = -\frac{1}{2\sqrt{2}} \cos 2x + c.$$

(*iii*) $\int \sin^{-1} (\cos x)\, dx$

$$= \int \sin^{-1}\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[\because \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\right]$$

$$= \int \left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2} \int 1 . dx - \int x\, dx$$

$$= \frac{\pi x}{2} - \frac{x^2}{2} + c.$$

(*iv*) $\int \frac{\sin x}{1 + \sin x}\, dx$

$$= \int \left(\frac{1 + \sin x - 1}{1 + \sin x}\right) dx \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \int \left(\frac{1 + \sin x}{1 + \sin x} - \frac{1}{1 + \sin x}\right) dx = \int 1\, dx - \int \frac{1}{1 + \sin x}\, dx$$

$$= \int 1 . dx - \int \frac{1}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x}\, dx \qquad \text{[On rationalization]}$$

$$= \int 1 . dx - \int \frac{1 - \sin x}{1 - \sin^2 x}\, dx$$

$$= \int 1\, dx - \int \frac{1 - \sin x}{\cos^2 x}\, dx$$

$$= \int 1 . dx - \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx \qquad \left[\begin{array}{l}\because \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \cos^2 A = 1 - \sin^2 A\end{array}\right]$$

$$= \int 1 . dx - \int \sec^2 x\, dx + \int \sec x \tan x\, dx$$

$$= x - \tan x + \sec x + c.$$

Example 28. *Evaluate the following integrals :*

(*i*) $\int \sin 2x \cos 3x\, dx$ (*ii*) $\int \frac{2 \cos 2x}{\cos^2 x \sin^2 x}\, dx$

(*iii*) $\int \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x}\, dx$ (*iv*) $\int \sin^2 \left[\tan^{-1}\left(\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}\right)\right] d\theta$

(*v*) $\int \tan^{-1} (\sec x + \tan x)\, dx$.

Solution. (*i*) $\int \sin 2x \cos 3x\, dx$

$$= \frac{1}{2} \int 2 \cos 3x \sin 2x\, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int [\sin(3x+2x) - \sin(3x-2x)]\,dx$$

$$[\because\ 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{1}{2}\int (\sin 5x - \sin x)\,dx = \frac{1}{2}\left[\int \sin 5x\,dx - \int \sin x\,dx\right]$$

$$= \frac{1}{2}\left[\frac{-\cos 5x}{5} + \cos x\right] + c = -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + c.$$

(*ii*) $\displaystyle\int \frac{2\cos 2x}{\cos^2 x\sin^2 x}\,dx$

$$= 2\int \left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x\sin^2 x}\right)dx \qquad [\because\ \cos 2A = \cos^2 A - \sin^2 A]$$

$$= 2\int \left(\frac{\cos^2 x}{\cos^2 x\sin^2 x} - \frac{\sin^2 x}{\cos^2 x\sin^2 x}\right)dx$$

$$= 2\int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}\right)dx = 2\left[\int \operatorname{cosec}^2 x\,dx - \int \sec^2 x\,dx\right]$$

$$= -2\cot x - 2\tan x + c = -2[\cot x + \tan x] + c.$$

(*iii*) $\displaystyle\int \left(\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\right)dx$

$$= \int \left(\frac{1 - 2\sin^2 x + 2\sin^2 x}{\cos^2 x}\right)dx \qquad [\because\ \cos 2A = 1 - 2\sin^2 A]$$

$$= \int \frac{1}{\cos^2 x}\,dx = \int \sec^2 dx = \tan x + c.$$

(*iv*) $\displaystyle\int \sin^2\left[\tan^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)\right]d\theta$

$$= \int \sin^2\left[\tan^{-1}\left(\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right)\right]d\theta \qquad \left[\begin{array}{l}\because\ 1-\cos 2A = 2\sin^2 A\\ \quad\ 1+\cos 2A = 2\cos^2 A\end{array}\right]$$

$$= \int \sin^2\left[\tan^{-1}(\sqrt{\tan^2\theta})\right]d\theta$$

$$= \int \sin^2\left[\tan^{-1}(\tan\theta)\right]d\theta = \int \sin^2\theta\,d\theta$$

$$= \int \left(\frac{1-\cos 2\theta}{2}\right)d\theta \qquad \left[\begin{array}{l}\because\quad 1-\cos 2A = 2\sin^2 A\\ \Rightarrow\ \left(\dfrac{1-\cos 2A}{2}\right) = \sin^2 A\end{array}\right]$$

$$= \frac{1}{2}\int 1\,d\theta - \frac{1}{2}\int \cos 2\theta\,d\theta = \frac{\theta}{2} - \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + c.$$

(v) $\int \tan^{-1}(\sec x + \tan x)\,dx$

$$= \int \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)dx = \int \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)dx$$

$$= \int \tan^{-1}\left(\frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right)dx$$

$$\left[\begin{array}{l} \because\ \cos^2 A + \sin^2 A = 1 \\ \Rightarrow\ \cos^2\frac{A}{2} + \sin^2\frac{A}{2} = 1 \\ \cos 2A = \cos^2 A - \sin^2 A \\ \Rightarrow\ \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} \\ \sin 2A = 2\sin A\cos A \\ \Rightarrow\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \end{array}\right]$$

$$= \int \tan^{-1}\left[\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right]dx$$

$$= \int \tan^{-1}\left[\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right]dx$$

$$= \int \tan^{-1}\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]dx$$

$$\left[\text{Dividing numerator and denominator by } \cos\frac{x}{2}\right]$$

$$= \int \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]dx \qquad \left[\because\ \tan\left(\frac{\pi}{4} + A\right) = \left(\frac{1+\tan A}{1-\tan A}\right)\right]$$

$$= \int \left(\frac{\pi}{4} + \frac{x}{2}\right)dx = \frac{\pi}{4}\int 1\,dx + \frac{1}{2}\int x\,dx$$

$$= \frac{\pi x}{4} + \frac{1}{2}\left(\frac{x^2}{2}\right) + c = \frac{1}{4}\pi x + \frac{x^2}{4} + c.$$

EXERCISE FOR PRACTICE

Integrate the following functions w.r.t. x Q. (1–7) :

1. (*i*) $x^{5/4}$ (*ii*) x^{-5} (*iii*) x^{-7}
2. (*i*) $\sqrt[3]{x^2}$ (*ii*) $\dfrac{(x+1)^2}{x\sqrt{x}}$
3. (*i*) 2^x (*ii*) $\left(x+\dfrac{1}{x}\right)$
4. (*i*) e^{2x} (*ii*) $x^2\left(1-\dfrac{1}{x^2}\right)$
5. (*i*) $(2x-1)(x^2+1)$ (*ii*) $\left(\dfrac{1}{x^2}+\sec^2 x+5x\right)$
6. (*i*) 3^{x+2} (*ii*) $\sqrt{e^x}$
7. (*i*) $(1+x)\sqrt{1-x}$ (*ii*) $\dfrac{x^4+1}{x^2+1}$

Evaluate the following integrals Q. (8–20) :

8. $\displaystyle\int \frac{x}{x-3}\,dx$
9. $\displaystyle\int \left(x+\frac{1}{x}\right)^2 (x-7)\,dx$
10. $\displaystyle\int \frac{1}{\sqrt{3x+2}+\sqrt{3x-2}}\,dx$
11. $\displaystyle\int \frac{1+2x^2}{x^2(1+x^2)}\,dx$
12. $\displaystyle\int (e^x+e^{-x})^2\,dx$
13. $\displaystyle\int (2-5x)(3+2x)(1-x)\,dx$
14. (*i*) $\displaystyle\int (2\sin x+x^2)\,dx$ (*ii*) $\displaystyle\int (\sin x+\cos x)^2\,dx$
15. (*i*) $\displaystyle\int (\sec^2 x+\operatorname{cosec}^2 x)\,dx$ (*ii*) $\displaystyle\int (3\sin x-2\cos x+4\sec^2 x-5\operatorname{cosec}^2 x)\,dx$
16. (*i*) $\displaystyle\int \sqrt{1-\cos 2x}\,dx$ (*ii*) $\displaystyle\int \sqrt{1+\cos 2x}\,dx$
17. $\displaystyle\int (2\tan x-3\cot x)^2\,dx$
18. $\displaystyle\int \frac{1}{1+\cos 3x}\,dx$
19. $\displaystyle\int \frac{\sin 2x}{\cos x}\,dx$
20. $\displaystyle\int \cos^2 nx\,dx.$

Answers

1. (*i*) $\dfrac{4}{9}x^{9/4}+c$ (*ii*) $-\dfrac{1}{4x^4}+c$ (*iii*) $\dfrac{-1}{6x^6}+c$
2. (*i*) $\dfrac{3}{5}x^{5/3}+c$ (*ii*) $\dfrac{2}{3}x^{3/2}+4x^{1/2}-2x^{-1/2}+c$
3. (*i*) $\dfrac{2^x}{\log 2}+c$ (*ii*) $\dfrac{x^2}{2}+\log|x|+c$
4. (*i*) $\dfrac{1}{2}e^{2x}+c$ (*ii*) $\dfrac{x^3}{3}-x+c$

5. (*i*) $\frac{x^4}{2} - \frac{x^3}{3} + x^2 - x + c$ (*ii*) $-\frac{1}{x} + \tan x + \frac{5}{2}x^2 + c$

6. (*i*) $9\left(\frac{3^x}{\log 3}\right) + c$ (*ii*) $2\sqrt{e^x} + c$

7. (*i*) $\frac{-4}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + c$ (*ii*) $\frac{x^3}{3} - x + 2\tan^{-1} x + c$

8. $x + 3\log|x-3| + c$

9. $\frac{x^4}{4} - \frac{7}{3}x^3 + x^2 + \log|x| + \frac{7}{x} - 14x + c$

10. $\frac{1}{18}[(3x+2)^{3/2} - (3x-2)^{3/2}] + c$

11. $-\frac{1}{x} + \tan^{-1} x + c$

12. $\frac{e^{2x}}{2} - \frac{1}{2}e^{-2x} + 2x + c$

13. $6x - \frac{17x^2}{2} + \frac{x^3}{3} + \frac{5x^4}{2} + c$

14. (*i*) $-2\cos x + \frac{x^3}{3} + c$ (*ii*) $x - \frac{1}{2}\cos 2x + c$

15. (*i*) $\tan x - \cot x + c$ (*ii*) $-3\cos x - 2\sin x + 4\tan x + 5\cot x + c$

16. (*i*) $-\sqrt{2}\cos x + c$ (*ii*) $\sqrt{2}\sin x + c$

17. $4\tan x - 9\cot x - 25x + c$

18. $\frac{1}{3}\tan\frac{3x}{2} + c$

19. $-2\cos x + c$

20. $\frac{x}{2} + \frac{\sin 2nx}{4nx} + c.$

2

Integration by Substitution—I

2.1 INTRODUCTION

In the previous chapter, we have considered the problems on integration of functions in standard forms and the problems involving combinations of these functions. Often, we come across functions which appear to be quite simple, but do not fit in the category of standard forms. So, we need to develop a method of integration for those functions which, otherwise, cannot be evaluated directly by the use of standard formulae. These integrals can be converted to standard form by the substitution of a new variable. These substitutions reduce the complicated integrals to the standard forms which can be easily integrated.

The method of evaluating an integral by reducing it to standard form by some suitable substitution is called integration by substitution'.

2.2 CHANGE OF INDEPENDENT VARIABLE

Theorem 1. *If the independent variable 'x' in the integral* $\int f(x)\,dx$ *be changed to 'z' by putting* $x = \phi(z)$ *; where* $\phi(z)$ *possesses continuous derivative* $\phi'(z)$*, then*

$$\int f(x)\,dx = \int f[\phi(z)]\,\phi'(z)\,.\,dz.$$

Proof. Let us consider that

$$I = \int f(x)\,dx,$$

then, $$\frac{dI}{dx} = f(x) \qquad ...(1)$$

Assume that ; $x = \phi(z)$ such that

$$\frac{dx}{dz} = \phi'(z) \qquad ...(2)$$

$\therefore$ We may write that

$$\frac{dI}{dz} = \frac{dI}{dx}\cdot\frac{dx}{dz} \qquad \text{[Using equations (1) and (2)]}$$

$$= f(x)\,\phi'(z)$$

$$= f[\phi(z)]\,\phi'(z) \qquad ...(3) \quad [\because\ x = \phi(z)]$$

$$= \text{a function of } z.$$

Integrating both sides of equation (3) w.r.t. z ; we have

$$I = \int f[\phi(z)]\, \phi'(z) \,.\, dz$$

$\Rightarrow$ $\int f(x)\, dx = \int f[\phi(z)] \,.\, \phi'(z) \,.\, dz.$ $\quad [\because\ I = \int f(x)\, dx]$

Remark 1. To evaluate $\int f(x)\, dx$ by the substitution $x = \phi(z)$.

Working Rule :

(*i*) In the integrand, put $x = \phi(z)$ and $dx = \phi'(z) \,.\, dz$.

(*ii*) Evaluate the resulting integral in z.

(*iii*) Express the result in terms of x from $x = \phi(z)$.

Remark 2. There are no hard and fast rules for making suitable substitutions. Experience is the best guide in this matter and one learns only through practice. However, some useful suggestions for standard substitutions are given below :

(*i*) If integrand is of the form $f(ax + b)$, put $ax + b = z$

$\Rightarrow$ $d(ax + b) = dz$

$\Rightarrow$ $(a + 0)\, dx = dz$

$\Rightarrow$ $dx = \dfrac{1}{a} \,.\, dz.$

(*ii*) If the integrand is of the form $x^{n-1} \,.\, f(x^n)$, put $x^n = z$

$\Rightarrow$ $dx^n = dz$

$\Rightarrow$ $nx^{n-1}\, dx = dz$

$\Rightarrow$ $x^{n-1}\, dx = \dfrac{1}{n} \,.\, dz.$

(*iii*) If integrand is of the form $[f(x)]^n \,.\, f'(x)$, put $f(x) = z$

$\Rightarrow$ $d[f(x)] = dz$

$\Rightarrow$ $f'(x)\, dx = dz.$

(*iv*) If integrand is of the form $\dfrac{f'(x)}{f(x)}$, put $f(x) = z$

$\Rightarrow$ $d[f(x)] = dz$

$\Rightarrow$ $f'(x)\, dx = dz$

(*v*) If integrand involves e^x, put $e^x = z$

$\Rightarrow$ $d(e^x) = dz$

$\Rightarrow$ $e^x \,.\, dx = dz.$

2.3 TWO IMPORTANT FORMS OF INTEGRALS

Theorem 2. *The integral of a fraction whose numerator is the derivative of the denominator is log | (denominator) |.*

i.e., $$\int \frac{f'(x)}{f(x)} \,.\, dx = \log\, |f(x)| + c.$$

Proof. Let us put $f(x) = z$

$\Rightarrow$ $d\,[f(x)] = dz$

$\Rightarrow$ $f'(x)\, dx = dz$

$\Rightarrow$ $\displaystyle\int \frac{f'(x)}{f(x)}\, dx = \int \frac{dz}{z} = \log z = \log\, [f(x)].$

Theorem 3. *The integral of the product of a function raised to an index other than (– 1) and its derivative is obtained by increasing the index of the function by 1 and dividing by the new index.*

i.e., $$\int [f(x)]^n \cdot f'(x) \cdot dx = \frac{[f(x)]^{n+1}}{n+1} + c\ ;\ n \neq -1.$$

Proof. Put $f(x) = z$

$\Rightarrow \quad d[f(x)] = dz$

$\Rightarrow \quad f'(x)\,dx = dz$

$$\Rightarrow \quad \int [f(x)]^n \cdot f'(x)\,dx = \int z^n \cdot dz$$

$$= \left[\frac{z^{n+1}}{n+1}\right] + c \qquad [\because\ n \neq -1]$$

$$= \frac{[f(x)]^{n+1}}{n+1} + c.$$

2.4 SOME STANDARD INTEGRALS

Prove that :

1. $\int \tan x\,dx = \log|\sec x| + c$

2. $\int \cot x\,dx = \log|\sin x| + c$

3. $\int \sec x\,dx = \log|\sec x + \tan x| + c$

$= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

4. $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + c$

$= \log\left|\tan\frac{x}{2}\right| + c.$

Proof. 1. Let $I = \int \tan x\,dx$

$$\Rightarrow \quad I = \int \frac{\sin x}{\cos x}\,dx \qquad \ldots(1)$$

Putting $\cos x = z \Rightarrow -\sin x\,dx = dz$

$$\Rightarrow \quad dx = \frac{-1}{\sin x}\,dz$$

$\therefore$ We have
$$I = \int \frac{\sin x}{\cos x}\cdot\left(\frac{-1}{\sin x}\right)\cdot dz = -\int \frac{1}{z}\cdot dz$$

$$= -\log|z| + c = -\log|\cos x| + c \qquad [\because\ z = \cos x]$$

$$= \log|(\cos x)^{-1}| + c \qquad [\because\ n \log m = \log m^n]$$

$$= \log|\sec x| + c.$$

2. Let $I = \int \cot x\,dx$

$$\Rightarrow \quad I = \int \frac{\cos x}{\sin x}\,dx$$

Putting $\sin x = z$

$$\Rightarrow \quad \cos x\,dx = dz \Rightarrow dx = \frac{1}{\cos x}\cdot dz$$

$\therefore$ We have $\quad I = \int \frac{\cos x}{z} \cdot \frac{1}{\cos x} \cdot dz = \int \frac{1}{z} \cdot dz$

$= \log |z| + c$

$= \log |\sin x| + c.$ $\quad [\because \; z = \sin x]$

3. Let $\quad I = \int \sec x \, dx$

$\Rightarrow \quad I = \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} \cdot dx$

[Multiply and divided by $(\sec x + \tan x)$]

Putting $\quad (\sec x + \tan x) = z$

$\Rightarrow \quad (\sec x \tan x + \sec^2 x) \, dx = dz$

$\Rightarrow \quad dx = \frac{1}{(\sec x \tan x + \sec^2 x)} \cdot dz$

$\therefore$ We have $\quad I = \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} \cdot \frac{1}{\sec x (\sec x + \tan x)} \cdot dz$

$= \int \frac{1}{z} \cdot dz$

$= \log |z| + c$

$= \log |\sec x + \tan x| + c \quad$ ...(1) $\quad [\because \; z = (\sec x + \tan x)]$

Also, we may write that

$$\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x}$$

$$= \frac{1 + \sin x}{\cos x}$$

$$= \frac{\left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}\right)}{\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)} \qquad \left[\begin{array}{l} \because \; \cos^2 A + \sin^2 A = 1 \\ \sin 2A = 2 \sin A \cos A \\ \cos 2A = \cos^2 A - \sin^2 A \end{array}\right]$$

$$= \frac{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2}{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)} = \frac{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)}{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)}$$

$$= \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \qquad \left[\text{Dividing numerator and denominator by } \cos \frac{x}{2}\right]$$

$\therefore \quad \sec x + \tan x = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \quad$ ...(2)

Putting this value of $(\sec x + \tan x)$ in (1), we get

$$I = \log \left| \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \right| + c.$$

4. Let $\quad I = \int \operatorname{cosec} x \, dx.$

$$\Rightarrow \quad I = \int \frac{\operatorname{cosec} x (\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)} \cdot dx$$

[Multiply and divided by $(\operatorname{cosec} x - \cot x)$]

Putting $\quad \operatorname{cosec} x - \cot x = z$

$\Rightarrow \quad (-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x) \, dx = dz$

$$\Rightarrow \quad dx = \frac{1}{\operatorname{cosec} x (\operatorname{cosec} x - \cot x)} \cdot dz$$

$\therefore$ We have
$$I = \int \frac{\operatorname{cosec} x (\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)} \cdot \frac{1}{\operatorname{cosec} x (\operatorname{cosec} x - \cot x)} \cdot dz$$

$$= \int \frac{1}{z} \cdot dz$$

$$= \log |z| + c$$

$$= \log |\operatorname{cosec} x - \cot x| + c \qquad \ldots(1) \quad [\because \; z = \operatorname{cosec} x - \cot x]$$

Also, we may write that

$$\operatorname{cosec} x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1 - \cos x}{\sin x}$$

$$= \frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \qquad \left[\begin{array}{l} \because \; 1 - \cos 2A = 2 \sin^2 A \\ \Rightarrow \; 1 - \cos A = 2 \sin^2 \frac{A}{2} \\ \text{and } \; \sin 2A = 2 \sin A \cos A \end{array}\right]$$

$$\therefore \quad \operatorname{cosec} x - \cot x = \tan \frac{x}{2} \qquad \ldots(2)$$

Putting this value in (1), we get

$$I = \int \operatorname{cosec} x \, dx = \log \left| \tan \frac{x}{2} \right| + c.$$

Remark. Let $\quad I = \int \operatorname{cosec} x \, dx = \int \frac{1}{\sin x} \, dx \qquad [\because \; \sin 2A = 2 \sin A \cos A]$

$$= \int \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \cdot dx$$

$\left[\text{Dividing numerator and denominator by } \cos \frac{x}{2}\right]$

$$= \int \frac{\sec^2 \frac{x}{2}}{2 \tan \frac{x}{2}} \, dx = \int \frac{\frac{1}{2} \sec^2 \frac{x}{2}}{\tan \frac{x}{2}} \cdot dx \qquad \left[\because \; \frac{d}{dx}\left(\tan \frac{x}{2}\right) = \frac{1}{2} \sec^2 \frac{x}{2}\right]$$

$$= \log \left(\tan \frac{x}{2}\right) + c = \log \left| \tan \frac{x}{2} \right| + c$$

2.5 IMPORTANT NOTE

The students are advised to remember the following results :

(i) $\int \sin x\, dx = -\cos x + c$

(ii) $\int \cos x\, dx = \sin x + c$

(iii) $\int \tan x\, dx = \log |\sec x| + c$

(iv) $\int \cot x\, dx = \log |\sin x| + c$

(v) $\int \sec x\, dx = \log |\sec x + \tan x| + c = \log \left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

(vi) $\int \operatorname{cosec} x\, dx = \log |\operatorname{cosec} x - \cot x| + c = \log \left|\tan \frac{x}{2}\right| + c.$

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following integrals :*

(i) $\int \frac{6x - 8}{3x^2 - 8x + 5} dx$ (ii) $\int (3x + 5)^7 . dx$

(iii) $\int \frac{1}{(5 - 3x)^4} dx$ (iv) $\int 2x \sin (x^2 + 1) . dx$

(v) $\int \frac{(\log x)^2}{x} . dx$ (vi) $\int \frac{1}{(x + x \log x)} . dx.$

Solution. (i) Let $I = \int \frac{6x - 8}{3x^2 - 8x + 5} . dx$

Putting $3x^2 - 8x + 5 = z \Rightarrow (6x - 8)\, dx = dz$

$\therefore$ We have
$$I = \int \frac{1}{z} . dz$$
$$= \log |z| + c$$
$$= \log |3x^2 - 8x + 5| + c. \qquad [\because z = 3x^2 - 8x + 5]$$

Note. If the numerator is the derivative of the denominator or a constant multiple of the derivative of the denominator, put the denominator equal to z.

(ii) Let $I = \int (3x + 5)^7 . dx$

Putting $3x + 5 = z \Rightarrow 3 . dx = dz \Rightarrow dx = \frac{1}{3} . dz$

$\therefore$ We have
$$I = \int \frac{1}{3} . (z)^7 . dz = \frac{1}{3} \int z^7 . dz$$
$$= \frac{1}{3} \frac{z^8}{8} + c = \frac{1}{24} . z^8 + c$$
$$= \frac{(3x + 5)^8}{24} + c. \qquad [\because z = 3x + 5]$$

(*iii*) Let $\quad I = \int \frac{1}{(5-3x)^4} \, . \, dx$

Putting $\quad 5 - 3x = z$

$\Rightarrow \quad -3\, dx = dz \;\Rightarrow\; dx = -\frac{1}{3} \, . \, dz$

$\therefore$ We have
$$I = \int -\frac{1}{3} \, . \, \frac{1}{z^4} \, . \, dz$$
$$= -\frac{1}{3} \int \frac{1}{z^4} \, . \, dz = -\frac{1}{3} \int z^{-4} \, . \, dz$$
$$= -\frac{1}{3} \, . \, \frac{z^{-3}}{-3} + c = \frac{1}{9} \, . \, \frac{1}{z^3} + c$$
$$= \frac{1}{9} \, . \, \frac{1}{(5-3x)^3} + c. \qquad [\because \; z = (5-3x)]$$

(*iv*) Let $\quad I = \int 2x \sin(x^2 + 1) \, . \, dx$

Putting $\quad x^2 + 1 = z \;\Rightarrow\; 2x\, dx = dz$

$\therefore$ We have
$$I = \int \sin z \, . \, dz = -\cos z + c$$
$$= -\cos(x^2 + 1) + c. \qquad [\because \; z = x^2 + 1]$$

(*v*) Let $\quad I = \int \frac{(\log x)^2}{x} \, . \, dx$

Putting $\quad \log x = z \;\Rightarrow\; \frac{1}{x} \, . \, dx = dz$

$\therefore$ We have
$$I = \int z^2 \, . \, dz = \frac{z^3}{3} + c$$
$$= \frac{1}{3} \, . \, [\log(x)]^3 + c. \qquad [\because \; z = \log x]$$

(*vi*) Let $\quad I = \int \frac{1}{x + x \log x} \, dx = \int \frac{1}{x(1 + \log x)} \, . \, dx$

Putting $\quad (1 + \log x) = z \;\Rightarrow\; \frac{1}{x} \, . \, dx = dz$

$\therefore$ We have
$$I = \int \frac{1}{z} \, . \, dz$$
$$= \log |z| + c$$
$$= \log |1 + \log x| + c. \qquad [\because \; z = 1 + \log x]$$

Example 2. *Evaluate the following integrals :*

(*i*) $\int \cos(2x + 3). \, dx$ (*ii*) $\int \frac{\sec^2 (\log x)}{x} . \, dx$

(*iii*) $\int \frac{e^{m \tan^{-1} x}}{1 + x^2} . \, dx$ (*iv*) $\int \frac{\sec^2 x}{5 + \tan x} . \, dx$

(v) $\int \frac{1}{1+e^{-x}} \cdot dx$ (vi) $\int \frac{9 \operatorname{cosec}^2 x}{1+\cot x} \cdot dx$

(vii) $\int \frac{e^x+1}{e^x-1} \cdot dx.$

Solution. (i) Let $I = \int \cos(2x+3) \cdot dx$

Putting $2x+3 = z$

$\Rightarrow \quad 2 \cdot dx = dz \;\Rightarrow\; dx = \frac{1}{2} \cdot dz$

$\therefore$ We have $I = \int \frac{1}{2} \cos z \cdot dz$

$= \frac{1}{2} \int \cos z \cdot dz = \frac{1}{2} \sin z + c$

$= \frac{1}{2} \sin(2x+3) + c.$ $[\because \; z = 2x+3]$

(ii) Let $I = \int \frac{\sec^2(\log x)}{x} \cdot dx$

Putting $\log x = z \;\Rightarrow\; \frac{1}{x} \cdot dx = dz$

$\therefore$ We have $I = \int \sec^2 z \cdot dz = \tan z + c$

$= \tan(\log x) + c.$ $[\because \; z = \log x]$

(iii) Let $I = \int \frac{e^{m \tan^{-1} x}}{1+x^2} \, dx$

Putting $\tan^{-1} x = z \;\Rightarrow\; \frac{1}{1+x^2} \cdot dx = dz$

$\therefore$ We have $I = \int e^{mz} \cdot dz = \frac{e^{mz}}{m} + c$

$= \frac{e^{m \tan^{-1} x}}{m} + c.$ $[\because \; z = \tan^{-1} x]$

(iv) Let $I = \int \frac{\sec^2 x}{5+\tan x} \, dx$

Putting $\tan x = z \;\Rightarrow\; \sec^2 x \, dx = dz$

$\therefore$ We have $I = \int \frac{1}{5+z} \cdot dz = \log |5+z| + c$

$= \log |5+\tan x| + c.$ $[\because \; z = \tan x]$

(v) Let $I = \int \frac{1}{1+e^{-x}} \cdot dx = \int \frac{1}{1+\frac{1}{e^x}} \cdot dx = \int \frac{1}{\frac{e^x+1}{e^x}} \cdot dx$

$\therefore \quad I = \int \frac{e^x}{e^x+1} \cdot dx$

Now, putting $1 + e^x = z \Rightarrow e^x\,dx = dz$

$\therefore$ We have $I = \int \frac{1}{z} \cdot dz = \log |z| + c$

$= \log |1 + e^x| + c.$ $[\because z = 1 + e^x]$

(*vi*) Let $I = \int \frac{9\,\text{cosec}^2 x}{1 + \cot x}\,dx$

Putting $1 + \cot x = z$

$\Rightarrow$ $-\text{cosec}^2 x\,dx = dz \Rightarrow \text{cosec}^2 x\,dx = -dz$

$\therefore$ We have $I = \int -\frac{9}{z} \cdot dz = -9 \int \frac{1}{z} \cdot dz$

$= -9 \log |z| + c$

$= -9 \log |1 + \cot x| + c.$ $[\because z = 1 + \cot x]$

(*vii*) Let $I = \int \frac{e^x + 1}{e^x - 1} \cdot dx$

[Dividing numerator and denominator by $e^{x/2}$].

$$\Rightarrow \qquad I = \int \frac{e^{x/2} + \frac{1}{e^{x/2}}}{e^{x/2} - \frac{1}{e^{x/2}}} \cdot dx = \int \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} \cdot dx$$

Putting $e^{x/2} - e^{-x/2} = z \Rightarrow \left(\frac{1}{2} e^{x/2} + \frac{1}{2} e^{-x/2}\right) \cdot dx = dz$

$\Rightarrow \frac{1}{2}(e^{x/2} + e^{-x/2})\,dx = dz \Rightarrow (e^{x/2} + e^{-x/2}) \cdot dx = 2\,dz$

$\therefore$ We have $I = \int \frac{2}{z} \cdot dz = 2 \int \frac{1}{z} \cdot dz$

$= 2 \log |z| + c$

$= 2 \log |e^{x/2} - e^{-x/2}| + c.$ $[\because z = e^{x/2} - e^{-x/2}]$

Example 3. *Evaluate the following integrals :*

(*i*) $\int \frac{1 - \tan x}{1 + \tan x} \cdot dx$ (*ii*) $\int \frac{1}{\sqrt{x}\,(1 + \sqrt{x})} \cdot dx$

(*iii*) $\int 4x\sqrt{3 - x^2} \cdot dx$ (*iv*) $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}} \cdot dx$

(*v*) $\int \sqrt{ax + b} \cdot dx$ (*vi*) $\int (4x + 2)\left(\sqrt{x^2 + x + 1}\right) \cdot dx.$

Solution. (*i*) Let $I = \int \frac{1 - \tan x}{1 + \tan x} \cdot dx = \int \frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}} \cdot dx$

$\therefore$ $I = \int \frac{\cos x - \sin x}{\cos x + \sin x} \cdot dx$

Putting $\cos x + \sin x = z$

$\Rightarrow \quad (-\sin x + \cos x) \,.\, dx = dz \quad \Rightarrow \quad (\cos x - \sin x)\, dx = dz$

$\therefore$ We have $\quad I = \int \frac{1}{z} \,.\, dz$

$= \log |z| + c$

$= \log |\cos x + \sin x| + c. \qquad [\because \; z = \cos x + \sin x]$

(*ii*) Let $\quad I = \int \frac{1}{\sqrt{x}\,(1+\sqrt{x})} \,.\, dx$

Putting $\quad 1 + \sqrt{x} = z \quad \Rightarrow \quad 0 + \frac{1}{2} x^{-1/2} \,.\, dx = dz$

$\Rightarrow \quad \frac{1}{2\sqrt{x}}\, dx = dz \quad \Rightarrow \quad \frac{1}{\sqrt{x}}\, dx = 2\, dz$

$\therefore$ We have $\quad I = 2 \int \frac{1}{z} \,.\, dz$

$= 2 \log |z| + c$

$= 2 \log |1 + \sqrt{x}| + c. \qquad [\because \; z = 1 + \sqrt{x}]$

(*iii*) Let $\quad I = \int 4x \sqrt{3 - x^2} \,.\, dx = \int \sqrt{3 - x^2} \,.\, 4x dx$

Putting $\quad 3 - x^2 = z \quad \Rightarrow \quad (0 - 2x)\, dx = dz \quad \Rightarrow \quad -2x\, dx = dz$

$\Rightarrow \quad 2x\, dx = -dz \quad \Rightarrow \quad 4x\, dx = -2dz$

$\therefore$ We have $\quad I = \int \sqrt{z} \,.\, (-2\, dz). = -2 \int z^{1/2} \,.\, dz$

$$= -2\, \frac{z^{1/2+1}}{(1/2+1)} + c = -2\, \frac{z^{3/2}}{\frac{3}{2}} + c$$

$$= -\frac{4}{3} z^{3/2} + c$$

$$= -\frac{4}{3} (3 - x^2)^{3/2} + c. \qquad [\because \; z = (3 - x^2)]$$

(*iv*) Let $\quad I = \int \frac{1}{\sqrt{x} + \sqrt[3]{x}} \,.\, dx = \int \frac{1}{x^{1/2} + x^{1/3}} \,.\, dx$

Putting $\quad x^{1/6} = z \quad \Rightarrow \quad x = z^6 \quad \Rightarrow \quad dx = 6z^5 \,.\, dz$

$\therefore$ We have

$$I = \int \frac{1}{z^3 + z^2} \,.\, 6z^5 \,.\, dz$$

$$= 6 \int \frac{z^5}{z^2(z+1)} \,.\, dz = 6 \int \frac{z^3}{(z+1)}\, dz$$

$$= 6 \int \left[(z^2 - z + 1) - \frac{1}{z+1} \right] . \, dz$$

$$\begin{array}{r|l}
z+1 \,\big)\; z^3 & (\, z^2 - z + 1 \\
z^3 + z^2 & \\
- \quad - & \\
\hline
-z^2 & \\
-z^2 - z & \\
+ \quad + & \\
\hline
z & \\
z + 1 & \\
- \quad - & \\
\hline
-1 &
\end{array}$$

$$= 6\left[\int z^2 . dz - \int z . dz + \int 1 . dz - \int \frac{1}{z+1} . dz\right]$$

$$= 6\left[\frac{z^3}{3} - \frac{z^2}{2} + z - \log|z+1|\right] + c$$

$$= \frac{6z^3}{3} - \frac{6z^2}{2} + 6z - 6\log|z+1| + c$$

$$= 2z^3 - 3z^2 + 6z - 6\log|z+1| + c$$

$$= 2(x^{1/6})^3 - 3(x^{1/6})^2 + 6x^{1/6} - 6\log|x^{1/6}+1| + c \qquad [\because\ z = x^{1/6}]$$

$$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6\log|x^{1/6}+1| + c.$$

(*v*) Let $\quad I = \int \sqrt{ax+b}\ . dx$

$\Rightarrow \quad I = \int (ax+b)^{1/2} . dx$

Putting $\quad ax + b = z \ \Rightarrow\ a . dx + 0 = dz$

$\Rightarrow \quad a . dx = dz \ \Rightarrow\ dx = \frac{1}{a} . dz$

$\therefore$ We have $\quad I = \int z^{1/2} . \frac{1}{a} . dz = \frac{1}{a}\int z^{1/2} . dz$

$$= \frac{1}{a}\frac{z^{1/2+1}}{\frac{1}{2}+1} + c = \frac{1}{a} . \frac{z^{3/2}}{\frac{3}{2}} + c$$

$$= \frac{2}{3a} z^{3/2} + c$$

$$= \frac{2}{3a}(ax+b)^{3/2} + c \qquad [\because\ z = ax + b].$$

(*vi*) Let $\quad I = \int (4x+2)\sqrt{x^2+x+1}\ . dx$

$\Rightarrow \quad I = \int 2(2x+1) . \sqrt{x^2+x+1}\ . dx$

Putting $\quad x^2 + x + 1 = z \ \Rightarrow\ (2x + 1 + 0)\, dx = dz \ \Rightarrow\ (2x+1)\, dx = dz$

$\therefore$ We have $\quad I = 2\int \sqrt{z}\ . dz = 2\int z^{1/2} . dz$

$$= 2\left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] + c = 2\left[\frac{z^{3/2}}{\frac{3}{2}}\right] + c$$

$$= \frac{4}{3} z^{3/2} + c$$

$$= \frac{4}{3}(x^2+x+1)^{3/2} + c. \qquad [\because\ z = x^2 + x + 1]$$

Example 4. *Evaluate the following :*

(i) $\int \frac{x^2}{(2+3x^3)^3} \cdot dx$ (ii) $\int \frac{1}{x(\log x)^p} \cdot dx \,;\, x > 0.$

(iii) $\int \frac{(1+\sqrt{x})^n}{\sqrt{x}} \cdot dx$ (iv) $\int \sec^4 x \,.\, \tan x \, dx.$

Solution. (i) Let $\quad I = \int \frac{x^2}{(2+3x^3)^3} \cdot dx$

Putting $\quad 2 + 3x^3 = z \;\Rightarrow\; 0 + 9x^2\,dx = dz \;\Rightarrow\; x^2\,dx = \frac{1}{9} \cdot dz$

$\therefore$ We have

$$I = \frac{1}{9}\int \frac{1}{z^3} \cdot dz = \frac{1}{9}\int z^{-3} \cdot dz$$

$$= \frac{1}{9}\frac{z^{-3+1}}{-3+1} + c = \frac{1}{9} \cdot \frac{z^{-2}}{-2} + c$$

$$= -\frac{1}{18} \cdot \frac{1}{z^2} + c$$

$$= -\frac{1}{18(2+3x^3)^2} + c. \qquad [\because\; z = 2 + 3x^3]$$

(ii) Let $\quad I = \int \frac{1}{x(\log x)^p} \cdot dx \,;\, x > 0.$

Putting $\quad \log x = z \;\Rightarrow\; \frac{1}{x} \cdot dx = dz$

$\therefore$ We have

$$I = \int \frac{1}{z^p} \cdot dz = \int z^{-p} \cdot dz$$

$$= \frac{z^{-p+1}}{-p+1} + c = \frac{z^{1-p}}{1-p} + c$$

$$= \frac{(\log x)^{1-p}}{1-p} + c. \qquad [\because\; z = \log x]$$

(iii) Let $\quad I = \int \frac{(1+\sqrt{x})^n}{\sqrt{x}} \cdot dx$

Putting $\quad 1 + \sqrt{x} = z$

$\Rightarrow \quad 1 + x^{1/2} = z \;\Rightarrow\; \left(0 + \frac{1}{2}x^{-1/2}\right) dx = dz$

$\Rightarrow \quad \frac{1}{2\sqrt{x}}\,dx = dz \;\Rightarrow\; \frac{1}{\sqrt{x}}\,dx = 2dz$

$\therefore$ We have

$$I = \int z^n \,.\, 2dz = 2\int z^n \,.\, dz$$

$$= 2\,\frac{z^{n+1}}{n+1} + c$$

$$= \frac{2(1+\sqrt{x})^{n+1}}{n+1} + c. \qquad [\because \ z = 1 + \sqrt{x}]$$

(*iv*) Let $\quad I = \int \sec^4 x \tan x \, dx.$

$\Rightarrow \quad I = \int \sec^3 x \,.\, \sec x \tan x \,.\, dx$

Putting $\quad \sec x = z \ \Rightarrow \ (\sec x \tan x) \,.\, dx = dz$

$\therefore$ We have $\quad I = \int z^3 \,.\, dz = \frac{z^4}{4} + c$

$$= \frac{\sec^4 x}{4} + c. \qquad [\because \ z = \sec x]$$

Example 5. *Evaluate the following :*

(*i*) $\int \frac{e^{2x}}{e^x + 1} \,.\, dx$ (*ii*) $\int \frac{\tan x}{\log (\sec x)} \,.\, dx$

(*iii*) $\int \frac{1}{x \log x \,[\log (\log x)]} \,.\, dx$ (*iv*) $\int \frac{\cos x - \sin x}{\cos x + \sin x} \, dx$

(*v*) $\int \frac{\sqrt{\tan x}}{\sin x \cos x} \,.\, dx$ (*vi*) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

(*vii*) $\int \sin^3 x \cos x \,.\, dx$ (*viii*) $\int \frac{x^8}{(1 - x^3)^{1/3}} \,.\, dx.$

Solution. (*i*) Let $\quad I = \int \frac{e^{2x}}{e^x + 1} \,.\, dx$

$\Rightarrow \quad I = \int \frac{e^x \,.\, e^x}{e^x + 1} \,.\, dx$

Putting $\quad e^x + 1 = z \ \Rightarrow \ e^x = (z - 1)$

$\Rightarrow \quad (e^x + 0) \,.\, dx = dz \ \Rightarrow \ e^x \,.\, dx = dz.$

$\therefore$ We have $\quad I = \int \frac{(z-1)}{z} \,.\, dz = \int \left(1 - \frac{1}{z}\right) dz$

$$= \int 1 \,.\, dz - \int \frac{1}{z} \,.\, dz = z - \log |z| + c$$

$$= (e^x + 1) - \log |e^x + 1| + c. \qquad [\because \ z = e^x + 1]$$

(*ii*) Let $\quad I = \int \frac{\tan x}{\log \sec x} \,.\, dx$

Putting $\quad \log \sec x = z$

$\Rightarrow \quad \frac{1}{\sec x} \,.\, (\sec x \tan x) \,.\, dx = dz \ \Rightarrow \ \tan x \, dx = dz$

$\therefore$ We have $\quad I = \int \frac{1}{z} \,.\, dz = \log |z| + c$

$$= \log |\log \sec x| + c. \qquad [\because \ z = \log \sec x]$$

(*iii*) Let $\quad I = \int \dfrac{1}{x \log x [\log(\log x)]} \cdot dx$

Putting $\quad \log(\log x) = z$

$\Rightarrow \quad \dfrac{1}{\log x} \cdot \dfrac{1}{x} \cdot dx = dz \quad \Rightarrow \quad \dfrac{1}{x \log x}\, dx = dz$

$\therefore$ We have $\quad I = \int \dfrac{1}{[\log(\log x)] \cdot (x \log x)} \cdot dx$

$$= \int \frac{1}{z} \cdot dz = \log |z| + c$$

$$= \log |\log(\log x)| + c. \qquad [\because\ z = \log(\log x)]$$

(*iv*) Let $\quad I = \int \dfrac{\cos x - \sin x}{\cos x + \sin x} \cdot dx$

Putting $\quad \cos x + \sin x = z$

$\Rightarrow \quad (-\sin x + \cos x)\, dx = dz \quad \Rightarrow \quad (\cos x - \sin x)\, dx = dz$

$\therefore$ We have $\quad I = \int \dfrac{1}{z} \cdot dz = \log |z| + c$

$$= \log |\cos x + \sin x| + c. \qquad [\because\ z = \cos x + \sin x]$$

(*v*) Let $\quad I = \int \dfrac{\sqrt{\tan x}}{\sin x \cos x} \cdot dx$

[Multiply and divide the denominator by $\cos x$]

$$= \int \frac{\sqrt{\tan x}}{\left(\dfrac{\sin x}{\cos x}\right) \cdot \cos^2 x} \cdot dx = \int \frac{\sqrt{\tan x}}{\tan x} \cdot \sec^2 x\, dx$$

$$= \int \frac{1}{\sqrt{\tan x}} \cdot \sec^2 x\, dx = \int (\tan x)^{-1/2} \cdot \sec^2 x\, dx$$

Putting $\quad \tan x = z \quad \Rightarrow \quad \sec^2 x\, dx = dz$

$\therefore$ We have $\quad I = \int z^{-1/2} \cdot dz = \left[\dfrac{z^{-\frac{1}{2}+1}}{-\dfrac{1}{2}+1}\right] + c$

$$= 2 z^{1/2} + c = 2\sqrt{z} + c$$

$$= 2\sqrt{\tan x} + c. \qquad [\because\ z = \tan x]$$

(*vi*) Let $\quad I = \int \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \cdot dx$

Putting $\quad e^x + e^{-x} = z \quad \Rightarrow \quad (e^x - e^{-x})\, dx = dz$

$\therefore$ We have $\quad I = \int \dfrac{1}{z} \cdot dz = \log |z| + c$

$$= \log |e^x + e^{-x}| + c. \qquad [\because\ z = e^x + e^{-x}]$$

(*vii*) Let $\quad I = \int \sin^3 x \cos x\, dx$

Putting $\quad \sin x = z \quad \Rightarrow \quad \cos x\, dx = dz$

$\therefore$ We have $\quad I = \int z^3 \,.\, dz = \dfrac{z^4}{4} + c$

$$= \frac{1}{4}\sin^4 x + c. \qquad [\because\ z = \sin x]$$

(*viii*) Let $\quad I = \int \dfrac{x^8}{(1-x^3)^{1/3}} \,.\, dx$

Putting $\quad (1 - x^3) = z \quad \Rightarrow \quad x^3 = 1 - z$

$\Rightarrow \quad 3x^2\, dx = -dz \quad \Rightarrow \quad x^2\, dx = -\dfrac{1}{3}\, dz$

$\therefore$ We have

$$I = \int \frac{x^8}{(1-x^3)^{1/3}} \,.\, dx = \int \frac{x^6}{(1-x^3)^{1/3}} \,.\, x^2\, dx$$

$$= \int \frac{(x^3)^2}{(1-x^3)^{1/3}} \,.\, x^2\, dx$$

$$= \int \frac{(1-z)^2}{z^{1/3}}\left(-\frac{1}{3}\right) dz \qquad [\because\ x^3 = 1 - z]$$

$$= -\frac{1}{3}\int \frac{1+z^2-2z}{z^{1/3}}\, dz \qquad [\because\ (a-b)^2 = a^2 + b^2 - 2ab]$$

$$= -\frac{1}{3}\left[\int \frac{1}{z^{1/3}}\, dz + \int \frac{z^2}{z^{1/3}} \,.\, dz - \int \frac{2z}{z^{1/3}} \,.\, dz\right]$$

$$= -\frac{1}{3}\left[\int z^{-1/3} \,.\, dz + \int z^{5/3}\, dz - 2\int z^{2/3} \,.\, dz\right]$$

$$= -\frac{1}{3}\left[\frac{z^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + \frac{z^{\frac{5}{3}+1}}{\frac{5}{3}+1} - 2 \,.\, \frac{z^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right] + c$$

$$= -\frac{1}{3}\left[\frac{3}{2}z^{2/3} + \frac{3}{8}z^{8/3} - 2 \,.\, \frac{3}{5}z^{5/3}\right] + c$$

$$= -\frac{1}{2}z^{2/3} - \frac{1}{8}z^{8/3} + \frac{2}{5}z^{5/3} + c$$

$$= -\frac{1}{2}(1-x^3)^{2/3} - \frac{1}{8}(1-x^3)^{8/3} + \frac{2}{5}(1-x)^{5/3} + c. \quad [\because\ z = 1 - x^3]$$

Example 6. *Evaluate the following integrals :*

(*i*) $\int \dfrac{1}{x \log x} \,.\, dx \qquad$ (*ii*) $\int \dfrac{e^x}{3+e^x} \,.\, dx \qquad$ (*iii*) $\int \dfrac{x^{e-1}+e^{x-1}}{x^e+e^x} \,.\, dx.$

Solution. (*i*) Let $\quad I = \int \dfrac{1}{x \log x}\, dx$

Putting $\quad \log x = z \quad \Rightarrow \quad \dfrac{1}{x} \,.\, dx = dz$

$\therefore$ We have $\quad I = \int \frac{1}{z} \cdot dz = \log |z| + c$

$= \log |\log x| + c. \qquad [\because \; z = \log x]$

(*ii*) Let $\quad I = \int \frac{e^x}{3 + e^x} \cdot dx$

Putting $\quad 3 + e^x = z \;\Rightarrow\; (0 + e^x)\, dx = dz \;\Rightarrow\; e^x\, dx = dz$

$\therefore$ We have $\quad I = \int \frac{1}{z} \cdot dz = \log |z| + c$

$= \log |3 + e^x| + c. \qquad [\because \; z = 3 + e^x]$

(*iii*) Let $\quad I = \int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} \cdot dx \qquad$ [Multiply and divided by e]

$\Rightarrow \quad I = \frac{1}{e} \int \frac{e\,(x^{e-1} + e^{x-1})}{x^e + e^x} \cdot dx = \frac{1}{e} \int \frac{(ex^{e-1} + e^x)}{x^e + e^x} \cdot dx$

Putting $\quad x^e + e^x = z \;\Rightarrow\; (ex^{e-1} + e^x) \cdot dx = dz$

$\therefore$ We have $\quad I = \frac{1}{e} \int \frac{1}{z} \cdot dz = \frac{1}{e} \log |z| + c$

$= \frac{1}{e} \log |x^e + e^x| + c. \qquad [\because \; z = x^e + e^x]$

Example 7. *Evaluate the following :*

(*i*) $\int e^{3 \log x} (x^4 + 1)^{-1} \cdot dx \qquad$ (*ii*) $\int \tan^3 x \sec^2 x \cdot dx$

(*iii*) $\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \cdot dx \qquad$ (*iv*) $\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} \cdot dx$

(*v*) $\int \sqrt{\sin 2x} \cdot \cos 2x \cdot dx \qquad$ (*vi*) $\int \frac{\cos x}{\sqrt{1 + \sin x}} \cdot dx.$

Solution. (*i*) Let $\quad I = \int e^{3 \log x} (x^4 + 1)^{-1} \cdot dx = \int e^{\log x^3} \cdot \frac{1}{x^4 + 1} \cdot dx$

$= \int \frac{x^3}{x^4 + 1} \cdot dx \qquad [\because \; e^{\log x^3} = x^3]$

Putting $\quad x^4 + 1 = z \;\Rightarrow\; (4x^3 + 0)\, dx = dz$

$\Rightarrow \quad 4x^3\, dx = dz \;\Rightarrow\; x^3\, dx = \frac{1}{4} dz$

$\therefore$ We have $\quad I = \int \frac{1}{z} \cdot \frac{1}{4} \cdot dz = \frac{1}{4} \int \frac{1}{z} \cdot dz$

$= \frac{1}{4} \log |z| + c = \frac{1}{4} \log |x^4 + 1| + c.$

(*ii*) Let $\quad I = \int \tan^3 x \sec^2 x\, dx$

Putting $\quad \tan x = z \;\Rightarrow\; \sec^2 x\, dx = dz$

$\therefore$ We have $\quad I = \int z^3 \cdot dz = \frac{z^4}{4} + c$

$$= \frac{1}{4}\tan^4 x + c. \qquad [\because\ z = \tan x]$$

(iii) Let $\quad I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \, . \, dx.$

Putting $\quad e^{2x} + e^{-2x} = z \quad \Rightarrow \quad (2e^{2x} - 2e^{-2x}) \, . \, dx = dz$

$\Rightarrow \quad 2(e^{2x} - e^{-2x})\, dx = dz \quad \Rightarrow \quad (e^{2x} - e^{-2x})\, dx = \frac{1}{2}\, dz$

$\therefore$ We have $\quad I = \frac{1}{2}\int \frac{1}{z}\, dz = \frac{1}{2}\log |z| + c$

$$= \frac{1}{2}\log |e^{2x} + e^{-2x}| + c. \qquad [\because\ z = e^{2x} + e^{-2x}]$$

(iv) Let $\quad I = \int \frac{\tan^4 \sqrt{x}\sec^2 \sqrt{x}}{\sqrt{x}} \, . \, dx$

Putting $\quad \tan \sqrt{x} = z \quad \Rightarrow \quad \sec^2 \sqrt{x} \, . \, \frac{d}{dx}(\sqrt{x}) = dz$

$\Rightarrow \quad \sec^2 \sqrt{x} \, . \, \frac{1}{2} \, . \, \frac{1}{\sqrt{x}}\, dx = dz \quad \Rightarrow \quad \frac{\sec^2 \sqrt{x}}{\sqrt{x}} = 2\, dz$

$\therefore$ We have $\quad I = 2\int z^4 \, . \, dz = 2\,\frac{z^5}{5} + c$

$$= 2\,\frac{\tan^5 \sqrt{x}}{5} + c \qquad [\because\ z = \tan \sqrt{x}]$$

$$= \frac{2}{5}\tan^5 \sqrt{x} + c.$$

(v) Let $\quad I = \int \sqrt{\sin 2x} \, . \, \cos 2x \, . \, dx$

Putting $\quad \sin 2x = z \quad \Rightarrow \quad 2\cos 2x\, dx = dz \quad \Rightarrow \quad \cos 2x\, dx = \frac{1}{2}\, dz$

$\therefore$ We have $\quad I = \frac{1}{2}\int \sqrt{z} \, . \, dz = \frac{1}{2}\int z^{1/2}\, dz$

$$= \frac{1}{2}\,\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{1}{2} \, . \, \frac{2}{3}\, z^{3/2} + c$$

$$= \frac{1}{3}(\sin 2x)^{3/2} + c. \qquad [\because\ z = \sin 2x]$$

(vi) Let $\quad I = \int \frac{\cos x}{\sqrt{1 + \sin x}}\, dx$

Putting $\quad 1 + \sin x = z \quad \Rightarrow \quad (0 + \cos x)\, dx = dz \quad \Rightarrow \quad \cos x\, dx = dz$

$\therefore$ We have $\quad I = \int \frac{1}{\sqrt{z}} \, . \, dz = \int z^{-1/2} \, . \, dz$

$$= \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{2}{1} \cdot z^{1/2} + c$$

$$= 2(1 + \sin x)^{1/2} + c \qquad [\because \ z = 1 + \sin x]$$

$$= 2\sqrt{1 + \sin x} + c.$$

Example 8. *Evaluate :*

(i) $\int \frac{(2 + 3x)}{(3 - 2x)} \cdot dx$ (ii) $\int \frac{1}{x\sqrt{x^6 - 1}} \cdot dx$

(iii) $\int (4x + 2)\sqrt{x^2 + x + 1}$ (iv) $\int \frac{\sec x}{\log(\sec x + \tan x)} \cdot dx.$

Solution. (i) Let $I = \int \frac{2 + 3x}{3 - 2x} \cdot dx$

Putting $(3 - 2x) = z \Rightarrow -2x = z - 3$

$$\Rightarrow \qquad x = \left(\frac{3 - z}{2}\right) \Rightarrow dx = -\frac{1}{2}dz$$

$\therefore$ We have $I = \int \frac{2 + 3\left(\frac{3 - z}{2}\right)}{z}\left(-\frac{1}{2}dz\right)$

$$= -\frac{1}{2}\int \frac{2 + \frac{9 - 3z}{2}}{z} \cdot dz = -\frac{1}{4}\int\left(\frac{13 - 3z}{z}\right) \cdot dz$$

$$= -\frac{1}{4}\left[\int \frac{13}{z} \cdot dz - \int \frac{3z}{z}dz\right] = -\frac{13}{4}\int \frac{1}{z}dz + \frac{3}{4}\int 1 . dz$$

$$= -\frac{13}{4} \log|z| + \frac{3}{4}z + c$$

$$= -\frac{13}{4} \log|(3 - 2x)| + \frac{3}{4}(3 - 2x) + c. \qquad [\because \ z = 3 - 2x]$$

(ii) Let $I = \int \frac{1}{x\sqrt{x^6 - 1}} \cdot dx$

$\Rightarrow$ $I = \int \frac{x^2}{x^3\sqrt{x^6 - 1}} \cdot dx$ [Multiply and divided by x^2]

Putting $x^3 = z$

$\Rightarrow$ $3x^2\,dx = dz \Rightarrow x^2\,dx = \frac{1}{3}dz$

$\therefore$ We have $I = \frac{1}{3}\int \frac{1}{z\sqrt{z^2 - 1}}dz$

$$\left[\begin{array}{l} \because \ \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}} \\ \Rightarrow \int \frac{1}{x\sqrt{x^2 - 1}} \cdot dx = \sec^{-1}x + c \end{array}\right]$$

$$= \frac{1}{3}\sec^{-1} z + c$$

$$= \frac{1}{3}\sec^{-1} x^3 + c. \qquad [\because\ z = x^3]$$

(iii) Let $\quad I = \int (4x+2)\sqrt{x^2+x+1}\ .\ dx$

Putting $\quad x^2 + x + 1 = z \quad \Rightarrow \quad (2x + 1 + 0)\, dx = dz$

$\Rightarrow \quad (2x+1)\, dx = dz \quad \Rightarrow \quad 2(2x+1)\, dx = 2dz$

$\Rightarrow \quad (4x+2)\, dx = 2dz.$

$\therefore$ We have $\quad I = \int \sqrt{z}\ .\ 2dz = 2\int z^{1/2}\, dz$

$$= 2\left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] + c = \frac{4}{3} z^{3/2} + c$$

$$= \frac{4}{3}(x^2+x+1)^{3/2} + c. \qquad [\because\ z = x^2+x+1]$$

(iv) Let $\quad I = \int \frac{\sec x}{\log(\sec x + \tan x)}\ .\ dx$

Putting $\quad \log(\sec x + \tan x) = z$

$\Rightarrow \quad \frac{1}{(\sec x + \tan x)}\ .\ (\sec x \tan x + \sec^2 x)\, dx = dz \quad \Rightarrow \quad \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\, dx = dz$

$\Rightarrow \quad \sec x\, dx = dz$

$\therefore$ We have $\quad I = \int \frac{1}{z}\ .\ dz = \log|z| + c$

$$= \log|\log(\sec x + \tan x)| + c. \qquad [\because\ z = \log(\sec x + \tan x)]$$

Example 9. *Evaluate the following integrals :*

(i) $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\ .\ dx$ $\qquad$ *(ii)* $\int \frac{2^x}{1-4^x}\, dx$

(iii) $\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}\, dx$ $\qquad$ *(iv)* $\int \frac{1}{\cos^2 x\,(1-\tan x)^2}\, dx$

(v) $\int \frac{\sin x}{(1+\cos x)^2}\ .\ dx$ $\qquad$ *(vi)* $\int \frac{\sqrt{1+x}}{\sqrt{1-x}}\ .\ dx$ $\qquad$ *(vii)* $\int \frac{1+\cos x}{1-\cos x}\, dx.$

Solution. *(i)* Let $I = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\ .\ dx$

Putting $\quad \sin^{-1} x = z \quad \Rightarrow \quad \frac{1}{\sqrt{1-x^2}}\ .\ dx = dz$

$\therefore$ We have $\quad I = \int e^z\ .\ dz = e^z + c$

$$= e^{\sin^{-1}x} + c. \qquad [\because\ z = \sin^{-1} x]$$

(*ii*) Let $\quad I = \int \frac{2^x}{\sqrt{1-4^x}} \, . \, dx$

Putting $\quad 2^x = z \;\Rightarrow\; 2^x \log 2 \, . \, dx = dz \qquad \left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$

$\Rightarrow \quad 2^x \, dx = \frac{1}{\log 2} \, . \, dz$

$\therefore$ We have $\quad I = \int \frac{1}{\sqrt{1-z^2}} \cdot \frac{1}{\log 2} \, dz \qquad [\because \; 4^x = (2^2)^x = 2^{2x} = (2^x)^2 = z^2]$

$$= \frac{1}{\log 2} \int \frac{1}{\sqrt{1-z^2}} \, dz = \frac{1}{\log 2} \sin^{-1} z + c$$

$$= \frac{1}{\log 2} \, . \, \sin^{-1} 2^x + c. \qquad [\because \; z = 2^x]$$

(*iii*) Let $\quad I = \int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} \, . \, dx$

$\Rightarrow \quad I = \frac{1}{2} \int \frac{2\cos x - 3\sin x}{3\cos x + 2\sin x} \, . \, dx$

Putting $\quad 3\cos x + 2\sin x = z \;\Rightarrow\; (-3\sin x + 2\cos x)\, dx = dz$

$\Rightarrow \quad (2\cos x - 3\sin x)\, dx = dz$

$\therefore$ We have $\quad I = \frac{1}{2} \int \frac{1}{z} \, . \, dz = \frac{1}{2} \log |z| + c$

$$= \frac{1}{2} \log |3\cos x + 2\sin x| + c. \qquad [\because \; z = 3\cos x + 2\sin x]$$

(*iv*) Let $\quad I = \int \frac{1}{\cos^2 x\,(1-\tan x)^2} \, . \, dx = \int \frac{\sec^2 x}{(1-\tan x)^2} \, . \, dx$

Putting $\quad (1-\tan x) = z \;\Rightarrow\; -\sec^2 x \, dx = dz \;\Rightarrow\; \sec^2 x \, dx = -dz$

$\therefore$ We have $\quad I = -\int \frac{1}{z^2} \, . \, dz = -\int z^{-2} \, . \, dz = -\left[\frac{z^{-2+1}}{-2+1}\right] + c$

$$= -\left[-\frac{1}{z}\right] + c = \frac{1}{z} + c$$

$$= \frac{1}{(1-\tan x)} + c. \qquad [\because \; z = 1 - \tan x]$$

(*v*) Let $\quad I = \int \frac{\sin x}{(1+\cos x)^2} \, . \, dx$

Putting $\quad (1+\cos x) = z \;\Rightarrow\; -\sin x \, dx = dz \;\Rightarrow\; \sin x \, dx = -dz.$

$\therefore$ We have $\quad I = -\int \frac{1}{z^2} \, . \, dz = -\int z^{-2} \, . \, dz$

$$= -\left[\frac{z^{-2+1}}{-2+1}\right] + c = -\left[-\frac{1}{z}\right] + c = \frac{1}{z} + c$$

$$= \frac{1}{1+\cos x} + c. \qquad [\because \; z = 1 + \cos x]$$

(*vi*) Let $\quad I = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot dx = \int \left(\frac{\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}} \right) \cdot dx$ [On rationalization]

$$= \int \frac{1+x}{\sqrt{1-x^2}} \cdot dx$$

$$= \int \frac{1}{\sqrt{1-x^2}} \cdot dx + \int \frac{x}{\sqrt{1-x^2}} \cdot dx \qquad \text{...(1)}$$

$$= \sin^{-1} x + \int \frac{x}{\sqrt{1-x^2}} \cdot dx$$

Putting $\quad 1 - x^2 = z \Rightarrow (0 - 2x)\, dx = dz$

$\Rightarrow \quad -2x\, dx = dz \Rightarrow x\, dx = -\frac{1}{2} dz$

$\therefore$ We have

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{z}} \cdot \left(-\frac{1}{2} dz \right) . = -\frac{1}{2} \int z^{-1/2} \cdot dz$$

$$= -\frac{1}{2} \left[\frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right] + c = -\frac{1}{2} \left[\frac{z^{1/2}}{1/2} \right] + c = -\sqrt{z} + c$$

$\therefore \quad \int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + c.$ $\quad [\because z = 1 - x^2]$

Putting this value in equation (1), we get

$$I = \sin^{-1} x + \left(-\sqrt{1-x^2} \right) + c$$

$$= \sin^{-1} x - \sqrt{1-x^2} + c.$$

(*vii*) Let $\quad I = \int \frac{1 + \cos x}{1 - \cos x} \cdot dx$

$$= \int \frac{2\cos^2 \frac{x}{2}}{2\sin^2 \frac{x}{2}} \cdot dx \qquad \left[\begin{array}{l} \because 1 + \cos 2A = 2\cos^2 A \\ \quad 1 - \cos 2A = 2\sin^2 A \end{array} \right]$$

$$= \int \cot^2 \frac{x}{2} dx$$

$$= \int \left(\text{cosec}^2 \frac{x}{2} - 1 \right) dx \qquad [\because \text{cosec}^2 A - \cot^2 A = 1]$$

$$= \int \text{cosec}^2 \frac{x}{2} dx - \int 1 \cdot dx = \frac{-\cot(x/2)}{1/2} - x + c$$

$$= -2\cot \frac{x}{2} - x + c.$$

Example 10. *Evaluate the following integrals :*

(i) $\int \frac{a}{b+ce^x} \cdot dx$ (ii) $\int \frac{1-\cot x}{1+\cot x} \cdot dx$

(iii) $\int \frac{1+\log x}{3+x\log x} dx$ (iv) $\int \frac{\sin(2\tan^{-1} x)}{1+x^2} \cdot dx$

(v) $\int \frac{\tan x}{a+b\tan^2 x} \cdot dx$ (vi) $\int \frac{1}{1-\tan x} \cdot dx.$

Solution. (i) Let $I = \int \frac{a}{b+ce^x} \cdot dx$

$\therefore$ $I = \int \frac{a\,e^{-x}}{be^{-x}+c} \cdot dx$ [Dividing numerator and denominator by e^x]

Putting $b\,e^{-x} + c = z \Rightarrow -b\,e^{-x}\,dx = dz \Rightarrow e^{-x}\,dx = -\frac{1}{b} \cdot dz$

$\therefore$ We have $I = \int \frac{a}{z} \cdot \left(-\frac{1}{b} dz\right) = -\frac{a}{b} \int \frac{1}{z} \cdot dz = -\frac{a}{b} \log |z| + c_1$

$= -\frac{a}{b} \log |b\,e^{-x} + c| + c_1.$ [$\because$ $z = b\,e^{-x} + c$]

(ii) Let $I = \int \frac{1-\cot x}{1+\cot x} \cdot dx = \int \frac{1-\frac{\cos x}{\sin x}}{1+\frac{\cos x}{\sin x}} \cdot dx$

$= \int \frac{\sin x - \cos x}{\sin x + \cos x} \cdot dx$

Putting $\sin x + \cos x = z \Rightarrow (\cos x - \sin x)\,dx = dz$

$\Rightarrow -(\sin x - \cos x)\,dx = dz \Rightarrow (\sin x - \cos x)\,dx = -dz$

$\therefore$ We have $I = -\int \frac{1}{z} \cdot dz = -\log |z| + c$

$= -\log |\sin x + \cos x| + c.$ [$\because$ $z = \sin x + \cos x$]

(iii) Let $I = \int \frac{1+\log x}{3+x\log x} \cdot dx$

Putting $(3 + x\log x) = z \Rightarrow \left[0 + x \cdot \frac{1}{x} + \log x \cdot 1\right] dx = dz$

$\Rightarrow (1 + \log x)\,dx = dz$

$\therefore$ We have $I = \int \frac{1}{z} \cdot dz = \log |z| + c$

$= \log |3 + x\log x| + c.$ [$\because$ $z = 3 + x\log x$]

(iv) Let $I = \int \frac{\sin(2\tan^{-1} x)}{1+x^2} \cdot dx$

Putting $\tan^{-1} x = z \Rightarrow \frac{1}{1+x^2} dx = dz$

$\therefore$ We have $\quad I = \int \sin 2z \,.\, dz = -\dfrac{\cos 2z}{2} + c$

$$= -\frac{1}{2}\cdot\left(\frac{1-\tan^2 z}{1+\tan^2 z}\right) + c \qquad \left[\because \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}\right]$$

$$= -\frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right) + c. \qquad \left[\begin{array}{l}\because \tan^{-1} x = z \\ \Rightarrow \quad x = \tan z\end{array}\right]$$

(*v*) Let $\quad I = \int \dfrac{\tan x}{a + b\tan^2 x}\,.\,dx$

$$= \int \frac{\dfrac{\sin x}{\cos x}}{a + b\dfrac{\sin^2 x}{\cos^2 x}}\,.\,dx = \int \frac{\dfrac{\sin x}{\cos x}}{\dfrac{a\cos^2 x + b\sin^2 x}{\cos^2 x}}\,dx$$

$\therefore \quad I = \int \dfrac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\,.\,dx$

Putting $\quad a\cos^2 x + b\sin^2 x = z$

$\Rightarrow \quad (2a\cos x\,(-\sin x) + 2b\sin x\cos x)\,dx = dz$

$\Rightarrow \quad (-2a\sin x\cos x + 2b\sin x\cos x)\,dx = dz$

$\Rightarrow \quad 2(b-a)\sin x\cos x\,dx = dz$

$\Rightarrow \quad \sin x\cos x\,dx = \dfrac{1}{2(b-a)}\,.\,dz$

$\therefore$ We have $\quad I = \int \dfrac{1}{z}\cdot\dfrac{1}{2(b-a)}\,.\,dz = \dfrac{1}{2(b-a)}\int \dfrac{1}{z}\,.\,dz$

$$= \frac{1}{2(b-a)}\,.\,\log|z| + c$$

$$= \frac{1}{2(b-a)}\log|a\cos^2 x + b\sin^2 x| + c. \qquad [\because \; z = a\cos^2 x + b\sin^2 x]$$

(*vi*) Let $\quad I = \int \dfrac{1}{1-\tan x}\,.\,dx = \int \dfrac{1}{1-\dfrac{\sin x}{\cos x}}\,.\,dx = \int \dfrac{\cos x}{\cos x - \sin x}\,.\,dx$

$$= \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x}\,.\,dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int \frac{(\cos x - \sin x) - (-\sin x - \cos x)}{(\cos x - \sin x)}\,.\,dx$$

[Add and subtract $\sin x$ to the numerator]

$$= \frac{1}{2}\int 1\,.\,dx - \frac{1}{2}\int \frac{-\sin x - \cos x}{\cos x - \sin x}\,.\,dx$$

$$= \frac{x}{2} - \frac{1}{2}\int \frac{-\sin x - \cos x}{\cos x - \sin x}\,dx + c \qquad \text{...(1)}$$

Putting $\quad \cos x - \sin x = z \;\Rightarrow\; (-\sin x - \cos x)\,dx = dz$

$\therefore$ We have $\quad I = \dfrac{x}{2} - \dfrac{1}{2}\int \dfrac{1}{z}\,.\,dz + c$

$$= \frac{x}{2} - \frac{1}{2} \log |z| + c$$

$$= \frac{x}{2} - \frac{1}{2} \log |\cos x - \sin x| + c. \qquad [\because \; z = \cos x - \sin x]$$

Example 11. *Evaluate the following integrals :*

(i) $\int \frac{4(x+1)(x+\log x)^3}{x} \, . \, dx$ (ii) $\int \operatorname{cosec} x \log (\operatorname{cosec} x - \cot x) \, . \, dx$

(iii) $\int \frac{1}{x^2} \sin\left(\frac{1}{x}\right) . \, dx$ (iv) $\int \sqrt{e^x - 1} \, . \, dx.$

Solution. (i) Let $\quad I = \int \frac{4(x+1)(x+\log x)^3}{x} \, . \, dx$

$$= \int 4\left(\frac{x+1}{x}\right) . (x + \log x)^3 \, . \, dx = 4 \int \left(1 + \frac{1}{x}\right)(x + \log x)^3 \, . \, dx$$

Putting $\quad x + \log x = z \;\Rightarrow\; \left(1 + \frac{1}{x}\right) dx = dz$

$\therefore$ We have $\quad I = 4 \int z^3 \, . \, dz$

$$= 4 \, . \, \frac{z^4}{4} + c = z^4 + c$$

$$= (x + \log x)^4 + c. \qquad [\because \; z = x + \log x]$$

(ii) Let $\quad I = \int \operatorname{cosec} x \log (\operatorname{cosec} x - \cot x) \, . \, dx$

Putting $\quad \log (\operatorname{cosec} x - \cot x) = z$

$\Rightarrow \quad \frac{1}{(\operatorname{cosec} x - \cot x)} \, . \, (-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x) \, dx = dz$

$\Rightarrow \quad \frac{\operatorname{cosec} x (\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)} \, . \, dx = dz$

$\Rightarrow \quad \operatorname{cosec} x \, dx = dz$

$\therefore$ We have $\quad I = \int z \, . \, dz = \frac{z^2}{2} + c$

$$= \frac{1}{2} [\log (\operatorname{cosec} x - \cot x)]^2 + c. \qquad [\because \; z = \log (\operatorname{cosec} x - \cot x)]$$

(iii) Let $\quad I = \int \frac{1}{x^2} \sin\left(\frac{1}{x}\right) . \, dx$

Putting $\quad \frac{1}{x} = z$

$\Rightarrow \quad -\frac{1}{x^2} \, dx = dz \;\Rightarrow\; \frac{1}{x^2} \, dx = -\, dz$

$\therefore$ We have $\quad I = \int -\sin z \, dz = -\int \sin z \, dz = -(-\cos z) + c = \cos z + c$

$$= \cos\left(\frac{1}{x}\right) + c. \qquad \left[\because \; z = \frac{1}{x}\right]$$

(*iv*) Let $\quad I = \int \sqrt{e^x - 1}\,.\,dx$

Putting $\sqrt{e^x - 1} = z$

$\Rightarrow \quad e^x - 1 = z^2 \quad \Rightarrow \quad e^x\,.\,dx = 2zdz$

$\Rightarrow \quad dx = \dfrac{2zdz}{e^x} \quad \Rightarrow \quad dx = \dfrac{2zdz}{1+z^2}$

$\therefore$ We have $\quad I = \int z\,.\,\dfrac{2z}{1+z^2}\,.\,dz \qquad [\because \; z^2 + 1 = e^x]$

$$= 2\int \frac{z^2}{1+z^2}\,dz$$

$$= 2\int \frac{1+z^2-1}{1+z^2}\,.\,dz \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= 2\int \left(1 - \frac{1}{1+z^2}\right).\,dz$$

$$= 2\left[\int 1\,.\,dz - \int \frac{1}{1+z^2}\,.\,dz\right] = 2\,[z - \tan^{-1} z] + c$$

$$= 2\left[\sqrt{e^x - 1} - \tan^{-1}\sqrt{e^x - 1}\right] + c. \qquad \left[\because \; z = \sqrt{e^x - 1}\right]$$

Example 12. *Evaluate the following integrals :*

(*i*) $\displaystyle\int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}\,.\,dx$ (*ii*) $\displaystyle\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x}\,.\,dx$

(*iii*) $\displaystyle\int \frac{x + \sqrt{x+1}}{x+2}\,.\,dx$ (*iv*) $\displaystyle\int \frac{1}{\left(\sqrt{x} + \sqrt{x+1}\right)}\,.\,dx.$

Solution. (*i*) Let $\quad I = \displaystyle\int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}\,.\,dx$

Putting $\quad \tan^{-1} x^4 = z \quad \Rightarrow \quad \dfrac{1}{1+(x^4)^2}\,.\,4x^3\,.\,dx = dz$

$\Rightarrow \quad \dfrac{1}{1+x^8}\,.\,4x^3\,dx = dz \quad \Rightarrow \quad \dfrac{x^3}{1+x^8}\,dx = \dfrac{1}{4}\,dz$

$\therefore$ We have $\quad I = \dfrac{1}{4}\displaystyle\int \sin z\,.\,dz = \frac{1}{4}(-\cos z) + c$

$$= -\frac{1}{4}\cos(\tan^{-1} x^4) + c. \qquad [\because \; z = \tan^{-1} x^4]$$

(*ii*) Let $\quad I = \displaystyle\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x}\,.\,dx$

Putting $\quad a^2 \sin^2 x + b^2 \cos^2 x = z$

$\Rightarrow \quad (2a^2 \sin x\,.\,\cos x - 2b^2 \cos x\,.\,\sin x)\,dx = dz$

$\Rightarrow \quad 2 \sin x \cos x\,(a^2 - b^2)\,.\,dx = dz$

$\Rightarrow \quad \sin 2x\,.\,dx = \dfrac{1}{(a^2 - b^2)}\,.\,dz$

$\therefore$ We have $$I = \int \frac{1}{z} \cdot \frac{1}{(a^2 - b^2)} \cdot dz$$
$$= \frac{1}{(a^2 - b^2)} \int \frac{1}{z} \cdot dz = \frac{1}{(a^2 - b^2)} \log |z| + c$$
$$= \frac{1}{(a^2 - b^2)} \log |a^2 \sin^2 x + b^2 \cos^2 x| + c.$$
$$[\because \; z = a^2 \sin^2 x + b^2 \cos^2 x]$$

(*iii*) Let $$I = \int \frac{x + \sqrt{x+1}}{x+2} \cdot dx$$

Putting $\sqrt{x+1} = z$

$\Rightarrow$ $x + 1 = z^2 \;\Rightarrow\; x = z^2 - 1 \;\Rightarrow\; dx = 2z\, dz$

$\therefore$ We have $$I = \int \frac{[(z^2-1) + z]}{(z^2 - 1 + 2)} \cdot 2z\, dz$$
$$= 2 \int \frac{(z^2 - 1 + z)\, z}{z^2 + 1} \cdot dz$$
$$= 2 \int \frac{z^3 + z^2 - z}{z^2 + 1} \cdot dz$$
$$= 2 \int \left[(z+1) - \frac{2z+1}{z^2+1} \right] \cdot dz$$

$$\begin{array}{r|l} z^2+1 \,\big)\, z^3 + z^2 - z \,\big(\, z+1 & \\ z^3 \quad\;\; + z & \\ - \quad\;\;\; - & \\ \hline z^2 - 2z & \\ z^2 \quad + 1 & \\ - \quad\;\; - & \\ \hline -2z - 1 & \end{array}$$

$$= 2 \int \left(z + 1 - \frac{2z}{z^2+1} - \frac{1}{z^2+1} \right) \cdot dz$$
$$= 2 \int z \,.\, dz + 2 \int 1 \,.\, dz - 2 \int \frac{2z}{z^2+1} \,.\, dz - 2 \int \frac{1}{z^2+1} \,.\, dz$$
$$= 2 \,.\, \frac{z^2}{2} + 2z - 2 \log |z^2 + 1| - 2 \tan^{-1} z + c$$
$$= z^2 + 2z - 2 \log |z^2 + 1| - 2 \tan^{-1} z + c \qquad [\because (z = \sqrt{x+1})]$$
$$= (x+1) + 2\sqrt{x+1} - 2 \log |x + 2| - 2 \tan^{-1} \sqrt{x+1} + c.$$

(*iv*) Let $$I = \int \frac{1}{\left(\sqrt{x} + \sqrt{x+1}\right)} \cdot dx$$
$$= \int \frac{1}{\left(\sqrt{x} + \sqrt{x+1}\right)} \times \frac{\left(\sqrt{x} - \sqrt{x+1}\right)}{\left(\sqrt{x} - \sqrt{x+1}\right)} \cdot dx \qquad \text{[On rationalization]}$$
$$= \int \frac{\left(\sqrt{x} - \sqrt{x+1}\right)}{x - (x+1)} \cdot dx = \int \frac{\sqrt{x} - \sqrt{x+1}}{-1} \cdot dx$$
$$= -\int \sqrt{x}\, dx + \int \sqrt{x+1} \cdot dx = -\int x^{1/2} \cdot dx + \int (x+1)^{1/2} \cdot dx$$

$$= -\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] + \left[\frac{(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] + c$$

$$= -\frac{2}{3}x^{3/2} + \frac{2}{3}(x+1)^{3/2} + c.$$

Example 13. *Evaluate the following integrals :*

(i) $\int \frac{\sin x}{\sin (x-a)} \cdot dx$

(ii) $\int \frac{1}{\sin (x-a) \cos (x-b)} \cdot dx \; ; a-b \neq (2n+1)\frac{\pi}{2} \; ; n \in Z$

(iii) $\int \frac{1}{\sin (x-a) \sin (x-b)} \cdot dx$ (iv) $\int \frac{\sin 2x}{\sin 4x} \cdot dx$

(v) $\int \frac{1}{\sqrt{\sin^3 x \sin (x+a)}} \cdot dx$ (vi) $\int \frac{1}{\cos (x-a) \cos (x-b)} \cdot dx$

(vii) $\int \frac{\sin (x+a)}{\sin (x+b)} \cdot dx.$

Solution. (i) Let $I = \int \frac{\sin x}{\sin (x-a)} \cdot dx$

Putting $x - a = z$

$\Rightarrow$ $x = z + a \Rightarrow dx = dz$

$\therefore$ We have $I = \int \frac{\sin (z+a)}{\sin z} \cdot dz$ $[\because \sin (A+B) = \sin A \cos B + \cos A \sin B]$

$$= \int \frac{\sin z \cos a + \cos z \sin a}{\sin z} \cdot dz$$

$$= \int \frac{\sin z \cos a}{\sin z} \cdot dz + \int \frac{\cos z \sin a}{\sin z} \cdot dz$$

$$= \int \cos a \cdot dz + \int \sin a \cot z \cdot dz$$

$$= z \cos a + \sin a \log |\sin z| + c$$

$$= (x-a) \cos a + \sin a \log |\sin (x-a)| + c.$$ $[\because z = x - a]$

(ii) Let $I = \int \frac{1}{\sin (x-a) \cos (x-b)} \cdot dx \; ; (a-b) \neq (2n+1)\frac{\pi}{2} \; ; n \in Z$

[Multiply and divided by $\cos (a-b)$]

$\Rightarrow$ $I = \frac{1}{\cos (a-b)} \int \frac{\cos (a-b)}{\sin (x-a) \cos (x-b)} \cdot dx$

$$= \frac{1}{\cos (a-b)} \int \frac{\cos [(x-b)-(x-a)]}{\sin (x-a) \cos (x-b)} \cdot dx$$

$$\left[\begin{array}{l} \because (x-b)-(x-a) = -b+a = (a-b) \\ \because \cos (A-B) = \cos A \cos B + \sin A \sin B \end{array}\right]$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(x-a)\cos(x-b) + \sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)} \, . \, dx$$

$$= \frac{1}{\cos(a-b)} \int \left[\frac{\cos(x-a)\cos(x-b)}{\sin(x-a)\cos(x-b)} + \frac{\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)} \right] . \, dx$$

$$= \frac{1}{\cos(a-b)} \int [\cot(x-a) + \tan(x-b)] \, . \, dx$$

$$= \frac{1}{\cos(a-b)} [\log|\sin(x-a)| - \log|\cos(x-b)|] + c$$

$$\left[\because \log m - \log n = \log \frac{m}{n} \right]$$

$$= \frac{1}{\cos(a-b)} \left[\log \left| \frac{\sin(x-a)}{\sin(x-b)} \right| \right] + c.$$

(iii) Let $I = \int \frac{1}{\sin(x-a)\sin(x-b)} \, . \, dx$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)} \, . \, dx$$

[Multiply and divided by $\sin(b-a)$.]

$$= \frac{1}{\sin(b-a)} \int \frac{\sin[(x-a)-(x-b)]}{\sin(x-a)\sin(x-b)} \, . \, dx$$

$$\left[\begin{array}{l} \because (x-a)-(x-b) = -a+b = (b-a) \\ \because \sin(A-B) = \sin A \cos B - \cos A \sin B \end{array} \right]$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)} \, . \, dx$$

$$= \frac{1}{\sin(b-a)} \int \left[\frac{\sin(x-a)\cos(x-b)}{\sin(x-a)\sin(x-b)} - \frac{\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)} \right] . \, dx$$

$$= \frac{1}{\sin(b-a)} \left[\int \cot(x-b) \, . \, dx - \int \cot(x-a) \, . \, dx \right]$$

$$= \frac{1}{\sin(b-a)} [\log|\sin(x-b)| - \log|\sin(x-a)|] + c$$

$$= \frac{1}{\sin(b-a)} \left[\log \left| \frac{\sin(x-b)}{\sin(x-a)} \right| \right] + c.$$

(iv) Let $I = \int \frac{\sin 2x}{\sin 4x} \, . \, dx$

$$= \int \frac{\sin 2x}{2 \sin 2x \cos 2x} \, . \, dx = \frac{1}{2} \int \sec 2x \, dx \qquad [\because \sin 2A = 2 \sin A \cos A]$$

$$= \frac{1}{2} \left[\frac{\log|\sec 2x + \tan 2x|}{2} \right] + c$$

$$= \frac{1}{4} \log|\sec 2x + \tan 2x| + c.$$

(*v*) Let $\quad I = \int \frac{1}{\sqrt{\sin^3 x \sin(x+a)}} \, . \, dx$

$= \int \frac{\sqrt{\sin x}}{\sqrt{\sin^4 x \, . \sin(x+a)}} \, . \, dx$ [Multiply and divided by $\sqrt{\sin x}$]

$= \int \frac{1}{\sin^2 x} \cdot \sqrt{\frac{\sin x}{\sin(x+a)}} \, . \, dx$

Putting $\quad \frac{\sin(x+a)}{\sin x} = z$

$\Rightarrow \quad \frac{\sin x \cos(x+a) - \cos x \sin(x+a)}{\sin^2 x} \, . \, dx = dz$

[$\because$ $\sin(A-B) = \sin A \cos B - \cos A \sin B$]

$\Rightarrow \quad \frac{\sin[x-(x+a)]}{\sin^2 x} \, . \, dx = dz \quad \Rightarrow \quad \frac{\sin(-a)}{\sin^2 x} \, dx = dz$

$\Rightarrow \quad -\frac{\sin a}{\sin^2 x} \, . \, dx = dz$

$\Rightarrow \quad \frac{1}{\sin^2 x} \, . \, dx = -\frac{1}{\sin a} \, . \, dz$

$\therefore$ We have $\quad I = \int \frac{1}{\sqrt{z}} \left(-\frac{1}{\sin a}\right) . \, dz = -\frac{1}{\sin a} \int z^{-1/2} \, . \, dz$

$= -\frac{1}{\sin a} \left[\frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right] + c = -\frac{2}{\sin a} z^{1/2} + c$

$= -\frac{2}{\sin a} \cdot \sqrt{\frac{\sin(x+a)}{\sin x}} + c.$ $\quad \left[\because \; z = \frac{\sin(x+a)}{\sin x}\right]$

(*vi*) Let $\quad I = \int \frac{1}{\cos(x-a)\cos(x-b)} \, . \, dx$

$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} \, . \, dx$

[Multiply and divided by $\sin(a-b)$]

$= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)} \, . \, dx$

[$\because$ $(x-b)-(x-a) = -b+a = (a-b)$]

$= \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)} \, . \, dx$

[$\because$ $\sin(A-B) = \sin A \cos B - \cos A \sin B$]

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)} \right] . dx$$

$$= \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)] . dx$$

$$= \frac{1}{\sin(a-b)} \left[\int \tan(x-b) . dx - \int \tan(x-a) . dx \right]$$

$$= \frac{1}{\sin(a-b)} [-\log|\cos(x-b)| + \log|\cos(x-a)|] + c$$

$$= \frac{1}{\sin(a-b)} \left[\log \left| \frac{\cos(x-a)}{\cos(x-b)} \right| \right] + c. \qquad \left[\because \log m - \log n = \log \frac{m}{n} \right]$$

(*vii*) Let $I = \int \frac{\sin(x+a)}{\sin(x+b)} . dx$

Putting $x + b = z \Rightarrow x = z - b \Rightarrow dx = dz$

$\therefore$ We have

$$I = \int \frac{\sin(z-b+a)}{\sin z} . dz$$

$$= \int \frac{\sin z \cos(a-b) + \cos z \sin(a-b)}{\sin z} . dz$$

$[\because \sin(A + B) = \sin A \cos B + \cos A \sin B]$

$$= \int \left[\frac{\sin z \cos(a-b)}{\sin z} + \frac{\cos z \sin(a-b)}{\sin z} \right] . dz$$

$$= \int \cos(a-b) . dz + \int \cot z \sin(a-b) . dz$$

$$= \cos(a-b) \int 1 . dz + \sin(a-b) \int \cot z \, dz$$

$$= z \cos(a-b) + \sin(a-b) . \log|\sin z| + c$$

$$= (x+b)\cos(a-b) + \sin(a-b)\log|\sin(x+b)| + c. \qquad [\because z = (x+b)]$$

Example 14. *Evaluate the following integrals :*

(*i*) $\int \sqrt{\frac{1+\sin x}{1-\sin x}} . dx$ (*ii*) $\int \frac{1}{a \sin x + b \cos x} . dx.$

Solution. (*i*) Let $I = \int \sqrt{\frac{1+\sin x}{1-\sin x}} . dx$

$$= \int \frac{\sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}}{\sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}}} \qquad \left[\begin{array}{l} \because \cos^2 A + \sin^2 A = 1 \\ \sin 2A = 2 \sin A \cos A \end{array} \right]$$

$$= \int \frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}} \cdot dx \qquad \left[\because \begin{array}{l}(a+b)^2 = a^2+b^2+2ab \\ (a-b)^2 = a^2+b^2-2ab\end{array}\right]$$

$$\Rightarrow \qquad I = \int \frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)} \cdot dx$$

Putting $\cos\frac{x}{2} - \sin\frac{x}{2} = z \quad \Rightarrow \quad \left(-\frac{1}{2}\sin\frac{x}{2} - \frac{1}{2}\cos\frac{x}{2}\right) \cdot dx = dz$

$$\Rightarrow \quad -\frac{1}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)dx = dz \quad \Rightarrow \quad \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)dx = -2dz$$

$\therefore$ We have $I = \int \frac{1}{z} \cdot (-2\,dz) = -2\int \frac{1}{z} \cdot dz = -2\log|z| + c$

$$= -2\log\left|\cos\frac{x}{2} - \sin\frac{x}{2}\right| + c. \qquad \left[\because z = \cos\frac{x}{2} - \sin\frac{x}{2}\right]$$

(ii) Let $I = \int \frac{1}{a\sin x + b\cos x} \cdot dx$

Putting $a = r\cos\theta$ and $b = r\sin\theta$

On squaring and adding, we get

$$a^2 + b^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow \qquad a^2 + b^2 = r^2 \quad \Rightarrow \quad r = \sqrt{a^2+b^2}$$

On dividing, we get

$$\frac{b}{a} = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$\therefore$ We have

$$I = \int \frac{1}{r\cos\theta\sin x + r\sin\theta\cos x} \cdot dx$$

$$= \frac{1}{r}\int \frac{1}{\sin(x+\theta)} \cdot dx \qquad [\because \sin(A+B) = \sin A\cos B + \cos A\sin B]$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\theta) \cdot dx$$

$$= \frac{1}{r}\log\left|\tan\left(\frac{x+\theta}{2}\right)\right| + c = \frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right| + c$$

$$= \frac{1}{r}\log\left|\tan\frac{x}{2} + \frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right| + c$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\frac{x}{2} + \frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right| + c.$$

2.6 INTEGRALS OF THE FORM : $\int \sin^m x \cos^n x\, dx$

In order to evaluate the integrals of the form $\int \sin^m x \cos^n x\, dx$, we may use the following rules :

(i) If $m, n \in \mathbf{N}$ and m is odd, then the substitution $z = \cos x$ is used.

(ii) If $m, n \in \mathbf{N}$ and n is odd, then the substitution $z = \sin x$ is used.

(iii) If $m, n \in \mathbf{N}$ and both m and n are odd, then either $z = \sin x$ or $z = \cos x$ is used. It is advisable to use $z = \sin x$ if $m \geq n$ and $z = \cos x$ if $n \geq m$.

(iv) If $m, n \in \mathbf{N}$ and m and n are even, then $\sin^m x$ and $\cos^n x$ are expressed in terms of sines and cosines of multiples of x.

(v) If $m, n \in \mathbf{Q}$ and $(m + n)$ is a negative even integer, then the substitution $z = \tan x$ is used.

Let us try to understand the above mentioned Rules with the help of following solved examples :

Example 15. *Evaluate the following integrals :*

(i) $\int \sin^5 x \cos^4 x\, dx$ (ii) $\int \sin^3 x \cos^3 x\, dx$

(iii) $\int \sin^5 x \cos^3 x\, dx$ (iv) $\int \sin^4 x \cos^3 x\, dx$

(v) $\int \sin^2 x \cos^2 x\, dx$ (vi) $\int \sin^2 x \cos^4 x\, dx$.

Solution. (i) Let $\mathrm{I} = \int \sin^5 x \cos^4 x\, dx$ [Here m is odd]

Putting $z = \cos x$

$\Rightarrow$ $dz = -\sin x\, dx \Rightarrow -dz = \sin x\, dx$

$\therefore$ We have $\mathrm{I} = \int \sin^4 x \,.\, \sin x \,.\, \cos^4 x\, dx$

$= \int (\sin^2 x)^2 \cos^4 x \,.\, \sin x\, dx$

$= \int (1 - \cos^2 x)^2 \,.\, \cos^4 x \sin x\, dx$ [$\because \sin^2 \mathrm{A} + \cos^2 \mathrm{A} = 1$]

Now by using substitution, we have

$$\mathrm{I} = \int (1 - z^2)^2 \,.\, z^4 (-dz). = -\int z^4 (1 - z^2)^2\, dz$$

$$= -\int z^4 (1 + z^4 - 2z^2) \,.\, dz \qquad [\because (a - b)^2 = a^2 + b^2 - 2ab]$$

$$= -\int (z^4 + z^8 - 2z^6) \,.\, dz = -\left[\frac{z^5}{5} + \frac{z^9}{9} - \frac{2z^7}{7}\right] + c$$

$$= -\frac{1}{5} z^5 - \frac{1}{9} z^9 + \frac{2}{7} z^7 + c$$

$$= -\frac{1}{5} \cos^5 x - \frac{1}{9} \cos^9 x + \frac{2}{7} \cos^7 x + c. \qquad [\because z = \cos x]$$

(ii) Let $\mathrm{I} = \int \sin^3 x \cos^3 x\, dx$

$= \int \sin^3 x \cos^2 x \,.\, \cos x\, dx$

$= \int \sin^3 x\,(1-\sin^2 x)\cos x\,dx$ $[\because \sin^2 A + \cos^2 A = 1]$

Putting $z = \sin x \Rightarrow dz = \cos x\,dx.$

$\therefore$ We have $I = \int z^3(1-z^2)\,dz = \int (z^3 - z^5)\,dz$

$= \dfrac{z^4}{4} - \dfrac{z^6}{6} + c$

$= \dfrac{\sin^4 x}{4} - \dfrac{\sin^6 x}{6} + c$ $[\because z = \sin x]$

$= \dfrac{1}{4}\sin^4 x - \dfrac{1}{6}\sin^6 x + c.$

(*iii*) Let $I = \int \sin^5 x \cos^3 x\,dx$

$= \int \sin^5 x \,.\, \cos^2 x \cos x\,dx$

$= \int \sin^5 x\,(1-\sin^2 x)\cos x\,dx$ $[\because \sin^2 A + \cos^2 A = 1]$

Putting $z = \sin x \Rightarrow dz = \cos x\,dx$

$\therefore$ We have $I = \int z^5(1-z^2)\,dz = \int (z^5 - z^7)\,dz$

$= \dfrac{z^6}{6} - \dfrac{z^8}{8} + c$

$= \dfrac{1}{6}\sin^6 x - \dfrac{1}{8}\sin^8 x + c.$ $[\because z = \sin x]$

(*iv*) Let $I = \int \sin^4 x \cos^3 x\,dx = \int \sin^4 x \,.\, \cos^2 x \,.\, \cos x\,dx$

$= \int \sin^4 x\,(1-\sin^2 x)\cos x\,dx$ $[\because \sin^2 A + \cos^2 A = 1]$

Putting $z = \sin x \Rightarrow dz = \cos x\,dx$

$\therefore$ We have $I = \int z^4(1-z^2)\,dz = \int (z^4 - z^6)\,dz$

$= \dfrac{z^5}{5} - \dfrac{z^7}{7} + c$

$= \dfrac{1}{5}\sin^5 x - \dfrac{1}{7}\sin^7 x + c.$ $[\because z = \sin x]$

(*v*) Let $I = \int \sin^2 x \cos^2 x\,dx = \int (\sin x \cos x)^2 \,.\, dx$

$= \dfrac{1}{4}\int 4(\sin x \cos x)^2\,dx$ [Multiply and divided by 4]

$= \dfrac{1}{4}\int (2\sin x \cos x)^2\,dx$

$= \dfrac{1}{4}\int (\sin 2x)^2 \,.\, dx$ $[\because 2\sin A \cos A = \sin 2A]$

$$= \frac{1}{4}\int \frac{1-\cos 4x}{2}\,dx \qquad \left[\begin{array}{l} \because \quad \cos 2A = 1 - 2\sin^2 A \\ \Rightarrow \quad \sin^2 A = \dfrac{1-\cos 2A}{2} \\ \Rightarrow \sin^2 2A = \dfrac{1-\cos 4A}{2} \end{array}\right]$$

$$= \frac{1}{8}\int 1\,.\,dx - \frac{1}{8}\int \cos 4x\,.\,dx = \frac{1}{8}x - \frac{1}{8}\frac{\sin 4x}{4} + c$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + c.$$

(*vi*) Let $\quad I = \int \sin^2 x \cos^4 x\,dx$

$$= \int \sin^2 x\,(\cos^2 x)^2\,.\,dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 .\,dx \qquad \left[\begin{array}{l} \because \quad \cos 2A = 1 - 2\sin^2 A \\ \Rightarrow \quad \sin^2 A = \dfrac{1-\cos 2A}{2} \\ \text{and } \cos 2A = 2\cos^2 A - 1 \\ \Rightarrow \quad \cos^2 A = \dfrac{1+\cos 2A}{2} \end{array}\right]$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos^2 2x + 2\cos 2x}{4}\right).\,dx \qquad [\because \ (a+b)^2 = a^2 + b^2 + 2ab]$$

$$= \frac{1}{8}\int (1-\cos 2x)(1+\cos^2 2x + 2\cos 2x)\,.\,dx$$

$$= \frac{1}{8}\int (1 + \cos^2 2x + 2\cos 2x - \cos 2x - \cos^3 2x - 2\cos^2 2x)\,.\,dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\,.\,dx$$

$$= \frac{1}{8}\int 1\,.\,dx + \frac{1}{8}\int \cos 2x\,.\,dx - \frac{1}{8}\int \frac{1+\cos 4x}{2}\,.\,dx - \frac{1}{8}\int \cos^2 2x\,.\,\cos 2x\,dx$$

$$= \frac{1}{8}\int 1\,.\,dx + \frac{1}{8}\int \cos 2x\,dx - \frac{1}{16}\int 1\,.\,dx - \frac{1}{16}\int \cos 4x\,dx$$

$$- \frac{1}{8}\int (1-\sin^2 2x)\cos 2x\,dx$$

For last integral ;
Putting $\quad \sin 2x = z$

$$\Rightarrow \quad 2\cos 2x\,dx = dz \quad \Rightarrow \cos 2x\,dx = \frac{1}{2}\,dz.$$

$\therefore$ We have
$$I = \frac{1}{8}x + \frac{1}{8}\frac{\sin 2x}{2} - \frac{1}{16}x - \frac{1}{16}\frac{\sin 4x}{4} - \frac{1}{8}\int (1-z^2)\frac{dz}{2}.$$

$$= \frac{1}{8}x + \frac{1}{16}\sin 2x - \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{16}\left(z - \frac{z^3}{3}\right) + c$$

$$= \frac{x}{8} + \frac{\sin 2x}{16} - \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin 2x}{16} + \frac{\sin^3 2x}{48} + c \qquad [\because \ z = \sin 2x]$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + c.$$

Example 16. *Evaluate the following integrals :*

(*i*) $\int \sin^2 x \cos^5 x \, dx$ (*ii*) $\int \cos^5 x \, . \, e^{\log \sin x} \, . \, dx$

(*iii*) $\int \sin^3 x \sqrt{\cos x} \, dx.$

Solution. (*i*) Let $I = \int \sin^2 x \cos^5 x \, dx$ [Here $n = 5$ is odd]

$$= \int \sin^2 x \cos^4 x \, . \cos x \, dx$$

$$= \int \sin^2 x \, . \, (\cos^2 x)^2 \, . \cos x \, dx$$

$$= \int \sin^2 x \, (1 - \sin^2 x)^2 \, . \cos x \, dx$$

Putting $z = \sin x \Rightarrow dz = \cos x \, dx$

$\therefore$ We have $I = \int z^2 (1 - z^2)^2 \, . \, dz$

$$= \int z^2 (1 + z^4 - 2z^2) \, dz = \int (z^2 + z^6 - 2z^4) \, dz$$

$[\because \ (a - b)^2 = a^2 + b^2 - 2ab]$

$$= \frac{z^3}{3} + \frac{z^7}{7} - \frac{2z^5}{5} + c$$

$$= \frac{1}{3} \sin^3 x + \frac{1}{7} \sin^7 x - \frac{2}{5} \sin^5 x + c. \qquad [\because \ z = \sin x]$$

(*ii*) Let $I = \int \cos^5 x \, e^{\log \sin x} \, dx$

$$= \int \cos^5 x \sin x \, dx \qquad [\because \ e^{\log f(x)} = f(x)]$$

Putting $z = \cos x$

$\Rightarrow$ $dz = - \sin x \, dx \Rightarrow - dz = \sin x \, dx$

$\therefore$ We have $I = - \int z^5 \, . \, dz = - \frac{z^6}{6} + c$

$$= - \frac{\cos^6 x}{6} + c. \qquad [\because \ z = \cos x]$$

(*iii*) Let $I = \int \sin^3 x \, . \sqrt{\cos x} \, dx$

$$= \int \sin^2 x \, . \sin x \sqrt{\cos x} \, dx$$

Putting $z = \cos x$

$\Rightarrow$ $dz = - \sin x \, dx \Rightarrow - dz = \sin x \, dx$

$\therefore$ We have $I = \int (1 - \cos^2 x) \sqrt{\cos x} \, . \sin x \, dx$

$$= \int (1-z^2)\sqrt{z}\,(-dz) = -\int (z^{1/2} - z^{5/2})\,.\,dz$$

$$= -\left[\frac{z^{1/2+1}}{\frac{1}{2}+1} - \frac{z^{5/2+1}}{\frac{5}{2}+1}\right] + c = -\left[\frac{z^{3/2}}{\frac{3}{2}} - \frac{z^{7/2}}{\frac{7}{2}}\right] + c$$

$$= -\left[\frac{2}{3}z^{3/2} - \frac{2}{7}z^{7/2}\right] + c$$

$$= -\frac{2}{3}\cos^{3/2} x + \frac{2}{7}\cos^{7/2} x + c.$$

Example 17. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{\sin^{3/2} x \cos^{5/2} x}\,.\,dx$ (*ii*) $\int \sin^{2/3} x \cos^3 x\, dx$

(*iii*) $\int \sec^{3/4} x \operatorname{cosec}^{5/4} x\, dx$ (*iv*) $\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,.\,dx$

(*v*) $\int \frac{1}{\sin^3 x \cos^5 x}\,.\,dx.$

Solution. (*i*) Let $I = \int \frac{1}{\sin^{3/2} x \cos^{5/2} x}\,.\,dx$

$\Rightarrow$ $I = \int \sin^{-\frac{3}{2}} x \cos^{-\frac{5}{2}} x\, dx$

Here $\left(-\frac{3}{2}\right) + \left(-\frac{5}{2}\right) = -4$ which is a negative even integer.

$\therefore$ $I = \int \frac{1}{\sin^{3/2} x \cos^{5/2} x}\,.\,dx$

$$= \int \frac{1}{\frac{\sin^{3/2} x}{\cos^{3/2} x}\,.\cos^{3/2} x\,.\cos^{5/2} x}\,.\,dx$$

[$\because$ Multiply and divide the denominator by $\cos^{3/2} x$]

$$= \int \frac{1}{\tan^{3/2} x \cos^4 x}\,.\,dx$$

$$= \int \frac{\sec^4 x}{\tan^{3/2} x}\,dx = \int \frac{\sec^2 x\,.\sec^2 x}{\tan^{3/2} x}\,dx$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x\, dx}{\tan^{3/2} x}\,dx \qquad [\because \sec^2 A - \tan^2 A = 1]$$

Now, putting $z = \tan x \Rightarrow dz = \sec^2 x\, dx$

$\therefore$ We have $I = \int \frac{1+z^2}{z^{3/2}}\,.\,dz$

$$= \int \frac{1}{z^{3/2}}\,dz + \int \frac{z^2}{z^{3/2}}\,dz = \int z^{-3/2}\,dz + \int z^{1/2}\,.\,dz$$

$$= \frac{z^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{z^{-1/2}}{-\frac{1}{2}} + \frac{z^{3/2}}{\frac{3}{2}} + c$$

$$= -2\,\frac{1}{\sqrt{z}} + \frac{2}{3}\,z^{3/2} + c$$

$$= -\frac{2}{\sqrt{\tan x}} + \frac{2}{3}\tan^{3/2} x + c. \qquad [\because\ z = \tan x]$$

(*ii*) Let $\quad I = \int \sin^{2/3} x \cos^3 x\, dx = \int \sin^{2/3} x \cos^2 x \,.\, \cos x\, dx$

$$= \int \sin^{2/3} x\,(1 - \sin^2 x) \cos x\, dx \qquad [\because\ \sin^2 A + \cos^2 A = 1]$$

Putting $\quad z = \sin x \;\Rightarrow\; dz = \cos x\, dx$

$\therefore$ We have $\quad I = \int z^{2/3}(1 - z^2)\, dz = \int (z^{2/3} - z^{8/3})\, dz$

$$= \frac{z^{\frac{2}{3}+1}}{\frac{2}{3}+1} - \frac{z^{\frac{8}{3}+1}}{\frac{8}{3}+1} + c = \frac{z^{5/3}}{\frac{5}{3}} - \frac{z^{11/3}}{\frac{11}{3}} + c$$

$$= \frac{3}{5}\sin^{5/3} x - \frac{3}{11}\sin^{11/3} x + c. \qquad [\because\ z = \sin x]$$

(*iii*) Let $\quad I = \int \sec^{3/4} x \operatorname{cosec}^{5/4} x\, dx$

$$= \int \frac{1}{\cos^{3/4} x \sin^{5/4} x}\,.\,dx$$

Here $\quad \frac{3}{4} + \frac{5}{4} = 2$, which is a positive even integer.

$\therefore$

$$I = \int \frac{1}{\cos^{3/4} x \sin^{5/4} x}\,.\,dx$$

$$I = \int \frac{1}{\frac{\sin^{5/4} x}{\cos^{5/4} x}\,.\cos^{5/4} x \cos^{3/4} x}\,.\,dx$$

[$\because$ Multiply and divide the denominator by $\cos^{5/4} x$].

$$= \int \frac{1}{\tan^{5/4} x \cos^2 x}\,dx = \int \frac{\sec^2 x}{\tan^{5/4} x}\,dx$$

Now, putting $\quad \tan x = z \;\Rightarrow\; \sec^2 x\, dx = dz$

$\therefore$ We have $\quad I = \int \frac{1}{z^{5/4}}\,dz$

$$= \int z^{-5/4}\,dz = \frac{z^{-5/4+1}}{-\frac{5}{4}+1} + c = \frac{z^{-1/4}}{-\frac{1}{4}} + c = -4\,z^{-1/4} + c$$

$$= \frac{-4}{\tan^{1/4} x} + c. \qquad [\because \ z = \tan x]$$

(*iv*) Let $\quad I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} \, dx$

$$= \int \frac{\sqrt{\tan x}}{\frac{\sin x}{\cos x} . \cos^2 x} \, dx$$

[Multiply and divide the denominator by $\cos x$]

$$= \int \frac{\sqrt{\tan x} . \sec^2 x}{\tan x} \, dx$$

$$= \int \frac{1}{\sqrt{\tan x}} . \sec^2 x \, dx$$

Now, putting $\quad z = \tan x \Rightarrow dz = \sec^2 x \, dx$

$\therefore$ We have $\quad I = \int \frac{1}{\sqrt{z}} . dz = \int z^{-1/2} . dz$

$$= \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{z^{1/2}}{\frac{1}{2}} + c = 2\sqrt{z} + c$$

$$= 2\sqrt{\tan x} + c. \qquad [\because \ z = \tan x]$$

(*v*) Let $\quad I = \int \frac{1}{\sin^3 x \cos^5 x} . dx$

$$= \int \sin^{-3} x \cos^{-5} x \, dx$$

Here $\quad (-3) + (-5) = -8$, which is a negative even integer.

$\therefore \quad I = \int \frac{1}{\sin^3 x \cos^5 x} \, dx$

$$= \int \frac{1}{\frac{\sin^3 x}{\cos^3 x} . \cos^3 x \cos^5 x} . dx$$

[Multiply and divide the denominator by $\cos^3 x$]

$$= \int \frac{1}{\tan^3 x . \cos^8 x} . dx$$

$$= \int \frac{\sec^8 x}{\tan^3 x} \, dx = \int \frac{\sec^6 x . \sec^2 x}{\tan^3 x} \, dx$$

$$= \int \frac{(1 + \tan^2 x)^3 . \sec^2 x}{\tan^3 x} . dx \qquad [\because \ \sec^2 A - \tan^2 A = 1]$$

Now, putting $\tan x = z \Rightarrow \sec^2 x\, dx = dz$

$\therefore$ We have $\quad I = \int \frac{(1+z^2)^3}{z^3} \cdot dz$

$$= \int \frac{(1+3z^2+3z^4+z^6)}{z^3} \cdot dz \qquad [\because \ (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3]$$

$$= \int \left[\frac{1}{z^3} + \frac{3z^2}{z^3} + \frac{3z^4}{z^3} + \frac{z^6}{z^3}\right] \cdot dz = \int \left(z^{-3} + \frac{3}{z} + 3z + z^3\right) \cdot dz$$

$$= \int z^{-3} \cdot dz + 3 \int \frac{1}{z} dz + 3 \int z \cdot dz + \int z^3 \cdot dz$$

$$= \frac{z^{-2}}{-2} + 3 \log |z| + \frac{3z^2}{2} + \frac{z^4}{4} + c$$

$$= -\frac{1}{2z^2} + 3 \log |z| + \frac{3}{2} z^2 + \frac{1}{4} z^4 + c$$

$$= -\frac{1}{2\tan^2 x} + 3 \log |\tan x| + \frac{3}{2} \tan^2 x + \frac{1}{4} \tan^4 x + c. \quad [\because \ z = \tan x]$$

Example 18. *Evaluate the following integrals :*

(i) $\int \sec^6 x\, dx$ *(ii)* $\int \sqrt{\tan x} \cdot (1 + \tan^2 x)\, dx.$

Solution. *(i)* Let $I = \int \sec^6 x\, dx = \int \sec^4 x \cdot \sec^2 x\, dx$

$$= \int (\sec^2 x)^2 \sec^2 x\, dx$$

$$= \int (1 + \tan^2 x)^2 \cdot \sec^2 x\, dx \qquad [\because \ \sec^2 A - \tan^2 A = 1]$$

Now, putting $\quad z = \tan x \Rightarrow dz = \sec^2 x\, dx$

$\therefore$ We have $\quad I = \int (1 + z^2)^2 \cdot dz$

$$= \int (1 + z^4 + 2z^2)\, dz \qquad [\because \ (a+b)^2 = a^2 + b^2 + 2ab]$$

$$= \int 1 \cdot dz + \int z^4 \cdot dz + 2 \int z^2 \cdot dz$$

$$= z + \frac{z^5}{5} + 2\frac{z^3}{3} + c$$

$$= \tan x + \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + c. \qquad [\because \ z = \tan x]$$

(ii) Let $\quad I = \int \sqrt{\tan x}\, (1 + \tan^2 x)\, dx$

$$= \int \sqrt{\tan x} \cdot \sec^2 x\, dx \qquad [\because \ \sec^2 A - \tan^2 A = 1]$$

Putting $\quad z = \tan x \Rightarrow dz = \sec^2 x\, dx$

$\therefore$ We have $\quad I = \int \sqrt{z}\, dz$

$$= \int z^{1/2}\, dz = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{z^{3/2}}{3/2} + c = \frac{2}{3} z^{3/2} + c = \frac{2}{3} \tan^{3/2} x + c. \qquad [\because\ z = \tan x]$$

Example 19. *Evaluate the following integrals :*

(i) $\int \cos^4 x\, dx$ *(ii)* $\int \sin^4 x\, dx.$

Solution. *(i)* Let $I = \int \cos^4 x\, dx$

$$= \int (\cos^2 x)^2\, dx$$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 . dx \qquad \left[\begin{array}{l} \because\ \cos 2A = 2\cos^2 A - 1 \\ \Rightarrow\ \cos^2 A = \dfrac{1+\cos 2A}{2} \end{array}\right]$$

$$= \frac{1}{4} \int (1 + \cos^2 2x + 2\cos 2x) . dx \qquad [\because\ (a+b)^2 = a^2 + b^2 + 2ab]$$

$$= \frac{1}{4} \int \left[1 + \left(\frac{1+\cos 4x}{2}\right) + 2\cos 2x\right] . dx$$

$$= \frac{1}{4} \int \left(1 + \frac{1}{2} + \frac{1}{2}\cos 4x + 2\cos 2x\right) . dx$$

$$= \frac{1}{4} \int \left(\frac{3}{2} + \frac{1}{2}\cos 4x + 2\cos 2x\right) . dx$$

$$= \frac{3}{8} \int 1 . dx + \frac{1}{8} \int \cos 4x . dx + \frac{1}{2} \int \cos 2x\, dx$$

$$= \frac{3}{8} x + \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + c.$$

(ii) Let $I = \int \sin^4 x\, dx$

$$= \int (\sin^2 x)^2 . dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 . dx \qquad \left[\begin{array}{l} \because\ \cos 2A = 1 - 2\sin^2 A \\ \Rightarrow\ \sin^2 A = \dfrac{1-\cos 2A}{2} \end{array}\right]$$

$$= \frac{1}{4} \int (1 + \cos^2 2x - 2\cos 2x) . dx \qquad [\because\ (a-b)^2 = a^2 + b^2 - 2ab]$$

$$= \frac{1}{4} \int \left[1 + \left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right] . dx$$

$$= \frac{1}{4} \int \left(1 + \frac{1}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right) . dx$$

$$= \frac{1}{4}\int\left(\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right).\,dx$$

$$= \frac{3}{8}\int 1.\,dx + \frac{1}{8}\int \cos 4x\,dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{3x}{8} + \frac{1}{8}\left(\frac{\sin 4x}{4}\right) - \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + c$$

$$= \frac{3x}{8} + \frac{\sin 4x}{32} - \frac{1}{4}\sin 2x + c.$$

2.7 SOME SPECIAL INTEGRALS

Theorem 1. *Prove that*

(*i*) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$ (*ii*) $\int \frac{1}{x^2-a^2}.\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(*iii*) $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$

Proof. (*i*) Let $I = \int \frac{1}{a^2-x^2}.\,dx$

$$= \int \frac{1}{(a-x)(a+x)}.\,dx \qquad [\because (a^2-b^2) = (a-b)(a+b)]$$

$$= \frac{1}{2a}\int \frac{2a}{(a-x)(a+x)}.\,dx \qquad \text{[Multiply and divided by } 2a]$$

$$= \frac{1}{2a}\int \frac{(a+x)+(a-x)}{(a-x)(a+x)}.\,dx \qquad [\because (a+x)+(a-x) = a+x+a-x = 2a]$$

$$= \frac{1}{2a}\int\left(\frac{1}{a-x}+\frac{1}{a+x}\right).\,dx$$

$$= \frac{1}{2a}\int \frac{1}{a-x}.\,dx + \frac{1}{2a}\int \frac{1}{a+x}.\,dx$$

$$= \frac{1}{2a}\frac{\log|a-x|}{(-1)} + \frac{1}{2a}\frac{\log|a+x|}{1} + c$$

$$= \frac{1}{2a}[\log|a+x| - \log|a-x|] + c$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c \qquad \left[\because \log_e m - \log_e n = \log_e \frac{m}{n}\right]$$

$$\therefore \int \frac{1}{a^2-x^2}.\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c.$$

(*ii*) Let $I = \int \frac{1}{x^2-a^2}.\,dx$

$$= \int \frac{1}{(x-a)(x+a)}.\,dx \qquad [\because (a^2-b^2) = (a-b)(a+b)]$$

$= \frac{1}{2a}\int \frac{2a}{(x-a)(x+a)} \cdot dx$ [Multyply and divided by $2a$]

$= \frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x-a)(x+a)} \cdot dx$ [$\because$ $(x+a)-(x-a)=x+a-x+a=2a$]

$= \frac{1}{2a}\int \left(\frac{1}{x-a}-\frac{1}{x+a}\right) \cdot dx$

$= \frac{1}{2a}\int \frac{1}{x-a} \cdot dx - \frac{1}{2a}\int \frac{1}{x+a} \cdot dx$

$= \frac{1}{2a}\log|x-a| - \frac{1}{2a}\log|x+a| + c$

$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$ $\left[\because \log_e m - \log_e n = \log_e \frac{m}{n}\right]$

$\therefore \int \frac{1}{x^2-a^2} \cdot dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c.$

(*iii*) Let $\quad I = \int \frac{1}{a^2+x^2} \cdot dx$

Put $\quad x = a\tan\theta$

$\Rightarrow \quad \frac{x}{a} = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{x}{a}\right).$

and $\quad x = a\tan\theta \quad \Rightarrow \quad dx = a\sec^2\theta\, d\theta$

$\therefore \quad I = \int \frac{a\sec^2\theta}{a^2+a^2\tan^2\theta} d\theta = \frac{a}{a^2}\int \frac{\sec^2\theta}{1+\tan^2\theta} \cdot d\theta$

$= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta} \cdot d\theta$ [$\because$ $\sec^2 A - \tan^2 A = 1$]

$= \frac{1}{a}\int 1 \cdot d\theta = \frac{1}{a} \cdot \theta + c$

$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ $\left[\because \theta = \tan^{-1}\frac{x}{a}\right]$

$\therefore \int \frac{1}{a^2+x^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$

Theorem 2. *Prove that :*

(*i*) $\int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c$ (*ii*) $\int \frac{1}{\sqrt{x^2-a^2}} \cdot dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$

(*iii*) $\int \frac{1}{\sqrt{a^2+x^2}} \cdot dx = \log\left|x+\sqrt{a^2+x^2}\right| + c$

(*iv*) $\int \frac{1}{x\sqrt{x^2-a^2}} \cdot dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c.$

Proof. (*i*) Let $\quad I = \int \frac{1}{\sqrt{a^2 - x^2}} \, . \, dx.$

Put $\quad x = a \sin\theta \;\Rightarrow\; \theta = \sin^{-1}\frac{x}{a}$

$\Rightarrow \quad dx = a\cos\theta \, d\theta$

$$\therefore \quad I = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} \, . \, a\cos\theta \, d\theta = \int \frac{a\cos\theta}{a\sqrt{1-\sin^2\theta}} \, . \, d\theta$$

$$= \int \frac{\cos\theta}{\sqrt{\cos^2\theta}} \, . \, d\theta = \int \frac{\cos\theta}{\cos\theta} \, d\theta$$

$$= \int 1 \, . \, d\theta = \theta + c$$

$$= \sin^{-1}\frac{x}{a} + c \qquad \left[\because \; \theta = \sin^{-1}\frac{x}{a}\right]$$

$$\therefore \quad \int \frac{1}{\sqrt{a^2 - x^2}} \, . \, dx = \sin^{-1}\frac{x}{a} + c.$$

(*ii*) Let $\quad I = \int \frac{1}{\sqrt{x^2 - a^2}} \, . \, dx$

Put $\quad x = a\sec\theta \;\Rightarrow\; \sec\theta = \frac{x}{a}$

$\Rightarrow \quad dx = a\sec\theta\tan\theta \, d\theta$

$$\therefore \quad I = \int \frac{1}{\sqrt{a^2\sec^2\theta - a^2}} \, . \, a\sec\theta\tan\theta \, . \, d\theta$$

$$= \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2(\sec^2\theta - 1)}} \, . \, d\theta$$

$$= \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}} \, . \, d\theta \qquad [\because \; \sec^2 A - \tan^2 A = 1]$$

$$= \int \frac{\sec\theta\tan\theta}{\tan\theta} \, . \, d\theta = \int \sec\theta \, d\theta$$

$$= \log|\sec\theta + \tan\theta| + c_1$$

$$= \log\left|\sec\theta + \sqrt{\sec^2\theta - 1}\right| + c_1 = \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + c_1$$

$$\left[\because \; \sec\theta = \frac{x}{a}\right]$$

$$= \log\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + c_1$$

$$= \log \left| x + \sqrt{x^2 - a^2} \right| - \log |a| + c_1$$

$$= \log \left| x + \sqrt{x^2 - a^2} \right| + c \qquad [\because \; c = c_1 - \log |a|]$$

$$\therefore \quad \int \frac{1}{\sqrt{x^2 - a^2}} \, . \, dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c.$$

(*iii*) Let $\quad I = \int \frac{1}{\sqrt{a^2 + x^2}} \, . \, dx$

Put $\quad x = a \tan \theta \; \Rightarrow \; \tan \theta = \frac{x}{a}$

$\Rightarrow \quad dx = a \sec^2 \theta \, d\theta$

$$\therefore \quad I = \int \frac{1}{\sqrt{a^2 + a^2 \tan^2 \theta}} \, . \, a \sec^2 \theta \, d\theta$$

$$= \int \frac{a \sec^2 \theta}{\sqrt{a^2 (1 + \tan^2 \theta)}} \, . \, d\theta$$

$$= \int \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta}} \, . \, d\theta \qquad [\because \; \sec^2 A - \tan^2 A = 1]$$

$$= \int \frac{\sec^2 \theta}{\sec \theta} \, d\theta = \int \sec \theta \, d\theta$$

$$= \log |\sec \theta + \tan \theta| + c_1$$

$$= \log \left| \sqrt{1 + \tan^2 \theta} + \tan \theta \right| + c_1$$

$$= \log \left| \sqrt{1 + \frac{x^2}{a^2}} + \frac{x}{a} \right| + c_1 \qquad \left[\because \; \tan \theta = \frac{x}{a}\right]$$

$$= \log \left| \frac{\sqrt{a^2 + x^2} + x}{a} \right| + c_1$$

$$= \log \left| \sqrt{a^2 + x^2} + x \right| - \log |a| + c_1$$

$$= \log \left| x + \sqrt{a^2 + x^2} \right| + c \qquad [\because \; c = c_1 - \log |a|]$$

$$\therefore \quad \int \frac{1}{\sqrt{a^2 + x^2}} \, . \, dx = \log \left| x + \sqrt{a^2 + x^2} \right| + c.$$

(*iv*) Let $\quad I = \int \frac{1}{x\sqrt{x^2 - a^2}} \, . \, dx$

Put $\quad x = a \sec \theta \; \Rightarrow \; \sec \theta = \frac{x}{a} \; \Rightarrow \; \theta = \sec^{-1} \frac{x}{a}$

$\Rightarrow \qquad dx = a \sec\theta \tan\theta \, d\theta$

$$\therefore \qquad I = \int \frac{1}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}} \cdot a\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{a\sec\theta\tan\theta}{a\sec\theta\sqrt{a^2(\sec^2\theta - 1)}} \cdot d\theta$$

$$= \int \frac{\tan\theta}{a\sqrt{\tan^2\theta}}\, d\theta \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$$= \frac{1}{a}\int \frac{\tan\theta}{\tan\theta}\, d\theta = \frac{1}{a}\int 1 \,.\, d\theta = \frac{1}{a}\,.\,\theta + c$$

$$= \frac{1}{a}\sec^{-1}\frac{x}{a} + c \qquad \left[\because \theta = \sec^{-1}\frac{x}{a}\right]$$

$$\therefore \quad \int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c.$$

Remember :

Some useful substitutions are here given below which provides a great help to the reader for evaluating special integrals :

Expression	**Substitution**
(*i*) $(a^2 + x^2)$	$x = a\tan\theta$ or $a\cot\theta$
(*ii*) $(a^2 - x^2)$	$x = a\sin\theta$ or $a\cos\theta$
(*iii*) $(x^2 - a^2)$	$x = a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$
(*iv*) $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
(*v*) $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
(*vi*) $\sqrt{\frac{x-\alpha}{\beta-x}}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$
(*vii*) $\sqrt{\frac{x-\alpha}{x-\beta}}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$
(*viii*) $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$
(*ix*) $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2\cos^2\theta.$

Example 20. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{4 + 9x^2} \cdot dx$ (*ii*) $\int \frac{1}{9x^2 - 7} \cdot dx$

(*iii*) $\int \frac{1}{1 + 2x^2} \cdot dx$ (*iv*) $\int \frac{1}{9x^2 - 4} \cdot dx$

(*v*) $\int \frac{1}{50 + 2x^2} \cdot dx$ (*vi*) $\int \frac{1}{16 - 9x^2} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{1}{(4+9x^2)} \cdot dx = \frac{1}{9}\int \frac{1}{\left(\frac{4}{9}+x^2\right)} \cdot dx$

$$= \frac{1}{9}\int \frac{1}{\left[\left(\frac{2}{3}\right)^2 + x^2\right]} \cdot dx \qquad \left[\text{By using} \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{9} \cdot \frac{1}{2/3} \cdot \tan^{-1}\frac{x}{(2/3)} + c$$

$$= \frac{1}{9} \cdot \frac{3}{2} \tan^{-1}\left(\frac{3x}{2}\right) + c = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + c.$$

(*ii*) Let $\quad I = \int \frac{1}{9x^2-7} \cdot dx = \frac{1}{9}\int \frac{1}{x^2 - \frac{7}{9}} \cdot dx$

$$= \frac{1}{9}\int \frac{1}{\left[x^2 - \left(\frac{\sqrt{7}}{3}\right)^2\right]} \cdot dx$$

$$\left[\text{By using} \int \frac{1}{x^2-a^2} \cdot dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{9} \cdot \frac{1}{2\left(\frac{\sqrt{7}}{3}\right)} \cdot \log\left|\frac{x-\sqrt{7}/3}{x+\sqrt{7}/3}\right| + c = \frac{1}{9} \cdot \frac{3}{2\sqrt{7}}\log\left|\frac{3x-\sqrt{7}}{3x+\sqrt{7}}\right| + c$$

$$= \frac{1}{6\sqrt{7}}\log\left|\frac{3x-\sqrt{7}}{3x+\sqrt{7}}\right| + c.$$

(*iii*) Let $\quad I = \int \frac{1}{1+2x^2} \cdot dx = \frac{1}{2}\int \frac{1}{\left(\frac{1}{2}+x^2\right)} \cdot dx$

$$= \frac{1}{2}\int \frac{1}{\left[\left(\frac{1}{\sqrt{2}}\right)^2 + x^2\right]} \cdot dx \qquad \left[\text{By using} \int \frac{1}{a^2+x^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{2} \cdot \frac{1}{(1/\sqrt{2})} \cdot \tan^{-1}\left(\frac{x}{(1/\sqrt{2})}\right) + c = \frac{1}{2} \cdot \frac{\sqrt{2}}{1}\tan^{-1}\left(\frac{\sqrt{2}x}{1}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{2}x\right) + c.$$

(*iv*) Let $\quad I = \int \frac{1}{9x^2 - 4} \cdot dx = \frac{1}{9} \int \frac{1}{x^2 - \frac{4}{9}} \cdot dx$

$$= \frac{1}{9} \int \frac{1}{x^2 - \left(\frac{2}{3}\right)^2} \cdot dx \qquad \left[\text{By using} \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c\right]$$

$$= \frac{1}{9} \cdot \frac{1}{2\left(\frac{2}{3}\right)} \cdot \log \left| \frac{x - 2/3}{x + 2/3} \right| + c = \frac{1}{9} \cdot \frac{3}{4} \log \left| \frac{3x-2}{3x+2} \right| + c$$

$$= \frac{1}{12} \log \left| \frac{3x-2}{3x+2} \right| + c.$$

(*v*) Let $\quad I = \int \frac{1}{50 + 2x^2} \cdot dx$

$$= \frac{1}{2} \int \frac{1}{\frac{50}{2} + x^2} \cdot dx = \frac{1}{2} \int \frac{1}{25 + x^2} \cdot dx$$

$$= \frac{1}{2} \int \frac{1}{(5)^2 + x^2} \cdot dx \qquad \left[\text{By using} \int \frac{1}{a^2 + x^2} \cdot dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \frac{1}{2} \cdot \frac{1}{5} \tan^{-1} \frac{x}{5} + c$$

$$= \frac{1}{10} \tan^{-1} \frac{x}{5} + c.$$

(*vi*) Let $\quad I = \int \frac{1}{16 - 9x^2} \cdot dx = \frac{1}{9} \int \frac{1}{\frac{16}{9} - x^2} \cdot dx$

$$= \frac{1}{9} \int \frac{1}{\left[\left(\frac{4}{3}\right)^2 - x^2\right]} \cdot dx \qquad \left[\text{By using} \int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c\right]$$

$$= \frac{1}{9} \cdot \frac{1}{2\left(\frac{4}{3}\right)} \cdot \log \left| \frac{\frac{4}{3} + x}{\frac{4}{3} - x} \right| + c = \frac{1}{9} \cdot \frac{3}{8} \log \left| \frac{4+3x}{4-3x} \right| + c$$

$$= \frac{1}{24} \log \left| \frac{4+3x}{4-3x} \right| + c.$$

Example 21. *Evaluate the following :*

(*i*) $\int \frac{1}{a^2 - b^2x^2} \cdot dx$ (*ii*) $\int \frac{1}{32 - 2x^2} \cdot dx$

(*iii*) $\int \frac{x^2}{x^6 - a^6} \cdot dx$ (*iv*) $\int \frac{1}{1 - 2x^2} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{1}{a^2 - b^2x^2} \cdot dx = \frac{1}{b^2} \int \frac{1}{\frac{a^2}{b^2} - x^2} \cdot dx$

$$= \frac{1}{b^2} \int \frac{1}{\left(\frac{a}{b}\right)^2 - x^2} \cdot dx \qquad \left[\text{By using} \int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c\right]$$

$$= \frac{1}{b^2} \cdot \frac{1}{\left(\frac{2a}{b}\right)} \cdot \log \left| \frac{\frac{a}{b} + x}{\frac{a}{b} - x} \right| + c = \frac{1}{b^2} \cdot \frac{b}{2a} \log \left| \frac{a + bx}{a - bx} \right| + c$$

$$= \frac{1}{2ab} \log \left| \frac{a + bx}{a - bx} \right| + c.$$

(*ii*) Let $I = \int \frac{1}{(32 - 2x^2)} \cdot dx = \frac{1}{2} \int \frac{1}{16 - x^2} \cdot dx$

$$= \frac{1}{2} \int \frac{1}{[(4)^2 - x^2]} \cdot dx \qquad \left[\text{By using} \int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c\right]$$

$$= \frac{1}{2} \cdot \frac{1}{2(4)} \cdot \log \left| \frac{4+x}{4-x} \right| + c$$

$$= \frac{1}{16} \log \left| \frac{4+x}{4-x} \right| + c.$$

(*iii*) Let $I = \int \frac{x^2}{x^6 - a^6} \cdot dx$

$$= \frac{1}{3} \int \frac{3x^2}{x^6 - a^6} \cdot dx \qquad \text{[Multiply and divided by 3]}$$

Put $z = x^3 \Rightarrow dz = 3x^2 \cdot dx$

$\therefore$ $I = \frac{1}{3} \int \frac{1}{z^2 - a^6} \cdot dz$

$$= \frac{1}{3} \int \frac{1}{z^2 - (a^3)^2} \cdot dz \qquad \left[\text{By using} \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c\right]$$

$$= \frac{1}{3} \cdot \frac{1}{2a^3} \cdot \log \left| \frac{z - a^3}{z + a^3} \right| + c$$

$$= \frac{1}{6a^3} \log \left| \frac{x^3 - a^3}{x^3 + a^3} \right| + c. \qquad [\because\ z = x^3]$$

(*iv*) Let $I = \int \frac{1}{1 - 2x^2} \cdot dx = \frac{1}{2} \int \frac{1}{(1/2 - x^2)} \cdot dx$

$$= \frac{1}{2} \int \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2 - x^2} \cdot dx \qquad \left[\text{By using} \int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + c\right]$$

$$= \frac{1}{2} \cdot \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)} \cdot \log \left| \frac{1/\sqrt{2} + x}{1/\sqrt{2} - x} \right| + c$$

$$= \frac{1}{2\sqrt{2}} \log \left| \frac{1 + \sqrt{2}x}{1 - \sqrt{2}x} \right| + c.$$

Example 22. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{1 + 5 \sin^2 x} \cdot dx$ (*ii*) $\int \frac{x^4}{1 + x^2} \cdot dx$

(*iii*) $\int \frac{3x}{1 + 2x^4} \cdot dx$ (*iv*) $\int \frac{x^2 - 1}{x^2 + 4} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{1}{1 + 5 \sin^2 x} \cdot dx$

[Dividing numerator and denominator by $\cos^2 x$]

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{5 \sin^2 x}{\cos^2 x}} \cdot dx = \int \frac{\sec^2 x}{\sec^2 x + 5 \tan^2 x} \cdot dx$$

$$= \int \frac{\sec^2 x}{1 + \tan^2 x + 5 \tan^2 x} \cdot dx \qquad [\because\ \sec^2 A - \tan^2 A = 1]$$

$$= \int \frac{\sec^2 x}{1 + 6 \tan^2 x} \cdot dx$$

Put $z = \tan x \Rightarrow dz = \sec^2 x\, dx$

$\therefore$ $I = \int \frac{1}{1 + 6z^2} \cdot dz = \frac{1}{6} \int \frac{1}{\left(\frac{1}{6} + z^2\right)} \cdot dz$

$$= \frac{1}{6} \int \frac{1}{\left(\frac{1}{\sqrt{6}}\right)^2 + z^2} \cdot dz \qquad \left[\text{By using} \int \frac{1}{a^2 + x^2} \cdot dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \frac{1}{6} \cdot \frac{1}{1/\sqrt{6}} \cdot \tan^{-1}\left(\frac{z}{1/\sqrt{6}}\right) + c = \frac{\sqrt{6}}{6} \tan^{-1}\left(\sqrt{6}z\right) + c$$

$$= \frac{1}{\sqrt{6}} \tan^{-1}\left(\sqrt{6} \tan x\right) + c. \qquad [\because \ z = \tan x]$$

(*ii*) Let $\quad I = \int \frac{x^4}{1+x^2} \cdot dx$

$$= \int \frac{x^4 - 1 + 1}{x^2 + 1} \cdot dx \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \int \left(\frac{x^4 - 1}{x^2 + 1} + \frac{1}{x^2 + 1}\right) \cdot dx$$

$$= \int \left[\frac{(x^2 - 1)(x^2 + 1)}{x^2 + 1} + \frac{1}{x^2 + 1}\right] \cdot dx \qquad [\because \ a^2 - b^2 = (a + b)(a - b)]$$

$$= \int (x^2 - 1) \cdot dx + \int \frac{1}{1 + x^2} \cdot dx \qquad \left[\text{By using} \int \frac{1}{a^2 + x^2} \cdot dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \frac{x^3}{3} - x + \tan^{-1} x + c.$$

(*iii*) Let $\quad I = \int \frac{3x}{1 + 2x^4} \cdot dx = 3 \int \frac{x}{1 + 2(x^2)^2} \cdot dx$

$$= \frac{3}{2} \int \frac{2x}{1 + 2(x^2)^2} \cdot dx \qquad \text{[Multyply and divided by 2]}$$

Put $\quad z = x^2 \Rightarrow dz = 2x \, dx$

$\therefore \quad I = \frac{3}{2} \int \frac{1}{1 + 2z^2} \cdot dz = \frac{3}{4} \int \frac{1}{\frac{1}{2} + z^2} \cdot dz$

$$= \frac{3}{4} \int \frac{1}{\left[\left(1/\sqrt{2}\right)^2 + z^2\right]} \cdot dz \qquad \left[\text{By using} \int \frac{1}{a^2 + x^2} \cdot dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \frac{3}{4} \cdot \frac{1}{\left(1/\sqrt{2}\right)} \tan^{-1}\left[\frac{z}{\left(1/\sqrt{2}\right)}\right] + c = \frac{3\sqrt{2}}{4} \tan^{-1}\left(\sqrt{2}z\right) + c$$

$$= \frac{3}{2\sqrt{2}} \tan^{-1}\left(\sqrt{2}x^2\right) + c. \qquad [\because \ z = x^2]$$

(*iv*) Let $\quad I = \int \frac{x^2 - 1}{x^2 + 4} \cdot dx = \int \frac{x^2 + 4 - 5}{x^2 + 4} \cdot dx$

$$= \int \left(\frac{x^2 + 4}{x^2 + 4} - \frac{5}{x^2 + 4}\right) \cdot dx = \int \left(1 - \frac{5}{x^2 + 4}\right) \cdot dx$$

$$= \int 1 \, . \, dx - 5 \int \frac{1}{x^2 + (2)^2} \, . \, dx \qquad \left[\text{By using} \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= x - 5 \, . \left(\frac{1}{2}\right) \tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= x - \frac{5}{2} \tan^{-1}\left(\frac{x}{2}\right) + c.$$

Example 23. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{(4 \sin^2 x + 5 \cos^2 x)} \, . \, dx$ (*ii*) $\int \frac{1}{(3 + 2 \cos^2 x)} \, dx$

(*iii*) $\int \frac{1}{x[7 + 3 (\log x)^2]} \, . \, dx$

Solution. (*i*) Let $I = \int \frac{1}{(4 \sin^2 x + 5 \cos^2 x)} \, . \, dx$

[Dividing numerator and denominator by $\cos^2 x$]

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{4 \sin^2 x}{\cos^2 x} + \frac{5 \cos^2 x}{\cos^2 x}} \, . \, dx = \int \frac{\sec^2 x}{4 \tan^2 x + 5} \, . \, dx$$

Put $z = \tan x \Rightarrow dz = \sec^2 x \, dx$

$\therefore$

$$I = \int \frac{1}{(4z^2 + 5)} \, . \, dz = \frac{1}{4} \int \frac{1}{\left(z^2 + \frac{5}{4}\right)} \, . \, dz$$

$$= \frac{1}{4} \int \frac{1}{\left[z^2 + \left(\frac{\sqrt{5}}{2}\right)^2\right]} \, . \, dz \qquad \left[\text{By using} \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \frac{1}{4} \cdot \frac{1}{(\sqrt{5}/2)} \tan^{-1}\left[\frac{z}{(\sqrt{5}/2)}\right] + c = \frac{1}{4} \cdot \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{2z}{\sqrt{5}}\right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2 \tan x}{\sqrt{5}}\right) + c. \qquad [\because \; z = \tan x]$$

(*ii*) Let $I = \int \frac{1}{3 + 2 \cos^2 x} \, . \, dx$ [Dividing numerator and denominator by $\cos^2 x$]

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{3}{\cos^2 x} + \frac{2 \cos^2 x}{\cos^2 x}} \, . \, dx = \int \frac{\sec^2 x}{3 \sec^2 x + 2} \, . \, dx$$

$$= \int \frac{\sec^2 x}{3 (1 + \tan^2 x) + 2} \, . \, dx = \int \frac{\sec^2 x}{3 + 3 \tan^2 x + 2} \, . \, dx$$

$[\because \; \sec^2 A - \tan^2 A = 1]$

$$= \int \frac{\sec^2 x}{5 + 3\tan^2 x}\, dx$$

Put $\quad z = \tan x \Rightarrow dz = \sec^2 x\, dx$

$\therefore$
$$I = \int \frac{1}{5 + 3z^2} \cdot dz = \frac{1}{3}\int \frac{1}{\left(\frac{5}{3} + z^2\right)} \cdot dz$$

$$= \frac{1}{3}\int \frac{1}{\left(\frac{\sqrt{5}}{\sqrt{3}}\right)^2 + z^2} \cdot dz \qquad \left[\text{By using} \int \frac{1}{a^2 + x^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{3} \cdot \frac{1}{\left(\frac{\sqrt{5}}{\sqrt{3}}\right)} \cdot \tan^{-1}\left(\frac{z}{\sqrt{5}/\sqrt{3}}\right) + c = \frac{1}{3} \cdot \frac{\sqrt{3}}{\sqrt{5}} \cdot \tan^{-1}\left(\frac{\sqrt{3}z}{\sqrt{5}}\right) + c$$

$$= \frac{1}{\sqrt{15}} \cdot \tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{5}}\right) + c. \qquad [\because z = \tan x]$$

(iii) Let
$$I = \int \frac{1}{x[7 + 3(\log x)^2]} \cdot dx = \int \frac{1/x}{7 + 3(\log x)^2} \cdot dx$$

Put $\quad z = \log x \Rightarrow dz = \frac{1}{x} \cdot dx$

$\therefore$
$$I = \int \frac{1}{7 + 3z^2} \cdot dz = \frac{1}{3}\int \frac{1}{\left(\frac{7}{3} + z^2\right)} \cdot dz$$

$$= \frac{1}{3}\int \frac{1}{\left[\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2 + z^2\right]} dz \qquad \left[\text{By using} \int \frac{1}{a^2 + x^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{3} \cdot \left(\frac{1}{\frac{\sqrt{7}}{\sqrt{3}}}\right) \cdot \tan^{-1}\left(\frac{z}{\frac{\sqrt{7}}{\sqrt{3}}}\right) + c = \frac{1}{3} \cdot \frac{\sqrt{3}}{\sqrt{7}} \tan^{-1}\left(\frac{\sqrt{3}z}{\sqrt{7}}\right) + c$$

$$= \frac{1}{\sqrt{21}} \tan^{-1}\left(\frac{\sqrt{3}\log x}{\sqrt{7}}\right) + c. \qquad [\because z = \log x]$$

Example 24. *Evaluate the following integral :*

(i) $\displaystyle\int \frac{1}{1 + 3\sin^2 x + 8\cos^2 x} \cdot dx \qquad$ *(ii)* $\displaystyle\int \frac{e^{-x}}{16 + 9e^{-2x}} \cdot dx.$

Solution. *(i)* Let $I = \displaystyle\int \frac{1}{1 + 3\sin^2 x + 8\cos^2 x} \cdot dx$

[Dividing numerator and denominator by $\cos^2 x$]

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{3\sin^2 x}{\cos^2 x} + \frac{8\cos^2 x}{\cos^2 x}} \, . \, dx$$

$$= \int \frac{\sec^2 x}{(\sec^2 x + 3\tan^2 x + 8)} \, . \, dx$$

$$= \int \frac{\sec^2 x}{(1 + \tan^2 x + 3\tan^2 x + 8)} \, . \, dx = \int \frac{\sec^2 x}{(9 + 4\tan^2 x)} \, . \, dx$$

$[\because \ \sec^2 A - \tan^2 A = 1]$

Put $\quad z = \tan x \ \Rightarrow \ dz = \sec^2 x \, dx$

$$\therefore \quad I = \int \frac{1}{9 + 4z^2} \, . \, dz = \frac{1}{4} \int \frac{1}{\left(\frac{9}{4} + z^2\right)} \, . \, dz$$

$$= \frac{1}{4} \int \frac{1}{\left(\frac{3}{2}\right)^2 + z^2} \, . \, dz \qquad \left[\text{By using } \int \frac{1}{a^2 + x^2} \, . \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \frac{1}{4} \cdot \frac{1}{\left(\frac{3}{2}\right)} \, . \tan^{-1} \left(\frac{z}{\frac{3}{2}}\right) + c = \frac{1}{4} \left(\frac{2}{3}\right) . \tan^{-1} \left(\frac{2z}{3}\right) + c$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{2\tan x}{3}\right) + c. \qquad [\because \ z = \tan x]$$

(*ii*) Let $\quad I = \int \frac{e^{-x}}{16 + 9e^{-2x}} \, . \, dx = \int \frac{e^{-x}}{16 + (3e^{-x})^2} \, . \, dx$

Put $\quad z = 3e^{-x}$

$\Rightarrow \quad dz = -3e^{-x} \, . \, dx \ \Rightarrow \ -\frac{1}{3} dz = e^{-x} \, . \, dx$

$$\therefore \quad I = \int \frac{1}{4^2 + z^2} \cdot \left(-\frac{1}{3} dz\right)$$

$$= -\frac{1}{3} \int \frac{1}{[(4)^2 + z^2]} \, . \, dz \qquad \left[\text{By using } \int \frac{1}{a^2 + x^2} \, . \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= -\frac{1}{3} \cdot \left(\frac{1}{4}\right) . \tan^{-1} \left(\frac{z}{4}\right) + c$$

$$= -\frac{1}{12} \tan^{-1} \left(\frac{3e^{-x}}{4}\right) + c. \qquad [\because \ z = 3e^{-x}]$$

Example 25. *Evaluate the following integrals :*

(i) $\int \frac{1}{\sqrt{15-8x^2}} \cdot dx$ (ii) $\int \frac{1}{\sqrt{16x^2+25}} \cdot dx$

(iii) $\int \frac{1}{x\sqrt{a^2x^2-b^2}} \cdot dx$ (iv) $\int \frac{1}{\sqrt{16x^2-9}} \cdot dx$

(v) $\int \frac{e^x}{e^{2x}-4} \cdot dx$ (vi) $\int \frac{1}{\sqrt{17-\frac{x^2}{12}}} \cdot dx.$

Solution. (i) Let $I = \int \frac{1}{\sqrt{15-8x^2}} \cdot dx = \frac{1}{\sqrt{8}} \int \frac{1}{\sqrt{\frac{15}{8}-x^2}} \cdot dx$

$$= \frac{1}{\sqrt{8}} \int \frac{1}{\sqrt{\left(\sqrt{\frac{15}{8}}\right)^2 - x^2}} \cdot dx \qquad \left[\text{By using } \int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{\sqrt{8}} \cdot \sin^{-1}\left(\frac{x}{\frac{\sqrt{15}}{\sqrt{8}}}\right) + c$$

$$= \frac{1}{2\sqrt{2}} \cdot \sin^{-1}\left(\frac{\sqrt{8}x}{\sqrt{15}}\right) + c.$$

(ii) Let $I = \int \frac{1}{\sqrt{16x^2+25}} \cdot dx = \frac{1}{4}\int \frac{1}{\sqrt{x^2+\left(\frac{25}{16}\right)}} \cdot dx$

$$= \frac{1}{4}\int \frac{1}{\sqrt{x^2+\left(\frac{5}{4}\right)^2}} \cdot dx \qquad \left[\text{By using } \int \frac{1}{\sqrt{x^2+a^2}} \cdot dx = \log\left|x+\sqrt{a^2+x^2}\right| + c\right]$$

$$= \frac{1}{4}\log\left|x+\sqrt{x^2+\left(\frac{5}{4}\right)^2}\right| + c_1$$

$$= \frac{1}{4}\log\left|x+\sqrt{x^2+\frac{25}{16}}\right| + c_1 = \frac{1}{4}\log\left|x+\frac{\sqrt{16x^2+25}}{4}\right| + c_1$$

$$= \frac{1}{4}\log\left|\frac{4x+\sqrt{16x^2+25}}{4}\right| + c_1$$

$$= \frac{1}{4}\log\left|4x+\sqrt{16x^2+25}\right| - \frac{1}{4}\log|4| + c_1 \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$= \frac{1}{4}\log\left|4x+\sqrt{16x^2+25}\right|+c. \qquad \left[\text{where}: c = c_1 - \frac{1}{4}\log|4|\right]$$

(*iii*) Let $\quad I = \int \frac{1}{x\sqrt{a^2x^2-b^2}}\,.\,dx = \frac{1}{a}\int \frac{1}{x\sqrt{x^2-\frac{b^2}{a^2}}}\,.\,dx$

$$= \frac{1}{a}\int \frac{1}{x\sqrt{x^2-(b/a)^2}}\,.\,dx \qquad \left[\text{By using} \int \frac{1}{x\sqrt{x^2-a^2}}\,.\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+c\right]$$

$$= \frac{1}{a}\cdot\frac{1}{(b/a)}\sec^{-1}\left(\frac{x}{b/a}\right)+c = \frac{1}{a}\cdot\frac{a}{b}\cdot\sec^{-1}\left(\frac{ax}{b}\right)+c$$

$$= \frac{1}{b}\sec^{-1}\left(\frac{ax}{b}\right)+c.$$

(*iv*) Let $I = \int \frac{1}{\sqrt{16x^2-9}}\,.\,dx = \frac{1}{4}\int \frac{1}{\sqrt{x^2-\frac{9}{16}}}\,.\,dx$

$$= \frac{1}{4}\int \frac{1}{\sqrt{x^2-\left(\frac{3}{4}\right)^2}}\,.\,dx \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2-a^2}}\,.\,dx = \log\left|x+\sqrt{x^2-a^2}\right|+c\right]$$

$$= \frac{1}{4}\log\left|x+\sqrt{x^2-\left(\frac{3}{4}\right)^2}\right|+c_1 = \frac{1}{4}\log\left|x+\sqrt{x^2-\frac{9}{16}}\right|+c_1$$

$$= \frac{1}{4}\log\left|x+\frac{\sqrt{16x^2-9}}{4}\right|+c_1 = \frac{1}{4}\log\left|\frac{4x+\sqrt{16x^2-9}}{4}\right|+c_1$$

$$= \frac{1}{4}\left[\log\left|4x+\sqrt{16x^2-9}\right|\right]-\frac{1}{4}\log|4|+c_1 \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$= \frac{1}{4}\log\left|4x+\sqrt{16x^2-9}\right|+c. \qquad \left[\text{where}: c = c_1 - \frac{1}{4}\log|4|\right]$$

(*v*) Let $I = \int \frac{e^x}{e^{2x}-4}\,.\,dx$

Put $\quad z = e^x \Rightarrow dz = e^x\,.\,dx$

$\therefore \quad I = \int \frac{1}{z^2-4}\,.\,dz$

$$= \int \frac{1}{z^2-(2)^2}\,.\,dz \qquad \left[\text{By using} \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c\right]$$

$$= \frac{1}{2(2)}\log\left|\frac{z-2}{z+2}\right|+c \qquad [\because z = e^x]$$

$$= \frac{1}{4} \log \left| \frac{e^x - 2}{e^x + 2} \right| + c.$$

(vi) Let $I = \int \frac{1}{\sqrt{17 - \frac{x^2}{12}}} \, . \, dx$

$$= \int \frac{1}{\sqrt{\frac{204 - x^2}{12}}} \, . \, dx = \int \frac{\sqrt{12}}{\sqrt{204 - x^2}} \, . \, dx$$

$$= \sqrt{12} \int \frac{1}{\sqrt{(\sqrt{204})^2 - x^2}} \, . \, dx \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}} . \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c \right]$$

$$= \sqrt{12} \sin^{-1}\left(\frac{x}{\sqrt{204}}\right) + c.$$

Example 26. *Evaluate the following integrals :*

(i) $\int \frac{2^x}{\sqrt{1 - 4^x}} \, . \, dx$ (ii) $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 9}} \, . \, dx$

(iii) $\int \frac{e^{3x}}{4 e^{6x} - 9} \, . \, dx$ (iv) $\int \sqrt{\frac{a - x}{a + x}} \, . \, dx$

(v) $\int \frac{x^2}{\sqrt{x^6 - 1}} \, . \, dx$ (vi) $\int \frac{1}{\sqrt{1 - e^{2x}}} \, . \, dx.$

Solution. (i) Let $I = \int \frac{2^x}{\sqrt{1 - 4^x}} \, . \, dx$

Put $z = 2^x$

$\Rightarrow \quad dz = 2^x \log 2 \, . \, dx \quad \Rightarrow \quad \frac{1}{\log 2} \, . \, dz = 2^x \, . \, dx$

$\therefore \quad I = \int \frac{1}{\sqrt{1 - z^2}} \cdot \left(\frac{1}{\log 2}\right) . \, dz$

$$= \frac{1}{\log 2} \int \frac{1}{\sqrt{(1)^2 - z^2}} \, dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}} . \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c \right]$$

$$= \frac{1}{\log 2} \, . \, \sin^{-1}(z) + c = \frac{1}{\log 2} \, . \, \sin^{-1}(2^x) + c. \qquad [\because \; z = 2^x]$$

(ii) Let $I = \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 9}} \, . \, dx$

Put $z = \tan x \;\Rightarrow\; dz = \sec^2 x \, dx$

$\Rightarrow \quad I = \int \frac{1}{\sqrt{z^2 + (3)^2}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 + x^2}} \cdot dx = \log \left| x + \sqrt{a^2 + x^2} \right| + c\right]$

$= \log \left| z + \sqrt{z^2 + 9} \right| + c$

$= \log \left| \tan x + \sqrt{\tan^2 x + 9} \right| + c. \qquad [\because \; z = \tan x]$

(*iii*) Let $I = \int \frac{e^{3x}}{4\,e^{6x} - 9} \cdot dx = \int \frac{e^{3x}}{4\,(e^{3x})^2 - 9} \cdot dx$

Put $\quad z = e^{3x}$

$\Rightarrow \quad dz = 3\,e^{3x} \cdot dx \;\Rightarrow\; \frac{1}{3}\,dz = e^{3x} \cdot dx$

$\therefore \quad I = \int \frac{1}{4z^2 - 9} \cdot \left(\frac{1}{3}\,dz\right) = \frac{1}{3} \int \frac{1}{4z^2 - 9} \cdot dz$

$= \frac{1}{12} \int \frac{1}{(z^2 - 9/4)} \cdot dz$

$= \frac{1}{12} \int \frac{1}{\left[z^2 - \left(\frac{3}{2}\right)^2\right]} \cdot dz \qquad \left[\text{By using} \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + c\right]$

$= \frac{1}{12} \cdot \left[\frac{1}{2(3/2)}\right] \cdot \log \left| \frac{z - 3/2}{z + 3/2} \right| + c = \frac{1}{36} \cdot \log \left| \frac{2z - 3}{2z + 3} \right| + c$

$= \frac{1}{36} \cdot \log \left| \frac{2\,e^{3x} - 3}{2\,e^{3x} + 3} \right| + c. \qquad [\because \; z = e^{3x}]$

(*iv*) Let $\quad I = \int \sqrt{\frac{a - x}{a + x}} \cdot dx$

$= \int \frac{\sqrt{a - x}}{\sqrt{a + x}} \times \frac{\sqrt{a - x}}{\sqrt{a - x}} \cdot dx \qquad$ [On rationalization]

$= \int \frac{(a - x)}{\sqrt{a^2 - x^2}} \cdot dx = \int \frac{a}{\sqrt{a^2 - x^2}} \cdot dx - \int \frac{x}{\sqrt{a^2 - x^2}}\,dx$

$= a \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx - \int \frac{x}{\sqrt{a^2 - x^2}}\,dx$

Put $\quad z = a^2 - x^2$

$\Rightarrow \quad dz = -\,2x\,dx \;\Rightarrow\; -\frac{1}{2}\,dz = x\,dx$

$\therefore \quad I = a \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx - \int \frac{1}{\sqrt{z}} \cdot \left(-\frac{1}{2}\right) dz$

$$= a \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx + \frac{1}{2} \int z^{-1/2} \cdot dz$$

$$= a \sin^{-1} \frac{x}{a} + \frac{1}{2} \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = a \sin^{-1} \frac{x}{a} + z^{1/2} + c$$

$$= a \sin^{-1} \frac{x}{a} + \sqrt{z} + c$$

$$= a \sin^{-1} \frac{x}{a} + \sqrt{a^2 - x^2} + c. \qquad [\because \ z = a^2 - x^2]$$

(*v*) Let $\quad I = \int \frac{x^2}{\sqrt{x^6 - 1}} \cdot dx = \int \frac{x^2}{\sqrt{(x^3)^2 - 1}} \cdot dx$

Put $\quad z = x^3$

$\Rightarrow \quad dz = 3x^2 \, dx \ \Rightarrow \ \frac{1}{3} dz = x^2 \, dx$

$\therefore \quad I = \frac{1}{3} \int \frac{1}{\sqrt{z^2 - 1}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c\right]$

$$= \frac{1}{3} \log \left| z + \sqrt{z^2 - 1} \right| + c$$

$$= \frac{1}{3} \log \left| x^3 + \sqrt{x^6 - 1} \right| + c. \qquad [\because \ z = x^3]$$

(*vi*) Let $\quad I = \int \frac{1}{\sqrt{1 - e^{2x}}} \cdot dx$

$$= \int \frac{e^{-x}}{\sqrt{e^{-2x}(1 - e^{2x})}} \cdot dx = \int \frac{e^{-x}}{\sqrt{e^{-2x} - 1}} \cdot dx$$

[Multiplying numerator and denominator by e^{-x}]

Put $\quad z = e^{-x}$

$\Rightarrow \quad dz = -e^{-x} \, dx \ \Rightarrow \ -dz = e^{-x} \, dx.$

$\therefore \quad I = \int \frac{1}{\sqrt{z^2 - 1}} (-dz)$

$= -\int \frac{1}{\sqrt{z^2 - 1}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c\right]$

$$= -\log \left| z + \sqrt{z^2 - 1} \right| + c$$

$$= -\log \left| e^{-x} + \sqrt{e^{-2x} - 1} \right| + c. \qquad [\because \ z = e^{-x}]$$

Example 27. *Evaluate the following integrals :*

(*i*) $\int \frac{2x^3}{4+x^8}\,dx$ (*ii*) $\int \frac{1}{x\sqrt{(\log x)^2-5}}\,.\,dx$ (*iii*) $\int \frac{a^x}{1-a^{2x}}\,.\,dx.$

Solution. (*i*) Let $I = \int \frac{2x^3}{4+x^8}\,.\,dx = \int \frac{2x^3}{4+(x^4)^2}\,.\,dx$

Put $z = x^4$

$\Rightarrow \quad dz = 4x^3\,dx \Rightarrow \frac{dz}{4} = x^3.\,dx$

$\therefore \quad I = \int \frac{2}{(4+z^2)}\cdot\frac{dz}{4}$

$= \frac{1}{2}\int \frac{1}{(2^2+z^2)}\,.\,dz$ $\left[\text{By using} \int \frac{1}{a^2+x^2}\,.\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$

$= \frac{1}{2}\cdot\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{z}{2}\right)+c = \frac{1}{4}\tan^{-1}\left(\frac{x^4}{2}\right)+c.$ $[\because \ z = x^4]$

(*ii*) Let $I = \int \frac{1}{x\sqrt{(\log x)^2-5}}\,.\,dx = \int \frac{1/x}{\sqrt{(\log x)^2-5}}\,.\,dx$

Put $z = \log x \Rightarrow dz = \frac{1}{x}\,.\,dx$

$\therefore \quad I = \int \frac{1}{\sqrt{z^2-5}}\,.\,dz$

$= \int \frac{1}{\sqrt{z^2-(\sqrt{5})^2}}\,.\,dz$ $\left[\text{By using} \int \frac{1}{\sqrt{x^2-a^2}}\,.\,dx = \log\left|\,x+\sqrt{x^2-a^2}\,\right|+c\right]$

$= \log\left|\,z+\sqrt{z^2-5}\,\right|+c$

$= \log\left|\,\log x+\sqrt{(\log x)^2-5}\,\right|+c.$ $[\because \ z = \log x]$

(*iii*) Let $I = \int \frac{a^x}{1-a^{2x}}\,.\,dx = \int \frac{a^x}{\sqrt{1^2-(a^x)^2}}\,.\,dx$

Put $z = a^x$

$\Rightarrow \quad dz = a^x\log a\,dx \Rightarrow \frac{1}{\log a}\,.\,dz = a^x\,.\,dx$

$\therefore \quad I = \int \frac{1}{\sqrt{1-z^2}}\cdot\left(\frac{1}{\log a}\,.\,dz\right)$

$= \frac{1}{\log a}\int \frac{1}{\sqrt{1^2-z^2}}\,.\,dz$ $\left[\text{By using} \int \frac{1}{\sqrt{a^2-x^2}}\,.\,dx = \sin^{-1}\frac{x}{a}+c\right]$

$= \frac{1}{\log a}\,.\,\sin^{-1}z+c = \frac{1}{\log a}\,.\,\sin^{-1}(a^x)+c.$ $[\because \ z = a^x]$

Example 28. *Evaluate the following integrals :*

(*i*) $\int \frac{\sin x}{\sqrt{4\cos^2 x - 1}} \cdot dx$ (*ii*) $\int \frac{\sec^2 x}{\sqrt{16 + \tan^2 x}} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{\sin x}{\sqrt{4\cos^2 x - 1}} \cdot dx$

Put $z = \cos x$

$\Rightarrow$ $dz = -\sin x\, dx \Rightarrow -dz = \sin x\, dx$

$\therefore$ $$I = \int \frac{1}{\sqrt{4z^2 - 1}} \cdot (-dz) = -\frac{1}{2}\int \frac{1}{\left[\sqrt{z^2 - \frac{1}{4}}\right]} dz$$

$$= -\frac{1}{2}\int \frac{1}{\left[\sqrt{z^2 - \left(\frac{1}{2}\right)^2}\right]} \cdot dz$$

$$\left[\text{By using} \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c\right]$$

$$= -\frac{1}{2} \cdot \log\left|z + \sqrt{z^2 - \frac{1}{4}}\right| + c_1 = -\frac{1}{2}\log\left|z + \frac{\sqrt{4z^2 - 1}}{2}\right| + c_1$$

$$= -\frac{1}{2}\log\left|\frac{2z + \sqrt{4z^2 - 1}}{2}\right| + c_1$$

$$= -\frac{1}{2}\left[\log\left|2z + \sqrt{4z^2 - 1}\right| - \log|2|\right] + c_1$$

$$= -\frac{1}{2}\log\left|2z + \sqrt{4z^2 - 1}\right| + \frac{1}{2}\log|2| + c_1$$

$$= -\frac{1}{2}\log\left|2z + \sqrt{4z^2 - 1}\right| + c \qquad \left[\text{where} : c = c_1 + \frac{1}{2}\log|2|\right]$$

$[\because \ z = \cos x]$

$$= -\frac{1}{2}\log\left|2\cos x + \sqrt{4\cos^2 x - 1}\right| + c.$$

(*ii*) Let $I = \int \frac{\sec^2 x}{\sqrt{16 + \tan^2 x}} \cdot dx$

Put $z = \tan x \Rightarrow dz = \sec^2 x\, dx$

$\therefore$ $$I = \int \frac{1}{\sqrt{16 + z^2}}\, dz = \int \frac{1}{\sqrt{4^2 + z^2}} \cdot dz$$

$$\left[\text{By using} \int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \log\left|x + \sqrt{a^2 + x^2}\right| + c\right]$$

$$= \log \left| z + \sqrt{z^2 + 16} \right| + c$$

$$= \log \left| \tan x + \sqrt{\tan^2 x + 16} \right| + c. \qquad [\because \; z = \tan x]$$

EXERCISE FOR PRACTICE

Evaluate the following :

1. $\int \frac{1 - \sin x}{x + \cos x}\, dx$
2. $\int \frac{(\log x)^3}{x}\, dx$
3. $\int \frac{\cos x - \sin x}{1 + \sin 2x}\, dx$
4. $\int \frac{\sin x}{1 + \cos x}\, dx$
5. $\int \frac{\cos x}{1 + \sin 2x}\, dx$
6. $\int x \sin x^2\, dx$
7. $\int \frac{\sin x}{(a + b\cos x)^2}\, dx$
8. $\int \frac{(\tan^{-1} x)^3}{1 + x^2}\, dx$
9. $\int \frac{1}{\sqrt{1 - \cos 2x}}\, dx$
10. $\int \frac{1 + \sin 2x}{x + \sin^2 x}\, dx$
11. $\int \frac{e^{4x} - 1}{e^{4x} + 1}\, dx$
12. $\int \frac{\cos x - \sin x}{\sqrt{1 + \sin 2x}}\, dx$
13. $\int \tan^3 x\, dx$
14. $\int \frac{\sin (x + a)}{\sin (x - a)}\, dx$
15. $\int \frac{\cos 2x}{\cos x}\, dx$
16. $\int \sin^5 x \cos^3 x\, dx$
17. $\int \tan 3x \tan 2x \tan x\, dx$
18. $\int \sin^4 x \cos^3 x\, dx$
19. $\int \frac{1}{4 + 25x^2}\, dx$
20. $\int \frac{\sin \theta}{1 - 4\cos^2 \theta}\, d\theta$

Answers

1. $\log | x + \cos x | + c$
2. $\frac{(\log x)^4}{4} + c$
3. $\frac{-1}{\cos x + \sin x} + c$
4. $-\log (1 + \cos x) + c$
5. $\tan^{-1} (\sin x) + c$
6. $-\frac{1}{2} \cos x^2 + c$
7. $\frac{1}{b\,(a + b\cos x)} + c$
8. $\frac{1}{4} (\tan^{-1} x)^4 + c$
9. $\frac{1}{\sqrt{2}} \log | \operatorname{cosec} x - \cot x | + c$
10. $\log | x + \sin^2 x | + c$
11. $\frac{1}{2} \log (e^{2x} + e^{-2x}) + c$
12. $\log | \sin x + \cos x | + c$
13. $\frac{1}{2} \tan^2 x - \log | \sec x | + c$
14. $(x - a) \cos 2a + \sin 2a \log | \sin (x - a) | + c$
15. $-\log | \sin x + \tan x | + 2 \sin x + c$
16. $\frac{1}{6} \sin^6 x - \frac{1}{8} \sin^8 x + c$
17. $\frac{1}{3} \log | \sec 3x | - \frac{1}{2} \log | \sec 2x | - \log | \sec x | + c$
18. $\frac{1}{5} \sin^5 x - \frac{\sin^7 x}{7} + c$
19. $\frac{1}{10} \tan^{-1} \left(\frac{5x}{2} \right) + c$
20. $-\frac{1}{4} \log \left| \frac{1 + 2\cos \theta}{1 - 2\cos \theta} \right| + c.$

3

Integration by Substitution—II

3.1 INTEGRALS OF SOME FUNCTIONS CONTAINING A QUADRATIC POLYNOMIAL

Evaluation of $\int \frac{1}{ax^2+bx+c}\,.\,dx$, $\int \frac{px+q}{ax^2+bx+c}\,.\,dx$, $\int \frac{1}{\sqrt{ax^2+bx+c}}\,.\,dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,.\,dx$.

The quadratic expression $ax^2 + bx + c$ may or may not be expressible as the product of linear factors. In case, $ax^2 + bx + c$ is easily reducible to two linear factors, then these integrals can be easily evaluated by using the method "Integration by partial fractions", which we shall study later : Thus ;

(*a*) If $ax^2 + bx + c$ is easily reducible to linear factors, then either the method of this section or the method of integration by partial fractions can be used. Out of these two methods, the method of integration by partial fractions would be found more convenient and easy.

(*b*) If $ax^2 + bx + c$ is not easily reducible to linear factors, then the method of integration by partial fractions is not applied and the method of this section is the only choice at our disposal.

3.1.1. Working Rule for Evaluating $\int \frac{1}{ax^2+bx+c}\,.\,dx$.

Step I. First take a common and make the co-efficient of x^2 unity, or by multiplying and dividing by it.

Step II. Complete the square by adding and subtracting the square of the half of co-efficient of x *i.e.* $\left(\frac{1}{2}\text{ co-eff. of } x\right)^2$.

Step III. Express $(ax^2 + bx + c)$ as the sum or difference of two squares. Then to integrate.

Step IV. Use the suitable formula from the following :

$$\int \frac{1}{a^2+x^2}\,dx,\ \int \frac{1}{a^2-x^2}\,dx \text{ and } \int \frac{1}{x^2-a^2}\,dx.$$

3.1.2. Working Rule for Evaluating $\int \frac{1}{\sqrt{ax^2+bx+c}}\,.\,dx$.

Step I. Make the co-efficient of x^2 unity by taking $\frac{1}{\sqrt{|a|}}$ out of the integral sign.

Step II. Complete the square by adding and subtracting the square of the half of co-efficient of x. *i.e.* $\left(\frac{1}{2}\text{ co-efficient of } x\right)^2$.

Step III. Express the integrand as :

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,.\,dx = \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{(x+\alpha)^2 \pm \beta^2}}\,.\,dx.$$

Step IV. Then to integrate, use the suitable formula from the following :

$$\int \frac{1}{\sqrt{a^2+x^2}}\,.\,dx,\ \int \frac{1}{\sqrt{x^2-a^2}}\,dx \text{ and } \int \frac{1}{\sqrt{a^2-x^2}}\,.\,dx.$$

3.1.3. Working Rule for Evaluating $\int \frac{px+q}{ax^2+bx+c}\,.\,dx$.

Step I. Put $(px+q) = \lambda\left[\frac{d}{dx}(ax^2+bx+c)\right] + \mu$ and find the values of λ and μ by comparing the co-efficients of x and the constant terms.

Step II. Given integral takes the form as :

$$\int \frac{px+q}{ax^2+bx+c}\,.\,dx = \lambda\int \frac{2ax+b}{ax^2+bx+c}\,.\,dx + \mu\int \frac{1}{ax^2+bx+c}\,.\,dx.$$

Step III. Evaluate the first integral by putting $z = ax^2+bx+c$.

Step IV. Evaluate the second integral by using the method discussed in Article. (3.1.1)

3.1.4. Working Rule for Evaluating $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,.\,dx$.

Step I. Put $(px+q) = \lambda\left[\frac{d}{dx}(ax^2+bx+c)\right] + \mu$ and find the values of λ and μ by comparing the co-efficients of x and the constant term.

Step II. Given integral takes the form as :

i.e.,
$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,.\,dx = \lambda\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,.\,dx + \mu\int \frac{1}{\sqrt{ax^2+bx+c}}\,.\,dx.$$

Step III. Evaluate the first integral by putting $z = ax^2+bx+c$.

Step IV. Evaluate the second integral by using the method discussed in Article. (3.1.2)

3.1.5. TYPE : Integrals of the Form $\int \frac{1}{ax^2+bx+c}\,.\,dx$.

[**Note.** For Working Rule Please Refer to Article 3.1.1]

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following integrals :*

(i) $\int \frac{1}{9x^2-12x+8}\,.\,dx$ (ii) $\int \frac{1}{1-6x-9x^2}\,.\,dx$

(iii) $\int \frac{1}{9x^2+6x+10} \cdot dx$ (iv) $\int \frac{1}{x^2+4x+8} \cdot dx$

(v) $\int \frac{1}{5-8x-x^2} \cdot dx$ (vi) $\int \frac{1}{4x^2-4x+3} \cdot dx.$

Solution. (i) Let $I = \int \frac{1}{9x^2-12x+8} \cdot dx$

$$= \frac{1}{9}\int \frac{1}{\left(x^2-\frac{4}{3}x+\frac{8}{9}\right)} \cdot dx \qquad \text{[Take 9 as common from the denominator]}$$

$$= \frac{1}{9}\int \frac{1}{\left(x^2-\frac{4}{3}x+\frac{4}{9}\right)+\frac{4}{9}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{4}{9} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{4}{9}\end{array}\right]$$

$$= \frac{1}{9}\int \frac{1}{\left(x-\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2} \cdot dx$$

Put $x - \frac{2}{3} = z \Rightarrow dx = dz$

$$\therefore \quad I = \frac{1}{9}\int \frac{1}{z^2+\left(\frac{2}{3}\right)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2+a^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{1}{9} \cdot \frac{1}{2/3}\tan^{-1}\left(\frac{z}{2/3}\right) + c = \frac{1}{9} \cdot \frac{3}{2}\tan^{-1}\left(\frac{3z}{2}\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left[\frac{3\left(x-\frac{2}{3}\right)}{2}\right] + c \qquad \left[\because z = \left(x-\frac{2}{3}\right)\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x-2}{2}\right) + c.$$

(ii) Let $I = \int \frac{1}{1-6x-9x^2} \cdot dx$

$$= -\frac{1}{9}\int \frac{1}{\left(x^2+\frac{2}{3}x-\frac{1}{9}\right)} \cdot dx \qquad \left[\text{Take } -\frac{1}{9} \text{ as common from the denom.}\right]$$

$$= -\frac{1}{9}\int \frac{1}{\left(x^2+\frac{2}{3}x+\frac{1}{9}\right)-\frac{2}{9}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{9} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{9}\end{array}\right]$$

$$= -\frac{1}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2-\frac{2}{9}} \cdot dx$$

Put $x + \frac{1}{3} = z \Rightarrow dx = dz$

$$\therefore \quad I = -\frac{1}{9}\int \frac{1}{z^2 - \left(\frac{\sqrt{2}}{3}\right)^2} . dz \qquad \left[\text{By using} \int \frac{1}{x^2 - a^2} . dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= -\frac{1}{9} . \frac{1}{2.\left(\frac{\sqrt{2}}{3}\right)} \log\left|\frac{z - (\sqrt{2}/3)}{z + (\sqrt{2}/3)}\right| + c$$

$$= -\frac{1}{9} . \frac{3}{2\sqrt{2}} . \log\left|\frac{x + \frac{1}{3} - \frac{\sqrt{2}}{3}}{x + \frac{1}{3} + \frac{\sqrt{2}}{3}}\right| + c \qquad \left[\because \ z = x + \frac{1}{3}\right]$$

$$= -\frac{1}{6\sqrt{2}} \log\left|\frac{3x + 1 - \sqrt{2}}{3x + 1 + \sqrt{2}}\right| + c.$$

(*iii*) Let $I = \int \frac{1}{9x^2 + 6x + 10} . dx = \frac{1}{9}\int \frac{1}{\left(x^2 + \frac{2}{3}x + \frac{10}{9}\right)} . dx$

$$= \frac{1}{9}\int \frac{1}{\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) + \left(\frac{10}{9} - \frac{1}{9}\right)} . dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{9} \text{ to the denom.} \\ \because \quad \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{9}\end{array}\right]$$

$$= \frac{1}{9}\int \frac{1}{\left(x + \frac{1}{3}\right)^2 + (1)^2} . dx$$

Put $x + \frac{1}{3} = z \Rightarrow dx = dz$

$$\therefore \quad I = \frac{1}{9}\int \frac{1}{z^2 + 1^2} . dz \qquad \left[\text{By using} \int \frac{1}{x^2 + a^2} . dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{9} . \tan^{-1}\left(\frac{z}{1}\right) + c$$

$$= \frac{1}{9} . \tan^{-1}\left(x + \frac{1}{3}\right) + c \qquad \left[\because \ z = x + \frac{1}{3}\right]$$

$$= \frac{1}{9} . \tan^{-1}\left(\frac{3x + 1}{3}\right) + c.$$

(*iv*) Let $\quad I = \int \frac{1}{3 + 2x - x^2} \cdot dx$

$$= - \int \frac{1}{x^2 - 2x - 3} \cdot dx$$

$$= - \int \frac{1}{(x^2 - 2x + 1) - 4} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract 1 to the denom.} \\ \because \quad \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = 1\end{array}\right]$$

$$= - \int \frac{1}{(x-1)^2 - (2)^2} \cdot dx$$

Put $(x - 1) = z \;\Rightarrow\; dx = dz$

$$\therefore \quad I = - \int \frac{1}{z^2 - (2)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= - \frac{1}{2(2)} \log\left|\frac{z-2}{z+2}\right| + c$$

$$= - \frac{1}{4} \log\left|\frac{x-1-2}{x-1+2}\right| + c \qquad [\because \; z = (x-1)]$$

$$= - \frac{1}{4} \log\left|\frac{x-3}{x+1}\right| + c.$$

(*v*) Let $\quad I = \int \frac{1}{5 - 8x - x^2} \cdot dx$

$$= - \int \frac{1}{x^2 + 8x - 5} \cdot dx$$

$$= - \int \frac{1}{(x^2 + 8x + 16) - 21} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract 16 to the denom.} \\ \because \quad \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = 16\end{array}\right]$$

$$= - \int \frac{1}{(x+4)^2 - (\sqrt{21})^2} \cdot dx$$

Put $(x + 4) = z \;\Rightarrow\; dx = dz$

$$\therefore \quad I = - \int \frac{1}{z^2 - (\sqrt{21})^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= - \frac{1}{2 \cdot \sqrt{21}} \log\left|\frac{z - \sqrt{21}}{z + \sqrt{21}}\right| + c$$

$$= - \frac{1}{2 \cdot \sqrt{21}} \log\left|\frac{x + 4 - \sqrt{21}}{x + 4 + \sqrt{21}}\right| + c \qquad [\because \; z = (x + 4)]$$

(*vi*) Let $\quad I = \int \frac{1}{4x^2 - 4x + 3} \cdot dx$

$$= \frac{1}{4}\int \frac{1}{x^2 - x + \frac{3}{4}} \cdot dx$$

$$= \frac{1}{4}\int \frac{1}{\left(x^2 - x + \frac{1}{4}\right) + \frac{2}{4}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \quad \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \frac{1}{4}\int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}} \cdot dx$$

Put $\left(x - \frac{1}{2}\right) = z \Rightarrow dx = dz$

$$\therefore \quad I = \frac{1}{4}\int \frac{1}{z^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{4}\left(\frac{1}{\frac{1}{\sqrt{2}}}\right)\tan^{-1}\left(\frac{z}{1/\sqrt{2}}\right) + c = \left(\frac{\sqrt{2}}{4}\right)\tan^{-1}\left(\sqrt{2}z\right) + c$$

$$\left[\because \ z = \left(x - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left[\sqrt{2}\left(x - \frac{1}{2}\right)\right] + c$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{2x - 1}{\sqrt{2}}\right) + c.$$

Example 2. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{x^2 - x + 1} \cdot dx$ (*ii*) $\int \frac{1}{3x^2 + 13x - 10} \cdot dx$

(*iii*) $\int \frac{1}{x^2 + 4x + 8} \cdot dx$ (*iv*) $\int \frac{1}{2x^2 + x - 1} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{1}{x^2 - x + 1} \cdot dx$

$$= \int \frac{1}{\left(x^2 - x + \frac{1}{4}\right) + \left(1 - \frac{1}{4}\right)} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \quad \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)} \cdot dx$$

Put $\left(x - \frac{1}{2}\right) = z \Rightarrow dx = dz$

$\therefore \quad I = \int \frac{1}{z^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$

$= \frac{1}{(\sqrt{3}/2)} \tan^{-1}\left(\frac{z}{\sqrt{3}/2}\right) + c$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x - 1/2}{\sqrt{3}/2}\right) + c \qquad \left[\because z = x - \frac{1}{2}\right]$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x - 1}{\sqrt{3}}\right) + c.$

(ii) Let $I = \int \frac{1}{3x^2 + 13x - 10} \cdot dx = \frac{1}{3} \int \frac{1}{\left(x^2 + \frac{13}{3}x - \frac{10}{3}\right)} \cdot dx$

$= \frac{1}{3} \int \frac{1}{x^2 + \frac{13}{3}x + \left(\frac{13}{6}\right)^2 - \left(\frac{13}{6}\right)^2 - \left(\frac{10}{3}\right)} \cdot dx$

$\left[\begin{array}{l}\text{Add and subtract} \left(\frac{13}{6}\right)^2 \text{ to the denom.} \\ \because \quad \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \left(\frac{13}{6}\right)^2\end{array}\right]$

$= \frac{1}{3} \int \frac{1}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{169}{36} + \frac{10}{3}\right)} \cdot dx = \frac{1}{3} \int \frac{1}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{289}{36}\right)} \cdot dx$

$= \frac{1}{3} \int \frac{1}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2} \cdot dx$

Put $x + \frac{13}{6} = z \Rightarrow dx = dz$

$\therefore \quad I = \frac{1}{3} \int \frac{1}{z^2 - \left(\frac{17}{6}\right)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + c\right]$

$= \frac{1}{3} \cdot \frac{1}{2\left(\frac{17}{6}\right)} \cdot \log\left|\frac{z - \frac{17}{6}}{z + \frac{17}{6}}\right| + c$

$$= \frac{1}{17}\log\left|\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right| + c = \frac{1}{17}\log\left|\frac{x-4/6}{x+5}\right| + c \qquad \left[\because\ z = x+\frac{13}{6}\right]$$

$$= \frac{1}{17}\log\left|\frac{x-\frac{2}{3}}{x+5}\right| + c = \frac{1}{17}\log\left|\frac{3x-2}{3(x+5)}\right| + c.$$

(iii) Let $\quad I = \int \frac{1}{x^2+4x+8}\,.\,dx$

$$= \int \frac{1}{(x^2+4x+4)+(8-4)}\,.\,dx \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.}\\ \because \quad \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \frac{1}{(x+2)^2+(2)^2}\,.\,dx$$

Put $\quad x+2 = z \Rightarrow dx = dz$

$\therefore$
$$I = \int \frac{1}{z^2+(2)^2}\,.\,dz \qquad \left[\text{By using } \int \frac{1}{x^2+a^2}\,.\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{z}{2}\right) + c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + c. \qquad [\because\ z = x+2]$$

(iv) Let $\quad I = \int \frac{1}{2x^2+x-1}\,.\,dx$

$$= \frac{1}{2}\int \frac{1}{\left(x^2+\frac{x}{2}-\frac{1}{2}\right)}\,.\,dx = \frac{1}{2}\int \frac{1}{\left[x^2+\frac{x}{2}+\left(\frac{1}{4}\right)^2\right]-\left(\frac{1}{4}\right)^2-\frac{1}{2}}\,.\,dx$$

$$\left[\begin{array}{l}\text{Add and subtract}\left(\frac{1}{4}\right)^2\text{ to the denom.}\\ \because \quad \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \left(\frac{1}{4}\right)^2\end{array}\right]$$

$$= \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2-\left(\frac{1}{16}+\frac{1}{2}\right)}\,.\,dx = \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2-\left(\frac{9}{16}\right)}\,.\,dx$$

Put $\quad x+\frac{1}{4} = z \Rightarrow dx = dz$

$$\therefore \qquad I = \frac{1}{2}\int \frac{1}{z^2 - \left(\frac{3}{4}\right)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{2}\left[\frac{1}{2\left(\frac{3}{4}\right)}\right] \cdot \log\left|\frac{z - 3/4}{z + 3/4}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{x + \frac{1}{4} - \frac{3}{4}}{x + \frac{1}{4} + \frac{3}{4}}\right| + c = \frac{1}{3}\log\left|\frac{x - \frac{1}{2}}{x+1}\right| + c \qquad \left[\because \ z = x + \frac{1}{4}\right]$$

$$= \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + c.$$

Example 3. *Evaluate the following integrals :*

(*i*) $\int \frac{x}{x^4 + x^2 + 1} \cdot dx$ (*ii*) $\int \frac{e^x}{e^{2x} + 6e^x + 5} \cdot dx$

(*iii*) $\int \frac{x}{3x^4 - 18x^2 + 11} \cdot dx$ (*iv*) $\int \frac{1}{x(x^n + 1)} \cdot dx$

(*v*) $\int \frac{1}{x(x^5 + 1)} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{x}{x^4 + x^2 + 1} \cdot dx = \int \frac{x}{(x^2)^2 + x^2 + 1} \cdot dx$

Put $\qquad x^2 = z$

$$\Rightarrow \qquad 2x\,dx = dz \ \Rightarrow \ x\,dx = \frac{1}{2}dz$$

$$\therefore \qquad I = \int \frac{1}{z^2 + z + 1} \cdot \left(\frac{1}{2}dz\right) = \frac{1}{2}\int \frac{1}{z^2 + z + 1} \cdot dz$$

$$= \frac{1}{2}\int \frac{1}{\left(z^2 + z + \frac{1}{4}\right) + \left(1 - \frac{1}{4}\right)} \cdot dz$$

$$\left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \qquad \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \frac{1}{2}\int \frac{1}{\left(z + \frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)} \cdot dz = \frac{1}{2}\int \frac{1}{\left(z + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \cdot dz$$

Again substitute : $\left(z+\frac{1}{2}\right)=y \Rightarrow dz=dy$

$\therefore$ $$I=\frac{1}{2}\int \frac{1}{y^2+\left(\frac{\sqrt{3}}{2}\right)^2}\cdot dy \qquad \left[\text{By using } \int \frac{1}{x^2+a^2}\cdot dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$=\frac{1}{2}\cdot\frac{1}{(\sqrt{3}/2)}\cdot\tan^{-1}\left(\frac{y}{\sqrt{3}/2}\right)+c$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z+1/2}{\sqrt{3}/2}\right)+c=\frac{1}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\cdot\left(z+\frac{1}{2}\right)+c$$

$[\because\ y=z+1/2]$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\cdot\left(x^2+\frac{1}{2}\right)+c \qquad [\because\ z=x^2]$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c.$$

(*ii*) Let $$I=\int \frac{e^x}{e^{2x}+6e^x+5}\cdot dx$$

Put $e^x=z \Rightarrow e^x\,dx=dz$

$\therefore$ $$I=\int \frac{1}{z^2+6z+5}\cdot dz$$

$$=\int \frac{1}{(z^2+6z+9)+(5-9)}\cdot dz \qquad \left[\begin{array}{l}\text{Add and subtract 9 to the denom.}\\ \because \quad \left(\frac{1}{2}\text{ co-eff. of } x\right)^2=9\end{array}\right]$$

$$=\int \frac{1}{(z+3)^2-(2)^2}\cdot dz$$

Put $z+3=y \Rightarrow dz=dy$

$\therefore$ $$I=\int \frac{1}{y^2-(2)^2}\cdot dy \qquad \left[\text{By using } \int \frac{1}{x^2-a^2}\cdot dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c\right]$$

$$=\frac{1}{2(2)}\log\left|\frac{y-2}{y+2}\right|+c$$

$$=\frac{1}{4}\log\left|\frac{z+3-2}{z+3+2}\right|+c \qquad [\because\ z+3=y]$$

$$=\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+c. \qquad [\because\ z=e^x]$$

(*iii*) Let $$I=\int \frac{x}{3x^4-18x^2+11}\cdot dx=\int \frac{x}{3(x^2)^2-18x^2+11}\cdot dx$$

Put $\quad x^2 = z$

$\Rightarrow \quad 2x\,dx = dz \;\Rightarrow\; x\,dx = \frac{1}{2}\,dz$

$\therefore \quad I = \int \frac{1}{3z^2 - 18z + 11}\cdot\left(\frac{1}{2}\,dz\right) = \frac{1}{2}\int \frac{1}{3z^2 - 18z + 11}\,dz$

$$= \frac{1}{6}\int \frac{1}{z^2 - 6z + \frac{11}{3}}\cdot dz$$

$$= \frac{1}{6}\int \frac{1}{(z^2 - 6z + 9) + \left(\frac{11}{3} - 9\right)}\cdot dz$$

$\left[\begin{array}{l}\text{Add and subtract 9 to the denom.} \\ \because \quad \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 9\end{array}\right]$

$$= \frac{1}{6}\int \frac{1}{(z-3)^2 - \frac{16}{3}}\cdot dz$$

Put $\quad z - 3 = y \;\Rightarrow\; dz = dy$

$\therefore \quad I = \frac{1}{6}\int \frac{1}{y^2 - \left(\frac{4}{\sqrt{3}}\right)^2}\cdot dy$

$\left[\text{By using } \int \frac{1}{x^2 - a^2}\cdot dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$

$$= \frac{1}{6}\cdot\frac{1}{2\left(\frac{4}{\sqrt{3}}\right)}\cdot\log\left|\frac{y - 4/\sqrt{3}}{y + 4/\sqrt{3}}\right| + c$$

$$= \frac{\sqrt{3}}{48}\log\left|\frac{z - 3 - 4/\sqrt{3}}{z - 3 + 4/\sqrt{3}}\right| + c \qquad [\because\; y = z - 3]$$

$$= \frac{\sqrt{3}}{48}\log\left|\frac{x^2 - 3 - 4/\sqrt{3}}{x^2 - 3 + 4/\sqrt{3}}\right| + c \qquad [\because\; x^2 = z]$$

$$= \frac{\sqrt{3}}{48}\log\left|\frac{\sqrt{3}x^2 - 3\sqrt{3} - 4}{\sqrt{3}x^2 - 3\sqrt{3} + 4}\right| + c.$$

(*iv*) Let $\quad I = \int \frac{1}{x\,(x^n + 1)}\cdot dx$

Put $\quad x^n + 1 = z \;\Rightarrow\; x^n = z - 1$

$\Rightarrow \quad nx^{n-1}\cdot dx = dz \;\Rightarrow\; \frac{n\,.\,x^n}{x}\,dx = dz$

$\Rightarrow \quad \frac{1}{x}dx = \frac{1}{nx^n}\cdot dz \quad \Rightarrow \quad \frac{1}{x}dx = \frac{1}{n(z-1)}\cdot dz$

$\therefore \quad I = \int \frac{1}{z}\cdot\frac{1}{n(z-1)}\cdot dz = \frac{1}{n}\int \frac{1}{z^2 - z}\cdot dz$

$$= \frac{1}{n}\int \frac{1}{\left(z^2 - z + \frac{1}{4}\right) - \frac{1}{4}}\cdot dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \quad \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \frac{1}{n}\int \frac{1}{\left(z - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\cdot dz$$

Put $z - \frac{1}{2} = y \Rightarrow dz = dy$

$$\therefore \quad I = \frac{1}{n}\int \frac{1}{y^2 - \left(\frac{1}{2}\right)^2}\cdot dy \qquad \left[\text{By using } \int \frac{1}{x^2 - a^2}\cdot dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{n}\cdot\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{y - 1/2}{y + 1/2}\right| + c$$

$$= \frac{1}{n}\log\left|\frac{z - 1/2 - 1/2}{z - 1/2 + 1/2}\right| + c = \frac{1}{n}\log\left|\frac{z-1}{z}\right| + c \qquad [\because \; y = z - 1/2]$$

$$= \frac{1}{n}\log\left|\frac{x^n + 1 - 1}{x^n + 1}\right| + c \qquad [\because \; z = x^n + 1]$$

$$= \frac{1}{n}\log\left|\frac{x^n}{x^n + 1}\right| + c.$$

(*v*) Let $I = \int \frac{1}{x(x^5 + 1)}\cdot dx$

Put $x^5 + 1 = z \quad \Rightarrow \quad x^5 = z - 1$

$\Rightarrow \quad 5x^4\,dx = dz \quad \Rightarrow \quad 5.\frac{1}{x}x^5\,dx = dz$

$\Rightarrow \quad \frac{1}{x}dx = \frac{1}{5x^5}dz \quad \Rightarrow \quad \frac{1}{x}dx = \frac{1}{5(z-1)}dz$

$\therefore \quad I = \int \frac{1}{z}\cdot\frac{1}{5(z-1)}\cdot dz = \frac{1}{5}\int \frac{1}{z^2 - z}\cdot dz$

$$= \frac{1}{5}\int \frac{1}{\left(z^2 - z + \frac{1}{4}\right) - \left(\frac{1}{4}\right)}\cdot dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \quad \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \frac{1}{5} \int \frac{1}{\left(z - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \cdot dz$$

Put $z - \frac{1}{2} = y \Rightarrow dz = dy$

$$\therefore \quad I = \frac{1}{5} \int \frac{1}{y^2 - \left(\frac{1}{2}\right)^2} \cdot dy \qquad \left[\text{By using} \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c\right]$$

$$= \frac{1}{5} \cdot \frac{1}{2(1/2)} \cdot \log \left| \frac{y - 1/2}{y + 1/2} \right| + c$$

$$= \frac{1}{5} \log \left| \frac{z - 1/2 - 1/2}{z - 1/2 + 1/2} \right| + c = \frac{1}{5} \log \left| \frac{z-1}{z} \right| + c \qquad [\because \; y = z - 1/2]$$

$$= \frac{1}{5} \log \left| \frac{x^5 + 1 - 1}{x^5 + 1} \right| + c \qquad [\because \; z = x^5 + 1]$$

$$= \frac{1}{5} \log \left| \frac{x^5}{x^5 + 1} \right| + c.$$

Example 4. *Evaluate :*

(*i*) $\int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} \cdot dx$ (*ii*) $\int \frac{x^2}{x^6 - a^6} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} \cdot dx$

Put $\sin x = z \Rightarrow \cos x \, dx = dz$

$$\therefore \quad I = \int \frac{1}{z^2 + 4z + 5} \cdot dz$$

$$= \int \frac{1}{(z^2 + 4z + 4) + (5 - 4)} \cdot dz \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.} \\ \because \quad \left(\frac{1}{2} \text{coeff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \frac{1}{(z+2)^2 + (1)^2} \cdot dz$$

Put $z + 2 = y \Rightarrow dz = dy$

$$\therefore \quad I = \int \frac{1}{y^2 + 1^2} \cdot dy \qquad \left[\text{By using} \int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \tan^{-1} \left(\frac{y}{1}\right) + c$$

$$= \tan^{-1} (z + 2) + c \qquad [\because \; y = z + 2]$$

$$= \tan^{-1} (\sin x + 2) + c. \qquad [\because \; z = \sin x]$$

(*ii*) Let $\quad I = \int \frac{x^2}{x^6 - a^6} \cdot dx = \int \frac{x^2}{(x^3)^2 - (a^3)^2} \cdot dx$

Put $\quad x^3 = z$

$\Rightarrow \quad 3x^2\, dx = dz \quad \Rightarrow \quad x^2\, dx = \frac{1}{3}\, dz$

$\therefore \quad I = \int \frac{1}{z^2 - (a^3)^2} \cdot \left(\frac{1}{3} dz\right)$

$= \frac{1}{3} \int \frac{1}{z^2 - (a^3)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + c\right]$

$= \frac{1}{3} \cdot \frac{1}{2(a^3)} \cdot \log \left| \frac{z - a^3}{z + a^3} \right| + c \qquad [\because \ z = x^3]$

$= \frac{1}{6a^3} \log \left| \frac{x^3 - a^3}{x^3 + a^3} \right| + c.$

3.1.6. TYPE II : Integrals of the form : $\int \frac{1}{\sqrt{ax^2 + bx + c}}$. dx.

[**Note.** For Working Rule please refer to Article 3.1.2]

SOME SOLVED EXAMPLES

Example 5. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{\sqrt{x^2 - 2x + 4}} \cdot dx$ (*ii*) $\int \frac{1}{\sqrt{(x-1)(x-2)}} \cdot dx$

(*iii*) $\int \frac{1}{\sqrt{7 - 6x - x^2}} \cdot dx$ (*iv*) $\int \frac{1}{2x^2 + 3x - 2} \cdot dx$

(*v*) $\int \frac{1}{\sqrt{(x-a)(x-b)}} \cdot dx$ (*vi*) $\int \frac{1}{\sqrt{3x^2 + 5x + 7}} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{1}{\sqrt{x^2 - 2x + 4}} \cdot dx$

$= \int \frac{1}{\sqrt{(x^2 - 2x + 1) + (4 - 1)}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract 1 to the denom.}\\ \because \quad \left(\frac{1}{2} \text{ coeff. of } x\right)^2 = 1\end{array}\right]$

$= \int \frac{1}{\sqrt{(x-1)^2 + (\sqrt{3})^2}} \cdot dx$

Put $\quad x - 1 = z \ \Rightarrow \ dx = dz$

$\therefore \quad I = \int \frac{1}{\sqrt{z^2 + (\sqrt{3})^2}} \cdot dz \qquad \left[\text{By using } \int \frac{1}{\sqrt{x^2 + a^2}} \cdot dx = \log \left| x + \sqrt{x^2 + a^2} \right| + c\right]$

$$= \log \left| z + \sqrt{z^2 + (\sqrt{3})^2} \right| + c$$

$$= \log \left| x - 1 + \sqrt{(x-1)^2 + 3} \right| + c = \log \left| x - 1 + \sqrt{x^2 + 1 - 2x + 3} \right| + c \qquad [\because \; z = x - 1]$$

$$= \log \left| (x-1) + \sqrt{x^2 - 2x + 4} \right| + c.$$

(*ii*) Let $\quad I = \int \frac{1}{\sqrt{(x-1)(x-2)}} \cdot dx = \int \frac{1}{\sqrt{x^2 - 3x + 2}} \cdot dx$

$$= \int \frac{1}{\sqrt{\left(x^2 - 3x + \frac{9}{4}\right) + \left(2 - \frac{9}{4}\right)}} \cdot dx \qquad \left[\begin{array}{l} \text{Add and subtract} \left(\frac{9}{4}\right) \text{to the denom.} \\ \because \quad \left(\frac{1}{2}\text{co-eff. of } x\right)^2 = \frac{9}{4} \end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} \cdot dx$$

Put $\left(x - \frac{3}{2}\right) = z \;\Rightarrow\; dx = dz$

$\therefore \quad I = \int \frac{1}{\sqrt{z^2 - \left(\frac{1}{2}\right)^2}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c\right]$

$$= \log \left| z + \sqrt{z^2 - (1/2)^2} \right| + c$$

$$= \log \left| \left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}} \right| + c = \log \left| \left(x - \frac{3}{2}\right) + \sqrt{\left(x^2 - 3x + \frac{9}{4} - \frac{1}{4}\right)} \right| + c$$

$$\left[\because \; z = \left(x - \frac{3}{2}\right)\right]$$

$$= \log \left| \left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2} \right| + c.$$

(*iii*) Let $\quad I = \int \frac{1}{\sqrt{7 - 6x - x^2}} \cdot dx = \int \frac{1}{\sqrt{7 - (x^2 + 6x)}} \cdot dx$

$$= \int \frac{1}{\sqrt{7 + 9 - (x^2 + 6x + 9)}} \cdot dx \qquad \left[\begin{array}{l} \text{Add and subtract 9 to the denom.} \\ \because \quad \left(\frac{1}{2}\text{co-eff. of } x\right)^2 = 9 \end{array}\right]$$

$$= \int \frac{1}{\sqrt{16 - (x+3)^2}} \cdot dx$$

Put $x + 3 = z \Rightarrow dx = dz$

$\therefore \quad I = \int \frac{1}{\sqrt{(4)^2 - z^2}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right) + c\right]$

$= \sin^{-1}\left(\frac{z}{4}\right) + c$

$= \sin^{-1} \frac{(x+3)}{4} + c. \qquad [\because \ z = x + 3]$

(*iv*) Let $I = \int \frac{1}{\sqrt{2x^2 + 3x - 2}} \cdot dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x^2 + \frac{3}{2}x - 1}} \cdot dx$

$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) - \frac{9}{16} - 1}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{9}{16} \text{ to the denom.} \\ \because \quad \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{9}{16}\end{array}\right]$

$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x + \frac{3}{4}\right)^2 - \frac{25}{16}}} \cdot dx$

Put $x + \frac{3}{4} = z \Rightarrow dx = dz$

$\therefore \quad I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{z^2 - \left(\frac{5}{4}\right)^2}} \cdot dz$

$\left[\text{By using} \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log\left| x + \sqrt{x^2 - a^2} \right| + c\right]$

$= \frac{1}{\sqrt{2}} \log\left| z + \sqrt{z^2 - \left(\frac{5}{4}\right)^2} \right| + c$

$= \frac{1}{\sqrt{2}} \log\left| x + \frac{3}{4} + \sqrt{\left(x + \frac{3}{4}\right)^2 - \frac{25}{16}} \right| + c \qquad \left[\because \ z = \left(x + \frac{3}{4}\right)\right]$

$= \frac{1}{\sqrt{2}} \log\left| x + \frac{3}{4} + \sqrt{x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{25}{16}} \right| + c$

$= \frac{1}{\sqrt{2}} \log\left| x + \frac{3}{4} + \sqrt{x^2 + \frac{3}{2}x - 1} \right| + c.$

(*v*) Let $\quad I = \int \frac{1}{\sqrt{(x-a)(x-b)}} \cdot dx = \int \frac{1}{\sqrt{x^2-(a+b)x+ab}} \cdot dx$

$$= \int \frac{1}{\sqrt{x^2-(a+b)x+\frac{(a+b)^2}{4}+ab-\frac{(a+b)^2}{4}}} \cdot dx$$

$$\left[\begin{array}{l}\text{Add and subtract } \left(\frac{a+b}{2}\right)^2 \text{ to the denom.} \\ \because \quad \left(\frac{1}{2}\text{coeff. of } x\right)^2 = \left(\frac{a+b}{2}\right)^2\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}} \cdot dx$$

Put $\left(x-\frac{a+b}{2}\right) = z \quad \Rightarrow \quad dx = dz$

$\therefore \quad I = \int \frac{1}{\sqrt{z^2-\left(\frac{a-b}{2}\right)^2}} \cdot dz$

$$\left[\text{By using } \int \frac{1}{\sqrt{x^2-a^2}} \cdot dx = \log\left|x+\sqrt{x^2-a^2}\right|+c\right]$$

$$= \log\left|z+\sqrt{z^2-\left(\frac{a-b}{2}\right)^2}\right|+c$$

$$= \log\left|\left(x-\frac{a+b}{2}\right)+\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}\right|+c \quad \left[\because\ z=\left(x-\frac{a+b}{2}\right)\right]$$

$$= \log\left|\left(x-\frac{a+b}{2}\right)+\sqrt{x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a-b)^2}{4}}\right|+c$$

$$= \log\left|\left(x-\frac{a+b}{2}\right)+\sqrt{x^2-(a+b)x+ab}\right|+c$$

$$= \log\left|\left(x-\frac{a+b}{2}\right)+\sqrt{(x-a)(x-b)}\right|+c.$$

(*vi*) Let $\quad I = \int \frac{1}{\sqrt{3x^2+5x+7}} \cdot dx = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{x^2+\frac{5}{3}x+\frac{7}{3}}} \cdot dx$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x^2 + \frac{5}{3}x + \frac{25}{36}\right) + \frac{7}{3} - \frac{25}{36}}} \cdot dx$$

$$\left[\begin{array}{l} \text{Add and subtract } \frac{25}{36} \text{ to the denom.} \\ \because \quad \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{25}{36} \end{array}\right]$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x + \frac{5}{6}\right)^2 + \frac{59}{36}}} \cdot dx$$

Put $\quad x + \frac{5}{6} = z \Rightarrow dx = dz$

$\therefore \quad$ $$I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{z^2 + \left(\frac{\sqrt{59}}{6}\right)^2}} \cdot dz$$

$$\left[\text{By using } \int \frac{1}{\sqrt{x^2 + a^2}} \cdot dx = \log\left| x + \sqrt{x^2 + a^2} \right| + c\right]$$

$$= \frac{1}{\sqrt{3}} \log\left| z + \sqrt{z^2 + \left(\frac{\sqrt{59}}{6}\right)^2} \right| + c$$

$$= \frac{1}{\sqrt{3}} \log\left| \left(x + \frac{5}{6}\right) + \sqrt{\left(x + \frac{5}{6}\right)^2 + \frac{59}{36}} \right| + c$$

$$= \frac{1}{\sqrt{3}} \log\left| \left(x + \frac{5}{6}\right) + \sqrt{x^2 + \frac{5}{3}x + \frac{25}{36} + \frac{59}{36}} \right| + c$$

$$= \frac{1}{\sqrt{3}} \log\left| \left(x + \frac{5}{6}\right) + \sqrt{x^2 + \frac{5}{3}x + \frac{7}{3}} \right| + c.$$

Example 6. *Evaluate the following integrals :*

(i) $\int \frac{1}{\sqrt{x(1-2x)}} \cdot dx$ *(ii)* $\int \frac{1}{\sqrt{(8 + 3x - x^2)}} \cdot dx$

(iii) $\int \frac{1}{\sqrt{2 - 4x + x^2}} \cdot dx$ *(iv)* $\int \frac{1}{\sqrt{x^2 + 12x + 11}} \cdot dx.$

Solution. *(i)* Let $I = \int \frac{1}{\sqrt{x(1-2x)}} \cdot dx = \int \frac{1}{\sqrt{x - 2x^2}} \cdot dx$

$$= \int \frac{1}{\sqrt{-2\left(x^2 - \frac{1}{2}x\right)}} \cdot dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{1}{16} - \left(x^2 - \frac{1}{2}x + \frac{1}{16}\right)}} \cdot dx$$

$$\left[\begin{array}{l}\text{Add and subtract } \frac{1}{16} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{co-eff. of } x\right)^2 = \frac{1}{16}\end{array}\right]$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}} \cdot dx$$

Put $\left(x - \frac{1}{4}\right) = z \Rightarrow dx = dz$

$\therefore$ $$I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2 - z^2}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{z}{1/4}\right) + c$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x - 1/4}{1/4}\right) + c \qquad \left[\because z = \left(x - \frac{1}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}(4x - 1) + c.$$

(*ii*) Let $$I = \int \frac{1}{\sqrt{8 + 3x - x^2}} \cdot dx = \int \frac{1}{\sqrt{8 - (x^2 - 3x)}} \cdot dx$$

$$= \int \frac{1}{\sqrt{\left(8 + \frac{9}{4}\right) - \left(x^2 - 3x + \frac{9}{4}\right)}} \cdot dx \left[\begin{array}{l}\text{Add and subtract } \frac{9}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{co-eff. of } x\right)^2 = \frac{9}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\frac{41}{4} - \left(x - \frac{3}{2}\right)^2}} \cdot dx$$

Put $\left(x - \frac{3}{2}\right) = z \Rightarrow dx = dz$

$\therefore$ $$I = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - z^2}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \sin^{-1}\left(\frac{z}{\frac{\sqrt{41}}{2}}\right) + c = \sin^{-1}\left[\frac{x - 3/2}{\sqrt{41}/2}\right] + c \qquad \left[\because\ z = \left(x - \frac{3}{2}\right)\right]$$

$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right) + c.$$

(*iii*) Let $\quad I = \int \frac{1}{\sqrt{2 - 4x + x^2}} \cdot dx = \int \frac{1}{\sqrt{x^2 - 4x + 2}} \cdot dx$

$$= \int \frac{1}{\sqrt{(x^2 - 4x + 4) + 2 - 4}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.}\\ \because \left(\frac{1}{2}\text{co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \frac{1}{\sqrt{(x-2)^2 - 2}} \cdot dx$$

Put $x - 2 = z \Rightarrow dx = dz$

$$\therefore \quad I = \int \frac{1}{\sqrt{z^2 - (\sqrt{2})^2}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log\left| x + \sqrt{x^2 - a^2} \right| + c\right]$$

$$= \log\left| z + \sqrt{z^2 - (\sqrt{2})^2} \right| + c$$

$$= \log\left| z + \sqrt{z^2 - 2} \right| + c = \log\left| (x-2) + \sqrt{(x-2)^2 - 2} \right| + c \qquad [\because\ z = x - 2]$$

$$= \log\left| (x-2) + \sqrt{x^2 - 4x + 4 - 2} \right| + c$$

$$= \log\left| (x-2) + \sqrt{x^2 - 4x + 2} \right| + c.$$

(*iv*) Let $I = \int \frac{1}{\sqrt{x^2 + 12x + 11}} \cdot dx$

$$= \int \frac{1}{\sqrt{(x^2 + 12x + 36) + 11 - 36}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract 36 to the denom.}\\ \because \left(\frac{1}{2}\text{co-eff. of } x\right)^2 = 36\end{array}\right]$$

$$= \int \frac{1}{\sqrt{(x+6)^2 - 25}} \cdot dx$$

Put $(x + 6) = z \Rightarrow dx = dz$

$$\therefore \quad I = \int \frac{1}{\sqrt{z^2 - (5)^2}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log\left| x + \sqrt{x^2 - a^2} \right| + c\right]$$

$$= \log\left| z + \sqrt{z^2 - (5)^2} \right| + c$$

$$= \log\left| (x+6) + \sqrt{(x+6)^2 - 25} \right| + c \qquad [\because\ z = x + 6]$$

$$= \log\left| (x+6) + \sqrt{x^2 + 12x + 11} \right| + c.$$

Example 7. *Evaluate the following integrals :*

(*i*) $\int \frac{2x}{\sqrt{1-x^2-x^4}} \cdot dx$ (*ii*) $\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}} \cdot dx$

(*iii*) $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}} \cdot dx$ (*iv*) $\int \frac{x^2}{\sqrt{1-x^6}} \cdot dx$

(*v*) $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} \cdot dx$ (*vi*) $\int \sqrt{\frac{x}{a^3-x^3}} \cdot dx$

(*vii*) $\int \frac{\sin 2x}{\sqrt{\sin^4 x + 4\sin^2 x - 2}} \cdot dx$ (*viii*) $\int \frac{1}{\sqrt{(1-x^2)[9+(\sin^{-1} x)^2]}} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{2x}{\sqrt{1-x^2-x^4}} \cdot dx$

Put $x^2 = z \Rightarrow 2x\,dx = dz$

$\therefore \quad I = \int \frac{1}{\sqrt{1-z-z^2}} \cdot dz = \int \frac{1}{\sqrt{1-(z^2+z)}} \cdot dz$

$$= \int \frac{1}{\sqrt{\left(1+\frac{1}{4}\right)-\left(z^2+z+\frac{1}{4}\right)}} \cdot dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ coeff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\frac{5}{4}-\left(z+\frac{1}{2}\right)^2}} \cdot dz$$

Put $z + \frac{1}{2} = y \Rightarrow dz = dy$

$$\therefore \quad I = \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - y^2}} \cdot dy \qquad \left[\text{By using } \int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \sin^{-1}\left(\frac{y}{\sqrt{5}/2}\right) + c$$

$$= \sin^{-1}\left(\frac{z+1/2}{\sqrt{5}/2}\right) + c \qquad \left[\because \left(z+\frac{1}{2}\right) = y\right]$$

$$= \sin^{-1}\left(\frac{2z+1}{\sqrt{5}}\right) + c$$

$$= \sin^{-1}\left(\frac{2x^2+1}{\sqrt{5}}\right) + c. \qquad [\because \; z = x^2]$$

(*ii*) Let $\quad I = \int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}} \cdot dx$

Put $\quad \sin x = z \Rightarrow \cos x\, dx = dz$

$\therefore \quad I = \int \frac{1}{\sqrt{z^2 - 2z - 3}} \cdot dz$

$= \int \frac{1}{\sqrt{(z^2 - 2z + 1) - 3 - 1}} \cdot dz$ $\quad \left[\begin{array}{l}\text{Add and subtract 1 to the denom.} \\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 1\end{array}\right]$

$= \int \frac{1}{\sqrt{(z-1)^2 - 4}} \cdot dz$

Put $\quad z - 1 = y \Rightarrow dz = dy$

$\therefore \quad I = \int \frac{1}{\sqrt{y^2 - (2)^2}} \cdot dy$ $\quad \left[\text{By using } \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c\right]$

$= \log\left|y + \sqrt{y^2 - 2^2}\right| + c = \log\left|y + \sqrt{y^2 - 4}\right| + c$

$= \log\left|(z-1) + \sqrt{(z-1)^2 - 4}\right| + c \qquad [\because \; y = z - 1]$

$= \log\left|(z-1) + \sqrt{z^2 - 2z + 1 - 4}\right| + c = \log\left|(z-1) + \sqrt{z^2 - 2z - 3}\right| + c$

$= \log\left|(\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3}\right| + c. \qquad [\because \; z = \sin x]$

(*iii*) Let $\quad I = \int \frac{\sec^2 x}{\sqrt{16 + \tan^2 x}} \cdot dx = \int \frac{\sec^2 x}{\sqrt{(4)^2 + \tan^2 x}} \cdot dx$

Put $\quad \tan x = z \Rightarrow \sec^2 x\, dx = dz$

$\therefore \quad I = \int \frac{1}{\sqrt{(4)^2 + z^2}} \cdot dz$ $\quad \left[\text{By using } \int \frac{1}{\sqrt{a^2 + x^2}} \cdot dx = \log\left|x + \sqrt{x^2 + a^2}\right| + c\right]$

$= \log\left|z + \sqrt{z^2 + (4)^2}\right| + c = \log\left|z + \sqrt{z^2 + 16}\right| + c$

$= \log\left|\tan x + \sqrt{\tan^2 x + 16}\right| + c. \qquad [\because \; z = \tan x]$

(*iv*) Let $I = \int \frac{x^2}{\sqrt{1 - x^6}} \cdot dx = \int \frac{x^2}{\sqrt{1 - (x^3)^2}} \cdot dx$

Put $\quad x^3 = z$

$\Rightarrow \quad 3x^2\, dx = dz \Rightarrow x^2 dx = \frac{1}{3}\, dz$

$\therefore \quad I = \int \frac{1}{\sqrt{1 - z^2}} \cdot \left(\frac{1}{3} dz\right)$

$$= \frac{1}{3}\int \frac{1}{\sqrt{(1)^2 - z^2}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{z}{1}\right) + c$$

$$= \frac{1}{3}\sin^{-1}(x^3) + c. \qquad [\because \; z = x^3]$$

(*v*) Let $\quad I = \int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} \cdot dx$

Put $\quad e^x = z \;\Rightarrow\; e^x\,dx = dz$

$\therefore \quad I = \int \frac{1}{\sqrt{5 - 4z - z^2}} \cdot dz = \int \frac{1}{\sqrt{5 - (z^2 + 4z)}} \cdot dz$

$$= \int \frac{1}{\sqrt{5 + 4 - (z^2 + 4z + 4)}} \cdot dz \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.} \\ \because \left(\frac{1}{2}\text{co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \frac{1}{\sqrt{9 - (z+2)^2}} \cdot dz$$

Put $\quad z + 2 = y \;\Rightarrow\; dz = dy$

$\therefore \quad I = \int \frac{1}{\sqrt{(3)^2 - y^2}} \cdot dy \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c\right]$

$$= \sin^{-1}\left(\frac{y}{3}\right) + c$$

$$= \sin^{-1}\left(\frac{z+2}{3}\right) + c \qquad [\because \; y = z + 2]$$

$$= \sin^{-1}\left(\frac{e^x+2}{3}\right) + c. \qquad [\because \; z = e^x]$$

(*vi*) Let $\quad I = \int \sqrt{\frac{x}{a^3 - x^3}} \cdot dx = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} \cdot dx$

$$= \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}} \cdot dx$$

Put $\quad x^{3/2} = z$

$\Rightarrow \quad \frac{3}{2}x^{1/2}\,dx = dz \;\Rightarrow\; \sqrt{x}\,dx = \frac{2}{3}\,dz$

$\therefore \quad I = \int \frac{1}{\sqrt{(a^{3/2})^2 - z^2}} \cdot \left(\frac{2}{3}dz\right)$

$$= \frac{2}{3}\int \frac{1}{\sqrt{(a^{3/2})^2 - z^2}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{z}{a^{3/2}}\right) + c$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + c \qquad [\because \; z = x^{3/2}]$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c.$$

(*vii*) Let $\quad I = \int \frac{\sin 2x}{\sqrt{\sin^4 x + 4\sin^2 x - 2}} \, . \, dx$

Put $\sin^2 x = z$

$\Rightarrow \; 2 \sin x \cos x \, dx = dz \;\Rightarrow\; \sin 2x \, dx = dz \qquad [\because \; 2 \sin A \cos A = \sin 2A]$

$$\therefore \qquad I = \int \frac{1}{\sqrt{z^2 + 4z - 2}} \, . \, dz$$

$$= \int \frac{1}{\sqrt{(z^2 + 4z + 4) - 2 - 4}} \, . \, dz \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.} \\ \because \; \left(\frac{1}{2} \text{co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \frac{1}{\sqrt{(z+2)^2 - 6}} \, . \, dz$$

Put $\quad z + 2 = y \;\Rightarrow\; dz = dy$

$$\therefore \qquad I = \int \frac{1}{\sqrt{y^2 - (\sqrt{6})^2}} \, . \, dy \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2 - a^2}} \, . \, dx = \log\left| x + \sqrt{x^2 - a^2} \right| + c\right]$$

$$= \log\left| y + \sqrt{y^2 - (\sqrt{6})^2} \right| + c$$

$$= \log\left| (z+2) + \sqrt{(z+2)^2 - 6} \right| + c \qquad [\because \; y = z + 2]$$

$$= \log\left| (z+2) + \sqrt{z^2 + 4z + 4 - 6} \right| + c = \log\left| (z+2) + \sqrt{z^2 + 4z - 2} \right| + c$$

$$= \log\left| (\sin^2 x + 2) + \sqrt{\sin^4 x + 4\sin^2 x - 2} \right| + c.$$

(*viii*) Let $\; I = \int \frac{1}{\sqrt{(1 - x^2)[9 + (\sin^{-1} x)^2]}} \, . \, dx$

$$= \int \frac{1}{\sqrt{[9 + (\sin^{-1} x)^2]} \; \sqrt{1 - x^2}} \, . \, dx$$

Put $\sin^{-1} x = z \;\Rightarrow\; \frac{1}{\sqrt{1 - x^2}} \, . \, dx = dz$

$$\therefore \qquad I = \int \frac{1}{\sqrt{9 + z^2}} \, . \, dz$$

$$= \int \frac{1}{\sqrt{(3)^2 + z^2}} \cdot dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2 + a^2}} \cdot dx = \log \left| x + \sqrt{x^2 + a^2} \right| + c\right]$$

$$= \log \left| z + \sqrt{z^2 + 9} \right| + c$$

$$= \log \left| \sin^{-1} x + \sqrt{(\sin^{-1} x)^2 + 9} \right| + c. \qquad [\because z = \sin^{-1} x]$$

Example 8. *Evaluate the following integrals :*

(i) $\int \sqrt{1 + \operatorname{cosec} x} \cdot dx$ *(ii)* $\int \sqrt{\sec x - 1} \cdot dx.$

Solution. *(i)* Let $I = \int \sqrt{1 + \operatorname{cosec} x} \cdot dx$

$$= \int \sqrt{1 + \frac{1}{\sin x}} \cdot dx = \int \sqrt{\frac{\sin x + 1}{\sin x}} \cdot dx$$

$$= \int \sqrt{\frac{(1 + \sin x)(1 - \sin x)}{\sin x (1 - \sin x)}} \cdot dx \qquad \text{[On rationalization]}$$

$$= \int \sqrt{\frac{1 - \sin^2 x}{\sin x - \sin^2 x}} \cdot dx \qquad [\because (a + b)(a - b) = a^2 - b^2]$$

$$= \int \frac{\cos x}{\sqrt{\sin x - \sin^2 x}} \cdot dx \qquad [\because \sin^2 A + \cos^2 A = 1]$$

Put $\sin x = z \Rightarrow \cos x \, dx = dz$

$$\therefore \quad I = \int \frac{1}{\sqrt{z - z^2}} \cdot dz = \int \frac{1}{\sqrt{-(z^2 - z)}} \cdot dz$$

$$= \int \frac{1}{\sqrt{\frac{1}{4} - \left(z^2 - z + \frac{1}{4}\right)}} \cdot dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(z - \frac{1}{2}\right)^2}} \cdot dz$$

Put $\left(z - \frac{1}{2}\right) = y \Rightarrow dz = dy$

$$\therefore \quad I = \int \frac{1}{\sqrt{(1/2)^2 - y^2}} \cdot dy \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \sin^{-1}\left(\frac{y}{1/2}\right) + c$$

$$= \sin^{-1}\left(\frac{z-1/2}{1/2}\right) + c \qquad [\because \ y = z - 1/2]$$

$$= \sin^{-1}(2z-1) + c$$

$$= \sin^{-1}(2\sin x - 1) + c. \qquad [\because \ z = \sin x]$$

(*ii*) Let $\quad I = \int \sqrt{\sec x - 1}\, .\, dx = \int \sqrt{\frac{1}{\cos x} - 1}\, .\, dx = \int \sqrt{\frac{1-\cos x}{\cos x}}\, .\, dx$

$$= \int \sqrt{\frac{(1-\cos x)(1+\cos x)}{\cos x\,(1+\cos x)}}\, .\, dx \qquad \text{[On rationalization]}$$

$$= \int \sqrt{\frac{1-\cos^2 x}{\cos x + \cos^2 x}}\, .\, dx \qquad [\because \ (a-b)(a+b) = a^2 - b^2]$$

$$= \int \frac{\sin x}{\sqrt{\cos^2 x + \cos x}}\, .\, dx \qquad [\because \ \sin^2 A + \cos^2 A = 1]$$

Put $\quad \cos x = z$

$\Rightarrow \quad -\sin x\, dx = dz \ \Rightarrow \ \sin x\, dx = -dz$

$\therefore \quad I = \int \frac{1}{\sqrt{z^2+z}}\,(-dz)$

$$= -\int \frac{1}{\sqrt{\left(z^2+z+\frac{1}{4}\right)-\frac{1}{4}}}\, .\, dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \ \left(\frac{1}{2}\text{ coeff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= -\int \frac{1}{\sqrt{\left(z+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, .\, dz$$

Put $\left(z+\frac{1}{2}\right) = y \ \Rightarrow \ dz = dy$

$\therefore \quad I = -\int \frac{1}{\sqrt{y^2 - \left(\frac{1}{2}\right)^2}}\, .\, dy \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2-a^2}}\, .\, dx = \log\left|x + \sqrt{x^2-a^2}\right| + c\right]$

$$= -\log\left|y + \sqrt{y^2 - \left(\frac{1}{2}\right)^2}\right| + c$$

$$= -\log\left|\left(z+\frac{1}{2}\right) + \sqrt{\left(z+\frac{1}{2}\right)^2 - \frac{1}{4}}\right| + c \qquad \left[\because \ y = \left(z+\frac{1}{2}\right)\right]$$

$$= -\log\left|\left(z+\frac{1}{2}\right) + \sqrt{z^2+z}\right| + c$$

$$= -\log\left|\left(\cos x+\frac{1}{2}\right) + \sqrt{\cos^2 x + \cos x}\right| + c. \qquad [\because \ z = \cos x]$$

3.1.7. Integrals of the Form $\int \frac{px+q}{ax^2+bx+c} \cdot dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}} \cdot dx$

[**Note.** For Working Rule Please Refer to Article 3.1.3 and 3.1.4]

SOME SOLVED EXAMPLES

Example 9. *Evaluate the following integrals :*

(*i*) $\int \frac{2x-3}{3x^2+4x+5} \cdot dx$ (*ii*) $\int \frac{x-1}{3x^2-4x+3} \cdot dx$

(*iii*) $\int \frac{2x-3}{x^2+3x-18} \cdot dx$ (*iv*) $\int \frac{4x+1}{x^2+3x+2} \cdot dx$

(*v*) $\int \frac{2x}{2+x-x^2} \cdot dx$ (*vi*) $\int \frac{x}{x^2+x+1} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{2x-3}{3x^2+4x+5} \cdot dx$

Let us write

$$(2x-3) = \lambda\left[\cdot\frac{d}{dx}(3x^2+4x+5)\right] + \mu$$

$$\Rightarrow \quad 2x-3 = \lambda(6x+4) + \mu \qquad ...(1)$$

$$\Rightarrow \quad 2x-3 = 6\lambda x + (4\lambda+\mu)$$

Comparing the co-efficients of x and the constant terms, we have

$$2 = 6\lambda \Rightarrow \lambda = \frac{1}{3}$$

and $$-3 = 4\lambda + \mu$$

$$\Rightarrow \quad \mu = -3 - 4\lambda = -3 - 4\left(\frac{1}{3}\right)$$

$$\therefore \quad \mu = -3 - \frac{4}{3} = -\frac{13}{3}$$

Putting the values of λ and μ in (1) :

$$\Rightarrow \quad (2x-3) = \frac{1}{3}(6x+4) - \frac{13}{3}.$$

$$\therefore \quad I = \int \frac{\frac{1}{3}(6x+4) - \frac{13}{3}}{3x^2+4x+5} \cdot dx$$

$$= \frac{1}{3}\int\left(\frac{6x+4}{3x^2+4x+5} - \frac{13}{3x^2+4x+5}\right) \cdot dx$$

$$= \frac{1}{3}\int \frac{6x+4}{3x^2+4x+5} \cdot dx - \frac{13}{3}\int \frac{1}{3x^2+4x+5} \cdot dx$$

$$\Rightarrow \qquad I = \frac{1}{3} I_1 - \frac{13}{3} I_2 \qquad ...(2)$$

Now $\quad I_1 = \int \frac{6x+4}{3x^2+4x+5} \cdot dx \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right]$

$$= \log|3x^2+4x+5| + c_1 \qquad ...(3)$$

and $\quad I_2 = \int \frac{1}{3x^2+4x+5} \cdot dx$

$$= \frac{1}{3}\int \frac{1}{\left(x^2 + \frac{4}{3}x + \frac{5}{3}\right)} \cdot dx$$

$$= \frac{1}{3}\int \frac{1}{\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + \left(\frac{5}{3} - \frac{4}{9}\right)} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{4}{9} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{coeff. of } x\right)^2 = \frac{4}{9}\end{array}\right]$$

$$= \frac{1}{3}\int \frac{1}{\left(x + \frac{2}{3}\right)^2 + \frac{11}{9}} \cdot dx$$

Put $\left(x + \frac{2}{3}\right) = z \;\Rightarrow\; dx = dz$

$$\therefore \qquad I_2 = \frac{1}{3}\int \frac{1}{z^2 + \left(\frac{\sqrt{11}}{3}\right)^2} \cdot dz \qquad \left[\text{By using} \int \frac{1}{x^2+a^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt{11/3}} \cdot \tan^{-1}\left(\frac{z}{\sqrt{11/3}}\right) + c_2 = \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{x + 2/3}{\sqrt{11/3}}\right) + c_2$$

$$= \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{3x+2}{\sqrt{11}}\right) + c_2 \qquad ...(4)$$

$\therefore$ From equation (2),

$$I = \frac{1}{3} I_1 - \frac{13}{3} I_2$$

$$= \frac{1}{3}[\log|3x^2+4x+5| + c_1] - \frac{13}{3}\left[\frac{1}{\sqrt{11}}\tan^{-1}\frac{3x+2}{\sqrt{11}} + c_2\right]$$

[Using (3) and (4)]

$$= \frac{1}{3}\log|3x^2+4x+5| + \frac{1}{3}c_1 - \frac{13}{3\sqrt{11}}\tan^{-1}\frac{3x+2}{\sqrt{11}} - \frac{13}{3}c_2$$

$$= \frac{1}{3}\log|3x^2+4x+5| - \frac{13}{3\sqrt{11}}\tan^{-1}\frac{3x+2}{\sqrt{11}} + c$$

where : $c = \left(\frac{1}{3}c_1 - \frac{13}{3}c_2\right)$

(ii) Let $\quad I = \int \frac{x-1}{3x^2-4x+3} \cdot dx$

Let us write :

$$x-1=\lambda\left[\frac{d}{dx}(3x^2-4x+3)\right]+\mu$$

$$\Rightarrow \quad x-1=\lambda(6x-4)+\mu=6\lambda x-4\lambda+\mu \qquad \text{...(1)}$$

Comparing the co-efficients of x and the constant terms, we have

$$1=6\lambda \Rightarrow \lambda=\frac{1}{6}$$

and
$$-1=-4\lambda+\mu \Rightarrow \mu=-1+4\lambda=-1+\frac{4}{6}=-\frac{1}{3}$$

Putting the values of λ and μ in (1),

$$(x-1)=\frac{1}{6}(6x-4)-\frac{1}{3}$$

$$\therefore \quad I=\int\frac{\frac{1}{6}(6x-4)-\frac{1}{3}}{(3x^2-4x+3)}\cdot dx$$

$$=\frac{1}{6}\int\frac{6x-4}{3x^2-4x+3}\cdot dx-\frac{1}{3}\int\frac{1}{3x^2-4x+3}\cdot dx$$

$$\Rightarrow \quad I=\frac{1}{6}I_1-\frac{1}{3}I_2 \qquad \text{...(2)}$$

Now
$$I_1=\int\frac{6x-4}{3x^2-4x+3}\cdot dx \qquad \left[\because \int\frac{f'(x)}{f(x)}\cdot dx=\log|f(x)|+c\right]$$

$$\Rightarrow \quad I_1=\log|3x^2-4x+3|+c_1 \qquad \text{...(3)}$$

and
$$I_2=\int\frac{1}{(3x^2-4x+3)}\cdot dx=\frac{1}{3}\int\frac{1}{\left(x^2-\frac{4}{3}x+1\right)}\cdot dx$$

$$=\frac{1}{3}\int\frac{1}{\left(x^2-\frac{4}{3}x+\frac{4}{9}\right)+\left(1-\frac{4}{9}\right)}\cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{4}{9} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{coeff. of } x\right)^2=\frac{4}{9}\end{array}\right]$$

$$=\frac{1}{3}\int\frac{1}{\left(x-\frac{2}{3}\right)^2+\frac{5}{9}}\cdot dx$$

Put $\left(x-\frac{2}{3}\right)=z \Rightarrow dx=dz$

$$\therefore \quad I_2=\frac{1}{3}\int\frac{1}{z^2+\left(\frac{\sqrt{5}}{3}\right)^2}\cdot dz \qquad \left[\text{By using} \int\frac{1}{x^2+a^2}\cdot dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$=\frac{1}{3}\cdot\frac{1}{(\sqrt{5}/3)}\cdot\tan^{-1}\left(\frac{z}{\sqrt{5}/3}\right)+c_2 = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x-2/3}{\sqrt{5}/3}\right)+c_2$$

$$= \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + c_2 \qquad ...(4)$$

$\therefore$ From equation (2),

$$I = \frac{1}{6} I_1 - \frac{1}{3} I_2$$

i.e.,
$$I = \frac{1}{6}\left[\log|3x^2 - 4x + 3| + c_1\right] - \frac{1}{3}\left[\frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + c_2\right] \quad \text{[Using (3) and (4)]}$$

$$\Rightarrow \quad I = \frac{1}{6}\log|3x^2 - 4x + 3| + \frac{1}{6}c_1 - \frac{1}{3\sqrt{5}} \tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) - \frac{1}{3}c_2$$

$$\Rightarrow \quad I = \frac{1}{6}\log|3x^2 - 4x + 3| - \frac{1}{3\sqrt{5}} \tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + c.$$

where, $c = \left(\frac{1}{6}c_1 - \frac{1}{3}c_2\right)$.

(*iii*) Let $\quad I = \int \frac{2x-3}{(x^2+3x-18)} \cdot dx$

Let us write :

$$(2x-3) = \lambda\left[\frac{d}{dx}(x^2+3x-18)\right] + \mu$$

$$\Rightarrow \quad (2x-3) = \lambda(2x+3) + \mu \qquad ...(1)$$

$$\Rightarrow \quad (2x-3) = 2\lambda x + (3\lambda + \mu)$$

Comparing the co-efficients of x and the constant terms, we have

$$2 = 2\lambda \Rightarrow \lambda = 1$$

and $\quad -3 = 3\lambda + \mu \Rightarrow \mu = -3 - 3\lambda = -3 - 3(1)$

$\Rightarrow \quad \mu = -6.$

Putting the values of λ and μ in (1),

$$(2x-3) = (2x+3) - 6$$

$$\therefore \quad I = \int \frac{(2x+3)-6}{(x^2+3x-18)} \cdot dx$$

$$= \int \frac{2x+3}{x^2+3x-18} \cdot dx - 6\int \frac{1}{x^2+3x-18} \cdot dx$$

$$\Rightarrow \quad I = I_1 - 6I_2 \qquad ...(2)$$

Now $\quad I_1 = \int \frac{2x+3}{x^2+3x-18} \cdot dx$

$$I_1 = \log|x^2+3x-18| + c_1 \qquad ...(3) \quad \left[\because \int \frac{f'(x)}{f(x)} \cdot dx = \log|f(x)| + c\right]$$

and $\quad I_2 = \int \frac{1}{x^2+3x-18} \cdot dx$

$$= \int \frac{1}{\left(x^2+3x+\frac{9}{4}\right)-18-\frac{9}{4}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{9}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2}\text{ coeff. of } x\right)^2 = \frac{9}{4}\end{array}\right]$$

$$= \int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \frac{81}{4}} \cdot dx$$

Put $\left(x+\frac{3}{2}\right) = z \Rightarrow dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{z^2 - \left(\frac{9}{2}\right)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2-a^2} \cdot dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{2 \cdot \left(\frac{9}{2}\right)} \cdot \log\left|\frac{z-9/2}{z+9/2}\right| + c_2$$

$$= \frac{1}{9}\log\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right| + c_2 \qquad \left[\because z = x + \frac{3}{2}\right]$$

$$= \frac{1}{9}\log\left|\frac{x-3}{x+6}\right| + c_2 \qquad \text{...(4)}$$

$\therefore$ From equation (2),

$$I = I_1 - 6I_2$$

$$= [\log|x^2+3x-18| + c_1] - 6\left[\frac{1}{9}\log\left|\frac{x-3}{x+6}\right| + c_2\right] \qquad \text{[Using (3) and (4)]}$$

$$= \log|x^2+3x-18| + c_1 - \frac{2}{3}\log\left|\frac{x-3}{x+6}\right| - 6c_2$$

$$= \log|x^2+3x-18| - \frac{2}{3}\log\left|\frac{x-3}{x+6}\right| + c \qquad \text{where : } c = (c_1 - 6c_2)$$

(*iv*) Let $I = \int \frac{4x+1}{x^2+3x+2} \cdot dx$

Let us write :

$$4x+1 = \lambda\left[\frac{d}{dx}(x^2+3x+2)\right] + \mu$$

$$\Rightarrow \quad 4x+1 = \lambda(2x+3) + \mu \qquad \text{...(1)}$$

$$\Rightarrow \quad 4x+1 = 2\lambda x + (3\lambda + \mu)$$

Comparing coefficients of x and the constant terms, we have

$$4 = 2\lambda \Rightarrow \lambda = 2$$

and $\quad 1 = 3\lambda + \mu \Rightarrow \mu = 1 - 3\lambda = 1 - 3(2) = -5.$

Putting the values of λ and μ in (1),

$$4x + 1 = 2(2x + 3) - 5$$

$$\therefore \quad I = \int \frac{2(2x+3)-5}{x^2+3x+2} \cdot dx$$

$$= 2\int \frac{(2x+3)}{x^2+3x+2} \cdot dx - 5\int \frac{1}{x^2+3x+2} \cdot dx$$

$$\Rightarrow \quad I = 2I_1 - 5I_2 \qquad \ldots(2)$$

Now $\quad I_1 = \int \frac{2x+3}{x^2+3x+2} \cdot dx$

$$\Rightarrow \quad I_1 = \log|x^2+3x+2| + c_1 \qquad \ldots(3) \qquad \left[\because \int \frac{f'(x)}{f(x)} \cdot dx = \log|f(x)| + c\right]$$

and $\quad I_2 = \int \frac{1}{x^2+3x+2} \cdot dx$

$$= \int \frac{1}{\left(x^2+3x+\frac{9}{4}\right)+\left(2-\frac{9}{4}\right)} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{9}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{9}{4}\end{array}\right]$$

$$= \int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \frac{1}{4}} \cdot dx$$

Put $\quad x + \frac{3}{2} = z \;\Rightarrow\; dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{z^2 - \left(\frac{1}{2}\right)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2-a^2} \cdot dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{2\left(\frac{1}{2}\right)} \cdot \log\left|\frac{z-1/2}{z+1/2}\right| + c_2$$

$$= \log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + c_2 \qquad \left[\because \; z = x + \frac{3}{2}\right]$$

$$= \log\left|\frac{x+1}{x+2}\right| + c_2 \qquad \ldots(4)$$

$\therefore$ From equation (2),

$$I = 2I_1 - 5I_2$$

$$= 2\,[\log|x^2+3x+2| + c_1] - 5\left[\log\left|\frac{x+1}{x+2}\right| + c_2\right] \qquad \text{[Using (3) and (4)]}$$

$$= 2\log|x^2+3x+2| + 2c_1 - 5\log\left|\frac{x+1}{x+2}\right| - 5c_2$$

$\therefore \quad I = 2\log|x^2+3x+2| - 5\log\left|\frac{x+1}{x+2}\right| + c \qquad$ where : $c = (2c_1 - 5c_2)$.

(*v*) Let $\quad I = \int \frac{2x}{2+x-x^2}\,.\,dx$

Let us write : $2x = \lambda\left[\frac{d}{dx}(2+x-x^2)\right] + \mu$

$$2x = \lambda(1-2x) + \mu = -2\lambda x + \lambda + \mu \qquad \text{...(1)}$$

Comparing the coefficients of x and the constant terms, we have

$$2 = -2\lambda \;\Rightarrow\; \lambda = -1$$

and $\quad 0 = \lambda + \mu \;\Rightarrow\; \mu = -\lambda \;\Rightarrow\; \mu = -(-1) = 1$

Putting the values of λ and μ in (1),

$$2x = -(1-2x) + 1$$

$\therefore \quad I = \int \frac{-(1-2x)+1}{2+x-x^2}\,.\,dx$

$\Rightarrow \quad I = \int \frac{-(1-2x)}{2+x-x^2}\,.\,dx + \int \frac{1}{2+x-x^2}\,.\,dx$

$\Rightarrow \quad I = -I_1 + I_2 \qquad$ ...(2)

Now $\quad I_1 = \int \frac{1-2x}{2+x-x^2}\,.\,dx$

$\Rightarrow \quad I_1 = \log|2+x-x^2| + c_1 \qquad$ ...(3) $\left[\because \int \frac{f'(x)}{f(x)}\,.\,dx = \log|f(x)| + c\right]$

and $\quad I_2 = \int \frac{1}{2+x-x^2}\,.\,dx = \int \frac{1}{2-(x^2-x)}\,.\,dx$

$$= \int \frac{1}{\left(2+\frac{1}{4}\right)-\left(x^2-x+\frac{1}{4}\right)}\,.\,dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ coeff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\frac{9}{4}-\left(x-\frac{1}{2}\right)^2}\,.\,dx$$

Put $\left(x-\frac{1}{2}\right) = z \;\Rightarrow\; dx = dz$

$\therefore \quad I_2 = \int \frac{1}{\left(\frac{3}{2}\right)^2 - z^2} \cdot dz \qquad \left[\text{By using} \int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c\right]$

$$= \frac{1}{2 \cdot \left(\frac{3}{2}\right)} \log \left| \frac{\frac{3}{2} + z}{\frac{3}{2} - z} \right| + c_2 = \frac{1}{3} \log \left| \frac{\frac{3}{2} + x - \frac{1}{2}}{\frac{3}{2} - x + \frac{1}{2}} \right| + c_2 \qquad \left[\because\ z = x - \frac{1}{2}\right]$$

$\Rightarrow \quad I_2 = \frac{1}{3} \log \left| \frac{x+1}{2-x} \right| + c_2 \qquad \ldots(4)$

$\therefore$ From equation (2),

$$I = -I_1 + I_2$$

i.e., $\quad I = -[\log |2 + x - x^2| + c_1] + \left[\frac{1}{3} \log \left| \frac{x+1}{2-x} \right| + c_2\right] \qquad$ [Using (3) and (4)]

$$= -\log |2 + x - x^2| - c_1 + \frac{1}{3} \log \left| \frac{x+1}{2-x} \right| + c_2$$

$$= -\log |2 + x - x^2| + \frac{1}{3} \log \left| \frac{x+1}{2-x} \right| + c. \qquad \text{where : } c = (c_2 - c_1)$$

(*vi*) Let $\quad I = \int \frac{x}{x^2 + x + 1} \cdot dx$

Let us write :

$$x = \lambda \left[\frac{d}{dx}(x^2 + x + 1)\right] + \mu$$

$\Rightarrow \quad x = \lambda (2x + 1) + \mu \qquad \ldots(1)$

$\Rightarrow \quad x = 2\lambda x + (\lambda + \mu)$

Comparing the co-efficients of x and the constant terms, we have

$$1 = 2\lambda \ \Rightarrow\ \lambda = \frac{1}{2}$$

and $\quad 0 = \lambda + \mu \ \Rightarrow\ \mu = -\lambda \ \Rightarrow\ \mu = -\frac{1}{2}$

Putting the values of λ and μ in (1),

$$x = \frac{1}{2}(2x + 1) - \frac{1}{2}$$

$\therefore \quad I = \int \frac{\frac{1}{2}(2x+1) - \frac{1}{2}}{(x^2 + x + 1)} \cdot dx$

$$= \frac{1}{2} \int \frac{2x + 1}{x^2 + x + 1} \cdot dx - \frac{1}{2} \int \frac{1}{x^2 + x + 1} \cdot dx$$

$\Rightarrow \quad I = \frac{1}{2} I_1 - \frac{1}{2} I_2 \qquad \ldots(2)$

Now $\quad I_1 = \int \frac{2x+1}{x^2+x+1} \cdot dx$

$= \log |x^2+x+1| + c_1 \qquad \text{...(3)} \qquad \left[\because \int \frac{f'(x)}{f(x)} \cdot dx = \log |f(x)| + c\right]$

and $\quad I_2 = \int \frac{1}{x^2+x+1} \cdot dx$

$$= \int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\left(1-\frac{1}{4}\right)} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \cdot dx$$

Put $\quad x + \frac{1}{2} = z \Rightarrow dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{z^2+\left(\frac{\sqrt{3}}{2}\right)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2+a^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{\sqrt{3}/2} \cdot \tan^{-1} \frac{z}{\sqrt{3}/2} + c_2$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{x+\frac{1}{2}}{\sqrt{3}/2} + c_2 \qquad \left[\because z = \left(x+\frac{1}{2}\right)\right]$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c_2 \qquad \text{...(4)}$$

$\therefore$ From equation (2),

$$I = \frac{1}{2} I_1 - \frac{1}{2} I_2$$

$$= \frac{1}{2} [\log |x^2+x+1| + c_1] - \frac{1}{2}\left[\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c_2\right]$$

[Using (3) and (4)]

$$= \frac{1}{2} \log |x^2+x+1| + \frac{1}{2} c_1 - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{1}{2} c_2$$

$$= \frac{1}{2} \log |x^2+x+1| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c.$$

where : $c = \left(\frac{1}{2} c_1 - \frac{1}{2} c_2\right)$.

Example 10. *Evaluate the following integrals :*

(*i*) $\int \frac{2x-1}{2x^2+2x+1} \cdot dx$ (*ii*) $\int \frac{5x-2}{3x^2+2x+1} \cdot dx$

(*iii*) $\int \frac{2x+5}{x^2-x-2} \cdot dx$ (*iv*) $\int \frac{x^2+x+1}{x^2-x} \cdot dx$

(*v*) $\int \frac{x^2+5x+3}{x^2+3x+2} \cdot dx$ (*vi*) $\int \frac{x^2+x+1}{x^2-x+1} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{2x-1}{2x^2+2x+1} \cdot dx$

Let us write :

$$2x-1 = \lambda\left[\frac{d}{dx}(2x^2+2x+1)\right] + \mu$$

$$\Rightarrow \quad 2x-1 = \lambda(4x+2)+\mu \quad \text{...(1)}$$

$$\Rightarrow \quad 2x-1 = 4\lambda x + (2\lambda+\mu)$$

Comparing the co-efficients of x and the constant terms, we have

$$2 = 4\lambda \;\Rightarrow\; \lambda = \frac{1}{2}$$

and

$$-1 = 2\lambda+\mu \;\Rightarrow\; \mu = -1-2\lambda = -1-2\left(\frac{1}{2}\right) = -2.$$

Putting the values of λ and μ in (1),

$$2x-1 = \frac{1}{2}(4x+2)-2.$$

$$\therefore \quad I = \int \frac{\frac{1}{2}(4x+2)-2}{2x^2+2x+1} \cdot dx$$

$$= \frac{1}{2}\int \frac{4x+2}{2x^2+2x+1} \cdot dx - 2\int \frac{1}{2x^2+2x+1} \cdot dx$$

$$\Rightarrow \quad I = \frac{1}{2} I_1 - 2 I_2 \quad \text{...(2)}$$

Now

$$I_1 = \int \frac{4x+2}{2x^2+2x+1} \cdot dx$$

$$= \log|2x^2+2x+1| + c_1 \quad \text{...(3)} \quad \left[\because \int \frac{f'(x)}{f(x)} \cdot dx = \log|f(x)| + c\right]$$

and

$$I_2 = \int \frac{1}{2x^2+2x+1} \cdot dx = \frac{1}{2}\int \frac{1}{x^2+x+\frac{1}{2}} \cdot dx$$

$$= \frac{1}{2}\int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)} \cdot dx$$

$$\left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2}\text{ coeff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}} \cdot dx$$

Put $\left(x+\frac{1}{2}\right) = z \Rightarrow dx = dz$

$\therefore$ $$I_2 = \frac{1}{2}\int \frac{1}{z^2+\left(\frac{1}{2}\right)^2} \cdot dz \qquad \left[\text{By using} \int \frac{1}{x^2+a^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{1}{2}\left(\frac{1}{1/2}\right) \cdot \tan^{-1}\frac{z}{(1/2)} + c_2$$

$$= \tan^{-1}\left(\frac{x+1/2}{1/2}\right) + c_2 \qquad \left[\because z = x+\frac{1}{2}\right]$$

$$= \tan^{-1}(2x+1) + c_2 \qquad \text{...(4)}$$

$\therefore$ From equation (2),

$$I = \frac{1}{2} I_1 - 2I_2$$

i.e., $$I = \frac{1}{2}[\log|2x^2+2x+1| + c_1] - 2[\tan^{-1}(2x+1)+c_2] \quad \text{[Using (3) and (4)]}$$

$$= \frac{1}{2}\log|2x^2+2x+1| + \frac{1}{2}c_1 - 2\tan^{-1}(2x+1) - 2c_2$$

$$= \frac{1}{2}\log|2x^2+2x+1| - 2\tan^{-1}(2x+1) + c \qquad \text{where : } c = \left(\frac{1}{2}c_1 - 2c_2\right).$$

(*ii*) Let $$I = \int \frac{5x-2}{3x^2+2x+1} \cdot dx$$

Let us write :

$$5x-2 = \lambda\left[\frac{d}{dx}(3x^2+2x+1)\right] + \mu$$

$$\Rightarrow \quad 5x-2 = \lambda(6x+2) + \mu \qquad \text{...(1)}$$

$$\Rightarrow \quad 5x-2 = 6\lambda x + 2\lambda + \mu$$

Comparing the co-efficients of x and the constant terms, we have

$$5 = 6\lambda \Rightarrow \lambda = \frac{5}{6}$$

and $$-2 = 2\lambda + \mu \Rightarrow \mu = -2 - 2\lambda$$

$$\Rightarrow \quad \mu = -2-2\left(\frac{5}{6}\right) = -2-\frac{10}{6} = -2-\frac{5}{3} = -\frac{11}{3}$$

Putting the values of λ and μ in (1),

$$5x-2 = \frac{5}{6}(6x+2) - \frac{11}{3}$$

$\therefore$ $$I = \int \frac{\frac{5}{6}(6x+2)-\frac{11}{3}}{(3x^2+2x+1)} \cdot dx$$

$$= \frac{5}{6}\int \frac{6x+2}{(3x^2+2x+1)} \cdot dx - \frac{11}{3}\int \frac{1}{3x^2+2x+1} \cdot dx$$

$\Rightarrow \qquad I = \frac{5}{6} I_1 - \frac{11}{3} I_2$ ...(2)

Now $\quad I_1 = \int \frac{6x+2}{3x^2+2x+1} \cdot dx$

$= \log |3x^2+2x+1| + c_1$ ...(3) $\left[\because \int \frac{f'(x)}{f(x)} \cdot dx = \log|f(x)| + c\right]$

and $\quad I_2 = \int \frac{1}{3x^2+2x+1} \cdot dx$

$$= \frac{1}{3}\int \frac{1}{\left(x^2 + \frac{2x}{3} + \frac{1}{3}\right)} \cdot dx$$

$$= \frac{1}{3}\int \frac{1}{\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) + \frac{1}{3} - \frac{1}{9}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{9} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{1}{9}\end{array}\right]$$

$$= \frac{1}{3}\int \frac{1}{\left(x + \frac{1}{3}\right)^2 + \frac{2}{9}} \cdot dx$$

Put $\quad x + \frac{1}{3} = z \Rightarrow dx = dz$

$\therefore \quad I_2 = \frac{1}{3}\int \frac{1}{z^2 + \left(\frac{\sqrt{2}}{3}\right)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$

$$= \frac{1}{3} \cdot \left(\frac{1}{\sqrt{2}/3}\right) \cdot \tan^{-1}\left(\frac{z}{\sqrt{2}/3}\right) + c_2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x + 1/3}{\sqrt{2}/3}\right) + c_2 \qquad \left[\because z = x + \frac{1}{3}\right]$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c_2 \qquad ...(4)$$

$\therefore$ From equation (2),

$$I = \frac{5}{6} I_1 - \frac{11}{3} I_2$$

i.e., $\quad I = \frac{5}{6}[\log|3x^2+2x+1| + c_1] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c_2\right]$

[Using (3) and (4)]

$$= \frac{5}{6}\log|3x^2+2x+1| + \frac{5}{6}c_1 - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) - \frac{11}{3}c_2$$

$$= \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c.$$

where : $c = \left(\frac{5}{6}c_1 - \frac{11}{3}c_2\right)$.

(*iii*) Let $\quad I = \int \frac{2x+5}{x^2 - x - 2} \cdot dx$

Let us write :

$$2x + 5 = \lambda \left[\frac{d}{dx}(x^2 - x - 2) \right] + \mu$$

$$\Rightarrow \quad 2x + 5 = \lambda(2x - 1) + \mu \quad \quad ...(1)$$

$$\Rightarrow \quad 2x + 5 = 2\lambda x - \lambda + \mu$$

Comparing the co-efficients of x and the constant terms, we have

$$2 = 2\lambda \Rightarrow \lambda = 1$$

and $\quad 5 = -\lambda + \mu \Rightarrow \mu = 5 + \lambda = 5 + 1 = 6$

Putting the values of λ and μ in (1),

$$(2x + 5) = (2x - 1) + 6$$

$$\therefore \quad I = \int \frac{(2x-1)+6}{x^2 - x - 2} \cdot dx$$

$$= \int \frac{2x-1}{x^2-x-2} \cdot dx + 6\int \frac{1}{x^2-x-2} \cdot dx$$

$$\Rightarrow \quad I = I_1 + 6I_2 \quad \quad ...(2)$$

Now $\quad I_1 = \int \frac{2x-1}{x^2 - x - 2} \cdot dx$

$$= \log |x^2 - x - 2| + c_1 \quad ...(3) \quad \left[\because \int \frac{f'(x)}{f(x)} \cdot dx = \log |f(x)| + c \right]$$

and $\quad I_2 = \int \frac{1}{x^2 - x - 2} \cdot dx$

$$= \int \frac{1}{\left(x^2 - x + \frac{1}{4}\right) - 2 - \frac{1}{4}} \cdot dx \quad \left[\begin{array}{l} \text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{1}{4} \end{array} \right]$$

$$= \int \frac{1}{\left(x - \frac{1}{2}\right)^2 - \frac{9}{4}} \cdot dx$$

Put $\quad x - \frac{1}{2} = z \Rightarrow dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{z^2 - \left(\frac{3}{2}\right)^2} \cdot dz \quad \left[\text{By using} \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c \right]$$

$$= \frac{1}{2 \cdot \left(\frac{3}{2}\right)} \log \left| \frac{z - 3/2}{z + 3/2} \right| + c_2 = \frac{1}{3} \log \left| \frac{x - \frac{1}{2} - \frac{3}{2}}{x - \frac{1}{2} + \frac{3}{2}} \right| + c_2 \quad \left[\because z = \left(x - \frac{1}{2}\right) \right]$$

$$= \frac{1}{3} \log \left| \frac{x-2}{x+1} \right| + c_2 \quad \quad ...(4)$$

$\therefore$ From equation (2),

$$I = I_1 + 6I_2$$

i.e., $$I = [\log |x^2 - x - 2| + c_1] + 6\left[\frac{1}{3}\log\left|\frac{x-2}{x+1}\right| + c_2\right]$$ [Using (3) and (4)]

$$= \log |x^2 - x - 2| + c_1 + 2\log\left|\frac{x-2}{x+1}\right| + 6c_2$$

$$= \log |x^2 - x - 2| + 2\log\left|\frac{x-2}{x+1}\right| + c.$$ where : $c = (c_1 + 6c_2)$.

(*iv*) Let $I = \int \frac{x^2 + x + 1}{x^2 - x} \cdot dx$ [Dividing the numerator by the denominator]

$$= \int \left(1 + \frac{2x+1}{x^2 - x}\right) \cdot dx$$

$$= \int 1 . dx + \int \frac{2x+1}{x^2 - x} \cdot dx$$

$$= x + \int \frac{2x+1}{x^2 - x} \cdot dx$$

$$\begin{array}{r} x^2 - x \overline{\smash{)}\, x^2 + x + 1\,}(\,1 \\ +x^2 - x \\ -\quad + \\ \hline 2x + 1 \\ \hline \end{array}$$

$\Rightarrow$ $I = x + I_1$ ...(1)

$\therefore$ $I_1 = \int \frac{2x+1}{x^2 - x} \cdot dx$

Let us write :

$$2x + 1 = \lambda\left[\frac{d}{dx}(x^2 - x)\right] + \mu$$

$\Rightarrow$ $2x + 1 = \lambda(2x - 1) + \mu$ ...(2)

$\Rightarrow$ $2x + 1 = 2\lambda x - \lambda + \mu$

Comparing the coefficients of x and the constant terms, we have

$2 = 2\lambda \quad \Rightarrow \lambda = 1$

and $1 = -\lambda + \mu \;\Rightarrow\; \mu = 1 + \lambda = 1 + 1 = 2$

Putting the values of λ and μ in (2),

$(2x + 1) = (2x - 1) + 2.$

$\therefore$ $I_1 = \int \frac{(2x-1)+2}{x^2 - x} \cdot dx$

$\Rightarrow$ $I_1 = \int \frac{2x-1}{x^2 - x} . dx + 2\int \frac{1}{x^2 - x} \cdot dx$

$\Rightarrow$ $I_1 = I_2 + 2I_3$ ...(3)

Now $I_2 = \int \frac{2x-1}{x^2 - x} \cdot dx$

$= \log |x^2 - x| + c_1$ ...(4) $\left[\because \int \frac{f'(x)}{f(x)} . dx = \log|f(x)| + c\right]$

and $$I_3 = \int \frac{1}{x^2 - x} \cdot dx$$

$$= \int \frac{1}{\left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and Subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \cdot dx$$

Put $\left(x - \frac{1}{2}\right) = z \Rightarrow dx = dz$

$\therefore$ $$I_3 = \int \frac{1}{z^2 - \left(\frac{1}{2}\right)^2} \cdot dz \qquad \left[\text{By using} \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c\right]$$

$$= \frac{1}{2 \cdot \left(\frac{1}{2}\right)} \log \left| \frac{z - \frac{1}{2}}{z + \frac{1}{2}} \right| + c_2 = \log \left| \frac{x - \frac{1}{2} - \frac{1}{2}}{x - \frac{1}{2} + \frac{1}{2}} \right| + c_2 \qquad \left[\because z = x - \frac{1}{2}\right]$$

$$= \log \left| \frac{x-1}{x} \right| + c_2 \qquad \text{...(5)}$$

$\therefore$ From equation (3),

$$I_1 = I_2 + 2I_3$$

$$= \log |x^2 - x| + c_1 + 2\left[\log\left|\frac{x-1}{x}\right| + c_2\right] \qquad \text{[Using (4) and (5)]}$$

$$= \log |x^2 - x| + c_1 + 2\log\left|\frac{x-1}{x}\right| + 2c_2$$

$$= \log |x^2 - x| + 2\log\left|\frac{x-1}{x}\right| + c \qquad \text{...(6)}$$

where : $c = (c_1 + 2c_2)$

$\therefore$ From equations (1) and (6), we have

$$I = x + \log |x^2 - x| + 2\log\left|\frac{x-1}{x}\right| + c.$$

(*v*) Let $$I = \int \frac{x^2 + 5x + 3}{x^2 + 3x + 2} \cdot dx$$

$\therefore$ $$I = \int \left(1 + \frac{2x+1}{x^2 + 3x + 2}\right) \cdot dx$$

[Dividing the numerator by the denominator]

$$\Rightarrow \qquad I = \int 1 . dx + \int \frac{2x+1}{x^2+3x+2} . dx$$

$$\Rightarrow \qquad I = x + I_1 \qquad \ldots(1)$$

Now $\qquad I_1 = \int \frac{2x+1}{x^2+3x+2} . dx$

$$\begin{array}{r|l} x^2+3x+2 & x^2+5x+3 \; (1 \\ & x^2+3x+2 \\ & - \quad - \quad - \\ \hline & \quad 2x+1 \end{array}$$

Let us write :

$$(2x+1) = \lambda \left[\frac{d}{dx} (x^2+3x+2) \right] + \mu$$

$$\Rightarrow \qquad 2x+1 = \lambda(2x+3) + \mu \qquad \ldots(2)$$

$$\Rightarrow \qquad 2x+1 = 2\lambda x + 3\lambda + \mu$$

Comparing the co-efficients of x and the constant terms,

$$2 = 2\lambda \;\Rightarrow\; \lambda = 1$$

and $\qquad 1 = 3\lambda + \mu \;\Rightarrow\; \mu = 1 - 3\lambda = 1 - 3(1) = -2$

Putting the values of λ and μ in (2),

$$(2x+1) = (2x+3) - 2.$$

$$\therefore \qquad I_1 = \int \frac{(2x+3)-2}{(x^2+3x+2)} . dx$$

$$\Rightarrow \qquad I_1 = \int \frac{2x+3}{x^2+3x+2} . dx - 2 \int \frac{1}{x^2+3x+2} . dx$$

$$\Rightarrow \qquad I_1 = I_2 - 2I_3 \qquad \ldots(3)$$

Now $\qquad I_2 = \int \frac{2x+3}{x^2+3x+2} . dx$

$$= \log |x^2+3x+2| + c_1 \qquad \ldots(4) \qquad \left[\because \int \frac{f'(x)}{f(x)} . dx = \log |f(x)| + c \right]$$

and $\qquad I_3 = \int \frac{1}{x^2+3x+2} . dx$

$$= \int \frac{1}{\left(x^2+3x+\frac{9}{4}\right)+\left(2-\frac{9}{4}\right)} . dx \qquad \left[\begin{array}{l} \text{Add and Subtract } \frac{9}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{9}{4} \end{array} \right]$$

$$= \int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \frac{1}{4}} . dx$$

Put $\left(x+\frac{3}{2}\right) = z \;\Rightarrow\; dx = dz$

$$\therefore \qquad I_3 = \int \frac{1}{z^2 - \left(\frac{1}{2}\right)^2} . dz \qquad \left[\text{By using} \int \frac{1}{x^2-a^2} . dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c \right]$$

$$= \frac{1}{2\left(\frac{1}{2}\right)} \cdot \log\left|\frac{z-\frac{1}{2}}{z+\frac{1}{2}}\right| + c_2 = \log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + c_2 \qquad \left[\because\ z = \left(x+\frac{3}{2}\right)\right]$$

$$= \log\left|\frac{x+1}{x+2}\right| + c_2 \qquad ...(5)$$

$\therefore$ From equation (3),

$$I_1 = I_2 - 2I_3$$

$$\therefore \qquad I_1 = \log|x^2+3x+2| + c_1 - 2\left[\log\left|\frac{x+1}{x+2}\right| + c_2\right] \qquad \text{[Using (4) and (5)]}$$

$$= \log|x^2+3x+2| + c_1 - 2\log\left|\frac{x+1}{x+2}\right| - 2c_2$$

$$= \log|x^2+3x+2| - 2\log\left|\frac{x+1}{x+2}\right| + c \qquad ...(6)$$

where : $c = (c_1 - 2\,c_2)$

$\therefore$ From equations (1) and (6), we have

$$I = x + \log|x^2+3x+2| - 2\log\left|\frac{x+1}{x+2}\right| + c.$$

(*vi*) Let $\quad I = \int \frac{x^2+x+1}{x^2-x+1} \cdot dx$

Please try yourself.

[**Hint.** Divide the numerator by the denominator, we have $I = \int\left(1 + \frac{2x}{x^2-x+1}\right) . dx$]

[**Ans** : $x + \log|x^2-x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$]

Example 11. *Evaluate the following integrals :*

(*i*) $\int \frac{2\sin 2\phi - \cos\phi}{6-\cos^2\phi - 4\sin\phi} \cdot d\phi \qquad$ (*ii*) $\int \frac{1}{2e^{2x}+3e^x+1} \cdot dx$

(*iii*) $\int \frac{x^2(x^4+4)}{x^2+4} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{2\sin 2\phi - \cos\phi}{6-\cos^2\phi - 4\sin\phi} \cdot d\phi$

$$= \int \frac{2(2\sin\phi\cos\phi) - \cos\phi}{6-(1-\sin^2\phi) - 4\sin\phi} \cdot d\phi \qquad \left[\begin{array}{l}\because\ 2\sin A\cos A = \sin 2A \\ \sin^2 A + \cos^2 A = 1\end{array}\right]$$

$$= \int \frac{4\sin\phi\cos\phi - \cos\phi}{5+\sin^2\phi - 4\sin\phi} \cdot d\phi$$

$$= \int \frac{(4\sin\phi - 1)\cos\phi}{\sin^2\phi - 4\sin\phi + 5} \cdot d\phi$$

Put $\quad \sin\phi = z \;\Rightarrow\; \cos\phi\, d\phi = dz$

$\therefore \quad I = \int \frac{(4z-1)}{(z^2-4z+5)}\,.\,dz$

Let us write :

$$4z - 1 = \lambda\left[\frac{d}{dz}(z^2-4z+5)\right] + \mu$$

$\Rightarrow \quad 4z - 1 = \lambda(2z-4) + \mu \qquad \ldots(1)$

$\Rightarrow \quad 4z - 1 = 2\lambda z - 4\lambda + \mu$

Comparing the co-efficients of z and the constant terms, we have

$$4 = 2\lambda \;\Rightarrow\; \lambda = 2$$

and $\quad -1 = -4\lambda + \mu \;\Rightarrow\; \mu = -1 + 4\lambda = -1 + 4(2) = -1 + 8 = 7$

Putting the values of λ and μ in (1),

$$(4z-1) = 2(2z-4) + 7$$

$\therefore \quad I = \int \frac{2(2z-4)+7}{z^2-4z+5}\,.\,dz$

$\Rightarrow \quad I = 2\int \frac{(2z-4)}{z^2-4z+5}\,.\,dz + 7\int \frac{1}{z^2-4z+5}\,.\,dz$

$\Rightarrow \quad I = 2\,I_1 + 7\,I_2 \qquad \ldots(2)$

Now $\quad I_1 = \int \frac{2z-4}{z^2-4z+5}\,.\,dz \qquad \left[\because \int \frac{f'(x)}{f(x)}\,.\,dx = \log|f(x)| + c\right]$

$= \log|z^2-4z+5| + c_1 \qquad \ldots(3) \quad [\because\; z = \sin\phi]$

$= \log|\sin^2\phi - 4\sin\phi + 5| + c_1$

and $\quad I_2 = \int \frac{1}{z^2-4z+5}\,.\,dz$

$= \int \frac{1}{(z^2-4z+4)+(5-4)}\,.\,dz \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.}\\ \because\; \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 4\end{array}\right]$

$= \int \frac{1}{(z-2)^2+1}\,.\,dz$

Put $\quad z - 2 = y \;\Rightarrow\; dz = dy$

$\therefore \quad I_2 = \int \frac{1}{y^2+1^2}\,.\,dy \qquad \left[\text{By using} \int \frac{1}{x^2+a^2}\,.\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$

$= \tan^{-1} y + c_2$

$= \tan^{-1}(z-2) + c_2 \qquad [\because\; y = z - 2]$

$= \tan^{-1}(\sin\phi - 2) + c_2 \qquad \ldots(4) \quad [\because\; z = \sin\phi]$

$\therefore$ From equation (2),

$I = 2I_1 + 7I_2$

$= 2\,[\log|\sin^2\phi - 4\sin\phi + 5| + c_1] + 7\,[\tan^{-1}(\sin\phi - 2) + c_2]$

[Using (3) and (4)]

$= 2\log|\sin^2\phi - 4\sin\phi + 5| + 2c_1 + 7\tan^{-1}(\sin\phi - 2) + 7c_2$

$$= 2 \log |\sin^2 \phi - 4 \sin \phi + 5| + 7 \tan^{-1} (\sin \phi - 2) + c.$$

where : $c = (2c_1 + 7c_2)$.

(*ii*) Let $\quad I = \int \dfrac{1}{2e^{2x} + 3e^x + 1} \cdot dx$

[Dividing the numerator and the denominator by e^{2x}]

$$= \int \frac{\frac{1}{e^{2x}}}{\left(\frac{2e^{2x}}{e^{2x}} + \frac{3e^x}{e^{2x}} + \frac{1}{e^{2x}}\right)} \cdot dx = \int \frac{e^{-2x}}{2 + 3e^{-x} + e^{-2x}} \cdot dx$$

Put $\quad e^{-x} = z$

$\Rightarrow \quad - e^{-x}\, dx = dz \;\Rightarrow\; e^{-x}\, dx = - dz$

$\therefore \quad I = \int \dfrac{e^{-x} \,.\, e^{-x}}{2 + 3e^{-x} + e^{-2x}} \cdot dx = \int \dfrac{z}{2 + 3z + z^2} (-dz)$

$$= - \int \frac{z}{(z^2 + 3z + 2)} \cdot dz$$

Let us write :

$$z = \lambda \left[\frac{d}{dx} (z^2 + 3z + 2)\right] + \mu$$

$\Rightarrow \quad z = \lambda(2z + 3) + \mu \quad$ ...(1)

$\Rightarrow \quad z = 2\lambda z + 3\lambda + \mu$

Comparing the co-efficients of z and the constant terms, we have

$$1 = 2\lambda \quad \Rightarrow \lambda = \frac{1}{2}$$

and $\quad 0 = 3\lambda + \mu \;\Rightarrow\; \mu = - 3\lambda = - 3\left(\frac{1}{2}\right) = -\frac{3}{2}$

Putting the values of λ and μ in (1),

$$z = \frac{1}{2}(2z + 3) - \frac{3}{2}$$

$\therefore \quad I = - \int \dfrac{\frac{1}{2}(2z + 3) - \frac{3}{2}}{(z^2 + 3z + 2)} \cdot dz$

$\Rightarrow \quad I = - \dfrac{1}{2} \int \dfrac{2z + 3}{z^2 + 3z + 2} \cdot dz + \dfrac{3}{2} \int \dfrac{1}{z^2 + 3z + 2} \cdot dz$

$\Rightarrow \quad I = - \dfrac{1}{2} I_1 + \dfrac{3}{2} I_2 \quad$ ...(2)

Now $\quad I_1 = \int \dfrac{2z + 3}{(z^2 + 3z + 2)} \cdot dz \qquad \left[\because \int \dfrac{f'(x)}{f(x)} . dx = \log |f(x)| + c\right]$

$$= \log |z^2 + 3z + 2| + c_1$$

$$= \log |e^{-2x} + 3e^{-x} + 2| + c_1 = \log \left|\frac{1}{e^{2x}} + \frac{3}{e^x} + 2\right| + c_1 \qquad [\because z = e^{-x}]$$

$$= \log\left|\frac{2e^{2x}+3e^x+1}{e^{2x}}\right| + c_1 \qquad ...(3)$$

and $$I_2 = \int \frac{1}{z^2+3z+2}\,.\,dz$$

$$= \int \frac{1}{\left(z^2+3z+\frac{9}{4}\right)-\frac{9}{4}+2}\,.\,dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{9}{4} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{9}{4}\end{array}\right]$$

$$= \int \frac{1}{\left(z+\frac{3}{2}\right)^2-\frac{1}{4}}\,.\,dz$$

Put $\left(z+\frac{3}{2}\right) = y \;\Rightarrow\; dz = dy$

$$\therefore \quad I_2 = \int \frac{1}{y^2-\left(\frac{1}{2}\right)^2}\,.\,dy \qquad \left[\text{By using } \int \frac{1}{x^2-a^2}\,.\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{y-\frac{1}{2}}{y+\frac{1}{2}}\right| + c_2 = \log\left|\frac{z+\frac{3}{2}-\frac{1}{2}}{z+\frac{3}{2}+\frac{1}{2}}\right| + c_2 \qquad \left[\because\; y = z+\frac{3}{2}\right]$$

$$= \log\left|\frac{z+1}{z+2}\right| + c_2 = \log\left|\frac{e^{-x}+1}{e^{-x}+2}\right| + c_2 = \log\left|\frac{1/e^x+1}{1/e^x+2}\right| + c_2$$

$$= \log\left|\frac{1+e^x}{1+2e^x}\right| + c_2 \qquad ...(4)$$

$\therefore$ From equation (2),

$$I = -\frac{1}{2}I_1 + \frac{3}{2}I_2$$

$$= -\frac{1}{2}\left[\log\left(\frac{1+3e^x+2e^{2x}}{e^{2x}}\right)+c_1\right] + \frac{3}{2}\left[\log\left(\frac{1+e^x}{1+2e^x}\right)+c_2\right] \quad \text{[Using (2) and (4)]}$$

$$= -\frac{1}{2}\log\left(\frac{1+3e^x+2e^{2x}}{e^{2x}}\right) - \frac{1}{2}c_1 + \frac{3}{2}\log\left(\frac{1+e^x}{1+2e^x}\right) + \frac{3}{2}c_2$$

$$= -\frac{1}{2}\log\left(\frac{1+3e^x+2e^{2x}}{e^{2x}}\right) + \frac{3}{2}\log\left(\frac{1+e^x}{1+2e^x}\right) + c.$$

where : $c = \left(-\frac{1}{2}c_1 + \frac{3}{2}c_2\right)$.

(*iii*) Let $\quad I = \int \frac{x^2(x^4+4)}{x^2+4}\,.\,dx$

$$= \int \frac{x^6 + 4x^2}{x^2 + 4} \cdot dx \qquad \text{[Divide the numerator by the denominator]}$$

$$\therefore \quad I = \int \left[(x^4 - 4x^2 + 20) - \frac{80}{x^2 + 4} \right] . dx$$

$$= \int x^4 . dx - 4 \int x^2 . dx$$

$$+ 20 \int 1 . dx - 80 \int \frac{1}{x^2 + 4} . dx$$

$$\left[\text{By using} \int \frac{1}{x^2 + a^2} . dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \right]$$

$$= \frac{x^5}{5} - \frac{4x^3}{3} + 20x - 80 . \frac{1}{2} \tan^{-1} \frac{x}{2} + c$$

$$= \frac{1}{5} x^5 - \frac{4}{3} x^3 + 20x - 40 \tan^{-1} \frac{x}{2} + c.$$

$$\begin{array}{r|l} x^2 + 4 & x^6 + 4x^2 \;(\; x^4 - 4x^2 + 20 \\ & x^6 + 4x^4 \\ & - \quad - \\ \hline & -4x^4 + 4x^2 \\ & -4x^4 - 16x^2 \\ & + \quad + \\ \hline & 20x^2 \\ & 20x^2 + 80 \\ & - \quad - \\ \hline & -80 \end{array}$$

Example 12. *Evaluate the following integrals :*

$(i) \int \frac{6x + 7}{\sqrt{(x - 5)(x - 4)}} . dx$ $\qquad (ii) \int \frac{2x + 3}{\sqrt{x^2 + 4x + 1}} . dx$

$(iii) \int \frac{x}{\sqrt{8 + x - x^2}} . dx$ $\qquad (iv) \int \frac{5 - 2x}{\sqrt{6 + x - x^2}} . dx.$

Solution. (*i*) Let $I = \int \frac{6x + 7}{\sqrt{(x - 5)(x - 4)}} . dx = \int \frac{6x + 7}{\sqrt{x^2 - 9x + 20}} . dx$

Let us write :

$$6x + 7 = \lambda . \frac{d}{dx} [x^2 - 9x + 20] + \mu .$$

$$\Rightarrow \quad 6x + 7 = \lambda (2x - 9) + \mu \qquad ...(1)$$

$$\Rightarrow \quad 6x + 7 = 2\lambda x - 9\lambda + \mu$$

Comparing the co-efficients of x and the constant terms, we have

$$6 = 2\lambda \;\Rightarrow\; \lambda = 3$$

and $\quad 7 = -9\lambda + \mu \;\Rightarrow\; \mu = 7 + 9\lambda = 7 + 9(3) = 34.$

Putting the values of λ and μ in (1),

$$6x + 7 = 3(2x - 9) + 34$$

$$\therefore \quad I = \int \frac{3(2x - 9) + 34}{\sqrt{x^2 - 9x + 20}} . dx$$

$$= 3 \int \frac{2x - 9}{\sqrt{x^2 - 9x + 20}} . dx + 34 \int \frac{1}{\sqrt{x^2 - 9x + 20}} . dx$$

$$\Rightarrow \quad I = 3 I_1 + 34 I_2 \qquad ...(2)$$

Now $\quad I_1 = \int \frac{2x - 9}{\sqrt{x^2 - 9x + 20}} \, . \, dx$

Put $\quad z = x^2 - 9x + 20 \Rightarrow dz = 2x - 9$

$$\therefore \quad I_1 = \int \frac{1}{\sqrt{z}} \, . \, dz = \int z^{-1/2} \, . \, dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1 = 2z^{1/2} + c_1$$

$$= 2\sqrt{x^2 - 9x + 20} + c_1 \qquad \text{...(3)} \quad [\because \; z = x^2 - 9x + 20]$$

and
$$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}} \, . \, dx$$

$$= \int \frac{1}{\sqrt{\left(x^2 - 9x + \frac{81}{4}\right) - \frac{81}{4} + 20}} \, . \, dx$$

$$\left[\begin{array}{l}\text{Add and subtract} \left(\frac{81}{4}\right) \text{to the denom.} \\ \because \; \left(\frac{1}{2} \text{co-eff. of } x\right)^2 = \frac{81}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}} \, . \, dx$$

Put $\quad z = \left(x - \frac{9}{2}\right) \Rightarrow dz = dx$

$$\therefore \quad I_2 = \int \frac{1}{\sqrt{z^2 - \left(\frac{1}{2}\right)^2}} \, . \, dx \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2 - a^2}} \, . \, dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c\right]$$

$$= \log \left| z + \sqrt{z^2 - \left(\frac{1}{2}\right)^2} \right| + c_2$$

$$= \log \left| \left(x - \frac{9}{2}\right) + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}} \right| + c_2 \qquad \left[\because \; z = x - \frac{9}{2}\right]$$

$$= \log \left| \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right| + c_2 \qquad \text{...(4)}$$

$\therefore$ From equation (2),

$$I = 3\, I_1 + 34\, I_2 \qquad \text{[Using (3) and (4)]}$$

$$= 3\left(2\sqrt{x^2 - 9x + 20} + c_1\right) + 34\left[\log \left| \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right| + c_2\right]$$

$$= 6\sqrt{x^2 - 9x + 20} + 34 \log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right| + c$$

where : $c = 3c_1 + 34c_2$.

(*ii*) Let $I = \int \frac{2x+3}{\sqrt{x^2+4x+1}} \,.\, dx$

Let us write :

$$2x + 3 = \lambda \,.\, \frac{d}{dx}(x^2 + 4x + 1) + \mu.$$

$$\Rightarrow \quad 2x + 3 = \lambda(2x + 4) + \mu \qquad \ldots(1)$$

$$\Rightarrow \quad 2x + 3 = 2\lambda x + 4\lambda + \mu$$

Comparing the co-efficients of x and the constant terms, we have

$$2\lambda = 2 \Rightarrow \lambda = 1$$

and $\quad 4\lambda + \mu = 3 \Rightarrow \mu = 3 - 4\lambda = 3 - 4(1) = -1.$

Putting the values of λ and μ in (1),

$$(2x + 3) = (2x + 4) - 1$$

$$\therefore \quad I = \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}} \,.\, dx$$

$$\Rightarrow \quad I = \int \frac{2x+4}{\sqrt{x^2+4x+1}} \,.\, dx - \int \frac{1}{\sqrt{x^2+4x+1}} \,.\, dx$$

$$\Rightarrow \quad I = I_1 - I_2 \qquad \ldots(2)$$

Now $\quad I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+1}} \,.\, dx$

Put $\quad x^2 + 4x + 1 = z \Rightarrow (2x + 4) \,.\, dx = dz$

$$\therefore \quad I_1 = \int \frac{1}{\sqrt{z}} \,.\, dz$$

$$= \int z^{-1/2} \,.\, dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2z^{1/2} + c_1$$

$$= 2\sqrt{x^2 + 4x + 1} + c_1 \qquad \ldots(3) \quad [\because \; z = x^2 + 4x + 1]$$

and $\quad I_2 = \int \frac{1}{\sqrt{x^2+4x+1}} \,.\, dx$

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+1-4}} \,.\, dx \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.}\\ \because \left(\frac{1}{2}\text{co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \frac{1}{\sqrt{(x+2)^2 - (\sqrt{3})^2}} \,.\, dx$$

Put $\quad x + 2 = z \Rightarrow \quad dx = dz$

$\therefore \qquad I_2 = \int \frac{1}{\sqrt{z^2 - (\sqrt{3})^2}} \cdot dz$

$$\left[\text{By using } \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c\right]$$

$$= \log \left| z + \sqrt{z^2 - (\sqrt{3})^2} \right| + c_2$$

$$= \log \left| (x+2) + \sqrt{(x+2)^2 - 3} \right| + c_2 \qquad [\because \ z = (x+2)]$$

$$= \log \left| (x+2) + \sqrt{x^2 + 4x + 1} \right| + c_2 \qquad \text{...(4)}$$

$\therefore$ From equation (2),

$$I = I_1 - I_2$$

$$= \left(2\sqrt{x^2 + 4x + 1} + c_1\right) - \left(\log \left| (x+2) + \sqrt{x^2 + 4x + 1} \right| + c_2\right)$$

[Using (3) and (4)]

$$= 2\sqrt{x^2 + 4x + 1} - \log \left| (x+2) + \sqrt{x^2 + 4x + 1} \right| + c. \qquad \text{where : } c = (c_1 - c_2).$$

(iii) Let $\quad I = \int \frac{x}{\sqrt{8 + x - x^2}} \cdot dx$

Let us write :

$$x = \lambda \cdot \frac{d}{dx}(8 + x - x^2) + \mu$$

$$\Rightarrow \qquad x = \lambda(1 - 2x) + \mu \qquad \text{...(1)}$$

$$\Rightarrow \qquad x = -2\lambda x + \lambda + \mu$$

Comparing the co-efficients of x and the constant terms, we have

$$1 = -2\lambda \ \Rightarrow \ \lambda = -\frac{1}{2}$$

and $\qquad 0 = \lambda + \mu \ \Rightarrow \ \mu = -\lambda = -\left(-\frac{1}{2}\right) = \frac{1}{2}.$

Putting the values of λ and μ in (1),

$$x = -\frac{1}{2}(1 - 2x) + \frac{1}{2}$$

$$\therefore \qquad I = \int \frac{-\frac{1}{2}(1-2x) + \frac{1}{2}}{\sqrt{8 + x - x^2}} \cdot dx$$

$$= -\frac{1}{2}\int \frac{1 - 2x}{\sqrt{8 + x - x^2}} \cdot dx + \frac{1}{2}\int \frac{1}{\sqrt{8 + x - x^2}} \cdot dx$$

$$\Rightarrow \qquad I = -\frac{1}{2} I_1 + \frac{1}{2} I_2 \qquad \text{...(2)}$$

Now $\quad I_1 = \int \frac{1-2x}{\sqrt{8+x-x^2}} \cdot dx$

Put $\quad 8 + x - x^2 = z \Rightarrow (1-2x)\,dx = dz$

$$\therefore \quad I_1 = \int \frac{1}{\sqrt{z}} \cdot dz = \int z^{-1/2} \cdot dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1$$

$$= 2z^{1/2} + c_1$$

$$= 2\sqrt{8+x-x^2} + c_1 \qquad \text{...(3)} \quad [\because \; z = 8+x-x^2]$$

and $$I_2 = \int \frac{1}{\sqrt{8+x-x^2}} \cdot dx = \int \frac{1}{\sqrt{8-(x^2-x)}} \cdot dx$$

$$= \int \frac{1}{\sqrt{\left(8+\frac{1}{4}\right)-\left(x^2-x+\frac{1}{4}\right)}} \cdot dx$$

$$\left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \; \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\frac{33}{4}-\left(x-\frac{1}{2}\right)^2}} \cdot dx$$

Put $\quad x - \frac{1}{2} = z \Rightarrow dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{\sqrt{\left(\frac{\sqrt{33}}{2}\right)^2 - z^2}} \cdot dz \qquad \left[\text{By using } \int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \sin^{-1}\frac{(x-1/2)}{\frac{\sqrt{33}}{2}} + c_2$$

$$= \sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right) + c_2 \qquad \text{...(4)}$$

$\therefore$ From equation (2),

$$I = -\frac{1}{2} I_1 + \frac{1}{2} I_2 \qquad \text{[Using (3) and (4)]}$$

$$= -\frac{1}{2}\left(2\sqrt{8+x-x^2} + c_1\right) + \frac{1}{2}\left[\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right) + c_2\right]$$

$$= -\sqrt{8+x-x^2} - \frac{1}{2}c_1 + \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right) + \frac{1}{2}c_2$$

$$= -\sqrt{8+x-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right) + c. \qquad \text{where : } c = \left(-\frac{1}{2}c_1 + \frac{1}{2}c_2\right)$$

(iv) Let $I = \int \frac{5-2x}{\sqrt{6+x-x^2}} \, . \, dx$

Let us write :

$$5 - 2x = \lambda \, . \, \frac{d}{dx}(6 + x - x^2) + \mu$$

$$\Rightarrow \quad 5 - 2x = \lambda(1 - 2x) + \mu \qquad ...(1)$$

$$\Rightarrow \quad 5 - 2x = -2\lambda x + \lambda + \mu$$

Comparing the co-efficients of x and the constant terms, we have

$$-2 = -2\lambda \;\Rightarrow\; \lambda = 1$$

and $5 = \lambda + \mu \;\Rightarrow\; \mu = 5 - \lambda = 5 - 1 = 4.$

Putting the values of λ and μ in (1),

$$(5 - 2x) = 1(1 - 2x) + 4$$

$$\therefore \quad I = \int \frac{(1-2x)+4}{\sqrt{6+x-x^2}} \, . \, dx$$

$$= \int \frac{1-2x}{\sqrt{6+x-x^2}} \, . \, dx + 4\int \frac{1}{\sqrt{6+x-x^2}} \, . \, dx$$

$$\Rightarrow \quad I = I_1 + 4\,I_2 \qquad ...(2)$$

Now $I_1 = \int \frac{1-2x}{\sqrt{6+x-x^2}} \, . \, dx$

Put $6 + x - x^2 = z \;\Rightarrow\; (1 - 2x)\,dx = dz$

$$\therefore \quad I_1 = \int \frac{1}{\sqrt{z}} \, . \, dz = \int z^{-1/2} \, . \, dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1$$

$$= 2\sqrt{z} + c_1$$

$$= 2\sqrt{6+x-x^2} + c_1 \qquad ...(3) \quad [\because \; z = (6 + x - x^2)]$$

and

$$I_2 = \int \frac{1}{\sqrt{6+x-x^2}} \, . \, dx = \int \frac{1}{\sqrt{6-(x^2-x)}} \, . \, dx$$

$$= \int \frac{1}{\sqrt{\left(6+\frac{1}{4}\right)-\left(x^2-x+\frac{1}{4}\right)}} \, . \, dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \; \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}} \, . \, dx$$

Put $\left(x - \frac{1}{2}\right) = z \Rightarrow dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 - z^2}} \cdot dz \qquad \left[\text{By using } \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \sin^{-1}\frac{z}{5/2} + c_2 = \sin^{-1}\frac{2z}{5} + c_2$$

$$= \sin^{-1}\frac{2(x - 1/2)}{5} + c_2 \qquad \left[\because z = \left(x - \frac{1}{2}\right)\right]$$

$$= \sin^{-1}\left(\frac{2x-1}{5}\right) + c_2 \qquad \text{...(4)}$$

$\therefore$ From equation (2),

$$I = I_1 + 4\, I_2$$

$$= \left(2\sqrt{6 + x - x^2} + c_1\right) + 4\left[\sin^{-1}\left(\frac{2x-1}{5}\right) + c_2\right] \qquad \text{[Using (3) and (4)]}$$

$$= 2\sqrt{6 + x - x^2} + c_1 + 4\sin^{-1}\left(\frac{2x-1}{5}\right) + 4c_2$$

$$= 2\sqrt{6 + x - x^2} + 4\sin^{-1}\left(\frac{2x-1}{5}\right) + c. \qquad \text{where : } c = (c_1 + 4c_2)$$

Example 13. *Evaluate the following integrals :*

(*i*) $\int \frac{5x+3}{\sqrt{x^2+4x+10}} \cdot dx$ (*ii*) $\int \frac{x+3}{\sqrt{5-4x-x^2}} \cdot dx$

(*iii*) $\int \frac{x+2}{\sqrt{x^2-1}} \cdot dx$ (*iv*) $\int \sqrt{\frac{a-x}{a+x}} \cdot dx$

(*v*) $\int \sqrt{\frac{1+x}{x}} \cdot dx$ (*vi*) $\int \frac{ax^3+bx}{x^4+c^2} \cdot dx$

(*vii*) $\int x\sqrt{\frac{a^2-x^2}{a^2+x^2}} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{5x+3}{\sqrt{x^2+4x+10}} \cdot dx$

Let us write :

$$5x + 3 = \lambda \cdot \frac{d}{dx}[x^2 + 4x + 10] + \mu$$

$$\Rightarrow \quad 5x + 3 = \lambda(2x + 4) + \mu \qquad \text{...(1)}$$

$$\Rightarrow \quad 5x + 3 = 2\lambda x + 4\lambda + \mu$$

Comparing the co-efficients of x and the constant terms, we have

$$5 = 2\lambda \Rightarrow \lambda = \frac{5}{2}$$

and $$3 = 4\lambda + \mu \Rightarrow \mu = 3 - 4\lambda = 3 - 4\left(\frac{5}{2}\right) = -7$$

Putting the values of λ and μ in (1),

$$(5x+3) = \frac{5}{2}(2x+4) - 7.$$

$$\therefore \qquad I = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}} \, . \, dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}} \, . \, dx - 7\int \frac{1}{\sqrt{x^2+4x+10}} \, . \, dx$$

$$\Rightarrow \qquad I = \frac{5}{2} I_1 - 7 I_2$$

Now $\qquad I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}} \, . \, dx$

Put $x^2 + 4x + 10 = z \;\Rightarrow\; (2x+4) \, . \, dx = dz$

$$\therefore \qquad I_1 = \int \frac{1}{\sqrt{z}} \, . \, dz = \int z^{-1/2} \, . \, dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1 = 2\sqrt{z} + c_1$$

$$= 2\sqrt{x^2+4x+10} + c_1 \qquad \ldots(3) \quad [\because \; z = (x^2+4x+10)]$$

and $\qquad I_2 = \int \frac{1}{\sqrt{x^2+4x+10}} \, . \, dx$

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+(10-4)}} \, . \, dx$$

$$\left[\begin{array}{l}\text{Add and subtract 4 to the denom.} \\ \because \; \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \frac{1}{\sqrt{(x+2)^2+6}} \, . \, dx$$

Put $x + 2 = z \;\Rightarrow\; dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{\sqrt{z^2+(\sqrt{6})^2}} \, . \, dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2+a^2}} \, . \, dx = \log\left|x+\sqrt{x^2+a^2}\right| + c\right]$$

$$= \log\left|z+\sqrt{z^2+6}\right| + c_2$$

$$= \log\left|x+2+\sqrt{(x+2)^2+6}\right| + c_2$$

$$= \log\left|x+2+\sqrt{x^2+4x+10}\right| + c_2 \qquad \ldots(4)$$

$\therefore$ From equation (2),

$$I = \frac{5}{2} I_1 - 7 I_2$$

$$= \frac{5}{2}\left[2\sqrt{x^2+4x+10}+c_1\right] - 7\left[\log\left|(x+2)+\sqrt{x^2+4x+10}\right|+c_2\right]$$

[Using (3) and (4)]

$$= 5\sqrt{x^2+4x+10} + \frac{5}{2}c_1 - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| - 7c_2$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + c.$$

where $c = \left(\frac{5}{2}c_1 - 7c_2\right)$.

(*ii*) Let $\quad I = \int \frac{x+3}{\sqrt{5-4x-x^2}} \cdot dx$

Let us write :

$$x + 3 = \lambda \cdot \frac{d}{dx}[5 - 4x - x^2] + \mu$$

$$\Rightarrow \quad x + 3 = \lambda(-4-2x) + \mu \qquad \text{...(1)}$$

$$\Rightarrow \quad x + 3 = -2\lambda x - 4\lambda + \mu$$

Comparing the co-efficients of x and the constant terms, we have

$$1 = -2\lambda \quad \Rightarrow \lambda = -\frac{1}{2}$$

and $\quad 3 = -4\lambda + \mu \Rightarrow \mu = 3 + 4\lambda = 3 + 4\left(-\frac{1}{2}\right) = 1$

Putting the values of λ and μ in (1),

$$(x+3) = -\frac{1}{2}(-4-2x) + 1$$

$$\therefore \quad I = \int \frac{-\frac{1}{2}(-4-2x)+1}{\sqrt{5-4x-x^2}} \cdot dx$$

$$= -\frac{1}{2}\int \frac{(-4-2x)}{\sqrt{5-4x-x^2}} \cdot dx + \int \frac{1}{\sqrt{5-4x-x^2}} \cdot dx$$

$$\Rightarrow \quad I = -\frac{1}{2} I_1 + I_2 \qquad \text{...(2)}$$

Now $\quad I_1 = \int \frac{(-4-2x)}{\sqrt{5-4x-x^2}} \cdot dx$

Put $\quad 5 - 4x - x^2 = z \Rightarrow (-4-2x)\,dx = dz$

$$\therefore \quad I_1 = \int \frac{1}{\sqrt{z}} \cdot dz = \int z^{-1/2} \cdot dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1 = 2\sqrt{z} + c_1$$

$$= 2\sqrt{5-4x-x^2} + c_1 \qquad \text{...(3)} \quad [\because z = (5-4x-x^2)]$$

and $\quad I_2 = \int \frac{1}{\sqrt{5-4x-x^2}} \cdot dx$

$$= \int \frac{1}{\sqrt{5-(4x+x^2)}} \cdot dx$$

$$= \int \frac{1}{\sqrt{(5+4)-(4x+x^2+4)}} \cdot dx \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \frac{1}{\sqrt{9-(x+2)^2}} \cdot dx$$

Put $x+2=z \Rightarrow dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{\sqrt{(3)^2-z^2}} \cdot dz \qquad \left[\text{By using } \int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \sin^{-1}\frac{z}{3} + c_2$$

$$= \sin^{-1}\left(\frac{x+2}{3}\right) + c_2 \qquad \ldots(4)$$

$\therefore$ From equation (2),

$$I = -\frac{1}{2} I_1 + I_2$$

$$= -\frac{1}{2}\left[2\sqrt{5-4x-x^2} + c_1\right] + \left[\sin^{-1}\left(\frac{x+2}{3}\right) + c_2\right] \qquad \text{[Using (3) and (4)]}$$

$$= -\sqrt{5-4x-x^2} - \frac{1}{2}c_1 + \sin^{-1}\left(\frac{x+2}{3}\right) + c_2$$

$$= -\sqrt{5-4x-x^2} + \sin^{-1}\left(\frac{x+2}{3}\right) + c. \qquad \text{where } c = \left(-\frac{1}{2}c_1 + c_2\right).$$

(*iii*) Let $I = \int \frac{x+2}{\sqrt{x^2-1}} \cdot dx$

Let us write :

$$x + 2 = \lambda \cdot \frac{d}{dx}(x^2-1) + \mu$$

$$\Rightarrow \quad x + 2 = \lambda(2x) + \mu \qquad \ldots(1)$$

Comparing the co-efficients of x and the constant terms, we have

$$1 = 2\lambda \Rightarrow \lambda = \frac{1}{2}$$

and $$2 = \mu \Rightarrow \mu = 2$$

Putting the values of λ and μ in (1),

$$x + 2 = \frac{1}{2}(2x) + 2$$

$$\therefore \quad I = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}} \cdot dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}} \cdot dx + 2\int \frac{1}{\sqrt{x^2-1}} \cdot dx$$

$\Rightarrow \quad I = \frac{1}{2} I_1 + 2 I_2$ ...(2)

Now $\quad I_1 = \int \frac{2x}{\sqrt{x^2 - 1}} \cdot dx$

Put $x^2 - 1 = z \Rightarrow 2x\,dx = dz$

$\therefore \quad I_1 = \int \frac{1}{\sqrt{z}} \cdot dz = \int z^{-1/2} \cdot dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1 = 2\sqrt{z} + c_1$

$= 2\sqrt{x^2 - 1} + c_1$ ...(3) $[\because \; z = (x^2 - 1)]$

and $\quad I_2 = \int \frac{1}{\sqrt{x^2 - (1)^2}} \cdot dx \quad \left[\text{By using } \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log\left| x + \sqrt{x^2 - a^2} \right| + c\right]$

$= \log\left| x + \sqrt{x^2 - 1} \right| + c_2$ ...(4)

$\therefore$ From equation (2),

$I = \frac{1}{2} I_1 + 2 I_2$

$= \frac{1}{2}\left[2\sqrt{x^2 - 1} + c_1\right] + 2\left[\log\left| x + \sqrt{x^2 - 1} \right| + c_2\right]$ [Using (3) and (4)]

$= \sqrt{x^2 - 1} + \frac{1}{2} c_1 + 2 \log\left| x + \sqrt{x^2 - 1} \right| + 2 c_2$

$= \sqrt{x^2 - 1} + 2 \log\left| x + \sqrt{x^2 - 1} \right| + c.$ where : $c = \left(\frac{1}{2} c_1 + 2c_2\right)$

(*iv*) Let $\quad I = \int \sqrt{\frac{a - x}{a + x}} \cdot dx$

$= \int \sqrt{\frac{a - x}{a + x} \times \frac{a - x}{a - x}} \cdot dx$ [On rationalization]

$= \int \sqrt{\frac{(a - x)^2}{(a^2 - x^2)}} \cdot dx$ $[\because \; (a - b)(a + b) = a^2 - b^2]$

$= \int \frac{a - x}{\sqrt{a^2 - x^2}} \cdot dx$

$= a \int \frac{1}{\sqrt{a^2 - x^2}}\, dx - \int \frac{x}{\sqrt{a^2 - x^2}} \cdot dx$

$= a\, I_1 - I_2$ ...(1)

Now $\quad I_1 = \int \frac{1}{\sqrt{a^2 - x^2}}\, dx \quad \left[\text{By using } \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1} \frac{x}{a} + c\right]$

$= \sin^{-1} \frac{x}{a} + c_1$ ...(2)

and $\quad \mathrm{I}_2 = \int \frac{x}{\sqrt{a^2 - x^2}} \cdot dx$

Put $\quad a^2 - x^2 = z$

$\Rightarrow \quad -2x\,dx = dz \ \Rightarrow \ x\,dx = -\frac{1}{2}dz$

$\therefore \quad \mathrm{I}_2 = \int \frac{1}{\sqrt{z}} \cdot \left(-\frac{1}{2}dz\right) = -\frac{1}{2}\int z^{-1/2} \cdot dz$

$$= -\frac{1}{2}\frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_2 = -\frac{1}{2} \cdot 2z^{1/2} + c_2 = -\sqrt{z} + c_2$$

$$= -\sqrt{a^2 - x^2} + c_2 \qquad \text{...(3)} \quad [\because \ z = a^2 - x^2]$$

$\therefore$ From equation (1),

$$\mathrm{I} = a\,\mathrm{I}_1 - \mathrm{I}_2$$

$$= a\left[\sin^{-1}\frac{x}{a} + c_1\right] - \left[-\sqrt{a^2 - x^2} + c_2\right] \qquad \text{[Using (2) and (3)]}$$

$$= a\sin^{-1}\frac{x}{a} + ac_1 + \sqrt{a^2 - x^2} - c_2$$

$$= a\sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2} + c \qquad \text{where : } c = (ac_1 - c_2)$$

(*v*) Let $\quad \mathrm{I} = \int \sqrt{\frac{1+x}{x}} \cdot dx$

$$= \int \sqrt{\frac{1+x}{x} \times \frac{1+x}{1+x}}\, dx \qquad \text{[Multiply and divided by } (1 + x)]$$

$$= \int \sqrt{\frac{(1+x)^2}{x(1+x)}} \cdot dx = \int \frac{1+x}{\sqrt{x + x^2}} \cdot dx$$

Let us write :

$$1 + x = \lambda \cdot \frac{d}{dx}(x + x^2) + \mu$$

$\Rightarrow \quad 1 + x = \lambda(1 + 2x) + \mu \qquad$ ...(1)

$\Rightarrow \quad 1 + x = 2\lambda x + \lambda + \mu$

Comparing the coefficients of x and the constant terms, we have

$$1 = 2\lambda \ \Rightarrow \ \lambda = \frac{1}{2}$$

and $\quad 1 = \lambda + \mu \ \Rightarrow \ \mu = 1 - \lambda = 1 - \frac{1}{2} = \frac{1}{2}$.

Putting the values of λ and μ in (1),

$$1 + x = \frac{1}{2}(1 + 2x) + \frac{1}{2}.$$

$$\therefore \qquad I = \int \frac{\frac{1}{2}(1+2x)+\frac{1}{2}}{\sqrt{x+x^2}} \, . \, dx$$

$$= \frac{1}{2}\int \frac{1+2x}{\sqrt{x+x^2}}\, dx + \frac{1}{2}\int \frac{1}{\sqrt{x+x^2}} \, . \, dx$$

$$\Rightarrow \qquad I = \frac{1}{2}\ I_1 + \frac{1}{2}\ I_2 \qquad \text{...(2)}$$

Now $\quad I_1 = \int \frac{1+2x}{\sqrt{x+x^2}}\, dx$

Put $\quad x + x^2 = z \;\Rightarrow\; (1+2x)\, dx = dz$

$$\therefore \qquad I_1 = \int \frac{1}{\sqrt{z}} \, . \, dz = \int z^{-1/2} \, . \, dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1 = 2\sqrt{z} + c_1$$

$$= 2\sqrt{x+x^2} + c_1 \qquad \text{...(3)}$$

and $\quad I_2 = \int \frac{1}{\sqrt{x+x^2}} \, . \, dx$

$$= \int \frac{1}{\sqrt{\left(x^2+x+\frac{1}{4}\right)-\frac{1}{4}}} \, . \, dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}} \, . \, dx$$

Put $\quad x + \frac{1}{2} = z \;\Rightarrow\; dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{\sqrt{z^2-\left(\frac{1}{2}\right)^2}} \, . \, dz \qquad \left[\text{By using} \int \frac{1}{\sqrt{x^2-a^2}} \, . \, dx = \log\left| x + \sqrt{x^2-a^2}\right| + c\right]$$

$$= \log\left| z + \sqrt{z^2-\left(\frac{1}{2}\right)^2}\right| + c_2 = \log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}}\right| + c_2$$

$$= \log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + c_2 \qquad \text{...(4)}$$

$\therefore$ From equation (2),

$$I = \frac{1}{2}\ I_1 + \frac{1}{2}\ I_2 \qquad \text{[Using (3) and (4)]}$$

$$= \frac{1}{2}\left(2\sqrt{x+x^2} + c_1\right) + \frac{1}{2}\left[\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + c_2\right]$$

$$= \sqrt{x+x^2} + \frac{1}{2}c_1 + \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + \frac{1}{2}c_2$$

$$= \sqrt{x+x^2} + \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + c. \qquad \text{where : } \left(c = \frac{1}{2}c_1 + \frac{1}{2}c_2\right)$$

(vi) Let $I = \int \frac{ax^3+bx}{x^4+c^2}\,dx = \int \frac{ax^3}{x^4+c^2}\,.\,dx + \int \frac{bx}{x^4+c^2}\,.\,dx$

$$= a\int \frac{x^3}{x^4+c^2}\,.\,dx + b\int \frac{x}{x^4+c^2}\,.\,dx$$

$$\Rightarrow \quad I = a\,I_1 + b\,I_2 \qquad \text{...(1)}$$

Now $I_1 = \int \frac{x^3}{x^4+c^2}\,.\,dx$

$$= \frac{1}{4}\int \frac{4x^3}{x^4+c^2}\,.\,dx \qquad \text{[Multiply and divided by 4]}$$

$$= \frac{1}{4}\log\left|x^4+c^2\right| + c_1 \qquad \text{...(2)} \quad \left[\because \int \frac{f'(x)}{f(x)}\,.\,dx = \log|f(x)| + c\right]$$

and $I_2 = \int \frac{x}{x^4+c^2}\,.\,dx$

$$= \frac{1}{2}\int \frac{2x}{(x^2)^2+c^2}\,dx \qquad \text{[Multiply and divided by 2]}$$

Put $x^2 = z \Rightarrow 2x\,dx = dz$

$$= \frac{1}{2}\int \frac{1}{z^2+c^2}\,dz \qquad \left[\text{By using} \int \frac{1}{x^2+a^2}\,.\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{2}\cdot\frac{1}{c}\tan^{-1}\left(\frac{z}{c}\right) + c_2$$

$$= \frac{1}{2c}\tan^{-1}\left(\frac{x^2}{c}\right) + c_2 \qquad \text{...(3)}$$

$\therefore$ From equation (1),

$$I = a\,I_1 + b\,I_2 \qquad \text{[Using (2) and (3)]}$$

$$= a\left[\frac{1}{4}\log|x^4+c^2| + c_1\right] + b\left[\frac{1}{2c}\tan^{-1}\left(\frac{x^2}{c}\right) + c_2\right]$$

$$= \frac{a}{4}\log|x^4+c^2| + ac_1 + \frac{b}{2c}\tan^{-1}\left(\frac{x^2}{c}\right) + bc_2$$

$$= \frac{a}{4}\log|x^4+c^2| + \frac{b}{2c}\tan^{-1}\left(\frac{x^2}{c}\right) + c_0. \qquad \text{where : } c_0 = (ac_1 + bc_2)$$

(*vii*) Let $I = \int x\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}\,.\,dx$

Put $x^2 = z$

$\Rightarrow \quad 2x\,dx = dz \;\Rightarrow\; x\,dx = \dfrac{1}{2}\,dz$

$\therefore \quad I = \dfrac{1}{2}\int \sqrt{\dfrac{a^2 - z}{a^2 + z}}\,.\,dz$

$= \dfrac{1}{2}\int \sqrt{\dfrac{a^2 - z}{a^2 + z} \times \dfrac{a^2 - z}{a^2 - z}}\,.\,dz$ [On rationalization]

$= \dfrac{1}{2}\int \sqrt{\dfrac{(a^2 - z)^2}{a^4 - z^2}}\,.\,dz = \dfrac{1}{2}\int \dfrac{a^2 - z}{\sqrt{a^4 - z^2}}\,.\,dz$ [$\because$ $(a - b)(a + b) = a^2 - b^2$]

$= \dfrac{a^2}{2}\int \dfrac{1}{\sqrt{a^4 - z^2}}\,.\,dz - \dfrac{1}{2}\int \dfrac{z}{\sqrt{a^4 - z^2}}\,.\,dz$

$\Rightarrow \quad I = \dfrac{a^2}{2}\,I_1 - \dfrac{1}{2}\,I_2$ ...(1)

Now $\quad I_1 = \int \dfrac{1}{\sqrt{a^4 - z^2}}\,.\,dz$

$= \int \dfrac{1}{\sqrt{(a^2)^2 - z^2}}\,.\,dz$ $\left[\text{By using} \int \dfrac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\dfrac{x}{a} + c\right]$

$= \sin^{-1}\left(\dfrac{z}{a^2}\right) + c_1$ ...(2)

and $\quad I_2 = \int \dfrac{z}{\sqrt{a^4 - z^2}}\,.\,dz$

Put $a^4 - z^2 = y$

$\Rightarrow \quad -\,2z dz = dy \;\Rightarrow\; z\,dz = -\,\dfrac{1}{2}\,dy$

$\therefore \quad I_2 = \int \dfrac{1}{\sqrt{y}}\left(-\,\dfrac{1}{2}\,dy\right)$

$= -\,\dfrac{1}{2}\int \dfrac{1}{\sqrt{y}}\,dy = -\,\dfrac{1}{2}\int y^{-1/2}\,.\,dy = -\,\dfrac{1}{2}\,.\,\dfrac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_2$

$= -\,\dfrac{1}{2}\,.\,(2\sqrt{y}) + c_2 = -\,\sqrt{y} + c_2$

$= -\,\sqrt{a^4 - z^2} + c_2$ ...(3) [$\because$ $y = a^4 - z^2$]

$\therefore$ From equation (1),

$$I = \frac{a^2}{2} I_1 - \frac{1}{2} I_2$$

$$= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{z}{a^2}\right) + c_1\right] - \frac{1}{2}\left[-\sqrt{a^4 - z^2} + c_2\right] \qquad \text{[Using (2) and (3)]}$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{z}{a^2}\right) + \frac{a^2}{2}\cdot c_1 + \frac{1}{2}\sqrt{a^4 - z^2} - \frac{1}{2}c_2$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{z}{a^2}\right) + \frac{1}{2}\sqrt{a^4 - z^2} + c \qquad \text{where } c = \left(\frac{a^2}{2}c_1 - \frac{1}{2}c_2\right)$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4 - x^4} + c. \qquad [\because\ z = x^2]$$

Example 14. *Evaluate the following integrals :*

(*i*) $\int \frac{x^3 + x + 1}{x^2 - 1}\cdot dx$ (*ii*) $\int \frac{x^2 + 5x + 3}{x^2 + 3x + 2}\cdot dx$

(*iii*) $\int \frac{x^2}{x^2 + 6x + 12}\cdot dx$ (*iv*) $\int \frac{x^3 + x^2 + 2x + 1}{x^2 - x + 1}\cdot dx$

(*v*) $\int \frac{x + 1}{\sqrt{4 + 5x - x^2}}\cdot dx.$

Solution. (*i*) Let $I = \int \frac{x^3 + x + 1}{x^2 - 1}\cdot dx$ [Dividing numerator by the denominator]

$$\begin{array}{r|l} x^2 - 1 & x^3 + x + 1 \,(\, x \\ & x^3 - x \\ & -\quad + \\ \hline & 2x + 1 \end{array}$$

$\therefore$
$$I = \int \left(x + \frac{2x + 1}{x^2 - 1}\right)\cdot dx$$

$$= \int x\cdot dx + \int \frac{2x + 1}{x^2 - 1}\cdot dx$$

$$= \int x\cdot dx + \int \frac{2x}{x^2 - 1}dx + \int \frac{1}{x^2 - 1}\cdot dx$$

$\left[\text{By using } \int \frac{f'(x)}{f(x)}\cdot dx = \log|f(x)| + c \text{ in the second integral}\right.$

$\left.\text{and } \int \frac{1}{x^2 - a^2}\cdot dx = \frac{1}{2a}\log\left|\frac{x - a}{x + a}\right| + c \text{ in the third integral}\right]$

$$= \frac{x^2}{2} + \log|x^2 - 1| + \frac{1}{2}\log\left|\frac{x - 1}{x + 1}\right| + c.$$

(*ii*) Let $I = \int \frac{x^2 + 5x + 3}{x^2 + 3x + 2}\cdot dx$ [Dividing the numerator by the denominator]

$$\therefore \quad I = \int \left(1 + \frac{2x+1}{x^2+3x+2}\right) . dx$$

$$x^2+3x+2 \,\overline{)\, x^2+5x+3 \,}\, 1$$
$$\underline{\begin{array}{r} x^2+3x+2 \\ - \quad - \quad - \end{array}}$$
$$2x+1 .$$

[Note this step]

$$= \int 1 . dx + \int \frac{2x+1}{x^2+3x+2} . dx$$

$$= \int 1 . dx + \int \frac{2x+3-2}{x^2+3x+2} . dx$$

$$= \int 1 . dx + \int \frac{2x+3}{x^2+3x+2} - 2 \int \frac{1}{x^2+3x+2} . dx$$

$$\left[\because \int \frac{f'(x)}{f(x)} . dx = \log |f(x)| + c\right]$$

$$\Rightarrow \quad I = x + \log |x^2+3x+2| - 2I_1 + c_1 \qquad ...(1)$$

Now $\quad I_1 = \int \frac{1}{x^2+3x+2} . dx$

$$= \int \frac{1}{\left(x^2+3x+\frac{9}{4}+2-\frac{9}{4}\right)} . dx$$

$$\left[\begin{array}{l}\text{Add and subtract } \frac{9}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{9}{4}\end{array}\right]$$

$$= \int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \frac{1}{4}} . dx$$

Put $\quad x + \frac{3}{2} = z \;\Rightarrow\; dx = dz$

$$\therefore \quad I_1 = \int \frac{1}{z^2 - \left(\frac{1}{2}\right)^2} dz \qquad \left[\text{By using} \int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{2\left(\frac{1}{2}\right)} . \log \left|\frac{z-\frac{1}{2}}{z+\frac{1}{2}}\right| + c_2$$

$$= \log \left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + c_2 \qquad \left[\because z = \left(x+\frac{3}{2}\right)\right]$$

$$= \log \left|\frac{x+1}{x+2}\right| + c_2 \qquad ...(2)$$

Putting this value in equation (1), we have

$$I = x + \log |x^2+3x+2| - 2\left[\log \left|\frac{x+1}{x+2}\right| + c_2\right] + c_1$$

$$= x + \log|x^2 + 3x + 2| - 2\log\left|\frac{x+1}{x+2}\right| - 2c_2 + c_1$$

$$= x + \log|x^2 + 3x + 2| - 2\log\left|\frac{x+1}{x+2}\right| + c \qquad \text{where : } c = (c_1 - 2c_2).$$

(*iii*) Let $\quad I = \int \frac{x^2}{x^2+6x+12}\,.\,dx$ [Dividing the numerator by the denominator]

$$x^2+6x+12 \,\big)\, x^2 \,\big(\, 1$$
$$\underline{x^2+6x+12}$$
$$\underline{-\quad -\quad -}$$
$$-6x-12$$

$\therefore \quad I = \int \left[1 - \frac{(6x+12)}{x^2+6x+12}\right].\,dx$

$$= \int 1.\,dx - 6\int \frac{x+2}{x^2+6x+12}\,.\,dx$$

$$= \int 1.\,dx - \frac{6}{2}\int \frac{2x+4}{x^2+6x+12}\,.\,dx$$

[Multiply and divided by 2 the integrand in second integral]

$$= \int 1.\,dx - 3\int \frac{2x+6-2}{x^2+6x+12}\,.\,dx \qquad \text{[Note this step]}$$

$$= \int 1.\,dx - 3\int \frac{2x+6}{x^2+6x+12}\,.\,dx + 6\int \frac{1}{x^2+6x+12}\,.\,dx$$

$$= x - 3\log|x^2+6x+12| + 6\int \frac{1}{x^2+6x+12}\,.\,dx$$

$$\left[\because \int \frac{f'(x)}{f(x)}\,.\,dx = \log|f(x)| + c\right]$$

$$= x - 3\log|x^2+6x+12| + 6\int \frac{1}{x^2+6x+9+12-9}\,.\,dx$$

$$\left[\begin{array}{l}\text{Add and subtract 9 to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 9\end{array}\right]$$

$$= x - 3\log|x^2+6x+12| + 6\int \frac{1}{(x+3)^2+(\sqrt{3})^2}\,.\,dx$$

$$\left[\text{By using} \int \frac{1}{x^2+a^2}\,.\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= x - 3\log|x^2+6x+12| + 6\,.\,\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+3}{\sqrt{3}} + c$$

$$= x - 3\log|x^2+6x+12| + 2\sqrt{3}\tan^{-1}\frac{x+3}{\sqrt{3}} + c.$$

(*iv*) Let $\quad I = \int \frac{x^3+x^2+2x+1}{x^2-x+1}\,.\,dx$ [Dividing the numerator by the denominator]

$$\therefore \quad I = \int \left[(x+2) + \frac{3x-1}{x^2-x+1}\right] . dx$$

$$= \int (x+2) . dx + \int \frac{3x-1}{x^2-x+1} . dx$$

$$= \int x . dx + 2\int 1 . dx + \int \frac{3x-1}{x^2-x+1} . dx$$

$$\begin{array}{r|l} x^2-x+1 & x^3+x^2+2x+1 \; (x+2 \\ & x^3-x^2+x \\ & \underline{-\quad +\quad -} \\ & 2x^2+x+1 \\ & 2x^2-2x+2 \\ & \underline{-\quad +\quad -} \\ & 3x-1 \end{array}$$

$$\Rightarrow \quad I = \frac{x^2}{2} + 2x + I_1 \qquad ...(1)$$

Now $\quad I_1 = \int \frac{3x-1}{x^2-x+1} . dx$

Let us write :

$$3x - 1 = \lambda \frac{d}{dx}(x^2 - x + 1) + \mu$$

$$\Rightarrow \quad 3x - 1 = \lambda(2x-1) + \mu \qquad ...(2)$$

$$\Rightarrow \quad 3x - 1 = 2\lambda x - \lambda + \mu$$

Comparing the co-efficients of x and the constant terms, we have

$$3 = 2\lambda \;\Rightarrow\; \lambda = \frac{3}{2}$$

and
$$-1 = -\lambda + \mu \;\Rightarrow\; \mu = \lambda - 1 = \frac{3}{2} - 1 = \frac{1}{2}$$

Putting the values of λ and μ in (2),

$$(3x-1) = \frac{3}{2}(2x-1) + \frac{1}{2}$$

$$\therefore \quad I_1 = \int \frac{\frac{3}{2}(2x-1) + \frac{1}{2}}{x^2-x+1} . dx$$

$$= \frac{3}{2}\int \frac{2x-1}{x^2-x+1} . dx + \frac{1}{2}\int \frac{1}{x^2-x+1} . dx$$

$$= \frac{3}{2} I_2 + \frac{1}{2} I_3 \qquad ...(3)$$

Now
$$I_2 = \int \frac{2x-1}{x^2-x+1} . dx \qquad \left[\because \int \frac{f'(x)}{f(x)} . dx = \log|f(x)| + c\right]$$

$$= \log|x^2 - x + 1| + c_1 \qquad ...(4)$$

and
$$I_3 = \int \frac{1}{x^2-x+1} dx$$

$$= \int \frac{1}{x^2 - x + \frac{1}{4} + 1 - \frac{1}{4}} dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} dx$$

Put $\quad x - \frac{1}{2} = z \Rightarrow dx = dz$

$\therefore \quad I_3 = \int \frac{1}{z^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \cdot dz \qquad \left[\text{By using } \int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{z}{\sqrt{3}/2}\right) + c_2$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x - 1/2}{\sqrt{3}/2}\right) + c_2 \qquad [\because z = (x - 1/2)]$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c_2 \qquad \ldots(5)$$

$\therefore$ From equation (3),

$$I_1 = \frac{3}{2} I_2 + \frac{1}{2} I_3$$

$$= \frac{3}{2}\left[\log|x^2 - x + 1| + c_1\right] + \frac{1}{2}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c_2\right]$$

[Using (4) and (5)]

$$= \frac{3}{2}\log|x^2 - x + 1| + \frac{3}{2}c_1 + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{2}c_2$$

$$= \frac{3}{2}\log|x^2 - x + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c \quad \text{where}: c = \left(\frac{3}{2}c_1 + \frac{1}{2}c_2\right)$$

Substituting this value of I_1 in (1), we have

$$I = \frac{x^2}{2} + 2x + \frac{3}{2}\log|x^2 - x + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + c.$$

(v) Let $\quad I = \int \frac{x+1}{\sqrt{4 + 5x - x^2}} \cdot dx$

Let us write :

$$x + 1 = \lambda \cdot \frac{d}{dx}(4 + 5x - x^2) + \mu$$

$$\Rightarrow \quad x + 1 = \lambda(5 - 2x) + \mu \qquad \ldots(1)$$

$$\Rightarrow \quad x + 1 = -2\lambda x + 5\lambda + \mu$$

Comparing the co-efficient of x and the constant terms, we have

$$1 = -2\lambda \Rightarrow \lambda = -\frac{1}{2}$$

and $\quad 1 = 5\lambda + \mu \Rightarrow \mu = 1 - 5\lambda = 1 - 5\left(-\frac{1}{2}\right) = \frac{7}{2}$

Putting the values of λ and μ in equation (1),

$$x + 1 = -\frac{1}{2}(5 - 2x) + \frac{7}{2}$$

$\therefore \qquad I = \int \dfrac{-\frac{1}{2}(5-2x)+\frac{7}{2}}{\sqrt{4+5x-x^2}}\,.\,dx$

$= -\dfrac{1}{2}\int \dfrac{5-2x}{\sqrt{4+5x-x^2}}\,dx + \dfrac{7}{2}\int \dfrac{1}{\sqrt{4+5x-x^2}}\,.\,dx$

$\Rightarrow \qquad I = -\dfrac{1}{2}\,I_1 + \dfrac{7}{2}\,I_2 \qquad \ldots(2)$

Now $\qquad I_1 = \int \dfrac{5-2x}{\sqrt{4+5x-x^2}}\,dx$

Put $\quad 4+5x-x^2 = z \;\Rightarrow\; (5-2x)\,dx = dz$

$\therefore \qquad I_1 = \int \dfrac{1}{\sqrt{z}}\,dz = \int z^{-1/2}\,dz = \dfrac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1 = 2\sqrt{z} + c_1$

$= 2\sqrt{4+5x-x^2} + c_1 \qquad \ldots(3) \quad [\because\; z = (4+5x-x^2)]$

and $\qquad I_2 = \int \dfrac{1}{\sqrt{4+5x-x^2}}\,.\,dx$

$= \int \dfrac{1}{\sqrt{4-(x^2-5x)}}\,.\,dx$

$= \int \dfrac{1}{\sqrt{4+\frac{25}{4}-\left(x^2-5x+\frac{25}{4}\right)}}\,.\,dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{25}{4} \text{ to the denom.}\\ \because\; \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{25}{4}\end{array}\right]$

$= \int \dfrac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{5}{2}\right)^2}}\,.\,dx$

Put $\quad x - \dfrac{5}{2} = z \;\Rightarrow\; dx = dz$

$\therefore \qquad I_2 = \int \dfrac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - z^2}}\,.\,dz \qquad \left[\text{By using} \int \dfrac{1}{\sqrt{a^2-x^2}}\,.\,dx = \sin^{-1}\dfrac{x}{a} + c\right]$

$= \sin^{-1}\dfrac{z}{\frac{\sqrt{41}}{2}} + c_2$

$= \sin^{-1}\left(\dfrac{x-5/2}{\sqrt{41}/2}\right) + c_2 \qquad \left[\because\; z = \left(x-\dfrac{5}{2}\right)\right]$

$= \sin^{-1}\left(\dfrac{2x-5}{\sqrt{41}}\right) + c_2 \qquad \ldots(4)$

$\therefore$ From equation (2),

$$I = -\frac{1}{2} I_1 + \frac{7}{2} I_2$$

$$= -\frac{1}{2}\left[2\sqrt{4+5x-x^2} + c_1\right] + \frac{7}{2}\left[\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) + c_2\right] \quad \text{[Using (3) and (4)]}$$

$$= -\sqrt{4+5x-x^2} - \frac{1}{2}c_1 + \frac{7}{2}\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) + \frac{7}{2}c_2$$

$$= -\sqrt{4+5x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) + c \qquad \text{where : } c = \left(-\frac{1}{2}c_1 + \frac{7}{2}c_2\right).$$

3.2. INTEGRATION OF SOME SPECIAL TYPES OF TRIGONOMETRIC FUNCTIONS

3.2.1. Integrals of the Form

$$\int \frac{1}{a\sin x + b\cos x}\,dx, \int \frac{1}{a + b\sin x}\,dx, \int \frac{1}{a + b\cos x}\,dx, \int \frac{1}{a\sin x + b\cos x + c}\,dx$$

Working Rule :

Step I. Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.

Step II. Replace $(1 + \tan^2 x/2)$ in the numerator by $\sec^2 \frac{x}{2}$.

Step III. Put $\tan \frac{x}{2} = z$ so that $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$.

This substitution reduces the integral in the form $\int \dfrac{1}{az^2 + bz + c} \cdot dz$

Step IV. Evaluate the integral in the step (*iii*) by using methods discussed earlier in the Article 3.1.1.

3.2.2. Integrals of the Form $\int \dfrac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$

Working Rule :

Stpe I. Write :

Numerator = λ [Differentiation of denominator] + μ (Denominator)

i.e., $\quad a\sin x + b\cos x = \lambda \dfrac{d}{dx}(c\sin x + d\cos x) + \mu(c\sin x + d\cos x)$.

Step II. Obtain the values of λ and μ by equating the coefficients of $\sin x$ and $\cos x$ on both sides.

Step III. The given integral

$$= \int \frac{\lambda \frac{d}{dx}(c\sin x + d\cos x)}{c\sin x + d\cos x}\,dx + \int \mu\,dx.$$

Step IV. The second integral is μx and evaluate the first integral by substituting $z = (c \sin x + d \cos x)$.

3.2.3. Integrals of the Form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Working Rule :

Step I. Write :

Numerator = q (Differentiation of denominator) + p (Denominator) + r

i.e., $(a \sin x + b \cos x + c) = q \frac{d}{dx} (d \sin x + e \cos x + f) + p (d \sin x + e \cos x + f) + r$

Step II. Obtain the values of q and p by equating the coefficients of $\sin x$ and $\cos x$ and the constant terms on both sides.

Step III. Given integral

$$= \int p\, dx + \int \frac{q \frac{d}{dx}(d \sin x + e \cos x + f)}{d \sin x + e \cos x + f} dx + \int \frac{r}{d \sin x + e \cos x + f} \cdot dx.$$

Step IV. The first integral is px. Evaluate the second integral by putting $z = (d \sin x + e \cos x + f)$ and the third integral by putting

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \text{ and } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \text{ and then taking } z = \tan \frac{x}{2}.$$

SOME SOLVED EXAMPLES

Example 15. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{4 + 5 \sin x} \cdot dx$ (*ii*) $\int \frac{1}{4 \cos x - 1} \cdot dx$

(*iii*) $\int \frac{1}{1 + \sin x + \cos x} dx$ (*iv*) $\int \frac{1}{2 + \cos x} \cdot dx$

(*v*) $\int \frac{1}{a + b \cos x} \cdot dx$ (*vi*) $\int \frac{1}{3 + 2 \sin x + \cos x} \cdot dx.$

Solution. (*i*) Let $I = \int \frac{1}{4 + 5 \sin x} \cdot dx$

Putting $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$, we have

$$I = \int \frac{1}{4 + 5\left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)} \cdot dx$$

$$= \int \frac{1}{\dfrac{4 + 4\tan^2 \frac{x}{2} + 10 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} \,.\, dx = \int \frac{\left(1 + \tan^2 \frac{x}{2}\right)}{4 + 4 \tan^2 \frac{x}{2} + 10 \tan \frac{x}{2}} \,.\, dx$$

$$= \frac{1}{2} \int \frac{\sec^2 \frac{x}{2}}{2 \tan^2 \frac{x}{2} + 5 \tan \frac{x}{2} + 2} \,.\, dx \qquad [\because \ \sec^2 A - \tan^2 A = 1]$$

Put $\tan \frac{x}{2} = z$

$\Rightarrow \quad \frac{1}{2} \sec^2 \frac{x}{2}\, dx = dz \quad \Rightarrow \quad \sec^2 \frac{x}{2}\, dx = 2dz$

$$\therefore \qquad I = \frac{1}{2} \int \frac{1}{(2z^2 + 5z + 2)} \,.\, 2dz = \int \frac{1}{(2z^2 + 5z + 2)}\, dz$$

$$= \frac{1}{2} \int \frac{1}{\left(z^2 + \frac{5}{2} z + 1\right)}\, dz$$

$$= \frac{1}{2} \int \frac{1}{z^2 + \frac{5}{2} z + \frac{25}{16} + 1 - \frac{25}{16}}\, dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{25}{16} \text{ to the denom.} \\ \because \ \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{25}{16}\end{array}\right]$$

$$= \frac{1}{2} \int \frac{1}{\left(z + \frac{5}{4}\right)^2 - \frac{9}{16}}\, dz$$

Put $z + \frac{5}{4} = y \quad \Rightarrow \quad dz = dy$

$$\therefore \qquad I = \frac{1}{2} \int \frac{1}{y^2 - \left(\frac{3}{4}\right)^2}\, dy \qquad \left[\text{By using } \int \frac{1}{x^2 - a^2} \,.\, dx = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + c\right]$$

$$= \frac{1}{2} \cdot \frac{1}{2\left(\frac{3}{4}\right)} \log \left| \frac{y - \frac{3}{4}}{y + \frac{3}{4}} \right| + c$$

$$= \frac{1}{2\left(\frac{3}{2}\right)} \log \left| \frac{z + \frac{5}{4} - \frac{3}{4}}{z + \frac{5}{4} + \frac{3}{4}} \right| + c \qquad \left[\because \ y = \left(z + \frac{5}{4}\right)\right]$$

$$= \frac{1}{3}\log\left|\frac{z+\frac{1}{2}}{z+2}\right| + c = \frac{1}{3}\log\left|\frac{2z+1}{2z+4}\right| + c$$

$$= \frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}+4}\right| + c. \qquad \left[\because\ z = \tan\frac{x}{2}\right]$$

(ii) Let $\quad \mathrm{I} = \int \frac{1}{4\cos x - 1}\,dx$

Putting $\cos x = \left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)$, we have

$$\mathrm{I} = \int \frac{1}{4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)-1}\,.\,dx$$

$$= \int \frac{1}{\frac{4-4\tan^2\frac{x}{2}-1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\,dx = \int \frac{1+\tan^2\frac{x}{2}}{-5\tan^2\frac{x}{2}+3}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{-5\tan^2\frac{x}{2}+3}\,dx \qquad [\because\ \sec^2 \mathrm{A} - \tan^2 \mathrm{A} = 1]$$

Put $\quad \tan\frac{x}{2} = z$

$\Rightarrow\ \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz \ \Rightarrow\ \sec^2\frac{x}{2}\,dx = 2dz$

$\therefore \quad \mathrm{I} = \int \frac{1}{-5z^2+3}\,(2dz)$

$$= -\frac{2}{5}\int \frac{1}{z^2-\frac{3}{5}}\,dz = -\frac{2}{5}\int \frac{1}{z^2-\left(\sqrt{\frac{3}{5}}\right)^2}\,dz$$

$$\left[\text{By using} \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= -\frac{2}{5} \cdot \frac{1}{2\sqrt{\frac{3}{5}}} \log \left| \frac{z - \sqrt{\frac{3}{5}}}{z + \sqrt{\frac{3}{5}}} \right| + c = -\frac{1}{5} \cdot \sqrt{\frac{5}{3}} \log \left| \frac{z - \sqrt{\frac{3}{5}}}{z + \sqrt{\frac{3}{5}}} \right| + c$$

$$= -\frac{1}{\sqrt{15}} \log \left| \frac{\tan \frac{x}{2} - \sqrt{\frac{3}{5}}}{\tan \frac{x}{2} + \sqrt{\frac{3}{5}}} \right| + c. \qquad \left[\because \ z = \tan \frac{x}{2}\right]$$

(*iii*) Let $\quad I = \int \frac{1}{1 + \sin x + \cos x} \, dx$

Putting $\quad \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$, we have

$$I = \int \frac{1}{1 + \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} \, dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} \, dx = \int \frac{1 + \tan^2 \frac{x}{2}}{2 + 2 \tan \frac{x}{2}} \, dx$$

$$= \int \frac{\sec^2 \frac{x}{2}}{2 + 2 \tan \frac{x}{2}} \, dx \qquad [\because \ \sec^2 A - \tan^2 A = 1]$$

Put $\quad \tan \frac{x}{2} = z$

$\Rightarrow \quad \frac{1}{2} \sec^2 \frac{x}{2} \, dx = dz \quad \Rightarrow \quad \sec^2 \frac{x}{2} \, dx = 2dz$

$\therefore \quad I = \int \frac{1}{2 + 2z} \, (2dz)$

$$= \int \frac{1}{z + 1} \, dz \qquad \left[\because \ \int \frac{1}{x} \, dx = \log |x| + c\right]$$

$$= \log |z + 1| + c$$

$$= \log \left| \tan \frac{x}{2} + 1 \right| + c. \qquad \left[\because \ z = \tan \frac{x}{2}\right]$$

(*iv*) Let $\quad I = \int \frac{1}{2 + \cos x} \, dx$

Putting $\quad \cos x = \left(\dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}}\right)$, we have

$$I = \int \frac{1}{2+\left(\dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}}\right)}\,dx$$

$$= \int \frac{\left(1+\tan^2\dfrac{x}{2}\right)}{2\left(1+\tan^2\dfrac{x}{2}\right)+\left(1-\tan^2\dfrac{x}{2}\right)}\,dx$$

$$= \int \frac{\sec^2\dfrac{x}{2}}{\tan^2\dfrac{x}{2}+3}\,dx \qquad [\because\ \sec^2 A - \tan^2 A = 1]$$

Put $\quad \tan\dfrac{x}{2} = z$

$\Rightarrow \quad \dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx = dz \ \Rightarrow\ \sec^2\dfrac{x}{2}\,dx = 2\,dz$

$\therefore \quad I = \displaystyle\int \frac{1}{z^2+3}\,(2\,dz)$

$$= 2\int \frac{1}{z^2+(\sqrt{3})^2}\,dz \qquad \left[\text{By using } \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= 2\,.\,\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right)+c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\dfrac{x}{2}}{\sqrt{3}}\right)+c. \qquad \left[\because\ z = \tan\frac{x}{2}\right]$$

(*v*) Let $\quad I = \displaystyle\int \frac{1}{a+b\cos x}\,dx$

Putting $\quad \cos x = \left(\dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}}\right)$, we have

$$I = \int \frac{1}{a+b\left(\dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}}\right)}\,dx = \int \frac{1+\tan^2\dfrac{x}{2}}{a+a\tan^2\dfrac{x}{2}+b-b\tan^2\dfrac{x}{2}}\,dx$$

$$= \int \frac{1+\tan^2 \frac{x}{2}}{(a+b)+(a-b)\tan^2 \frac{x}{2}}\, dx$$

$$\therefore \qquad I = \int \frac{\sec^2 \frac{x}{2}}{(a+b)+(a-b)\tan^2 \frac{x}{2}}\, dx \qquad [\because \sec^2 A - \tan^2 A = 1]$$

Put $\tan \frac{x}{2} = z$

$$\Rightarrow \frac{1}{2}\sec^2 \frac{x}{2}\, dx = dz \Rightarrow \sec^2 \frac{x}{2}\, dx = 2\, dz$$

$$\therefore \qquad I = 2 \int \frac{1}{(a+b)+(a-b)z^2}\, dz$$

$$= \left(\frac{2}{a-b}\right) \int \frac{1}{\frac{a+b}{a-b} + z^2}\, dz$$

Now there may be following possibilities :

Case I : When $a > b$, then

$$I = \frac{2}{a-b} \int \frac{1}{\left(\frac{a+b}{a-b}\right) + z^2}\, dz \qquad \left[\because \frac{a+b}{a-b} > 0\right]$$

$$= \frac{2}{a-b} \int \frac{1}{\left(\sqrt{\frac{a+b}{a-b}}\right)^2 + z^2}\, dz \qquad \left[\text{By using} \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{2}{a-b} \cdot \frac{1}{\sqrt{\left(\frac{a+b}{a-b}\right)}} \cdot \tan^{-1}\left(\frac{z}{\sqrt{\frac{a+b}{a-b}}}\right) + c$$

$$= \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\frac{\sqrt{a-b}}{\sqrt{a+b}} \tan \frac{x}{2}\right) + c. \qquad \left[\because z = \tan \frac{x}{2}\right]$$

Case II : When $a < b$, then

$$I = \frac{2}{b-a} \int \frac{1}{\left(\sqrt{\frac{b+a}{b-a}}\right)^2 - z^2}\, dz \qquad \left[\text{By using} \int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

$$= \frac{2}{b-a} \cdot \frac{1}{2 \cdot \sqrt{\frac{b+a}{b-a}}} \log \left| \frac{\sqrt{\frac{b+a}{b-a}} + z}{\sqrt{\frac{b+a}{b-a}} - z} \right| + c$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \left| \frac{\sqrt{b+a} + \sqrt{b-a} \,.\, z}{\sqrt{b+a} - \sqrt{b-a}\, z} \right| + c$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \left| \frac{\sqrt{b+a} + \sqrt{b-a} \tan \frac{x}{2}}{\sqrt{b+a} - \sqrt{b-a} \tan \frac{x}{2}} \right| + c. \qquad \left[\because\ z = \tan \frac{x}{2}\right]$$

Case III : When $a = b$, then

$$\mathrm{I} = \int \frac{1}{a + b \cos x}\, dx = \int \frac{1}{a + a \cos x}\, dx$$

$$= \frac{1}{a} \int \frac{1}{1 + \cos x}\, dx = \frac{1}{a} \int \frac{1}{2 \cos^2 \frac{x}{2}}\, dx \qquad \left[\begin{array}{l} \because\ 1 + \cos 2\mathrm{A} = 2 \cos^2 \mathrm{A} \\ \Rightarrow\ 1 + \cos \mathrm{A} = 2 \cos^2 \frac{\mathrm{A}}{2} \end{array}\right]$$

$$= \frac{1}{2a} \int \sec^2 \frac{x}{2}\, dx = \frac{1}{2a} \left(\frac{\tan x/2}{1/2} \right) + c$$

$$= \frac{1}{a} \tan \frac{x}{2} + c.$$

(*vi*) Let $\mathrm{I} = \int \frac{1}{3 + 2 \sin x + \cos x}\, dx$

Putting $\sin x = \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)$ and $\cos x = \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)$, we have

$$\mathrm{I} = \int \frac{1}{3 + 2 \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)}\, dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{3 + 3 \tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}}\, dx = \int \frac{\sec^2 \frac{x}{2}}{2 \tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 4}\, dx$$

Put $\tan \frac{x}{2} = z$

$$\Rightarrow\ \frac{1}{2} \sec^2 \frac{x}{2}\, dx = dz \ \Rightarrow\ \sec^2 \frac{x}{2}\, dx = 2dz$$

$$\therefore\qquad \mathrm{I} = \int \frac{1}{2z^2 + 4z + 4} \,.\, (2dz) = \int \frac{1}{z^2 + 2z + 2} \,.\, dz$$

$$= \int \frac{1}{(z^2 + 2z + 1) + (2 - 1)}\, dz \qquad \left[\begin{array}{l} \text{Add and subtract 1 to the denom.} \\ \because\ \left(\frac{1}{2} \text{ co-eff. of } x \right)^2 = 1 \end{array}\right]$$

$$= \int \frac{1}{(z+1)^2 + (1)^2}\, dz$$

Put $z + 1 = y \Rightarrow dz = dy$

$\therefore \quad I = \int \frac{1}{y^2 + (1)^2}\, dy \qquad \left[\text{By using} \int \frac{1}{x^2 + a^2} . dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$

$= \tan^{-1} y + c \qquad [\because \; y = (z + 1)]$

$= \tan^{-1}(z + 1) + c$

$= \tan^{-1}\left(\tan \frac{x}{2} + 1\right) + c. \qquad \left[\because \; z = \tan \frac{x}{2}\right]$

Example 16. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{\sin x + \sqrt{3} \cos x}\, dx$ (*ii*) $\int \frac{1 + \sin x}{\sin x (1 + \cos x)}\, dx$

(*iii*) $\int \frac{1}{1 - 2 \sin x}\, dx$ (*iv*) $\int \frac{\cos \alpha \cos x + 1}{\cos \alpha + \cos x}\, dx$

(*v*) $\int \frac{1}{a + b \sin x}\, dx.$

Solution. (*i*) Let $I = \int \frac{1}{\sin x + \sqrt{3} \cos x}\, dx$

Putting $\sin x = \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)$ and $\cos x = \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)$, we have

$$I = \int \frac{1}{\left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right) + \sqrt{3}\left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)} . dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2} + \sqrt{3} - \sqrt{3} \tan^2 \frac{x}{2}} . dx$$

$$= \int \frac{\sec^2 \frac{x}{2}}{2 \tan \frac{x}{2} + \sqrt{3} - \sqrt{3} \tan^2 \frac{x}{2}}\, dx \qquad [\because \; \sec^2 A - \tan^2 A = 1]$$

$$= \frac{1}{\sqrt{3}} \int \frac{\sec^2 \frac{x}{2}}{\frac{2}{\sqrt{3}} \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}}\, dx$$

Put $\tan \frac{x}{2} = z$

$\Rightarrow \quad \frac{1}{2} \sec^2 \frac{x}{2}\, dx = dz \Rightarrow \sec^2 \frac{x}{2}\, dx = 2dz$

$$\therefore \quad I = \frac{1}{\sqrt{3}}\int \frac{1}{\frac{2}{\sqrt{3}}z + 1 - z^2}(2dz) = \frac{2}{\sqrt{3}}\int \frac{1}{1 - \left(z^2 - \frac{2}{\sqrt{3}}z\right)}\,dz$$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{\left(1+\frac{1}{3}\right) - \left(z^2 - \frac{2}{\sqrt{3}}z + \frac{1}{3}\right)}\,dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{3} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{co-eff. of } x\right)^2 = \frac{1}{3}\end{array}\right]$$

$$= \frac{2}{\sqrt{3}}\int \frac{1}{\frac{4}{3} - \left(z - \frac{1}{\sqrt{3}}\right)^2}\,dz$$

Put $z - \frac{1}{\sqrt{3}} = y \Rightarrow dz = dy$

$$\therefore \quad I = \frac{2}{\sqrt{3}}\int \frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2 - y^2}\,dy \qquad \left[\text{By using} \int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{1}{2\left(\frac{2}{\sqrt{3}}\right)}\log\left|\frac{\frac{2}{\sqrt{3}} + y}{\frac{2}{\sqrt{3}} - y}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{\frac{2}{\sqrt{3}} + z - \frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}} - z + \frac{1}{\sqrt{3}}}\right| + c \qquad \left[\because \left(z - \frac{1}{\sqrt{3}}\right) = y\right]$$

$$= \frac{1}{2}\log\left|\frac{z + \frac{1}{\sqrt{3}}}{\sqrt{3} - z}\right| + c = \frac{1}{2}\log\left|\frac{\sqrt{3}z + 1}{3 - \sqrt{3}z}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{1 + \sqrt{3}\tan\frac{x}{2}}{3 - \sqrt{3}\tan\frac{x}{2}}\right| + c. \qquad \left[\because z = \tan\frac{x}{2}\right]$$

(*ii*) Let $\quad I = \int \frac{1 + \sin x}{\sin x\,(1 + \cos x)}\,dx$

Putting $\sin x = \left(\frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right)$ and $\cos x = \left(\frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right)$, we have

$$\mathrm{I} = \int \frac{1 + \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)\left[1 + \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]}\, dx$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2} + 2\tan\frac{x}{2}\right)}{\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)\left[1+\tan^2\frac{x}{2} + 1 - \tan^2\frac{x}{2}\right]} \,.\, dx$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right)\left(1+\tan^2\frac{x}{2}\right)}{4\tan\frac{x}{2}}\, dx$$

$$= \int \frac{\left(1+\tan\frac{x}{2}\right)^2 \sec^2\frac{x}{2}}{4\tan\frac{x}{2}}\, dx \qquad [\because\ \sec^2 A - \tan^2 A = 1]$$

Put $\tan\frac{x}{2} = z$

$\Rightarrow\ \frac{1}{2}\sec^2\frac{x}{2}\, dx = dz \ \Rightarrow\ \sec^2\frac{x}{2}\, dx = 2\, dz$

$\therefore \qquad \mathrm{I} = \int \frac{(1+z)^2}{4z}\,(2dz) = \int \frac{1+z^2+2z}{2z}\, dz$

$$= \frac{1}{2}\int\left(\frac{1}{z} + z + 2\right) dz = \frac{1}{2}\left[\int \frac{1}{z}\, dz + \int z dz + \int 2dz\right]$$

$$= \frac{1}{2}\left[\log|z| + \frac{z^2}{2} + 2z\right] + c \qquad \left[\because\ \int \frac{1}{x}\, dx = \log|x| + c\right]$$

$$= \frac{1}{2}\left[\log\left|\tan\frac{x}{2}\right| + \frac{\tan^2 x/2}{2} + 2\tan\frac{x}{2}\right] + c. \qquad \left[\because\ z = \tan\frac{x}{2}\right]$$

(*iii*) Let $\mathrm{I} = \int \frac{1}{1 - 2\sin x}\, dx$

Putting $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, we have

$$I = \int \frac{1}{1 - 2\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\, dx = \int \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2} - 4\tan\frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2} - 4\tan\frac{x}{2}}\, dx \qquad [\because \sec^2 A - \tan^2 A = 1]$$

Put $\tan\frac{x}{2} = z$

$$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\, dx = dz \Rightarrow \sec^2\frac{x}{2}\, dx = 2dz$$

$$\therefore \qquad I = \int \frac{1}{1+z^2-4z}(2dz) = 2\int \frac{1}{z^2-4z+1}\, dz$$

$$= 2\int \frac{1}{(z^2-4z+4)+1-4}\, dz \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= 2\int \frac{1}{(z-2)^2-3}\, dz$$

Put $z - 2 = y \Rightarrow dz = dy$

$$\therefore \qquad I = 2\int \frac{1}{y^2-(\sqrt{3})^2}\, dy \qquad \left[\text{By using } \int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= 2\,\frac{1}{2(\sqrt{3})}\log\left|\frac{y-\sqrt{3}}{y+\sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{z-2-\sqrt{3}}{z-2+\sqrt{3}}\right| + c \qquad [\because y = z-2]$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{\tan\frac{x}{2}-2-\sqrt{3}}{\tan\frac{x}{2}-2+\sqrt{3}}\right| + c. \qquad \left[\because z = \tan\frac{x}{2}\right]$$

(*iv*) Let $I = \int \frac{\cos\alpha\cos x + 1}{\cos\alpha + \cos x}\, dx$

$$= \int \frac{\cos\alpha\cos x + \cos^2\alpha + 1 - \cos^2\alpha}{\cos\alpha + \cos x}\, dx$$

[Add and subtract $\cos^2\alpha$ to the numerator.]

$$= \int \frac{\cos\alpha(\cos x + \cos\alpha) + \sin^2\alpha}{(\cos\alpha + \cos x)} \cdot dx \qquad [\because \cos^2 A + \sin^2 A = 1]$$

$$= \int \left[\frac{\cos\alpha(\cos\alpha+\cos x)}{(\cos\alpha+\cos x)} + \frac{\sin^2\alpha}{(\cos\alpha+\cos x)}\right].\,dx$$

$$= \cos\alpha \int 1.\,dx + \sin^2\alpha \int \frac{1}{\cos\alpha+\cos x}\,dx$$

$$\Rightarrow \quad I = \cos\alpha\,.\,x + \sin^2\alpha\,.\,I_1 \qquad \ldots(1)$$

Now $\quad I_1 = \int \frac{1}{\cos\alpha+\cos x}\,dx$

Putting $\cos x = \left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)$, we have

$$I_1 = \int \frac{1}{\cos\alpha + \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\,dx = \int \frac{1+\tan^2\frac{x}{2}}{\cos\alpha+\cos\alpha\,.\tan^2\frac{x}{2}+1-\tan^2\frac{x}{2}}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{\cos\alpha+\cos\alpha\tan^2\frac{x}{2}+1-\tan^2\frac{x}{2}}\,dx \qquad [\because \sec^2 A - \tan^2 A = 1]$$

Put $\quad \tan\frac{x}{2} = z$

$$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz \Rightarrow \sec^2\frac{x}{2} = 2dz.$$

$$\therefore \quad I_1 = \int \frac{1}{\cos\alpha+\cos\alpha\,z^2+1-z^2}\,(2dz)$$

$$= 2\int \frac{1}{z^2\cos\alpha - z^2 + 1 + \cos\alpha}\,dz = 2\int \frac{1}{(1+\cos\alpha)-(1-\cos\alpha)\,z^2}\,dz$$

$$= \frac{2}{1-\cos\alpha}\int \frac{1}{\dfrac{1+\cos\alpha}{1-\cos\alpha} - z^2}\,dz$$

$$= \frac{2}{2\sin^2\frac{\alpha}{2}}\int \frac{1}{\dfrac{2\cos^2\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}} - z^2}\,dz \qquad \left[\begin{array}{l}\because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ 1+\cos A = 2\cos^2\frac{A}{2} \\ \quad 1-\cos 2A = 2\sin^2 A \\ \Rightarrow\ 1-\cos A = 2\sin^2\frac{A}{2}\end{array}\right]$$

$$= \frac{1}{\sin^2\frac{\alpha}{2}}\int \frac{1}{\cot^2\frac{\alpha}{2} - z^2}\,dz \qquad \left[\text{By using} \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

$$= \frac{1}{\sin^2\frac{\alpha}{2}}\cdot\frac{1}{2\cot\frac{\alpha}{2}}\cdot\log\left|\frac{\cot\frac{\alpha}{2}+z}{\cot\frac{\alpha}{2}-z}\right| + c$$

$$= \frac{1}{2\left(\sin^2\frac{\alpha}{2}\right)\left(\dfrac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\right)} \cdot \log\left|\frac{\cot\frac{\alpha}{2}+\tan\frac{x}{2}}{\cot\frac{\alpha}{2}-\tan\frac{x}{2}}\right| + c \qquad \left[\because\ z = \tan\frac{x}{2}\right]$$

$$= \frac{1}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} \cdot \log\left|\frac{\cot\frac{\alpha}{2}+\tan\frac{x}{2}}{\cot\frac{\alpha}{2}-\tan\frac{x}{2}}\right| + c$$

$$= \frac{1}{\sin\alpha} \cdot \log\left|\frac{\cot\frac{\alpha}{2}+\tan\frac{x}{2}}{\cot\frac{\alpha}{2}-\tan\frac{x}{2}}\right| + c. \qquad \left[\begin{array}{l}\because\ 2\sin A\cos A = \sin 2A \\ \Rightarrow 2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A\end{array}\right]$$

(*v*) Let $\quad I = \int \frac{1}{a + b\sin x}\,dx$

Putting $\quad \sin x = \left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)$, we have

$$I = \int \frac{1}{a + b\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\,dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{a + a\tan^2\frac{x}{2} + 2b\tan\frac{x}{2}}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{a + a\tan^2\frac{x}{2} + 2b\tan\frac{x}{2}}\,dx \qquad [\because\ \sec^2 A - \tan^2 A = 1]$$

Put $\quad \tan\frac{x}{2} = z$

$$\Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz \quad \Rightarrow \quad \sec^2\frac{x}{2}\,dx = 2dz$$

$$\therefore \qquad I = \int \frac{1}{a + az^2 + 2bz} \cdot 2dz = \frac{2}{a}\int \frac{1}{\left(z^2 + \frac{2bz}{a} + 1\right)}\,dz$$

$$= \frac{2}{a}\int \frac{1}{\left(z^2 + \frac{2b}{a}z + \frac{b^2}{a^2}\right) + 1 - \frac{b^2}{a^2}}\,dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{b^2}{a^2} \text{ to the denom.} \\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{b^2}{a^2}\end{array}\right]$$

$$= \frac{2}{a}\int \frac{1}{\left(z + \frac{b}{a}\right)^2 + \frac{a^2 - b^2}{a^2}}\,dz$$

Put $z + \frac{b}{a} = y \;\Rightarrow\; dz = dy$

$$\therefore \qquad I = \frac{2}{a}\int \frac{1}{y^2 + \frac{a^2 - b^2}{a^2}}\,dy$$

Now the following possibilities are there :

Case I : When $a > b$, then $\left[\because \frac{a^2 - b^2}{a^2} > 0\right]$

$$I = \frac{2}{a}\int \frac{1}{y^2 + \frac{a^2 - b^2}{a^2}}\,dz$$

$$= \frac{2}{a}\int \frac{1}{y^2 + \left(\sqrt{\frac{a^2 - b^2}{a^2}}\right)^2}\,dz \qquad \left[\text{By using } \int \frac{1}{x^2 + a^2}\,.\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{2}{a}\cdot\frac{1}{\frac{\sqrt{a^2 - b^2}}{a}}\,.\,\tan^{-1}\left(\frac{y}{\frac{\sqrt{a^2 - b^2}}{a}}\right) + c$$

$$= \frac{2}{\sqrt{a^2 - b^2}}\,.\,\tan^{-1}\left(\frac{z + \frac{b}{a}}{\frac{\sqrt{a^2 - b^2}}{a}}\right) + c \qquad \left[\because y = \left(z + \frac{b}{a}\right)\right]$$

$$= \frac{2}{\sqrt{a^2 - b^2}}\,.\,\tan^{-1}\left(\frac{az + b}{\sqrt{a^2 - b^2}}\right) + c$$

$$= \frac{2}{\sqrt{a^2 - b^2}}\,.\,\tan^{-1}\left(\frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}}\right) + c. \qquad \left[\because z = \tan\frac{x}{2}\right]$$

Case II : When $a < b$, then

$$I = \frac{2}{a}\int \frac{1}{y^2 + \frac{a^2 - b^2}{a^2}}\, dz = \frac{2}{a}\int \frac{1}{y^2 - \left(\frac{b^2 - a^2}{a^2}\right)}\, dz$$

$$= \frac{2}{a}\int \frac{1}{y^2 - \left(\frac{\sqrt{b^2 - a^2}}{a}\right)^2}\, dz \quad \left[\text{By using} \int \frac{1}{x^2 - a^2}\cdot dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{2}{a}\cdot\left(\frac{1}{2\,\frac{\sqrt{b^2 - a^2}}{a}}\right)\cdot \log\left|\frac{y - \frac{\sqrt{b^2 - a^2}}{a}}{y + \frac{\sqrt{b^2 - a^2}}{a}}\right| + c$$

$$= \frac{1}{\sqrt{b^2 - a^2}}\log\left|\frac{z + \frac{b}{a} - \frac{\sqrt{b^2 - a^2}}{a}}{z + \frac{b}{a} + \frac{\sqrt{b^2 - a^2}}{a}}\right| + c \qquad \left[\because\ y = \left(z + \frac{b}{a}\right)\right]$$

$$= \frac{1}{\sqrt{b^2 - a^2}}\log\left|\frac{az + b - \sqrt{b^2 - a^2}}{az + b + \sqrt{b^2 - a^2}}\right| + c$$

$$= \frac{1}{\sqrt{b^2 - a^2}}\log\left|\frac{a\tan\frac{x}{2} + b - \sqrt{b^2 - a^2}}{a\tan\frac{x}{2} + b + \sqrt{b^2 - a^2}}\right| + c. \qquad \left[\because\ z = \tan\frac{x}{2}\right]$$

Case III : When $a = b$, then

$$I = \int \frac{1}{a + b\sin x}\, dx$$

$$= \int \frac{1}{a + a\sin x}\, dx = \frac{1}{a}\int \frac{1}{(1 + \sin x)}\, dx$$

$$= \frac{1}{a}\int \frac{1}{(1 + \sin x)}\cdot\frac{1 - \sin x}{1 - \sin x}\cdot dx \qquad \text{[On rationalization]}$$

$$= \frac{1}{a}\int \frac{1 - \sin x}{1 - \sin^2 x}\, dx \qquad [\because\ (a + b)(a - b) = a^2 - b^2]$$

$$= \frac{1}{a}\int \frac{1 - \sin x}{\cos^2 x}\, dx \qquad [\because\ \sin^2 A + \cos^2 A = 1]$$

$$= \frac{1}{a}\int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx = \frac{1}{a}\int (\sec^2 x - \sec x \tan x)\, dx$$

$$= \frac{1}{a}(\tan x - \sec x) + c.$$

Example 17. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{2+\cos x-\sin x}\,dx$ (*ii*) $\int \frac{1}{5+4\cos x}\,dx$.

Solution. (*i*) Let $I = \int \frac{1}{2+\cos x-\sin x}\,dx$

Putting $\sin x = \left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)$ and $\cos x = \left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)$, we have

$$I = \int \frac{1}{2+\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)-\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\,dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{2+2\tan^2\frac{x}{2}+1-\tan^2\frac{x}{2}-2\tan\frac{x}{2}}\,dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{\left(\tan^2\frac{x}{2}-2\tan\frac{x}{2}+3\right)}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{\left(\tan^2\frac{x}{2}-2\tan\frac{x}{2}+3\right)}\,dx \qquad [\because\ \sec^2 A-\tan^2 A = 1]$$

Put $\tan\frac{x}{2} = z$

$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz \Rightarrow \sec^2\frac{x}{2}\,dx = 2dz$

$\therefore$ $$I = \int \frac{1}{(z^2-2z+3)}\cdot(2dz) = 2\int \frac{1}{(z^2-2z+3)}\,dz$$

$$= 2\int \frac{1}{(z^2-2z+1)+(3-1)}\,dz \qquad \left[\begin{array}{l}\text{Add and subtract 1 to the denom.}\\ \because \left(\frac{1}{2}\text{ coeff. of } x\right)^2 = 1\end{array}\right]$$

$$= 2\int \frac{1}{(z-1)^2+(\sqrt{2})^2}\,dz$$

Put $z-1 = y \Rightarrow dz = dy$

$$= 2\int \frac{1}{y^2 + (\sqrt{2})^2}\, dy \qquad \left[\text{By using} \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= 2 \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + c$$

$$= \sqrt{2}\tan^{-1}\left(\frac{z-1}{\sqrt{2}}\right) + c \qquad [\because\ y = (z-1)]$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2} - 1}{\sqrt{2}}\right) + c. \qquad \left[\because\ z = \tan\frac{x}{2}\right]$$

(ii) Let $\quad \mathrm{I} = \int \frac{1}{5 + 4\cos x}\, dx$

Putting $\quad \cos x = \left(\frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right)$, we have

$$\mathrm{I} = \int \frac{1}{5 + 4\left(\frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right)}\, dx = \int \frac{\left(1 + \tan^2\frac{x}{2}\right)}{5 + 5\tan^2\frac{x}{2} + 4 - 4\tan^2\frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} + 9}\, dx \qquad [\because\ \sec^2 \mathrm{A} - \tan^2 \mathrm{A} = 1]$$

Put $\quad \tan\frac{x}{2} = z$

$$\Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}\, dx = dz \quad \Rightarrow \quad \sec^2\frac{x}{2}\, dx = 2dz$$

$$\therefore \quad \mathrm{I} = \int \frac{1}{z^2 + 9}(2dz) = 2\int \frac{1}{z^2 + (3)^2}\, dz$$

$$= 2\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{z}{3}\right) + c \qquad \left[\text{By using} \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{3}\right) + c. \qquad \left[\because\ z = \tan\frac{x}{2}\right]$$

Example 18. *Evaluate the following integrals :*

(i) $\int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x}\, dx$ *(ii)* $\int \frac{4 \sin x + 5 \cos x}{5 \sin x + 4 \cos x}\, dx.$

Solution. *(i)* Let $I = \int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x}\, dx$

Let λ and μ be constants such that

$$2 \sin x + 3 \cos x = \lambda \frac{d}{dx} (3 \sin x + 4 \cos x) + \mu (3 \sin x + 4 \cos x)$$

$$\Rightarrow \quad 2 \sin x + 3 \cos x = \lambda (3 \cos x - 4 \sin x) + \mu(3 \sin x + 4 \cos x) \qquad ...(A)$$

$$\Rightarrow \quad 2 \sin x + 3 \cos x = (3\mu - 4\lambda) \sin x + (4\mu + 3\lambda) \cos x$$

Comparing the co-efficients of sin x and cos x on both sides, we have

$$3\mu - 4\lambda = 2 \qquad ...(1)$$

and

$$4\mu + 3\lambda = 3 \qquad ...(2)$$

Multiplying (1) by 3 and (2) by 4, we get

$$9\mu - 12\lambda = 6 \qquad ...(3)$$

$$16\mu + 12\lambda = 12 \qquad ...(4)$$

Adding (3) and (4) ; we get

$$25\mu = 18 \quad \Rightarrow \quad \mu = \frac{18}{25}$$

Using this value in (1) ; we get

$$3\left(\frac{18}{25}\right) - 4\lambda = 2 \quad \Rightarrow \quad \frac{54}{25} - 2 = 4\lambda$$

$$\Rightarrow \quad 4\lambda = \frac{4}{25} \quad \Rightarrow \quad \lambda = \frac{1}{25}$$

$\therefore$ Equation (A) becomes :

$$2 \sin x + 3 \cos x = \frac{1}{25} (3 \cos x - 4 \sin x) + \frac{18}{25} (3 \sin x + 4 \cos x)$$

$$\therefore \quad I = \int \frac{\frac{18}{25}(3 \sin x + 4 \cos x) + \frac{1}{25}(3 \cos x - 4 \sin x)}{(3 \sin x + 4 \cos x)}\, dx$$

$$= \frac{18}{25} \int 1 . dx + \frac{1}{25} \int \frac{3 \cos x - 4 \sin x}{3 \sin x + 4 \cos x}\, dx$$

$$= \frac{18}{25} x + \frac{1}{25} \log |3 \sin x + 4 \cos x| + c. \qquad \left[\because \int \frac{f'(x)}{f(x)} . dx = \log |f(x)| + c\right]$$

(ii) Let $I = \int \frac{4 \sin x + 5 \cos x}{5 \sin x + 4 \cos x}\, dx$

Let λ and μ be constants such that

$$(4 \sin x + 5 \cos x) = \lambda \frac{d}{dx} (5 \sin x + 4 \cos x) + \mu (5 \sin x + 4 \cos x)$$

$$\Rightarrow \quad (4 \sin x + 5 \cos x) = \lambda (5 \cos x - 4 \sin x) + \mu (5 \sin x + 4 \cos x) \qquad ...(A)$$

$$\Rightarrow \quad (4 \sin x + 5 \cos x) = (5\mu - 4\lambda) \sin x + (4\mu + 5\lambda) \cos x$$

Comparing the co-efficients of $\sin x$ and $\cos x$ on both sides, we get

$$5\mu - 4\lambda = 4 \qquad ...(1)$$

and

$$4\mu + 5\lambda = 5 \qquad ...(2)$$

Multiplying (2) by 4 and (1) by 5, we get

$$25\mu - 20\lambda = 20 \qquad ...(3)$$

$$16\mu + 20\lambda = 20 \qquad ...(4)$$

Adding (3) and (4), we get

$$41\mu = 40 \quad \Rightarrow \quad \mu = \frac{40}{41}$$

Using this value in (3), we get

$$25\left(\frac{40}{41}\right) - 20\lambda = 20 \quad \Rightarrow \quad \frac{1000}{41} - 20 = 20\lambda$$

$$\Rightarrow \quad \frac{180}{41} = 20\lambda \quad \Rightarrow \quad \lambda = \frac{9}{41}$$

$\therefore$ Equation (A) becomes :

$$(4\sin x + 5\cos x) = \frac{9}{41}(5\cos x - 4\sin x) + \frac{40}{41}(5\sin x + 4\cos x)$$

$$\therefore \quad I = \int \frac{\frac{40}{41}(5\sin x + 4\cos x) + \frac{9}{41}(5\cos x - 4\sin x)}{(5\sin x + 4\cos x)}\,dx$$

$$= \frac{40}{41}\int 1 . dx + \frac{9}{41}\int \frac{5\cos x - 4\sin x}{5\sin x + 4\cos x}\,dx \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

$$= \frac{40}{41}x + \frac{9}{41}\log|5\sin x + 4\cos x| + c.$$

Example 19. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{x + \sqrt{a^2 - x^2}}\,dx$ (*ii*) $\int \frac{1}{a + b\cot x}\,dx.$

Solution. (*i*) Let $I = \int \frac{1}{x + \sqrt{a^2 - x^2}}\,dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

$$\therefore \quad I = \int \frac{a\cos\theta\,d\theta}{a\sin\theta + \sqrt{a^2 - a^2\sin^2\theta}}\,d\theta$$

$$= \int \frac{a\cos\theta}{a\sin\theta + a\sqrt{1 - \sin^2\theta}}\,d\theta$$

$$= \int \frac{\cos\theta}{\sin\theta + \sqrt{\cos^2\theta}}\,d\theta = \int \frac{\cos\theta}{\sin\theta + \cos\theta}\,d\theta \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \frac{1}{2}\int \frac{2\cos\theta}{\sin\theta + \cos\theta}\,d\theta \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int \frac{\cos\theta + \cos\theta}{\sin\theta + \cos\theta}\, d\theta$$

$$= \frac{1}{2}\int \frac{(\sin\theta + \cos\theta) + (\cos\theta - \sin\theta)}{(\sin\theta + \cos\theta)}\, d\theta$$

[Add and subtract $\sin\theta$ to the numerator]

$$= \frac{1}{2}\int 1 . d\theta + \frac{1}{2}\int \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}\, d\theta \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right]$$

$$= \frac{1}{2}\theta + \frac{1}{2}\log|\sin\theta + \cos\theta| + c \qquad \left[\begin{array}{l}\because\ x = a\sin\theta \Rightarrow \dfrac{x}{a} = \sin\theta \\ \Rightarrow \theta = \sin^{-1}\dfrac{x}{a}\end{array}\right]$$

$$= \frac{1}{2}\sin^{-1}\frac{x}{a} + \frac{1}{2}\log\left|\frac{x}{a} + \sqrt{1 - \frac{x^2}{a^2}}\right| + c \qquad \left[\begin{array}{l}\because\ \cos\theta = \sqrt{1 - \sin^2\theta} \\ \qquad = \sqrt{1 - \dfrac{x^2}{a^2}}\end{array}\right]$$

$$= \frac{1}{2}\sin^{-1}\frac{x}{a} + \frac{1}{2}\log\left|\frac{x}{a} + \frac{\sqrt{a^2 - x^2}}{a}\right| + c.$$

(*ii*) Let $\quad I = \int \frac{1}{a + b\cot x}\, dx = \int \frac{1}{a + b\dfrac{\cos x}{\sin x}}\, dx$

$$= \int \frac{\sin x}{a\sin x + b\cos x}\, dx$$

Let λ and μ be constants such that

$$\sin x = \lambda \frac{d}{dx}(a\sin x + b\cos x) + \mu(a\sin x + b\cos x)$$

$\Rightarrow \quad \sin x = \lambda(a\cos x - b\sin x) + \mu(a\sin x + b\cos x)$ ...(A)

$\Rightarrow \quad \sin x = (a\mu - b\lambda)\sin x + (b\mu + a\lambda)\cos x$

Comparing the co-efficients of $\sin x$ and $\cos x$ on both sides, we get

$a\mu - b\lambda = 1$ ...(1)

and $\quad b\mu - a\lambda = 0$ ...(2)

Solving (1) and (2), we have

$$\frac{\mu}{a} = \frac{\lambda}{-b} = \frac{1}{a^2 + b^2}$$

$$\Rightarrow \quad \mu = \frac{a}{a^2 + b^2} \quad \text{and} \quad \lambda = \frac{-b}{a^2 + b^2}$$

Using these values in (A), we get

$$\sin x = \frac{-b}{a^2 + b^2}(a\cos x - b\sin x) + \frac{a}{a^2 + b^2}(a\sin x + b\cos x)$$

$$\therefore \quad I = \int \frac{\frac{-b}{a^2+b^2}(a\cos x - b\sin x) + \frac{a}{a^2+b^2}(a\sin x + b\cos x)}{(a\sin x + b\cos x)}\,dx$$

$$= \frac{-b}{a^2+b^2}\int \frac{a\cos x - b\sin x}{a\sin x + b\cos x}\,dx + \frac{a}{a^2+b^2}\int 1\,.\,dx \quad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|+c\right]$$

$$= \frac{-b}{a^2+b^2}\log|a\sin x + b\cos x| + \frac{a}{a^2+b^2}\,x + c.$$

Example 20. *Evaluate the following integrals :*

(i) $\int \frac{\sin x}{\sin x - \cos x}\,dx$ *(ii)* $\int \frac{\cos x}{\cos x + \sin x}\,dx.$

Solution. *(i)* Let $I = \int \frac{\sin x}{\sin x - \cos x}\,dx$

$$= \frac{1}{2}\int \frac{2\sin x}{\sin x - \cos x}\,dx$$ [Multiply and divided by 2]

$$= \frac{1}{2}\int \frac{\sin x + \sin x}{\sin x - \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\sin x - \cos x)}{(\sin x - \cos x)}\,dx$$

[Add and subtract cos x to the numerator]

$$= \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x - \cos x}\,dx + \frac{1}{2}\int 1\,.\,dx \quad \left[\because \int \frac{f'(x)}{f(x)}\,.\,dx = \log|f(x)|+c\right]$$

$$= \frac{1}{2}\log|\sin x - \cos x| + \frac{1}{2}x + c.$$

(ii) Let $I = \int \frac{\cos x}{\cos x + \sin x}\,dx$

$$= \frac{1}{2}\int \frac{2\cos x}{\cos x + \sin x}\,dx$$ [Multiply and divided by 2]

$$= \frac{1}{2}\int \frac{\cos x + \cos x}{\cos x + \sin x}\,dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (-\sin x + \cos x)}{\cos x + \sin x}\,dx$$

[Add and subtract sin x to the numerator]

$$= \frac{1}{2}\int 1\,.\,dx + \frac{1}{2}\int \frac{(-\sin x + \cos x)}{\cos x + \sin x}\,dx \quad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|+c\right]$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\cos x + \sin x| + c.$$

Example 21. *Evaluate the following integrals :*

(i) $\int \frac{\cos x+2\sin x+3}{4\cos x+5\sin x+6}\cdot dx$ (ii) $\int \frac{5\cos x+6}{2\cos x+\sin x+3}\,dx$

(iii) $\int \frac{3\cos x+2}{\sin x+2\cos x+3}\,dx$ (iv) $\int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}\,dx.$

Solution. (i) Let $I=\int \frac{\cos x+2\sin x+3}{4\cos x+5\sin x+6}\,dx$

Let p, q, r be the constants, such that

$$\cos x+2\sin x+3=p(4\cos x+5\sin x+6)+q\frac{d}{dx}(4\cos x+5\sin x+6)+r$$

$$\Rightarrow \quad \cos x+2\sin x+3=p(4\cos x+5\sin x+6)+q(-4\sin x+5\cos x)+r \quad \text{...(1)}$$

$$\Rightarrow \quad \cos x+2\sin x+3=(4p+5q)\cos x+(5p-4q)\sin x+6p+r$$

Comparing the coefficients of $\cos x$, $\sin x$ and the constant terms, we have

$$4p+5q=1 \quad \text{...(2)}$$

$$5p-4q=2 \quad \text{...(3)}$$

$$6p+r=3 \quad \text{...(4)}$$

Multiplying (2) by 4 and (3) by 5 and adding, we have

$$41p=14 \Rightarrow p=\frac{14}{41}$$

Using this value of p in (2), we get

$$4\left(\frac{14}{41}\right)+5q=1 \Rightarrow 5q=1-\frac{56}{41}$$

$$\Rightarrow \quad q=-\frac{15}{5\times 41} \Rightarrow q=-\frac{3}{41}$$

Now using the value of p in (4) ; we get

$$6\left(\frac{14}{41}\right)+r=3 \Rightarrow r=3-\frac{84}{41} \Rightarrow r=\frac{39}{41}$$

Substituting values of p, q and r in (1), we get

$$\cos x+2\sin x+3=\frac{14}{41}(4\cos x+5\sin x+6)+\left(\frac{-3}{41}\right)(-4\sin x+5\cos x)+\frac{39}{41}$$

$$\therefore \quad I=\frac{14}{41}\int \frac{(4\cos x+5\sin x+6)}{(4\cos x+5\sin x+6)}dx+\left(\frac{-3}{41}\right)\int \frac{(-4\sin x+5\cos x)}{4\cos x+5\sin x+6}dx$$

$$+\frac{39}{41}\int \frac{1}{4\cos x+5\sin x+6}dx$$

$$\left[\because \int \frac{f'(x)}{f(x)}dx=\log|f(x)|+c\right]$$

$$\Rightarrow \quad I=\frac{14}{41}x-\frac{3}{41}\log|4\cos x+5\sin x+6|+\frac{39}{41}\int \frac{1}{4\cos x+5\sin x+6}dx \quad \text{...(5)}$$

Let $I_1=\int \frac{1}{4\cos x+5\sin x+6}dx$

Putting $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, we have

$$I_1 = \int \frac{1}{4\left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+5\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+6}\,dx$$

$$= \int \frac{1+\tan^2\frac{x}{2}}{4-4\tan^2\frac{x}{2}+10\tan\frac{x}{2}+6+6\tan^2\frac{x}{2}}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{2\tan^2\frac{x}{2}+10\tan\frac{x}{2}+10}\,dx \qquad [\because\ \sec^2 A - \tan^2 A = 1]$$

Put $\tan\dfrac{x}{2} = z$

$\Rightarrow\ \dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx = dz \ \Rightarrow\ \sec^2\dfrac{x}{2}\,dx = 2dz$

$$\therefore \qquad I_1 = \int \frac{1}{2z^2+10z+10}(2dz) = \int \frac{1}{z^2+5z+5}\,dz$$

$$= \int \frac{1}{\left(z^2+5z+\frac{25}{4}\right)+\left(5-\frac{25}{4}\right)}\,dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{25}{4} \text{ to the denom.}\\ \because\ \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{25}{4}\end{array}\right]$$

$$= \int \frac{1}{\left(z+\frac{5}{2}\right)^2-\left(\frac{5}{4}\right)}\,dz$$

Put $\left(z+\dfrac{5}{2}\right) = y \ \Rightarrow\ dz = dy$

$$\therefore \qquad I_1 = \int \frac{1}{y^2-\left(\frac{\sqrt{5}}{2}\right)^2}\,dy \qquad \left[\text{By using } \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c\right]$$

$$= \frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\log\left|\frac{y-\frac{\sqrt{5}}{2}}{y+\frac{\sqrt{5}}{2}}\right|+c = \frac{1}{\sqrt{5}}\log\left|\frac{z+\frac{5}{2}-\frac{\sqrt{5}}{2}}{z+\frac{5}{2}+\frac{\sqrt{5}}{2}}\right|+c \qquad \left[\because\ y = \left(z+\frac{5}{2}\right)\right]$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{\tan\frac{x}{2} + \frac{5-\sqrt{5}}{2}}{\tan\frac{x}{2} + \frac{5+\sqrt{5}}{2}} \right| + c \qquad \left[\because\ z = \tan\frac{x}{2}\right]$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{2\tan\frac{x}{2} + (5-\sqrt{5})}{2\tan\frac{x}{2} + (5+\sqrt{5})} \right| + c \qquad ...(6)$$

By using (5) and (6), we get

$$I = \frac{14}{41}x - \frac{3}{41}\log|4\cos x + 5\sin x + 6| + \frac{39}{41\sqrt{5}} \log \left| \frac{2\tan\frac{x}{2} + (5-\sqrt{5})}{2\tan\frac{x}{2} + (5+\sqrt{5})} \right| + c.$$

(*ii*) Let $\quad I = \int \frac{5\cos x + 6}{2\cos x + \sin x + 3}\, dx$

Let p, q, r be the constants, such that

$$5\cos x + 6 = p(2\cos x + \sin x + 3) + q \cdot \frac{d}{dx}(2\cos x + \sin x + 3) + r$$

$\Rightarrow \quad 5\cos x + 6 = p(2\cos x + \sin x + 3) + q(-2\sin x + \cos x) + r \qquad ...(1)$

$\Rightarrow \quad 5\cos x + 6 = (2p + q)\cos x + (p - 2q)\sin x + 3p + r$

Comparing the co-efficients of $\cos x$, $\sin x$ and the constant terms, we have

$2p + q = 5 \qquad ...(2)$

$p - 2q = 0 \qquad ...(3)$

$3p + r = 6 \qquad ...(4)$

From (3) : $\quad p - 2q = 0 \Rightarrow p = 2q$

From (2) : $\quad 2p + q = 5 \Rightarrow 2(2q) + q = 5$

$\Rightarrow \quad 4q + q = 5 \Rightarrow q = 1$ and $\therefore\ p = 2$

From (4) : $\quad 3p + r = 6$

$\Rightarrow \quad 3(2) + r = 6 \Rightarrow r = 6 - 6 \Rightarrow r = 0$

Substituting values of p, q and r in (1), we get

$5\cos x + 6 = 2(2\cos x + \sin x + 3) + 1(-2\sin x + \cos x) + 0$

$$\therefore \quad I = 2\int \frac{(2\cos x + \sin x + 3)}{(2\cos x + \sin x + 3)}\, dx + \int \frac{(-2\sin x + \cos x)}{(2\cos x + \sin x + 3)}\, dx$$

$$\left[\because\ \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right]$$

$\Rightarrow \quad I = 2x + \log|2\cos x + \sin x + 3| + c.$

(*iii*) Let $\quad I = \int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\, dx$

Let p, q, r be the constants, such that

$$3\cos x + 2 = p(\sin x + 2\cos x + 3) + q \cdot \frac{d}{dx}(\sin x + 2\cos x + 3) + r$$

$\Rightarrow \quad 3\cos x + 2 = p(\sin x + 2\cos x + 3) + q(\cos x - 2\sin x) + r \quad ...(1)$

$\Rightarrow \quad 3\cos x + 2 = (2p + q)\cos x + (p - 2q)\sin x + 3p + r.$

Comparing the coefficients of cos x, sin x and the constant terms :

$$2p + q = 3 \quad ...(2)$$

$$p - 2q = 0 \quad ...(3)$$

$$3p + r = 2 \quad ...(4)$$

On solving (2), (3) and (4), we get

$$p = \frac{6}{5},\ q = \frac{3}{5},\ r = -\frac{8}{5}$$

Substituting values of p, q and r in (1), we get

$$3\cos x + 2 = \frac{6}{5}(\sin x + 2\cos x + 3) + \frac{3}{5}(\cos x - 2\sin x) - \frac{8}{5}$$

$$\therefore \quad I = \frac{6}{5}\int \frac{(\sin x + 2\cos x + 3)}{(\sin x + 2\cos x + 3)}\,dx + \frac{3}{5}\int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}\,dx - \frac{8}{5}\int \frac{1}{(\sin x + 2\cos x + 3)}\,dx$$

$$= \frac{6}{5}x + \frac{3}{5}\log|\sin x + 2\cos x + 3| - \frac{8}{5}\int \frac{1}{(\sin x + 2\cos x + 3)}\,dx \quad ...(5)$$

Let $\quad I_1 = \int \frac{1}{\sin x + 2\cos x + 3}\,dx$

Putting $\quad \sin x = \dfrac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$ and $\cos x = \dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$, we have

$$I_1 = \int \frac{1}{\left(\dfrac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right) + 2\left(\dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right) + 3}\,dx$$

$$= \int \frac{1 + \tan^2\frac{x}{2}}{2\tan\frac{x}{2} + 2 - 2\tan^2\frac{x}{2} + 3\left(1 + \tan^2\frac{x}{2}\right)}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 5}\,dx \qquad [\because \sec^2 A - \tan^2 A = 1]$$

Put $\quad \tan\frac{x}{2} = z$

$\Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz \quad \Rightarrow \quad \sec^2\frac{x}{2}\,dx = 2dz$

$\therefore \quad I_1 = \int \frac{1}{z^2+2z+5}(2dz) = 2\int \frac{1}{z^2+2z+5}\,dz$

$= 2\int \frac{1}{(z^2+2z+1)+(5-1)}\,dz \qquad \left[\begin{array}{l}\text{Add and subtract 1 to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 1\end{array}\right]$

$= 2\int \frac{1}{(z+1)^2+(2)^2}\,dz$

Put $\quad (z+1) = y \;\Rightarrow\; dz = dy$

$\therefore \quad I_1 = 2\int \frac{1}{y^2+(2)^2}\,dy \qquad \left[\text{By using} \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$

$= 2\,.\,\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)+c$

$= \tan^{-1}\left(\frac{z+1}{2}\right)+c \qquad [\because\; y = (z+1)]$

$= \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+c \qquad ...(6) \quad \left[\because\; z = \tan\frac{x}{2}\right]$

By using (5) and (6), we get

$$I = \frac{6}{5}x + \frac{3}{5}\log|\sin x + 2\cos x + 3| + \left(\frac{-8}{5}\right)\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+c$$

$$= \frac{6}{5}x + \frac{3}{5}\log|\sin x + 2\cos x + 3| - \frac{8}{5}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+c.$$

(*iv*) Let $\quad I = \int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}\,dx$

Let p, q, r be the constants, such that

$(3+2\cos x+4\sin x) = p(2\sin x+\cos x+3) + q\frac{d}{dx}(2\sin x+\cos x+3)+r$

$\Rightarrow \quad (3+2\cos x+4\sin x) = p(2\sin x+\cos x+3) + q(2\cos x-\sin x)+r \qquad ...(1)$

$\Rightarrow \quad (3+2\cos x+4\sin x) = (2p-q)\sin x + (p+2q)\cos x + 3p + r$

Comparing the co-efficients of $\cos x$, $\sin x$ and the constant terms, we have

$2p - q = 4 \qquad ...(2)$

$p + 2q = 2 \qquad ...(3)$

$3p + r = 3 \qquad ...(4)$

Multiplying (2) by 2 and (3) by 1 and adding, we get

$$5p = 10 \Rightarrow p = 2$$

From (2) : $2(2) - q = 4 \Rightarrow 4 - q = 4 \Rightarrow q = 0$

From (4) : $3(2) + r = 3 \Rightarrow r = -3$

Substituting the values of p, q and r in (1), we get

$$(3 + 2\cos x + 4\sin x) = 2(2\sin x + \cos x + 3) + 0(2\cos x - \sin x) - 3$$

$$\therefore \quad I = 2\int \frac{(2\sin x + \cos x + 3)}{(2\sin x + \cos x + 3)}\,dx + \int \frac{0}{(2\sin x + \cos x + 3)}\,dx - 3\int \frac{1}{(2\sin x + \cos x + 3)}\,dx$$

$$\Rightarrow \quad I = 2x - 3\int \frac{1}{2\sin x + \cos x + 3}\,dx \qquad \ldots(5)$$

Let $\quad I_1 = \int \frac{1}{2\sin x + \cos x + 3}\,dx$

Putting $\sin x = \dfrac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$ and $\cos x = \dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$, we have

$$\therefore \quad I_1 = \int \frac{1}{2\left(\dfrac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right) + \left(\dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right) + 3}\,dx$$

$$= \int \frac{1 + \tan^2\frac{x}{2}}{4\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2} + 3 + 3\tan^2\frac{x}{2}}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{2\tan^2\frac{x}{2} + 4\tan\frac{x}{2} + 4}\,.\,dx \qquad [\because \sec^2 A - \tan^2 A = 1]$$

Put $\quad \tan\frac{x}{2} = z$

$$\Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz \Rightarrow \sec^2\frac{x}{2}\,dx = 2dz$$

$$\therefore \quad I_1 = \int \frac{1}{2z^2 + 4z + 4}\,(2dz)$$

$$= \int \frac{1}{z^2 + 2z + 2}\,dz$$

$$= \int \frac{1}{(z^2 + 2z + 1) + (2 - 1)}\,dz \qquad \left[\begin{array}{l}\text{Add and subtract 1 to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 1\end{array}\right]$$

$\Rightarrow \quad I_1 = \int \frac{1}{(z+1)^2 + 1^2}\, dz$

Put $\quad z + 1 = y \Rightarrow dz = dy$

$\therefore \quad I_1 = \int \frac{1}{y^2 + (1)^2}\, dy \qquad \left[\text{By using } \int \frac{1}{x^2+a^2}\,.\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$

$= \tan^{-1}\left(\frac{y}{1}\right) + c$

$= \tan^{-1}(z+1) + c \qquad [\because\ y = (z+1)]$

$\Rightarrow \quad I_1 = \tan^{-1}\left(\tan\frac{x}{2} + 1\right) + c \qquad \ldots(6) \quad \left[\because\ z = \tan\frac{x}{2}\right]$

By using (5) and (6), we get

$$I = 2x - 3\tan^{-1}\left(\tan\frac{x}{2} + 1\right) + c.$$

EXERCISE FOR PRACTICE

Integrate the following functions w.r.t. x Q. (1—10).

1. (i) $2x(x^2+2)^{3/2}$ (ii) $\dfrac{x}{\sqrt{2x^2+3}}$

2. (i) $\dfrac{\log x^2}{x}$ (ii) $\dfrac{x+3}{x^2+6x+4}$

3. (i) $\dfrac{1-\cot x}{1+\cot x}$ (ii) $\dfrac{e^x}{e^x+3}$

4. (i) $x\sqrt{x+2}$ (ii) $\sin^3(2x+1)$

5. (i) $\sin 4x \cos 7x$ (ii) $(e^x+1)^2 \,.\, e^x$

6. (i) $\dfrac{1+\cos x}{1-\cos x}$ (ii) $\dfrac{1}{x\log x}$

7. (i) $\dfrac{e^x+1}{e^x+x}$ (ii) $\dfrac{1}{16x^2-25}$

8. (i) $\dfrac{(1+\sqrt{x})^n}{\sqrt{x}}$ (ii) $\dfrac{\sec^2(2\tan^{-1}x)}{1+x^2}$

9. (i) $\sec^3 x \tan x$ (ii) $\sec x \log(\sec x + \tan x)$

10. (i) $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$ (ii) $\dfrac{1}{\sqrt{x}}\cos\sqrt{x}$.

Evaluate the following integrals Q. (11–30)

11. (i) $\displaystyle\int \frac{e^{2x}}{e^{2x}-2}\, dx$ (ii) $\displaystyle\int \frac{x^2}{9+4x^2}\, dx$

12. (i) $\displaystyle\int \frac{1}{x^2-10x+34}\, dx$ (ii) $\displaystyle\int \frac{1}{3+2x-x^2}\, dx$

13. (i) $\displaystyle\int \frac{1+\sin x}{\sqrt{x-\cos x}}\, dx$ (ii) $\displaystyle\int x^3\cos x^4\, dx$

14. (*i*) $\int \frac{\sec^2 x}{5\tan^2 x - 12\tan x + 14}\, dx$ (*ii*) $\int \frac{2x-3}{x^2+3x-18}\, dx$

15. (*i*) $\int \frac{e^x}{e^{2x}+6e^x+5}\, dx$ (*ii*) $\int \frac{1}{e^x+e^{-x}}\, dx$

16. (*i*) $\int \frac{1}{\sqrt{4-2x-x^2}}\, dx$ (*ii*) $\int \frac{1}{\sqrt{7-6x-x^2}}\, dx$

17. $\int \frac{\sin 8x}{\sqrt{9+\sin^4 x}}\, dx$

18. $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\, dx$

19. $\int \frac{x^3+x+1}{x^2-1}\, dx$

20. $\int \frac{2x+5}{x^2-x-2}\, dx$

21. $\int \frac{2x+1}{\sqrt{x^2+4x+3}}\, dx$

22. $\int \frac{2x+3}{\sqrt{x^2+4x+5}}\, dx$

23. $\int \frac{1}{13+3\cos x+4\sin x}\, dx$

24. $\int \frac{1}{\cos x-\sin x}\, dx$

25. $\int \frac{5\cos x+6}{2\cos x+\sin x+3}\, dx$

26. $\int \frac{\sin x+\cos x}{3\sin x+4\cos x+1}\, dx$

27. $\int \frac{1}{\sqrt{8+3x-x^2}}\, dx$

28. $\int \frac{1}{\sqrt{81-16x^2}}\, dx$

29. $\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}\, dx$

30. $\int \frac{2x+5}{\sqrt{x^2+3x+2}}\, dx.$

Answers

1. (*i*) $\frac{2}{5}(x^2+2)^{5/2}+c$ (*ii*) $\frac{1}{2}\sqrt{2x^2+3}+c$

2. (*i*) $(\log x)^2+c$ (*ii*) $\frac{1}{2}\log|x^2+6x+4|+c$

3. (*i*) $-\log|\sin x+\cos x|+c$ (*ii*) $\log|3+e^x|+c$

4. (*i*) $\frac{2}{5}(x+2)^{5/2}-\frac{4}{3}(x+2)^{3/2}+c$ (*ii*) $-\frac{3}{8}\cos(2x+1)+\frac{1}{24}\cos(6x+3)+c$

5. (*i*) $-\frac{1}{22}\cos 11x+\frac{1}{6}\cos 3x+c$ (*ii*) $\frac{1}{3}(e^x+1)^3+c$

6. (*i*) $-2\cot\frac{x}{2}+c$ (*ii*) $\log|\log x|+c$

7. (*i*) $\log|e^x+x|+c$ (*ii*) $\frac{1}{40}\log\left|\frac{4x-5}{4x+5}\right|+c$

8. (*i*) $\frac{2(1+\sqrt{x})^{n+1}}{n+1}+c$ (*ii*) $\frac{1}{2}\tan(2\tan^{-1}x)+c$

9. (*i*) $\frac{\sec^3 x}{3}+c$ (*ii*) $\frac{1}{2}[\log(\sec x+\tan x)]^2+c$

10. (*i*) $e^{\sin^{-1} x}+c$ (*ii*) $2\sin\sqrt{x}+c$

11. (*i*) $\frac{1}{2}\log|e^{2x}-2|+c$ (*ii*) $\frac{1}{4}x-\frac{3}{8}\tan^{-1}\left(\frac{2x}{3}\right)+c$

12. (*i*) $\frac{1}{3}\tan^{-1}\left(\frac{x-5}{3}\right)+c$ (*ii*) $\frac{1}{4}\log\left|\frac{1+x}{3-x}\right|+c$

13. (*i*) $2\sqrt{x}-\cos x+c$ (*ii*) $\frac{1}{4}\sin x^4+c$

14. (*i*) $\frac{1}{\sqrt{34}}\tan^{-1}\left(\frac{5\tan x-6}{\sqrt{34}}\right)+c$ (*ii*) $\log|x^2+3x-18|-\frac{2}{3}\log\left|\frac{x-3}{x+6}\right|+c$

15. (*i*) $\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+c$ (*ii*) $\tan^{-1}(e^x)+c$

16. (*i*) $\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right)+c$ (*ii*) $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

17. $\frac{1}{4}\log\left|\sin^2 4x+\sqrt{9+\sin^4 x}\right|+c$ **18.** $\sin^{-1}\left(\frac{\sin x}{2}\right)+c$

19. $\frac{x^2}{2}+\log|x^2-1|+\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+c$ **20.** $\log|x^2-x-2|+2\log\left|\frac{x-2}{x+2}\right|+c$

21. $2\sqrt{x^2+4x+3}-3\log|x+2+\sqrt{x^2+4x+3}|+c$

22. $2\sqrt{x^2+4x+5}-\log|x+2+\sqrt{x^2+4x+5}|+c$

23. $\frac{1}{6}\tan^{-1}\left(\frac{5\tan\frac{x}{2}+2}{6}\right)+c$ **24.** $\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\tan\frac{x}{2}+1}{\sqrt{2}-\tan\frac{x}{2}-1}\right|+c$

25. $2x+\log|2\cos x+\sin x+3|+c$

26. $\frac{7}{25}x-\frac{1}{25}\log|3\sin x+4\cos x+1|-\frac{7}{50\sqrt{6}}\log\left|\frac{\sqrt{\frac{8}{3}}-1+\tan\frac{x}{2}}{\sqrt{\frac{8}{3}}+1-\tan\frac{x}{2}}\right|+c$

27. $\sin^{-1}\frac{2x-5}{\sqrt{41}}+c$ **28.** $\frac{1}{4}\sin^{-1}\left(\frac{4x}{9}\right)+c$

29. $-\frac{5}{13}x+\frac{12}{13}\log|3\cos x+2\sin x|+c$

30. $2\sqrt{x^2+3x+2}+2\log\left|x+\frac{3}{2}+\sqrt{x^2+3x+2}\right|+c.$

4

Integration By Parts

4.1. INTRODUCTION

This is one of the most useful integration device which is applied in a variety of circumstances. Particularly, it is useful for integrals where the integrates are functions, any one of which may be algebraic, exponential, logarithmic, trigonometric functions.

4.2. INTEGRATION BY PARTS

Theorem 1. *If u and v are two functions of x, then :*

$$\int uv\,dx = u\left(\int v\,dx\right) - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

i.e., Integral of the product of two functions = Ist functions × Integral of 2nd function – Integral of (Diff. co-eff. of Ist × Integral of 2nd).

Proof. Let $f(x)$ and $g(x)$ be any two functions

Then, by the product rule of differentiation, we have

$$\frac{d}{dx}[f(x)\,.\,g(x)] = f(x)\,.\,\frac{d}{dx}[g(x)] + g(x)\,\frac{d}{dx}[f(x)]$$

Integrating both sides, we get

$$f(x)\,.\,g(x) = \int\left(f(x)\,.\,\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]\right)dx$$

$$= \int\left(f(x)\,.\,\frac{d}{dx}[g(x)]\right)dx + \int\left(g(x)\,.\,\frac{d}{dx}[f(x)]\right)dx$$

$$\Rightarrow \quad \int\left(f(x)\frac{d}{dx}[g(x)]\right)dx = f(x)\,.\,g(x) - \int\left(g(x)\,.\,\frac{d}{dx}[f(x)]\right)dx$$

Let $\quad u = f(x)$

and $\quad v = \frac{d}{dx}[g(x)] \Rightarrow \int v\,dx = g(x)$

$$\therefore \quad \int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx.$$

Integration with the help of the above rule is called the integration by parts. Following points must be kept in mind, whenever making the use of the above formula.

(*i*) If the integrate contains both functions integrable, take that function as the first which can be finished by repeated differentiation and the other as the second. *e.g.*

$$\text{in} \int x^2 \sin x \, dx \text{ or in} \int x^2 e^x \, dx.$$

Take x^2 as the first function and the other as the second.

(*ii*) If the integrated contains one unintegrable, *e.g.*, inverse circular function or a logarithmic function, take unintegrable as the first function and other as second. *e.g.*

$$\text{in} \int x^2 \log x \, dx, \int x \sin^{-1} x \, dx.$$

Take $\log x$, $\sin^{-1} x$ as the first function and other as second respectively.

(*iii*) If the integrate contains only one unintegrable function, then, take the given function as the first function and unity as the second.

(*iv*) If the integrate contains both the functions integrable and none can be finished by repeated differentiation, then take any one as the first function and other as the second. Repeat the rule of integration by parts.

(*v*) We can also choose the first function as the function which comes first in the word ILATE.

where :

I — stands for the inverse trigonometric functions ($\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc.)

L — stands for logarithmic functions.

A — stands for algebraic functions.

T — stands for trigonometric functions.

E — stands for exponential functions.

Remark. It is also worth mentioning that integration by parts is not applicable to the product of function in all cases. For example, the method does not work for $\int \sqrt{x} \sin x \, dx$.

The reason is that there does not exist any function whose derivative is $\sqrt{x} \sin x$.

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following integrals :*

(*i*) $\int x e^{2x} \, dx$ (*ii*) $\int x^2 e^{-x} \, dx$

(*iii*) $\int x e^x \, dx$ (*iv*) $\int x^3 e^x \, dx$

(*v*) $\int x^2 e^{3x} \, dx$ (*vi*) $\int x^2 e^{-3x} \, dx.$

Solution. (*i*) Let $I = \int \underset{\text{I}}{x} \underset{\text{II}}{e^{2x}} \, dx$

Integrating by parts, we get

$$I = x \int e^{2x} \, dx - \int \left[\frac{d}{dx}(x) \int e^{2x} \, dx \right] dx$$

$$= \frac{x e^{2x}}{2} - \int \frac{1 . e^{2x}}{2} \, dx = \frac{x e^{2x}}{2} - \frac{1}{2} \int e^{2x} \, dx$$

$$= \frac{x\,e^{2x}}{2} - \frac{1}{2}\frac{e^{2x}}{2} + c$$

$$= \frac{x\,e^{2x}}{2} - \frac{1}{4}\,e^{2x} + c.$$

(*ii*) Let $\mathrm{I} = \int \underset{\mathrm{I}}{x^2}\,\underset{\mathrm{II}}{e^{-x}}\,dx$

Integrating by parts, we get

$$\mathrm{I} = x^2 \int e^{-x}\,dx - \int \left[\frac{d}{dx}(x^2)\,.\int e^{-x}\,dx\right] dx$$

$$= \frac{x^2\,e^{-x}}{(-1)} - \int 2x\,.\frac{e^{-x}}{(-1)}\,dx$$

$$= -x^2\,e^{-x} + 2\int \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{e^{-x}}\,dx \qquad \text{[Integrating again by parts]}$$

$$= -x^2\,e^{-x} + 2\left[x\int e^{-x}\,dx - \int\left(\frac{d}{dx}(x)\int e^{-x}\,dx\right)dx\right]$$

$$= -x^2\,e^{-x} + 2\left[\frac{x\,e^{-x}}{-1} - \int \frac{1.\,e^{-x}}{-1}\,dx\right]$$

$$= -x^2\,e^{-x} - 2x\,e^{-x} + 2\int e^{-x}\,dx = -x^2\,e^{-x} - 2x\,e^{-x} + \frac{2\,e^{-x}}{-1} + c$$

$$= -e^{-x}\,(x^2 + 2x + 2) + c.$$

(*iii*) Let $\quad \mathrm{I} = \int \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{e^{x}}\,dx$

Integrating by parts, we get

$$\mathrm{I} = x\int e^x\,dx - \int\left[\frac{d}{dx}(x)\,.\int e^x\,dx\right]dx$$

$$= x\,e^x - \int 1\,.\,e^x\,dx = x\,e^x - e^x + c$$

$$= e^x\,(x - 1) + c.$$

(*iv*) Let $\quad \mathrm{I} = \int \underset{\mathrm{I}}{x^3}\,\underset{\mathrm{II}}{e^{x}}\,dx$

Integrating by parts, we get

$$\mathrm{I} = x^3\int e^x\,dx - \int\left[\frac{d}{dx}(x^3)\int e^x\,dx\right]dx$$

$$= x^3\,e^x - \int 3x^2\,e^x\,dx$$

$$= x^3\,e^x - 3\int \underset{\mathrm{I}}{x^2}\,\underset{\mathrm{II}}{e^{x}}\,dx \qquad \text{[Integrating again by parts]}$$

$$= x^3\,e^x - 3\left[x^2\int e^x\,dx - \int\left(\frac{d}{dx}(x^2)\,.\int e^x\,dx\right)dx\right]$$

$$= x^3 e^x - 3\left[x^2 e^x - \int 2x\, e^x\, dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6 \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^x}\, dx \qquad \text{[Integrating again by parts]}$$

$$= x^3 e^x - 3x^2 e^x + 6\left[x \int e^x\, dx - \int \left(\frac{d}{dx}(x) \cdot \int e^x\, dx\right) dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6\left[x\, e^x - \int 1 . e^x\, dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6x\, e^x - 6 \int e^x\, dx$$

$$= x^3 e^x - 3x^2 e^x + 6x\, e^x - 6\, e^x + c$$

$$= e^x (x^3 - 3x^2 + 6x - 6) + c.$$

(*v*) Let $\quad I = \int \underset{\text{I}}{x^2}\, \underset{\text{II}}{e^{3x}}\, dx$

Integrating by parts, we get

$$I = x^2 \int e^{3x}\, dx - \int \left[\frac{d}{dx}(x^2) \int e^{3x}\, dx\right] dx$$

$$= \frac{x^2\, e^{3x}}{3} - \int 2x \frac{e^{3x}}{3}\, dx$$

$$= \frac{x^2\, e^{3x}}{3} - \frac{2}{3} \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^{3x}}\, dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{x^2\, e^{3x}}{3} - \frac{2}{3}\left[x . \int e^{3x}\, dx - \int \left(\frac{d}{dx}(x) . \int e^{3x}\, dx\right) dx\right]$$

$$= \frac{x^2\, e^{3x}}{3} - \frac{2}{3}\left[x . \frac{e^{3x}}{3} - \int 1 . \frac{e^{3x}}{3}\, dx\right]$$

$$= \frac{x^2\, e^{3x}}{3} - \frac{2}{9}\, x\, e^{3x} + \frac{2}{9} \int e^{3x}\, dx = \frac{x^2\, e^{3x}}{3} - \frac{2}{9}\, x\, e^{3x} + \frac{2}{9}\left(\frac{e^{3x}}{3}\right) + c$$

$$= \frac{1}{3} x^2\, e^{3x} - \frac{2}{9}\, x\, e^{3x} + \frac{2}{27}\, e^{3x} + c.$$

(*vi*) Let $\quad I = \int \underset{\text{I}}{x^2}\, \underset{\text{II}}{e^{-3x}}\, dx$

Integrating by parts, we get

$$I = x^2 \int e^{-3x}\, dx - \int \left[\frac{d}{dx}(x^2) . \int e^{-3x}\, dx\right] dx$$

$$= \frac{x^2\, e^{-3x}}{(-3)} - \int 2x \frac{e^{-3x}}{(-3)}\, dx$$

$$= -\frac{1}{3} x^2\, e^{-3x} + \frac{2}{3} \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^{-3x}}\, dx \qquad \text{[Integrating again by parts]}$$

$$= -\frac{1}{3} x^2\, e^{-3x} + \frac{2}{3}\left[x . \int e^{-3x}\, dx - \int \left(\frac{d}{dx}(x) . \int e^{-3x}\, dx\right) dx\right]$$

$$= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[x \cdot \frac{e^{-3x}}{-3} - \int 1 \cdot \frac{e^{-3x}}{-3}\,dx\right]$$

$$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}x e^{-3x} + \frac{2}{9}\int e^{-3x}\,dx$$

$$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}x e^{-3x} + \frac{2}{9}\left(\frac{e^{-3x}}{-3}\right) + c$$

$$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}x e^{-3x} - \frac{2}{27}e^{-3x} + c$$

$$= -\frac{1}{3}e^{-3x}\left[x^2 + \frac{2}{3}x + \frac{2}{9}\right] + c.$$

Example 2. *Evaluate the following integrals :*

(*i*) $\int x \sin x\,dx$ (*ii*) $\int x \cos^2 x\,dx$

(*iii*) $\int x \sin^3 x\,dx$ (*iv*) $\int x \sin 3x\,dx$

(*v*) $\int \frac{x}{\cos^2 x}\,dx$ (*vi*) $\int x^2 \cos x\,dx.$

Solution. (*i*) Let $I = \int \underset{\text{I}}{x}\ \underset{\text{II}}{\sin} x\,dx$

Integrating by parts, we get

$$I = x \cdot \int \sin x\,dx - \int\left[\frac{d}{dx}(x) \cdot \int \sin x\,dx\right]dx$$

$$= x(-\cos x) - \int 1 \cdot (-\cos x)\,dx = -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c.$$

Remark. Observe that while finding the integral of the second function, we did not add a constant of integration. If we write the integral of the second function $\sin x$ as $-\cos x + c_1$, where c_1 is any constant, then ;

$$\int x \sin x\,dx = x(-\cos x + c_1) - \int(-\cos x + c_1)\,dx$$

$$= x(-\cos x + c_1) + \int \cos x\,dx - c_1\int dx$$

$$= -x\cos x + c_1 x + \sin x - c_1 x + c$$

$$= -x\cos x + \sin x + c$$

This shows that adding a constant to the integral of the second function is superfluous so far as the final result is concerned while applying the device of integration by parts.

(*ii*) Let $I = \int x \cos^2 x\,dx$

$$= \int x\left(\frac{1 + \cos 2x}{2}\right)dx \qquad [\because \cos 2A = 2\cos^2 A - 1]$$

$$= \frac{1}{2}\int x\,dx + \frac{1}{2}\int \underset{\text{I}}{x}\ \underset{\text{II}}{\cos}\ 2x\,dx$$

Integrating by parts, we get

$$I = \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{2}\left[x \cdot \int \cos 2x\, dx - \int \left(\frac{d}{dx}(x) \cdot \int \cos 2x\, dx\right) dx\right]$$

$$= \frac{1}{4}x^2 + \frac{1}{2}\left[x\,\frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2}\, dx\right]$$

$$= \frac{1}{4}x^2 + \frac{1}{4}x \sin 2x - \frac{1}{4}\int \sin 2x\, dx = \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x - \frac{1}{4}\left(-\frac{\cos 2x}{2}\right) + c$$

$$= \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + c.$$

(*iii*) Let $I = \int x \sin^3 x\, dx$

$$= \int x\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right) dx \qquad \left[\begin{array}{l} \because\ \sin 3A = 3\sin A - 4\sin^3 A \\ \Rightarrow \sin^3 A = \dfrac{3}{4}\sin A - \dfrac{1}{4}\sin 3A \end{array}\right]$$

$$= \frac{3}{4}\int \underset{\text{I}}{x}\, \underset{\text{II}}{\sin x}\, dx - \frac{1}{4}\int \underset{\text{I}}{x}\, \underset{\text{II}}{\sin}\, 3x\, dx$$

Integrating by parts, we get

$$I = \frac{3}{4}\left[x \cdot \int \sin x\, dx - \int \left(\frac{d}{dx}(x) \cdot \int \sin x\, dx\right) dx\right]$$

$$- \frac{1}{4}\left[x \int \sin 3x\, dx - \int \left(\frac{d}{dx}(x) \int \sin 3x\, dx\right) dx\right]$$

$$= \frac{3}{4}\left[x(-\cos x) - \int 1(-\cos x)\, dx\right] - \frac{1}{4}\left[x\,\frac{(-\cos 3x)}{3} - \int 1 \cdot \frac{(-\cos 3x)}{3}\, dx\right]$$

$$= -\frac{3}{4}x\cos x + \frac{3}{4}\int \cos x\, dx + \frac{1}{12}x\cos 3x - \frac{1}{12}\int \cos 3x\, dx$$

$$= -\frac{3}{4}x\cos x + \frac{3}{4}\sin x + \frac{1}{12}x\cos 3x - \frac{1}{12}\left(\frac{\sin 3x}{3}\right) + c$$

$$= -\frac{3}{4}x\cos x + \frac{3}{4}\sin x + \frac{1}{12}x\cos 3x - \frac{1}{36}\sin 3x + c.$$

(*iv*) Let $I = \int \underset{\text{I}}{x}\, \underset{\text{II}}{\sin}\, 3x\, dx$

Integrating by parts, we get

$$I = x\int \sin 3x\, dx - \int \left[\frac{d}{dx}(x) \cdot \int \sin 3x\, dx\right] dx$$

$$= x\left(-\frac{\cos 3x}{3}\right) - \int 1 \cdot \left(-\frac{\cos 3x}{3}\right) dx$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\, dx = -\frac{1}{3}x\cos 3x + \frac{1}{3}\left(\frac{\sin 3x}{3}\right) + c$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + c.$$

(*v*) Let $\quad I = \int \frac{x}{\cos^2 x}\, dx = \int \underset{\text{I}}{x} \underset{\text{II}}{\sec^2 x}\, dx$

Integrating by parts, we get

$$I = x \int \sec^2 x\, dx - \int \left(\frac{d}{dx}(x) \int \sec^2 x\, dx\right) dx$$

$$= x \tan x - \int 1 \,.\, \tan x\, dx = x \tan x - (-\log |\cos x|) + c$$

$$= x \tan x + \log |\cos x| + c.$$

(*vi*) Let $\quad I = \int \underset{\text{I}}{x^2} \underset{\text{II}}{\cos x}\, dx$

Integrating by parts, we get

$$I = x^2 \int \cos x\, dx - \int \left[\frac{d}{dx}(x^2) \,.\, \int \cos x\, dx\right] dx$$

$$= x^2 \sin x - \int 2x \sin x\, dx$$

$$= x^2 \sin x - 2 \int \underset{\text{I}}{x} \underset{\text{II}}{\sin x}\, dx \qquad \text{[Integrating again by parts]}$$

$$= x^2 \sin x - 2\left[x \,.\, \int \sin x\, dx - \int \left(\frac{d}{dx}(x) \int \sin x\, dx\right) dx\right]$$

$$= x^2 \sin x - 2x(-\cos x) + 2 \int 1 \,.\, (-\cos x)\, dx$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c.$$

Example 3. *Evaluate the following integrals :*

(*i*) $\int (x^2 + x + 1)\, e^x\, dx$ (*ii*) $\int (2 + 3x^2) \cos 3x\, dx$

(*iii*) $\int (1 - x^2) \sin 2x\, dx$ (*iv*) $\int x^2 \cos^2 x\, dx$

(*v*) $\int (1 + x^2) \cos 2x\, dx$ (*vi*) $\int x \cos^3 x\, dx.$

Solution. (*i*) Let $I = \int \underset{\text{I}}{(x^2 + x + 1)} \underset{\text{II}}{e^x}\, dx$

Integrating by parts, we get

$$I = (x^2 + x + 1) \,.\, \int e^x\, dx - \int \left[\frac{d}{dx}(x^2 + x + 1) \,.\, \int e^x\, dx\right] dx$$

$$= (x^2 + x + 1)\, e^x - \int \underset{\text{I}}{(2x + 1)} \underset{\text{II}}{e^x}\, dx \qquad \text{[Integrating again by parts]}$$

$$= (x^2 + x + 1)\, e^x - \left[(2x + 1) \,.\, \int e^x\, dx - \int \left\{\frac{d}{dx}(2x + 1) \,.\, \int e^x\, dx\right\} dx\right]$$

$$= (x^2 + x + 1)\, e^x - (2x + 1)\, e^x + \int 2 \,.\, e^x\, dx$$

$$= (x^2 + x + 1)\, e^x - (2x + 1)\, e^x + 2e^x + c$$

$$= e^x (x^2 + x + 1 - 2x - 1 + 2) + c$$

$$= e^x (x^2 - x + 2) + c.$$

(*ii*) Let $\mathrm{I} = \int (2 + 3\underset{\mathrm{I}}{x^2}) \underset{\mathrm{II}}{\cos}\, 3x\, dx$

Integrating by parts, we get

$$\mathrm{I} = (2 + 3x^2) \int \cos 3x\, dx - \int \left[\frac{d}{dx}(2+3x^2) \int \cos 3x\, dx\right] dx$$

$$= (2 + 3x^2)\left(\frac{\sin 3x}{3}\right) - \int 6x \,.\, \frac{\sin 3x}{3}\, dx$$

$$= \frac{1}{3}(2 + 3x^2) \sin 3x - 2 \int \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{\sin}\, 3x\, dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{1}{3}(2+3x^2)\sin 3x - 2\left[x \,.\int \sin 3x\, dx - \int \left(\frac{d}{dx}(x) \,.\int \sin 3x\, dx\right) dx\right]$$

$$= \frac{1}{3}(2+3x^2)\sin 3x - 2\left[x\left(-\frac{\cos 3x}{3}\right) - \int 1\left(-\frac{\cos 3x}{3}\right) dx\right]$$

$$= \frac{1}{3}(2+3x^2)\sin 3x + \frac{2}{3}x\cos 3x - \frac{2}{3}\int \cos 3x\, dx$$

$$= \frac{1}{3}(2+3x^2)\sin 3x + \frac{2}{3}x\cos 3x - \frac{2}{3}\,\frac{\sin 3x}{3} + c$$

$$= \frac{1}{3}(2+3x^2)\sin 3x + \frac{2}{3}x\cos 3x - \frac{2}{9}\sin 3x + c$$

$$= \left(\frac{2}{3} + x^2 - \frac{2}{9}\right)\sin 3x + \frac{2}{3}x\cos 3x + c$$

$$= \left(x^2 + \frac{4}{9}\right)\sin 3x + \frac{2}{3}x\cos 3x + c.$$

(*iii*) Let $\mathrm{I} = \int \underset{\mathrm{I}}{(1 - x^2)} \underset{\mathrm{II}}{\sin}\, 2x\, dx$

Integrating by parts, we get

$$\mathrm{I} = (1 - x^2) \int \sin 2x\, dx - \int \left[\frac{d}{dx}(1-x^2) \int \sin 2x\, dx\right] dx$$

$$= (1 - x^2)\left(-\frac{\cos 2x}{2}\right) - \int (-2x)\left(-\frac{\cos 2x}{2}\right) dx$$

$$= \frac{(x^2-1)(\cos 2x)}{2} - \int \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{\cos}\, 2x\, dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{1}{2}(x^2-1)\cos 2x - \left[x \,.\int \cos 2x\, dx - \int \left(\frac{d}{dx}(x) \,.\int \cos 2x\, dx\right) dx\right]$$

$$= \frac{1}{2}(x^2-1)\cos 2x - \left[x\left(\frac{\sin 2x}{2}\right) - \int 1 .\, \frac{\sin 2x}{2}\, dx\right]$$

$$= \frac{1}{2}(x^2-1)\cos 2x - \frac{1}{2}x\sin 2x + \frac{1}{2}\int \sin 2x\, dx$$

$$= \frac{1}{2}(x^2 - 1)\cos 2x - \frac{1}{2}x \sin 2x + \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + c$$

$$= \frac{1}{2}(x^2 - 1)\cos 2x - \frac{1}{2}x \sin 2x - \frac{1}{4}\cos 2x + c$$

$$= \left(\frac{x^2}{2} - \frac{1}{2} - \frac{1}{4}\right)\cos 2x - \frac{1}{2}x \sin 2x + c$$

$$= \left(\frac{x^2}{2} - \frac{3}{4}\right)\cos 2x - \frac{1}{2}x \sin 2x + c.$$

(*iv*) Let $\quad I = \int x^2 \cos^2 x \, dx = \int x^2 \left(\frac{1+\cos 2x}{2}\right) dx \qquad [\because \cos 2A = 2\cos^2 A - 1]$

$$= \frac{1}{2}\int x^2 \, dx + \frac{1}{2}\int \underset{\text{I}}{x^2} \underset{\text{II}}{\cos} 2x \, dx$$

Integrating by parts, we get

$$I = \frac{1}{2}\left(\frac{x^3}{3}\right) + \frac{1}{2}\left[x^2 \int \cos 2x \, dx - \int \left(\frac{d}{dx}(x^2) . \int \cos 2x \, dx\right) dx\right]$$

$$= \frac{x^3}{6} + \frac{1}{2}\left[x^2 \frac{\sin 2x}{2} - \int 2x \frac{\sin 2x}{2} dx\right]$$

$$= \frac{x^3}{6} + \frac{1}{4}x^2 \sin 2x - \frac{1}{2}\int \underset{\text{I}}{x} \underset{\text{II}}{\sin} 2x \, dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x - \frac{1}{2}\left[x . \int \sin 2x \, dx - \int \left\{\frac{d}{dx}(x) \int \sin 2x \, dx\right\} dx\right]$$

$$= \frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x - \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - \int 1 . \left(-\frac{\cos 2x}{2}\right) dx\right]$$

$$= \frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x + \frac{1}{4}x \cos 2x - \frac{1}{4}\int \cos 2x \, dx$$

$$= \frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x + \frac{1}{4}x \cos 2x - \frac{1}{4}\frac{\sin 2x}{2} + c$$

$$= \frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x + \frac{1}{4}x \cos 2x - \frac{1}{8}\sin 2x + c$$

$$= \frac{1}{6}x^3 + \left(\frac{2x^2 - 1}{8}\right)\sin 2x + \frac{1}{4}x \cos 2x + c.$$

(*v*) Let $\quad I = \int \underset{\text{I}}{(1 + x^2)} \underset{\text{II}}{\cos} 2x \, dx$

Integrating by parts, we get

$$I = (1 + x^2) \int \cos 2x \, dx - \int \left[\frac{d}{dx}(1 + x^2) . \int \cos 2x \, dx\right] dx$$

$$= (1 + x^2)\frac{\sin 2x}{2} - \int 2x . \frac{\sin 2x}{2} dx$$

$$= \frac{1}{2}(1+x^2)\sin 2x - \int \underset{\text{I}}{x} \underset{\text{II}}{\sin 2x}\, dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{1}{2}(1+x^2)\sin 2x - \left[x \cdot \int \sin 2x\, dx - \int \left\{\frac{d}{dx}(x) \cdot \int \sin 2x\, dx\right\} dx\right]$$

$$= \frac{1}{2}(1+x^2)\sin 2x - \left[x\left(-\frac{\cos 2x}{2}\right) - \int 1 \cdot \left(-\frac{\cos 2x}{2}\right) dx\right]$$

$$= \frac{1}{2}(1+x^2)\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{1}{2}(1+x^2)\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{2}\frac{\sin 2x}{2} + c$$

$$= \frac{1}{2}(1+x^2)\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + c$$

$$= \left(\frac{1}{2} + \frac{x^2}{2} - \frac{1}{4}\right)\sin 2x + \frac{1}{2}x\cos 2x + c$$

$$= \left(\frac{2x^2+1}{4}\right)\sin 2x + \frac{1}{2}x\cos 2x + c.$$

(*vi*) Let $\quad I = \int x\cos^3 x\, dx$

$$= \int x\left(\frac{1}{4}\cos 3x + \frac{3}{4}\cos x\right) dx \qquad \left[\begin{array}{l} \because\ \cos 3A = 4\cos^3 A - 3\cos A \\ \Rightarrow\ 4\cos^3 A = \cos 3A + 3\cos A \\ \Rightarrow\ \cos^3 A = \frac{1}{4}\cos 3A + \frac{3}{4}\cos A \end{array}\right]$$

$$= \frac{1}{4}\int \underset{\text{I}}{x} \underset{\text{II}}{\cos 3x}\, dx + \frac{3}{4}\int \underset{\text{I}}{x} \underset{\text{II}}{\cos x}\, dx$$

Integrating by parts, we get

$$I = \frac{1}{4}\left[x\int \cos 3x\, dx - \int\left\{\frac{d}{dx}(x)\int \cos 3x\, dx\right\} dx\right]$$

$$+ \frac{3}{4}\left[x\int \cos x\, dx - \int\left\{\frac{d}{dx}(x)\int \cos x\, dx\right\} dx\right]$$

$$= \frac{1}{4}\left[x\frac{\sin 3x}{3} - \int 1 \cdot \frac{\sin 3x}{3}\, dx\right] + \frac{3}{4}\left[x\sin x - \int 1 \cdot \sin x\, dx\right]$$

$$= \frac{1}{12}x\sin 3x - \frac{1}{12}\int \sin 3x\, dx + \frac{3}{4}x\sin x - \frac{3}{4}\int \sin x\, dx$$

$$= \frac{1}{12}x\sin 3x - \frac{1}{12}\left(-\frac{\cos 3x}{3}\right) + \frac{3}{4}x\sin x - \frac{3}{4}\quad - \cos x) + c$$

$$= \frac{1}{12}x\sin 3x + \frac{1}{36}\cos 3x + \frac{3}{4}x\sin x + \frac{3}{4}\cos x + c.$$

Example 4. *Evaluate the following integrals :*

(*i*) $\int (1 - x^2) \log x \, dx$ (*ii*) $\int x \log (x + 3) \, dx$

(*iii*) $\int \log (1 + x^2) \, dx$ (*iv*) $\int x^3 \log x \, dx$

(*v*) $\int x^3 \log 2x \, dx$ (*vi*) $\int \frac{\log x}{x^n} \, dx.$

Solution. (*i*) Let $I = \int \underset{\text{II}}{(1 - x^2)} \underset{\text{I}}{\log x} \, dx$

Integrating by parts, we get

$$I = \log x \,.\, \int (1 - x^2) dx - \int \left\{ \frac{d}{dx} (\log x) \,.\, \int (1 - x^2) dx \right\} dx$$

$$= \log x \left(x - \frac{x^3}{3} \right) - \int \frac{1}{x} \left(x - \frac{x^3}{3} \right) dx = \left(\frac{3x - x^3}{3} \right) \log x - \int \left(1 - \frac{x^2}{3} \right) dx$$

$$= \left(\frac{3x - x^3}{3} \right) \log x - \int 1 \,.\, dx + \frac{1}{3} \int x^2 \, dx = \left(\frac{3x - x^3}{3} \right) \log x - x + \frac{1}{3} \,.\, \frac{x^3}{3} + c$$

$$= \left(\frac{3x - x^3}{3} \right) \log x - x + \frac{x^3}{9} + c.$$

(*ii*) Let $I = \int \underset{\text{II}}{x} \log \underset{\text{I}}{(x + 3)} \, dx$

Integrating by parts, we get

$$I = \log (x + 3) \int x \, dx - \int \left\{ \frac{d}{dx} (\log (x + 3)) \int x \, dx \right\} dx$$

$$= \log (x + 3) \,.\, \frac{x^2}{2} - \int \frac{1}{x + 3} \left(\frac{x^2}{2} \right) dx$$

$$= \frac{x^2}{2} \log (x + 3) - \frac{1}{2} \int \frac{x^2}{x + 3} \, dx.$$

Note. If the integrate is a rational function whose numerator and denominator are polynomials and degree of numerator is greater than or equal to that of the denominator, first divide the numerator by the denominator and then use the result :

$$\left[\frac{\text{Numerator}}{\text{Denominator}} = \text{Quetient} + \frac{\text{Remainder}}{\text{Denominator}} \right]$$

$$= \frac{x^2}{2} \log (x + 3) - \frac{1}{2} \int \left[x - \frac{3x}{x + 3} \right] dx$$

$$= \frac{x^2}{2} \log (x + 3) - \frac{1}{2} \int x \, dx + \frac{3}{2} \int \frac{x}{x + 3} \, dx$$

$$= \frac{x^2}{2} \log (x + 3) - \frac{1}{2} \,.\, \frac{x^2}{2} + \frac{3}{2} \int \frac{(x + 3 - 3)}{x + 3} \, dx$$

[Add and subtract 3 to the numerator]

$x + 3 \,)\, x^2 \,(\, x$
$+ x^2 + 3x$
$- \quad -$
$- 3x$

$$= \frac{x^2}{2}\log(x+3) - \frac{1}{4}x^2 + \frac{3}{2}\left[\int 1 . dx - \int \frac{3}{x+3}dx\right]$$

$$= \frac{x^2}{2}\log(x+3) - \frac{1}{4}x^2 + \frac{3}{2}x - \frac{9}{2}\int \frac{1}{x+3}dx$$

$$= \frac{x^2}{2}\log(x+3) - \frac{1}{4}x^2 + \frac{3}{2}x - \frac{9}{2}\log(x+3) + c.$$

(*iii*) Let $\quad I = \int \log(1+x^2)\,dx$

$$= \int \underset{\text{II}}{1} . \log \underset{\text{I}}{(1+x^2)}\,dx$$

[Note this step, here, we are taking unity as second function]

Integrating by parts, we get

$$I = \log(1+x^2)\int 1 . dx - \int\left\{\frac{d}{dx}(\log(1+x^2)) . \int 1 . dx\right\}dx$$

$$= \log(1+x^2) . x - \int \frac{1}{1+x^2}(2x) . x\,dx$$

$$= x\log(1+x^2) - 2\int \frac{x^2}{1+x^2}dx$$

$$= x\log(1+x^2) - 2\int \frac{(1+x^2-1)}{1+x^2}dx \quad \text{[Add and subtract 1 to the numerator]}$$

$$= x\log(1+x^2) - 2\left[\int 1 . dx - \int \frac{1}{1+x^2}dx\right]$$

$$= x\log(1+x^2) - 2x + 2\int \frac{1}{1+x^2}dx$$

$$\left[\text{By using} \int \frac{1}{a^2+x^2} . dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= x\log(1+x^2) - 2x + 2\tan^{-1}x + c.$$

(*iv*) Let $\quad I = \int \underset{\text{II}}{x^3} . \underset{\text{I}}{\log x}\,dx$

Integrating by parts, we get

$$I = \log x\int x^3\,dx - \int\left\{\frac{d}{dx}(\log x)\int x^3\,dx\right\}dx$$

$$= \log x . \frac{x^4}{4} - \int \frac{1}{x} . \frac{x^4}{4}dx = \frac{1}{4}x^4\log x - \frac{1}{4}\int x^3\,dx$$

$$= \frac{1}{4}x^4\log x - \frac{1}{4}\left(\frac{x^4}{4}\right) + c$$

$$= \frac{1}{4}x^4\log x - \frac{1}{16}x^4 + c.$$

(*v*) Let $\quad I = \int \underset{\text{II}}{x^3} \log \underset{\text{I}}{2x}\, dx.$

Integrating by parts, we get

$$I = \log 2x \cdot \int x^3\, dx - \int \left\{\frac{d}{dx}(\log 2x) \cdot \int x^3\, dx\right\} dx$$

$$= \log 2x \cdot \frac{x^4}{4} - \int \frac{1}{2x}(2) \cdot \frac{x^4}{4}\, dx$$

$$= \frac{1}{4}x^4 \log 2x - \frac{1}{4}\int x^3\, dx = \frac{1}{4}x^4 \log 2x - \frac{1}{4} \cdot \frac{x^4}{4} + c$$

$$= \frac{1}{4}x^4 \log 2x - \frac{1}{16}x^4 + c.$$

(*vi*) Let $\quad I = \int \frac{\log x}{x^n}\, dx$

$$= \int \underset{\text{II}}{x^{-n}} \log \underset{\text{I}}{x}\, dx$$

Integrating by parts, we get

$$I = \log x \int x^{-n}\, dx - \int \left\{\frac{d}{dx}(\log x) \cdot \int x^{-n}\, dx\right\} dx$$

$$= \log x \cdot \frac{x^{-n+1}}{(-n+1)} - \int \frac{1}{x} \cdot \frac{x^{-n+1}}{(-n+1)}\, dx$$

$$= \frac{1}{(1-n)} \log x \cdot x^{-n+1} - \frac{1}{(1-n)}\int x^{-n}\, dx$$

$$= \frac{\log x}{(1-n)} \cdot \frac{1}{x^{n-1}} - \frac{1}{(1-n)} \frac{x^{-n+1}}{(-n+1)} + c$$

$$= \frac{\log x}{(1-n)x^{n-1}} - \frac{1}{(1-n)^2} \cdot \frac{1}{x^{n-1}} + c$$

$$= \frac{1}{(1-n)x^{n-1}}\left[\log x - \frac{1}{(1-n)}\right] + c.$$

Example 5. *Evaluate the following integrals :*

(*i*) $\int x \log(1 + x)\, dx$ (*ii*) $\int x (\log x)^2\, dx$

(*iii*) $\int (\log x)^2\, dx$ (*iv*) $\int x^2 a^{3x}\, dx$

(*v*) $\int \frac{\log x}{(1+x)^2}\, dx$ (*vi*) $\int x \tan^{-1} x\, dx.$

Solution. (*i*) Let $I = \int \underset{\text{II}}{x} \log \underset{\text{I}}{(1+x)}\, dx$

Integrating by parts, we get

$$I = \log(1 + x) \int x - \int \left\{\frac{d}{dx}(\log(1+x)) \int .x\, dx\right\} dx$$

$$= \log(1+x) \cdot \frac{x^2}{2} - \int \frac{1}{1+x} \cdot \frac{x^2}{2}\, dx = \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int \frac{x^2}{1+x}\, dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int \frac{x^2-1+1}{(x+1)}\, dx$$

[Add and subtract 1 to the numerator]

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int \left(\frac{(x-1)(x+1)}{x+1} + \frac{1}{x+1}\right) dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int \left[(x-1) + \frac{1}{x+1}\right] dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int x\, dx + \frac{1}{2}\int 1 . dx - \frac{1}{2}\int \frac{1}{x+1}\, dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2} \cdot \frac{x^2}{2} + \frac{1}{2} . x - \frac{1}{2}\log|x+1| + c$$

$$= \frac{x^2}{2}\log(1+x) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\log|x+1| + c.$$

(*ii*) Let $I = \int \underset{\text{II}}{x} \underset{\text{I}}{(\log x)^2}\, dx$

Integrating by parts, we get

$$I = (\log x)^2 \int x\, dx - \int \left\{\frac{d}{dx}(\log x)^2 . \int x\, dx\right\} dx$$

$$= (\log x)^2 . \frac{x^2}{2} - \int 2\log x . \frac{1}{x} . \frac{x^2}{2}\, dx$$

$$= \frac{1}{2}x^2(\log x)^2 - \int \underset{\text{II}}{x} \underset{\text{I}}{\log x}\, dx$$

[Integrating again by parts]

$$= \frac{1}{2}x^2(\log x)^2 - \left[\log x . \int x\, dx - \int \left\{\frac{d}{dx}(\log x) . \int x\, dx\right\} dx\right]$$

$$= \frac{1}{2}x^2(\log x)^2 - \left[\log x . \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2}\, dx\right]$$

$$= \frac{1}{2}x^2(\log x)^2 - \frac{1}{2}x^2\log x + \frac{1}{2}\int x\, dx$$

$$= \frac{1}{2}x^2(\log x)^2 - \frac{1}{2}x^2\log x + \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

$$= \frac{1}{2}x^2\log x[\log x - 1] + \frac{x^2}{4} + c.$$

(*iii*) Let $I = \int (\log x)^2\, dx$

$$= \int \underset{\text{II}}{1} . \underset{\text{I}}{(\log x)^2}\, dx$$

[Taking unity as the second function]

Integrating by parts, we get

$$I = (\log x)^2 . \int 1 . dx - \int \left\{ \frac{d}{dx} (\log x)^2 . \int 1 . dx \right\} dx$$

$$= (\log x)^2 . x - \int 2 \log x . \frac{1}{x} . x \, dx$$

$$= x (\log x)^2 - 2 \int \log x \, dx$$

$$= x (\log x)^2 - 2 \int \underset{\text{II}}{1} . \underset{\text{I}}{\log x} \, dx \qquad \text{[Integrating again by parts].}$$

$$= x (\log x)^2 - 2 \left[\log x . \int 1 . dx - \int \left\{ \frac{d}{dx} (\log x) . \int 1 \, dx \right\} dx \right]$$

$$= x (\log x)^2 - 2 \left[\log x . x - \int \frac{1}{x} . x \, dx \right]$$

$$= x (\log x)^2 - 2x \log x + 2 \int 1 . dx$$

$$= x (\log x)^2 - 2x \log x + 2x + c.$$

(*iv*) Let $\quad I = \int \underset{\text{I}}{x^2} \, \underset{\text{II}}{a^{3x}} \, dx$

Integrating by parts, we get

$$I = x^2 \int a^{3x} \, dx - \int \left\{ \frac{d}{dx} (x^2) . \int a^{3x} \, dx \right\} dx$$

$$= x^2 . \frac{a^{3x}}{3 \log a} - \int 2x . \frac{a^{3x}}{3 \log a} \, dx$$

$$= \frac{x^2 a^{3x}}{3 \log a} - \frac{2}{3 \log a} \int \underset{\text{I}}{x} . \underset{\text{II}}{a^{3x}} \, dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{x^2 a^{3x}}{3 \log a} - \frac{2}{3 \log a} \left[x . \int a^{3x} \, dx - \int \left\{ \frac{d}{dx} (x) . \int a^{3x} \, dx \right\} dx \right]$$

$$= \frac{x^2 a^{3x}}{3 \log a} - \frac{2}{3 \log a} \left[x . \frac{a^{3x}}{3 \log a} - \int 1 . \frac{a^{3x}}{3 \log a} \, dx \right]$$

$$= \frac{x^2 a^{3x}}{3 \log a} - \frac{2}{9} \frac{x \, a^{3x}}{(\log a)^2} + \frac{2}{9 (\log a)^2} \int a^{3x} \, dx$$

$$= \frac{x^2 a^{3x}}{3 \log a} - \frac{2}{9 (\log a)^2} x \, a^{3x} + \frac{2}{9 (\log a)^2} . \frac{a^{3x}}{3 \log a} + c$$

$$= \frac{x^2 a^{3x}}{3 \log a} - \frac{2x \, a^{3x}}{9 (\log a)^2} + \frac{2 \, a^{3x}}{27 (\log a)^3} + c.$$

(*v*) Let $\quad I = \int \frac{\log x}{(1+x)^2} \, dx$

$$= \int \underset{\text{I}}{\log x} . \underset{\text{II}}{\frac{1}{(1+x)^2}} \, dx$$

Integrating by parts, we get

$$I = \log x \cdot \int \frac{1}{(1+x)^2}\,dx - \int \left\{\frac{d}{dx}(\log x) \cdot \int \frac{1}{(1+x)^2}\,dx\right\} dx$$

$$= \log x \cdot \frac{(1+x)^{-2+1}}{-2+1} - \int \frac{1}{x} \cdot \frac{(1+x)^{-2+1}}{(-2+1)}\,dx$$

$$= -\log x \cdot (1+x)^{-1} + \int \frac{1}{x}(1+x)^{-1}\,dx$$

$$= -\log x \cdot \frac{1}{(1+x)} + \int \frac{1}{x(1+x)}\,dx$$

$$= -\frac{\log x}{(1+x)} + \int \left[\frac{1}{x} - \frac{1}{1+x}\right] dx \qquad \left[\because \frac{1}{x} - \frac{1}{1+x} = \frac{1+x-x}{x(1+x)} = \frac{1}{x(1+x)}\right]$$

$$= -\frac{\log x}{(1+x)} + \int \frac{1}{x}\,dx - \int \frac{1}{1+x}\,dx$$

$$= -\frac{\log x}{(1+x)} + \log|x| - \log|1+x| + c \qquad \left[\because \log m - \log n = \log \frac{m}{n}\right]$$

$$= -\frac{\log x}{(1+x)} + \log\left|\frac{x}{1+x}\right| + c.$$

(*vi*) Let $\quad I = \int \underset{\text{II}}{x} \cdot \underset{\text{I}}{\tan^{-1} x}\,dx$

Integrating by parts, we get

$$I = \tan^{-1} x \cdot \int x\,dx - \int \left\{\frac{d}{dx}(\tan^{-1} x) \cdot \int x\,dx\right\} dx$$

$$= \tan^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} \cdot dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,dx$$

[Add and subtract 1 to the numerator]

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int 1 \cdot dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$

$$\left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + c = \left(\frac{x^2+1}{2}\right)\tan^{-1} x - \frac{x}{2} + c.$$

Example 6. *Evaluate the following integrals :*

(i) $\int x^2 \log(1+x)\, dx$ (ii) $\int \log\left(x+\sqrt{a^2+x^2}\right) dx$

(iii) $\int \sin^{-1} x\, dx$ (iv) $\int \tan^{-1} x\, dx$

(v) $\int \sec^{-1} x\, dx$ (vi) $\int 2x^3 e^{x^2}\, dx.$

Solution. (i) Let $I = \int \underset{\text{II}}{x^2} \log \underset{\text{I}}{(1+x)}\, dx$

Integrating by parts, we get

$$I = \log(1+x) \cdot \int x^2\, dx - \int \left\{\frac{d}{dx}(\log(1+x)) \cdot \int x^2\, dx\right\} dx$$

$$= \log(1+x) \cdot \frac{x^3}{3} - \int \frac{1}{1+x} \cdot \frac{x^3}{3}\, dx = \frac{x^3}{3}\log(1+x) - \frac{1}{3}\int \frac{x^3}{1+x}\, dx$$

See Note. Given in the example 4 (*ii*).

$$= \frac{x^3}{3}\log(1+x) - \frac{1}{3}\int\left[x^2 - x + 1 - \frac{1}{1+x}\right] dx$$

$$= \frac{x^3}{3}\log(1+x)$$

$$- \frac{1}{3}\left[\int x^2\, dx - \int x\, dx + \int 1 . dx - \int \frac{1}{1+x} dx\right]$$

$$= \frac{x^3}{3}\log(1+x) - \frac{1}{3}\left[\frac{x^3}{3} - \frac{x^2}{2} + x - \log|1+x|\right] + c$$

$$= \frac{x^3}{3}\log(1+x) - \frac{x^3}{9} + \frac{x^2}{6} - \frac{x}{3} + \frac{1}{3}\log|1+x| + c.$$

$$\begin{array}{r|l}
x+1 \,)\; x^3 & (\, x^2 - x + 1 \\
x^3 + x^2 & \\
- \quad - & \\ \hline
-x^2 & \\
-x^2 - x & \\
+ \quad + & \\ \hline
x & \\
x + 1 & \\
- \quad - & \\ \hline
-1 &
\end{array}$$

(*ii*) Let $I = \int \log\left(x+\sqrt{a^2+x^2}\right) dx$

$$= \int \underset{\text{II}}{1} \cdot \underset{\text{I}}{\log\left(x+\sqrt{a^2+x^2}\right)} dx$$ [Taking unity as second function]

Integrating by parts, we get

$$I = \log\left(x+\sqrt{a^2+x^2}\right) . \int 1 . dx - \int \left\{\frac{d}{dx}\left[\log\left(x+\sqrt{a^2+x^2}\right)\right] . \int 1 . dx\right\} dx$$

$$= \log\left(x+\sqrt{a^2+x^2}\right) . x - \int \frac{1}{x+\sqrt{a^2+x^2}}\left(1 + \frac{1}{2\sqrt{a^2+x^2}} . 2x\right) . x\, dx$$

$$= x\log\left(x+\sqrt{a^2+x^2}\right) - \int \frac{1}{x+\sqrt{a^2+x^2}}\left(1+\frac{x}{\sqrt{a^2+x^2}}\right) x\, dx$$

$$= x\log\left(x+\sqrt{a^2+x^2}\right)-\int \frac{1}{(x+\sqrt{a^2+x^2})}\left(\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}}\right)x\,dx$$

$$= x\log\left(x+\sqrt{a^2+x^2}\right)-\int \frac{x}{\sqrt{a^2+x^2}}\,dx$$

$$= x\log\left(x+\sqrt{a^2+x^2}\right)-\int (a^2+x^2)^{-1/2}\,.\,x\,dx$$

$$= x\log\left(x+\sqrt{a^2+x^2}\right)-\frac{1}{2}\int (a^2+x^2)^{-1/2}\,.\,2x\,dx \qquad \text{[Multiply and divided by 2]}$$

$$= x\log\left(x+\sqrt{a^2+x^2}\right)-\frac{1}{2}\cdot\frac{(a^2+x^2)^{-1/2+1}}{-\frac{1}{2}+1}+c$$

$$\left[\because \int [f(x)]^n\,.\,f'(x)\,dx=\frac{[f(x)]^{n+1}}{n+1}+c\right]$$

$$= x\log\left(x+\sqrt{a^2+x^2}\right)-\frac{1}{2}\,\frac{(a^2+x^2)^{1/2}}{\frac{1}{2}}+c$$

$$= x\log\left(x+\sqrt{a^2+x^2}\right)-\sqrt{a^2+x^2}+c.$$

(*iii*) Let $\quad I=\int \sin^{-1}x\,dx$

Put $\quad x=\sin\theta \Rightarrow \sin^{-1}x=\theta$

$\Rightarrow \quad dx=\cos\theta\,d\theta$

$\therefore \quad I=\int \underset{\mathrm{I}}{\theta}\,\underset{\mathrm{II}}{\cos\theta}\,d\theta$

Integrating by parts, we get

$$I=\theta\,.\int\cos\theta\,d\theta-\int\left\{\frac{d}{d\theta}(\theta)\,.\int\cos\theta\,d\theta\right\}d\theta$$

$$=\theta\sin\theta-\int 1\,.\,\sin\theta\,d\theta$$

$$=\theta\sin\theta+\cos\theta+c$$

$$=\theta\sin\theta+\sqrt{1-\sin^2\theta}+c \qquad \left[\begin{array}{l}\because \sin^2 A+\cos^2 A=1\\ \Rightarrow \cos^2 A=1-\sin^2 A\\ \Rightarrow \cos A=\sqrt{1-\sin^2 A}\end{array}\right]$$

$$=x\sin^{-1}x+\sqrt{1-x^2}+c. \qquad [\because\ x=\sin\theta]$$

(*iv*) Let $\quad I=\int \tan^{-1}x\,dx$

Put $\quad x=\tan\theta \Rightarrow \tan^{-1}x=\theta$

$\Rightarrow \quad dx=\sec^2\theta\,d\theta$

$\therefore \quad I=\int \underset{\mathrm{I}}{\theta}\,\underset{\mathrm{II}}{\sec^2\theta}\,d\theta$

Integrating by parts, we get

$$\mathrm{I} = \theta \,.\int \sec^2\theta \, d\theta - \int \left\{\frac{d}{d\theta}(\theta) . \int \sec^2\theta \, d\theta\right\} d\theta$$

$$= \theta \tan\theta - \int 1 . \tan\theta \, d\theta = \theta \tan\theta - \log|\sec\theta| + c$$

$$= \theta \tan\theta - \log\left|\sqrt{1+\tan^2\theta}\right| + c \qquad \begin{bmatrix} \because \sec^2 A - \tan^2 A = 1 \\ \Rightarrow \sec^2 A = 1 + \tan^2 A \\ \Rightarrow \sec A = \sqrt{1+\tan^2 A} \end{bmatrix}$$

$$= x \tan^{-1} x - \log\left|\sqrt{1+x^2}\right| + c \qquad [\because x = \tan\theta]$$

$$= x \tan^{-1} x - \frac{1}{2}\log|1+x^2| + c.$$

(*v*) Let $\quad \mathrm{I} = \int \sec^{-1} x \, dx$

Put $\quad x = \sec\theta \ \Rightarrow\ \sec^{-1} x = \theta$

$\Rightarrow \quad dx = \sec\theta \tan\theta \, d\theta$

$\therefore \quad \mathrm{I} = \int \underset{\mathrm{I}}{\theta} . \underset{\mathrm{II}}{\sec\theta \tan\theta} \, d\theta$

Integrating by parts, we get

$$= \theta \,.\int \sec\theta \tan\theta \, d\theta - \int \left\{\frac{d}{d\theta}(\theta) . \int \sec\theta \tan\theta \, d\theta\right\} d\theta$$

$$= \theta \sec\theta - \int 1 . \sec\theta \, d\theta = \theta \sec\theta - \log|\sec\theta + \tan\theta| + c$$

$$= \theta \sec\theta - \log\left|\sec\theta + \sqrt{\sec^2\theta - 1}\right| + c \qquad \begin{bmatrix} \because \sec^2 A - \tan^2 A = 1 \\ \Rightarrow \tan^2 A = \sec^2 A - 1 \\ \Rightarrow \tan A = \sqrt{\sec^2 A - 1} \end{bmatrix}$$

$$= x \sec^{-1} x - \log\left|x + \sqrt{x^2 - 1}\right| + c. \qquad [\because x = \sec\theta]$$

(*vi*) Let $\quad \mathrm{I} = \int 2x^3 e^{x^2} dx = \int (2x)\, x^2 e^{x^2} dx$

Put $\quad x^2 = z \ \Rightarrow\ 2x \, dx = dz$

$\therefore \quad \mathrm{I} = \int \underset{\mathrm{I}}{z} \underset{\mathrm{II}}{e^z} \, dz$

Integrating by parts, we get

$$\mathrm{I} = z \,.\int e^z \, dz - \int \left\{\frac{d}{dz}(z) . \int e^z \, dz\right\} dz$$

$$= z e^z - \int 1 . e^z \, dz$$

$$= z e^z - e^z + c = x^2 e^{x^2} - e^{x^2} + c \qquad [\because z = x^2]$$

$$= e^{x^2}(x^2 - 1) + c.$$

Example 7. *Evaluate the following integrals :*

(*i*) $\int \frac{\log(x+2)}{(x+2)^2}\,dx$ (*ii*) $\int e^x \cos x\,dx$

(*iii*) $\int \frac{x-\sin x}{1-\cos x}\,dx$ (*iv*) $\int \sec^3 x\,dx$

(*v*) $\int \cos^{-1} x\,dx$ (*vi*) $\int (\sin^{-1} x)^2\,dx.$

Solution. (*i*) Let $I = \int \frac{\log(x+2)}{(x+2)^2}\,dx = \int \log(x+2) \cdot \frac{1}{(x+2)^2}\,dx$

$$= \int \underset{\text{II}}{(x+2)^{-2}} \cdot \underset{\text{I}}{\log(x+2)}\,dx$$

Integrating by parts, we get

$$I = \log(x+2) \cdot \int (x+2)^{-2}\,dx - \int \left\{\frac{d}{dx}\log(x+2) \cdot \int (x+2)^{-2}\,dx\right\} dx$$

$$= \log(x+2) \cdot \frac{(x+2)^{-2+1}}{(-2+1)} - \int \frac{1}{x+2} \cdot \frac{(x+2)^{-2+1}}{-2+1}\,dx$$

$$= -\frac{\log(x+2)}{(x+2)} + \int \frac{1}{(x+2)^2}\,dx = -\frac{\log(x+2)}{x+2} + \int (x+2)^{-2}\,dx$$

$$= -\frac{\log(x+2)}{x+2} + \frac{(x+2)^{-2+1}}{(-2+1)} + c$$

$$= -\frac{\log(x+2)}{x+2} - \frac{1}{(x+2)} + c.$$

(*ii*) Let $I = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\cos x}\,dx$ ...(1)

Integrating by parts, we get

$$I = \cos x \cdot \int e^x\,dx - \int \left\{\frac{d}{dx}(\cos x) \cdot \int e^x\,dx\right\} dx$$

$$= \cos x\, e^x - \int (-\sin x)\, e^x\,dx$$

$$= e^x \cos x + \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin x}\,dx \qquad \text{[Integrating again by parts]}$$

$$= e^x \cos x + \sin x \cdot \int e^x\,dx - \int \left\{\frac{d}{dx}(\sin x) \cdot \int e^x\,dx\right\} dx$$

$$= e^x \cos x + \sin x\, e^x - \int \cos x\, e^x\,dx$$

$\Rightarrow$ $I = e^x \cos x + e^x \sin x - I$ [$\because$ By using (1)]

$\Rightarrow$ $2I = e^x (\cos x + \sin x)$

$\Rightarrow$ $I = \frac{e^x}{2}(\sin x + \cos x) + c.$

(*iii*) Let $\quad I = \int \frac{x - \sin x}{1 - \cos x}\, dx$

$$= \int \frac{x}{1-\cos x}\, dx - \int \frac{\sin x}{1-\cos x}\, dx$$

$$= \int \frac{x}{2\sin^2 \frac{x}{2}}\, dx - \int \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}}\, dx$$

$$= \frac{1}{2} \int \underset{\text{I}}{x}\, \underset{\text{II}}{\text{cosec}^2} \frac{x}{2}\, dx - \int \cot \frac{x}{2}\, dx$$

$\left[\begin{array}{l} \because \sin 2A = 2 \sin A \cos A \\ \Rightarrow \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \\ \quad\text{and} \\ \cos 2A = 1 - 2 \sin^2 A \\ \Rightarrow 2 \sin^2 A = 1 - \cos 2A \\ \Rightarrow 2 \sin^2 \frac{A}{2} = 1 - \cos A \end{array}\right]$

Integrating by parts, we get

$$I = \frac{1}{2}\left[x \,.\int \text{cosec}^2 \frac{x}{2}\, dx - \int \left\{\frac{d}{dx}(x) \,.\int \text{cosec}^2 \frac{x}{2}\, dx\right\} dx\right] - \int \cot \frac{x}{2}\, dx$$

$$= \frac{1}{2}\left[x \,.\left(-2\cot\frac{x}{2}\right) - \int 1 .\left(-2 \cot \frac{x}{2}\right) dx\right] - \int \cot \frac{x}{2}\, dx + c$$

$$= -x \cot \frac{x}{2} + \int \cot \frac{x}{2}\, dx - \int \cot \frac{x}{2}\, dx + c = -x \cot \frac{x}{2} + c.$$

(*iv*) Let $\quad I = \int \sec^3 x\, dx \qquad ...(1)$

$$= \int \underset{\text{I}}{\sec x} \,.\, \underset{\text{II}}{\sec^2 x}\, dx$$

Integrating by parts, we get

$$I = \sec x \,.\int \sec^2 x\, dx - \int \left\{\frac{d}{dx}(\sec x) \,.\int \sec^2 x\, dx\right\} dx$$

$$= \sec x \,.\, \tan x - \int (\sec x \tan x) \tan x\, dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x\, dx \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx \qquad [\because \text{ By using (1)}]$$

$\Rightarrow \quad I = \sec x \tan x - I + \log |\sec x + \tan x| + c$

$\Rightarrow \quad 2I = \sec x \tan x + \log |\sec x + \tan x| + c$

$\Rightarrow \quad I = \frac{1}{2} (\sec x \tan x + \log |\sec x + \tan x| + c).$

(*v*) Let $\quad I = \int \cos^{-1} x\, dx$

$$= \int \underset{\text{II}}{1} .\, \underset{\text{I}}{\cos^{-1} x}\, dx \qquad \text{[Taking unity as second function]}$$

Integrating by parts, we get

$$I = \cos^{-1} x \,.\int 1 . dx - \int \left\{ \frac{d}{dx}(\cos^{-1} x) . \int .1 . dx \right\} dx$$

$$= \cos^{-1} x \,.\, x - \int \frac{-1}{\sqrt{1-x^2}} . x\, dx$$

$$= x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\, dx$$

Put $1 - x^2 = z \Rightarrow -2x\, dx = dz \Rightarrow x\, dx = -\frac{1}{2}\, dz$

$\therefore$

$$I = x \cos^{-1} x + \int \frac{1}{\sqrt{z}}\left(-\frac{1}{2}\, dz\right) = x \cos^{-1} x - \frac{1}{2}\int z^{-1/2}\, dz$$

$$= x \cos^{-1} x - \frac{1}{2}\left[\frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] + c$$

$$= x \cos^{-1} x - z^{1/2} + c = x \cos^{-1} x - \sqrt{z} + c$$

$$= x \cos^{-1} x - \sqrt{1-x^2} + c. \qquad [\because\ 1 - x^2 = z]$$

(*vi*) Let $I = \int (\sin^{-1} x)^2\, dx$

Put $\sin^{-1} x = z \Rightarrow x = \sin z \Rightarrow dx = \cos z\, dz$

$\therefore$ $I = \int \underset{\text{I}}{z^2} . \underset{\text{II}}{\cos z}\, dz$

Integrating by parts, we get

$$I = z^2 . \int \cos z\, dz - \int \left\{ \frac{d}{dz}(z^2) . \int \cos z\, dz \right\} dz$$

$$= z^2 (\sin z) - \int 2z . \sin z\, dz$$

$$= z^2 \sin z - 2 \int \underset{\text{I}}{z} \underset{\text{II}}{\sin z}\, dz \qquad \text{[Integrating again by parts]}$$

$$= z^2 \sin z - 2\left[z . \int \sin z\, dz - \int \left\{ \frac{d}{dz}(z) . \int \sin z\, dz \right\} dz\right]$$

$$= z^2 \sin z - 2\left[z(-\cos z) - \int 1 . (-\cos z)\, dz\right]$$

$$= z^2 \sin z + 2z \cos z - 2 \int \cos z\, dz = z^2 \sin z + 2z \cos z - 2 \sin z + c$$

$$= z^2 \sin z + 2z \sqrt{1 - \sin^2 z} - 2 \sin z + c \qquad \left[\begin{array}{l} \because\ \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \cos^2 A = 1 - \sin^2 A \\ \Rightarrow \cos A = \sqrt{1 - \sin^2 A} \end{array}\right]$$

$$= (\sin^{-1} x)^2 \,.\, x + 2 \sin^{-1} x \sqrt{1-x^2} - 2x + c \qquad [\because \ \sin^{-1} x = z]$$

$$= x (\sin^{-1} x)^2 + 2 \sqrt{1-x^2} \,.\, \sin^{-1} x - 2x + c.$$

Example 8. *Evaluate the following integrals :*

(*i*) $\int \sin \sqrt{x}\, dx$ (*ii*) $\int x \sin^{-1} x\, dx$

(*iii*) $\int x \cos^{-1} x\, dx$ (*iv*) $\int x \cot^{-1} x\, dx$

(*v*) $\int x \operatorname{cosec}^{-1} x\, dx$ (*vi*) $\int \cot^{-1} x\, dx$

(*vii*) $\int \sin^3 \sqrt{x}\, dx$ (*viii*) $\int \sec^{-1} \sqrt{x}\, dx$

(*ix*) $\int \cos x \log \sin x\, dx.$

Solution. (*i*) Let $\mathrm{I} = \int \sin \sqrt{x}\, dx$

Put $\sqrt{x} = z \Rightarrow \dfrac{1}{2\sqrt{x}} dx = dz \Rightarrow \dfrac{1}{2z} dx = dz \Rightarrow dx = 2z\, dz$

$\therefore \quad \mathrm{I} = \int (\sin z)(2z\, dz) = 2 \int \underset{\mathrm{I}}{z} \underset{\mathrm{II}}{\sin z}\, dz$

Integrating by parts, we get

$$\mathrm{I} = 2\left[z \,.\, \int \sin z\, dz - \int \left\{\frac{d}{dz}(z) \,.\, \int \sin z\, dz\right\} dz\right]$$

$$= 2\left[z(-\cos z) - \int 1 \,.\, (-\cos z)\, dz\right]$$

$$= -2z \cos z + 2 \int \cos z\, dz = -2z \cos z + 2 \sin z + c$$

$$= -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + c. \qquad [\because \ z = \sqrt{x}\,]$$

(*ii*) Let $\mathrm{I} = \int \underset{\mathrm{II}}{x} \underset{\mathrm{I}}{\sin^{-1} x}\, dx$ ...(1)

Integrating by parts, we get

$$\mathrm{I} = \sin^{-1} x \,.\, \int x \,.\, dx - \int \left\{\frac{d}{dx}(\sin^{-1} x) \,.\, \int x \,.\, dx\right\} dx$$

$$= \sin^{-1} x \,.\, \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \,.\, \frac{x^2}{2}\, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}}\, dx$$

$\Rightarrow \quad \mathrm{I} = \dfrac{x^2}{2} \sin^{-1} x - \dfrac{1}{2} \mathrm{I}_1$ ...(2)

where $\mathrm{I}_1 = \int \dfrac{x^2}{\sqrt{1-x^2}}\, dx$

Put $x = \sin \theta \Rightarrow x^2 = \sin^2 \theta$

$\Rightarrow \quad dx = \cos \theta\, d\theta$

$$\therefore \qquad I_1 = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$$

$$= \int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} . \cos\theta \, d\theta \qquad [\because \ \sin^2 A + \cos^2 A = 1]$$

$$= \int \frac{\sin^2\theta}{\cos\theta} \cos\theta \, d\theta = \int \sin^2\theta \, d\theta$$

$$= \int \frac{1-\cos 2\theta}{2} \, d\theta = \frac{1}{2}\int 1 . d\theta - \frac{1}{2}\int \cos 2\theta \, d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{2} . \frac{\sin 2\theta}{2} + c_1 \qquad \left[\begin{array}{l} \because \ \cos 2A = 1 - 2\sin^2 A \\ \Rightarrow 2\sin^2 A = 1 - \cos 2A \\ \Rightarrow \sin^2 A = \left(\dfrac{1-\cos 2A}{2}\right) \end{array}\right]$$

$$= \frac{1}{2}\theta - \frac{1}{4}(2\sin\theta\cos\theta) + c_1 \qquad [\because \ \sin 2A = 2\sin A \cos A]$$

$$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + c_1$$

$$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta} + c_1 \qquad \left[\begin{array}{l} \because \ \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \cos^2 A = 1 - \sin^2 A \\ \Rightarrow \cos A = \sqrt{1-\sin^2 A} \end{array}\right]$$

$$\Rightarrow \qquad I_1 = \frac{1}{2}\sin^{-1} x - \frac{1}{2} x\sqrt{1-x^2} + c_1 \qquad [\because \ x = \sin\theta]$$

Putting this value of I_1 in equation (2), we have

$$I = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left[\frac{1}{2}\sin^{-1} x - \frac{1}{2} x\sqrt{1-x^2} + c_1\right]$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\left[\sin^{-1} x - x\sqrt{1-x^2}\right] - \frac{1}{2}c_1$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\left[\sin^{-1} x - x\sqrt{1-x^2}\right] + c. \qquad \text{where } c = -\frac{1}{2}c_1.$$

(iii) **Let** $I = \int \underset{\text{II}}{x} \underset{\text{I}}{\cos^{-1} x} \, dx$ ...(1)

Integrating by parts, we get

$$I = \cos^{-1} x \int x . dx - \int \left\{\frac{d}{dx}(\cos^{-1} x) . \int x \, dx\right\} dx$$

$$= \cos^{-1} x . \frac{x^2}{2} - \int \frac{-1}{\sqrt{1-x^2}} . \frac{x^2}{2} \, dx = \frac{x^2}{2}\cos^{-1} x + \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} \, dx$$

$\Rightarrow \qquad \mathrm{I} = \dfrac{x^2}{2}\cos^{-1}x + \dfrac{1}{2}\mathrm{I}_1 \qquad \ldots(2)$

where $\quad \mathrm{I}_1 = \displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

Put $\quad x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$$\therefore \quad \mathrm{I}_1 = \int \frac{\cos^2\theta}{\sqrt{1-\cos^2\theta}}(-\sin\theta\, d\theta)$$

$$= -\int \frac{\cos^2\theta}{\sqrt{\sin^2\theta}}\sin\theta\, d\theta \qquad \begin{bmatrix} \because \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \sin^2 A = 1 - \cos^2 A \end{bmatrix}$$

$$= -\int \frac{\cos^2\theta}{\sin\theta}\sin\theta\, d\theta = -\int \cos^2\theta\, d\theta$$

$$= -\int \frac{1+\cos 2\theta}{2}\, d\theta \qquad \begin{bmatrix} \because \cos 2A = 2\cos^2 A - 1 \\ \Rightarrow 2\cos^2 A = 1 + \cos 2A \\ \Rightarrow \cos^2 A = \dfrac{1+\cos 2A}{2} \end{bmatrix}$$

$$= -\frac{1}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta\, d\theta$$

$$= -\frac{1}{2}\theta - \frac{1}{2}\frac{\sin 2\theta}{2} + c_1 = -\frac{1}{2}\theta - \frac{1}{4}(2\sin\theta\cos\theta) + c_1$$

$$= -\frac{1}{2}\theta - \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta} + c_1 \qquad \begin{bmatrix} \because \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \cos^2 A = 1 - \sin^2 A \\ \Rightarrow \cos A = \sqrt{1-\sin^2 A} \end{bmatrix}$$

$$\Rightarrow \quad \mathrm{I}_1 = -\frac{1}{2}\cos^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + c_1$$

Putting this value of I_1 in equation (2), we have

$$\therefore \quad \mathrm{I} = \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\left[-\frac{1}{2}\cos^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + c_1\right]$$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{1}{4}\cos^{-1}x - \frac{1}{4}x\sqrt{1-x^2} + \frac{1}{2}c_1$$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{1}{4}\cos^{-1}x - \frac{1}{4}x\sqrt{1-x^2} + c \qquad \text{where } c = \frac{1}{2}c_1.$$

(*iv*) Let $\quad \mathrm{I} = \displaystyle\int \underset{\mathrm{II}}{x}\,\underset{\mathrm{I}}{\cot^{-1}x}\, dx$

Integrating by parts, we get

$$\mathrm{I} = \cot^{-1}x \,.\int x\, dx - \int\left\{\frac{d}{dx}(\cot^{-1}x)\,.\int x\, dx\right\}dx$$

$$= \cot^{-1} x \cdot \frac{x^2}{2} - \int \left(\frac{-1}{1+x^2}\right) \cdot \frac{x^2}{2}\, dx = \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} \int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} \int \frac{1+x^2-1}{1+x^2}\, dx \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} \int \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right) dx$$

$$= \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} \int 1 . dx - \frac{1}{2} \int \frac{1}{1+x^2}\, dx \qquad \left[\because \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \frac{x^2}{2} \cot^{-1} x + \frac{1}{2} x - \frac{1}{2} \tan^{-1} x + c.$$

(*v*) Let $\quad \mathrm{I} = \int \underset{\mathrm{II}}{x} \underset{\mathrm{I}}{\operatorname{cosec}^{-1} x}\, dx \qquad \ldots(1)$

Integrating by parts, we get

$$\mathrm{I} = \operatorname{cosec}^{-1} x \cdot \int x \cdot dx - \int \left\{\frac{d}{dx} (\operatorname{cosec}^{-1} x) . \int x\, dx\right\} dx$$

$$= \operatorname{cosec}^{-1} x \cdot \frac{x^2}{2} - \int \left(\frac{-1}{x\sqrt{x^2-1}}\right) \cdot \frac{x^2}{2}\, dx = \frac{x^2}{2} \operatorname{cosec}^{-1} x + \frac{1}{2} \int \frac{x}{\sqrt{x^2-1}}\, dx$$

$$\Rightarrow \qquad \mathrm{I} = \frac{x^2}{2} \operatorname{cosec}^{-1} x + \frac{1}{2} \mathrm{I}_1 \qquad \ldots(2)$$

where $\quad \mathrm{I}_1 = \int \frac{x}{\sqrt{x^2-1}}\, dx$

Put $x^2 - 1 = z \;\Rightarrow\; 2x\, dx = dz \;\Rightarrow\; x\, dx = \frac{1}{2} dz$

$$\therefore \qquad \mathrm{I}_1 = \int \frac{1}{\sqrt{z}} \left(\frac{1}{2} dz\right) = \frac{1}{2} \int z^{-1/2}\, dz$$

$$= \frac{1}{2} \left[\frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] + c_1 = \frac{1}{2} \cdot \frac{z^{1/2}}{1/2} + c_1 = \sqrt{z} + c_1$$

$$\Rightarrow \qquad \mathrm{I}_1 = \sqrt{x^2-1} + c_1 \qquad [\because\; z = x^2 - 1]$$

Putting this value of I_1 in equation (2), we have

$$\mathrm{I} = \frac{x^2}{2} \operatorname{cosec}^{-1} x + \frac{1}{2}\left[\sqrt{x^2-1} + c_1\right] = \frac{x^2}{2} \operatorname{cosec}^{-1} x + \frac{1}{2} \sqrt{x^2-1} + \frac{1}{2} c_1$$

$$= \frac{x^2}{2} \operatorname{cosec}^{-1} x + \frac{1}{2} \sqrt{x^2-1} + c. \qquad \text{where } c = \frac{1}{2} c_1.$$

(*vi*) Let $\quad I = \int \cot^{-1} x \, dx$

$= \int \underset{\text{II}}{1} . \underset{\text{I}}{\cot^{-1} x} \, dx \quad$ [Taking unity as the second function]

Integrating by parts, we get

$$I = \cot^{-1} x . \int 1 . dx - \int \left\{ \frac{d}{dx} (\cot^{-1} x) . \int 1 . dx \right\} dx$$

$$= \cot^{-1} x . x - \int \left(\frac{-1}{1+x^2} \right) x \, dx = x \cot^{-1} x + \int \frac{x}{1+x^2} \, dx$$

$$= x \cot^{-1} x + \frac{1}{2} \int \frac{2x}{1+x^2} \, dx \quad \text{[Multiply and divided by 2]}$$

$$= x \cot^{-1} x + \frac{1}{2} \log |1 + x^2| + c. \quad \left[\because \int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c \right]$$

(*vii*) Let $\quad I = \int \sin^3 \sqrt{x} \, dx$

Put $\sqrt{x} = z \Rightarrow \frac{1}{2\sqrt{x}} dx = dz \Rightarrow \frac{1}{2z} dx = dz \Rightarrow dx = 2z \, dz$

$\therefore \quad I = \int (\sin^3 z) \, 2z \, dz = 2 \int z \sin^3 z \, dz$

$$= 2 \int z \left(\frac{3}{4} \sin z - \frac{1}{4} \sin 3z \right) dz \quad \left[\begin{array}{l} \because \sin 3A = 3 \sin A - 4 \sin^3 A \\ \Rightarrow 4 \sin^3 A = 3 \sin A - \sin 3A \\ \Rightarrow \sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin 3A \end{array} \right]$$

$$= \frac{3}{2} \int \underset{\text{I}}{z} \, \underset{\text{II}}{\sin z} \, dz - \frac{1}{2} \int \underset{\text{I}}{z} \, \underset{\text{II}}{\sin 3z} \, dz$$

Integrating by parts, we get

$$I = \frac{3}{2} \left[z . \int \sin z \, dz - \int \left\{ \frac{d}{dz}(z) . \int \sin z \, dz \right\} dz \right]$$

$$- \frac{1}{2} \left[z . \int \sin 3z \, dz - \int \left\{ \frac{d}{dz}(z) . \int \sin 3z \, dz \right\} dz \right]$$

$$= \frac{3}{2} \left[z . (-\cos z) - \int 1 . (-\cos z) \, dz \right] - \frac{1}{2} \left[z . \left(\frac{-\cos 3z}{3} \right) - \int 1 . \left(\frac{-\cos 3z}{3} \right) dz \right]$$

$$= -\frac{3}{2} z \cos z + \frac{3}{2} \int \cos z \, dz + \frac{1}{6} z \cos 3z - \frac{1}{6} \int \cos 3z \, dz$$

$$= -\frac{3}{2} z \cos z + \frac{3}{2} \sin z + \frac{1}{6} z \cos 3z - \frac{1}{6} \left(\frac{\sin 3z}{3} \right) + c$$

$$= -\frac{3}{2} z \cos z + \frac{3}{2} \sin z + \frac{1}{6} z \cos 3z - \frac{1}{18} \sin 3z + c$$

$$= -\frac{3}{2}\sqrt{x}\cos\sqrt{x} + \frac{3}{2}\sin\sqrt{x} + \frac{1}{6}\sqrt{x}\cos 3\sqrt{x} - \frac{1}{18}\sin 3\sqrt{x} + c.$$

(viii) Let $\quad I = \int \sec^{-1}\sqrt{x}\, dx \quad \text{...(1)}$

Put $\quad \sqrt{x} = z \Rightarrow \frac{1}{2\sqrt{x}}\, dx = dz \Rightarrow \frac{1}{2z}\, dx = dz \Rightarrow dx = 2z\, dz$

$$\therefore \quad I = \int (\sec^{-1} z)\, 2z\, dz$$

$$= 2\int \underset{\text{II}}{z}\, \underset{\text{I}}{\sec^{-1} z}\, dz \quad \text{...(2)}$$

Integrating by parts, we get

$$I = 2\left[\sec^{-1} z \,.\int z\, dz - \int \left\{\frac{d}{dz}(\sec^{-1} z).\int z\, dz\right\} dz\right]$$

$$= 2\left[\sec^{-1} z \,.\frac{z^2}{2} - \int \frac{1}{z\sqrt{z^2-1}}.\frac{z^2}{2}\, dz\right]$$

$$= z^2 \sec^{-1} z - \int \frac{z}{\sqrt{z^2-1}}\, dz$$

$$\Rightarrow \quad I = z^2 \sec^{-1} z - I_1 \quad \text{...(3)}$$

where $\quad I_1 = \int \frac{z}{\sqrt{z^2-1}}\, dz$

Put $\quad z^2 - 1 = y \Rightarrow 2z\, dz = dy \Rightarrow z\, dz = \frac{1}{2}\, dy$

$$\therefore \quad I_1 = \int \frac{1}{\sqrt{y}}\left(\frac{1}{2}dy\right) = \frac{1}{2}\int y^{-1/2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] + c_1 = \frac{1}{2}\frac{y^{1/2}}{1/2} + c_1 = \sqrt{y} + c_1$$

$$= \sqrt{z^2-1} + c_1 \quad [\because\ y = z^2 - 1]$$

Putting this value of I_1 in equation (3), we have

$$I = z^2 \sec^{-1} z - \left(\sqrt{z^2-1} + c_1\right) = x \sec^{-1}\sqrt{x} - \sqrt{x-1} - c_1$$

$$= x \sec^{-1}\sqrt{x} - \sqrt{x-1} + c \quad \text{where } c = -c_1.$$

(ix) Let $\quad I = \int \underset{\text{II}}{\cos x}\,.\,\underset{\text{I}}{\log \sin x}\, dx$

Integrating by parts, we get

$$= \log \sin x \,.\int \cos x\, dx - \int \left\{\frac{d}{dx}(\log \sin x).\int \cos x\, dx\right\} dx$$

$$= (\log \sin x)(\sin x) - \int \left(\frac{1}{\sin x} \cos x\right). \sin x \, dx$$

$$= \sin x (\log \sin x) - \int \cos x \, dx$$

$$= \sin x (\log \sin x) - \sin x + c.$$

Example 9. *Evaluate the following integrals :*

(*i*) $\int x \sin x \sin 2x \sin 3x \, dx$ (*ii*) $\int \cos \sqrt{x} \, dx$

(*iii*) $\int \sin^{-1} (3x - 4x^3) \, dx$ (*iv*) $\int x \tan^{-1} x \, dx$

(*v*) $\int x \sec^{-1} x \, dx$.

Solution. (*i*) Let $I = \int x \sin x \sin 2x \sin 3x \, dx$

$$= \frac{1}{2} \int x (2 \sin 3x \sin 2x) \sin x \, dx$$ [Multiply and divided by 2]

$$= \frac{1}{2} \int x [\cos (3x - 2x) - \cos (3x + 2x)] \sin x \, dx$$

[$\because$ $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$]

$$= \frac{1}{2} \int x [\cos x - \cos 5x] \sin x \, dx$$

$$= \frac{1}{2} \int (x \cos x \sin x - x \cos 5x \sin x) \, dx$$

$$= \frac{1}{4} \int x (2 \cos x \sin x) \, dx - \frac{1}{4} \int x (2 \cos 5x \sin x) \, dx$$

[Multiply and divided by 2 again]

$$= \frac{1}{4} \int x \sin 2x \, dx - \frac{1}{4} \int x [\sin (5x + x) - \sin (5x - x)] \, dx$$

[$\because$ $\sin 2A = 2 \sin A \cos A$]

[$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$]

$$= \frac{1}{4} \int \underset{\text{I}}{x} \underset{\text{II}}{\sin 2x} - \frac{1}{4} \int \underset{\text{I}}{x} \underset{\text{II}}{\sin 6x} \, dx + \frac{1}{4} \int \underset{\text{I}}{x} \underset{\text{II}}{\sin 4x} \, dx$$

Integrating by parts, we get

$$I = \frac{1}{4}\left[x . \int \sin 2x \, dx - \int \left\{\frac{d}{dx}(x) . \int \sin 2x \, dx\right\} dx\right]$$

$$- \frac{1}{4}\left[x . \int \sin 6x \, dx - \int \left\{\frac{d}{dx}(x) . \int \sin 6x \, dx\right\} dx\right]$$

$$+ \frac{1}{4}\left[x . \int \sin 4x \, dx - \int \left\{\frac{d}{dx}(x) . \int \sin 4x . dx\right\} dx\right]$$

$$= \frac{1}{4}\left[x.\left(-\frac{\cos 2x}{2}\right) - \int 1.\left(-\frac{\cos 2x}{2}\right)dx\right] - \frac{1}{4}\left[x.\left(-\frac{\cos 6x}{6}\right) - \int 1.\left(-\frac{\cos 6x}{6}\right)dx\right]$$

$$+ \frac{1}{4}\left[x.\left(-\frac{\cos 4x}{4}\right) - \int 1.\left(-\frac{\cos 4x}{4}\right)dx\right]$$

$$= -\frac{1}{8}x\cos 2x + \frac{1}{8}\int \cos 2x\,dx + \frac{1}{24}x\cos 6x - \frac{1}{24}\int \cos 6x\,dx$$

$$-\frac{1}{16}x\cos 4x + \frac{1}{16}\int \cos 4x\,dx$$

$$= -\frac{1}{8}x\cos 2x + \frac{1}{8}\left(\frac{\sin 2x}{2}\right) + \frac{1}{24}x\cos 6x - \frac{1}{24}\left(\frac{\sin 6x}{6}\right)$$

$$-\frac{1}{16}x\cos 4x + \frac{1}{16}\left(\frac{\sin 4x}{4}\right) + c$$

$$= -\frac{1}{8}x\cos 2x + \frac{1}{16}\sin 2x + \frac{1}{24}x\cos 6x - \frac{1}{144}\sin 6x - \frac{1}{16}x\cos 4x + \frac{1}{64}\sin 4x + c.$$

(*ii*) Let $\quad I = \int \cos\sqrt{x}\,dx$

Put $\quad \sqrt{x} = z \;\Rightarrow\; \frac{1}{2\sqrt{x}}\,dx = dz \;\Rightarrow\; \frac{1}{2z}\,dx = dz \;\Rightarrow\; dx = 2z\,dz$

$\therefore \quad I = \int (\cos z)\,2z\,dz$

$$= 2\int \underset{\text{I}}{z}\,\underset{\text{II}}{\cos}\,z\,dz$$

Integrating by parts, we get

$$I = 2\left[z.\int \cos z\,dz - \int\left\{\frac{d}{dz}(z).\int \cos z\,dz\right\}dz\right]$$

$$= 2\left[z\sin z - \int 1.\sin z\,dz\right]$$

$$= 2z\sin z - 2(-\cos z) + c = 2z\sin z + 2\cos z + c$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + c. \qquad [\because\; z = \sqrt{x}]$$

(*iii*) Let $\quad I = \int \sin^{-1}(3x - 4x^3)\,dx$

Put $\quad x = \sin\theta \;\Rightarrow\; \sin^{-1}x = \theta$

$\Rightarrow \quad dx = \cos\theta\,d\theta$

$\therefore \quad I = \int \sin^{-1}(3\sin\theta - 4\sin^3\theta)\cos\theta\,d\theta \qquad [\because\; \sin 3A = 3\sin A - 4\sin^3 A]$

$$= \int \sin^{-1}(\sin 3\theta)\cos\theta\,d\theta = \int 3\theta\cos\theta\,d\theta$$

$$= 3\int \underset{\text{I}}{\theta}\,\underset{\text{II}}{\cos}\,\theta\,d\theta$$

Integrating by parts, we get

$$I = 3\left[\theta.\int \cos\theta\,d\theta - \int\left\{\frac{d}{d\theta}(\theta).\int \cos\theta\,d\theta\right\}d\theta\right]$$

$$= 3\left[\theta \sin\theta - \int 1 . \sin\theta \, d\theta\right]$$

$$= 3\theta \sin\theta - 3(-\cos\theta) + c = 3\theta\sin\theta + 3\cos\theta + c$$

$$= 3\theta\sin\theta + 3\sqrt{1-\sin^2\theta} + c$$

$$= (3\sin^{-1}x)\,x + 3\sqrt{1-x^2} + c \qquad \left[\begin{array}{l} \because \sin^2 A + \cos^2 A = 1 \\ \cos^2 A = 1 - \sin^2 A \\ \cos A = \sqrt{1-\sin^2 A} \end{array}\right]$$

$$= 3x\sin^{-1}x + 3\sqrt{1-x^2} + c. \qquad [\because x = \sin\theta]$$

(*iv*) Let $\quad I = \int \underset{\text{II}}{x}\,\underset{\text{I}}{\tan^{-1}x}\,dx$

Integrating by parts, we get

$$I = \tan^{-1}x . \int x\,dx - \int\left\{\frac{d}{dx}(\tan^{-1}x) . \int x . dx\right\}dx$$

$$= \tan^{-1}x . \frac{x^2}{2} - \int \frac{1}{1+x^2} . \frac{x^2}{2}\,dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(\frac{x^2+1-1}{x^2+1}\right)dx \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int 1 . dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$

$$\left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c.$$

(*v*) Let $\quad I = \int \underset{\text{II}}{x}\,\underset{\text{I}}{\sec^{-1}x}\,dx \qquad ...(1)$

Integrating by parts, we get

$$I = \sec^{-1}x . \int x\,dx - \int\left\{\frac{d}{dx}(\sec^{-1}x) . \int x . dx\right\}dx$$

$$= \sec^{-1}x . \frac{x^2}{2} - \int \frac{1}{x\sqrt{x^2-1}} . \frac{x^2}{2}\,dx = \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\int \frac{x}{\sqrt{x^2-1}}\,dx$$

$$\Rightarrow \qquad I = \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}I_1 \qquad ...(2)$$

where $\quad I_1 = \int \dfrac{x}{\sqrt{x^2-1}}\,dx$

Put $x^2 - 1 = z \Rightarrow 2x\,dx = dz \Rightarrow x\,dx = \frac{1}{2}dz$

$\therefore \quad I_1 = \int \frac{1}{\sqrt{z}} \cdot \left(\frac{1}{2}dz\right) = \frac{1}{2}\int z^{-1/2}\,dz$

$$= \frac{1}{2}\left[\frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] + c_1 = \frac{1}{2}\left[\frac{z^{1/2}}{1/2}\right] + c_1 = \sqrt{z} + c_1$$

$\Rightarrow \quad I_1 = \sqrt{x^2 - 1} + c_1 \qquad \left[\because\ z = \sqrt{x^2-1}\right]$

Putting this value of I_1 in equation (2), we have

$$I = \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\left[\sqrt{x^2-1} + c_1\right]$$

$$= \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\sqrt{x^2-1} - \frac{1}{2}c_1$$

$$= \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\sqrt{x^2-1} + c. \qquad \text{where } c = -\frac{1}{2}c_1.$$

Example 10. *Evaluate the following integrals :*

(i) $\int x^2 \sin^{-1} x\,dx$ *(ii)* $\int x^2 \cos^{-1} x\,dx$

(iii) $\int x^2 \cot^{-1} x\,dx$ *(iv)* $\int x^2 \operatorname{cosec}^{-1} x\,dx$

(v) $\int x^2 \tan^{-1} x\,dx.$

Solution. *(i)* Let $I = \int \underset{\text{II}}{x^2}\, \underset{\text{I}}{\sin^{-1}x}\,dx$ ...(1)

Integrating by parts, we get

$$I = \sin^{-1}x \int x^2\,dx - \int \left\{\frac{d}{dx}(\sin^{-1}x) \cdot \int x^2\,dx\right\} dx$$

$$= \sin^{-1}x \cdot \frac{x^3}{3} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^3}{3}\,dx = \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

$\Rightarrow \quad I = \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}I_1$ ...(2)

where $\quad I_1 = \int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{x^2}{\sqrt{1-x^2}} \cdot x\,dx$

Put $\quad \sqrt{1-x^2} = z$

$\Rightarrow \quad 1 - x^2 = z^2 \Rightarrow x^2 = 1 - z^2 \Rightarrow 2x\,dx = -2z\,dz \Rightarrow x\,dx = -z\,dz$

$\therefore \quad I_1 = \int \frac{(1-z^2)}{z}(-z\,dz) = \int (z^2 - 1)\,dz$

$$= \int z^2\,dz - \int 1 \cdot dz = \frac{z^3}{3} - z + c_1$$

$$= \frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2} + c_1 \qquad \left[\because\ z = \sqrt{1-x^2}\right]$$

$$\Rightarrow \qquad I_1 = \frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} + c_1$$

Putting this value of I_1 in equation (2), we have

$$I = \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\left[\frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} + c_1\right]$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{9}(1-x^2)^{3/2} + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{3}c_1$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{9}(1-x^2)^{3/2} + \frac{1}{3}\sqrt{1-x^2} + c_1. \qquad \text{where } c_1 = -\frac{1}{3}c_1.$$

(*ii*) Let $\qquad I = \int \underset{\text{II}}{x^2} \underset{\text{I}}{\cos^{-1}x}\, dx \qquad ...(1)$

Integrating by parts, we get

$$I = \cos^{-1}x \,.\, \int x^2\, dx - \int \left\{\frac{d}{dx}(\cos^{-1}x) \,.\, \int x^2 \,.\, dx\right\} dx$$

$$= \cos^{-1}x \,.\, \frac{x^3}{3} - \int \left(\frac{-1}{\sqrt{1-x^2}}\right) . \frac{x^3}{3}\, dx = \frac{x^3}{3}\cos^{-1}x + \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\, dx$$

$$\Rightarrow \qquad I = \frac{x^3}{3}\cos^{-1}x + \frac{1}{3} I_1 \qquad ...(2)$$

where $\qquad I_1 = \int \frac{x^3}{\sqrt{1-x^2}}\, dx = \int \frac{x^2}{\sqrt{1-x^2}}\, x\, dx$

Put $\qquad \sqrt{1-x^2} = z$

$\Rightarrow \qquad 1 - x^2 = z^2 \;\Rightarrow\; x^2 = 1 - z^2 \;\Rightarrow\; 2x\, dx = -2z\, dz \;\Rightarrow\; x\, dx = -z\, dz$

$\therefore \qquad I_1 = \int \frac{(1-z^2)}{z}(-z\, dz) = \int (z^2 - 1)\, dz$

$$= \int z^2\, dz - \int 1 \,.\, dz = \frac{z^3}{3} - z + c_1$$

$$= \frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2} + c_1 \qquad \left[\because\ z = \sqrt{1-x^2}\right]$$

$$I_1 = \frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} + c_1$$

Putting this value of I_1 in equation (2), we have

$$I = \frac{x^3}{3}\cos^{-1}x + \frac{1}{3}\left[\frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} + c_1\right]$$

$$= \frac{x^3}{3}\cos^{-1}x + \frac{1}{9}(1-x^2)^{3/2} - \frac{1}{3}\sqrt{1-x^2} + \frac{1}{3}c_1$$

$$= \frac{x^3}{3}\cos^{-1}x + \frac{1}{9}(1-x^2)^{3/2} - \frac{1}{3}\sqrt{1-x^2} + c. \qquad \text{where } c = \frac{1}{3}c_1.$$

(*iii*) Let $\quad I = \int \underset{\text{II}}{x^2}\, \underset{\text{I}}{\cot^{-1} x}\, dx \qquad \text{...(1)}$

Integrating by parts, we get

$$I = \cot^{-1}x \,.\int x^2\,dx - \int \left\{\frac{d}{dx}(\cot^{-1}x)\,.\int x^2\,dx\right\}dx$$

$$= \cot^{-1}x\,.\,\frac{x^3}{3} - \int\left(\frac{-1}{1+x^2}\right).\frac{x^3}{3}\,dx = \frac{x^3}{3}\cot^{-1}x + \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$$

NOTE THIS STEP : See note example 4(*ii*).

$$= \frac{x^3}{3}\cot^{-1}x + \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx$$

$$\begin{array}{r} x^2+1\,\big)\; x^3 \;\big(\, x \\ +x^3 + x \\ -\quad - \\ \hline -x \end{array}$$

$$= \frac{x^3}{3}\cot^{-1}x + \frac{1}{3}\int x\,dx - \frac{1}{3}\int \frac{x}{1+x^2}\,dx$$

$$= \frac{x^3}{3}\cot^{-1}x + \frac{1}{3}\int x\,dx - \frac{1}{6}\int \frac{2x}{1+x^2}\,dx \quad \text{[Multiply and divided by 2]}$$

$$= \frac{x^3}{3}\cot^{-1}x + \frac{1}{3}.\left(\frac{x^2}{2}\right) - \frac{1}{6}\log|1+x^2| + c$$

$$\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

$$= \frac{x^3}{3}\cot^{-1}x + \frac{x^2}{6} - \frac{1}{6}\log|1+x^2| + c.$$

(*iv*) Let $\quad I = \int \underset{\text{II}}{x^2}\, \underset{\text{I}}{\operatorname{cosec}^{-1} x}\, dx \qquad \text{...(1)}$

Integrating by parts, we get

$$I = \operatorname{cosec}^{-1}x \int x^2\,dx - \int\left\{\frac{d}{dx}(\operatorname{cosec}^{-1}x)\,.\int x^2\,dx\right\}dx$$

$$= \operatorname{cosec}^{-1}x\,.\,\frac{x^3}{3} - \int\left(\frac{-1}{x\sqrt{x^2-1}}\right).\frac{x^3}{3}\,dx = \frac{x^3}{3}\operatorname{cosec}^{-1}x + \frac{1}{3}\int\frac{x^2}{\sqrt{x^2-1}}\,dx$$

$$\Rightarrow \qquad I = \frac{x^3}{3}\operatorname{cosec}^{-1}x + \frac{1}{3}I_1 \qquad \text{...(2)}$$

where $\quad I_1 = \int \frac{x^2}{\sqrt{x^2-1}}\,dx$

Put $\quad x = \operatorname{cosec}\theta \;\Rightarrow\; dx = -\operatorname{cosec}\theta\cot\theta\,d\theta$

$$\therefore \qquad I_1 = \int \frac{\operatorname{cosec}^2\theta}{\sqrt{\operatorname{cosec}^2\theta - 1}}\,.\,(-\operatorname{cosec}\theta\cot\theta\,d\theta)$$

$$= -\int \frac{\operatorname{cosec}^3\theta \cot\theta}{\sqrt{\cot^2\theta}}\, d\theta \qquad [\because \operatorname{cosec}^2 A - \cot^2 A = 1]$$

$$= -\int \operatorname{cosec}^3\theta\, d\theta \qquad ...(3)$$

$$= -\int \underset{\text{I}}{\operatorname{cosec}\theta}\, \underset{\text{II}}{\operatorname{cosec}^2\theta}\, d\theta \qquad \text{[Integrating again by parts]}$$

$$= -\left[\operatorname{cosec}\theta \,.\int \operatorname{cosec}^2\theta\, d\theta - \int \left\{\frac{d}{d\theta}(\operatorname{cosec}\theta)\,.\int \operatorname{cosec}^2\theta\, d\theta\right\} d\theta\right]$$

$$= -\left[\operatorname{cosec}\theta\,(-\cot\theta) - \int (-\operatorname{cosec}\theta\cot\theta)(-\cot\theta)\, d\theta\right]$$

$$= \operatorname{cosec}\theta\cot\theta + \int \operatorname{cosec}\theta\cot^2\theta\, d\theta$$

$$= \operatorname{cosec}\theta\cot\theta + \int \operatorname{cosec}\theta\,(\operatorname{cosec}^2\theta - 1)\, d\theta \qquad [\because \operatorname{cosec}^2 A - \cot^2 A = 1]$$

$$= \operatorname{cosec}\theta\cot\theta + \int \operatorname{cosec}^3\theta\, d\theta - \int \operatorname{cosec}\theta\, d\theta$$

$$\Rightarrow \quad I_1 = \operatorname{cosec}\theta\cot\theta - I_1 - \log|\operatorname{cosec}\theta - \cot\theta| + c_1 \qquad [\because \text{ Using equation (3)}]$$

$$\Rightarrow \quad 2I_1 = \operatorname{cosec}\theta\cot\theta - \log|\operatorname{cosec}\theta - \cot\theta| + c_1$$

$$\Rightarrow \quad I_1 = \frac{1}{2}\operatorname{cosec}\theta\cot\theta - \frac{1}{2}\log|\operatorname{cosec}\theta - \cot\theta| + \frac{1}{2}c_1 \qquad \left[\begin{array}{l}\because \operatorname{cosec}^2 A - \cot^2 A = 1\\ \cot^2 A = \operatorname{cosec}^2 A - 1\\ \cot A = \sqrt{\operatorname{cosec}^2 A - 1}\end{array}\right]$$

$$\Rightarrow \quad I_1 = \operatorname{cosec}\theta\sqrt{\operatorname{cosec}^2\theta - 1} - \frac{1}{2}\log\left|\operatorname{cosec}\theta - \sqrt{\operatorname{cosec}^2\theta - 1}\right| + \frac{1}{2}c_1$$

$$\Rightarrow \quad I_1 = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\log\left|x - \sqrt{x^2-1}\right| + \frac{1}{2}c_1 \qquad [\because \operatorname{cosec}\theta = x]$$

Putting this value of I_1 in equation (2), we have

$$I = \frac{x^3}{3}\operatorname{cosec}^{-1}x + \frac{1}{3}\left[\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\log\left|x-\sqrt{x^2-1}\right| + \frac{1}{2}c_1\right]$$

$$= \frac{x^3}{3}\operatorname{cosec}^{-1}x + \frac{1}{6}x\sqrt{x^2-1} - \frac{1}{6}\log\left|x-\sqrt{x^2-1}\right| + \frac{1}{6}c_1$$

$$= \frac{x^3}{3}\operatorname{cosec}^{-1}x + \frac{1}{6}x\sqrt{x^2-1} - \frac{1}{6}\log\left|x-\sqrt{x^2-1}\right| + c. \qquad \text{where } c = \frac{1}{6}c_1.$$

(*v*) Let $I = \int \underset{\text{II}}{x^2}\, \underset{\text{I}}{\tan^{-1}x}\, dx$

Integrating by parts, we get

$$I = \tan^{-1}x\,.\int x^2\, dx - \int \left\{\frac{d}{dx}(\tan^{-1}x)\,.\int x^2\, dx\right\} dx$$

$$= \tan^{-1} x \cdot \frac{x^3}{3} - \int \left(\frac{1}{1+x^2}\right) \cdot \frac{x^3}{3}\, dx = \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2}\, dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left(x - \frac{x}{x^2+1}\right) dx \qquad \text{[Note this step]}$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int x\, dx + \frac{1}{3} \int \frac{x}{x^2+1}\, dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int x\, dx + \frac{1}{6} \int \frac{2x}{x^2+1}\, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3}\left(\frac{x^2}{2}\right) + \frac{1}{6} \log |x^2+1| + c \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log |f(x)| + c\right]$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{6} x^2 + \frac{1}{6} \log |x^2+1| + c.$$

Example 11. *Evaluate the following integrals :*

(i) $\int \cos^{-1} \frac{1}{x}\, dx$ (ii) $\int \frac{\log(\log x)}{x}\, dx$

(iii) $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$ (iv) $\int \sin x \log \cos x\, dx.$

Solution. (i) Let $I = \int \cos^{-1} \frac{1}{x}$

$$= \int \sec^{-1} x\, dx \qquad \left[\because \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \int \underset{\text{II}}{1} \cdot \underset{\text{I}}{\sec^{-1}} x\, dx \qquad \text{[Taking unity as second function]}$$

Integrating by parts, we get

$$I = \sec^{-1} x \cdot \int 1 \cdot dx - \int \left\{\frac{d}{dx}(\sec^{-1} x) \cdot \int 1 \cdot dx\right\} dx$$

$$= \sec^{-1} x \cdot x - \int \left(\frac{1}{x\sqrt{x^2-1}}\right) x\, dx$$

$$= x \sec^{-1} x - \int \frac{1}{\sqrt{x^2-1}} \cdot dx \qquad \left[\because \int \frac{1}{\sqrt{x^2-a^2}}\, dx = \log\left|x + \sqrt{x^2-a^2}\right| + c\right]$$

$$= x \sec^{-1} x - \log\left|x + \sqrt{x^2-1}\right| + c.$$

(ii) Let $I = \int \frac{\log(\log x)}{x}\, dx$

Put $\log x = z \Rightarrow \frac{1}{x} dx = dz$

$\therefore \qquad I = \int \log z \, dz$

$= \int \underset{II}{1} . \underset{I}{\log z} \, dz$ [Taking unity as second function]

Integrating by parts, we get

$$I = \log z . \int 1 . dz - \int \left\{ \frac{d}{dz} (\log z) . \int 1 . dz \right\} dz$$

$$= \log z . z - \int \frac{1}{z} . z \, dz = z \log z - \int 1 . dz$$

$$= z \log z - z + c = \log x \log (\log x) - \log x + c \qquad [\because \; z = \log x]$$

$$= \log x [\log (\log x) - 1] + c.$$

(*iii*) Let $\qquad I = \int \frac{\sin^{-1} x}{(1 - x^2)^{3/2}} dx$

Put $\qquad \sin^{-1} x = z \;\Rightarrow\; x = \sin z \;\Rightarrow\; dx = \cos z \, dz$

$$\therefore \qquad I = \int \frac{z}{(1 - \sin^2 z)^{3/2}} \cos z \, dz \qquad [\because \; \sin^2 A + \cos^2 A = 1]$$

$$= \int \frac{z}{(\cos^2 z)^{3/2}} \cos z \, dz = \int \frac{z}{\cos^3 z} \cos z \, dz$$

$$= \int \underset{I}{z} \, \underset{II}{\sec^2 z} \, dz \qquad \left[\because \; \frac{1}{\cos A} = \sec A\right]$$

Integrating by parts, we get

$$I = z . \int \sec^2 z \, dz - \int \left\{ \frac{d}{dz} (z) . \int \sec^2 z \, dz \right\} dz$$

$$= z . \tan z - \int 1 . \tan z \, dz = z \tan z + \log |\cos z| + c$$

$$= z \frac{\sin z}{\cos z} + \log |\cos z| + c$$

$$= z . \frac{\sin z}{\sqrt{1 - \sin^2 z}} + \log \left| \sqrt{1 - \sin^2 z} \right| + c \qquad \left[\begin{array}{l} \because \; \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \; \cos^2 A = 1 - \sin^2 A \\ \Rightarrow \; \cos A = \sqrt{1 - \sin^2 A} \end{array}\right]$$

$$= (\sin^{-1} x) . \frac{x}{\sqrt{1 - x^2}} + \log \left| \sqrt{1 - x^2} \right| + c \qquad [\because \; \sin^{-1} x = z]$$

$$= \frac{x}{\sqrt{1 - x^2}} \sin^{-1} x + \log \left| \sqrt{1 - x^2} \right| + c.$$

(*iv*) Let $\qquad I = \int \sin x \log \cos x \, dx$

Put $\qquad \cos x = z \;\Rightarrow\; - \sin x \, dx = dz \;\Rightarrow\; \sin x \, dx = - dz$

$\therefore \qquad I = \int \log z \, (- dz)$

$= - \int \underset{II}{1} . \underset{I}{\log z} \, dz$ [Taking unity as second function]

$$= -\left[\log z \,.\int 1.\,dz - \int\left\{\frac{d}{dz}(\log z).\int 1.\,dz\right\}dz\right]$$

$$= -\left[\log z \,.\, z - \int \frac{1}{z}.\,z\,dz\right]$$

$$= -\left[z\log z - \int 1.\,dz\right] = -z\log z + \int 1.\,dz$$

$$= -z\log z + z + c = -z\,[\log z - 1] + c$$

$$= -\cos x\,[\log(\cos x) - 1] + c. \qquad [\because\ z = \cos x]$$

Example 12. *Evaluate the following integrals :*

(i) $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$ (ii) $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

(iii) $\int \frac{x^2\tan^{-1}x}{1+x^2}\,dx$ (iv) $\int \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)dx$

(v) $\int \sin^{-1}\left(\sqrt{\frac{x}{a+x}}\right)dx$ (vi) $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

(vii) $\int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx.$

Solution. (i) Let $I = \int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\therefore \quad I = \int \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)dx$$

$$= \int \cos^{-1}(\cos 2\theta)\,dx \qquad \left[\because\ \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}\right]$$

$$= \int 2\theta\,dx = 2\int \tan^{-1}x\,dx \qquad [\because\ \theta = \tan^{-1}x]$$

$$= 2\int \underset{\text{II}}{1}.\underset{\text{I}}{\tan^{-1}x}\,dx \qquad \text{[Taking unity as the IInd function]}$$

$$= 2\left[\tan^{-1}x.\int 1.\,dx - \int\left\{\frac{d}{dx}(\tan^{-1}x).\int 1.\,dx\right\}dx\right]$$

$$= 2\left[\tan^{-1}x.\,x - \int\left(\frac{1}{1+x^2}\right)x\,dx\right]$$

$$= 2\left[x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}\,dx\right] \qquad [\because\ \text{Multiply and divided by 2}]$$

$$= 2x\tan^{-1}x - \log|1+x^2| + c. \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

(ii) Let $\quad I = \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Put $\quad x = \sin\theta \Rightarrow \theta = \sin^{-1}x \Rightarrow dx = \cos\theta\, d\theta$

$\therefore \quad I = \int \frac{(\sin\theta)\,\theta}{\sqrt{1-\sin^2\theta}} . \cos\theta\, d\theta$

$$= \int \frac{\theta\sin\theta}{\sqrt{\cos^2\theta}}\cos\theta\, d\theta \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$I = \int \underset{\text{I}}{\theta}\,\underset{\text{II}}{\sin}\,\theta\, d\theta$$

Integrating by parts, we get

$$I = \theta . \int \sin\theta\, d\theta - \int\left\{\frac{d}{d\theta}(\theta) . \int \sin\theta\, d\theta\right\}d\theta$$

$$= \theta(-\cos\theta) - \int 1.(-\cos\theta)\,d\theta = -\theta\cos\theta + \int \cos\theta\, d\theta$$

$$= -\theta\cos\theta + \sin\theta + c$$

$$= -\theta\left(\sqrt{1-\sin^2\theta}\right) + \sin\theta + c \qquad \begin{bmatrix}\because \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \cos^2 A = 1 - \sin^2 A \\ \Rightarrow \cos A = \sqrt{1-\sin^2 A}\end{bmatrix}$$

$$= -\sin^{-1}x\left(\sqrt{1-x^2}\right) + x + c \qquad [\because \theta = \sin^{-1}x]$$

$$= -\sqrt{1-x^2} . \sin^{-1}x + x + c$$

(iii) Let $\quad I = \int \frac{x^2\tan^{-1}x}{1+x^2}dx$

Put $\quad \tan^{-1}x = z \Rightarrow x = \tan z \Rightarrow \frac{1}{1+x^2}dx = dz$

$\therefore \quad I = \int (\tan^2 z)\, z\, dz = \int z\tan^2 z\, dz$

$$= \int z(\sec^2 z - 1)\,dz \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$$= \int \underset{\text{I}}{z}\,\underset{\text{II}}{\sec^2} z\, dz - \int z\, dz$$

Integrating by parts, we get

$$I = z . \int \sec^2 z\, dz - \int\left\{\frac{d}{dz}(z) . \int \sec^2 z\, dz\right\}dz - \frac{z^2}{2} + c$$

$$= z\tan z - \int 1.\tan z\,dz - \frac{z^2}{2} + c = z\tan z - \log|\sec z| - \frac{z^2}{2} + c$$

$$= z\tan z - \log\left|\sqrt{1+\tan^2 z}\right| - \frac{z^2}{2} + c \qquad \left[\begin{array}{l} \because\ \sec^2 A - \tan^2 A = 1 \\ \Rightarrow\ \sec^2 A = 1 + \tan^2 A \\ \Rightarrow\ \sec A = \sqrt{1+\tan^2 A} \end{array}\right]$$

$$= \tan^{-1} x\,.\,x - \log\left|\sqrt{1+x^2}\right| - \frac{(\tan^{-1} x)^2}{2} + c$$

$$= x\tan^{-1} x - \log\left|\sqrt{1+x^2}\right| - \frac{1}{2}(\tan^{-1} x)^2 + c.$$

(*iv*) Let $\quad I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

Put $\quad x = \cos\theta \;\Rightarrow\; \theta = \cos^{-1} x \;\Rightarrow\; dx = -\sin\theta\,d\theta$

$\therefore \quad I = \int \tan^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)(-\sin\theta\,d\theta)$

$$= -\int \tan^{-1}\left(\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\right)\sin\theta\,d\theta \qquad \left[\begin{array}{l} \because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ 1+\cos A = 2\cos^2\frac{A}{2} \\ \text{and } 1-\cos 2A = 2\sin^2 A \\ \Rightarrow\ 1-\cos A = 2\sin^2\frac{A}{2} \end{array}\right]$$

$$= -\int \tan^{-1}(\tan\theta/2)\sin\theta\,d\theta$$

$$= -\int \frac{\theta}{2}\sin\theta\,d\theta$$

$$= -\frac{1}{2}\int \underset{\text{I}}{\theta}\,\underset{\text{II}}{\sin}\,\theta\,d\theta$$

Integrating by parts, we get

$$= -\frac{1}{2}\left[\theta\,.\int\sin\theta\,d\theta - \int\left\{\frac{d}{d\theta}(\theta)\,.\int\sin\theta\,d\theta\right\}.\,d\theta\right]$$

$$= -\frac{1}{2}\left[\theta\,.(-\cos\theta) - \int 1.(-\cos\theta)\,d\theta\right] = \frac{1}{2}\theta\,.\cos\theta - \frac{1}{2}\int\cos\theta\,d\theta$$

$$= \frac{1}{2}\theta\,.\cos\theta - \frac{1}{2}\sin\theta + c$$

$$= \frac{1}{2}\theta\cos\theta - \frac{1}{2}\sqrt{1-\cos^2\theta} + c \qquad \left[\begin{array}{l} \because\ \sin^2 A + \cos^2 A = 1 \\ \Rightarrow\ \sin^2 A = 1-\cos^2 A \\ \Rightarrow\ \sin A = \sqrt{1-\cos^2 A} \end{array}\right]$$

$$= \frac{1}{2} x \cos^{-1} x - \frac{1}{2}\sqrt{1-x^2} + c. \qquad [\because \ x = \cos\theta]$$

(*v*) Let $\quad I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

Put $\sin^{-1}\sqrt{\frac{x}{a+x}} = \theta \Rightarrow \sqrt{\frac{x}{a+x}} = \sin\theta \Rightarrow \frac{x}{a+x} = \sin^2\theta$ [Squaring both sides]

$$\Rightarrow \quad x = a\sin^2\theta + x\sin^2\theta$$

$$\Rightarrow \quad x(1-\sin^2\theta) = a\sin^2\theta$$

$$\Rightarrow \quad x = \frac{a\sin^2\theta}{\cos^2\theta} \qquad [\because \ \sin^2 A + \cos^2 A = 1]$$

$$\Rightarrow \quad x = a\tan^2\theta \Rightarrow \theta = \tan^{-1}\sqrt{\frac{x}{a}}$$

$$\Rightarrow \quad dx = 2a\tan\theta\sec^2\theta\, d\theta.$$

$$\therefore \quad I = \int \theta\,.\,(2a\tan\theta\sec^2\theta\, d\theta)$$

$$= 2a\int \underset{\text{I}}{\theta}\,.\,\underset{\text{II}}{(\tan\theta\sec^2\theta)}\, d\theta$$

Integrating by parts, we get

$$I = 2a\left[\theta\,.\int(\tan\theta\sec^2\theta)\,d\theta - \int\left\{\frac{d}{d\theta}(\theta)\,.\int(\tan\theta\sec^2\theta)\,d\theta\right\}d\theta\right]$$

$$\left[\because \ \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= 2a\left[\theta\,.\,\frac{\tan^2\theta}{2} - \int 1\,.\,\frac{\tan^2\theta}{2}\,d\theta\right] = a\,\theta\tan^2\theta - a\int\tan^2\theta\,d\theta$$

$$= a\,\theta\tan^2\theta - a\int(\sec^2\theta - 1)\,d\theta \qquad [\because \ \sec^2 A - \tan^2 A = 1]$$

$$= a\,\theta\tan^2\theta - a\int\sec^2\theta\,d\theta + a\int 1\,.\,d\theta = a\,\theta\tan^2\theta - a\tan\theta + a\,\theta + c$$

$$= a\,.\,\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - a\,.\sqrt{\frac{x}{a}} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$$

$$\left[\begin{array}{l}\because \ x = a\tan^2\theta \Rightarrow \tan^2\theta = \dfrac{x}{a} \\ \Rightarrow \ \tan\theta = \sqrt{\dfrac{x}{a}} \Rightarrow \theta = \tan^{-1}\sqrt{\dfrac{x}{a}}\end{array}\right]$$

$$= (x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + c.$$

(*vi*) Let $\quad I = \int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

Put $\quad x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\therefore \quad I = \int \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta \qquad \left[\because \sin 2A = \frac{2\tan A}{1+\tan^2 A}\right]$$

$$= \int 2\theta\sec^2\theta\, d\theta$$

$$= 2\int \underset{\text{I}}{\theta}\, \underset{\text{II}}{\sec^2\theta}\, d\theta$$

Integrating by parts, we get

$$I = 2\left[\theta . \int \sec^2\theta . d\theta - \int\left\{\frac{d}{d\theta}(\theta) . \int \sec^2\theta\, d\theta\right\}d\theta\right]$$

$$= 2\left[\theta . \tan\theta - \int 1 . \tan\theta\, d\theta\right] = 2\theta\tan\theta + 2\log|\cos\theta| + c$$

$$= 2\theta\tan\theta + 2\log\left|\frac{1}{\sec\theta}\right| + c$$

$$= 2\theta\tan\theta + 2\log\left|\frac{1}{\sqrt{1+\tan^2\theta}}\right| + c \qquad \left[\begin{array}{l}\because \cos A = \dfrac{1}{\sec A} \\ \text{and } \sec^2 A - \tan^2 A = 1 \\ \Rightarrow \sec^2 A = 1 + \tan^2 A \\ \Rightarrow \sec A = \sqrt{1+\tan^2 A}\end{array}\right]$$

$$= 2\tan^{-1}x . x + 2\log\left|\frac{1}{\sqrt{1+x^2}}\right| + c = 2x\tan^{-1}x + 2\log|(1+x^2)^{-1/2}| + c$$

$$[\because \ x = \tan\theta]$$

$$= 2x\tan^{-1}x + 2\left(-\frac{1}{2}\right)\log|1+x^2| + c \qquad [\because \ \log m^n = n\log m]$$

$$= 2x\tan^{-1}x - \log|1+x^2| + c.$$

(*vii*) Let $\quad I = \int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx$

$$= \int \underset{\text{II}}{1} . \underset{\text{I}}{\log(\log x)}\, dx + \int \frac{1}{(\log x)^2}\, dx \qquad \text{[Taking unity as second function]}$$

Integrating by parts, we get

$$I = \log(\log x) . \int 1 . dx - \int\left\{\frac{d}{dx}(\log(\log x)) . \int 1 . dx\right\}dx + \int \frac{1}{(\log x)^2}\, dx$$

$$= \log(\log x) . x - \int\left(\frac{1}{\log x} . \frac{1}{x}\right) . x\, dx + \int \frac{1}{(\log x)^2}\, dx$$

$$= x \log(\log x) - \int \frac{1}{\log x}\,dx + \int \frac{1}{(\log x)^2}\,dx$$

$$= x \log(\log x) - \int \underset{\text{II}}{1} . \underset{\text{I}}{(\log x)^{-1}}\,dx + \int \frac{1}{(\log x)^2}\,dx$$

[Taking unity as second function. Integrating again by parts.]

$$= x \log(\log x) - \left[(\log x)^{-1} . \int 1 . dx - \int \left\{\frac{d}{dx}((\log x)^{-1}) . \int 1 . dx\right\} dx\right] + \int \frac{1}{(\log x)^2}\,dx$$

$$= x \log(\log x) - \left[(\log x)^{-1} . x - \int \left((-1)(\log x)^{-2} . \frac{1}{x}\right) . x\,dx\right] + \int \frac{1}{(\log x)^2}\,dx$$

$$= x \log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2}\,dx + \int \frac{1}{(\log x)^2}\,dx + c$$

$$= x \log(\log x) - \frac{x}{\log x} + c.$$

Example 13. *Evaluate the following integrals :*

(*i*) $\int \frac{-x \cos^{-1} x}{\sqrt{1-x^2}}\,dx$ (*ii*) $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

(*iii*) $\int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) dx$ (*iv*) $\int \cos\left(2 \cot^{-1} \sqrt{\frac{1-x}{1+x}}\right) dx$

(*v*) $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx.$

Solution. (*i*) Let $I = \int \frac{-x \cos^{-1} x}{\sqrt{1-x^2}}\,dx$

Put $\cos^{-1} x = \theta \;\Rightarrow\; x = \cos\theta \;\Rightarrow\; \frac{-1}{\sqrt{1-x^2}}\,dx = d\theta$

$\therefore \quad I = \int \underset{\text{I}}{\theta} \underset{\text{II}}{\cos\theta}\,d\theta$

Integrating by parts, we get

$$I = \theta . \int \cos\theta\,d\theta - \int \left\{\frac{d}{d\theta}(\theta) . \int \cos\theta\,d\theta\right\} d\theta$$

$$= \theta \sin\theta - \int 1 . \sin\theta\,d\theta$$

$$= \theta \sin\theta - (-\cos\theta) + c = \theta \sin\theta + \cos\theta + c$$

$$= \theta\sqrt{1-\cos^2\theta} + \cos\theta + c \qquad \left[\begin{array}{l} \because \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \sin^2 A = 1 - \cos^2 A \\ \Rightarrow \sin A = \sqrt{1-\cos^2 A} \end{array}\right]$$

$$= \cos^{-1} x \sqrt{1-x^2} + x + c$$

$$= \sqrt{1-x^2}\cos^{-1} x + x + c.$$

(*ii*) Let $\quad I = \int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

Put $\quad x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$\therefore \quad I = \int \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\sec^2\theta\, d\theta$

$$= \int \tan^{-1}(\tan 2\theta)\sec^2\theta\, d\theta \qquad \left[\because \tan 2A = \frac{2\tan A}{1-\tan^2 A}\right]$$

$$= \int 2\theta \sec^2\theta\, d\theta$$

$$= 2\int \underset{\text{I}}{\theta}\,.\underset{\text{II}}{\sec^2\theta}\, d\theta$$

Integrating by parts, we get

$$= 2\left[\theta\,.\int \sec^2\theta\, d\theta - \int\left\{\frac{d}{d\theta}(\theta)\,.\int \sec^2\theta\, d\theta\right\}d\theta\right]$$

$$= 2\left[\theta\,.\tan\theta - \int 1\,.\tan\theta\, d\theta\right] = 2\theta\tan\theta - 2\log|\sec\theta| + c$$

$$= 2\theta\tan\theta - 2\log\left|\sqrt{1+\tan^2\theta}\right| + c \qquad \left[\begin{array}{l}\because \sec^2 A - \tan^2 A = 1\\ \Rightarrow \sec^2 A = 1+\tan^2 A\\ \Rightarrow \sec A = \sqrt{1+\tan^2 A}\end{array}\right]$$

$$= 2\theta\tan\theta - 2\left(\frac{1}{2}\right)\log|1+\tan^2\theta| + c \qquad [\because x = \tan\theta]$$

$$= 2\theta\tan\theta - \log|1+\tan^2\theta| + c = 2\tan^{-1}x\,.\,x - \log|1+x^2| + c$$

$$= 2x\tan^{-1}x - \log|1+x^2| + c.$$

(*iii*) Let $\quad I = \int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) dx$ **(Please try yourself)**

[**Hint.** Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$. [**Ans :** $3x\tan^{-1}x - \frac{3}{2}\log|1+x^2| + c$.]

$\tan 3\theta = \left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$]

(*iv*) Let $\quad I = \int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right) dx$

Put $\quad x = \cos\theta \Rightarrow \theta = \cos^{-1}x \Rightarrow dx = -\sin\theta\, d\theta$

$\therefore \quad I = \int \cos\left(2\cot^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)(-\sin\theta\, d\theta)$

$$= -\int \cos\left(2\cot^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right).\sin\theta\,d\theta \qquad \left[\begin{array}{l}\because\ 1-\cos 2A = 2\sin^2 A \\ \Rightarrow 1-\cos A = 2\sin^2\frac{A}{2} \\ \text{and } 1+\cos 2A = 2\cos^2 A \\ \Rightarrow 1+\cos A = 2\cos^2\frac{A}{2}\end{array}\right]$$

$$= -\int \cos\left[2\cot^{-1}\left(\tan\frac{\theta}{2}\right)\right]\sin\theta\,d\theta$$

$$= -\int \cos\left[2\cot^{-1}\left\{\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}\right]\sin\theta\,d\theta$$

$$= -\int \cos\left[2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]\sin\theta\,d\theta = -\int \cos(\pi-\theta)\sin\theta\,d\theta$$

$$= -\int (-\cos\theta)\sin\theta\,d\theta \qquad [\because\ \cos(180^\circ-\theta) = -\cos\theta]$$

$$= \int \cos\theta\,(-\sin\theta)\,d\theta$$

$$= -\frac{\cos^2\theta}{2}+c \qquad \left[\because\ \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}+c\right]$$

$$= -\frac{x^2}{2}+c. \qquad [\because\ x = \cos\theta]$$

(*v*) Let $\quad I = \int \dfrac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx \qquad ...(1)$

$$= \int \frac{\sin^{-1}\sqrt{x}-\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}\,dx \qquad \left[\because\ \sin^{-1}\theta+\cos^{-1}\theta = \frac{\pi}{2}\right]$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x}-\frac{\pi}{2}\right)dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x} - \int 1\,.\,dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x} - x + c$$

$\Rightarrow \qquad I = \dfrac{4}{\pi}I_1 - x + c \qquad ...(2)$

where $\qquad I_1 = \int \sin^{-1}\sqrt{x}\,dx$

Put $\sin^{-1}\sqrt{x} = z \;\Rightarrow\; \sqrt{x} = \sin z \;\Rightarrow\; x = \sin^2 z$

$\Rightarrow \qquad dx = 2\sin z\cos z\,dz \;\Rightarrow\; dx = \sin 2z\,dz \qquad [\because\ 2\sin A\cos A = \sin 2A]$

$\therefore \qquad I_1 = \int \underset{\text{I}}{z}\,.\underset{\text{II}}{\sin}\,2z\,dz$

Integrating by parts, we get

$$I_1 = z \cdot \int \sin 2z\, dz - \int \left\{\frac{d}{dz}(z) \cdot \int \sin 2z\, dz\right\} dz$$

$$= z \cdot \left(-\frac{\cos 2z}{2}\right) - \int 1 \cdot \left(-\frac{\cos 2z}{2}\right) dz = -\frac{1}{2} z \cos 2z + \frac{1}{2}\int \cos 2z\, dz$$

$$= -\frac{1}{2} z \cos 2z + \frac{1}{2}\left(\frac{\sin 2z}{2}\right) = -\frac{1}{2} z \cos 2z + \frac{1}{4} \sin 2z$$

$$= -\frac{1}{2} z\left(1 - 2\sin^2 z\right) + \frac{1}{4} \cdot 2 \sin z \cos z \qquad [\because \cos 2A = 1 - 2\sin^2 A]$$

$$= -\frac{1}{2} z\left(1 - 2\sin^2 z\right) + \frac{1}{2} \sin z \sqrt{1 - \sin^2 z} \qquad \left[\begin{array}{l} \because \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \cos^2 A = 1 - \sin^2 A \\ \Rightarrow \cos A = \sqrt{1 - \sin^2 A} \end{array}\right]$$

$$\Rightarrow \quad I_1 = -\frac{1}{2} \sin^{-1}\sqrt{x}\,(1 - 2x) + \frac{1}{2}\sqrt{x}\,\sqrt{1 - x} \qquad \left[\because z = \sin^{-1}\sqrt{x}\right]$$

Putting this value of I_1 in equation (2), we have

$$I = \frac{4}{\pi}\left[-\frac{1}{2}\left(\sin^{-1}\sqrt{x}\right)(1 - 2x) + \frac{1}{2}\sqrt{x}\,\sqrt{1 - x}\right] - x + c$$

$$= \frac{2}{\pi}\left[(2x - 1)\sin^{-1}\sqrt{x} + \sqrt{x - x^2}\right] - x + c.$$

4.3. INTEGRALS OF THE FORM $\int e^{ax} \sin bx\, dx$, $\int e^{ax} \cos bx\, dx$, $\int e^{kx} [kf(x) + f'(x)]\, dx$

Theorem 1. *Prove that :*

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{(a^2 + b^2)} (a \sin bx - b \cos bx) + c.$$

Proof. Let $\quad I = \int \underset{\text{II}}{e^{ax}}\ \underset{\text{I}}{\sin}\ bx\, dx \qquad$ ...(1)

Integrating by parts, we get

$$I = \sin bx \cdot \int e^{ax}\, dx - \int \left\{\frac{d}{dx}(\sin bx) \cdot \int e^{ax}\, dx\right\} dx$$

$$= \sin bx \cdot \frac{e^{ax}}{a} - \int b \cos bx \cdot \frac{e^{ax}}{a} \cdot dx$$

$$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a} \int \underset{\text{II}}{e^{ax}}\ \underset{\text{I}}{\cos}\ bx\, dx \qquad \text{(Integrating again by parts)}$$

$$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a}\left[\cos bx \cdot \int e^{ax}\, dx - \int \left\{\frac{d}{dx}(\cos bx) \cdot \int e^{ax}\, dx\right\} dx\right]$$

$$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a}\left[\cos bx \cdot \frac{e^{ax}}{a} - \int (-b \sin bx) \cdot \frac{e^{ax}}{a}\, dx\right]$$

$$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx\, dx$$

$\Rightarrow$ $$I = \frac{e^{ax}}{a^2}[a \sin bx - b \cos bx] - \frac{b^2}{a^2} I \qquad \text{[By using equation (1)]}$$

$\Rightarrow$ $$\left(1 + \frac{b^2}{a^2}\right) I = \frac{e^{ax}}{a^2} [a \sin bx - b \cos bx]$$

$\Rightarrow$ $$\left(\frac{a^2 + b^2}{a^2}\right) I = \frac{e^{ax}}{a^2} [a \sin bx - b \cos bx]$$

$\Rightarrow$ $$I = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c.$$

Theorem 2. *Prove that :*

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c.$$

Proof. Let $I = \int \underset{\text{II}}{e^{ax}} \underset{\text{I}}{\cos}\, bx\, dx$ ...(1)

Integrating by parts, we get

$$I = \cos bx \cdot \int e^{ax}\, dx - \int \left\{\frac{d}{dx}(\cos bx) \cdot \int e^{ax}\, dx\right\} \cdot dx$$

$$= \cos bx \cdot \frac{e^{ax}}{a} - \int (-b \sin bx) \cdot \frac{e^{ax}}{a}\, dx$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \int \underset{\text{II}}{e^{ax}} \underset{\text{I}}{\sin}\, bx\, dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a}\left[\sin bx \cdot \int e^{ax}\, dx - \int \left\{\frac{d}{dx}(\sin bx) \cdot \int e^{ax}\, dx\right\} dx\right]$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a}\left[\sin bx \cdot \frac{e^{ax}}{a} - \int b \cos bx \cdot \frac{e^{ax}}{a}\, dx\right]$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx\, dx$$

$\Rightarrow$ $$I = \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I \qquad \text{[By using equation (1)]}$$

$\Rightarrow$ $$\left(1 + \frac{b^2}{a^2}\right) I = \frac{e^{ax}}{a^2} (a \cos bx + b \sin bx)$$

$\Rightarrow$ $$\left(\frac{a^2 + b^2}{a^2}\right) I = \frac{e^{ax}}{a^2} (a \cos bx + b \sin bx)$$

$\therefore \quad I = \dfrac{e^{ax}}{(a^2+b^2)}(a \cos bx + b \sin bx) + c.$

Theorem 3. *Prove that :*

$$\int e^{kx}\,[k\,f(x) + f'(x)]\,dx = e^{kx} f(x) + c.$$

Proof. Let $I = \int e^{kx}\,[k\,f(x) + f'(x)]\,dx$

$$= k \int \underset{\text{II}}{e^{kx}}\, \underset{\text{I}}{f(x)}\,dx + \int e^{kx} f'(x)\,dx$$

Integrating by parts, we get

$$I = k \cdot \left[f(x) \cdot \int e^{kx}\,.\,dx - \int \left\{\frac{d}{dx}(f(x)) \cdot \int e^{kx}\,dx\right\} dx\right] + \int e^{kx} f'(x)\,dx$$

$$= k\left[f(x) \cdot \frac{e^{kx}}{k} - \int f'(x) \cdot \frac{e^{kx}}{k}\,dx\right] + \int e^{kx} f'(x)\,dx$$

$$= f(x)\,e^{kx} - \int e^{kx} f'(x)\,dx + \int e^{kx} f'(x)\,dx + c$$

$$= e^{kx} f(x) + c.$$

Remark. In the above theorem, if $k = 1$ then, we have :

$$\int e^x\,[f(x) + f'(x)]\,dx = e^x f(x) + c.$$

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following integrals :*

(i) $\int e^x \sin x\,dx$ (ii) $\int e^{2x} \sin 3x\,dx$

(iii) $\int e^{-2x} \sin x\,dx$ (iv) $\int e^{-x} \cos x\,dx$

(v) $\int e^x \cos 2x\,dx$ (vi) $\int e^{2x} \cos^2 x\,dx.$

Solution. (i) Let $I = \int \underset{\text{II}}{e^x}\, \underset{\text{I}}{\sin}\, x\,dx$ ...(1)

Integrating by parts, we get

$$= \sin x \cdot \int e^x\,dx - \int \left\{\frac{d}{dx}(\sin x) \cdot \int e^x\,dx\right\} dx$$

$$= \sin x\,e^x - \int \cos x\,e^x\,dx$$

$$= e^x \sin x - \int \underset{\text{II}}{e^x}\, \underset{\text{I}}{\cos}\, x\,dx \qquad \text{[Integrating again by parts]}$$

$$= e^x \sin x - \left[\cos x \cdot \int e^x\,dx - \int \left\{\frac{d}{dx}(\cos x) \cdot \int e^x\,dx\right\} dx\right].$$

$$= e^x \sin x - \left[\cos x\, e^x - \int (-\sin x)\, e^x\, dx\right]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$$

$\Rightarrow$ $$I = e^x (\sin x - \cos x) - I$$ [By using equation (1)]

$\Rightarrow$ $$2I = e^x (\sin x - \cos x)$$

$\Rightarrow$ $$I = \frac{e^x}{2} (\sin x - \cos x) + c.$$

(*ii*) Let $I = \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\sin} 3x\, dx$ ...(1)

Integrating by parts, we get

$$I = \sin 3x \,.\, \int e^{2x}\, dx - \int \left\{\frac{d}{dx}(\sin 3x) \,.\, \int e^{2x}\, dx\right\} dx$$

$$= \sin 3x\, \frac{e^{2x}}{2} - \int (3 \cos 3x) \frac{e^{2x}}{2}\, dx$$

$$= \frac{e^{2x}}{2} \sin 3x - \frac{3}{2} \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\cos} 3x \,.\, dx$$ (Integrating again by parts)

$$= \frac{e^{2x}}{2} \sin 3x - \frac{3}{2}\left[\cos 3x \,.\, \int e^{2x}\, dx - \int \left\{\frac{d}{dx}(\cos 3x) \,.\, \int e^{2x}\, dx\right\} dx\right]$$

$$= \frac{e^{2x}}{2} \sin 3x - \frac{3}{2}\left[\cos 3x \,.\, \frac{e^{2x}}{2} - \int (-3 \sin 3x) \,.\, \frac{e^{2x}}{2}\, dx\right]$$

$$= \frac{e^{2x}}{2} \sin 3x - \frac{3}{4} e^{2x} \cos 3x - \frac{9}{4} \int e^{2x} \sin 3x\, dx$$

$\Rightarrow$ $$I = \frac{e^{2x}}{4} (2 \sin 3x - 3 \cos 3x) - \frac{9}{4} I$$ [By using equation (1)]

$\Rightarrow$ $$\left(1 + \frac{9}{4}\right) I = \frac{e^{2x}}{4} (2 \sin 3x - 3 \cos 3x)$$

$\Rightarrow$ $$\frac{13}{4} I = \frac{e^{2x}}{4} (2 \sin 3x - 3 \cos 3x)$$

$\Rightarrow$ $$I = \frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + c.$$

(*iii*) Let $I = \int \underset{\text{II}}{e^{-2x}} \underset{\text{I}}{\sin} x\, dx$ ...(1)

Integrating by parts, we get

$$I = \sin x \,.\, \int e^{-2x}\, dx - \int \left\{\frac{d}{dx}(\sin x) \,.\, \int e^{-2x}\, dx\right\} dx$$

$$= \sin x \,.\, \frac{e^{-2x}}{-2} - \int \cos x \,.\, \frac{e^{-2x}}{-2}\, dx$$

$$= -\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\int \underset{\text{II}}{e^{-2x}} \underset{\text{I}}{\cos} x\, dx \qquad \text{[Integrating again by parts]}$$

$$= -\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\left[\cos x \, . \int e^{-2x}\, dx - \int \left\{\frac{d}{dx}(\cos x) \, . \int e^{-2x}\, dx\right\} dx\right]$$

$$= -\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\left[\cos x \, . \frac{e^{-2x}}{-2} - \int (-\sin x) \, . \frac{e^{-2x}}{-2}\, dx\right]$$

$$= -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x - \frac{1}{4}\int e^{-2x}\sin x\, dx$$

$$\Rightarrow \quad \text{I} = -\frac{1}{4}e^{-2x}[2\sin x + \cos x] - \frac{1}{4}\text{I} \qquad \text{[By using equation (1)]}$$

$$\Rightarrow \quad \left(1 + \frac{1}{4}\right)\text{I} = -\frac{1}{4}e^{-2x}(2\sin x + \cos x)$$

$$\Rightarrow \quad \frac{5}{4}\text{I} = -\frac{1}{4}e^{-2x}(2\sin x + \cos x)$$

$$\Rightarrow \quad \text{I} = -\frac{1}{5}e^{-2x}(2\sin x + \cos x) + c.$$

(*iv*) Let $\quad \text{I} = \int \underset{\text{II}}{e^{-x}} \underset{\text{I}}{\cos} x\, dx \qquad$ ...(1)

Integrating by parts, we get

$$\text{I} = \cos x \, . \int e^{-x}\, dx - \int \left\{\frac{d}{dx}(\cos x) \, . \int e^{-x}\, dx\right\} dx$$

$$= \cos x \, . \frac{e^{-x}}{-1} - \int (-\sin x) \, . \frac{e^{-x}}{-1}\, dx$$

$$= -e^{-x}\cos x - \int \underset{\text{II}}{e^{-x}} \underset{\text{I}}{\sin} x\, dx \qquad \text{(Integrating again by parts)}$$

$$= -e^{-x}\cos x - \left[\sin x \, . \int e^{-x}\, dx - \int \left\{\frac{d}{dx}(\sin x) \, . \int e^{-x}\, dx\right\} dx\right]$$

$$= -e^{-x}\cos x - \left[\sin x \, . \frac{e^{-x}}{-1} - \int \cos x \, . \frac{e^{-x}}{-1}\, dx\right]$$

$$= -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x\, dx$$

$$\Rightarrow \quad \text{I} = -e^{-x}\cos x + e^{-x}\sin x - \text{I} \qquad \text{[By using equation (1)]}$$

$$\Rightarrow \quad 2\text{I} = -e^{-x}[\cos x - \sin x]$$

$$\Rightarrow \quad \text{I} = -\frac{1}{2}e^{-x}[\cos x - \sin x] + c.$$

(*v*) Let $\quad \text{I} = \int \underset{\text{II}}{e^{x}} \underset{\text{I}}{\cos} 2x\, dx \qquad$ ...(1)

Integrating by parts, we get

$$\text{I} = \cos 2x \, . \int e^{x}\, dx - \int \left\{\frac{d}{dx}(\cos 2x) \, . \int e^{x}\, dx\right\} dx$$

$$= \cos 2x\, e^x - \int (-2 \sin 2x)\, e^x\, dx$$

$$= e^x \cos 2x + 2 \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin} 2x\, dx \qquad \text{[Integrating again by parts]}$$

$$= e^x \cos 2x + 2 \left[\sin 2x \,.\, \int e^x\, dx - \int \left\{ \frac{d}{dx} (\sin 2x) \,.\, \int e^x\, dx \right\} dx \right]$$

$$= e^x \cos 2x + 2 \left[\sin 2x \,.\, e^x - \int (2 \cos 2x) e^x\, dx \right]$$

$$= e^x \cos 2x + 2\, e^x \sin 2x - 4 \int e^x \cos 2x\, dx$$

$$\Rightarrow \quad I = e^x (\cos 2x + 2 \sin 2x) - 4I \qquad \text{[By using equation (1)]}$$

$$\Rightarrow \quad 5I = e^x (\cos 2x + 2 \sin 2x)$$

$$\Rightarrow \quad I = \frac{1}{5} e^x (\cos 2x + 2 \sin 2x) + c.$$

(*vi*) Let $I = \int e^{2x} \cos^2 x\, dx$ ...(1)

$$= \int e^{2x} \left(\frac{1 + \cos 2x}{2} \right) dx \qquad \left[\begin{array}{l} \because\ \cos 2A = 2 \cos^2 A - 1 \\ \Rightarrow\ 1 + \cos 2A = 2 \cos^2 A \\ \Rightarrow\ \cos^2 A = \dfrac{1 + \cos 2A}{2} \end{array} \right]$$

$$= \frac{1}{2} \int e^{2x}\, dx + \frac{1}{2} \int e^{2x} \cos 2x\, dx = \frac{1}{2} \frac{e^{2x}}{2} + \frac{1}{2} I_1$$

$$\Rightarrow \quad I = \frac{e^{2x}}{4} + \frac{1}{2} I_1 \qquad \text{...(2)}$$

where $I_1 = \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\cos} 2x\, dx$ ...(3)

Integrating by parts, we get

$$I_1 = \cos 2x \,.\, \int e^{2x}\, dx - \int \left\{ \frac{d}{dx} (\cos 2x) \,.\, \int e^{2x}\, dx \right\} dx$$

$$= \cos 2x \,.\, \frac{e^{2x}}{2} - \int (-2 \sin 2x) \frac{e^{2x}}{2}\, dx$$

$$= \frac{e^{2x}}{2} \cos 2x + \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\sin} 2x\, dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{e^{2x}}{2} \cos 2x + \sin 2x \,.\, \int e^{2x}\, dx - \int \left\{ \frac{d}{dx} (\sin 2x) \,.\, \int e^{2x}\, dx \right\} dx$$

$$= \frac{e^{2x}}{2} \cos 2x + \sin 2x \,.\, \frac{e^{2x}}{2} - \int (2 \cos 2x) \frac{e^{2x}}{2}\, dx$$

$$= \frac{e^{2x}}{2} (\cos 2x + \sin 2x) - \int e^{2x} \cos 2x\, dx$$

$$\Rightarrow \quad I_1 = \frac{e^{2x}}{2}(\cos 2x + \sin 2x) - I_1 \qquad \text{[By using equation (3)]}$$

$$\Rightarrow \quad 2I_1 = \frac{e^{2x}}{2}(\cos 2x + \sin 2x)$$

$$\Rightarrow \quad I_1 = \frac{e^{2x}}{4}(\cos 2x + \sin 2x)$$

Putting this value of I_1 in equation (2), we have

$$I = \frac{e^{2x}}{4} + \frac{1}{2}\left[\frac{e^{2x}}{4}(\cos 2x + \sin 2x)\right] + c$$

$$= \frac{e^{2x}}{4} + \frac{e^{2x}}{8}(\cos 2x + \sin 2x) + c.$$

Example 2. *Evaluate the following integrals :*

(i) $\int e^{ax} \sin(bx + c)\, dx$ (ii) $\int e^{ax} \cos(bx + c)\, dx$

(iii) $\int \sin(\log x)\, dx$ (iv) $\int x^2 e^{x^3} \cos x^3\, dx$

(v) $\int e^{\sin x} \sin 2x\, dx$ (vi) $\int e^{-x} \cos 4x \cos 2x\, dx$

(vii) $\int \frac{e^{a \tan^{-1} x}}{(1+x^2)^{3/2}}\, dx.$

Solution. (i) Let $I = \int \underset{\mathrm{I}}{e^{ax}}\ \underset{\mathrm{II}}{\sin}\,(bx + c)\, dx$ ...(1)

Integrating by parts, we get

Remark. If the integrand contains both the functions integrable and none can be finished by repeated differentiation, then take any one as the first function and other as the second one. Repeat the rule of 'Integration by parts'.

$$I = e^{ax} . \int \sin(bx+c)\, dx - \int \left\{\frac{d}{dx}(e^{ax}) . \int \sin(bx+c)\, dx\right\} dx$$

$$= e^{ax}\left(-\frac{\cos(bx+c)}{b}\right) - \int a\, e^{ax}\left(-\frac{\cos(bx+c)}{b}\right) dx$$

$$= -\frac{e^{ax}}{b}\cos(bx+c) + \frac{a}{b}\int \underset{\mathrm{I}}{e^{ax}}\ \underset{\mathrm{II}}{\cos(bx+c)}\, dx \qquad \text{[Integrating again by parts]}$$

$$= -\frac{e^{ax}}{b}\cos(bx+c) + \frac{a}{b}\left[e^{ax} . \int \cos(bx+c) - \int\left\{\frac{d}{dx}(e^{ax}) . \int \cos(bx+c)\, dx\right\} dx\right]$$

$$= -\frac{e^{ax}}{b}\cos(bx+c) + \frac{a}{b}\left[e^{ax}\frac{\sin(bx+c)}{b} - \int a\, e^{ax} . \frac{\sin(bx+c)}{b}\, dx\right]$$

$$= -\frac{e^{ax}}{b}\cos(bx+c) + \frac{a}{b^2}e^{ax}\sin(bx+c) - \frac{a^2}{b^2}\int e^{ax} . \sin(bx+c)\, dx$$

$$\Rightarrow \quad I = \frac{e^{ax}}{b}[a \sin(bx+c) - b\cos(bx+c)] - \frac{a^2}{b^2} I \qquad \text{[By using equation (1)]}$$

$$\Rightarrow \quad \left(1 + \frac{a^2}{b^2}\right) I = \frac{e^{ax}}{b^2}[a \sin(bx+c) - b\cos(bx+c)]$$

$$\Rightarrow \quad \left(\frac{a^2+b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}[a \sin(bx+c) - b\cos(bx+c)]$$

$$\therefore \quad I = \frac{e^{ax}}{a^2+b^2}[a \sin(bx+c) - b\cos(bx+c)] + C \qquad ...(2)$$

Note. The value of I can also be expressed alternatively as follows :

Put $\qquad a = r\cos\theta,\ b = r\sin\theta,\ r > 0$

Square and add, we have :

$$a^2 + b^2 = r^2(\sin^2\theta + \cos^2\theta)$$

$$\Rightarrow \quad r = \sqrt{a^2+b^2} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

On dividing, $\quad \dfrac{b}{a} = \dfrac{r\sin\theta}{r\cos\theta}$

$$\Rightarrow \quad \tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$\therefore$ Equation (2) becomes

$$I = \frac{e^{ax}}{r^2}[r\cos\theta\sin(bx+c) - r\sin\theta\cos(bx+c)] + C$$

$$= \frac{e^{ax}}{r^2}\,.\,r[\cos\theta\sin(bx+c) - \sin\theta\cos(bx+c)] + C$$

$$= \frac{e^{ax}}{r}\sin(bx+c-\theta) + C \qquad [\because \sin(A-B) = \sin A\cos B - \cos A\sin B]$$

$$\therefore \quad I = \frac{e^{ax}}{\sqrt{a^2+b^2}}\,.\,\sin\left(bx + c - \tan^{-1}\frac{b}{a}\right) + C.$$

(*ii*) Let $I = \int \underset{\text{I}}{e^{ax}} \underset{\text{II}}{\cos(bx+c)}\, dx \qquad ...(1)$

Integrating by parts, we get

$$I = e^{ax}\,.\int\cos(bx+c)\,dx - \int\left\{\frac{d}{dx}(e^{ax})\,.\int\cos(bx+c)\,dx\right\}dx$$

$$= e^{ax}\,.\,\frac{\sin(bx+c)}{b} - \int a\,e^{ax}\,.\,\frac{\sin(bx+c)}{b}\,dx$$

$$= \frac{e^{ax}}{b}\sin(bx+c) - \frac{a}{b}\int \underset{\text{I}}{e^{ax}}\ \underset{\text{II}}{\sin(bx+c)}\,dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{e^{ax}}{b}\sin(bx+c) - \frac{a}{b}\left[e^{ax}\,.\int\sin(bx+c)\,dx - \int\left\{\frac{d}{dx}(e^{ax})\,.\int\sin(bx+c)\,dx\right\}dx\right]$$

$$= \frac{e^{ax}}{b}\sin(bx+c) - \frac{a}{b}\left[e^{ax}\left(-\frac{\cos(bx+c)}{b}\right) - \int a\,e^{ax}\left(-\frac{\cos(bx+c)}{b}\right)dx\right]$$

$$= \frac{e^{ax}}{b}\sin(bx+c) + \frac{a\,e^{ax}}{b^2}\cos(bx+c) - \frac{a^2}{b^2}\int e^{ax}\cos(bx+c)\,dx$$

$$\Rightarrow \quad \mathrm{I} = \frac{e^{ax}}{b^2}[b\sin(bx+c) + a\cos(bx+c)] - \frac{a^2}{b^2}\mathrm{I} \qquad \text{[By using equation (1)]}$$

$$\Rightarrow \quad \left(1+\frac{a^2}{b^2}\right)\mathrm{I} = \frac{e^{ax}}{b^2}[b\sin(bx+c) + a\cos(bx+c)]$$

$$\Rightarrow \quad \left(\frac{a^2+b^2}{b^2}\right)\mathrm{I} = \frac{e^{ax}}{b^2}[b\sin(bx+c) + a\cos(bx+c)]$$

$$\Rightarrow \quad \mathrm{I} = \frac{e^{ax}}{(a^2+b^2)}[b\sin(bx+c) + a\cos(bx+c)] + \mathrm{C}. \qquad ...(2)$$

Note. The value of I can also be expressed alternatively as follows :

Put $\quad a = r\cos\theta,\ b = r\sin\theta,\ r > 0$

Square and add, we have

$$a^2 + b^2 = r^2(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow \quad r = \sqrt{a^2+b^2} \qquad [\because \ \sin^2 \mathrm{A} + \cos^2 \mathrm{A} = 1]$$

On dividing, $\quad \dfrac{b}{a} = \dfrac{r\sin\theta}{r\cos\theta}$

$$\Rightarrow \quad \tan\theta = \frac{b}{a} \ \Rightarrow\ \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$\therefore$ Equation (2) becomes

$$\mathrm{I} = \frac{e^{ax}}{r^2}[r\sin\theta\sin(bx+c) + r\cos\theta\cos(bx+c)] + \mathrm{C}$$

$$= \frac{e^{ax}}{r^2}\,.\,r\,[\sin\theta\sin(bx+c) + \cos\theta\cos(bx+c)] + \mathrm{C}$$

$$= \frac{e^{ax}}{r}[\cos(bx+c-\theta)] + \mathrm{C} \qquad [\because \ \cos(\mathrm{A}-\mathrm{B}) = \cos\mathrm{A}\cos\mathrm{B} + \sin\mathrm{A}\sin\mathrm{B}]$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\left[\cos\left(bx+c-\tan^{-1}\frac{b}{a}\right)\right] + \mathrm{C}.$$

(*iii*) Let $\quad \mathrm{I} = \int \sin(\log x)\,dx \qquad ...(1)$

Put $\quad \log x = z \ \Rightarrow\ x = e^z \ \Rightarrow\ dx = e^z\,dz$

$$\therefore \quad \mathrm{I} = \int \sin z\,.\,e^z\,dz$$

$$\Rightarrow \quad \mathrm{I} = \int \underset{\mathrm{I}}{e^z}\ \underset{\mathrm{II}}{\sin z}\,dz \qquad ...(2)$$

Integrating by parts, we get

$$\mathrm{I} = e^z \,.\int \sin z\,dz - \int \left\{\frac{d}{dz}(e^z).\int \sin z\,dz\right\} dz$$

$$= e^z\,.(-\cos z) - \int e^z\,.(-\cos z)\,dz$$

$$= -e^z \cos z + \int \underset{\mathrm{I}}{e^z}\ \underset{\mathrm{II}}{\cos z}\,dz \qquad \text{[Integrating again by parts]}$$

$$= -e^z \cos z + \left[e^z\,.\int \cos z\,dz - \int \left\{\frac{d}{dz}(e^z).\int \cos z\,dz\right\} dz\right]$$

$$= -e^z \cos z + \left[e^z \sin z - \int e^z \sin z\,dz\right]$$

$\Rightarrow$ $\mathrm{I} = -e^z \cos z + e^z \sin z - \mathrm{I}$ [By using equation (2)]

$\Rightarrow$ $2\mathrm{I} = e^z(\sin z - \cos z)$

$\Rightarrow$ $\mathrm{I} = \dfrac{e^z}{2}(\sin z - \cos z) + c$

$\Rightarrow$ $\mathrm{I} = \dfrac{e^{\log x}}{2}[\sin(\log x) - \cos(\log x)] + c$ [$\because$ $z = \log x$]

$\Rightarrow$ $\mathrm{I} = \dfrac{x}{2}[\sin(\log x) - \cos(\log x)] + c.$ [$\because$ $e^{(\log f(x))} = f(x)$]

(*iv*) Let $\mathrm{I} = \int x^2 e^{x^3} \cos x^3\,dx$ ...(1)

Put $x^2 = z \Rightarrow 3x^2\,dx = dz \Rightarrow x^2\,dx = \dfrac{1}{3}dz$

$\therefore$ $\mathrm{I} = \int e^z \cos z\,.\left(\dfrac{1}{3}dz\right)$

$\Rightarrow$ $\mathrm{I} = \dfrac{1}{3}\int \underset{\mathrm{I}}{e^z}\ \underset{\mathrm{II}}{\cos z}\,dz$ ...(2)

Integrating by parts, we get

$$\mathrm{I} = \frac{1}{3}\left[e^z\,.\int \cos z\,dz - \int \left\{\frac{d}{dz}(e^z).\int \cos z\,dz\right\} dz\right]$$

$$= \frac{1}{3}\left[e^z \sin z - \int e^z \sin z\,dz\right]$$

$$= \frac{1}{3}e^z \sin z - \frac{1}{3}\int \underset{\mathrm{I}}{e^z}\ \underset{\mathrm{II}}{\sin z}\,dz \qquad \text{[Integrating again by parts]}$$

$$= \frac{1}{3}e^z \sin z - \frac{1}{3}\left[e^z\,.\int \sin z\,dz - \int \left\{\frac{d}{dz}(e^z).\int \sin z\,dz\right\} dz\right]$$

$$= \frac{1}{3}e^z \sin z - \frac{1}{3}\left[e^z(-\cos z) - \int e^z(-\cos z)\,dz\right]$$

$$= \frac{1}{3}e^z \sin z + \frac{1}{3}e^z \cos z - \frac{1}{3}\int e^z \cos z\,dz$$

$\Rightarrow \quad I = -\frac{1}{3} e^z \sin z + \frac{1}{3} e^z \cos z - \frac{1}{3} I$ [By using equation (2)]

$\Rightarrow \quad \left(1+\frac{1}{3}\right) I = \frac{1}{3} e^z [\cos z - \sin z]$

$\Rightarrow \quad \frac{4}{3} I = \frac{1}{3} e^z [\cos z - \sin z]$

$\Rightarrow \quad I = \frac{1}{4} e^z (\cos z - \sin z) + c$

$\Rightarrow \quad I = \frac{1}{4} e^{x^3} (\cos x^3 - \sin x^3) + c.$ $[\because \ z = x^3]$

(*v*) Let $\quad I = \int e^{\sin x} \sin 2x \, dx$

$= \int e^{\sin x} . 2 \sin x \cos x \, dx$ $[\because \ \sin 2A = 2 \sin A \cos A]$

Put $\sin x = z \Rightarrow \cos x \, dx = dz$

$\therefore \quad I = \int e^z . (2z \, dz)$

$= 2 \int \underset{\text{I}}{z} \underset{\text{II}}{e^z} \, dz$

Integrating by parts, we get

$$I = 2\left[z . \int e^z \, dz - \int \left\{\frac{d}{dz}(z) . \int e^z \, dz\right\} dz\right]$$

$$= 2\left[z e^z - \int 1 . e^z \, dz\right] = 2z e^z - 2 e^z + c = 2 e^z (z - 1) + c$$

$= 2 e^{\sin x} (\sin x - 1) + c.$ $[\because \ z = \sin x]$

(*vi*) Let $I = \int e^{-x} \cos 4x \cos 2x \, dx$ ...(1)

$= \frac{1}{2} \int e^{-x} (2 \cos 4x \cos 2x) \, dx$ [Multiply and divided by 2]

$= \frac{1}{2} \int e^{-x} [\cos (4x + 2x) + \cos (4x - 2x)] \, dx$

$[\because \ 2 \cos A \cos B = \cos (A + B) + \cos (A - B)]$

$= \frac{1}{2} \int e^{-x} (\cos 6x + \cos 2x) \, dx$

$= \frac{1}{2} \int e^{-x} \cos 6x \, dx + \frac{1}{2} \int e^{-x} \cos 2x \, dx$

$\Rightarrow \quad I = \frac{1}{2} I_1 + \frac{1}{2} I_2 \text{(say)}$ ...(2)

where $\quad I_1 = \int \underset{\text{I}}{e^{-x}} \underset{\text{II}}{\cos 6x} \, dx$ ...(3)

Integrating by parts, we get

$$I_1 = e^{-x} . \int \cos 6x \, dx - \int \left\{ \frac{d}{dx}(e^{-x}) . \int \cos 6x \, dx \right\} dx$$

$$= e^{-x} \frac{\sin 6x}{6} - \int (-e^{-x}) . \frac{\sin 6x}{6} \, dx$$

$$= \frac{1}{6} e^{-x} \sin 6x + \frac{1}{6} \int \underset{\text{I}}{e^{-x}} \underset{\text{II}}{\sin 6x} \, dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{1}{6} e^{-x} \sin 6x + \frac{1}{6} \left[e^{-x} . \int \sin 6x \, dx - \int \left\{ \frac{d}{dx}(e^{-x}) . \int \sin 6x \, dx \right\} dx \right]$$

$$= \frac{1}{6} e^{-x} \sin 6x + \frac{1}{6} \left[e^{-x} . \left(\frac{-\cos 6x}{6} \right) - \int (-e^{-x}) . \left(\frac{-\cos 6x}{6} \right) dx \right]$$

$$= \frac{1}{6} e^{-x} \sin 6x - \frac{1}{36} e^{-x} \cos 6x - \frac{1}{36} \int e^{-x} \cos 6x \, dx$$

$$\Rightarrow \quad I_1 = \frac{1}{36} e^{-x} [6 \sin 6x - \cos 6x] - \frac{1}{36} I_1 \qquad \text{[By using equation (3)]}$$

$$\Rightarrow \quad \left(1 + \frac{1}{36} \right) I_1 = \frac{1}{36} e^{-x} [6 \sin 6x - \cos 6x]$$

$$\Rightarrow \quad \frac{37}{36} I_1 = \frac{1}{36} e^{-x} (6 \sin 6x - \cos 6x)$$

$$\Rightarrow \quad I_1 = \frac{1}{37} e^{-x} (6 \sin 6x - \cos 6x) \qquad \text{...(4)}$$

and

$$I_2 = \int \underset{\text{I}}{e^{-x}} \underset{\text{II}}{\cos 2x} \, dx \qquad \text{...(5)}$$

Integrating by parts, we get

$$I_2 = e^{-x} . \int \cos 2x \, dx - \int \left\{ \frac{d}{dx}(e^{-x}) . \int \cos 2x \, dx \right\} dx$$

$$= e^{-x} \frac{\sin 2x}{2} - \int (-e^{-x}) . \frac{\sin 2x}{2} \, dx$$

$$= \frac{1}{2} e^{-x} \sin 2x + \frac{1}{2} \int \underset{\text{I}}{e^{-x}} \underset{\text{II}}{\sin 2x} \, dx \qquad \text{[Integrating again by parts]}$$

$$= \frac{1}{2} e^{-x} \sin 2x + \frac{1}{2} \left[e^{-x} . \int \sin 2x \, dx - \int \left\{ \frac{d}{dx}(e^{-x}) . \int \sin 2x \, dx \right\} dx \right]$$

$$= \frac{1}{2} e^{-x} \sin 2x + \frac{1}{2} \left[e^{-x} . \left(\frac{-\cos 2x}{2} \right) - \int (-e^{-x}) . \left(\frac{-\cos 2x}{2} \right) dx \right]$$

$$= \frac{1}{2} e^{-x} \sin 2x - \frac{1}{4} e^{-x} \cos 2x - \frac{1}{4} \int e^{-x} \cos 2x \, dx$$

$$\Rightarrow \quad I_2 = \frac{1}{4} e^{-x} [2 \sin 2x - \cos 2x] - \frac{1}{4} I_2 \qquad [\because \text{ By using equation (5)}]$$

$\Rightarrow \quad \left(1+\dfrac{1}{4}\right)I_2 = \dfrac{1}{4}e^{-x}[2\sin 2x - \cos 2x]$

$\Rightarrow \quad \dfrac{5}{4}I_2 = \dfrac{1}{4}e^{-x}(2\sin 2x - \cos 2x)$

$\Rightarrow \quad I_2 = \dfrac{1}{5}e^{-x}(2\sin 2x - \cos 2x)$ ...(6)

$\therefore$ From equation (2),

$$I = \frac{1}{2}I_1 + \frac{1}{2}I_2$$

$\Rightarrow \quad I = \dfrac{1}{2}\left[\dfrac{1}{37}e^{-x}(6\sin 6x - \cos 6x)\right] + \dfrac{1}{2}\left[\dfrac{1}{5}e^{-x}(2\sin 2x - \cos 2x)\right] + c$

[Using equations (4) and (6)]

$\therefore \quad I = e^{-x}\left[\dfrac{1}{74}(6\sin 6x - \cos 6x) + \dfrac{1}{10}(2\sin 2x - \cos 2x)\right] + c.$

(*vii*) Let $\quad I = \displaystyle\int \frac{e^{a\tan^{-1}x}}{(1+x^2)^{3/2}}\,dx$ ...(1)

Put $\tan^{-1}x = z \Rightarrow x = \tan z$

$\Rightarrow \quad \dfrac{1}{1+x^2}\,dx = dz$

$\therefore \quad I = \displaystyle\int \frac{e^{a\tan^{-1}x}}{\sqrt{1+x^2}\,.(1+x^2)}\,dx = \int \frac{e^{az}}{\sqrt{1+\tan^2 z}}\,dz$

$= \displaystyle\int \frac{e^{az}}{\sec z}\,dz \qquad \left[\begin{array}{l}\because \sec^2 A - \tan^2 A = 1\\ \Rightarrow \sec^2 A = 1 + \tan^2 A\end{array}\right]$

$I = \displaystyle\int \underset{\text{I}}{e^{az}}\ \underset{\text{II}}{\cos z}\,dz$ ...(2)

Integrating by parts, we get

$$I = e^{az}\,.\int \cos z\,dz - \int\left\{\frac{d}{dz}(e^{az})\,.\int \cos z\,dz\right\}dz$$

$$= e^{az}\sin z - \int a\,e^{az}\sin z\,dz$$

$= e^{az}\sin z - a\displaystyle\int \underset{\text{I}}{e^{az}}\ \underset{\text{II}}{\sin z}\,dz$ [Integrating again by parts]

$$= e^{az}\sin z - a\left[e^{az}\,.\int \sin z\,dz - \int\left\{\frac{d}{dz}(e^{az})\,.\int \sin z\,dz\right\}dz\right]$$

$$= e^{az}\sin z - a\left[e^{az}\,.(-\cos z) - \int a\,e^{az}(-\cos z)\,dz\right]$$

$$= e^{az}\sin z + a\,e^{az}\cos z - a^2\int e^{az}\cos z\,dz$$

$\Rightarrow \quad I = e^{az}(\sin z + a\cos z) - a^2 I$ [By using equation (2)]

$\Rightarrow \quad (1 + a^2)\, \mathrm{I} = e^{az} (\sin z + a \cos z)$

$$\Rightarrow \quad \mathrm{I} = \frac{1}{(1+a^2)} e^{az} (\sin z + a \cos z) + c$$

$$\Rightarrow \quad \mathrm{I} = \frac{1}{(1+a^2)} e^{a \tan^{-1} x} [\sin (\tan^{-1} x) + a \cos (\tan^{-1} x)] + c. \qquad [\because \ z = \tan^{-1} x]$$

Example 3. *Evaluate the following integrals :*

(*i*) $\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$ (*ii*) $\int e^x \left(\frac{x+1}{(2+x)^2}\right) dx$

(*iii*) $\int e^{2x} \left(\frac{2x-1}{4x^2}\right) dx$ (*iv*) $\int e^x \left(\frac{x^2+1}{(x+1)^2}\right) dx.$

Solution. (*i*) Let $\mathrm{I} = \int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

$$= \int \underset{\mathrm{I}}{\frac{1}{x}} \underset{\mathrm{II}}{e^x} dx - \int \frac{1}{x^2} e^x dx$$

Integrating the first integral by parts, we get

$$\mathrm{I} = \frac{1}{x} . \int e^x dx - \int \left\{\frac{d}{dx}\left(\frac{1}{x}\right) . \int e^x dx\right\} dx - \int \frac{1}{x^2} e^x dx$$

$$= \frac{1}{x} e^x - \int \left(-\frac{1}{x^2}\right) e^x dx - \int \frac{1}{x^2} e^x dx$$

$$= \frac{1}{x} e^x + \int \frac{1}{x^2} e^x dx - \int \frac{1}{x^2} e^x dx + c = \frac{1}{x} e^x + c.$$

(*ii*) Let $\mathrm{I} = \int e^x \left(\frac{x+1}{(2+x)^2}\right) dx$

$$= \int e^x \left(\frac{x+2-1}{(2+x)^2}\right) dx \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \int e^x \left(\frac{x+2}{(x+2)^2} - \frac{1}{(x+2)^2}\right) dx$$

$$= \int \underset{\mathrm{II}}{e^x} . \underset{\mathrm{I}}{\frac{1}{(x+2)}} dx - \int e^x . \frac{1}{(x+2)^2} dx$$

Integrating the first integral by parts, we get

$$\mathrm{I} = \left(\frac{1}{x+2}\right) . \int e^x dx - \int \left\{\frac{d}{dx}\left(\frac{1}{x+2}\right) . \int e^x dx\right\} dx - \int e^x . \frac{1}{(x+2)^2} dx$$

$$= \left(\frac{1}{x+2}\right) e^x - \int \left(-\frac{1}{(x+2)^2}\right) e^x dx - \int e^x \frac{1}{(x+2)^2} dx$$

$$= e^x\left(\frac{1}{x+2}\right) + \int e^x \frac{1}{(x+2)^2}\,dx - \int e^x \frac{1}{(x+2)^2}\,dx + c$$

$$= \frac{e^x}{x+2} + c.$$

(*iii*) Let $\quad I = \int e^{2x}\left(\frac{2x-1}{4x^2}\right)dx$

Put $\quad 2x = z \;\Rightarrow\; 2dx = dz \;\Rightarrow\; dx = \frac{1}{2}\,dz$

$$\therefore \quad I = \int e^z\left(\frac{z-1}{z^2}\right)\left(\frac{1}{2}\,dz\right) = \frac{1}{2}\int e^z\left(\frac{z}{z^2} - \frac{1}{z^2}\right)dz$$

$$= \frac{1}{2}\int \underset{\text{II}}{e^z} \cdot \underset{\text{I}}{\frac{1}{z}}\,dz - \frac{1}{2}\int e^z \cdot \frac{1}{z^2}\,dz$$

Integrating the first integral by parts, we get

$$I = \frac{1}{2}\left[\left(\frac{1}{z}\right)\cdot\int e^z\,dz - \int\left\{\frac{d}{dz}\left(\frac{1}{z}\right)\cdot\int e^z\,.dz\right\}dz\right] - \frac{1}{2}\int e^z \cdot \frac{1}{z^2}\,dz$$

$$= \frac{1}{2}\left[\left(\frac{1}{z}\right)e^z - \int\left(-\frac{1}{z^2}\right)e^z\,dz\right] - \frac{1}{2}\int e^z \cdot \frac{1}{z^2}\,dz$$

$$= \frac{1}{2}\frac{e^z}{z} + \frac{1}{2}\int e^z \cdot \frac{1}{z^2}\,dz - \frac{1}{2}\int e^z \cdot \frac{1}{z^2}\,dz + c = \frac{e^z}{2z} + c$$

$$= \frac{e^{2x}}{2(2x)} + c = \frac{e^{2x}}{4x} + c \qquad [\because\; z = 2x]$$

(*iv*) Let $\quad I = \int e^x\left(\frac{x^2+1}{(x+1)^2}\right)dx$

$$= \int e^x\left(\frac{x^2+2-1}{(x+1)^2}\right)dx \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \int e^x\left(\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}\right)dx = \int e^x\left(\frac{(x-1)(x+1)}{(x+1)^2} + \frac{2}{(x+1)^2}\right)dx$$

$$= \int \underset{\text{II}}{e^x}\,\underset{\text{I}}{\left(\frac{x-1}{x+1}\right)}dx + \int e^x \cdot \frac{2}{(x+1)^2}\,dx$$

Integrating the first integral by parts, we get

$$I = \left(\frac{x-1}{x+1}\right)\cdot\int e^x\,dx - \int\left\{\frac{d}{dx}\left(\frac{x-1}{x+1}\right)\cdot\int e^x\,dx\right\}dx + \int e^x \cdot \frac{2}{(x+1)^2}\,dx$$

$$= \left(\frac{x-1}{x+1}\right)e^x - \int\left[\frac{(x+1).1-(x-1).1}{(x+1)^2}\right].e^x\,dx + \int e^x \cdot \frac{2}{(x+1)^2}\,dx$$

$$= \left(\frac{x-1}{x+1}\right)e^x - \int e^x \cdot \frac{2}{(x+1)^2}\,dx + \int e^x \cdot \frac{2}{(x+1)^2}\,dx + c$$

$$= \left(\frac{x-1}{x+1}\right) \cdot e^x + c.$$

Example 4. *Evaluate the following integrals :*

(*i*) $\int e^x \left(\frac{2-x}{(1-x)^2}\right) dx$ (*ii*) $\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$

(*iii*) $\int e^x \sec x\,(1 + \tan x)\,dx$ (*iv*) $\int e^x [\sec x + \log(\sec x + \tan x)]\,dx$

(*v*) $\int e^{2x} (-\sin x + 2\cos x)\,dx$ (*vi*) $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx.$

Solution. (*i*) Let $I = \int e^x \left(\frac{2-x}{(1-x)^2}\right) dx = \int e^x \left(\frac{1-x+1}{(1-x)^2}\right) dx$

$$= \int e^x \left(\frac{1-x}{(1-x)^2} + \frac{1}{(1-x)^2}\right) \cdot dx = \int e^x \left(\frac{1}{(1-x)} + \frac{1}{(1-x)^2}\right) dx$$

$$= \int \underset{\text{II}}{e^x} \cdot \underset{\text{I}}{\frac{1}{(1-x)}}\,dx + \int e^x \cdot \frac{1}{(1-x)^2}\,dx$$

Integrating the first integral by parts, we get

$$I = \left(\frac{1}{1-x}\right) \cdot \int e^x\,dx - \int \left\{\frac{d}{dx}\left(\frac{1}{1-x}\right) \cdot \int e^x\,dx\right\} dx + \int e^x \cdot \frac{1}{(1-x)^2}\,dx$$

$$= \left(\frac{1}{1-x}\right)e^x - \int \left[\frac{-1}{(1-x)^2}\right](-1)\,e^x\,dx + \int e^x \cdot \frac{1}{(1-x)^2}\,dx$$

$$= \left(\frac{1}{1-x}\right)e^x - \int \frac{e^x}{(1-x)^2}\,dx + \int \frac{e^x}{(1-x)^2}\,dx + c$$

$$= \frac{e^x}{1-x} + c.$$

(*ii*) Let $I = \int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$

$$= \int \underset{\text{II}}{e^x} \cdot \underset{\text{I}}{\frac{1}{x^2}}\,dx - \int e^x \cdot \frac{2}{x^3}\,dx$$

Integrating the first integral by parts, we get

$$= \frac{1}{x^2} \cdot \int e^x\,dx - \int \left\{\frac{d}{dx}\left(\frac{1}{x^2}\right) \cdot \int e^x\,dx\right\} dx - \int e^x \cdot \frac{2}{x^3}\,dx$$

$$= \frac{1}{x^2}e^x - \int \frac{-2}{x^3}e^x\,dx - \int e^x \cdot \frac{2}{x^3}\,dx$$

$$= \frac{e^x}{x^2} + \int \frac{2}{x^3} e^x \, dx - \int e^x \cdot \frac{2}{x^3} \, dx + c$$

$$= \frac{e^x}{x^2} + c.$$

(*iii*) Let $\mathrm{I} = \int e^x \sec x \, (1 + \tan x) \, dx$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sec x} \, dx + \int e^x \sec x \tan x \, dx$$

Integrating the first integral by parts, we get

$$= \sec x \int e^x \, dx - \int \left\{ \frac{d}{dx} (\sec x) \cdot \int e^x \, dx \right\} dx + \int e^x \sec x \tan x \, dx$$

$$= \sec x \cdot e^x - \int \sec x \tan x \cdot e^x \, dx + \int e^x \sec x \tan x \, dx.$$

$$= e^x \sec x - \int e^x \sec x \tan x \, dx + \int e^x \sec x \tan x \, dx + c$$

$$= e^x \sec x + c.$$

(*iv*) Let $\mathrm{I} = \int e^x \, [\sec x + \log (\sec x + \tan x)] \, dx$

$$= \int e^x \sec x \, dx + \int \underset{\text{II}}{e^x} \underset{\text{I}}{\log (\sec x + \tan x)} \, dx$$

Integrating the second integral by parts, we get

$$\mathrm{I} = \int e^x \sec x \, dx + \log (\sec x + \tan x) \cdot \int e^x \, dx - \int \left\{ \frac{d}{dx} [\log (\sec x + \tan x)] \cdot \int e^x \, dx \right\} dx$$

$$= \int e^x \sec x \, dx + \log (\sec x + \tan x) \cdot e^x - \int \left[\frac{1}{(\sec x + \tan x)} (\sec x \tan x + \sec^2 x) \cdot e^x \right] dx$$

$$= \int e^x \sec x \, dx + e^x \log (\sec x + \tan x) - \int \frac{\sec x \, (\sec x + \tan x)}{(\sec x + \tan x)} e^x \, dx$$

$$= \int e^x \sec x \, dx + e^x \log (\sec x + \tan x) - \int e^x \sec x \, dx$$

$$= e^x \cdot \log (\sec x + \tan x) + c.$$

(*v*) Let $\mathrm{I} = \int e^{2x} (-\sin x + 2 \cos x) \, dx = \int e^{2x} (-\sin x) \, dx + 2 \int e^{2x} \cos x \, dx$

$$= 2 \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\cos x} \, dx - \int e^{2x} \sin x \, dx$$

Integrating the first integral by parts, we get

$$\mathrm{I} = 2 \left[\cos x \cdot \int e^{2x} \, dx - \int \left\{ \frac{d}{dx} (\cos x) \cdot \int e^{2x} \, dx \right\} dx - \int e^{2x} \sin x \, dx \right]$$

$$= 2\left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \cdot \frac{e^{2x}}{2}\, dx - \int e^{2x} \sin x\, dx\right]$$

$$= e^{2x} \cos x + \int e^{2x} \sin x\, dx - \int e^{2x} \sin x\, dx + c$$

$$= e^{2x} \cos x + c.$$

(*vi*) Let $\quad I = \int e^x \left(\frac{1 - \sin x}{1 - \cos x}\right) dx$

$$= \int e^x \left(\frac{1}{(1 - \cos x)} - \frac{\sin x}{(1 - \cos x)}\right) dx = \int \frac{e^x}{(1 - \cos x)}\, dx - \int e^x \frac{\sin x}{(1 - \cos x)}\, dx$$

$$= \int \frac{e^x}{2 \sin^2 \frac{x}{2}}\, dx - \int e^x \left(\frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}}\right) dx \qquad \left[\begin{array}{l} \because\ 1 - \cos 2A = 2 \sin^2 A \\ \Rightarrow\ 1 - \cos A = 2 \sin^2 \frac{A}{2} \\ \sin 2A = 2 \sin A \cos A \\ \Rightarrow\ \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \end{array}\right]$$

$$= \frac{1}{2} \int e^x \operatorname{cosec}^2 \frac{x}{2}\, dx - \int e^x \cot \frac{x}{2}\, dx$$

$$= -\int \underset{\text{II}}{e^x}\ \underset{\text{I}}{\cot \frac{x}{2}}\, dx + \frac{1}{2} \int e^x \operatorname{cosec}^2 \frac{x}{2}\, dx$$

Integrating the first integral by parts, we get

$$= -\left[\cot \frac{x}{2} \cdot \int e^x\, dx - \int \left\{\frac{d}{dx}\left(\cot \frac{x}{2}\right) \cdot \int e^x\, dx\right\} dx\right] + \frac{1}{2} \int e^x \operatorname{cosec}^2 \frac{x}{2}\, dx$$

$$= -\left[\cot \frac{x}{2} \cdot e^x - \int \left(-\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2}\right) e^x\, dx\right] + \frac{1}{2} \int e^x \operatorname{cosec}^2 \frac{x}{2}\, dx$$

$$= -e^x \cot \frac{x}{2} - \frac{1}{2} \int e^x \operatorname{cosec}^2 \frac{x}{2}\, dx + \frac{1}{2} \int e^x \operatorname{cosec}^2 \frac{x}{2}\, dx + c$$

$$= -e^x \cot \frac{x}{2} + c.$$

Example 5. *Evaluate the following integrals :*

(*i*) $\int \frac{2 + \sin 2x}{1 + \cos 2x} e^x\, dx$ $\qquad$ (*ii*) $\int e^x (\sin x + \cos x)\, dx$

(*iii*) $\int \frac{x}{(x + 1)^2} e^x\, dx$ $\qquad$ (*iv*) $\int [\sin (\log x) + \cos (\log x)]\, dx$

(*v*) $\int \frac{\log x}{(1 + \log x)^2}\, dx$ $\qquad$ (*vi*) $\int e^x \left(\frac{\sin x \cos x + 1}{\cos^2 x}\right) dx.$

Solution. (*i*) Let $I = \int e^x \cdot \left(\frac{2 + \sin 2x}{1 + \cos 2x}\right) dx$

$$= \int e^x \cdot \frac{2 + 2 \sin x \cos x}{2 \cos^2 x}\, dx \qquad \left[\begin{array}{l} \because\ \sin 2A = 2 \sin A \cos A \\ \quad 1 + \cos 2A = 2 \cos^2 A \end{array}\right]$$

$$= \int e^x \cdot \left(\frac{2}{2\cos^2 x} + \frac{2 \sin x \cos x}{2 \cos^2 x} \right) dx$$

$$= \int e^x (\sec^2 x + \tan x)\, dx = \int e^x \sec^2 x\, dx + \int e^x \tan x\, dx$$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\tan x}\, dx + \int e^x \sec^2 x\, dx$$

Integrating the first integral by parts, we get

$$\text{I} = \tan x \,.\, \int e^x\, dx - \int \left\{ \frac{d}{dx} (\tan x) \,.\, \int e^x\, dx \right\} dx + \int e^x \sec^2 x\, dx$$

$$= \tan x \,.\, e^x - \int \sec^2 x \,.\, e^x\, dx + \int e^x \sec^2 x\, dx + c$$

$$= e^x \tan x - \int e^x \sec^2 x\, dx + \int e^x \sec^2 x\, dx + c$$

$$= e^x \tan x + c.$$

(*ii*) Let $\text{I} = \int e^x (\sin x + \cos x)\, dx.$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin x}\, dx + \int e^x \cos x\, dx$$

Integrating the first integral by parts, we get

$$\text{I} = \sin x \,.\, \int e^x\, dx - \int \left\{ \frac{d}{dx} (\sin x) \,.\, \int e^x\, dx \right\} dx + \int e^x \cos x\, dx$$

$$= \sin x \,.\, e^x - \int \cos x\, e^x\, dx + \int e^x \cos x\, dx + c$$

$$= e^x \sin x - \int e^x \cos x\, dx + \int e^x \cos x\, dx + c$$

$$= e^x \sin x + c.$$

(*iii*) Let $\text{I} = \int \frac{x\, e^x}{(x+1)^2}\, dx$

$$= \int e^x \left[\frac{x+1-1}{(x+1)^2} \right] dx$$ **[Add and subtract 1 to the numerator]**

$$= \int e^x \left[\frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2} \right] dx = \int e^x \left[\frac{1}{x+1} - \frac{1}{(x+1)^2} \right] dx$$

$$= \int \underset{\text{II}}{e^x} \,.\, \underset{\text{I}}{\frac{1}{(x+1)}}\, dx - \int e^x \,.\, \frac{1}{(x+1)^2}\, dx$$

Integrating the first integral by parts, we get

$$\text{I} = \frac{1}{(x+1)} \,.\, \int e^x\, dx - \int \left\{ \frac{d}{dx} \left(\frac{1}{x+1} \right) . \int e^x\, dx \right\} dx - \int e^x \,.\, \frac{1}{(x+1)^2}\, dx$$

$$= \frac{1}{x+1} \,.\, e^x - \int (-1) . \frac{1}{(x+1)^2} e^x\, dx - \int e^x \,.\, \frac{1}{(x+1)^2}\, dx + c$$

$$= \frac{e^x}{x+1} + \int e^x \cdot \frac{1}{(x+1)^2}\,dx - \int e^x \frac{1}{(x+1)^2}\,dx + c$$

$$= \frac{e^x}{x+1} + c.$$

(*iv*) Let $\quad I = \int [\sin(\log x) + \cos(\log x)]\,dx$

Put $\quad \log x = z \Rightarrow x = e^z \Rightarrow dx = e^z\,dz$

$\therefore \quad I = \int (\sin z + \cos z)\,e^z\,dz$

$$= \int \underset{\text{II}}{e^z}\ \underset{\text{I}}{\sin z}\,dz + \int e^z \cos z\,dz$$

Integrating the first integral by parts, we get

$$I = \sin z \cdot \int e^z\,dz - \int \left\{\frac{d}{dz}(\sin z) \cdot \int e^z\,dz\right\}dz + \int e^z \cos z\,dz$$

$$= \sin z \cdot e^z - \int \cos z \cdot e^z\,dz + \int e^z \cos z\,dz$$

$$= e^z \sin z - \int e^z \cos z\,dz + \int e^z \cos z\,dz + c$$

$$= e^z \sin z + c$$

$= e^{\log x} \sin(\log x) + c \qquad [\because\ z = \log x]$

$= x \sin(\log x) + c. \qquad [\because\ e^{\log f(x)} = f(x)]$

(*v*) Let $\quad I = \int \frac{\log x}{(1+\log x)^2}\,dx$ **(Please try yourself.)**

[**Hint.** Put $\log x = z \Rightarrow x = e^z \Rightarrow dx = e^z\,dz$ **[Ans.** $\frac{x}{(\log x + 1)} + c.$]

$\therefore \quad I = \int \frac{z}{(1+z)^2}\,e^z\,dz]$

(*vi*) Let $\quad I = \int e^x \left(\frac{\sin x \cos x + 1}{\cos^2 x}\right)dx = \int e^x \left(\frac{\sin x \cos x}{\cos^2 x} + \frac{1}{\cos^2 x}\right)dx$

$$= \int \underset{\text{II}}{e^x}\ \underset{\text{I}}{\tan x}\,dx + \int e^x \sec^2 x\,dx$$

Integrating the first integral by parts, we get

$$I = \tan x \cdot \int e^x\,dx - \int \left\{\frac{d}{dx}(\tan x) \cdot \int e^x\,dx\right\}dx + \int e^x \sec^2 x\,dx$$

$$= \tan x \cdot e^x - \int \sec^2 x \cdot e^x\,dx + \int e^x \sec^2 x + c$$

$$= e^x \tan x + c.$$

Example 6. *Evaluate the following integrals :*

(*i*) $\int e^x (\tan x + \log \sec x)\,dx$ (*ii*) $\int e^x (\cot x + \log \sin x)\,dx$

(*iii*) $\int e^{2x} \left(\frac{\sin 4x - 2}{1 - \cos 4x}\right)dx.$

Solution. (*i*) Let $I = \int e^x (\tan x + \log \sec x)\, dx$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\log \sec x}\, dx + \int e^x \tan x\, dx$$

Integrating the first integral by parts, we get

$$I = \log \sec x \,.\, \int e^x\, dx - \int \left\{\frac{d}{dx}(\log \sec x) \,.\, \int e^x\, dx\right\} . dx + \int e^x \tan x\, dx$$

$$= \log \sec x \,.\, e^x - \int \frac{1}{\sec x}(\sec x \tan x) \,.\, e^x\, dx + \int e^x \tan x\, dx$$

$$= e^x \log \sec x - \int e^x \tan x\, dx + \int e^x \tan x\, dx + c$$

$$= e^x \log \sec x + c.$$

(*ii*) Let $\quad I = \int e^x (\cot x + \log \sin x)\, dx$

$$= \int e^x \cot x\, dx + \int e^x \log \sin x\, dx$$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\log \sin x}\, dx + \int e^x \cot x\, dx$$

Integrating the first integral by parts, we get

$$I = \log \sin x \,.\, \int e^x\, dx - \int \left\{\frac{d}{dx}(\log \sin x) \,.\, \int e^x\, dx\right\} . dx + \int e^x \cot x\, dx$$

$$= \log \sin x \,.\, e^x - \int \frac{1}{\sin x}(\cos x) \,.\, e^x\, dx + \int e^x \cot x\, dx$$

$$= e^x \log \sin x - \int e^x \cot x\, dx + \int e^x \cot x\, dx + c$$

$$= e^x \log (\sin x) + c.$$

(*iii*) Let $\quad I = \int e^{2x} \left(\frac{\sin 4x - 2}{1 - \cos 4x}\right) dx$

$$= \int e^{2x} \left(\frac{2 \sin 2x \cos 2x - 2}{2 \sin^2 2x}\right) dx \qquad \left[\begin{array}{l} \because\ \sin 2A = 2 \sin A \cos A \\ \Rightarrow\ \sin 4A = 2 \sin 2A \cos 2A \\ \quad 1 - \cos 2A = 2 \sin^2 A \\ \Rightarrow\ 1 - \cos 4A = 2 \sin^2 2A \end{array}\right]$$

$$= \int e^{2x} \left(\frac{2 \sin 2x \cos 2x}{2 \sin^2 2x} - \frac{2}{2 \sin^2 2x}\right) dx$$

$$= \int e^{2x} (\cot 2x - \text{cosec}^2\, 2x)\, dx$$

$$= \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\cot 2x}\, dx - \int e^{2x} \text{cosec}^2\, 2x\, dx$$

Integrating the first integral by parts, we get

$$I = \cot 2x \,.\, \int e^{2x}\, dx - \int \left\{\frac{d}{dx}(\cot 2x) \,.\, \int e^{2x}\, dx\right\} dx - \int e^{2x} \text{cosec}^2\, 2x\, dx$$

$$= \cot 2x \cdot \frac{e^{2x}}{2} - \int (-2\operatorname{cosec}^2 2x) \cdot \frac{e^{2x}}{2}\, dx - \int e^{2x} \operatorname{cosec}^2 2x\, dx$$

$$= \frac{e^{2x}}{2} \cot 2x + \int e^{2x} \operatorname{cosec}^2 2x - \int e^{2x} \operatorname{cosec}^2 2x + c$$

$$= \frac{e^{2x}}{2} \cot 2x + c.$$

Example 7. *Evaluate the following integrals :*

(i) $\int \cos 2x \cdot \log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) dx$ (ii) $\int \frac{\sin^{-1} x}{x^2}\, dx$

(iii) $\int e^x \left(\frac{\sqrt{1-x^2}\, \sin^{-1} x + 1}{\sqrt{1-x^2}}\right) dx$ (iv) $\int x \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right) dx.$

Solution. (i) Let $I = \int \underset{\text{II}}{\cos 2x} \cdot \underset{\text{I}}{\log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)} dx$

Integrating by parts, we get

$$I = \log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) \cdot \int \cos 2x\, dx - \int \left[\left\{\frac{d}{dx} \log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)\right\} \cdot \int \cos 2x\, dx\right] dx$$

$$= \log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) \cdot \frac{\sin 2x}{2} - \int \frac{1}{\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)} \cdot \frac{d}{dx}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) \cdot \frac{\sin 2x}{2} \cdot dx$$

$$= \frac{\sin 2x}{2} \log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) - \int \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) \cdot \left(\frac{2}{(\cos x - \sin x)^2}\right)^{**} \cdot \frac{\sin 2x}{2}\, dx$$

$$^{**}\frac{d}{dx}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right).$$

$$= \frac{(\cos x - \sin x)\frac{d}{dx}(\cos x + \sin x) - (\cos x + \sin x)\frac{d}{dx}(\cos x - \sin x)}{(\cos x - \sin x)^2}$$

$$= \frac{(\cos x - \sin x)(-\sin x + \cos x) - (\cos x + \sin x)(-\sin x - \cos x)}{(\cos x - \sin x)^2}$$

$$= \frac{(\cos x - \sin x)^2 + (\cos x + \sin x)^2}{(\cos x - \sin x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x - 2\sin x \cos x + \cos^2 x + \sin^2 x + 2\sin x \cos x}{(\cos x - \sin x)^2}$$

$$= \frac{2}{(\cos x - \sin x)^2}. \qquad [\because \cos^2 A + \sin^2 A = 1]$$

$$= \frac{\sin 2x}{2} \log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) - \int \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) \cdot \frac{2}{(\cos x - \sin x)^2} \cdot \frac{\sin 2x}{2}\, dx$$

$$= \frac{\sin 2x}{2} \log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) - \int \frac{1}{(\cos^2 x - \sin^2 x)} \cdot \sin 2x\, dx$$

$[\because \cos^2 A - \sin^2 A = \cos 2A]$

$$= \frac{\sin 2x}{2} \log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) - \int \frac{\sin 2x}{\cos 2x}\, dx$$

$$= \frac{\sin 2x}{2} \log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) - \int \tan 2x\, dx$$

$$= \frac{\sin 2x}{2} \cdot \log\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) + \log \frac{|\cos 2x|}{2} + c.$$

(*ii*) Let $I = \int \frac{\sin^{-1} x}{x^2}\, dx$ ...(1)

$$= \int \underset{\text{II}}{x^{-2}} \underset{\text{I}}{\sin^{-1} x}\, dx$$

Integrating by parts, we get

$$I = \sin^{-1} x \cdot \int x^{-2}\, dx - \int \left\{\frac{d}{dx}(\sin^{-1} x) \cdot \int x^{-2}\, dx\right\} dx$$

$$= \sin^{-1} x \cdot \frac{x^{-2+1}}{-2+1} - \int \left(\frac{1}{\sqrt{1-x^2}}\right) \cdot \frac{x^{-2+1}}{-2+1}\, dx$$

$$= -\frac{\sin^{-1} x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\, dx$$

$\Rightarrow$ $I = -\frac{\sin^{-1} x}{x} + I_1$ ...(2)

where $I_1 = \int \frac{1}{x\sqrt{1-x^2}}\, dx$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\therefore$ $I_1 = \int \frac{1}{\sin\theta\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$ $\left[\begin{array}{l} \because \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \cos^2 A = 1 - \sin^2 A \\ \Rightarrow \cos A = \sqrt{1-\sin^2 A} \end{array}\right]$

$$= \int \frac{\cos\theta}{\sin\theta \cdot \cos\theta}\, d\theta = \int \operatorname{cosec}\theta\, d\theta$$

$$= \log|\operatorname{cosec}\theta - \cot\theta| = \log\left|\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right| = \log\left|\frac{1}{\sin\theta} - \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right|$$

$I_1 = \log\left|\frac{1-\sqrt{1-x^2}}{x}\right|$ $[\because x = \sin\theta]$

Putting this value of I_1 in equation (2), we get

$$I = -\frac{\sin x}{x} + \log\left|\frac{1-\sqrt{1-x^2}}{x}\right| + c.$$

(*iii*) Let $\quad I = \int e^x \left(\frac{\sqrt{1-x^2}\sin^{-1}x + 1}{\sqrt{1-x^2}}\right) dx$

$$= \int e^x \left(\frac{\sqrt{1-x^2}\sin^{-1}x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right) dx = \int e^x \left(\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}\right) dx$$

$$= \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin^{-1}x}\, dx + \int e^x \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$

Integrating the first integral by parts, we get

$$= \sin^{-1}x \cdot \int e^x\, dx - \int \left\{\frac{d}{dx}(\sin^{-1}x) \cdot \int e^x\, dx\right\} dx + \int e^x \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \sin^{-1}x \cdot e^x - \int \frac{1}{\sqrt{1-x^2}} e^x\, dx + \int e^x \cdot \frac{1}{\sqrt{1-x^2}}\, dx + c$$

$$= e^x \sin^{-1}x + c.$$

(*iv*) Let $\quad I = \int x \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right) dx$

Put $\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right) = \theta$

$$\Rightarrow \quad \frac{1}{2}\sqrt{\frac{2a-x}{a}} = \sin\theta \quad \Rightarrow \quad \sqrt{\frac{2a-x}{a}} = 2\sin\theta \quad \Rightarrow \quad \frac{2a-x}{a} = 4\sin^2\theta$$

(Squaring both sides)

$\Rightarrow \quad 2a - x = 4a\sin^2\theta$

$\Rightarrow \quad 2a - 4a\sin^2\theta = x$

$\Rightarrow \quad x = 2a(1 - 2\sin^2\theta) = 2a\cos 2\theta \qquad [\because \cos 2A = 1 - 2\sin^2 A]$

$\Rightarrow \quad dx = -4a\sin 2\theta\, d\theta$

$\therefore \quad I = \int (2a\cos 2\theta)\,\theta \cdot (-4a\sin 2\theta)\, d\theta$

$$= -4a^2 \int \theta \cdot 2\sin 2\theta\cos 2\theta\, d\theta$$

$$= -4a^2 \int \underset{\text{I}}{\theta} \cdot \underset{\text{II}}{\sin 4\theta}\, d\theta \qquad \left[\begin{array}{l}\because \sin 2A = 2\sin A\cos A \\ \Rightarrow \sin 4A = 2\sin 2A\cos 2A\end{array}\right]$$

Integrating by parts, we get

$$I = -4a^2\left[\theta \cdot \int \sin 4\theta\, d\theta - \int \left\{\frac{d}{dx}(\theta) \cdot \int \sin 4\theta\, d\theta\right\} d\theta\right]$$

$$= -4a^2\left[\theta.\left(-\frac{\cos 4\theta}{4}\right) - \int 1.\left(-\frac{\cos 4\theta}{4}\right)d\theta\right]$$

$$= +a^2\,\theta\cos 4\theta - a^2\int \cos 4\theta\,d\theta$$

$$= a^2\,\theta\,(2\cos^2 2\theta - 1) - a^2 . \frac{\sin 4\theta}{4} \qquad \left[\begin{array}{l}\because\ \cos 2A = 2\cos^2 A - 1\\ \Rightarrow\ \cos 4A = 2\cos^2 2A - 1\end{array}\right]$$

$$= 2a^2\,\theta\cos^2 2\theta - a^2\,\theta - \frac{a^2}{4}\,(2\sin 2\theta\cos 2\theta)$$

$$= 2a^2\,\theta\cos^2 2\theta - a^2\,\theta - \frac{a^2}{2}\left(\sqrt{1-\cos^2 2\theta}\,.\cos 2\theta\right) \qquad \left[\begin{array}{l}\because\quad x = 2a\cos 2\theta\\ \Rightarrow\ \cos 2\theta = \dfrac{x}{2a}\\ \Rightarrow\quad 2\theta = \cos^{-1}\left(\dfrac{x}{2a}\right)\end{array}\right]$$

$$= 2a^2 . \frac{1}{2}\cos^{-1}\left(\frac{x}{2a}\right).\left(\frac{x}{2a}\right)^2 - \frac{a^2}{2}\cos^{-1}\left(\frac{x}{2a}\right) - \frac{a^2}{2}\left(\sqrt{1-\left(\frac{x}{2a}\right)^2}\,.\frac{x}{2a}\right) + c$$

$$= \frac{x^2}{4}\cos^{-1}\left(\frac{x}{2a}\right) - \frac{a^2}{2}\cos^{-1}\left(\frac{x}{2a}\right) - \frac{ax}{4}\sqrt{\frac{4a^2-x^2}{4a^2}} + c$$

$$= \left(\frac{x^2}{4} - \frac{a^2}{2}\right)\cos^{-1}\left(\frac{x}{2a}\right) - \frac{x\sqrt{4a^2-x^2}}{8} + c$$

$$= \left(\frac{x^2-2a^2}{4}\right)\cos^{-1}\left(\frac{x}{2a}\right) - \frac{x}{8}\sqrt{4a^2-x^2} + c.$$

Example 8. *Evaluate the following integrals :*

(i) $\int e^{-x/2} . \dfrac{\sqrt{1-\sin x}}{1+\cos x}\,dx$ *(ii)* $\int \left[\log(1+\cos x) - x\tan\dfrac{x}{2}\right]dx.$

Solution. *(i)* Let $I = \int e^{-x/2} . \dfrac{\sqrt{1-\sin x}}{1+\cos x}\,dx$ $\left[\begin{array}{l}\because\ \sin^2 A + \cos^2 A = 1\\ \Rightarrow\ 1+\cos 2A = 2\cos^2 A\\ \Rightarrow\ 1+\cos A = 2\cos^2\dfrac{A}{2}\end{array}\right]$

$$= \int e^{-x/2}\,\frac{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}{2\cos^2\frac{x}{2}}\,dx$$

$$= \int e^{-x/2} \frac{\sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{2\cos^2\frac{x}{2}}\, dx = \int e^{-\frac{x}{2}} \left(\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{2\cos^2\frac{x}{2}}\right) dx$$

$$= \int e^{-\frac{x}{2}} \left(\frac{\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} - \frac{\sin\frac{x}{2}}{2\cos^2\frac{x}{2}}\right) dx = \int e^{-\frac{x}{2}} \left(\frac{1}{2}\sec\frac{x}{2} - \frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2}\right) dx$$

$$= \frac{1}{2}\int \underset{\text{II}}{e^{-x/2}}\, \underset{\text{I}}{\sec\frac{x}{2}}\, dx - \frac{1}{2}\int e^{-x/2} \sec\frac{x}{2}\tan\frac{x}{2}\, dx$$

Integrating the first integral by parts, we get

$$\text{I} = \frac{1}{2}\left[\sec\frac{x}{2} \cdot \int e^{-x/2}\, dx - \int \left\{\frac{d}{dx}\left(\sec\frac{x}{2}\right) \cdot \int e^{-x/2}\, dx\right\} dx\right] - \frac{1}{2}\int e^{-x/2} \sec\frac{x}{2}\tan\frac{x}{2}\, dx$$

$$= \frac{1}{2}\left[\sec\frac{x}{2} \cdot \left(\frac{e^{-x/2}}{-1/2}\right) - \int \left(\sec\frac{x}{2}\tan\frac{x}{2}\right)\left(\frac{1}{2}\right) \cdot \left(\frac{e^{-x/2}}{-1/2}\right)\right] - \frac{1}{2}\int e^{-x/2} \sec\frac{x}{2}\tan\frac{x}{2}\, dx$$

$$= -e^{-x/2}\sec\frac{x}{2} + \frac{1}{2}\int e^{-x/2}\sec\frac{x}{2}\tan\frac{x}{2}\, dx - \frac{1}{2}\int e^{-x/2}\sec\frac{x}{2}\tan\frac{x}{2}\, dx$$

$$= -e^{-x/2}\sec\frac{x}{2} + c.$$

(*ii*) Let $\text{I} = \int \left[\log(1 + \cos x) - x\tan\frac{x}{2}\right] dx$

$$= \int \underset{\text{I}}{\log(1 + \cos x)} \,.\, \underset{\text{II}}{1}\, dx - \int x\tan\frac{x}{2}\, dx \qquad \text{[Taking unity as second function]}$$

Integrating the first integral by parts, we get

$$\text{I} = \log(1 + \cos x) \,.\, \int 1 \,.\, dx - \int \left\{\frac{d}{dx}\log(1 + \cos x) \,.\, \int 1 \,.\, dx\right\} dx - \int x\tan\frac{x}{2}\, dx$$

$$= \log(1 + \cos x) \,.\, x - \int \frac{1}{(1 + \cos x)} \,.\, (-\sin x) \,.\, x\, dx - \int x\tan\frac{x}{2}\, dx$$

$$= x\log(1 + \cos x) + \int \frac{x\sin x}{1 + \cos x}\, dx - \int x\tan\frac{x}{2}\, dx$$

$$= x\log(1 + \cos x) + \int \frac{x \,.\, 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2 \frac{x}{2}}\, dx - \int x\tan\frac{x}{2}\, dx$$

$$\left[\begin{array}{l} \because\ \sin 2\text{A} = 2\sin\text{A}\cos\text{A} \\ \quad 1 + \cos 2\text{A} = 2\cos^2\text{A} \\ \Rightarrow\ 1 + \cos\text{A} = 2\cos^2\frac{\text{A}}{2} \end{array}\right]$$

$$= x \log (1 + \cos x) + \int x \tan \frac{x}{2}\, dx - \int x \tan \frac{x}{2}\, dx$$

$$= x \log (1 + \cos x) + c.$$

Example 9. *Evaluate the following integrals :*

(i) $\int \frac{x}{1 - \sin x}\, dx$ (ii) $\int \frac{x}{1 + \sin x}\, dx.$

Solution. (i) Let $I = \int \frac{x}{1 - \sin x}\, dx$

$$= \int \frac{x}{1 - \cos\left(\frac{\pi}{2} - x\right)}\, dx \qquad [\because \cos (90° - \theta) = \sin \theta]$$

$$= \int \frac{x}{2 \sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\, dx \qquad \left[\begin{array}{l} \because 1 - \cos 2A = 2 \sin^2 A \\ \Rightarrow 1 - \cos A = 2 \sin^2 \frac{A}{2} \end{array}\right]$$

$$= \frac{1}{2} \int \underset{\text{I}}{x} \underset{\text{II}}{\operatorname{cosec}^2} \left(\frac{\pi}{4} - \frac{x}{2}\right) dx$$

Integrating by parts, we get

$$I = \frac{1}{2}\left[x \,.\int \operatorname{cosec}^2\left(\frac{\pi}{4} - \frac{x}{2}\right) dx - \int \left\{\frac{d}{dx}(x) \,.\int \operatorname{cosec}^2\left(\frac{\pi}{4} - \frac{x}{2}\right) dx\right\} dx\right]$$

$$= \frac{1}{2}\left[x \,.\left\{\frac{-\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)}{-\frac{1}{2}}\right\} - \int 1 \,.\left\{\frac{-\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)}{-\frac{1}{2}}\right\} dx\right]$$

$$= x \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) - \int \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) dx = x \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) - \frac{\log\left|\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\right|}{-\frac{1}{2}} + c$$

$$= x \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) + 2 \log\left|\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + c.$$

(ii) Let $I = \int \frac{x}{1 + \sin x}\, dx$

$$= \int \frac{x}{1 + \cos\left(\frac{\pi}{2} - x\right)}\, dx \qquad [\because \cos (90° - \theta) = \sin \theta]$$

$$= \int \frac{x}{2 \cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\, dx \qquad \left[\begin{array}{l} \because 1 + \cos 2A = 2 \cos^2 A \\ \Rightarrow 1 + \cos A = 2 \cos^2 \frac{A}{2} \end{array}\right]$$

$$= \frac{1}{2} \int \underset{\text{I}}{x} \underset{\text{II}}{\sec^2} \left(\frac{\pi}{4} - \frac{x}{2}\right) dx$$

Integrating by parts, we get

$$I = \frac{1}{2}\left[x \cdot \int \sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right)dx - \int \left\{\frac{d}{dx}(x) \cdot \int \sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right)dx\right\}dx\right]$$

$$= \frac{1}{2}\left[x \cdot \left\{\frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)}{-\frac{1}{2}}\right\} - \int 1 \cdot \left\{\frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)}{-\frac{1}{2}}\right\}dx\right]$$

$$= -x \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + \int \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)dx$$

$$= -x \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + \log\frac{\left|\sec\left(\frac{\pi}{4} - \frac{x}{2}\right)\right|}{-1/2} + c$$

$$= -x \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) - 2\log\left|\sec\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + c.$$

4.4. THREE STANDARD INTEGRALS $\int \sqrt{a^2 - x^2}\,dx$, $\int \sqrt{a^2 + x^2}\,dx$ and $\int \sqrt{x^2 - a^2}\,dx$

Theorem 4. *Prove that :*

(*i*) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(*ii*) $\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + c$

(*iii*) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c.$

Proof. (*i*) Let $I = \int \sqrt{a^2 - x^2}\,dx$ ...(1)

$$= \int \underset{\text{I}}{\sqrt{a^2 - x^2}} \cdot \underset{\text{II}}{1}\,dx$$ [Taking unity as second function]

Integrating by parts, we get

$$I = \sqrt{a^2 - x^2} \cdot \int 1 \cdot dx - \int \left\{\frac{d}{dx}\left(\sqrt{a^2 - x^2}\right) \cdot \int 1 \cdot dx\right\}dx$$

$$= \sqrt{a^2 - x^2} \cdot x - \int \frac{1}{2\sqrt{a^2 - x^2}}(-2x) \cdot x\,dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}}\,dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{(a^2 - x^2 - a^2)}{\sqrt{a^2 - x^2}}\,dx$$ [Add and subtract a^2 to the numerator]

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\,dx + \int \frac{a^2}{\sqrt{a^2 - x^2}}\,dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\,dx + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}\,dx$$

$$\Rightarrow \quad I = x\sqrt{a^2 - x^2} - I + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}\,dx \qquad [\because \text{ By using equation (1)}]$$

$$\Rightarrow \quad 2I = x\sqrt{a^2 - x^2} + a^2 . \sin^{-1}\frac{x}{a} + c_1 \qquad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$\Rightarrow \quad I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c.$$

$$\therefore \quad \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c.$$

(*ii*) Let $\quad I = \int \sqrt{a^2 + x^2}\,dx \qquad \ldots(1)$

$$= \int \underset{\text{I}}{\sqrt{a^2 + x^2}} . \underset{\text{II}}{1}\,dx \qquad \text{[Taking unity as second function]}$$

Integrating by parts, we get

$$I = \sqrt{a^2 + x^2} . \int 1 . dx - \int \left\{\frac{d}{dx}\left(\sqrt{a^2 + x^2}\right) . \int 1 . dx\right\} dx$$

$$= \sqrt{a^2 + x^2} . x - \int \frac{1}{2\sqrt{a^2 + x^2}}(2x) . x\,dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}}\,dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2 - a^2}{\sqrt{a^2 + x^2}}\,dx \qquad [\text{Add and subtract } a^2 \text{ to the numerator}]$$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2}{\sqrt{a^2 + x^2}}\,dx + \int \frac{a^2}{\sqrt{a^2 + x^2}}\,dx$$

$$= x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\,dx + a^2 \int \frac{1}{\sqrt{a^2 + x^2}}\,dx$$

$$\Rightarrow \quad I = x\sqrt{a^2 + x^2} - I + a^2 . \int \frac{1}{\sqrt{a^2 + x^2}}\,dx \qquad \text{[By using equation (1)]}$$

$$\Rightarrow \quad 2I = x\sqrt{a^2 + x^2} + a^2 . \log\left|x + \sqrt{a^2 + x^2}\right| + c_1$$

$$\left[\because \int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \log\left|x + \sqrt{a^2 + x^2}\right| + c\right]$$

$$\Rightarrow \qquad I = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}.\log\left|x+\sqrt{a^2+x^2}\right| + c$$

$$\therefore \quad \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + c.$$

(*iii*) Let $I = \int \sqrt{x^2-a^2}\,dx$ ...(1)

$$= \int \underset{\text{I}}{\sqrt{x^2-a^2}}\,.\,\underset{\text{II}}{1}\,dx \qquad \text{(Taking unity as second function)}$$

Integrating by parts, we get

$$I = \sqrt{x^2-a^2}\,.\int 1.\,dx - \int\left\{\frac{d}{dx}\left(\sqrt{x^2-a^2}\right).\int 1.\,dx\right\}dx$$

$$= \sqrt{x^2-a^2}\,.\,x - \int \frac{1}{2\sqrt{x^2-a^2}}\,(2x)\,.\,x\,dx = x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}\,dx$$

[Add and subtract a^2 to the numerator]

$$= x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)}{\sqrt{x^2-a^2}}\,dx - \int \frac{a^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$\Rightarrow \qquad I = x\sqrt{x^2-a^2} - I - a^2\int \frac{1}{\sqrt{x^2-a^2}}\,dx \qquad [\because \text{ By using equation (1)}]$$

$$\Rightarrow \qquad 2I = x\sqrt{x^2-a^2} - a^2.\log\left|x+\sqrt{x^2-a^2}\right| + c_1$$

$$\left[\because \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$\Rightarrow \qquad I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}.\log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\therefore \quad \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c.$$

Remark. The above integrals (*i*), (*ii*) and (*iii*) can also be evaluated by making trigonometric substitutions $x = a\sin\theta$, $x = a\tan\theta$ and $x = a\sec\theta$ respectively.

4.5. INTEGRALS OF THE FORM $\int \sqrt{ax^2+bx+c}\ dx$ and $\int (px+q)\sqrt{ax^2+bx+c}\ .\ dx$

4.5.1. Working Rule for the Evaluation of $\int \sqrt{ax^2+bx+c}\ .\ dx$

(*i*) First take a as common and make the co-efficient of x^2 unity.

(*ii*) Then complete the square in terms of x^2 and x by adding and subtracting the square of half the co-efficient of x.

(*iii*) After applying these two steps we get the integrand in one of the standard forms as discussed above.

4.5.2. Working Rule for the Evaluation of $\int (px+q)\sqrt{ax^2+bx+c}\ .\ dx$

(*i*) Express $(px + q)$ as :

$$(px+q) = A\frac{d}{dx}(ax^2+bx+c) + B.$$

$$\Rightarrow \quad (px+q) = A(2ax+b) + B.$$

(*ii*) Find the values of A and B by equating the co-efficients of x and constant terms on both sides.

(*iii*) Now, the given integral becomes :

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx = A\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + B\int \sqrt{ax^2+bx+c}\,dx$$

(*iv*) Evaluate the first integral by putting $z = (ax^2 + bx + c)$ and the second integral by using the method discussed in Article (4.5.1).

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following integrals* :

(*i*) $\int \sqrt{9-x^2}\ dx$ (*ii*) $\int \sqrt{2-3x^2}\ dx$

(*iii*) $\int \sqrt{4x^2-64}\ dx$ (*iv*) $\int \sqrt{4x^2+9}\ dx$

(*v*) $\int \sqrt{1+\frac{x^2}{9}}\ dx$ (*vi*) $\int \sqrt{16x^2-49}\ dx.$

Solution. (*i*) Let $I = \int \sqrt{9-x^2}\ dx = \int \sqrt{(3)^2-x^2}\ dx$

$$\left[\because \text{ By using } \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{x}{2}\sqrt{(3)^2-x^2} + \frac{(3)^2}{2}\sin^{-1}\frac{x}{3} + c$$

$$= \frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + c.$$

(*ii*) Let $I = \int \sqrt{2-3x^2}\, dx$

$$= \sqrt{3}\int \sqrt{\frac{2}{3}-x^2}\, dx = \sqrt{3}\int \sqrt{\left(\sqrt{\frac{2}{3}}\right)^2 - x^2}\, dx$$

$$\left[\because \text{ By using } \int \sqrt{a^2-x^2}\,.\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c\right]$$

$$= \sqrt{3}\left[\frac{x}{2}\sqrt{\left(\sqrt{2/3}\right)^2 - x^2} + \frac{\left(\sqrt{2/3}\right)^2}{2}\sin^{-1}\frac{x}{\sqrt{2/3}} + c\right]$$

$$= \frac{\sqrt{3}x}{2}\sqrt{\frac{2}{3}-x^2} + \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}x}{\sqrt{2}}\right) + c$$

$$= \frac{x}{2}\sqrt{2-3x^2} + \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}x}{\sqrt{2}}\right) + c.$$

(*iii*) Let $I = \int \sqrt{4x^2-64}\,.\,dx$

$$= \int 2\sqrt{x^2-16}\,dx = 2\int \sqrt{x^2-(4)^2}\,dx$$

$$\left[\because \text{ By using } \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$= 2\left[\frac{x}{2}\sqrt{(x)^2-(4)^2} - \frac{(4)^2}{2}\log\left|x+\sqrt{x^2-4^2}\right| + c\right]$$

$$= \frac{x}{2}\sqrt{x^2-16} - 16\log\left|x+\sqrt{x^2-16}\right| + c.$$

Note. Whenever we multiply or divide the constant c by another constant quantity, then we write it as c again because the product or the quotient is again constant.

(*iv*) Let $I = \int \sqrt{4x^2+9}\,dx$

$$= \int 2\sqrt{\left(x^2+\frac{9}{4}\right)}\,dx = 2\int \sqrt{x^2+\left(\frac{3}{2}\right)^2}\,dx$$

$$\left[\because \text{ By using } \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c\right]$$

$$= 2\left[\frac{x}{2}\sqrt{x^2+\left(\frac{3}{2}\right)^2} + \frac{(3/2)^2}{2}\log\left|x+\sqrt{x^2+\left(\frac{3}{2}\right)^2}\right| + c\right]$$

$$= x\sqrt{x^2+\frac{9}{4}} + \frac{9}{4}\log\left|x+\sqrt{x^2+\frac{9}{4}}\right| + c$$

$$= \frac{x}{2}\sqrt{4x^2+9} + \frac{9}{4}\log\left|\frac{2x+\sqrt{4x^2+9}}{2}\right| + c.$$

(*v*) Let $\quad I = \int \sqrt{1+\frac{x^2}{9}}\, dx$

$$= \int \sqrt{\frac{9+x^2}{9}}\, dx = \frac{1}{3}\int \sqrt{(3)^2+x^2}\, dx$$

$$\left[\because \text{ By using } \int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + c\right]$$

$$= \frac{1}{3}\left[\frac{x}{2}\sqrt{(3)^2+x^2} + \frac{(3)^2}{2}\log\left|x+\sqrt{(3)^2+(x)^2}\right| + c\right]$$

$$= \frac{x}{6}\sqrt{9+x^2} + \frac{3}{2}\log\left|x+\sqrt{9+x^2}\right| + c.$$

(*vi*) Let $\quad I = \int \sqrt{16x^2-49}\, .\, dx$

$$= \int 4\sqrt{x^2-\frac{49}{16}}\, dx = 4\int \sqrt{x^2-\left(\frac{7}{4}\right)^2}\, dx$$

$$\left[\because \text{ By using } \int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$= 4\left[\frac{x}{2}\sqrt{x^2-\left(\frac{7}{4}\right)^2} - \frac{(7/4)^2}{2}\log\left|x+\sqrt{x^2-\left(\frac{7}{4}\right)^2}\right| + c\right]$$

$$= 4\left[\frac{x}{2}\sqrt{x^2-\frac{49}{16}} - \frac{49}{32}\log\left|x+\sqrt{x^2-\frac{49}{16}}\right| + c\right]$$

$$= 4\left[\frac{x}{8}\sqrt{16x^2-49} - \frac{49}{32}\log\left|\frac{4x+\sqrt{16x^2-49}}{16}\right| + c\right]$$

$$= \frac{x}{2}\sqrt{16x^2-49} - \frac{49}{8}\log\left|\frac{4x+\sqrt{16x^2-49}}{16}\right| + c.$$

Example 2. *Evaluate the following integrals :*

(*i*) $\int \sqrt{8+7x^2}\, dx$ (*ii*) $\int \sqrt{3x^2+4}\, dx$

(*iii*) $\int \sqrt{17x^2+11}\, dx$ (*iv*) $\int \sqrt{5-\frac{x^2}{12}}\, dx.$

Solution. (*i*) Let $I = \int \sqrt{8+7x^2}\, dx$

$$= \sqrt{7}\int \sqrt{\frac{8}{7}+x^2}\ dx = \sqrt{7}\int \sqrt{\left(\sqrt{8/7}\right)^2+x^2}\ dx$$

$$\left[\because \text{ By using} \int \sqrt{a^2+x^2}\ dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right|+c\right]$$

$$= \sqrt{7}\left[\frac{x}{2}\sqrt{\left(\sqrt{\frac{8}{7}}\right)^2+x^2}+\frac{\left(\sqrt{8/7}\right)^2}{2}\log\left|x+\sqrt{\left(\sqrt{\frac{8}{7}}\right)^2+x^2}\right|\right]+c$$

$$= \sqrt{7}\left[\frac{x}{2}\sqrt{\frac{8}{7}+x^2}+\frac{8}{14}\log\left|x+\sqrt{\frac{8}{7}+x^2}\right|\right]+c$$

$$= \sqrt{7}\left[\frac{x}{2}\sqrt{\frac{8+7x^2}{7}}+\frac{8}{14}\log\left|x+\sqrt{\frac{8+7x^2}{7}}\right|\right]+c$$

$$= \frac{x}{2}\sqrt{8+7x^2}+\frac{4\sqrt{7}}{7}\log\left|\frac{\sqrt{7}x+\sqrt{8+7x^2}}{\sqrt{7}}\right|+c.$$

(*ii*) Let $I = \int \sqrt{3x^2+4}\ .\ dx$

$$= \sqrt{3}\int \sqrt{x^2+\frac{4}{3}}\ dx = \sqrt{3}\int \sqrt{x^2+\left(\frac{2}{\sqrt{3}}\right)^2}\ dx$$

$$\left[\because \text{ By using} \int \sqrt{x^2+a^2}\ dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c\right]$$

$$= \sqrt{3}\left[\frac{x}{2}\sqrt{x^2+\left(\frac{2}{\sqrt{3}}\right)^2}+\frac{(2/\sqrt{3})^2}{2}\log\left|x+\sqrt{x^2+\left(\frac{2}{\sqrt{3}}\right)^2}\right|\right]+c$$

$$= \sqrt{3}\left[\frac{x}{2}\sqrt{x^2+\frac{4}{3}}+\frac{2}{3}\log\left|x+\sqrt{x^2+\frac{4}{3}}\right|\right]+c$$

$$= \frac{x}{2}\sqrt{3x^2+4}+\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{3}x+\sqrt{3x^2+4}}{\sqrt{3}}\right|+c.$$

(*iii*) Let $\quad I = \int \sqrt{17x^2-11}\ dx$

$$= \sqrt{17}\int \sqrt{x^2-\frac{11}{17}}\ dx = \sqrt{17}\int \sqrt{x^2-\left(\sqrt{\frac{11}{17}}\right)^2}\ dx$$

$$\left[\because \text{ By using} \int \sqrt{x^2-a^2}\ dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c\right]$$

$$= \sqrt{17}\left[\frac{x}{2}\sqrt{x^2-\left(\sqrt{\frac{11}{17}}\right)^2} - \frac{\left(\sqrt{11/17}\right)^2}{2}\log\left|x+\sqrt{x^2-\left(\sqrt{\frac{11}{17}}\right)^2}\right|\right]+c$$

$$= \sqrt{17}\left[\frac{x}{2}\sqrt{x^2-\frac{11}{17}} - \frac{11}{34}\log\left|x+\sqrt{x^2-\frac{11}{17}}\right|\right]+c$$

$$= \frac{x}{2}\sqrt{17x^2-11} - \frac{11\sqrt{17}}{34}\log\left|\frac{\sqrt{17}x+\sqrt{17x^2-11}}{\sqrt{17}}\right|+c.$$

(iv) Let $\quad I = \int \sqrt{5-\frac{x^2}{12}}\;.\,dx$

$$= \frac{1}{\sqrt{12}}\int\sqrt{60-x^2}\,dx = \frac{1}{\sqrt{12}}\int\sqrt{(\sqrt{60})^2-x^2}\,dx$$

$$\left[\because \text{ By using } \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c\right]$$

$$= \frac{1}{\sqrt{12}}\left[\frac{x}{2}\sqrt{(\sqrt{60})^2-x^2}+\frac{\left(\sqrt{60}\right)^2}{2}\sin^{-1}\frac{x}{\sqrt{60}}\right]+c$$

$$= \frac{x}{2\sqrt{12}}\sqrt{60-x^2}+\frac{30}{\sqrt{12}}\sin^{-1}\frac{x}{\sqrt{60}}+c$$

$$= \frac{x}{4\sqrt{3}}\sqrt{60-x^2}+\frac{15}{\sqrt{3}}\sin^{-1}\frac{x}{\sqrt{60}}+c.$$

Example 3. *Evaluate the following integrals :*

(i) $\int \frac{\sqrt{9+(\log x)^2}}{x}\,dx$ *(ii)* $\int e^x\sqrt{e^{2x}+4}\,dx$

(iii) $\int \cos x\sqrt{9-\sin^2 x}\,dx.$

Solution. *(i)* Let $I = \int \frac{\sqrt{9+(\log x)^2}}{x}\,dx$

Put $\quad \log x = z \;\Rightarrow\; \frac{1}{x}\,dx = dz$

$\therefore \quad I = \int\sqrt{9+z^2}\;.\,dz = \int\sqrt{(3)^2+z^2}\,dz$

$$\left[\because \text{ By using } \int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right|+c\right]$$

$$= \frac{z}{2}\sqrt{(3)^2+z^2}+\frac{(3)^2}{2}\log\left|z+\sqrt{(3)^2+z^2}\right|+c$$

$$= \frac{z}{2}\sqrt{9+z^2}+\frac{9}{2}\log\left|z+\sqrt{9+z^2}\right|+c$$

$$= \frac{\log x}{2}\sqrt{9+(\log x)^2} + \frac{9}{2}\log\left|\log x + \sqrt{9+(\log x)^2}\right| + c. \qquad [\because\ z = \log x]$$

(*ii*) Let $\quad I = \int e^x \sqrt{e^{2x}+4}\ dx$

Put $\quad e^x = z \Rightarrow e^x\,dx = dz$

$\therefore \quad I = \int \sqrt{z^2+(2)^2}\ dz$

$$\left[\because \text{ By using } \int \sqrt{x^2+a^2}\ dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c\right]$$

$$= \frac{z}{2}\sqrt{z^2+(2)^2} + \frac{(2)^2}{2}\log\left|z+\sqrt{z^2+(2)^2}\right| + c$$

$$= \frac{z}{2}\sqrt{z^2+4} + 2\log\left|z+\sqrt{z^2+4}\right| + c$$

$$= \frac{e^x}{2}\sqrt{e^{2x}+4} + 2\log\left|e^x+\sqrt{e^{2x}+4}\right| + c. \qquad [\because\ z = e^x]$$

(*iii*) Let $\quad I = \int \cos x\sqrt{9-\sin^2 x}\ dx$

Put $\quad \sin x = z \Rightarrow \cos x\,dx = dz$

$\therefore \quad I = \int \sqrt{9-z^2}\ dz = \int \sqrt{(3)^2-z^2}\ dz$

$$\left[\because \text{ By using } \int \sqrt{a^2-x^2}\ dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \frac{z}{2}\sqrt{(3)^2-z^2} + \frac{(3)^2}{2}\sin^{-1}\left(\frac{z}{3}\right) + c = \frac{z}{2}\sqrt{9-z^2} + \frac{9}{2}\sin^{-1}\left(\frac{z}{3}\right) + c$$

$$= \frac{\sin x}{2}\sqrt{9-\sin^2 x} + \frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right) + c. \qquad [\because\ z = \sin x]$$

Example 4. *Evaluate the following integrals :*

(*i*) $\int \sqrt{x^2-4x+2}\ dx$ (*ii*) $\int \sqrt{1-4x-x^2}\ dx$

(*iii*) $\int \sqrt{x^2+2x+5}\ dx$ (*iv*) $\int \sqrt{7x-10-x^2}\ dx$

(*v*) $\int \sqrt{2x^2+3x+4}\ dx$ (*vi*) $\int \sqrt{2ax-x^2}\ dx$

(*vii*) $\int \sqrt{5-4x-x^2}\ dx.$

Solution. (*i*) Let $I = \int \sqrt{x^2-4x+2}\ dx$

$\therefore \quad I = \int \sqrt{(x^2-4x+4)+(2-4)}\ dx \qquad \left[\begin{array}{l}\text{Add and subtract 4,} \\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 4\end{array}\right]$

$$= \int \sqrt{(x-2)^2 - \left(\sqrt{2}\right)^2}\ dx$$

Put $x - 2 = z \Rightarrow dx = dz$

$$\therefore \quad I = \int \sqrt{z^2 - \left(\sqrt{2}\right)^2}\ dz$$

$$\left[\because \text{ By using} \int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c\right]$$

$$= \frac{z}{2}\sqrt{z^2 - 2} - \frac{2}{2}\log\left|z + \sqrt{z^2 - 2}\right| + c$$

$$= \frac{(x-2)}{2}\sqrt{(x-2)^2 - 2} - \log\left|(x-2) + \sqrt{(x-2)^2 - 2}\right| + c \qquad [\because\ z = (x-2)]$$

$$= \frac{(x-2)}{2}\sqrt{x^2 - 4x + 2} - \log\left|(x-2) + \sqrt{x^2 - 4x + 2}\right| + c.$$

(*ii*) Let $I = \int \sqrt{1 - 4x - x^2}\ dx$

$$= \int \sqrt{1 - (x^2 + 4x + 4) + 4}\ .\ dx \qquad \left[\begin{array}{l}\text{Add and subtract 4,} \\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \sqrt{\left(\sqrt{5}\right)^2 - (x+2)^2}\ dx$$

Put $(x + 2) = z \Rightarrow dx = dz$

$$\therefore \quad I = \int \sqrt{\left(\sqrt{5}\right)^2 - z^2}\ dz$$

$$\left[\because \text{ By using} \int \sqrt{a^2 - x^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{z}{2}\sqrt{\left(\sqrt{5}\right)^2 - z^2} + \frac{\left(\sqrt{5}\right)^2}{2}\sin^{-1}\frac{z}{\sqrt{5}} + c$$

$$= \left(\frac{x+2}{5}\right)\sqrt{5 - (x+2)^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c \qquad [\because\ z = (x+2)]$$

$$= \left(\frac{x+2}{5}\right)\sqrt{1 - 4x - x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c.$$

(*iii*) Let $I = \int \sqrt{x^2 + 2x + 5}\ dx$

$$= \int \sqrt{x^2 + 2x + 5 - 4 + 4}\ dx \qquad \left[\begin{array}{l}\text{Add and subtract 4,} \\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \sqrt{(x^2 + 2x + 1) + 4}\ dx = \int \sqrt{(x+1)^2 + 2^2}\ dx$$

Put $x + 1 = z \quad \Rightarrow \quad dx = dz$

$\therefore \qquad \text{I} = \int \sqrt{z^2 + 2^2}\; dz$

$$\left[\because \text{ By using} \int \sqrt{x^2 + a^2}\; dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left| x + \sqrt{x^2 + a^2}\right| + c\right]$$

$$= \frac{z}{2}\sqrt{z^2 + 4} + \frac{4}{2}\log\left| z + \sqrt{z^2 + 4}\right| + c \qquad [\because \; z = (x+1)]$$

$$= \left(\frac{x+1}{2}\right)\sqrt{(x+1)^2 + 4} + 2\log\left|(x+1) + \sqrt{(x+1)^2 + 4}\right| + c$$

$$= \left(\frac{x+1}{2}\right)\sqrt{x^2 + 2x + 5} + 2\log\left|(x+1) + \sqrt{x^2 + 2x + 5}\right| + c$$

(*iv*) Let $\quad \text{I} = \int \sqrt{7x - 10 - x^2}\; dx = \int \sqrt{-(x^2 - 7x + 10)}\; dx$

$$= \int \sqrt{-\left(x^2 - 7x + 10 + \frac{49}{4} - \frac{49}{4}\right)}\; dx \qquad \left[\begin{array}{l}\text{Add and subtract } \dfrac{49}{4}, \\ \because \left(\dfrac{1}{2}\text{ co-eff. of } x\right)^2 = \dfrac{49}{4}\end{array}\right]$$

$$= \int \sqrt{-\left\{\left(x - \frac{7}{2}\right)^2 - \frac{9}{4}\right\}}\; dx = \int \sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{7}{2}\right)^2}\; dx$$

Put $\left(x - \frac{7}{2}\right) = z \quad \Rightarrow \quad dx = dz$

$\therefore \qquad \text{I} = \int \sqrt{\left(\frac{3}{2}\right)^2 - z^2}\; dz$

$$\left[\because \text{ By using} \int \sqrt{a^2 - x^2}\; dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \frac{z}{2}\sqrt{\left(\frac{3}{2}\right)^2 - z^2} + \frac{(3/2)^2}{2}\sin^{-1}\left(\frac{z}{3/2}\right) + c$$

$$= \frac{z}{2}\sqrt{\frac{9}{4} + z^2} + \frac{9}{8}\sin^{-1}\left(\frac{2z}{3}\right) + c$$

$$= \frac{(x - 7/2)}{2}\sqrt{\frac{9}{4} + \left(x - \frac{7}{2}\right)^2} + \frac{9}{8}\sin^{-1}\left(\frac{2(x - 7/2)}{3}\right) + c$$

$$= \frac{2x - 7}{4}\sqrt{7x^2 - 10x - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x - 7}{3}\right) + c.$$

(*v*) Let $\quad \text{I} = \int \sqrt{2x^2 + 3x + 4}\; dx = \sqrt{2}\int \sqrt{\left(x^2 + \frac{3}{2}x + 2\right)}\; dx$

$$= \sqrt{2} \int \sqrt{\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) + \left(2 - \frac{9}{16}\right)}\, dx \qquad \left[\begin{array}{l}\text{Add and subtract } \dfrac{9}{16}, \\ \because \left(\dfrac{1}{2}\text{ co-eff. of } x\right)^2 = \dfrac{9}{16}\end{array}\right]$$

$$= \sqrt{2} \int \sqrt{\left(x + \frac{3}{4}\right)^2 + \frac{23}{16}}\, .\, dx$$

Put $\left(x + \frac{3}{4}\right) = z \;\Rightarrow\; dx = dz$

$$\therefore \qquad I = \sqrt{2} \int \sqrt{z^2 + \left(\frac{\sqrt{23}}{4}\right)^2}\, dz$$

$$\left[\because \text{ By using} \int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left| x + \sqrt{x^2 + a^2}\right| + c\right]$$

$$= \sqrt{2}\left[\frac{z}{2}\sqrt{z^2 + \left(\frac{\sqrt{23}}{4}\right)^2} + \frac{\left(\sqrt{23}/4\right)^2}{2}\log\left| z + \sqrt{z^2 + \left(\frac{\sqrt{23}}{4}\right)^2}\right|\right] + c$$

$$= \sqrt{2}\left[\frac{z}{2}\sqrt{z^2 + \frac{23}{16}} + \frac{23}{32}\log\left| z + \sqrt{z^2 + \frac{23}{16}}\right|\right] + c$$

$$= \frac{\left(x + \frac{3}{4}\right)}{\sqrt{2}}\sqrt{\left(x + \frac{3}{4}\right)^2 + \frac{23}{16}} + \frac{23\sqrt{2}}{32}\log\left|\left(x + \frac{3}{4}\right) + \sqrt{\left(x + \frac{3}{4}\right)^2 + \frac{23}{16}}\right| + c$$

$$\left[\because\; z = \left(x + \frac{3}{4}\right)\right]$$

$$= \frac{4x + 3}{4\sqrt{2}}\sqrt{x^2 + \frac{3}{2}x + 2} + \frac{23\sqrt{2}}{32}\log\left|\frac{4x + 3}{4} + \sqrt{x^2 + \frac{3}{2}x + 2}\right| + c$$

$$= \frac{4x + 3}{8}\sqrt{2x^2 + 3x + 4} + \frac{23\sqrt{2}}{32}\log\left|\frac{4x + 3}{4} + \frac{\sqrt{2x^2 + 3x + 4}}{\sqrt{2}}\right| + c.$$

(*vi*) Let $\quad I = \sqrt{2ax - x^2}\; dx$

$$\therefore \qquad I = \sqrt{-(x^2 - 2ax + a^2) + a^2}\; dx \qquad \left[\begin{array}{l}\text{Add and subtract } a^2 \\ \because \left(\dfrac{1}{2}\text{ co-eff. of } x\right)^2 = a^2\end{array}\right]$$

$$= \sqrt{-(x - a)^2 + a^2}\; dx$$

$$= \sqrt{a^2 - (x - a)^2}\; dx$$

Put $x - a = z \Rightarrow dx = dz$

$\therefore \quad I = \int \sqrt{a^2 - z^2}\, dz$

$$\left[\because \text{ By using } \int \sqrt{a^2 - z^2}\, dz = \frac{z}{2}\sqrt{a^2 - z^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{z}{a}\right) + c\right]$$

$$= \frac{z}{2}\sqrt{a^2 - z^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{z}{a}\right) + c$$

$$= \left(\frac{x-a}{2}\right)\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c \qquad [\because z = (x-a)]$$

$$= \left(\frac{x-a}{2}\right)\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c.$$

(*vii*) Let $I = \int \sqrt{5 - 4x - x^2}\, dx = \int \sqrt{-(x^2 + 4x - 5)}\, dx$

$$= \int \sqrt{-(x^2 + 4x + 4 - 5 - 4)}\, dx \qquad \left[\begin{array}{l}\text{Add and subtract 4,} \\ \because \left(\frac{1}{2}\text{ coeff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int \sqrt{-[(x+2)^2 - 9]}\, dx = \int \sqrt{(3)^2 - (x+2)^2}\, dx$$

Put $x + 2 = z \Rightarrow dx = dz$

$$= \int \sqrt{(3)^2 - z^2}\, dz$$

$$\left[\because \text{ By using } \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \frac{z}{2}\sqrt{9 - z^2} + \frac{(3)^2}{2}\sin^{-1}\left(\frac{z}{3}\right) + c$$

$$= \left(\frac{x+2}{2}\right)\sqrt{9 - (x+2)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + c \qquad [\because z = (x+2)]$$

$$= \left(\frac{x+2}{2}\right)\sqrt{5 - 4x - x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + c.$$

Example 5. *Evaluate the following integrals :*

(*i*) $\int (2x - 5)\sqrt{2 + 3x - x^2}\, dx$ (*ii*) $\int (x + 1)\sqrt{x^2 - x + 1}\, dx$

(*iii*) $\int (x + 2)\sqrt{2x^2 + 2x + 1}\, dx$ (*iv*) $\int (2x + 3)\sqrt{x^2 + 4x + 3}\, dx$

(*v*) $\int (6x + 5)\sqrt{6 + x - x^2}\, dx$ (*vi*) $\int x^2\sqrt{a^6 - x^6}\, dx$

(*vii*) $\int \cos x\,(\sin x - 1)\sqrt{4 - \sin^2 x}\, dx$ (*viii*) $\int x\sqrt{\frac{1+x}{1-x}}\, dx$

(*ix*) $\int (x - 5)\sqrt{x^2 + x}\, dx.$

Solution. (*i*) Let $I = \int (2x-5)\sqrt{2+3x-x^2}\, dx$ ...(1)

Let $(2x-5) = A\dfrac{d}{dx}(2+3x-x^2) + B$

$\Rightarrow$ $(2x-5) = A(3-2x) + B$ ...(2)

Equating the co-efficients of x and the constant term on both sides, we have

$$2 = -2A \Rightarrow A = -1$$

and $-5 = 3A + B \Rightarrow -5 = 3(-1) + B \Rightarrow -5 + 3 = B$

$$\Rightarrow B = -2$$

$\therefore$ Equation (2) becomes

$$(2x-5) = -1(3-2x) + (-2).$$

Putting this value of $(2x-5)$ in equation (1), we have

$$I = \int [-1(3-2x) - 2] . \sqrt{2+3x-x^2}\, dx$$

$\Rightarrow$ $I = -\int (3-2x)\sqrt{2+3x-x^2}\, dx - 2\int \sqrt{2+3x-x^2}\, dx$

$\therefore$ $I = -I_1 - 2I_2$ (say) ...(3)

where $I_1 = \int (3-2x)\sqrt{2+3x-x^2}\, dx$

Put $2+3x-x^2 = z \Rightarrow (3-2x)\, dx = dz$

$\therefore$ $I_1 = \int \sqrt{z}\, dz = \dfrac{(z)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c_1 = \dfrac{2}{3} z^{3/2} + c_1$

$\Rightarrow$ $I_1 = \dfrac{2}{3}(2+3x-x^2)^{3/2} + c_1$ ...(4)

and $I_2 = \int \sqrt{2+3x-x^2}\, dx = \int \sqrt{-(x^2-3x-2)}\, dx$

$$= \int \sqrt{-\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} - 2\right)}\, dx \qquad \left[\text{Add and subtract } \frac{9}{4}, \because \left(\frac{1}{2}\text{ coeff. of } x\right)^2 = \frac{9}{4}\right]$$

$$= \int \sqrt{-\left[\left(x-\frac{3}{2}\right)^2 - \frac{17}{4}\right]}\, dx = \int \sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2}\, dx$$

$$\left[\because \text{By using } \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\right]$$

$$= \left(\frac{x-\frac{3}{2}}{2}\right)\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2} + \frac{\left(\frac{\sqrt{17}}{2}\right)^2}{2}\sin^{-1}\frac{\left(x-\frac{3}{2}\right)}{\frac{\sqrt{17}}{2}} + c_2$$

$\Rightarrow \quad I_2 = \frac{2x-3}{4}\sqrt{2+3x-x^2} + \frac{17}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c_2$ ...(5)

$\therefore$ From equation (3),

$I = -I_1 - 2I_2$ [Using equations (4) and (5)]

$$= -\left[\frac{2}{3}(2+3x-x^2)^{3/2} + c_1\right] - 2\left[\frac{2x-3}{4}\sqrt{2+3x-x^2} + \frac{17}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c_2\right]$$

$$= -\frac{2}{3}(2+3x-x^2)^{3/2} - c_1 - \frac{(2x-3)}{2}\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) - 2c_2$$

$$\therefore \quad I = -\frac{2}{3}(2+3x-x^2)^{3/2} - \frac{(2x-3)}{2}\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$$

where $c = -c_1 - 2c_2$.

(*ii*) Let $\quad I = \int (x+1)\sqrt{x^2-x+1}\ .\ dx$ ...(1)

Let $\quad (x+1) = A\frac{d}{dx}(x^2-x+1) + B$

$\Rightarrow \quad x+1 = A(2x-1) + B$ ...(2)

Equating the co-efficients of x and the constant term on both sides, we have

$$1 = 2A \Rightarrow A = \frac{1}{2}$$

and $$1 = -A + B \Rightarrow 1 = -\frac{1}{2} + B \Rightarrow B = \frac{3}{2}$$

$\therefore$ Equation (2) becomes

$$(x+1) = \frac{1}{2}(2x-1) + \frac{3}{2}$$

Putting this value of $(x+1)$ in equation (1), we have

$$I = \int\left[\frac{1}{2}(2x-1) + \frac{3}{2}\right]\sqrt{x^2-x+1}\ dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int (2x-1)\sqrt{x^2-x+1}\ dx + \frac{3}{2}\int \sqrt{x^2-x+1}\ dx$$

$\Rightarrow \quad I = \frac{1}{2}I_1 + \frac{3}{2}I_2$ (say) ...(3)

where $\quad I_1 = \int (2x-1)\sqrt{x^2-x+1}\ dx$

Put $x^2 - x + 1 = z \Rightarrow (2x-1)\,dx = dz$

$$\therefore \quad I_1 = \int \sqrt{z}\ dz = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c_1 = \frac{2}{3}z^{3/2} + c_1$$

$\Rightarrow \quad I_1 = \frac{2}{3}(x^2-x+1)^{3/2} + c_1$ ...(4) $[\because\ z = x^2 - x + 1]$

and $\quad I_2 = \int \sqrt{x^2 - x + 1}\,.\,dx$

$$= \int \sqrt{\left(x^2 - x + \frac{1}{4}\right) + \left(1 - \frac{1}{4}\right)}\,.\,dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4}, \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}\,dx = \int \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$\left[\because \text{ By using } \int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c\right]$$

$$= \frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right| + c_2$$

$$\Rightarrow I_2 = \frac{(2x - 1)}{4}\sqrt{x^2 - x + 1} + \frac{3}{8}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x + 1}\right| + c_2 \qquad \ldots(5)$$

$\therefore$ From equation (3),

$$I = \frac{1}{2}I_1 + \frac{3}{2}I_2$$

$$= \frac{1}{2}\left[\frac{2}{3}(x^2 - x + 1)^{3/2} + c_1\right] \qquad \text{[Using equations (4) and (5)]}$$

$$+ \frac{3}{2}\left[\frac{(2x - 1)}{4}\sqrt{x^2 - x + 1} + \frac{3}{8}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x + 1}\right| + c_2\right]$$

$$= \frac{1}{3}(x^2 - x + 1)^{3/2} + \frac{1}{2}c_1 + \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1}$$

$$+ \frac{9}{16}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x + 1}\right| + \frac{3}{2}c_2$$

$$= \frac{1}{3}(x^2 - x + 1)^{3/2} + \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1} + \frac{9}{16}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x + 1}\right| + C$$

where $C = \frac{1}{2}c_1 + \frac{3}{2}c_2$.

(*iii*) Let $\quad I = \int (x + 2)\sqrt{2x^2 + 2x + 1}\,dx \qquad \ldots(1)$

Let $(x + 2) = A\,\frac{d}{dx}(2x^2 + 2x + 1) + B$

$\Rightarrow \quad (x + 2) = A(4x + 2) + B \qquad \ldots(2)$

Equating the co-efficients of x and the constant term on both sides, we have

$$1 = 4A \quad \Rightarrow \quad A = \frac{1}{4}$$

and $\quad 2 = 2A + B \;\Rightarrow\; 2 = 2\left(\frac{1}{4}\right) + B \;\Rightarrow\; 2 - \frac{1}{2} = B \;\Rightarrow\; B = \frac{3}{2}$

$\therefore$ Equation (2) becomes

$$x + 2 = \frac{1}{4}(4x + 2) + \frac{3}{2}.$$

Putting this value of $(x + 2)$ in equation (1), we have

$$= \int \left[\frac{1}{4}(4x+2) + \frac{3}{2}\right]\sqrt{2x^2+2x+1}\, dx$$

$$\Rightarrow \quad I = \frac{1}{4}\int (4x+2)\sqrt{2x^2+2x+1}\, dx + \frac{3}{2}\int \sqrt{2x^2+2x+1}\, dx$$

$$\Rightarrow \quad I = \frac{1}{4} I_1 + \frac{3}{2} I_2 \quad \text{(say)} \qquad \ldots(3)$$

where $\quad I_1 = \int (4x+2)\sqrt{2x^2+2x+1}\, dx$

Put $(2x^2 + 2x + 1) = z \;\Rightarrow\; (4x + 2)\, dx = dz$

$$\therefore \quad I_1 = \int \sqrt{z}\, dz = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c_1 = \frac{2}{3} z^{3/2} + c_1$$

$$\Rightarrow \quad I_1 = \frac{2}{3}(2x^2 + 2x + 1)^{3/2} + c_1 \qquad \ldots(4) \quad [\because\; z = (2x^2 + 2x + 1)]$$

and $\quad I_2 = \int \sqrt{2x^2+2x+1}\, dx = \int \sqrt{2}\sqrt{x^2+x+\frac{1}{2}}\, dx$

$$= \sqrt{2}\int \sqrt{\left(x^2+x+\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)}\, dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4}, \\ \because\; \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \sqrt{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{1}{4}}\, dx = \sqrt{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\, dx$$

$$\left[\because\; \text{By using} \int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2+a^2}\right| + c\right]$$

$$= \sqrt{2}\left[\frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} + \frac{(1/2)^2}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right|\right] + c_2$$

$$= \frac{(2x+1)}{4}\sqrt{2x^2+2x+1} + \frac{\sqrt{2}}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+\frac{1}{2}}\right| + c_2 \qquad \ldots(5)$$

$\therefore$ From equation (3),

$$I = \frac{1}{4} I_1 + \frac{3}{2} I_2$$

$$= \frac{1}{4}\left[\frac{2}{3}(2x^2 + 2x + 1)^{3/2} + c_1\right] \qquad \text{[Using equations (4) and (5)]}$$

$$+ \frac{3}{2}\left[\frac{(2x+1)}{4}\sqrt{2x^2+2x+1} + \frac{1}{4\sqrt{2}}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+\frac{1}{2}}\right| + c_2\right]$$

$$= \frac{1}{6}(2x^2+2x+1)^{3/2} + \frac{1}{4}c_1 + \frac{3}{8}(2x+1)\sqrt{2x^2+2x+1}$$

$$+ \frac{3}{8\sqrt{2}}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+\frac{1}{2}}\right| + \frac{3}{2}c_2$$

$$\Rightarrow \quad I = \frac{1}{6}(2x^2+2x+1)^{3/2} + \frac{3}{8}(2x+1)\sqrt{2x^2+2x+1}$$

$$+ \frac{3}{8\sqrt{2}}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+\frac{1}{2}}\right| + c$$

where $c = \frac{1}{4}c_1 + \frac{3}{2}c_2$.

(*iv*) Let $\quad I = \int (2x+3)\sqrt{x^2+4x+3}\ dx \qquad \ldots(1)$

Let $\quad (2x+3) = A\frac{d}{dx}(x^2+4x+3) + B$

$\Rightarrow \quad 2x + 3 = A(2x+4) + B \qquad \ldots(2)$

Equating the co-efficients of x and constant terms on both sides, we have

$$2 = 2A \Rightarrow A = 1$$

and $\quad 3 = 4A + B \Rightarrow 3 = 4(1) + B \Rightarrow B = -1$

$\therefore$ Equation (2) becomes

$$(2x+3) = 1\,.\,(2x+4) - 1.$$

Putting this value of $(2x+3)$ in equation (1), we have

$$I = \int [(2x+4)-1].\sqrt{x^2+4x+3}\ .\ dx$$

$$\Rightarrow \quad I = \int (2x+4)\sqrt{x^2+4x+3}\ dx - \int \sqrt{x^2+4x+3}\ dx$$

$\Rightarrow \quad I = I_1 - I_2$ (say) $\qquad \ldots(3)$

where $\quad I_1 = \int (2x+4)\sqrt{x^2+4x+3}\ dx$

Put $\quad (x^2+4x+3) = z \Rightarrow (2x+4)\,dx = dz$

$$\therefore \quad I_1 = \int \sqrt{z}\ dz = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c_1 = \frac{2}{3}z^{3/2} + c_1$$

$\Rightarrow \quad I_1 = \frac{2}{3}(x^2+4x+3)^{3/2} + c_1$ ...(4) $[\because \ z = (x^2+4x+3)]$

and $\quad I_2 = \int \sqrt{x^2+4x+3}\ dx$

$= \int \sqrt{(x^2+4x+4)+(3-4)}\ dx$ $\left[\begin{array}{l}\text{Add and subtract } 4, \\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 4\end{array}\right]$

$= \int \sqrt{(x+2)^2-(1)^2}\ dx$

$\left[\because \text{ By using } \int \sqrt{x^2-a^2}\ dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c\right]$

$= \frac{(x+2)}{2}\sqrt{(x+2)^2-1^2} - \frac{(1)^2}{2}\log\left|(x+2)+\sqrt{(x+2)^2-1^2}\right| + c_2$

$\Rightarrow \quad I_2 = \frac{(x+2)}{2}\sqrt{x^2+4x+3} - \frac{1}{2}\log\left|(x+2)+\sqrt{x^2+4x+3}\right| + c_2$ ...(5)

$\therefore$ From equation (3),

$I = I_1 - I_2$

$= \frac{2}{3}(x^2+4x+3)^{3/2} + c_1 - \frac{(x+2)}{2}\sqrt{x^2+4x+3}$

$+ \frac{1}{2}\log\left|(x+2)+\sqrt{x^2+4x+3}\right| - c_2$

[Using equations (4) and (5)]

$= \frac{2}{3}(x^2+4x+3)^{3/2} - \frac{(x+2)}{2}\sqrt{x^2+4x+3}$

$+ \frac{1}{2}\log\left|(x+2)+\sqrt{x^2+4x+3}\right| + c$

where $c = c_1 - c_2$.

(*v*) Let $\quad I = \int (6x+5)\sqrt{6+x-x^2}\ dx$ ...(1)

Let $(6x+5) = A\frac{d}{dx}(6+x-x^2) + B$

$\Rightarrow \quad (6x+5) = A(1-2x) + B$ ...(2)

Equating the co-efficients of x and the constant terms on both sides, we have

$6 = -2A \Rightarrow A = -3$

and $\quad 5 = A + B \Rightarrow 5 = -3 + B \Rightarrow B = 8$

$\therefore$ Equation (2) becomes

$(6x+5) = -3(1-2x) + 8$

Putting this value of $(6x+5)$ in equation (1), we have

$I = \int [-3(1-2x)+8]\sqrt{6+x-x^2}\ dx$

$= -3\int (1-2x)\sqrt{6+x-x^2}\ dx + 8\int \sqrt{6+x-x^2}\ dx$

$$\Rightarrow \qquad I = -3I_1 + 8I_2 \qquad \text{(say)} \qquad \ldots(3)$$

where $I_1 = \int (1-2x)\sqrt{6+x-x^2}\,dx$

Put $(6+x-x^2) = z \Rightarrow (1-2x)\,dx = dz$

$$\therefore \qquad I_1 = \int \sqrt{z}\,dz = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c_1 = \frac{2}{3}z^{3/2} + c_1 \qquad [\because z = (6+x-x^2)]$$

$$\Rightarrow \qquad I_1 = \frac{2}{3}(6+x-x^2)^{3/2} + c_1 \qquad \ldots(4)$$

and
$$I_2 = \int \sqrt{6+x-x^2}\,dx = \int \sqrt{6-(x^2-x)}$$

$$= \int \sqrt{6-\left(x^2-x+\frac{1}{4}\right)+\frac{1}{4}}\,dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4},\\ \because \left(\frac{1}{2}\text{ coeff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}\,dx = \int \sqrt{\left(\frac{5}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

$$\left[\because \text{By using } \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{\left(x-\frac{1}{2}\right)}{2}\sqrt{\left(\frac{5}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} + \frac{\left(\frac{5}{2}\right)^2}{2}\sin^{-1}\frac{\left(x-\frac{1}{2}\right)}{\frac{5}{2}} + c_2$$

$$\Rightarrow \qquad I_2 = \frac{(2x-1)}{4}\sqrt{6+x-x^2} + \frac{25}{8}\sin^{-1}\left(\frac{2x-1}{5}\right) + c_2 \qquad \ldots(5)$$

$\therefore$ From equation (3),

$$I = -3I_1 + 8I_2$$

$$= -3\left[\frac{2}{3}(6+x-x^2)^{3/2} + c_1\right] + 8\left[\frac{(2x-1)}{4}\sqrt{6+x-x^2} + \frac{25}{8}\sin^{-1}\left(\frac{2x-1}{5}\right) + c_2\right]$$

[Using equations (4) and (5)]

$$= -2(6+x-x^2)^{3/2} - 3c_1 + 2(2x-1)\sqrt{6+x-x^2} + 25\sin^{-1}\left(\frac{2x-1}{5}\right) + 8c_2$$

$$= -2(6+x-x^2)^{3/2} + 2(2x-1)\sqrt{6+x-x^2} + 25\sin^{-1}\left(\frac{2x-1}{5}\right) + c$$

where $c = -3c_1 + 8c_2$.

(*vi*) Let $I = \int x^2\sqrt{a^6-x^6}\,dx$

$$= \int x^2\sqrt{(a^3)^2-(x^3)^2}\,dx$$

Put $\quad x^3 = z \Rightarrow 3x^2\,dx = dz \Rightarrow x^2\,dx = \frac{1}{3}\,dz$

$\therefore \quad I = \int \sqrt{(a^3)^2 - z^2}\left(\frac{1}{3}dz\right) = \frac{1}{3}\int \sqrt{(a^3)^2 - z^2}\,dz$

$$\left[\because \text{ By using } \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{3}\left[\frac{z}{2}\sqrt{(a^3)^2 - z^2} + \frac{(a^3)^2}{2}\sin^{-1}\left(\frac{z}{a^3}\right)\right] + c$$

$$= \frac{x^3}{6}\sqrt{a^6 - x^6} + \frac{a^6}{6}\sin^{-1}\left(\frac{x^3}{a^3}\right) + c.$$

(*vii*) Let $I = \int \cos x\,(\sin x - 1)\sqrt{4 - \sin^2 x}\,dx$

Put $\sin x = z \Rightarrow \cos x\,dx = dz$

$\therefore \quad I = \int (z-1)\sqrt{4 - z^2}\,dz$

$$= \int z\sqrt{4 - z^2}\,dz - \int \sqrt{4 - z^2}\,dz$$

$$= -\frac{1}{2}\int (4 - z^2)^{1/2}(-2z\,dz) - \int \sqrt{(2)^2 - z^2}\,dz$$

[Multiply and divide the first integral by (– 2)]

$$= -\frac{1}{2}\left[\frac{(4 - z^2)^{3/2}}{3/2}\right] - \left[\frac{z}{2}\sqrt{(2)^2 - z^2} + \frac{(2)^2}{2}\sin^{-1}\left(\frac{z}{2}\right)\right] + c$$

$$\left[\begin{aligned} &\because \text{ By using } \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c \\ &\text{and } \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c\end{aligned}\right]$$

$$= -\frac{1}{3}(4 - z^2)^{3/2} - \frac{z}{2}\sqrt{4 - z^2} - 2\sin^{-1}\left(\frac{z}{2}\right) + c$$

$$= -\frac{1}{3}(4 - \sin^2 x)^{3/2} - \frac{\sin x}{2}\sqrt{4 - \sin^2 x} - 2\sin^{-1}\left(\frac{\sin x}{2}\right) + c. \quad [\because\ z = \sin x]$$

(*viii*) Let $I = \int x\sqrt{\frac{1+x}{1-x}}\,dx$

$$= \int x\,\frac{\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx \qquad \text{[On rationalization]}$$

$$= \int \frac{x(1+x)}{\sqrt{1-x^2}}\,.\,dx = \int \frac{x + x^2}{\sqrt{1-x^2}}\,dx = -\int \frac{-x - x^2}{\sqrt{1-x^2}}\,dx$$

$$= -\int \frac{(1 - x^2) - (x + 1)}{\sqrt{1-x^2}}\,dx \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= -\int \frac{(1-x^2)}{\sqrt{1-x^2}}\,dx + \int \frac{x+1}{\sqrt{1-x^2}}\,dx$$

$$= -\int \sqrt{1-x^2}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= -\int \sqrt{(1)^2-x^2}\,dx + \int x\,(1-x^2)^{-1/2}\,dx + \int \frac{1}{\sqrt{(1)^2-x^2}}\,dx$$

$$= -\int \sqrt{(1)^2-x^2}\,dx - \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx + \int \frac{1}{\sqrt{(1)^2-x^2}}\,dx$$

[Multiply and divided the second integral by (– 2)]

$$\left[\begin{array}{l} \because \text{ By using:} \\ \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c \\ \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c \\ \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c \end{array}\right]$$

$$= -\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x - \frac{1}{2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)} + \sin^{-1}x + c$$

$$= -\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x - \sqrt{1-x^2} + c$$

$$= -\frac{(x+2)}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + c.$$

(*ix*) Let $\quad I = \int (x-5)\sqrt{x^2+x}\,dx \quad$ ...(1)

Let $\quad (x-5) = A\,\frac{d}{dx}(x^2+x) + B$

$\Rightarrow \quad (x-5) = A\,(2x+1) + B \quad$ ...(2)

Equating the co-efficients of x and the constant terms, on both sides, we have

$$1 = 2A \;\Rightarrow\; A = \frac{1}{2}$$

and $$-5 = A + B \;\Rightarrow\; -5 = \frac{1}{2} + B \;\Rightarrow\; B = -\frac{11}{2}$$

$\therefore$ Equation (2) becomes

$$(x-5) = \frac{1}{2}(2x+1) - \frac{11}{2}$$

Putting this value of $(x-5)$ in equation (1), we have

$$I = \int \left[\frac{1}{2}(2x+1) - \frac{11}{2}\right]\sqrt{x^2+x}\ dx$$

$$= \frac{1}{2}\int (2x+1)\sqrt{x^2+x}\ dx - \frac{11}{2}\int \sqrt{x^2+x}\ dx$$

$$\Rightarrow \quad I = \frac{1}{2} I_1 - \frac{11}{2} I_2 \quad \text{(say)} \qquad ...(3)$$

where $I_1 = \int (2x+1)\sqrt{x^2+x}\ dx$

Put $x^2 + x = z \Rightarrow (2x+1)\,dx = dz$

$$\therefore \quad I_1 = \int \sqrt{z}\ dz = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c_1 = \frac{2}{3} z^{3/2} + c_1$$

$$\Rightarrow \quad I_1 = \frac{2}{3}(x^2+x)^{3/2} + c_1 \qquad ...(4) \quad [\because z = x^2 + x]$$

and $I_2 = \int \sqrt{x^2+x}\ dx$

$$= \int \sqrt{\left(x^2+x+\frac{1}{4}\right) - \frac{1}{4}}\ dx \qquad \left[\text{Add and subtract } \frac{1}{4},\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\right]$$

$$= \int \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\ dx$$

$$\left[\because \text{By using } \int \sqrt{x^2-a^2}\ dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2-a^2}\right| + c\right]$$

$$= \frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} - \frac{(1/2)^2}{2}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + c_2$$

$$\Rightarrow \quad I_2 = \frac{(2x+1)}{4}\sqrt{x^2+x} - \frac{1}{8}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right| + c_2 \qquad ...(5)$$

$\therefore$ From equation (3)

$$I = \frac{1}{2} I_1 - \frac{11}{2} I_2 \qquad \text{[Using equations (4) and (5)]}$$

$$= \frac{1}{2}\left[\frac{2}{3}(x^2+x)^{3/2} + c_1\right] - \frac{11}{2}\left[\frac{(2x+1)}{4}\sqrt{x^2+x} - \frac{1}{8}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right| + c_2\right]$$

$$= \frac{1}{3}(x^2+x)^{3/2} + \frac{1}{2}c_1 - \frac{11(2x+1)}{8}\sqrt{x^2+x} + \frac{11}{16}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right| - \frac{11}{2}c_2$$

$$= \frac{1}{3}(x^2+x)^{3/2} - \frac{11}{8}(2x+1)\sqrt{x^2+x} + \frac{11}{16}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right| + c$$

where $c = \frac{1}{2}c_1 - \frac{11}{2}c_2$.

EXERCISE FOR PRACTICE

Evaluate the following integrals Q. (1—10).

1. (i) $\int x\, e^{-x}\, dx$ (ii) $\int x^2\, e^{3x}\, dx$

2. (i) $\int \log x\, dx$ (ii) $\int (1 - x^2) \log x\, dx$

3. (i) $\int x^2 \cos x\, dx$ (ii) $\int x \sin 2x\, dx$

4. (i) $\int (1 + 5x)\, e^{2x}\, dx$ (ii) $\int x \log x^2\, dx$

5. (i) $\int x \sin x \cos x\, dx$ (ii) $\int (2 + 7x) \cos 6x\, dx$

6. (i) $\int (\log x)^2\, dx$ (ii) $\int \dfrac{\log (x + 2)}{(x + 2)^2}\, dx$

7. (i) $\int x \cos^3 x\, dx$ (ii) $\int x^2 \log (1 + x)\, dx$

8. (i) $\int \dfrac{\log x}{x^2}\, dx$ (ii) $\int x \,.\, 2^x\, dx$

9. (i) $\int x^{2n-1} \cos x^n\, dx$ (ii) $\int 2x^3\, e^{x^2}\, dx$

10. (i) $\int e^{\sqrt{x}}\, dx$ (ii) $\int \sin^3 \sqrt{x}\, dx.$

Evaluate the following integrals Q. (11—35).

11. (i) $\int (e^{\log x} + \sin x) \cos x\, dx$ (ii) $\int x \operatorname{cosec} x \cot x\, dx$

12. (i) $\int \cos^{-1} x\, dx$ (ii) $\int \operatorname{cosec}^{-1} x\, dx$

13. (i) $\int (\sin^{-1} x)^2\, dx$ (ii) $\int x\, (\tan^{-1} x^2)\, dx$

14. (i) $\int \dfrac{x - \sin x}{1 - \cos x}\, dx$ (ii) $\int \cos^{-1} \left(\dfrac{1 - x^2}{1 + x^2} \right) dx$

15. (i) $\int e^x \left(\dfrac{x - 1}{x^2} \right) dx$ (ii) $\int e^{ax} \sin x\, dx$

16. (i) $\int (x + 1)\, e^x \log (x\, e^x)\, dx$ (ii) $\int x \left(\dfrac{\sec 2x - 1}{\sec 2x + 1} \right) dx$

17. (i) $\int e^x \left[\sin^{-1} x + \dfrac{1}{\sqrt{1 - x^2}} \right] dx$ (ii) $\int \cos^{-1} (4x^3 - 3x)\, dx$

18. (i) $\int e^x (\cot x + \log \sin x)\, dx$ (ii) $\int e^{x/2} \sin \left(\dfrac{x}{2} + \dfrac{\pi}{4} \right) dx$

19. $\int (x + 1)\, e^x \log (x\, e^x)\, dx$ **20.** $\int \dfrac{\sin^{-1} x}{(1 - x^2)^{3/2}}\, dx.$

21. $\int \sin (\log x)\, dx$ **22.** $\int e^{-x/2} \dfrac{\sqrt{1 - \sin x}}{1 + \cos x}\, dx$

23. $\int \sqrt{9-x^2}\ dx$

24. $\int \sqrt{5x^2-20}\ dx$

25. $\int 2x\sqrt{x^4-36}\ dx$

26. $\int (x-5)\sqrt{x^2-x}\ dx$

27. $\int x\sqrt{1+x+x^2}\ dx$

28. $\int \sqrt{8+2x-x^2}\ dx$

29. $\int \sqrt{4-3x-2x^2}\ dx$

30. $\int (6x+5)\sqrt{6+x-x^2}\ dx$

31. $\int (4x+1)\sqrt{x^2-x-2}\ dx$

32. $\int e^{2x}\left(\dfrac{2x-1}{4x^2}\right)dx$

33. $\int \dfrac{e^x(x^2+x+1)}{(x+1)^2}\ dx$

34. $\int e^{2x}\sin x\cos x\, dx$

35. $\int \sin^{-1}\sqrt{\dfrac{x}{a+x}}\ dx.$

Answers

1. (*i*) $-xe^{-x}-e^{-x}+c$ (*ii*) $\dfrac{1}{27}e^{3x}(9x^2-6x+2)+c$

2. (*i*) $x\log x-x+c$ (*ii*) $\left(x-\dfrac{x^3}{3}\right)\log x-x+\dfrac{x^3}{9}+c$

3. (*i*) $x^2\sin x+2x\cos x-2\sin x+c$ (*ii*) $-\dfrac{x}{2}\cos 2x+\dfrac{1}{4}\sin 2x+c$

4. (*i*) $\dfrac{e^{2x}}{4}(10x-3)+c$ (*ii*) $x^2\log x-\dfrac{x^2}{2}+c$

5. (*i*) $-x\dfrac{\cos 2x}{4}+\dfrac{\sin 2x}{8}+c$ (*ii*) $\dfrac{2+7x}{6}\sin 6x+\dfrac{7}{36}\cos 6x+c$

6. (*i*) $x(\log x)^2-2x\log x+2x+c$ (*ii*) $-\dfrac{\log(x+2)+1}{x+2}+c$

7. (*i*) $\dfrac{1}{4}\left[3x\sin x+3\cos x+\dfrac{x\sin 3x}{3}+\dfrac{\cos 3x}{9}\right]+c$

(*ii*) $-\dfrac{1}{(x+1)}[\log(x+1)+1]+c$

8. (*i*) $-\dfrac{(1+\log x)}{x}+c$ (*ii*) $\dfrac{2^x(x\log 2-1)}{(\log 2)^2}+c$

9. (*i*) $\dfrac{1}{n}x^n\sin x^n+\dfrac{1}{n}\cos x^n+c$ (*ii*) $e^{x^2}(x^2-1)+c$

10. (*i*) $2e^{\sqrt{x}}\left(\sqrt{x}-1\right)+c$ (*ii*) $-3x^{3/2}\cos^3\sqrt{x}+6x^{1/3}\sin^3\sqrt{x}+6\cos^3\sqrt{x}+c$

11. (*i*) $x\sin x+\cos x+\dfrac{1}{2}\sin^2 x+c$ (*ii*) $-x\operatorname{cosec} x+\log|\operatorname{cosec} x-\cot x|+c$

12. (*i*) $x\cos^{-1}x-\sqrt{1-x^2}+c$ (*ii*) $x\operatorname{cosec}^{-1}x-\log\left|x-\sqrt{x^2-1}\right|+c$

13. (*i*) $(\sin^{-1}x)^2x+2\sqrt{1-x^2}\sin^{-1}x-2x+c$ (*ii*) $\dfrac{1}{2}x^2\tan^{-1}x^2-\dfrac{1}{4}\log(1+x^4)+c$

14. (*i*) $-x\cot\dfrac{x}{2}+c$ (*ii*) $2x\tan^{-1}x-\log(1+x^2)+c$

15. (i) $\dfrac{e^x}{x} + c$ (ii) $\dfrac{e^{ax}}{a^2 + b^2}\,(a \sin bx - b \cos bx) + c$

16. (i) $x\, e^x\, [\log (xe^x) - 1] + c$ (ii) $x \tan x - \log \sec x - \dfrac{x}{2} + c$

17. (i) $e^x \sin^{-1} x + c$ (ii) $3x \cos^{-1} x - 3\sqrt{1 - x^2} + c$

18. (i) $e^x \log \sin x + c$ (ii) $\sqrt{2}\, e^{x/2} \sin \dfrac{x}{2} + c$

19. $x\, e^x\, [\log (x\, e^x) - 1] + c$

20. $\dfrac{x}{\sqrt{1 - x^2}} \sin^{-1} x + \dfrac{1}{2} \log |1 - x^2| + c$

21. $\dfrac{x}{2}\,[\sin (\log x) - \cos (\log x)] + c$

22. $e^{-x/2} \sec \dfrac{x}{2} + c$

23. $\dfrac{1}{2}\, x \sqrt{9 - x^2} + \dfrac{9}{2} \sin^{-1}\left(\dfrac{x}{3}\right) + c$

24. $\sqrt{5}\left[\dfrac{x}{2}\sqrt{x^2 - 4} - \log\left(x + \sqrt{x^2 - 4}\right)\right] + c$

25. $\dfrac{x^2}{2}\sqrt{x^4 - 36} - 18 \log \left| x^2 + \sqrt{x^4 - 36} \right| + c$

26. $\dfrac{1}{3}(x^2 - x)^{3/2} - \dfrac{9}{8}(2x - 1)\sqrt{x^2 - x} + \dfrac{9}{16} \log \left| x - \dfrac{1}{2} + \sqrt{x^2 - x} \right| + c$

27. $\dfrac{1}{3}(x^2 + x - 1)^{3/2} - \dfrac{2x + 1}{8}\sqrt{x^2 + x + 1} + \dfrac{3}{16} \log \left| \left(x + \dfrac{1}{2}\right) + \sqrt{x^2 + x + 1} \right| + c$

28. $\dfrac{x - 1}{2}\sqrt{8 + 2x - x^2} + \dfrac{9}{2} \sin^{-1}\left(\dfrac{x - 1}{3}\right) + c$

29. $\dfrac{(4x + 3)}{8}\sqrt{4 - 3x - 2x^2} + \dfrac{41\sqrt{2}}{32} \sin^{-1} \dfrac{(4x + 3)}{\sqrt{41}} + c$

30. $-2\,(6 + x - x^2)^{3/2} + 2(2x - 1)\sqrt{6 + x - x^2} + 25 \sin^{-1} \dfrac{(2x - 1)}{5} + c$

31. $\dfrac{4}{3}(x^2 - x - 2)^{3/2} + \dfrac{3}{4}(2x - 1)\sqrt{x^2 - x - 2} - \dfrac{27}{8} \log \left| \left(x - \dfrac{1}{2}\right) + \sqrt{x^2 - x - 2} \right| + c$

32. $\dfrac{e^{2x}}{4x} + c$

33. $\dfrac{x\, e^x}{x + 1} + c$

34. $\dfrac{e^{2x}}{8}(\sin 2x - \cos 2x) + c$

35. $(x + a) \tan^{-1} \sqrt{\dfrac{x}{a}} - \sqrt{ax} + c.$

5

Integration by Partial Fractions

5.1 PARTIAL FRACTIONS

5.1.1 Definition : Rational Function

If $f(x)$ and $g(x)$ are two polynomials, then $\dfrac{f(x)}{g(x)}$ is called a rational algebraic function or rational function of x.

5.1.2 Definition : Proper Rational Fraction

Any rational algebraic fraction is said to be a proper rational algebraic fraction if degree of numerator is less than the degree of denominator.

i.e., if deg. $f(x) <$ deg. $g(x)$, then $\dfrac{f(x)}{g(x)}$ is called proper rational fraction.

e.g., $\dfrac{3x-1}{x^2-1}, \dfrac{x^2+1}{(x-1)^2(x+1)}$ etc.

5.1.3 Definition : Improper Rational Fraction

Any rational algebraic function is said to be an improper rational algebraic fraction if degree of numerator is greater than or equal to the degree of the denominator.

i.e., if deg. $f(x) \geq$ deg. $g(x)$, then $\dfrac{f(x)}{g(x)}$ is called an improper rational fraction.

If $\dfrac{f(x)}{g(x)}$ is an improper rational fraction, then, we divide $f(x)$ by $g(x)$ so that the rational fraction $\dfrac{f(x)}{g(x)}$ is expressed in the form $\dfrac{f(x)}{g(x)} = u(x) + \dfrac{v(x)}{g(x)}$.

where : $u(x)$ and $v(x)$ are polynomials such that the degree of $v(x)$ is less than the degree of $g(x)$.

$\therefore$ Any improper fraction can be expressed as the sum of a polynomial and a proper fraction.

e.g., $\dfrac{x^3+x^2+1}{x^2-5x+6}$ is an improper fraction.

$\therefore \quad \dfrac{x^3+x^2+1}{x^2-5x+6} = (x+6) + \dfrac{24x-35}{x^2-5x+6}$ [On dividing (x^3+x^2+1) by (x^2-5x+6)]

5.2 RESOLUTION OF A FRACTION INTO PARTIAL FRACTIONS

We know the method of finding the sum of two or more algebraic fractions by reducing the denominators of fractions to a common denominator, which is their L.C.M.

The reverse process of breaking up a single fraction into simpler fractions whose denominators are the factors of the denominator of the given fraction is called the Resolution of a fraction into its partial fractions.

e.g., $\dfrac{2x}{x^2-1} = \dfrac{1}{x-1} + \dfrac{1}{x+1}$

Here, $\dfrac{1}{x-1}$ and $\dfrac{1}{x+1}$ are the partial fractions of the fraction $\dfrac{2x}{x^2-1}$.

The resolution of $\dfrac{f(x)}{g(x)}$ into partial fractions depends mainly upon the nature of the factors of $g(x)$.

Case I. When the denominator has any linear non-repeated factor of the form $(x-a)$, then there exists a partial fractions of the form $\dfrac{A}{(x-a)}$.

Case II. When the denominator has some or all linear factors repeated n-times, like $(x-a)^n$, then there exists partial fractions of the form $\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-a)^3} + \ldots\ldots$ n terms.

Case III. When some of the factors of denominator are quadratic but non-repeating like (ax^2+bx+c), then there exists partial fractions of the form $\dfrac{Ax+B}{ax^2+bx+c}$.

where A and B are constants to be determined by comparing co-efficients of like powers of x in the numerator on both sides.

Case IV. When some of the factors of denominator are quadratic and repeating n-times like $(ax^2+bx+c)^n$, then there exists partial fractions of the form

$$\frac{Ax+B}{(ax^2+bx+c)} + \frac{Cx+D}{(ax^2+bx+c)^2} + \ldots\ldots \text{ } n\text{-terms.}$$

where A, B, C, D, ... are constants to be determined by comparing co-efficients of like powers of x in the numerator on both sides.

The following Table 5.1 indicates the types of simple partial fractions that are to be associated with what kind of rational fractions.

TABLE 5.1

S. No.	Form of rational fraction	Form of the partial fraction
1.	$\dfrac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-b)}$
2.	$\dfrac{px+q}{(x-b)^2}$	$\dfrac{A}{(x-b)}+\dfrac{B}{(x-b)^2}$
3.	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-b)}+\dfrac{C}{(x-c)}$
4.	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-b)}$
5.	$\dfrac{px+q}{ax^2+bx+c}$	$\dfrac{Ax+B}{ax^2+bx+c}$
6.	$\dfrac{px^2+qx+r}{(ax^2+bx+c)^2}$	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{Cx+D}{(ax^2+bx+c)^2}$

The values of A, B, C, D, ... can be obtained by comparing the co-efficients of like powers of x in the numerator on both sides.

Remark 1. Before resolving a given fraction into partial fractions we must see that numerator of the given fraction is of the lower degree than the denominator. *i.e.,* the fraction must be a proper rational fraction.

If it is not so, the first divide the numerator by the denominator and express the fraction as the sum of a polynomial and a proper rational fraction.

Remark 2. If a rational fraction contains only even powers of x in both the numerator and denominator, then to resolve it into partial fractions, we proceed as follows :

Put $x^2 = z$ in the given rational fraction. Resolve the rational fraction obtained above into partial fractions by using one of the appropriate methods discussed above. Finally replace z by x^2.

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following integrals :*

(*i*) $\displaystyle\int \frac{2x+5}{(x^2-x-2)}\,dx$ (*ii*) $\displaystyle\int \frac{2x+1}{(x+1)(x-2)}\,dx$

(*iii*) $\displaystyle\int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx$ (*iv*) $\displaystyle\int \frac{3x+4}{(x^2-5x+6)}\,dx$

(*v*) $\displaystyle\int \frac{3x+2}{(x-1)(2x+3)}\,dx$ (*vi*) $\displaystyle\int \frac{x+1}{(x^2+4x-5)}\,dx.$

Solution. (*i*) Let $\displaystyle I = \int \frac{2x+5}{x^2-x-2}\,dx$

$$\therefore \quad I = \int \frac{2x+5}{(x-2)(x+1)}\,dx \qquad ...(1)$$

Let $\dfrac{2x+5}{(x-2)(x+1)} = \dfrac{A}{(x-2)} + \dfrac{B}{(x+1)}$ ...(2)

Multiplying both sides by $(x-2)(x+1)$, we get

$$(2x+5) = A(x+1) + B(x-2) \quad ...(3)$$

$$(x+1) = 0 \Rightarrow x = -1$$

$$(x-2) = 0 \Rightarrow x = 2$$

Put $x = -1$ in (3), we get

$$2(-1) + 5 = A(-1+1) + B(-1-2)$$

$$\Rightarrow \quad 3 = -3B \Rightarrow B = -1$$

Put $x = 2$ in (3), we get

$$2(2) + 5 = A(2+1) + B(2-2)$$

$$\Rightarrow \quad 9 = 3A \Rightarrow A = 3.$$

$\therefore$ Equation (2) becomes

$$\frac{2x+5}{(x-2)(x+1)} = \frac{3}{x-2} - \frac{1}{x+1}$$

$$\therefore \quad I = \int \left(\frac{3}{x-2} - \frac{1}{x+1}\right) dx = 3\int \frac{1}{x-2}\,dx - \int \frac{1}{x+1}\,dx$$

$$= 3 \log |x-2| - \log |x+1| + c. \qquad \left[\because \int \frac{1}{x}\,dx = \log|x| + c\right]$$

(*ii*) Let $I = \displaystyle\int \frac{2x+1}{(x+1)(x-2)}\,dx$ ...(1)

Let $\dfrac{2x+1}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$ ...(2)

Multiplying both sides by $(x+1)(x-2)$, we get

$$2x + 1 = A(x-2) + B(x+1) \quad ...(3)$$

$$x - 2 = 0 \Rightarrow x = 2$$

$$x + 1 = 0 \Rightarrow x = -1$$

Put $x = 2$ in (3), we get

$$2(2) + 1 = A(2-2) + B(2+1)$$

$$\Rightarrow \quad 5 = 3B \Rightarrow B = 5/3$$

Put $x = -1$ in (3), we get

$$2(-1) + 1 = A(-1-2) + B(-1+1)$$

$$\Rightarrow \quad -1 = -3A \Rightarrow A = \frac{1}{3}.$$

$\therefore$ Equation (2) becomes

$$\frac{2x+1}{(x+1)(x-2)} = \frac{1/3}{x+1} + \frac{5/3}{x-2} = \frac{1}{3(x+1)} + \frac{5}{3(x-2)}$$

$$\therefore \quad I = \int \left(\frac{1}{3(x+1)} + \frac{5}{3(x-2)}\right) dx$$

$$= \frac{1}{3}\int \frac{1}{x+1}\,dx + \frac{5}{3}\int \frac{1}{x-2}\,dx$$

$$= \frac{1}{3}\log|x+1| + \frac{5}{3}\log|x-2| + c.$$

(*iii*) Let $\quad I = \int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx \quad$ ...(1)

Let $\quad \frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3} \quad$ ...(2)

Multiplying both sides by $(x-1)(x+2)(x-3)$, we get

$$2x-1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2) \quad \text{...(3)}$$

$$x-1=0 \Rightarrow x=1$$

$$x+2=0 \Rightarrow x=-2$$

$$x-3=0 \Rightarrow x=3$$

Put $x = 1$ in (3), we get

$$2(1)-1 = A(1+2)(1-3) + B(1-1)(1-3) + C(1-1)(1+2)$$

$$\Rightarrow \quad 1 = A(3)(-2) \Rightarrow A = -\frac{1}{6}.$$

Put $x = -2$ in (3), we get

$$2(-2)+1 = A(-2+2)(-2-3) + B(-2-1)(-2-3) + C(-2-1)(-2+2)$$

$$\Rightarrow \quad -5 = B(-3)(-5) \Rightarrow B = -\frac{1}{3}.$$

Put $x = 3$ in (3), we get

$$2(3)-1 = A(3+2)(3-3) + B(3-1)(3-3) + C(3-1)(3+2)$$

$$\Rightarrow \quad 5 = C(2)(5) \Rightarrow C = \frac{1}{2}$$

Substituting the values of A, B and C in (2), we have

$$\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{-1/6}{x-1} + \frac{-1/3}{x+2} + \frac{1/2}{x-3} = \frac{-1}{6(x-1)} - \frac{1}{3(x+2)} + \frac{1}{2(x-3)}$$

$$\therefore \quad I = \int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx = \int \left[\frac{-1}{6(x-1)} - \frac{1}{3(x+2)} + \frac{1}{2(x-3)}\right] dx$$

$$= -\frac{1}{6}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{1}{x+2}\,dx + \frac{1}{2}\int \frac{1}{x-3}\,dx$$

$$= -\frac{1}{6}\log|x-1| - \frac{1}{3}\log|x+2| + \frac{1}{2}\log|x-3| + c.$$

(*iv*) Let $\quad I = \int \frac{3x+4}{x^2-5x+6}\,dx$

$$\therefore \quad I = \int \frac{3x+4}{(x-2)(x-3)}\,dx \quad \text{...(1)}$$

Let $\dfrac{3x+4}{(x-2)(x-3)} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$ ...(2)

Multiplying both sides by $(x-2)(x-3)$, we get

$$3x + 4 = A(x-3) + B(x-2) \quad ...(3)$$

$$x - 2 = 0 \Rightarrow x = 3 \,;\, x - 3 = 0 \Rightarrow x = 3$$

Put $x = 2$ in (3), we get

$$3(2) + 4 = A(2-3) + B(2-2) \Rightarrow 10 = -A \Rightarrow A = -10$$

Put $x = 3$ in (3), we get

$$3(3) + 4 = A(3-3) + B(3-2) \Rightarrow 13 = B \Rightarrow B = 13$$

Substituting the values of A and B in equation (2), we have

$$\frac{3x+4}{(x-2)(x-3)} = \frac{-10}{x-2} + \frac{13}{x-3}$$

$$\therefore \qquad I = \int \frac{3x+4}{(x-2)(x-3)}\,dx = \int \left(\frac{-10}{x-2} + \frac{13}{x-3}\right) dx$$

$$= -10 \int \frac{1}{x-2}\,dx + 13 \int \frac{1}{x-3}\,dx$$

$$= -10 \log |x-2| + 13 \log |x-3| + c.$$

(*v*) Let $I = \displaystyle\int \frac{3x+2}{(x-1)(2x+3)}\,dx$ ...(1)

Let $\dfrac{3x+2}{(x-1)(2x+3)} = \dfrac{A}{x-1} + \dfrac{B}{2x+3}$ ...(2)

Multiplying both sides by $(x-1)(2x+3)$, we get

$$(3x + 2) = A(2x + 3) + B(x - 1) \quad ...(3)$$

$$x - 1 = 0 \Rightarrow x = 1$$

$$2x + 3 = 0 \Rightarrow 2x = -3 \Rightarrow x = -\frac{3}{2}$$

Put $x = 1$ in (3), we get

$$3(1) + 2 = A(2.1 + 3) + B(1-1) \Rightarrow 5 = 5A \Rightarrow A = 1$$

Put $x = -\dfrac{3}{2}$ in (3), we get

$$3\left(-\frac{3}{2}\right) + 2 = A\left[2\left(-\frac{3}{2}\right) + 3\right] + B\left(-\frac{3}{2} - 1\right) \Rightarrow -\frac{9}{2} + 2 = A\,(0) + B\left(-\frac{5}{2}\right)$$

$$\Rightarrow \qquad -\frac{5}{2} = -\frac{5}{2}B \Rightarrow B = 1.$$

Substituting the values of A and B in equation (2), we have

$$\frac{3x+2}{(x-1)(2x+3)} = \frac{1}{x-1} + \frac{1}{2x+3}$$

$$\therefore \quad I = \int \frac{3x+2}{(x-1)(2x+3)}\,dx = \int \left[\frac{1}{x-1} + \frac{1}{2x+3}\right] dx$$

$$= \int \frac{1}{x-1}\,dx + \int \frac{1}{2x+3}\,dx$$

$$= \log|x-1| + \log|2x+3| + c.$$

(*vi*) Let $\quad I = \int \frac{x+1}{x^2+4x-5}\,dx \qquad \ldots(1)$

$$\therefore \quad I = \int \frac{x+1}{(x+5)(x-1)}\,dx$$

Let $\frac{x+1}{(x+5)(x-1)} = \frac{A}{(x+5)} + \frac{B}{(x-1)} \qquad \ldots(2)$

Multiplying both sides by $(x+5)(x-1)$, we get

$$x+1 = A(x-1) + B(x+5) \qquad \ldots(3)$$

$$x-1 = 0 \Rightarrow x = 1$$

$$x+5 = 0 \Rightarrow x = -5$$

Put $x = 1$ in (3), we get

$$1+1 = A(1-1) + B(1+5) \Rightarrow 2 = 6B \Rightarrow B = \frac{1}{3}.$$

Put $x = -5$ in (3), we get

$$-5+1 = A(-5-1) + B(-5+5) \Rightarrow -4 = -6A \Rightarrow A = \frac{2}{3}$$

Substituting the values of A and B in equation (2), we have

$$\frac{x+1}{(x+5)(x-1)} = \frac{2/3}{x+5} + \frac{1/3}{x-1} = \frac{2}{3(x+5)} + \frac{1}{3(x-1)}$$

$$\therefore \quad I = \int \frac{x+1}{(x+5)(x-1)}\,dx = \int \left[\frac{2}{3(x+5)} + \frac{1}{3(x-1)}\right] dx$$

$$= \frac{2}{3}\int \frac{1}{x+5}\,dx + \frac{1}{3}\int \frac{1}{x-1}\,dx$$

$$= \frac{2}{3}\log|x+5| + \frac{1}{3}\log|x-1| + c.$$

Example 2. *Evaluate the following integrals :*

(*i*) $\int \frac{x}{(x-1)(x-2)(x-3)}\,dx$ (*ii*) $\int \frac{2x}{x^2+3x+2}\,dx$

(*iii*) $\int \frac{3x-1}{(x-2)^2}\,dx$ (*iv*) $\int \frac{2x+3}{x^2+7x+10}\,dx$

(*v*) $\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$ (*vi*) $\int \frac{x^2+2}{(x^2+1)(x^2+4)}\,dx.$

Solution. (*i*) Let $I = \int \frac{x}{(x-1)(x-2)(x-3)}\,dx \qquad \ldots(1)$

Let $\quad \frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \qquad \ldots(2)$

Multiplying both sides by $(x-1)(x-2)(x-3)$, we get

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)x-2) \quad \text{...(3)}$$

$$x-1=0 \Rightarrow x=1$$

$$x-2=0 \Rightarrow x=2$$

$$x-3=0 \Rightarrow x=3$$

Put $x = 1$ in (3), we get

$$1 = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$$

$$\Rightarrow \quad 1 = A(-1)(-2) \Rightarrow A = \frac{1}{2}$$

Put $x = 2$ in (3), we get

$$2 = A(2-2)(2-3) + B(2-1)(2-3) + C(2-1)(2-2)$$

$$\Rightarrow \quad 2 = B(1)(-1) \Rightarrow B = -2$$

Put $x = 3$ in (3), we get

$$3 = A(3-2)(3-3) + B(3-1)(3-3) + C(3-1)(3-2)$$

$$\Rightarrow \quad 3 = C(2)(1) \Rightarrow C = \frac{3}{2}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{1/2}{x-1} + \frac{(-2)}{x-2} + \frac{3/2}{x-3} = \frac{1}{2(x-1)} - \frac{2}{x-2} + \frac{3}{2(x-3)}$$

$$\therefore \quad I = \int \left[\frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}\right] dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}\,dx - 2\int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{1}{x-3}\,dx$$

$$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + c.$$

(*ii*) Let $\quad I = \int \frac{2x}{x^2+3x+2}\,dx$

$$\therefore \quad I = \int \frac{2x}{(x+1)(x+2)}\,dx \quad \text{...(1)}$$

Let $\dfrac{2x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$ ...(2)

Multiplying both sides by $(x+1)(x+2)$, we get

$$2x = A(x+2) + B(x+1) \quad \text{...(3)}$$

$$x+1=0 \Rightarrow x=-1$$

$$x+2=0 \Rightarrow x=-2$$

Put $x = -1$ in (3), we get

$$2(-1) = A(-1+2) + B(-1+1) \Rightarrow -2 = A \Rightarrow A = -2$$

Put $x = -2$ in (3), we get

$$2(-2) = A(-2+2) + B(-2+1) \Rightarrow -4 = -B \Rightarrow B = 4$$

Substituting the values of A and B in equation (2), we have

$$\frac{2x}{(x+1)(x+2)} = \frac{-2}{x+1} + \frac{4}{x+2}$$

$\therefore \quad I = \int \frac{2x}{x^2+3x+2}\,dx = \int \left(\frac{-2}{x+1} + \frac{4}{x+2}\right) dx$

$= -2 \int \frac{1}{x+1}\,dx + 4 \int \frac{1}{x+2}\,dx$

$= -2 \log |x+1| + 4 \log |x+2| + c.$

(*iii*) Let $\quad I = \int \frac{3x-1}{(x-2)^2}\,dx$ ....(1)

Let $\quad \frac{3x-1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$ ...(2)

Multiplying both sides by $(x-2)^2$, we get

$3x - 1 = A(x-2) + B$ ...(3)

$x - 2 = 0 \Rightarrow x = 2$

Put $x = 2$ in (3), we get

$3(2) - 1 = A(2-2) + B \Rightarrow 5 = B \Rightarrow B = 5$

Equating the co-efficients of x on both sides of equation (3), we have

$3 = A \Rightarrow A = 3$

Substituting the values of A and B in equation (2), we have

$\frac{3x-1}{(x-2)^2} = \frac{3}{x-2} + \frac{5}{(x-2)^2}$

$\therefore \quad I = \int \frac{3x-1}{(x-2)^2}\,dx = \int \left(\frac{3}{x-2} + \frac{5}{(x-2)^2}\right) dx$

$= 3 \int \frac{1}{x-2}\,dx + 5 \int (x-2)^{-2}\,dx$

$= 3 \log |x-2| + 5\,\frac{(x-2)^{-2+1}}{-2+1} + c \qquad \left[\because \int [f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$

$= 3 \log |x-2| - \frac{5}{x-2} + c$

(*iv*) Let $\quad I = \int \frac{2x+3}{x^2+7x+10}\,dx$

$\therefore \quad I = \int \frac{2x+3}{(x+2)(x+5)}\,dx$ ...(1)

Let $\frac{2x+3}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2}$ ...(2)

Multiplying both sides by $(x+5)(x+2)$, we get

$2x + 3 = A(x+2) + B(x+5)$ ...(3)

$x + 2 = 0 \Rightarrow x = -2$

$x + 5 = 0 \Rightarrow x = -5$

Put $x = -2$ in (3), we have

$$2(-2) + 3 = A(-2+2) + B(-2+5) \Rightarrow -1 = 3B \Rightarrow B = \frac{-1}{3}$$

Put $x = -5$ in (3), we have

$$2(-5) + 3 = A(-5+2) + B(-5+5) \Rightarrow -7 = -3A \Rightarrow A = \frac{7}{3}$$

Substituting the values of A and B in equation (2), we have

$$\frac{2x+3}{(x+5)(x+2)} = \frac{7/3}{x+5} + \frac{-1/3}{x+2} = \frac{7}{3(x+5)} - \frac{1}{3(x+2)}$$

$$\therefore \qquad I = \int \frac{2x+3}{x^2+7x+10}\,dx = \int \left[\frac{7}{3(x+5)} - \frac{1}{3(x+2)}\right] dx$$

$$= \frac{7}{3}\int \frac{1}{x+5}\,dx - \frac{1}{3}\int \frac{1}{x+2}\,dx$$

$$= \frac{7}{3}\log|x+5| - \frac{1}{3}\log|x+2| + c.$$

(v) Let $\qquad I = \int \frac{3x-2}{(x+1)^2(x+3)}\,dx \qquad \ldots(1)$

Let $\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)} \qquad \ldots(2)$

Multiplying both sides by $(x+1)^2(x+3)$, we get

$$(3x-2) = A(x+1)(x+3) + B(x+3) + C(x+1)^2 \qquad \ldots(3)$$

$$x + 1 = 0 \Rightarrow x = -1$$

$$x + 3 = 0 \Rightarrow x = -3$$

Put $x = -1$ in (3), we get

$$[3(-1) - 2] = A(-1+1)(-1+3) + B(-1+3) + C(-1+1)^2$$

$$\Rightarrow \qquad -5 = 2B \Rightarrow B = -\frac{5}{2}$$

Put $x = -3$ in (3), we get

$$[3(-3) - 2] = A(-3+1)(-3+3) + B(-3+3) + C(-3+1)^2$$

$$\Rightarrow \qquad -9 - 2 = 4C \Rightarrow C = \frac{-11}{4}$$

Comparing co-efficients of x^2 on both sides of (3), we get

$$0 = A + C \Rightarrow A = -C$$

$$\Rightarrow \qquad A = -\left(\frac{-11}{4}\right) = \frac{11}{4}.$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{3x-2}{(x-1)^2(x+3)} = \frac{11/4}{(x+1)} + \frac{-5/2}{(x+1)^2} + \frac{-11/4}{(x+3)} = \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2} - \frac{11}{4(x+3)}$$

$$\therefore \qquad I = \int \frac{3x-2}{(x-1)^2(x+3)}\,dx = \int \left[\frac{11}{4(x+1)} - \frac{5}{2(x+1)^2} - \frac{11}{4(x+3)}\right] dx$$

$$= \frac{11}{4}\int \frac{1}{x+1}\,dx - \frac{5}{2}\int (x+1)^{-2}\,dx - \frac{11}{4}\int \frac{1}{x+3}\,dx$$

$$= \frac{11}{4}\log|x+1| - \frac{5}{2}\frac{(x+1)^{-2+1}}{(-2+1)} - \frac{11}{4}\log|x+3| + c$$

$$\left[\because \int [f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= \frac{11}{4}\log|x+1| - \frac{11}{4}\log|x+3| + \frac{5}{2(x+1)} + c$$

$$= \frac{11}{4}\log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + c. \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

(*vi*) Let $\quad I = \int \frac{x^2+2}{(x^2+1)(x^2+4)}\,dx \quad$ ...(1)

Put $\quad x^2 = z \quad$ [See remark 2 in Art. 4.2]

$$\therefore \quad \frac{x^2+2}{(x^2+1)(x^2+4)} = \frac{z+2}{(z+1)(z+4)}$$

Let $\quad \frac{(z+2)}{(z+1)(z+4)} = \frac{A}{(z+1)} + \frac{B}{(z+4)} \quad$ ...(2)

Multiplying both sides by $(z+1)(z+4)$, we get

$$(z+2) = A(z+4) + B(z+1) \quad ...(3)$$

$$z+1 = 0 \Rightarrow z = -1$$

$$z+4 = 0 \Rightarrow z = -4$$

Put $z = -1$ in (3), we get

$$(-1+2) = A(-1+4) + B(-1+1) \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$$

Put $z = -4$ in (3), we get

$$(-4+2) = A(-4+4) + B(-4+1) \Rightarrow -2 = -3B \Rightarrow B = \frac{2}{3}$$

Substituting values of A and B in equation (2), we have

$$\frac{z+2}{(z+1)(z+4)} = \frac{1/3}{(z+1)} + \frac{2/3}{(z+4)} = \frac{1}{3(z+1)} + \frac{2}{3(z+4)}$$

or $\quad \frac{x^2+2}{(x^2+1)(x^2+4)} = \frac{1}{3(x^2+1)} + \frac{2}{3(x^2+4)} \quad [\because z = x^2]$

$$\therefore \quad I = \int \frac{x^2+2}{(x^2+1)(x^2+4)}\,dx = \int\left[\frac{1}{3(x^2+1)} + \frac{2}{3(x^2+4)}\right]dx$$

$$= \frac{1}{3}\int \frac{1}{x^2+1}\,dx + \frac{2}{3}\int \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{3}\tan^{-1}x + \frac{2}{3}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + c \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{3}\left[\tan^{-1}x + \tan^{-1}\frac{x}{2}\right] + c.$$

Example 3. *Evaluate the following integrals :*

(*i*) $\int \frac{11x+6}{(2x+1)(5x+3)}\,dx$ (*ii*) $\int \frac{x^2}{(x-1)(x-2)(x-3)}\,dx$

(*iii*) $\int \frac{x}{(x-1)(x^2+1)}\,dx$ (*iv*) $\int \frac{x^4+1}{x^3-x^2+x-1}\,dx.$

Solution. (*i*) Let $I = \int \frac{11x+6}{(2x+1)(5x+3)}\,dx$ ...(1)

Let $\frac{11x+6}{(2x+1)(5x+3)} = \frac{A}{2x+1} + \frac{B}{5x+3}$ ...(2)

Multiplying both sides by $(2x+1)(5x+3)$, we get

$$11x + 6 = A(5x + 3) + B(2x + 1) \qquad ...(3)$$

$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2},\ 5x + 3 = 0 \Rightarrow x = -\frac{3}{5}$$

Put $x = -\frac{1}{2}$ in equation (3), we get

$$\left[11\left(\frac{-1}{2}\right)+6\right] = A\left[5\left(\frac{-1}{2}\right)+3\right] + B\left[2\left(\frac{-1}{2}\right)+1\right]$$

$$\Rightarrow \quad \frac{-11}{2}+6 = A\left[\frac{-5}{2}+3\right] \quad \Rightarrow \quad \frac{1}{2} = \frac{A}{2} \quad \Rightarrow \quad A = 1$$

Put $x = -\frac{3}{5}$ in equation (3), we get

$$\left[11\left(\frac{-3}{5}\right)+6\right] = A\left[5\left(\frac{-3}{5}\right)+3\right] + B\left[2\left(\frac{-3}{5}\right)+1\right]$$

$$\Rightarrow \quad \frac{-33}{5}+6 = B\left[\frac{-6}{5}+1\right] \quad \Rightarrow \quad \frac{-3}{5} = \frac{-B}{5} \quad \Rightarrow \quad B = 3$$

Substituting the values of A and B in equation (3), we have

$$\frac{11x+6}{(2x+1)(5x+3)} = \frac{1}{2x+1} + \frac{3}{5x+3}$$

$$\therefore \quad I = \int \frac{11x+6}{(2x+1)(5x+3)}\,dx = \int \left[\frac{1}{2x+1} + \frac{3}{5x+3}\right] dx$$

$$= \int \frac{1}{2x+1}\,dx + 3\int \frac{1}{5x+3}\,dx$$

$$= \frac{\log|2x+1|}{2} + \frac{3\log|5x+3|}{5} + c \qquad \left[\because \int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c\right]$$

$$= \frac{1}{2}\log|2x+1| + \frac{3}{5}\log|5x+3| + c.$$

(*ii*) Let $\quad I = \int \frac{x^2}{(x+1)(x+2)(x+3)}\,dx$ ...(1)

Let $$\frac{x^2}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \quad \text{...(2)}$$

Multiplying both sides by $(x + 1)(x + 2)(x + 3)$, we get

$$x^2 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2) \quad \text{...(3)}$$

$$x + 1 = 0 \Rightarrow x = -1,$$
$$x + 2 = 0 \Rightarrow x = -2,$$
$$x + 3 = 0 \Rightarrow x = -3.$$

Put $x = -1$ in (3), we get

$$(-1)^2 = A(-1+2)(-1+3) + B(-1+1)(-1+3) + C(-1+1)(-1+2)$$

$$\Rightarrow \quad 1 = 2A \Rightarrow A = \frac{1}{2}$$

Put $x = -2$ in (3), we get

$$(-2)^2 = A(-2+2)(-2+3) + B(-2+1)(-2+3) + C(-2+1)(-2+2)$$

$$\Rightarrow \quad 4 = -B \Rightarrow B = -4$$

Put $x = -3$ in (3), we get

$$(-3)^2 = A(-3+2)(-3+3) + B(-3+1)(-3+3) + C(-3+1)(-3+2)$$

$$\Rightarrow \quad 9 = 2C \Rightarrow C = \frac{9}{2}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{x^2}{(x+1)(x+2)(x+3)} = \frac{1/2}{x+1} + \frac{-4}{x+2} + \frac{9/2}{x+3} = \frac{1}{2(x+1)} - \frac{4}{x+2} + \frac{9}{2(x+3)}$$

$$\therefore \quad I = \int \frac{x^2}{(x+1)(x+2)(x+3)}\,dx = \int \left[\frac{1}{2(x+1)} - \frac{4}{x+2} + \frac{9}{2(x+3)}\right] dx$$

$$= \frac{1}{2}\int \frac{1}{x+1}\,dx - 4\int \frac{1}{x+2}\,dx + \frac{9}{2}\int \frac{1}{x+3}\,dx$$

$$= \frac{1}{2}\log|x+1| - 4\log|x+2| + \frac{9}{2}\log|x+3| + c.$$

(*iii*) Let $$I = \int \frac{x}{(x-1)(x^2+1)}\,dx \quad \text{...(1)}$$

Let $$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \quad \text{...(2)}$$

Multiplying both sides by $(x-1)(x^2+1)$, we get

$$x = A(x^2+1) + (Bx+C)(x-1) \quad \text{...(3)}$$

Put $x = 1$ in (3), we get

$$1 = A[1+1] + (B+C)(1-1) \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

Equating the co-efficients of x^2 on both sides of equation (3), we get

$$0 = A + B \Rightarrow B = -A \Rightarrow B = -\frac{1}{2}$$

Equating the constant terms on both sides of equation (3), we get

$$0 = A - C \Rightarrow A = C \Rightarrow C = \frac{1}{2}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{x}{(x-1)(x^2+1)} = \frac{1/2}{x-1} + \frac{\left(-\frac{1}{2}\right)x + \frac{1}{2}}{x^2+1} = \frac{1}{2(x-1)} - \frac{1}{2}\left(\frac{x-1}{x^2+1}\right)$$

$$\therefore \quad I = \int \frac{x}{(x-1)(x^2+1)}\,dx = \int \left[\frac{1}{2(x-1)} - \frac{1}{2}\left(\frac{x-1}{x^2+1}\right)\right] dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

[Multiply and divide the second integral by 2]

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + c. \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c \quad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

(*iv*) Let $I = \int \frac{x^4+1}{x^3-x^2+x-1}\,dx$ ...(1)

Since, the degree of the numerator is greater than the degree of the denominator, therefore by actual division, we have

$$\therefore \quad I = \int \frac{x^4+1}{x^3-x^2+x-1}\,dx$$

$$= \int \left[x + 1 + \frac{2}{x^3-x^2+x-1}\right] dx \qquad ...(2)$$

$$\begin{array}{r|l} x^3 - x^2 + x - 1 \,)\; x^4 + 1 & (x+1 \\ x^4 - x^3 + x^2 - x & \\ - \quad + \quad - \quad + & \\ \hline x^3 - x^2 + x + 1 & \\ x^3 - x^2 + x - 1 & \\ - \quad + \quad - \quad + & \\ \hline 2 & \end{array}$$

Now $\frac{2}{x^3-x^2+x-1} = \frac{2}{(x-1)(x^2+1)}$ $\quad \left[\begin{array}{l} \because x^3 - x^2 + x - 1 \\ = x^2(x-1) + 1(x-1) \\ = (x-1)(x^2+1) \end{array}\right]$

Consider $\frac{2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$ ...(3)

Multiplying both sides by $(x-1)(x^2+1)$, we get

$$2 = A(x^2+1) + (Bx+C)(x-1) \qquad ...(4)$$

Put $x = 1$ in (4), we get

$$2 = A(1+1) + (B+C)(1-1) \Rightarrow 2 = 2A \Rightarrow A = 1.$$

Equating the co-efficients of x^2 on both sides of equation (4), we get

$$0 = A + B \Rightarrow B = -A \Rightarrow B = -1$$

Equating the constant terms on both sides of equation (4), we get

$$2 = A - C \Rightarrow C = A - 2 \Rightarrow C = -1.$$

Substituting the values of A, B and C in equation (3), we have

$$\frac{2}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{-x-1}{x^2+1}$$

$$\therefore \quad I = \int \left[x + 1 + \frac{2}{(x-1)(x^2+1)}\right] dx = \int \left[x + 1 + \frac{1}{x-1} - \frac{x+1}{x^2+1}\right] dx$$

$$= \int x\,dx + \int 1\,dx + \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$= \frac{x^2}{2} + x + \log|x-1| - \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$= \frac{x^2}{2} + x + \log|x-1| - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1^2}\,dx$$

[Multiplying and divided by 2]

$$= \frac{x^2}{2} + x + \log|x-1| - \frac{1}{2}\log|x^2+1| - \tan^{-1} x + c.$$

$$\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1} x + c\right]$$

Example 4. *Evaluate the following integrals :*

(i) $\displaystyle\int \frac{x^2 - x - 1}{x^3 - x^2 - 6x}\,dx$ (ii) $\displaystyle\int \frac{12x^2 - 2x - 9}{(4x^2 - 1)(x + 3)}\,dx$

(iii) $\displaystyle\int \frac{x^2 + x + 1}{x^2(x + 2)}\,dx$ (iv) $\displaystyle\int \frac{e^x}{e^{2x} - 4}\,dx$

(v) $\displaystyle\int \frac{x^2 + 8x + 4}{x^3 - 4x}\,dx$ (vi) $\displaystyle\int \frac{bx + c}{(x-p)(x-q)(x-r)}\,dx.$

Solution. (i) Let $I = \displaystyle\int \frac{x^2 - x - 1}{x^3 - x^2 - 6x}\,dx$ ...(1)

We have $\displaystyle\frac{x^2 - x - 1}{x^3 - x^2 - 6x} = \frac{x^2 - x - 1}{x(x^2 - x - 6)} = \frac{x^2 - x - 1}{x(x-3)(x+2)}$

Let $\displaystyle\frac{x^2 - x - 1}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+2)}$ ...(2)

Multiplying both sides by $x(x - 3)(x + 2)$, we get

$$x^2 - x - 1 = A(x-3)(x+2) + Bx(x+2) + Cx(x-3) \quad ...(3)$$

$$x = 0, x - 3 = 0 \Rightarrow x = 3, x + 2 = 0 \Rightarrow x = -2$$

Put $x = 0$ in (3), we get

$$-1 = A(0-3)(0+2) + B.0(0+2) + C.0(0-3)$$

$$\Rightarrow \quad -1 = -6A \quad \Rightarrow \quad A = \frac{1}{6}.$$

Put $x = 3$ in (3), we get

$$[(-3)^2 - 3 - 1] = A(3-3)(3+2) + 3B(3+2) + 3C(3-3)$$

$$\Rightarrow \quad 9 - 3 - 1 = 3B(5) \quad \Rightarrow \quad 5 = 15B \quad \Rightarrow \quad B = 1/3.$$

Put $x = -2$ in (3), we get

$$[(-2)^2 - (-2) - 1] = A(-2-3)(-2+2) + B(-2)(-2+2) + C(-2)(-2-3)$$

$$\Rightarrow \quad 4 + 2 - 1 = 10C \quad \Rightarrow \quad 5 = 10C \quad \Rightarrow \quad C = \frac{1}{2}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{x^2 - x - 1}{x(x-3)(x-2)} = \frac{1/6}{x} + \frac{1/3}{(x-3)} + \frac{1/2}{(x+2)} = \frac{1}{6x} + \frac{1}{3(x-3)} + \frac{1}{2(x+2)}$$

$$\therefore \quad I = \int \frac{x^2 - x - 1}{x(x-3)(x-2)}\,dx = \int \left[\frac{1}{6x} + \frac{1}{3(x-3)} + \frac{1}{2(x+2)}\right] dx$$

$$= \frac{1}{6}\int \frac{1}{x}\,dx + \frac{1}{3}\int \frac{1}{x-3}\,dx + \frac{1}{2}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{6}\log|x| + \frac{1}{3}\log|x-3| + \frac{1}{2}\log|x+2| + c$$

(*ii*) Let $\quad I = \int \dfrac{12x^2 - 2x - 9}{(4x^2-1)(x+3)}\,dx$

$$\Rightarrow \quad I = \int \frac{12x^2 - 2x - 9}{(2x+1)(2x-1)(x+3)}\,dx \qquad \text{...(1)}$$

Let $$\frac{12x^2 - 2x - 9}{(2x+1)(2x-1)(x+3)} = \frac{A}{(2x+1)} + \frac{B}{(2x-1)} + \frac{C}{(x+3)} \qquad \text{...(2)}$$

Multiplying both sides by $(2x+1)(2x-1)(x+3)$, we get

$$12x^2 - 2x - 9 = A(2x-1)(x+3) + B(2x+1)(x+3) + C(2x+1)(2x-1) \qquad \text{...(3)}$$

$$2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}, \qquad x + 3 = 0 \quad \Rightarrow \quad x = -3$$

$$2x + 1 = 0 \quad \Rightarrow \quad x = -\frac{1}{2}.$$

$\therefore$ Put $x = \dfrac{1}{2}$ in (3), we get

$$12\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) - 9 = A\left[2\left(\frac{1}{2}\right) - 1\right]\left(\frac{1}{2} + 3\right) + B\left[2\left(\frac{1}{2}\right) + 1\right]\left(\frac{1}{2} + 3\right) + C\left[2\left(\frac{1}{2}\right) + 1\right]\left[2\left(\frac{1}{2}\right) - 1\right]$$

$\Rightarrow \quad 3 - 1 - 9 = B\,(2)\left(\frac{7}{2}\right) \quad \Rightarrow \quad -7 = 7B \quad \Rightarrow \quad B = -1$

Put $x = \frac{-1}{2}$ in (3), we get

$$12\left(\frac{-1}{2}\right)^2 - 2\left(\frac{-1}{2}\right) - 9 = A\left[2\left(\frac{-1}{2}\right) - 1\right]\left(\frac{-1}{2} + 3\right) + B\left[2\left(\frac{-1}{2}\right) + 1\right]\left(\frac{-1}{2} + 3\right)$$
$$+ C\left[2\left(\frac{-1}{2}\right) + 1\right]\left[2\left(\frac{-1}{2}\right) - 1\right]$$

$\Rightarrow \quad 3 + 1 - 9 = A(-2)\left(\frac{5}{2}\right) \quad \Rightarrow \quad -5 = -5A \quad \Rightarrow \quad A = 1$

Put $x = -3$ in (3), we get

$$12(-3)^2 - 2(-3) - 9 = A[2(-3) - 1](-3 + 3) + B[2(-3) + 1](-3 + 3)$$
$$+ C[2(-3) + 1][2(-3) - 1]$$

$\Rightarrow \quad 108 + 6 - 9 = C(-5)(-7) \quad \Rightarrow \quad 105 = 35C \quad \Rightarrow \quad C = 3.$

Substituting the values of A, B and C in equation (2), we get

$$\frac{12x^2 - 2x - 9}{(2x+1)(2x-1)(x+3)} = \frac{1}{(2x+1)} + \frac{-1}{(2x-1)} + \frac{3}{x+3}$$

$$\therefore \quad I = \int \frac{12x^2 - 2x - 9}{(2x+1)(2x-1)(x+3)}\,dx = \int \left[\frac{1}{2x+1} - \frac{1}{2x-1} + \frac{3}{x+3}\right] dx$$

$$= \int \frac{1}{2x+1}\,dx - \int \frac{1}{2x-1}\,dx + 3\int \frac{1}{x+3}\,dx$$

$$= \frac{\log|2x+1|}{2} - \frac{\log|2x-1|}{2} + 3\log|x+3| + C \qquad \left[\because \int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c\right]$$

$$= \frac{1}{2}\log\left|\frac{2x+1}{2x-1}\right| + 3\log|x+3| + c. \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

(*iii*) Let $I = \int \frac{x^2 + x + 1}{x^2(x+2)}\,dx$ ...(1)

Let $\frac{x^2 + x + 1}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$ ...(2)

Multiplying both sides by $x^2(x + 2)$, we get

$$x^2 + x + 1 = Ax(x + 2) + B(x + 2) + Cx^2 \qquad ...(3)$$

$$x = 0$$

$$x + 2 = 0 \quad \Rightarrow \quad x = -2$$

Put $x = 0$ in equation (3), we get

$$1 = 0 + B(0 + 2) + 0 \quad \Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$$

Put $x = -2$ in equation (3), we get

$$(-2)^2 + (-2) + 1 = A(-2)(-2 + 2) + B(-2 + 2) + C(-2)^2$$

$\Rightarrow \quad 4 - 2 + 1 = 4C \Rightarrow C = \dfrac{3}{4}$

Equating co-efficients of x^2 on both sides of (3), we have

$$1 = A + C \Rightarrow A = 1 - C \Rightarrow A = 1 - \frac{3}{4} \Rightarrow A = \frac{1}{4}.$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{x^2+x+1}{x^2(x+2)} = \frac{1/4}{x} + \frac{1/2}{x^2} + \frac{3/4}{x+2} = \frac{1}{4x} + \frac{1}{2x^2} + \frac{3}{4(x+2)}$$

$$\therefore \quad I = \int \frac{x^2+x+1}{x^2(x+2)}\,dx = \int \left[\frac{1}{4x} + \frac{1}{2x^2} + \frac{3}{4(x+2)}\right] dx$$

$$= \frac{1}{4}\int \frac{1}{x}\,dx + \frac{1}{2}\int x^{-2}\,dx + \frac{3}{4}\int \frac{1}{x+2}\,dx \qquad \left[\because \int [f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= \frac{1}{4}\log|x| + \frac{1}{2}\frac{(x^{-2+1})}{(-2+1)} + \frac{3}{4}\log|x+2| + c \qquad \left[\because \int \frac{1}{x}\,dx = \log|x| + c\right]$$

$$= \frac{1}{4}\log|x| - \frac{1}{2x} + \frac{3}{4}\log|x+2| + c. \qquad \left[\because \int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c\right]$$

(*iv*) Let $\quad I = \displaystyle\int \frac{e^x}{e^{2x}-4}\,dx$

Put $\quad z = e^x \Rightarrow dz = e^x\,dx$

$$\therefore \quad I = \int \frac{1}{z^2-4}\,dz = \int \frac{1}{(z-2)(z+2)}\,dz \qquad \text{...(1)}$$

Let
$$\frac{1}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2} \qquad \text{...(2)}$$

Multiplying both sides by $(z-2)(z+2)$, we get

$$1 = A(z+2) + B(z-2) \qquad \text{...(3)}$$

$z - 2 = 0 \Rightarrow z = 2$

$z + 2 = 0 \Rightarrow z = -2$

Put $z = -2$ in (3), we get

$$1 = A(-2+2) + B(-2-2) \Rightarrow 1 = -4B \Rightarrow B = -\frac{1}{4}.$$

Put $z = 2$ in (3), we get

$$1 = A(2+2) + B(2-2) \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$$

Substituting the values of A and B in (2), we get

$$\frac{1}{(z-2)(z+2)} = \frac{1/4}{(z-2)} + \frac{-1/4}{(z+2)} = \frac{1}{4(z-2)} - \frac{1}{4(z+2)}$$

$$\therefore \quad I = \int \frac{1}{(z-2)(z+2)}\,dz = \int \left[\frac{1}{4(z-2)} - \frac{1}{4(z+2)}\right] dz$$

$$= \frac{1}{4}\int \frac{1}{z-2}dz - \frac{1}{4}\int \frac{1}{z+2}dz = \frac{1}{4}\log|z-2| - \frac{1}{4}\log|z+2| + c$$

$$= \frac{1}{4}\log\left|\frac{z-2}{z+2}\right| + c \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$[\because z = e^x]$$

$$= \frac{1}{4}\log\left|\frac{e^x-2}{e^x+2}\right| + c.$$

Alternatively :

Let $\quad I = \int \frac{e^x}{e^{2x}-4}dx$

Put $\quad e^x = z \Rightarrow e^x\,dx = dz$

$\therefore \quad I = \int \frac{1}{z^2-4}dz = \int \frac{1}{z^2-2^2}dz$

$$= \frac{1}{2(2)}\log\left|\frac{z-2}{z+2}\right| + c \qquad \left[\because \text{By using} \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{4}\log\left|\frac{e^x-2}{e^x+2}\right| + c. \qquad [\because z = e^x]$$

(*v*) Let $\quad I = \int \frac{x^2+8x+4}{x^3-4x}dx = \int \frac{x^2+8x+4}{x(x^2-4)}dx = \int \frac{x^2+8x+4}{x(x+2)(x-2)}dx \quad ...(1)$

Let $\quad \frac{x^2+8x+4}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \quad ...(2)$

Multiplying both sides by $x(x+2)(x-2)$, we get

$$x^2 + 8x + 4 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2) \quad ...(3)$$

$$x = 0, x-2 = 0 \Rightarrow x = 2, x+2 = 0 \Rightarrow x = -2$$

Put $x = 0$ in (3), we get

$$4 = A(0+2)(0-2) + 0 + 0 \Rightarrow 4 = -4A \Rightarrow A = -1$$

Put $x = 2$ in (3), we get

$$(2)^2 + 8(2) + 4 = A(2+2)(2-2) + B.2(2-2) + C.2(2+2)$$

$$\Rightarrow \quad 4 + 16 + 4 = 8C \Rightarrow 24 = 8C \Rightarrow C = 3.$$

Put $x = -2$ in (3), we get

$$(-2)^2 + 8(-2) + 4 = A(-2+2)(-2-2) + B(-2)(-2-2) + C(-2)(-2+2)$$

$$\Rightarrow \quad 4 - 16 + 4 = 8B \Rightarrow -8 = 8B \Rightarrow B = -1.$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{x^2+8x+4}{x(x^2-4)} = \frac{-1}{x} + \frac{-1}{x+2} + \frac{3}{x-2}$$

$$\therefore \quad I = \int \frac{x^2+8x+4}{x(x^2-4)}dx = \int \left[-\frac{1}{x} - \frac{1}{x+2} + \frac{3}{x-2}\right]dx$$

$$= -\int \frac{1}{x}dx - \int \frac{1}{x+2}dx + 3\int \frac{1}{x-2}dx$$

$$= -\log|x| - \log|x+2| + 3\log|x-2| + c.$$

(*vi*) Let $\quad I = \int \frac{bx + c}{(x - p)(x - q)(x - r)}\, dx \quad$ ...(1)

Let $\quad \frac{bx + c}{(x - p)(x - q)(x - r)} = \frac{A}{x - p} + \frac{B}{x - q} + \frac{C}{x - r} \quad$ ...(2)

Multiplying both sides by $(x - p)(x - q)(x - r)$, we get

$(bx + c) = A(x - q)(x - r) + B(x - p)(x - r) + C(x - p)(x - q) \quad$ ...(3)

$$x - p = 0 \Rightarrow x = p$$
$$x - q = 0 \Rightarrow x = q$$
$$x - r = 0 \Rightarrow x = r$$

Put $x = p$ in (3), we get

$$bp + c = A(p - q)(p - r) + B(p - p)(p - r) + C(p - p)(p - q)$$
$$\Rightarrow \quad bp + c = A(p - q)(p - r)$$
$$\Rightarrow \quad A = \frac{bp + c}{(p - q)(p - r)}$$

Put $x = q$ in (3), we get

$$bq + c = A(q - q)(q - r) + B(q - p)(q - r) + C(q - p)(q - q)$$
$$\Rightarrow \quad bq + c = B(q - p)(q - r)$$
$$\Rightarrow \quad B = \frac{bq + c}{(q - p)(q - r)}$$

Put $x = r$ in (3), we get

$$br + c = A(r - q)(r - r) + B(r - p)(r - r) + C(r - p)(r - q)$$
$$\Rightarrow \quad br + c = C(r - p)(r - q)$$
$$\Rightarrow \quad C = \frac{br + c}{(r - p)(r - q)}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{bx + c}{(x - p)(x - q)(x - r)} = \frac{\frac{bp + c}{(p - q)(p - r)}}{x - p} + \frac{\frac{bq + c}{(q - p)(q - r)}}{x - q} + \frac{\frac{br + c}{(r - p)(r - q)}}{(x - r)}$$

$$\therefore \quad I = \int \frac{bx + c}{(x - p)(x - q)(x - r)}\, dx = \int \left[\frac{(bp + c)}{(p - q)(p - r)(x - p)} + \frac{(bq + c)}{(q - p)(q - r)(x - q)} + \frac{(br + c)}{(r - p)(r - q)(x - r)} \right] . dx$$

$$= \frac{bp + c}{(p - q)(p - r)} \int \frac{1}{x - p}\, dx + \frac{bq + c}{(q - p)(q - r)} . \int \frac{1}{x - q}\, dx + \frac{br + c}{(r - p)(r - q)} \int \frac{1}{x - r}\, dx$$

$$= \frac{bp + c}{(p - q)(p - r)} \log |x - p| + \frac{bq + c}{(q - p)(q - r)} \log |x - q| + \frac{br + c}{(r - p)(r - q)} \log |x - r| + c.$$

Example 5. *Evaluate the following integrals* :

(*i*) $\int \frac{2x}{(x^2 + 1)(x^2 + 3)}\, dx \qquad$ (*ii*) $\int \frac{x^3}{(x - 1)(x - 2)}\, dx$

(*iii*) $\int \frac{8}{(x + 2)(x^2 + 4)}\, dx \qquad$ (*iv*) $\int \frac{x^3 + x + 1}{x^2 - 1}\, dx$

(*v*) $\int \frac{2}{(1 - x)(1 + x^2)}\, dx \qquad$ (*vi*) $\int \frac{x^2 + x + 1}{(x - 1)^3}\, dx.$

Solution. (*i*) Let $\quad I = \int \frac{2x}{(x^2+1)(x^2+3)}\, dx$ ..(1)

Put $\quad x^2 = z \Rightarrow 2x\, dx = dz$

$\therefore \quad I = \int \frac{1}{(z+1)(z+3)}\, dz$

Let $\quad \frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$ ...(2)

Multiplying both sides by $(z+1)(z+3)$, we get

$$1 = A(z+3) + B(z+1) \quad ...(3)$$

$$z+1=0 \Rightarrow z=-1, z+3=0 \Rightarrow z=-3$$

Put $z = -1$ in (3), we get

$$1 = A(-1+3) + B(-1+1) \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

Put $z = -3$ in (3), we get

$$1 = A(-3+3) + B(-3+1) \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$$

Substituting the values of A and B in equation (2), we have

$$\frac{1}{(z+1)(z+3)} = \frac{1/2}{(z+1)} + \frac{-1/2}{(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$\therefore \quad I = \int \frac{1}{(z+1)(z+3)}\, dz = \frac{1}{2}\int \frac{1}{z+1}\, dz - \frac{1}{2}\int \frac{1}{z+3}\, dz$$

$$= \frac{1}{2}\log|z+1| - \frac{1}{2}\log|z+3| + c$$

$$= \frac{1}{2}\log\left|\frac{z+1}{z+3}\right| + c \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + c. \qquad [\because z = x^2]$$

(*ii*) Let $\quad I = \int \frac{x^3}{(x-1)(x-2)}\, dx = \int \frac{x^3}{x^2-3x+2}\, dx$ ...(1)

Here the degree of numerator is greater than the degree of denominator. Therefore, dividing the numerator by the denominator.

$$\therefore \int \frac{x^3}{x^2-3x+2}\, dx = \int \left[(x+3) + \frac{7x-6}{x^2-3x+2}\right] dx$$

$$I = \int \left[(x+3) + \frac{7x-6}{(x-1)(x-2)}\right] dx \quad ...(2)$$

Let $\frac{7x-6}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$ ...(3)

$$\begin{array}{r|l} x^2-3x+2 & x^3 \quad (x+3 \\ & +x^3 - 3x^2 + 2x \\ & - \quad + \quad - \\ \hline & 3x^2 - 2x \\ & +3x^2 - 9x + 6 \\ & - \quad + \quad - \\ \hline & 7x - 6 \end{array}$$

$[\because (x^2-3x+2) = (x-1)(x-2)]$

Multiplying both sides by $(x-1)(x-2)$, we get

$$7x - 6 = A(x-2) + B(x-1) \quad \ldots(4)$$

$$x - 1 = 0 \Rightarrow x = 1, x - 2 = 0 \Rightarrow x = 2$$

Put $x = 1$ in (4), we get

$$7(1) - 6 = A(1-2) + B(1-1) \Rightarrow 1 = -A \Rightarrow A = -1$$

Put $x = 2$ in (4), we get

$$7(2) - 6 = A(2-2) + B(2-1) \Rightarrow 8 = B \Rightarrow B = 8$$

Substituting the values of A and B in equation (3), we get

$$\frac{7x-6}{x^2-3x+2} = \frac{-1}{x-1} + \frac{8}{x-2}$$

$$\therefore \quad I = \int \left[(x+3) - \frac{1}{x-1} + \frac{8}{x-2}\right] dx$$

$$= \int x\,dx + 3\int 1.dx - 1\int \frac{1}{x-1}dx + 8\int \frac{1}{x-2}dx$$

$$= \frac{x^2}{2} + 3x - \log|x-1| + 8\log|x-2| + c.$$

(iii) Let $\quad I = \int \frac{8}{(x+2)(x^2+4)}dx \quad \ldots(1)$

Let $\frac{8}{(x+2)(x^2+4)} = \frac{A}{(x+2)} + \frac{Bx+C}{(x^2+4)} \quad \ldots(2)$

Multiplying both sides by $(x+2)(x^2+4)$, we get

$$8 = A(x^2+4) + (Bx+C)(x+2) \quad \ldots(3)$$

$$x + 2 = 0 \Rightarrow x = -2$$

Put $x = -2$ in (3), we get

$$8 = A[(-2)^2 + 4] + [B(-2) + C][-2+2] \Rightarrow 8 = 8A \Rightarrow A = 1$$

Equating the co-efficients of x^2 on both sides of equation (3), we have

$$0 = A + B \Rightarrow B = -A \Rightarrow B = -1$$

Equating the constant terms on both sides of equation (3), we have

$$8 = 4A + 2C \Rightarrow 2C = 8 - 4 \Rightarrow 2C = 4 \Rightarrow C = 2.$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{-x+2}{x^2+4}$$

$$\therefore \quad I = \int \frac{8}{(x+2)(x^2+4)}dx = \int \left(\frac{1}{x+2} + \frac{2-x}{x^2+4}\right)dx$$

$$= \int \frac{1}{x+2}dx + 2\int \frac{1}{x^2+4}dx - \int \frac{x}{x^2+4}dx$$

$$= \int \frac{1}{x+2}dx + 2\int \frac{1}{x^2+2^2}dx - \frac{1}{2}\int \frac{2x}{x^2+4}dx$$

[Multiply and divide the third integral by 2]

$$= \log|x+2| + 2 \cdot \frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{1}{2}\log|x^2+4| + c$$

$$\left[\begin{array}{l} \because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c \\ \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c \end{array}\right]$$

$$= \log|x+2| + \tan^{-1}\frac{x}{2} - \frac{1}{2}\log|x^2+4| + c.$$

(*iv*) Let $\quad I = \int \frac{x^3+x+1}{x^2-1}\,dx$

Here the degree of numerator is greater than the degree of denominator. Therefore, dividing the numerator by the denominator.

$$\therefore \qquad I = \int \left(x + \frac{2x+1}{x^2-1}\right) dx \qquad \text{...(1)}$$

$$\begin{array}{r|l} x^2-1 & x^3+x+1 \;(\,x \\ & +x^3-x \\ & -\quad + \\ \hline & 2x+1 \end{array}$$

Let $\quad \frac{2x+1}{x^2-1} = \frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \qquad \text{...(2)}$

Multiplying both sides by $(x+1)(x-1)$, we get

$$2x+1 = A(x-1) + B(x+1) \qquad \text{...(3)}$$

$$x - 1 = 0 \quad \Rightarrow \quad x = 1$$

$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$

Put $x = 1$ in (3), we get

$$2(1) + 1 = A(1-1) + B(1+1) \;\Rightarrow\; 3 = 2B \;\Rightarrow\; B = \frac{3}{2}.$$

Put $x = -1$ in (3), we get

$$2(-1) + 1 = A(-1-1) + B(-1+1) \;\Rightarrow\; -1 = -2A \;\Rightarrow\; A = \frac{1}{2}$$

Substituting the values of A and B in equation (2), we have

$$\frac{2x+1}{(x+1)(x-1)} = \frac{1/2}{(x+1)} + \frac{3/2}{(x-1)} = \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

$$\therefore \qquad I = \int \left[x + \frac{1}{2(x+1)} + \frac{3}{2(x-1)}\right] dx$$

$$= \int x \cdot dx + \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{3}{2}\int \frac{1}{x-1}\,dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + c.$$

(*v*) Let $\qquad I = \int \frac{2}{(1-x)(1+x^2)}\,dx \qquad \text{...(1)}$

Let $\qquad \frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{(1+x^2)} \qquad \text{...(2)}$

Multiplying both sides by $(1-x)(1+x^2)$, we get

$$2 = A(1+x^2) + (Bx+C)(1-x) \quad ...(3)$$

$$1-x=0 \Rightarrow x=1$$

Put $x = 1$ in (3), we get

$$2 = A(1+1) + (B.1+C)(1-1) \Rightarrow 2 = 2A \Rightarrow A = 1$$

Equating the co-efficients of x^2 on both sides of equation (3), we have

$$0 = A - B \Rightarrow A = B \Rightarrow B = 1$$

Equating the constant terms on both sides of equation (3), we get

$$2 = A + C \Rightarrow C = 2 - A \Rightarrow C = 2 - 1 \Rightarrow C = 1$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\therefore \quad I = \int \frac{2}{(1-x)(1+x^2)}\,dx = \int \left(\frac{1}{1-x} + \frac{x+1}{1+x^2}\right) dx$$

$$= \int \frac{1}{1-x}\,dx + \int \frac{x+1}{1+x^2}\,dx$$

$$= \int \frac{1}{1-x}\,dx + \int \frac{x}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

$$= \int \frac{1}{1-x}\,dx + \frac{1}{2}\int \frac{2x}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

[Multiply and divide the IInd integral by 2]

$$= \log|1-x| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + c.$$

$$\left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c \qquad \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

(*vi*) Let $\quad I = \int \frac{x^2+x+1}{(x-1)^3}\,dx \quad ...(1)$

Let $\quad \frac{x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \quad ...(2)$

Multiplying both sides by $(x-1)^3$, we get

$$x^2 + x + 1 = A(x-1)^2 + B(x-1) + C \quad ...(3)$$

$$x - 1 = 0 \Rightarrow x = 1$$

Put $x = 1$ in (3), we get

$$[(1)^2 + 1 + 1] = A(1-1)^2 + B(1-1) + C \Rightarrow 3 = C \Rightarrow C = 3$$

Equating the co-efficients of x^2 on both sides of equation (3), we have

$$1 = A \Rightarrow A = 1$$

Equating the constant terms on both sides of equation (3), we have

$$1 = A - B + C \Rightarrow 1 = 1 - B + 3 \Rightarrow B = 3.$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{x^2+x+1}{(x-1)^3}=\frac{1}{x-1}+\frac{3}{(x-1)^2}+\frac{3}{(x-1)^3}$$

$$\therefore \quad I=\int\frac{x^2+x+1}{(x-1)^3}dx=\int\left[\frac{1}{x-1}+\frac{3}{(x-1)^2}+\frac{3}{(x-1)^3}\right]dx$$

$$=\int\frac{1}{x-1}dx+3\int\frac{1}{(x-1)^2}dx+3\int\frac{1}{(x-1)^3}dx$$

$$=\int\frac{1}{x-1}dx+3\int(x-1)^{-2}dx+3\int(x-1)^{-3}dx$$

$$=\log|x-1|+3\frac{(x-1)^{-2+1}}{(-2+1)}+3\frac{(x-1)^{-3+1}}{(-3+1)}+c$$

$$\left[\because \int[f(x)]^n\,dx=\left|\frac{[f(x)]^{n+1}}{n+1}\right|+c\right]$$

$$=\log|x-1|-\frac{3}{x-1}-\frac{3}{2(x-1)^2}+c.$$

Example 6. *Evaluate the following integrals :*

(i) $\int\frac{x^2+x-1}{x^2+x-6}dx$ (ii) $\int\frac{x^3}{(x-1)(x-2)(x-3)}dx$

(iii) $\int\frac{3x-2}{(x+3)(x+1)^2}dx$ (iv) $\int\frac{x^2}{(x^2+1)(x^2+2)(x^2+3)}dx.$

Solution. (i) Let $I=\int\frac{x^2+x-1}{x^2+x-6}dx$ ...(1)

Since the integrand is an improper fraction, therefore by dividing numerator with the denominator, we get

$$I=\int\frac{x^2+x-1}{x^2+x-6}dx=\int\left[1+\frac{5}{x^2+x-6}\right]dx \quad ...(2)$$

$$\begin{array}{r|l} x^2+x-6 & x^2+x-1 \;(1 \\ & +x^2+x-6 \\ & -\quad-\quad+ \\ \hline & 5 \end{array}$$

We have $\frac{5}{x^2+x-6}=\frac{5}{x^2+3x-2x-6}=\frac{5}{(x+3)(x-2)}$

Let $\frac{5}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}$ ...(3)

Multiplying both sides by $(x+3)(x-2)$, we get

$$5=A(x-2)+B(x+3) \quad ...(4)$$

$$x+3=0 \Rightarrow x=-3, x-2=0 \Rightarrow x=2$$

Put $x=-3$ in (4), we get

$$5=A(-3-2)+B(-3+3) \Rightarrow 5=-5A \Rightarrow A=-1$$

Put $x=2$ in (4), we get

$$5=A(2-2)+B(2+3) \Rightarrow 5=5B \Rightarrow B=1$$

Substituting the values of A and B in equation (3), we have

$$\frac{5}{(x+3)(x-2)}=\left(\frac{-1}{x+3}+\frac{1}{x-2}\right)$$

$\therefore \quad I = \int \frac{x^2+x-1}{x^2+x-6}\,dx = \int \left[1+\frac{5}{x^2+x-6}\right] dx$

$= \int \left[1+\left(\frac{-1}{x+3}\right)+\left(\frac{1}{x-2}\right)\right] dx$

$= \int 1.\,dx - \int \frac{1}{x+3}\,dx + \int \frac{1}{x-2}\,dx$

$= x - \log|x+3| + \log|x-2| + c$

$\therefore \quad I = x + \log\left|\frac{x-2}{x+3}\right| + c. \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$

(*ii*) Let $\quad I = \int \frac{x^3}{(x-1)(x-2)(x-3)}\,dx$

or $\quad I = \int \frac{x^3}{(x^3-6x^2+11x-6)}\,dx$

Since the integrand is an improper fraction, therefore, by dividing the numerator with the denominator, we get

$$\begin{array}{r|l} x^3-6x^2+11x-6 & x^3 \quad (1 \\ & \underline{\overset{-}{x^3} \overset{+}{-6x^2} \overset{-}{+11x} \overset{+}{-6}} \\ & 6x^2-11x+6 \end{array}$$

$I = \int \left[1+\frac{6x^2-11x+6}{(x-1)(x-2)(x-3)}\right] dx$

or $\quad I = \int 1.\,dx + \int \frac{6x^2-11x+6}{(x-1)(x-2)(x-3)}\,dx \quad$ ...(1)

Let $\quad \frac{6x^2-11x+6}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3} \quad$ ...(2)

Multiplying both sides by $(x-1)(x-2)(x-3)$, we get

$6x^2-11x+6 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad$ ...(3)

$x-1=0 \Rightarrow x=1, x-2=0 \Rightarrow x=2, x-3=0 \Rightarrow x=3.$

Put $x = 1$ in (3), we get

$[6(1)^2 - 11(1) + 6] = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$

$\Rightarrow \quad 6-11+6 = 2A \Rightarrow A = 1/2$

Put $x = 2$ in (3), we get

$[6(2)^2 - 11(2) + 6] = A(2-2)(2-3) + B(2-1)(2-3) + C(2-1)(2-2)$

$\Rightarrow \quad (24-22+6) = -B \Rightarrow B = -8$

Put $x = 3$ in (3), we get

$[6(3)^2 - 11(3) + 6] = A(3-2)(3-3) + B(3-1)(3-3) + C(3-1)(3-2)$

$\Rightarrow \quad (54-33+6) = 2C \Rightarrow C = \frac{27}{2}$

Substituting the values of A, B and C in equation (2), we have

$$\frac{6x^2-11x+6}{(x-1)(x-2)(x-3)} = \frac{1/2}{x-1}+\frac{-8}{x-2}+\frac{27/2}{x-3} = \frac{1}{2(x-1)}-\frac{8}{x-2}+\frac{27}{2(x-3)}$$

$$\therefore \quad I = \int 1 . dx + \int \left[\frac{1}{2(x-1)} - \frac{8}{x-2} + \frac{27}{2(x-3)} \right] dx$$

$$= \int 1 . dx + \frac{1}{2} \int \frac{1}{x-1} dx - 8 \int \frac{1}{x-2} dx + \frac{27}{2} \int \frac{1}{x-3} dx$$

$$= x + \frac{1}{2} \log |x-1| - 8 \log |x-2| + \frac{27}{2} \log |x-3| + c.$$

(*iii*) Let $I = \int \frac{(3x-2)}{(x+3)(x+1)^2} dx$ ...(1)

Let $\frac{(3x-2)}{(x+3)(x+1)^2} = \frac{A}{x+3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ ..(2)

Multiplying both sides by $(x + 3)(x + 1)^2$, we get

$$3x - 2 = A(x + 1)^2 + B(x + 1)(x + 3) + C(x + 3) \quad ...(3)$$

$$x + 1 = 0 \Rightarrow x = -1, x + 3 = 0 \Rightarrow x = -3$$

Put $x = -1$ in (3), we get

$$[3(-1) - 2] = A[(-1) + 1]^2 + B[-1 + 1](-1 + 3) + C(-1 + 3)$$

$$\Rightarrow \quad -5 = 2C \Rightarrow C = -\frac{5}{2}$$

Put $x = -3$ in (3), we get

$$[3(-3) - 2] = A(-3 + 1)^2 + B(-3 + 1)(-3 + 3) + C(-3 + 3)$$

$$\Rightarrow \quad -11 = 4A \Rightarrow A = -\frac{11}{4}$$

Equating the co-efficients of x^2 on both sides of equation (3), we get

$$0 = A + B \Rightarrow B = -A \Rightarrow B = \frac{11}{4}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{3x-2}{(x+3)(x+1)^2} = \frac{-11/4}{x+3} + \frac{11/4}{x+1} + \frac{-5/2}{(x+1)^2} = \frac{-11}{4(x+3)} + \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2}$$

$$\therefore \quad I = \int \frac{3x-2}{(x+3)(x+1)^2} dx = \int \left[\frac{-11}{4(x+3)} + \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2} \right] dx$$

$$= -\frac{11}{4} \int \frac{1}{x+3} dx + \frac{11}{4} \int \frac{1}{x+1} dx - \frac{5}{2} \int \frac{1}{(x+1)^2} dx$$

$$= -\frac{11}{4} \log |x+3| + \frac{11}{4} \log |x+1| - \frac{5}{2} \frac{(x+1)^{-2+1}}{(-2+1)} + c$$

$$= \frac{11}{4} \log \left| \frac{x+1}{x+3} \right| + \frac{5}{2(x+1)} + c. \qquad \left[\because \log m - \log n = \log \left(\frac{m}{n} \right) \right]$$

(*iv*) Let $I = \int \frac{x^2}{(x^2+1)(x^2+2)(x^2+3)} dx$ ...(1)

Let $\frac{x^2}{(x^2+1)(x^2+2)(x^2+3)} = \frac{z}{(z+1)(z+2)(z+3)} = \frac{A}{(z+1)} + \frac{B}{(z+2)} + \frac{C}{(z+3)}$

...(2) [Put $x^2 = z$]

Multiplying both sides by $(z + 1)(z + 2)(z + 3)$, we get

$$z = A(z + 2)(z + 3) + B(z + 1)(z + 3) + C(z + 1)(z + 2) \quad ...(3)$$

$$z + 1 = 0 \Rightarrow z = -1, z + 2 = 0 \Rightarrow z = -2, z + 3 = 0 \Rightarrow z = -3$$

Put $z = -1$ in (3), we get

$$-1 = A(-1 + 2)(-1 + 3) + B(-1 + 1)(-1 + 3) + C(-1 + 1)(-1 + 2)$$

$$\Rightarrow \quad -1 = 2A \Rightarrow A = -\frac{1}{2}$$

Put $z = -2$ in (3), we get

$$-2 = A(-2 + 2)(-2 + 3) + B(-2 + 1)(-2 + 3) + C(-2 + 1)(-2 + 2)$$

$$\Rightarrow \quad -2 = -B \Rightarrow B = 2$$

Put $z = -3$ in (3), we get

$$-3 = A(-3 + 2)(-3 + 3) + B(-3 + 1)(-3 + 3) + C(-3 + 1)(-3 + 2)$$

$$\Rightarrow \quad -3 = 2C \Rightarrow C = -\frac{3}{2}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{z}{(z+1)(z+2)(z+3)} = \frac{-1/2}{(z+1)} + \frac{2}{(z+2)} + \frac{-3/2}{(z+3)}$$

or

$$\frac{x^2}{(x^2+1)(x^2+2)(x^2+3)} = -\frac{1}{2(x^2+1)} + \frac{2}{(x^2+2)} - \frac{3}{2(x^2+3)} \quad [\because z = x^2]$$

$$\therefore \quad I = \int \frac{x^2}{(x^2+1)(x^2+2)(x^2+3)}\,dx = \int \left[-\frac{1}{2(x^2+1)} + \frac{2}{(x^2+2)} - \frac{3}{2(x^2+3)}\right] dx$$

$$= -\frac{1}{2}\int \frac{1}{x^2+1}\,dx + 2\int \frac{1}{x^2+(\sqrt{2})^2}\,dx - \frac{3}{2}\int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$\left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= -\frac{1}{2}\tan^{-1}x + 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{3}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + c$$

$$= -\frac{1}{2}\tan^{-1}x + \sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{\sqrt{3}}{2}\tan^{-1}\frac{x}{\sqrt{3}} + c.$$

Example 7. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{(x+1)^2(x^2+1)}\,dx$ (*ii*) $\int \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

(*iii*) $\int \frac{5x^3+18x^2-10x-6}{x(x+3)(5x-2)}\,dx$ (*iv*) $\int \frac{1}{x^4-1}\,dx$

(*v*) $\int \frac{1}{x(x^4-1)}\,dx$ (*vi*) $\int \frac{x^2}{x^2-4x+3}\,dx.$

Solution. (*i*) Let $I = \int \frac{1}{(x+1)^2(x^2+1)}\,dx \quad ...(1)$

Let $$\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)} \qquad ...(2)$$

Multiplying both sides by $(x+1)^2(x^2+1)$, we get

$$1 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 \qquad ...(3)$$

Put $x = -1$ in (3), we get

$$1 = A[-1+1][(-1)^2+1] + B[(-1)^2+1] + [C(-1)+D][-1+1]^2$$

$$\Rightarrow \quad 1 = B(1+1) \;\Rightarrow\; 1 = 2B \;\Rightarrow\; B = \frac{1}{2}.$$

Equating co-efficients of x^3 on both sides of equation (3), we have

$$0 = A + C \;\Rightarrow\; A + C = 0 \qquad ...(4)$$

Equating co-efficients of x^2 on both sides of equation (3), we have

$$0 = A + B + 2C + D \qquad ...(5)$$

Equating co-efficients of x on both sides of equation (3), we have

$$0 = A + C + 2D \qquad ...(6)$$

Solving equations (4), (5) and (6), we have

$$A = \frac{1}{2},\; C = -\frac{1}{2},\; D = 0.$$

Substituting the values of A, B, C and D in equation (2), we have

$$\frac{1}{(x+1)^2(x^2+1)} = \frac{1/2}{(x+1)} + \frac{1/2}{(x+1)^2} + \frac{-\frac{1}{2}x+0}{(x^2+1)} = \frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} - \frac{x}{2(x^2+1)}$$

$$\therefore \quad I = \int \frac{1}{(x+1)^2(x^2+1)}\,dx = \int \left[\frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} - \frac{x}{2(x^2+1)}\right] dx$$

$$= \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{(x+1)^2}\,dx - \frac{1}{2}\int \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int (x+1)^{-2}\,dx - \frac{1}{4}\int \frac{2x}{x^2+1}\,dx$$

[Multiply and divide the third integral by 2]

$$= \frac{1}{2}\log|x+1| + \frac{1}{2}\frac{(x+1)^{-2+1}}{(-2+1)} - \frac{1}{4}\log|x^2+1| + c$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\log|x^2+1| + c. \qquad \left[\begin{array}{l} \because \displaystyle\int [f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c \\ \displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c \end{array}\right]$$

(*ii*) Let $$I = \int \frac{x^2}{(x^2+1)(x^2+4)}\,dx \qquad ...(1)$$

Putting $x^2 = z$

$\therefore$ We have $$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{z}{(z+1)(z+4)}$$

Let $\dfrac{z}{(z+1)(z+4)} = \dfrac{A}{(z+1)} + \dfrac{B}{(z+4)}$ ...(2)

Multiplying both sides by $(z + 1)(z + 4)$, we get

$$z = A(z+4) + B(z+1) \quad ...(3)$$

$$z + 1 = 0 \Rightarrow z = -1$$

$$z + 4 = 0 \Rightarrow z = -4$$

Put $z = -1$ in (3), we get

$$-1 = A(-1+4) + B(-1+1) \Rightarrow -1 = 3A \Rightarrow A = -\frac{1}{3}$$

Put $z = -4$ in (3), we get

$$-4 = A(-4+4) + B(-4+1) \Rightarrow -4 = -3B \Rightarrow B = \frac{4}{3}$$

Substituting the values of A and B in equation (2), we have

$$\frac{z}{(z+1)(z+4)} = \frac{-1/3}{(z+1)} + \frac{4/3}{(z+4)} = \frac{-1}{3(z+1)} + \frac{4}{3(z+4)}$$

$$\Rightarrow \quad \frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)} \qquad [\because\ x^2 = z]$$

$$\therefore \quad I = \int \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \int \left(-\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}\right) dx$$

$$= -\frac{1}{3}\int \frac{1}{x^2+1^2}\,dx + \frac{4}{3}\int \frac{1}{x^2+2^2}\,dx$$

$$= -\frac{1}{3}\tan^{-1} x + \frac{4}{3}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + c \qquad \left[\because\ \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= -\frac{1}{3}\tan^{-1} x + \frac{2}{3}\tan^{-1}\frac{x}{2} + c.$$

(*iii*) Let $I = \displaystyle\int \frac{5x^3 + 18x^2 - 10x - 6}{x(x+3)(5x-2)}\,dx$

$$= \int \frac{5x^3 + 18x^2 - 10x - 6}{(5x^3 + 13x^2 - 6x)}\,dx \quad ...(1)$$

Since the integrand is not a proper fraction, therefore by actual division, we have

$$\therefore \quad \int \frac{5x^3 + 18x^2 - 10x - 6}{5x^3 + 13x^2 - 6x}\,dx$$

$$= \int \left(1 + \frac{5x^2 - 4x - 6}{5x^3 + 13x^2 - 6x}\right) dx \quad ...(2)$$

$$\begin{array}{r} 5x^3 + 13x^2 - 6x \,\big)\, 5x^3 + 18x^2 - 10x - 6 \,\big(\, 1 \\ 5x^3 + 13x^2 - 6x \\ - \quad - \quad + \\ \hline 5x^2 - 4x - 6 \end{array}$$

Let $\dfrac{5x^2 - 4x - 6}{x(x+3)(5x-2)} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{5x-2}$ ...(3)

Multiplying both sides by $x(x + 3)(5x - 2)$, we have

$$5x^2 - 4x - 6 = A(x+3)(5x-2) + Bx(5x-2) + Cx(x+3) \quad ...(4)$$

$$x = 0,\ x + 3 = 0 \Rightarrow x = -3$$

$$5x - 2 = 0 \quad \Rightarrow \quad 5x = +2 \quad \Rightarrow \quad x = +\frac{2}{5}$$

Put $x = 0$ in (4), we get

$$-6 = A(0 + 3)(0 - 2) + 0 + 0 \quad \Rightarrow \quad -6 = -6A \quad \Rightarrow \quad A = 1$$

Put $x = -3$ in (4), we get

$$5(-3)^2 - 4(-3) - 6 = A(-3 + 3)[5(-3) - 2] + B(-3)[5(-3) - 2] + C(-3)[-3 + 3]$$

$$\Rightarrow \quad 45 + 12 - 6 = 51B \quad \Rightarrow \quad 51 = 51B \quad \Rightarrow \quad B = 1$$

Put $x = \frac{2}{5}$ in (4), we get

$$5\left(\frac{2}{5}\right)^2 - 4\left(\frac{2}{5}\right) - 6 = A\left(\frac{2}{5} + 3\right)\left[5\left(\frac{2}{5}\right) - 2\right] + B\left(\frac{2}{5}\right)\left[5\left(\frac{2}{5}\right) - 2\right] + C\left(\frac{2}{5}\right)\left(\frac{2}{5} + 3\right)$$

$$\Rightarrow \quad \frac{4}{5} - \frac{8}{5} - 6 = C\left(\frac{34}{25}\right) \quad \Rightarrow \quad -\frac{34}{5} = \frac{34}{25}C \quad \Rightarrow \quad C = -5$$

Substituting the values A, B and C in equation (3), we have

$$\frac{5x^2 - 4x - 6}{x(x + 3)(5x - 2)} = \frac{1}{x} + \frac{1}{x + 3} + \frac{-5}{(5x - 2)}$$

$$\therefore \quad I = \int \left(1 + \frac{1}{x} + \frac{1}{x + 3} - \frac{5}{(5x - 2)}\right) dx$$

$$= \int 1 . dx + \int \frac{1}{x} dx + \int \frac{1}{x + 3} dx - 5 \int \frac{1}{5x - 2} dx$$

$$\left[\begin{array}{l} \because \int \frac{1}{x} dx = \log|x| + c \\ \int \frac{1}{(ax + b)} dx = \frac{\log|ax + b|}{a} + c \end{array}\right]$$

$$= x + \log|x| + \log|x + 3| - \frac{5 \log|5x - 2|}{5} + c$$

$$= x + \log|x| + \log|x + 3| - \log|5x - 2| + c.$$

(*iv*) Let $\quad I = \int \frac{1}{x^4 - 1} dx = \int \frac{1}{(x^2 - 1)(x^2 + 1)} dx$

$$= \int \frac{1}{(x + 1)(x - 1)(x^2 + 1)} dx \qquad \text{...(1)}$$

Let $\quad \frac{1}{(x + 1)(x - 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 1} \qquad$ ...(2)

Multiplying both sides by $(x + 1)(x - 1)(x^2 + 1)$, we get

$$1 = A(x - 1)(x^2 + 1) + B(x + 1)(x^2 + 1) + (Cx + D)(x + 1)(x - 1) \qquad \text{...(3)}$$

$$x - 1 = 0 \quad \Rightarrow \quad x = 1$$

$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$

Put $x = 1$ in (3), we get

$$1 = A(1 - 1)(1 + 1) + B(1 + 1)(1 + 1) + (C + D)(1 + 1)(1 - 1)$$

$$\Rightarrow \quad 1 = 4B \quad \Rightarrow \quad B = \frac{1}{4}$$

Put $x = -1$ in (3), we get

$$1 = A(-1-1)[(-1)^2+1] + B(-1+1)[(-1)^2+1] + [C(-1)+D](-1+1)(-1-1)$$

$\Rightarrow \quad 1 = A(-4) \Rightarrow A = -\frac{1}{4}$

Equating the co-efficients of x^3 on both sides of equation (3), we have

$$0 = A + B + C \Rightarrow 0 = -\frac{1}{4} + \frac{1}{4} + C \Rightarrow C = 0.$$

Equating the constant terms on both sides of equation (3), we have

$$1 = -A + B - D \Rightarrow 1 = -\frac{1}{4} + \frac{1}{4} - D \Rightarrow D = -1$$

Substituting the values of A, B, C and D in equation (2), we have

$$\frac{1}{x^4-1} = \frac{-1/4}{x+1} + \frac{1/4}{x-1} + \frac{-1}{x^2+1} = -\frac{1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{x^2+1}$$

$$\therefore \quad I = \int \frac{1}{x^4-1}\,dx = \int \left(-\frac{1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{(x^2+1)}\right) dx$$

$$= -\frac{1}{4}\int \frac{1}{x+1}\,dx + \frac{1}{4}\int \frac{1}{x-1}\,dx - \int \frac{1}{x^2+1^2}\,dx$$

$$\left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= -\frac{1}{4}\log|x+1| + \frac{1}{4}\log|x-1| - \tan^{-1}x + c$$

$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \tan^{-1}x + c. \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

(v) Let $\quad I = \int \frac{1}{x(x^4-1)}\,dx \quad$ ...(1)

Put $\quad x^4 = z \Rightarrow 4x^3\,dx = dz \Rightarrow dx = \frac{1}{4x^3}\,dz$

$$\therefore \quad I = \int \frac{1}{x(z-1)} \cdot \frac{1}{4x^3}\,dz = \frac{1}{4}\int \frac{1}{x^4(z-1)}\,dz = \frac{1}{4}\int \frac{1}{z(z-1)}\,dz$$

Let $\quad \frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} \quad$ ...(2)

Multiplying both sides by $z(z-1)$, we get

$1 = A(z-1) + Bz \quad$ ...(3)

$z = 0, z - 1 = 0 \Rightarrow z = 1$

Put $z = 0$ in (3), we get

$1 = A(0-1) + 0 \Rightarrow -A = 1 \Rightarrow A = -1$

Put $z = 1$ in (3), we get

$1 = A(1-1) + B \Rightarrow B = 1$

Substituting the values of A and B in equation (2), we have

$$\frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}$$

$$\therefore \quad I = \frac{1}{4}\int \frac{1}{z(z-1)}\,dz = \frac{1}{4}\int\left(-\frac{1}{z}+\frac{1}{z-1}\right)dz$$

$$= -\frac{1}{4}\int \frac{1}{z}\,dz + \frac{1}{4}\int \frac{1}{z-1}\,dz = -\frac{1}{4}\log|z| + \frac{1}{4}\log|z-1| + c$$

$$= \frac{1}{4}\log\left|\frac{z-1}{z}\right| + c \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$= \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + c. \qquad [\because z = x^4]$$

(*vi*) Let $I = \int \frac{x^2}{x^2-4x+3}\,dx$

The integrand is not a proper fraction, therefore by actual division, we have

$$\therefore \quad I = \int \frac{x^2}{x^2-4x+3}\,dx = \int\left(1+\frac{4x-3}{x^2-4x+3}\right)dx \qquad ...(1)$$

$$\begin{array}{r|l} x^2-4x+3 & x^2 \quad (1 \\ & x^2-4x+3 \\ & - \quad + \quad - \\ \hline & 4x-3 \end{array}$$

We have $\frac{4x-3}{x^2-4x+3} = \frac{4x-3}{(x-1)(x-3)}$

Let $\frac{4x-3}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$ ...(2)

Multiplying both sides by $(x-1)(x-3)$, we have

$$4x-3 = A(x-3) + B(x-1) \qquad ...(3)$$

$$x-1=0 \Rightarrow x=1$$

$$x-3=0 \Rightarrow x=3$$

Put $x = 1$ in (3), we get

$$4(1)-3 = A(1-3) + B(1-1) \Rightarrow 1 = -2A \Rightarrow A = -\frac{1}{2}$$

Put $x = 3$ in (3), we get

$$4(3)-3 = A(3-3) + B(3-1) \Rightarrow 9 = 2B \Rightarrow B = \frac{9}{2}$$

Substituting the values of A and B in equation (2), we have

$$\frac{4x-3}{(x-1)(x-3)} = \frac{-1/2}{(x-1)} + \frac{9/2}{(x-3)} = \frac{-1}{2(x-1)} + \frac{9}{2(x-3)}$$

$$\therefore \quad I = \int\left[1+\frac{4x-3}{x^2-4x+3}\right]dx = \int\left[1-\frac{1}{2(x-1)}+\frac{9}{2(x-3)}\right]dx$$

$$= \int 1.dx - \frac{1}{2}\int\frac{1}{x-1}\,dx + \frac{9}{2}\int\frac{1}{x-3}\,dx$$

$$= x - \frac{1}{2}\log|x-1| + \frac{9}{2}\log|x-3| + c.$$

Example 8. *Evaluate the following integrals :*

(i) $\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$ (ii) $\int \frac{3x+5}{(x^3-x^2-x+1)}\,dx$

(iii) $\int \frac{5x^2+18x+17}{(x+1)^2(2x+3)}\,dx$ (iv) $\int \frac{1}{x(x^n+1)}\,dx$

(v) $\int \frac{\cos x}{(1-\sin x)(2-\sin x)}\,dx$ (vi) $\int \frac{x^4+x^3+2x^2+4x+1}{x(x+1)}\,dx.$

Solution. (i) Let $I = \int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$ ...(1)

Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+3)}$ ...(2)

Multiplying both sides by $(x-1)^2(x+3)$, we get

$$x^2+1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2 \quad ...(3)$$

$$x-1=0 \Rightarrow x=1$$

$$x+3=0 \Rightarrow x=-3$$

Put $x = 1$ in (3), we get

$$(1)^2+1 = A(1-1)(1+3) + B(1+3) + C(1-1)^2$$

$$\Rightarrow \quad 2 = 4B \Rightarrow B = \frac{1}{2}$$

Put $x = -3$ in (3), we get

$$(-3)^2+1 = A(-3-1)(-3+3) + B(-3+3) + C(-3-1)^2$$

$$\Rightarrow \quad 10 = 16C \Rightarrow C = \frac{5}{8}.$$

Equating the co-efficients of x^2 on both sides of equation (3), we get

$$1 = A + C \Rightarrow A = 1 - C \Rightarrow A = 1 - \frac{5}{8} \Rightarrow A = \frac{3}{8}.$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{3/8}{x-1} + \frac{1/2}{(x-1)^2} + \frac{5/8}{(x+3)} = \frac{3}{8(x-1)} + \frac{1}{2(x-1)^2} + \frac{5}{8(x+3)}$$

$$\therefore \quad I = \int \frac{x^2+1}{(x-1)^2(x+3)}\,dx = \int \left[\frac{3}{8(x-1)} + \frac{1}{2(x-1)^2} + \frac{5}{8(x+3)}\right] dx$$

$$= \frac{3}{8}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int (x-1)^{-2}\,dx + \frac{5}{8}\int \frac{1}{x+3}\,dx$$

$$= \frac{3}{8}\log|x-1| + \frac{1}{2}\frac{(x-1)^{-2+1}}{(-2+1)} + \frac{5}{8}\log|x+3| + c$$

$$= \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + c.$$

(*ii*) Let $\quad I = \int \frac{3x+5}{x^3 - x^2 - x + 1}\,dx$

$$= \int \frac{3x+5}{(x^2-1)(x-1)}\,dx \qquad \left[\begin{aligned} \because\ & x^3 - x^2 - x + 1 \\ &= x^2(x-1) - 1(x-1) \\ &= (x^2-1)(x-1) \\ &= (x+1)(x-1)(x-1) \\ &= (x+1)(x-1)^2 \end{aligned}\right]$$

$$\Rightarrow \qquad I = \int \frac{3x+5}{(x+1)(x-1)^2}\,dx \qquad \ldots(1)$$

Let $\frac{3x+5}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ ...(2)

Multiplying both sides by $(x+1)(x-1)^2$, we get

$$3x + 5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1) \qquad \ldots(3)$$

$$x - 1 = 0 \ \Rightarrow\ x = 1$$

$$x + 1 = 0 \ \Rightarrow\ x = -1$$

Put $x = -1$ in (3), we get

$$3(-1) + 5 = A(-1-1)^2 + B(-1+1)(-1-1) + C(-1+1)$$

$$\Rightarrow \qquad 2 = 4A \ \Rightarrow\ A = \frac{1}{2}$$

Put $x = 1$ in (3), we get

$$3(1) + 5 = A(1-1)^2 + B(1+1)(1-1) + C(1+1) \ \Rightarrow\ 8 = 2C \ \Rightarrow\ C = 4.$$

Equating the coefficients of x^2 on both sides of equation (3), we have

$$0 = A + B \ \Rightarrow\ B = -A \ \Rightarrow\ B = -\frac{1}{2}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{3x+5}{(x+1)(x-1)^2} = \frac{1/2}{(x+1)} + \frac{-1/2}{(x-1)} + \frac{4}{(x-1)^2} = \frac{1}{2(x+1)} - \frac{1}{2(x-1)} + \frac{4}{(x-1)^2}$$

$$\therefore \qquad I = \int \frac{3x+5}{(x+1)(x-1)^2}\,dx = \int \left[\frac{1}{2(x+1)} - \frac{1}{2(x-1)} + \frac{4}{(x-1)^2}\right] dx$$

$$= \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x-1}\,dx + 4\int \frac{1}{(x-1)^2}\,dx$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{2}\log|x-1| + 4\,\frac{(x-1)^{-2+1}}{-2+1} + c$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + c. \qquad \left[\because\ \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

(*iii*) Let $\quad I = \int \frac{5x^2 + 18x + 17}{(x+1)^2(2x+3)}\,dx$ ...(1)

Let $\dfrac{5x^2+18x+17}{(x+1)^2(2x+3)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(2x+3)}$ ...(2)

Multiplying both sides by $(x+1)^2(2x+3)$, we get

$$5x^2 + 18x + 17 = A(x+1)(2x+3) + B(2x+3) + C(x+1)^2 \quad ...(3)$$

$$x + 1 = 0 \Rightarrow x = -1$$

$$2x + 3 = 0 \Rightarrow 2x = -3 \Rightarrow x = -\frac{3}{2}$$

Put $x = -1$ in (3), we get

$$5(-1)^2 + 18(-1) + 17 = A(-1+1)[2(-1)+3] + B[2(-1)+3] + C(-1+1)^2$$

$$\Rightarrow \quad 5 - 18 + 17 = B \Rightarrow B = 4.$$

Put $x = -\dfrac{3}{2}$ in (3), we get

$$5\left(-\frac{3}{2}\right)^2 + 18\left(-\frac{3}{2}\right) + 17 = A\left(-\frac{3}{2}+1\right)\left[2\left(-\frac{3}{2}\right)+3\right] + B\left[2\left(-\frac{3}{2}\right)+3\right] + C\left(-\frac{3}{2}+1\right)^2$$

$$\Rightarrow \quad 5\left(\frac{9}{4}\right) - 27 + 17 = 0 + 0 + C\left(-\frac{1}{2}\right)^2$$

$$\Rightarrow \quad \frac{45}{4} - 10 = \frac{1}{4}C \Rightarrow \frac{5}{4} = \frac{1}{4}C \Rightarrow C = 5$$

Equating the coefficients of x^2 on both sides of equation (3), we get

$$5 = 2A + C \Rightarrow 2A = 5 - C \Rightarrow 2A = 0 \Rightarrow A = 0.$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{5x^2+18x+17}{(x+1)^2(2x+3)} = \frac{0}{(x+1)} + \frac{4}{(x+1)^2} + \frac{5}{(2x+3)}$$

$$\therefore \quad I = \int \frac{5x^2+18x+17}{(x+1)^2(2x+3)}\,dx = \int \left[\frac{4}{(x+1)^2} + \frac{5}{(2x+3)}\right] dx$$

$$= 4\int (x+1)^{-2}\,dx + 5\int \frac{1}{2x+3}\,dx$$

$$= \frac{4(x+1)^{-2+1}}{-2+1} + \frac{5\log|2x+3|}{2} + c \qquad \left[\because \int [f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c \quad \int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c\right]$$

$$= \frac{-4}{x+1} + \frac{5}{2}\log|2x+3| + c.$$

(*iv*) Let $\quad I = \displaystyle\int \frac{1}{x(x^n+1)}\,dx$

$$\Rightarrow \quad I = \int \frac{x^{n-1}}{x^n(x^n+1)}\,dx \qquad \text{[Multiply and divided by } x^{n-1}]$$

Put $x^n = z \Rightarrow nx^{n-1}\,dx = dz \Rightarrow x^{n-1}\,dx = \frac{1}{n}\,dz$

$$\therefore \quad I = \int \frac{1}{z(z+1)} \cdot \frac{1}{n}\,dz = \frac{1}{n}\int \frac{1}{z(z+1)}\,dz$$

Let $\frac{1}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$

Multiplying both sides by $z(z + 1)$, we get

$$1 = A(z + 1) + Bz$$

Put $z = 0$ in (3), we get

$$1 = A(0 + 1) + 0 \Rightarrow A = 1$$

Put $z = -1$ in (3), we get

$$1 = A(-1 + 1) + B(-1) \Rightarrow B = -1$$

Substituting the values of A and B in equation (2), we have

$$\frac{1}{z(z+1)} = \frac{1}{z} + \frac{-1}{z+1}$$

$$\therefore \quad I = \frac{1}{n}\int \frac{1}{z(z+1)}\,dz = \frac{1}{n}\int \left[\frac{1}{z} - \frac{1}{z+1}\right] dz$$

$$= \frac{1}{n}\int \frac{1}{z}\,dz - \frac{1}{n}\int \frac{1}{z+1}\,dz = \frac{1}{n}\log|z| - \frac{1}{n}\log|z+1| + c$$

$$= \frac{1}{n}\log\left|\frac{z}{z+1}\right| + c \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + c. \qquad [\because z = x^n]$$

Note. This question may be asked in several forms. *i.e.*, by changing the values of n, we have different forms of the same question.

e.g., when $n = 4, 5, 6, 3, 8$ etc.

The reader may advised to solve this problem by taking different values of n on the same steps.

(*v*) Let $I = \int \frac{\cos x}{(1-\sin x)(2-\sin x)}\,dx$

Put $\sin x = z \Rightarrow \cos x\,dx = dz$

$$\therefore \quad I = \int \frac{1}{(1-z)(2-z)}\,dz \qquad ...(1)$$

$$\text{Let} \quad \frac{1}{(1-z)(2-z)} = \frac{A}{(1-z)} + \frac{B}{(2-z)} \qquad ...(2)$$

Multiplying both sides by $(1 - z)(2 - z)$, we get

$$1 = A(2 - z) + B(1 - z) \qquad ...(3)$$

$$1 - z = 0 \Rightarrow z = 1$$

$$2 - z = 0 \Rightarrow z = 2.$$

Put $z = 1$ in (3), we get

$$1 = A(2 - 1) + B(1 - 1) \Rightarrow 1 = A \Rightarrow A = 1$$

Put $z = 2$ in (3), we get

$$1 = A(2-2) + B(1-2) \Rightarrow 1 = -B \Rightarrow B = -1$$

Substituting the values of A and B in equation (2), we have

$$\therefore \quad \frac{1}{(1-z)(2-z)} = \frac{1}{(1-z)} + \frac{-1}{(2-z)}$$

$$\therefore \quad I = \int \frac{1}{(1-z)(2-z)}\,dz = \int \left[\frac{1}{1-z} - \frac{1}{2-z}\right] dz$$

$$= \int \frac{1}{1-z}\,dz - \int \frac{1}{2-z}\,dz = -\log|1-z| + \log|2-z| + c$$

$$= \log\left|\frac{2-z}{1-z}\right| + c \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + c. \qquad [\because z = \sin x]$$

(*vi*) Let $\quad I = \int \dfrac{x^4 + x^3 + 2x^2 + 4x + 1}{x(x+1)}\,dx$

Since the integrand is not a proper fraction, therefore by actual division, we have

$$I = \int \left[(x^2+2) + \frac{2x+1}{x^2+x}\right] dx \quad \text{...(1)}$$

Let $\quad \dfrac{2x+1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1} \quad$ ...(2)

Multiplying both sides by $x(x+1)$, we get

$$2x + 1 = A(x+1) + Bx \quad \text{...(3)}$$

Put $x = 0$ in (3), we get

$$1 = A(0+1) + 0 \Rightarrow A = 1$$

$$\begin{array}{r|l} x^2+x \,\big)\; x^4+x^3+2x^2+4x+1 \,\big(& x^2+2 \\ +x^4+x^3 \qquad\qquad\quad & \\ -\quad\; - \qquad\qquad\quad & \\ \hline 2x^2+4x+1 & \\ +2x^2+2x \quad\; & \\ -\quad\; - \quad\;\; & \\ \hline 2x+1 & \end{array}$$

Put $x = -1$ in (3), we get

$$2(-1) + 1 = A(-1+1) + B(-1) \Rightarrow -1 = -B \Rightarrow B = 1$$

Substituting the values of A and B in equation (2), we have

$$\frac{2x+1}{x(x+1)} = \frac{1}{x} + \frac{1}{x+1}$$

$$\therefore \quad I = \int \frac{x^4 + x^3 + 2x^2 + 4x + 1}{x(x+1)}\,dx = \int \left(x^2 + 2 + \frac{2x+1}{x^2+x}\right) dx$$

$$= \int \left(x^2 + 2 + \frac{1}{x} + \frac{1}{x+1}\right) dx = \int x^2\,dx + 2\int 1.\,dx + \int \frac{1}{x}\,dx + \int \frac{1}{x+1}\,dx$$

$$= \frac{x^3}{3} + 2x + \log|x| + \log|x+1| + c.$$

Example 9. *Evaluate the following integrals :*

(*i*) $\displaystyle\int \frac{8}{(x+2)(x^2+4)}\,dx$ $\qquad$ (*ii*) $\displaystyle\int \frac{1}{x^3+x^2+x}\,dx$

(*iii*) $\displaystyle\int \frac{5x}{(x+1)(x^2+9)}\,dx$ $\qquad$ (*iv*) $\displaystyle\int \frac{1}{x^3+x^2+x+1}\,dx$

(v) $\int \frac{2x}{x^3 - 1}\, dx$ (vi) $\int \frac{2x^3}{(x^2+1)^2}\, dx$

(vii) $\int \frac{1}{x^4 - a^4}\, dx.$

Solution. (i) Let $I = \int \frac{8}{(x+2)(x^2+4)}\, dx$ ...(1)

Let $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$ ...(2)

Multiplying both sides by $(x + 2)(x^2 + 4)$, we get

$$8 = A(x^2 + 4) + (Bx + C)(x + 2) \quad ...(3)$$

$$x + 2 = 0 \quad \Rightarrow \quad x = -2$$

Put $x = -2$ in (3), we get

$$8 = A[(-2)^2 + 4] + [B(-2) + C][-2 + 2] \Rightarrow 8 = 8A \Rightarrow A = 1$$

Equating the co-efficients of x^2 on both sides of equation (3), we have

$$0 = A + B \Rightarrow B = -A \Rightarrow B = -1$$

Equating the constant terms on both sides of equation (3), we have

$$8 = 4A + 2C$$

$$\Rightarrow 8 - 4A = 2C \quad \Rightarrow 8 - 4 = 2C \Rightarrow C = 2$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{-x+2}{x^2+4}$$

$$\therefore \quad I = \int \frac{8}{(x+2)(x^2+4)}\, dx = \int \left[\frac{1}{x+2} + \frac{-x+2}{x^2+4}\right] dx$$

$$= \int \frac{1}{x+2}\, dx - \int \frac{x}{x^2+4}\, dx + 2\int \frac{1}{x^2+4}\, dx$$

$$= \int \frac{1}{x+2}\, dx - \frac{1}{2}\int \frac{2x}{x^2+4}\, dx + 2\int \frac{1}{x^2+2^2}\, dx$$

[Multiply and divide the second integral by 2]

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + 2.\frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

$$\left[\begin{array}{l} \because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c \\ \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c \\ \int \frac{1}{ax+b}\, dx = \frac{\log|ax+b|}{a} + c \end{array}\right]$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + \tan^{-1}\frac{x}{2} + c.$$

(*ii*) Let $\quad I = \int \frac{1}{x^3 + x^2 + x}\, dx = \int \frac{1}{x(x^2 + x + 1)}\, dx$ ...(1)

Let $\quad \frac{1}{x(x^2 + x + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1}$ ...(2)

Multiplying both sides by $x(x^2 + x + 1)$, we get

$1 = A(x^2 + x + 1) + (Bx + C)x$ ...(3)

Put $x = 0$ in equation (3), we get

$1 = A(0 + 0 + 1) + 0 \;\Rightarrow\; A = 1$

Equating the co-efficients of x^2 on both sides of equation (3), we get

$0 = A + B \;\Rightarrow\; B = -A \;\Rightarrow\; B = -1$

Equating the co-efficients of x on both sides of equation (3), we get

$0 = A + C \;\Rightarrow\; C = -A \;\Rightarrow\; C = -1$

Substituting the values of A, B and C in equation (2), we have

$$\frac{1}{x(x^2 + x + 1)} = \frac{1}{x} + \frac{-x-1}{x^2 + x + 1} = \frac{1}{x} - \frac{x+1}{x^2 + x + 1}$$

$\therefore \quad I = \int \frac{1}{x(x^2 + x + 1)}\, dx = \int \left[\frac{1}{x} - \frac{x+1}{x^2 + x + 1}\right] dx$

$\Rightarrow \quad I = \int \frac{1}{x}\, dx - \int \frac{x+1}{x^2 + x + 1}\, dx = \log|x| - \int \frac{x+1}{x^2 + x + 1}\, dx$ ...(4)

Let $\quad x + 1 = \lambda \frac{d}{dx}(x^2 + x + 1) + \mu$

$\Rightarrow \quad x + 1 = \lambda(2x + 1) + \mu$

$\Rightarrow \quad x + 1 = 2\lambda x + (\lambda + \mu)$ ...(5)

Equating the co-efficients of x on both sides of equation (5), we get

$1 = 2\lambda \;\Rightarrow\; \lambda = \frac{1}{2}$

Equating the constant terms on both sides of equation (5), we get

$1 = \lambda + \mu \;\Rightarrow\; \mu = 1 - \lambda \;\Rightarrow\; \mu = 1 - \frac{1}{2} \;\Rightarrow\; \mu = \frac{1}{2}$

$$\therefore \int \frac{x+1}{x^2 + x + 1}\, dx = \int \frac{\left(\frac{1}{2}(2x+1) + \frac{1}{2}\right)}{(x^2 + x + 1)}\, dx$$

$$= \frac{1}{2}\int \frac{2x+1}{x^2 + x + 1}\, dx + \frac{1}{2}\int \frac{1}{x^2 + x + 1}\, dx$$

$$= \frac{1}{2}\log|x^2 + x + 1| + \frac{1}{2}\int \frac{1}{x^2 + x + 1}\, dx \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right]$$

$$= \frac{1}{2}\log|x^2+x+1| + \frac{1}{2}\int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\left(1-\frac{1}{4}\right)}\,dx$$

$\left[\begin{array}{l}\text{Add and sub. } \frac{1}{4} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$

$$= \frac{1}{2}\log|x^2+x+1| + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{2}\log|x^2+x+1| + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\log|x^2+x+1| + \frac{1}{2}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\cdot\tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c$$

$$\left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{1}{2}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

$\therefore$ From equation (4), we have

$$I = \log|x| - \frac{1}{2}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c.$$

(*iii*) Let $\quad I = \int \frac{5x}{(x+1)(x^2+9)}\,dx$ ...(1)

Let $\quad \frac{5x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$ ...(2)

Multiplying both sides by $(x+1)(x^2+9)$, we get

$$5x = A(x^2+9) + (Bx+C)(x+1) \quad \text{...(3)}$$

Put $x = -1$ in (3), we get

$$5(-1) = A[(-1)^2+9] + [B(-1)+C](-1+1) \Rightarrow -5 = 10A \Rightarrow A = -\frac{1}{2}$$

Equating the co-efficients of x^2 on both sides of equation (3), we get

$$0 = A + B \Rightarrow B = -A \Rightarrow B = \frac{1}{2}$$

Equating the constant terms on both sides of equation (3), we get

$$0 = 9A + C \Rightarrow C = -9A \Rightarrow -9\left(-\frac{1}{2}\right) \Rightarrow C = \frac{9}{2}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{5x}{(x+1)(x^2+9)} = \frac{\left(-\frac{1}{2}\right)}{(x+1)} + \frac{\frac{1}{2}x+\frac{9}{2}}{(x^2+9)} = \frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)}$$

$$\therefore \quad I = \int \frac{5x}{(x+1)(x^2+9)}\,dx = \int\left[-\frac{1}{2(x+1)} + \frac{x+9}{2(x^2+9)}\right]dx$$

$$= -\frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{x}{x^2+9}\,dx + \frac{9}{2}\int \frac{1}{x^2+9}\,dx$$

$$= -\frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{4}\int \frac{2x}{x^2+9}\,dx + \frac{9}{2}\int \frac{1}{x^2+3^2}\,dx$$

[Multiply and divide the second integral by 2]

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{9}{2}\cdot\frac{1}{3}\tan^{-1}\frac{x}{3} + c$$

$$\left[\begin{array}{l} \because \int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c \\ \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c \\ \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c \end{array}\right]$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{3}{2}\tan^{-1}\frac{x}{3} + c.$$

(*iv*) Let $I = \int \frac{1}{x^3+x^2+x+1}\,dx$

$$\therefore \quad I = \int \frac{1}{(x+1)(x^2+1)}\,dx \quad \ldots(1) \qquad \left[\begin{array}{l} \because x^3+x^2+x+1 \\ = x^2(x+1) + 1(x+1) \\ = (x+1)(x^2+1) \end{array}\right]$$

Let $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ ...(2)

Multiplying both sides by $(x+1)(x^2+9)$, we get

$$1 = A(x^2+1) + (Bx+C)(x+1) \quad \ldots(3)$$

$$x+1=0 \Rightarrow x=-1$$

Put $x = -1$ in (3), we get

$$1 = A[(-1)^2+1] + [B(-1)+C][-1+1] \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

Equating the co-efficients of x^2 on both sides of equation (3), we get

$$0 = A + B \Rightarrow B = -A \Rightarrow B = -\frac{1}{2}$$

Equating the constant terms on both sides of equation (3), we get

$$1 = A + C \Rightarrow C = 1 - A \Rightarrow C = 1 - \frac{1}{2} \Rightarrow C = \frac{1}{2}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{1}{(1+x)(1+x^2)} = \frac{\frac{1}{2}}{1+x} + \frac{-\frac{1}{2}x+\frac{1}{2}}{(1+x^2)} = \frac{1}{2(1+x)} + \frac{-x+1}{2(1+x^2)}$$

$$\therefore \quad I = \int \frac{1}{(1+x)(1+x^2)}\,dx = \int \left[\frac{1}{2(1+x)} + \frac{-x+1}{2(1+x^2)}\right] dx$$

$$= \frac{1}{2}\int \frac{1}{1+x}\,dx - \frac{1}{2}\int \frac{x}{1+x^2}\,dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{1+x}\,dx - \frac{1}{4}\int \frac{2x}{1+x^2}\,dx + \frac{1}{2}\int \frac{1}{1^2+x^2}\,dx$$

[Multiply and divide the second integral by 2]

$$= \frac{1}{2}\log|1+x| - \frac{1}{4}\log|1+x^2| + \frac{1}{2}\tan^{-1} x + c.$$

$$\left[\begin{array}{l} \because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c \\ \because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c \end{array}\right]$$

(v) Let $I = \int \frac{2x}{x^3-1}\,dx$ $\quad [\because (a^3-b^3) = (a-b)(a^2+ab+b^2)]$

$$= \int \frac{2x}{(x-1)(x^2+x+1)}\,dx \qquad ...(1)$$

Let $$\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)} \qquad ...(2)$$

Multiplying both sides by $(x-1)(x^2+x+1)$, we get

$$2x = A(x^2+x+1) + (Bx+C)(x-1) \qquad ...(3)$$

Put $x = 1$ in (3), we get

$$2 = A[(1)^2+1+1] + (B+C)(1-1) \Rightarrow 2 = 3A \Rightarrow A = \frac{2}{3}$$

Equating co-efficients of x^2 on both sides of equation (3), we get

$$0 = A + B \Rightarrow B = -A \Rightarrow B = -\frac{2}{3}$$

Equating the constant terms on both sides of equation (3), we get

$$0 = A - C \Rightarrow C = A \Rightarrow C = \frac{2}{3}$$

Substituting the values of A, B and C in equation (2), we get

$$\frac{2x}{(x-1)(x^2+x+1)} = \frac{\frac{2}{3}}{(x-1)} + \frac{\left(-\frac{2}{3}x+\frac{2}{3}\right)}{(x^2+x+1)} = \frac{2}{3(x-1)} - \frac{2(x-1)}{3(x^2+x+1)}$$

$$\therefore \quad I = \int \frac{2x}{(x-1)(x^2+x+1)}\,dx = \int \left[\frac{2}{3(x-1)} - \frac{2(x-1)}{3(x^2+x+1)}\right] dx$$

$$= \frac{2}{3}\int \frac{1}{x-1}\,dx - \frac{2}{3}\int \frac{(x-1)}{x^2+x+1}\,dx$$

$$= \frac{2}{3}\log|x-1| - \frac{2}{3}\int \frac{x-1}{x^2+x+1}\,dx \qquad \ldots(4)$$

Let $\quad x - 1 = \lambda \dfrac{d}{dx}(x^2+x+1) + \mu$

$\Rightarrow \quad x - 1 = \lambda(2x+1) + \mu \qquad \ldots(5)$

$\Rightarrow \quad x - 1 = 2\lambda x + (\lambda + \mu)$

Equating the like power terms of x on both sides, we have

$$1 = 2\lambda \Rightarrow \lambda = \frac{1}{2}$$

and $$-1 = \lambda + \mu \Rightarrow \mu = -1 - \lambda \Rightarrow \mu = -1 - \frac{1}{2} \Rightarrow \mu = -\frac{3}{2}$$

$\therefore$ Equation (5) becomes

$$(x-1) = \frac{1}{2}(2x+1) - \frac{3}{2}$$

$$\int \frac{x-1}{x^2+x+1}\,dx = \int \frac{\frac{1}{2}(2x+1) - \frac{3}{2}}{x^2+x+1}\,dx$$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$= \frac{1}{2}\log|x^2+x+1| - \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$= \frac{1}{2}\log|x^2+x+1| - \frac{3}{2}\int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\left(1-\frac{1}{4}\right)}\,dx$$

$$\left[\begin{array}{l}\text{Add and substract } \frac{1}{4} \text{ to the denom} \\ \because \left(\frac{1}{2}\text{co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \frac{1}{2}\log|x^2+x+1| - \frac{3}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\log|x^2+x+1| - \frac{3}{2}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c_1$$

$$\left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{2}\log|x^2+x+1| - \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c_1$$

$\therefore$ From equation (4), we have

$$I = \frac{2}{3}\log|x-1| - \frac{2}{3}\left[\frac{1}{2}\log|x^2+x+1| - \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c_1\right]$$

$$= \frac{2}{3}\log|x-1| - \frac{1}{3}\log|x^2+x+1| - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{-2}{3}c_1$$

$$= \frac{2}{3}\log|x-1| - \frac{1}{3}\log|x^2+x+1| - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c.$$

where $c = -\frac{2}{3}c_1$.

(*vi*) Let $\quad I = \int \frac{2x^3}{(x^2+1)^2}\,dx \quad$ ...(1)

Let $\quad \frac{2x^3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2} \quad$ ...(2)

Multiplying both sides by $(x^2+1)^2$, we get

$$2x^3 = (Ax+B)(x^2+1) + (Cx+D)$$

$\Rightarrow \quad 2x^3 = Ax^3 + Bx^2 + (A+C)x + B + D \quad$ ...(3)

Equating the co-efficients of x^3, x^2, x and the constant terms on both sides of equation (3), we get

$$A = 2 \Rightarrow A = 2$$
$$B = 0 \Rightarrow B = 0$$
$$A + C = 0 \Rightarrow C = -A \Rightarrow C = -2$$
$$B + D = 0 \Rightarrow D = -B \Rightarrow D = 0$$

Substituting the values of A, B, C and D in equation (2), we have

$$\frac{2x^3}{(x^2+1)^2} = \frac{2x}{x^2+1} + \frac{-2x}{(x^2+1)^2} = \frac{2x}{x^2+1} - \frac{2x}{(x^2+1)^2}$$

$\therefore \quad I = \int \frac{2x}{x^2+1}\,dx - \int \frac{2x}{(x^2+1)^2}\,dx$

Put $\quad x^2 = z \Rightarrow 2x\,dx = dz$

$\therefore \quad I = \int \frac{1}{z+1}\,dz - \int \frac{1}{(z+1)^2}\,dz$

$$= \log|z+1| - \frac{(z+1)^{-2+1}}{-2+1} + c$$

$$= \log|z+1| + \frac{1}{(z+1)} + c$$

$$= \log|x^2+1| + \frac{1}{x^2+1} + c. \qquad [\because z = x^2]$$

(*vii*) Let $\quad I = \int \frac{1}{x^4 - a^4}\, dx = \int \frac{1}{(x^2 + a^2)(x^2 - a^2)}\, dx$

$\Rightarrow \quad I = \int \frac{1}{(x-a)(x+a)(x^2+a^2)}\, dx$

Let $\quad \frac{1}{(x-a)(x+a)(x^2+a^2)} = \frac{A}{x-a} + \frac{B}{x+a} + \frac{Cx+D}{x^2+a^2}$ (Please try yourself)

[**Hint.** See exampe 5 (*iv*)] $\left[\textbf{Ans. } \frac{1}{4a^3} \log\left|\frac{x-a}{x+a}\right| - \frac{1}{2a^3} \tan^{-1}\frac{x}{a} + c\right]$

Example 10. *Evaluate the following integrals :*

(*i*) $\int \frac{x^3+2}{(x-1)(x-2)^3}\, dx$ (*ii*) $\int \frac{x+1}{x(1+x e^x)}\, dx$

(*iii*) $\int \frac{5x^2+20x+6}{x^3+2x^2+x}\, dx$ (*iv*) $\int \frac{x^2+x+1}{(x-1)^4}\, dx$

(*v*) $\int \frac{\sec^2 x}{\tan^3 x + 4 \tan x}\, dx.$

Solution. (*i*) Let $I = \int \frac{x^3+2}{(x-1)(x-2)^3}\, dx$ ...(1)

Let $\frac{x^3+2}{(x-1)(x-2)^3} = \frac{A}{x-1} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$ ...(2)

Multiplying both sides by $(x-1)(x-2)^3$, we get

$$x^3 + 2 = A(x-2)^3 + B(x-1)(x-2)^2 + C(x-1)(x-2) + D(x-1) \quad ...(3)$$

$$x - 1 = 0 \Rightarrow x = 1, x - 2 = 0 \Rightarrow x = 2$$

Put $x = 1$ in equation (3), we get

$$[(1)^3 + 2] = A(1-2)^3 + B(1-1)(1-2)^2 + C(1-1)(1-2) + D(1-1)$$

$$\Rightarrow \quad 3 = -A \Rightarrow A = -3$$

Put $x = 2$ in (3), we get

$$[(2)^3 + 2] = A(2-2)^3 + B(2-1)(2-2)^2 + C(2-1)(2-2) + D(2-1)$$

$$\Rightarrow \quad 10 = D \Rightarrow D = 10$$

Equating the co-efficients of x^3 on both sides of equation (3), we get

$$1 = A + B \quad \Rightarrow B = 1 - A \Rightarrow B = 1 - (-3) \Rightarrow B = 4.$$

Equating the co-efficients of x^2 on both sides of equation (3), we get

$$0 = -6A - 5B + C$$

$$\Rightarrow \quad C = 6A + 5B \Rightarrow C = 6(-3) + 5(4) \Rightarrow C = -18 + 20 \Rightarrow C = 2.$$

Substituting the values of A, B, C and D in equation (2), we have

$$\frac{x^3+2}{(x-1)(x-2)^3} = \frac{-3}{x-1} + \frac{4}{x-2} + \frac{2}{(x-2)^2} + \frac{10}{(x-2)^3}$$

$$\therefore \quad I = \int \frac{x^3+2}{(x-1)(x-2)^3}\, dx = \int \left[\frac{-3}{x-1} + \frac{4}{(x-2)} + \frac{2}{(x-2)^2} + \frac{10}{(x-2)^3}\right] dx$$

$$= -3\int \frac{1}{x-1}\,dx + 4\int \frac{1}{x-2}\,dx + 2\int (x-2)^{-2}\,dx + 10\int (x-2)^{-3}\,dx$$

$$= -3\log|x-1| + 4\log|x-2| + 2\,\frac{(x-2)^{-2+1}}{(-2+1)} + 10\,\frac{(x-2)^{-3+1}}{(-3+1)} + c$$

$$= -3\log|x-1| + 4\log|x-2| - \frac{2}{x-2} - \frac{5}{(x-2)^2} + c.$$

(*ii*) Let $\quad I = \int \frac{x+1}{x(1+x\,e^x)}\,dx$

$$= \int \frac{(x+1)\,e^x}{x\,e^x(1+x\,e^x)}\,dx \qquad \text{...(1) [Multiply and divided by } e^x]$$

Put $\quad x\,e^x = z$

$\Rightarrow\ (x\,.\,e^x + e^x)\,dx = dz \ \Rightarrow\ (x+1)\,e^x\,dx = dz$

$\therefore \quad I = \int \frac{1}{z(1+z)}\,dz$

Let $\quad \frac{1}{z(1+z)} = \frac{A}{z} + \frac{B}{1+z}$ ...(2)

Multiplying both sides by $z(1+z)$, we get

$1 = A(1+z) + Bz$ ...(3)

Put $z = 0$ in (3), we get : $1 = A(1+0) \ \Rightarrow\ A = 1$

Put $z = -1$ in (3), we get : $1 = A[1 + (-1)] + B(-1) \ \Rightarrow\ 1 = -B \ \Rightarrow\ B = -1$

Substituting the values of A and B in equation (2), we have

$$\frac{1}{z(1+z)} = \frac{1}{z} + \frac{-1}{(1+z)}$$

$$\therefore \quad I = \int \frac{1}{z(1+z)}\,dz = \int \left[\frac{1}{z} - \frac{1}{1+z}\right] dz$$

$$= \int \frac{1}{z}\,dz - \int \frac{1}{1+z}\,dz = \log|z| - \log|1+z| + c$$

$$= \log\left|\frac{z}{1+z}\right| + c \qquad \left[\because\ \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \log\left|\frac{x\,e^x}{1+x\,e^x}\right| + c. \qquad [\because\ z = x\,e^x]$$

(*iii*) Let $\quad I = \int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx = \int \frac{5x^2+20x+6}{x\,(x^2+2x+1)}\,.\,dx$

$$= \int \frac{5x^2+20x+6}{x(x+1)^2}\,dx \qquad \text{...(1)}$$

Let $\quad \frac{5x^2+20x+6}{x\,(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ ...(2)

Multiplying both sides by $x(x+1)^2$, we get

$5x^2 + 20x + 6 = A(x+1)^2 + Bx(x+1) + Cx$ ...(3)

$$x + 1 = 0 \Rightarrow x = -1.$$

Put $x = 0$ in equation (3), we get

$$6 = A(0+1)^2 \Rightarrow 6 = A \Rightarrow A = 6$$

Put $x = -1$ in equation (3), we get

$$5(-1)^2 + 20(-1) + 6 = C(-1) \Rightarrow 5 - 20 + 6 = -C \Rightarrow C = 9$$

Now, equating co-efficients of x^2 on both sides of equation (3), we get

$$5 = A + B \Rightarrow B = 5 - A \Rightarrow B = 5 - 6 \Rightarrow B = -1$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{5x^2+20x+6}{x(x+1)^2} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$$

$$\therefore \quad I = \int \frac{5x^2+20x+6}{x(x+1)^2}\,dx = \int \left[\frac{1}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right].dx$$

$$= 6\int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx + 9\int (x+1)^{-2}.dx$$

$$= 6\log|x| - \log|x+1| + 9\,\frac{(x+1)^{-2+1}}{-2+1} + c$$

$$= 6\log|x| - \log|x+1| - \frac{9}{x+1} + c$$

(*iv*) Let $\quad I = \int \dfrac{x^2+x+1}{(x-1)^4}\,dx$

Put $\quad x - 1 = z \Rightarrow x = z + 1 \Rightarrow dx = dz$

$$\therefore \quad I = \int \frac{(z+1)^2+(z+1)+1}{z^4}\,dz = \int \frac{z^2+2z+1+z+1+1}{z^4}\,dz$$

$$= \int \frac{z^2+3z+3}{z^4}\,dz = \int\left[\frac{z^2}{z^4}+\frac{3z}{z^4}+\frac{3}{z^4}\right]dz = \int\left[\frac{1}{z^2}+\frac{3}{z^3}+\frac{3}{z^4}\right]dz$$

$$= \int z^{-2}\,dz + 3\int z^{-3}\,dz + 3\int z^{-4}\,dz$$

$$= \frac{z^{-2+1}}{(-2+1)} + 3.\frac{z^{-3+1}}{(-3+1)} + 3.\frac{z^{-4+1}}{(-4+1)} + c$$

$$= -\frac{1}{z} - \frac{3}{2z^2} - \frac{1}{z^3} + c$$

$$= -\frac{1}{x-1} - \frac{3}{2(x-1)^2} - \frac{1}{(x-1)^3} + c. \qquad [\because\ z = (x-1)]$$

(*v*) Let $\quad I = \int \dfrac{\sec^2 x}{\tan^3 x + 4\tan x}\,dx \qquad ...(1)$

Put $\quad \tan x = z \Rightarrow \sec^2 x\,dx = dz$

$$\therefore \quad I = \int \frac{1}{z^3+4z}\,dz$$

Let $\dfrac{1}{z(z^2+4)} = \dfrac{A}{z} + \dfrac{Bz+C}{z^2+4} \qquad ...(2)$

Multiplying both sides by $z(z^2 + 4)$, we get

$$1 = A(z^2 + 4) + (Bz + C)z \qquad ...(3)$$

Put $z = 0$ in (3), we get : $1 = 4A \Rightarrow A = \frac{1}{4}$

Equating the co-efficients of z^2 on both sides of equation (3), we get

$$0 = A + B \Rightarrow B = -A \Rightarrow B = -\frac{1}{4}.$$

Equating the co-efficients of z on both sides of equation (3), we get

$$0 = C \Rightarrow C = 0$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{1}{z(z^2+4)} = \frac{\frac{1}{4}}{z} + \frac{-\frac{1}{4}z + 0}{z^2 + 4} = \frac{1}{4z} - \frac{1}{4}\left(\frac{z}{z^2+4}\right)$$

$$\therefore \quad I = \int \frac{1}{z(z^2+4)}\,dz = \int \left[\frac{1}{4z} - \frac{1}{4}\left(\frac{z}{z^2+4}\right)\right] dz$$

$$= \frac{1}{4}\int \frac{1}{z}\,dz - \frac{1}{4}\int \frac{z}{z^2+4}$$

$$= \frac{1}{4}\int \frac{1}{z}\,dz - \frac{1}{8}\int \frac{2z}{z^2+4}\,dz \quad \text{[Multiply and divide the second integral by 2]}$$

$$= \frac{1}{4}\log|z| - \frac{1}{8}\log|z^2 + 4| + c \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

$$= \frac{1}{4}\log|\tan x| - \frac{1}{8}\log|\tan^2 x + 4| + c. \qquad [\because z = \tan x]$$

Example 11. *Evaluate the following integrals :*

(i) $\int \frac{\sin\theta\cos\theta}{\cos^2\theta - \cos\theta - 2}\,d\theta$ (ii) $\int \frac{1 - \cos\theta}{\cos\theta\,(1 + \cos\theta)}\,d\theta$

(iii) $\int \frac{1}{\sin\theta - \sin 2\theta}\,d\theta$ (iv) $\int \frac{1}{\sin x\,(3 + 2\cos x)}\,dx$

(v) $\int \frac{\sec x}{1 + \operatorname{cosec} x}\,dx$ (vi) $\int \frac{1 + \sin x}{\sin x\,(1 + \cos x)}\,dx.$

Solution. (i) Let $I = \int \frac{\sin\theta\cos\theta}{\cos^2\theta - \cos\theta - 2}\,d\theta$

Put $\cos\theta = z \Rightarrow -\sin\theta\,d\theta = dz \Rightarrow \sin\theta\,d\theta = -dz$

$$\therefore \quad I = \int \frac{z}{z^2 - z - 2}(-dz) = +\int \frac{-z}{z^2 - z - 2}\,dz$$

$$= \int \frac{-z}{(z-2)(z+1)}\,dz \qquad ...(1)$$

Let $$\frac{-z}{(z-2)(z+1)} = \frac{A}{(z-2)} + \frac{B}{z+1} \qquad ...(2)$$

Multiplying both sides by $(z - 2)(z + 1)$, we get

$$-z = A(z + 1) + B(z - 2) \qquad ...(3)$$

$$z+1=0 \Rightarrow z=-1$$
$$z-2=0 \Rightarrow z=+2$$

Put $z=-1$ in (3), we get

$$-(-1)=A(-1+1)+B(-1-2) \Rightarrow 1=-3B \Rightarrow B=-\frac{1}{3}$$

Put $z=2$ in (3), we get

$$-2=A(2+1)+B(2-2) \Rightarrow -2=3A \Rightarrow A=-\frac{2}{3}.$$

Substituting the values of A and B in equation (2), we have

$$\frac{-z}{(z-2)(z+1)}=\frac{-2/3}{(z-2)}+\frac{-1/3}{(z+1)}=-\frac{2}{3(z-2)}-\frac{1}{3(z+1)}$$

$$\therefore \quad I=\int \frac{-z}{(z-2)(z+1)}\,dz=\int\left[-\frac{2}{3(z-2)}-\frac{1}{3(z+1)}\right]dz$$

$$=-\frac{2}{3}\int\frac{1}{z-2}\,dz-\frac{1}{3}\int\frac{1}{z+1}\,dz=-\frac{2}{3}\log|z-2|-\frac{1}{3}\log|z+1|+c$$

$$=-\frac{2}{3}\log|\cos\theta-2|-\frac{1}{3}\log|\cos\theta+1|+c. \qquad [\because z=\cos\theta]$$

(*ii*) Let $$I=\int\frac{(1-\cos\theta)}{\cos\theta\,(1+\cos\theta)}\,d\theta \qquad ...(1)$$

Put $\cos\theta=z$

$$\therefore \quad \frac{1-\cos\theta}{\cos\theta\,(1+\cos\theta)}=\frac{1-z}{z(1+z)}$$

Let $$\frac{1-z}{z(1+z)}=\frac{A}{z}+\frac{B}{1+z} \qquad ...(2)$$

Multiplying both sides by $z(1+z)$, we get

$$1-z=A(1+z)+Bz \qquad ...(3)$$

Put $z=0$ in (3), we get : $1=A(1+0)+0 \Rightarrow A=1$

Put $z=-1$ in (3), we get : $1-(-1)=A[1+(-1)]+B(-1) \Rightarrow 2=-B \Rightarrow B=-2$

Substituting the values of A and B in equation (2), we have

$$\frac{1-z}{z(1+z)}=\frac{1}{z}+\frac{-2}{1+z}=\frac{1}{z}-\frac{2}{1+z}$$

$$\therefore \quad \frac{1-\cos\theta}{\cos\theta\,(1+\cos\theta)}=\frac{1}{\cos\theta}-\frac{2}{1+\cos\theta} \qquad [\because z=\cos\theta]$$

$$=\sec\theta-\frac{2}{2\cos^2\frac{\theta}{2}} \qquad \left[\begin{array}{l}\because 1+\cos 2A=2\cos^2 A\\ \Rightarrow 1+\cos A=2\cos^2\frac{A}{2}\end{array}\right]$$

$$=\sec\theta-\sec^2\frac{\theta}{2}$$

$$\therefore \quad I=\int\frac{1-\cos\theta}{\cos\theta\,(1+\cos\theta)}\,d\theta=\int\left[\sec\theta-\sec^2\frac{\theta}{2}\right]d\theta$$

$$= \int \sec\theta \, d\theta - \int \sec^2 \frac{\theta}{2} \, d\theta$$

$$= \log |\sec\theta + \tan\theta| - \frac{\tan\theta/2}{(1/2)} + c$$

$$= \log |\sec\theta + \tan\theta| - 2\tan\frac{\theta}{2} + c.$$

(*iii*) Let $\quad I = \int \frac{1}{\sin\theta - \sin 2\theta} \, d\theta$

$$= \int \frac{1}{\sin\theta - 2\sin\theta\cos\theta} \, d\theta \qquad [\because \ \sin 2A = 2\sin A\cos A]$$

$$= \int \frac{1}{\sin\theta(1 - 2\cos\theta)} \, d\theta$$

$$= \int \frac{\sin\theta}{\sin^2\theta(1 - 2\cos\theta)} \, d\theta \qquad \text{[Multiply and divided by } \sin\theta]$$

$$= \int \frac{\sin\theta}{(1 - \cos^2\theta)(1 - 2\cos\theta)} \, d\theta \qquad \begin{bmatrix} \because \ \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \ \sin^2 A = 1 - \cos^2 A \end{bmatrix}$$

Put $\quad \cos\theta = z$

$\Rightarrow \quad -\sin\theta \, d\theta = dz \ \Rightarrow \ \sin\theta \, d\theta = -dz$

$$\therefore \quad I = \int \frac{1}{(1 - z^2)(1 - 2z)} (-dz) \qquad \ldots(1)$$

$$= \int \frac{-1}{(1 - z^2)(1 - 2z)} \, dz = \int \frac{-1}{(1 - z)(1 + z)(1 - 2z)} \, dz$$

Let $\dfrac{-1}{(1 - z)(1 + z)(1 - 2z)} = \dfrac{A}{(1 - z)} + \dfrac{B}{(1 + z)} + \dfrac{C}{(1 - 2z)}$ ...(2)

Multiplying both sides by $(1 - z)(1 + z)(1 - 2z)$, we get

$$-1 = A(1 + z)(1 - 2z) + B(1 - z)(1 - 2z) + C(1 - z)(1 + z) \qquad \ldots(3)$$

$1 - z = 0 \ \Rightarrow \ z = 1$

$1 + z = 0 \ \Rightarrow \ z = -1$

$1 - 2z = 0 \ \Rightarrow \ z = \frac{1}{2}$

Put $z = 1$ in (3), we get

$$-1 = A(1 + 1)(1 - 2) + B(1 - 1)(1 - 2) + C(1 - 1)(1 + 1)$$

$\Rightarrow \quad -1 = -2A \ \Rightarrow \ A = \frac{1}{2}$

Put $z = -1$ in (3), we get

$$-1 = A(-1 + 1)[1 - 2(-1)] + B[1 - (-1)][1 - 2(-1)] + C[1 - (-1)][1 + (-1)]$$

$\Rightarrow \quad -1 = 6B \ \Rightarrow \ B = -\frac{1}{6}$

Put $z = \frac{1}{2}$ in (3), we get

$$-1 = A\left(1 + \frac{1}{2}\right)\left(1 - 2.\frac{1}{2}\right) + B\left(1 - \frac{1}{2}\right)\left(1 - 2.\frac{1}{2}\right) + C\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)$$

$$\Rightarrow \quad -1 = C\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \Rightarrow -1 = \frac{3}{4}C \Rightarrow C = -\frac{4}{3}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{-1}{(1-z)(1+z)(1-2z)} = \frac{1/2}{(1-z)} + \frac{-1/6}{(1+z)} + \frac{-4/3}{(1-2z)} = \frac{1}{2(1-z)} - \frac{1}{6(1+z)} - \frac{4}{3(1-2z)}$$

$$\therefore \quad I = \int \frac{-1}{(1-z)(1+z)(1-2z)}\,dz = \int\left[\frac{1}{2(1-z)} - \frac{1}{6(1+z)} - \frac{4}{3(1-2z)}\right]dz$$

$$= \frac{1}{2}\int \frac{1}{1-z}\,dz - \frac{1}{6}\int \frac{1}{1+z}\,dz - \frac{4}{3}\int \frac{1}{1-2z}\,dz$$

$$= \frac{1}{2}\log|1-z| - \frac{1}{6}\log|1+z| - \frac{4}{3}\,\frac{\log|1-2z|}{-2} + c$$

$$\left[\because \int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c\right]$$

$$= \frac{1}{2}\log|1-z| - \frac{1}{6}\log|1+z| + \frac{2}{3}\log|1-2z| + c$$

$$\therefore \quad I = \frac{1}{2}\log|1-\cos x| - \frac{1}{6}\log|1+\cos x| + \frac{2}{3}\log|1-2\cos x| + c. \qquad [\because z = \cos x]$$

(*iv*) Let $I = \int \frac{1}{\sin x\,(3+2\cos x)}\,dx$

$$= \int \frac{\sin x}{\sin^2 x\,(3+2\cos x)}\,dx \qquad \text{[Multiply and divided by } \sin x\text{]}$$

$$= \int \frac{\sin x}{(1-\cos^2 x)(3+2\cos x)}\,dx \qquad \left[\begin{array}{l}\because \sin^2 A + \cos^2 A = 1\\ \Rightarrow \sin^2 A = 1-\cos^2 A\end{array}\right]$$

$$I = \int \frac{\sin x}{(1-\cos x)(1+\cos x)(3+2\cos x)}\,dx$$

Put $\cos x = z \Rightarrow -\sin x\,dx = dz \Rightarrow \sin x\,dx = -dz$

$$\therefore \quad I = \int \frac{-1}{(1-z)(1+z)(3+2z)}\,dz \qquad ...(1)$$

Let $$\frac{-1}{(1-z)(1+z)(3+2z)} = \frac{A}{(1-z)} + \frac{B}{(1+z)} + \frac{C}{(3+2z)} \qquad ...(2)$$

Multiplying both sides by $(1-z)(1+z)(3+2z)$, we get

$$-1 = A(1+z)(3+2z) + B(1-z)(3+2z) + C(1-z)(1+z) \qquad ...(3)$$

$1 + z = 0 \Rightarrow z = -1$

$1 - z = 0 \Rightarrow z = 1$

$3 + 2z = 0 \Rightarrow 2z = -3 \Rightarrow z = -\frac{3}{2}$

Put $z = -1$ in (3), we get

$$-1 = A[+1+(-1)][3+2(-1)] + B[1-(-1)][3+2(-1)] + C[+1+(-1)][1-(-1)]$$

$$\Rightarrow \quad -1 = B(2)(1) \Rightarrow B = -\frac{1}{2}$$

Put $z = 1$ in (3), we get

$$-1 = A(1+1)(3+2) + B(1-1)(3+2) + C(1+1)(1-1) \Rightarrow -1 = 10A \Rightarrow A = -\frac{1}{10}$$

Put $z = -\frac{3}{2}$ in (3), we get

$$-1 = A\left[1+\left(-\frac{3}{2}\right)\right]\left[3+2\left(-\frac{3}{2}\right)\right] + B\left[1-\left(-\frac{3}{2}\right)\right]\left[3+2\left(-\frac{3}{2}\right)\right] + C\left[1+\left(-\frac{3}{2}\right)\right]\left[1-\left(-\frac{3}{2}\right)\right]$$

$$\Rightarrow \quad -1 = C\left(-\frac{1}{2}\right)\left(\frac{5}{2}\right) \Rightarrow -1 = -\frac{5}{4}C \Rightarrow C = \frac{4}{5}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{-1}{(1-z)(1+z)(3+2z)} = \frac{-1/10}{(1-z)} + \frac{-1/2}{(1+z)} + \frac{4/5}{(3+2z)} = \frac{-1}{10(1-z)} - \frac{1}{2(1+z)} + \frac{4}{5(3+2z)}$$

$$\therefore \quad I = \int \frac{-1}{(1-z)(1+z)(3+2z)}\,dz = \int \left[\frac{-1}{10(1-z)} - \frac{1}{2(1+z)} + \frac{4}{5(3+2z)}\right] dz$$

$$= \frac{-1}{10}\int \frac{1}{1-z}\,dz - \frac{1}{2}\int \frac{1}{1+z}\,dz + \frac{4}{5}\int \frac{1}{3+2z}\,dz$$

$$= \frac{-1}{10}\log\frac{|1-z|}{(-1)} - \frac{1}{2}\log|1+z| + \frac{4}{5}\,\frac{\log|3+2z|}{2} + c$$

$$\left[\because \int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c\right]$$

$$= \frac{1}{10}\log|1-z| - \frac{1}{2}\log|1+z| + \frac{2}{5}\log|3+2z| + c$$

$$= \frac{1}{10}\log|1-\cos x| - \frac{1}{2}\log|1+\cos x| + \frac{2}{5}\log|3+2\cos x| + c. \qquad [\because z = \cos x]$$

(v) Let $I = \int \frac{\sec x}{1+\operatorname{cosec} x}\,dx$

$$= \int \frac{\frac{1}{\cos x}}{1+\frac{1}{\sin x}}\,dx = \int \frac{\sin x}{\cos x\,(1+\sin x)}\,dx$$

$$= \int \frac{\sin x \cos x}{\cos^2 x\,(1+\sin x)}\,dx \qquad \text{[Multiply and divided by } \cos x]$$

$$= \int \frac{\sin x \cos x}{(1-\sin^2 x)(1+\sin x)}\,dx \qquad \left[\begin{array}{l}\because \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \cos^2 A = 1 - \sin^2 A\end{array}\right]$$

$$I = \int \frac{\sin x \cos x}{(1-\sin x)(1+\sin x)^2}\,dx$$

Put $\sin x = z \Rightarrow \cos x\, dx = dz$

$$\therefore \quad I = \int \frac{z}{(1-z)(1+z)^2}\, dz \qquad ...(1)$$

Let $$\frac{z}{(1-z)(1+z)^2} = \frac{A}{(1-z)} + \frac{B}{(1+z)} + \frac{C}{(1+z)^2} \qquad ...(2)$$

Multiplying both sides by $(1-z)(1+z)^2$, we get

$$z = A(1+z)^2 + B(1-z)(1+z) + C(1-z) \qquad ...(3)$$

$1 - z = 0 \Rightarrow z = 1$

$1 + z = 0 \Rightarrow z = -1$

Put $z = 1$ in (3), we get

$$1 = A(1+1)^2 + B(1-1)(1+1) + C(1-1) \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$$

Put $z = -1$ in (3), we get

$$-1 = A[1+(-1)]^2 + B[1-(-1)][1+(-1)] + C[1-(-1)]$$

$$\Rightarrow \quad -1 = 2C \Rightarrow C = -\frac{1}{2}$$

Equating the constant terms on both sides of equation (3), we have

$$0 = A + B + C \Rightarrow 0 = \frac{1}{4} + B - \frac{1}{2} \Rightarrow B = \frac{1}{4}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{z}{(1-z)(1+z)^2} = \frac{1/4}{(1-z)} + \frac{1/4}{(1+z)} + \frac{-1/2}{(1+z)^2} = \frac{1}{4(1-z)} + \frac{1}{4(1+z)} - \frac{1}{2(1+z)^2}$$

$$\therefore \quad I = \int \frac{z}{(1-z)(1+z)^2}\, dz = \int \left[\frac{1}{4(1-z)} + \frac{1}{4(1+z)} - \frac{1}{2(1+z)^2}\right] dz$$

$$= \frac{1}{4}\int \frac{1}{1-z}\, dz + \frac{1}{4}\int \frac{1}{1+z}\, dz - \frac{1}{2}\int (1+z)^{-2}\, dz$$

$$= \frac{1}{4}\frac{\log|1-z|}{(-1)} + \frac{1}{4}\log|1+z| - \frac{1}{2}\frac{(1+z)^{-2+1}}{(-2+1)} + c$$

$$= -\frac{1}{4}\log|1-z| + \frac{1}{4}\log|1+z| + \frac{1}{2(1+z)} + c$$

$$= \frac{1}{4}\log\left|\frac{1+z}{1-z}\right| + \frac{1}{2(1+z)} + c \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{1}{4}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{2(1+\sin x)} + c. \qquad [\because z = \sin x]$$

(*vi*) Let $$I = \int \frac{1+\sin x}{\sin x(1+\cos x)}\, dx$$

$$\Rightarrow \quad I = \int \frac{1}{\sin x(1+\cos x)}\, dx + \int \frac{\sin x}{\sin x(1+\cos x)}\, dx$$

$$\Rightarrow \quad I = I_1 + I_2 \quad \text{(say)} \qquad ...(1)$$

where $$I_1 = \int \frac{1}{\sin x\,(1+\cos x)}\,dx$$

$$= \int \frac{\sin x}{\sin^2 x\,(1+\cos x)}\,dx \qquad \text{[Multiply and divide by } \sin x]$$

$$= \int \frac{\sin x}{(1-\cos^2 x)(1+\cos x)}\,dx \qquad \left[\begin{array}{l}\because\ \sin^2 A + \cos^2 A = 1\\ \Rightarrow\ \sin^2 A = 1-\cos^2 A\end{array}\right]$$

$$= \int \frac{\sin x}{(1-\cos x)(1+\cos x)^2}\,dx$$

Put $\cos x = z \ \Rightarrow\ -\sin x\,dx = dz \ \Rightarrow\ \sin x\,dx = -dz$

$$\therefore \quad I_1 = \int \frac{1}{(1-z)(1+z)^2}\,(-dz) = \int \frac{-1}{(1-z)(1+z)^2}\,dz$$

Let $$\frac{-1}{(1-z)(1+z)^2} = \frac{A}{(1-z)} + \frac{B}{(1+z)} + \frac{C}{(1+z)^2} \qquad \text{...(2)}$$

Multiplying both sides by $(1-z)(1+z)^2$, we get

$$-1 = A(1+z)^2 + B(1-z)(1+z) + C(1-z) \qquad \text{...(3)}$$

$$1-z=0 \ \Rightarrow\ z=1,\ 1+z=0 \ \Rightarrow\ z=-1$$

Put $z = 1$ in (3), we get

$$-1 = A(1+1)^2 + B(1-1)(1+1) + C(1-1) \ \Rightarrow\ -1 = 4A \ \Rightarrow\ A = -\frac{1}{4}$$

Put $z = -1$ in (3), we get

$$-1 = A\,[1+(-1)]^2 + B\,[1-(-1)]\,[1+(-1)] + C\,[1-(-1)]$$

$$\Rightarrow \quad -1 = 2C \ \Rightarrow\ C = -\frac{1}{2}$$

Equating constant terms on both sides of equation (3), we get

$$-1 = A + B + C$$

$$-1 = -\frac{1}{4} + B - \frac{1}{2}$$

$$-1 = -\frac{3}{4} + B \ \Rightarrow\ B = -1 + \frac{3}{4} \ \Rightarrow\ B = -\frac{1}{4}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{-1}{(1-z)(1+z)^2} = \frac{-1/4}{(1-z)} + \frac{-1/4}{(1+z)} + \frac{-1/2}{(1+z)^2} = -\frac{1}{4(1-z)} - \frac{1}{4(1+z)} - \frac{1}{2(1+z)^2}$$

$$\therefore \quad I_1 = \int \frac{-1}{(1-z)(1+z)^2}\,dz = \int \left[\frac{-1}{4(1-z)} - \frac{1}{4(1+z)} - \frac{1}{2(1+z)^2}\right] dz$$

$$= -\frac{1}{4}\int \frac{1}{1-z}\,dz - \frac{1}{4}\int \frac{1}{1+z}\,dz - \frac{1}{2}\int (1+z)^{-2}\,dz$$

$$= -\frac{1}{4}\,\frac{\log|1-z|}{(-1)} - \frac{1}{4}\log|1+z| - \frac{1}{2}\,\frac{(1+z)^{-2+1}}{(-2+1)} + c_1$$

$$= \frac{1}{4}\log|1-z| - \frac{1}{4}\log|1+z| + \frac{1}{2(1+z)} + c_1$$

$$= \frac{1}{4}\log\left|\frac{1-z}{1+z}\right| + \frac{1}{2(1+z)} + c_1 \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \quad I_1 = \frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos x}\right| + \frac{1}{2(1+\cos x)} + c_1 \qquad \text{...(4)} \; [\because z = \cos x]$$

and $$I_2 = \int \frac{1}{1+\cos x}\, dx$$

$$= \int \frac{1}{2\cos^2 \frac{x}{2}}\, dx \qquad \left[\begin{array}{l}\because 1+\cos 2A = 2\cos^2 A \\ \Rightarrow 1+\cos A = 2\cos^2 \frac{A}{2}\end{array}\right]$$

$$= \frac{1}{2}\int \sec^2 \frac{x}{2}\, dx = \frac{1}{2}\cdot\frac{\tan\frac{x}{2}}{1/2} + c_2$$

$$\Rightarrow \quad I_2 = \tan\frac{x}{2} + c_2 \qquad \text{...(5)}$$

$\therefore$ From equation (1), we have

$$I = I_1 + I_2$$

$$\Rightarrow \quad I = \frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos x}\right| + \frac{1}{2(1+\cos x)} + c_1 + \tan\frac{x}{2} + c_2 \qquad \text{[Using (4) and (5)]}$$

$$\Rightarrow \quad I = \frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos x}\right| + \frac{1}{2(1+\cos x)} + \tan\frac{x}{2} + c. \qquad \text{where } c = c_1 + c_2.$$

Example 12. *Evaluate the following integrals* :

(i) $\displaystyle\int \frac{2\sin 2\theta - \cos\theta}{6 - \cos^2\theta - 4\sin\theta}\, d\theta$ (ii) $\displaystyle\int \frac{(3\sin x - 2)\cos x}{5 - \cos^2 x - 4\sin x}\, dx$

(iii) $\displaystyle\int \frac{x^4}{x^4 - 16}\, dx$ (iv) $\displaystyle\int \frac{\tan x + \tan^3 x}{1 + \tan^3 x}\, dx.$

Solution. (i) Let $I = \displaystyle\int \frac{2\sin 2\theta - \cos\theta}{6 - \cos^2\theta - 4\sin\theta}\, d\theta$

$$= \int \frac{2(2\sin\theta\cos\theta) - \cos\theta}{6 - (1 - \sin^2\theta) - 4\sin\theta}\, d\theta \qquad \left[\begin{array}{l}\because \sin 2A = 2\sin A\cos A \\ \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \cos^2 A = 1 - \sin^2 A\end{array}\right]$$

$$= \int \frac{(4\sin\theta - 1)\cos\theta}{5 - 4\sin\theta + \sin^2\theta}\, d\theta = \int \frac{(4\sin\theta - 1)\cos\theta}{\sin^2\theta - 4\sin\theta + 5}\, d\theta$$

Put $\sin\theta = z \Rightarrow \cos\theta\, d\theta = dz$

$$\therefore \quad I = \int \frac{(4z-1)}{z^2 - 4z + 5}\, dz$$

Since denominator is not factorizable.

$\therefore$ Let $4z - 1 = \lambda \dfrac{d}{dz}[z^2 - 4z + 5] + \mu$

$\Rightarrow$ $(4z - 1) = \lambda(2z - 4) + \mu$ ...(1)

$\Rightarrow$ $4z - 1 = 2\lambda z - 4\lambda + \mu$

Equating the co-efficients of like power terms of z and the constant terms on both sides, we have

$$4 = 2\lambda \Rightarrow \lambda = 2$$

and $\quad -1 = -4\lambda + \mu \Rightarrow \mu = -1 + 4\lambda = -1 + 4(2) = 7 \Rightarrow \mu = 7$

$\therefore$ Equation (1) becomes $(4z - 1) = 2(2z - 4) + 7$

$$\therefore \quad I = \int \frac{(4z-1)}{z^2 - 4z + 5}\,dz = \int \frac{2(2z-4)+7}{z^2-4z+5}\,dz$$

$$= 2\int \frac{2z-4}{z^2-4z+5}\,dz + 7\int \frac{1}{z^2-4z+5}\,dz$$

$$= 2\log|z^2 - 4z + 5| + 7\int \frac{1}{z^2-4z+5}\,dz \qquad \left[\because \int \frac{f'(z)}{f(z)}\,dz = \log|f(z)| + c\right]$$

$$= 2\log|z^2 - 4z + 5| + 7\int \frac{1}{(z^2-4z+4)+(5-4)}\,dz$$

$$\left[\begin{array}{l}\text{Add \& subtract 4 to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } z\right)^2 = 4\end{array}\right]$$

$$= 2\log|z^2 - 4z + 5| + 7\int \frac{1}{(z-2)^2+(1)^2}\,dz$$

$$= 2\log|z^2 - 4z + 5| + 7\,.\,\frac{1}{1}\tan^{-1}\frac{(z-2)}{1} + c$$

$$\left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= 2\log|z^2 - 4z + 5| + 7\tan^{-1}(z-2) + c$$

$$= 2\log|\sin^2\theta - 4\sin\theta + 5| + 7\tan^{-1}(\sin\theta - 2) + c. \qquad [\because z = \sin\theta]$$

(*ii*) Let $\quad I = \int \frac{(3\sin x - 2)\cos x}{5 - \cos^2 x - 4\sin x}\,dx$

$$= \int \frac{(3\sin x - 2)\cos x}{5 - (1 - \sin^2 x) - 4\sin x}\,dx \qquad \left[\begin{array}{l}\because \sin^2 A + \cos^2 A = 1\\ \Rightarrow \cos^2 A = 1 - \sin^2 A\end{array}\right]$$

$$= \int \frac{(3\sin x - 2)\cos x}{\sin^2 x - 4\sin x + 4}\,dx \qquad ...(1)$$

Put $\quad \sin x = z \Rightarrow \cos x\,dx = dz$

$$\therefore \quad I = \int \frac{(3z-2)}{z^2-4z+4}\,dz = \int \frac{3z-2}{(z-2)^2}\,dz$$

Let $\frac{3z-2}{(z-2)^2} = \frac{A}{(z-2)} + \frac{B}{(z-2)^2}$ ...(2)

Multiplying both sides by $(z-2)^2$, we get

$(3z - 2) = A(z - 2) + B$ ...(3)

Put $z = 2$ in (3), we get

$3(2) - 2 = A(2 - 2) + B \Rightarrow \underline{4 = B} \Rightarrow B = 4$

Equating the constant terms on both sides of equation (3), we get

$$-2 = -2A + B \quad \Rightarrow \quad -2 = -2A + 4 \quad \Rightarrow \quad -2A = -6 \quad \Rightarrow \quad A = 3$$

Substituting the values of A and B in equation (2), we have

$$\frac{3z-2}{(z-2)^2} = \frac{3}{z-2} + \frac{4}{(z-2)^2}$$

$$\therefore \qquad I = \int \frac{3z-2}{(z-2)^2}\,dz = \int \left[\frac{3}{z-2} + \frac{4}{(z-2)^2}\right] dz = 3\int \frac{1}{z-2}\,dz + 4\int (z-2)^{-2}\,dz$$

$$= 3\log|z-2| + \frac{4(z-2)^{-2+1}}{(-2+1)} + c = 3\log|z-2| - \frac{4}{z-2} + c$$

$$= 3\log|\sin x - 2| - \frac{4}{(\sin x - 2)} + c. \qquad [\because\ z = \sin x]$$

(*iii*) Let $I = \int \frac{x^4}{x^4 - 16}\,dx$

Since the integrand is not a proper fraction, therefore, by actual division, we have

$$I = \int \frac{x^4}{x^4-16}\,dx = \int \left(1 + \frac{16}{x^4-16}\right) dx$$

$$\Rightarrow \qquad I = \int 1.\,dx + \int \frac{16}{x^4-16}\,dx \qquad \text{...(1)}$$

$$\begin{array}{r|l} x^4 - 16 & x^4 \ (1 \\ & +x^4 - 16 \\ & -\quad + \\ \hline & 16 \end{array}$$

We have $\frac{16}{x^4-16} = \frac{16}{(x^2)^2 - (4)^2} = \frac{16}{(x^2-4)(x^2+4)}$

$\therefore$ Let Put $x^2 = z$

$$\frac{16}{(z-4)(z+4)} = \frac{A}{z-4} + \frac{B}{z+4} \qquad \text{...(2)}$$

Multiplying both sides by $(z-4)(z+4)$, we get

$$16 = A(z+4) + B(z-4) \qquad \text{...(3)}$$

$$z + 4 = 0 \quad \Rightarrow \quad z = -4,\ z - 4 = 0 \quad \Rightarrow \quad z = 4$$

Put $z = -4$ in (3), we get

$$16 = A(-4+4) + B(-4-4) \quad \Rightarrow \quad 16 = -8B \quad \Rightarrow \quad B = -2$$

Put $z = 4$ in (3), we get

$$16 = A(4+4) + B(4-4) \quad \Rightarrow \quad 16 = 8A \quad \Rightarrow \quad A = 2$$

Substituting the values of A and B in equation (2), we have

$$\frac{16}{(z-4)(z+4)} = \frac{2}{z-4} + \frac{-2}{z+4}$$

or

$$\frac{16}{(x^2-4)(x^2+4)} = \frac{2}{x^2-4} - \frac{2}{x^2+4} \qquad \text{...(4)} \quad [\because\ z = x^2]$$

$$\therefore \qquad I = \int \frac{x^4}{x^4-16}\,dx = \int \left(1 + \frac{16}{x^4-16}\right) dx \qquad [\because\ \text{By using (4)}]$$

$$= \int \left(1 + \frac{2}{x^2-4} - \frac{2}{x^2+4}\right) dx$$

$$= \int 1 . dx + 2\int \frac{1}{x^2 - 2^2} dx - 2\int \frac{1}{x^2 + 2^2} dx$$

$$= x + 2 . \frac{1}{2(2)} \log\left|\frac{x-2}{x+2}\right| - 2 . \frac{1}{2} \tan^{-1}\frac{x}{2} + c$$

$$= x + \frac{1}{2}\log\left|\frac{x-2}{x+2}\right| - \tan^{-1}\frac{x}{2} + c. \qquad \left[\because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c \atop \int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

(iv) Let $I = \int \frac{\tan x + \tan^3 x}{1 + \tan^3 x} dx = \int \frac{\tan x (1 + \tan^2 x)}{1 + \tan^3 x} dx$ $\quad \left[\because \sec^2 A - \tan^2 A = 1 \atop \Rightarrow \sec^2 A = 1 + \tan^2 A\right]$

$$\Rightarrow \qquad I = \int \frac{\tan x \sec^2 x}{1 + \tan^3 x} dx \qquad ...(1)$$

Put $\tan x = z \;\Rightarrow\; \sec^2 x\, dx = dz$

$$\therefore \qquad I = \int \frac{z}{1 + z^3} dz$$

We have $\frac{z}{1+z^3} = \frac{z}{(1+z)(1-z+z^2)}$ $\quad [\because (a^3 + b^3) = (a + b)(a^2 - ab + b^2)]$

$$\therefore \text{ Let } \frac{z}{(1+z)(1-z+z^2)} = \frac{A}{1+z} + \frac{Bz + C}{(z^2 - z + 1)} \qquad ...(2)$$

Multiplying both sides by $(1 + z)(1 - z + z^2)$, we get

$$z = A(z^2 - z + 1) + (Bz + C)(1 + z) \qquad ...(3)$$

Put $z = -1$ in (3), we get

$$-1 = A[(-1)^2 - (-1) + 1] + [B(-1) + C][1 + (-1)]$$

$$\Rightarrow \qquad -1 = A(1 + 1 + 1) \;\Rightarrow\; -1 = 3A \;\Rightarrow\; A = -\frac{1}{3}.$$

Equating co-efficients of z^2 on both sides of equation (3), we get

$$0 = A + B \;\Rightarrow\; B = -A \;\Rightarrow\; B = -\left(-\frac{1}{3}\right) = \frac{1}{3}$$

Equating the constant terms on both sides of equation (3), we get

$$0 = A + C \;\Rightarrow\; C = -A \;\Rightarrow\; C = \frac{1}{3}.$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{z}{(1+z)(1-z+z^2)} = \frac{-\frac{1}{3}}{(1+z)} + \frac{\frac{1}{3}z + \frac{1}{3}}{(z^2 - z + 1)} = \frac{-1}{3(1+z)} + \frac{1}{3}\frac{z+1}{(z^2 - z + 1)}$$

$$\therefore \qquad I = \int \frac{z}{1+z^3} dz = \int \left[\frac{-1}{3(1+z)} + \frac{(z+1)}{3(z^2 - z + 1)}\right] dz$$

$$= -\frac{1}{3}\int \frac{1}{1+z} dz + \frac{1}{3}\int \frac{(z+1)}{z^2 - z + 1} dz$$

$$= -\frac{1}{3}\int \frac{1}{1+z}\,dz + \frac{1}{6}\int \frac{2z+2}{z^2-z+1}\,dz$$

[Multiply and divide the second integral by 2]

$$= -\frac{1}{3}\int \frac{1}{1+z}\,dz + \frac{1}{6}\int \frac{2z-1+3}{z^2-z+1}\,dz$$ [Note this step]

$$= -\frac{1}{3}\int \frac{1}{1+z}\,dz + \frac{1}{6}\int \frac{2z-1}{z^2-z+1}\,dz + \frac{3}{6}\int \frac{1}{z^2-z+1}\,dz$$

$$= -\frac{1}{3}\log|1+z| + \frac{1}{6}\log|z^2-z+1| + \frac{1}{2}\int \frac{1}{z^2-z+1}\,dz$$

$$\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

$$= -\frac{1}{3}\log|1+z| + \frac{1}{6}\log|z^2-z+1| + \frac{1}{2}\int \frac{1}{z^2-z+\frac{1}{4}+1-\frac{1}{4}}\,dz$$

$$\left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denominator}\\ \because \left(\frac{1}{2}\text{ co-eff. of } z\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= -\frac{1}{3}\log|1+z| + \frac{1}{6}\log|z^2-z+1| + \frac{1}{2}\int \frac{1}{\left(z-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dz$$

$$= -\frac{1}{3}\log|1+z| + \frac{1}{6}\log|z^2-z+1| + \frac{1}{2}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\frac{\left(z-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c$$

$$\left[\because \text{By using} \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= -\frac{1}{3}\log|1+z| + \frac{1}{6}\log|z^2-z+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2z-1}{\sqrt{3}}\right) + c$$

$$\therefore\quad I = -\frac{1}{3}\log|1+\tan x| + \frac{1}{6}\log|\tan^2 x - \tan x + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x - 1}{\sqrt{3}}\right) + c.$$

$[\because\ z = \tan x]$

Example 13. *Evaluate the following integrals :*

(*i*) $\displaystyle\int \frac{1}{x\,[6(\log x)^2 + 7\log x + 2]}\,dx$ (*ii*) $\displaystyle\int \frac{\sin x}{\sin 4x}\,dx$

(*iii*) $\displaystyle\int \frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3}\,dx$ (*iv*) $\displaystyle\int \frac{1}{x\log x\,(2+\log x)}\,dx$

(v) $\int \frac{\sqrt{\cos x}}{\sin x}\, dx$ (vi) $\int \frac{a^x}{a^{2x} - 6a^x + 5}\, dx$

(vii) $\int \frac{1}{2e^{2x} + 3e^x + 1}\, dx.$

Solution. (i) Let $I = \int \frac{1}{x[6(\log x)^2 + 7\log x + 2]}\, dx$ ...(1)

Put $\log x = z \Rightarrow \frac{1}{x}\, dx = dz$

$\therefore \quad I = \int \frac{1}{6z^2 + 7z + 2}\, dz = \int \frac{1}{(2z+1)(3z+2)}\, dz$

Let $\frac{1}{(2z+1)(3z+2)} = \frac{A}{(2z+1)} + \frac{B}{(3z+2)}$ ...(2)

Multiplying both sides by $(2z + 1)(3z + 2)$, we get

$$1 = A(3z + 2) + B(2z + 1) \qquad ...(3)$$

$$2z + 1 = 0 \Rightarrow z = -\frac{1}{2}$$

$$3z + 2 = 0 \Rightarrow z = -\frac{2}{3}$$

Put $z = -\frac{1}{2}$ in (3), we get

$$1 = A\left[3\left(-\frac{1}{2}\right) + 2\right] + B\left[2\left(-\frac{1}{2}\right) + 1\right]$$

$$\Rightarrow \quad 1 = A\left(2 - \frac{3}{2}\right) \Rightarrow 1 = \frac{1}{2}A \Rightarrow A = 2$$

Put $z = \left(-\frac{2}{3}\right)$ in (3), we get

$$1 = A\left[3\left(-\frac{2}{3}\right) + 2\right] + B\left[2\left(-\frac{2}{3}\right) + 1\right]$$

$$\Rightarrow \quad 1 = B\left(-\frac{4}{3} + 1\right) \Rightarrow 1 = -\frac{1}{3}B \Rightarrow B = -3.$$

Substituting the values of A and B in equation (2), we have

$$\frac{1}{(2z+1)(3z+2)} = \frac{2}{2z+1} + \frac{-3}{3z+2} = \left(\frac{2}{2z+1} - \frac{3}{3z+2}\right)$$

$$\therefore \quad I = \int \frac{1}{(2z+1)(3z+2)}\, dz = \int \left[\frac{2}{2z+1} - \frac{3}{3z+2}\right] dz$$

$$= 2\int \frac{1}{2z+1}\, dz - 3\int \frac{1}{3z+2}\, dz$$

$$= 2\,\frac{\log|2z+1|}{2} - 3\,\frac{\log|3z+2|}{3} + c \qquad \left[\because \int \frac{1}{ax+b}\, dx = \frac{\log|ax+b|}{a} + c\right]$$

$$= \log|2z+1| - \log|3z+2| + c$$

$$= \log\left|\frac{2z+1}{3z+2}\right| + c \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \log\left|\frac{2\log x+1}{3\log x+2}\right| + c. \qquad [\because z = \log x]$$

(*ii*) Let $I = \int \frac{\sin x}{\sin 4x}\,dx$

$$\therefore \quad I = \int \frac{\sin x}{2\sin 2x\cos 2x}\,dx = \int \frac{\sin x}{2(2\sin x\cos x)\cos 2x}\,dx \qquad \left[\begin{array}{l}\because \sin 2A = 2\sin A\cos A\\ \Rightarrow \sin 4A = 2\sin 2A\cos 2A\end{array}\right]$$

$$= \int \frac{1}{4\cos x(1-2\sin^2 x)}\,dx \qquad [\because \cos 2A = 1 - 2\sin^2 A]$$

$$= \frac{1}{4}\int \frac{\cos x}{\cos^2 x(1-2\sin^2 x)}\,dx \qquad \text{[Multiply and divide by } \cos x]$$

$$\Rightarrow \quad I = \frac{1}{4}\int \frac{\cos x}{(1-\sin^2 x)(1-2\sin^2 x)}\,dx \qquad \left[\begin{array}{l}\because \sin^2 A + \cos^2 A = 1\\ \Rightarrow \cos^2 A = 1 - \sin^2 A\end{array}\right]$$

Put $\sin x = z \;\Rightarrow\; \cos x\,dx = dz$

$$\therefore \quad I = \frac{1}{4}\int \frac{1}{(1-z^2)(1-2z^2)}\,dz$$

Let $\dfrac{1}{(1-z^2)(1-2z^2)} = \dfrac{1}{(1-y)(1-2y)} = \dfrac{A}{(1-y)} + \dfrac{B}{(1-2y)}$ ...(2) [Put $z^2 = y$]

Multiplying both sides by $(1-y)(1-2y)$, we get

$$1 = A(1-2y) + B(1-y) \qquad \text{...(3)}$$

$$1 - y = 0 \;\Rightarrow\; y = 1,\; 1 - 2y = 0 \;\Rightarrow\; y = \frac{1}{2}$$

Put $y = 1$ in (3), we get

$$1 = A(1-2) + B(1-1) \;\Rightarrow\; 1 = -A \;\Rightarrow\; A = -1$$

Put $y = \frac{1}{2}$ in (3), we get

$$1 = A\left[1 - 2\left(\frac{1}{2}\right)\right] + B\left[1 - \frac{1}{2}\right] \;\Rightarrow\; 1 = \frac{1}{2}B \;\Rightarrow\; B = 2$$

Substituting the values of A and B in equation (2), we have

$$\frac{1}{(1-y)(1-2y)} = \frac{-1}{(1-y)} + \frac{2}{(1-2y)}$$

or

$$\frac{1}{(1-z^2)(1-2z^2)} = \frac{-1}{(1-z^2)} + \frac{2}{(1-2z^2)} \qquad [\because y = z^2]$$

$$\therefore \quad I = \frac{1}{4}\int \frac{1}{(1-z^2)(1-2z^2)}\,dz = \frac{1}{4}\int \left[\frac{-1}{(1-z^2)} + \frac{2}{(1-2z^2)}\right]dz$$

$$= -\frac{1}{4}\int \frac{1}{1-z^2}\,dz + \frac{1}{4}\int \frac{2}{2\left(\frac{1}{2} - z^2\right)}\,dz$$

$$= -\frac{1}{4}\int \frac{1}{1^2 - z^2}\,dz + \frac{1}{4}\int \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2 - z^2}\,dz$$

$$= -\frac{1}{4}\cdot\frac{1}{2}\log\left|\frac{1+z}{1-z}\right| + \frac{1}{4}\cdot\frac{1}{2\left(\frac{1}{\sqrt{2}}\right)}\log\left|\frac{\frac{1}{\sqrt{2}}+z}{\frac{1}{\sqrt{2}}-z}\right| + c$$

$$\left[\because \int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

$$= -\frac{1}{8}\log\left|\frac{1+z}{1-z}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}z}{1-\sqrt{2}z}\right| + c$$

$$= -\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + c. \qquad [\because\ z = \sin x]$$

(*iii*) Let $I = \int \frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3}\,dx$ ...(1)

Let $\frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3} = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)^3}$ ...(2)

Multiplying both sides by $(x^2 + 2)^3$, we get

$x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4 = (Ax + B)(x^2 + 2)^2 + (Cx + D)(x^2 + 2) + (Ex + F).$

$= (Ax + B)(x^4 + 2x^2 + 4) + (Cx^3 + Dx^2 + 2Cx + D) + (Ex + F)$

$= Ax^5 + Bx^4 + (4A + C)x^3 + (4B + D)x^2 + (4A + 2C + E)x + (4B + 2D + F)$

...(3)

Equating the co-efficients of x^5 on both sides of equation (3), we get

$1 = A \Rightarrow A = 1$

Equating the co-efficients of x^4 on both sides of equation (3), we get

$-1 = B \Rightarrow B = -1$

Equating the co-efficients of x^3 on both sides of equation (3), we get

$4 = 4A + C \Rightarrow C = 4 - 4A \Rightarrow C = 4 - 4(1) \Rightarrow C = 0$

Equating the co-efficients of x^2 on both sides of equation (3), we get

$-4 = 4B + D \Rightarrow D = -4 - 4B \Rightarrow D = 0$

Equating the co-efficients of x on both sides of equation (3), we get

$8 = 4A + 2C + E \Rightarrow 8 = 4(1) + 2(0) + E \Rightarrow E = 4$

Equating the constant terms on both sides of equation (3), we get

$-4 = 4B + 2D + F \Rightarrow -4 = 4(-1) + 2(0) + F \Rightarrow F = 0$

Substituting the values of A, B, C, D, E and F in equation (2), we have

$$\frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3} = \frac{x-1}{x^2+2} + \frac{0x+0}{(x^2+2)^2} + \frac{4x+0}{(x^2+2)^3}$$

$\therefore \quad I = \int \frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3}\,dx = \int \left[\frac{x-1}{x^2+2} + \frac{4x}{(x^2+2)^3}\right] dx$

$$= \int \frac{x}{x^2+2}\,dx - \int \frac{1}{x^2+2}\,dx + 2\int \frac{2x}{(x^2+2)^3}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{x^2+2}\,dx - \int \frac{1}{x^2+\left(\sqrt{2}\right)^2}\,dx + 2\int (x^2+2)^{-3}\,2x\,dx$$

[Multiply and divide the first integral by 2]

$$= \frac{1}{2}\log|x^2+2| - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + 2\,\frac{(x^2+2)^{-3+1}}{(-3+1)} + c$$

$$\left[\begin{array}{l} \because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c \\ \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c \\ \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c \end{array}\right]$$

$$= \frac{1}{2}\log|x^2+2| - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{1}{(x^2+2)^2} + c.$$

(*iv*) Let $\quad I = \int \frac{1}{x\log x\,[2+\log x]}\,dx$

Put $\log x = z \;\Rightarrow\; \frac{1}{x}\,dx = dz$

$\therefore \quad I = \int \frac{1}{z(2+z)}\,dz \qquad \text{...(1)}$

Let $\frac{1}{z(2+z)} = \frac{A}{z} + \frac{B}{(2+z)} \qquad \text{...(2)}$

Multiplying both sides by $z(2+z)$, we get

$1 = A(2+z) + Bz \qquad \text{...(3)}$

Put $z = 0$ in (3), we get

$$1 = A(2+0) \;\Rightarrow\; A = \frac{1}{2}$$

Put $z = -2$ in (3), we get

$$1 = A[2+(-2)] + B(-2) \;\Rightarrow\; 1 = -2B \;\Rightarrow\; B = -\frac{1}{2}$$

Substituting the values of A and B in equation (2), we have

$$\frac{1}{z(2+z)} = \frac{1/2}{z} + \frac{-1/2}{(2+z)} = \frac{1}{2z} - \frac{1}{2(2+z)}$$

$\therefore \quad I = \int \frac{1}{z(2+z)}\,dz = \int \left[\frac{1}{2z} - \frac{1}{2(2+z)}\right] dz$

$= \frac{1}{2}\int \frac{1}{z}\,dz - \frac{1}{2}\int \frac{1}{2+z}\,dz = \frac{1}{2}\log|z| - \frac{1}{2}\log|2+z| + c$

$= \frac{1}{2}\log\left|\frac{z}{2+z}\right| + c \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$

$= \frac{1}{2}\log\left|\frac{\log x}{2+\log x}\right| + c. \qquad [\because z = \log x]$

(*v*) Let $\quad I = \int \frac{\sqrt{\cos x}}{\sin x}\,dx$

$= \int \frac{\sqrt{\cos x}\,.\,\sin x}{\sin^2 x}\,dx$ [Multiply and divide by $\sin x$]

$= \int \frac{\sqrt{\cos x}\,\sin x}{(1-\cos^2 x)}\,dx \qquad \left[\begin{array}{l}\because \sin^2 A + \cos^2 A = 1 \\ \Rightarrow \sin^2 A = 1 - \cos^2 A\end{array}\right]$

Put $\cos x = z^2 \Rightarrow \sqrt{\cos x} = z \Rightarrow -\sin x\,dx = 2z\,dz \Rightarrow \sin x\,dx = -2z\,dz$

$\therefore \quad I = \int \frac{z}{(1-z^4)}(-2z\,dz) = \int \frac{2z^2}{(z^4-1)}\,dz = \int \frac{z^2+z^2}{(z^4-1)}\,dz$ [Note this step]

$= \int \frac{z^2+1+z^2-1}{(z^2+1)(z^2-1)}\,dz$ [Add and subtract 1 to the numerator]

$= \int \frac{1}{z^2-1}\,dz + \int \frac{1}{z^2+1}\,dz$

$= \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + \tan^{-1} z + c \qquad \left[\begin{array}{l}\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c \\ \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\end{array}\right]$

$= \frac{1}{2}\log\left|\frac{\sqrt{\cos x}-1}{\sqrt{\cos x}+1}\right| + \tan^{-1}\left(\sqrt{\cos x}\right) + c. \qquad [\because z = \sqrt{\cos x}\,]$

(*vi*) Let $\quad I = \int \frac{a^x}{a^{2x} - 6a^x + 5}\,dx$

Put $\quad a^x = z \Rightarrow a^x \log a\,dx = dz \Rightarrow a^x\,dx = \frac{1}{\log a}\,.\,dz$

$\therefore \quad I = \int \frac{1}{z^2-6z+5}\cdot\left(\frac{1}{\log a}\,dz\right) = \frac{1}{\log a}\int \frac{1}{z^2-6z+5}\,dz$

$\Rightarrow \quad I = \frac{1}{\log a}\int \frac{1}{(z-1)(z-5)}\,dz \qquad \left[\begin{array}{l}\because z^2-6z+5 \\ = z^2-5z-z+5 \\ = z(z-5)-1(z-5) \\ = (z-1)(z-5)\end{array}\right]$

Let $\dfrac{1}{(z-1)(z-5)} = \dfrac{A}{(z-1)} + \dfrac{B}{(z-5)}$

Multiplying both sides by $(z-1)(z-5)$, we get

$$1 = A(z-5) + B(z-1)$$

$$z - 1 = 0 \Rightarrow z = 1, z - 5 = 0 \Rightarrow z = 5$$

Put $z = 1$ in (3), we get

$$1 = A(1-5) + B(1-1) \Rightarrow 1 = -4A \Rightarrow A = -\frac{1}{4}$$

Put $z = 5$ in (3), we get

$$1 = A(5-5) + B(5-1) \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$$

Substituting the values of A and B in equation (2), we have

$$\frac{1}{(z-1)(z-5)} = \frac{-1/4}{(z-1)} + \frac{1/4}{(z-5)} = -\frac{1}{4(z-1)} + \frac{1}{4(z-5)}$$

$$\therefore \quad I = \frac{1}{\log a}\int \frac{1}{(z-1)(z-5)}\,dz = \frac{1}{\log a}\int\left[-\frac{1}{4(z-1)} + \frac{1}{4(z-5)}\right]dz$$

$$= \frac{1}{\log a}\left[-\frac{1}{4}\int \frac{1}{z-1}\,dz + \frac{1}{4}\int \frac{1}{z-5}\,dz\right]$$

$$= \frac{1}{\log a}\left[-\frac{1}{4}\log|z-1| + \frac{1}{4}\log|z-5|\right] + c = \frac{1}{4\log a}\left[\log\left|\frac{z-5}{z-1}\right|\right] + c$$

$$\therefore \quad I = \frac{1}{4\log a}\left[\log\left|\frac{a^x-5}{a^x-1}\right|\right] + c. \qquad [\because\ z = a^x]$$

(*vii*) Let $\quad I = \displaystyle\int \frac{1}{2e^{2x} + 3e^x + 1}\,dx$

Put $\quad e^x = z \Rightarrow e^x\,dx = dz \Rightarrow dx = \dfrac{1}{e^x}\,dz = \dfrac{1}{z}\,dz$

$$\therefore \quad I = \int \frac{1}{z(2z^2+3z+1)}\,dz$$

$$\Rightarrow \quad I = \int \frac{1}{z(z+1)(2z+1)}\,dz \qquad \left[\begin{aligned} \because\ & 2z^2+3z+1 \\ &= 2z^2+2z+z+1 \\ &= 2z(z+1)+1(z+1) \\ &= (z+1)(2z+1)\end{aligned}\right]$$

Let $\dfrac{1}{z(z+1)(2z+1)} = \dfrac{A}{z} + \dfrac{B}{z+1} + \dfrac{C}{2z+1}$

Multiplying both sides by $z(z+1)(2z+1)$, we get

$$1 = A(z+1)(2z+1) + Bz(2z+1) + Cz(z+1).$$

$$z = 0, z+1 = 0 \Rightarrow z = -1, 2z+1 = 0 \Rightarrow z = -\frac{1}{2}$$

Put $z = 0$ in (3), we get

$$1 = A(0+1)(0+1) \Rightarrow A = 1$$

Put $z = -1$ in (3), we get

$$1 = A(-1+1)[2(-1)+1] + B(-1)[2(-1)+1] + C(-1)[-1+1]$$

$$\Rightarrow \quad 1 = B \Rightarrow B = 1$$

Put $z = -\frac{1}{2}$ in (3), we get

$$1 = A\left[-\frac{1}{2}+1\right]\left[2\left(-\frac{1}{2}\right)+1\right] + B\left(-\frac{1}{2}\right)\left[2\left(-\frac{1}{2}\right)+1\right] + C\left(-\frac{1}{2}\right)\left[-\frac{1}{2}+1\right]$$

$$\Rightarrow \quad 1 = C\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) \Rightarrow 1 = -\frac{1}{4}C \Rightarrow C = -4$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{1}{z(z+1)(2z+1)} = \frac{1}{z} + \frac{1}{z+1} - \frac{4}{(2z+1)}$$

$$\therefore \quad I = \int \frac{1}{z(z+1)(2z+1)}\,dz = \int \left[\frac{1}{z} + \frac{1}{z+1} - \frac{4}{(2z+1)}\right] dz$$

$$= \int \frac{1}{z}\,dz + \int \frac{1}{z+1}\,dz - 4\int \frac{1}{2z+1}\,dz$$

$$= \log|z| + \log|z+1| - 4\,\frac{\log|2z+1|}{2} + c$$

$$\left[\because \int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c\right]$$

$$= \log|z| + \log|z+1| - 2\log|2z+1| + c$$

$$\therefore \quad I = \log|e^x| + \log|e^x+1| - 2\log|2e^x+1| + c \qquad [\because z = e^x]$$

$$\Rightarrow \quad I = x + \log|e^x+1| - 2\log|2e^x+1| + c. \qquad [\because \log e^{f(x)} = f(x)]$$

5.3 SOME SPECIAL TYPE OF INTEGRALS

5.3.1 Integral of the Form $\int \frac{x^2-1}{x^4+kx^2+1}\,dx,\ \int \frac{x^2+1}{x^4+kx^2+1}\,dx,\ \int \frac{1}{x^4+kx^2+1}\,dx,$

where k be any real number.

This type of integrals are evaluated by dividing the numerator and denominator by x^2 and then putting $\left(x+\frac{1}{x}\right)$ or $\left(x-\frac{1}{x}\right)$ equal to z.

Working Rule :

(*i*) Divide numerator and denominator by x^2.

(*ii*) If $\left(1-\frac{1}{x^2}\right)$ occur in the numerator, then we put $\left(x+\frac{1}{x}\right) = z$, because $\left(1-\frac{1}{x^2}\right)dx = dz$.

(*iii*) If $\left(1+\frac{1}{x^2}\right)$ occur in the numerator, then we put $\left(x-\frac{1}{x}\right) = z$, because $\left(1+\frac{1}{x^2}\right)dx = dz$.

(*iv*) These substitutions reduces the integral in one of the following forms :

$$\int \frac{1}{x^2+a^2}\,dx,\ \int \frac{1}{x^2-a^2}\,dx,\ \int \frac{1}{a^2-x^2}\,dx.$$

(*v*) Then solve by using the appropriate formulae.

5.3.2 Integral of the Form $\int \frac{f(x)}{P\sqrt{Q}}\,dx$,

where : P and Q both are linear functions of x.

i.e., $$\int \frac{1}{\text{linear}\sqrt{\text{linear}}}\,dx \quad i.e., \quad \int \frac{1}{(ax+b)\sqrt{cx+d}}\,dx$$

To evaluate this type of integrals, we put

$$Q = z^2 \quad i.e., \quad \sqrt{\text{linear}} = z.$$

5.3.3 Integrals of the Form $\int \frac{f(x)}{P\sqrt{Q}}\,dx$,

where : P is a quadratic expression and Q is a linear expression.

i.e., $$\frac{1}{\text{Quadratic}\sqrt{\text{linear}}}\,dx \quad i.e., \quad \int \frac{1}{(ax^2+bx+c)\sqrt{px+q}}\,dx$$

To evaluate this type of integrals we put,

$$Q = z^2 \quad i.e., \quad \sqrt{\text{linear}} = z.$$

Remark. If we have the polynomial of degree greater than or equal to that of rational linear factor instead of 1 in numerator then first divide the numerator by the rational linear factor until the degree of the remainder is less than that of the rational factor and the proceed further.

5.3.4 Integrals of the Form $\int \frac{f(x)}{P\sqrt{Q}}\,dx$,

where : P is a linear expression and Q is a quadratic expression.

i.e., $$\int \frac{1}{\text{linear}\sqrt{\text{quadratic}}}\,dx \quad i.e., \quad \int \frac{1}{(ax+b)\sqrt{px^2+qx+r}}\,dx$$

To evaluate this type of integrals, we put

$$P = \frac{1}{z} \quad i.e., \quad \text{linear} = \frac{1}{z}.$$

5.3.5 Integrals of the Form $\int \frac{f(x)}{P\sqrt{Q}}\,dx$,

where : P and Q both are pure quadratic expression in x.

i.e., $$\int \frac{1}{\text{Pure quadratic}\sqrt{\text{Pure quadratic}}}\,dx$$

i.e., $$\int \frac{1}{(ax^2+b)\sqrt{px^2+q}}\,dx$$

To evaluate this type of integrals, we put

$$x = \frac{1}{z} \text{ and then } Q = z^2.$$

i.e., $\quad x = \frac{1}{z}$ and then in the resulting integral.

Put $\sqrt{\text{Pure quadratic}} = z$.

Remark. Pure quadratic : A quadratic expression of the form $ax^2 + b$ (*i.e.*, a quadratic expression in which co-efficient of x is zero) is called a Pure Quadratic.

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following integrals :*

(*i*) $\int \frac{x^2 - 1}{x^4 + x^2 + 1}\, dx$ $\qquad$ (*ii*) $\int \frac{x^2 + 4}{x^4 + 16}\, dx$

(*iii*) $\int \frac{1}{x^4 + 1}\, dx$ $\qquad$ (*iv*) $\int \frac{x^2}{x^4 + 1}\, dx.$

Solution. (*i*) Let $I = \int \frac{x^2 - 1}{x^4 + x^2 + 1}\, dx$

$$\therefore \quad I = \int \frac{\frac{x^2}{x^2} - \frac{1}{x^2}}{\frac{x^4}{x^2} + \frac{x^2}{x^2} + \frac{1}{x^2}}\, dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}}\, dx \qquad \left[\begin{array}{l}\text{Dividing the numerator} \\ \text{and denominator by } x^2\end{array}\right]$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 1}\, dx \qquad \left[\begin{array}{l}\because \left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \\ \Rightarrow \left[\left(x + \frac{1}{x}\right)^2 - 1\right] = \left(x^2 + \frac{1}{x^2} + 1\right)\end{array}\right]$$

Put $x + \frac{1}{x} = z \;\Rightarrow\; \left(1 - \frac{1}{x^2}\right) dx = dz$

$$\therefore \quad I = \int \frac{1}{z^2 - 1}\, dz = \int \frac{1}{z^2 - 1^2}\, dz$$

$$= \frac{1}{2(1)} \log\left|\frac{z - 1}{z + 1}\right| + c \qquad \left[\because \text{ By using } \int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + c\right]$$

$$= \frac{1}{2} \log\left|\frac{\left(x + \frac{1}{x}\right) - 1}{\left(x + \frac{1}{x}\right) + 1}\right| + c$$

$$= \frac{1}{2} \log\left|\frac{x^2 + 1 - x}{x^2 + 1 + x}\right| + c.$$

(*ii*) Let $I = \int \frac{x^2+4}{x^4+16}\,dx$

$$\Rightarrow \quad I = \int \frac{\frac{x^2}{x^2}+\frac{4}{x^2}}{\frac{x^4}{x^2}+\frac{16}{x^2}}\,dx = \int \frac{\left(1+\frac{4}{x^2}\right)}{\left(x^2+\frac{16}{x^2}\right)}\,dx$$ [Dividing the numerator and denominator by x^2]

$$= \int \frac{\left(1+\frac{4}{x^2}\right)}{\left(x^2+\frac{16}{x^2}-8+8\right)}\,dx$$ [Add and subtract 8 to the denominator]

$$= \int \frac{\left(1+\frac{4}{x^2}\right)}{\left(x-\frac{4}{x}\right)^2+8}\,dx$$ $\left[\because \left(x-\frac{4}{x}\right)^2 = x^2+\frac{16}{x^2}-2(x)\left(\frac{4}{x}\right) \Rightarrow \left(x-\frac{4}{x}\right)^2 = \left(x^2+\frac{16}{x^2}-8\right)\right]$

Put $\left(x-\frac{4}{x}\right) = z \Rightarrow \left(1+\frac{4}{x^2}\right)dx = dz$

$$\therefore \quad I = \int \frac{1}{z^2+8}\,dz = \int \frac{1}{z^2+\left(\sqrt{8}\right)^2}\,dz$$ $\left[\because \text{By using} \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$

$$= \frac{1}{\sqrt{8}}\tan^{-1}\frac{z}{\sqrt{8}}+c = \frac{1}{\sqrt{8}}\tan^{-1}\left(\frac{x-\frac{4}{x}}{\sqrt{8}}\right)+c$$ $\left[\because z = \left(x-\frac{4}{x}\right)\right]$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-4}{2\sqrt{2}x}\right)+c.$$

(*iii*) Let $I = \int \frac{1}{x^4+1}\,dx$ ...(1)

$$= \frac{1}{2}\int \frac{2}{x^4+1}\,dx$$ [Multiply and divided by 2]

$$= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}\,dx$$ [Add and subtract x^2 to the numerator]

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}\,dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}I_1 - \frac{1}{2}I_2 \text{ (say)}$$ ...(2)

where : $I_1 = \int \frac{x^2+1}{x^4+1}\,dx$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and denominator by } x^2\end{array}\right]$$

$$\Rightarrow \qquad I_1 = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+2}\,dx \qquad \left[\begin{array}{l}\because \left(x-\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}-2\\ \Rightarrow \left[\left(x-\frac{1}{x}\right)^2+2\right] = x^2+\frac{1}{x^2}\end{array}\right]$$

Put $\left(x-\frac{1}{x}\right) = z \;\Rightarrow\; \left(1+\frac{1}{x^2}\right)dx = dz$

$$\therefore \; I_1 = \int \frac{1}{z^2+2}\,dz = \int \frac{1}{z^2+\left(\sqrt{2}\right)^2}\,dz \qquad \left[\because \text{ By using } \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+c_1 = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)+c_1 \qquad \left[\because\; z = x-\frac{1}{x}\right]$$

$$\Rightarrow \qquad I_1 = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+c_1 \qquad \text{...(3)}$$

and

$$I_2 = \int \frac{x^2-1}{x^4+1}\,dx = \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}\,dx \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and denominator by } x^2\end{array}\right]$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-2}\,dx \qquad \left[\begin{array}{l}\because \left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2\\ \Rightarrow \left(x+\frac{1}{x}\right)^2-2 = x^2+\frac{1}{x^2}\end{array}\right]$$

Put $\left(x+\frac{1}{x}\right) = z \;\Rightarrow\; \left(1-\frac{1}{x^2}\right)dx = dz$

$$\therefore \qquad I_2 = \int \frac{1}{z^2-2}\,dz = \int \frac{1}{z^2-\left(\sqrt{2}\right)^2}\,dz$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right|+c_2 \qquad \left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c\right]$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+c_2 \qquad \left[\because\; z = \left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow \quad I_2 = \frac{1}{2\sqrt{2}} \log \left| \frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x} \right| + c_2 \qquad \text{...(4)}$$

$\therefore$ From equation (2), we have

$$I = \frac{1}{2} I_1 - \frac{1}{2} I_2$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + c_1\right] - \frac{1}{2}\left[\frac{1}{2\sqrt{2}} \log \left| \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right| + c_2\right]$$

[Using (3) and (4)]

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + \frac{1}{2}c_1 - \frac{1}{4\sqrt{2}} \log \left| \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right| - \frac{1}{2}c_2$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}} \log \left| \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right| + c.$$

where $c = \left(\frac{1}{2}c_1 - \frac{1}{2}c_2\right)$

(*iv*) Let $I = \int \frac{x^2}{x^4+1}\, dx$ ...(1)

$$= \frac{1}{2}\int \frac{2x^2}{x^4+1}\, dx$$ [Multiply and divided by 2]

$$= \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)}{x^4+1}\, dx$$ [Add and subtract 1 to the numerator]

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}\, dx + \frac{1}{2}\int \frac{x^2-1}{x^4+1}\, dx$$

$$\Rightarrow \quad I = \frac{1}{2} I_1 + \frac{1}{2} I_2 \qquad \text{...(2)}$$

Proceed further exactly on the same steps of part (*iii*).

$$\left[\textbf{Ans.} \quad \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + \frac{1}{4\sqrt{2}} \log \left| \frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x} \right| + c.\right] \qquad \text{where } c = \frac{1}{2}c_1 + \frac{1}{2}c_2$$

Example 2. *Evaluate the following integrals* :

(*i*) $\int \frac{x^2}{x^4+x^2+1}\, dx$ (*ii*) $\int \sqrt{\cot x}\, dx$

(*iii*) $\int \sqrt{\tan x}\, dx$ (*iv*) $\int \frac{x^3+1}{x^4+1}\, dx.$

Solution. (*i*) Let $I = \int \frac{x^2}{x^4+x^2+1}\, dx$ ...(1)

$$= \frac{1}{2}\int \frac{2x^2}{x^4+x^2+1}\, dx$$ [Multiply and divided by 2]

$$= \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)}{x^4+x^2+1}\,dx \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+x^2+1}\,dx + \frac{1}{2}\int \frac{x^2-1}{x^4+x^2+1}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2} I_1 + \frac{1}{2} I_2 \qquad \ldots(2)$$

where $$I_1 = \int \frac{x^2+1}{x^4+x^2+1}\,dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+1+\frac{1}{x^2}\right)}\,dx \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and denominator by } x^2\end{array}\right]$$

$$I_1 = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left[\left(x-\frac{1}{x}\right)^2+2\right]+1}\,dx \qquad \left[\begin{array}{l}\because \left(x-\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}-2\\ \Rightarrow \left(x-\frac{1}{x}\right)^2+2 = \left(x^2+\frac{1}{x^2}\right)\end{array}\right]$$

Put $\left(x-\frac{1}{x}\right) = z \;\Rightarrow\; \left(1+\frac{1}{x^2}\right)dx = dz$

$$\therefore \quad I_1 = \int \frac{1}{z^2+2+1}\,dz = \int \frac{1}{z^2+\left(\sqrt{3}\right)^2}\,dz$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}} + c_1 \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right) + c_1 \qquad \left[\because z = \left(x-\frac{1}{x}\right)\right]$$

$$\Rightarrow \quad I_1 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + c_1 \qquad \ldots(3)$$

and $$I_2 = \int \frac{x^2-1}{x^4+x^2+1}\,dx = \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+1+\frac{1}{x^2}\right)}\,dx \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and denominator by } x^2\end{array}\right]$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)+1}\,dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-2+1}\,dx \qquad \left[\begin{array}{l}\because \left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2\\ \Rightarrow \left[\left(x+\frac{1}{x}\right)^2-2\right] = \left(x^2+\frac{1}{x^2}\right)\end{array}\right]$$

Put $\left(x+\frac{1}{x}\right)=z \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dz$

$\therefore \quad I_2 = \int \frac{1}{z^2-1}\,dz$

$= \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c_2 \qquad \left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$

$= \frac{1}{2}\log\left|\frac{\left(x+\frac{1}{x}\right)-1}{\left(x+\frac{1}{x}\right)+1}\right| + c_2 \qquad \left[\because z = \left(x+\frac{1}{x}\right)\right]$

$\Rightarrow \quad I_2 = \frac{1}{2}\log\left|\frac{x^2+1-x}{x^2+1+x}\right| + c_2 \qquad ...(4)$

$\therefore$ From equation (2), we have

$I = \frac{1}{2}I_1 + \frac{1}{2}I_2$

$= \frac{1}{2}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + c_1\right] + \frac{1}{2}\left[\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + c_2\right] \qquad$ [Using (3) and (4)]

$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + \frac{1}{2}c_1 + \frac{1}{4}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + \frac{1}{2}c_2$

$\therefore \quad I = \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + \frac{1}{4}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + c. \qquad$ where $c = \left(\frac{1}{2}c_1 + \frac{1}{2}c_2\right)$

(*ii*) Let $I = \int \sqrt{\cot x}\,dx$

Put $\sqrt{\cot x} = z \Rightarrow \cot x = z^2 \Rightarrow x = \cot^{-1} z^2 \Rightarrow dx = \frac{-2z}{1+z^4}\,dz$

$\therefore \quad I = \int z.\left(\frac{-2z}{1+z^4}\right)dz = -\int \frac{2z^2}{1+z^4}\,dz = -\int \frac{z^2+z^2}{z^4+1}\,dz \qquad ...(1)$

$= -\int \frac{(z^2+1)+(z^2-1)}{z^4+1}\,dz \qquad$ [Add and subtract 1 to the numerator]

$= -\int \frac{z^2+1}{z^4+1}\,dz - \int \frac{z^2-1}{z^4+1}\,dz$

$\Rightarrow \quad I = -I_1 - I_2$ (say) $\qquad ...(2)$

where $\quad I_1 = \int \frac{z^2+1}{z^4+1}\,dz = \int \frac{1+\frac{1}{z^2}}{z^2+\frac{1}{z^2}}\,dz \qquad$ [Dividing the numerator and denominator by z^2]

$$= \int \frac{\left(1+\frac{1}{z^2}\right)}{\left(z-\frac{1}{z}\right)^2+2}\, dz \qquad \left[\because \left(z-\frac{1}{z}\right)^2 = z^2+\frac{1}{z^2}-2 \Rightarrow \left[\left(z-\frac{1}{z}\right)^2+2\right]=\left(z^2+\frac{1}{z^2}\right)\right]$$

Put $\left(z-\frac{1}{z}\right)=y \Rightarrow \left(1+\frac{1}{z^2}\right)dz = dy$

$$\therefore \quad I_1 = \int \frac{1}{y^2+2}\,dy = \int \frac{1}{y^2+\left(\sqrt{2}\right)^2}\,dy$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}} + c_1 \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2}}\right) + c_1 \qquad \left[\because y=\left(z-\frac{1}{z}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z^2-1}{\sqrt{2}z}\right) + c_1$$

$$\therefore \quad I_1 = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot x-1}{\sqrt{2\cot x}}\right) + c_1 \qquad \ldots(3) \quad \left[\because z=\sqrt{\cot x}\right]$$

Now $$I_2 = \int \frac{z^2-1}{z^4+1}\,dz = \int \frac{\left(1-\frac{1}{z^2}\right)}{\left(z^2+\frac{1}{z^2}\right)}\,dz \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and denominator by } z^2\end{array}\right]$$

$$= \int \frac{\left(1-\frac{1}{z^2}\right)}{\left(z+\frac{1}{z}\right)^2-2}\, dz \qquad \left[\because \left(z+\frac{1}{z}\right)^2 = z^2+\frac{1}{z^2}+2 \Rightarrow \left[\left(z+\frac{1}{z}\right)^2-2\right]=\left(z^2+\frac{1}{z^2}\right)\right]$$

Put $\left(z+\frac{1}{z}\right)=y \Rightarrow \left(1-\frac{1}{z^2}\right)dz = dy$

$$\therefore \quad I_2 = \int \frac{1}{y^2-2}\,dy = \int \frac{1}{y^2-\left(\sqrt{2}\right)^2}\,dy$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{y-\sqrt{2}}{y+\sqrt{2}}\right| + c_2 \qquad \left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c\right]$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{z+\frac{1}{z}-\sqrt{2}}{z+\frac{1}{z}+\sqrt{2}}\right| + c_2 = \frac{1}{2\sqrt{2}}\log\left|\frac{z^2-\sqrt{2}z+1}{z^2+\sqrt{2}z+1}\right| + c_2 \qquad \left[\because y=\left(z+\frac{1}{z}\right)\right]$$

$$\Rightarrow \quad I_2 = \frac{1}{2\sqrt{2}} \log \left| \frac{\cot x - \sqrt{2\cot x} + 1}{\cot x + \sqrt{2\cot x} + 1} \right| + c_2 \qquad ...(4) \left[\because z = \sqrt{\cot x}\right]$$

$\therefore$ From equation (2), we have

$$I = -I_1 - I_2$$

$$= -\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\cot x - 1}{\sqrt{2\cot x}}\right) + c_1\right] - \left[\frac{1}{2\sqrt{2}} \log \left| \frac{\cot x - \sqrt{2\cot x} + 1}{\cot x + \sqrt{2\cot x} + 1} \right| + c_2\right]$$

[Using (3) and (4)]

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\cot x - 1}{\sqrt{2\cot x}}\right) - c_1 - \frac{1}{2\sqrt{2}} \log \left| \frac{\cot x - \sqrt{2\cot x} + 1}{\cot x + \sqrt{2\cot x} + 1} \right| - c_2$$

$$\therefore \quad I = -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\cot x - 1}{\sqrt{2\cot x}}\right) - \frac{1}{2\sqrt{2}} \log \left| \frac{\cot x - \sqrt{2\cot x} + 1}{\cot x + \sqrt{2\cot x} + 1} \right| + c$$

where $c = -c_1 - c_2$.

(*iii*) Let $I = \int \sqrt{\tan x}\, dx$ ...(1)

Put $\sqrt{\tan x} = z \Rightarrow \tan x = z^2 \Rightarrow \sec^2 x\, dx = 2z\, dz$

$$\Rightarrow \quad dx = \frac{1}{\sec^2 x} 2z\, dz = \frac{2z}{(1+\tan^2 x)}\, dz = \frac{2z}{1+z^4}\, dz$$

$$\therefore \quad I = \int z \cdot \left(\frac{2z}{1+z^4}\right) dz = \int \frac{2z^2}{z^4+1}\, dz = \int \frac{z^2+z^2}{z^4+1}\, dz$$

[Note this step]

$$= \int \frac{(z^2+1)+(z^2-1)}{z^4+1}\, dz$$

[Add and subtract 1 to the numerator]

$$= \int \frac{z^2+1}{z^4+1}\, dz + \int \frac{z^2-1}{z^4+1}\, dz$$

$$\Rightarrow \quad I = I_1 + I_2 \quad \text{(say)} \qquad ...(2)$$

[Proceed further by yourself as in part (*ii*)]

$$\left[\textbf{Ans. } \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1} \right| + c\right]$$

(*iv*) Let $I = \int \frac{x^3+1}{x^4+1}\, dx = \int \frac{x^3}{x^4+1}\, dx + \int \frac{1}{x^4+1}\, dx$ ...(1)

$$\Rightarrow \quad I = I_1 + I_2 \quad \text{(say)} \qquad ...(2)$$

where $I_1 = \int \frac{x^3}{x^4+1}\, dx$

$$= \frac{1}{4} \int \frac{4x^3}{x^4+1}\, dx$$

[Multiply and divided by 4]

$$\Rightarrow \qquad I_1 = \frac{1}{4}\log|x^4+1| + c_1 \qquad \text{...(3)} \left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

and
$$I_2 = \int \frac{1}{x^4+1}dx = \frac{1}{2}\int \frac{2}{x^4+1}dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int \frac{1+1}{x^4+1}dx$$

$$= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}dx \qquad \text{[Add and subtract } x^2 \text{ to the numerator]}$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}dx$$

$$I_2 = \frac{1}{2}I_3 - \frac{1}{2}I_4 \text{ (say)} \qquad \text{...(4)}$$

where
$$I_3 = \int \frac{x^2+1}{x^4+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx \qquad \left[\begin{array}{l}\text{Dividing numerator}\\ \text{and denominator by } x^2\end{array}\right]$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+2}dx \qquad \left[\begin{array}{l}\because \left(x-\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}-2\\ \Rightarrow \left[\left(x-\frac{1}{x}\right)^2+2\right] = x^2+\frac{1}{x^2}\end{array}\right]$$

Put $\left(x-\frac{1}{x}\right) = z \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dz$

$$\therefore \qquad I_3 = \int \frac{1}{z^2+2}dz = \int \frac{1}{z^2+\left(\sqrt{2}\right)^2}dz$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} + c_3 \qquad \left[\because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + c_3 \qquad \left[\because z = x - \frac{1}{x}\right]$$

$$\Rightarrow \qquad I_3 = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + c_3 \qquad \text{...(5)}$$

and
$$I_4 = \int \frac{x^2-1}{x^4+1}dx = \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}dx \qquad \left[\begin{array}{l}\text{Dividing numerator}\\ \text{and denominator by } x^2\end{array}\right]$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 2}\, dx \qquad \left[\begin{array}{l} \because \left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \\ \Rightarrow \left[\left(x + \frac{1}{x}\right)^2 - 2\right] = \left(x^2 + \frac{1}{x^2}\right) \end{array}\right]$$

Put $\left(x + \frac{1}{x}\right) = z \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dz$

$$\therefore \quad I_4 = \int \frac{1}{z^2 - 2}\, dz = \int \frac{1}{z^2 - \left(\sqrt{2}\right)^2}\, dz \qquad \left[\because \int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{2\sqrt{2}} \log\left|\frac{z - \sqrt{2}}{z + \sqrt{2}}\right| + c_4$$

$$= \frac{1}{2\sqrt{2}} \log\left|\frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}}\right| + c_4 \qquad \left[\because z = \left(x + \frac{1}{x}\right)\right]$$

$$\Rightarrow \quad I_4 = \frac{1}{2\sqrt{2}} \log\left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right| + c_4 \qquad \text{...(6)}$$

$\therefore$ From equation (4), we have

$$I_2 = \frac{1}{2} I_3 - \frac{1}{2} I_4$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) + c_3\right] - \frac{1}{2}\left[\frac{1}{2\sqrt{2}} \log\left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right| + c_4\right]$$

[Using equations (5) and (6)]

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) + \frac{1}{2} c_3 - \frac{1}{4\sqrt{2}} \log\left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right| - \frac{1}{2} c_4$$

$$\Rightarrow \quad I_2 = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}} \log\left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right| + c_2 \qquad \text{...(7)}$$

where $c_2 = \left(\frac{1}{2} c_3 - \frac{1}{2} c_4\right)$

$\therefore$ From equation (2), we have

$$I = I_1 + I_2 \qquad \text{[Using equations (3) and (7)]}$$

$$= \frac{1}{4} \log|x^4 + 1| + c_1 + \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}} \log\left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right| + c_2$$

$$\Rightarrow \quad I = \frac{1}{4} \log|x^4 + 1| + \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}} \log\left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right| + c$$

where $c = (c_1 + c_2)$

Example 3. *Evaluate the following integrals :*

(i) $\int \frac{1}{\sin^4\theta + \cos^4\theta}\, d\theta$ (ii) $\int \frac{x^2 - 3x + 1}{x^4 - x^2 + 1}\, dx$

(iii) $\int \frac{x^2 + a^2}{x^4 + a^2x^2 + a^4}\, dx$ (iv) $\int \frac{x^2 + a^2}{x^4 + a^4}\, dx$

(v) $\int \frac{1}{\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x}\, dx$ (vi) $\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx.$

Solution. (i) Let $I = \int \frac{1}{\sin^4\theta + \cos^4\theta}\, d\theta$

Dividing the numerator and denominator by $\cos^4\theta$, we have

$$I = \int \frac{\frac{1}{\cos^4\theta}}{\frac{\sin^4\theta}{\cos^4\theta} + \frac{\cos^4\theta}{\cos^4\theta}}\, d\theta = \int \frac{\sec^4\theta}{\tan^4\theta + 1}\, d\theta$$

$$= \int \frac{\sec^2\theta \,.\, \sec^2\theta}{(\tan^4\theta + 1)}\, d\theta = \int \frac{(1 + \tan^2\theta)\sec^2\theta}{(\tan^4\theta + 1)}\, d\theta$$

Put $\tan\theta = z \Rightarrow \sec^2\theta\, d\theta = dz$

$$\therefore \quad I = \int \frac{(z^2 + 1)}{(z^4 + 1)}\, dz = \int \frac{\left(1 + \frac{1}{z^2}\right)}{\left(z^2 + \frac{1}{z^2}\right)}\, dz \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and denominator by } z^2\end{array}\right]$$

$$= \int \frac{\left(1 + \frac{1}{z^2}\right)}{\left(z - \frac{1}{z}\right)^2 + 2}\, dz \qquad \left[\begin{array}{l}\because \left(z - \frac{1}{z}\right)^2 = z^2 + \frac{1}{z^2} - 2\\ \Rightarrow \left[\left(z - \frac{1}{z}\right)^2 + 2\right] = \left(z^2 + \frac{1}{z^2}\right)\end{array}\right]$$

Put $\left(z - \frac{1}{z}\right) = y \Rightarrow \left(1 + \frac{1}{z^2}\right) dz = dy$

$$\therefore \quad I = \int \frac{1}{y^2 + 2}\, dy = \int \frac{1}{y^2 + \left(\sqrt{2}\right)^2}\, dy$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + c \qquad \left[\because \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{z - \frac{1}{z}}{\sqrt{2}}\right) + c = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{z^2 - 1}{\sqrt{2}z}\right) + c \qquad \left[\because y = \left(z - \frac{1}{z}\right)\right]$$

$$\therefore \quad I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan^2\theta - 1}{\sqrt{2}\tan\theta}\right) + c. \qquad [\because z = \tan\theta]$$

(*ii*) Let $\quad I = \int \frac{x^2 - 3x + 1}{x^4 - x^2 + 1}\,dx$ ...(1)

$$= \int \frac{x^2 + 1}{x^4 - x^2 + 1}\,dx - \int \frac{3x}{x^4 - x^2 + 1}\,dx$$

$\Rightarrow \quad I = I_1 - I_2$ ...(2)

where $\quad I_1 = \int \frac{x^2 + 1}{x^4 - x^2 + 1}\,dx = \int \frac{1 + \frac{1}{x^2}}{x^2 - 1 + \frac{1}{x^2}}\,dx$ $\left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and denominator by } x^2\end{array}\right]$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2}\right) - 1}\,dx = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 2 - 1}\,dx$$

$$\left[\begin{array}{l}\because \left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2\\ \Rightarrow \left(x - \frac{1}{x}\right)^2 + 2 = \left(x^2 + \frac{1}{x^2}\right)\end{array}\right]$$

Put $\quad \left(x - \frac{1}{x}\right) = z \;\Rightarrow\; \left(1 + \frac{1}{x^2}\right)dx = dz$

$$= \int \frac{1}{z^2 + 1}\,dz = \tan^{-1} z + c_1 \qquad \left[\because \int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$\Rightarrow \quad I_1 = \tan^{-1}\left(x - \frac{1}{x}\right) + c_1$ ...(3) $\left[\because z = \left(x - \frac{1}{x}\right)\right]$

and $\quad I_2 = \int \frac{3x}{x^4 - x^2 + 1}\,dx$

Put $\quad x^2 = y \;\Rightarrow\; 2x\,dx = dy \;\Rightarrow\; x\,dx = \frac{1}{2}\,dy$

$\therefore \quad I_2 = \int \frac{3}{y^2 - y + 1}\left(\frac{1}{2}\,dy\right) = \frac{3}{2}\int \frac{1}{y^2 - y + 1}\,dy$

$$= \frac{3}{2}\int \frac{1}{\left(y^2 - y + \frac{1}{4}\right) + \left(1 - \frac{1}{4}\right)}\,dy \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } y\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \frac{3}{2}\int \frac{1}{\left(y - \frac{1}{2}\right)^2 + \frac{3}{4}}\,dy$$

$$= \frac{3}{2}\int \frac{1}{\left(y - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dy$$

$$= \frac{3}{2} \cdot \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1} \frac{\left(y - \frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c_2 \qquad \left[\because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \sqrt{3} \tan^{-1} \left(\frac{2y - 1}{\sqrt{3}}\right) + c_2$$

$$\Rightarrow \quad I_2 = \sqrt{3} \tan^{-1} \left(\frac{2x^2 - 1}{\sqrt{3}}\right) + c_2 \qquad \text{...(4)} \quad [\because \ y = x^2]$$

$\therefore$ From equation (2), we have

$$I = I_1 + I_2$$

$$= \tan^{-1} \left(\frac{x^2 - 1}{x}\right) + c_1 - \sqrt{3} \tan^{-1} \left(\frac{2x^2 - 1}{\sqrt{3}}\right) - c_2$$

[Using equations (3) and (4)]

$$\therefore \quad I = \tan^{-1} \left(\frac{x^2 - 1}{x}\right) - \sqrt{3} \tan^{-1} \left(\frac{2x^2 - 1}{\sqrt{3}}\right) + c. \qquad \text{where } c = (c_1 - c_2)$$

(*iii*) Let $I = \int \frac{x^2 + a^2}{x^4 + a^2x^2 + a^4} dx = \int \frac{\left(1 + \frac{a^2}{x^2}\right)}{\left(x^2 + a^2 + \frac{a^4}{x^2}\right)} dx$ $\left[\begin{array}{l}\text{Dividing the numerator} \\ \text{and denominator by } x^2\end{array}\right]$

Put $\left(x - \frac{a^2}{x}\right) = z \Rightarrow \left(1 + \frac{a^2}{x^2}\right) dx = dz$

$$\therefore \quad I = \int \frac{1}{(z^2 + 2a^2 + a^2)} dz \qquad \left[\begin{array}{l} \because \left(x - \frac{a^2}{x}\right)^2 = x^2 + \frac{a^4}{x^2} - 2a^2 \\ \Rightarrow \left[\left(x - \frac{a^2}{x}\right)^2 + 2a^2\right] = \left(x^2 + \frac{a^4}{x^2}\right) \\ \Rightarrow (z^2 + 2a^2) = \left(x^2 + \frac{a^4}{x^2}\right) \end{array}\right]$$

$$= \int \frac{1}{z^2 + 3a^2} dz = \int \frac{1}{z^2 + \left(\sqrt{3}a\right)^2} dz$$

$$= \frac{1}{\sqrt{3}a} \cdot \tan^{-1} \left(\frac{z}{\sqrt{3}a}\right) + c \qquad \left[\because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \frac{1}{\sqrt{3}a} \tan^{-1} \left(\frac{\left(x - \frac{a^2}{x}\right)}{\sqrt{3}a}\right) + c$$

$$= \frac{1}{\sqrt{3}a} \tan^{-1}\left(\frac{x^2 - a^2}{\sqrt{3}ax}\right) + c.$$

(iv) Let $\quad I = \int \frac{x^2 + a^2}{x^4 + a^4}\, dx = \int \frac{\left(1 + \frac{a^2}{x^2}\right)}{\left(x^2 + \frac{a^4}{x^2}\right)}\, dx \qquad \left[\begin{array}{l}\text{Dividing the numerator} \\ \text{and denominator by } x^2\end{array}\right]$

Put $\quad \left(x - \frac{a^2}{x}\right) = z \;\Rightarrow\; \left(1 + \frac{a^2}{x^2}\right) dx = dz$

$\therefore \qquad I = \int \frac{1}{z^2 + 2a^2}\, dz \qquad \left[\begin{array}{l}\because \left(x - \frac{a^2}{x}\right)^2 = x^2 + \frac{a^4}{x^2} - 2a^2 \\ \Rightarrow \left[\left(x - \frac{a^2}{x}\right)^2 + 2a^2\right] = \left(x^2 + \frac{a^4}{x^2}\right) \\ \Rightarrow (z^2 + 2a^2) = \left(x^2 + \frac{a^4}{x^2}\right)\end{array}\right]$

$$= \int \frac{1}{z^2 + \left(\sqrt{2}a\right)^2}\, dz$$

$= \frac{1}{\sqrt{2}a} \tan^{-1}\left(\frac{z}{\sqrt{2}a}\right) + c \qquad \left[\because \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$

$$= \frac{1}{\sqrt{2}a} \tan^{-1}\left(\frac{x - a^2/x}{\sqrt{2}a}\right) + c$$

$$= \frac{1}{\sqrt{2}a} \tan^{-1}\left(\frac{x^2 - a^2}{\sqrt{2}ax}\right) + c.$$

(v) Let $\quad I = \int \frac{1}{\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x}\, dx$

$= \int \frac{\frac{1}{\cos^4 x}}{\frac{\cos^4 x}{\cos^4 x} - \frac{\cos^2 x \sin^2 x}{\cos^4 x} + \frac{\sin^4 x}{\cos^4 x}}\, dx \qquad \left[\begin{array}{l}\text{Dividing the numerator} \\ \text{and denominator by } \cos^4 x\end{array}\right]$

$$= \int \frac{\sec^4 x}{1 - \tan^2 x + \tan^4 x}\, dx = \int \frac{\sec^2 x \,.\, \sec^2 x}{(\tan^2 x)^2 - \tan^2 x + 1}\, dx$$

$= \int \frac{(1 + \tan^2 x) \sec^2 x}{(\tan^2 x)^2 - \tan^2 x + 1}\, dx \qquad \left[\begin{array}{l}\because \sec^2 A - \tan^2 A = 1 \\ \Rightarrow \sec^2 A = (1 + \tan^2 A)\end{array}\right]$

Put $\quad \tan x = z \;\Rightarrow\; \sec^2 x\, dx = dz$

$\therefore \qquad I = \int \frac{(1 + z^2)}{z^4 - z^2 + 1}\, dz = \int \frac{z^2 + 1}{z^4 - z^2 + 1}\, dz$

$\therefore \quad I = \int \frac{\left(1+\frac{1}{z^2}\right)}{\left(z^2 - 1 + \frac{1}{z^2}\right)} dz = \int \frac{\left(1+\frac{1}{z^2}\right)}{\left(z^2 + \frac{1}{z^2} - 1\right)} dz \qquad$ [Dividing the numerator and denominator by z^2]

Put $\left(z - \frac{1}{z}\right) = y \Rightarrow \left(1 + \frac{1}{z^2}\right) dz = dy$

$$\therefore \quad I = \int \frac{1}{y^2 + 2 - 1} dy \qquad \left[\begin{array}{l} \because \left(z - \frac{1}{z}\right)^2 = z^2 + \frac{1}{z^2} - 2 \\ \Rightarrow \left[\left(z - \frac{1}{z}\right)^2 + 2\right] = \left(z^2 + \frac{1}{z^2}\right) \\ \Rightarrow (y^2 + 2) = \left(z^2 + \frac{1}{z^2}\right) \end{array}\right]$$

$$= \int \frac{1}{y^2 + 1^2} dy = \tan^{-1} y + c \qquad \left[\because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$$

$$= \tan^{-1}\left(z - \frac{1}{z}\right) + c \qquad \left[\because y = \left(z - \frac{1}{z}\right)\right]$$

$$= \tan^{-1}\left(\frac{z^2 - 1}{z}\right) + c$$

$$= \tan^{-1}\left(\frac{\tan^2 x - 1}{\tan x}\right) + c. \qquad [\because z = \tan x]$$

(*vi*) Let $I = \int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx = \int \left(\sqrt{\tan x} + \frac{1}{\sqrt{\tan x}}\right) dx = \int \frac{\tan x + 1}{\sqrt{\tan x}} dx$

Put $\sqrt{\tan x} = z \Rightarrow \tan x = z^2 \Rightarrow x = \tan^{-1} z^2 \Rightarrow dx = \frac{2z}{1 + z^4} dz$

$$\therefore \quad I = \int \frac{z^2 + 1}{z} \cdot \frac{2z}{z^4 + 1} dz$$

$$= 2 \int \frac{z^2 + 1}{z^4 + 1} dz = 2 \int \frac{\left(1 + \frac{1}{z^2}\right)}{\left(z^2 + \frac{1}{z^2}\right)} dz \qquad \text{[Dividing the numerator and denominator by } z^2\text{]}$$

Put $\left(z - \frac{1}{z}\right) = y \Rightarrow \left(1 + \frac{1}{z^2}\right) dz = dy$

$$\therefore \quad I = 2 \int \frac{1}{y^2 + 2} dy$$

$$= 2\int \frac{1}{y^2+\left(\sqrt{2}\right)^2}\,dy \qquad \left[\begin{array}{l} \because \left(z-\frac{1}{z}\right)^2 = z^2+\frac{1}{z^2}-2 \\ \Rightarrow \left[\left(z-\frac{1}{z}\right)^2+2\right] = \left(z^2+\frac{1}{z^2}\right) \\ \Rightarrow (y^2+2) = \left(z^2+\frac{1}{z^2}\right) \end{array}\right]$$

$$= 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\cdot\left(\frac{y}{\sqrt{2}}\right)+c \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \sqrt{2}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2}}\right)+c = \sqrt{2}\tan^{-1}\left(\frac{z^2-1}{\sqrt{2}z}\right)+c \qquad \left[\because y = \left(z-\frac{1}{z}\right)\right]$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2}\tan x}\right)+c. \qquad [\because z = \tan x]$$

Example 4. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{(2x+3)\sqrt{4x+5}}\,dx$ (*ii*) $\int \frac{1}{(x-3)\sqrt{x+1}}\,dx$

(*iii*) $\int \frac{1}{(2x+3)\sqrt{x+5}}\,dx$ (*iv*) $\int \frac{x^2}{(x+3)\sqrt{3x+4}}\,dx$

(*v*) $\int \frac{1}{(x-1)\sqrt{2x+3}}\,dx.$

Solution. (*i*) Let $I = \int \frac{1}{(2x+3)\sqrt{4x+5}}\,dx$

Put $\sqrt{4x+5} = z$ [Squaring on both sides]

$$\Rightarrow \quad 4x+5 = z^2 \;\Rightarrow\; x = \left(\frac{z^2-5}{4}\right) \;\Rightarrow\; 4\,dx = 2z\,dz \;\Rightarrow\; dx = \frac{z}{2}\,dz$$

$$\therefore \qquad I = \int \frac{1}{\left[2\left(\frac{z^2-5}{4}\right)+3\right].z}\cdot\left(\frac{z}{2}\,dz\right) = \int \frac{1}{\left(\frac{z^2-5}{2}+3\right)}\left(\frac{1}{2}\,dz\right)$$

$$= \int \frac{1}{z^2+1}\,dz = \int \frac{1}{z^2+1^2}\,dz$$

$$= \tan^{-1}z+c \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \tan^{-1}\left(\sqrt{4x+5}\right)+c. \qquad \left[\because z = \sqrt{4x+5}\right]$$

(*ii*) Let $\quad I = \int \frac{1}{(x-3)\sqrt{x+1}}\, dx$

Put $\sqrt{x+1} = z$ [Squaring on both sides]

$\Rightarrow \quad x + 1 = z^2 \Rightarrow x = (z^2 - 1) \Rightarrow dx = 2z\, dz$

$$\therefore \quad I = \int \frac{1}{(z^2-1-3)\,z} \cdot (2z\, dz)$$

$$= 2\int \frac{1}{z^2-4}\, dz = 2\int \frac{1}{z^2-2^2}\, dz$$

$$\therefore \quad I = 2 \cdot \frac{1}{2(2)} \log\left|\frac{z-2}{z+2}\right| + c \qquad \left[\because \int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right| + c. \qquad \left[\because z = \sqrt{x+1}\right]$$

(*iii*) Let $\quad I = \int \frac{1}{(2x+3)\sqrt{x+5}}\, dx$

Put $\sqrt{x+5} = z$ (Squaring on both sides)

$\Rightarrow \quad x + 5 = z^2 \Rightarrow x = z^2 - 5 \Rightarrow dx = 2z\, dz$

$$\therefore \quad I = \int \frac{1}{[2(z^2-5)+3].\,z}\,(2z\, dz) = 2\int \frac{1}{2z^2-7}\, dz$$

$$= \frac{2}{2}\int \frac{1}{z^2-\left(\frac{7}{2}\right)}\, dz = \int \frac{1}{z^2-(\sqrt{7/2})^2}\, dz$$

$$\Rightarrow \quad I = \frac{1}{2\left(\sqrt{7/2}\right)} \cdot \log\left|\frac{z-\sqrt{7/2}}{z+\sqrt{7/2}}\right| + c \qquad \left[\because \int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{\sqrt{14}}\log\left|\frac{\sqrt{x+5}-\sqrt{7/2}}{\sqrt{x+5}+\sqrt{7/2}}\right| + c. \qquad \left[\because z = \sqrt{x+5}\right]$$

(*iv*) Let $\quad I = \int \frac{x^2}{(x+3)\sqrt{3x+4}}\, dx$

Put $\sqrt{3x+4} = z$ (Squaring on both sides)

$$\Rightarrow \quad 3x + 4 = z^2 \Rightarrow x = \frac{z^2-4}{3} \Rightarrow dx = \frac{2}{3}z\, dz$$

$$\therefore \quad I = \int \frac{\left(\frac{z^2-4}{3}\right)^2}{\left(\frac{z^2-4}{3}+3\right).\,z} \cdot \left(\frac{2}{3}z\, dz\right) = \frac{2}{3}\int \frac{\left(\frac{z^4+16-8z^2}{9}\right)}{\left(\frac{z^2-4+9}{3}\right)}\, dz$$

$$\Rightarrow \quad I = \frac{2}{9}\int \frac{z^4-8z^2+16}{(z^2+5)}\, dz$$

Since the degree of numerator is greater than the degree of denominator, therefore by long division, we have

$$\begin{array}{l|l} z^2+5 & z^4-8z^2+16 \ (z^2-13 \\ & \underline{\ \ +z^4+5z^2} \\ & \quad - \quad - \\ & \quad -13z^2+16 \\ & \quad \underline{-13z^2-65} \\ & \qquad + \qquad + \\ & \qquad\qquad 81 \end{array}$$

$$\mathrm{I} = \frac{2}{9}\int\left[(z^2-13)+\frac{81}{z^2+5}\right]dz$$

$$= \frac{2}{9}\left[\int z^2\,dz - 13\int 1.dz + 81\int \frac{1}{z^2+\left(\sqrt{5}\right)^2}dz\right]$$

$$= \frac{2}{9}\left[\frac{z^3}{3} - 13z + 81.\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{z}{\sqrt{5}}\right)\right] + c \qquad \left[\because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{2z^3}{27} - \frac{26}{9}z + \frac{18}{\sqrt{5}}\tan^{-1}\left(\frac{z}{\sqrt{5}}\right) + c$$

$$\therefore \quad \mathrm{I} = \frac{2}{27}(3x+4)^{3/2} - \frac{26}{9}\sqrt{3x+4} + \frac{18}{\sqrt{5}}\tan^{-1}\left(\sqrt{\frac{3x+4}{5}}\right) + c. \qquad \left[\because z = \sqrt{3x+4}\right]$$

(*v*) Let $\quad \mathrm{I} = \int \frac{1}{(x-1)\sqrt{2x+3}}\,dx$

Put $\sqrt{2x+3} = z$ [Squaring on both sides]

$$\Rightarrow \quad 2x+3 = z^2 \;\Rightarrow\; x = \frac{z^2-3}{2} \;\Rightarrow\; dx = \frac{1}{2}(2z\,dz) \;\Rightarrow\; dx = z\,dz$$

$$\therefore \quad \mathrm{I} = \int \frac{1}{\left(\frac{z^2-3}{2}-1\right).z}(z\,dz)$$

$$= 2\int \frac{1}{(z^2-3-2)}dz = 2\int \frac{1}{z^2-5}dz = 2\int \frac{1}{z^2-\left(\sqrt{5}\right)^2}dz$$

$$= 2.\frac{1}{2\sqrt{5}}\log\left|\frac{z-\sqrt{5}}{z+\sqrt{5}}\right| + c \qquad \left[\because \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c\right]$$

$$= \frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{2x+3}-\sqrt{5}}{\sqrt{2x+3}+\sqrt{5}}\right| + c. \qquad \left[\because z = \sqrt{2x+3}\right]$$

Example 5. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{(x^2-3x+2)\sqrt{x-1}}\,dx$ (*ii*) $\int \frac{\sqrt{x}}{1+x}\,dx$

(*iii*) $\int \frac{x^3}{(x-1)\sqrt{x+2}}\,dx$ (*iv*) $\int \frac{1}{x^2\sqrt{x+1}}\,dx$

(*v*) $\int \frac{x+2}{(x^2+2x+2)\sqrt{x+1}}\,dx.$

Solution. (*i*) Let $I = \int \frac{1}{(x^2 - 3x + 2)\sqrt{x-1}}\, dx$

Put $\quad \sqrt{x-1} = z$ [Squaring on both sides]

$\Rightarrow \quad x - 1 = z^2 \Rightarrow x = z^2 + 1 \Rightarrow dx = 2z\, dz$

$$\therefore \quad I = \int \frac{1}{[(z^2+1)^2 - 3(z^2+1) + 2] \cdot z} (2z\, dz)$$

$$= 2\int \frac{1}{z^4 + 1 + 2z^2 - 3z^2 - 3 + 2}\, dz$$

$$= 2\int \frac{1}{z^4 - z^2}\, dz = 2\int \frac{1}{z^2(z^2-1)}\, dz$$

Let $\quad \frac{1}{z^2(z^2-1)} = \frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{(y-1)}$ ...(1) [Put $z^2 = y$]

Multiplying both sides by $y(y-1)$, we get

$$1 = A(y-1) + By$$

Put $y = 0$, we get

$$1 = A(0-1) \Rightarrow A = -1$$

Put $y = 1$, we get

$$1 = A(1-1) + B \Rightarrow B = 1$$

Substituting the values of A and B in (1), we have

$$\frac{1}{y(y-1)} = \frac{-1}{y} + \frac{1}{(y-1)}$$

or $\quad \frac{1}{z^2(z^2-1)} = \frac{-1}{z^2} + \frac{1}{(z^2-1)}$ $[\because \ y = z^2]$

$$\therefore \quad I = 2\int \frac{1}{z^2(z^2-1)}\, dz = 2\int \left[-\frac{1}{z^2} + \frac{1}{z^2-1}\right] dz$$

$$= 2\left[\int \frac{1}{z^2-1}\, dz - \int \frac{1}{z^2}\, dz\right]$$

$$= 2\int \frac{1}{z^2 - 1^2}\, dz - 2\int z^{-2}\, dz$$

$$= 2 \cdot \frac{1}{2(1)} \log\left|\frac{z-1}{z+1}\right| - 2\frac{z^{-2+1}}{(-2+1)} + c$$

$$\left[\because \int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \log\left|\frac{z-1}{z+1}\right| + \frac{2}{z} + c$$

$$= \log\left|\frac{\sqrt{x-1}-1}{\sqrt{x-1}+1}\right| + \frac{2}{\sqrt{x-1}} + c. \qquad \left[\because \ z = \sqrt{x-1}\right]$$

(*ii*) Let $\quad I = \int \frac{\sqrt{x}}{x+1}\,dx = \int \frac{\sqrt{x}}{(x+1)} \times \frac{\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{x}{(x+1)\sqrt{x}}\,dx$ **[Rationalisation]**

Put $\quad \sqrt{x} = z$ **[Squaring on both sides]**

$\Rightarrow \quad x = z^2 \;\Rightarrow\; dx = 2z\,dz$

$$\therefore \quad I = \int \frac{z^2}{(z^2+1).z}\,.(2z\,dz) = 2\int \frac{z^2}{z^2+1}\,dz$$

$$= 2\int \frac{z^2+1-1}{z^2+1}\,dz \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= 2\int \left[\frac{z^2+1}{z^2+1} - \frac{1}{z^2+1}\right] dz = 2\int 1.\,dz - 2\int \frac{1}{z^2+1^2}\,dz$$

$$= 2z - 2\tan^{-1} z + c \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + c. \qquad \left[\because z = \sqrt{x}\right]$$

(*iii*) Let $\quad I = \int \frac{x^3}{(x-1)\sqrt{x+2}}\,dx$

Put $\sqrt{x+2} = z$ **[Squaring on both sides]**

$\Rightarrow \quad x + 2 = z^2 \;\Rightarrow\; x = z^2 - 2 \;\Rightarrow\; dx = 2z\,dz$

$$\therefore \quad I = \int \frac{(z^2-2)^3}{(z^2-2-1).z}\,.(2z\,dz)$$

$$= 2\int \frac{[z^6 - 8 - 6z^2(z^2-2)]}{z^2-3}\,dz \qquad [\because (a-b)^3 = a^3 - b^3 - 3ab\,(a-b)]$$

$$= 2\int \frac{z^6 - 6z^4 + 12z^2 - 8}{z^2-3}\,dz$$

Since the degree of numerator is greater than the degree of denominator, therefore by long division, we have

$$\therefore \quad I = 2\int \left[(z^4 - 3z^2 + 3) + \frac{1}{z^2-3}\right].\,dz$$

$$= 2\int z^4\,dz - 6\int z^2\,dz + 6\int 1.\,dz + 2\int \frac{1}{z^2-\left(\sqrt{3}\right)^2}\,dz$$

$$= \frac{2z^5}{5} - \frac{6z^3}{3} + 6z + 2.\frac{1}{2\sqrt{3}}\log\left|\frac{z-\sqrt{3}}{z+\sqrt{3}}\right| + c$$

$$\left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$\begin{array}{r|l}
z^2-3 & z^6 - 6z^4 + 12z^2 - 8 \;\big(\; z^4 - 3z^2 + 3 \\
 & +z^6 - 3z^4 \\
 & \underline{-\quad +} \\
 & -3z^4 + 12z^2 - 8 \\
 & -3z^4 + 9z^2 \\
 & \underline{+\quad -} \\
 & 3z^2 - 8 \\
 & 3z^2 - 9 \\
 & \underline{-\quad +} \\
 & +1
\end{array}$$

$$= \frac{2}{5}(x+2)^{5/2} - 2(x+2)^{3/2} + 6\sqrt{x+2} + \frac{1}{\sqrt{3}} \log \left| \frac{\sqrt{x+2} - \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} \right| + c. \qquad [\because\ z = \sqrt{x+2}]$$

(*iv*) Let $I = \int \frac{1}{x^2 \sqrt{x+1}}\, dx$ ...(1)

Put $\sqrt{x+1} = z$ [Squaring on both sides]

$\Rightarrow \quad x + 1 = z^2 \ \Rightarrow\ x = z^2 - 1 \ \Rightarrow\ dx = 2z\, dz$

$$\therefore \quad I = \int \frac{1}{(z^2-1)^2 . z}(2z\, dz)$$

$$= 2\int \frac{1}{(z^2-1)^2}\, dz = 2\int \frac{1}{[(z-1)(z+1)]^2}\, dz$$

Let $\frac{1}{(z-1)^2 (z+1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{(z+1)} + \frac{D}{(z+1)^2}$ ...(2)

Multiplying both sides by $(z-1)^2 (z+1)^2$, we get

$$1 = A(z-1)(z+1)^2 + B(z+1)^2 + C(z+1)(z-1)^2 + D(z-1)^2 \qquad \text{...(3)}$$

$$z - 1 = 0 \ \Rightarrow\ z = 1, z + 1 = 0 \ \Rightarrow\ z = -1$$

Put $z = -1$ in (3), we get

$$1 = A[(-1)-1][-1+1]^2 + B[-1+1]^2 + C[-1+1][-1-1]^2 + D[-1-1]^2$$

$$\Rightarrow \quad 1 = 4D \ \Rightarrow\ D = \frac{1}{4}.$$

Put $z = 1$ in (3), we get

$$1 = A(1-1)(1+1)^2 + B(1+1)^2 + C(1+1)(1-1)^2 + D(1-1)^2$$

$$\Rightarrow \quad 1 = 4B \ \Rightarrow\ B = \frac{1}{4}$$

Equating the co-efficients of z^3 on both sides of equation (3), we get

$$0 = A + C \ \Rightarrow\ C = -A$$

Equating the constant terms on both sides of equation (3), we get

$$1 = -A + B + C + D \qquad [\because\ C = -A]$$

$$\Rightarrow \quad 1 = -A + B - A + D$$

$$\Rightarrow \quad 1 = -2A + \frac{1}{4} + \frac{1}{4} \ \Rightarrow\ 1 - \frac{1}{2} = -2A \ \Rightarrow\ A = -\frac{1}{4}$$

$$\therefore \quad C = -A \ \Rightarrow\ C = \frac{1}{4}$$

Substituting the values of A, B, C and D in equation (2), we have

$$\frac{1}{(z-1)^2 (z+1)^2} = \frac{-1/4}{(z-1)} + \frac{1/4}{(z-1)^2} + \frac{1/4}{(z+1)} + \frac{1/4}{(z+1)^2}$$

$$= -\frac{1}{4(z-1)} + \frac{1}{4(z-1)^2} + \frac{1}{4(z+1)} + \frac{1}{4(z+1)^2}$$

$$\therefore \quad I = 2\int \frac{1}{(z-1)^2(z+1)^2}\,dz = 2\int \left[-\frac{1}{4(z-1)} + \frac{1}{4(z-1)^2} + \frac{1}{4(z+1)} + \frac{1}{4(z+1)^2}\right].\,dz$$

$$= -\frac{1}{2}\int \frac{1}{z-1}\,dz + \frac{1}{2}\int \frac{1}{(z-1)^2}\,dz + \frac{1}{2}\int \frac{1}{z+1}\,dz + \frac{1}{2}\int \frac{1}{(z+1)^2}\,dz$$

$$= -\frac{1}{2}\log|z-1| + \frac{1}{2}\frac{(z-1)^{-2+1}}{(-2+1)} + \frac{1}{2}\log|z+1| + \frac{1}{2}\frac{(z+1)^{-2+1}}{(-2+1)} + c$$

$$= -\frac{1}{2}\log|z-1| - \frac{1}{2(z-1)} + \frac{1}{2}\log|z+1| - \frac{1}{2(z+1)} + c$$

$$= -\frac{1}{2}\left[\log\left|\frac{z-1}{z+1}\right| + \frac{1}{z-1} + \frac{1}{z+1}\right] + c = -\frac{1}{2}\left[\log\left|\frac{z-1}{z+1}\right| + \frac{2z}{z^2-1}\right] + c$$

$$= -\frac{1}{2}\log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| - \frac{\sqrt{x+1}}{x} + c. \qquad \left[\because\ z = \sqrt{x+1}\right]$$

(*v*) Let $I = \int \frac{x+2}{(x^2+2x+2)\sqrt{x+1}}\,dx$

Put $\sqrt{x+1} = z$ [Squaring on both sides]

$\Rightarrow \quad x + 1 = z^2 \;\Rightarrow\; x = z^2 - 1 \;\Rightarrow\; dx = 2z\,dz$

$$\therefore \quad I = \int \frac{z^2-1+2}{[(z^2-1)^2 + 2(z^2-1) + 2].\,z}\,(2z\,dz)$$

$$= \int \frac{2(z^2+1)}{z^4 + 1 - 2z^2 + 2z^2 - 2 + 2}\,dz = 2\int \frac{(z^2+1)}{(z^4+1)}\,dz$$

$$= 2\int \frac{\left(1+\frac{1}{z^2}\right)}{\left(z^2+\frac{1}{z^2}\right)}\,dz \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and the denominator by } z^2\end{array}\right]$$

Put $z - \frac{1}{z} = y \;\Rightarrow\; \left(1+\frac{1}{z^2}\right)dz = dy$ $\qquad \left[\begin{array}{l}\because\ \left(z-\frac{1}{z}\right)^2 = z^2 + \frac{1}{z^2} - 2\\ \Rightarrow \left[\left(z-\frac{1}{z}\right)^2 + 2\right] = \left(z^2+\frac{1}{z^2}\right)\end{array}\right]$

$$= 2\int \frac{1}{y^2+2}\,dy = 2\int \frac{1}{y^2 + \left(\sqrt{2}\right)^2}\,dy$$

$$= 2\,.\,\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + c \qquad \left[\because\ \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \sqrt{2}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2}}\right) + c = \sqrt{2}\tan^{-1}\left(\frac{z^2-1}{\sqrt{2}z}\right) + c \qquad \left[\because\ y = \left(z-\frac{1}{z}\right)\right]$$

$$= \sqrt{2}\tan^{-1}\left(\frac{x+1-1}{\sqrt{2}\sqrt{x+1}}\right) + c \qquad \left[\because z = \sqrt{x+1}\right]$$

$$= \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2(x+1)}}\right) + c.$$

Example 6. *Evaluate the following integrals :*

(*i*) $\int \frac{1}{(x+1)\sqrt{x^2-1}}\,dx$ (*ii*) $\int \frac{1}{(x^2-4)\sqrt{x+1}}\,dx$

(*iii*) $\int \frac{1}{(x+1)\sqrt{1-2x-x^2}}\,dx$ (*iv*) $\int \frac{1}{(x+1)\sqrt{x^2+x+1}}\,dx$

(*v*) $\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$ (*vi*) $\int \frac{1}{\sqrt{2e^x-1}}\,dx.$

Solution. (*i*) Let $\quad I = \int \frac{1}{(x+1)\sqrt{x^2-1}}\,dx$

Put $\quad (x+1) = \frac{1}{z} \Rightarrow x = \left(\frac{1}{z}-1\right) \Rightarrow dx = -\frac{1}{z^2}\,dz$

$$\therefore \quad I = \int \frac{1}{\left(\frac{1}{z}\right)\sqrt{\left(\frac{1}{z}-1\right)^2-1}}\left(-\frac{1}{z^2}dz\right) = \int \frac{1}{\frac{1}{z}\sqrt{\frac{1}{z^2}+1-\frac{2}{z}-1}}\cdot\left(-\frac{1}{z^2}dz\right)$$

$$= \int \frac{1}{\left(\frac{1}{z^2}\right)\sqrt{1-2z}}\cdot\left(-\frac{1}{z^2}dz\right) = -\int \frac{1}{\sqrt{1-2z}}\,dz$$

$$= -\int (1-2z)^{-1/2}\,dz$$

$$= -\frac{(1-2z)^{-1/2+1}}{\left(-\frac{1}{2}+1\right)(-2)} + c \qquad \left[\because \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1).a} + c\right]$$

$$= \sqrt{1-2z} + c = \sqrt{1-\frac{2}{x+1}} + c \qquad \left[\because z = \frac{1}{x+1}\right]$$

$$= \sqrt{\frac{x-1}{x+1}} + c.$$

(*ii*) Let $\quad I = \int \frac{1}{(x^2-4)\sqrt{x+1}}\,dx \qquad ...(1)$

Put $\quad \sqrt{x+1} = z \qquad$ [Squaring on both sides]

$\Rightarrow \quad x+1 = z^2 \Rightarrow x = z^2-1 \Rightarrow dx = 2z\,dz$

$$\therefore \quad I = \int \frac{1}{[(z^2-1)^2-4].z}\,.(2z\,dz)$$

$$= 2\int \frac{1}{z^4+1-2z^2-4}\,dz$$

$\therefore \qquad I = 2\int \frac{1}{z^4 - 2z^2 - 3}\,dz$

Let $\quad \frac{1}{z^4 - 2z^2 - 3} = \frac{1}{y^2 - 2y - 3} = \frac{1}{(y-3)(y+1)}$ [Put $z^2 = y$]

$\Rightarrow \quad \frac{1}{(y-3)(y+1)} = \frac{A}{y-3} + \frac{B}{y+1}$ ...(2)

Multiplying both sides $(y-3)(y+1)$, we get

$1 = A(y+1) + B(y-3)$ ...(3)

$y + 1 = 0 \Rightarrow y = -1, y - 3 = 0 \Rightarrow y = 3$

Put $y = -1$ in (3), we get

$$1 = A(-1+1) + B(-1-3) \Rightarrow 1 = -4B \Rightarrow B = -\frac{1}{4}$$

Put $y = 3$ in (3), we get

$$1 = A(3+1) + B(3-3) \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$$

Substituting the values of A and B in (2), we have

$$\frac{1}{(y-3)(y+1)} = \frac{1/4}{(y-3)} + \frac{-1/4}{(y+1)} = \frac{1}{4(y-3)} - \frac{1}{4(y+1)}$$

or $\quad \frac{1}{(z^2-3)(z^2+1)} = \frac{1}{4(z^2-3)} - \frac{1}{4(z^2+1)}$ [$\because \; y = z^2$]

$\therefore \qquad I = 2\int \frac{1}{(z^2-3)(z^2+1)}\,dz = 2\int \left[\frac{1}{4(z^2-3)} - \frac{1}{4(z^2+1)}\right] dz$

$$= \frac{1}{2}\int \frac{1}{z^2-3}\,dz - \frac{1}{2}\int \frac{1}{z^2+1}\,dz = \frac{1}{2}\int \frac{1}{z^2 - \left(\sqrt{3}\right)^2}\,dz - \frac{1}{2}\int \frac{1}{z^2+1^2}\,dz$$

$$= \frac{1}{2}\cdot\frac{1}{2\sqrt{3}}\log\left|\frac{z-\sqrt{3}}{z+\sqrt{3}}\right| - \frac{1}{2}\cdot\tan^{-1} z + c$$

$$\left[\begin{array}{l} \because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c \\ \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c \end{array}\right]$$

$$= \frac{1}{4\sqrt{3}}\log\left|\frac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}\right| - \frac{1}{2}\tan^{-1}\left(\sqrt{x+1}\right) + c.$$

(*iii*) Let $\quad I = \int \frac{1}{(x+1)\sqrt{1-2x-x^2}}\,dx$

Put $(x+1) = \frac{1}{z} \Rightarrow x = \frac{1}{z} - 1 \Rightarrow dx = -\frac{1}{z^2}\,dz$

$\therefore \qquad I = \int \frac{1}{\left(\frac{1}{z}\right)\sqrt{1 - 2\left(\frac{1}{z}-1\right) - \left(\frac{1}{z}-1\right)^2}}\cdot\left(-\frac{1}{z^2}\,dz\right)$

$$= \int \frac{1}{\left(\frac{1}{z}\right)\sqrt{1-\frac{2}{z}+2-\frac{1}{z^2}-1+\frac{2}{z}}}\left(-\frac{1}{z^2}\,dz\right)$$

$$= -\int \frac{1}{z\sqrt{2-\frac{1}{z^2}}}\,dz = -\int \frac{1}{\sqrt{2z^2-1}}\,dz$$

$$= -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{z^2-1/2}}\,dz = -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{z^2-\left(\frac{1}{\sqrt{2}}\right)^2}}\,dz$$

$$= -\frac{1}{\sqrt{2}}\cdot\log\left|z+\sqrt{z^2-\frac{1}{2}}\right|+c \qquad \left[\because \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right|+c\right]$$

$$= -\frac{1}{\sqrt{2}}\log\left|\frac{1}{x+1}+\sqrt{\left(\frac{1}{x+1}\right)^2-\frac{1}{2}}\right|+c = -\frac{1}{\sqrt{2}}\log\left|\frac{1}{x+1}+\frac{\sqrt{1-2x-x^2}}{\sqrt{2}(x+1)}\right|+c$$

$$\left[\because\ z=\frac{1}{x+1}\right]$$

$$= -\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-2x-x^2}}{\sqrt{2}(x+1)}\right|+c.$$

(*iv*) Let $\quad I = \int \frac{1}{(x+1)\sqrt{x^2+x+1}}\,dx$

Put $\quad x+1=\frac{1}{z} \Rightarrow x=\frac{1}{z}-1 \Rightarrow dx=-\frac{1}{z^2}\,dz$

$$\therefore \qquad I = \int \frac{1}{\left(\frac{1}{z}\right)\sqrt{\left(\frac{1}{z}-1\right)^2+\left(\frac{1}{z}-1\right)+1}}\left(-\frac{1}{z^2}\,dz\right)$$

$$= -\int \frac{1}{z\sqrt{\frac{1}{z^2}+1-\frac{2}{z}+\frac{1}{z}-1+1}}\,dz = -\int \frac{1}{z\sqrt{\frac{1-z+z^2}{z^2}}}\,dz$$

$$= -\int \frac{1}{\sqrt{z^2-z+1}}\,dz \qquad \text{[Note this step]}$$

$$= -\int \frac{1}{\sqrt{\left(z^2-z+\frac{1}{4}\right)+1-\frac{1}{4}}}\,dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4}\text{, to the denom}\\ \because \left(\frac{1}{2}\text{ co-eff. of } z\right)^2=\frac{1}{4}\end{array}\right]$$

$$= -\int \frac{1}{\sqrt{\left(z-\frac{1}{2}\right)^2+\frac{3}{4}}}\,dz = -\int \frac{1}{\sqrt{\left(z-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}\,dz$$

$$= -\log\left|\left(z-\frac{1}{2}\right)+\sqrt{\left(z-\frac{1}{2}\right)^2+\frac{3}{4}}\right|+c$$

$$\left[\because \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|+c\right]$$

$$= -\log\left|\left(z-\frac{1}{2}\right)+\sqrt{z^2-z+1}\right|+c$$

$$= -\log\left|\left(\frac{1}{x+1}-\frac{1}{2}\right)+\sqrt{\left(\frac{1}{x+1}\right)^2-\left(\frac{1}{x+1}\right)+1}\right|+c \qquad \left[\because z=\frac{1}{x+1}\right]$$

$$= -\log\left|\frac{1-x}{2(x+1)}+\sqrt{\frac{1-(x+1)+(x+1)^2}{(x+1)^2}}\right|+c$$

$$= -\log\left|\frac{1-x}{2(x+1)}+\frac{\sqrt{x^2+x+1}}{x+1}\right|+c.$$

(*v*) Let $\quad I = \int \frac{1}{x^2\sqrt{x^2+1}}\,dx$

Put $\quad x = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2}\,dz$

$\therefore \quad I = \int \frac{1}{\left(\frac{1}{z^2}\right)\sqrt{\frac{1}{z^2}+1}}\left(-\frac{1}{z^2}\right)dz = -\int \frac{z}{\sqrt{z^2+1}}\,dz$

Put $(z^2+1) = y \Rightarrow 2z\,dz = dy \Rightarrow z\,dz = \frac{1}{2}\,dy$

$\therefore \quad I = -\int \frac{1}{\sqrt{y}}\left(\frac{1}{2}\,dy\right) = -\frac{1}{2}\int y^{-1/2}\,dy = -\frac{1}{2}\left[\frac{y^{-1/2+1}}{\left(-\frac{1}{2}+1\right)}\right]+c$

$$= -y^{1/2}+c = -\sqrt{y}+c = -\sqrt{z^2+1}+c \qquad [\because y=(z^2+1)]$$

$$= -\sqrt{\frac{1}{x^2}+1}+c \qquad \left[\because z=\frac{1}{x}\right]$$

$$= -\frac{\sqrt{1+x^2}}{x}+c.$$

(*vi*) Let $\quad I = \int \frac{1}{\sqrt{2e^x - 1}}\, dx = \int \frac{e^x}{e^x\sqrt{2e^x - 1}}\, dx$ [Multiply and divided by e^x]

Put $\sqrt{2e^x - 1} = z$ (Squaring on both sides)

$$\Rightarrow \quad 2e^x - 1 = z^2 \quad \Rightarrow \quad e^x = \left(\frac{z^2+1}{2}\right)$$

$$\Rightarrow \quad 2e^x\, dx = 2z\, dz \quad \Rightarrow \quad e^x\, dx = z\, dz$$

$$\therefore \quad I = \int \frac{1}{\left(\frac{z^2+1}{2}\right).\, z} \,.\, (z\, dz) = \int \frac{2}{z^2+1}\, dz = 2\int \frac{1}{z^2+1^2}\, dz$$

$$= 2\tan^{-1} z + c \qquad \left[\because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= 2\tan^{-1}\left(\sqrt{2e^x - 1}\right) + c. \qquad \left[\because\ z = \sqrt{2e^x - 1}\right]$$

Example 7. *Evaluate the following integrals* :

(*i*) $\int \frac{1}{(x+1)\sqrt{x^2+1}}\, dx$ (*ii*) $\int \frac{x}{(x^2+4)\sqrt{x^2+1}}\, dx$

(*iii*) $\int \frac{1}{(x^2-1)\sqrt{x^2+1}}\, dx$ (*iv*) $\int \frac{\sqrt{1+x^2}}{1-x^2}\, dx$

(*v*) $\int \frac{1}{(2x^2+3)\sqrt{3x^2-1}}\, dx$ (*vi*) $\int \frac{x}{(x^2+a^2)\sqrt{x^2+b^2}}\, dx$

(*vii*) $\int \frac{1+x^{-2/3}}{1+x}\, dx.$

Solution. (*i*) Let $I = \int \frac{1}{(x+1)\sqrt{x^2+1}}\, dx$

Put $\quad x + 1 = \frac{1}{z} \quad \Rightarrow \quad x = \left(\frac{1}{z} - 1\right) \quad \Rightarrow \quad dx = -\frac{1}{z^2}\, dz$

$$\therefore \quad I = \int \frac{1}{\left(\frac{1}{z}\right)\sqrt{\left(\frac{1}{z}-1\right)^2 + 1}}\left(-\frac{1}{z^2}\, dz\right) = -\int \frac{1}{z\sqrt{\frac{1}{z^2} + 1 - \frac{2}{z} + 1}}\, dz$$

$$= -\int \frac{1}{z\sqrt{\frac{1}{z^2} - \frac{2}{z} + 2}}\, dz = -\int \frac{1}{\sqrt{1 - 2z + 2z^2}}\, dz$$

$$= -\int \frac{1}{\sqrt{2z^2 - 2z + 1}}\, dz = -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{z^2 - z + \frac{1}{2}}}\, dz$$

$$= -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(z^2 - z + \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)}}\, dz \qquad \left[\begin{array}{l} \text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2}\text{ co-eff. of } z\right)^2 = \frac{1}{4} \end{array}\right]$$

$$= -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(z - \frac{1}{2}\right)^2 + \frac{1}{4}}}\, dz = -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(z - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}}\, dz$$

$$= -\frac{1}{\sqrt{2}} \log \left| \left(z - \frac{1}{2}\right) + \sqrt{\left(z - \frac{1}{2}\right)^2 + \frac{1}{4}} \right| + c$$

$$\left[\because \int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \log \left| x + \sqrt{x^2 + a^2} \right| + c\right]$$

$$= -\frac{1}{\sqrt{2}} \log \left| \left(z - \frac{1}{2}\right) + \frac{\sqrt{2z^2 - 2z + 1}}{\sqrt{2}} \right| + c$$

$$= -\frac{1}{\sqrt{2}} \log \left| \left(\frac{1}{x+1} - \frac{1}{2}\right) + \frac{\sqrt{2\left(\frac{1}{x+1}\right)^2 - 2\left(\frac{1}{x+1}\right) + 1}}{\sqrt{2}} \right| + c \qquad \left[\because z = \frac{1}{x+1}\right]$$

$$= -\frac{1}{\sqrt{2}} \log \left| \frac{1-x}{2(x+1)} + \frac{\sqrt{x^2+1}}{\sqrt{2}\,(x+1)} \right| + c.$$

(*ii*) Let $I = \int \frac{x}{(x^2+4)\sqrt{x^2+1}}\, dx$

Put $\sqrt{x^2+1} = z$ (Squaring on both sides)

$\Rightarrow \quad x^2 + 1 = z^2 \;\Rightarrow\; x^2 = z^2 - 1 \;\Rightarrow\; 2x\,dx = 2z\,dz \;\Rightarrow\; x\,dx = z\,dz$

$$\therefore \quad I = \int \frac{1}{(z^2 - 1 + 4)\,.\,z}\,(z\,dz)$$

$$= \int \frac{1}{z^2 + 3}\, dz = \int \frac{1}{z^2 + \left(\sqrt{3}\right)^2}\, dz$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z}{\sqrt{3}}\right) + c \qquad \left[\because \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{x^2+1}}{\sqrt{3}}\right) + c \qquad \left[\because z = \sqrt{x^2+1}\right]$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\sqrt{\frac{x^2+1}{3}}\right) + c.$$

(*iii*) Let $\quad I = \int \frac{1}{(x^2-1)\sqrt{x^2+1}}\, dx$

Put $\quad x = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2}\, dz$

$\therefore \quad I = \int \frac{1}{\left(\frac{1}{z^2}-1\right)\sqrt{\frac{1}{z^2}+1}}\left(-\frac{1}{z^2}\, dz\right)$

$$= -\int \frac{1}{\left(\frac{1-z^2}{z^2}\right)\frac{\sqrt{1+z^2}}{z}}\left(\frac{1}{z^2}\right) dz = -\int \frac{z}{(1-z^2)\sqrt{1+z^2}}\, dz$$

Put $\quad \sqrt{1+z^2} = y$ **(Squaring on both sides)**

$\Rightarrow \quad 1 + z^2 = y^2 \Rightarrow z^2 = y^2 - 1 \Rightarrow 2z\, dz = 2y\, dy \Rightarrow z\, dz = y\, dy$

$\therefore \quad I = -\int \frac{1}{[1-(y^2-1)].\, y}\,(y\, dy)$

$$= -\int \frac{1}{2-y^2}\, dy = -\int \frac{1}{(\sqrt{2})^2 - y^2}\, dy$$

$$= -\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2}+y}{\sqrt{2}-y}\right| + c \qquad \left[\because \int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

$\therefore \quad I = -\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2}+\sqrt{1+z^2}}{\sqrt{2}-\sqrt{1+z^2}}\right| + c \qquad \left[\because y = \sqrt{1+z^2}\right]$

$$= -\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2}+\sqrt{1+\frac{1}{x^2}}}{\sqrt{2}-\sqrt{1+\frac{1}{x^2}}}\right| + c \qquad \left[\because z = \frac{1}{x}\right]$$

$$= -\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2}+\sqrt{x^2+1}}{\sqrt{2}-\sqrt{x^2+1}}\right| + c.$$

(*iv*) Let $\quad I = \int \frac{\sqrt{1+x^2}}{1-x^2}\, dx \qquad \ldots(1)$

$$= \int \frac{\sqrt{1+x^2}}{1-x^2} \times \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\, dx$$ **[Rationalisation]**

$$= \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^2}}\,dx = \int \frac{2-(1-x^2)}{(1-x^2)\sqrt{1+x^2}}\,dx \qquad \text{[Note this step]}$$

$$= \int \frac{2}{(1-x^2)\sqrt{1+x^2}}\,dx - \int \frac{(1-x^2)}{(1-x^2)\sqrt{1+x^2}}\,dx$$

$$= 2\int \frac{1}{(1-x^2)\sqrt{1+x^2}}\,dx - \int \frac{1}{\sqrt{1+x^2}}\,dx$$

$\Rightarrow$ $\quad I = 2I_1 - I_2$ (say) ...(2)

where $\quad I_1 = \int \frac{1}{(1-x^2)\sqrt{1+x^2}}\,dx$

Put $\quad x = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2}\,dz$

$$\therefore \quad I_1 = \int \frac{1}{\left(1-\frac{1}{z^2}\right)\sqrt{1+\frac{1}{z^2}}}\left(-\frac{1}{z^2}\,dz\right)$$

$$= -\int \frac{1}{\left(\frac{z^2-1}{z^2}\right)\sqrt{\frac{z^2+1}{z^2}}}\left(\frac{1}{z^2}\,dz\right) = -\int \frac{z}{(z^2-1)\sqrt{z^2+1}}\,dz$$

Put $\sqrt{z^2+1} = y$ (Squaring on both sides)

$\Rightarrow \quad z^2+1 = y^2 \Rightarrow z^2 = y^2-1 \Rightarrow 2z\,dz = 2y\,dy \Rightarrow z\,dz = y\,dy$

$$\therefore \quad I_1 = -\int \frac{1}{(y^2-1-1)y}\,(y\,dy)$$

$$= -\int \frac{1}{y^2-2}\,dy = -\int \frac{1}{y^2-\left(\sqrt{2}\right)^2}\,dy$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{y-\sqrt{2}}{y+\sqrt{2}}\right| + c_1 \qquad \left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{z^2+1}-\sqrt{2}}{\sqrt{z^2+1}+\sqrt{2}}\right| + c_1 \qquad \left[\because y = \sqrt{z^2+1}\right]$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{2}}{\sqrt{\frac{1}{x^2}+1}+\sqrt{2}}\right| + c_1 \qquad \left[\because z = \frac{1}{x}\right]$$

$$\Rightarrow \quad I_1 = -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2}x}{\sqrt{1+x^2}+\sqrt{2}x}\right| + c_1 \qquad ...(3)$$

and $\quad I_2 = \int \frac{1}{\sqrt{1+x^2}}\,dx \qquad \left[\because \int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left|x+\sqrt{a^2+x^2}\right| + c\right]$

$$\Rightarrow \quad I_2 = \log\left|x+\sqrt{1+x^2}\right| + c_2 \quad \text{...(4)}$$

$\therefore$ From equation (2), we have

$$I = 2I_1 - I_2$$

$$= 2\left[-\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2}x}{\sqrt{1+x^2}+\sqrt{2}x}\right| + c_1\right] - \left[\log\left|x+\sqrt{1+x^2}\right| + c_2\right]$$

[Using equations (3) and (4)]

$$= -\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2}x}{\sqrt{1+x^2}+\sqrt{2}x}\right| + 2c_1 - \log\left|x+\sqrt{1+x^2}\right| - c_2$$

$$= -\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2}x}{\sqrt{1+x^2}+\sqrt{2}x}\right| - \log\left|x+\sqrt{1+x^2}\right| + c. \quad \text{where } c = (2c_1 - c_2)$$

(v) Let $I = \int \frac{1}{(2x^2+3)\sqrt{3x^2-1}}\,dx$

Put $x = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2}\,dz$

$$\therefore \quad I = \int \frac{1}{\left(\frac{2}{z^2}+3\right)\sqrt{\frac{3}{z^2}-1}}\left(-\frac{1}{z^2}\,dz\right)$$

$$= -\int \frac{1}{\left(\frac{2+3z^2}{z^2}\right)\sqrt{\frac{3-z^2}{z^2}}}\left(\frac{1}{z^2}\,dz\right) = -\int \frac{z}{(2+3z^2)\sqrt{3-z^2}}\,dz$$

Put $\sqrt{3-z^2} = y$ [Squaring on both sides]

$$\Rightarrow \quad 3 - z^2 = y^2 \Rightarrow 3 - y^2 = z^2 \Rightarrow -2z\,dz = 2y\,dy \Rightarrow -z\,dz = y\,dy$$

$$\therefore \quad I = \int \frac{1}{[2+3(3-y^2)].\,y}\,(y\,dy)$$

$$= \int \frac{1}{11-3y^2}\,dy = \frac{1}{3}\int \frac{1}{\left(\frac{11}{3}-y^2\right)}\,dy = \frac{1}{3}\int \frac{1}{\left(\sqrt{11/3}\right)^2 - y^2}\,dy$$

$$= \frac{1}{3}\cdot\frac{1}{2\sqrt{11/3}}\log\left|\frac{\sqrt{11/3}+y}{\sqrt{11/3}-y}\right| + c \quad \left[\because \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

$$= \frac{1}{2\sqrt{33}}\log\left|\frac{\sqrt{11/3}+\sqrt{3-z^2}}{\sqrt{11/3}-\sqrt{3-z^2}}\right| + c \quad \left[\because y = \sqrt{3-z^2}\right]$$

$$= \frac{1}{2\sqrt{33}}\log\left|\frac{\sqrt{11/3}+\sqrt{3-\frac{1}{x^2}}}{\sqrt{11/3}-\sqrt{3-\frac{1}{x^2}}}\right| + c \quad \left[\because z = \frac{1}{x}\right]$$

$$= \frac{1}{2\sqrt{33}} \log \left| \frac{\frac{\sqrt{11}}{\sqrt{3}} + \frac{\sqrt{3x^2-1}}{x}}{\frac{\sqrt{11}}{\sqrt{3}} - \frac{\sqrt{3x^2-1}}{x}} \right| + c$$

$$= \frac{1}{2\sqrt{33}} \log \left| \frac{\sqrt{11}x + \sqrt{9x^2-3}}{\sqrt{11}x - \sqrt{9x^2-3}} \right| + c.$$

(vi) Let $\quad I = \int \frac{x}{(x^2+a^2)\sqrt{x^2+b^2}}\,dx$

Put $\sqrt{x^2+b^2} = z$ [Squaring on both sides]

$\Rightarrow \quad x^2+b^2 = z^2 \quad \Rightarrow \quad x^2 = z^2 - b^2 \quad \Rightarrow \quad 2x\,dx = 2z\,dz \quad \Rightarrow \quad x\,dx = z\,dz$

$\therefore \quad I = \int \frac{1}{(z^2-b^2+a^2)\,z} \cdot (z\,dz) = \int \frac{1}{z^2-(b^2-a^2)}\,dz.$

Now, the following cases arises :

Case I : When $a^2 > b^2 \Rightarrow b^2 - a^2 < 0$

$$\therefore \quad I = \int \frac{1}{z^2 + \left(\sqrt{a^2-b^2}\right)^2}\,dz$$

$$= \frac{1}{\sqrt{a^2-b^2}} \tan^{-1}\left(\frac{z}{\sqrt{a^2-b^2}}\right) + c \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{\sqrt{a^2-b^2}} \tan^{-1}\left(\sqrt{\frac{x^2+b^2}{a^2-b^2}}\right) + c. \qquad \left[\because z = \sqrt{x^2+b^2}\right]$$

Case II : When $a^2 < b^2 \Rightarrow b^2 - a^2 > 0$

$$\therefore \quad I = \int \frac{1}{z^2 - \left(\sqrt{b^2-a^2}\right)^2}\,dz$$

$$= \frac{1}{2\sqrt{b^2-a^2}} \log \left| \frac{z - \sqrt{b^2-a^2}}{z + \sqrt{b^2-a^2}} \right| + c \quad \left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$\therefore \quad I = \frac{1}{2\sqrt{b^2-a^2}} \log \left| \frac{\sqrt{x^2+b^2} - \sqrt{b^2-a^2}}{\sqrt{x^2+b^2} + \sqrt{b^2-a^2}} \right| + c. \qquad \left[\because z = \sqrt{x^2+b^2}\right]$$

(vii) Let $\quad I = \int \frac{1+x^{-2/3}}{1+x}\,dx$ ...(1)

Put $\quad x = z^3 \Rightarrow z = x^{1/3} \Rightarrow dx = 3z^2\,dz$

$$\therefore \quad I = \int \frac{1+(z^3)^{-2/3}}{1+z^3}\,(3z^2\,dz)$$

$$= 3 \int \frac{\left(1+\frac{1}{z^2}\right)}{(1+z^3)} (z^2\, dz) = 3 \int \frac{z^2+1}{1+z^3}\, dz$$

$$\Rightarrow \qquad I = 3 \int \frac{z^2+1}{(1+z)(1-z+z^2)}\, dz \qquad [\because \ (a^3+b^3) = (a+b)(a^2-ab+b^2)]$$

Let $\dfrac{z^2+1}{(1+z)(1-z+z^2)} = \dfrac{A}{(1+z)} + \dfrac{(Bz+C)}{(1-z+z^2)}$ ...(2)

Multiplying both sides by $(1+z)(1-z+z^2)$, we get

$$z^2 + 1 = A(1 - z + z^2) + (Bz + c)(1 + z) \qquad ...(3)$$

Put $z = -1$ in equation (3), we get

$$[(-1)^2 + 1] = A\,[1 - (-1) + (-1)^2] + [B\,(-1) + C]\,[1 + (-1)]$$

$$\Rightarrow \qquad 2 = 3A \ \Rightarrow \ A = \frac{2}{3}$$

Equating the co-efficients of z^2 on both sides of equation (3), we get

$$1 = A + B \ \Rightarrow \ 1 = \frac{2}{3} + B \ \Rightarrow \ B = \frac{1}{3}$$

Equating the constant terms on both sides of equation (3), we get

$$1 = A + C \ \Rightarrow \ 1 = \frac{2}{3} + C \ \Rightarrow \ C = \frac{1}{3}$$

Substituting the values of A, B and C in equation (2), we have

$$\frac{z^2+1}{(1+z)(1-z+z^2)} = \frac{2/3}{(1+z)} + \frac{\frac{1}{3}z + \frac{1}{3}}{(1-z+z^2)} = \frac{2}{3(1+z)} + \frac{1}{3}\left(\frac{z+1}{z^2-z+1}\right)$$

$$\therefore \qquad I = 3 \int \frac{z^2+1}{(1+z)(1-z+z^2)}\, dz = 3 \int \left[\frac{2}{3(1+z)} + \frac{1}{3}\left(\frac{z+1}{z^2-z+1}\right)\right] dz$$

$$= 2 \int \frac{1}{1+z}\, dz + \int \frac{z+1}{z^2-z+1}\, dz$$

$$= 2 \log|1+z| + \int \frac{z+1}{z^2-z+1}\, dz$$

$$= 2 \log|1+z| + \frac{1}{2} \int \frac{2z+2}{z^2-z+1}\, dz \qquad \text{[Multiply and divide by 2]}$$

$$= 2 \log|1+z| + \frac{1}{2} \int \frac{2z-1+3}{z^2-z+1}\, dz \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= 2 \log|1+z| + \frac{1}{2} \int \frac{2z-1}{z^2-z+1}\, dz + \frac{3}{2} \int \frac{1}{z^2-z+1}\, dz$$

$$= 2\log|1+z| + \frac{1}{2}\int \frac{2z-1}{z^2-z+1}\,dz + \frac{3}{2}\int \frac{1}{\left(z^2-z+\frac{1}{4}\right)+\left(1-\frac{1}{4}\right)}\,dz$$

$$\left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{co-eff. of } z\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= 2\log|1+z| + \frac{1}{2}\int \frac{2z-1}{z^2-z+1}\,dz + \frac{3}{2}\int \frac{1}{\left(z-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dz$$

$$= 2\log|1+z| + \frac{1}{2}\log|z^2-z+1| + \frac{3}{2}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{z-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$\left[\begin{array}{l}\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\\ \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\end{array}\right]$$

$$= 2\log|1+z| + \frac{1}{2}\log|z^2-z+1| + \sqrt{3}\tan^{-1}\left(\frac{2z-1}{\sqrt{3}}\right) + c$$

$$= 2\log|1+x^{1/3}| + \frac{1}{2}\log|x^{2/3}-x^{1/3}+1| + \sqrt{3}\tan^{-1}\left(\frac{2x^{1/3}-1}{\sqrt{3}}\right) + c.$$

$$[\because\ z = x^{1/3}]$$

Example 8. *Evaluate the following integrals* :

(*i*) $\int \frac{x^2+1}{x^4-x^2+1}\,dx$ (*ii*) $\int \frac{x^2-1}{x^4+1}\,dx$

(*iii*) $\int \frac{1}{x^4+3x^2+1}\,dx$ (*iv*) $\int \frac{x+2}{\sqrt{x^2+2x+5}}\,dx.$

Solution. (*i*) Let $I = \int \frac{x^2+1}{x^4-x^2+1}\,dx$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2-1+\frac{1}{x^2}\right)}\,dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)-1}\,dx$$

$$\left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and the denominator by } x^2\end{array}\right]$$

Put $x - \frac{1}{x} = z \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dz$

$$\left[\begin{array}{l}\because \left(x-\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}-2\\ \Rightarrow \left(x-\frac{1}{x}\right)^2+2 = \left(x^2+\frac{1}{x^2}\right)\\ \Rightarrow z^2+2 = \left(x^2+\frac{1}{x^2}\right)\end{array}\right]$$

$\therefore \quad I = \int \frac{1}{z^2+2-1} dz = \int \frac{1}{z^2+1} dz$

$= \tan^{-1} z + c \qquad \left[\because \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c\right]$

$= \tan^{-1}\left(x - \frac{1}{x}\right) + c. \qquad \left[\because z = \left(x - \frac{1}{x}\right)\right]$

(*ii*) Let $\quad I = \int \frac{x^2-1}{x^4+1} dx = \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)} dx \qquad$ [Dividing the numerator and the denominator by x^2]

Put $x + \frac{1}{x} = z \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dz$

$$\left[\begin{aligned} &\because \left(x+\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \\ &\Rightarrow \left(x+\frac{1}{x}\right)^2 - 2 = \left(x^2 + \frac{1}{x^2}\right) \\ &\Rightarrow z^2 - 2 = \left(x^2 + \frac{1}{x^2}\right) \end{aligned}\right]$$

$\therefore \quad I = \int \frac{1}{z^2-2} dz$

$= \int \frac{1}{z^2 - (\sqrt{2})^2} dz$

$= \frac{1}{2\sqrt{2}} \log\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + c \qquad \left[\because \int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c\right]$

$= \frac{1}{2\sqrt{2}} \log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + c \qquad \left[\because z = \left(x + \frac{1}{x}\right)\right]$

$= \frac{1}{2\sqrt{2}} \log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + c.$

(*iii*) Let $\quad I = \int \frac{1}{x^4+3x^2+1} dx$

$\therefore \quad = \frac{1}{2}\int \frac{2}{x^4+3x^2+1} dx = \frac{1}{2}\int \frac{1+1}{x^4+3x^2+1} dx \qquad$ [Multiply and divided by 2]

$= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+3x^2+1} dx \qquad$ [Add and subtract x^2 to the numerator]

$= \frac{1}{2}\int \frac{x^2+1}{x^4+3x^2+1} dx - \frac{1}{2}\int \frac{x^2-1}{x^4+3x^2+1} dx$

$\Rightarrow \quad I = \frac{1}{2} I_1 - \frac{1}{2} I_2 \qquad$ ...(1)

where $$I_1 = \int \frac{x^2+1}{x^4+3x^2+1}\,dx$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+3+\frac{1}{x^2}\right)}\,dx \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and the denominator by } x^2\end{array}\right]$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)+3}\,dx$$

Put $\left(x-\frac{1}{x}\right) = z \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dz$ $\left[\begin{array}{l}\because \left(x-\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}-2\\ \Rightarrow \left(x-\frac{1}{x}\right)^2+2 = \left(x^2+\frac{1}{x^2}\right)\\ \Rightarrow z^2+2 = \left(x^2+\frac{1}{x^2}\right)\end{array}\right]$

$$\therefore \quad I_1 = \int \frac{1}{z^2+2+3}\,dz = \int \frac{1}{z^2+\left(\sqrt{5}\right)^2}\,dz$$

$$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{z}{\sqrt{5}}\right)+c_1 \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{5}}\right)+c_1 \qquad \left[\because z = \left(x-\frac{1}{x}\right)\right]$$

$$\Rightarrow \quad I_1 = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right)+c_1 \qquad \ldots(2)$$

and $$I_2 = \int \frac{x^2-1}{x^4+3x^2+1}\,dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+3+\frac{1}{x^2}\right)}\,dx \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and the denominator by } x^2\end{array}\right]$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)+3}\,dx$$

$$\left[\because \left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2 \Rightarrow \left(x+\frac{1}{x}\right)^2 - 2 = \left(x^2+\frac{1}{x^2}\right) \Rightarrow (y^2-2) = \left(x^2+\frac{1}{x^2}\right)\right]$$

Put $\left(x+\frac{1}{x}\right) = y \Rightarrow \left(1-\frac{1}{x^2}\right) dx = dy$

$$\therefore \quad I_2 = \int \frac{1}{y^2-2+3}\, dy$$

$$= \int \frac{1}{y^2+1}\, dy = \tan^{-1} y + c_2 \qquad \left[\because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \tan^{-1}\left(x+\frac{1}{x}\right) + c_2 \qquad \left[\because y = \left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow \quad I_2 = \tan^{-1}\left(\frac{x^2+1}{x}\right) + c_2 \qquad \text{...(3)}$$

$\therefore$ From equation (1), we have

$$I = \frac{1}{2} I_1 - \frac{1}{2} I_2$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right)+c_1\right] - \frac{1}{2}\left[\tan^{-1}\left(\frac{x^2+1}{x}\right)+c_2\right]$$

[Using equations (2) and (3)]

$$= \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right) + \frac{1}{2}c_1 - \frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right) - \frac{1}{2}c_2$$

$$= \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right) + c. \qquad \text{where } c = \frac{1}{2}c_1 - \frac{1}{2}c_2$$

(*iv*) Let $I = \int \frac{x+2}{\sqrt{x^2+2x+5}}\, dx$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+5}}\, .\, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int \frac{2x+2+2}{\sqrt{x^2+2x+5}}\, dx \qquad \text{[Note this step]}$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+5}}\, dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+5}}\, dx$$

$$= \frac{1}{2}\int (x^2+2x+5)^{-1/2}(2x+2)\, dx + \int \frac{1}{\sqrt{x^2+2x+5}}\, dx$$

$$= \frac{1}{2}\frac{(x^2+2x+5)^{-1/2+1}}{-\frac{1}{2}+1} + \int \frac{1}{\sqrt{x^2+2x+5}}\, dx \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}+c\right]$$

$$= \sqrt{x^2+2x+5} + \int \frac{1}{\sqrt{(x^2+2x+1)+(5-1)}}\,dx \quad \left[\begin{array}{l}\text{Completing the square in the} \\ \text{second integral by adding \&} \\ \text{subtract}\left(\frac{1}{2}\text{ co-eff. of } x\right)^2 \text{ i.e., } 1\end{array}\right]$$

$$= \sqrt{x^2+2x+5} + \int \frac{1}{\sqrt{(x+1)^2+2^2}}\,dx \quad \left[\because \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right| + c\right]$$

$$= \sqrt{x^2+2x+5} + \log\left|(x+1)+\sqrt{(x+1)^2+2^2}\right| + c$$

$$= \sqrt{x^2+2x+5} + \log\left|(x+1)+\sqrt{x^2+2x+5}\right| + c.$$

Example 9. *Evaluate the following integrals :*

(i) $\int \frac{1}{(x+1)\sqrt{x^2+5x+6}}\,dx$ (ii) $\int \frac{1}{(x+2)\sqrt{x^2+6x+7}}\,dx$

(iii) $\int \frac{x^2+2x+3}{(x+1)\sqrt{x^2+1}}\,dx$ (iv) $\int \frac{1}{x}\frac{\sqrt{1-x}}{\sqrt{1+x}}\,dx.$

Solution. (i) Let $I = \int \frac{1}{(x+1)\sqrt{x^2+5x+6}}\,dx$

Put $(x+1) = \frac{1}{z} \Rightarrow x = \left(\frac{1}{z}-1\right) \Rightarrow dx = -\frac{1}{z^2}\,dz$

$$\therefore \quad I = \int \frac{1}{\frac{1}{z}\sqrt{\left(\frac{1}{z}-1\right)^2+5\left(\frac{1}{z}-1\right)+6}}\left(-\frac{1}{z^2}\,dz\right)$$

$$= -\int \frac{1}{\sqrt{\frac{1}{z^2}+1-\frac{2}{z}+\frac{5}{z}-5+6}}\left(\frac{1}{z}\,dz\right) = -\int \frac{1}{\sqrt{\frac{1}{z^2}+\frac{3}{z}+2}}\left(\frac{1}{z}\,dz\right)$$

$$= -\int \frac{1}{\sqrt{2z^2+3z+1}}\,dz = -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(z^2+\frac{3}{2}z+\frac{1}{2}\right)}}\,dz$$

$$= -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(z^2+\frac{3}{2}z+\frac{9}{16}\right)+\left(\frac{1}{2}-\frac{9}{16}\right)}}\,dz$$

$$\left[\begin{array}{l}\text{Add and subtract } \frac{9}{16} \text{ to the denom.} \\ \because \left(\frac{1}{2}\text{ co-eff. of } z\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}\end{array}\right]$$

$$= -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(z+\frac{3}{4}\right)^2-\frac{1}{16}}}\,dz = -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(z+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}}\,dz$$

$$= -\frac{1}{\sqrt{2}} \log \left| \left(z + \frac{3}{4}\right) + \sqrt{\left(z + \frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2} \right| + c$$

$$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c\right]$$

$$= -\frac{1}{\sqrt{2}} \log \left| \left(\frac{4z+3}{4}\right) + \sqrt{\frac{2z^2 + 3z + 1}{2}} \right| + c$$

$$\therefore \qquad I = -\frac{1}{\sqrt{2}} \log \left| \frac{4\left(\frac{1}{x+1}\right) + 3}{4} + \sqrt{\frac{2\left(\frac{1}{x+1}\right)^2 + 3\left(\frac{1}{x+1}\right) + 1}{2}} \right| + c$$

$$\left[\because z = \left(\frac{1}{x+1}\right)\right]$$

$$= -\frac{1}{\sqrt{2}} \log \left| \frac{3x+7}{4(x+1)} + \frac{\sqrt{x^2 + 5x + 6}}{\sqrt{2}\,(x+1)} \right| + c.$$

(*ii*) Let $I = \int \frac{1}{(x+2)\sqrt{x^2 + 6x + 7}}\, dx$

Put $(x+2) = \frac{1}{z} \Rightarrow x = \frac{1}{z} - 2 \Rightarrow dx = -\frac{1}{z^2}\, dz$

$$\therefore \qquad I = \int \frac{1}{\left(\frac{1}{z}\right)\sqrt{\left(\frac{1}{z} - 2\right)^2 + 6\left(\frac{1}{z} - 2\right) + 7}} \left(-\frac{1}{z^2}\, dz\right)$$

$$= -\int \frac{1}{\sqrt{\frac{1}{z^2} - \frac{4}{z} + 4 + \frac{6}{z} - 12 + 7}} \left(\frac{1}{z}\, dz\right)$$

$$= -\int \frac{1}{\sqrt{\frac{1}{z^2} + \frac{2}{z} - 1}} \left(\frac{1}{z}\, dz\right) = -\int \frac{1}{\sqrt{1 + 2z - z^2}}\, dz$$

$$= -\int \frac{1}{\sqrt{-(z^2 - 2z - 1)}}\, dz$$

$$= -\int \frac{1}{\sqrt{-(z^2 - z + 1 - 1 - 1)}}\, dz \qquad \left[\begin{array}{l}\textbf{Add and subtract 1 to the denom.} \\ \because \left(\frac{1}{2}\,\textbf{co-eff. of } z\right)^2 = 1\end{array}\right]$$

$$= -\int \frac{1}{\sqrt{-(z-1)^2 + \left(\sqrt{2}\right)^2}}\, dz = -\int \frac{1}{\sqrt{\left(\sqrt{2}\right)^2 - (z-1)^2}}\, dz$$

$$= -\sin^{-1}\left(\frac{z-1}{\sqrt{2}}\right) + c \qquad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= -\sin^{-1}\left(\frac{\frac{1}{x+2} - 1}{\sqrt{2}}\right) + c \qquad \left[\because z = \frac{1}{x+2}\right]$$

$$= -\sin^{-1}\left(\frac{-x-1}{\sqrt{2}\,(x+2)}\right) + c.$$

(*iii*) Let $I = \int \frac{x^2 + 2x + 3}{(x+1)\sqrt{x^2+1}}\, dx$ ...(1)

$$= \int \frac{(x^2 + 2x + 1) + 2}{(x+1)\sqrt{x^2+1}}\, dx \qquad \text{[Note this step]}$$

$$= \int \frac{(x+1)^2 + 2}{(x+1)\sqrt{x^2+1}}\, dx = \int \frac{(x+1)}{\sqrt{x^2+1}}\, dx + 2\int \frac{1}{(x+1)\sqrt{x^2+1}}\, dx$$

$$= \int \frac{x}{\sqrt{x^2+1}}\, dx + \int \frac{1}{\sqrt{x^2+1}}\, dx + 2\int \frac{1}{(x+1)\sqrt{x^2+1}}\, dx$$

$\Rightarrow$ $I = I_1 + I_2 + 2I_3$ ...(2)

where $I_1 = \int \frac{x}{\sqrt{x^2+1}}\, dx$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2+1}}\, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int (x^2+1)^{-1/2}\,(2x)\, dx$$

$$= \frac{1}{2}\,\frac{(x^2+1)^{-1/2+1}}{\left(-\frac{1}{2}+1\right)} + c_1 \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$\Rightarrow$ $I_1 = \sqrt{x^2+1} + c_1$ ...(3)

and $I_2 = \int \frac{1}{\sqrt{x^2+1}}\, dx$

$\Rightarrow$ $I_2 = \log\left|x + \sqrt{x^2+1}\right| + c_2$ ...(4) $\left[\because \int \frac{1}{\sqrt{x^2+a^2}}\, dx = \log\left|x + \sqrt{x^2+a^2}\right| + c\right]$

and $I_3 = \int \frac{1}{(x+1)\sqrt{x^2+1}}\, dx$

Put $(x+1)=\frac{1}{z} \Rightarrow x=\frac{1}{z}-1 \Rightarrow dx=-\frac{1}{z^2}\,dz$

$$\therefore \quad I_3=\int \frac{1}{\left(\frac{1}{z}\right)\sqrt{\left(\frac{1}{z}-1\right)^2+1}}\left(-\frac{1}{z^2}dz\right)=-\int \frac{1}{\sqrt{\frac{1}{z^2}-\frac{2}{z}+1+1}}\left(\frac{1}{z}dz\right)$$

$$=-\int \frac{1}{\sqrt{1-2z+2z^2}}\,dz=-\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{z^2-z+\frac{1}{2}}}\,dz$$

$$=-\int \frac{1}{\sqrt{\left(z^2-z+\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)}}\,dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } z\right)^2=\frac{1}{4}\end{array}\right]$$

$$=-\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(z-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}\,dz$$

$$=-\frac{1}{\sqrt{2}}\log\left|\left(z-\frac{1}{2}\right)+\sqrt{\left(z-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right|+c_3$$

$$\left[\because \int \frac{1}{\sqrt{x^2+a^2}}\,dx=\log\left|x+\sqrt{x^2+a^2}\right|+c\right]$$

$$=-\frac{1}{\sqrt{2}}\log\left|\frac{2z-1}{2}+\sqrt{z^2-z+\frac{1}{2}}\right|+c_3$$

$$=-\frac{1}{\sqrt{2}}\log\left|\frac{2\left(\frac{1}{x+1}\right)-1}{2}+\sqrt{\left(\frac{1}{x+1}\right)^2-\left(\frac{1}{x+1}\right)+\frac{1}{2}}\right|+c_3 \qquad \left[\because z=\frac{1}{x+1}\right]$$

$$=-\frac{1}{\sqrt{2}}\log\left|\frac{1-x}{2(1+x)}+\frac{\sqrt{2-2(x+1)+(x+1)^2}}{\sqrt{2}(x+1)}\right|+c_3$$

$$\Rightarrow \quad I_3=-\frac{1}{\sqrt{2}}\log\left|\frac{1-x}{2(1+x)}+\frac{\sqrt{x^2+1}}{\sqrt{2}(x+1)}\right|+c_3 \qquad \text{...(5)}$$

$\therefore$ From equation (2), we have

$$I=I_1+I_2+2I_3$$

$$=\sqrt{x^2+1}+c_1+\log\left|x+\sqrt{x^2+1}\right|+c_2$$

$$+2\left[-\frac{1}{\sqrt{2}}\log\left|\frac{1-x}{2(1+x)}+\frac{\sqrt{x^2+1}}{\sqrt{2}(x+1)}\right|+c_3\right]$$

[Using equation (3), (4) and (5)]

$$\therefore \quad = \sqrt{x^2+1} + \log\left|x + \sqrt{x^2+1}\right| - \sqrt{2}\log\left|\frac{1-x}{2(x+1)} + \frac{\sqrt{x^2+1}}{\sqrt{2}(x+1)}\right| + c.$$

where $c = (c_1 + c_2 - 2c_3)$

(*iv*) Let $\quad I = \int \frac{1\sqrt{1-x}}{x\sqrt{1+x}}\,dx$

$$= \int \frac{1\sqrt{1-x}}{x\sqrt{1+x}} \times \frac{\sqrt{1-x}}{\sqrt{1-x}}\,dx \qquad \text{[Rationalisation]}$$

$$= \int \frac{(1-x)}{x\sqrt{(1+x)(1-x)}}\,dx = \int \frac{1-x}{x\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{x\sqrt{1-x^2}} - \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{x\sqrt{1-x^2}} - \sin^{-1}x + c \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c\right]$$

Put $\quad \frac{1}{x} = z \;\Rightarrow\; x = \frac{1}{z} \;\Rightarrow\; dx = -\frac{1}{z^2}\,dz$

$$= \int \frac{1}{\left(\frac{1}{z}\right)\sqrt{1-\frac{1}{z^2}}}\left(-\frac{1}{z^2}\,dz\right) - \sin^{-1}x + c$$

$$= -\int \frac{1}{\sqrt{z^2-1}}\,dz - \sin^{-1}x + c$$

$$= -\log\left|z + \sqrt{z^2-1}\right| - \sin^{-1}x + c$$

$$\left[\because \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x + \sqrt{x^2-a^2}\right| + c\right]$$

$$= -\log\left|\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right| - \sin^{-1}x + c \qquad \left[\because z = \frac{1}{x}\right]$$

$$= -\log\left|\frac{1+\sqrt{1-x^2}}{x}\right| - \sin^{-1}x + c.$$

EXERCISE FOR PRACTICE

Integrate the following functions w.r.t. x :

1. $\frac{2x}{x^2+3x+2}$
2. $\frac{2x-1}{(x-1)(x+2)(x+3)}$
3. $\frac{1}{(x-1)^2(x-2)}$
4. $\frac{x}{(x-1)(x^2+4)}$

5. $\dfrac{1}{1+3e^x+2e^{2x}}$

6. $\dfrac{1}{(e^x-1)^2}$

7. $\dfrac{\sin x}{\cos x\,(1+\cos^2 x)}$

8. $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

9. $\dfrac{x^2}{x^4-1}$

10. $\dfrac{x^2-1}{x^4+1}$

11. $\dfrac{1}{(2x+3)\sqrt{4x+5}}$

12. $\dfrac{1}{(1+x^2)\sqrt{2+x^2}}$

13. $\dfrac{1}{(x^2-1)\sqrt{x^2+1}}$

14. $\dfrac{1}{\sin^4 x+\cos^4 x}$

15. $\dfrac{x^2+a^2}{x^4+a^4}$

16. $\dfrac{1}{(x^2+1)\sqrt{x}}$

17. $\dfrac{1}{x\sqrt{ax-x^2}}$

18. $\dfrac{x^2+1}{x^4+7x^2+1}$

19. $\dfrac{5}{(x^2+1)(x+2)}$

20. $\dfrac{x}{(x^2-a^2)(x^2-b^2)}$.

Answers

1. $4\log|x+2|-2\log|x+1|+c$

2. $-\dfrac{1}{6}\log|x-1|-\dfrac{1}{3}\log|x+2|+\dfrac{1}{2}\log|x-3|+c$

3. $\log\left|\dfrac{x-2}{x-1}\right|+\dfrac{1}{x-1}+c$

4. $\dfrac{1}{5}\log|x-1|-\dfrac{1}{10}\log|x^2+4|+\dfrac{2}{5}\tan^{-1}\dfrac{x}{2}+c$

5. $x+\log(1+e^x)-2\log(1+2e^x)+c$

6. $\log\left|\dfrac{e^x}{e^x-1}\right|-\dfrac{1}{e^x-1}+c$

7. $\dfrac{1}{2}\log(1+\cos^2 x)-\log|\cos x|+c$

8. $x+\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{x}{\sqrt{3}}-3\tan^{-1}\left(\dfrac{x}{2}\right)+c$

9. $\dfrac{1}{4}\log\left|\dfrac{x-1}{x+1}\right|+\dfrac{1}{2}\tan^{-1}x+c$

10. $\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|+c$

11. $\tan^{-1}\left(\sqrt{4x+5}\right)+c$

12. $-\tan^{-1}\left(\dfrac{\sqrt{2+x^2}}{x}\right)+c$

13. $-\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{\sqrt{2}x+\sqrt{x^2+1}}{\sqrt{2}x-\sqrt{x^2+1}}\right|+c$

14. $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{\tan^2\theta-1}{\sqrt{2}\tan\theta}\right)$

15. $\dfrac{1}{\sqrt{2}a}\tan^{-1}\left(\dfrac{x^2-a^2}{\sqrt{2}\,ax}\right)+c$

16. $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{x-1}{\sqrt{2x}}\right)-\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{x-\sqrt{2x}+1}{x+\sqrt{2x}+1}\right|+c$

17. $-\dfrac{2}{a}\sqrt{\dfrac{a-x}{x}}+c$

18. $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x^2-1}{3x}\right)+c$

19. $2\tan^{-1}x-\dfrac{1}{2}\log|x^2+1|+\log|x+2|+c$

20. $\dfrac{1}{2(a^2-b^2)}\log\left|\dfrac{x^2-a^2}{x^2-b^2}\right|+c.$

6

Definite Integral as the Limit of a Sum

6.1. DEFINITION : DEFINITE INTEGRAL AS THE LIMIT OF A SUM

If $f(x)$ is a continuous and single-valued function in the closed interval $[a, b]$, then

$$\lim_{h \to 0} h[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + \overline{n-1}\, h)]$$

where : $\quad h = \dfrac{b-a}{n}$ or $nh = b - a$

is called the definite integral of $f(x)$ between the limits a and b and is written as :

$$\int_a^b f(x)\, dx$$

Thus : $$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + \overline{n-1}\, h)]$$

where : $nh = b - a$

Or $$\int_a^b f(x)\, dx = \lim \sum_{r=0}^{n-1} hf(a + rh)$$

where : $h \to 0$, $n \to \infty$, $nh = b - a$

Note 1. The above method of evaluating $\int_a^b f(x)\, dx$

i.e., $$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + \overline{n-1}\, h)]$$

where : $\quad nh = b - a$

is called integration by summation or integration from definition or integration from first principles or integration by *ab*-nitio.

Note 2. Proceed to the limit as $h \to 0$ only after putting $nh = b - a$.

6.2. DEFINITION : WALLI'S FORMULA

If $f(x)$ be a function such that :

(*i*) $f(x)$ is continuous function defined on $[a, b]$ and

(*ii*) $f(x)$ does not change its sign on $[a, b]$.

Then, $$\lim_{r \to 1} (r - 1)\, [f(a)\,.\, a + f(ar)\,.\, ar + f(ar^2)\,.\, ar^2 + \ldots\ldots + f(ar^{n-1})\, ar^{n-1}]$$

where : $r^n = \dfrac{b}{a}$, $(a \neq 0)$

This definition is useful to integrate the functions of the form $\int_a^b x^n \, dx$.

where : n is a positive integer.

Remark : The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable.

If the independent variable is denoted by t or u or y instead of x, we simply write the integral as $\int_a^b f(t)\, dt$ or $\int_a^b f(u)\, du$ or $\int_a^b f(y)\, dy$ instead of $\int_a^b f(x)\, dx$. Hence, the value of integration is called a dummy variable.

6.3. SOME USEFUL RESULTS HELPFUL FOR THE EVALUATION OF INTEGRALS BY SUMMATION

1. $1 + 2 + 3 + \ldots\ldots + (n - 1) = \dfrac{n(n-1)}{2}$
2. $1 + 2 + 3 + \ldots\ldots + n = \dfrac{n(n+1)}{2}$
3. $1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}$
4. $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
5. $1^3 + 2^3 + 3^3 + \ldots\ldots + (n-1)^3 = \dfrac{n^2(n-1)^2}{4} = \left[\dfrac{n(n-1)}{2}\right]^2$
6. $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \dfrac{n^2(n+1)^2}{4} = \left[\dfrac{n(n+1)}{2}\right]^2$
7. $a + ar + ar^2 + \ldots\ldots + ar^{n-1} = \dfrac{a(r^n - 1)}{(r-1)}$, $r \neq 1, r > 1$.
8. $\sin a + \sin (a + h) + \sin (a + 2h) + \ldots.. + \sin (a + \overline{n-1}\, h)$

$$= \frac{\sin\left[a + \left(\dfrac{n-1}{2}\right)h\right]\sin\left(\dfrac{nh}{2}\right)}{\sin\left(\dfrac{h}{2}\right)}$$

9. $\cos a + \cos(a + h) + \cos(a + 2h) + \ldots\ldots + \cos (a + \overline{n-1}\, h)$

$$= \frac{\cos\left[a + \left(\dfrac{n-1}{2}\right)h\right]\sin\left(\dfrac{nh}{2}\right)}{\sin\left(\dfrac{h}{2}\right)}$$

6.4. IMPORTANT NOTE

A check on the answer :

If $f(x)$ is positive in the interval (a, b), then $\int_a^b f(x)\, dx$ is also positive.

$\therefore \qquad \int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a+(n-1)h\}]$

where : $nh = b - a$.

Now, each term of the series on the R.H.S. of above equation is positive.

Therefore their sum and therefore the limit of their sum is also positive.

Similarly, if $f(x)$ is negative in the interval (a, b), then $\int_a^b f(x)\,dx$ is also negative.

The above note enables us to say whether the value of a definite integral should be positive or negative and, therefore, serves as a check on the answer.

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following definite integrals as limit of sums :*

(i) $\int_1^2 x\,dx$ (ii) $\int_0^3 (x+4)\,dx$

(iii) $\int_a^b x\,dx$ (iv) $\int_0^5 (x+1)\,dx$

(v) $\int_0^2 (2x+1)\,dx$ (vi) $\int_{-1}^1 (x+3)\,dx$

(vii) $\int_2^7 (5x-9)\,dx$ (viii) $\int_0^4 2x\,dx.$

Solution. (i) Let $I = \int_1^2 x\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 1$, $b = 2$, $f(x) = x$

Let $\qquad h = \dfrac{b-a}{n} = \dfrac{2-1}{n} = \dfrac{1}{n} \Rightarrow nh = 1 ; n \in N$

$$\therefore \quad I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a+(n-1)h\}]$$

$$= \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f\{1+(n-1)h\}] \qquad [\because a = 1]$$

$$= \lim_{h\to 0} h[1 + (1+h) + (1+2h) + \ldots\ldots + \{1+(n-1)h\}] \qquad [\because f(x) = x]$$

$$= \lim_{h\to 0} h[(1+1+1+\ldots\ldots+1) + h\{1+2+3+\ldots\ldots+(n-1)\}]$$

$$= \lim_{h\to 0} h[n + h\{1+2+3+\ldots\ldots+(n-1)\}]$$

$$= \lim_{h\to 0} h\left[n + h.\frac{n(n-1)}{2}\right] \qquad \left[\because [1+2+3+\ldots\ldots+(n-1)] = \frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0}\left[nh + \frac{nh(nh-h)}{2}\right]$$

$$= \lim_{h\to 0}\left[1 + \frac{1(1-h)}{2}\right] \qquad [\because nh = 1]$$

$$= 1 + \frac{1}{2} = \frac{3}{2}.$$

(*ii*) Let $\quad I = \int_0^3 (x + 4)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 0,\ b = 3,\ f(x) = x + 4$

Let $\quad h = \dfrac{b-a}{n} = \dfrac{3-0}{n} = \dfrac{3}{n} \quad \Rightarrow \quad nh = 3,\ n \in \mathrm{N}.$

$$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}] \qquad [\because\ a = 0]$$

$$= \lim_{h \to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h \to 0} h[(0+4) + (h+4) + (2h+4) + \ldots\ldots + \{(n-1)h + 4\}] \quad [\because\ f(x) = (x+4)]$$

$$= \lim_{h \to 0} h[(4 + 4 + 4 + \ldots\ldots + 4) + \{h + 2h + \ldots\ldots + (n-1)h\}]$$

$$= \lim_{h \to 0} h[4n + h\{1 + 2 + 3 + \ldots\ldots + (n-1)\}]$$

$$= \lim_{h \to 0} h\left[4n + h \cdot \frac{n(n-1)}{2}\right] \qquad \left[\because\ [1+2+3+\ldots\ldots+(n-1)] = \frac{n(n-1)}{2}\right]$$

$$= \lim_{h \to 0}\left[4nh + \frac{nh(nh-h)}{2}\right] = \lim_{h \to 0}\left[4(3) + \frac{3(3-h)}{2}\right] \qquad [\because\ nh = 3]$$

$$= 12 + \frac{3(3-0)}{2} = 12 + \frac{9}{2} = \frac{33}{2}.$$

(*iii*) Let $\quad I = \int_a^b x\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = a,\ b = b,\ f(x) = x$

Let $\quad h = \dfrac{b-a}{n} \quad \Rightarrow \quad nh = b - a,\ n \in \mathrm{N}.$

$$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[a + (a+h) + (a+2h) + \ldots\ldots + \{a + (n-1)h\}] \qquad [\because\ f(x) = x]$$

$$= \lim_{h \to 0} h[(a + a + a + \ldots\ldots + a) + h(1 + 2 + 3 + \ldots\ldots + (n-1))]$$

$$= \lim_{h \to 0} h\left[na + h \cdot \frac{n(n-1)}{2}\right] \qquad \left[\because\ [1+2+3+\ldots\ldots+(n-1)] = \frac{n(n-1)}{2}\right]$$

$$= \lim_{h \to 0}\left[nha + \frac{nh(nh-h)}{2}\right]$$

$$= \lim_{h \to 0}\left[(b-a)a + \frac{(b-a)(b-a-h)}{2}\right] \qquad [\because\ nh = (b-a)]$$

$$= a(b-a) + \frac{(b-a)(b-a)}{2}$$

$$= (b-a)\left[a + \frac{b-a}{2}\right] = \frac{(b-a)(b+a)}{2} = \frac{b^2-a^2}{2}.$$

(*iv*) Let $\quad I = \int_0^5 (x+1)\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 0$, $b = 5$, $f(x) = x + 1$.

Let $\quad h = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n} \;\Rightarrow\; nh = 5, n \in N.$

$$\therefore \quad I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a+(n-1)h\}]$$

$$= \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0+(n-1)h\}] \qquad [\because\ a = 0]$$

$$= \lim_{h\to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h\to 0} h[(0+1) + (h+1) + (2h+1) + \ldots\ldots + \{(n-1)h+1\}] \qquad [\because\ f(x) = (x+1)]$$

$$= \lim_{h\to 0} h[(1+1+1+\ldots\ldots+1) + \{h + 2h + \ldots\ldots + (n-1)h\}]$$

$$= \lim_{h\to 0} h[n + h\{1+2+3+\ldots\ldots+(n-1)\}]$$

$$= \lim_{h\to 0} h\left[n + h\cdot\frac{n(n-1)}{2}\right] \qquad \left[\because\ [1+2+3+\ldots\ldots+(n-1)] = \frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0}\left[nh + \frac{nh(nh-h)}{2}\right] = \lim_{h\to 0}\left[5 + \frac{5(5-h)}{2}\right] \qquad [\because\ nh = 5]$$

$$= 5 + \frac{25}{2} = \frac{35}{2}.$$

(*v*) Let $\quad I = \int_0^2 (2x+1)\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 0$, $b = 2$, $f(x) = (2x + 1)$.

Let $\quad h = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} \;\Rightarrow\; nh = 2.$

$$\therefore \quad I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a+(n-1)h\}]$$

$$= \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0+(n-1)h\}] \qquad [\because\ a = 0]$$

$$= \lim_{h\to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h\to 0} h[\{2(0)+1\} + (2h+1) + (2(2h)+1\} + \ldots\ldots + \{2(n-1)h+1\}]$$

$$[\because\ f(x) = (2x+1)]$$

$$= \lim_{h\to 0} h[1 + (2h+1) + (4h+1) + 2h(n-1) + 1]$$

$$= \lim_{h\to 0} h[(1+1+1+\ldots\ldots+1) + 2h[1+2+3+\ldots\ldots(n-1)]]$$

$= \lim_{h \to 0} h\left[n + 2h\frac{n(n-1)}{2}\right]$ $\left[\because\ [1+2+3+......+(n-1)] = \frac{n(n-1)}{2}\right]$

$= \lim_{h \to 0} [nh + nh(nh - h]$

$= \lim_{h \to 0} [2 + 2(2 - h)] = 2 + 4 = 6.$ $[\because\ nh = 2]$

(*vi*) Let $\text{I} = \int_{-1}^{1} (x+3)\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = -1$, $b = 1$, $f(x) = x + 3$.

Let $h = \frac{b-a}{n} = \frac{1-(-1)}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in \text{N}.$

$\therefore$ $\text{I} = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a + (n-1)h)]$

$= \lim_{h \to 0} h[f(-1) + f(-1+h) + f(-1+2h) + ... + f(-1+(n-1)h)]$

$[\because\ a = -1]$

$= \lim_{h \to 0} h[(-1+3) + (-1+h+3) + (-1+2h+3) + ... + \{-1+(n-1)h+3\}]$

$[\because\ f(x) = x + 3]$

$= \lim_{h \to 0} h[2 + (h+2) + (2h+2) + ... + \{(n-1)h + 2\}]$

$= \lim_{h \to 0} h[(2+2+2+......+2) + h\{1+2+3+......+(n-1)\}]$

$= \lim_{h \to 0} h\left[2n + h\,.\frac{n(n-1)}{2}\right]$ $\left[\because\ [1+2+3+......+(n-1)] = \frac{n(n-1)}{2}\right]$

$= \lim_{h \to 0}\left[2nh + \frac{nh(nh-h)}{2}\right]$

$= \lim_{h \to 0}\left[2(2) + \frac{2(2-h)}{2}\right]$ $[\because\ nh = 2]$

$= 4 + (2 - 0) = 6.$

(*vii*) Let $\text{I} = \int_{2}^{7} (5x - 9)\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 2$, $b = 7$, $f(x) = 5x - 9$.

Let $h = \frac{b-a}{n} = \frac{7-2}{n} = \frac{5}{n} \Rightarrow nh = 5, n \in \text{N}.$

$\therefore$ $\text{I} = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)]$

$= \lim_{h \to 0} h[f(2) + f(2+h) + f(2+2h) + ... + f(2+(n-1)h)]$ $[\because\ a = 2]$

$= \lim_{h \to 0} h[5(2) - 9) + \{5(2+h) - 9\} + \{5(2+2h) - 9\} + ... + \{5(2+(n-1)h) - 9\}]$

$[\because\ f(x) = (5x - 9)]$

$= \lim_{h \to 0} h[1 + (5h + 1) + (10h + 1) + \ldots\ldots + \{5(n-1)h + 1\}]$

$= \lim_{h \to 0} h[(1 + 1 + 1 + \ldots\ldots + 1) + 5h\{1 + 2 + 3 + \ldots\ldots + (n-1)\}]$

$= \lim_{h \to 0} h\left[n + 5h \cdot \frac{n(n-1)}{2}\right]$ $\left[\because\ [1+2+3+\ldots\ldots+(n-1)] = \frac{n(n-1)}{2}\right]$

$= \lim_{h \to 0}\left[nh + \frac{5nh(nh-h)}{2}\right] = \lim_{h \to 0}\left[5 + \frac{5.5(5-h)}{2}\right]$ $[\because\ nh = 5]$

$= 5 + \frac{25(5-0)}{2} = 5 + \frac{125}{2} = \frac{135}{2}$.

(*viii*) Let $I = \int_0^4 2x\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 0$, $b = 4$, $f(x) = 2x$

Let $h = \frac{b-a}{n} = \frac{4-0}{n} \Rightarrow nh = 4, n \in N$.

$\therefore\ I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$

$= \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}]$ $[\because\ a = 0]$

$= \lim_{h \to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$

$= \lim_{h \to 0} h[\{2(0) + 2(h) + 2(2h) + \ldots\ldots + (n-1)2h]$ $[\because\ f(x) = 2x]$

$= \lim_{h \to 0} h[2h(1 + 2 + 3 + \ldots\ldots + (n-1))]$

$= \lim_{h \to 0} h\left[2h \cdot \frac{n(n-1)}{2}\right]$ $\left[\because\ [1+2+3+\ldots\ldots+(n-1)] = \frac{n(n-1)}{2}\right]$

$= \lim_{h \to 0} [nh(nh-h)]$

$= 4(4-0)] = 16.$ $[\because\ nh = 4]$

Example 2. *Evaluate the following integrals as limit of sums :*

(*i*) $\int_0^2 (x+3)\,dx$ (*ii*) $\int_3^5 (2-x)\,dx$

(*iii*) $\int_1^2 (2x+3)\,dx$ (*iv*) $\int_1^3 (2x+1)\,dx$

(*v*) $\int_0^2 (x+4)\,dx$ (*vi*) $\int_2^4 (2x-1)\,dx$.

Solution. (*i*) Let $I = \int_0^2 (x+3)\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 0$, $b = 2$, $f(x) = x + 3$.

Let $h = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in N$.

$\therefore \quad I = \lim\limits_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$

$= \lim\limits_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}]$ $\quad [\because \; a = 0]$

$= \lim\limits_{h \to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$

$= \lim\limits_{h \to 0} h[3 + (h+3) + (2h+3) + \ldots\ldots + \{(n-1)h + 3\}]$ $\quad [\because \; f(x) = (x+3)]$

$= \lim\limits_{h \to 0} h[(3 + 3 + 3 + \ldots\ldots + 3) + \{h + 2h + \ldots\ldots + (n-1)h\}]$

$= \lim\limits_{h \to 0} h[3n + h\{[1 + 2 + 3 + \ldots\ldots + (n-1)]\}]$

$= \lim\limits_{h \to 0} h\left[3n + h \cdot \dfrac{n(n-1)}{2}\right]$ $\quad \left[\because \; 1 + 2 + 3 + \ldots\ldots + (n-1) = \dfrac{n(n-1)}{2}\right]$

$= \lim\limits_{h \to 0}\left[3nh + \dfrac{nh(nh-h)}{2}\right] = \lim\limits_{h \to 0}\left[3(2) + \dfrac{2(2-h)}{2}\right]$ $\quad [\because \; nh = 2]$

$= 6 + 2 = 8.$

(*ii*) Let $I = \int_3^5 (2-x)\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 3, b = 5, f(x) = (2-x)$.

Let $h = \dfrac{b-a}{n} = \dfrac{5-3}{n} = \dfrac{2}{n} \Rightarrow nh = 2, n \in \mathbb{N}$

$\therefore \quad I = \lim\limits_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$

$= \lim\limits_{h \to 0} h[f(3) + f(3+h) + f(3+2h) + \ldots\ldots + f\{3 + (n-1)h\}]$ $\quad [\because \; a = 3]$

$= \lim\limits_{h \to 0} h[(2-3) + \{2 - (3+h)\} + \{2 - (3+2h)\} + \ldots\ldots + \{2 - (3 + (n-1)h)\}]$

$[\because \; f(x) = (2-x)]$

$= \lim\limits_{h \to 0} h[-1 + (-1-h) + (-1-2h) + \ldots\ldots + \{-1 - (n-1)h\}]$

$= \lim\limits_{h \to 0} h[(-1 -1 -1 \ldots. -1) - \{h + 2h + 3h + \ldots\ldots + (n-1)h\}]$

$= \lim\limits_{h \to 0} h[-n - h\{1 + 2 + 3 + \ldots\ldots + (n-1)h\}]$

$= \lim\limits_{h \to 0} h\left[-n - h \cdot \dfrac{n(n-1)}{2}\right]$ $\quad \left[\because \; 1 + 2 + 3 + \ldots\ldots + (n-1) = \dfrac{n(n-1)}{2}\right]$

$= \lim\limits_{h \to 0}\left[-nh - \dfrac{nh(nh-h)}{2}\right] = \lim\limits_{h \to 0}\left[-2 - \dfrac{2(2-h)}{2}\right]$ $\quad [\because \; nh = 2]$

$= -2 - 2 = -4.$

(*iii*) Let $I = \int_1^2 (2x+3)\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 1, b = 2, f(x) = (2x+3)$.

Let $h = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n} \Rightarrow nh = 1, n \in \mathbf{N}$

$\therefore \quad \mathrm{I} = \lim\limits_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)]$

$= \lim\limits_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f(1+(n-1)h)]$ $\quad [\because\ a = 1]$

$= \lim\limits_{h \to 0} h[\{2(1)+3\} + \{2(1+h)+3\} + \{2(1+2h)+3\} + \ldots\ldots + f\{2(1+(n-1)h)+3\}]$

$[\because\ f(x) = (2x+3)]$

$= \lim\limits_{h \to 0} h[5 + (5+2h) + (5+4h) + \ldots\ldots + \{5 + 2(n-1)h\}]$

$= \lim\limits_{h \to 0} h[(5+5+5+\ldots..+5) + \{2h + 4h + \ldots\ldots + 2(n-1)h\}]$

$= \lim\limits_{h \to 0} h[5n + 2h[1+2+3+\ldots\ldots+(n-1)]]$

$= \lim\limits_{h \to 0} h\left[5n + 2h \cdot \frac{n(n-1)}{2}\right]$ $\quad \left[\because\ 1+2+3+\ldots\ldots+(n-1) = \frac{n(n-1)}{2}\right]$

$= \lim\limits_{h \to 0} [5nh + nh(nh-h)] = \lim\limits_{h \to 0} [5 + 1(1-h)]$ $\quad [\because\ nh = 1]$

$= 5 + 1 = 6.$

(*iv*) Let $\mathrm{I} = \int_1^3 (2x+1)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 1$, $b = 3$, $f(x) = (2x+1)$.

Let $h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in \mathbf{N}$

$\therefore \quad \mathrm{I} = \lim\limits_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)]$

$= \lim\limits_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f(1+(n-1)h)]$ $\quad [\because\ a = 1]$

$= \lim\limits_{h \to 0} h[\{2(1)+1\} + \{2(1+h)+1\} + \{2(1+2h)+1\} + \ldots\ldots + \{2(1+(n-1)h)+1\}]$

$[\because\ f(x) = (2x+1)]$

$= \lim\limits_{h \to 0} h[3 + (3+2h) + (3+4h) + \ldots\ldots + \{3 + 2(n-1)h\}]$

$= \lim\limits_{h \to 0} h[(3+3+3+\ldots..+3) + \{2h + 4h + \ldots\ldots + 2(n-1)h\}]$

$= \lim\limits_{h \to 0} h[3n + 2h[\{1+2+3+\ldots\ldots+(n-1)\}]]$

$\left[\because\ [1+2+3+\ldots\ldots+(n-1)] = \frac{n(n-1)}{2}\right]$

$= \lim\limits_{h \to 0} h\left[3n + 2h \cdot \frac{n(n-1)}{2}\right]$

$= \lim\limits_{h \to 0} [3nh + nh(nh-h)] = \lim\limits_{h \to 0} [3(2) + 2(2-h)]$ $\quad [\because\ nh = 2]$

$= 6 + 4 = 10.$

(*v*) Let $I = \int_0^2 (x+4)\, dx$.

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 0$, $b = 2$, $f(x) = (x+4)$.

Let $h = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in N$.

$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$

$= \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}]$ $\quad [\because a = 0]$

$= \lim_{h \to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$

$= \lim_{h \to 0} h[(0+4) + (h+4) + (2h+4) + \ldots\ldots + \{(n-1)h + 4\}]$ $\quad [\because f(x) = (x+4)]$

$= \lim_{h \to 0} h[(4+4+4+\ldots..+4) + \{h + 2h + 3h + \ldots\ldots + (n-1)h\}]$

$= \lim_{h \to 0} h[4n + h\{1+2+3+\ldots\ldots+(n-1)\}]$

$= \lim_{h \to 0} h\left[4n + h \cdot \frac{n(n-1)}{2}\right]$ $\quad \left[\because \; [1+2+3+\ldots\ldots+(n-1)] = \frac{n(n-1)}{2}\right]$

$= \lim_{h \to 0}\left[4nh + \frac{nh(nh-h)}{2}\right] = \lim_{h \to 0}\left[4(2) + \frac{2(2-h)}{2}\right]$ $\quad [\because nh = 2]$

$= 8 + 2 = 10.$

(*vi*) Let $I = \int_2^4 (2x-1)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 2$, $b = 4$, $f(x) = (2x-1)$.

Let $h = \frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in N$.

$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$

$= \lim_{h \to 0} h[f(2) + f(2+h) + f(2+2h) + \ldots\ldots + f\{2 + (n-1)h\}]$ $\quad [\because a = 2]$

$= \lim_{h \to 0} h[\{2(2)-1\} + \{2(2+h)-1\} + \{2(2+2h)-1\} + \ldots\ldots + \{2(2+(n-1)h) - 1\}]$

$[\because f(x) = (2x-1)]$

$= \lim_{h \to 0} h[3 + (3+2h) + (3+4h) + \ldots\ldots + \{3 + 2(n-1)h\}]$

$= \lim_{h \to 0} h[(3+3+3+\ldots..+3) + \{2h + 4h + \ldots\ldots + 2(n-1)h\}]$

$= \lim_{h \to 0} h[3n + 2h[1+2+3+\ldots\ldots+(n-1)h]]$

$= \lim_{h \to 0} h\left[3n + 2h \cdot \frac{n(n-1)}{2}\right]$ $\quad \left[\because \; 1+2+3+\ldots\ldots+(n-1) = \frac{n(n-1)}{2}\right]$

$= \lim_{h \to 0} [3nh + nh(nh-h)] = \lim_{h \to 0} [3(2) + 2(2-h)]$ $\quad [\because nh = 2]$

$= 6 + 4 = 10.$

Example 3. *Evaluate the following integrals as the limit of sums :*

(*i*) $\int_a^b x^2\,dx$ (*ii*) $\int_1^2 x^2\,dx$

(*iii*) $\int_0^2 (x^2+3)\,dx$ (*iv*) $\int_1^3 (2x^2+5x)\,dx$

(*v*) $\int_1^3 (x^2+5x)\,dx$.

Solution. (*i*) Let $I = \int_a^b x^2\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = a,\ b = b,\ f(x) = x^2$.

Let $h = \dfrac{b-a}{n} \Rightarrow nh = b - a,\ n \in N$

$$\therefore\ I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h\to 0} h[a^2 + (a+h)^2 + (a+2h)^2 + \ldots\ldots + \{a + (n-1)h\}^2] \qquad [\because\ f(x) = x^2]$$

$$= \lim_{h\to 0} h[a^2 + (a^2 + h^2 + 2ah) + (a^2 + (2h)^2 + 4ah) + \ldots\ldots + \{a^2 + (n-1)^2h^2 + 2a(n-1)h\}]$$

$$= \lim_{h\to 0} h[(a^2 + a^2 + a^2 + \ldots\ldots + a^2) + \{h^2 + (2h)^2 + \ldots\ldots + (n-1)^2 h^2\} + \{2ah + 4ah + \ldots\ldots + 2a(n-1)h\}]$$

$$= \lim_{h\to 0} h[na^2 + h^2\{1^2 + 2^2 + \ldots\ldots + (n-1)^2\} + 2ah\{1 + 2 + \ldots\ldots + (n-1)\}]$$

$$= \lim_{h\to 0} h\left[na^2 + h^2 \cdot \frac{n(n-1)(2n-1)}{6} + 2ah \cdot \frac{n(n-1)}{2}\right]$$

$$\left[\begin{array}{l}\because\ [1+2+3+\ldots\ldots+(n-1)] = \dfrac{n(n-1)}{2} \\ \quad [1^2+2^2+3^2+\ldots\ldots+(n-1)^2] = \dfrac{n(n-1)(2n-1)}{6}\end{array}\right]$$

$$= \lim_{h\to 0}\left[(nh)a^2 + \frac{nh(nh-h)(2nh-h)}{6} + anh(nh-h)\right]$$

$$= \lim_{h\to 0}\left[(b-a)a^2 + \frac{(b-a)(b-a-h)\{2(b-a)-h\}}{6} + a(b-a)(b-a-h)\right] \qquad [\because\ nh = (b-a)]$$

$$= \left[(b-a)a^2 + \frac{(b-a)(b-a)\,2(b-a)}{6} + a(b-a)(b-a)\right]$$

$$= \left[(b-a)a^2 + \frac{1}{3}(b-a)^3 + a(b-a)^2\right] = \frac{(b-a)}{3}\,[3a^2 + (b-a)^2 + 3a(b-a)]$$

$$= \frac{b-a}{3}\,[3a^2 + b^2 + a^2 - 2ab + 3ab - 3a^2] = \frac{(b-a)}{3}\,(a^2 + ab + b^2)$$

$$= \frac{1}{3}(b^3 - a^3). \qquad [\because\ (a^3 - b^3) = (a-b)(a^2 + ab + b^2)]$$

(*ii*) Let $I = \int_1^2 x^2\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 1$, $b = 2$, $f(x) = x^2$

Let $h = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n} \Rightarrow nh = 1, n \in \mathrm{N}.$

$$\therefore\ I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)]$$

$$= \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f(1+(n-1)h)] \qquad [\because\ a = 1]$$

$$= \lim_{h\to 0} h[1^2 + (1+h)^2 + (1+2h)^2 + \ldots\ldots + \{1+(n-1)h\}^2] \qquad [\because\ f(x) = x^2]$$

$$= \lim_{h\to 0} h[1 + (1+h^2+2h) + (1+(2h)^2+4h) + \ldots\ldots + \{1+(n-1)^2h^2+2(n-1)h\}]$$

$$= \lim_{h\to 0} h[(1+1+1+\ldots\ldots+1) + \{h^2+(2h)^2+\ldots\ldots+(n-1)^2h^2\} + \{2h+4h+\ldots\ldots+2(n-1)h\}]$$

$$= \lim_{h\to 0} h[n + h^2(1^2+2^2+3^2+\ldots\ldots(n-1)^2) + \{2h+4h+\ldots\ldots+2(n-1)h\}]$$

$$= \lim_{h\to 0} h[n + h^2(1^2+2^2+3^2+\ldots\ldots(n-1)^2) + 2h\{1+2+3+\ldots\ldots+(n-1)\}]$$

$$= \lim_{h\to 0} h\left[n + h^2 \cdot \frac{n(n-1)(2n-1)}{6} + 2h \cdot \frac{n(n-1)}{2}\right]$$

$$\left[\begin{array}{l} \because\ [1+2+3+\ldots\ldots+(n-1)] = \dfrac{n(n-1)}{2} \\ [1^2+2^2+3^2+\ldots\ldots+(n-1)^2] = \dfrac{n(n-1)(2n-1)}{6} \end{array}\right]$$

$$= \lim_{h\to 0}\left[nh + \frac{nh(nh-h)(2nh-h)}{6} + nh(nh-h)\right] \qquad [\because\ nh = 1]$$

$$= \lim_{h\to 0}\left[1 + \frac{1(1-h)(2-h)}{6} + 1(1-h)\right]$$

$$= 1 + \frac{1}{3} + 1 = \frac{3+1+3}{3} = \frac{7}{3}.$$

(*iii*) Let $I = \int_0^2 (x^2+3)\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 0$, $b = 2$, $f(x) = (x^2 + 3)$.

Let $h = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in \mathrm{N}.$

$$\therefore\ I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)]$$

$$= \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f(0+(n-1)h)] \qquad [\because\ a = 0]$$

$$= \lim_{h\to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f((n-1)h)]$$

$$= \lim_{h\to 0} h[(0+3) + (h^2+3) + \{(2h)^2+3\} + \ldots\ldots + \{((n-1)h)^2+3\}] \qquad [\because\ f(x) = (x^2+3)]$$

$$= \lim_{h\to 0} h[(3+3+3+\ldots\ldots+3) + \{h^2+(2h)^2+\ldots\ldots+(n-1)^2 h^2\}]$$

$$= \lim_{h\to 0} h[3n + h^2(1^2+2^2+3^2+\ldots\ldots+(n-1)^2]$$

$$= \lim_{h\to 0} h\left[3n + h^2 \cdot \frac{n(n-1)(2n-1)}{6}\right]$$

$$\left[\because\ [1^2+2^2+3^2+\ldots\ldots+(n-1)^2] = \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[3nh + \frac{nh(nh-h)(2nh-h)}{6}\right] = \lim_{h\to 0}\left[3(2) + \frac{2(2-h)(2.2-h)}{6}\right] \qquad [\because\ nh=2]$$

$$= 6 + \frac{2(2-0)(4-0)}{6} = 6 + \frac{16}{6} = 6 + \frac{8}{3} = \frac{26}{3}.$$

(*iv*) Let $I = \int_1^3 (2x^2+5x)\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 1$, $b = 3$, $f(x) = (2x^2+5x)$.

Let $\quad h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in N.$

$$\therefore\ I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a+(n-1)h\}]$$

$$= \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f\{1+(n-1)h\}] \qquad [\because\ a=1]$$

$$= \lim_{h\to 0} h[\{2(1)^2+5(1)\} + \{2(1+h)^2+5(1+h)\} + \{2(1+2h)^2+5(1+2h)\}$$
$$+ \ldots\ldots + \{2(1+(n-1)h)^2 + 5(1+(n-1)h)\}]$$
$$[\because\ f(x) = (2x^2+5x)]$$

$$= \lim_{h\to 0} h[7 + \{2(1+h^2+2h)+5+5h\} + \{2(1+(2h)^2+2(2h))+5+10h\} + \ldots\ldots$$
$$+ \{2(1+(n-1)^2h^2 + 2(n-1)h + 5 + 5(n-1)h\}]$$

$$= \lim_{h\to 0} h[7 + (7+2h^2+9h) + \{7+2(2h)^2+18h\} + \ldots\ldots + \{7+2(n-1)^2h^2+9(n-1)h\}]$$

$$= \lim_{h\to 0} h[\{7+7+7+\ldots\ldots+7) + 2h^2\{1^2+2^2+\ldots\ldots+(n-1)^2\}$$
$$+ 9h\{1+2+\ldots\ldots+(n-1)\}]$$

$$\left[\begin{array}{l}\because\ [1+2+3+\ldots\ldots+(n-1)] = \dfrac{n(n-1)}{2} \\ [1^2+2^2+3^2+\ldots\ldots+(n-1)^2] = \dfrac{n(n-1)(2n-1)}{6}\end{array}\right]$$

$$= \lim_{h\to 0} h\left[7n + 2h^2 \cdot \frac{n(n-1)(2n-1)}{6} + 9h \cdot \frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0}\left[7nh + \frac{nh(nh-h)(2nh-h)}{3} + \frac{9}{2}nh(nh-h)\right]$$

$$= \lim_{h\to 0}\left[7(2) + \frac{2(2-h)(2.2-h)}{3} + \frac{9}{2}.2(2-h)\right] \qquad [\because\ nh = 2]$$

$$= 14 + \frac{16}{3} + 18 = \frac{42+16+54}{3} = \frac{112}{3}.$$

(v) Let $I = \int_1^3 (x^2 + 5x)\, dx$.

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 1$, $b = 3$, $f(x) = (x^2 + 5x)$

Let $\qquad h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in N.$

$$\therefore\ I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a+(n-1)h\}]$$

$$= \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f\{1+(n-1)h\}] \qquad [\because\ a = 1]$$

$$= \lim_{h\to 0} h[\{1+5(1)\} + \{(1+h)^2 + 5(1+h)\} + \{(1+2h)^2 + 5(1+2h)\} + \ldots\ldots + \{(1+(n-1)h)^2 + 5(1+(n-1)h)\}]$$

$$[\because\ f(x) = (x^2 + 5x)]$$

$$= \lim_{h\to 0} h[6 + (1 + h^2 + 2h + 5 + 5h) + (1 + (2h)^2 + 2(2h) + 5 + 10h) + \ldots\ldots + \{1 + (n-1)^2h^2 + 2(n-1)h + 5 + 5(n-1)h\}]$$

$$= \lim_{h\to 0} h[6 + (6 + h^2 + 7h) + (6 + (2h)^2 + 14h) + \ldots\ldots + \{6 + (n-1)^2 h^2 + 7(n-1)h\}]$$

$$= \lim_{h\to 0} h[\{6 + 6 + 6 + \ldots\ldots + 6) + (h^2 + (2h)^2 + \ldots\ldots + (n-1)^2 h^2) + \{7h + 14h + \ldots\ldots + 7(n-1)h\}]$$

$$= \lim_{h\to 0} h[6n + h^2\{1^2 + 2^2 + \ldots\ldots + (n-1)^2\} + 7h\{1 + 2 + \ldots\ldots + (n-1)\}]$$

$$= \lim_{h\to 0} h\left[6n + h^2.\frac{n(n-1)(2n-1)}{6} + 7h.\frac{n(n-1)}{2}\right]$$

$$\left[\begin{array}{l}\because\ [1+2+3+\ldots\ldots+(n-1)] = \dfrac{n(n-1)}{2} \\ [1^2+2^2+3^2+\ldots\ldots+(n-1)^2] = \dfrac{n(n-1)(2n-1)}{6}\end{array}\right]$$

$$= \lim_{h\to 0}\left[6nh + \frac{nh(nh-h)(2nh-h)}{6} + \frac{7nh(nh-h)}{2}\right]$$

$$= \lim_{h\to 0}\left[6(2) + \frac{2(2-h)(2.2-h)}{6} + \frac{7.2(2-h)}{2}\right] \qquad [\because\ nh = 2]$$

$$= 12 + \frac{8}{3} + 14 = 26 + \frac{8}{3} = \frac{78+8}{3} = \frac{86}{3}.$$

Example 4. *Evaluate the following integrals as the limit of sums :*

(i) $\int_0^2 (x^2 + 2)\, dx$ $\qquad$ (ii) $\int_1^2 (x^2 - 1)\, dx$

(iii) $\int_1^3 (2x^2 + 5)\, dx$ $\qquad$ (iv) $\int_1^4 (x^2 + 3x)\, dx$

(v) $\int_0^3 (2x^2 + 3)\, dx$ $\qquad$ (vi) $\int_1^3 (x^2 + x)\, dx$.

Solution. (*i*) Let $I = \int_0^2 (x^2 + 2)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 0$, $b = 2$, $f(x) = (x^2 + 2)$.

Let $\quad h = \dfrac{b-a}{n} = \dfrac{2-0}{n} = \dfrac{2}{n} \Rightarrow nh = 2 \,;\, n \in \mathbb{N}.$

$$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}] \qquad [\because \ a = 0]$$

$$= \lim_{h \to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h \to 0} h[(0+2) + (h^2+2) + \{(2h)^2 + 2\} + \ldots\ldots + \{(n-1)^2 h^2 + 2\}]$$

$$[\because \ f(x) = (x^2 + 2)]$$

$$= \lim_{h \to 0} h[(2 + 2 + 2 + \ldots\ldots + 2) + \{h^2 + (2h)^2 + \ldots\ldots + (n-1)^2 h^2\}]$$

$$= \lim_{h \to 0} h[2n + h^2(1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2)]$$

$$= \lim_{h \to 0} h\left[2n + h^2 \cdot \frac{n(n-1)(2n-1)}{6}\right]$$

$$\left[\because \ [1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2] = \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h \to 0}\left[2nh + \frac{nh(nh-h)(2nh-h)}{6}\right]$$

$$= \lim_{h \to 0}\left[2(2) + \frac{2(2-h)(2.2-h)}{6}\right] \qquad [\because \ nh = 2]$$

$$= 4 + \frac{(2-0)(4-0)}{3} = 4 + \frac{8}{3} = \frac{12+8}{3} = \frac{20}{3}.$$

(*ii*) Let $I = \int_1^2 (x^2 - 1)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 1$, $b = 2$, $f(x) = (x^2 - 1)$.

Let $\quad h = \dfrac{b-a}{n} = \dfrac{2-1}{n} = \dfrac{1}{n} \Rightarrow nh = 1, n \in \mathbb{N}.$

$$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f\{1 + (n-1)h\}] \qquad [\because \ a = 1]$$

$$= \lim_{h \to 0} h[(1^2 - 1) + \{(1+h)^2 - 1\} + \{(1+2h)^2 - 1\} + \ldots\ldots + \{(1 + (n-1)h)^2 - 1\}]$$

$$[\because \ f(x) = (x^2 - 1)]$$

$$= \lim_{h \to 0} h[0 + (1 + h^2 + 2h - 1) + \{1 + (2h)^2 + 2(2h) - 1\} + \ldots\ldots + \{1 + (n-1)^2 h^2 + 2(n-1)h - 1\}]$$

$$= \lim_{h \to 0} h[(h^2 + 2h) + \{(2h)^2 + 2(2h)\} + \ldots\ldots + \{(n-1)^2 h^2 + 2(n-1)h\}]$$

$$= \lim_{h \to 0} h[\{h^2 + 4h^2 + \ldots\ldots + (n-1)^2 h^2\} + \{2h + 2(2h) + \ldots\ldots + 2(n-1)h\}]$$

$$= \lim_{h \to 0} h\{h^2 (1^2 + 2^2 + \ldots\ldots + (n-1)^2) + 2h\{1 + 2 + \ldots\ldots + (n-1)\}\}$$

$$= \lim_{h \to 0} h\left[h^2 \cdot \frac{n(n-1)(2n-1)}{6} + \frac{2h \cdot n(n-1)}{2}\right]$$

$$\left[\begin{aligned} \because\ [1+2+3+\ldots\ldots+(n-1)] &= \frac{n(n-1)}{2} \\ [1^2+2^2+3^2+\ldots\ldots+(n-1)^2] &= \frac{n(n-1)(2n-1)}{6} \end{aligned}\right]$$

$$= \lim_{h \to 0}\left[\frac{nh(nh-h)(2nh-h)}{6} + nh(nh-h)\right]$$

$$= \lim_{h \to 0}\left[\frac{1(1-h)(2-h)}{6} + 1(1-h)\right] \qquad [\because\ nh = 1]$$

$$= \frac{1}{3} + 1 = \frac{4}{3}.$$

(*iii*) Let $I = \int_1^3 (2x^2 + 5)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 1$, $b = 3$, $f(x) = (2x^2 + 5)$

Let $h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in N.$

$$\therefore\ I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)]$$

$$= \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f(1+(n-1)h)] \qquad [\because\ a = 1]$$

$$= \lim_{h \to 0} h[\{2(1)^2+5\} + \{2(1+h)^2+5\} + \{2(1+2h)^2+5\} + \ldots\ldots + \{2(1+(n-1)h)^2+5\}]$$

$$[\because\ f(x) = (2x^2+5)]$$

$$= \lim_{h \to 0} h[7 + \{2(1+h^2+2h)+5\} + \{2(1+(2h)^2+2(2h))+5\} + \ldots\ldots + \{2(1+(n-1)^2h^2+2(n-1)h)+5\}]$$

$$= \lim_{h \to 0} h[7 + (7+2h^2+4h) + (7+2(2h)^2+8h) + \ldots\ldots + \{7+2(n-1)^2h^2+4(n-1)h\}]$$

$$= \lim_{h \to 0} h[(7+7+7+\ldots\ldots+7) + \{2h^2+2(2h)^2+\ldots\ldots+2(n-1)^2h^2\} + \{4h+8h+\ldots\ldots+4(n-1)h\}]$$

$$= \lim_{h \to 0} h[7n + 2h^2\{1^2+2^2+\ldots\ldots+(n-1)^2\} + 4h\{1+2+\ldots\ldots+(n-1)\}]$$

$$= \lim_{h \to 0} h\left[7n + 2h^2 \cdot \frac{n(n-1)(2n-1)}{6} + 4h \cdot \frac{n(n-1)}{2}\right]$$

$$\left[\begin{aligned} \because\ [1+2+3+\ldots\ldots+(n-1)] &= \frac{n(n-1)}{2} \\ [1^2+2^2+3^2+\ldots\ldots+(n-1)^2] &= \frac{n(n-1)(2n-1)}{6} \end{aligned}\right]$$

$$= \lim_{h \to 0}\left[7nh + \frac{nh(nh-h)(2nh-h)}{3} + 2nh(nh-h)\right]$$

$$= \lim_{h \to 0} \left[7(2) + \frac{2(2-h)(2.2-h)}{3} + 2(2)(2-h) \right] \qquad [\because \ nh = 2]$$

$$= 14 + \frac{16}{3} + 8 = 22 + \frac{16}{3} = \frac{66+16}{3} = \frac{82}{3}.$$

(*iv*) Let $I = \int_1^4 (x^2 + 3x)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 1$, $b = 4$, $f(x) = (x^2 + 3x)$.

Let $h = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n} \Rightarrow nh = 3, n \in N.$

$$\therefore \ I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f\{1 + (n-1)h\}] \qquad [\because \ a = 1]$$

$$= \lim_{h \to 0} h[\{1^2 + 3(1)\} + \{(1+h)^2 + 3(1+h)\} + \{(1+2h)^2 + 3(1+2h)\} + \ldots\ldots + \{(1+(n-1)h)^2 + 3(1+(n-1)h)\}]$$

$$[\because \ f(x) = (x^2 + 3x)]$$

$$= \lim_{h \to 0} h[4 + (1 + h^2 + 2h + 3 + 3h) + \{1 + (2h)^2 + 2(2h) + 3 + 6h\} + \ldots\ldots + \{1 + (n-1)^2h^2 + 2(n-1)h + 3 + 3(n-1)h\}]$$

$$= \lim_{h \to 0} h[4 + (4 + h^2 + 5h) + (4 + (2h)^2 + 10h) + \ldots\ldots + \{4 + (n-1)^2h^2 + 5(n-1)h\}]$$

$$= \lim_{h \to 0} h[(4 + 4 + 4 + \ldots\ldots + 4) + \{h^2 + (2h)^2 + \ldots\ldots + (n-1)^2 h^2\} + \ldots\ldots + \{5h + 10h + \ldots\ldots + 5(n-1)h\}]$$

$$= \lim_{h \to 0} h[4n + h^2 \{1^2 + 2^2 + \ldots\ldots + (n-1)^2\} + 5h\{1 + 2 + \ldots\ldots + (n-1)\}]$$

$$= \lim_{h \to 0} h\left[4n + h^2 . \frac{n(n-1)(2n-1)}{6} + 5h . \frac{n(n-1)}{2}\right]$$

$$\left[\begin{array}{l} \because \ [1 + 2 + 3 + \ldots\ldots + (n-1)] = \dfrac{n(n-1)}{2} \\ \quad [1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2] = \dfrac{n(n-1)(2n-1)}{6} \end{array}\right]$$

$$= \lim_{h \to 0} \left[4nh + \frac{nh(nh-h)(2nh-h)}{6} + \frac{5nh(nh-h)}{2}\right]$$

$$= \lim_{h \to 0} \left[4(3) + \frac{3(3-h)(2.3-h)}{6} + \frac{5(3)(3-h)}{2}\right] \qquad [\because \ nh = 3]$$

$$= 12 + 9 + \frac{45}{2} = 21 + \frac{45}{2} = \frac{42+45}{2} = \frac{87}{2}.$$

(*v*) Let $\quad I = \int_0^3 (2x^2 + 3)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 0$, $b = 3$, $f(x) = (2x^2 + 3)$.

Let $\qquad h = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n} \Rightarrow nh = 3, n \in N.$

$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a + h) + f(a + 2h) + ... + f(a + (n - 1)h)]$

$= \lim_{h \to 0} h[f(0) + f(0 + h) + f(0 + 2h) + ... + f(0 + (n - 1)h)] \quad [\because a = 0]$

$= \lim_{h \to 0} h[f(0) + f(h) + f(2h) + ... + f((n - 1)h)]$

$= \lim_{h \to 0} h[\{2(0) + 3\} + (2h^2 + 3) + \{2(2h)^2 + 3\} + ... + \{2((n - 1)h)^2 + 3\}]$

$[\because f(x) = (2x^2 + 3)]$

$= \lim_{h \to 0} h[3 + (2h^2 + 3) + \{2(4h^2) + 3\} + ... + \{2(n - 1)^2 h^2 + 3\}]$

$= \lim_{h \to 0} h[(3 + 3 + 3 + ... + 3) + 2h^2\{1^2 + 2^2 + ... + (n - 1)^2\}]$

$= \lim_{h \to 0} h\left[3n + 2h^2 \cdot \frac{n(n-1)(2n-1)}{6}\right]$

$\left[\because [1^2 + 2^2 + 3^2 + ... + (n - 1)^2] = \frac{n(n-1)(2n-1)}{6}\right]$

$= \lim_{h \to 0}\left[3nh + \frac{nh(nh-h)(2nh-h)}{3}\right]$

$= \lim_{h \to 0}\left[3(3) + \frac{3(3-h)(2.3-h)}{3}\right] \quad [\because nh = 3]$

$= 9 + 18 = 27.$

(*vi*) Let $\quad I = \int_1^3 (x^2 + x)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 1, b = 3, f(x) = (x^2 + x)$.

Let $\quad h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n} \quad \Rightarrow \quad nh = 2, n \in N.$

$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a + h) + f(a + 2h) + ... + f(a + (n - 1)h)]$

$= \lim_{h \to 0} h[f(1) + f(1 + h) + f(1 + 2h) + ... + f(1 + (n - 1)h)] \quad [\because a = 1]$

$= \lim_{h \to 0} h[(1^2 + 1) + \{(1 + h)^2 + (1 + h)\} + \{(1 + 2h)^2 + (1 + 2h)\} +$

$...... + \{(1 + (n - 1)h)^2 + 1 + (n - 1)h\}]$

$[\because f(x) = (x^2 + x)]$

$= \lim_{h \to 0} h[2 + (1 + h^2 + 2h + 1 + h) + \{1 + 4h^2 + 2(2h) + 1 + 2h\} +$

$...... + \{1 + (n - 1)^2h^2 + 2(n - 1)h + 1 + (n - 1)h\}]$

$= \lim_{h \to 0} h[2 + (2 + h^2 + 3h) + (2 + 4h^2 + 6h) + ... + \{2 + (n - 1)^2h^2 + 3(n - 1)h\}]$

$= \lim_{h \to 0} h[(2 + 2 + 2 + ... + 2) + \{h^2 + (2h)^2 + ... + (n - 1)^2\}$

$+ \{3h + 6h + ... + 3(n - 1)h\}]$

$= \lim_{h \to 0} h[2n + h^2\{1^2 + 2^2 + 3^2 + ... + (n - 1)^2\} + 3h\{1 + 2 + 3 + ... + (n - 1)\}]$

$$= \lim_{h \to 0} h\left[2n + h^2 \cdot \frac{n(n-1)(2n-1)}{6} + 3h \cdot \frac{n(n-1)}{2}\right]$$

$$\left[\because\ [1+2+3+\ldots\ldots+(n-1)] = \frac{n(n-1)}{2} \quad [1^2+2^2+3^2+\ldots\ldots+(n-1)^2] = \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h \to 0}\left[2nh + \frac{nh(nh-h)(2nh-h)}{6} + \frac{3nh(nh-h)}{2}\right]$$

$$= \lim_{h \to 0}\left[2(2) + \frac{2(2-h)(2.2-h)}{6} + \frac{3(2)(2-h)}{2}\right] \qquad [\because\ nh = 2]$$

$$= 4 + \frac{2(2-0)(4-0)}{6} + 3(2-0)$$

$$= 4 + \frac{8}{3} + 6 = 10 + \frac{8}{3} = \frac{30+8}{3} = \frac{38}{3}.$$

Example 5. *Evaluate the following integrals as the limit of sums :*

(i) $\int_1^4 (x^2 - x)\, dx$ (ii) $\int_1^3 (2x^2 + x + 9)\, dx$

(iii) $\int_0^2 (x^2 + 1)\, dx$ (iv) $\int_1^2 x^3\, dx$

(v) $\int_0^1 (3x^2 + 2x + 1)\, dx$.

Solution. (i) Let $I = \int_1^4 (x^2 - x)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 1$, $b = 4$, $f(x) = (x^2 - x)$.

Let $h = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n} \Rightarrow nh = 3, n \in N$.

$$\therefore\ I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f\{1 + (n-1)h\}] \qquad [\because\ a = 1]$$

$$= \lim_{h \to 0} h[\{(1)^2 - 1\} + \{(1+h)^2 - (1+h)\} + \{(1+2h)^2 - (1+2h)\} + \ldots\ldots + \{(1+(n-1)h)^2 - (1+(n-1)h)\}]$$

$$[\because\ f(x) = (x^2 - x)]$$

$$= \lim_{h \to 0} h[0 + (1 + h^2 + 2h - 1 - h) + (1 + (2h)^2 + 2(2h) - 1 - 2h) + \ldots\ldots + \{1 + (n-1)^2h^2 + 2(n-1)h - 1 - (n-1)h\}]$$

$$= \lim_{h \to 0} h[(h^2 + h) + \{(2h)^2 + (2h)\} + \ldots\ldots + \{(n-1)^2 h^2 + (n-1)h\}]$$

$$= \lim_{h \to 0} h[\{h^2 + (2h)^2 + \ldots\ldots + (n-1)^2 h^2\} + \{h + 2h + \ldots\ldots + (n-1)h\}]$$

$$= \lim_{h \to 0} h[h^2 \{1^2 + 2^2 + \ldots\ldots + (n-1)^2\} + h\{1 + 2 + \ldots\ldots + (n-1)\}]$$

$$= \lim_{h \to 0} h \left[h^2 \cdot \frac{n(n-1)(2n-1)}{6} + h \cdot \frac{n(n-1)}{2} \right]$$

$$\left[\begin{array}{l} \because \ [1+2+3+......+(n-1)] = \dfrac{n(n-1)}{2} \\ [1^2+2^2+3^2+......+(n-1)^2] = \dfrac{n(n-1)(2n-1)}{6} \end{array} \right]$$

$$= \lim_{h \to 0} \left[\frac{nh(nh-h)(2nh-h)}{6} + \frac{nh(nh-h)}{2} \right]$$

$$= \lim_{h \to 0} \left[\frac{3(3-h)(2.3-h)}{6} + \frac{3(3-h)}{2} \right] \qquad [\because \ nh = 3]$$

$$= 9 + \frac{9}{2} = \frac{18+9}{2} = \frac{27}{2}.$$

(*ii*) Let $I = \int_1^3 (2x^2 + x + 9)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 1$, $b = 3$, $f(x) = (2x^2 + x + 9)$.

Let $h = \dfrac{b-a}{n} = \dfrac{3-1}{n} = \dfrac{2}{n} \Rightarrow nh = 2, n \in N$.

$$\therefore \ I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + ... + f\{1 + (n-1)h\}] \qquad [\because \ a = 1]$$

$$= \lim_{h \to 0} h[\{2(1)^2 + 1 + 9\} + \{2(1+h)^2 + (1+h) + 9\} + \{2(1+2h)^2 + (1+2h) + 9\} + ... + \{2(1+(n-1)h)^2 + (1+(n-1)h) + 9\}]$$

$$[\because \ f(x) = (2x^2 + x + 9)]$$

$$= \lim_{h \to 0} h[12 + \{2(1 + h^2 + 2h) + 1 + h + 9\} + \{2(1 + 4h^2 + 2(2h)) + 1 + 2h + 9\} + ... + \{2(1 + (n-1)^2h^2 + 2(n-1)h) + 1 + (n-1)h + 9\}]$$

$$= \lim_{h \to 0} h[12 + (12 + 2h^2 + 5h) + (12 + 2(2h)^2 + 10h) + ... + \{12 + 2(n-1)^2 h^2 + 5(n-1)h\}]$$

$$= \lim_{h \to 0} h[(12 + 12 + 12 + ... + 12) + \{2h^2 + 2(2h)^2 + ... + 2(n-1)^2 h^2\} + \{5h + 10h + ... + 5(n-1)h\}]$$

$$= \lim_{h \to 0} h[12n + 2h^2 \{1^2 + 2^2 + ... + (n-1)^2\} + 5h\{1 + 2 + ... + (n-1)\}]$$

$$= \lim_{h \to 0} h \left[12n + 2h^2 \cdot \frac{n(n-1)(2n-1)}{6} + 5h \cdot \frac{n(n-1)}{2} \right]$$

$$\left[\begin{array}{l} \because \ [1+2+3+......+(n-1)] = \dfrac{n(n-1)}{2} \\ [1^2+2^2+3^2+......+(n-1)^2] = \dfrac{n(n-1)(2n-1)}{6} \end{array} \right]$$

$$= \lim_{h \to 0} \left[12nh + \frac{nh(nh-h)(2nh-h)}{3} + \frac{5nh(nh-h)}{2} \right]$$

$$= \lim_{h \to 0} \left[12(2) + \frac{2(2-h)(2.2-h)}{3} + \frac{5(2)(2-h)}{2} \right] \qquad [\because \ nh = 2]$$

$$= 24 + \frac{16}{3} + 10 = 34 + \frac{16}{3} = \frac{102+16}{3} = \frac{118}{3}.$$

(iii) Let $I = \int_0^2 (x^2 + 1)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 0$, $b = 2$, $f(x) = (x^2 + 1)$.

Let $\qquad h = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in \mathrm{N}.$

$$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}] \qquad [\because \ a = 0]$$

$$= \lim_{h \to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h \to 0} h[(0+1) + (h^2+1) + \{(2h)^2+1\} + \ldots\ldots + \{(n-1)^2 h^2 + 1\}]$$

$$[\because \ f(x) = (x^2+1)]$$

$$= \lim_{h \to 0} h[(1+1+1+\ldots\ldots+1) + \{h^2 + (2h)^2 + \ldots\ldots + (n-1)^2 h^2\}]$$

$$= \lim_{h \to 0} h[n + h^2\{1^2 + 2^2 + \ldots\ldots + (n-1)^2\}]$$

$$= \lim_{h \to 0} h\left[n + h^2 \cdot \frac{n(n-1)(2n-1)}{6}\right]$$

$$\left[\because \ [1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2] = \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h \to 0} \left[nh + \frac{nh(nh-h)(2nh-h)}{6}\right]$$

$$= \lim_{h \to 0} \left[2 + \frac{2(2-h)(2.2-h)}{6}\right] \qquad [\because \ nh = 2]$$

$$= 2 + \frac{(2-0)(4-0)}{3} = 2 + \frac{8}{3} = \frac{6+8}{3} = \frac{14}{3}.$$

(iv) Let $I = \int_1^2 x^3\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 1$, $b = 2$, $f(x) = x^3$.

Let $\quad h = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n} \Rightarrow nh = 1, n \in \mathrm{N}.$

$$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f\{1 + (n-1)h\}] \qquad [\because \ a = 1]$$

$$= \lim_{h \to 0} h[1^3 + (1+h)^3 + (1+2h)^3 + \ldots\ldots + \{1 + (n-1)h\}^3] \qquad [\because \ f(x) = x^3]$$

$$= \lim_{h \to 0} h[1 + (1 + 3h + 3h^2 + h^3) + (1 + 6h + 12h^2 + 8h^3) + \ldots\ldots + \{1 + 3(n-1)h + 3(n-1)^2 h^2 + (n-1)^3 h^3\}]$$

$$= \lim_{h \to 0} h\,[(1 + 1 + 1 + \ldots\ldots + 1) + \{3h + 6h + \ldots\ldots + 3(n-1)h\}$$

$$+ \{3h^2 + 12h^2 + \ldots\ldots + 3(n-1)^2 h^2\} + \{h^3 + 8h^3 + \ldots\ldots + (n-1)^3 h^3\}]$$

$$= \lim_{h \to 0} h[n + 3h\{1 + 2 + \ldots\ldots + (n-1)\} + 3h^2\{1^2 + 2^2 + \ldots\ldots + (n-1)^2\}$$

$$+ h^3\{1^3 + 2^3 + \ldots\ldots + (n-1)^3\}]$$

$$= \lim_{h \to 0} h\left[n + 3h \cdot \frac{n(n-1)}{2} + 3h^2 \cdot \frac{n(n-1)(2n-1)}{6} + h^3 \cdot \frac{n^2(n-1)^2}{4}\right]$$

$$\left[\begin{array}{l} \because\ [1+2+3+\ldots\ldots+(n-1)] = \dfrac{n(n-1)}{2} \\ [1^2+2^2+3^2+\ldots\ldots+(n-1)^2] = \dfrac{n(n-1)(2n-1)}{6} \\ [1^3+2^3+3^3+\ldots\ldots+(n-1)^3] = \dfrac{n^2(n-1)^2}{4} \end{array}\right]$$

$$= \lim_{h \to 0}\left[nh + \frac{3nh(nh-h)}{2} + \frac{nh(nh-h)(2nh-h)}{2} + \frac{n^2h^2(nh-h)^2}{4}\right]$$

$$= \lim_{h \to 0}\left[1 + \frac{3(1-h)}{2} + \frac{1(1-h)(2-h)}{2} + \frac{1(1-h)^2}{4}\right] \qquad [\because\ nh = 1]$$

$$= 1 + \frac{3(1-0)}{2} + \frac{(1-0)(2-0)}{2} + \frac{(1-0)^2}{4}$$

$$= 1 + \frac{3}{2} + 1 + \frac{1}{4} = 2 + \frac{3}{2} + \frac{1}{4} = \frac{8+6+1}{4} = \frac{15}{4}.$$

(*v*) Let $I = \int_0^1 (3x^2 + 2x + 1)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 0$, $b = 1$, $f(x) = (3x^2 + 2x + 1)$.

Let $h = \dfrac{b-a}{n} = \dfrac{1-0}{n} = \dfrac{1}{n} \Rightarrow nh = 1, n \in \mathbb{N}$.

$$\therefore\ I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}] \qquad [\because\ a = 0]$$

$$= \lim_{h \to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h \to 0} h[\{3(0)^2 + 2(0) + 1\} + \{3h^2 + 2h + 1\} + \{3(2h)^2 + 2(2h) + 1\} +$$

$$\ldots\ldots + \{3(n-1)^2h^2 + 2(n-1)h + 1\}]$$

$$[\because\ f(x) = (3x^2 + 2x + 1)]$$

$$= \lim_{h \to 0} h[(1 + 1 + 1 + \ldots\ldots + 1) + \{3h^2 + 3(2h)^2 + \ldots\ldots + 3(n-1)^2h^2\} +$$

$$+ \{2h + 2(2h) + \ldots\ldots + 2(n-1)h\}]$$

$$= \lim_{h \to 0} h[n + 3h^2\{1^2 + 2^2 + \ldots\ldots + (n-1)^2\} + 2h\{1 + 2 + \ldots\ldots + (n-1)\}]$$

$$= \lim_{h \to 0} h\left[n + 3h^2 \cdot \frac{n(n-1)(2n-1)}{6} + 2h \cdot \frac{n(n-1)}{2}\right]$$

$$\left[\because \; 1 + 2 + 3 + \ldots\ldots + (n-1) = \frac{n(n-1)}{2} \atop 1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h \to 0}\left[nh + \frac{nh(nh-h)(2nh-h)}{2} + nh(nh-h)\right]$$

$$= \lim_{h \to 0}\left[1 + \frac{1(1-h)(2-h)}{2} + 1(1-h)\right] \qquad [\because \; nh = 1]$$

$$= 1 + 1 + 1 = 3.$$

Example 6. *Evaluate by summation, the definite integral :*

(i) $\int_a^b e^x \, dx$ (ii) $\int_0^4 e^x \, dx$

(iii) $\int_0^5 e^{-2x} \, dx$ (iv) $\int_0^2 e^x \, dx$.

Solution. (i) Let $I = \int_a^b e^x \, dx$.

Comparing I with $\int_a^b f(x) \, dx$, we get : $a = a$, $b = b$, $f(x) = e^x$.

Let $h = \frac{b-a}{n} = \frac{b-a}{n} \Rightarrow nh = b - a, n \in N$.

$$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)]$$

$$= \lim_{h \to 0} h[e^a + e^{a+h} + e^{a+2h} + \ldots\ldots + e^{a+(n-1)h}] \qquad [\because \; f(x) = e^x]$$

Since $e^a + e^{a+h} + e^{a+2h} + \ldots\ldots + e^{a+(n-1)h}$ is a G.P. (of n-terms), where first term is e^a and common ratio is e^h.

$$\therefore \quad \text{Sum} = \frac{e^a(e^{nh}-1)}{e^h - 1} \qquad \left[\because \; \frac{a(r^n-1)}{r-1}\right]$$

$$= \lim_{h \to 0} h\left[\frac{e^a(e^{nh}-1)}{e^h-1}\right] = \lim_{h \to 0}\left[\frac{he^a(e^{b-a}-1)}{e^h-1}\right] \qquad [\because \; nh = (b-a)]$$

$$= \lim_{h \to 0}\left(\frac{h}{e^h-1}\right)(e^b - e^a) = \lim_{h \to 0} \frac{1}{\left(\frac{e^h-1}{h}\right)} \cdot (e^b - e^a)$$

$$= e^b - e^a. \qquad \left[\because \; \lim_{h \to 0}\left(\frac{e^h-1}{h}\right) = 1\right]$$

(ii) Let $I = \int_0^4 e^x \, dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 0$, $b = 4$, $f(x) = e^x$.

Let $\quad h = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n} \Rightarrow nh = 4, n \in \mathrm{N}.$

$$\therefore \quad \mathrm{I} = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}] \qquad [\because a = 0]$$

$$= \lim_{h \to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h \to 0} h[e^0 + e^h + e^{2h} + \ldots\ldots + e^{(n-1)h}] \qquad [\because f(x) = e^x]$$

$$= \lim_{h \to 0} h\left[\frac{1\{1-(e^h)^n\}}{1-e^h}\right] \qquad \left[\because S_n = \frac{a(1-r^n)}{1-r}\right]$$

$$= \lim_{h \to 0} \frac{h(1-e^{nh})}{1-e^h}$$

$$= \lim_{h \to 0} \frac{h(1-e^4)}{1-e^h} = (1-e^4) \cdot \lim_{h \to 0} \frac{h}{(1-e^h)} \qquad [\because nh = 4]$$

$$= (1-e^4) \cdot \lim_{h \to 0} \frac{1}{\left(\frac{e^h-1}{h}\right)} = (e^4-1) \times \frac{1}{1} = (e^4-1). \qquad \left[\because \lim_{h \to 0} \frac{e^h-1}{h} = 1\right]$$

(*iii*) Let $\quad \mathrm{I} = \int_0^5 e^{-2x}\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 0$, $b = 5$, $f(x) = e^{-2x}$

Let $\quad h = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n} \Rightarrow nh = 5, n \in \mathrm{N}.$

$$\therefore \quad \mathrm{I} = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}] \qquad [\because a = 0]$$

$$= \lim_{h \to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h \to 0} h[e^0 + e^{-2h} + e^{-4h} + \ldots\ldots + e^{-2(n-1)h}] \qquad [\because f(x) = e^{-2x}]$$

$$= \lim_{h \to 0} h[1 + e^{-2h} + (e^{-2h})^2 + \ldots\ldots + (e^{-2h})^{n-1}]$$

$$= \lim_{h \to 0} h\left[\frac{1(1-(e^{-2h})^n}{1-e^{-2h}}\right] \qquad \left[\because S_n = \frac{a(1-r^n)}{1-r}\right]$$

$$= \lim_{h \to 0} \frac{h(1-e^{-2nh})}{1-e^{-2h}} = \lim_{h \to 0} \frac{h(1-e^{-2(5)})}{1-e^{-2h}} \qquad [\because nh = 5]$$

$$= \lim_{h\to 0} \frac{h(e^{-10}-1)}{(e^{-2h}-1)} = (e^{-10}-1) \cdot \lim_{h\to 0} \frac{1}{\left(\frac{e^{-2h}-1}{h}\right)}$$

$$= (e^{-10}-1) \cdot \lim_{h\to 0} \frac{1}{\frac{e^{-2h}-1}{-2h} \times (-2)}$$ [Multiply and divided by (– 2)]

$$= (e^{-10}-1) \frac{1}{1(-2)} = -\frac{1}{2}(e^{-10}-1).$$ $\left[\because \lim_{h\to 0} \frac{e^h-1}{h} = 1\right]$

(iv) Let $I = \int_0^2 e^x \, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 0,\ b = 2,\ f(x) = e^x$.

Let $h = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} \Rightarrow nh = 2,\ n \in \mathrm{N}.$

$\therefore$ $I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)h)]$

$= \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f(0+(n-1)h)]$ $[\because a = 0]$

$= \lim_{h\to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f((n-1)h)]$

$= \lim_{h\to 0} h[e^0 + e^h + e^{2h} + \ldots\ldots + e^{(n-1)h}]$ $[\because f(x) = e^x]$

$= \lim_{h\to 0} h\left[e^0 \left\{\frac{(e^h)^n - 1}{e^h - 1}\right\}\right]$ $\left[\because a + ar + ar^2 + \ldots\ldots + ar^{n-1} = a\left(\frac{r^n-1}{r-1}\right)\right]$

$= \lim_{h\to 0}\left[\frac{h(e^{nh}-1)}{e^h-1}\right] = \lim_{h\to 0}\left[\frac{h(e^2-1)}{e^h-1}\right]$ $[\because nh = 2]$

$$= (e^2-1) \lim_{h\to 0} \left(\frac{1}{\frac{e^h-1}{h}}\right)$$

$$= (e^2-1) \times \frac{1}{1} = (e^2-1).$$ $\left[\because \lim_{h\to 0} \frac{e^h-1}{h} = 1\right]$

Example 7. *Evaluate by summation, the definite integral :*

(i) $\int_{-1}^{1} e^x \, dx$ *(ii)* $\int_a^b e^{\lambda x} \, dx$

(iii) $\int_0^1 e^{2-3x} \, dx$ *(iv)* $\int_0^4 (x + e^{2x}) \, dx.$

Solution. *(i)* Let $I = \int_{-1}^{1} e^x \, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = -1,\ b = 1,\ f(x) = e^x$.

Let $h = \frac{b-a}{n} = \frac{1-(-1)}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in N.$

$\therefore$ $I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)]$

$= \lim_{h \to 0} h[f(-1) + f(-1+h) + f(-1+2h) + ... + f(-1+(n-1)h)]$ $[\because a = -1]$

$= \lim_{h \to 0} h[e^{-1} + (e^{-1+h}) + (e^{-1+2h}) + ... + (e^{-1+(n-1)h})]$ $[\because f(x) = e^x]$

$= \lim_{h \to 0} h[e^{-1} + e^{-1}.e^h + e^{-1}.e^{2h} + ... + e^{-1}.e^{(n-1)h}]$

$= \lim_{h \to 0} he^{-1}[1 + e^h + e^{2h} + ... + e^{(n-1)h}]$

$= \lim_{h \to 0} \frac{h}{e}\left[\frac{(e^h)^n - 1}{e^h - 1}\right] = \lim_{h \to 0} \frac{h}{e}\left[\frac{e^{nh} - 1}{e^h - 1}\right]$ $\left[\because \text{For a G.P., } S_n = \frac{a(r^n - 1)}{r - 1}\right]$

$= \lim_{h \to 0} \frac{h}{e}\left[\frac{e^2 - 1}{e^h - 1}\right] = \frac{e^2-1}{e} . \lim_{h \to 0}\left(\frac{h}{e^h - 1}\right)$ $[\because nh = 2]$

$= \frac{e^2-1}{e} \lim_{h \to 0} \left(\frac{1}{\frac{e^h - 1}{h}}\right) = \frac{e^2-1}{e} \times \frac{1}{1} = e - \frac{1}{e}$.

(*ii*) Let $I = \int_a^b e^{\lambda x}\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = a, b = b, f(x) = e^{\lambda x}$.

Let $h = \frac{b-a}{n} \Rightarrow nh = b - a, n \in N.$

$\therefore$ $I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)]$

$= \lim_{h \to 0} h[e^{\lambda a} + e^{\lambda(a+h)} + e^{\lambda(a+2h)} + ... + e^{\lambda(a+(n-1)h)}]$ $[\because f(x) = e^{\lambda x}]$

$= \lim_{h \to 0} h[e^{\lambda a} + e^{\lambda a} . e^{\lambda h} + e^{\lambda a} . e^{2\lambda h} + ... + e^{\lambda a} . e^{\lambda(n-1)h}]$

$= \lim_{h \to 0} he^{\lambda a}[1 + e^{\lambda h} + e^{2\lambda h} + ... + e^{\lambda(n-1)h}]$

$= \lim_{h \to 0} he^{\lambda a}\left[\frac{1(1-(e^{\lambda h})^n}{1 - e^{\lambda h}}\right]$ $\left[\because \text{For a G.P., } S_n = \frac{a(1-r^n)}{1-r}\right]$

$= e^{\lambda a} \lim_{h \to 0} h\left[\frac{1 - e^{\lambda nh}}{1 - e^{\lambda h}}\right] = e^{\lambda a} \lim_{h \to 0} h\left[\frac{1 - e^{\lambda(b-a)}}{1 - e^{\lambda h}}\right]$ $[\because nh = (b-a)]$

$= e^{\lambda a} (e^{\lambda(b-a)} - 1) \lim_{h \to 0}\left(\frac{h}{e^{\lambda h} - 1}\right)$

$= e^{\lambda a} [e^{\lambda(b-a)} - 1] . \lim_{h \to 0}\left(\frac{1}{\frac{e^{\lambda h} - 1}{\lambda h} \times \lambda}\right)$ [Multiply and divided by λ]

$\left[\because \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = 1\right]$

$$= (e^{\lambda b} - e^{\lambda a}) \cdot \frac{1}{1.\lambda} = \frac{1}{\lambda}(e^{\lambda b} - e^{\lambda a}).$$

(iii) Let $I = \int_0^1 e^{2-3x}\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = 0$, $b = 1$, $f(x) = e^{2-3x}$.

Let $h = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} \Rightarrow nh = 1, n \in N.$

$$\therefore \quad I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}] \qquad [\because \ a = 0]$$

$$= \lim_{h\to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h\to 0} h[e^{2-3(0)} + e^{2-3h} + e^{2-3(2h)} + \ldots\ldots + e^{2-3(n-1)h}] \qquad [\because \ f(x) = e^{2-3x}]$$

$$= \lim_{h\to 0} h[e^2 + e^2 \cdot e^{-3h} + e^2 \cdot e^{-3(2h)} + \ldots\ldots + e^2 \cdot e^{-3(n-1)h}]$$

$$= \lim_{h\to 0} h\, e^2[1 + e^{-3h} + e^{-3(2h)} + \ldots\ldots + e^{-3(n-1)h}]$$

$$= \lim_{h\to 0} he^2\left[\frac{(e^{-3h})^n - 1}{e^{-3h} - 1}\right] \qquad \left[\because \ \text{For a G.P., } S_n = \frac{a(r^n - 1)}{r - 1}\right]$$

$$= \lim_{h\to 0} he^2\left[\frac{e^{-3nh} - 1}{e^{-3h} - 1}\right]$$

$$= e^2 \lim_{h\to 0} h\left[\frac{e^{-3} - 1}{e^{-3h} - 1}\right] = e^2(e^{-3} - 1) \cdot \lim_{h\to 0}\left(\frac{h}{e^{-3h} - 1}\right) \qquad [\because \ nh = 1]$$

$$= (e^{-1} - e^2) \cdot \lim_{h\to 0}\left[\frac{1}{\left(\frac{e^{-3h} - 1}{-3h}\right) \times (-3)}\right] \qquad \text{[Multiply and divided by } (-3)]$$

$$= (e^{-1} - e^2) \cdot \frac{1}{1(-3)} = \frac{1}{3}(e^2 - e^{-1}).$$

(iv) Let $I = \int_0^4 (x + e^{2x})\,dx$

Comparing I with $\int_0^4 (x + e^{2x})\,dx$, we get : $a = 0$, $b = 4$, $f(x) = x + e^{2x}$.

Let $h = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n} \Rightarrow nh = 4, n \in N.$

$$\therefore \quad I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}] \qquad [\because \ a = 0]$$

$$= \lim_{h\to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h\to 0} h[(0 + e^0) + (h + e^{2h}) + \{2h + e^{2(2h)}\} + \ldots\ldots + \{(n-1)h + e^{2(n-1)h}\}]$$

$$[\because \ f(x) = (x + e^{2x})]$$

$$= \lim_{h \to 0} h[\{h + 2h + ... + (n-1)h\} + \{1 + e^{2h} + e^{2(2h)} + ... + e^{2(n-1)h}\}]$$

$$= \lim_{h \to 0} h[h\{1 + 2 + ... + (n-1)\} + \left\{1 . \frac{(e^{2h})^n - 1}{e^{2h} - 1}\right\}$$

$$= \lim_{h \to 0} h\left[h . \frac{n(n-1)}{2} + \frac{e^{2nh} - 1}{e^{2h} - 1}\right] \qquad \left[\because [1+2+3+......+(n-1)] = \frac{n(n-1)}{2}\right]$$

$$= \lim_{h \to 0} \frac{nh(nh-h)}{2} + \lim_{h \to 0} \frac{h(e^{2nh} - 1)}{e^{2h} - 1}$$

$$= \lim_{h \to 0} \frac{4(4-h)}{2} + \lim_{h \to 0} \frac{h(e^{2(4)} - 1)}{e^{2h} - 1} \qquad [\because \ nh = 4]$$

$$= \frac{16}{2} + (e^8 - 1) \lim_{h \to 0} \left(\frac{1}{\frac{e^{2h} - 1}{2h} \times 2}\right) \qquad \text{[Multiply and divided by 2]}$$

$$= 8 + (e^8 - 1) . \frac{1}{1(2)} \qquad \left[\because \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = 1\right]$$

$$= 8 + \frac{1}{2}(e^8 - 1) = \frac{16 + e^8 - 1}{2} = \frac{15 + e^8}{2}.$$

Example 8. *Evaluate the following integrals as the limit of a sum :*

(i) $\int_a^b e^{-x}\, dx$ *(ii)* $\int_2^4 2^x\, dx$ *(iii)* $\int_0^1 (e^x + 1)\, dx.$

Solution. (*i*) Let $I = \int_a^b e^{-x}\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = a$, $b = b$, $f(x) = e^{-x}$.

Let $h = \dfrac{b-a}{n} \Rightarrow nh = b - a, n \in N$.

$$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a + (n-1)h)]$$

$$= \lim_{h \to 0} h[e^{-a} + e^{-(a+h)} + e^{-(a+2h)} + ... + e^{-\{a+(n-1)h\}}] \qquad [\because \ f(x) = e^{-x}]$$

$$= \lim_{h \to 0} h[e^{-a} + e^{-a} . e^{-h} + e^{-a} . e^{-2h} + ... + e^{-a} . e^{-(n-1)h}]$$

$$= \lim_{h \to 0} he^{-a}[1 + e^{-h} + e^{-2h} + ... + e^{-(n-1)h}]$$

$$= \lim_{h \to 0} he^{-a}\left[\frac{1 - (e^{-h})^n}{1 - e^{-h}}\right] = \lim_{h \to 0} he^{-a}\left[\frac{1 - e^{-nh}}{1 - e^{-h}}\right]$$

$$= e^{-a} . \lim_{h \to 0} h\left[\frac{1 - e^{-nh}}{1 - e^{-h}}\right] = e^{-a} . \lim_{h \to 0} h\left[\frac{1 - e^{-(b-a)}}{1 - e^{-h}}\right] \qquad [\because \ nh = (b-a)]$$

$$= e^{-a}(1 - e^{-(b-a)}) . \lim_{h \to 0}\left(\frac{h}{1 - e^{-h}}\right) = (e^{-a} - e^{-b})(1) \qquad \left[\because \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = 1\right]$$

$$= e^{-a} - e^{-b}.$$

(*ii*) Let $\quad I = \int_2^4 2^x \, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 2$, $b = 4$, $f(x) = 2^x$.

Let $\quad h = \frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n} \Rightarrow nh = 2, n \in N.$

$$\therefore \quad I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h\to 0} h[f(2) + f(2+h) + f(2+2h) + \ldots\ldots + f\{2 + (n-1)h\}] \qquad [\because a = 2]$$

$$= \lim_{h\to 0} h[2^2 + 2^{2+h} + 2^{2+2h} + \ldots\ldots + 2^{2+(n-1)h}] \qquad [\because f(x) = 2^x]$$

$$= \lim_{h\to 0} h[2^2 + 2^2 . 2^h + 2^2 . 2^{2h} + \ldots\ldots + 2^2 . 2^{(n-1)h}]$$

$$= \lim_{h\to 0} 4h[1 + 2^h + 2^{2h} + \ldots\ldots + 2^{(n-1)h}]$$

$$= \lim_{h\to 0} 4h\left[\frac{(2^h)^n - 1}{2^h - 1}\right] = \lim_{h\to 0} 4h\left[\frac{2^{nh} - 1}{2^h - 1}\right]$$

$$= \lim_{h\to 0} 4h\left[\frac{2^2 - 1}{2^h - 1}\right] = \lim_{h\to 0} 4\left[\frac{4-1}{\frac{2^h - 1}{h}}\right] \qquad [\because nh = 2]$$

$$= \frac{12}{\log 2}. \qquad \left[\because \lim_{h\to 0} \frac{2^h - 1}{h} = \log 2\right]$$

(*iii*) Let $\quad I = \int_0^1 (e^x + 1)\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = 0$, $b = 1$, $f(x) = (e^x + 1)$.

Let $\quad h = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} \Rightarrow nh = 1, n \in N.$

$$\therefore \quad I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$$

$$= \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots\ldots + f\{0 + (n-1)h\}] \qquad [\because a = 0]$$

$$= \lim_{h\to 0} h[f(0) + f(h) + f(2h) + \ldots\ldots + f\{(n-1)h\}]$$

$$= \lim_{h\to 0} h[(e^0 + 1) + (e^h + 1) + (e^{2h} + 1) + \ldots\ldots + \{e^{(n-1)h} + 1\}] \quad [\because f(x) = (e^x + 1)]$$

$$= \lim_{h\to 0} h[(1 + 1 + 1 + \ldots\ldots + 1) + \{e^0 + e^h + e^{2h} + \ldots\ldots + e^{(n-1)h}\}]$$

$$= \lim_{h\to 0} h\left[n + \frac{1-(e^h)^n}{1-e^h}\right] = \lim_{h\to 0}\left[nh + \frac{h(1-e^{nh})}{1-e^h}\right]$$

$$= \lim_{h\to 0}\left[1 + \frac{h(1-e)}{1-e^h}\right] \qquad [\because nh = 1]$$

$$= 1 + (1-e)\lim_{h\to 0}\left(\frac{1}{\frac{1-e^h}{h}}\right) = 1 + (e-1)\lim_{h\to 0}\left(\frac{1}{\frac{e^h-1}{h}}\right)$$

$$= [1 + (e-1).1] = e. \qquad \left[\because \lim_{h\to 0}\left(\frac{e^h-1}{h}\right) = 1\right]$$

Example 9. *Evaluate the following as limit of sum :*

(i) $\int_a^b \sin x\,dx$ (ii) $\int_a^b \cos x\,dx$

(iii) $\int_0^{\pi/2} \cos x\,dx$ (iv) $\int_0^{\pi/2} \sin x\,dx$

(v) $\int_a^b \frac{1}{x}\,dx$ (vi) $\int_a^b \frac{1}{x^2}\,dx$

(vii) $\int_a^b \frac{1}{\sqrt{x}}\,dx$ (viii) $\int_a^b \sin^2 x\,dx.$

Solution. (i) Let $I = \int_a^b \sin x\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = a$, $b = b$, $f(x) = \sin x$.

Let $h = \frac{b-a}{n} \Rightarrow nh = b - a,\ n \in N.$

$\therefore\ I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$

$= \lim_{h\to 0} h[\sin a + \sin(a+h) + \sin(a+2h) + \ldots\ldots + \sin\{a + (n-1)h\}]$

$[\because f(x) = \sin x]$

$$= \lim_{h\to 0} h\left[\frac{\sin\left[a + \left(\frac{n-1}{2}\right)h\right].\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}\right]$$

$$\left[\because [\sin a + \sin(a+h) + \sin(a+2h) + \ldots\ldots + \sin(a + \overline{n-1}\,h)] = \frac{\sin\left[a + \left(\frac{n-1}{2}\right)h\right].\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}\right]$$

$$= \lim_{h\to 0}\left[2\left(\frac{\frac{h}{2}}{\sin\frac{h}{2}}\right).\sin\left\{a + \frac{1}{2}(n-1)h\right\}.\sin\left(\frac{nh}{2}\right)\right] \qquad \text{[Multiply and divided by 2]}$$

$$= \lim_{h\to 0}\left[\left(\frac{\frac{h}{2}}{\sin\frac{h}{2}}\right).2\sin\left\{a + \frac{nh}{2} - \frac{h}{2}\right\}\sin\left(\frac{nh}{2}\right)\right]$$

$$= \lim_{h \to 0} \left[\left(\frac{\left(\frac{h}{2}\right)}{\sin \frac{h}{2}} \right) . 2 \sin \left\{ a + \frac{b-a}{2} - \frac{h}{2} \right\} \sin \left(\frac{b-a}{2} \right) \right] \qquad [\because \ nh = (b-a)]$$

$$= \lim_{h \to 0} \left(\frac{\frac{h}{2}}{\sin \frac{h}{2}} \right) . \lim_{h \to 0} \left[2 \sin \left(\frac{a+b}{2} - \frac{h}{2} \right) \right] . \sin \left(\frac{b-a}{2} \right)$$

$$= 1 . 2 \ \sin \left(\frac{a+b}{2} \right) \sin \left(\frac{b-a}{2} \right) = 2 \sin \left(\frac{b+a}{2} \right) \sin \left(\frac{b-a}{2} \right) \qquad \left[\because \ \lim_{x \to 0} \frac{\sin x}{x} = 1 \right]$$

$$= \cos \left(\frac{b+a-b+a}{2} \right) - \cos \left(\frac{b+a+b-a}{2} \right)$$

$$[\because \ 2 \sin A \sin B = \cos (A - B) - \cos (A + B)]$$

$= \cos a - \cos b.$

(*ii*) Let $I = \int_a^b \cos x \, dx$

Comparing I with $\int_a^b f(x) \, dx$, we get : $a = a$, $b = b$, $f(x) = \cos x$.

Let $h = \frac{b-a}{n} \Rightarrow nh = (b-a), n \in N$.

$\therefore \quad I = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a + (n-1)h)]$

$= \lim_{h \to 0} h[\cos a + \cos(a+h) + \cos(a+2h) + \ldots\ldots + \cos(a+(n-1)h)]$

$[\because \ f(x) = \cos x]$

$$= \lim_{h \to 0} h \left[\frac{\cos \left[a + \left(\frac{n-1}{2} \right) h \right] . \sin \left(\frac{nh}{2} \right)}{\sin \left(\frac{h}{2} \right)} \right]$$

$$\left[\because \ [\cos a + \cos(a+h) + \cos(a+2h) + \ldots\ldots + \cos(a + \overline{n-1}\,h)] = \frac{\cos \left[a + \left(\frac{n-1}{2} \right) h \right] . \sin \left(\frac{nh}{2} \right)}{\sin \left(\frac{h}{2} \right)} \right]$$

$$= \lim_{h \to 0} \left[\left(\frac{\frac{h}{2}}{\sin \left(\frac{h}{2} \right)} \right) 2 \cos \left\{ a + \frac{nh}{2} - \frac{h}{2} \right\} \sin \left(\frac{nh}{2} \right) \right] \qquad \text{[Multiply and divided by 2]}$$

$$= \lim_{h \to 0} \left[\left(\frac{\frac{h}{2}}{\sin \left(\frac{h}{2} \right)} \right) 2 \cos \left\{ a + \frac{b-a}{2} - \frac{h}{2} \right\} \sin \left(\frac{b-a}{2} \right) \right] \qquad [\because \ nh = (b-a)]$$

$$= \lim_{h \to 0} \left(\frac{1}{\frac{\sin\left(\frac{h}{2}\right)}{h/2}} \right) . \ 2 \lim_{h \to 0} \left[\cos\left(\frac{b+a}{2} - \frac{h}{2} \right) \right] . \sin\left(\frac{b-a}{2} \right)$$

$$= 2 \cos\left(\frac{b+a}{2} \right) . \sin\left(\frac{b-a}{2} \right)$$

$$= \sin\left(\frac{b+a+b-a}{2} \right) - \sin\left(\frac{b+a-b+a}{2} \right)$$

$[\because \ 2 \cos A \sin B = \sin (A + B) - \sin (A - B)]$

$$= \sin b - \sin a.$$

(*iii*) Let $\quad I = \int_0^{\pi/2} \cos x \, dx$

Comparing I with $\int_a^b f(x) \, dx$, we get : $a = 0, b = \frac{\pi}{2}$, $f(x) = \cos x$.

Proceed as part (*ii*), we get

$$I = \sin b - \sin a$$

Put $\quad a = 0, b = \frac{\pi}{2}$

$$I = \sin \frac{\pi}{2} - \sin 0 = 1 - 0 = 1.$$

(*iv*) Let $I = \int_0^{\pi/2} \sin x \, dx$

Comparing I with $\int_a^b f(x) \, dx$, we get : $a = 0, b = \frac{\pi}{2}$, $f(x) = \sin x$.

Proceed as part (*i*), we get

$$I = \cos a - \cos b$$

Put $\quad a = 0, b = \frac{\pi}{2}$

$$I = \cos 0 - \cos \frac{\pi}{2}$$

$$= 1 - 0 = 1.$$

(*v*) Let $\quad I = \int_a^b \frac{1}{x} \, dx \qquad \text{...(1)}$

Comparing I with $\int_a^b f(x) \, dx$, we get : $a = a, b = b, f(x) = \frac{1}{x}$ and $r^n = \frac{b}{a}$, $a \neq 0$

$\therefore$ By Walli's Formula,

$$\int_a^b f(x) \, dx = \lim_{r \to 1} (r - 1) \, [af(a) + arf(ar) + ar^2 f(ar^2) + \ldots\ldots + ar^{n-1} f(ar^{n-1})]$$

$$\therefore \qquad I = \lim_{r \to 1} (r - 1) \left[a \, . \, \frac{1}{a} + ar \, . \, \frac{1}{ar} + (ar^2) \, . \, \frac{1}{(ar^2)} + \ldots\ldots + (ar^{n-1}) \, . \, \frac{1}{(ar^{n-1})} \right]$$

$$\left[\because \ f(x) = \frac{1}{x} \right]$$

$$= \lim_{r \to 1} (r-1)[1+1+1+\ldots\ldots+1]$$

$\Rightarrow$ $$I = \lim_{r \to 1} (r-1) \cdot n \qquad \ldots(2)$$

Now $r^n = \dfrac{b}{a}$

Taking log on both sides, $\log r^n = \log \dfrac{b}{a} \;\Rightarrow\; n \log r = \log\left(\dfrac{b}{a}\right)$

$\Rightarrow$ $$n = \frac{\log\left(\frac{b}{a}\right)}{\log r} \qquad \ldots(3)$$

Substituting this value of n in equation (2), we get

$$I = \lim_{r \to 1} \frac{(r-1)\log\left(\frac{b}{a}\right)}{\log r} = \log\left(\frac{b}{a}\right) \cdot \lim_{r \to 1}\left(\frac{r-1}{\log r}\right)$$

Put $r = 1 + h \;\Rightarrow\; h \to 0$ as $r \to 1$

$\therefore$ $$I = \log\left(\frac{b}{a}\right) \cdot \lim_{h \to 0} \frac{(1+h-1)}{\log(1+h)} = \log\left(\frac{b}{a}\right) \cdot \lim_{h \to 0} \frac{h}{\log(1+h)}$$

$$= \log\left(\frac{b}{a}\right) \cdot \lim_{h \to 0}\left[\frac{1}{\frac{1}{h}\log(1+h)}\right]$$

$$= \log\left(\frac{b}{a}\right) \cdot 1 = \log b - \log a.$$

(*vi*) Let $I = \int_a^b \dfrac{1}{x^2}\,dx$

Comparing I with $\int_a^b f(x)\,dx$, we get : $a = a$, $b = b$, $f(x) = \dfrac{1}{x^2}$ and $r^n = \dfrac{b}{a}$

$\therefore$ By Walli's Formula,

$$\int_a^b f(x)\,dx = \lim_{r \to 1} (r-1)\,[af(a) + arf(ar) + ar^2 f(ar^2) + \ldots\ldots + ar^{n-1} \cdot f(ar^{n-1})]$$

$\therefore$ $$I = \lim_{r \to 1}(r-1)\left[a \cdot \frac{1}{a^2} + ar \cdot \frac{1}{(ar)^2} + ar^2 \cdot \frac{1}{(ar^2)^2} + \ldots\ldots + ar^{n-1} \cdot \frac{1}{(ar^{n-1})^2}\right]$$

$$\left[\because\; f(x) = \frac{1}{x^2}\right]$$

$$= \lim_{r \to 1}(r-1)\left[\frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \ldots\ldots + \frac{1}{ar^{n-1}}\right]$$

$$= \lim_{r \to 1}(r-1)\left[\frac{\frac{1}{a}\left[\frac{1}{r^n} - 1\right]}{\frac{1}{r} - 1}\right] \qquad \left[\because\; \text{In a G.P., sum } S_n = \frac{a(r^n - 1)}{r-1}\right]$$

$$= \lim_{r \to 1} (r-1) \left[\frac{\frac{1}{a}\left[\frac{1-r^n}{r^n}\right]}{\frac{1-r}{r}} \right] = \lim_{r \to 1} \frac{(r-1).r}{(1-r)} \left[\frac{1}{a} \left(\frac{1-\frac{b}{a}}{\frac{b}{a}} \right) \right] \qquad \left[\because\ r^n = \frac{b}{a} \right]$$

$$= \lim_{r \to 1} \frac{r(r-1)}{1-r} \cdot \frac{1}{a} \left[\frac{a}{b} - 1 \right]$$

$$= \lim_{r \to 1} -r \left[\frac{1}{b} - \frac{1}{a} \right] = -1 \left(\frac{1}{b} - \frac{1}{a} \right) = \frac{1}{a} - \frac{1}{b}.$$

(*vii*) Let $I = \int_a^b \frac{1}{\sqrt{x}}\, dx$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = a$, $b = b$, $f(x) = \frac{1}{\sqrt{x}}$, and $r^n = \frac{b}{a}$

$\therefore$ By Walli's Formula,

$$\int_a^b f(x)\, dx = \lim_{r \to 1} (r-1)\, [af(a) + arf(ar) + ar^2 f(ar^2) + \ldots\ldots + ar^{n-1} f(ar^{n-1})]$$

$$\therefore \qquad I = \lim_{r \to 1} (r-1) \left[a \cdot \frac{1}{\sqrt{a}} + ar \cdot \frac{1}{\sqrt{ar}} + ar^2 \cdot \frac{1}{\sqrt{ar^2}} + \ldots\ldots + ar^{n-1} \cdot \frac{1}{\sqrt{ar^{n-1}}} \right]$$

$$\left[\because\ f(x) = \frac{1}{\sqrt{x}} \right]$$

$$= \lim_{r \to 1} (r-1) \left[\sqrt{a} + \sqrt{ar} + \sqrt{ar^2} + \ldots\ldots + \sqrt{ar^{n-1}} \right]$$

$$= \lim_{r \to 1} (r-1) \left[\frac{\sqrt{a}\left\{ \left(\sqrt{r}\right)^n - 1 \right\}}{\sqrt{r} - 1} \right] \qquad \left[\because\ \text{In a G.P., sum } S_n = \frac{a(r^n - 1)}{r - 1} \right]$$

$$= \lim_{r \to 1} \left[\frac{\left(\sqrt{r}+1\right)\left(\sqrt{r}-1\right)}{\sqrt{r}-1} \right] \cdot \left[\sqrt{a} \cdot \left(\sqrt{r}\right)^n - 1 \right]$$

$$= \lim_{r \to 1} \left(\sqrt{r}+1\right) \cdot \sqrt{a} \cdot \left[\sqrt{\frac{b}{a}} - 1 \right] \qquad \left[\because\ r^n = \frac{b}{a} \right]$$

$$= (1+1)\sqrt{a} \cdot \left[\frac{\sqrt{b} - \sqrt{a}}{\sqrt{a}} \right] = 2\left(\sqrt{b} - \sqrt{a}\right).$$

(*viii*) Let $I = \int_a^b \sin^2 x\, dx = \frac{1}{2} \int_a^b (1 - \cos 2x)\, dx \qquad [\because\ 1 - \cos 2A = 2 \sin^2 A]$

Comparing I with $\int_a^b f(x)\, dx$, we get : $a = a$, $b = b$, $f(x) = (1 - \cos 2x)$.

Let $h = \dfrac{b-a}{n} \Rightarrow nh = (b-a), n \in \mathbb{N}$.

$\therefore \quad I = \lim\limits_{h \to 0} \dfrac{h}{2}\,[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\{a + (n-1)h\}]$

$$= \lim_{h \to 0} \frac{h}{2}\left[1(1-\cos 2a) + \{1 - \cos 2(a+h)\} + \ldots\ldots + \left\{1 - \cos 2\left(a + \overline{n-1}\,h\right)\right\}\right]$$

$[\because \ f(x) = (1 - \cos 2x)]$

$$= \lim_{h \to 0} \frac{h}{2}\left[(1+1+1+\ldots\ldots+1) - \left\{\cos 2a + \cos 2(a+h) + \ldots\ldots + \cos 2\left(a + \overline{n-1}\,h\right)\right\}\right]$$

$$= \lim_{h \to 0} \frac{h}{2}\left[n - \frac{\cos\left(2a + \overline{n-1}\,h\right)\sin nh}{\sin h}\right]$$

$$= \lim_{h \to 0} \frac{1}{2}\left[nh - \frac{h\cos\left(2a + \overline{n-1}\,h\right)\sin nh}{\sin h}\right]$$

$$= \lim_{h \to 0} \frac{1}{2}\left[(b-a) - \frac{\cos[2a + b - a - h]\sin(b-a)}{\left(\dfrac{\sin h}{h}\right)}\right] \qquad [\because \ nh = (b-a)$$

$$= \frac{1}{2}\left[(b-a) - \frac{\cos(b+a)\sin(b-a)}{1}\right]$$

$$= \frac{1}{2}\left[(b-a) - \frac{1}{2}\,.\,2\cos(b+a)\sin(b-a)\right] \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\left[(b-a) - \frac{1}{2}\{\sin(b+a+b-a) - \sin(b+a-b+a)\}\right]$$

$[\because \ 2\cos A \sin B = \sin(A+B) - \sin(A-B)]$

$$= \frac{1}{2}\left[(b-a) - \frac{1}{2}\{\sin 2b - \sin 2a\}\right]$$

$$= \frac{1}{2}\left[(b-a) - \frac{1}{2}\{2\sin b\cos b - 2\sin a\cos a\}\right] \qquad [\because \ \sin 2A = 2\sin A\cos A]$$

$$= \frac{1}{2}(b-a) - \frac{1}{2}(\sin a\cos a - \sin b\cos b).$$

EXERCISE FOR PRACTICE

Evaluate the following integrals as the limit of a sum :

1. $\displaystyle\int_0^5 (x+1)\,dx$

2. $\displaystyle\int_3^5 (2-x)\,dx$

3. $\displaystyle\int_2^3 (2x^2+1)\,dx$

4. $\displaystyle\int_a^b e^{-x}\,dx$

5. $\displaystyle\int_1^3 (x^2+x)\,dx$

6. $\displaystyle\int_0^2 e^{3x+1}\,dx$

7. $\int_0^4 (x + e^{2x})\, dx$

8. $\int_0^{\pi/2} \sin x\, dx$

9. $\int_0^3 (x^2 + 1)\, dx$

10. $\int_2^3 e^{-x}\, dx$

11. $\int_0^5 (x + 1)\, dx$

12. $\int_1^2 (x^2 - 1)\, dx$

Answers

1. $\frac{35}{2}$	2. -4	3. 8	4. $e^{-a} - e^{-b}$
5. $\frac{38}{3}$	6. $\frac{e(e^6 - 1)}{3}$	7. $8 + \frac{e^8}{2}$	8. 1
9. 12	10. $e^{-2} - e^{-3}$	11. $\frac{35}{2}$	12. $\frac{4}{3}$.

7

Definite Integral by Using Indefinite Integral

7.1. INTRODUCTION

The concept of definite integrals links the area to other important concepts such as lengths, graphs of the different curves, volume, density, probability and work etc. Definite integral is very useful in calculating areas bounded by curves, arc length, volumes, velocity, length, work, moment of inertia etc.

We will also see that the Indefinite Integral and the Definite Integral are closely related to each other.

7.2. FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Statement. If $f(x)$ is a continuous function defined on closed interval $[a, b]$, and if $F(x)$ be the primitive or antiderivative of $f(x)$ *i.e.*, $\frac{d}{dx}\{F(x)\} = f(x)$, then the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)\, dx$ and is equal to $[F(b) - F(a)]$.

i.e., $$\int_a^b f(x)\, dx = [F(x)]_a^b = [F(b) - F(a)].$$

Here, a and b are called the limits of integration, 'a' is called the lower limit and 'b' is called the upper limit.

The interval $[a, b]$ is called the interval of integration.

Theorem 1. *If $f(x)$ is a continuous and single valued function of x in the interval $[a, b]$ where a and b are finite and $a < b$, then*

$$\lim_{h \to 0} h[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + (n - 1)h)] = F(b) - F(a).$$

i.e., $$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof. Let AB be the graph of the curve $y = f(x)$. LA and MB are the ordinates $x = a$, $x = b$.

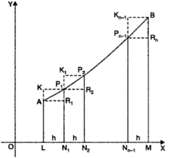
Then, $$S = \text{area ALMB}$$
$$= F(b) - F(a) \qquad \ldots(1)$$

where $F'(x) = f(x)$.

Divide LM into n equal parts, each equal to h such that

$$nh = \text{LM} = \text{OM} - \text{OL}$$
$$= b - a. \qquad \ldots(2)$$

by means of points $N_1, N_2, \ldots\ldots, N_{n-1}$ and through these points draw $N_1P_1, N_2P_2, \ldots\ldots, N_{n-1}P_{n-1}$ perpendiculars to x-axis meeting the curve in points $P_1, P_2, \ldots\ldots, P_{n-1}$.



Complete the two sets of rectangles $AN_1, P_1N_2, \ldots\ldots, P_{n-1}M$ and $LP_1, N_1P_2, \ldots\ldots, N_{n-1}B$.

Abscissae of the points $A, P_1, P_2, \ldots\ldots, P_{n-1}$, B are :

$a, a + h, a + 2h, \ldots\ldots, a + (n - 1)h, a + nh = b$ [By using equation (2)]

Their respective ordinates are :

$$\text{LA} = f(a),\ N_1P_1 = f(a + h),\ N_2P_2 = f(a + 2h), \ldots\ldots,$$
$$\ldots\ldots, N_{n-1}P_{n-1} = f(a + \overline{n-1}\,h),\ \text{MB} = f(a + nh)$$

Now sum of the areas of inscribed rectangles $AN_1, P_1N_2, \ldots\ldots, P_{n-1}M$.

$$= h\,f(a) + h\,f(a + h) + h\,f(a + 2h) + \ldots\ldots + h\,f(a + \overline{n-1}\,h)$$
$$= h\,[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + \overline{n-1}\,h)]$$
$$= S_n \text{ (say)} \qquad \ldots(3)$$

Sum of the areas of circumscribed rectangles $LP_1, N_1P_2, \ldots\ldots, N_{n-1}B$.

$$= h\,f(a + h) + h\,f(a + 2h) + \ldots\ldots + h\,f(a + nh)$$
$$= h\,[f(a + h) + f(a + 2h) + \ldots\ldots + f(a + nh)]$$
$$= h\,[f(a + h) + f(a + 2h) + \ldots\ldots + f(a + \overline{n-1}\,h)] + h\,f(a + nh)$$
$$= h\,[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + \overline{n-1}\,h)] + h\,f(a + nh) - h\,f(a)$$

[Add and subtract $h\,f(a)$]

$$= h\,[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + \overline{n-1}\,h) + h\,[f(b) - f(a)]$$

[$\because\ nh = b - a \Rightarrow a + nh = b$]

$$= S_n + h\,[f(b) - f(a)] \qquad \ldots(4) \text{ [By using equation (3)]}$$

From the fig, it is obvious that area ALMB lies between the sum of the areas of inscribed and circumscribed rectangles.

$\therefore$ S lies between S_n and $S_n + h\,[f(b) - f(a)]$

As $h \to 0$ $\qquad S = \lim\limits_{h\to 0} S_n = \int_a^b f(x)\,dx.$

$$= \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a + \overline{n-1}\,h)] \qquad \ldots(5)$$

$\therefore$ From equations (1) and (5) ;

Equating the two values of S, *i.e.*, area ALMB, we get

$$\lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a + \overline{n-1}\,h)] = F(b) - F(a).$$

i.e., $\qquad \int_a^b f(x)\,dx = F(b) - F(a).$

where $\qquad F'(x) = f(x).$

Remark 1. In the $\int_a^b f(x)\,dx$, it does not matter which antiderivative is used to evaluate definite integral, because :

Let us take $F(x) + c$ instead of $F(x)$ as the antiderivative of $f(x)$.

$$\therefore \qquad \int_a^b f(x)\,dx = [F(x) + c]_a^b$$

$$= [F(b) + c] - [F(a) + c]$$

$$= F(b) - F(a).$$

As the constant of integration disappears so $\int_a^b f(x)\,dx$ has a definite value $[F(b) - F(a)]$.

In other words, to evaluate the definite integral there is no need to keep constant of integration.

Remark 2. This theorem is very useful, because it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum. Though the fundamental theorem of integral calculus has eased the process of evaluating definite integrals, but this has involved the search of a function, whose derivative is the integrand in the given definite integral. In other words, we shall find the definite integral of a function by first finding its indefinite integral.

7.3. EVALUATION OF DEFINITE INTEGRAL BY CHANGING LIMITS AFTER SUBSTITUTION

Method to evaluate $\int_a^b f(x)\,dx$ by the substitution $x = g(z)$.

In the integrand put $x = g(z) \;\Rightarrow\; dx = g'(z)\,dz$

$\therefore$ when $x = a$ and $x = b$. Then find the corresponding values of z say $g(a)$ and $g(b)$, respectively.

Note. While evaluating the definite integral by substitution, the limits of integration must also be changed.

Remember 1. It is a common mistake of not changing the limits of integration while evaluating the integral in z between the old limits a and b which are the values of x and not of y.

Remember 2. The new limits $g(a)$ and $g(b)$ in evaluating the definite integral $\int_a^b f(x)\,dx$ by substituting $x = g(z)$ must be taken in such a manner that as x increases continuously from a to b, z increases from $g(a)$ to $g(b)$ or decreases from $g(a)$ to $g(b)$ continuously.

SOME SOLVED EXAMPLES

Example 1. *Evaluate the following definite integrals :*

(*i*) $\int_{-4}^{-1} \frac{1}{x}\, dx$ (*ii*) $\int_{1}^{5} x^2\, dx$

(*iii*) $\int_{2}^{3} \frac{1}{x}\, dx$ (*iv*) $\int_{1}^{2} x^2\, dx$

(*v*) $\int_{-5}^{5} x\, dx$ (*vi*) $\int_{2}^{5} (x^3 + x)\, dx.$

Solution. (*i*) Let $I = \int_{-4}^{-1} \frac{1}{x}\, dx = \int_{-4}^{-1} \frac{1}{x}\, dx = \Big[\log|x|\Big]_{-4}^{-1}$

$= \log|-1| - \log|-4| = \log 1 - \log 4$

$= 0 - \log 4 = -\log 4.$ $[\because \log 1 = 0]$

(*ii*) Let $I = \int_{1}^{5} x^2\, dx = \int_{1}^{5} x^2\, dx = \left[\frac{x^3}{3}\right]_1^5$

$= \frac{(5)^3}{3} - \frac{(1)^3}{3} = \frac{125}{3} - \frac{1}{3}$

$= \frac{124}{3}.$

(*iii*) Let $I = \int_{2}^{3} \frac{1}{x}\, dx$

$= \int_{2}^{3} \frac{1}{x}\, dx = \Big[\log|x|\Big]_2^3 = \log|3| - \log|2| = \log 3 - \log 2$

$= \log \frac{3}{2}.$ $\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$

(*iv*) Let $I = \int_{1}^{2} x^2\, dx$

$= \int_{1}^{2} x^2\, dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{(2)^3}{3} - \frac{(1)^3}{3}$

$= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$

(*v*) Let $I = \int_{-5}^{5} x\, dx$

$= \int_{-5}^{5} x\, dx = \left[\frac{x^2}{2}\right]_{-5}^{5} = \frac{(5)^2}{2} - \frac{(-5)^2}{2}$

$= \frac{25}{2} - \frac{25}{2} = 0.$

(*vi*) Let $\quad I = \int_2^5 (x^3 + x)\, dx$

$$= \int_2^5 (x^3 + x)\, dx = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_2^5 = \left[\frac{(5)^4}{4} + \frac{(5)^2}{2}\right] - \left[\frac{(2)^4}{4} + \frac{(2)^2}{2}\right]$$

$$= \frac{625}{4} + \frac{25}{2} - \frac{16}{4} - \frac{4}{2} = \frac{625}{4} + \frac{25}{2} - 4 - 2$$

$$= \frac{625}{4} + \frac{25}{2} - 6 = \frac{625 + 50 - 24}{4} = \frac{651}{4}.$$

Example 2. *Evaluate the following definite integrals :*

(*i*) $\int_0^1 \frac{1}{2x - 3}\, dx$ (*ii*) $\int_{-1}^1 (x + 1)\, dx$

(*iii*) $\int_0^8 x^{5/3}\, dx$ (*iv*) $\int_0^4 (x + x^{3/2})\, dx$

(*v*) $\int_1^2 \frac{x^2 - 3x + 2}{x^4}\, dx$ (*vi*) $\int_0^4 \left(\sqrt{x} - 2x + x^2\right) dx.$

Solution. (*i*) Let $I = \int_0^1 \frac{1}{2x - 3}\, dx$

$$= \int_0^1 \frac{1}{2x - 3}\, dx = \left[\frac{\log |(2x - 3)|}{2}\right]_0^1 = \frac{1}{2}\Big[\log |(2x - 3)|\Big]_0^1$$

$$= \frac{1}{2}\,[\log |2(1) - 3| - \log |2(0) - 3|]$$

$$= \frac{1}{2}\,[\log |-1| - \log |-3|] = \frac{1}{2}\,[\log 1 - \log 3]$$

$$= \frac{1}{2}\,(0 - \log 3) = -\frac{1}{2}\log 3. \qquad [\because \log 1 = 0]$$

(*ii*) Let $\quad I = \int_{-1}^1 (1 + x)\, dx$

$$= \int_{-1}^1 (1 + x)\, dx = \left[x + \frac{x^2}{2}\right]_{-1}^1 = \left[1 + \frac{(1)^2}{2}\right] - \left[(-1) + \frac{(-1)^2}{2}\right]$$

$$= \left(1 + \frac{1}{2}\right) - \left(-1 + \frac{1}{2}\right) = \frac{3}{2} - \left(-\frac{1}{2}\right)$$

$$= \frac{3}{2} + \frac{1}{2} = 2.$$

(*iii*) Let $\quad I = \int_0^8 x^{5/3}\, dx$

$$= \int_0^8 x^{5/3}\, dx = \left[\frac{x^{\frac{5}{3}+1}}{\frac{5}{3} + 1}\right]_0^8 = \left[\frac{x^{\frac{8}{3}}}{\frac{8}{3}}\right]_0^8 = \frac{3}{8}\,[(8)^{8/3} - (0)^{8/3}]$$

$$= \frac{3}{8}[(2^3)^{8/3} - 0] = \frac{3}{8}[2^8] = \frac{3}{8} \times 256$$
$$= 96.$$

(*iv*) Let $\quad I = \int_0^4 (x + x^{3/2})\, dx$

$$= \int_0^4 (x + x^{3/2})\, dx = \left[\frac{x^2}{2} + \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_0^4 = \left[\frac{x^2}{2} + \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^4$$

$$= \left[\frac{(4)^2}{2} + \frac{2}{5}(4)^{5/2}\right] - [0 + 0]$$

$$= \frac{16}{2} + \frac{2}{5}(32) = 8 + \frac{64}{5} = \frac{40 + 64}{5} \qquad [\because \; 4^{5/2} = (2^2)^{5/2} = 2^5 = 32]$$

$$= \frac{104}{5}.$$

(*v*) Let $\quad I = \int_1^2 \frac{x^2 - 3x + 2}{x^4}\, dx$

$$= \int_1^2 \left(\frac{x^2}{x^4} - \frac{3x}{x^4} + \frac{2}{x^4}\right) dx = \int_1^2 \left(\frac{1}{x^2} - \frac{3}{x^3} + \frac{2}{x^4}\right) dx$$

$$= \int_1^2 (x^{-2} - 3x^{-3} + 2x^{-4})\, dx = \left[\frac{x^{-2+1}}{-2+1} - 3\frac{x^{-3+1}}{-3+1} + 2\frac{x^{-4+1}}{-4+1}\right]_1^2$$

$$= \left[\frac{x^{-1}}{-1} - 3\frac{x^{-2}}{-2} + 2\frac{x^{-3}}{-3}\right]_1^2 = \left[-\frac{1}{x} + \frac{3}{2x^2} - \frac{2}{3x^3}\right]_1^2$$

$$= \left[-\frac{1}{2} + \frac{3}{2(2)^2} - \frac{2}{3(2)^3}\right] - \left[-\frac{1}{1} + \frac{3}{2(1)^2} - \frac{2}{3(1)^3}\right]$$

$$= \left[-\frac{1}{2} + \frac{3}{8} - \frac{1}{12}\right] - \left[-1 + \frac{3}{2} - \frac{2}{3}\right] = \left[\frac{-12 + 9 - 2}{24}\right] - \left(-\frac{1}{6}\right)$$

$$= -\frac{5}{24} + \frac{1}{6} = -\frac{1}{24}.$$

(*vi*) Let $\quad I = \int_0^4 \left(\sqrt{x} - 2x + x^2\right) dx$

$$= \int_0^4 \left(\sqrt{x} - 2x + x^2\right) dx = \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 2\frac{x^2}{2} + \frac{x^3}{3}\right]_0^4$$

$$= \left[\frac{x^{3/2}}{3/2} - x^2 + \frac{x^3}{3}\right]_0^4 = \left[\frac{2}{3}(4)^{3/2} - (4)^2 + \frac{(4)^3}{3}\right] - [0 - 0 + 0]$$

$$= \left[\frac{2(8)}{3} - \frac{16}{1} + \frac{64}{3}\right] = \frac{16 - 48 + 64}{3}$$

$$= \frac{32}{3}.$$

Example 3. *Evaluate the following definite integrals :*

(i) $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\, dx$ (ii) $\int_0^\pi \sin \frac{x}{2}\, dx$

(iii) $\int_0^1 \frac{1}{\sqrt{1+x} + \sqrt{x}}\, dx$ (iv) $\int_0^{\pi/2} (\sin x + \cos x)\, dx$

(v) $\int_0^\pi \cos x\, dx$ (vi) $\int_0^{\pi/2} \cos 2x\, dx.$

Solution. (i) Let $I = \int_1^2 (4x^3 - 5x^2 + 6x + 9)\, dx$

$$= \int_1^2 (4x^3 - 5x^2 + 6x + 9)\, dx = \left[\frac{4x^4}{4} - \frac{5x^3}{3} + \frac{6x^2}{2} + 9x\right]_1^2$$

$$= \left[x^4 - \frac{5}{3}x^3 + 3x^2 + 9x\right]_1^2$$

$$= \left[(2)^4 - \frac{5}{3}(2)^3 + 3(2)^2 + 9(2)\right] - \left[(1)^4 - \frac{5}{3}(1)^3 + 3(1)^2 + 9(1)\right]$$

$$= \left[16 - \frac{40}{3} + 12 + 18\right] - \left[1 - \frac{5}{3} + 3 + 9\right] = 46 - \frac{40}{3} - 13 + \frac{5}{3} = 33 - \frac{35}{3}$$

$$= \frac{99 - 35}{3} = \frac{64}{3}.$$

(ii) Let $I = \int_0^\pi \sin \frac{x}{2}\, dx$

$$= \int_0^\pi \sin \frac{x}{2}\, dx = \left[-\frac{\cos x/2}{1/2}\right]_0^\pi = -2\left[\cos \frac{x}{2}\right]_0^\pi$$

$$= -2\left[\cos \frac{\pi}{2} - \cos 0\right] = -2(0 - 1)$$

$$= 2.$$

(iii) Let $I = \int_0^1 \frac{1}{\sqrt{1+x} + \sqrt{x}}\, dx$

$$= \int_0^1 \frac{1}{\sqrt{1+x} + \sqrt{x}} \times \frac{\sqrt{1+x} - \sqrt{x}}{\sqrt{1+x} - \sqrt{x}}\, dx$$ **[Rationalization]**

$$= \int_0^1 \frac{\sqrt{1+x} - \sqrt{x}}{(1+x-x)}\, dx = \int_0^1 \left(\sqrt{1+x} - \sqrt{x}\right) dx$$

$$= \left[\frac{(1+x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^1 = \left[\frac{2}{3}(1+x)^{3/2} - \frac{2}{3}x^{3/2}\right]_0^1$$

$$= \left[\frac{2}{3}(1+1)^{3/2} - \frac{2}{3}(1)^{3/2}\right] - \left[\frac{2}{3}(1+0)^{3/2} - \frac{2}{3}(0)^{3/2}\right]$$

$$= \frac{2}{3}[(2)^{3/2} - 1] - \frac{2}{3}[1 - 0] = \frac{2}{3}\left(2\sqrt{2}\right) - \frac{2}{3} - \frac{2}{3} = \frac{4\sqrt{2}}{3} - \frac{4}{3}$$

$$= \frac{4}{3}\left(\sqrt{2} - 1\right).$$

(*iv*) Let $\quad I = \int_0^{\pi/2} (\sin x + \cos x)\, dx$

$$= \int_0^{\pi/2} (\sin x + \cos x)\, dx = [-\cos x + \sin x]_0^{\pi/2}$$

$$= \left[-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] - [-\cos 0 + \sin 0] = (0 + 1) - (-1 + 0)$$

$$= 1 + 1 = 2.$$

(*v*) Let $\quad I = \int_0^{\pi} \cos x\, dx$

$$= \int_0^{\pi} \cos x\, dx = \Big[\sin x\Big]_0^{\pi} = \sin \pi - \sin 0 \qquad [\because \ \sin 180° = 0]$$

$$= 0.$$

(*vi*) Let $\quad I = \int_0^{\pi/2} \cos 2x\, dx$

$$= \int_0^{\pi/2} \cos 2x\, dx = \left[\frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$= \frac{1}{2}[\sin \pi - \sin 0] = 0.$$

Example 4. *Evaluate the following definite integrals :*

(*i*) $\int_0^{\pi/2} \sin^2 x\, dx$ (*ii*) $\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin^2 x}\, dx$

(*iii*) $\int_0^{\pi/2} \sin 2x\, dx$ (*iv*) $\int_0^{\pi/4} \frac{1}{\cos^2 x}\, dx$

(*v*) $\int_4^5 e^x\, dx$ (*vi*) $\int_0^{\pi/2} \frac{\sin x}{\sqrt{1 + \cos x}}\, dx.$

Solution. (*i*) Let $I = \int_0^{\pi/2} \sin^2 x \, dx$

$$= \int_0^{\pi/2} \left(\frac{1 - \cos 2x}{2} \right) dx \qquad \left[\begin{array}{l} \because \ 1 - \cos 2A = 2 \sin^2 A \\ \Rightarrow \ \dfrac{1 - \cos 2A}{2} = \sin^2 A \end{array} \right]$$

$$= \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx = \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi/2}$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - \frac{\sin 2\left(\frac{\pi}{2}\right)}{2} \right] - \frac{1}{2} \left[0 - \frac{\sin 0}{2} \right] = \frac{1}{2} \left[\frac{\pi}{2} - \frac{\sin \pi}{2} \right] - 0$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{4} . \qquad [\because \ \sin \pi = \sin 180^\circ = 0]$$

(*ii*) Let $I = \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin^2 x} \, dx$

$$= \int_{\pi/6}^{\pi/2} \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \, dx = \int_{\pi/6}^{\pi/2} \operatorname{cosec} x \cot x \, dx$$

$$= \Big[- \operatorname{cosec} x \Big]_{\pi/6}^{\pi/2} = - \left[\operatorname{cosec} \frac{\pi}{2} - \operatorname{cosec} \frac{\pi}{6} \right]$$

$$= - [1 - 2] = 1.$$

(*iii*) Let $I = \int_0^{\pi/2} \sin 2x \, dx$

$$= \int_0^{\pi/2} \sin 2x \, dx = \left[- \frac{\cos 2x}{2} \right]_0^{\pi/2} = - \frac{1}{2} \left[\cos 2\left(\frac{\pi}{2}\right) - \cos 2\,(0) \right]$$

$$= - \frac{1}{2} \, [\cos \pi - \cos 0]$$

$$= - \frac{1}{2} \, (- 1 - 1) = 1. \qquad [\because \ \cos \pi = \cos 180^\circ = -1]$$

(*iv*) Let $I = \int_0^{\pi/4} \frac{1}{\cos^2 x} \, dx$

$$= \int_0^{\pi/4} \sec^2 x \, dx = [\tan x]_0^{\pi/4} = \left[\tan \frac{\pi}{4} - \tan 0 \right]$$

$$= 1 - 0 = 1. \qquad \left[\because \ \tan \frac{\pi}{4} = \tan 45^\circ = 1 \right]$$

(*v*) Let $\quad I = \int_4^5 e^x \, dx$

$$= \int_4^5 e^x \, dx = \left[e^x\right]_4^5 = e^5 - e^4$$

$$= e^4 (e - 1).$$

(*vi*) Let $\quad I = \int_0^{\pi/2} \frac{\sin x}{\sqrt{1 + \cos x}} \, dx$

$$= \int_0^{\pi/2} \frac{\sin x}{\sqrt{1 + \cos x}} \, dx = \int_0^{\pi/2} \frac{\sin x}{\sqrt{2\cos^2 \frac{x}{2}}} \, dx \qquad \left[\begin{array}{l} \because \; 1 + \cos 2A = 2\cos^2 A \\ \Rightarrow 1 + \cos A = 2\cos^2 \frac{A}{2} \end{array}\right]$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\cos \frac{x}{2}} \, dx \qquad \left[\begin{array}{l} \because \; \sin 2A = 2 \sin A \cos A \\ \Rightarrow \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \end{array}\right]$$

$$= \frac{2}{\sqrt{2}} \int_0^{\pi/2} \sin \frac{x}{2} \, dx = \sqrt{2} \left[-\frac{\cos x/2}{1/2} \right]_0^{\pi/2} = -2\sqrt{2} \left[\cos \frac{x}{2} \right]_0^{\pi/2}$$

$$= -2\sqrt{2} \left[\cos \frac{\pi}{4} - \cos 0° \right] = -2\sqrt{2} \left[\frac{1}{\sqrt{2}} - 1 \right] \qquad \left[\because \; \cos \frac{\pi}{4} = \cos 45° = \frac{1}{\sqrt{2}} \right]$$

$$= -2 + 2\sqrt{2}.$$

Example 5. *Evaluate the following definite integrals :*

(*i*) $\int_0^{\pi/2} \sin x \sin 2x \, dx$ (*ii*) $\int_0^1 \frac{1}{1 + x^2} \, dx$

(*iii*) $\int_2^3 \frac{1}{x^2 - 1} \, dx$ (*iv*) $\int_0^1 \frac{1}{\sqrt{1 - x^2}} \, dx$

(*v*) $\int_3^4 \frac{1}{x^2 - 4} \, dx$ (*vi*) $\int_0^{\pi/2} \frac{\sin^2 x}{(1 + \cos x)^2} \, dx.$

Solution. (*i*) Let $I = \int_0^{\pi/2} \sin x \sin 2x \, dx$ [Multiply and divided by 2]

$$= \frac{1}{2} \int_0^{\pi/2} 2 \sin 2x \sin x \, dx$$

$$= \frac{1}{2} \int_0^{\pi/2} [\cos (2x - x) - \cos (2x + x)] \, dx$$

$$[\because \; 2 \sin A \sin B = \cos (A - B) - \cos (A + B)]$$

$$= \frac{1}{2} \int_0^{\pi/2} (\cos x - \cos 3x) \, dx = \frac{1}{2} \left[\sin x - \frac{\sin 3x}{3} \right]_0^{\pi/2}$$

$$= \frac{1}{2} \left[\sin \frac{\pi}{2} - \frac{\sin 3(\pi/2)}{3} \right] - \frac{1}{2} \left[\sin 0° - \frac{\sin 0°}{3} \right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{3}(-1)\right] - \frac{1}{2}[0 - 0] \qquad \left[\because \sin\frac{3\pi}{2} = \sin\left(\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1\right]$$

$$= \frac{1}{2}\left[\frac{4}{3}\right] = \frac{2}{3}.$$

(*ii*) Let $\quad I = \int_0^1 \frac{1}{1+x^2}\,dx$

$$= \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1 \qquad \left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= [\tan^{-1}(1) - \tan^{-1}(0)] = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

(*iii*) Let $\quad I = \int_2^3 \frac{1}{x^2-1}\,dx$

$$= \int_2^3 \frac{1}{x^2-1}\,dx = \left[\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right]_2^3 \qquad \left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{3-1}{3+1}\right) - \log\left(\frac{2-1}{2+1}\right)\right] = \frac{1}{2}\left[\log\left(\frac{2}{4}\right) - \log\left(\frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{1}{2}\right) - \log\left(\frac{1}{3}\right)\right] = \frac{1}{2}\log\left(\frac{1}{2}\times\frac{3}{1}\right) \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{1}{2}\log\frac{3}{2}.$$

(*iv*) Let $\quad I = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1} x\right]_0^1 \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}.$$

(*v*) Let $\quad I = \int_3^4 \frac{1}{x^2-4}\,dx$

$$= \int_3^4 \frac{1}{x^2-2^2}\,dx = \frac{1}{2(2)}\left[\log\left|\frac{x-2}{x+2}\right|\right]_3^4 \qquad \left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{4}\left[\log\left(\frac{4-2}{4+2}\right) - \log\left(\frac{3-2}{3+2}\right)\right] = \frac{1}{4}\left[\log\frac{1}{3} - \log\frac{1}{5}\right]$$

$$= \frac{1}{4}\log\left[\frac{1}{3}\times\frac{5}{1}\right] \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{1}{4}\log\frac{5}{3}.$$

(*vi*) Let $\mathrm{I} = \int_0^{\pi/2} \frac{\sin^2 x}{(1+\cos x)^2}\,dx = \int_0^{\pi/2}\left(\frac{\sin x}{1+\cos x}\right)^2 dx$

$$= \int_0^{\pi/2}\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)^2 dx \qquad \left[\begin{array}{l}\because \sin 2\mathrm{A} = 2\sin\mathrm{A}\cos\mathrm{A} \\ \Rightarrow \sin\mathrm{A} = 2\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{A}}{2} \\ \because 1+\cos 2\mathrm{A} = 2\cos^2\mathrm{A} \\ \Rightarrow 1+\cos\mathrm{A} = 2\cos^2\frac{\mathrm{A}}{2}\end{array}\right]$$

$$= \int_0^{\pi/2}\tan^2\frac{x}{2}\,dx = \int_0^{\pi/2}\left(\sec^2\frac{x}{2} - 1\right)dx \qquad [\because \sec^2\mathrm{A} - \tan^2\mathrm{A} = 1]$$

$$= \left[\frac{\tan\frac{x}{2}}{1/2} - x\right]_0^{\pi/2} = \left[2\tan\frac{x}{2} - x\right]_0^{\pi/2} = \left[2\tan\left(\frac{\pi/2}{2}\right) - \frac{\pi}{2}\right] - [2\tan 0^\circ - 0^\circ]$$

$$= \left[2\tan\frac{\pi}{4} - \frac{\pi}{2}\right] - 0$$

$$= 2 - \frac{\pi}{2}. \qquad \left[\because \tan\frac{\pi}{4} = \tan 45^\circ = 1\right]$$

Example 6. *Evaluate the following definite integrals :*

(*i*) $\int_0^{\pi/4} \tan^2 x\,dx$ (*ii*) $\int_0^{\pi/2} \cos^2 x\,dx$

(*iii*) $\int_0^{\pi/2} \cos^4 x\,dx$ (*iv*) $\int_0^{\pi/4} \sin^4 x\,dx$

(*v*) $\int_0^{\infty} \frac{1}{a^2+b^2x^2}\,dx$ (*vi*) $\int_0^{\pi/6} \cos x\cos 2x\,dx$.

Solution. (*i*) Let $\mathrm{I} = \int_0^{\pi/4}\tan^2 x\,dx$

$$= \int_0^{\pi/4}(\sec^2 x - 1)\,dx \qquad [\because \sec^2\mathrm{A} - \tan^2\mathrm{A} = 1]$$

$$= [\tan x - x]_0^{\pi/4} = \left[\tan\frac{\pi}{4} - \frac{\pi}{4}\right] - [\tan 0^\circ - 0^\circ]$$

$$= \left(1 - \frac{\pi}{4}\right). \qquad \left[\because \tan\frac{\pi}{4} = \tan 45^\circ = 1\right]$$

(*ii*) Let $I = \int_0^{\pi/2} \cos^2 x\, dx = \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right) dx$ $\left[\begin{array}{l} \because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ \dfrac{1+\cos 2A}{2} = \cos^2 A \end{array}\right]$

$$= \frac{1}{2}\int_0^{\pi/2} (1+\cos 2x)\, dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin 0^\circ}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0\right] = \frac{\pi}{4}.$$ $[\because\ \sin \pi = \sin 180^\circ = 0]$

(*iii*) Let $I = \int_0^{\pi/2} \cos^4 x\, dx$ $\left[\begin{array}{l} \because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ \dfrac{1+\cos 2A}{2} = \cos^2 A \end{array}\right]$

$$= \int_0^{\pi/2} (\cos^2 x)^2\, dx = \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 + \cos^2 2x + 2\cos 2x)\, dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 + \frac{1+\cos 4x}{2} + 2\cos 2x\right) dx$$ $\left[\begin{array}{l} \because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ 1+\cos 4A = 2\cos^2 2A \\ \Rightarrow\ \left(\dfrac{1+\cos 4A}{2}\right) = \cos^2 2A \end{array}\right]$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 + \frac{1}{2} + \frac{1}{2}\cos 4x + 2\cos 2x\right) dx = \frac{1}{4}\int_0^{\pi/2} \left(\frac{3}{2} + \frac{1}{2}\cos 4x + 2\cos 2x\right) dx$$

$$= \frac{3}{8}\int_0^{\pi/2} 1 . dx + \frac{1}{8}\int_0^{\pi/2} \cos 4x\, dx + \frac{1}{2}\int_0^{\pi/2} \cos 2x\, dx$$

$$= \frac{3}{8}\Big[x\Big]_0^{\pi/2} + \frac{1}{8}\left[\frac{\sin 4x}{4}\right]_0^{\pi/2} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{3}{8}\left(\frac{\pi}{2} - 0\right) + \frac{1}{32}[\sin 2\pi - \sin 0^\circ] + \frac{1}{4}[\sin \pi - \sin 0^\circ]$$

$$= \frac{3\pi}{16} + \frac{1}{32}(0-0) + \frac{1}{4}(0-0)$$

$$= \frac{3\pi}{16}.$$

(*iv*) Let $I = \int_0^{\pi/4} \sin^4 x\, dx = \int_0^{\pi/4} (\sin^2 x)^2\, dx$

$$= \int_0^{\pi/4} \left(\frac{1-\cos 2x}{2}\right)^2 dx$$ $\left[\begin{array}{l} \because\ 1-\cos 2A = 2\sin^2 A \\ \Rightarrow\ \left(\dfrac{1-\cos 2A}{2}\right) = \sin^2 A \end{array}\right]$

$$= \frac{1}{4}\int_0^{\pi/4}(1 + \cos^2 2x - 2\cos 2x)\,dx$$

$$= \frac{1}{4}\int_0^{\pi/4}\left[1 + \frac{1+\cos 4x}{2} - 2\cos 2x\right]dx \qquad \left[\begin{array}{l} \because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ 1+\cos 4A = 2\cos^2 2A \\ \Rightarrow\ \left(\dfrac{1+\cos 4A}{2}\right) = \cos^2 2A \end{array}\right]$$

$$= \frac{1}{4}\int_0^{\pi/4}\left[1 + \frac{1}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right]dx = \frac{1}{4}\int_0^{\pi/4}\left(\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right)dx$$

$$= \frac{3}{8}\int_0^{\pi/4} 1.\,dx + \frac{1}{8}\int_0^{\pi/4}\cos 4x\,dx - \frac{1}{2}\int_0^{\pi/4}\cos 2x\,dx$$

$$= \frac{3}{8}[x]_0^{\pi/4} - \frac{1}{8}\left[\frac{\sin 4x}{4}\right]_0^{\pi/4} - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{3}{8}\left(\frac{\pi}{4} - 0\right) - \frac{1}{32}[\sin\pi - \sin 0^\circ] - \frac{1}{4}\left[\sin\frac{\pi}{2} - \sin 0^\circ\right]$$

$$= \frac{3\pi}{32} - \frac{1}{32}(0-0) - \frac{1}{4}(1-0)$$

$$= \frac{3\pi}{32} - \frac{1}{4}.$$

(*v*) Let $\quad I = \int_0^\infty \frac{1}{a^2 + b^2x^2}\,dx = \frac{1}{b^2}\int_0^\infty \frac{1}{\left(\frac{a}{b}\right)^2 + x^2}\,dx$

$$= \frac{1}{b^2}\cdot\left[\frac{1}{\left(\frac{a}{b}\right)}\tan^{-1}\frac{x}{\frac{a}{b}}\right]_0^\infty \qquad \left[\because\ \int\frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{b^2}\times\frac{b}{a}\left[\tan^{-1}\frac{bx}{a}\right]_0^\infty = \frac{1}{ab}[\tan^{-1}\infty - \tan^{-1}0] = \frac{1}{ab}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{2ab}.$$

(*vi*) Let $\quad I = \int_0^{\pi/6}\cos x\cos 2x\,dx$

$$= \frac{1}{2}\int_0^{\pi/6} 2\cos 2x\cos x\,dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int_0^{\pi/6}[\cos(2x+x) + \cos(2x-x)]\,dx$$

$[\because\ 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$

$$= \frac{1}{2}\int_0^{\pi/6} (\cos 3x + \cos x)\, dx$$

$$= \frac{1}{2}\left[\frac{\sin 3x}{3}\right]_0^{\pi/6} + \frac{1}{2}[\sin x]_0^{\pi/6}$$

$$= \frac{1}{6}\left[\sin 3\left(\frac{\pi}{6}\right) - \sin 3(0^\circ)\right] + \frac{1}{2}\left[\sin\frac{\pi}{6} - \sin 0^\circ\right]$$

$$= \frac{1}{6}[1-0] + \frac{1}{2}\left[\frac{1}{2} - 0\right] = \frac{1}{6} + \frac{1}{4} \qquad \left[\begin{array}{l} \because\ \sin\frac{3\pi}{6} = \sin\frac{\pi}{2} = 1 \\ \sin\frac{\pi}{6} = \sin 30^\circ = \frac{1}{2} \end{array}\right]$$

$$= \frac{2+3}{12} = \frac{5}{12}.$$

Example 7. *Evaluate the following definite integrals :*

(i) $\int_0^{\pi} \sin^3 x\, dx$ (ii) $\int_0^{\pi/2} \cos^3 x\, dx$

(iii) $\int_0^{\pi/2} \sin^4 x\, dx$ (iv) $\int_0^{\pi}\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right) dx.$

Solution. (i) Let $I = \int_0^{\pi} \sin^3 x\, dx$

$$= \int_0^{\pi} \frac{1}{4}\,[3\sin x - \sin 3x]\, dx \qquad \left[\begin{array}{l} \because\ \sin 3A = 3\sin A - 4\sin^3 A \\ \Rightarrow\ 4\sin^3 A = 3\sin A - \sin 3A \\ \Rightarrow\ \sin^3 A = \frac{1}{4}[3\sin A - \sin 3A] \end{array}\right]$$

$$= \frac{1}{4}\int_0^{\pi} (3\sin x - \sin 3x)\, dx = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\left(-3\cos\pi + \frac{\cos 3\pi}{3}\right) - \left(-3\cos 0^\circ + \frac{\cos 0^\circ}{3}\right)\right]$$

$$= \frac{1}{4}\left[-3(-1) + \frac{(-1)}{3}\right] - \frac{1}{4}\left[-3 + \frac{1}{3}\right]$$

$$= \frac{1}{4}\left[3 - \frac{1}{3}\right] - \frac{1}{4}\left[\frac{-9+1}{3}\right] = \frac{1}{4}\left[\frac{9-1}{3}\right] - \frac{1}{4}\left[-\frac{8}{3}\right]$$

$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$$

(ii) Let $I = \int_0^{\pi/2} \cos^3 x\, dx$

$$= \int_0^{\pi/2} \frac{1}{4}\,[\cos 3x + 3\cos x]\, dx \qquad \left[\begin{array}{l} \because\ \cos 3A = 4\cos^3 A - 3\cos A \\ \Rightarrow\ 4\cos^3 A = \cos 3A + 3\cos A \\ \Rightarrow\ \cos^3 A = \frac{1}{4}[\cos 3A + 3\cos A] \end{array}\right]$$

$$= \frac{1}{4}\int_0^{\pi/2} (\cos 3x + 3\cos x)\, dx$$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(\frac{\sin 3\left(\frac{\pi}{2}\right)}{3} + 3\sin\pi\right) - \left(\frac{\sin 0^\circ}{3} + 3\sin 0^\circ\right)\right]$$

$$\left[\because\ \sin\frac{3\pi}{2} = \sin\left(\pi + \frac{\pi}{2}\right) = -1\right]$$

$$= \frac{1}{4}\left[\left(-\frac{1}{3} + 3\right) - (0+0)\right] = \frac{1}{4}\left[\frac{-1+9}{3}\right]$$

$$= \frac{2}{3}.$$

(*iii*) Let $I = \int_0^{\pi/2} \sin^4 x\, dx = \int_0^{\pi/2} (\sin^2 x)^2\, dx$

$$= \int_0^{\pi/2}\left(\frac{1-\cos 2x}{2}\right)^2 dx \qquad \left[\begin{array}{l}\because\ 1-\cos 2A = 2\sin^2 A\\ \Rightarrow \left(\dfrac{1-\cos 2A}{2}\right) = \sin^2 A\end{array}\right]$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 + \cos^2 2x - 2\cos 2x)\, dx$$

$$= \frac{1}{4}\int_0^{\pi/2}\left(1 + \frac{1+\cos 4x}{2} - 2\cos 2x\right) dx \qquad \left[\begin{array}{l}\because\ 1+\cos 2A = 2\cos^2 A\\ \Rightarrow 1+\cos 4A = 2\cos^2 2A\\ \Rightarrow \left(\dfrac{1+\cos 4A}{2}\right) = \cos^2 2A\end{array}\right]$$

$$= \frac{1}{4}\int_0^{\pi/2}\left(1 + \frac{1}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right) dx = \frac{1}{4}\int_0^{\pi/2}\left(\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right) dx$$

$$= \frac{3}{8}\int_0^{\pi/2} 1.\, dx + \frac{1}{8}\int_0^{\pi/2} \cos 4x\, dx - \frac{1}{2}\int_0^{\pi/2} \cos 2x\, dx$$

$$= \frac{3}{8}\left[x\right]_0^{\pi/2} + \frac{1}{8}\left[\frac{\sin 4x}{4}\right]_0^{\pi/2} - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{3}{8}\left[\frac{\pi}{2} - 0\right] + \frac{1}{32}\left[\sin 4\left(\frac{\pi}{2}\right) - \sin 0^\circ\right] - \frac{1}{4}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0^\circ\right]$$

$$= \frac{3\pi}{16} + \frac{1}{32}[0-0] - \frac{1}{4}[0-0]$$

$$= \frac{3\pi}{16}.$$

(*iv*) Let $\quad I = \int_0^\pi \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$

$$= \int_0^\pi \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx = \int_0^\pi (-\cos x)\, dx$$

$$= -\int_0^\pi \cos x \, dx \qquad \left[\begin{array}{l} \because \cos 2A = \cos^2 A - \sin^2 A \\ \Rightarrow \cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \\ \Rightarrow (-\cos A) = \left(\sin^2 \frac{A}{2} - \cos^2 \frac{A}{2} \right) \end{array} \right]$$

$$= -[\sin x]_0^\pi = -[\sin \pi - \sin 0^\circ] = -[0 - 0]$$

$$= 0.$$

Example 8. *Evaluate the following definite integrals :*

(*i*) $\int_0^{\pi/2} \sin^3 x \, dx$ (*ii*) $\int_0^{\pi/4} 2 \tan^3 x \, dx$

(*iii*) $\int_0^{\pi/4} \sqrt{1 - \sin 2x} \, dx$ (*iv*) $\int_0^{\pi/2} \sqrt{1 + \sin x} \, dx$

(*v*) $\int_{\pi/6}^{\pi/4} \text{cosec } x \, dx$ (*vi*) $\int_0^1 \frac{1-x}{1+x} \, dx.$

Solution. (*i*) Let $I = \int_0^{\pi/2} \sin^3 x \, dx$

$$= \int_0^{\pi/2} \frac{1}{4} [3 \sin x - \sin 3x] \, dx$$

$$= \frac{1}{4} \int_0^{\pi/2} (3 \sin x - \sin 3x) \, dx \qquad \left[\begin{array}{l} \because \sin 3A = 3 \sin A - 4 \sin^3 A \\ \Rightarrow 4 \sin^3 A = 3 \sin A - \sin 3A \\ \Rightarrow \sin^3 A = \frac{1}{4} [3 \sin A - \sin 3A \end{array} \right]$$

$$= \frac{3}{4} \int_0^{\pi/2} \sin x \, dx - \frac{1}{4} \int_0^{\pi/2} \sin 3x \, dx = \frac{3}{4} [-\cos x]_0^{\pi/2} - \frac{1}{4} \left[-\frac{\cos 3x}{3} \right]_0^{\pi/2}$$

$$= -\frac{3}{4} \left[\cos \frac{\pi}{2} - \cos 0^\circ \right] + \frac{1}{12} \left[\cos 3\left(\frac{\pi}{2}\right) - \cos 0^\circ \right]$$

$$= -\frac{3}{4} [0 - 1] + \frac{1}{12} [0 - 1] = \frac{3}{4} - \frac{1}{12} = \frac{9 - 1}{12} = \frac{8}{12}$$

$$= \frac{2}{3}.$$

(*ii*) Let $\quad I = \int_0^{\pi/4} 2\tan^3 x\, dx = 2\int_0^{\pi/4} \tan x \,.\, \tan^2 x\, dx$

$= 2\int_0^{\pi/4} \tan x\,(\sec^2 x - 1)\, dx \qquad [\because \sec^2 A - \tan^2 A = 1]$

$= 2\int_0^{\pi/4} \tan x \sec^2 x\, dx - 2\int_0^{\pi/4} \tan x\, dx$

$= 2\left[\dfrac{\tan^2 x}{2}\right]_0^{\pi/4} - 2\Big[\log|\sec x|\Big]_0^{\pi/4} \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \dfrac{[f(x)]^{n+1}}{n+1} + c\right]$

$= \left[\tan^2\left(\dfrac{\pi}{4}\right) - \tan^2 0^\circ\right] - 2\left[\log\left|\sec\left(\dfrac{\pi}{4}\right)\right| - \log|\sec 0^\circ|\right]$

$= [(1)^2 - 0] - 2\,[\log(\sqrt{2}) - \log 1]$

$= 1 - 2\log\sqrt{2} + 0 = 1 - 2\log\sqrt{2} = 1 - \log\left(\sqrt{2}\right)^2 \qquad [\because m\log n = \log n^m]$

$= 1 - \log 2.$

(*iii*) Let $\quad I = \int_0^{\pi/4} \sqrt{1 - \sin 2x}\; dx \qquad \left[\begin{array}{l} \because \cos^2 A + \sin^2 A = 1 \\ \sin 2A = 2\sin A\cos A \end{array}\right]$

$= \int_0^{\pi/4} \sqrt{1 - \sin 2x}\; dx = \int_0^{\pi/4} \sqrt{\cos^2 x + \sin^2 x - 2\sin x\cos x}\;.\, dx$

$= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2}\; dx$

$= \int_0^{\pi/4} (\cos x - \sin x)\, dx$

$= \Big[(\sin x + \cos x)\Big]_0^{\pi/4} \qquad \left[\begin{array}{l} \because \sin\dfrac{\pi}{4} = \sin 45^\circ = \dfrac{1}{\sqrt{2}} \\ \cos\dfrac{\pi}{4} = \cos 45^\circ = \dfrac{1}{\sqrt{2}} \end{array}\right]$

$= \left[\left(\sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4}\right) - (\sin 0^\circ + \cos 0^\circ)\right]$

$= \left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\right) - (0 + 1) = \dfrac{2}{\sqrt{2}} - 1 \;=\; \sqrt{2} - 1.$

(*iv*) Let $\quad I = \int_0^{\pi/2} \sqrt{1 + \sin x}\; dx = \int_0^{\pi/2} \sqrt{1 + \sin x}\; dx$

$= \int_0^{\pi/2} \sqrt{\left(\cos^2\dfrac{x}{2} + \sin^2\dfrac{x}{2} + 2\sin\dfrac{x}{2}\cos\dfrac{x}{2}\right)}\; dx \qquad \left[\begin{array}{l} \because \cos^2 A + \sin^2 A = 1 \\ \Rightarrow \cos^2\dfrac{A}{2} + \sin^2\dfrac{A}{2} = 1 \\ \sin 2A = 2\sin A\cos A \\ \Rightarrow \sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} \end{array}\right]$

$$= \int_0^{\pi/2} \sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\, dx = \int_0^{\pi/2} \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) . \,dx$$

$$= \int_0^{\pi/2} \cos\frac{x}{2}\, dx + \int_0^{\pi/2} \sin\frac{x}{2}\, dx = \left[\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} + \left[\frac{-\cos\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2}$$

$$= 2\left[\sin\frac{\pi}{4} - \sin 0^\circ\right] - 2\left[\cos\frac{\pi}{4} - \cos 0^\circ\right]$$

$$= 2\left[\frac{1}{\sqrt{2}} - 0\right] - 2\left[\frac{1}{\sqrt{2}} - 1\right] \qquad \left[\begin{array}{l} \because\ \sin\frac{\pi}{4} = \sin 45^\circ = \frac{1}{\sqrt{2}} \\ \cos\frac{\pi}{4} = \cos 45^\circ = \frac{1}{\sqrt{2}} \end{array}\right]$$

$$= \sqrt{2} - \sqrt{2} + 2 = 2.$$

(*v*) Let $\quad I = \int_{\pi/6}^{\pi/4} \operatorname{cosec} x\, dx$

$$= \int_{\pi/6}^{\pi/4} \operatorname{cosec} x\, dx = \Big[\log|\operatorname{cosec} x - \cot x|\Big]_{\pi/6}^{\pi/4}$$

$$= \left[\log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right|\right] - \left[\log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|\right]$$

$$= \log|\sqrt{2} - 1| - \log|2 - \sqrt{3}|$$

$$= \log\left|\frac{\sqrt{2}-1}{2-\sqrt{3}}\right|. \qquad \left[\because\ \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

Alternative Method :

$$\int_{\pi/6}^{\pi/4} \operatorname{cosec} x\, dx = \Big[\log|\operatorname{cosec} x - \cot x|\Big]_{\pi/6}^{\pi/4}$$

$$= \left[\log\left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right|\right]_{\pi/6}^{\pi/4} = \left[\log\left|\frac{1-\cos x}{\sin x}\right|\right]_{\pi/6}^{\pi/4}$$

$$= \log\left|\frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}\right| - \log\left|\frac{1-\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}\right|$$

$$= \log\left|\frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right| - \log\left|\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right| = \log|\sqrt{2}-1| - \log|2-\sqrt{3}|$$

$$= \log\left|\frac{\sqrt{2}-1}{2-\sqrt{3}}\right|. \qquad \left[\because\ \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

(*vi*) Let $\quad \mathrm{I} = \int_0^1 \frac{1-x}{1+x}\, dx = -\int_0^1 \frac{x-1}{x+1}\, dx$

$= -\int_0^1 \left(\frac{x+1-2}{x+1}\right) dx$ [Add and subtract 1 to the numerator]

$= -\int_0^1 \left[\frac{x+1}{x+1} - \frac{2}{x+1}\right] dx = -\int_0^1 1\, dx + 2\int_0^1 \frac{1}{x+1}\, dx$

$= -\Big[x\Big]_0^1 + 2\Big[\log|x+1|\Big]_0^1$

$= -(1-0) + 2\,[\log(1+1) - \log(0+1)]$

$= -1 + 2\log 2 - 2\log 1$ $\qquad [\because \log 1 = 0]$

$= -1 + 2\log 2 = -1 + \log(2)^2$ $\qquad [\because m\log n = \log n^m]$

$= -1 + \log 4.$

Example 9. *Evaluate the following definite integrals :*

(*i*) $\int_{\pi/4}^{\pi/2} \sqrt{1+\sin 2x}\, dx$ $\qquad$ (*ii*) $\int_0^{\pi/2} \sqrt{1+\cos x}\, dx$

(*iii*) $\int_{\pi/4}^{\pi/2} \cot x\, dx$ $\qquad$ (*iv*) $\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\, dx$

(*v*) $\int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx$ $\qquad$ (*vi*) $\int_0^{\pi/2} \sqrt{1-\cos 2x}\, dx.$

Solution. (*i*) Let $\mathrm{I} = \int_{\pi/4}^{\pi/2} \sqrt{1+\sin 2x}\, dx$

$= \int_{\pi/4}^{\pi/2} \sqrt{\cos^2 x + \sin^2 x + 2\sin x\cos x}\, dx$ $\qquad \left[\begin{array}{l} \because \cos^2 \mathrm{A} + \sin^2 \mathrm{A} = 1 \\ \sin 2\mathrm{A} = 2\sin \mathrm{A}\cos \mathrm{A} \end{array}\right]$

$= \int_{\pi/4}^{\pi/2} \sqrt{(\cos x + \sin x)^2}\, dx = \int_{\pi/4}^{\pi/2} (\cos x + \sin x)\, dx$

$= \int_{\pi/4}^{\pi/2} \cos x\, dx + \int_{\pi/4}^{\pi/2} \sin x\, dx$

$= \Big[\sin x\Big]_{\pi/4}^{\pi/2} + \Big[-\cos x\Big]_{\pi/4}^{\pi/2}$

$= \left[\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right] - \left[\cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right] = \left(1 - \frac{1}{\sqrt{2}}\right) - \left(0 - \frac{1}{\sqrt{2}}\right)$

$= 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 1.$

(*ii*) Let $\quad \mathrm{I} = \int_0^{\pi/2} \sqrt{1+\cos x}\, dx$

$= \int_0^{\pi/2} \sqrt{2\cos^2 \frac{x}{2}}\, dx$ $\qquad \left[\begin{array}{l} \because 1 + \cos 2\mathrm{A} = 2\cos^2 \mathrm{A} \\ \Rightarrow 1 + \cos \mathrm{A} = 2\cos^2 \dfrac{\mathrm{A}}{2} \end{array}\right]$

$$= \sqrt{2}\int_0^{\pi/2} \cos\frac{x}{2}\, dx = \sqrt{2}\left[\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} = 2\sqrt{2}\left[\sin\left(\frac{\pi/2}{2}\right) - \sin 0^\circ\right]$$

$$= 2\sqrt{2}\left[\sin\frac{\pi}{4} - \sin 0^\circ\right] = 2\sqrt{2}\left[\frac{1}{\sqrt{2}} - 0\right]$$

$$= 2.$$

(iii) Let $\quad I = \int_{\pi/4}^{\pi/2} \cot x\, dx$

$$= \int_{\pi/4}^{\pi/2} \cot x\, dx = \Big[\log|(\sin x)|\Big]_{\pi/4}^{\pi/2} = \left[\log\left(\sin\frac{\pi}{2}\right) - \log\left(\sin\frac{\pi}{4}\right)\right]$$

$$= \left[\log 1 - \log\frac{1}{\sqrt{2}}\right] = 0 - \log(2)^{-1/2} = -\left(-\frac{1}{2}\right)\log 2 \qquad [\because\ n\log m = \log m^n]$$

$$= \frac{1}{2}\log 2.$$

(iv) Let $\quad I = \int_{\pi/4}^{\pi/2} \sqrt{1 - \sin 2x}\, dx \qquad \left[\begin{array}{l}\because\ \cos^2 A + \sin^2 A = 1 \\ \sin 2A = 2\sin A\cos A\end{array}\right]$

$$= \int_{\pi/4}^{\pi/2} \sqrt{\cos^2 x + \sin^2 x - 2\sin x\cos x}\, dx$$

$$= \int_{\pi/4}^{\pi/2} \sqrt{(\cos x - \sin x)^2}\, dx = \int_{\pi/4}^{\pi/2} |\cos x - \sin x|\, dx \qquad \left[\because\ \sqrt{x^2} = |x|\right]$$

$$= \int_{\pi/4}^{\pi/2} -(\cos x - \sin x)\, dx \qquad \left[\begin{array}{l}\because\ \cos x < \sin x \quad \text{for } \frac{\pi}{4} < x < \frac{\pi}{2} \\ \therefore\ \cos x - \sin x < 0 \\ \Rightarrow |\cos x - \sin x| = -(\cos x - \sin x)\end{array}\right]$$

$$= \int_{\pi/4}^{\pi/2} \sin x\, dx - \int_{\pi/4}^{\pi/2} \cos x\, dx$$

$$= \Big[-\cos x\Big]_{\pi/4}^{\pi/2} - \Big[\sin x\Big]_{\pi/4}^{\pi/2} = -\left[\cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right] - \left[\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right]$$

$$= -\left[0 - \frac{1}{\sqrt{2}}\right] - \left[1 - \frac{1}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} - 1$$

$$= \sqrt{2} - 1.$$

(v) Let $\quad I = \int_0^{\pi/4} \sqrt{1 + \sin 2x}\, dx$

$$= \int_0^{\pi/4} \sqrt{\cos^2 x + \sin^2 x + 2\sin x\cos x}\, dx \qquad \left[\begin{array}{l}\because\ \cos^2 A + \sin^2 A = 1 \\ \sin 2A = 2\sin A\cos A\end{array}\right]$$

$$= \int_0^{\pi/4} \sqrt{(\cos x + \sin x)^2}\, dx = \int_0^{\pi/4} (\cos x + \sin x)\, dx$$

$$= \int_0^{\pi/4} \cos x\, dx + \int_0^{\pi/4} \sin x\, dx = \Big[\sin x\Big]_0^{\pi/4} + \Big[-\cos x\Big]_0^{\pi/4}$$

$$= \left[\sin\frac{\pi}{4} - \sin 0^\circ\right] - \left[\cos\frac{\pi}{4} - \cos 0^\circ\right] = \left(\frac{1}{\sqrt{2}} - 0\right) - \left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1 = 1.$$

(*vi*) Let $I = \int_0^{\pi/2} \sqrt{1 - \cos 2x}\, dx$

$$= \int_0^{\pi/2} \sqrt{2\sin^2 x}\, dx = \sqrt{2}\int_0^{\pi/2} \sin x\, dx \qquad [\because\ 1 - \cos 2A = 2\sin^2 A]$$

$$= \sqrt{2}\Big[-\cos x\Big]_0^{\pi/2} = -\sqrt{2}\Big[\cos x\Big]_0^{\pi/2} = -\sqrt{2}\left[\cos\frac{\pi}{2} - \cos 0^\circ\right]$$

$$= -\sqrt{2}\,[0 - 1] = \sqrt{2}.$$

Example 10. *Evaluate the following definite integrals :*

(*i*) $\int_0^{\pi} \frac{1}{1 + \sin x}\, dx$ (*ii*) $\int_0^{\pi/4} \sin 2x \sin 3x\, dx$

(*iii*) $\int_{-\pi/4}^{+\pi/4} \frac{1}{1 + \sin x}\, dx.$

Solution. (*i*) Let $I = \int_0^{\pi} \frac{1}{1 + \sin x}\, dx$

$$= \int_0^{\pi} \frac{1}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x}\, dx \qquad \text{[Rationalization]}$$

$$= \int_0^{\pi} \frac{1 - \sin x}{1 - \sin^2 x}\, dx = \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x}\, dx \qquad [\because\ \cos^2 A + \sin^2 A = 1]$$

$$= \int_0^{\pi} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}\right) dx = \int_0^{\pi} (\sec^2 x - \tan x \sec x)\, dx$$

$$= \int_0^{\pi} \sec^2 x\, dx - \int_0^{\pi} \tan x \sec x\, dx$$

$$= \Big[\tan x\Big]_0^{\pi} - \Big[\sec x\Big]_0^{\pi} = (\tan \pi - \tan 0^\circ) - (\sec \pi - \sec 0^\circ)$$

$$= (0 - 0) - (-1 - 1) = -(-2)$$

$$= 2.$$

(*ii*) Let $I = \int_0^{\pi/4} \sin 2x \sin 3x\, dx$

$$= \frac{1}{2}\int_0^{\pi/4} 2 \sin 3x \sin 2x\, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int_0^{\pi/4} [\cos(3x - 2x) - \cos(3x + 2x)]\, dx \qquad \left[\because\ 2\sin A \sin B = \cos(A - B) - \cos(A + B)\right]$$

$$= \frac{1}{2}\int_0^{\pi/4} \cos x\, dx - \frac{1}{2}\int_0^{\pi/4} \cos 5x\, dx = \frac{1}{2}[\sin x]_0^{\pi/4} - \frac{1}{2}\left[\frac{\sin 5x}{5}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{4} - \sin 0^\circ\right] - \frac{1}{10}\left[\sin\frac{5\pi}{4} - \sin 0^\circ\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} - 0\right] - \frac{1}{10}\left[-\frac{1}{\sqrt{2}} - 0\right] \qquad \left[\because\ \sin\frac{5\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{2\sqrt{2}} + \frac{1}{10\sqrt{2}} = \frac{5+1}{10\sqrt{2}} = \frac{6}{10\sqrt{2}} = \frac{3}{5\sqrt{2}}.$$

(*iii*) Let $\quad I = \int_{-\pi/4}^{\pi/4} \frac{1}{1 + \sin x}\, dx$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x}\, dx \qquad \textbf{[Rationalization]}$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1 - \sin x}{1 - \sin^2 x}\, dx = \int_{-\pi/4}^{\pi/4} \frac{1 - \sin x}{\cos^2 x}\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx = \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx - \int_{-\pi/4}^{\pi/4} \sec x \tan x\, dx$$

$$= \left[\tan x\right]_{-\pi/4}^{\pi/4} - \left[\sec x\right]_{-\pi/4}^{\pi/4} = \left[\tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)\right] - \left[\sec\frac{\pi}{4} - \sec\left(-\frac{\pi}{4}\right)\right]$$

$$= [1 - (-1)] - \left[\sqrt{2} - \sqrt{2}\right] = 1 + 1 - \sqrt{2} + \sqrt{2}$$

$$= 2.$$

Example 11. (*i*) *If* $\int_0^a \frac{1}{2 + 8x^2}\, dx = \frac{\pi}{16}$, *find the value of a.*

(*ii*) *If* $\int_a^b x^3\, dx = 0$ *and* $\int_a^b x^2\, dx = \frac{2}{3}$, *then find the values of a and b.*

(*iii*) *If* $\int_1^m (3x^2 + 2x + 1)\, dx = 11$, *find the value of m.*

(*iv*) *Evaluate* : $\int_0^4 \frac{2x + 3}{5x^2 + 1}\, dx.$

Solution. (*i*) Let $I = \int_0^a \frac{1}{2 + 8x^2}\, dx$

$$= \frac{1}{8}\int_0^a \frac{1}{\left(\frac{1}{4}+x^2\right)}\,dx = \frac{1}{8}\int_0^a \frac{1}{\left(\frac{1}{2}\right)^2 + x^2}\,dx$$

$$\left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{8}\cdot\left[\frac{1}{1/2}\tan^{-1}\frac{x}{1/2}\right]_0^a = \frac{2}{8}\left[\tan^{-1} 2x\right]_0^a$$

$$= \frac{1}{4}\left[\tan^{-1} 2a - \tan^{-1} 0\right] = \frac{1}{4}\tan^{-1} 2a.$$

Since it is given that $I = \frac{\pi}{16}$

$\therefore \quad \frac{1}{4}\tan^{-1} 2a = \frac{\pi}{16}$

$\Rightarrow \quad \tan^{-1} 2a = \frac{\pi}{4}$

$\Rightarrow \quad 2a = \tan\left(\frac{\pi}{4}\right) \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}.$

(*ii*) Let $\quad I_1 = \int_a^b x^3\,dx$

$$= \left[\frac{x^4}{4}\right]_a^b = \frac{1}{4}\left[x^4\right]_a^b = \frac{1}{4}[b^4 - a^4]$$

Since, it is given that, $I_1 = 0$

$\therefore \quad \frac{1}{4}[b^4 - a^4] = 0 \Rightarrow \quad b^4 - a^4 = 0$

$\Rightarrow \quad b^4 - a^4 = 0$

$\Rightarrow \quad (b^2 - a^2)(b^2 + a^2) = 0$

$\Rightarrow \quad (b-a)(b+a)(b^2+a^2) = 0 \qquad \ldots(1)$

Let $\quad I_2 = \int_a^b x^2\,dx$

$$= \int_a^b x^2\,dx = \left[\frac{x^3}{3}\right]_a^b = \frac{1}{3}\left[x^3\right]_a^b = \frac{1}{3}[b^3 - a^3]$$

Also, it is given that : $I_2 = \frac{2}{3}$.

$\therefore \quad \frac{1}{3}[b^3 - a^3] = \frac{2}{3} \Rightarrow b^3 - a^3 = 2. \qquad \ldots(2)$

Now, from equation (1), we have

$(b-a)(b+a)(b^2+a^2) = 0 \qquad \left[\begin{array}{l}\because (b-a) = 0 \Rightarrow b^3 - a^3 \neq 2 \\ \because (b^2 + a^2 \neq 0)\end{array}\right]$

$$= \left[\frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{\left(x + \frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} \right) \right]_0^1 \qquad \left[\because \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \right]$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) \right]_0^1 = \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{2+1}{\sqrt{3}} \right) - \tan^{-1} \left(\frac{0+1}{\sqrt{3}} \right) \right]$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1} \sqrt{3} - \tan^{-1} \frac{1}{\sqrt{3}} \right]$$

$$= \frac{2}{\sqrt{3}} \left[\frac{\pi}{3} - \frac{\pi}{6} \right] = \frac{2}{\sqrt{3}} \left(\frac{\pi}{6} \right) \qquad \left[\begin{array}{l} \because \ \tan \frac{\pi}{6} = \tan 30^\circ = \frac{1}{\sqrt{3}} \\ \quad \tan \frac{\pi}{3} = \tan 60^\circ = \sqrt{3} \end{array} \right]$$

$$= \frac{\pi}{3\sqrt{3}}.$$

(*iii*) Let $\quad I = \int_2^3 \frac{x}{x^2+1}\, dx$

$$= \frac{1}{2} \int_2^3 \frac{2x}{x^2+1}\, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2} \left[\log |(x^2+1)| \right]_2^3 \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log |f(x)| + c \right]$$

$$= \frac{1}{2} \left[\log (3^2+1) - \log (2^2+1) \right] = \frac{1}{2} [\log 10 - \log 5]$$

$$= \frac{1}{2} \log \left(\frac{10}{5} \right) = \frac{1}{2} \log 2. \qquad \left[\because \ \log m - \log n = \log \left(\frac{m}{n} \right) \right]$$

(*iv*) Let $\quad I = \int_0^1 \frac{2x+3}{5x^2+1}\, dx$

$$= \int_0^1 \left(\frac{2x}{5x^2+1} + \frac{3}{5x^2+1} \right) dx = \int_0^1 \frac{2x}{5x^2+1}\, dx + 3 \int_0^1 \frac{1}{5x^2+1}\, dx$$

$$= \frac{1}{5} \int_0^1 \frac{10x}{5x^2+1}\, dx + \frac{3}{5} \int_0^1 \frac{1}{x^2 + \left(\frac{1}{\sqrt{5}} \right)^2}\, dx \qquad \left[\begin{array}{l} \text{Multiply and divide} \\ \text{the first integral by 5} \end{array} \right]$$

$$= \frac{1}{5} \left[\log |5x^2+1| \right]_0^1 + \frac{3}{5} \left[\frac{1}{1/\sqrt{5}} \tan^{-1} \frac{x}{1/\sqrt{5}} \right]_0^1$$

$$\left[\begin{array}{l} \because \int \frac{f'(x)}{f(x)}\, dx = \log |f(x)| + c \\ \quad \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \end{array} \right]$$

$$= \frac{1}{5}\,[\log(5+1) - \log(0+1)] + \frac{3}{5}\,.\,\sqrt{5}\left[\tan^{-1}\sqrt{5}x\right]_0^1$$

$$= \frac{1}{5}[\log 6 - \log 1] + \frac{3}{\sqrt{5}}\left[\tan^{-1}\sqrt{5} - \tan^{-1} 0\right]$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}. \qquad [\because \ \log 1 = 0]$$

(v) Let $I = \int_0^2 \frac{5x+1}{x^2+4}\,dx$

$$= \int_0^2 \left(\frac{5x}{x^2+4} + \frac{1}{x^2+4}\right) dx = 5\int_0^2 \frac{x}{x^2+4}\,dx + \int_0^2 \frac{1}{x^2+4}\,dx$$

$$= \frac{5}{2}\int_0^2 \frac{2x}{x^2+4}\,dx + \int_0^2 \frac{1}{x^2+2^2}\,dx \qquad \begin{bmatrix}\text{Multiply and divide}\\ \text{the first integral by 2}\end{bmatrix}$$

$$= \frac{5}{2}\left[\log|x^2+4|\right]_0^2 + \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 \qquad \begin{bmatrix}\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\\ \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\end{bmatrix}$$

$$= \frac{5}{2}\left[\log|(2)^2+4| - \log|0+4|\right] + \frac{1}{2}\left[\tan^{-1}\frac{2}{2} - \tan^{-1} 0\right]$$

$$= \frac{5}{2}\,[\log 8 - \log 4] + \frac{1}{2}\,[\tan^{-1} 1 - 0]$$

$$= \frac{5}{2}\log\left(\frac{8}{4}\right) + \frac{1}{2}\left(\frac{\pi}{4}\right) \qquad \begin{bmatrix}\because \ \log m - \log n = \log\frac{m}{n}\\ \tan\frac{\pi}{4} = \tan 45^\circ = 1\end{bmatrix}$$

$$= \frac{5}{2}\log 2 + \frac{\pi}{8}.$$

(vi) Let $I = \int_0^{\pi/4} (2\sec^2 x + x^3 + 2)\,dx$

$$= 2\int_0^{\pi/4} \sec^2 x\,dx + \int_0^{\pi/4} x^3\,dx + 2\int_0^{\pi/4} 1\,.\,dx$$

$$= 2\left[\tan x\right]_0^{\pi/4} + \left[\frac{x^4}{4}\right]_0^{\pi/4} + 2\left[x\right]_0^{\pi/4}$$

$$= 2\left[\tan\frac{\pi}{4} - \tan 0^\circ\right] + \frac{1}{4}\left[\left(\frac{\pi}{4}\right)^4 - 0\right] + 2\left[\frac{\pi}{4} - 0\right]$$

$$= 2\,(1-0) + \frac{1}{4}\left[\frac{\pi^4}{256} - 0\right] + \frac{\pi}{2} = 2 + \frac{\pi^4}{1024} + \frac{\pi}{2}$$

$$= \frac{\pi^4}{1024} + \frac{\pi}{2} + 2.$$

Example 14. *Evaluate the following* :

(*i*) $\int_0^2 \frac{1}{4+x-x^2}\,dx$ (*ii*) $\int_0^2 \frac{1}{\sqrt{3+2x-x^2}}\,dx$

(*iii*) $\int_1^2 \frac{1}{\sqrt{x^2+4x+3}}\,dx$ (*iv*) $\int_0^4 \frac{1}{\sqrt{x^2+2x+3}}\,dx$

(*v*) $\int_{1/4}^{1/2} \frac{1}{\sqrt{x-x^2}}\,dx$ (*vi*) $\int_0^a \frac{1}{\sqrt{ax-x^2}}\,dx.$

Solution. (*i*) Let $I = \int_0^2 \frac{1}{4+x-x^2}\,dx = \int_0^2 \frac{1}{4-(x^2-x)}\,dx$

$$= \int_0^2 \frac{1}{\left(4+\frac{1}{4}\right)-\left(x^2-x+\frac{1}{4}\right)}\,dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denominator} \\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int_0^2 \frac{1}{\frac{17}{4}-\left(x-\frac{1}{2}\right)^2}\,dx = \int_0^2 \frac{1}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

$$= \left[\frac{1}{2.\left(\frac{\sqrt{17}}{2}\right)}.\log\left|\frac{\frac{\sqrt{17}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{17}}{2}-\left(x-\frac{1}{2}\right)}\right|\right]_0^2 \qquad \left[\because \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\left|\frac{\frac{\sqrt{17}}{2}+\left(2-\frac{1}{2}\right)}{\frac{\sqrt{17}}{2}-\left(2-\frac{1}{2}\right)}\right| - \log\left|\frac{\frac{\sqrt{17}}{2}+\left(0-\frac{1}{2}\right)}{\frac{\sqrt{17}}{2}-\left(0-\frac{1}{2}\right)}\right|\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\left|\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}}\right| - \log\left|\frac{\frac{\sqrt{17}}{2}-\frac{1}{2}}{\frac{\sqrt{17}}{2}+\frac{1}{2}}\right|\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\right) - \log\left(\frac{\sqrt{17}-1}{\sqrt{17}+1}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{\frac{\sqrt{17}+3}{\sqrt{17}-3}}{\frac{\sqrt{17}-1}{\sqrt{17}+1}}\right)\right] \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3} \times \frac{\sqrt{17}+1}{\sqrt{17}-1}\right) = \frac{1}{\sqrt{17}} \log\left[\frac{17+3\sqrt{17}+\sqrt{17}+3}{17-3\sqrt{17}-\sqrt{17}+3}\right]$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right) = \frac{1}{\sqrt{17}} \log\left[\frac{4(5+\sqrt{17})}{4(5-\sqrt{17})}\right]$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right).$$

(*ii*) Let $\quad I = \int_0^2 \frac{1}{\sqrt{3+2x-x^2}}\, dx = \int_0^2 \frac{1}{\sqrt{-(x^2-2x-3)}}\, dx$

$$= \int_0^2 \frac{1}{\sqrt{-[x^2-2x+1-3-1]}}\, dx \qquad \left[\begin{array}{l}\text{Add and subtract 1 to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 1\end{array}\right]$$

$$= \int_0^2 \frac{1}{\sqrt{-[(x-1)^2-4]}}\, dx = \int_0^2 \frac{1}{\sqrt{(2)^2-(x-1)^2}}\, dx$$

$$= \left[\sin^{-1}\left(\frac{x-1}{2}\right)\right]_0^2 \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \left[\sin^{-1}\left(\frac{2-1}{2}\right) - \sin^{-1}\left(\frac{0-1}{2}\right)\right]$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{6} + \frac{\pi}{6} \qquad \left[\because \sin\frac{\pi}{6} = \sin 30^\circ = \frac{1}{2}\right]$$

$$= \frac{2\pi}{6} = \frac{\pi}{3}.$$

(*iii*) Let $\quad I = \int_1^2 \frac{1}{\sqrt{x^2+4x+3}}\, dx$

$$= \int_1^2 \frac{1}{\sqrt{(x^2+4x+4)+(3-4)}}\, dx \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 4\end{array}\right]$$

$$= \int_1^2 \frac{1}{\sqrt{(x+2)^2-1}}\, dx \qquad \left[\because \int \frac{1}{\sqrt{x^2-a^2}}\, dx = \log\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$= \left[\log\left|(x+2)+\sqrt{(x+2)^2-1}\right|\right]_1^2 = \left[\log\left|(x+2)+\sqrt{x^2+4x+3}\right|\right]_1^2$$

$$= [\log|(2+2)+\sqrt{4+8+3}| - \log|(1+2)+\sqrt{1+4+3}|]$$

$= \log\left(4+\sqrt{15}\right) - \log\left(3+\sqrt{8}\right)$

$= \log\left(\dfrac{4+\sqrt{15}}{3+\sqrt{8}}\right).$ $\left[\because \log m - \log n = \log\left(\dfrac{m}{n}\right)\right]$

(*iv*) Let $\quad I = \displaystyle\int_0^4 \frac{1}{\sqrt{x^2+2x+3}}\,dx$

$= \displaystyle\int_0^4 \frac{1}{\sqrt{(x^2+2x+1)+(3-1)}}\,dx$ $\left[\begin{array}{l}\text{Add and subtract 1 to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = 1\end{array}\right]$

$= \displaystyle\int_0^4 \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}\,dx$ $\left[\because \displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|+c\right]$

$= \left[\log\left|(x+1)+\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}\right|\right]_0^4 = \log\left|(x+1)+\sqrt{x^2+2x+3}\right|_0^4$

$= [\log|(4+1)+\sqrt{16+8+3}| - \log|(0+1)+\sqrt{0+0+3}|]$

$= [\log(5+\sqrt{27}) - \log(1+\sqrt{3})]$

$= \log\left(\dfrac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$ $\left[\because \log m - \log n = \log\left(\dfrac{m}{n}\right)\right]$

$= \log\left(\dfrac{5+3\sqrt{3}}{1+\sqrt{3}} \times \dfrac{1-\sqrt{3}}{1-\sqrt{3}}\right)$ (Rationalization)

$= \log\left(\dfrac{5+3\sqrt{3}-5\sqrt{3}-9}{1-3}\right) = \log\left(\dfrac{-4-2\sqrt{3}}{-2}\right)$

$= \log(2+\sqrt{3}).$

(*v*) Let $\quad I = \displaystyle\int_{1/4}^{1/2} \frac{1}{\sqrt{x-x^2}}\,dx = \int_{1/4}^{1/2} \frac{1}{\sqrt{-(x^2-x)}}\,dx$

$= \displaystyle\int_{1/4}^{1/2} \frac{1}{\sqrt{-\left[x^2-x+\frac{1}{4}-\frac{1}{4}\right]}}\,dx$ $\left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$

$= \displaystyle\int_{1/4}^{1/2} \frac{1}{\sqrt{-\left[\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]}}\,dx = \int_{1/4}^{1/2} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}\,dx$

Put $x = 0$ in (2), we get

$$1 = B(0 + 1) \Rightarrow B = 1$$

Put $x = -1$ in (2), we get

$$1 = A(-1)(-1+1) + B(-1+1) + C(-1)^2$$

$$\Rightarrow \quad 1 = C \Rightarrow C = 1$$

Comparing the co-efficients of x on both sides of equation (2), we have

$$0 = A + B \Rightarrow A = -B \Rightarrow A = -1.$$

Substituting the values of A, B and C in equation (1), we have

$$\frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$$\therefore \quad I = \int_1^3 \frac{1}{x^2(x+1)}\,dx = \int_1^3 \left[\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right] dx$$

$$= -\int_1^3 \frac{1}{x}\,dx + \int_1^3 x^{-2}\,dx + \int_1^3 \frac{1}{x+1}\,dx$$

$$= -\left[\log|x|\right]_1^3 + \left[\frac{x^{-2+1}}{-2+1}\right]_1^3 + \left[\log|x+1|\right]_1^3$$

$$= -[\log 3 - \log 1] - 1\left[\frac{1}{x}\right]_1^3 + [\log|3+1| - \log|1+1|]$$

$$= -\log 3 - \left[\frac{1}{3} - \frac{1}{1}\right] + [\log 4 - \log 2]$$

$$= -\log 3 + \frac{2}{3} + \log\left(\frac{4}{2}\right) \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= -\log 3 + \frac{2}{3} + \log 2$$

$$= \frac{2}{3} + \log\frac{2}{3}.$$

(*iii*) Let $I = \int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx$

Let $\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(z+a^2)(z+b^2)} = \frac{A}{(z+a^2)} + \frac{B}{(z+b^2)}$ ...(1) [Put $(x^2 = z)$]

Multiplying both sides by $(z + a^2)(z + b^2)$, we get

$$1 = A(z + b^2) + B(z + a^2) \qquad ...(2)$$

Put $z = -a^2$ in (2), we get

$$1 = A(-a^2 + b^2) + B(-a^2 + a^2) \Rightarrow 1 = A(b^2 - a^2) \Rightarrow A = \frac{1}{b^2 - a^2}$$

Put $z = -b^2$ in (2), we get

$$1 = A(-b^2 + b^2) + B(-b^2 + a^2) \Rightarrow 1 = B(a^2 - b^2) \Rightarrow B = \frac{1}{a^2 - b^2}$$

Substituting the values of A and B in (1), we have

$$\frac{1}{(z+a^2)(z+b^2)} = \frac{\frac{1}{b^2-a^2}}{(z+a^2)} + \frac{\frac{1}{a^2-b^2}}{(z+b^2)} = \frac{1}{(b^2-a^2)(z+a^2)} - \frac{1}{(b^2-a^2)(z+b^2)}$$

or $$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right] \qquad [\because\ z = x^2]$$

$$\therefore \quad \mathrm{I} = \int_0^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\,.\,dx = \int_0^{\infty}\left[\frac{1}{b^2-a^2}\left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right)\right]dx$$

$$= \frac{1}{b^2-a^2}\left[\int_0^{\infty}\frac{1}{x^2+a^2}\,dx - \int_0^{\infty}\frac{1}{x^2+b^2}\,dx\right]$$

$$= \frac{1}{b^2-a^2}\left[\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)_0^{\infty} - \left(\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right)_0^{\infty}\right]$$

$$\left[\because\ \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{b^2-a^2}\left[\left(\frac{1}{a}\tan^{-1}\infty - \frac{1}{a}\tan^{-1}0^{\circ}\right) - \left(\frac{1}{b}\tan^{-1}\infty - \frac{1}{b}\tan^{-1}0^{\circ}\right)\right]$$

$$= \frac{1}{b^2-a^2}\left[\left(\frac{\pi}{2a}-0\right) - \left(\frac{\pi}{2b}-0\right)\right] = \frac{1}{b^2-a^2}\left[\frac{\pi}{2a} - \frac{\pi}{2b}\right]$$

$$= \frac{\pi}{b^2-a^2}\left(\frac{b-a}{2ab}\right) = \frac{\pi(b-a)}{2ab(b-a)(b+a)}$$

$$= \frac{\pi}{2ab(a+b)}\,.$$

(iv) Let $\quad \mathrm{I} = \int_1^2 \frac{1}{(x+1)(x+2)}\,dx$

Let $$\frac{1}{(x+1)(x+2)} = \frac{\mathrm{A}}{x+1} + \frac{\mathrm{B}}{x+2} \qquad \text{...(1)}$$

Multiplying both sides by $(x+1)(x+2)$, we get

$$1 = \mathrm{A}(x+2) + \mathrm{B}(x+1) \qquad \text{...(2)}$$

Put $x = -1$ in (2), we get

$$1 = \mathrm{A}(-1+2) + \mathrm{B}(-1+1) \ \Rightarrow\ 1 = \mathrm{A} \ \Rightarrow\ \mathrm{A} = 1$$

Put $x = -2$ in (2), we get

$$1 = \mathrm{A}(-2+2) + \mathrm{B}(-2+1) \ \Rightarrow\ 1 = -\mathrm{B} \ \Rightarrow\ \mathrm{B} = -1$$

Substituting the values of A and B in (1), we have

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{-1}{x+2}$$

$$= 5\int_1^2 1.\,dx + \frac{5}{2}\int_1^2 \frac{1}{x+1}\,dx - \frac{45}{2}\int_1^2 \frac{1}{x+3}\,dx$$

$$= 5\Big[x\Big]_1^2 + \frac{5}{2}\Big[\log|x+1|\Big]_1^2 - \frac{45}{2}\Big[\log|x+3|\Big]_1^2$$

$$= 5\,[2-1] + \frac{5}{2}\,[\log|2+1| - \log|1+1|] - \frac{45}{2}\,[\log|2+3| - \log|1+3|]$$

$$= 5 + \frac{5}{2}\,[\log 3 - \log 2] - \frac{45}{2}\,[\log 5 - \log 4] \qquad \left[\because\ \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= 5 + \frac{5}{2}\log\frac{3}{2} - \frac{45}{2}\log\frac{5}{4}.$$

Example 16. *Evaluate the following* :

(i) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx$ (ii) $\displaystyle\int_2^3 \frac{x^2+1}{(2x+1)(x^2-1)}\,dx$

(iii) $\displaystyle\int_0^1 \frac{x}{x^2+4x+3}\,dx$ (iv) $\displaystyle\int_0^1 \sqrt{x(1-x)}\,dx$

(v) $\displaystyle\int_1^2 \frac{1}{x(1+x^2)}\,dx.$

Solution. (i) Let $\mathrm{I} = \displaystyle\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx$

Put $\sin x = z \Rightarrow \cos x\,dx = dz$

When $x = 0 \Rightarrow z = \sin 0^\circ = 0$ and when $x = \pi/2 \Rightarrow z = \sin \pi/2 = 1.$

$$\therefore \quad \mathrm{I} = \int_0^1 \frac{1}{(1+z)(2+z)}\,dz$$

Let $\quad \dfrac{1}{(1+z)(2+z)} = \dfrac{\mathrm{A}}{1+z} + \dfrac{\mathrm{B}}{2+z}$ ...(1)

Multiplying both sides by $(1+z)(2+z)$, we get

$1 = \mathrm{A}(2+z) + \mathrm{B}(1+z)$ ...(2)

Put $z = -1$ in (2), we get

$1 = \mathrm{A}[2+(-1)] + \mathrm{B}[1+(-1)] \Rightarrow 1 = \mathrm{A} \Rightarrow \mathrm{A} = 1$

Put $z = -2$ in (2), we get

$1 = \mathrm{A}[2+(-2)] + \mathrm{B}[1+(-2)] \Rightarrow 1 = -\mathrm{B} \Rightarrow \mathrm{B} = -1.$

Substituting the values of A and B in equation (1), we have

$$\frac{1}{(1+z)(2+z)} = \frac{1}{1+z} + \frac{-1}{2+z}$$

$$\therefore \quad \mathrm{I} = \int_0^1 \frac{1}{(1+z)(2+z)}\,dz = \int_0^1 \left[\frac{1}{1+z} - \frac{1}{2+z}\right] dz$$

$$= \int_0^1 \frac{1}{1+z}\,dz - \int_0^1 \frac{1}{2+z}\,dz$$

$$= \left[\log|1+z|\right]_0^1 - \left[\log|2+z|\right]_0^1$$
$= [\log|1+1| - \log|1+0|] - [\log|2+1| + \log|2+0|]$
$= (\log 2 - \log 1) - (\log 3 - \log 2)$ $\quad [\because \log 1 = 0]$
$= \log 2 - \log 3 + \log 2 = 2\log 2 - \log 3$
$= \log(2)^2 - \log 3$ $\quad [\because n\log m = \log m^n]$

$$= \log 4 - \log 3 = \log\frac{4}{3}. \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

(*ii*) Let $\quad I = \int_2^3 \frac{x^2+1}{(2x+1)(x^2-1)}\,dx = \int_2^3 \frac{x^2+1}{(2x+1)(x-1)(x+1)}\,dx$

Let $\quad \frac{x^2+1}{(2x+1)(x-1)(x+1)} = \frac{A}{(2x+1)} + \frac{B}{(x-1)} + \frac{C}{(x+1)}$ ...(1)

Multiplying both sides by $(2x+1)(x-1)(x+1)$, we get

$x^2 + 1 = A(x-1)(x+1) + B(2x+1)(x+1) + C(2x+1)(x-1)$ ...(2)

Put $x = 1$ in (2), we get

$1 + 1 = A(1-1)(1+1) + B[2+1][1+1] + C[2+1][1-1]$

$\Rightarrow \quad 2 = 6B \;\Rightarrow\; B = \frac{1}{3}$

Put $x = -1$ in (2), we get

$[(-1)^2 + 1] = A[-1-1][-1+1] + B[2(-1)+1][-1+1] + C[2(-1)+1][-1-1]$

$\Rightarrow \quad 2 = C(-1)(-2) \;\Rightarrow\; 2 = 2C \;\Rightarrow\; C = 1$

Comparing the co-efficients of x^2 on both sides of equation (2), we have

$$1 = A + 2B + 2C \;\Rightarrow\; 1 = A + 2\left(\frac{1}{3}\right) + 2(1)$$

$$\Rightarrow \quad 1 - 2 - \frac{2}{3} = A \;\Rightarrow\; -1 - \frac{2}{3} = A \;\Rightarrow\; A = -\frac{5}{3}$$

Substituting the values of A, B and C in equation (1), we have

$$\frac{x^2+1}{(2x+1)(x-1)(x+1)} = \frac{-5/3}{(2x+1)} + \frac{1/3}{x-1} + \frac{1}{x+1} = -\frac{5}{3(2x+1)} + \frac{1}{3(x-1)} + \frac{1}{x+1}$$

$$\therefore \quad I = \int_2^3 \frac{x^2+1}{(2x+1)(x^2-1)}\,dx = \int_2^3 \left[-\frac{5}{3(2x+1)} + \frac{1}{3(x-1)} + \frac{1}{x+1}\right]dx$$

$$= -\frac{5}{3}\int_2^3 \frac{1}{2x+1}\,dx + \frac{1}{3}\int_2^3 \frac{1}{x-1}\,dx + \int_2^3 \frac{1}{x+1}\,dx$$

$$= -\frac{5}{3}\left[\frac{\log|2x+1|}{2}\right]_2^3 + \frac{1}{3}\left[\log|x-1|\right]_2^3 + \left[\log|x+1|\right]_2^3$$

$$= -\frac{5}{6}[\log|2(3)+1| - \log|2(2)+1|] + \frac{1}{3}[\log|3-1| - \log|2-1|]$$
$$+ [\log|3+1| - \log|2+1|]$$

Put $x = 0$ in equation (2), we get

$$1 = A(1+0) + (B.0 + C).0 \Rightarrow A = 1$$

Comparing co-efficients of x^2 on both sides of equation (2), we have

$$0 = A + B \Rightarrow B = -A \Rightarrow B = -1$$

Comparing co-efficients of x on both sides of equation (2), we have

$$0 = C \quad \Rightarrow C = 0$$

Substituting the values of A, B and C in equation (1), we have

$$\frac{1}{x(1+x^2)} = \frac{1}{x} + \frac{-x}{1+x^2}$$

$$\therefore \quad I = \int_1^2 \frac{1}{x(1+x^2)}\,dx = \int_1^2 \left[\frac{1}{x} - \frac{x}{1+x^2}\right] dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{1+x^2}\,dx$$

$$= \int_1^2 \frac{1}{x}\,dx - \frac{1}{2}\int_1^2 \frac{2x}{1+x^2}\,dx \qquad \left[\begin{array}{l}\text{Multiply and divide the}\\ \text{second integral by 2}\end{array}\right]$$

$$= \Big[\log|x|\Big]_1^2 - \frac{1}{2}\Big[\log|1+x^2|\Big]_1^2 \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

$$= [\log 2 - \log 1] - \frac{1}{2}[\log|1+2^2| - \log|1+1|]$$

$$= \log 2 - \frac{1}{2}[\log 5 - \log 2]$$

$$= \log 2 - \frac{1}{2}\log\frac{5}{2}. \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

Example 17. *Evaluate the following :*

(i) $\displaystyle\int_{\pi/2}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$ (ii) $\displaystyle\int_0^1 x\,e^x\,dx$

(iii) $\displaystyle\int_0^1 \tan^{-1} x\,dx$ (iv) $\displaystyle\int_1^2 \frac{\log x}{x^2}\,dx$

(v) $\displaystyle\int_0^{\pi/2} x^2 \cos 2x\,dx.$

Solution. (i) Let $\displaystyle I = \int_{\pi/2}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$

$$= \int_{\pi/2}^{\pi} e^x \left[\frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right] dx \qquad \left[\begin{array}{l}\because \sin 2A = 2\sin A\cos A\\ \Rightarrow \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\\ \quad 1-\cos 2A = 2\sin^2 A\\ \Rightarrow 1-\cos A = 2\sin^2\frac{A}{2}\end{array}\right]$$

$$= \int_{\pi/2}^{\pi} e^x \left[\frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right] dx$$

$$= \int_{\pi/2}^{\pi} e^x \left[\frac{1}{2}\operatorname{cosec}^2 \frac{x}{2} - \cot\frac{x}{2}\right] dx$$

$$\left[\because \int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c\right]$$

$$= \left[e^x \left(-\cot\frac{x}{2}\right)\right]_{\pi/2}^{\pi}$$

$$= \left[-e^\pi \cot\frac{\pi}{2} + e^{\pi/2}\cot\frac{\pi}{4}\right] \qquad \left[\begin{array}{l} \because \cot\frac{\pi}{2} = \cot 90^\circ = 0 \\ \cot\frac{\pi}{4} = \cot 45^\circ = 1 \end{array}\right]$$

$$= [0 + e^{\pi/2}] = e^{\pi/2}.$$

(*ii*) Let $\quad I = \int_0^1 x\, e^x\, dx$

Let $\quad \int x\, e^x\, dx = \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^x}\, dx$

Integrating by parts, we have

$$\int x\, e^x\, dx = x \,.\, \int e^x\, dx - \int \left\{\frac{d}{dx}(x) \,.\, \int e^x\, dx\right\} dx$$

$$= x\, e^x - \int 1 \,.\, e^x\, dx$$

$$= x\, e^x - e^x$$

$$\therefore \quad \int_0^1 x\, e^x\, dx = \left[x\, e^x - e^x\right]_0^1 = [(1 \,.\, e^1 - e^1) - (0 - e^0)]$$

$$= 1.$$

(*iii*) Let $\quad I = \int_0^1 \tan^{-1} x\, dx$

Let $\quad \int \tan^{-1} x\, dx = \int \underset{\text{II}}{1} \,.\, \underset{\text{I}}{\tan^{-1} x}\, dx$ [Taking unity as the second function]

Integrating by parts, we have

$$\int \tan^{-1} x\, dx = \tan^{-1} x \,.\, \int 1 \,.\, dx - \int \left\{\frac{d}{dx}(\tan^{-1} x) \,.\, \int 1 \,.\, dx\right\} dx$$

$$= \tan^{-1} x \,.\, x - \int \frac{1}{1+x^2} \,.\, x\, dx$$

$$= x \tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\, dx \qquad \text{[Multiply and divided by 2]}$$

$$= x \tan^{-1} x - \frac{1}{2}\,[\log|(1+x^2)|] \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right]$$

$$\therefore \quad I = \int_0^1 \tan^{-1} x\, dx = \left[x \tan^{-1} x - \frac{1}{2}\log|(1+x^2)|\right]_0^1$$

$$= \left[1 \tan^{-1} 1 - \frac{1}{2}\log(1+1)\right] - \left[0 - \frac{1}{2}\log(1+0)\right]$$

$$= \frac{1}{2}x^2 \sin 2x - \left[x \cdot \frac{(-\cos 2x)}{2} - \int 1 \cdot \frac{(-\cos 2x)}{2}\, dx\right]$$

$$= \frac{1}{2}x^2 \sin 2x + \frac{1}{2}x \cos 2x - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{1}{2}x^2 \sin 2x + \frac{1}{2}x \cos 2x - \frac{1}{2}\frac{\sin 2x}{2}$$

$$= \frac{1}{2}x^2 \sin 2x + \frac{1}{2}x \cos 2x - \frac{1}{4}\sin 2x$$

$$\therefore \quad I = \int_0^{\pi/2} x^2 \cos 2x\, dx = \left[\frac{1}{2}x^2 \sin 2x + \frac{1}{2}x \cos 2x - \frac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[2x^2 \sin 2x + 2x \cos 2x - \sin 2x\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[2\left(\frac{\pi}{2}\right)^2 \sin\left(2 \cdot \frac{\pi}{2}\right) + \frac{2\pi}{2}\cos\left(2 \cdot \frac{\pi}{2}\right) - \sin\left(2 \cdot \frac{\pi}{2}\right)\right] - \frac{1}{4}[0 + 0 - \sin 0^\circ]$$

$$= \frac{1}{4}\left[2\frac{\pi^2}{4}\sin \pi + \pi \cos \pi - \sin \pi\right] - 0$$

$$= \frac{1}{4}[0 + \pi(-1) - 0] = -\frac{\pi}{4}.$$

Example 18. *If* $\int_0^a \sqrt{x}\, dx = 2a\int_0^{\pi/2} \sin^3 x\, dx$, *find the value of* $\int_a^{a+1} x \,.\, dx$.

Solution. Let $I_1 = \int_0^a \sqrt{x}\, dx = \left[\dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^a = \dfrac{2}{3}\left[x^{3/2}\right]_0^a = \dfrac{2}{3}[a^{3/2} - 0]$

$$\Rightarrow \quad I_1 = \frac{2}{3}a^{3/2} \qquad \text{...(2)}$$

Let $I_2 = 2a\int_0^{\pi/2} \sin^3 x\, dx$ ...(3)

$$= 2a\int_0^{\pi/2} \frac{1}{4}[3 \sin x - \sin 3x]\, dx \qquad \left[\begin{array}{l} \because \ \sin 3A = 3\sin A - 4\sin^3 A \\ \Rightarrow 4\sin^3 A = 3\sin A - \sin 3A \\ \Rightarrow \sin^3 A = \frac{1}{4}[3\sin A - \sin 3A] \end{array}\right]$$

$$= \frac{6a}{4}\int_0^{\pi/2} \sin x\, dx - \frac{2a}{4}\int_0^{\pi/2} \sin 3x\, dx$$

$$= \frac{3a}{2}[-\cos x]_0^{\pi/2} - \frac{a}{2}\left[-\frac{\cos 3x}{3}\right]_0^{\pi/2} = -\frac{3a}{2}\left[\cos\frac{\pi}{2} - \cos 0^\circ\right] + \frac{a}{6}\left[\cos\frac{3\pi}{2} - \cos 0^\circ\right]$$

$$= -\frac{3a}{2}[0 - 1] + \frac{a}{6}[0 - 1]$$

$$= \frac{3a}{2} - \frac{a}{6} = \frac{9a - a}{6} = \frac{8a}{6} = \frac{4a}{3} \qquad \text{...(4)}$$

Example 20. *Evaluate the following :*

(i) $\int_0^{2\pi} e^{x/2} \sin\left(\frac{x}{2}+\frac{\pi}{4}\right) dx$ (ii) $\int_2^4 \frac{x^2+x}{\sqrt{2x+1}}\, dx$

(iii) $\int_0^{\pi/2} x \sin x\, dx$ (iv) $\int_0^{\pi/6} (2+3x^2) \cos 3x\, dx$

(v) $\int_{\pi/4}^{\pi/2} \cos 2x \log \sin x\, dx$ (vi) $\int_0^{\pi/2} x^2 \sin 3x\, dx$

(vii) $\int_1^3 \frac{\log x}{(x+1)^2}\, dx$ (viii) $\int_1^2 \log x\, dx.$

Solution. (i) Let $\mathrm{I} = \int_0^{2\pi} e^{x/2} \sin\left(\frac{x}{2}+\frac{\pi}{4}\right) dx$

Let $\mathrm{I}_1 = \int \underset{\mathrm{I}}{e^{x/2}} \underset{\mathrm{II}}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)} dx$

Integrating by parts, we have

$$\mathrm{I}_1 = e^{x/2} . \int \sin\left(\frac{x}{2}+\frac{\pi}{4}\right) dx - \int \left\{\frac{d}{dx}(e^{x/2}) . \int \sin\left(\frac{x}{2}+\frac{\pi}{4}\right) dx\right\} dx$$

$$= e^{x/2} . \left[\frac{-\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}{\frac{1}{2}}\right] - \int \frac{1}{2} e^{x/2} . \left[\frac{-\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}{\frac{1}{2}}\right] dx$$

$$= -2e^{x/2} \cos\left(\frac{x}{2}+\frac{\pi}{4}\right) + \int \underset{\mathrm{I}}{e^{x/2}} \underset{\mathrm{II}}{\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)} dx$$

Integrating by parts again, we have

$$\mathrm{I}_1 = -2e^{x/2} \cos\left(\frac{x}{2}+\frac{\pi}{4}\right) + \left[e^{x/2} . \int \cos\left(\frac{x}{2}+\frac{\pi}{4}\right) dx - \int \left\{\frac{d}{dx}(e^{x/2}) . \int \cos\left(\frac{x}{2}+\frac{\pi}{4}\right) dx\right\} dx\right]$$

$$= -2e^{x/2} \cos\left(\frac{x}{2}+\frac{\pi}{4}\right) + \left[e^{x/2} \left\{\frac{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}{\frac{1}{2}}\right\} - \int \frac{1}{2} e^{x/2} . \frac{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}{\frac{1}{2}} . dx\right]$$

$$= -2e^{x/2} \cos\left(\frac{x}{2}+\frac{\pi}{4}\right) + 2e^{x/2} \sin\left(\frac{x}{2}+\frac{\pi}{4}\right) - \int e^{x/2} \sin\left(\frac{x}{2}+\frac{\pi}{4}\right) dx$$

$$= -2e^{x/2} \cos\left(\frac{x}{2}+\frac{\pi}{4}\right) + 2e^{x/2} \sin\left(\frac{x}{2}+\frac{\pi}{4}\right) - \mathrm{I}_1$$

$$\Rightarrow \quad 2\mathrm{I}_1 = -2e^{x/2} \cos\left(\frac{x}{2}+\frac{\pi}{4}\right) + 2e^{x/2} \sin\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \mathrm{I}_1 = -e^{x/2} \left[\cos\left(\frac{x}{2}+\frac{\pi}{4}\right) - \sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\right]$$

$$\therefore \quad I = \int_0^{2\pi} e^{x/2} \sin\left(\frac{x}{2}+\frac{\pi}{4}\right) dx = -\left[e^{x/2}\left\{\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)-\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\right\}\right]_0^{2\pi}$$

$$= -\left[e^{2\pi/2}\left\{\cos\left(\frac{2\pi}{2}+\frac{\pi}{4}\right)-\sin\left(\frac{2\pi}{2}+\frac{\pi}{4}\right)\right\}\right]-\left[e^0\left\{\cos\left(0+\frac{\pi}{4}\right)-\sin\left(0+\frac{\pi}{4}\right)\right\}\right]$$

$$= -\left[e^{\pi}\left\{\cos\frac{5\pi}{4}-\sin\frac{5\pi}{4}\right\}\right]-\left[\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\right]$$

$$= -\left[e^{\pi}\left\{-\frac{1}{\sqrt{2}}-\left(-\frac{1}{\sqrt{2}}\right)\right\}\right]-\left[\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right] \qquad \left[\begin{array}{l}\because \cos\frac{5\pi}{4}=\cos\left(\pi+\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}\\ \quad \sin\frac{5\pi}{4}=\sin\left(\pi+\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}\end{array}\right]$$

$$= -e^{\pi}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)-\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)$$

$$= -e^{\pi}(0)-(0)=0.$$

(*ii*) Let $I = \int_2^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx$

Let $\quad I_1 = \int \underset{\text{I}}{(x^2+x)} \,.\, \underset{\text{II}}{(2x+1)^{-1/2}}\,dx$

Integrating by parts, we have

$$I_1 = (x^2+x)\,.\int(2x+1)^{-1/2}\,dx - \int\left\{\frac{d}{dx}(x^2+x)\,.\int(2x+1)^{-1/2}\,dx\right\}dx$$

$$= (x^2+x)\,.\,\frac{(2x+1)^{-\frac{1}{2}+1}}{2.\left(-\frac{1}{2}+1\right)} - \int \frac{(2x+1)\,.\,(2x+1)^{-\frac{1}{2}+1}}{2.\left(-\frac{1}{2}+1\right)}\,dx$$

$$\left[\because \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}+c\right]$$

$$= (x^2+x)\sqrt{2x+1} - \int (2x+1)(2x+1)^{1/2}\,dx$$

$$= (x^2+x)\sqrt{2x+1} - \int (2x+1)^{3/2}\,dx$$

$$= (x^2+x)\sqrt{2x+1} - \frac{(2x+1)^{\frac{3}{2}+1}}{2.\left(\frac{3}{2}+1\right)} = (x^2+x)\sqrt{2x+1} - \frac{1}{5}(2x+1)^{5/2}$$

$$\therefore \quad \int_2^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx = \left[(x^2+x)\sqrt{2x+1} - \frac{1}{5}(2x+1)^{5/2}\right]_2^4$$

$$= \left[(4^2 + 4)\sqrt{2(4)+1} - \frac{1}{5}[2(4)+1]^{5/2}\right] - \left[(2^2+2)\sqrt{2(2)+1} - \frac{1}{5}[2(2)+1]^{5/2}\right]$$

$$= \left[20\sqrt{9} - \frac{1}{5}(9)^{5/2}\right] - \left[6\sqrt{5} - \frac{1}{5}(5)^{5/2}\right]$$

$$= \left(20(3) - \frac{243}{5}\right) - 6\sqrt{5} + \frac{25\sqrt{5}}{5} \quad \left[\begin{array}{l} \because\ (9)^{5/2} = (9^{1/2})^5 = (3)^5 = 243 \\ (5)^{5/2} = (5^5)^{1/2} = \sqrt{3125} = \sqrt{625 \times 5} = 25\sqrt{5} \end{array}\right]$$

$$= 60 - \frac{243}{5} - 6\sqrt{5} + 5\sqrt{5}$$

$$= \frac{300 - 243}{5} - \sqrt{5} = \frac{57}{5} - \sqrt{5}\ .$$

(*iii*) Let $\quad I = \int_0^{\pi/2} x \sin x \, dx$

Let $\quad I_1 = \int \underset{\mathrm{I}}{x} \, . \underset{\mathrm{II}}{\sin x} \, dx$

Integrating by parts, we have

$$I_1 = x \, . \int \sin x \, dx - \int \left\{\frac{d}{dx}(x) \, . \int \sin x \, dx\right\} dx$$

$$= x(-\cos x) - \int 1 \, . (-\cos x) \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x$$

$$\therefore \qquad I = \int_0^{\pi/2} x \sin x \, dx = [-x \cos x + \sin x]_0^{\pi/2}$$

$$= \left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] - [-0 + \sin 0°] = \left[-\frac{\pi}{2}(0) + 1\right] - [0 + 0]$$

$$= 1.$$

(*iv*) Let $\quad I = \int_0^{\pi/6} (2 + 3x^2) \cos 3x \, dx$

Let $\quad I_1 = \int \underset{\mathrm{I}}{(2 + 3x^2)} . \underset{\mathrm{II}}{\cos 3x} \, dx$

Integrating by parts, we have

$$I_1 = (2 + 3x^2) \, . \int \cos 3x \, dx - \int \left\{\frac{d}{dx}(2 + 3x^2) \, . \int \cos 3x \, dx\right\} dx$$

$$= (2 + 3x^2) \, . \frac{\sin 3x}{3} - \int 6x \, . \frac{\sin 3x}{3} \, dx$$

$$= \frac{1}{3}(2 + 3x^2) \sin 3x - 2 \int \underset{\mathrm{I}}{x} \, \underset{\mathrm{II}}{\sin 3x} \, dx$$

Integrating again by parts, we have

$$I_1 = \frac{1}{3}(2 + 3x^2) \sin 3x - 2\left[x \, . \int \sin 3x \, dx - \int \left\{\frac{d}{dx}(x) \, . \int \sin 3x \, dx\right\} dx\right]$$

$$= \frac{1}{3}(2 + 3x^2) \sin 3x - 2\left[x \, . \left(-\frac{\cos 3x}{3}\right) - \int 1 \, . \left(-\frac{\cos 3x}{3}\right) dx\right]$$

$$= \frac{1}{3}(2+3x^2)\sin 3x + \frac{2}{3}x\cos 3x - \frac{2}{3}\int \cos 3x\,dx$$

$$= \frac{1}{3}(2+3x^2)\sin 3x + \frac{2}{3}x\cos 3x - \frac{2}{3}\left(\frac{\sin 3x}{3}\right)$$

$$= \frac{1}{3}(2+3x^2)\sin 3x + \frac{2}{3}x\cos 3x - \frac{2}{9}\sin 3x$$

$$= \frac{1}{9}(6+9x^2-2)\sin 3x + \frac{2}{3}x\cos 3x$$

$$= \frac{1}{9}(9x^2+4)\sin 3x + \frac{2}{3}x\cos 3x$$

$$\therefore \quad \int_0^{\pi/6}(2+3x^2)\cos 3x\,dx = \left[\frac{1}{9}(9x^2+4)\sin 3x + \frac{2}{3}x\cos 3x\right]_0^{\pi/6}$$

$$= \left[\frac{1}{9}\left\{9\left(\frac{\pi}{6}\right)^2+4\right\}\sin\frac{3\pi}{6} + \frac{2}{3}\left(\frac{\pi}{6}\right)\cos\frac{3\pi}{6}\right] - \left[\frac{1}{9}(0+4)\sin 0° + 0\right]$$

$$= \left(\frac{1}{9}\left[\frac{9\pi^2}{36}+4\right]\sin\frac{\pi}{2} + \frac{\pi}{9}\cos\frac{\pi}{2}\right) - 0$$

$$= \left[\frac{\pi^2}{36}+\frac{4}{9}+0\right] = \frac{1}{36}(\pi^2+16).$$

(*v*) Let $\quad I = \int_{\pi/4}^{\pi/2}\cos 2x\log(\sin x)\,dx$

Let $\quad I_1 = \int \underset{\text{I}}{\cos 2x}\ \underset{\text{II}}{\log(\sin x)}\,dx$

Integrating by parts, we have

$$I_1 = \log(\sin x)\,.\int\cos 2x\,dx - \int\left\{\frac{d}{dx}\{\log(\sin x)\}\,.\int\cos 2x\,dx\right\}dx$$

$$= \log(\sin x)\,.\,\frac{\sin 2x}{2} - \int\frac{1}{\sin x}\,.\cos x\,.\frac{\sin 2x}{2}\,dx$$

$[\because \quad \sin 2A = 2\sin A\cos A]$

$$= \frac{1}{2}\log(\sin x)\,.\sin 2x - \frac{1}{2}\int\frac{\cos x}{\sin x}\,.\,2\sin x\cos x\,dx$$

$$= \frac{1}{2}\log(\sin x)\,.\sin 2x - \int\cos^2 x\,dx \qquad \left[\begin{array}{l}\because\ 1+\cos 2A = 2\cos^2 A\\ \Rightarrow\ \frac{1}{2}(1+\cos 2A) = \cos^2 A\end{array}\right]$$

$$= \frac{1}{2}\log(\sin x)\,.\sin 2x - \frac{1}{2}\int(1+\cos 2x)\,dx$$

$$= \frac{1}{2}\log(\sin x)\,.\sin 2x - \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]$$

$$= \frac{1}{2}\left[\log(\sin x)\,.\sin 2x - x - \frac{\sin 2x}{2}\right]$$

$$\therefore \quad \text{I} = \int_{\pi/4}^{\pi/2} \cos 2x \log(\sin x)\, dx = \frac{1}{2}\left[\log(\sin x) . \sin 2x - x - \frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{2}\left[\log\left(\sin\frac{\pi}{2}\right)\sin 2\left(\frac{\pi}{2}\right) - \frac{\pi}{2} - \frac{\sin^2\left(\frac{\pi}{2}\right)}{2}\right]$$

$$- \frac{1}{2}\left[\log\left(\sin\frac{\pi}{4}\right)\sin 2\left(\frac{\pi}{4}\right) - \frac{\pi}{4} - \frac{\sin^2\left(\frac{\pi}{4}\right)}{2}\right]$$

$$= \frac{1}{2}\left[(\log 1)\sin\pi - \frac{\pi}{2} - \frac{1}{2}\sin\pi\right] - \frac{1}{2}\left[\log\left(\frac{1}{\sqrt{2}}\right)\sin\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right]$$

$$= \frac{1}{2}\left[0 - \frac{\pi}{2} - 0\right] - \frac{1}{2}\left[\log\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{4} - \frac{1}{2}\right]$$

$$= -\frac{\pi}{4} - \frac{1}{2}\log\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{8} + \frac{1}{4}$$

$$= -\frac{\pi}{8} + \frac{1}{4} - \frac{1}{2}\log\left(\frac{1}{\sqrt{2}}\right).$$

(*vi*) Let $\quad \text{I} = \int_0^{\pi/2} x^2 \sin 3x\, dx$

Let $\quad \text{I}_1 = \int \underset{\text{I}}{x^2}\ \underset{\text{II}}{\sin 3x}\, dx$

Integrating by parts, we have

$$\text{I}_1 = x^2 . \int \sin 3x\, dx - \int \left\{\frac{d}{dx}(x^2) . \int \sin 3x\, dx\right\} dx$$

$$= x^2 . \frac{(-\cos 3x)}{3} - \int 2x \frac{(-\cos 3x)}{3}\, dx$$

$$= -\frac{1}{3}x^2 \cos 3x + \frac{2}{3}\int \underset{\text{I}}{x}\ \underset{\text{II}}{\cos 3x}\, dx$$

Integrating by parts again, we have

$$\text{I}_1 = -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\left[x . \int \cos 3x\, dx - \int \left\{\frac{d}{dx}(x) . \int \cos 3x\, dx\right\} dx\right]$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\left[x . \frac{\sin 3x}{3} - \int \frac{1 . \sin 3x}{3}\, dx\right]$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x - \frac{2}{9}\int \sin 3x\, dx$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x - \frac{2}{9}\left(-\frac{\cos 3x}{3}\right)$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x$$

$$= \frac{1}{27}(-9x^2 + 2)\cos 3x + \frac{2}{9} x \sin 3x$$

$$\therefore \quad I = \int_0^{\pi/2} x^2 \sin 3x \, dx = \left[\frac{1}{27}(-9x^2+2)\cos 3x + \frac{2}{9} x \sin 3x\right]_0^{\pi/2}$$

$$= \left[\frac{1}{27}\left\{-9\left(\frac{\pi}{2}\right)^2 + 2\right\}\cos\frac{3\pi}{2} + \frac{2}{9}\left(\frac{\pi}{2}\right)\sin\frac{3\pi}{2}\right] - \left[\frac{1}{27}\{0+2\}\cos 0° + 0\right]$$

$$= \left[0 + \frac{\pi}{9}(-1)\right] - \frac{2}{27} \qquad \left[\because \cos\left(\frac{3\pi}{2}\right) = \cos\left(\pi + \frac{\pi}{2}\right) = 0, \quad \sin\left(\frac{3\pi}{2}\right) = \sin\left(\pi + \frac{\pi}{2}\right) = -1\right]$$

$$= -\frac{\pi}{9} - \frac{2}{27} = -\frac{1}{27}(3\pi + 2).$$

(*vii*) Let $I = \int_1^3 \frac{\log x}{(x+1)^2}\, dx$ ...(1)

Let $I_1 = \int \underset{\text{I}}{\log x} \, . \, \underset{\text{II}}{(x+1)^{-2}}\, dx$ ...(2)

Integrating by parts, we get

$$I_1 = \log x \, . \int (x+1)^{-2}\, dx - \int \left\{\frac{d}{dx}(\log x) \, . \int (x+1)^{-2}\, dx\right\} dx$$

$$= \log x \, . \frac{(x+1)^{-2+1}}{(-2+1)} - \int \frac{1}{x} \, . \frac{(x+1)^{-2+1}}{(-2+1)}\, dx$$

$$= -\frac{\log x}{x+1} + \int \frac{1}{x(x+1)}\, dx \qquad \text{...(3)}$$

Let $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$ ...(4)

Muliplying both sides by $x(x+1)$, we have

$$1 = A(x+1) + Bx \qquad \text{...(5)}$$

Put $x = 0$ in equation (5), we get

$$1 = A(0+1) + 0 \Rightarrow A = 1$$

Put $x = -1$ in equation (5), we get

$$1 = A(-1+1) + B(-1) \Rightarrow 1 = -B \Rightarrow B = -1$$

Substituting the values of A and B in equation (4), we have

$$\frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{x+1}$$

$\therefore$ From equation (3), we get

$$I_1 = -\frac{\log x}{x+1} + \int\left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= -\frac{\log x}{x+1} + \int \frac{1}{x}\, dx - \int \frac{1}{x+1}\, dx = -\frac{\log x}{x+1} + \log|x| - \log|x+1|$$

$$= -\frac{\log x}{x+1} + \log\left|\frac{x}{x+1}\right| \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\therefore \quad \int_1^3 \frac{\log x}{(x+1)^2}\,dx = \left[-\frac{\log(x)}{(x+1)} + \log\left|\frac{x}{x+1}\right|\right]_1^3$$

$$= \left[-\frac{\log 3}{3+1} + \log\left(\frac{3}{3+1}\right)\right] - \left[-\frac{\log 1}{1+1} + \log\left(\frac{1}{1+1}\right)\right]$$

$$= -\frac{\log 3}{4} + \log\left(\frac{3}{4}\right) - \left[0 + \log\left(\frac{1}{2}\right)\right]$$

$$= -\frac{1}{4}\log 3 + \log 3 - \log 4 - [\log 1 - \log 2]$$

$$= \frac{3}{4}\log 3 - \log 4 + \log 2 \qquad \left[\because \ \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{3}{4}\log 3 - 2\log 2 + \log 2 \qquad [\because \ \log 4 = \log 2^2 = 2\log 2]$$

$$= \frac{3}{4}\log 3 - \log 2.$$

(*viii*) Let $\quad I = \int_1^2 \log x \, . \, dx$

Let $\quad I_1 = \int \underset{II}{1} \, . \, \underset{I}{\log x}$ [Taking unity as the second function]

Integrating by parts, we have

$$= \log x \, . \int 1 \, . \, dx - \int \left\{\frac{d}{dx}(\log x) \, . \int 1 \, . \, dx\right\} dx$$

$$= \log x \, . \, x - \int \frac{1}{x} \, . \, x\,dx = x\log x - \int 1 \, . \, dx$$

$$= x\log x - x$$

$$\therefore \quad \int_1^2 \log x\,dx = \Big[x\log x - x\Big]_1^2$$

$$= [2\log 2 - 2] - [1\log 1 - 1]$$

$$= 2\log 2 - 2 - 0 + 1 = 2\log 2 - 1 \qquad [\because \ n\log m = \log m^n]$$

$$= \log 2^2 - 1 = \log 4 - 1$$

$$= -1 + \log 4.$$

Example 21. *If f(x) is of the form :*

$$f(x) = a + bx + cx^2$$

Show that : $\quad \int_0^1 f(x)\,dx = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right].$

Solution. It is given that

$$f(x) = cx^2 + bx + a \qquad ...(1)$$

$$\therefore \quad \int_0^1 f(x)\,dx = \int_0^1 (cx^2 + bx + a)\,dx$$

$$= c\int_0^1 x^2\,dx + b\int_0^1 x\,dx + a\int_0^1 1 \, . \, dx$$

$\therefore \quad I = \int_5^{17} \frac{1}{z}\left(\frac{1}{2}\,dz\right) = \frac{1}{2}\int_5^{17} \frac{1}{z}\,dz$

$= \frac{1}{2}\Big[\log|z|\Big]_5^{17}$

$= \frac{1}{2}\,[\log|17| - \log|5|] \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$

$= \frac{1}{2}\log\frac{17}{5}$.

(*v*) Let $\quad I = \int_1^2 3x\sqrt{5-x^2}\,dx$

Put $\quad 5 - x^2 = z \;\Rightarrow\; -2x\,dx = dz \;\Rightarrow\; x\,dx = -\frac{1}{2}\,dz$

When $\quad x = 1, z = 5 - (1)^2 = 5 - 1 = 4$ and when $x = 2, z = 5 - (2)^2 = 5 - 4 = 1$

$\therefore \quad I = 3\int_4^1 \sqrt{z}\,.\left(-\frac{1}{2}\,dz\right) = -\frac{3}{2}\int_4^1 z^{1/2}\,dz$

$= -\frac{3}{2}\left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_4^1 = \left(-\frac{3}{2}\right)\left(\frac{2}{3}\right)\left[z^{3/2}\right]_4^1 \qquad [\because (4)^{3/2} = (2^2)^{3/2} = (2)^3 = 8]$

$= -[(1)^{3/2} - (4)^{3/2}] = -[1 - 8] = 7.$

(*vi*) Let $\quad I = \int_0^1 \frac{x^2}{1+x^6}\,dx$

Put $\quad x^3 = z \;\Rightarrow\; 3x^2\,dx = dz \;\Rightarrow\; x^2\,dx = \frac{1}{3}\,dz$

When $\quad x = 0, z = 0$ and when $x = 1, z = (1)^3 = 1$

$\therefore \quad I = \int_0^1 \frac{1}{1+z^2}\left(\frac{1}{3}\,dz\right) = \frac{1}{3}\int_0^1 \frac{1}{1+z^2}\,dz \qquad \left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$

$= \frac{1}{3}\Big[\tan^{-1} z\Big]_0^1 = \frac{1}{3}\,[\tan^{-1} 1 - \tan^{-1} 0] \qquad \left[\because \tan\frac{\pi}{4} = \tan 45° = 1\right]$

$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{12}$.

Example 23. *Evaluate the following integrals :*

(*i*) $\int_0^1 \frac{x}{1+\sqrt{x}}\,dx$ (*ii*) $\int_0^1 \frac{e^x}{1+e^{2x}}\,dx$

(*iii*) $\int_0^1 \frac{24x^3}{(1+x^2)^4}\,dx$ (*iv*) $\int_1^2 \frac{3x}{9x^2-1}\,dx$

(*v*) $\int_0^1 x\sqrt{1-x^2}\,dx$ (*vi*) $\int_0^1 x(1-x)^5\,dx$.

Solution. (*i*) Let $\mathrm{I} = \int_0^1 \frac{x}{1+\sqrt{x}}\, dx$

Put $\sqrt{x} = z \Rightarrow x = z^2 \Rightarrow dx = 2z\, dz$

When $x = 0, z = 0$ and when $x = 1, z = \sqrt{1} = 1$

$$\therefore \quad \mathrm{I} = \int_0^1 \frac{z^2}{1+z}(2z\, dz) = 2\int_0^1 \frac{z^3}{(1+z)}\, dz$$

$$= 2\int_0^1 \frac{z^3+1-1}{z+1}\, dz \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= 2\int_0^1 \left[\frac{z^3+1}{z+1} - \frac{1}{z+1}\right] dz = 2\int_0^1 \frac{z^3+1}{z+1}\, dz - 2\int_0^1 \frac{1}{z+1}\, dz$$

Remember. If the integrand is a rational function whose numerator and denominator are polynomials and degree of numerator is greater than or equal to that of the denominator, first divide the numerator by the denominator and then use the result :

$$\left[\frac{\text{Numerator}}{\text{Denominator}} = \text{Quatient} + \frac{\text{Remainder}}{\text{Denominator}}\right]$$

$\therefore$ By actual division ;

$$= 2\int_0^1 (z^2 - z + 1)\, dz - 2\int_0^1 \frac{1}{z+1}\, dz$$

$$= 2\left[\frac{z^3}{3} - \frac{z^2}{2} + z\right]_0^1 - 2\left[\log|z+1|\right]_0^1$$

$$= 2\left[\frac{1}{3} - \frac{1}{2} + 1 - 0\right] - 2\,[\log(1+1) - \log(0+1)]$$

$$= 2\left[\frac{2-3+6}{6}\right] - 2\,[\log 2 - \log 1]$$

$$= \frac{5}{3} - 2\log 2.$$

$$\begin{array}{r l}
z+1 \,\big) & z^3+1 \quad \big(z^2 - z + 1 \\
 & z^3 + z^2 \\
 & - \quad - \\
\hline
 & -z^2 + 1 \\
 & -z^2 - z \\
 & + \quad + \\
\hline
 & +z + 1 \\
 & z + 1 \\
 & - \quad - \\
\hline
 & \times
\end{array}$$

(*ii*) Let $\mathrm{I} = \int_0^1 \frac{e^x}{1+e^{2x}}\, dx$

Put $e^x = z \Rightarrow e^x\, dx = dz$

When $x = 0 \Rightarrow z = e^0 = 1$ and when $x = 1 \Rightarrow z = e^1 = e$

$$\therefore \quad \mathrm{I} = \int_1^e \frac{1}{1+z^2}\, dz$$

$$= \left[\tan^{-1} z\right]_1^e \qquad \left[\because \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= [\tan^{-1} e - \tan^{-1} 1] = \tan^{-1} e - \frac{\pi}{4}.$$

(*iii*) Let $I = \int_0^1 \frac{24x^3}{(1+x^2)^4}\,dx = \int_0^1 \frac{12x^2 \,.\, 2x}{(1+x^2)^4}\,dx$ [Note this step]

Put $(1 + x^2) = z \;\Rightarrow\; 2x\,dx = dz$

When $x = 0, z = (0 + 1) = 1$ and when $x = 1, z = (1 + 1^2) = 2$

$$\therefore \quad I = \int_1^2 \frac{12(z-1)}{z^4}\,dz$$

$$= 12\int_1^2\left(\frac{z}{z^4} - \frac{1}{z^4}\right)dz = 12\int_1^2 \frac{1}{z^3}\,dz - 12\int_1^2 \frac{1}{z^4}\,dz$$

$$= 12\left[\frac{z^{-3+1}}{-3+1}\right]_1^2 - 12\left[\frac{z^{-4+1}}{-4+1}\right]_1^2 = -6\left[\frac{1}{z^2}\right]_1^2 + 4\left[\frac{1}{z^3}\right]_1^2$$

$$= -6\left[\frac{1}{2^2} - \frac{1}{1^2}\right] + 4\left[\frac{1}{2^3} - \frac{1}{1^3}\right]$$

$$= -6\left[\frac{1}{4} - 1\right] + 4\left[\frac{1}{8} - 1\right] = -6\left(-\frac{3}{4}\right) + 4\left(-\frac{7}{8}\right)$$

$$= \frac{9}{2} - \frac{7}{2} = \frac{2}{2} = 1.$$

(*iv*) Let $\quad I = \int_1^2 \frac{3x}{9x^2 - 1}\,dx$

Put $\quad 9x^2 - 1 = z \;\Rightarrow\; 18x\,dx = dz \;\Rightarrow\; 3x\,dx = \frac{1}{6}\,dz$

When $x = 1, z = 9(1)^2 - 1 = 9 - 1 = 8$ and when $x = 2, z = 9(2)^2 - 1 = 9(4) - 1 = 36 - 1 = 35$

$$\therefore \quad I = \int_8^{35} \frac{1}{z}\left(\frac{1}{6}\,dz\right) = \frac{1}{6}\int_8^{35} \frac{1}{z}\,dz$$

$$= \frac{1}{6}\Big[\log|z|\Big]_8^{35} = \frac{1}{6}\,[\log 35 - \log 8]$$

$$= \frac{1}{6}\log\frac{35}{8}. \qquad \left[\because\ \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

(*v*) Let $I = \int_0^1 x\sqrt{1-x^2}\,dx$

Put $\quad 1 - x^2 = z \;\Rightarrow\; -2x\,dx = dz \;\Rightarrow\; x\,dx = -\frac{1}{2}\,dz$

When $\quad x = 0, z = 1 - 0 = 1$ and when $x = 1, z = 1 - (1)^2 = 1 - 1 = 0$

$$\therefore \quad I = \int_1^0 \sqrt{z}\,.\left(-\frac{1}{2}\right)dz = -\frac{1}{2}\int_1^0 z^{1/2}\,dz$$

$$= -\frac{1}{2}\left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^0 = -\frac{1}{2}\times\frac{2}{3}\left[z^{3/2}\right]_1^0$$

$$= \left(\frac{\pi}{2} + 0\right) - (0 + 1) = \frac{\pi}{2} - 1.$$

(*ii*) Let $\quad I = \int_a^b \frac{\log x}{x}\, dx$

Put $\quad \log x = z \;\Rightarrow\; \frac{1}{x}\, dx = dz$

When $\quad x = a,\ z = \log a$ and when $x = b,\ z = \log b$

$\therefore \quad I = \int_{\log a}^{\log b} z\, dz = \left[\frac{z^2}{2}\right]_{\log a}^{\log b} = \frac{1}{2}\,[(\log b)^2 - (\log a)^2]$

$$= \frac{1}{2}\,[(\log b + \log a)(\log b - \log a)] \qquad \left[\begin{array}{l} \because\ \log m + \log n = \log m.n \\ \log m - \log n = \log\left(\dfrac{m}{n}\right) \end{array}\right]$$

$$= \frac{1}{2} \log (ba) \log\left(\frac{b}{a}\right).$$

(*iii*) Let $\quad I = \int_0^{\pi/2} \frac{\sin x \cos x}{1 + \sin^4 x}\, dx$

Put $\sin^2 x = z \;\Rightarrow\; 2 \sin x \cos x\, dx = dz \;\Rightarrow\; \sin x \cos x = \frac{1}{2}\, dz$

When $\quad x = 0,\ z = \sin^2 0 = 0$ and when $x = \frac{\pi}{2},\ z = \sin^2 \frac{\pi}{2} = (1)^2 = 1$

$\therefore \quad I = \int_0^1 \frac{1}{1+z^2}\left(\frac{1}{2}\, dz\right) = \frac{1}{2}\int_0^1 \frac{1}{1+z^2}\, dz$

$$= \frac{1}{2}\left[\tan^{-1} z\right]_0^1 \qquad \left[\because\ \int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{2}\,[\tan^{-1} 1 - \tan^{-1} 0] = \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{8}.$$

(*iv*) Let $\quad I = \int_1^2 x\sqrt{3x - 2}\, dx$

Put $\sqrt{3x-2} = z \;\Rightarrow\; (3x - 2) = z^2 \;\Rightarrow\; 3x = z^2 + 2 \;\Rightarrow\; 3\, dx = 2z\, dz \;\Rightarrow\; dx = \frac{2}{3} z\, dz$

When $\quad x = 1,\ z = \sqrt{3(1) - 2} = \sqrt{1} = 1$ and when $x = 2,\ z = \sqrt{3(2) - 2} = \sqrt{6 - 2} = \sqrt{4} = 2.$

$\therefore \quad I = \int_1^2 \left(\frac{z^2 + 2}{3}\right).z.\left(\frac{2}{3} z\, dz\right)$

$$= \frac{2}{9}\int_1^2 (z^4 + 2z^2)\, dz = \frac{2}{9}\left[\frac{z^5}{5} + \frac{2z^3}{3}\right]_1^2$$

$$= \frac{2}{9}\left[\frac{(2)^5}{5} + \frac{2(2)^3}{3}\right] - \frac{2}{9}\left[\frac{(1)^5}{5} + \frac{2(1)^3}{3}\right]$$

$$= \frac{2}{9}\left[\frac{32}{5} + \frac{16}{3}\right] - \frac{2}{9}\left[\frac{1}{5} + \frac{2}{3}\right] = \frac{2}{9}\left(\frac{96+80}{15}\right) - \frac{2}{9}\left(\frac{3+10}{15}\right) = \frac{2}{9}\left[\frac{176}{15} - \frac{13}{15}\right] = \frac{2}{9}\left[\frac{176-13}{15}\right]$$

$$= \frac{2}{9} \times \frac{163}{15} = \frac{326}{135}.$$

(*v*) Let $\quad I = \int_0^1 \cos^{-1} x \, dx$

Put $\quad \cos^{-1} x = z \ \Rightarrow\ x = \cos z \ \Rightarrow\ dx = -\sin z \, dz$

When $\quad x = 0, \cos z = 0 \ \Rightarrow\ z = \pi/2$ and when $x = 1, \cos z = 1 \ \Rightarrow\ z = 0$

$$\therefore \quad I = \int_{\pi/2}^{0} z(-\sin z \, dz) = -\int_{\pi/2}^{0} z \sin z \, dz$$

Let $\quad I_1 = \int \underset{\text{I}}{z} \ \underset{\text{II}}{\sin z} \, dz$

Integrating by parts, we have

$$I_1 = z \,.\, \int \sin z \, dz - \int \left\{\frac{d}{dz}(z) \,.\, \int \sin z \, dz\right\} dz$$

$$= z\,(-\cos z) - \int 1 \,.\, (-\cos z)\, dz = -z \cos z + \int \cos z \, dz$$

$$= -z \cos z + \sin z$$

$$\therefore \quad -\int_{\pi/2}^{0} z \sin z \, dz = -\left[-z \cos z + \sin z\right]_{\pi/2}^{0}$$

$$= \left[z \cos z - \sin z\right]_{\pi/2}^{0}$$

$$= [0 - \sin 0°] - \left[\frac{\pi}{2}\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right] = 0 - (0 - 1) = 1.$$

(*vi*) Let $\quad I = \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} \, dx$

Put $\quad \cos x = z \ \Rightarrow\ -\sin x \, dx = dz \ \Rightarrow\ \sin x \, dx = -dz$

When $\quad x = 0, z = \cos 0° = 1$ and when $x = \frac{\pi}{2}$, $z = \cos\frac{\pi}{2} = 0$

$$\therefore \quad I = \int_1^0 \frac{1}{1+z^2}(-dz) = -\int_1^0 \frac{1}{1+z^2} \, dz$$

$$= -\left[\tan^{-1} z\right]_1^0 \qquad \left[\because \int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= -[\tan^{-1} 0 - \tan^{-1} 1] = -\left(0 - \frac{\pi}{4}\right) = \frac{\pi}{4}.$$

Example 25. *Evaluate the following integrals :*

(i) $\int_0^{\pi/2} \sin 2x \tan^{-1} (\sin x)\, dx$ (ii) $\int_0^{\pi/2} \sqrt{\cos\theta}\, \sin^3\theta\, d\theta$

(iii) $\int_0^{\pi/2} \sqrt{\sin\theta}\, \cos^5\theta\, d\theta$ (iv) $\int_0^{\pi/2} \cos^3\theta\, (\sin\theta)^{1/4}\, d\theta$

(v) $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx.$

Solution. (i) Let $I = \int_0^{\pi/2} \sin 2x \tan^{-1} (\sin x)\, dx$

$\Rightarrow \quad I = \int_0^{\pi/2} 2 \sin x \cos x \tan^{-1} (\sin x)\, dx \qquad [\because \sin 2A = 2 \sin A \cos A]$

Put $\sin x = z \Rightarrow \cos x\, dx = dz$

When $x = 0, z = \sin 0° = 0$ and when $x = \frac{\pi}{2}, z = \sin\frac{\pi}{2} = 1$

$\therefore \quad I = \int_0^1 2z (\tan^{-1} z)\, dz = 2\int_0^1 z \tan^{-1} z\, dz$

Let $\quad I_1 = \int \underset{\text{II}}{z} \underset{\text{I}}{\tan^{-1} z}\, dz$

Integrating by parts, we get

$$I_1 = \tan^{-1} z \,.\int z\, dz - \int \left\{\frac{d}{dz}(\tan^{-1} z) . \int z\, dz\right\} dz$$

$$= \tan^{-1} z \,.\, \frac{z^2}{2} - \int \frac{1}{1+z^2} . \frac{z^2}{2}\, dz = \frac{1}{2} z^2 \tan^{-1} z - \frac{1}{2}\int \frac{z^2}{z^2+1}\, dz$$

$$= \frac{1}{2} z^2 \tan^{-1} z - \frac{1}{2}\int \left(\frac{z^2+1-1}{z^2+1}\right) dz \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \frac{1}{2} z^2 \tan^{-1} z - \frac{1}{2}\int 1 . dz + \frac{1}{2}\int \frac{1}{1+z^2}\, dz$$

$$= \frac{1}{2} z^2 \tan^{-1} z - \frac{1}{2} z + \frac{1}{2}\tan^{-1} z \qquad \left[\because \int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{2}\,[(z^2+1) \tan^{-1} z - z]$$

$$\therefore \quad I = 2\int_0^1 z \tan^{-1} z\, dz = 2\left[\frac{1}{2}\{(z^2+1)\tan^{-1} z - z\}\right]_0^1 = \left[(z^2+1)\tan^{-1} z - z\right]_0^1$$

$$= [(1+1) \tan^{-1} 1 - 1] - [(0+1) \tan^{-1} 0 - 0]$$

$$= [2 \tan^{-1} 1 - 1] - \tan^{-1} 0 = 2\,\frac{\pi}{4} - 1 - 0$$

$$= \frac{\pi}{2} - 1.$$

(ii) Let $\quad I = \int_0^{\pi/2} \sqrt{\cos\theta}\, .\, \sin^3\theta\, d\theta$

$$= \int_0^{\pi/2} \sqrt{\cos\theta}\, .\, \sin^2\theta \sin\theta\, d\theta \qquad \text{[Note this step]}$$

$= \int_0^{\pi/2} \sqrt{\cos\theta}\,.\,(1-\cos^2\theta)\,.\sin\theta\,d\theta$ $\quad [\because \sin^2 A + \cos^2 A = 1]$

Put $\cos\theta = z \Rightarrow -\sin\theta\,d\theta = dz \Rightarrow \sin\theta\,d\theta = -dz$

When $\theta = 0,\ z = \cos 0° = 1$ and when $\theta = \frac{\pi}{2},\ z = \cos\frac{\pi}{2} = 0$

$$\therefore \quad I = \int_1^0 \sqrt{z}\,(1-z^2)(-dz) = \int_1^0 \sqrt{z}\,(z^2-1)\,dz = \int_1^0 (z^{5/2} - z^{1/2})\,dz$$

$$= \left[\frac{z^{\frac{5}{2}+1}}{\frac{5}{2}+1} - \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^0 = \left[\frac{2}{7}z^{7/2} - \frac{2}{3}z^{3/2}\right]_1^0 = \left[(0-0) - \left(\frac{2}{7} - \frac{2}{3}\right)\right]$$

$$= -\left(\frac{6-14}{21}\right) = \frac{8}{21}.$$

(iii) Let $I = \int_0^{\pi/2} \sqrt{\sin\theta}\,.\cos^5\theta\,d\theta$

$= \int_0^{\pi/2} \sqrt{\sin\theta}\,.\,(\cos^2\theta)^2\,.\cos\theta\,d\theta$ [Note this step]

$= \int_0^{\pi/2} \sqrt{\sin\theta}\,.\,(1-\sin^2\theta)^2\cos\theta\,d\theta$ $\quad [\because \cos^2 A + \sin^2 A = 1]$

Put $\sin\theta = z \Rightarrow \cos\theta\,d\theta = dz$

When $\theta = 0,\ z = \sin 0° = 0$ and when $\theta = \frac{\pi}{2},\ z = \sin\frac{\pi}{2} = 1$

$$\therefore \quad I = \int_0^1 \sqrt{z}\,(1-z^2)^2\,dz$$

$= \int_0^1 \sqrt{z}\,(1 + z^4 - 2z^2)\,dz$ $\quad [\because (a-b)^2 = a^2 + b^2 - 2ab]$

$$= \int_0^1 (z^{1/2} + z^{9/2} - 2z^{5/2})\,dz$$

$$= \left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{z^{\frac{9}{2}+1}}{\frac{9}{2}+1} - \frac{2z^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right]_0^1 = \left[\frac{2}{3}z^{3/2} + \frac{2}{11}z^{11/2} - \frac{4}{7}z^{7/2}\right]_0^1$$

$$= \left[\frac{2}{3}(1)^{3/2} + \frac{2}{11}(1)^{11/2} - \frac{4}{7}(1)^{7/2}\right] - [0+0-0] = \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154 + 42 - 132}{231} = \frac{64}{231}.$$

(iv) Let $I = \int_0^{\pi/2} \cos^3\theta\,(\sin\theta)^{1/4}\,d\theta$

$= \int_0^{\pi/2} (\sin\theta)^{1/4}\,.\cos^2\theta\,.\cos\theta\,d\theta$ [Note this step]

$$= \int_0^{\pi/2} (\sin\theta)^{1/4} \, . \, (1 - \sin^2\theta) \, . \, \cos\theta \, d\theta \qquad [\because \; \sin^2 A + \cos^2 A = 1]$$

Put $\sin\theta = z \;\Rightarrow\; \cos\theta \, d\theta = dz$

When $\theta = 0, z = \sin 0° = 0$ and when $\theta = \dfrac{\pi}{2}$, $z = \sin\dfrac{\pi}{2} = 1$

$$\therefore \qquad I = \int_0^1 (z)^{1/4} (1 - z^2) \, dz = \int_0^1 (z^{1/4} - z^{9/4}) \, dz$$

$$= \left[\frac{z^{\frac{1}{4}+1}}{\frac{1}{4}+1} - \frac{z^{\frac{9}{4}+1}}{\frac{9}{4}+1} \right]_0^1 = \left[\frac{4}{5} z^{5/4} - \frac{4}{13} z^{13/4} \right]_0^1$$

$$= \left[\frac{4}{5} (1)^{5/4} - \frac{4}{13} (1)^{13/4} \right] - [0 - 0] = \left(\frac{4}{5} - \frac{4}{13} \right) - 0$$

$$= \frac{52 - 20}{65} = \frac{32}{65} \, .$$

(*v*) Let $\quad I = \displaystyle\int_0^1 \sin^{-1}\left(\frac{2x}{1 + x^2} \right) dx$

Put $\quad x = \tan\theta \;\Rightarrow\; dx = \sec^2\theta \, d\theta$

When $\quad x = 0, \tan\theta = 0 \;\Rightarrow\; \theta = 0$ and when $x = 1, \tan\theta = 1 \;\Rightarrow\; \theta = \dfrac{\pi}{4}$

$$\therefore \qquad I = \int_0^{\pi/4} \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta} \right) . \sec^2\theta \, d\theta \qquad \left[\because \; \sin 2A = \frac{2\tan A}{1 + \tan^2 A} \right]$$

$$= \int_0^{\pi/4} \sin^{-1} (\sin 2\theta) \sec^2\theta \, d\theta$$

$$\Rightarrow \qquad I = \int_0^{\pi/4} 2\theta \sec^2\theta \, d\theta = 2 \int_0^{\pi/4} \theta \sec^2\theta \, d\theta$$

Let $\quad I_1 = \displaystyle\int \underset{\text{I}}{\theta} \, \underset{\text{II}}{\sec^2\theta} \, d\theta$

Integrating by parts, we get

$$= \theta \, . \int \sec^2\theta \, d\theta - \int \left\{ \frac{d}{d\theta} (\theta) \, . \int \sec^2\theta \, d\theta \right\} d\theta$$

$$= \theta \tan\theta - \int 1 \, . \tan\theta \, d\theta = \theta \tan\theta - \log |\sec\theta|$$

$$\therefore \qquad I = 2 \int_0^{\pi/4} \theta \sec^2\theta \, d\theta = 2 \Big[\theta \tan\theta - \log |\sec\theta| \Big]_0^{\pi/4}$$

$$= 2 \left[\frac{\pi}{4} \tan\frac{\pi}{4} - \log\left(\sec\frac{\pi}{4} \right) \right] - 2 \, [0 - \log (\sec 0°)]$$

$$= 2 \left[\frac{\pi}{4} - \log\sqrt{2} \right] - 2(0 - \log 1)$$

$$= \frac{\pi}{2} - 2\log\sqrt{2} = \frac{\pi}{2} - \log\left(\sqrt{2}\right)^2 \qquad \left[\begin{array}{l}\because \log 1 = 0 \\ m\log n = \log n^m\end{array}\right]$$

$$= \frac{\pi}{2} - \log 2.$$

Example 26. *Evaluate the following integrals :*

(i) $\int_0^a \frac{x}{\sqrt{a^2+x^2}}\,dx$ (ii) $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}\,dx$

(iii) $\int_1^3 \frac{\cos(\log x)}{x}\,dx$ (iv) $\int_0^1 \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$

(v) $\int_0^{\pi/6}(1-\cos 3\theta)\sin 3\theta\,d\theta$ (vi) $\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx$

Solution. (i) Let $I = \int_0^a \frac{x}{\sqrt{a^2+x^2}}\,dx$

Put $\sqrt{a^2+x^2} = z \Rightarrow a^2+x^2 = z^2 \Rightarrow 2x\,dx = 2z\,dz \Rightarrow x\,dx = z\,dz$

When $x = 0, z = \sqrt{a^2+0} = \sqrt{a^2} = a$ and when $x = a$, $z = \sqrt{a^2+a^2} = \sqrt{2a^2} = \sqrt{2}\,a$

$$\therefore \quad I = \int_a^{\sqrt{2}a}\frac{1}{z}.(z.dz) = \int_a^{\sqrt{2}a}1.dz = \Big[z\Big]_a^{\sqrt{2}a} = [\sqrt{2}a - a]$$

$$= a(\sqrt{2}-1).$$

(ii) Let $I = \int_0^a \frac{x^4}{\sqrt{a^2-x^2}}\,dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

When $x = 0, a\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

and when $x = a, a\sin\theta = a \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$.

$$\therefore \quad I = \int_0^{\pi/2}\frac{a^4\sin^4\theta}{\sqrt{a^2-a^2\sin^2\theta}}.a\cos\theta\,d\theta$$

$$= \int_0^{\pi/2}\frac{a^5\sin^4\theta\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}\,d\theta = \int_0^{\pi/2}\frac{a^5\sin^4\theta\cos\theta}{a\sqrt{\cos^2\theta}}\,d\theta \quad [\because \sin^2 A + \cos^2 A = 1]$$

$$= a\int_0^{\pi/2}\sin^4\theta\,d\theta = a\int_0^{\pi/2}(\sin^2\theta)^2\,d\theta$$

$$= a^4\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta \qquad \left[\begin{array}{l}\because 1-\cos 2A = 2\sin^2 A \\ \frac{1-\cos 2A}{2} = \sin^2 A\end{array}\right]$$

$$= \frac{a^4}{4}\int_0^{\pi/2}(1+\cos^2 2\theta - 2\cos 2\theta)\,d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2}\left[1 + \frac{1}{2}(1+\cos 4\theta) - 2\cos 2\theta\right] d\theta \qquad \left[\begin{array}{l} \because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ 1+\cos 4A = 2\cos^2 2A \\ \Rightarrow\ \frac{1}{2}(1+\cos 4A) = \cos^2 2A \end{array}\right]$$

$$= \frac{a^4}{8}\int_0^{\pi/2}(2 + 1 + \cos 4\theta - 4\cos 2\theta)\, d\theta$$

$$= \frac{a^4}{8}\left[3\int_0^{\pi/2} 1.\, d\theta + \int_0^{\pi/2}\cos 4\theta\, d\theta - 4\int_0^{\pi/2}\cos 2\theta\, d\theta\right]$$

$$= \frac{a^4}{8}\left(3.\left[\theta\right]_0^{\pi/2} + \left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2} - 4\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}\right)$$

$$= \frac{a^4}{8}\left[3\left(\frac{\pi}{2} - 0\right) + \frac{1}{4}\left(\sin\frac{4\pi}{2} - \sin 0°\right) - 2\left(\sin\frac{2\pi}{2} - \sin 0°\right)\right]$$

$$= \frac{a^4}{8}\left[\frac{3\pi}{2} + \frac{1}{4}(0-0) - 2(0-0)\right]$$

$$= \frac{3a^4\pi}{16}.$$

(*iii*) Let $I = \int_1^3 \frac{\cos(\log x)}{x}\, dx$

Put $\log x = z \Rightarrow \frac{1}{x}\, dx = dz$

When $x = 1, z = \log 1 \Rightarrow z = 0$ and when $x = 3, z = \log 3 \Rightarrow z = \log 3$.

$$\therefore \quad I = \int_0^{\log 3}\cos z\, dz$$

$$= \left[\sin z\right]_0^{\log 3} = [\sin(\log 3) - \sin 0°] = \sin(\log 3) - 0$$

$$= \sin(\log 3).$$

(*iv*) Let $I = \int_0^1 \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\, dx = \int_0^1 \frac{x\tan^{-1}x}{(1+x^2)\sqrt{1+x^2}}\, dx$

Put $\tan^{-1}x = z \Rightarrow x = \tan z \Rightarrow \frac{1}{1+x^2}\, dx = dz$

When $x = 0, \tan z = 0 \Rightarrow z = 0$ and when $x = 1, \tan z = 1 \Rightarrow z = \frac{\pi}{4}$.

$$\therefore \quad I = \int_0^{\pi/4}\frac{z\tan z}{\sqrt{1+\tan^2 z}}\, dz$$

$$= \int_0^{\pi/4}\frac{z\tan z}{\sqrt{\sec^2 z}}\, dz = \int_0^{\pi/4}\frac{z\tan z}{\sec z}\, dz \qquad [\because\ \sec^2 A - \tan^2 A = 1]$$

$$= \int_0^{\pi/4} z \cdot \frac{\sin z}{\cos x} \cdot \frac{1}{\frac{1}{\cos z}}\, dz = \int_0^{\pi/4} z \sin z\, dz$$

Let $\quad I_1 = \int \underset{\text{I}}{z}\ \underset{\text{II}}{\sin z}\, dz$

Integrating by parts, we get

$$I_1 = z \cdot \int \sin z\, dz - \int \left\{\frac{d}{dz}(z) \cdot \int \sin z\, dz\right\} dz$$

$$= z(-\cos z) - \int 1 \cdot (-\cos z)\, dz$$

$$= -z\cos z + \int \cos z\, dz = -z\cos z + \sin z$$

$\therefore \quad I = \int_0^{\pi/4} z \sin z\, dz = \Big[-z\cos z + \sin z\Big]_0^{\pi/4}$

$$= \left[-\frac{\pi}{4}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right] - [0 + \sin 0°] = \left(-\frac{\pi}{4}\cdot\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\right) - 0$$

$$= \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} = \left(\frac{4-\pi}{4\sqrt{2}}\right).$$

(*v*) Let $\quad I = \int_0^{\pi/6} (1 - \cos 3\theta) \sin 3\theta\, d\theta$

Put $\cos 3\theta = z \Rightarrow -3\sin 3\theta\, d\theta = dz \Rightarrow \sin 3\theta\, d\theta = -\frac{1}{3}dz$

When $\quad \theta = 0, z = \cos 0 \Rightarrow z = 1$ and when $\theta = \frac{\pi}{6}, z = \cos\frac{3\pi}{6} \Rightarrow z = \cos\frac{\pi}{2} = 0$

$\therefore \quad I = \int_1^0 (1-z)\left(-\frac{1}{3}dz\right) = -\frac{1}{3}\int_1^0 (1-z)\, dz$

$$= -\frac{1}{3}\left[z - \frac{z^2}{2}\right]_1^0 = -\frac{1}{3}\left[(0-0) - \left(1 - \frac{1}{2}\right)\right] = -\frac{1}{3}\left[-\frac{1}{2}\right]$$

$$= \frac{1}{6}.$$

(*vi*) Let $\quad I = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}} \times \frac{\sqrt{1-\cos x}}{\sqrt{1-\cos x}}\, dx$

[Rationalization]

$$= \int_{\pi/3}^{\pi/2} \frac{\sqrt{1-\cos^2 x}}{(1-\cos x)^3}\, dx = \int_{\pi/3}^{\pi/2} \frac{\sqrt{\sin^2 x}}{(1-\cos x)^3}\, dx \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \int_{\pi/3}^{\pi/2} \frac{\sin x}{(1-\cos x)^3}\, dx$$

$$= 2a \int_0^{\pi/4} \sin^{-1} \sqrt{\frac{\tan^2 \theta}{1+\tan^2 \theta}} \tan \theta \sec^2 \theta \, d\theta$$

$$= 2a \int_0^{\pi/4} \sin^{-1} \sqrt{\frac{\tan^2 \theta}{\sec^2 \theta}} \tan \theta \sec^2 \theta \, d\theta \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$$= 2a \int_0^{\pi/4} \sin^{-1} \left(\frac{\tan \theta}{\sec \theta}\right) \tan \theta \sec^2 \theta \, d\theta = 2a \int_0^{\pi/4} \sin^{-1} \left[\frac{\frac{\sin \theta}{\cos \theta}}{\frac{1}{\cos \theta}}\right] \tan \theta \sec^2 \theta \, d\theta$$

$$= 2a \int_0^{\pi/4} \sin^{-1} (\sin \theta) \,.\, \tan \theta \sec^2 \theta \, d\theta = 2a \int_0^{\pi/4} \theta \,.\, \tan \theta \sec^2 \theta \, d\theta$$

Let $\quad I_1 = \int \underset{\text{I}}{\theta} \, \underset{\text{II}}{\tan \theta \sec^2 \theta} \, d\theta$

Integrating by parts, we get

$$I_1 = \theta \,.\, \int \tan \theta \sec^2 \theta \, d\theta - \int \left\{\frac{d}{d\theta}(\theta) \,.\, \int \tan \theta \sec^2 \theta \, d\theta\right\} d\theta$$

$$= \theta \,.\, \frac{\tan^2 \theta}{2} - \int 1 \,.\, \frac{\tan^2 \theta}{2} \, d\theta \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \frac{f(x)^{n+1}}{n+1} + c\right]$$

$$= \frac{1}{2} \theta \tan^2 \theta - \frac{1}{2} \int (\sec^2 \theta - 1) \, d\theta \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$$= \frac{1}{2} \theta \tan^2 \theta - \frac{1}{2} \int \sec^2 \theta \, d\theta + \frac{1}{2} \int 1 \,.\, d\theta = \frac{1}{2} \theta \tan^2 \theta - \frac{1}{2} \tan \theta + \frac{\theta}{2}$$

$$\therefore \quad I = 2a \int_0^{\pi/4} \theta \,.\, \tan \theta \sec^2 \theta \, d\theta = 2a \left[\frac{1}{2} \theta \tan^2 \theta - \frac{1}{2} \tan \theta + \frac{\theta}{2}\right]_0^{\pi/4}$$

$$= 2a \,.\, \frac{1}{2} \left[\theta \tan^2 \theta - \tan \theta + \theta\right]_0^{\pi/4} = a \left[\left(\frac{\pi}{4} \tan^2 \frac{\pi}{4} - \tan \frac{\pi}{4} + \frac{\pi}{4}\right) - (0 - \tan 0^\circ + 0)\right]$$

$$= a \left[\left(\frac{\pi}{4} - 1 + \frac{\pi}{4}\right) - (0)\right] = a \left[\frac{2\pi}{4} - 1\right]$$

$$= \frac{a\pi}{2} - a.$$

(iii) Let $I = \int_0^{\left(\frac{\pi}{2}\right)^{1/3}} x^2 \sin x^3 \, dx$

Put $\quad x^3 = z \;\Rightarrow\; 3x^2 \, dx = dz \;\Rightarrow\; x^2 \, dx = \frac{1}{3} dz$

When $\quad x = 0, z = (0)^3 = 0$ and when $x = \left(\frac{\pi}{2}\right)^{1/3}, z = \left[\left(\frac{\pi}{2}\right)^{1/3}\right]^3 \;\Rightarrow\; z = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\pi/2} \sin z \,.\, \left(\frac{1}{3} dz\right) = \frac{1}{3} \int_0^{\pi/2} \sin z \, dz$$

$$= [(9)^{5/4} - (1)^{5/4}] \qquad \left[\because\ 9^{5/4} = (3^2)^{5/4} = (3)^{5/2} = \left[(3)^{1/2}\right]^5 = \left(\sqrt{3}\right)^5 = 9\sqrt{3}\right]$$

$$= \left(9\sqrt{3} - 1\right).$$

(*iii*) Let $\quad I = \int_0^1 2x \cos^{-1} x^2\, dx$

Put $\quad x^2 = z \;\Rightarrow\; 2x\, dx = dz$

When $\quad x = 0,\ \ z = (0)^2 = 0$ and when $x = 1,\ \ z = (1)^2 = 1$

$\therefore \quad I = \int_0^1 \cos^{-1} z\, dz$

$$= \left[-\frac{1}{\sqrt{1-z^2}}\right]_0^1 = -\left[\frac{1}{\sqrt{1-(1)^2}} - \frac{1}{\sqrt{1-0}}\right]$$

$$= -[0-1] = 1.$$

(*iv*) Let $\quad I = \int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta$

Put $\quad \cos 2\theta = z \;\Rightarrow\; -2\sin 2\theta\, d\theta = dz \;\Rightarrow\; \sin 2\theta\, d\theta = -\frac{1}{2}\, dz$

When $\quad \theta = 0,\ \ z = \cos 0° = 1$ and when $\theta = \frac{\pi}{6},\ \ z = \cos 2\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$\therefore \quad I = \int_1^{1/2} (z)^{-3} \cdot \left(-\frac{1}{2}\, dz\right) = -\frac{1}{2}\int_1^{1/2} z^{-3}\, dz$

$$= -\frac{1}{2}\left[\frac{z^{-3+1}}{-3+1}\right]_1^{1/2} = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left[\frac{1}{z^2}\right]_1^{1/2}$$

$$= \frac{1}{4}\left[\frac{1}{(1/2)^2} - \frac{1}{(1)^2}\right] = \frac{1}{4}[4-1] = \frac{3}{4}.$$

(*v*) Let $\quad I = \int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\, dx$

Put $\quad \sin x = z \;\Rightarrow\; \cos x\, dx = dz$

When $\quad x = 0,\ \ z = \sin 0° = 0$ and when $x = \frac{\pi}{2},\ \ z = \sin\frac{\pi}{2} = 1$

$\therefore \quad I = \int_0^1 \frac{1}{1+z^2}\, dz$

$$= \left[\tan^{-1} z\right]_0^1 = [\tan^{-1} 1 - \tan^{-1} 0] \qquad \left[\because\ \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4}.$$

(*vi*) Let $I = \int_0^{(\pi^2)^{1/3}} \sqrt{x} \cos^2 (x^{3/2})\, dx$

Put $x^{3/2} = z \Rightarrow \frac{3}{2} x^{1/2}\, dx = dz \Rightarrow \sqrt{x}\, dx = \frac{2}{3} dz$

When $x = 0,\ z = (0)^{3/2} = 0$ and when $x = (\pi^2)^{1/3},\quad z = (\pi^{2/3})^{3/2} = \pi$

$$\therefore \quad I = \int_0^{\pi} \cos^2 z \left(\frac{2}{3} dz\right) = \frac{2}{3}\int_0^{\pi} \cos^2 z\, dz$$

$$= \frac{2}{3}\int_0^{\pi}\left(\frac{1+\cos 2z}{2}\right) dz = \frac{1}{3}\int_0^{\pi} (1+\cos 2z)\, dz \qquad \left[\begin{array}{l} \because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ \dfrac{1+\cos 2A}{2} = \cos^2 A \end{array}\right]$$

$$= \frac{1}{3}\left[z + \frac{\sin 2z}{2}\right]_0^{\pi} = \frac{1}{3}\left[\left(\pi + \frac{\sin 2\pi}{2}\right) - \left(0 + \frac{\sin 0^\circ}{2}\right)\right] = \frac{1}{3}[(\pi - 0) - (0 + 0)]$$

$$= \frac{\pi}{3}.$$

Example 29. *Evaluate the following integrals :*

(*i*) $\int_0^1 (\cos^{-1} x)^2\, dx$ (*ii*) $\int_2^4 \frac{x^2 + x}{\sqrt{2x+1}}\, dx$

(*iii*) $\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$ (*iv*) $\int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x \cot x}{1 + \operatorname{cosec}^2 x}\, dx$

(*v*) $\int_0^{(\pi/2)^{1/3}} x^5 \sin x^3\, dx$ (*vi*) $\int_0^{\pi/4} (\tan^2 \theta + \tan^4 \theta)\, d\theta.$

Solution. (*i*) Let $I = \int_0^1 (\cos^{-1} x)^2\, dx$

Put $\cos^{-1} x = \theta \Rightarrow x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

When $x = 0,\ \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ and when $x = 1,\ \cos\theta = 1 \Rightarrow \theta = 0.$

$$\therefore \quad I = \int_{\pi/2}^{0} \theta^2 (-\sin\theta\, d\theta) = -\int_{\pi/2}^{0} \theta^2 \sin\theta\, d\theta$$

Let $I_1 = \int \underset{\text{I}}{\theta^2}\ \underset{\text{II}}{\sin\theta}\, d\theta$

Integrating by parts, we have

$$I_1 = \theta^2 . \int \sin\theta . d\theta - \int \left\{\frac{d}{d\theta}(\theta^2) . \int \sin\theta\, d\theta\right\} d\theta$$

$$= \theta^2 (-\cos\theta) - \int 2\theta . (-\cos\theta)\, d\theta$$

$$= -\theta^2 \cos\theta + 2\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\cos\theta}\, d\theta$$

Integrating by parts again, we have

$$I_1 = -\theta^2\cos\theta + 2\left[\theta . \int \cos\theta\, d\theta - \int\left\{\frac{d}{d\theta}(\theta) . \int \cos\theta\, d\theta\right\} d\theta\right]$$

$$= \int_0^{\pi/4} \tan^{-1}(\tan 2\theta)\sec^2\theta\, d\theta \qquad \left[\because \tan 2A = \frac{2\tan A}{1-\tan^2 A}\right]$$

$$= \int_0^{\pi/4} 2\theta \sec^2\theta\, d\theta = 2\int_0^{\pi/4} \theta\sec^2\theta\, d\theta$$

Let $\quad I_1 = \int \underset{I}{\theta}\ \underset{II}{\sec^2\theta}\, d\theta$

Integrating by parts, we have

$$I_1 = \theta \,.\int \sec^2\theta\, d\theta - \int\left\{\frac{d}{d\theta}(\theta).\int \sec^2\theta\, d\theta\right\} d\theta$$

$$= \theta\tan\theta - \int 1 \,.\tan\theta\, d\theta = \theta\tan\theta - \log|\sec\theta|$$

$$\therefore \quad I = 2\int_0^{\pi/4}\theta\sec^2\theta\, d\theta = 2\Big[\theta\tan\theta - \log|\sec\theta|\Big]_0^{\pi/4}$$

$$= 2\left[\left(\frac{\pi}{4}\tan\frac{\pi}{4} - \log\left|\sec\frac{\pi}{4}\right|\right) - (0 - \log|\sec 0°|)\right]$$

$$= 2\left[\left(\frac{\pi}{4} - \log\sqrt{2}\right) - (0 - \log 1)\right] = \frac{\pi}{2} - 2\log\sqrt{2}$$

$$= \frac{\pi}{2} - 2\log(2)^{1/2} = \frac{\pi}{2} - 2\,\frac{1}{2}\log 2 \qquad [\because m\log n = \log n^m]$$

$$= \frac{\pi}{2} - \log 2.$$

(*iv*) Let $\quad I = \int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x\cot x}{1+\operatorname{cosec}^2 x}\, dx$

Put $\operatorname{cosec} x = z \Rightarrow -\operatorname{cosec} x\cot x\, dx = dz \Rightarrow \operatorname{cosec} x\cot x\, dx = -dz$

When $x = \frac{\pi}{2}$, $z = \operatorname{cosec}\frac{\pi}{2} \Rightarrow z = 1$ and when $x = \frac{\pi}{6}$, $z = \operatorname{cosec}\frac{\pi}{6} \Rightarrow z = 2.$

$$\therefore \quad I = \int_2^1 \frac{1}{1+z^2}(-dz) = -\int_2^1 \frac{1}{1+z^2}\, dz$$

$$= -\Big[\tan^{-1} z\Big]_2^1 \qquad \left[\because \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= -[\tan^{-1}(1) - \tan^{-1}(2)] = -\left[\frac{\pi}{4} - \tan^{-1} 2\right]$$

$$= \tan^{-1} 2 - \frac{\pi}{4}.$$

(*v*) Let $\quad I = \int_0^{\left(\frac{\pi}{2}\right)^{1/3}} x^5\sin x^3\, dx = \int_0^{\left(\frac{\pi}{2}\right)^{1/3}} x^3 \,.\, x^2\sin x^3\, dx$

Put $\quad x^3 = z \Rightarrow 3x^2\, dx = dz \Rightarrow x^2\, dx = \frac{1}{3}\, dz$

When $x = 0,\ \ z = (0)^3 = 0$ and when $x = \left(\frac{\pi}{2}\right)^{1/3}$, $\ z = \left[\left(\frac{\pi}{2}\right)^{1/3}\right]^3 = \frac{\pi}{2}$.

$$\therefore \qquad I = \int_0^{\pi/2} z \sin z \left(\frac{1}{3} dz\right) = \frac{1}{3}\int_0^{\pi/2} z \sin z \, dz$$

Let $\quad I_1 = \int \underset{\text{I}}{z} \underset{\text{II}}{\sin z} \, dz$

Integrating by parts, we have

$$I_1 = z \,.\int \sin z \, dz - \int \left\{\frac{d}{dz}(z) . \int \sin z \, dz\right\} dz$$

$$= z(-\cos z) - \int 1 . (-\cos z) \, dz = -z \cos z + \int \cos z \, dz$$

$$= -z \cos z + \sin z$$

$$\therefore \qquad I = \frac{1}{3}\int_0^{\pi/2} z \sin z \, dz = \frac{1}{3}\Big[-z \cos z + \sin z\Big]_0^{\pi/2}$$

$$= \frac{1}{3}\left[\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (0 + \sin 0^\circ)\right] = \frac{1}{3}[(0 + 1) - (0 + 0)]$$

$$= \frac{1}{3}.$$

(*vi*) Let $\quad I = \int_0^{\pi/4} (\tan^2\theta + \tan^4\theta) \, d\theta = \int_0^{\pi/4} \tan^2\theta\,(1 + \tan^2\theta) \, d\theta$

$$I = \int_0^{\pi/4} \tan^2\theta \sec^2\theta \, d\theta \qquad [\because \ \sec^2 A - \tan^2 A = 1]$$

Put $\quad \tan\theta = z \ \Rightarrow \ \sec^2\theta \, d\theta = dz$

When $\theta = 0,\ \ z = \tan 0^\circ = 0$ and when $\theta = \frac{\pi}{4}$, $\ z = \tan\frac{\pi}{4} = 1$

$$\therefore \qquad I = \int_0^1 z^2 \, dz = \left[\frac{z^3}{3}\right]_0^1 = \frac{1}{3}\Big[z^3\Big]_0^1 = \frac{1}{3}[(1)^3 - (0)^3]$$

$$= \frac{1}{3}(1 - 0) = \frac{1}{3}.$$

Example 30. *Evaluate the following definite integrals :*

(*i*) $\int_0^{\pi/6} (2 + 3x^2) \cos 3x \, dx$ $\qquad$ (*ii*) $\int_0^1 x \log\left(1 + \frac{x}{2}\right) dx$

(*iii*) $\int_0^1 x \tan^{-1} x \, dx$ $\qquad$ (*iv*) $\int_0^\infty \frac{x \tan^{-1} x}{(1 + x^2)^2} \, dx$

(*v*) $\int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$ $\qquad$ (*vi*) $\int_0^{\pi/2} \frac{1}{4 \sin^2 x + 5 \cos^2 x} \, dx.$

Solution. (*i*) Let $I = \int_0^{\pi/6} \underset{\text{I}}{(2 + 3x^2)} \underset{\text{II}}{\cos 3x} \, dx$

Integrating by parts, we have

$$I = \left[(2+3x^2)\cdot\frac{\sin 3x}{3}\right]_0^{\pi/6} - \int_0^{\pi/6} \frac{d}{dx}(2+3x^2)\cdot\frac{\sin 3x}{3}\,dx$$

$$= \frac{1}{3}\left[(2+3x^2)\sin 3x\right]_0^{\pi/6} - \frac{1}{3}\int_0^{\pi/6} 6x \sin 3x\,dx$$

$$= \frac{1}{3}\left[\left(2+3\left(\frac{\pi}{6}\right)^2\right)\sin\frac{3\pi}{6} - (2+0)\sin 0^\circ\right] - 2\int_0^{\pi/6} \underset{\text{I}}{x}\ \underset{\text{II}}{\sin 3x}\,dx$$

Integrating again by parts, we have

$$I = \frac{1}{3}\left[\left(2+\frac{\pi^2}{12}\right)-0\right] - 2\left[x\cdot\left(-\frac{\cos 3x}{3}\right)\right]_0^{\pi/6} + 2\int_0^{\pi/6}\frac{d}{dx}(x)\left(-\frac{\cos 3x}{3}\right)dx$$

$$= \frac{2}{3}+\frac{\pi^2}{36}+\frac{2}{3}\left[x\cos 3x\right]_0^{\pi/6} - \frac{2}{3}\int_0^{\pi/6} 1\,.\cos 3x\,dx$$

$$= \frac{2}{3}+\frac{\pi^2}{36}+\frac{2}{3}\left[\frac{\pi}{6}\cos\frac{3\pi}{6}-0\right] - \frac{2}{3}\left[\frac{\sin 3x}{3}\right]_0^{\pi/6}$$

$$= \frac{2}{3}+\frac{\pi^2}{36}+\frac{2}{3}[0] - \frac{2}{9}\left[\sin\frac{3\pi}{6} - \sin 0^\circ\right]$$

$$= \frac{2}{3}+\frac{\pi^2}{36}-\frac{2}{9} = \frac{24+\pi^2-8}{36} = \frac{\pi^2+16}{36}.$$

(*ii*) Let $I = \int_0^1 \underset{\text{II}}{x}\ \underset{\text{I}}{\log\left(1+\frac{x}{2}\right)}\,dx$

Integrating by parts, we have

$$I = \left[\log\left(1+\frac{x}{2}\right)\cdot\frac{x^2}{2}\right]_0^1 - \int_0^1 \frac{d}{dx}\left\{\log\left(1+\frac{x}{2}\right)\right\}\cdot\frac{x^2}{2}\,dx$$

$$= \left[\log\left(1+\frac{1}{2}\right)\cdot\frac{1}{2} - \log(1+0)\,.\,0\right] - \int_0^1 \frac{1}{\left(1+\frac{x}{2}\right)}\cdot\left(\frac{1}{2}\right)\cdot\frac{x^2}{2}\,dx$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{4}\int_0^1 \frac{x^2}{\left(\frac{2+x}{2}\right)}\,dx$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\int_0^1 \frac{x^2}{x+2}\,dx$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\int_0^1\left[(x-2)+\frac{4}{x+2}\right]dx$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\int_0^1 x-2+\frac{4}{x+2}\,dx$$

$$\begin{array}{r|l} x+2 & x^2 \quad (x-2 \\ & x^2+2x \\ & -\quad - \\ \hline & -2x \\ & -2x-4 \\ & +\quad + \\ \hline & 4 \end{array}$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\left[\frac{x^2}{2} - 2x + 4\log|x+2|\right]_0^1$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\left[\frac{(1)^2}{2} - 2(1) + 4\log|1+2| - 0 - 0 + 4\log|0+2|\right]$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\left[\frac{1}{2} - 2 + 4\log 3 - 4\log 2\right]$$

$$= \frac{1}{2}\log\frac{3}{2} - \frac{1}{2}\left[-\frac{3}{2} + 4\log\frac{3}{2}\right] = \frac{1}{2}\log\frac{3}{2} + \frac{3}{4} - 2\log\frac{3}{2}$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{3}{4} - \frac{3}{2}\log\frac{3}{2}.$$

(*iii*) Let $I = \int_0^1 \underset{\text{II}}{x}\ \underset{\text{I}}{\tan^{-1} x}\, dx$

Integrating by parts, we have

$$I = \left[\tan^{-1} x \cdot \frac{x^2}{2}\right]_0^1 - \int_0^1 \frac{d}{dx}(\tan^{-1} x)\cdot\frac{x^2}{2}\, dx$$

$$= \frac{1}{2}\left[x^2\tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\, dx$$

$$= \frac{1}{2}[1\tan^{-1} 1 - 0] - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2}\, dx \qquad \left[\begin{array}{l}\because \text{ Add and subtract} \\ 1 \text{ to the numerator}\end{array}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] - \frac{1}{2}\int_0^1\left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right)dx = \frac{\pi}{8} - \frac{1}{2}\int_0^1 1\, dx + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\, dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[x\right]_0^1 + \frac{1}{2}\left[\tan^{-1} x\right]_0^1 \qquad \left[\because \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{\pi}{8} - \frac{1}{2}[1-0] + \frac{1}{2}[\tan^{-1} 1 - \tan^{-1} 0] = \frac{\pi}{8} - \frac{1}{2} + \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8}$$

$$= \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}.$$

(*iv*) Let $I = \int_0^\infty \frac{x\tan^{-1} x}{(1+x^2)^2}\, dx$

Put $\tan^{-1} x = z \Rightarrow x = \tan z \Rightarrow \frac{1}{1+x^2}\, dx = dz$

When $x = 0$, $\tan z = 0 \Rightarrow z = 0$ and when $x = \infty$, $\tan z = \infty \Rightarrow z = \frac{\pi}{2}$

$\therefore \quad I = \sqrt{2}\int_{-1}^{1} \frac{1}{\sqrt{1-z^2}}\, dz$

$= \sqrt{2}\left[\sin^{-1} z\right]_{-1}^{1} \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a} + c\right]$

$= \sqrt{2}\ [\sin^{-1}(1) - \sin^{-1}(-1)] = \sqrt{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \sqrt{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$

$= \sqrt{2}\pi.$

(*vi*) Let $\quad I = \int_0^{\pi/2} \frac{1}{4\sin^2 x + 5\cos^2 x}\, dx$

$= \int_0^{\pi/2} \left(\frac{\frac{1}{\cos^2 x}}{\frac{4\sin^2 x}{\cos^2 x} + \frac{5\cos^2 x}{\cos^2 x}} \right) dx \qquad \left[\begin{array}{l}\text{Dividing the numerator and}\\ \text{the denominator by } \cos^2 x\end{array}\right]$

$= \int_0^{\pi/2} \frac{\sec^2 x}{4\tan^2 x + 5}\, dx$

Put $\quad \tan x = z \;\Rightarrow\; \sec^2 x\, dx = dz$

When $\quad x = 0,\ \ z = \tan 0° = 0$ and when $x = \frac{\pi}{2},\ \ z = \tan\frac{\pi}{2} = \infty.$

$\therefore \quad I = \int_0^{\infty} \frac{1}{4z^2+5}\, dz = \frac{1}{4}\int_0^{\infty} \frac{1}{z^2 + \frac{5}{4}}\, dz = \frac{1}{4}\int_0^{\infty} \frac{1}{z^2 + \left(\frac{\sqrt{5}}{2}\right)^2}\, dz$

$= \frac{1}{4}\left[\frac{1}{\frac{\sqrt{5}}{2}} \tan^{-1} \frac{z}{\frac{\sqrt{5}}{2}}\right]_0^{\infty} \qquad \left[\because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$

$= \frac{1}{4}\cdot\frac{2}{\sqrt{5}}\left[\tan^{-1}\frac{2z}{\sqrt{5}}\right]_0^{\infty} = \frac{1}{2\sqrt{5}}\ [\tan^{-1}\infty - \tan^{-1} 0]$

$= \frac{1}{2\sqrt{5}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4\sqrt{5}}.$

Example 31. *Evaluate the following integrals :*

(*i*) $\int_0^{\pi/2} \frac{1}{5 + 4\sin x}\, dx$ (*ii*) $\int_0^{\pi/2} \frac{1}{3 + 2\cos x}\, dx$

(*iii*) $\int_0^{\pi/2} \frac{1}{6 - \cos x}\, dx$ (*iv*) $\int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x}\, dx$

(*v*) $\int_0^{\pi/2} \frac{1}{1 - 2\sin x}\, dx$ (*vi*) $\int_0^{\pi/2} \frac{1}{7 + 2\sin x}\, dx$

$$= \frac{2}{3}\tan^{-1}\left[\frac{\left(3-\frac{4}{3}\right)}{1+3\left(\frac{4}{3}\right)}\right] == \frac{2}{3}\tan^{-1}\left[\frac{\frac{5}{3}}{1+4}\right]$$

$$\left[\because \tan^{-1} A - \tan^{-1} B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$$

$$= \frac{2}{3}\tan^{-1}\frac{1}{3}.$$

(ii) Let $\quad I = \int_0^{\pi/2} \frac{1}{3+2\cos x}\,dx$

Put $\quad z = \tan\frac{x}{2} \Rightarrow dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\Rightarrow \quad 2dz = \left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow \frac{2}{1+z^2}\,dz = dx \qquad [\because \sec^2 A - \tan^2 A = 1]$

Now $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} \Rightarrow \cos x = \dfrac{1-z^2}{1+z^2}$

When $\quad x = 0,\ z = \tan 0° = 0$ and when $x = \frac{\pi}{2},\ z = \tan\frac{\pi}{4} = 1$

$$\therefore \quad I = \int_0^1 \frac{1}{3+2\left(\frac{1-z^2}{1+z^2}\right)}\cdot\frac{2}{1+z^2}\,dz = \int_0^1 \frac{1+z^2}{(3+3z^2+2-2z^2)}\cdot\frac{2}{1+z^2}\,dz$$

$$= 2\int_0^1 \frac{1}{z^2+5}\,dz = 2\int_0^1 \frac{1}{z^2+\left(\sqrt{5}\right)^2}\,dz$$

$$= 2\left[\frac{1}{\sqrt{5}}\tan^{-1}\frac{z}{\sqrt{5}}\right]_0^1 \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{2}{\sqrt{5}}\left[\tan^{-1}\frac{1}{\sqrt{5}} - \tan^{-1}0°\right]$$

$$= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\right).$$

(iii) Let $\quad I = \int_0^{\pi/2} \frac{1}{6-\cos x}\,dx$

Put $\quad z = \tan\frac{x}{2} \Rightarrow dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\Rightarrow \quad 2dz = \left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow \frac{2}{1+z^2}\,dz = dx \qquad [\because \sec^2 A - \tan^2 A = 1]$

Now $\cos x = \dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}} \Rightarrow \cos x = \dfrac{1-z^2}{1+z^2}$

When $x = 0$, $z = \tan 0° = 0$ and when $x = \dfrac{\pi}{2}$, $z = \tan\dfrac{\pi}{4} = 1$.

$$\therefore \quad \text{I} = \int_0^1 \frac{1}{6-\left(\dfrac{1-z^2}{1+z^2}\right)}\cdot\frac{2}{1+z^2}\,dz = \int_0^1 \frac{1+z^2}{6+6z^2-1+z^2}\cdot\frac{2}{1+z^2}\,dz$$

$$= 2\int_0^1 \frac{1}{7z^2+5}\,dz = \frac{2}{7}\int_0^1 \frac{1}{z^2+\dfrac{5}{7}}\,dz = \frac{2}{7}\int_0^1 \frac{1}{z^2+\left(\sqrt{\dfrac{5}{7}}\right)^2}\,dz$$

$$= \frac{2}{7}\left[\frac{1}{\sqrt{\dfrac{5}{7}}}\tan^{-1}\frac{z}{\sqrt{\dfrac{5}{7}}}\right]_0^1 \qquad \left[\because \int\frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{2}{7}\cdot\frac{\sqrt{7}}{\sqrt{5}}\left[\tan^{-1}\frac{\sqrt{7}z}{\sqrt{5}}\right]_0^1 = \frac{2}{\sqrt{7}\sqrt{5}}\left[\tan^{-1}\frac{\sqrt{7}}{\sqrt{5}} - \tan^{-1}0\right]$$

$$= \frac{2}{\sqrt{35}}\tan^{-1}\left(\sqrt{\frac{7}{5}}\right).$$

(*iv*) Let $\text{I} = \displaystyle\int_0^{\pi} \frac{1}{(3+2\sin x+\cos x)}\,dx$

Put $z = \tan\dfrac{x}{2} \Rightarrow dz = \dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx$

$\Rightarrow \quad 2dz = \left(1+\tan^2\dfrac{x}{2}\right)dx \Rightarrow \dfrac{2}{1+z^2}\,dz = dx \qquad [\because \sec^2 A - \tan^2 A = 1]$

Now $\sin x = \dfrac{2\tan\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}}$, $\cos x = \dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}}$

$\Rightarrow \quad \sin x = \dfrac{2z}{1+z^2}$, $\cos x = \dfrac{1-z^2}{1+z^2}$

When $x = 0$, $z = \tan 0° = 0$ and when $x = \pi$, $z = \tan\dfrac{\pi}{2} = \infty$

$$\therefore \quad \text{I} = \int_0^{\infty} \frac{1}{\left[3+2\left(\dfrac{2z}{1+z^2}\right)+\dfrac{1-z^2}{1+z^2}\right]}\cdot\frac{2}{1+z^2}\,dz$$

$$= 2\left[\frac{1}{2\sqrt{3}}.\log\left|\frac{z-2-\sqrt{3}}{z-2+\sqrt{3}}\right|\right]_0^1 \qquad \left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{\sqrt{3}}\left[\log\left|\frac{1-2-\sqrt{3}}{1-2+\sqrt{3}}\right| - \log\left|\frac{0-2-\sqrt{3}}{0-2+\sqrt{3}}\right|\right]$$

$$= \frac{1}{\sqrt{3}}\left[\log\left|\frac{-1-\sqrt{3}}{-1+\sqrt{3}}\right| - \log\left|\frac{-2-\sqrt{3}}{-2+\sqrt{3}}\right|\right] = \frac{1}{\sqrt{3}}\left[\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \log\left(\frac{\sqrt{3}+2}{\sqrt{3}-2}\right)\right]$$

$$= \frac{1}{\sqrt{3}}\left[\log\left(\frac{\frac{\sqrt{3}+1}{\sqrt{3}-1}}{\frac{\sqrt{3}+2}{\sqrt{3}-2}}\right)\right] \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{1}{\sqrt{3}}\left[\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}-2}{\sqrt{3}+2}\right)\right] = \frac{1}{\sqrt{3}}\left[\log\left(\frac{3+\sqrt{3}-2\sqrt{3}-2}{3-\sqrt{3}+2\sqrt{3}-2}\right)\right]$$

$$= \frac{1}{\sqrt{3}}\log\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}\log\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\times\frac{1-\sqrt{3}}{1-\sqrt{3}}\right) \qquad \text{[Rationalization]}$$

$$= \frac{1}{\sqrt{3}}\log\left(\frac{1+3-2\sqrt{3}}{1-3}\right) = \frac{1}{\sqrt{3}}\log\left(\frac{4-2\sqrt{3}}{-2}\right) = \frac{1}{\sqrt{3}}\log[-(2-\sqrt{3})]$$

$$= \frac{1}{\sqrt{3}}\log(\sqrt{3}-2).$$

(*vi*) Let $\quad I = \int_0^{\pi/2} \frac{1}{7+2\sin x}\,dx$

Put $\quad z = \tan\frac{x}{2} \Rightarrow dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\Rightarrow \quad 2dz = \left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow \frac{2}{1+z^2}\,dz = dx \qquad [\because \sec^2 A - \tan^2 A = 1]$

Now $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{2z}{1+z^2}$

When $x = 0$, $z = \tan 0° = 0$ and when $x = \frac{\pi}{2}$, $z = \tan\frac{\pi}{4} = 1$

$$\therefore \qquad I = \int_0^1 \frac{1}{7+2\left(\frac{2z}{1+z^2}\right)}\cdot\frac{2}{1+z^2}\,dz = 2\int_0^1 \frac{1+z^2}{7+7z^2+4z}\cdot\frac{1}{1+z^2}\,dz$$

$$= 2\int_0^1 \frac{1}{7z^2+4z+7}\,dz = \frac{2}{7}\int_0^1 \frac{1}{\left(z^2+\frac{4}{7}z+1\right)}\,dz$$

$$\therefore \quad I = \frac{2}{7}\int_0^1 \frac{1}{\left(z^2+\frac{4}{7}z+\frac{4}{49}\right)+\left(1-\frac{4}{49}\right)}\,dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{4}{49} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } z\right)^2 = \frac{4}{49}\end{array}\right]$$

$$= \frac{2}{7}\int_0^1 \frac{1}{\left(z+\frac{2}{7}\right)^2+\frac{45}{49}}\,dz = \frac{2}{7}\int_0^1 \frac{1}{\left(z+\frac{2}{7}\right)^2+\left(\frac{\sqrt{45}}{7}\right)^2}\,dz$$

$$= \frac{2}{7}\left[\frac{1}{\frac{\sqrt{45}}{7}}\cdot\tan^{-1}\left(\frac{z+\frac{2}{7}}{\frac{\sqrt{45}}{7}}\right)\right]_0^1 \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{2}{7}\cdot\frac{7}{\sqrt{45}}\left[\tan^{-1}\frac{7z+2}{\sqrt{45}}\right]_0^1 = \frac{2}{\sqrt{45}}\left[\tan^{-1}\left(\frac{7+2}{3\sqrt{5}}\right)-\tan^{-1}\left(\frac{0+2}{3\sqrt{5}}\right)\right]$$

$$= \frac{2}{\sqrt{45}}\left[\tan^{-1}\left(\frac{9}{3\sqrt{5}}\right)-\tan^{-1}\left(\frac{2}{3\sqrt{5}}\right)\right] = \frac{2}{\sqrt{45}}\left[\tan^{-1}\frac{3}{\sqrt{5}}-\tan^{-1}\frac{2}{3\sqrt{5}}\right]$$

$$= \frac{2}{\sqrt{45}}\left[\tan^{-1}\left(\frac{\frac{3}{\sqrt{5}}-\frac{2}{3\sqrt{5}}}{1+\frac{3}{\sqrt{5}}\cdot\frac{2}{3\sqrt{5}}}\right)\right] \qquad \left[\because \tan^{-1}A-\tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$$

$$= \frac{2}{\sqrt{45}}\left[\tan^{-1}\left(\frac{\frac{9-2}{3\sqrt{5}}}{1+\frac{2}{5}}\right)\right] = \frac{2}{3\sqrt{5}}\left[\tan^{-1}\left(\frac{\frac{7}{3\sqrt{5}}}{\frac{7}{5}}\right)\right] = \frac{2}{3\sqrt{5}}\tan^{-1}\left(\frac{7}{3\sqrt{5}}\cdot\frac{5}{7}\right)$$

$$= \frac{2}{3\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}}{3}\right).$$

(*vii*) Let $I = \int_0^{\pi/2} \frac{1}{5\cos x+3\sin x}\,dx$

Put $\quad z = \tan\frac{x}{2} \quad \Rightarrow \quad dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\Rightarrow \quad 2dz = \left(1+\tan^2\frac{x}{2}\right)dx \quad \Rightarrow \quad \frac{2}{1+z^2}\,dz = dx \qquad [\because \sec^2 A - \tan^2 A = 1]$

Now $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\Rightarrow$ $\sin x = \dfrac{2z}{1+z^2}$, $\cos x = \dfrac{1-z^2}{1+z^2}$

When $x = 0$, $z = \tan 0° = 0$ and when $x = \dfrac{\pi}{2}$, $z = \tan\dfrac{\pi}{4} = 1$

$$\therefore \quad I = \int_0^1 \frac{1}{5\left(\frac{1-z^2}{1+z^2}\right)+3\left(\frac{2z}{1+z^2}\right)} \cdot \frac{2}{1+z^2}\, dz = 2\int_0^1 \frac{1+z^2}{5-5z^2+6z} \cdot \frac{1}{1+z^2}\, dz$$

$$= 2\int_0^1 \frac{1}{5-5z^2+6z} \cdot dz = -\frac{2}{5}\int_0^1 \frac{1}{z^2-\frac{6}{5}z-1}\, dz$$

$$= -\frac{2}{5}\int_0^1 \frac{1}{\left(z^2-\frac{6}{5}z+\frac{9}{25}\right)-\left(1+\frac{9}{25}\right)}\, dz$$

$$\left[\begin{array}{l}\text{Add and subtract } \frac{9}{25} \text{ to the denom.} \\ \because \left(\frac{1}{2}\text{co-eff. of } z\right)^2 = \frac{9}{25}\end{array}\right]$$

$$= -\frac{2}{5}\int_0^1 \frac{1}{\left(z-\frac{3}{5}\right)^2-\frac{34}{25}}\, dz = -\frac{2}{5}\int_0^1 \frac{1}{\left(z-\frac{3}{5}\right)^2-\left(\frac{\sqrt{34}}{5}\right)^2}\, dz$$

$$= -\frac{2}{5} \cdot \left[\frac{1}{2 \cdot \frac{\sqrt{34}}{5}} \log\left|\frac{z-\frac{3}{5}-\frac{\sqrt{34}}{5}}{z-\frac{3}{5}+\frac{\sqrt{34}}{5}}\right|\right]_0^1 \quad \left[\because \int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= -\frac{2}{5} \cdot \frac{5}{2\sqrt{34}}\left[\log\left|\frac{5z-3-\sqrt{34}}{5z-3+\sqrt{34}}\right|\right]_0^1$$

$$= -\frac{1}{\sqrt{34}}\left[\log\left|\frac{5-3-\sqrt{34}}{5-3+\sqrt{34}}\right| - \log\left|\frac{0-3-\sqrt{34}}{0-3+\sqrt{34}}\right|\right]$$

$$= -\frac{1}{\sqrt{34}}\left[\log\left|\frac{2-\sqrt{34}}{2+\sqrt{34}}\right| - \log\left|\frac{\sqrt{34}+3}{\sqrt{34}-3}\right|\right]$$

$$= -\frac{1}{\sqrt{34}}\left[\log\left|\frac{\sqrt{34}-2}{\sqrt{34}+2}\right| - \log\left|\frac{\sqrt{34}+3}{\sqrt{34}-3}\right|\right]$$

Put $z = \tan\frac{x}{2} \Rightarrow dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\Rightarrow \quad 2dz = \left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow \frac{2}{1+z^2}\,dz = dx$ $\quad [\because \sec^2 A - \tan^2 A = 1]$

Now $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{2z}{1+z^2}$

When $x = 0,\ z = \tan 0° = 0$ and when $x = \frac{\pi}{2},\ z = \tan\frac{\pi}{4} = 1$

$$\therefore \quad I = \int_0^1 \frac{1}{4+5\left(\frac{2z}{1+z^2}\right)}\cdot\frac{2}{1+z^2}\,dz$$

$$= 2\int_0^1 \frac{1+z^2}{4+4z^2+10z}\cdot\frac{1}{1+z^2}\,dz = 2\int \frac{1}{4z^2+10z+4}\,dz = \frac{1}{2}\int \frac{1}{z^2+\frac{5}{2}z+1}\,dz$$

$$\therefore \quad I = \frac{1}{2}\int_0^1 \frac{1}{\left(z^2+\frac{5}{2}z+\frac{25}{16}\right)+\left(1-\frac{25}{16}\right)}\,dz \quad \left[\begin{array}{l}\text{Add and subtract } \frac{25}{16} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } z\right)^2 = \frac{25}{16}\end{array}\right]$$

$$= \frac{1}{2}\int_0^1 \frac{1}{\left(z+\frac{5}{4}\right)^2-\frac{9}{16}}\,dz = \frac{1}{2}\int_0^1 \frac{1}{\left(z+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}\,dz$$

$$= \frac{1}{2}\left[\frac{1}{2\left(\frac{3}{4}\right)}\cdot\log\left|\frac{z+\frac{5}{4}-\frac{3}{4}}{z+\frac{5}{4}+\frac{3}{4}}\right|\right]_0^1 \quad \left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left[\log\left|\frac{4z+2}{4z+8}\right|\right]_0^1 = \frac{1}{3}\left[\log\left|\frac{4+2}{4+8}\right| - \log\left|\frac{0+2}{0+8}\right|\right]$$

$$= \frac{1}{3}\left[\log\left(\frac{6}{12}\right) - \log\left(\frac{2}{8}\right)\right] = \frac{1}{3}\left[\log\frac{1}{2} - \log\frac{1}{4}\right]$$

$$= \frac{1}{3}\log\left[\frac{1}{2}\times\frac{4}{1}\right] = \frac{1}{3}\log 2. \quad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

(x) Let $\quad I = \displaystyle\int_0^{\pi/2} \frac{1}{(2\cos x + 4\sin x)}\,dx$

Put $z = \tan\frac{x}{2} \Rightarrow dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\Rightarrow \quad 2dz = \left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow \frac{2}{1+z^2}\,dz = dx \qquad [\because \sec^2 A - \tan^2 A = 1]$

Now $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \quad \cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\Rightarrow \quad \sin x = \dfrac{2z}{1+z^2}, \quad \cos x = \dfrac{1-z^2}{1+z^2}$

When $x = 0, \; z = \tan 0° = 0$ and when $x = \frac{\pi}{2}, \; z = \tan\frac{\pi}{4} = 1$

$$\therefore \quad I = \int_0^1 \frac{1}{2\left(\frac{1-z^2}{1+z^2}\right)+4\left(\frac{2z}{1+z^2}\right)}\cdot\frac{2}{1+z^2}\,dz$$

$$= 2\int_0^1 \frac{1+z^2}{2-2z^2+8z}\cdot\frac{1}{1+z^2}\,dz = 2\int_0^1 \frac{1}{-2z^2+8z+2}\,dz = -\int_0^1 \frac{1}{z^2-4z-1}\,dz$$

$$= -\int_0^1 \frac{1}{(z^2-4z+4)+(-1-4)}\,dz \qquad \left[\begin{array}{l}\text{Add and subtract 4 to the denom.}\\ \because \left(\frac{1}{2}\text{co-eff. of } z\right)^2 = 4\end{array}\right]$$

$$= -\int_0^1 \frac{1}{(z-2)^2-5}\,dz = -\int_0^1 \frac{1}{(z-2)^2-\left(\sqrt{5}\right)^2}\,dz$$

$$= \int_0^1 \frac{1}{\left(\sqrt{5}\right)^2-(z-2)^2}\,dz$$

$$= \frac{1}{2\sqrt{5}}\left[\log\left|\frac{\sqrt{5}+z-2}{\sqrt{5}-z+2}\right|\right]_0^1 \qquad \left[\because \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c\right]$$

$$= \frac{1}{2\sqrt{5}}\left[\log\left|\frac{\sqrt{5}+1-2}{\sqrt{5}-1+2}\right| - \log\left|\frac{\sqrt{5}+0-2}{\sqrt{5}-0+2}\right|\right]$$

$$= \frac{1}{2\sqrt{5}}\left[\log\left|\frac{\sqrt{5}-1}{\sqrt{5}+1}\right| - \log\left|\frac{\sqrt{5}-2}{\sqrt{5}+2}\right|\right]$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{\frac{\sqrt{5}-1}{\sqrt{5}+1}}{\frac{\sqrt{5}-2}{\sqrt{5}+2}}\right| = \frac{1}{2\sqrt{5}}\log\left[\frac{\sqrt{5}-1}{\sqrt{5}+1}\times\frac{\sqrt{5}+2}{\sqrt{5}-2}\right]$$

$$= \frac{1}{2\sqrt{5}} \log\left[\frac{5-\sqrt{5}+2\sqrt{5}-2}{5+\sqrt{5}-2\sqrt{5}-2}\right] = \frac{1}{2\sqrt{5}} \log\left[\frac{3+\sqrt{5}}{3-\sqrt{5}}\right]$$

$$= \frac{1}{2\sqrt{5}} \log\left[\frac{3+\sqrt{5}}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}\right] \qquad \text{[Rationalization]}$$

$$= \frac{1}{2\sqrt{5}} \log\left[\frac{9+3\sqrt{5}+3\sqrt{5}+5}{9-5}\right] = \frac{1}{2\sqrt{5}} \log\left[\frac{14+6\sqrt{5}}{4}\right]$$

$$= \frac{1}{2\sqrt{5}} \log\left(\frac{7+3\sqrt{5}}{2}\right).$$

Example 32. *Evaluate the following integrals :*

(i) $\int_0^{\pi} \frac{1}{5+4\cos x}\,dx$ (ii) $\int_0^{\pi/2} \frac{\cos x}{3\cos x+\sin x}\,dx$

(iii) $\int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3}\,dx$ (iv) $\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$

(v) $\int_0^{\pi/4} \sec^4 x\,dx$.

Solution. (i) Let $I = \int_0^{\pi} \frac{1}{5+4\cos x}\,dx$

Put $z = \tan\frac{x}{2} \Rightarrow dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\Rightarrow$ $2dz = \left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow \frac{2}{1+z^2}\,dz = dx$ $[\because \sec^2 A - \tan^2 A = 1]$

Now $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{1-z^2}{1+z^2}$

When $x = 0$, $z = \tan 0° = 0$ and when $x = \pi$, $z = \tan\frac{\pi}{2} = \infty$.

$$\therefore \quad I = \int_0^{\infty} \frac{1}{5+4\left(\frac{1-z^2}{1+z^2}\right)} \cdot \frac{2}{1+z^2}\,dz = 2\int_0^{\infty} \frac{1+z^2}{5+5z^2+4-4z^2} \cdot \frac{1}{1+z^2}\,dz$$

$$= 2\int_0^{\infty} \frac{1}{z^2+9}\,dz = 2\int_0^{\infty} \frac{1}{z^2+3^2}\,dz$$

$$= 2 \cdot \left[\frac{1}{3}\tan^{-1}\frac{z}{3}\right]_0^{\infty} \qquad \left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{2}{3}\,[\tan^{-1}\infty - \tan^{-1} 0] = \frac{2}{3}\left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi}{3}.$$

Solution. (*i*) Let $I = \int_0^1 \underset{\text{II}}{x} \, \underset{\text{I}}{(\tan^{-1} x)^2} \, dx$

Integrating by parts, we have

$$= (\tan^{-1} x)^2 \cdot \frac{x^2}{2} - \int_0^1 \frac{d}{dx}(\tan^{-1} x)^2 \cdot \frac{x^2}{2} \, dx$$

$$= \left[\frac{x^2}{2}(\tan^{-1} x)^2\right]_0^1 - \int_0^1 \left(2 \tan^{-1} x \cdot \frac{1}{1+x^2}\right) \cdot \frac{x^2}{2} \, dx$$

$$= \left[\frac{1}{2}(\tan^{-1} 1)^2 - 0\right] - \int_0^1 \frac{x^2}{1+x^2} \tan^{-1} x \, dx$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)^2 - \int_0^1 \frac{1+x^2-1}{1+x^2} \tan^{-1} x \, dx \qquad \text{[Note this step]}$$

$$= \frac{1}{2}\left(\frac{\pi^2}{16}\right) - \int_0^1 \left(1 - \frac{1}{1+x^2}\right) \tan^{-1} x \, dx$$

$$= \frac{\pi^2}{32} - \int_0^1 \tan^{-1} x \, dx + \int_0^1 \frac{1}{1+x^2} \tan^{-1} x \, dx$$

$$= \frac{\pi^2}{32} + \int_0^1 \frac{1}{1+x^2} \tan^{-1} x \, dx - \int_0^1 1 \cdot (\tan^{-1} x) \, dx$$

$$\left[\because \int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= \frac{\pi^2}{32} + \left[\frac{(\tan^{-1} x)^2}{2}\right]_0^1 - \int_0^1 \underset{\text{II}}{1} \cdot \underset{\text{I}}{(\tan^{-1} x)} \, dx \qquad \text{[Integrating by parts]}$$

$$= \frac{\pi^2}{32} + \frac{1}{2}[(\tan^{-1} 1)^2 - (\tan^{-1} 0)^2] - \left[\left| \tan^{-1} x \cdot x \right|_0^1 - \int_0^1 \frac{d}{dx}(\tan^{-1} x) \cdot x \, dx\right]$$

$$= \frac{\pi^2}{32} + \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - 0\right] - \left[(1 \tan^{-1} 1 - 0) - \int_0^1 \frac{1}{1+x^2} x \, dx\right]$$

$$= \frac{\pi^2}{32} + \frac{1}{2}\left(\frac{\pi^2}{16}\right) - \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} \, dx$$

$$= \frac{\pi^2}{32} + \frac{\pi^2}{32} - \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2} \, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{2\pi^2}{32} - \frac{\pi}{4} - \frac{1}{2}\left[\log|1+x^2|\right]_0^1 \qquad \left[\because \int \frac{f'(x)}{f(x)} \, dx = \log|f(x)| + c\right]$$

$$= \frac{\pi^2}{16} - \frac{\pi}{4} - \frac{1}{2}[\log|1+1| - \log|1+0|] = \frac{\pi^2}{16} - \frac{\pi}{4} - \frac{1}{2}[\log 2 - \log 1]$$

Example 34. *Evaluate the following integrals :*

(i) $\int_0^{\pi/3} \frac{\cos x}{3+4\sin x}\,dx$ (ii) $\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$

(iii) $\int_1^2 \frac{1}{x(1+\log x)^2}\,dx$ (iv) $\int_0^{\pi/2} \frac{\cos^2\theta}{\cos^2\theta+4\sin^2\theta}\,d\theta.$

Solution. (i) Let $I = \int_0^{\pi/3} \frac{\cos x}{3+4\sin x}\,dx$

$= \frac{1}{4}\int_0^{\pi/3} \frac{4\cos x}{3+4\sin x}\,dx$ [Multiply and divided by 4]

$= \frac{1}{4}\Big[\log|3+4\sin x|\Big]_0^{\pi/3}$ $\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|+c\right]$

$= \frac{1}{4}\left[\log\left|3+4\sin\frac{\pi}{3}\right| - \log|3+4\sin 0°|\right]$

$= \frac{1}{4}\left[\log\left(3+4\cdot\frac{\sqrt{3}}{2}\right) - \log(3+0)\right] = \frac{1}{4}\left[\log\left(3+2\sqrt{3}\right) - \log 3\right]$

$= \frac{1}{4}\log\left(\frac{3+2\sqrt{3}}{3}\right).$ $\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$

(ii) Let $I = \int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$

Put $z = \tan\frac{x}{2} \Rightarrow dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\Rightarrow 2dz = \left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow \frac{2}{1+z^2}\,dz = dx$ [$\because \sec^2 A - \tan^2 A = 1$]

Now $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\Rightarrow \sin x = \frac{2z}{1+z^2}$, $\cos x = \frac{1-z^2}{1+z^2}$

When $x = 0$, $z = \tan 0° = 0$ and when $x = \frac{\pi}{2}$, $z = \tan\frac{\pi}{4} = 1$

$\therefore I = \int_0^1 \dfrac{\left(\frac{1-z^2}{1+z^2}\right)}{1+\left(\frac{1-z^2}{1+z^2}\right)+\left(\frac{2z}{1+z^2}\right)}\cdot\frac{2}{1+z^2}\,dz = 2\int_0^1 \frac{1-z^2}{1+z^2+1-z^2+2z}\cdot\frac{1}{1+z^2}\,dz$

$= 2\int_0^1 \frac{1-z^2}{(2z+2)(1+z^2)}\,dz = 2\int_0^1 \frac{(1-z)(1+z)}{2(z+1)(1+z^2)}\,dz$

$$= -\left[\sin^5 \frac{\pi}{2} \cos \frac{\pi}{2} - \sin^5 0° \cos 0°\right] + \int_0^{\pi/2} 5 \sin^4 \theta \cos \theta \,.\, \cos \theta \, d\theta$$

$$= -[0 - 0] + 5 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta \, d\theta$$

$$= 5 \int_0^{\pi/2} \sin^4 \theta \,.\, (1 - \sin^2 \theta) \, d\theta \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= 5 \int_0^{\pi/2} \sin^4 \theta \, d\theta - 5 \int_0^{\pi/2} \sin^6 \theta \, d\theta$$

$$= 5 \int_0^{\pi/2} \sin^4 \theta \, d\theta - 5I \qquad [\because \text{ By using equation (1)}]$$

$$\Rightarrow \quad 6I = 5 \int_0^{\pi/2} \sin^4 \theta \, d\theta$$

$$\therefore \quad I = \frac{5}{6} \int_0^{\pi/2} \sin^4 \theta \, d\theta.$$

Example 36. *Evaluate the following integrals :*

(i) $\int_0^{\pi/2} \frac{1}{a \cos x + b \sin x} \, dx$ (ii) $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1 + \cos x}}{(1 - \cos x)^{5/2}} \, dx$

(iii) $\int_0^1 x \sqrt{\frac{1 - x^2}{1 + x^2}} \, dx$ (iv) $\int_0^{\pi/2} \frac{1}{9 + 16 \cos^2 x} \, dx$

(v) $\int_0^{\pi/2} \frac{1}{4 + 9 \cos^2 x} \, dx$ (vi) $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} \, dx.$

Solution. (i) Let $I = \int_0^{\pi/2} \frac{1}{a \cos x + b \sin x} \, dx$

Put $a = r \cos \theta, b = r \sin \theta$

Squaring and adding, we get

$$a^2 + b^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$\Rightarrow \quad a^2 + b^2 = r^2 \quad \Rightarrow \quad r = \sqrt{a^2 + b^2}$$

Also $\frac{b}{a} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta \quad \Rightarrow \quad \theta = \tan^{-1} \frac{b}{a}$

$$\therefore \quad I = \int_0^{\pi/2} \frac{1}{r \cos \theta \cos x + r \sin \theta \sin x} \, dx$$

$$= \frac{1}{r} \int_0^{\pi/2} \frac{1}{\cos (x - \theta)} \, dx \qquad [\because \cos (A - B) = \cos A \cos B + \sin A \sin B]$$

$$= \frac{1}{\sqrt{a^2 + b^2}} \int_0^{\pi/2} \sec (x - \theta) \, d\theta$$

$$= \frac{1}{\sqrt{a^2 + b^2}} \Big[\log | \sec (x - \theta) + \tan (x - \theta) | \Big]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{a^2 + b^2}} \left[\log \left| \sec \left(\frac{\pi}{2} - \theta \right) + \tan \left(\frac{\pi}{2} - \theta \right) \right| - \log | \sec (0 - \theta) + \tan (0 - \theta) | \right]$$

$$= \frac{1}{\sqrt{a^2+b^2}} [\log |\operatorname{cosec}\theta + \cot\theta| - \log |\sec\theta - \tan\theta|]$$

$$\left[\begin{array}{l} \because \sec\left(\frac{\pi}{2}-A\right) = \operatorname{cosec} A \\ \tan\left(\frac{\pi}{2}-A\right) = \cot A \\ \sec(-\theta) = \sec\theta, \tan(-\theta) = -\tan\theta \end{array}\right]$$

$$= \frac{1}{\sqrt{a^2+b^2}} [\log |\operatorname{cosec}\theta + \cot\theta| - \log |\sec\theta - \tan\theta|]$$

$$= \frac{1}{\sqrt{a^2+b^2}} \left[\log\left|\sqrt{1+\cot^2\theta} + \cot\theta\right| - \log\left|\sqrt{1+\tan^2\theta} - \tan\theta\right|\right]$$

$$\left[\begin{array}{l} \because \operatorname{cosec}^2 A - \cot^2 A = 1 \\ \Rightarrow \operatorname{cosec}^2 A = 1 + \cot^2 A \\ \Rightarrow \operatorname{cosec} A = \sqrt{1+\cot^2 A} \\ \sec^2 A - \tan^2 A = 1 \\ \Rightarrow \sec^2 A = 1 + \tan^2 A \\ \Rightarrow \sec A = \sqrt{1+\tan^2 A} \end{array}\right]$$

$$= \frac{1}{\sqrt{a^2+b^2}} \left[\log\left|\sqrt{1+\frac{a^2}{b^2}} + \frac{a}{b}\right| - \log\left|\sqrt{1+\frac{b^2}{a^2}} - \frac{b}{a}\right|\right]$$

$$\left[\because \frac{b}{a} = \tan\theta \Rightarrow \frac{a}{b} = \cot\theta\right]$$

$$= \frac{1}{\sqrt{a^2+b^2}} \left[\log\left|\frac{\sqrt{b^2+a^2}+a}{b}\right| - \log\left|\frac{\sqrt{a^2+b^2}-b}{a}\right|\right]$$

$$= \frac{1}{\sqrt{a^2+b^2}} \left[\log\left|\frac{\sqrt{a^2+b^2}+a}{b} \times \frac{a}{\sqrt{a^2+b^2}-b}\right|\right]$$

$$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \frac{1}{\sqrt{a^2+b^2}} \left[\log\left|\frac{a}{b} \cdot \frac{\left(\sqrt{a^2+b^2}+a\right)}{\left(\sqrt{a^2+b^2}-b\right)}\right|\right].$$

(ii) Let $\quad I = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx$

$$= \int_{\pi/3}^{\pi/2} \frac{\sqrt{2\cos^2 \frac{x}{2}}}{\left(2\sin^2 \frac{x}{2}\right)^{5/2}}\, dx \qquad \left[\begin{array}{l} \because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ 1+\cos A = 2\cos^2 \frac{A}{2} \\ \quad 1-\cos 2A = 2\sin^2 A \\ \Rightarrow\ 1-\cos A = 2\sin^2 \frac{A}{2} \end{array}\right]$$

$$= \int_{\pi/3}^{\pi/2} \frac{\sqrt{2}\cos \frac{x}{2}}{4\sqrt{2}\sin^5 \frac{x}{2}}\, dx = \frac{1}{4}\int_{\pi/3}^{\pi/2} \frac{\cos \frac{x}{2}}{\sin^5 \frac{x}{2}}\, dx$$

Put $\sin \frac{x}{2} = z \;\Rightarrow\; \frac{1}{2}\cos \frac{x}{2}\, dx = dz \;\Rightarrow\; \cos \frac{x}{2}\, dx = 2dz$

When $x = \frac{\pi}{3},\ z = \sin \frac{\pi}{6} = \frac{1}{2}$ and when $x = \frac{\pi}{2},\ z = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore$ $$I = \frac{1}{4}\int_{1/2}^{1/\sqrt{2}} \frac{1}{z^5}(2dz) = \frac{1}{2}\int_{1/2}^{1/\sqrt{2}} z^{-5}\, dz$$

$$= \frac{1}{2}\left[\frac{z^{-5+1}}{-5+1}\right]_{1/2}^{1/\sqrt{2}} = -\frac{1}{8}\left[\frac{1}{z^4}\right]_{1/2}^{1/\sqrt{2}}$$

$$= -\frac{1}{8}\left[\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^4} - \frac{1}{\left(\frac{1}{2}\right)^4}\right] = -\frac{1}{8}\left[\frac{1}{\frac{1}{4}} - \frac{1}{\frac{1}{16}}\right] = -\frac{1}{8}[4-16]$$

$$= -\frac{1}{8}(-12) = \frac{3}{2}.$$

(*iii*) Let $I = \int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\, dx$

Put $x^2 = z \;\Rightarrow\; 2x\, dx = dz \;\Rightarrow\; x\, dx = \frac{1}{2}dz$

When $x = 0,\ z = (0)^2 = 0$ and when $x = 1,\ z = (1)^2 = 1$

$\therefore$ $$I = \int_0^1 \sqrt{\frac{1-z}{1+z}} \cdot \left(\frac{1}{2}dz\right) = \frac{1}{2}\int_0^1 \sqrt{\frac{1-z}{1+z}}\, dz$$

$$= \frac{1}{2}\int_0^1 \sqrt{\frac{1-z}{1+z} \times \frac{1-z}{1-z}}\, dz \qquad \text{[Rationalization]}$$

$$= \frac{1}{2}\int_0^1 \frac{\sqrt{(1-z)^2}}{\sqrt{(1+z)(1-z)}}\, dz = \frac{1}{2}\int_0^1 \frac{1-z}{\sqrt{1-z^2}}\, dz$$

$$= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-z^2}}\, dz - \frac{1}{2}\int_0^1 \frac{z}{\sqrt{1-z^2}}\, dz$$

$$= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-z^2}}\,dz + \frac{1}{4}\int_0^1 (-2z)(1-z^2)^{-1/2}\,dz$$

[Multiply and divide the second integral by 2]

$$= \frac{1}{2}\left[\sin^{-1} z\right]_0^1 + \frac{1}{4}\left[\frac{(1-z^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_0^1 \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c \quad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{2}[\sin^{-1} 1 - \sin^{-1} 0] + \frac{1}{4}\cdot\frac{2}{1}\left[(1-z^2)^{1/2}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] + \frac{1}{2}[(1-1)^{1/2} - (1-0)^{1/2}] = \frac{\pi}{4} + \frac{1}{2}[0-1]$$

$$= \frac{\pi}{4} - \frac{1}{2}.$$

(*iv*) Let $\quad I = \int_0^{\pi/2} \frac{1}{9 + 16\cos^2 x}\,dx$

$$= \int_0^{\pi/2} \frac{\frac{1}{\cos^2 x}}{\frac{9}{\cos^2 x} + \frac{16\cos^2 x}{\cos^2 x}}\,dx \qquad \left[\text{Dividing the numerator and the denominator by } \cos^2 x\right]$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{9\sec^2 x + 16}\,dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{9(1+\tan^2 x) + 16}\,dx \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{9\tan^2 x + 25}\,dx$$

Put $\quad \tan x = z \;\Rightarrow\; \sec^2 x\,dx = dz$

When $\quad x = 0,\; z = \tan 0° = 0$ and when $x = \frac{\pi}{2},\; z = \tan\frac{\pi}{2} = \infty$

$$\therefore \quad I = \int_0^\infty \frac{1}{9z^2 + 25}\,dz = \frac{1}{9}\int_0^\infty \frac{1}{z^2 + \frac{25}{9}}\,dz = \frac{1}{9}\int_0^\infty \frac{1}{z^2 + \left(\frac{5}{3}\right)^2}\,dz$$

$$= \frac{1}{9}\left[\frac{1}{5/3}\tan^{-1}\frac{z}{5/3}\right]_0^\infty \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{9}\cdot\frac{3}{5}\left[\tan^{-1}\frac{3z}{5}\right]_0^\infty = \frac{1}{15}[\tan^{-1}\infty - \tan^{-1} 0] = \frac{1}{15}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{30}.$$

$$= \left[\log\left(\sin\frac{\pi}{2}\right)e^{\pi/2} - \log\left(\sin\frac{\pi}{4}\right)e^{\pi/4}\right] - \int_{\pi/4}^{\pi/2}\frac{1}{\sin x}\cdot\cos x\cdot e^x\,dx + \int_{\pi/4}^{\pi/2}e^x\cot x\,dx$$

$$= \left[(\log 1)\,e^{\pi/2} - \left[\log\left(\frac{1}{\sqrt{2}}\right)\right]e^{\pi/4}\right] - \int_{\pi/4}^{\pi/2}e^x\cot x\,dx + \int_{\pi/4}^{\pi/2}e^x\cot x\,dx$$

$$= 0 - e^{\pi/4}\cdot\log\frac{1}{\sqrt{2}} = -e^{\pi/4}\log(2)^{-1/2} = -e^{\pi/4}\cdot\left(-\frac{1}{2}\right)\log 2$$

$$= \frac{1}{2}e^{\pi/4}\log 2.$$

(*v*) Let $I = \int_0^{\pi/4}\sqrt{\tan x}\,dx$ ...(1)

Put $\sqrt{\tan x} = z \Rightarrow \tan x = z^2 \Rightarrow \sec^2 x\,dx = 2z\,dz$

$\Rightarrow (1+\tan^2 x)\,dx = 2z\,dz$ $[\because \sec^2 A - \tan^2 A = 1]$

$\Rightarrow (1+z^4)\,dx = 2z\,dz \Rightarrow dx = \dfrac{2z}{1+z^4}\,dz$

When $x = 0,\ z = \sqrt{\tan 0^\circ} = \sqrt{0} = 0$ and when $x = \dfrac{\pi}{4},\ z = \sqrt{\tan\dfrac{\pi}{4}} = \sqrt{1} = 1.$

$$\therefore \quad I = \int_0^1 z\cdot\frac{2z}{1+z^4}\,dz$$

$$= \int_0^1\frac{2z^2}{z^4+1}\,dz = \int_0^1\frac{z^2+z^2}{z^4+1}\,dz \qquad \text{[Note this step]}$$

$$= \int_0^1\frac{z^2+1+z^2-1}{z^4+1}\,dz \qquad \text{[Add and subtract 1 to the numerator]}$$

$$= \int_0^1\frac{z^2+1}{z^4+1}\,dz + \int_0^1\frac{z^2-1}{z^4+1}\,dz$$

$$\Rightarrow \quad I = I_1 + I_2 \quad \text{(say)} \qquad ...(2)$$

Now $I_1 = \displaystyle\int\frac{z^2+1}{z^4+1}\,dz = \int\frac{1+\frac{1}{z^2}}{z^2+\frac{1}{z^2}}\,dz$ $\left[\begin{array}{l}\text{Dividing the numerator and}\\ \text{the denominator by } z^2\end{array}\right]$

Put $\left(z-\dfrac{1}{z}\right) = y \Rightarrow \left(1+\dfrac{1}{z^2}\right)dz = dy$

$\therefore$ $I_1 = \displaystyle\int\frac{1}{y^2+2}\,dy$

$= \displaystyle\int\frac{1}{y^2+\left(\sqrt{2}\right)^2}\,dy$

$$\left[\begin{array}{l}\because \quad \left(z-\frac{1}{z}\right) = y\\ \Rightarrow \quad \left(z-\frac{1}{z}\right)^2 = y^2\\ \Rightarrow z^2+\frac{1}{z^2}-2 = y^2\\ \Rightarrow \quad z^2+\frac{1}{z^2} = y^2+2\end{array}\right]$$

$$= \frac{1}{\sqrt{2}}\left[0-\left(-\frac{\pi}{2}\right)\right]+\frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \qquad [\because \ \log 1 = 0]$$

$$= \frac{\pi}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}\right) \qquad \text{(Rationalization)}$$

$$= \frac{\pi}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\log\left(\frac{\left(\sqrt{2}-1\right)^2}{1}\right)$$

$$= \frac{\pi}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\cdot 2\log\left(\sqrt{2}-1\right) \qquad [\because \ m\log n = \log n^m]$$

$$= \frac{\pi}{2\sqrt{2}}+\frac{1}{\sqrt{2}}\log\left(\sqrt{2}-1\right).$$

Example 38. *If p and q are non-zero integers, prove that :*

(i) $\int_0^\pi \cos px \cos qx \, dx = 0$ if $p \neq q$ $\quad$ (ii) $\int_0^\pi \sin px \sin qx \, dx = 0$ if $p \neq q$

$= \frac{\pi}{2}$ if $p = q$. $\quad$ $= \frac{\pi}{2}$ if $p = q$.

Solution. (i) If $p \neq q$; Let $\mathrm{I} = \int_0^\pi \cos px \cos qx \, dx$

$$= \frac{1}{2}\int_0^\pi 2\cos px \cos qx \, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int_0^\pi [\cos(p+q)x + \cos(p-q)x] \, dx$$

$$[\because \ 2\cos A\cos B = \cos(A+B)+\cos(A-B)]$$

$$= \frac{1}{2}\left[\int_0^\pi \cos(p+q)x \, dx + \int_0^\pi \cos(p-q)x \, dx\right] = \frac{1}{2}\left\{\left[\frac{\sin(p+q)x}{p+q}\right]_0^\pi + \left[\frac{\sin(p-q)x}{p-q}\right]_0^\pi\right\}$$

$$= \frac{1}{2(p+q)}[\sin(p+q)\pi - \sin 0^\circ] + \frac{1}{2(p-q)}[\sin(p-q)\pi - \sin 0^\circ]$$

$$\left[\begin{array}{l}\because \ \sin n\pi = 0, n \in Z \\ \quad (p+q),(p-q) \in Z\end{array}\right]$$

$$= \frac{1}{2(p+q)}[0-0] + \frac{1}{2(p-q)}[0-0] = 0.$$

If $p = q$;

$$\therefore \quad \mathrm{I} = \int_0^\pi \cos^2 px \, dx$$

$$= \int_0^\pi \left(\frac{1+\cos 2px}{2}\right) dx \qquad \left[\begin{array}{l}(1+\cos 2A = 2\cos^2 A \\ \Rightarrow \left(\frac{1+\cos 2A}{2}\right) = \cos^2 A\end{array}\right]$$

$$= \frac{1}{2}\int_0^{\pi}(1+\cos 2px)\,dx = \frac{1}{2}\left[x+\frac{\sin 2px}{2p}\right]_0^{\pi} = \frac{1}{2}\left[\left(\pi+\frac{\sin 2p\pi}{2p}\right)-\left(0+\frac{\sin 0^\circ}{2p}\right)\right]$$

$$= \frac{1}{2}[(\pi+0)-0] = \frac{\pi}{2}.$$

(ii) If $p \neq q$;

Let $I = \int_0^{\pi} \sin px \sin qx\, dx$

$$= \frac{1}{2}\int_0^{\pi} 2\sin px \sin qx\, dx \qquad \text{[Multiply and divided by 2]}$$

$$= \frac{1}{2}\int_0^{\pi} [\cos(p-q)x - \cos(p+q)x]\, dx$$

$$[\because\ 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2}\int_0^{\pi}\cos(p-q)x\,dx - \frac{1}{2}\int_0^{\pi}\cos(p+q)x\,dx = \frac{1}{2}\left[\frac{\sin(p-q)x}{p-q}\right]_0^{\pi} - \frac{1}{2}\left[\frac{\sin(p+q)x}{p+q}\right]_0^{\pi}$$

$$= \frac{1}{2(p-q)}[\sin(p-q)\pi - \sin 0^\circ] - \frac{1}{2(p+q)}[\sin(p+q)\pi - \sin 0^\circ]$$

$$= \frac{1}{2(p-q)}[0-0] - \frac{1}{2(p+q)}[0-0] = 0.$$

If $p = q$;

$$\therefore \quad I = \int_0^{\pi}\sin^2 px\,dx = \frac{1}{2}\int_0^{\pi}(1-\cos 2px)\,dx \qquad \left[\begin{array}{l}\because\ 1-\cos 2A = 2\sin^2 A \\ \Rightarrow \left(\dfrac{1-\cos 2A}{2}\right) = \sin^2 A\end{array}\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin 2px}{2p}\right]_0^{\pi} = \frac{1}{2}\left[\left(\pi - \frac{\sin 2p\pi}{2p}\right) - \left(0 - \frac{\sin 0^\circ}{2p}\right)\right]$$

$$= \frac{1}{2}[(\pi-0)-(0-0)]$$

$$= \frac{\pi}{2}.$$

EXERCISE FOR PRACTICE

Evaluate the following definite integrals : (1—16)

1. $\int_0^1 \frac{1}{2x-3}\,dx$

2. $\int_{-\pi/2}^{\pi/2} \cos x\,dx$

3. $\int_0^{\pi/2} \frac{1}{1+\cos\theta}\,d\theta$

4. $\int_0^{\pi/2} \sqrt{1-\cos 2x}\,dx$

5. $\int_0^{\pi/2} (x + 6\cos 2x)\,dx$

6. $\int_0^{\pi/4} \sin^4 x\,dx$

7. $\int_0^3 \sqrt{9-x^2}\,dx$

8. $\int_1^3 \frac{\log x}{(1+x)^2}\,dx$

9. $\int_0^{\pi/2} x \cos x \, dx$

10. $\int_1^2 \frac{x+3}{x(x+2)} \, dx$

11. $\int_0^1 \frac{x^2}{1+x^6} \, dx$

12. $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} \, dx$

13. $\int_0^1 \sin^{-1} x \, dx$

14. $\int_0^{\pi/2} \frac{1}{5+4\cos x} \, dx$

15. $\int_0^{\pi/4} \frac{1}{4+9\cos^2 x} \, dx$

16. $\int_0^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$

Answers

1. $-\frac{1}{2}\log 3$	2. 2	3. 1	4. $\sqrt{2}$
5. $\frac{\pi^2}{8}$	6. $\frac{3\pi}{32} - \frac{1}{4}$	7. $\frac{9\pi}{4}$	8. $\frac{3}{4}\log 3 - \log 2$
9. $\frac{\pi}{2} - 1$	10. $\frac{1}{2}\log 6$	11. $\frac{\pi}{12}$	12. $\frac{\pi}{4}$
13. $\left(\frac{\pi}{2} - 1\right)$	14. $\frac{2}{3}\tan^{-1}\frac{1}{3}$	15. $\frac{1}{2\sqrt{13}}\tan^{-1}\frac{2}{\sqrt{13}}$	16. 1

8

Properties of Definite Integrals

8.1. PROPERTIES OF DEFINITE INTEGRALS

Property 1. *Prove that* $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz.$

Proof. Let $\int f(x)\,dx = F(x)$

$\Rightarrow \qquad \int f(z)\,dz = F(z)$

$\therefore$ L.H.S. $\qquad = \int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a) \qquad$ ...(1)

R.H.S. $\qquad = \int_a^b f(z)\,dz = \Big[F(z)\Big]_a^b = F(b) - F(a) \qquad$ ...(2)

$\therefore$ From equations (1) and (2), we have

$$\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$$

The value of a definite integral remains unchanged if the variable is changed, provided the function and limits of integration remain the same.

Property 2. *Prove that* $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Proof. Let $\int f(x)\,dx = F(x)$

$\therefore$ L.H.S. $\qquad = \int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a) \qquad$ ...(1)

R.H.S. $\qquad = -\int_b^a f(x)\,dx = -\Big[F(x)\Big]_b^a = -[F(a) - F(b)] = F(b) - F(a) \qquad$ ...(2)

$\therefore$ From equations (1) and (2), we have

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

An interchange of the limits of integration only changes the sign of the integral.

Property 3. *Prove that* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ *where* $a < c < b$.

$$= \int_0^a f(a + x)\, dx$$

$$= \int_0^a f(x)\, dx \qquad [\because\ f(a + x) = f(x)]$$

= the first integral on R.H.S. of equation (1).

Now, take the third integral on the R.H.S. of equation (1), we have

Let $\quad I_1 = \int_{2a}^{3a} f(x)\, dx$

Put $\quad x = a + z \ \Rightarrow\ dx = dz$

When $\ x = 2a,\ z + a = 2a \ \Rightarrow\ z = a$ and when $x = 3a,\ z + a = 3a \ \Rightarrow\ z = 2a$

$$\therefore \quad I_1 = \int_{2a}^{3a} f(x)\, dx = \int_a^{2a} f(a + z)\, dz = \int_a^{2a} f(a + x)\, dx \qquad \left[\because\ \int_a^b f(x)\, dx = \int_a^b f(z)\, dz\right]$$

$$= \int_a^{2a} f(x)\, dx \qquad [\because\ f(a + x) = f(x)]$$

= the second integral on R.H.S. of equation (1)

= the first integral on R.H.S. of equation (1).

Similarly, every integral on R.H.S. of equation (1) is equal to the first integral on R.H.S. of equation (1).

$$\therefore \quad \int_0^{na} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(x)\, dx + \int_0^a f(x)\, dx \ \ldots\ldots\ n \text{ times.}$$

$$= n \int_0^a f(x)\, dx.$$

8.2. DEFINITION

Even Function

A function $f(x)$ is said to be an even function of x if $f(-x) = f(x)$.

Odd Function

A function $f(x)$ is said to be odd function of x if $f(-x) = -f(x)$.

Property 8. *Prove that :*

(i) If $f(x)$ is an even function of x, then $\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)\, dx$

(ii) If $f(x)$ is an odd function of x, then $\int_{-a}^a f(x)\, dx = 0.$

Proof. Since $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx, \quad a < c < b$

$$\therefore \quad \int_{-a}^a f(x)\, dx = \int_{-a}^0 f(x)\, dx + \int_0^a f(x)\, dx \qquad \ldots(1)$$

Now, take the first integral in the R.H.S. of equation (1), we have

Let $\quad I = \int_{-a}^0 f(x)\, dx$

Put $\quad x = -z \ \Rightarrow\ dx = -dz$

When $\ x = -a,\ -z = -a \ \Rightarrow\ z = a$ and when $x = 0,\ -z = 0 \ \Rightarrow\ z = 0$

Solution. (*i*) Let $\quad I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx \qquad \ldots(1)$

$$\Rightarrow \qquad I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\, dx \qquad \left[\because\ \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\therefore \qquad I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx \qquad \ldots(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\cos x + \sin x}\right] dx = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx$$

$$= \int_0^{\pi/2} 1 . dx = \left[x\right]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

$$\therefore \qquad I = \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{4} .$$

(*ii*) Let $\quad I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx \qquad \ldots(1)$

$$= \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\, dx \qquad \left[\because\ \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\therefore \qquad I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\, dx \qquad \ldots(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\cos x}{\sin x + \cos x} + \frac{\sin x}{\cos x + \sin x}\right] dx = \int_0^{\pi/2} \frac{\cos x + \sin x}{\cos x + \sin x}\, dx$$

$$= \int_0^{\pi/2} 1 . dx = \left[x\right]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

$$\therefore \qquad I = \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{4} .$$

(*iii*) Let $\quad I = \int_0^{\pi/2} \frac{1}{1 + \tan x}\, dx = \int_0^{\pi/2} \frac{1}{1 + \frac{\sin x}{\cos x}}\, dx$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx \qquad \ldots(1)$$

$$= \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)}\, dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx \qquad ...(2)$$

Adding equation (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx + \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\cos x}{\cos x + \sin x} + \frac{\sin x}{\sin x + \cos x}\right] dx = \int_0^{\pi/2} \left[\frac{\cos x + \sin x}{\cos x + \sin x}\right] dx$$

$$= \int_0^{\pi/2} 1 \,.\, dx = \Big[x\Big]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}\,.$$

(*iv*) Let $\quad I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \qquad ...(1)$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\, dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \qquad ...(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\right] . \, dx = \int_0^{\pi/2} \left[\frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\right] dx$$

$$= \int_0^{\pi/2} 1 \,.\, dx = \Big[x\Big]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}\,.$$

(*v*) Let $\quad I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \qquad ...(1)$

Example 2. *Evaluate the following integrals :*

(*i*) $\int_0^{\pi/2} \frac{\sqrt{cot\, x}}{\sqrt{cot\, x} + \sqrt{tan\, x}}\, dx$ (*ii*) $\int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\, dx$

(*iii*) $\int_0^{\pi/2} \frac{\sqrt{tan\, x}}{1+\sqrt{tan\, x}}\, dx$ (*iv*) $\int_0^{\pi/2} \frac{sin^{3/2}\, x}{sin^{3/2}\, x + cos^{3/2}\, x}\, dx$

(*v*) $\int_0^{\pi/2} \frac{1}{1+\sqrt{tan\, x}}\, dx$ (*vi*) $\int_0^{\pi/2} \frac{cos^5\, x}{sin^5\, x + cos^5\, x}\, dx$.

Solution. (*i*) Let $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$ ...(1)

$$= \int_0^{\pi/2} \frac{\sqrt{\cot\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cot\left(\frac{\pi}{2} - x\right)} + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$\therefore$ $I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}}\, dx$ ...(2)

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx + \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} + \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}}\right] dx = \int_0^{\pi/2} \left[\frac{\sqrt{\cot x} + \sqrt{\tan x}}{\sqrt{\cot x} + \sqrt{\tan x}}\right] dx$$

$$= \int_0^{\pi/2} 1 .\, dx = \left[x\right]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

$\therefore$ $I = \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{4}$.

(*ii*) Let $I = \int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\, dx$ ...(1)

$$= \int_a^b \frac{f(a+b-x)}{f[a+b-(a+b-x)] + f(a+b-x)}\, dx$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right]$$

$\therefore$ $I = \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}\, dx$ ...(2)

Adding equations (1) and (2), we get

$$2I = \int_a^b \frac{f(x)}{f(a+b-x)+f(x)}\, dx + \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}\, dx$$

$$= \int_a^b \left[\frac{f(x)}{f(a+b-x)+f(x)} + \frac{f(a+b-x)}{f(x)+f(a+b-x)}\right] dx$$

$$= \int_a^b \frac{f(x) + f(a+b-x)}{f(x) + f(a+b-x)}\, dx = \int_a^b 1 \,.\, dx = \left[x\right]_a^b = [b-a]$$

$$\therefore \quad I = \frac{1}{2}(b-a).$$

(*iii*) Let $\quad I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\, dx = \int_0^{\pi/2} \frac{\sqrt{\frac{\sin x}{\cos x}}}{1+\sqrt{\frac{\sin x}{\cos x}}}\, dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \qquad \text{...(1)}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \qquad \text{...(2)}$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\right] dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$= \int_0^{\pi/2} 1 \,.\, dx = \left[x\right]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

(*iv*) Let $\quad I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}\, dx \qquad \text{...(1)}$

$$= \int_0^{\pi/2} \frac{\sin^{3/2}\left(\frac{\pi}{2} - x\right)}{\sin^{3/2}\left(\frac{\pi}{2} - x\right) + \cos^{3/2}\left(\frac{\pi}{2} - x\right)}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x}\, dx \qquad \text{...(2)}$$

Example 3. *Evaluate the following integrals :*

(*i*) $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{1+\sqrt{\cot x}}\, dx$ (*ii*) $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}}\, dx$

(*iii*) $\int_0^{a} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\, dx$ (*iv*) $\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}}\, dx$

(*v*) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}\, dx$ (*vi*) $\int_0^{\pi/2} \frac{1}{1+\tan^3 x}\, dx.$

Solution. (*i*) Let $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{1+\sqrt{\cot x}}\, dx = \int_0^{\pi/2} \frac{\sqrt{\frac{\cos x}{\sin x}}}{1+\sqrt{\frac{\cos x}{\sin x}}}\, dx$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \qquad ...(1)$$

Proceed as in example 1(*iv*).

$\left[\textbf{Hint.} \text{ Using } \int_0^a f(x) = \int_0^a f(a-x)\, dx\right]$ $\left[\textbf{Ans. } \frac{\pi}{4}.\right]$

(*ii*) Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}}\, dx = \int_{\pi/6}^{\pi/3} \frac{1}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}\, dx$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \qquad ...(1)$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}\, dx$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right]$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}\, dx \qquad \left[\because \frac{\pi}{3}+\frac{\pi}{6} = \frac{2\pi+\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}\right]$$

$$\therefore \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \qquad ...(2)$$

Adding equations (1) and (2), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

$$= \int_{\pi/6}^{\pi/3} \left[\frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right] dx$$

$$= \int_{\pi/6}^{\pi/3} \left[\frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right] dx$$

$$= \int_{\pi/6}^{\pi/3} 1 . dx = \left[x \right]_{\pi/6}^{\pi/3} = \left[\frac{\pi}{3} - \frac{\pi}{6} \right] = \frac{\pi}{6}$$

$\therefore$ $$\mathrm{I} = \frac{1}{2} . \frac{\pi}{6} = \frac{\pi}{12} .$$

(*iii*) Let $$\mathrm{I} = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx \quad \ldots(1)$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{a-(a-x)}} dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx \right]$$

$\therefore$ $$\mathrm{I} = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx \quad \ldots(2)$$

Adding equations (1) and (2), we get

$$2\mathrm{I} = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx + \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx$$

$$= \int_0^a \left[\frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} + \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} \right] dx = \int_0^a \left[\frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} \right] dx$$

$$= \int_0^a 1 . dx = \left[x \right]_0^a = [a - 0] = a$$

$\therefore$ $$\mathrm{I} = \frac{1}{2} . (a) = \frac{a}{2} .$$

(*iv*) Let $$\mathrm{I} = \int_0^{\pi/2} \frac{1}{1 + \sqrt{\cot x}} dx = \int_0^{\pi/2} \frac{1}{1 + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}} dx$$

$\therefore$ $$\mathrm{I} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Proceed as in example 1(*v*).

$\left[\textbf{Hint.} \text{ Using } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx \right]$ $\left[\textbf{Ans. } \frac{\pi}{4} . \right]$

(*v*) Let $$\mathrm{I} = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx \quad \ldots(1)$$

$$= \int_0^{\pi/2} \frac{\sin^n \left(\frac{\pi}{2} - x \right)}{\sin^n \left(\frac{\pi}{2} - x \right) + \cos^n \left(\frac{\pi}{2} - x \right)} dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx \right]$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\, dx \quad \text{...(2)}$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx + \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin^n x}{\sin^n x + \cos^n x} + \frac{\cos^n x}{\cos^n x + \sin^n x} \right] dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} \right] dx$$

$$= \int_0^{\pi/2} 1 . dx = \left[x \right]_0^{\pi/2} = \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{4} .$$

(*vi*) Let $\quad I = \int_0^{\pi/2} \frac{1}{1 + \tan^3 x}\, dx = \int_0^{\pi/2} \frac{1}{1 + \frac{\sin^3 x}{\cos^3 x}}\, dx$

$$= \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\, dx \quad \text{...(1)}$$

$$= \int_0^{\pi/2} \frac{\cos^3 \left(\frac{\pi}{2} - x \right)}{\cos^3 \left(\frac{\pi}{2} - x \right) + \sin^3 \left(\frac{\pi}{2} - x \right)}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx \right]$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx \quad \text{...(2)}$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\, dx + \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\cos^3 x}{\cos^3 x + \sin^3 x} + \frac{\sin^3 x}{\sin^3 x + \cos^3 x} \right] dx$$

$$= \int_0^{\pi/2} \left[\frac{\cos^3 x + \sin^3 x}{\cos^3 x + \sin^3 x} \right] dx$$

$$= \int_0^{\pi/2} 1 . dx = \left[x \right]_0^{\pi/2} = \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{4} .$$

Example 4. *Evaluate the following integrals :*

(i) $\int_0^{\pi/2} \log (\tan x)\, dx$ (ii) $\int_0^{\pi/2} \log (\cot x)\, dx$

(iii) $\int_0^1 x(1-x)^n\, dx$ (iv) $\int_0^{\pi/4} \log (1 + \tan x)\, dx$

(v) *Show that :* $\int_0^{\pi/2} \phi(\sin x)\, dx = \int_0^{\pi/2} \phi(\cos x)\, dx$

(vi) $\int_0^1 \frac{\log (1+x)}{1+x^2}\, dx.$

Solution. (*i*) Let $I = \int_0^{\pi/2} \log (\tan x)\, dx$ ...(1)

$$= \int_0^{\pi/2} \log\left[\tan\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$\therefore$ $I = \int_0^{\pi/2} \log (\cot x)\, dx$ ...(2)

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \log (\tan x)\, dx + \int_0^{\pi/2} \log (\cot x)\, dx$$

$$= \int_0^{\pi/2} [\log (\tan x) + \log (\cot x)]\, dx$$

$$= \int_0^{\pi/2} \log (\tan x \,.\, \cot x)\, dx \qquad [\because \log m + \log n = \log (m.n)]$$

$$= \int_0^{\pi/2} \log 1 \,.\, dx = \int_0^{\pi/2} 0 \,.\, dx = 0 \qquad [\because \log 1 = 0]$$

$\therefore$ $I = 0.$

(*ii*) Let $I = \int_0^{\pi/2} \log (\cot x)\, dx$ ...(1)

$$= \int_0^{\pi/2} \log\left[\cot\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$\therefore$ $I = \int_0^{\pi/2} \log (\tan x)\, dx$ ...(2)

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \log (\cot x)\, dx + \int_0^{\pi/2} \log (\tan x)\, dx$$

$$= \int_0^{\pi/2} [\log (\cot x) + \log (\tan x)]\, dx$$

$$= \int_0^{\pi/2} \log (\cot x \,.\, \tan x)\, dx \qquad [\because \log m + \log n = \log (m.n)]$$

$$= \int_0^{\pi/2} \log 1 \,.\, dx$$ **[Ans.** 0]

Proceed as in part (*i*).

$$= \int_0^1 \left[\log\left(\frac{1-x}{x}\right) + \log\left(\frac{x}{1-x}\right)\right] dx$$

$$= \int_0^1 \log\left(\frac{1-x}{x} \cdot \frac{x}{1-x}\right) dx \qquad [\because \ \log m + \log n = \log m.n]$$

$$= \int_0^1 \log 1 \, . \, dx = \int_0^1 0 \, . \, dx = 0$$

$\therefore \quad 2I = 0 \Rightarrow I = 0.$

(*iv*) Let $\quad I = \int_0^\infty \frac{\log x}{1+x^2}\, dx$

Put $\quad x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

When $\quad x = 0, \ \tan\theta = 0 \Rightarrow \theta = 0$ and when $x = \infty, \ \tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\log(\tan\theta)}{1+\tan^2\theta} \, . \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{\log(\tan\theta)}{\sec^2\theta} \, . \sec^2\theta\, d\theta \qquad [\because \ \sec^2 A - \tan^2 A = 1]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log(\tan\theta)\, d\theta \qquad \ldots(1)$$

Proceed as in example 4(*i*). **[Ans.** 0.]

(*v*) Let $\quad I = \int_0^{\pi/2} \sin 2x \log(\tan x)\, dx \qquad \ldots(1)$

$$= \int_0^{\pi/2} \sin\left\{2\left(\frac{\pi}{2} - x\right)\right\} . \log\left\{\tan\left(\frac{\pi}{2} - x\right)\right\} . \, dx \quad \left[\because \ \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^{\pi/2} \sin(\pi - 2x) \log(\cot x)\, dx \qquad [\because \ \sin(\pi - A) = \sin A]$$

$$\therefore \quad I = \int_0^{\pi/2} \sin 2x \log(\cot x)\, dx \qquad \ldots(2)$$

Adding equation (1) and (2), we get

$$2I = \int_0^{\pi/2} \sin 2x \log(\tan x)\, dx + \int_0^{\pi/2} \sin 2x \log(\cot x)\, dx$$

$$= \int_0^{\pi/2} [\sin 2x \log(\tan x) + \sin 2x \log(\cot x)]\, dx$$

$$= \int_0^{\pi/2} \sin 2x\, [\log(\tan x) + \log(\cot x)]\, dx$$

$$= \int_0^{\pi/2} \sin 2x\, [\log(\tan x \, . \cot x)]\, dx \qquad [\because \ \log m + \log n = \log m.n]$$

$$= \int_0^{\pi/2} \sin 2x \log 1 \, . \, dx = \int_0^{\pi/2} 0 \, . \, dx = 0 \qquad [\because \ \log 1 = 0]$$

$I = 0.$

Adding equations (1) and (2), we get

$$2I = \int_0^5 \frac{\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[3]{5-x}}\,dx + \int_0^5 \frac{\sqrt[3]{5-x}}{\sqrt[3]{5-x}+\sqrt[3]{x}}\,dx$$

$$= \int_0^5 \left[\frac{\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[3]{5-x}} + \frac{\sqrt[3]{5-x}}{\sqrt[3]{5-x}+\sqrt[3]{x}}\right] dx = \int_0^5 \left[\frac{\sqrt[3]{x}+\sqrt[3]{5-x}}{\sqrt[3]{x}+\sqrt[3]{5-x}}\right].\,dx$$

$$= \int_0^5 1.\,dx = \Big[x\Big]_0^5 = [5-0] = 5$$

$$\therefore \quad I = \frac{5}{2}.$$

(*ii*) Let $I = \int_0^\pi \frac{x \tan x}{\sec x + \cos x}\,dx$

$$= \int_0^\pi \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \cos(\pi - x)}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^\pi \frac{(\pi - x)(-\tan x)}{-\sec x - \cos x}\,dx = \int_0^\pi \frac{(\pi - x)\tan x}{\sec x + \cos x}\,dx$$

$$= \int_0^\pi \frac{\pi \tan x}{\sec x + \cos x}\,dx - \int_0^\pi \frac{x \tan x}{\sec x + \cos x}\,dx$$

$$= \pi \int_0^\pi \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \cos x}\,.\,dx - I$$

$$\therefore \quad 2I = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x}\,dx$$

Put $\cos x = z \Rightarrow -\sin x\,dx = dz \Rightarrow \sin x\,dx = -dz$

When $x = 0,\ z = \cos 0° = 1$ and when $x = \pi,\ z = \cos \pi = -1$

$$\therefore \quad 2I = \pi \int_{+1}^{-1} \frac{1}{1+z^2}\,(-dz) = -\pi \int_{+1}^{-1} \frac{1}{1+z^2}\,dz$$

$$= \pi \int_{-1}^{+1} \frac{1}{1+z^2}\,dz \qquad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= \pi \Big[\tan^{-1} z\Big]_{-1}^{1} \qquad \left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \pi\,[\tan^{-1}(1) - \tan^{-1}(-1)] = \pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \pi\left[\frac{\pi}{2}\right]$$

$$\Rightarrow \quad 2I = \frac{\pi^2}{2} \Rightarrow I = \frac{\pi^2}{4}.$$

(*iii*) Let $\quad I = \int_0^{\pi/2} [2 \log \sin x - \log \sin 2x]\, dx \qquad \ldots(1)$

$$= \int_0^{\pi/2} \left[2 \log \left\{ \sin \left(\frac{\pi}{2} - x \right) \right\} - \log \left\{ \sin 2 \left(\frac{\pi}{2} - x \right) \right\} \right] dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx \right]$$

$$= \int_0^{\pi/2} [2 \log (\cos x) - \log \{\sin (\pi - 2x)\}]\, dx$$

$$= \int_0^{\pi/2} [2 \log (\cos x) - \log (\sin 2x)]\, dx \qquad \ldots(2) \quad [\because \sin (\pi - A) = \sin A]$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} [2 \log (\sin x) - \log (\sin 2x)]\, dx + \int_0^{\pi/2} [2 \log (\cos x) - \log (\sin 2x)]\, dx$$

$$= \int_0^{\pi/2} [2 \log (\sin x) - \log (\sin 2x) + 2 \log (\cos x) - \log (\sin 2x)]\, dx$$

$$= \int_0^{\pi/2} [2 (\log (\sin x) + \log (\cos x)) - 2 \log (\sin 2x)]\, dx$$

$$= 2 \int_0^{\pi/2} [\log (\sin x \,.\, \cos x) - \log (\sin 2x)]\, dx \qquad [\because \log m + \log n = \log (m.n)]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} [\log (\sin x \cos x) - \log (\sin 2x)\, dx$$

$$= \int_0^{\pi/2} \log \left[\frac{\sin x \cos x}{\sin 2x} \right] dx \qquad \left[\because \log m - \log n = \log \left(\frac{m}{n} \right) \right]$$

$$= \int_0^{\pi/2} \log \left[\frac{\sin x \cos x}{2 \sin x \cos x} \right] dx = \int_0^{\pi/2} \log \left(\frac{1}{2} \right) dx \qquad [\because \sin 2A = 2 \sin A \cos A]$$

$$= \log \left(\frac{1}{2} \right) \int_0^{\pi/2} 1 \,.\, dx$$

$$= \log (2)^{-1} \left[x \right]_0^{\pi/2} = -\log 2 \left[\frac{\pi}{2} - 0 \right] \qquad [\because \log m^n = n \log m]$$

$$= -\frac{\pi}{2} \log 2.$$

(*iv*) Let $\quad I = \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\, dx$

Let $\quad f(x) = x^3 \sin^4 x$

$\Rightarrow \quad f(-x) = (-x)^3 \sin^4 (-x) = -x^3 [\sin (-x)]^4$

$\qquad = -x^3 [-\sin x]^4 \qquad [\because \sin (-\theta) = -\sin \theta]$

$\qquad = -x^3 \sin^4 x$

$\qquad = -f(x)$

$\Rightarrow$ $f(x)$ is an odd function. $\qquad \left[\because \int_{-a}^{a} f(x)\, dx = 0; \text{ where } f(x) \text{ is an odd function} \right]$

$\therefore \quad \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x \, dx = 0.$

(*v*) Let $\quad I = \int_0^{\pi/2} \log(\sin x)\, dx \qquad ...(1)$

$= \int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$\therefore \quad I = \int_0^{\pi/2} \log(\cos x)\, dx \qquad ...(2)$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \log(\sin x)\, dx + \int_0^{\pi/2} \log(\cos x)\, dx$$

$$= \int_0^{\pi/2} [\log(\sin x) + \log(\cos x)]\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \,.\, \cos x)\,.\, dx \qquad [\because \log m + \log n = \log(m.n)]$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin x \cos x}{2}\right) dx \qquad \text{[Note this step] [Multily and divded by 2]}$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx \qquad [\because \sin 2A = 2 \sin A \cos A]$$

$$= \int_0^{\pi/2} [\log(\sin 2x) - \log 2]\, dx \qquad \left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$$

$$= \int_0^{\pi/2} \log(\sin 2x)\, dx - \int_0^{\pi/2} \log 2 \,.\, dx$$

Put $\quad 2x = z \;\Rightarrow\; 2dx = dz \;\Rightarrow\; dx = \frac{1}{2} dz$

When $x = 0$, $z = 2(0) = 0$ and when $x = \frac{\pi}{2}$, $z = 2\left(\frac{\pi}{2}\right) = \pi$

$$\therefore \quad 2I = \int_0^{\pi} \log(\sin z)\left(\frac{1}{2} dz\right) - \log 2 \int_0^{\pi/2} 1 \,.\, dx = \frac{1}{2}\int_0^{\pi} \log(\sin z)\, dz - \log 2\, [x]_0^{\pi/2}$$

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} \log(\sin z)\, dz - \log 2\left[\frac{\pi}{2} - 0\right]$$

$$= \int_0^{\pi/2} \log(\sin x)\, dx - \frac{\pi}{2} \log 2 \qquad \left[\because \int_a^b f(x)\,dx = \int_a^b f(z)\,dz\right]$$

$$= I - \frac{\pi}{2} \log 2$$

$$\Rightarrow \quad I = -\frac{\pi}{2} \log 2.$$

(*vi*) Let $\quad I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}}\, dx \qquad ...(1)$ [Rationalization]

$$= \int_{-a}^{a} \sqrt{\frac{(a-x)^2}{a^2 - x^2}}\, dx = \int_{-a}^{a} \frac{a-x}{\sqrt{a^2 - x^2}}\, dx$$

$$= \pi^2 - \pi \Big[\tan x - \sec x \Big]_0^\pi = \pi^2 - \pi\,[(\tan \pi - \sec \pi) - (\tan 0^\circ - \sec 0^\circ)]$$

$$= \pi^2 - \pi\,[(0 - (-1)) - (0 - 1)] = \pi^2 - \pi\,[1 + 1] = \pi^2 - 2\pi$$

$\Rightarrow$ $$I = \frac{\pi}{2}(\pi - 2).$$

(*iii*) Let $I = \int_0^\infty \frac{x}{(1+x)(1+x^2)}\,dx$

Put $x = \tan\theta \;\Rightarrow\; dx = \sec^2\theta\, d\theta$

When $x = 0$, $\tan\theta = 0 \Rightarrow \theta = 0$ and when $x = \infty$, $\tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$.

$\therefore$ $$I = \int_0^{\pi/2} \frac{\tan\theta}{(1+\tan\theta)(1+\tan^2\theta)} \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan\theta}{(1+\tan\theta)\sec^2\theta} \cdot \sec^2\theta\, d\theta \qquad [\because\ \sec^2 A - \tan^2 A = 1]$$

$$= \int_0^{\pi/2} \frac{\tan\theta}{1+\tan\theta}\, d\theta \qquad \text{...(1)}$$

$$= \int_0^{\pi/2} \frac{\tan\left(\frac{\pi}{2} - \theta\right)}{1+\tan\left(\frac{\pi}{2} - \theta\right)}\, d\theta \qquad \left[\because\ \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\cot\theta}{1+\cot\theta}\, d\theta = \int_0^{\pi/2} \frac{\frac{1}{\tan\theta}}{1+\frac{1}{\tan\theta}}\, d\theta$$

$\therefore$ $$I = \int_0^{\pi/2} \frac{1}{1+\tan\theta}\, d\theta \qquad \text{...(2)}$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\tan\theta}{1+\tan\theta}\, d\theta + \int_0^{\pi/2} \frac{1}{1+\tan\theta}\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{\tan\theta}{1+\tan\theta} + \frac{1}{1+\tan\theta}\right] d\theta = \int_0^{\pi/2} \left(\frac{1+\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/2} 1 \,.\, d\theta = \Big[\theta\Big]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

$\therefore$ $$I = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

$$= \frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{\pi}{2}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0^\circ}{2}\right)\right] = \frac{\pi}{2}[(\pi - 0) - (0 - 0)] = \frac{\pi^2}{2}$$

$\therefore \quad I = \frac{\pi^2}{4}.$

(ii) Let $\quad I = \int_0^{\pi} \sin^{2m} x \cos^{2n+1} x\,dx$

$$= \int_0^{\pi} \sin^{2m}(\pi - x)\cos^{2n+1}(\pi - x)\,.\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi} \sin^{2m} x\,(-\cos x)^{2n+1}\,.\,dx$$

$$= \int_0^{\pi} \sin^{2m} x\,(-\cos^{2n+1} x)\,dx \qquad [\because \ (-1)^{2n+1} = (-1)]$$

$$= -\int_0^{\pi} \sin^{2m} x \cos^{2n+1} x\,dx$$

$\Rightarrow \quad I = -\int_0^{\pi} \sin^{2m} x \cos^{2n+1} x\,dx = -I$

$\Rightarrow \quad I + I = 0 \ \Rightarrow \ 2I = 0 \ \Rightarrow \ I = 0.$

(iii) Let $\quad I = \int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin x}{2+\sin x}\right) dx$

Let $\quad f(x) = \log\left(\frac{2-\sin x}{2+\sin x}\right)$

$\Rightarrow \quad f(-x) = \log\left[\frac{2-\sin(-x)}{2+\sin(-x)}\right] = \log\left[\frac{2+\sin x}{2-\sin x}\right] \qquad [\because \ \sin(-\theta) = -\sin\theta]$

$$= \log\left(\frac{2-\sin x}{2+\sin x}\right)^{-1} = -\log\left(\frac{2-\sin x}{2+\sin x}\right) \qquad [\because \ m\log n = \log n^m]$$

$$= -f(x).$$

$\therefore$ **$f(x)$ is an odd function.**

Hence $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin x}{2+\sin x}\right) dx = 0. \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 0;\text{ if } f(x) \text{ is an odd function}\right]$

(iv) Let $\quad I = \int_0^a \frac{a}{\left(x + \sqrt{a^2 - x^2}\right)^2}\,dx$

Put $\quad x = a\sin\theta \quad \Rightarrow \quad dx = a\cos\theta\,d\theta$

When $\quad x = 0, \ a\sin\theta = 0 \quad \Rightarrow \quad \sin\theta = 0 \ \Rightarrow \ \theta = 0$

and when $\quad x = a, \ a\sin\theta = a \quad \Rightarrow \quad \sin\theta = 1 \ \Rightarrow \ \theta = \frac{\pi}{2}$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{\cos x + \sin x} \cdot dx$$

$$= \int_0^{\pi/2} \left[\frac{x}{\sin x + \cos x} + \frac{\left(\frac{\pi}{2} - x\right)}{\cos x + \sin x} \right] dx = \int_0^{\pi/2} \frac{x + \frac{\pi}{2} - x}{\sin x + \cos x} dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

Proceed as in part (*iv*).

[**Hint.** Multiply and dividing the denominator by $\sqrt{2}$] $\left[\textbf{Ans. } \frac{\pi}{2\sqrt{2}} \log\left(\sqrt{2} + 1\right)\right]$

Example 11. *Evaluate the following integrals* :

(*i*) $\int_0^{\pi} \frac{x}{1 + e \sin x} dx$ (*ii*) $\int_0^{1} \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) dx$

(*iii*) $\int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x} dx$ (*iv*) $\int_0^{\pi} \frac{x}{1 + \cos \alpha \sin x} dx$

(*v*) $\int_0^{\pi/2} \frac{1}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx$ (*vi*) $\int_0^{\pi} x \cos^2 x \, dx$.

Solution. (*i*) Let $I = \int_0^{\pi} \frac{x}{1 + e \sin x} dx$ ...(1)

$$= \int_0^{\pi} \frac{(\pi - x)}{1 + e \sin (\pi - x)} dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$\therefore \quad I = \int_0^{\pi} \frac{\pi - x}{1 + e \sin x} dx \qquad ...(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x}{1 + e \sin x} dx + \int_0^{\pi} \frac{\pi - x}{1 + e \sin x} dx = \int_0^{\pi} \left[\frac{x}{1 + e \sin x} + \frac{\pi - x}{1 + e \sin x}\right] dx$$

$$= \int_0^{\pi} \frac{x + \pi - x}{1 + e \sin x} dx = \pi \int_0^{\pi} \frac{1}{1 + e \sin x} dx \qquad [\because f(\pi - x) = f(x)]$$

$$\therefore \quad 2I = 2\pi \int_0^{\pi/2} \frac{1}{1 + e \sin x} dx \qquad \left[\because \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx \text{ ; if } f(2a - x) = f(x)\right]$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{1}{1 + e \sin x} dx \qquad [\text{Here, } 2a = \pi]$$

$$= \pi \int_0^{\pi/2} \frac{1}{1 + e \sin\left(\frac{\pi}{2} - x\right)} dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

Integrating the first integral by parts, we have

$$I = 3\left[\tan^{-1} x \,.\, x\right]_0^1 - 3\int_0^1 \frac{d}{dx}(\tan^{-1} x)\,.\, x\, dx - \pi\left[x\right]_{1/\sqrt{3}}^1$$

$$= 3\left[x \tan^{-1} x\right]_0^1 - 3\int_0^1 \frac{x}{1+x^2}\, dx - \pi\left[1 - \frac{1}{\sqrt{3}}\right]$$

$$= 3\,[1 \tan^{-1} 1 - 0] - \frac{3}{2}\int_0^1 \frac{2x}{1+x^2}\, dx - \pi\left[1 - \frac{1}{\sqrt{3}}\right] \quad \text{[Multiply and divided by 2]}$$

$$= 3\left[\frac{\pi}{4}\right] - \frac{3}{2}\left[\log|1+x^2|\right]_0^1 - \pi\left[1 - \frac{1}{\sqrt{3}}\right] \quad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right]$$

$$= \frac{3\pi}{4} - \pi + \frac{\pi}{\sqrt{3}} - \frac{3}{2}\,[\log|1+1| - \log|1+0|]$$

$$= \frac{\pi}{\sqrt{3}} - \frac{\pi}{4} - \frac{3}{2}\,[\log 2 - \log 1]$$

$$= \pi\left(\frac{1}{\sqrt{3}} - \frac{1}{4}\right) - \frac{3}{2}\log 2.$$

(*iii*) Let $$I = \int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x}\, dx \qquad \ldots(1)$$

$$= \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}\, dx \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$\therefore$ $$I = \int_0^{\pi/2} \frac{\cos^2 x}{1 + \cos x \sin x}\, dx \qquad \ldots(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x}\, dx + \int_0^{\pi/2} \frac{\cos^2 x}{1 + \cos x \sin x}\, dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin^2 x}{1 + \sin x \cos x} + \frac{\cos^2 x}{1 + \cos x \sin x}\right] dx = \int_0^{\pi/2} \left[\frac{\sin^2 x + \cos^2 x}{1 + \sin x \cos x}\right] dx$$

$$= \int_0^{\pi/2} \frac{1}{1 + \sin x \cos x}\, dx \quad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \int_0^{\pi/2} \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x}}\, dx \quad \left[\begin{array}{l}\text{Dividing the numerator and}\\ \text{the denominator by } \cos^2 x\end{array}\right]$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + \tan x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{1+\tan^2 x+\tan x}\, dx \qquad [\because \ \sec^2 A - \tan^2 A = 1]$$

Put $\tan x = z \Rightarrow \sec^2 x\, dx = dz$

When $x = 0$, $z = \tan 0° = 0$ and when $x = \frac{\pi}{2}$, $z = \tan\frac{\pi}{2} = \infty$

$$\therefore \quad 2I = \int_0^{\infty} \frac{1}{z^2+z+1}\, dz$$

$$= \int_0^{\infty} \frac{1}{\left(z^2+z+\frac{1}{4}\right)+\left(1-\frac{1}{4}\right)}\, dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because \left(\frac{1}{2}\text{ co-eff. of } z\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int_0^{\infty} \frac{1}{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}}\, dz = \int_0^{\infty} \frac{1}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\, dz$$

$$= \left[\frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{\left(z+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}\right]_0^{\infty} \qquad \left[\because \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)\right]_0^{\infty} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\infty - \tan^{-1}\left(\frac{0+1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{2}-\frac{\pi}{6}\right] = \frac{2}{\sqrt{3}}\left[\frac{3\pi-\pi}{6}\right] = \frac{2}{\sqrt{3}}\left[\frac{2\pi}{6}\right] = \frac{2\pi}{3\sqrt{3}}$$

$$\therefore \quad I = \frac{\pi}{3\sqrt{3}}.$$

(iv) Let $I = \int_0^{\pi} \frac{x}{1+\cos\alpha \sin x}\, dx$ ...(1)

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)}{1+\cos\alpha\sin(\pi - x)}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi - x}{1+\cos\alpha \sin x}\, dx$ ...(2)

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x}{1+\cos\alpha\sin x}\, dx + \int_0^{\pi} \frac{\pi - x}{1+\cos\alpha\sin x}\, dx$$

$$= \int_0^{\pi} \left[\frac{x}{1+\cos\alpha\sin x} + \frac{\pi - x}{1+\cos\alpha\sin x}\right] dx$$

$$= \int_0^{\pi} \left[\frac{x+\pi-x}{1+\cos\alpha\sin x}\right] dx = \int_0^{\pi} \frac{\pi}{1+\cos\alpha\sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{1}{1 + \cos\alpha\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}\,dx \qquad \left[\begin{array}{l}\because\ \sin 2A = 2\sin A\cos A\\ \Rightarrow\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\end{array}\right]$$

$$= \pi \int_0^{\pi} \frac{\dfrac{1}{\cos^2\frac{x}{2}}}{\dfrac{1}{\cos^2\frac{x}{2}} + \dfrac{\cos\alpha\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{\cos^2\frac{x}{2}}}\,dx \qquad \left[\begin{array}{l}\text{Dividing the numerator and}\\ \text{the denominator by } \cos^2\frac{x}{2}\end{array}\right]$$

$$= \pi \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2} + 2\cos\alpha\tan\frac{x}{2}}\,dx$$

$$= \pi \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{1 + \tan^2\frac{x}{2} + 2\cos\alpha\,.\tan\frac{x}{2}}\,dx \qquad [\because\ \sec^2 A - \tan^2 A = 1]$$

Put $\tan\frac{x}{2} = z \;\Rightarrow\; \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz \;\Rightarrow\; \sec^2\frac{x}{2}\,dx = 2dz$

When $x = 0$, $z = \tan 0° = 0$ and when $x = \pi$, $z = \tan\frac{\pi}{2} = \infty$

$$\therefore \qquad 2I = \pi \int_0^{\infty} \frac{1}{1 + z^2 + 2z\cos\alpha}\,.\,(2dz) = 2\pi \int_0^{\infty} \frac{1}{z^2 + 2z\cos\alpha + 1}\,dz$$

$$\Rightarrow \qquad I = \pi \int_0^{\infty} \frac{1}{(z^2 + 2z\cos\alpha + \cos^2\alpha) + (1 - \cos^2\alpha)}\,dz$$

$$\left[\begin{array}{l}\text{Add and subtract } \cos^2\alpha \text{ to the denom.}\\ \because\ \left(\frac{1}{2}\text{co-eff. of } z\right)^2 = \cos^2\alpha\end{array}\right]$$

$$= \pi \int_0^{\infty} \frac{1}{(z + \cos\alpha)^2 + \sin^2\alpha}\,dz \qquad [\because\ \sin^2 A + \cos^2 A = 1]$$

$$= \pi \left[\frac{1}{\sin\alpha}\,.\,\tan^{-1}\left(\frac{z + \cos\alpha}{\sin\alpha}\right)\right]_0^{\infty} \qquad \left[\because\ \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{\pi}{\sin\alpha}\left[\tan^{-1}\infty - \tan^{-1}\left(\frac{0 + \cos\alpha}{\sin\alpha}\right)\right] = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} - \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \alpha\right)\right\}\right] = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)\right] = \frac{\pi\alpha}{\sin\alpha}$$

$= \pi\alpha \operatorname{cosec}\alpha.$

$$\therefore \qquad I = \int_0^{\pi} (\pi - x) \cos^2 x \, dx \qquad \ldots(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi} x \cos^2 x \, dx + \int_0^{\pi} (\pi - x) \cos^2 x \, dx$$

$$= \int_0^{\pi} [x \cos^2 x + (\pi - x) \cos^2 x] \, dx$$

$$= \int_0^{\pi} \pi \cos^2 x \, dx$$

Let $\quad f(x) = \cos^2 x$

$\Rightarrow \quad f(\pi - x) = \cos^2 (\pi - x) = (-\cos x)^2 = \cos^2 x = f(x).$

$$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a - x) = f(x)\right]$$

$$\therefore \qquad 2I = 2\pi \int_0^{\pi/2} \cos^2 x \, dx \qquad [\text{Here } 2a = \pi]$$

$$\Rightarrow \qquad I = \pi \int_0^{\pi/2} \cos^2 x \, dx$$

$$= \pi \int_0^{\pi/2} \frac{1 + \cos 2x}{2} \, . \, dx \qquad \left[\begin{array}{l}\because \ 1 + \cos 2A = 2\cos^2 A \\ \Rightarrow \ \dfrac{1 + \cos 2A}{2} = \cos^2 A\end{array}\right]$$

$$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

$$= \frac{\pi}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{2}\left[\left(\frac{\pi}{2} + \frac{\sin 2\pi}{2}\right) - \left(0 + \frac{\sin 0^\circ}{2}\right)\right] = \frac{\pi}{2}\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right]$$

$$= \frac{\pi^2}{4} \, .$$

Example 12. *Evaluate the following integrals :*

(*i*) $\int_0^{\pi} x \log (\sin x) \, dx$ (*ii*) $\int_0^{\pi/2} x \cot x \, dx$

(*iii*) $\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} \, dx$ (*iv*) $\int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} \, dx$

(*v*) $\int_0^{\pi/2} \log (\tan \theta + \cot \theta) \, d\theta$ (*vi*) $\int_0^{\infty} \log\left(x + \frac{1}{x}\right) . \frac{1}{1 + x^2} \, dx$

(*vii*) $\int_0^{\pi} \log (1 + \cos x) \, dx$ (*viii*) $\int_0^{1} \frac{\log x}{\sqrt{1 - x^2}} \, dx.$

Solution. (*i*) Let $I = \int_0^{\pi} x \log (\sin x) \, dx \qquad \ldots(1)$

$$= \int_0^{\pi} (\pi - x) \log [\sin (\pi - x)] \, dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$$

$$\therefore \qquad I = \int_0^{\pi} (\pi - x) \log (\sin x) \, dx \qquad \ldots(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^\pi x \log(\sin x)\, dx + \int_0^\pi (\pi - x) \log(\sin x)\, dx$$

$$= \int_0^\pi [x \log(\sin x) + (\pi - x) \log(\sin x)]\, dx$$

$$= \int_0^\pi [x \log(\sin x) + \pi \log(\sin x) - x \log(\sin x)]\, dx$$

$$= \int_0^\pi \pi \log(\sin x)\, dx$$

$$\therefore \quad 2I = 2\pi \int_0^{\pi/2} \log(\sin x)\, dx \qquad \left[\because \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx, \text{ if } f(2a - x) = f(x)\right]$$

$$\Rightarrow \quad I = \pi\left[-\frac{\pi}{2}\log 2\right] \qquad \left[\begin{array}{l}\because \displaystyle\int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2 \\ \text{See Example 6}(v).\end{array}\right]$$

$$= \frac{\pi^2}{2} \log(2)^{-1} \qquad [\because \; m \log n = \log n^m]$$

$$= \frac{\pi^2}{2} \log \frac{1}{2}.$$

(*ii*) Let $I = \int_0^{\pi/2} \underset{\text{I}}{x^2}\ \underset{\text{II}}{\cot x}\, dx$

Integrating by parts, we have

$$= [x \,.\, \log \sin x]_0^{\pi/2} - \int_0^{\pi/2} \frac{d}{dx}(x).\, \log(\sin x)\, dx$$

$$= \left[\frac{\pi}{2}\log\left(\sin\frac{\pi}{2}\right) - 0\right] - \int_0^{\pi/2} 1 \,.\, \log(\sin x)\, dx$$

$$= \left(\frac{\pi}{2}\log 1\right) - \int_0^{\pi/2} \log(\sin x)\, dx$$

$$= 0 - \left(-\frac{\pi}{2}\log 2\right) \qquad \left[\begin{array}{l}\because \displaystyle\int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2 \\ \text{See Example 6}(v).\end{array}\right]$$

$$= \frac{\pi}{2}\log 2.$$

(*iii*) Let $I = \int_0^\pi \frac{x}{a^2\cos^2 x + b^2 \sin^2 x}\, dx$ ...(1)

$$= \int_0^\pi \frac{(\pi - x)}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$= \int_0^\pi \frac{(\pi - x)}{a^2(-\cos x)^2 + b^2 \sin^2 x}\, dx$$

$\therefore \quad I = \int_0^\pi \frac{(\pi - x)}{a^2\cos^2 x + b^2\sin^2 x}\, dx$ ...(2)

$$= \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= -\int_0^{\pi/2} \log(\sin\theta\cos\theta)\, d\theta.$$

$$= -\int_0^{\pi/2} [\log(\sin\theta) + \log(\cos\theta)]\, d\theta \qquad [\because \log(m.n) = \log m + \log n]$$

$$= -\int_0^{\pi/2} \log(\sin\theta)\, d\theta - \int_0^{\pi/2} \log(\cos\theta)\, d\theta$$

$$= -\int_0^{\pi/2} \log(\sin\theta)\, d\theta - \int_0^{\pi/2} \log\left[\cos\left(\frac{\pi}{2} - \theta\right)\right] d\theta$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= -\int_0^{\pi/2} \log(\sin\theta)\, d\theta - \int_0^{\pi/2} \log(\sin\theta)\, d\theta$$

$$= -2\int_0^{\pi/2} \log(\sin\theta)\, d\theta$$

$$= -2\left[-\frac{\pi}{2}\log 2\right] \qquad \left[\because \int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2 \;\; \text{See Example 6}(v).\right]$$

$\therefore$ $\quad I = \pi \log 2.$

(*vi*) Let $\quad I = \int_0^{\infty} \log\left(x + \frac{1}{x}\right) . \frac{1}{1+x^2}\, dx$

Put $\quad x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

When $\quad x = 0, \ \tan\theta = 0 \Rightarrow \theta = 0$ and when $x = \infty, \tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\pi/2} \log\left[\tan\theta + \frac{1}{\tan\theta}\right] . \left(\frac{1}{1+\tan^2\theta}\right) . \sec^2\theta\, d\theta \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$$= \int_0^{\pi/2} \log(\tan\theta + \cot\theta) . \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log(\tan\theta + \cot\theta)\, d\theta$$

Proceed as in part (v). **[Ans.** $\pi \log 2$.**]**

(*vii*) Let $\quad I = \int_0^{\pi} \log(1 + \cos x)\, dx \qquad ...(1)$

$$= \int_0^{\pi} \log[1 + \cos(\pi - x)]\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$I = \int_0^{\pi} \log[1 - \cos x]\, dx \qquad ...(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi} \log(1 + \cos x)\, dx + \int_0^{\pi} \log(1 - \cos x)\, dx$$

$$= \int_0^{\pi} [\log(1 + \cos x) + \log(1 - \cos x)]\, dx$$

$$= \int_0^{\pi} \log[(1 + \cos x)(1 - \cos x)]\, dx \qquad [\because \log m + \log n = \log(m.n)]$$

$$= \int_0^{\pi} \log(1 - \cos^2 x)\, dx$$

$$= \int_0^{\pi} \log(\sin^2 x)\, dx \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= 2\int_0^{\pi} \log(\sin x)\, dx$$

$$\therefore \quad I = 2\left(-\frac{\pi}{2}\log 2\right) \qquad \left[\because \int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2 \right.$$
See Example 6(*v*).]

$$\Rightarrow \quad I = -\pi \log 2.$$

(*viii*) Let $I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}}\, dx$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

When $x = 0,\ \sin\theta = 0 \Rightarrow \theta = 0$ and when $x = 1,\ \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\log(\sin\theta)}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\, d\theta = \int_0^{\pi/2} \frac{\log(\sin\theta)}{\sqrt{\cos^2\theta}} \cdot \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{\log(\sin\theta)}{\cos\theta} \cdot \cos\theta\, d\theta = \int_0^{\pi/2} \log(\sin\theta)\, d\theta$$

$$= -\frac{\pi}{2}\log 2. \qquad \left[\because \int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2 \right.$$
See Example 6(*v*).]

Example 13. *Evaluate the following integrals :*

(*i*) $\int_0^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta}\, d\theta,\ a > b > 0$ (*ii*) $\int_{-\pi}^{\pi} x^{10} \sin^7 x\, dx$

(*iii*) $\int_{-\pi}^{\pi} \sin^5 x \cos x\, dx$ (*iv*) $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx$

(*v*) $\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$ (*vi*) $\int_{-1}^{1} e^{|x|}\, dx$

(*vii*) $\int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\cot x}}\, dx.$

Solution. (*i*) Let $I = \int_0^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta}\, d\theta$ ...(1)

$$= \int_0^{2\pi} \frac{\sin 2(2\pi - \theta)}{a - b\cos(2\pi - \theta)}\, d\theta \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$= \int_0^{2\pi} \frac{\sin(4\pi - 2\theta)}{a - b\cos\theta}\, d\theta$$

$$\therefore \quad I = \int_0^{2\pi} \frac{-\sin 2\theta}{a - b\cos 2\theta}\, d\theta \quad \ldots(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta}\, d\theta - \int_0^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta}\, d\theta$$

$$\Rightarrow \quad 2I = 0$$

$$\Rightarrow \quad I = 0.$$

(*ii*) Let $\quad I = \int_{-\pi}^{\pi} x^{10} \sin^7 x\, dx$

Let $\quad f(x) = x^{10} \sin^7 x$

$\Rightarrow \quad f(-x) = (-x)^{10} [\sin(-x)]^7 = x^{10}(-\sin x)^7 = -x^{10}\sin^7 x = -f(x).$

$\therefore$ $f(x)$ is an odd function.

$\therefore \int_{-\pi}^{\pi} x^{10} \sin^7 x\, dx = 0 \qquad \left[\because \int_{-a}^{a} f(x)\, dx = 0; \text{ if } f(x) \text{ is an odd function}\right]$

(*iii*) Let $\quad I = \int_{-\pi}^{\pi} \sin^5 x \cos x\, dx$

Let $\quad f(x) = \sin^5 x \cos x$

$\Rightarrow \quad f(-x) = \sin^5(-x)\cos(-x) = [\sin(-x)]^5 \cos x = (-\sin x)^5 \cos x = -\sin^5 x \cos x = -f(x)$

$\Rightarrow$ $f(x)$ is an odd function.

$\therefore \int_{-\pi}^{\pi} \sin^5 x \cos x\, dx = 0. \qquad \left[\because \int_{-a}^{a} f(x)\, dx = 0; \text{ if } f(x) \text{ is an odd function}\right]$

(*iv*) Let $\quad I = \int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx$

Let $\quad f(x) = \log\left(\frac{2-x}{2+x}\right)$

$$\Rightarrow \quad f(-x) = \log\left[\frac{2-(-x)}{2+(-x)}\right] = \log\left(\frac{2+x}{2-x}\right)$$

$$= \log\left(\frac{2-x}{2+x}\right)^{-1} = -\log\left(\frac{2-x}{2+x}\right) = -f(x) \qquad [\because m \log n = \log n^m]$$

$\therefore$ $f(x)$ is an odd function. $\qquad \left[\because \int_{-a}^{a} f(x)\, dx = 0; \text{ if } f(x) \text{ is an odd function}\right]$

$\therefore \int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx = 0.$

(*v*) Let $\quad I = \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$

$$= \int_{-\pi/4}^{\pi/4} \log\left[\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)\right] dx \qquad [\text{Multiply and divided by } \sqrt{2}]$$

$$= \int_{-\pi/4}^{\pi/4} \log\left[\sqrt{2}\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right)\right] dx$$

$$I = \int_{-\pi/4}^{\pi/4} \log\left[\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right] dx \quad [\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$$

Put $x + \frac{\pi}{4} = \theta \Rightarrow dx = d\theta$

When $x = -\frac{\pi}{4}$, $\theta = -\frac{\pi}{4} + \frac{\pi}{4} = 0$ and when $x = \frac{\pi}{4}$, $\theta = \frac{\pi}{4} + \frac{\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$.

$$\therefore \quad I = \int_0^{\pi/2} \log\left(\sqrt{2}\sin\theta\right) d\theta$$

$$= \int_0^{\pi/2} \left[\log\sqrt{2} + \log(\sin\theta)\right] d\theta \quad [\because \log(m.n) = \log m + \log n]$$

$$= \log\sqrt{2}\int_0^{\pi/2} 1.\, d\theta + \int_0^{\pi/2} \log(\sin\theta)\, d\theta$$

$$= \frac{1}{2}\log 2\,[\theta]_0^{\pi/2} + \left(-\frac{\pi}{2}\log 2\right) \quad \left[\because \int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2 \text{ See Example 6(v).}\right]$$

$$= \frac{1}{2}\log 2\left[\frac{\pi}{2} - 0\right] - \frac{\pi}{2}\log 2$$

$$= \frac{\pi}{4}\log 2 - \frac{\pi}{2}\log 2$$

$$= -\frac{\pi}{4}\log 2.$$

(*vi*) Let $I = \int_{-1}^{1} e^{|x|}\, .\, dx$

Let $f(x) = e^{|x|}$

$\Rightarrow$ $f(-x) = e^{|-x|} = e^{|x|} = f(x)$.

$\Rightarrow$ $f(x)$ is an even function.

$$\therefore \quad I = \int_{-1}^{1} e^{|x|}\, dx = 2\int_0^1 e^{|x|}\, .\, dx = 2\int_0^1 e^x\, dx \quad [\because |x| = x, \text{ for } 0 \le x \le 1]$$

$$= 2\left[e^x\right]_0^1 = 2\,[e^1 - e^0]$$

$$= 2(e - 1).$$

(*vii*) Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\cot x}}\, dx$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\frac{\cos x}{\sin x}}}\, dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \quad \ldots(1)$$

Put $x = \left(\frac{\pi}{2} - z\right) \Rightarrow dx = -dz$

$$\therefore \quad \int_{-2}^{2} |x+1|\,dx = \int_{-2}^{-1} |x+1|\,dx + \int_{-1}^{2} |x+1|\,dx = \int_{-2}^{-1} -(x+1)\,dx + \int_{-1}^{2} (x+1)\,dx$$

$$= -\left[\frac{x^2}{2}+x\right]_{-2}^{-1} + \left[\frac{x^2}{2}+x\right]_{-1}^{2} = -\left[\left(\frac{1}{2}-1\right)-\left(\frac{4}{2}-2\right)\right]+\left[\left(\frac{4}{2}+2\right)-\left(\frac{1}{2}-1\right)\right]$$

$$= -\left[-\frac{1}{2}-0\right]+\left[4-\left(-\frac{1}{2}\right)\right] = \frac{1}{2}+4+\frac{1}{2} = 4+1$$

$$= 5.$$

(*ii*) Let $\quad I = \int_0^3 |3x-1|\,dx$

Since, we have $|3x-1| = \begin{cases} -(3x-1); \text{ if } 0 < x < \dfrac{1}{3} \\ (3x-1); \text{ if } \quad \dfrac{1}{3} < x < 3 \end{cases}$

$$\therefore \quad \int_0^3 |3x-1|.\,dx = \int_0^{1/3} |3x-1|\,dx + \int_{1/3}^{3} |3x-1|\,dx = \int_0^{1/3} -(3x-1)\,dx + \int_{1/3}^{3} (3x-1)\,dx$$

$$= -\left[\frac{3x^2}{2}-x\right]_0^{1/3} + \left[\frac{3x^2}{2}-x\right]_{1/3}^{3}$$

$$= -\left[\left(\frac{3}{2}\left(\frac{1}{3}\right)^2-\frac{1}{3}\right)-(0-0)\right]+\left[\left(\frac{3(3)^2}{2}-3\right)-\left(\frac{3}{2}\left(\frac{1}{3}\right)^2-\frac{1}{3}\right)\right]$$

$$= -\left[\frac{3}{2}.\frac{1}{9}-\frac{1}{3}\right]+\left[\left(\frac{3(9)}{2}-3\right)-\left(\frac{3}{2}.\frac{1}{9}-\frac{1}{3}\right)\right]$$

$$= -\left[\frac{1}{6}-\frac{1}{3}\right]+\left[\frac{27}{2}-3-\frac{1}{6}+\frac{1}{3}\right] = -\left(-\frac{1}{6}\right)+\left[\frac{81-18-1+2}{6}\right]$$

$$= \frac{1}{6}+\frac{64}{6} = \frac{65}{6}.$$

(*iii*) Let $\quad I = \int_2^8 |x-5|\,dx$

Since, we have $|x-5| = \begin{cases} -(x-5); \text{ if } 2 \le x < 5 \\ (x-5); \text{ if } \quad 5 < x \le 8 \end{cases}$

$$\therefore \quad \int_2^8 |x-5|.\,dx = \int_2^5 |x-5|\,dx + \int_5^8 |x-5|\,dx$$

$$= \int_2^5 -(x-5)\,dx + \int_5^8 (x-5)\,dx = \int_2^5 (5-x)\,dx + \int_5^8 (x-5)\,dx$$

$$= \left[5x-\frac{x^2}{2}\right]_2^5 + \left[\frac{x^2}{2}-5x\right]_5^8$$

$$= \left[\left(5(5)-\frac{25}{2}\right)-\left(5(2)-\frac{4}{2}\right)\right]+\left[\left(\frac{64}{2}-5(8)\right)-\left(\frac{25}{2}-5(5)\right)\right]$$

$$= \left[\left(25 - \frac{25}{2}\right) - (10 - 2)\right] + \left[(32 - 40) - \left(\frac{25}{2} - 25\right)\right]$$

$$= \frac{25}{2} - 8 + (-8) - \left(-\frac{25}{2}\right) = \frac{25}{2} - 8 - 8 + \frac{25}{2}$$

$$= 25 - 16 = 9.$$

(*iv*) Let $\quad I = \int_{-5}^{5} |x + 2|\, dx$

Since, we have

$$|x + 2| = \begin{cases} -(x+2); \text{ if } & x + 2 < 0 \\ (x+2); \text{ if } & x + 2 \geq 0 \end{cases}$$

$$\therefore \quad \int_{-5}^{5} |x + 2|.\, dx = \int_{-5}^{-2} |x + 2|\, dx + \int_{-2}^{5} |x + 2|\, dx$$

$$= \int_{-5}^{-2} -(x+2)\, dx + \int_{-2}^{5} (x+2)\, dx = -\int_{-5}^{-2} (x+2)\, dx + \int_{-2}^{5} (x+2)\, dx$$

$$= -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\left[\left(\frac{(-2)^2}{2} + 2(-2)\right) - \left(\frac{(-5)^2}{2} + 2(-5)\right)\right] + \left[\left(\frac{25}{2} + 10\right) - \left(\frac{(-2)^2}{2} + 2(-2)\right)\right]$$

$$= -\left[\left(\frac{4}{2} - 4\right) - \left(\frac{25}{2} - 10\right)\right] + \left[\left(\frac{25 + 20}{2}\right) - \left(\frac{4}{2} - 4\right)\right]$$

$$= -\left[-2 - \frac{5}{2}\right] + \left[\frac{45}{2} - (-2)\right] = -\left(-\frac{9}{2}\right) + \left(\frac{45}{2} + 2\right) = \frac{9}{2} + \frac{49}{2}$$

$$= \frac{58}{2} = 29.$$

(*v*) Let $\qquad I = \int_{1/4}^{1} |2x - 1|\,.\, dx$

Since, we have

$$|2x - 1| = \begin{cases} -(2x - 1); \text{ if } & \frac{1}{4} \leq x < \frac{1}{2} \\ (2x - 1); \text{ if } & \frac{1}{2} < x \leq 1 \end{cases}$$

$$\therefore \quad \int_{1/4}^{1} |2x - 1|.\, dx = \int_{1/4}^{1/2} |2x - 1|\, dx + \int_{1/2}^{1} |2x - 1|\, dx$$

$$= \int_{1/4}^{1/2} -(2x - 1)\, dx + \int_{1/2}^{1} (2x - 1)\, dx$$

$$= \int_{1/4}^{1/2} (1 - 2x)\, dx + \int_{1/2}^{1} (2x - 1)\, dx$$

$$\therefore \int_{-4}^{4}|x+1|\,dx = \int_{-4}^{-1}|x+1|\,dx + \int_{-1}^{4}|x+1|\,dx = \int_{-4}^{-1} -(x+1)\,dx + \int_{-1}^{4}(x+1)\,dx$$

$$= -\left[\frac{x^2}{2}+x\right]_{-4}^{-1} + \left[\frac{x^2}{2}+x\right]_{-1}^{4}$$

$$= -\left[\left(\frac{(-1)^2}{2}+(-1)\right)-\left(\frac{(-4)^2}{2}+(-4)\right)\right]+\left[\left(\frac{(4)^2}{2}+4\right)-\left(\frac{(-1)^2}{2}+(-1)\right)\right]$$

$$= -\left[\left(\frac{1}{2}-1\right)-(8-4)\right]+\left[12-\left(\frac{1}{2}-1\right)\right] = -\left(-\frac{1}{2}\right)+4+12-\left(-\frac{1}{2}\right) = \frac{1}{2}+16+\frac{1}{2}$$

$$= 16+1 = 17.$$

(*ii*) Let $\quad I = \int_{-1}^{1}|2x+1|\,dx$

Since, we have

$$|2x+1| = \begin{cases} -(2x+1); \text{ if } & -1 < x < -\dfrac{1}{2} \\ (2x+1); \text{ if } & -\dfrac{1}{2} \geq x \geq 1 \end{cases}$$

$$\therefore \int_{-1}^{1}|2x+1|\,dx = \int_{-1}^{-1/2}|2x+1|\,dx + \int_{-1/2}^{1}|2x+1|\,dx$$

$$= \int_{-1}^{-1/2} -(2x+1)\,dx + \int_{-1/2}^{1}(2x+1)\,dx$$

$$= -\left[x^2+x\right]_{-1}^{-1/2} + \left[x^2+x\right]_{-1/2}^{1}$$

$$= -\left[\left\{\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)\right\}-\left\{(-1)^2+(-1)\right\}\right]+\left[(1+1)-\left\{\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)\right\}\right]$$

$$= -\left[\frac{1}{4}-\frac{1}{2}-0\right]+\left[2-\left(\frac{1}{4}-\frac{1}{2}\right)\right] = \frac{1}{4}+2+\frac{1}{4} = \frac{1+8+1}{4}$$

$$= \frac{10}{4} = \frac{5}{2}.$$

(*iii*) Let $\quad I = \int_{0}^{2}|x^2+2x-3|\,dx$

Since, we have

$$|x^2+2x-3| = \begin{cases} -(x^2+2x-3); \text{ if } 0 < x < 1 \\ (x^2+2x-3); \text{ if } 1 \leq x \leq 2 \end{cases}$$

$$\therefore \int_{0}^{2}|x^2+2x-3|\,.\,dx = \int_{0}^{1}|x^2+2x-3|\,dx + \int_{1}^{2}|x^2+2x-3|\,dx$$

$$= \int_{0}^{1} -(x^2+2x-3)\,dx + \int_{1}^{2}(x^2+2x-3)\,dx$$

$$= -\left[\frac{x^3}{3}+x^2-3x\right]_{0}^{1} + \left[\frac{x^3}{3}+x^2-3x\right]_{1}^{2}$$

$$= -\left[\left(\frac{1}{3}+1-3\right)-(0)\right]+\left[\left\{\frac{(2)^3}{3}+(2)^2-3(2)\right\}-\left\{\frac{(1)^3}{3}+(1)^2-3(1)\right\}\right]$$

$$= -\left[\frac{1}{3}-2\right]+\left[\left(\frac{8}{3}+4-6\right)-\left(\frac{1}{3}+1-2\right)\right] = -\left(-\frac{5}{3}\right)+\frac{2}{3}+\frac{5}{3}=\frac{5}{3}+\frac{2}{3}+\frac{5}{3}$$

$$= \frac{12}{3} = 4.$$

(*iv*) Let $I = \int_1^4 (|x-1|+|x-2|+|x-3|)\,dx$

Let $f(x) = |x-1| + |x-2| + |x-3|$

$$\therefore \quad f(x) = \begin{cases} -(x-1)-(x-2)-(x-3);\text{ if } x<1 \\ (x-1)-(x-2)-(x-3);\text{ if } 1\le x<2 \\ (x-1)+(x-2)-(x-3);\text{ if } 2\le x<3 \\ (x-1)+(x+2)+(x-3);\text{ if } x\ge 3 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -3x+6;\text{ if } x<1 \\ -x+4;\text{ if } 1\le x<2 \\ x;\text{ if } 2\le x<3 \\ 3x-6;\text{ if } x\ge 3 \end{cases}$$

$$\therefore \quad I = \int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx$$

$$= \int_1^2 (-x+4)\,dx + \int_2^3 x\,dx + \int_3^4 (3x-6)\,dx$$

$$= \left[-\frac{x^2}{2}+4x\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 + \left[\frac{3x^2}{2}-6x\right]_3^4$$

$$= \left[\left\{\frac{-(2)^2}{2}+4(2)\right\}-\left\{\frac{-1}{2}+4\right\}\right]+\left[\frac{9}{2}-\frac{4}{2}\right]+\left[\left\{\frac{3(4)^2}{2}-6(4)\right\}-\left\{\frac{3(3)^2}{2}-6(3)\right\}\right]$$

$$= \left[(-2+8)-\left(\frac{7}{2}\right)\right]+\left[\frac{5}{2}\right]+\left[(24-24)-\left(\frac{27}{2}-18\right)\right]$$

$$= \left(6-\frac{7}{2}\right)+\frac{5}{2}+\left(0-\left(-\frac{9}{2}\right)\right) = 6-\frac{7}{2}+\frac{5}{2}+\frac{9}{2}$$

$$= \frac{12-7+5+9}{2} = \frac{19}{2}.$$

(*v*) Let $I = \int_0^3 |x-2|\,dx$

Since, we have

$$|x-2| = \begin{cases} -(x-2);\text{ if } x-2\le 0 \\ (x-2);\text{ if } x-2\ge 0 \end{cases}$$

$$= \int_0^3 |x-2|\,dx = \int_0^2 |x-2|\,dx + \int_2^3 |x-2|\,dx$$

$$= \int_0^2 -(x-2)\,dx + \int_2^3 (x-2)\,dx = \int_0^2 (2-x)\,dx + \int_2^3 (x-2)\,dx$$

$$= \left[2x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^3$$

$$= \left[\left(2(2) - \frac{(2)^2}{2}\right) - 0\right] + \left[\left(\frac{(3)^2}{2} - 2(3)\right) - \left(\frac{(2)^2}{2} - 2(2)\right)\right]$$

$$= \left[(4-2) + \left(\frac{9}{2} - 6\right) - (-2)\right] = 2 + \left(-\frac{3}{2}\right) + 2$$

$$= 4 - \frac{3}{2} = \frac{5}{2}.$$

(*vi*) Let $\quad I = \int_{-2}^{+2} |2x+3|\,dx$

Since, we have

$$|2x+3| = \begin{cases} -(2x+3); \text{ if } & -2 \le x \le -\dfrac{3}{2} \\ (2x+3); \text{ if } & -\dfrac{3}{2} \le x \le 2 \end{cases}$$

$$\therefore \quad \int_{-2}^{2} |2x+3|\,dx = \int_{-2}^{-3/2} |2x+3|\,dx + \int_{-3/2}^{2} |2x+3|\,dx$$

$$= \int_{-2}^{-3/2} -(2x+3)\,dx + \int_{-3/2}^{2} (2x+3)\,dx$$

$$= -\left[x^2 + 3x\right]_{-2}^{-3/2} + \left[x^2 + 3x\right]_{-3/2}^{2}$$

$$= -\left[\left\{\left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right)\right\} - \left\{(-2)^2 + 3(-2)\right\}\right] + \left[(4+6) - \left\{\left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right)\right\}\right]$$

$$= -\left[\left(\frac{9}{4} - \frac{9}{2}\right) - (4-6)\right] + \left[10 - \left(\frac{9}{4} - \frac{9}{2}\right)\right] = +\frac{9}{4} - 2 + 10 + \frac{9}{4} = \frac{9}{2} + 8 = \frac{9+16}{2}$$

$$= \frac{25}{2}.$$

(*vii*) Let $\quad I = \int_1^3 |x-4|\,dx$

Since, we have

$$|x-4| = \begin{cases} -(x-4); \text{ if } & -x < 4 \\ (x-4); \text{ if } & x \ge 4 \end{cases}$$

$$\therefore \quad \int_1^3 |x-4|\,dx = \int_1^3 -(x-4)\,.\,dx = \int_1^3 (4-x)\,dx$$

$$= \int_0^{2\pi} \left[\frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} + \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} \right] dx$$

$$= \int_0^{2\pi} \frac{2\pi \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$$

$$\therefore \quad I = \pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$$

$$= 2\pi \int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx \qquad \left[\because \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx; \text{ if } f(2a - x) = f(x) \right]$$

[Again using the same property as above]

$$= 4\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx \qquad \text{...(3)}$$

$$= 4\pi \int_0^{\pi/2} \frac{\sin^{2n}\left(\frac{\pi}{2} - x\right)}{\sin^{2n}\left(\frac{\pi}{2} - x\right) + \cos^{2n}\left(\frac{\pi}{2} - x\right)}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx \right]$$

$$= 4\pi \int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x}\, dx \qquad \text{...(4)}$$

Adding equations (3) and (4), we get

$$2I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx + 4\pi \int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x}\, dx$$

$$= 4\pi \int_0^{\pi/2} \left[\frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} + \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x} \right] dx$$

$$= 4\pi \int_0^{\pi/2} \left[\frac{\sin^{2n} x + \cos^{2n} x}{\sin^{2n} x + \cos^{2n} x} \right] dx$$

$$\Rightarrow \quad I = 2\pi \int_0^{\pi/2} 1 \,.\, dx = 2\pi \left[x\right]_0^{\pi/2} = 2\pi \left[\frac{\pi}{2} - 0 \right] = \pi^2$$

$$\Rightarrow \quad I = \pi^2.$$

(*iii*) Let $\quad I = \int_0^{\pi} \cos 2x \log(\sin x)\, dx$

$$\therefore \quad I = 2\int_0^{\pi/2} \underset{\text{II}}{\cos 2x} \underset{\text{I}}{\log(\sin x)}\, dx \qquad \left[\because \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx; \text{ if } f(2a - x) = f(x) \right]$$

Integrating by parts, we get

$$I = 2\left[\log(\sin x) \,.\, \frac{\sin 2x}{2} \right]_0^{\pi/2} - 2\int_0^{\pi/2} \frac{d}{dx}\{\log(\sin x)\} \,.\, \frac{\sin 2x}{2}\, dx$$

$$= [\sin 2x \log (\sin x)]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{\sin x} \cdot \cos x \cdot \sin 2x \, dx$$

$[\because \ \sin 2A = 2 \sin A \cos A]$

$$= \left[\sin \pi \log \left\{\sin\left(\frac{\pi}{2}\right)\right\} - \sin 0^\circ \log (\sin 0^\circ)\right] - \int_0^{\pi/2} \frac{\cos x}{\sin x} \cdot 2 \sin x \cos x \, dx$$

$$= (0 - 0) - 2 \int_0^{\pi/2} \cos^2 x \, dx$$

$$= -2 \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right) dx \qquad \left[\begin{array}{l} \because \ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow \ \dfrac{1+\cos 2A}{2} = \cos^2 A \end{array}\right]$$

$$= -\int_0^{\pi/2} (1 + \cos 2x) \, dx$$

$$= -\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = -\left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin 0^\circ}{2}\right)\right]$$

$$= -\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right]$$

$\therefore$ $$I = -\frac{\pi}{2}.$$

(*iv*) Let $$I = \int_{-1}^{1} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$$

Let $$f(x) = \frac{x \sin^{-1} x}{\sqrt{1-x^2}}$$

$\Rightarrow$ $$f(-x) = \frac{(-x)\sin^{-1}(-x)}{\sqrt{1-(-x)^2}} = \frac{(-x)(-\sin^{-1} x)}{\sqrt{1-x^2}} = \frac{x \sin^{-1} x}{\sqrt{1-x^2}} = f(x)$$

$\therefore$ $f(x)$ is an even function.

$\therefore$ $$I = \int_{-1}^{1} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx = 2\int_0^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx \qquad \left[\begin{array}{l} \because \ \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \, ; \\ \text{if } f(x) \text{ is an even function} \end{array}\right]$$

Put $x = \sin \theta \ \Rightarrow \ dx = \cos \theta \, d\theta$

When $x = 0 \, ; \sin \theta = 0 \ \Rightarrow \ \theta = 0$ and when $x = 1 \, ; \sin \theta = 1 \ \Rightarrow \ \theta = \frac{\pi}{2}$

$\therefore$ $$I = 2\int_0^{\pi/2} \frac{\theta \sin \theta}{\sqrt{1-\sin^2 \theta}} \cdot \cos \theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\theta \sin \theta \cos \theta}{\sqrt{\cos^2 \theta}} \, d\theta \qquad [\because \ \sin^2 A + \cos^2 A = 1]$$

$$= 2\int_0^{\pi/2} \underset{\text{I}}{\theta} \, \underset{\text{II}}{(\sin \theta)} \, d\theta$$

Integrating by parts, we get

$$= 2\left[\theta \,.\, (-\cos\theta)\right]_0^{\pi/2} - 2\int_0^{\pi/2} \frac{d}{d\theta}(\theta) \,.\, (-\cos\theta)\, d\theta$$

$$= -2\left[\theta\cos\theta\right]_0^{\pi/2} + 2\int_0^{\pi/2} \cos\theta\, d\theta = -2\left[\frac{\pi}{2}\cos\frac{\pi}{2} - 0\right] + 2\left[\sin\theta\right]_0^{\pi/2}$$

$$= -2\,[0 - 0] + 2\left[\sin\frac{\pi}{2} - \sin 0^\circ\right] = 0 + 2\,[1 - 0]$$

$$= 2.$$

(*v*) Let $\quad I = \displaystyle\int_{-\pi/4}^{\pi/4} \frac{\left(x + \frac{\pi}{4}\right)}{(2 - \cos 2x)}\, dx$

$$= \int_{-\pi/4}^{\pi/4} \frac{x}{(2 - \cos 2x)}\, dx + \int_{-\pi/4}^{\pi/4} \frac{\frac{\pi}{4}}{(2 - \cos 2x)}\, dx \qquad \text{[Note this step]}$$

$$= 0 + \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{1}{2 - \cos 2x}\, dx$$

$$\left[\begin{array}{r} \because \displaystyle\int_{-a}^{a} f(x)\, dx = 0; \text{ if } f(x) \text{ is an odd function} \\ \displaystyle\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx\,; \text{ if } f(x) \text{ is an even function} \\ \text{Here } \dfrac{x}{2 - \cos 2x} \text{ is an odd function} \\ \text{and } \quad \dfrac{1}{2 - \cos 2x} \text{ is an even function.} \end{array}\right]$$

$$= 2 \,.\, \frac{\pi}{4}\int_0^{\pi/4} \frac{1}{2 - \cos 2x}\, dx$$

$$= \frac{\pi}{2}\int_0^{\pi/4} \frac{1}{2 - \dfrac{1 - \tan^2 x}{1 + \tan^2 x}}\, dx \qquad \left[\because \ \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}\right]$$

$$= \frac{\pi}{2}\int_0^{\pi/4} \frac{1 + \tan^2 x}{2(1 + \tan^2 x) - (1 - \tan^2 x)}\, dx = \frac{\pi}{2}\int_0^{\pi/4} \frac{1 + \tan^2 x}{1 + 3\tan^2 x}\, dx$$

$\Rightarrow \qquad I = \dfrac{\pi}{2}\displaystyle\int_0^{\pi/2} \frac{\sec^2 x}{1 + 3\tan^2 x}\, dx \qquad \left[\begin{array}{l} \because \ \sec^2 A - \tan^2 A = 1 \\ \Rightarrow \sec^2 A = 1 + \tan^2 A \end{array}\right]$

Put $\qquad \tan x = z \ \Rightarrow \ \sec^2 x\, dx = dz$

When $x = 0\,;\ z = \tan 0^\circ = 0$ and when $x = \dfrac{\pi}{2}\,;\ z = \tan\dfrac{\pi}{2} = \infty$

$$I = \frac{\pi}{2}\int_0^{\infty} \frac{1}{1 + 3z^2}\, dz$$

$$= \frac{\pi}{6}\int_0^{\infty} \frac{1}{\frac{1}{3} + z^2}\, dz = \frac{\pi}{6}\int_0^{\infty} \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2 + z^2}\, dz$$

$$\therefore \qquad I = 2\pi \int_{+1}^{-1} \frac{1}{1+z^2}(-dz) = -2\pi \int_{1}^{-1} \frac{1}{1+z^2}\, dz$$

$$= 2\pi \int_{-1}^{1} \frac{1}{1+z^2}\, dz \qquad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= 2\pi \left[\tan^{-1} z\right]_{-1}^{1} \qquad \left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= 2\pi\,[\tan^{-1} 1 - \tan^{-1}(-1)] = 2\pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 2\pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right]$$

$$= 2\pi\left(\frac{\pi}{2}\right) = \pi^2.$$

Example 17. *Evaluate the following integrals :*

(i) $\int_{-\pi/2}^{\pi/2} |\sin x|\, dx$ (ii) $\int_0^{2\pi} |\sin x|\, dx$

(iii) $\int_{-1}^{3/2} |x \sin \pi x|\, dx$ (iv) $\int_0^{\pi} |\cos x|\, dx$

(v) $\int_0^{\pi/2} |\cos 2x|\, dx$ (vi) $\int_{-\pi/2}^{\pi/2} (\sin |x| - \cos |x|)\, dx$

(vii) $\int_0^{3/2} |x \cos \pi x|\, dx$ (viii) $\int_{-\pi/4}^{\pi/4} |\sin x|\, dx$

(ix) $\int_{1/e}^{e} |\log_e x|\, dx$ (x) $\int_0^{2\pi} |\cos x|\, dx$

(xi) $\int_{-1/2}^{1/2} \left| x \cos \frac{\pi}{2} x \right| dx.$

Solution. (i) Let $I = \int_{-\pi/2}^{\pi/2} |\sin x|\, dx$

Since, we have

$$|\sin x| = \begin{cases} -\sin x; \text{ if } & -\frac{\pi}{2} \le x < 0 \\ \sin x; \text{ if } & 0 < x \le \frac{\pi}{2} \end{cases}$$

$$\therefore \qquad I = \int_{-\pi/2}^{\pi/2} |\sin x|\, dx = \int_{-\pi/2}^{0} |\sin x|\, dx + \int_0^{\pi/2} |\sin x|\, dx$$

$$= \int_{-\pi/2}^{0} -\sin x\, dx + \int_0^{\pi/2} \sin x\, dx$$

$$= -\left[-\cos x\right]_{-\pi/2}^{0} + \left[-\cos x\right]_0^{\pi/2} = \left[\cos x\right]_{-\pi/2}^{0} - \left[\cos x\right]_0^{\pi/2}$$

$$= \left[\cos 0^\circ - \cos\left(-\frac{\pi}{2}\right)\right] - \left[\cos\frac{\pi}{2} - \cos 0^\circ\right] = [1-0] - [0-1]$$

$$= 1 + 1 = 2.$$

(*ii*) Let $\quad I = \int_0^{2\pi} |\sin x|\, dx$

Since, we have

$$|\sin x| = \begin{cases} -\sin x; \text{ if } & \pi \le x \le 2\pi \\ \sin x; \text{ if } & 0 \le x \le \pi \end{cases}$$

$\therefore \quad I = \int_0^{2\pi} |\sin x|\, dx = \int_0^{\pi} |\sin x|\, dx + \int_{\pi}^{2\pi} |\sin x|\, dx$

$$= \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} -\sin x\, dx$$

$$= \Big[-\cos x\Big]_0^{\pi} + \Big[\cos x\Big]_{\pi}^{2\pi} = -[(\cos\pi - \cos 0°)] + [\cos 2\pi - \cos\pi]$$

$$= -[-1-1] + [1-(-1)] = -[-2] + [1+1]$$

$$= 2 + 2 = 4.$$

(*iii*) Let $\quad I = \int_{-1}^{3/2} |x \sin \pi x|\, dx$

Since, we have if $-1 < x < \dfrac{3}{2} \Rightarrow -\pi < \pi x < \dfrac{3\pi}{2}$

$\therefore$ when, $-1 < x < 0 \Rightarrow -\pi < \pi x < 0$

$\Rightarrow \sin \pi x < 0$

$\Rightarrow x \sin \pi x > 0$

$\therefore \quad |x \sin \pi x| = x \sin \pi x \quad$ ...(1)

when, $\quad 0 < x < 1 \Rightarrow 0 < \pi x < \pi$

$\Rightarrow \sin \pi x > 0$

$\Rightarrow x \sin \pi x > 0$

$\therefore \quad |x \sin \pi x| = x \sin \pi x \quad$ ...(2)

when $1 < x < \dfrac{3}{2} \Rightarrow \pi < \pi x < \dfrac{3\pi}{2}$

$\Rightarrow \sin \pi x < 0$

$\Rightarrow x \sin \pi x < 0$

$\therefore \quad |x \sin \pi x| = -x \sin \pi x \quad$ ...(3)

$\therefore$ From equations (1) to (3), we have

$$|x \sin \pi x| = \begin{cases} x \sin \pi x; \text{ if } & -1 < x < 1 \\ -x \sin \pi x; \text{ if } & 1 < x < \dfrac{3}{2} \end{cases}$$

$\therefore \quad I = \int_{-1}^{3/2} |x \sin \pi x|\, dx = \int_{-1}^{0} |x \sin \pi x|\, dx + \int_0^1 |x \sin \pi x|\, dx + \int_1^{3/2} |x \sin \pi x|\, dx$

$$= \int_{-1}^{0} x \sin \pi x\, dx + \int_0^1 x \sin \pi x\, dx + \int_1^{3/2} -x \sin \pi x\, dx$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx;\ a < c < b\right]$$

$$I = \int_{-1}^{1} \underset{\text{I}}{x} \underset{\text{II}}{\sin \pi x}\, dx - \int_{1}^{3/2} \underset{\text{I}}{x} \underset{\text{II}}{\sin \pi x}\, dx$$

Integrating by parts, we get

$$= \left[x \cdot \left(-\frac{\cos \pi x}{\pi}\right)\right]_{-1}^{1} - \int_{-1}^{1} \frac{d}{dx}(x) \cdot \left(-\frac{\cos \pi x}{\pi}\right) dx - \left[x \cdot \left(-\frac{\cos \pi x}{\pi}\right)\right]_{1}^{3/2}$$

$$+ \int_{1}^{3/2} \frac{d}{dx}(x) \cdot \left(-\frac{\cos \pi x}{\pi}\right) dx$$

$$= -\frac{1}{\pi}[x \cos \pi x]_{-1}^{1} + \frac{1}{\pi}\int_{-1}^{1} \cos \pi x\, dx + \frac{1}{\pi}[x \cos \pi x]_{1}^{3/2} - \frac{1}{\pi}\int_{1}^{3/2} \cos \pi x\, dx$$

$$= -\frac{1}{\pi}[\cos \pi - \{-\cos(-\pi)\}] + \frac{1}{\pi}\left[\frac{\sin \pi x}{\pi}\right]_{-1}^{1} + \frac{1}{\pi}\left[\frac{3}{2}\cos\frac{3\pi}{2} - \cos \pi\right] - \frac{1}{\pi}\left[\frac{\sin \pi x}{\pi}\right]_{1}^{3/2}$$

$$= -\frac{1}{\pi}[(-1) + (-1)] + \frac{1}{\pi^2}[\sin \pi - \sin(-\pi)] + \frac{1}{\pi}\left[\frac{3}{2}(0) - (-1)\right] - \frac{1}{\pi^2}\left[\sin\frac{3\pi}{2} - \sin \pi\right]$$

$$= \frac{2}{\pi} + \frac{1}{\pi^2}[0 + 0] + \frac{1}{\pi}[0 + 1] - \frac{1}{\pi^2}[-1 - 0] = \frac{2}{\pi} + 0 + \frac{1}{\pi} + \frac{1}{\pi^2} = \frac{3}{\pi} + \frac{1}{\pi^2}$$

$$= \frac{3\pi + 1}{\pi^2}.$$

(*iv*) Let $I = \int_0^{\pi} |\cos x|\, dx$

Since, we have $|\cos x| = \begin{cases} \cos x\,; \text{if} \quad 0 \le x \le \dfrac{\pi}{2} \\ -\cos x\,; \text{if} \quad \dfrac{\pi}{2} \le x \le \pi \end{cases}$

$$\therefore \qquad I = \int_0^{\pi} |\cos x|\, dx = \int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx$$

$$= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx$$

$$= \left[\sin x\right]_0^{\pi/2} - \left[\sin x\right]_{\pi/2}^{\pi} = \left[\sin\frac{\pi}{2} - \sin 0°\right] - \left[\sin \pi - \sin\frac{\pi}{2}\right]$$

$$= [1 - 0] - [0 - 1] = 1 + 1 = 2.$$

(*v*) Let $\qquad I = \int_0^{\pi/2} |\cos 2x|\, dx$

Since, we have $|\cos 2x| = \begin{cases} \cos 2x\,; \text{if } 0 \le x < \dfrac{\pi}{4} \Rightarrow 0 \le 2x < \dfrac{\pi}{2} \\ -\cos 2x\,; \text{if } \dfrac{\pi}{4} < x \le \dfrac{\pi}{2} \Rightarrow \dfrac{\pi}{2} < 2x \le \pi \end{cases}$

$$\therefore \qquad I = \int_0^{\pi/2} |\cos 2x|\, dx = \int_0^{\pi/4} |\cos 2x|\, dx + \int_{\pi/4}^{\pi/2} |\cos 2x|\, dx$$

$$= \int_0^{\pi/4} \cos 2x\, dx + \int_{\pi/4}^{\pi/2} - \cos 2x\, dx$$

$$= \left[\frac{\sin 2x}{2}\right]_0^{\pi/4} - \left[\frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2} = \frac{1}{2}\left[\sin 2\left(\frac{\pi}{4}\right) - \sin 0°\right] - \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 2\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{2} - 0\right] - \frac{1}{2}\left[\sin \pi - \sin\frac{\pi}{2}\right] = \frac{1}{2}[1 - 0] - \frac{1}{2}[0 - 1]$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

(*vi*) Let $\quad I = \int_{-\pi/2}^{\pi/2} (\sin |x| - \cos |x|)\, dx$

$$= \int_{-\pi/2}^{\pi/2} \sin |x|\, dx - \int_{-\pi/2}^{\pi/2} \cos |x|\, dx$$

Since $\sin |x|$ and $\cos |x|$ are even functions.

$\therefore \quad I = 2\int_0^{\pi/2} \sin |x|\, dx - 2\int_0^{\pi/2} \cos |x|\, dx \quad \left[\because \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx;\ \text{if } f(x) \text{ is an even function}\right]$

$$= 2\int_0^{\pi/2} \sin x\, dx - 2\int_0^{\pi/2} \cos x\, dx \quad \left[\because |x| = x \text{ on } \left[0, \frac{\pi}{2}\right]\right]$$

$$= 2\left[-\cos x\right]_0^{\pi/2} - 2\left[\sin x\right]_0^{\pi/2}$$

$$= -2\left[\cos\frac{\pi}{2} - \cos 0°\right] - 2\left[\sin\frac{\pi}{2} - \sin 0°\right] = -2[0 - 1] - 2[1 - 0]$$

$$= 2 - 2 = 0.$$

(*vii*) Let $\quad I = \int_0^{3/2} |x \cos \pi x|\, dx$

Put $\quad \pi x = z \Rightarrow \pi dx = dz \Rightarrow dx = \frac{1}{\pi} dz$

When $\quad x = 0;\ z = 0$ and when $x = \frac{3}{2};\ z = \frac{3\pi}{2}$

$\therefore \quad I = \int_0^{3\pi/2} \left|\frac{z}{\pi} \cos z\right| . \frac{1}{\pi} dz$

$$= \frac{1}{\pi^2}\int_0^{3\pi/2} |z \cos z|\, dz$$

$$= \frac{1}{\pi^2}\int_0^{3\pi/2} |z| . |\cos z|\, dz$$

$$= \frac{1}{\pi^2}\int_0^{3\pi/2} z\, |\cos z|\, dz$$

Since, we have

$$|\cos z| = \begin{cases} \cos z\,;\text{ if } \cos z \geq 0 \text{ on } \left[0, \dfrac{\pi}{2}\right] \\ -\cos z\,;\text{ if } \cos z \leq 0 \text{ on } \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] \end{cases}$$

$$\therefore \quad I = \frac{1}{\pi^2}\left[\int_0^{\pi/2} z\,|\cos z|\,dz + \int_{\pi/2}^{3\pi/2} z\,|\cos z|\,dz\right] = \frac{1}{\pi^2}\left[\int_0^{\pi/2} z\cos z\,dz + \int_{\pi/2}^{3\pi/2} z\,(-\cos z)\,dz\right]$$

$$= \frac{1}{\pi^2}\left[\int_0^{\pi/2} \underset{\text{I}}{z}\ \underset{\text{II}}{\cos z}\,dz - \int_{\pi/2}^{3\pi/2} \underset{\text{I}}{z}\ \underset{\text{II}}{\cos z}\,dz\right]$$

Integrating by parts, we get

$$= \frac{1}{\pi^2}\left[z\,.\sin z\right]_0^{\pi/2} - \frac{1}{\pi^2}\int_0^{\pi/2}\sin z\,dz - \frac{1}{\pi^2}\left[z\sin z\right]_{\pi/2}^{3\pi/2} - \frac{1}{\pi^2}\int_{\pi/2}^{3\pi/2}\sin z\,dz$$

$$= \frac{1}{\pi^2}\left[\frac{\pi}{2}\sin\frac{\pi}{2} - 0\right] - \frac{1}{\pi^2}\left[\cos z\right]_0^{\pi/2} - \frac{1}{\pi^2}\left[\frac{3\pi}{2}\sin\frac{3\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2}\right] - \frac{1}{\pi^2}\left[\cos z\right]_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{\pi^2}\left[\frac{\pi}{2} - 0\right] - \frac{1}{\pi^2}\left[\cos\frac{\pi}{2} - \cos 0°\right] - \frac{1}{\pi^2}\left[-\frac{3\pi}{2} - \frac{\pi}{2}\right] - \frac{1}{\pi^2}\left[\cos\frac{3\pi}{2} - \cos\frac{\pi}{2}\right]$$

$$= \frac{1}{2\pi} - \frac{1}{\pi^2}[0 - 1] + \frac{3}{2\pi} + \frac{1}{2\pi} - \frac{1}{\pi^2}[0 - 0] = \frac{1}{2\pi} - \frac{1}{\pi^2} + \frac{3}{2\pi} + \frac{1}{2\pi}$$

$$= \frac{5}{2\pi} - \frac{1}{\pi^2} = \left(\frac{5\pi - 2}{2\pi^2}\right).$$

(*viii*) Let $\quad I = \int_{-\pi/4}^{\pi/4} |\sin x|\,dx$

Since, we have

$$|\sin x| = \begin{cases} \sin x\,;\text{ if } 0 \leq x \leq \dfrac{\pi}{4} \\ -\sin x\,;\text{ if } -\dfrac{\pi}{4} \leq x \leq 0 \end{cases}$$

$$\therefore \quad I = \int_{-\pi/4}^{\pi/4} |\sin x|\,dx = \int_{-\pi/4}^{0} |\sin x|\,dx + \int_0^{\pi/4} |\sin x|\,dx$$

$$= \int_{-\pi/4}^{0} -\sin x\,dx + \int_0^{\pi/4}\sin x\,dx$$

$$= -\left[-\cos x\right]_{-\pi/4}^{0} + \left[-\cos x\right]_0^{\pi/4} = \left[\cos x\right]_{-\pi/4}^{0} - \left[\cos x\right]_0^{\pi/4}$$

$$= \left[\cos 0° - \cos\left(-\frac{\pi}{4}\right)\right] - \left[\cos\frac{\pi}{4} - \cos 0°\right] = \left[1 - \frac{1}{\sqrt{2}}\right] - \left[\frac{1}{\sqrt{2}} - 1\right]$$

$$= 2 - \sqrt{2}\,.$$

(*ix*) Let $\quad \mathrm{I} = \int_{1/e}^{e} |\log_e x| \, . \, dx$

Since, we have

$$|\log_e x| = \begin{cases} -\log_e x ; \text{ if } & \frac{1}{e} < x < 1 \\ \log_e x ; \text{ if } & -1 < x < e \end{cases}$$

$\therefore$
$$\mathrm{I} = \int_{1/e}^{e} |\log_e x| . \, dx = \int_{1/e}^{1} -\log_e x \, dx + \int_1^e \log_e x \, dx$$

$$= -\int_{1/e}^{1} \log_e x \, dx + \int_1^e \log_e x \, dx$$

$$= -\Big[x(\log_e x - 1) \Big]_{1/e}^{1} + \Big[x(\log_e x - 1) \Big]_1^e$$

$$= -\left[1.(\log_e 1 - 1) - \left\{ \frac{1}{e} \log_e \left(\frac{1}{e} \right) - 1 \right\} \right] + [e(\log_e e - 1) - 1(\log_e 1 - 1)]$$

$$= -\left[(0-1) - \frac{1}{e}(-1-1) \right] + [e(1-1) - 1(0-1)]$$

$$= -\left[-1 + \frac{2}{e} \right] + [0+1] = 1 - \frac{2}{e} + 1$$

$$= 2 - \frac{2}{e} .$$

(*x*) Let $\quad \mathrm{I} = \int_0^{2\pi} |\cos x| \, dx$

Since, we have

$$|\cos x| = \begin{cases} \cos x ; \text{ when } 0 \le x \le \frac{\pi}{2} \\ -\cos x ; \text{ when } \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ \cos x ; \text{ when } \frac{3\pi}{2} \le x \le 2\pi \end{cases}$$

$\therefore$
$$\mathrm{I} = \int_0^{\pi/2} |\cos x| \, dx + \int_{\pi/2}^{3\pi/2} |\cos x| \, dx + \int_{3\pi/2}^{2\pi} |\cos x| \, dx$$

$$= \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{3\pi/2} -\cos x \, dx + \int_{3\pi/2}^{2\pi} \cos x \, dx$$

Note. $\because \int \underset{\text{II}}{1} . \underset{\text{I}}{\log_e x} \, dx$ [Taking unity as second function]

Integrating by parts, we have

$$= \log_e x \, . \, x - \int \frac{d}{dx}(\log_e x) \, . \, x \, dx = x \log_e x - \int \frac{1}{x} \, . \, x \, dx$$

$$= x \log_e x - \int 1 \, . \, dx = x \log_e x - x = x(\log_e x - 1)$$

$$= -[-2-6] + \left[3 + \frac{27}{2}\right] = 8 + 3 + \frac{27}{2} = 11 + \frac{27}{2} = \frac{22+27}{2}$$

$$= \frac{49}{2}.$$

(viii) $$\int_0^9 f(x)\,dx = \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^9 f(x)\,dx$$

$$= \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^3 f(x)\,dx + \int_3^9 f(x)\,dx$$

$$= \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^3 1.\,dx + \int_3^9 e^{x-3}\,dx$$

$$= \Big[-\cos x\Big]_0^{\pi/2} + \Big[x\Big]_{\pi/2}^3 + \Big[e^{x-3}\Big]_3^9$$

$$= -\left[\cos\frac{\pi}{2} - \cos 0°\right] + \left[3 - \frac{\pi}{2}\right] + [e^{9-3} - e^{3-3}]$$

$$= -[0-1] + \left[3 - \frac{\pi}{2}\right] + [e^6 - e^0] = 1 + 3 - \frac{\pi}{2} + e^6 - 1$$

$$= 3 - \frac{\pi}{2} + e^6.$$

Example 19. *Prove that :* $\int_0^{2a} f(x)\,dx = \int_0^{2a} f(2a-x)\,dx.$

Solution. Let $I = \int_0^{2a} f(x)\,dx$

Put $(2a - x) = z \Rightarrow -dx = dz \Rightarrow dx = -dz$

When $x = 0 \Rightarrow z = 2a - 0 \Rightarrow z = 2a$

and when $x = 2a \Rightarrow z = 2a - 2a \Rightarrow z = 0$

$\therefore$ $$I = \int_{2a}^0 f(2a-z)(-dz) = -\int_{2a}^0 f(2a-z)\,dz$$

$$= \int_0^{2a} f(2a-z)\,dz \qquad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$\Rightarrow$ $$I = \int_0^{2a} f(2a-x)\,dx. \qquad \left[\because \int_a^b f(x)\,dx = \int_a^b f(z)\,dz\right]$$

Example 20. *Evaluate :* $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx.$

Solution. Let $$I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx \qquad ...(1)$$

$$= \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi-x)} + e^{-\cos(\pi-x)}}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$\therefore$ $$I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\,dx \qquad ...(2) \quad [\because \cos(\pi - x) = -\cos x]$$

9

Area of Bounded Regions Using Definite Integrals

9.1 INTRODUCTION

We know the methods of evaluating definite integrals. In this chapter we shall develop the idea of calculating area of the bounded regions by using definite integral. The first step in finding the areas of bounded regions is to identify the region whose area is to be determined. For this, we shall require to draw rough sketch of the given function. The process of drawing rough sketch of a given function is called curve sketching. So, we first discuss drawing of rough sketch and symmetry of common curves.

9.2 STEPS TO DRAW A ROUGH SKETCH

Step I. Symmetry : The following rules are applied to ascertain whether the given curve is symmetrical about any line or not.

(*i*) **Symmetry about x-axis :** If the equation of the curve remains unaltered when y is changed to $-y$, then the curve is symmetrical about x-axis.
e.g., The curve of $y^2 = 4ax$ is symmetrical about x-axis.

(*ii*) **Symmetry about y-axis :** If the equation of the curve remains unaltered when x is changed to $-x$, then the curve is symmetrical about y-axis.
e.g., $x^4 - x^2 + 5y = 0$ is symmetrical about y-axis.

(*iii*) **Symmetry in opposite quadrants :** If the equation of the curve remains unaltered when both x and y are changed to $-x$ and $-y$ respectively, then the curve is symmetrical in opposite quadrants.
e.g., the curve $y = \sin x$ is symmetrical in opposite quadrants.

(*iv*) **Symmetry about the line y = x :** If the equation of the curve remains unaltered when both x and y are changed to y and x respectively, then the curve is symmetrical about the line $y = x$.
e.g., the curve $x^2 + y^2 = 8axy$ is symmetrical about $y = x$.

Step II. Origin : Find whether the curve is passing through the origin or not.

Step III. Axes intersection : The points where the curve intersects the axes are found out by putting $x = 0$ and $y = 0$ in turn, in the equation of the curve.

Step IV. Monotonicity : The regions where the function is strictly increasing or strictly decreasing are found out by studying the sign of the derivative.

Step V. Extreme values : The extreme points are the turning points of the curve. These points are very useful for curve sketching.

Step VI. Some additional points : Finally, some points are also found out on the curve. The graph of the function is drawn by free hand.

9.3 ROUGH SKETCHES OF SOME COMMON CURVES : i.e., (STRAIGHT LINE, CIRCLE, PARABOLA, ELLIPSE)

In the problems of finding areas of bounded regions, a circle or a parabola, or an ellipse or a straight line may be involved.

Let us discuss some of the basic characteristics of these curves.

9.3.1 I. Straight Line : Every first degree equation $ax + by + c = 0$ in x and y represents a straight line and every inequation of the form $ax + by \geq c$ or $ax + by \leq c$ represents the region on either side of the line $ax + by = c$.

The rough sketch of a straight line is as shown in the Fig. (9.1).

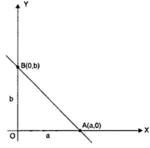


Fig. 9.1

9.3.2 II. Circle : (*i*) The equation of an circle having centre at the origin *i.e.*, (0, 0) and radius r is given by :

$$x^2 + y^2 = r^2$$

(*ii*) The equation of a circle having centre at (h, k) and radius r is given by :

$$(x - h)^2 + (y - k)^2 = r^2$$

(*iii*) The general equation of a circle is given by :

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

This represents the circle whose centre is at $(-g, -f)$ and radius equal to $\sqrt{g^2 + f^2 - c}$.

Note. (*i*) The circle represented by $x^2 + y^2 = r^2$ is symmetrical about both x-axis as well as y-axis as it contains only even powers of x and y.

(*ii*) The circle represented by $(x - h)^2 + (y - k)^2 = r^2$ is symmetrical about the lines

$$x = h \quad \text{and} \quad y = k.$$

The rough sketch of the circle in the standard form and in the central form refer to Fig. 9.2 (*a*) and 9.2 (*b*) respectively.

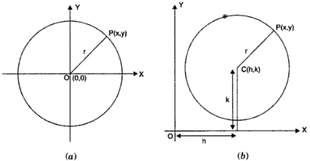


Fig. 9.2

9.3.3 III. Parabola : There are four standard forms of parabola with vertex at the origin and axis along either of co-ordinate axis.

1. Right Handed Parabola : The equation of this type of parabola is of the form

$$y^2 = 4ax \; ; a > 0.$$

For this Parabola :

(*i*) Vertex : (0, 0)
(*ii*) Focus : $(a, 0)$
(*iii*) Directrix : $x + a = 0$
(*iv*) Latus rectum : $4a$
(*v*) Axis : $y = 0$
(*vi*) Symmetry : It is symmetric about x-axis.

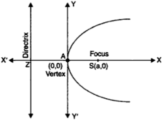


Fig. 9.3 (*a*)

2. Left Handed Parabola : The equation of this type of parabola is of the form

$$y^2 = -4ax \; ; a > 0.$$

For this Parabola :

(*i*) Vertex : (0, 0)
(*ii*) Focus : $(-a, 0)$
(*iii*) Directrix : $x - a = 0$
(*iv*) Latus rectum : $4a$
(*v*) Axis : $y = 0$
(*vi*) Symmetry : It is symmetric about x-axis.

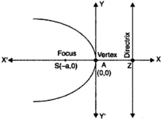


Fig. 9.3 (*b*)

3. Upward Parabola : The equation of this type of parabola is of the form

$$x^2 = 4ay, a > 0.$$

For this parabola :

(*i*) Vertex : (0, 0)
(*ii*) Focus : $(0, a)$
(*iii*) Directrix : $y + a = 0$
(*iv*) Latus rectum : $4a$
(*v*) Axis : $x = 0$
(*vi*) Symmetry : It is symmetric about y-axis.

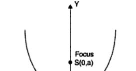


Fig. 9.3 (*c*)

4. Downward Parabola : The equation of this type of parabola is of the from $x^2 = -4ay$; $a > 0$.

For this parabola :

(*i*) Vertex : (0, 0)
(*ii*) Focus : $(0, -a)$
(*iii*) Directrix : $y - a = 0$
(*iv*) Latus rectum : $4a$
(*v*) Axis : $x = 0$
(*vi*) Symmetry : It is symmetric about y-axis.

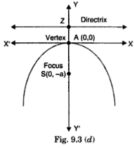


Fig. 9.3 (*d*)

9.3.4 IV. Ellipse : There are two standard forms of ellipse :

1. Foci on x-axis : The equation of this type of ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0.$$

For this ellipse :

(*i*) Centre : (0, 0)
(*ii*) Vertices : $(\pm a, 0)$
(*iii*) Foci : $(\pm ae, 0)$
(*iv*) Directrices : $x = \pm \frac{a}{e}$
(*v*) Major axis : $2a$
(*vi*) Minor axis : $2b$
(*vii*) Equation of major axis : $y = 0$
(*viii*) Equation of minor axis : $x = 0$
(*ix*) Latus rectum = $\frac{2b^2}{a}$
(*x*) Symmetry : It is symmetric about both axis.

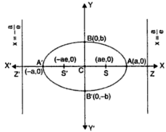


Fig. 9.4 (*a*)

2. Foci on y-axis : The equation of this type of ellipse is of the form :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0.$$

For this ellipse :

(*i*) Centre : (0, 0)
(*ii*) Vertices : $(0, \pm a)$
(*iii*) Foci : $(0, \pm ae)$
(*iv*) Directrices : $y = \pm \frac{a}{e}$
(*v*) Major axis = $2a$
(*vi*) Minor axis = $2b$
(*vii*) Equation of major axis : $x = 0$
(*viii*) Equation of minor axis : $y = 0$
(*ix*) Latus rectum = $\frac{2b^2}{a}$
(*x*) Symmetry : It is symmetric about both the axis.

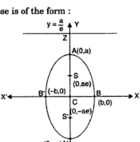


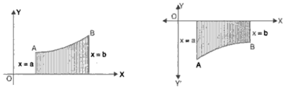
Fig. 9.4 (*b*)

The area bounded by the curve AB, the ordinates A and B, and the x-axis is often called 'the area under the curve AB'.

9.5 IMPORTANT REMARKS

1. The area under the curve $y = f(x)$, above x-axis between the ordinates $x = a$ and $x = b$ is given by

$$\int_a^b y \, . \, dx = \int_a^b f(x) \, dx \, .$$



(a) (b)

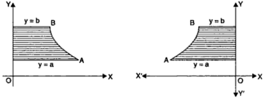
Fig. 9.6

2. The area bounded by the curve $y = f(x)$, below x-axis between the ordinates $x = a$ and $x = b$ is given by

$$\int_a^b - y \, dx = - \int_a^b y \, dx = - \int_a^b f(x) \, dx \, .$$

3. The area bounded by the curve $x = f(y)$, y-axis between the abscissae $y = a$ and $y = b$ is given by

$$\int_a^b x \, dy = \int_a^b f(y) \, dy \, .$$



(c) (d)

Fig. 9.6

4. The area bounded by the curve $x = f(y)$, y-axis between abscissae $y = a$ and $y = b$ is given by

$$\int_a^b - x \, dy = - \int_a^b f(y) \, dy \, .$$

5. If $f(x) \geq 0$ for $a \leq x \leq c$ and $f(x) \leq 0$ for $c \leq x \leq b$, then the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by

$$\int_a^c f(x)\,dx + \int_c^b -f(x)\,dx$$

$$= \int_a^c f(x)\,dx - \int_c^b f(x)\,dx.$$

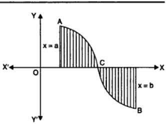


Fig. 9.6 (*e*)

Note. The rough sketch is drawn to see whether the region is above x-axis or is below x-axis or is partly above and partly below the x-axis. If it is difficult to draw the sketch of a function or is not specifically asked in the question, then we can avoid drawing the sketch and observe the sign of the function on the interval under consideration.

SOME SOLVED EXAMPLES

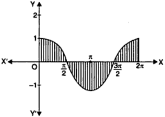
Example 1. *Find the area bounded by the curve $y = \cos x$, x-axis and the ordinates $x = 0$ and $x = 2\pi$.*

Solution. The equation of the given curve is $y = \cos x$.

Now $\cos x > 0$ when $x \in \left(0, \frac{\pi}{2}\right)$

$\cos x < 0$ when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\cos x > 0$ when $x \in \left(\frac{3\pi}{2}, 2\pi\right)$



$\Rightarrow$ $f(x)$ changes sign in the given interval.

Table of values of y corresponding to values of x from 0 to 2π is given by

x	0	$\pi/2$	π	$3\pi/2$	2π
y	1	0	-1	0	1

By joining these points with a free hand we obtain a rough sketch of the curve as shown in the figure.

$\therefore$ Required area = Area of shaded region.

$$= \int_0^{2\pi} |y|\,.\,dx = \int_0^{\pi/2} |y|\,.\,dx + \int_{\pi/2}^{3\pi/2} |y|\,.\,dx + \int_{3\pi/2}^{2\pi} |y|\,.\,dx$$

$$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/2} -\cos x\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$$

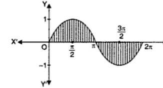
$$= \Big[\sin x\Big]_0^{\pi/2} - \Big[\sin x\Big]_{\pi/2}^{3\pi/2} + \Big[\sin x\Big]_{3\pi/2}^{2\pi}$$

$$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right] + \left[\sin 2\pi - \sin\frac{3\pi}{2}\right]$$

$$= (1 - 0) - (-1 - 1) + [0 - (-1)] = 1 + 2 + 1$$

$$= 4 \text{ sq. units.}$$

Example 2. *Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.*



Solution. The equation of the given curve is

$$y = \sin x.$$

Now $\quad \sin x \geq 0$, for $0 \leq x \leq \pi$

and $\quad \sin x \leq 0$, for $\pi \leq x \leq 2\pi$.

Thus the curve will lie above the x-axis for $0 \leq x \leq \pi$ and below the x-axis for $\pi \leq x \leq 2\pi$.

Table of values of y corresponding to values of x from 0 to 2π is given by

x	0	$\pi/2$	π	$3\pi/2$	2π
y	0	1	0	−1	0

By joining these points with a free hand we obtain a rough sketch of the curve as shown in the figure.

$\therefore$ Required area = Area of shaded region.

$$= \int_0^{2\pi} |y|\,.\,dx = \int_0^{\pi} |y|\,.\,dx + \int_{\pi}^{2\pi} |y|\,.\,dx$$

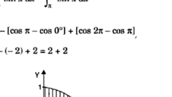
$$= \int_0^{\pi} y\,.\,dx + \int_{\pi}^{2\pi} -y\,.\,dx = \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$$

$$= \Big[-\cos x\Big]_0^{\pi} - \Big[-\cos x\Big]_{\pi}^{2\pi} = -[\cos\pi - \cos 0°] + [\cos 2\pi - \cos\pi]$$

$$= -[-1 - 1] + [1 - (-1)] = -(-2) + 2 = 2 + 2$$

$$= 4 \text{ sq. units.}$$

Example 3. *Draw a rough sketch of $y = \sin 2x$ and determine the area enclosed by the curve, x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.*



Solution. The equation of the given curve is

$$y = \sin 2x.$$

Since $\quad \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \;\Rightarrow\; \frac{\pi}{2} \leq 2x \leq \frac{3\pi}{2}$

Table of values of y corresponding to values of x from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$ is given by

x	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$	$7\pi/12$	$2\pi/3$	$3\pi/4$
y	1	0.87	0.5	0	– 0.5	– 0.87	– 1

By joining these points with a free hand we obtain a rough sketch of the curve as shown in the figure.

$\therefore$ Required area = Area of shaded region.

$$= \int_{\pi/4}^{3\pi/4} |y|.\,dx = \int_{\pi/4}^{\pi/2} |y|.\,dx + \int_{\pi/2}^{3\pi/4} |y|.\,dx$$

$$= \int_{\pi/4}^{\pi/2} y.\,dx + \int_{\pi/2}^{3\pi/4} -y.\,dx = \int_{\pi/4}^{\pi/2} \sin 2x\,dx - \int_{\pi/2}^{3\pi/4} \sin 2x\,dx$$

$$= \left[\frac{-\cos 2x}{2}\right]_{\pi/4}^{\pi/2} - \left[\frac{-\cos 2x}{2}\right]_{\pi/2}^{3\pi/4} = \frac{-1}{2}\Big[\cos 2x\Big]_{\pi/4}^{\pi/2} + \frac{1}{2}\Big[\cos 2x\Big]_{\pi/2}^{3\pi/4}$$

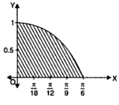
$$= \frac{-1}{2}\left[\cos \pi - \cos \frac{\pi}{2}\right] + \frac{1}{2}\left[\cos \frac{3\pi}{2} - \cos \pi\right] = \frac{-1}{2}\,[-1-0] + \frac{1}{2}\,[0-(-1)]$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \text{ sq. unit.}$$

Example 4. *Make a rough sketch of the function $y = \cos 3x$, $0 \le x \le \frac{\pi}{6}$ and determine the area enclosed between the curve and the co-ordinate axis.*

Solution. The equation of the given curve is

$$y = \cos 3x.$$



Since $\quad 0 \le x \le \frac{\pi}{6}$

$\Rightarrow \quad 0 \le 3x \le \frac{\pi}{2}$

$\Rightarrow \quad \cos 3x \ge 0.$

$\therefore$ The curve $y = \cos 3x$, $0 \le x \le \frac{\pi}{6}$ lies above the x-axis.

Table of values of y corresponding to values of x from 0 to $\frac{\pi}{6}$ is given by

x	$\pi/18$	$\pi/12$	$\pi/9$	$\pi/6$
y	0.87	0.71	0.5	0

By joining these points with a free hand we obtain a rough sketch of the curve as shown in the figure.

$\therefore$ Required area = Area between curve and the co-ordinate axis. *i.e.*, the area of the shaded region.

$$= \int_0^{\pi/6} y.\,dx = \int_0^{\pi/6} \cos 3x\,dx$$

By joining these points with a free hand, we obtain a rough sketch of the curve as shown in the figure.

$\therefore$ Required area = Area of the shaded region

$$= \int_0^2 x \, dy = \int_0^2 (2y - y^2) \, dy$$

$$= \left[\frac{2y^2}{2} - \frac{y^3}{3}\right]_0^2 = \left[y^2 - \frac{y^3}{3}\right]_0^2 = \left[\left((2)^2 - \frac{(2)^3}{3}\right) - (0 - 0)\right] = \left[\left(4 - \frac{8}{3}\right)\right] = \frac{12 - 8}{3}$$

$$= \frac{4}{3} \text{ sq. units.}$$

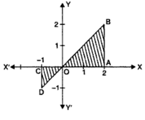
Example 12. *Find the area bounded by the curve $y = x$, x-axis and the ordinates $x = -1$, $x = 2$.*

Solution. The equation of the given curve is

$$y = x.$$

The graph of $y = x$ is a straight line passing through the origin and making an angle 45° with the x-axis.

The function $y = x$ is negative on $[-1, 0]$ and positive on $[0, 2]$.



$\therefore$ Required area = Area of the shaded region

= Area OCD + Area OAB

$$= \int_{-1}^0 |y| \, dx + \int_0^2 |y| \, dx$$

$$= \int_{-1}^0 -y \, dx + \int_0^2 y \, dx = -\int_{-1}^0 x \, dx + \int_0^2 x \, dx$$

$$= -\left[\frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^2}{2}\right]_0^2 = -\left[0 - \frac{(-1)^2}{2}\right] + \left[\frac{(2)^2}{2} - 0\right]$$

$$= \frac{1}{2} + 2 = \frac{5}{2} \text{ sq. units.}$$

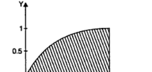
Example 13. *Make a rough sketch of the function $y = \sin 2x$, $0 \le x \le \frac{\pi}{4}$ and determine the area enclosed between the curve and the co-ordinate axis.*

Solution. The equation of the given curve is

$$y = \sin 2x$$

Since $0 \le x \le \frac{\pi}{4} \Rightarrow 0 \le 2x \le \frac{\pi}{2}$

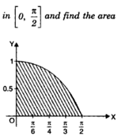
$\therefore$ $y = \sin 2x \ge 0$ when x increases from 0 to $\frac{\pi}{4}$.



$$= 8\int_0^4 x^{1/2}\,dx = 8\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^4 = 8.\frac{2}{3}\left[x^{3/2}\right]_0^4$$

$$= \frac{16}{3}\left[(4)^{3/2} - 0\right] = \frac{16}{3}[8-0] = \frac{128}{3} \text{ sq. units.}$$

Example 17. *Draw a rough sketch of the curve $y = \cos^2 x$ in $\left[0, \frac{\pi}{2}\right]$ and find the area enclosed by the curve, x-axis and the ordinates $x = 0$, $x = \frac{\pi}{2}$.*



Solution. The equation of the given curve is

$$y = \cos^2 x\,.\ \ \forall\, x \in \left[0, \frac{\pi}{2}\right]$$

Now $y = \cos^2 x \geq 0$, when x increases from 0 to $\frac{\pi}{2}$.

Table of values of y corresponding to the values of x from 0 to $\frac{\pi}{2}$ is given by

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	1	0.7	0.5	0.25	0

By joining these points with a free hand, we obtain a rough sketch of the curve as shown in the figure.

$\therefore$ Required area = Area of shaded region

$$= \int_0^{\pi/2} y.dx = \int_0^{\pi/2} \cos^2 x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} (1 + \cos^2 2x)\,dx \qquad \left[\begin{array}{l} \because\ 1 + \cos 2A = 2\cos^2 A \\ \Rightarrow\ \dfrac{1+\cos 2A}{2} = \cos^2 A \end{array}\right]$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\left\{\frac{\pi}{2} + \frac{\sin\pi}{2}\right\} - \left\{0 + \frac{\sin 0^\circ}{2}\right\}\right] = \frac{1}{2}\left[\left(\frac{\pi}{2} + 0\right) - (0+0)\right]$$

$$= \frac{\pi}{4} \text{ sq. units.}$$

Example 18. *Make a rough sketch of the function $y = 4 - x^2$, $0 \leq x \leq 2$ and determine the area enclosed between the curve and $x = 0$, $x = 2$ and x-axis.*

Solution. The given equation of the curve is

$$y = 4 - x^2.$$

The value of y is positive, as x increases from 0 to 2.

Solution. The given equation of the curve is

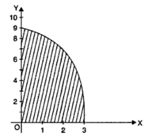
$$y = 9 - x^2$$

The value of $y = 9 - x^2$ is positive, when x increases from 0 to 3.

Table of values of y corresponding to the values of x from 0 to 3 is given by

x	0	1	2	3
y	9	8	5	0

By joining these points with a free hand, we obtain a rough sketch of the curve as shown in the figure.



$\therefore$ Required area = Area of shaded region

$$= \int_0^3 y.dx = \int_0^3 (9 - x^2)\,dx$$

$$= \left[9x - \frac{x^3}{3}\right]_0^3$$

$$= \left[\left\{9(3) - \frac{(3)^3}{3}\right\} - (0 - 0)\right]$$

$$= \left[\left(27 - \frac{27}{3}\right) - 0\right] = 27 - 9$$

= 18 sq. units.

Example 23. *Find the area bounded by the curve $xy = 4$, the x-axis and the lines $x = 1$, $x = 3$.*

Solution. The given equation of the curve is

$$xy = 4 \Rightarrow y = \frac{4}{x}$$

As x increases from 1 to 3, the value of $y = \frac{4}{x}$ decreases from $\frac{4}{1} = 4$ to $\frac{4}{3}$.

$\therefore$ The function $y = \frac{4}{x}$ is positive on [1, 3].

$\therefore$ Required area $= \int_1^3 y.dx = \int_1^3 \frac{4}{x}.dx = 4\int_1^3 \frac{1}{x}\,dx$

$$= 4\Big[\log|x|\Big]_1^3 = 4[\log|3| - \log|1|]$$

= 4 log 3 sq. units.

Example 24. *Draw a rough sketch of the curve $y = \cos^2 x$ in $[0, \pi]$ and find the area enclosed by the curve, the lines $x = 0$, $x = \pi$ and the x-axis.*

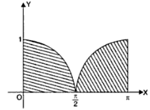
Solution. The given equation of the curve is

$$y = \cos^2 x.$$

Table of values of y corresponding to the values of x from 0 to π is given by

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	1	0.5	0	0.5	1

By joining these points with a free hand, we obtain a rough sketch of the curve as shown in the figure.



$\therefore$ Required area = Area of shaded region

$$= \int_0^\pi y.dx = \int_0^\pi \cos^2 x\, dx$$

$$\left[\begin{array}{l} \because\ 1+\cos 2A = 2\cos^2 A \\ \Rightarrow\ \dfrac{1+\cos 2A}{2} = \cos^2 A \end{array}\right]$$

$$= \int_0^\pi \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^\pi (1+\cos 2x)\, dx$$

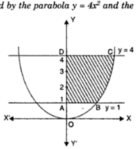
$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2}\left[\left(\pi + \frac{\sin 2\pi}{2}\right) - \left(0 + \frac{\sin 0^\circ}{2}\right)\right] = \frac{1}{2}\,[(\pi+0)-0]$$

$$= \frac{\pi}{2} \text{ sq. units.}$$

Example 25. *Find the area in first quadrant bounded by the parabola $y = 4x^2$ and the lines $x = 0$, $y = 1$ and $y = 4$.*

Solution. The equation of the given curve is :

$$y = 4x^2 \Rightarrow x^2 = \frac{1}{4}y$$



The given equation of the curve represents a parabola with vertex at the origin and axis is y-axis.

It is symmetrical about y-axis as it contains only the even powers of x.

$y = 1$ and $y = 4$ are the horizontal lines parallel to x-axis at a distance of 1 unit and 4 units from it respectively.

$\therefore$ Required area = Area of shaded region

= Area ABCD

$$= \int_1^4 |x|\, dy \qquad \left[\because\ x^2 = \frac{1}{4}y \Rightarrow x = \frac{1}{2}\sqrt{y}\right]$$

$$= \int_1^4 \frac{1}{2}\sqrt{y}\, dy = \frac{1}{2}\int_1^4 y^{1/2}\, dy$$

Table of values of y corresponding to the values of x from 0 to $\frac{\pi}{3}$ is given by

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	0	0.8	1	0.8

By joining these points with a free hand, we obtain a rough sketch of the given curve as shown in the figure.

$\therefore$ Required area = A_2 = Area of shaded region

$$= \int_0^{\pi/3} y \,.\, dx = \int_0^{\pi/3} \sin 2x \, dx$$

$$= \left[\frac{-\cos 2x}{2}\right]_0^{\pi/3} = \frac{-1}{2}\left[\cos 2x\right]_0^{\pi/3}$$

$$= \frac{-1}{2}\left[\cos\frac{2\pi}{3} - \cos 0\right] = \frac{-1}{2}\left[\frac{-1}{2} - 1\right] = \frac{-1}{2}\left(\frac{-3}{2}\right)$$

$$= \frac{3}{4} \text{ sq. units.}$$

$\therefore$ The ratio of the areas under the curves $y = \sin x$ and $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{3}$ is given by

$$A_1 : A_2 = \frac{1}{2} : \frac{3}{4} = 2 : 3.$$

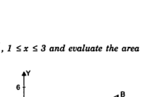
Example 27. *Sketch a rough graph of $y = 4\sqrt{x-1}$, $1 \le x \le 3$ and evaluate the area between the curve, x-axis and the line $x = 3$.*

Solution. The equation of the given curve is

$$y = 4\sqrt{x-1}.$$

Table of values of y corresponding to the values of x from 1 to 3 is given by

x	1	2	3
y	0	4	$4\sqrt{2} = 5.6$



By joining these points with a free hand, we obtain a rough sketch of the given curve as shown in the figure.

$\therefore$ Required area = Area of the shaded region

$$= \int_1^3 y \,.\, dx = \int_1^3 4\sqrt{x-1} \, dx$$

$$= 4\int_1^3 (x-1)^{1/2} \,.\, dx$$

$$= 4\left[\frac{(x-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^3 = 4\cdot\frac{2}{3}\left[(x-1)^{3/2}\right]_1^3 = \frac{8}{3}\left[(3-1)^{3/2}-(1-1)^{3/2}\right]$$

$$= \frac{8}{3}\left[2^{3/2}-0\right] = \frac{16\sqrt{2}}{3} \text{ sq. units.}$$

Example 28. *Draw a rough sketch of the function $y = \sqrt{x} + 1$ in [0, 4] and determine the area of the region enclosed by the curve, x-axis and the lines $x = 0$, $x = 4$.*

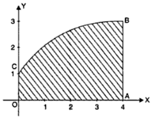
Solution. The given equation of the curve is

$$y = \sqrt{x} + 1.$$

Table of the values of y corresponding to the values of x from 0 to 4 is given by

x	0	1	2	3	4
y	1	2	2.4	2.7	3

By joining these points with a free hand, we obtain a rough sketch of the given curve as shown in the figure.



$\therefore$ Required area = Area of the shaded region

= Area OABC

$$= \int_0^4 y\,.\,dx = \int_0^4 (\sqrt{x}+1)\,dx$$

$$= \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + x\right]_0^4 = \left[\frac{2}{3}x^{3/2} + x\right]_0^4$$

$$= \left[\left\{\frac{2}{3}(4)^{3/2}+4\right\} - \{0+0\}\right] = \left[\frac{2}{3}(8)+4-0\right] = \left[\frac{16}{3}+4\right]$$

$$= \frac{16+12}{3} = \frac{28}{3} \text{ sq. units.}$$

Example 29. *Draw a rough sketch of the curve $y = \sqrt{3x+4}$ and find the area under the curve, above x-axis and between the lines $x = 0$ and $x = 4$.*

Solution. The equation of the given curve is

$$y = \sqrt{3x+4}.$$

The value of y is positive, when x increases from 0 to 4.

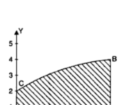


Table of values of y corresponding to the values of x from 0 to 4 is given by

x	0	1	2	3	4
y	2	2.6	3.1	3.6	4

By joining these points with a free hand, we obtain a rough sketch of the given curve as shown in the figure.

$\therefore$ Required area = Area of the shaded region

= Area OABC

$$= \int_0^4 y \, . \, dx = \int_0^4 \sqrt{3x+4}\, dx = \int_0^4 (3x+4)^{1/2} \, . \, dx$$

$$= \left[\frac{(3x+4)^{\frac{1}{2}+1}}{3.\left(\frac{1}{2}+1\right)} \right]_0^4 = \frac{2}{3} . \frac{1}{3} \left[(3x+4)^{3/2} \right]_0^4$$

$$= \frac{2}{9} [(3(4) + 4)^{3/2} - (3(0) + 4)^{3/2}]$$

$$= \frac{2}{9} [(16)^{3/2} - (4)^{3/2}] = \frac{2}{9} [64 - 8] = \frac{2}{9} \times 56$$

$$= \frac{112}{9} \text{ sq. units.}$$

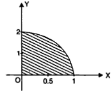
Example 30. *Draw a rough sketch of the function* $y = 2\sqrt{1-x^2}$, $x \in [0, 1]$ *and evaluate the area enclosed between the curve and the x-axis.*

Solution. The given equation of the curve is :

$$y = 2\sqrt{1-x^2}.$$

As $x \in [0, 1]$, therefore, value of y is non-negative.

Table of values of y corresponding to the values of x from 0 to 1 is given by



x	0	0.25	0.5	0.75	1
y	2	1.9	1.7	1.3	0

By joining these points with a free hand, we obtain a rough sketch of the given curve as shown in the figure.

$\therefore$ Required area = Area of shaded region

$$= \int_0^1 y \, . \, dx = \int_0^1 2\sqrt{1-x^2}\, dx$$

$$= 2 \int_0^1 \sqrt{(1)^2 - x^2} \, . \, dx$$

$$= 2\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_0^1 \qquad \left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\right]$$

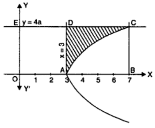
$$= 2\left[\frac{1\sqrt{1-1}}{2} + \frac{1}{2}\sin^{-1}1 - 0 - \frac{1}{2}\sin^{-1}0\right] = 2\left[0 + \frac{1}{2}\cdot\frac{\pi}{2} - 0 - 0\right]$$

$= \frac{\pi}{2}$ sq. units.

Example 31. *Find the area bounded by the curve $y^2 = 4a^2(x-3)$ and the lines $x = 3$, $y = 4a$.*

Solution. The given equation of the curve is

$$y^2 = 4a^2(x-3).$$



This equation represents a parabola with vertex at (3, 0) and axis along x-axis.

Its latus rectum is $4a^2$.

$x = 3$ is a line parallel to y-axis at a distance of 3 units from it and $y = 4a$ is a line parallel to x-axis at a distance of $4a$ units from it.

$\therefore$ Required area = Area of shaded region.

= Area ACD

= (Area OACDE)

− (Area OADE)

$$= \int_0^{4a}(x \text{ of parabola})\,.\,dy - \int_0^{4a}(x \text{ of line AD})\,dy$$

$$= \int_0^{4a}\left(\frac{y^2}{4a^2} + 3\right)dy - \int_0^{4a} 3\,.\,dy$$

$$= \left[\frac{y^3}{12a^2} + 3y\right]_0^{4a} - 3\left[y\right]_0^{4a} = \left[\left\{\frac{(4a)^3}{12a^2} + 3(4a)\right\} - \{0+0\}\right] - 3[4a - 0]$$

$$= \frac{64a^3}{12a^2} + 12a - 0 - 12a$$

$= \frac{16a}{3}$ sq. units.

Example 32. *Find the area lying above the x-axis and under the parabola $y = 4x - x^2$.*

Solution. The equation of the given curve is

$$y = 4x - x^2. \qquad \text{...(1)}$$

Now, to find the limits of integration, we have to find the points of intersection of the curve with the x-axis.

Put $y = 0$ in equation (1), we get

$$0 = 4x - x^2 \quad \Rightarrow \quad x^2 - 4x = 0$$

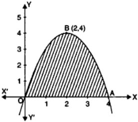
$$\Rightarrow \quad x(x-4) = 0 \quad \Rightarrow \quad x = 0, x = 4.$$

$\therefore$ The curve meets the x-axis in O(0, 0) and A(4, 0).

Table of the values of y corresponding to the values of x from 0 to 4 is given by

x	0	1	2	3	4
y	0	3	4	3	0

By joining these points, with a free hand, we obtain a rough sketch of the given curve as shown in the figure.



$\therefore$ Required area = Area of shaded region

= Area OAB

$$= \int_0^4 y \, . \, dx = \int_0^4 (4x - x^2)\, dx$$

$$= \left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_0^4$$

$$= \left[\left\{2(4)^2 - \frac{(4)^3}{3}\right\} - \{0 - 0\}\right]$$

$$= \left[32 - \frac{64}{3} - 0\right] = \frac{96 - 64}{3} = \frac{32}{3} \text{ sq. units.}$$

Example 33. *Sketch the graph of* $y = |\,x + 1\,|$. *Evaluate* $\int_{-4}^{2} |\,x + 1\,|\, dx$. *What does the value of this integral represent on the graph.*

Solution. We have $y = |\,x + 1\,|$

$$|\,x + 1\,| = \begin{Bmatrix} -(x+1); & \text{if } x + 1 < 0 \\ (x+1); & \text{if } x + 1 \geq 0 \end{Bmatrix}$$

$$\Rightarrow \qquad y = \begin{bmatrix} (-x-1); & \text{if } x < -1 \\ (x+1); & \text{if } x \geq -1 \end{bmatrix}$$

Table of values of y corresponding to the values of x is given by :

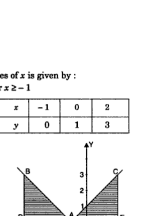
For $x \leq -1$

x	-4	-2	-1
y	3	1	0

For $x \geq -1$

x	-1	0	2
y	0	1	3

By joining these points with a free hand, we obtain a rough sketch of the given curve as shown in the figure.



$\therefore$ Required area = Area of shaded region

= (Area ABD) + (Area ACE)

$$= \int_{-1}^{3}(x+2)\,dx + \int_{-3}^{-1} -x\,dx$$

$$= \left[\frac{x^2}{2}+2x\right]_{-1}^{3} - \left[\frac{x^2}{2}\right]_{-3}^{-1}$$

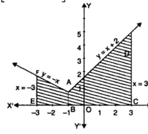
$$= \left[\left\{\frac{(3)^2}{2}+2(3)\right\} - \left\{\frac{(-1)^2}{2}+2(-1)\right\}\right] - \left[\frac{(-1)^2}{2} - \frac{(-3)^2}{2}\right]$$

$$= \left[\left(\frac{9}{2}+6\right) - \left(\frac{1}{2}-2\right)\right] - \left[\frac{1}{2}-\frac{9}{2}\right]$$

$$= \frac{9}{2}+6-\frac{1}{2}+2-\left(\frac{-8}{2}\right)$$

$$= \frac{8}{2}+8+\frac{8}{2} = 8+8$$

= 16 sq. units.



Example 35. *Find the area bounded by the parabola $x = 4 - y^2$ and the y-axis.*

Solution. The equation of the given curve is

$$x = 4 - y^2 \qquad \text{...(1)}$$

$$\Rightarrow \quad y^2 = 4 - x \quad \Rightarrow \quad y^2 = -(x-4)$$

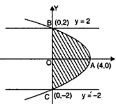
This equation represents a left handed parabola with vertex at (4, 0).

Now, to find the limit of integration, let us find out the intersection of the curve with the y-axis.

$\therefore$ Put $x = 0$ in equation (1), we get

$$0 = 4 - y^2 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

$\therefore$ The curve meets the y-axis at (0, – 2) and (0, 2).



The rough sketch of the curve is as shown in the figure.

$y = +2$ and $y = -2$ are the lines parallel to x-axis.

$\therefore$ Required area = Area of shaded region

$$= \int_{-2}^{2} |x|\,.\,dy = \int_{-2}^{2}(4-y^2)\,dy$$

$$= \left[4y - \frac{y^3}{3}\right]_{-2}^{2} = \left[\left\{4(2) - \frac{(2)^3}{3}\right\} - \left\{4(-2) - \frac{(-2)^3}{3}\right\}\right]$$

$$= \left[\left(8-\frac{8}{3}\right) - \left(-8+\frac{8}{3}\right)\right]$$

$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{48-16}{3}$$

$$= \frac{32}{3} \text{ sq. units.}$$

Example 36. *Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.*

Solution. As x increases from 0 to π, the functions $\cos^2 x$ and $\sin^2 x$ are both positive.

$\Rightarrow$ The curves $y = \sin^2 x$ and $y = \cos^2 x$ lie above the x-axis.

$\therefore$ Area under the curve $y = \sin^2 x$

$$\Rightarrow \qquad A_1 = \int_0^{\pi} y \,.\, dx = \int_0^{\pi} \sin^2 x \, dx \qquad \left[\begin{array}{l} \because \ 1 - \cos 2A = 2\sin^2 A \\ \Rightarrow \ \dfrac{1-\cos 2A}{2} = \sin^2 A \end{array}\right]$$

$$= \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx = \frac{1}{2}\int_0^{\pi} (1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}\left[\left\{\pi - \frac{\sin 2\pi}{2}\right\} - \left\{0 - \frac{\sin 0}{2}\right\}\right]$$

$$= \frac{1}{2}\left[(\pi - 0) - (0 - 0)\right] = \frac{\pi}{2} \text{ sq. units.}$$

The area under the curve $y = \cos^2 x$

$$A_2 = \int_0^{\pi} y \,.\, dx = \int_0^{\pi} \cos^2 x \, dx \qquad \left[\begin{array}{l} \because \ 1 + \cos 2A = 2\cos^2 A \\ \Rightarrow \ \dfrac{1+\cos 2A}{2} = \cos^2 A \end{array}\right]$$

$$= \int_0^{\pi} \left(\frac{1+\cos 2x}{2}\right) dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\left\{\pi + \frac{\sin 2\pi}{2}\right\} - \left\{0 + \frac{\sin 0}{2}\right\}\right]$$

$$= \frac{1}{2}\left[(\pi + 0) - (0 + 0)\right] = \frac{\pi}{2} \text{ sq. units.}$$

$$\therefore \qquad A_1 : A_2 = \frac{\pi}{2} : \frac{\pi}{2} \quad \Rightarrow \quad 1 : 1$$

$$\Rightarrow \qquad A_1 = A_2$$

$\therefore$ Area under the two given curves are equal.

Example 37. *Find the area bounded by the curve $x = at^2$, $y = 2at$ between the ordinates corresponding to $t = 1$ and $t = 2$.*

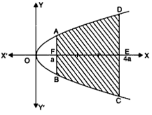
Solution. The given equations of the curve are

$$x = at^2$$

$$y = 2at \quad \Rightarrow \quad t = \frac{y}{2a}$$

$$\therefore \qquad x = at^2$$

$$\Rightarrow \qquad x = a \,.\, \left(\frac{y}{2a}\right)^2$$



$\Rightarrow \quad x = \dfrac{ay^2}{4a^2} \Rightarrow x = \dfrac{y^2}{4a}$

$\Rightarrow \quad y^2 = 4ax$ ...(1)

when $t = 1$, $x = at^2 \Rightarrow x = a(1)^2 \Rightarrow x = a$

when $t = 2$, $x = at^2 = a(2)^2 = 4a$

Clearly, the parabola is symmetrical about the x-axis.

The rough sketch of this parabola is as shown in the figure.

$\therefore$ Required area = Area of the shaded region.

= Area ABCD = 2 × Area ADEF

$$= 2\int_a^{4a} y \,.\, dx \qquad \left[\begin{array}{l} \because\ y^2 = 4ax \\ \Rightarrow\ y = \sqrt{4ax} = 2\sqrt{ax} \end{array}\right]$$

$$= 2\int_a^{4a} 2\sqrt{ax}\, dx$$

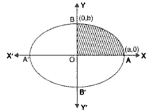
$$= 4\sqrt{a}\int_a^{4a} x^{1/2}\, dx = 4\sqrt{a}\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_a^{4a}$$

$$= 4\sqrt{a}\,.\,\frac{2}{3}\left[x^{3/2}\right]_a^{4a} = \frac{8\sqrt{a}}{3}\,[(4a)^{3/2} - a^{3/2}] = \frac{8\sqrt{a}}{3}\,[8a^{3/2} - a^{3/2}]$$

$$= \frac{8\sqrt{a}}{3}\,.\,7a\sqrt{a} = \frac{56a^2}{3} \text{ sq. units.}$$

Example 38. *Find the area under the ellipse :*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Solution. The equation of the given curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The ellipse is symmetrical about both the axes as all powers of x and y both are even in the given equation of curve.

The rough sketch of the curve is as shown in the figure.

$\therefore$ Required area = 4 × area of the ellipse in first quadrant.

= 4 × area OAB

$$= 4\int_0^a y \,.\, dx \qquad ...(1)$$

The given equation of the curve is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow \quad \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$\Rightarrow \quad y^2=\frac{b^2}{a^2}(a^2-x^2) \qquad [\because\ y \geq 0 \text{ in region OAB}]$$

$$\Rightarrow \quad y=\frac{b}{a}\sqrt{a^2-x^2}$$

Putting this value of y in equation (1), we have

$\therefore$ Required area $= \dfrac{4b}{a}\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx$

Put $x = a \sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

When $x = 0 \Rightarrow 0 = a\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0.$

and when $x = a \Rightarrow a = a\sin\theta \Rightarrow \sin\theta = 1 \Rightarrow \theta = \dfrac{\pi}{2}.$

$\therefore$ We have

$\therefore$ Required area $= \dfrac{4b}{a}\displaystyle\int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta}\,.\,a\cos\theta\,d\theta$

$$=\frac{4b}{a}\cdot\int_0^{\pi/2} a^2\sqrt{1-\sin^2\theta}\,.\,\cos\theta\,d\theta$$

$$=4ab\int_0^{\pi/2}\cos\theta\,.\,\cos\theta\,d\theta \qquad [\because\ \sin^2 A+\cos^2 A=1]$$

$$=4ab\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=4ab\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta \qquad \left[\begin{array}{l}\because\ 1+\cos 2\theta=2\cos^2\theta\\ \Rightarrow\ \left(\dfrac{1+\cos 2\theta}{2}\right)=\cos^2\theta\end{array}\right]$$

$$=2ab\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$=2ab\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}=2ab\left[\left\{\frac{\pi}{2}+\frac{\sin\pi}{2}\right\}-\left\{0+\frac{\sin 0^\circ}{2}\right\}\right]$$

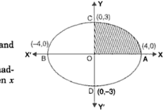
$$=2ab\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]$$

$= \pi ab$ sq. units.

Example 42. *Find the area of the region bounded by the ellipse* $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Solution. The equation of the given curve is

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$



As the equation contains only even powers of x and y, so the ellipse is symmetrical about both the axis.

Also the curve is symmetrical in opposite quadrants as the given equation remains unchanged when x and y are replaced by $-x$ and $-y$.

$$\therefore \quad \frac{x^2}{16}+\frac{y^2}{9}=1 \Rightarrow \frac{y^2}{9}=1-\frac{x^2}{16}$$

$$\Rightarrow \quad y^2=\frac{9}{16}(16-x^2) \Rightarrow y=\pm\frac{3}{4}\sqrt{16-x^2}$$

Table of values of y corresponding to the values of x satisfying the given equation is given by

x	0	4	-4	2	-2
y	± 3	0	0	± 2.5	± 2.5

By joining these points with a free hand, we obtain a rough sketch of the curve as shown in the figure.

$\therefore$ Required area = 4 × (Area of ellipse in the first quadrant.)

$$= 4\times(\text{Area OAC}) = 4\int_0^4 y\,.dx \qquad [\because\ y\ge 0]$$

$$= 4\int_0^4 \frac{3}{4}\sqrt{16-x^2}\,.dx = 3\int_0^4\sqrt{16-x^2}\,dx = 3\int_0^4\sqrt{(4)^2-x^2}\,.dx$$

$$= 3\,.\left[\frac{x\sqrt{16-x^2}}{2}+\frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$

$$\left[\because\ \int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c\right]$$

$$= 3\left[\left(\frac{4\sqrt{16-16}}{2}+8\sin^{-1}\frac{4}{4}\right)-\left(\frac{0}{2}+8\sin^{-1}0\right)\right]$$

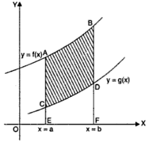
$$= 3\left[\left(0+8\,.\frac{\pi}{2}\right)-(0+0)\right]$$

$$= 3(4\pi) = 12\pi \text{ sq. units.}$$

9.6 AREA BETWEEN TWO CURVES

Let $y = f(x)$ and $y = g(x)$ be two functions such that $0 \le g(x) \le f(x)$ for $a \le x \le b$.
i.e., both the curves lie above the x-axis and the curve $y = f(x)$ lies above the curve $y = g(x)$.

The area between $y = f(x)$ and $y = g(x)$ for $a \le x \le b$ is as shown in the figure.



$\therefore$ Required area = Area of shaded region

= Area ABDC = Area ABFE − Area CDFE

= (Area bounded by $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$)

− (Area bounded by $y = g(x)$, x-axis and the ordinates $x = a$, $x = b$).

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,.\,dx$$

$\therefore$ Area between $y = f(x)$ and $y = g(x)$, $a \le x \le b$.

$$= \int_a^b (y_{\text{upper curve}} - y_{\text{lower curve}})\,.\,dx$$

Remark. Sometimes, area bounded between two curves is asked. In such cases, to find the ordinates a and b *i.e.*, the limits of integration, we find the points of intersection of the two curves.

SOME SOLVED EXAMPLES

Example 1. *Using integration, find the area of the region bounded by the curves* $y = x^2 + 2$, $y = x$, $x = 0$ *and* $x = 3$.

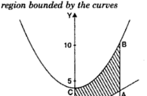
Solution. The given equation of curves are

$y = x^2 + 2$ ...(1)

$y = x$ ...(2)

$x = 0$ ...(3)

$x = 3$ ...(4)



Now, from equation (1), we have

$x^2 = y - 2$

Table of values of y corresponding to the values of x is given by

x	-2	2	0
y	0	1	0.5

By joining these points, we obtain a rough sketch of the given curves as shown in the figure.

From equation (1),

$$y = \frac{x^2}{4}$$

$$\Rightarrow \qquad x = 4y - 2$$

$$\Rightarrow \qquad x = 4\left(\frac{x^2}{4}\right) - 2 \;\Rightarrow\; x^2 - x - 2 = 0$$

$$\Rightarrow \qquad (x-2)(x+1) = 0 \;\Rightarrow\; x = 2, x = -1$$

For $x = 2$, $\qquad y = \frac{(2)^2}{4} = \frac{4}{4} = 1$

For $x = -1$, $\qquad y = \frac{(-1)^2}{4} = \frac{1}{4}$

$\therefore$ The points of intersection of the given parabola and the line are $(2, 1)$ and $\left(-1, \frac{1}{4}\right)$.

$\therefore$ Required area = Area of shaded region

= Area OABC

$$= \int_{-1}^{2} (y_{\text{upper curve}} - y_{\text{lower curve}}) \, . \, dx$$

$$= \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4}\right) dx = \frac{1}{4}\int_{-1}^{2} (x + 2 - x^2)\, dx \qquad \left[\begin{array}{l} \because \text{ From (1)}: y = \dfrac{x^2}{4} \\ \quad \text{From (2)}: y = \dfrac{x+2}{4} \end{array}\right]$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{4}\left[\left\{\frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3}\right\} - \left\{\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right\}\right]$$

$$= \frac{1}{4}\left[\left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right] = \frac{1}{4}\left[6 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}\right]$$

$$= \frac{1}{4}\left[8 - \frac{9}{3} - \frac{1}{2}\right] = \frac{1}{4}\left[8 - 3 - \frac{1}{2}\right] = \frac{1}{4}\left[5 - \frac{1}{2}\right]$$

$$= \frac{1}{4} \cdot \frac{9}{2} = \frac{9}{8} \text{ sq. units.}$$

Table of values of y corresponding to the values of x satisfying equation (2) is given by

x	0	0.5	1
y	0	1	2

By joining these points, we obtain a rough sketch of the given equation of curves as shown in the figure.

To find the points of intersection of equations (1) and (2).

Let us solve the given equations :

$$y^2 = 4x \quad \text{and} \quad y = 2x$$

$\Rightarrow \quad y^2 = 4x \;\Rightarrow\; 4x^2 = 4x$

$\Rightarrow \quad 4x(x-1) = 0 \;\Rightarrow\; x = 0, x = 1$

For $x = 0$, $\quad y^2 = 4(0) \;\Rightarrow\; y = 0$

For $x = 1$, $\quad y^2 = 4 \;\Rightarrow\; y = 2$

$\therefore$ The point of intersection of two given curves are (0, 0) and (1, 2).

$\therefore$ Required area = Area of the shaded region = Area OABC

$$= \int_0^1 (y_{\text{upper curve}} - y_{\text{lower curve}})\,.\,dx$$

$$= \int_0^1 (2\sqrt{x} - 2x)\,dx = 2\int_0^1 (x^{1/2} - x)\,dx$$

$$= 2\left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^2}{2}\right]_0^1 = 2\left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[\left\{\frac{2}{3}(1)^{3/2} - \frac{(1)^2}{2}\right\} - |0-0|\right] = 2\left[\frac{2}{3} - \frac{1}{2} - 0\right] = 2\left[\frac{4-3}{6}\right]$$

$$= \frac{1}{3} \text{ sq. units.}$$

Example 5. *Find the area of the region included between the parabola* $y = \dfrac{3x^2}{4}$ *and the line* $3x - 2y + 12 = 0$.

Solution. The given equations of the curve are

$$y = \frac{3x^2}{4} \qquad \ldots(1)$$

$$3x - 2y + 12 = 0 \qquad \ldots(2)$$

Equation (1) $y = \dfrac{3x^2}{4} \;\Rightarrow\; x^2 = \dfrac{4y}{3}$ represents an upward parabola with vertex at the origin (0, 0) and symmetrical about y-axis as it contains only even powers of x.

Equation (2) represents a straight line passing through the points (– 2, 3) and (4, 12).

$$= \frac{3}{4}\left[48 - \frac{64}{3} + 12 - \frac{8}{3}\right] = \frac{3}{4}\left[60 - \frac{72}{3}\right] = \frac{3}{4}\left[\frac{180 - 72}{3}\right]$$

$$= \frac{108}{4} = 27 \text{ sq. units.}$$

Example 6. *Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$.*

Solution. The given equations of the curves are

$$y^2 = 4ax \quad \text{...(1)}$$

$$y = mx \quad \text{...(2)}$$

Equation (1) represents the right handed parabola with vertex at the origin (0, 0) and symmetric to x-axis as it contains only even powers of y.

$y = mx$ represents a straight line passing through the origin and having the slope m.

To find the points of intersection of the parabola and the straight line we solve the equations (1) and (2).

From (1) and (2), we have

$$y^2 = 4ax$$

$$\Rightarrow \quad (mx)^2 = 4ax \Rightarrow m^2x^2 = 4ax$$

$$\Rightarrow \quad m^2x^2 - 4ax = 0 \Rightarrow x(m^2x - 4a) = 0$$

$$\Rightarrow \quad x = 0, \quad x = \frac{4a}{m^2}$$

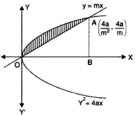
For $x = 0$, $\quad y = m(0) = 0$

For $x = \frac{4a}{m^2}$, $\quad y = m\left(\frac{4a}{m^2}\right) = \frac{4a}{m}$

$\therefore$ The points of intersection of the given curve and chord are (0, 0) and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.

The rough sketch of the curves are as shown in the figure.

$\therefore$ Required area = Area of the shaded region



$$= \int_0^{4a/m^2} (y_{\text{upper curve}} - y_{\text{lower curve}})\,.\,dx$$

$$= \int_0^{4a/m^2} (\sqrt{4ax} - mx)\, dx$$

$$= \left[\sqrt{4a}\,.\,\frac{x^{3/2}}{3/2} - \frac{mx^2}{2}\right]_0^{4a/m^2}$$

$$= \left[\frac{4\sqrt{a}}{3}\,x^{3/2} - \frac{mx^2}{2}\right]_0^{4a/m^2}$$

$$= \left[\left\{\frac{4\sqrt{a}}{3}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2\right\} - \{0 - 0\}\right]$$

$$= \left[\left\{\frac{3(3)^2}{2} - \frac{(3)^3}{3}\right\} - 0\right] = \left[\frac{27}{2} - 9\right] = \frac{27 - 18}{2}$$

$$= \frac{9}{2} \text{ sq. units.}$$

Example 8. *Find the area of the region included between the parabola $y^2 = x$ and the straight line $x + y = 2$.*

Solution. The given equations of the curve are

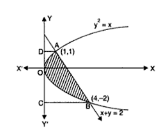
$$y^2 = x \qquad \text{...(1)}$$

$$x + y = 2 \qquad \text{...(2)}$$

Equation (1) $y^2 = x$, represents a parabola with vertex at the origin (0, 0) and is symmetric about x-axis as it contains only even powers of y.

Equation (2) represents a straight line passing through the points (1, 1) and (4, – 2).

Now, to find the points of intersection of the two curves, let us solve equations (1) and (2) simultaneously.



$\therefore$ From (1), $y^2 = x$

$\Rightarrow \quad y^2 + y = 2 \quad \Rightarrow \quad y^2 + y - 2 = 0$

$\Rightarrow \quad y^2 + 2y - y - 2 = 0 \quad \Rightarrow \quad (y + 2)(y - 1) = 0$

$y = -2, \quad y = 1$

For $y = -2, \quad x = (-2)^2 = 4$

For $y = 1, \quad x = (1)^2 = 1$

$\therefore$ The points of intersection of the two curves are (1, 1) and (4, – 2).

The rough sketch of these curves is as shown in the figure.

$\therefore$ Required area = Area of the shaded region

= Area OABO

$$= \int_{-2}^{1} (x_{\text{upper curve}} - x_{\text{lower curve}})\, dy$$

$$= \int_{-2}^{1} [(2 - y) - y^2]\, dy = \int_{-2}^{1} (2 - y - y^2)\, dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1} = \left[\left\{2(1) - \frac{(1)^2}{2} - \frac{(1)^3}{3}\right\} - \left\{2(-2) - \frac{(-2)^2}{2} - \frac{(-2)^3}{3}\right\}\right]$$

$$= \left[\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)\right] = \left(\frac{12 - 3 - 2}{6}\right) - \left(-6 + \frac{8}{3}\right) = \frac{7}{6} + 6 - \frac{8}{3}$$

$$= \frac{7 + 36 - 16}{6} = \frac{27}{6} = \frac{9}{2} \text{ sq. units.}$$

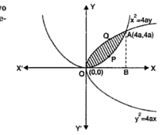
Example 9. *Find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$.*

Solution. The given equations of the curves are

$$y^2 = 16x \qquad \text{...(1)}$$

$$x = 4 \qquad \text{...(2)}$$

Now, to find the point of intersection of the two curves, let us solve the equations (1) and (2) simultaneously.



From (2), $\therefore\quad y = \frac{x^2}{4a}$

Putting this value of y in (1), we get

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow \quad \frac{x^4}{16a^2} = 4ax \quad \Rightarrow \quad x^4 - 64a^3x = 0$$

$$\Rightarrow \quad x(x^3 - 64a^3) = 0 \quad \Rightarrow \quad x = 0,\ x = 4a$$

For $x = 0$, $y = 0$

For $x = 4a$, $y = 4a$.

$\therefore$ The points of intersection of two curves are (0, 0) and $(4a, 4a)$.

$\therefore$ Required area = Area of the shaded region

= Area OPAQO

= Area OQAB – Area OPAB

$$= \int_0^{4a} (y_{\text{upper curve}} - y_{\text{lower curve}})\,.\,dx$$

$$= \int_0^{4a}\left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx = 2\sqrt{a}\int_0^{4a}\sqrt{x}\,dx - \frac{1}{4a}\int_0^{4a} x^2\,dx$$

$$= 2\sqrt{a}\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a} = 2\sqrt{a}\,.\,\frac{2}{3}\left[x^{3/2}\right]_0^{4a} - \frac{1}{12a}\left[x^3\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\,[(4a)^{3/2} - 0] - \frac{1}{12a}\,[(4a)^3 - 0]$$

$$= \frac{4\sqrt{a}}{3}\,.\,8a^{3/2} - \frac{1}{12a}\,.\,64a^3 = \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3} \text{ sq. units.}$$

Example 11. *Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x$.*

Solution. The given equations of the curve are :

$y = x^2$ ...(1)

$y = x$ ...(2)

Equation (1) represents a parabola with vertex at the origin (0, 0) and symmetric about y-axis as it contains only even powers of x.

Equation (2) represents a straight line passing through the origin.

Table of values of x and y satisfying equation (1) is given by

x	0	1	-1	2	-2
y	0	1	1	4	4

Table of values of x and y satisfying equation (2) is given by

x	0	1	2	3
y	0	1	2	3

By joining these points, we obtain a rough sketch of the given curves as shown in the figure.

Now, to find the points of intersection of the given curves, let us solve equations (1) and (2) simultaneously.

From (1) $y = x^2$

$\Rightarrow \quad x = x^2 \qquad [\because \; y = x]$

$\Rightarrow \quad x^2 - x = 0 \;\Rightarrow\; x(x-1) = 0$

$\Rightarrow \quad x = 0, x = 1$

For $x = 0$, $y = 0$

For $x = 1$, $y = 1$

$\therefore$ The points of intersection of the given curves are (0, 0) and (1, 1).

$\therefore$ Required area = Area of the shaded region

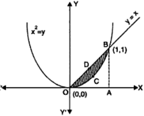
= Area OABO – Area OABCO

$$= \int_0^1 (y_{\text{upper curve}} - y_{\text{lower curve}})\, dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\left(\frac{(1)^2}{2} - \frac{(1)^3}{3}\right) - 0\right] = \frac{1}{2} - \frac{1}{3}$$

$$= \frac{3-2}{6} = \frac{1}{6} \text{ sq. units.}$$



Example 12. *Find the area of the region bounded by the parabola $y = x^2$ and lines $y = |x|$.*

Solution. The given equations of the curves are

$y = x^2 \qquad ...(1)$

$y = |x| \qquad ...(2)$

Equation (1), represents a parabola with vertex at the origin (0, 0) and symmetric to y-axis as it contains only even powers of x.

Equation (2), $y = |x|$ *i.e.*, $y = x$ and $y = -x$ represents two straight lines passing through the origin and making an angle of 45° and 135° with the positive direction of x-axis.

The rough sketch of these curves are as shown in the figure.

$\therefore$ Required area = Area of the shaded region

= 2(Area OMANO)

[$\because$ Both the curves are symmetrical about y-axis]

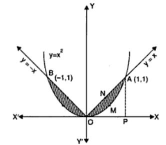
= 2[Area ONAP − Area OMAP]

$$= 2\int_0^1 (y_{\text{upper curve}} - y_{\text{lower curve}}) \cdot dx$$

$$= 2\int_0^1 (x - x^2)\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = 2\left(\frac{3-2}{6}\right)$$

$= \frac{1}{3}$ sq. units.



Example 13. *Find the area of the region bounded by the parabola $y^2 = 4ax$ and the line $x = a$.*

Solution. The given equations of the curves are

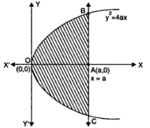
$$y^2 = 4ax \qquad \text{...(1)}$$

$$x = a \qquad \text{...(2)}$$

Equation (1) represents a parabola with vertex at the origin (0, 0) and symmetric about x-axis as it contains only even powers of y.

Equations (2), $x = a$ represents a straight line parallel to y-axis at a distance a units from it.

Now, to find the points of intersection of these curves put $x = a$ in equation (1), we have



$$y^2 = 4ax$$

$\Rightarrow$ $y^2 = 4a(a) = 4a^2$

$\Rightarrow$ $y = \pm 2a$

For $x = a$, $y = 2a$

For $x = a$, $y = -2a$

$\therefore$ The points of intersection of these curves are $(a, 2a)$ and $(a, -2a)$.

The rough sketch of these curves are as shown in the figure.

$\therefore$ Required area = Area of the shaded region

= Area OBC = 2(Area OAB)

$$= 2\int_0^a y \cdot dx = \int_0^a \sqrt{4ax}\,dx = 2 \cdot 2\sqrt{a}\int_0^a \sqrt{x}\,dx \qquad [\because\ y = \sqrt{4ax}]$$

$$= 4\sqrt{a}\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^a = 4\sqrt{a} \cdot \frac{2}{3}\left[x^{3/2}\right]_0^a$$

$$= \frac{8\sqrt{a}}{3}\,[a^{3/2} - 0] = \frac{8}{3}\,a^2 \text{ sq. units.}$$

Example 14. *Find the area of the bounded region bounded by the two parabolas $y = x^2$ and $x = y^2$.*

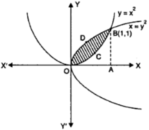
Solution. The given equations of the curves are

$$y = x^2 \quad ...(1)$$

$$x = y^2 \quad ...(2)$$

Equation (1) $y = x^2$ represents a parabola with vertex at the origin (0, 0) and symmetric about y-axis as it contains only even powers of x.

Equation (2) $x = y^2$ represents a parabola with vertex at the origin (0, 0) and symmetric about x-axis as it contains only even powers of y.



The rough sketch of these curves are as shown in the figure.

Now, to find the points of intersection of the two curves, let us solve equations (1) and (2).

$\therefore$ From (2) : $x = y^2$

$\Rightarrow$ $y = x^2$

$y = (y^2)^2 = y^4$

$\Rightarrow$ $y^4 - y = 0 \Rightarrow y(y^3 - 1) = 0$

$\Rightarrow$ $y = 0, y = 1$

For $y = 0$, $x = (0)^2 = 0$

For $y = 1$, $x = (1)^2 = 1$

$\therefore$ The points of intersection of the two curves are (0, 0) and (1, 1).

$\therefore$ Required area = Area OCBDO = Area of the shaded region

= Area OABD – Area OABC

$$= \int_0^1 (y_{\text{upper curve}} - y_{\text{lower curve}})\,.\,dx$$

$$= \int_0^1 (\sqrt{x} - x^2)\,dx$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1 = \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 = \left[\left\{\frac{2}{3}(1)^{3/2} - \frac{(1)^3}{3}\right\} - 0\right]$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq. units.}$$

Example 15. *Find the area of the region bounded by the parabola $y^2 = 2x + 1$ and the line $x - y - 1 = 0$.*

Solution. The given equations of the curves are

$$y^2 = 2x + 1$$

$$\Rightarrow \quad y^2 = 2\left(x + \frac{1}{2}\right) \quad ...(1)$$

$$x - y - 1 = 0 \quad ...(2)$$

Equation (1) represents a parabola with vertex at $\left(\frac{1}{2}, 0\right)$ and symmetrical about x-axis as it contains only even powers of y.

Clearly, equation (2) $x - y - 1 = 0$ represents a straight line.

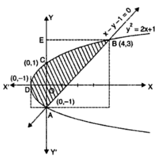
Table of values of x and y satisfying equation (1) is given by

x	0	1	4
y	± 1	± 1.73	± 3

Table of values of x and y satisfying equation (2) is given by

x	1	2	3	0	-1	4
y	0	1	2	-1	-2	3

By joining these points, we obtain a rough sketch of the curves as shown in the figure.



Now, to find the points of intersection of the two curves, let us solve equations (1) and (2).

From (1) $y^2 = 2x + 1$

$\Rightarrow \quad (x-1)^2 = 2x + 1 \qquad \left[\because \; x - y - 1 = 0 \;\Rightarrow\; y = x - 1\right]$

$\Rightarrow \quad x^2 - 2x + 1 = 2x + 1$

$\Rightarrow \quad x^2 - 4x = 0$

$\Rightarrow \quad x(x-4) = 0 \;\Rightarrow\; x = 0, x = 4.$

For $x = 0, \quad y = -1$

For $x = 4, \quad y = 3$

$\therefore$ The points of intersection of the two curves are $(0, -1)$ and $(4, 3)$.

$\therefore$ Required area = Area of the shaded region

= Area ABCD

= Area AOCD + Area ABECOA − Area BEC

$$= -\int_{-1}^{1} (x_{\text{parabola}})\, dy + \int_{-1}^{3} (x_{\text{line}})\, dy - \int_{1}^{3} (x_{\text{parabola}})\, dy$$

$$= -\int_{-1}^{1} \left(\frac{y^2-1}{2}\right) dy + \int_{-1}^{3} (y+1)\, dy - \int_{1}^{3} \left(\frac{y^2-1}{2}\right) dy$$

$$= \frac{-1}{2}\left[\frac{y^3}{3} - y\right]_{-1}^{1} + \left[\frac{y^2}{2} + y\right]_{-1}^{3} - \frac{1}{2}\left[\frac{y^3}{3} - y\right]_{1}^{3}$$

$$= \frac{-1}{2}\left[\left\{\frac{(1)^3}{3} - 1\right\} - \left\{\frac{(-1)^3}{3} - (-1)\right\}\right] + \left[\left\{\frac{(3)^2}{2} + 3\right\} - \left\{\frac{(-1)^2}{2} + (-1)\right\}\right]$$

$$- \frac{1}{2}\left[\left\{\frac{(3)^3}{3} - 3\right\} - \left\{\frac{(1)^3}{3} - 1\right\}\right]$$

$$= \frac{-1}{2}\left[\frac{1}{3} - 1 + \frac{1}{3} - 1\right] + \left[\frac{9}{2} + 3 - \frac{1}{2} + 1\right] - \frac{1}{2}\left[9 - 3 - \frac{1}{3} + 1\right]$$

$$= \frac{-1}{2}\left[\frac{2}{3}-2\right]+[4+4]-\frac{1}{2}\left[7-\frac{1}{3}\right]$$

$$= \frac{-1}{2}\left[\frac{2-6}{3}\right]+8-\frac{1}{2}\left[\frac{21-1}{3}\right]=\frac{2}{3}+8-\frac{10}{3}=8-\frac{8}{3}$$

$$= \frac{24-8}{3}=\frac{16}{3} \text{ sq. units.}$$

Example 16. *Find the area bounded by the curves $y = x$ and $y = x^3$.*

Solution. The given equations of the curves are

$$y = x \quad ...(1)$$

$$y = x^3 \quad ...(2)$$

Equation (1) $y = x$ represents a straight line passing through the origin and making an angle of 45° with x-axis.

To find the points of intersection of the two curves, let us solve equations (1) and (2) simultaneously.

$\therefore$ From (1) and (2), we have

$$x = x^3 \Rightarrow x^3 - x = 0$$

$\Rightarrow \quad x(x^2-1) = 0$

$\Rightarrow \quad x = 0, x = \pm 1$

For $x = 0$, $\quad y = 0$

For $x = -1$, $\quad y = -1$

For $x = 1$, $\quad y = 1$

$\therefore$ The points of intersection of the two curves are (0, 0), (– 1, – 1) and (1, 1).

Graph of $y = x^3$ and $y = x$.

Symmetry : The equations (1) and (2) remain unchanged on changing x to $-x$ and y to $-y$.

$\therefore$ The curves (1) and (2) are symmetrical in opposite quadrants.

The curves passes through the origin.

Monotonicity : $\quad y = x^3$

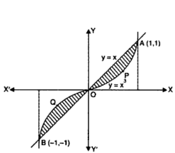
$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 \geq 0$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$$

$$\Rightarrow \quad \frac{dy}{dx} > 0 \Rightarrow 3x^2 > 0 \quad \text{for } x \neq 0.$$

$\therefore$ The curve is strictly increasing for $x \neq 0$.

Extreme values :



$$\frac{d^2y}{dx^2} = 6x$$

$$\Rightarrow \quad \left.\frac{d^2y}{dx^2}\right|_{x=0} = 6(0) = 0.$$

$\Rightarrow \qquad \dfrac{d^3y}{dx^3} = 6 \neq 0$ at $x = 0$.

$\therefore$ $x = 0$ gives a point of inflexion.

Table of values of x and y satisfying equation (1) is given by

x	0	-1	1	-2	2
y	0	-1	1	-8	8

By joining these points we obtain a rough sketch of the curve as shown in the figure.

$\therefore$ Required area = Area of the shaded region

$$= \text{Area BOQB} + \text{Area AOPA}$$

$$= \int_{-1}^{0} \left[(-y_{\text{line}}) - (-y_{\text{curve}})\right] . dx + \int_{0}^{1} \left[(y_{\text{line}} - y_{\text{curve}})\right] . dx$$

$$= \int_{-1}^{0} (-x + x^3)\, dx + \int_{0}^{1} (x - x^3)\, dx$$

$$= \left[-\frac{x^2}{2} + \frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$$

$$= \left[0 - \left\{-\frac{(-1)^2}{2} + \frac{(-1)^4}{4}\right\}\right] + \left[\left\{\frac{(1)^2}{2} - \frac{(1)^4}{4}\right\} - 0\right]$$

$$= \left[\frac{1}{2} - \frac{1}{4} + \frac{1}{2} - \frac{1}{4}\right] = 1 - \frac{2}{4} = \frac{4-2}{4} = \frac{2}{4}$$

$$= \frac{1}{2} \text{ sq. units.}$$

Example 17. *Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$ as x varies from 0 to $\frac{\pi}{2}$ and find the area of the region enclosed by the curves and the y-axis.*

Solution. The given equations of the curves are

$$y = \sin x \qquad \text{...(1)}$$

$$y = \cos x \qquad \text{...(2)}$$

Table of values of x and y satisfying equations (1) and (2) are given by

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = \sin x$	0	$\frac{1}{2} = 0.5$	$\frac{1}{\sqrt{2}} = 0.71$	$\frac{\sqrt{3}}{2} = 0.87$	1
$y = \cos x$	1	$\frac{\sqrt{3}}{2} = 0.87$	$\frac{1}{\sqrt{2}} = 0.71$	$\frac{1}{2} = 0.5$	0

Equation (2) represents a left handed parabola with vertex at (1, 0).

Now, to find the points of intersection of two curves, let us solve equations (1) and (2) simultaneously.

$\therefore \quad x + 1 = -x + 1$

$\Rightarrow \quad 2x = 0 \Rightarrow x = 0$

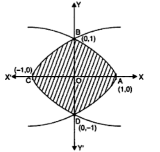
For $x = 0$, $y^2 = 0 + 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

$\therefore$ The points of intersection of two curves are (0, 1) and (0, – 1).

The rough sketch of the two curves is as shown in the figure.

By symmetry of parabolas :

Area OAB = Area OAD and Area OBC = Area OCD.



$\therefore$ Required area = Area of shaded region

$= 2(\text{Area OBC}) + 2(\text{Area OAB})$

$$= 2\int_{-1}^{0}\sqrt{x+1}\,dx + 2\int_{0}^{1}\sqrt{-x+1}\,dx$$

$$= 2\left[\frac{(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{-1}^{0} + 2\left[\frac{(-x+1)^{\frac{1}{2}+1}}{(-1)\left(\frac{1}{2}+1\right)}\right]_{0}^{1}$$

$$= 2.\frac{2}{3}\left[(x+1)^{3/2}\right]_{-1}^{0} - 2.\frac{2}{3}\left[(-x+1)^{3/2}\right]_{0}^{1}$$

$$= \frac{4}{3}[(0+1)^{3/2} - (-1+1)^{3/2}] - \frac{4}{3}[(-1+1)^{3/2} - (0+1)^{3/2}] = \frac{4}{3}(1-0) - \frac{4}{3}(0-1)$$

$$= \frac{4}{3} + \frac{4}{3} = \frac{8}{3} \text{ sq. units.}$$

Example 21. *Find the area of the region common to the two parabolas $y = 2x^2$ and $x^2 = y - 4$.*

Solution. The given equations of the curves are

$$y = 2x^2 \qquad ...(1)$$

$$x^2 = y - 4 \qquad ...(2)$$

Equation (1) represents a parabola opening upward with vertex at the origin (0, 0).

Equation (2) also represents an upward parabola with vertex at (0, 4).

Table of values of x and y satisfying equations (1) and (2) is given by

x	0	– 1	1	2	– 2
$y = 2x^2$	0	2	2	8	8
$y = x^2 + 4$	4	5	5	8	8

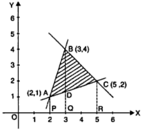
By joining these points with a free hand, we obtain a rough sketch of the curves as shown in the figure.

$\Rightarrow \quad y - 1 = \frac{3}{1}(x-2)$

$\Rightarrow \quad y - 1 = 3x - 6$

$\Rightarrow \quad 3x - y - 5 = 0 \quad \ldots(1)$

$\left[\because\right.$ Equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by $(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ $\left.\right]$



Equation of line BC is given by

$$(y-4) = \frac{2-4}{5-3}(x-3)$$

$\Rightarrow \quad (y-4) = \frac{-2}{2}(x-3)$

$\Rightarrow \quad y - 4 = -x + 3$

$\Rightarrow \quad x + y - 7 = 0 \quad \ldots(2)$

Equation of line AC is given by

$$(y-1) = \frac{2-1}{5-2}(x-2)$$

$\Rightarrow \quad (y-1) = \frac{1}{3}(x-2)$

$\Rightarrow \quad 3y - 3 = x - 2$

$\Rightarrow \quad x - 3y + 1 = 0 \quad \ldots(3)$

$\therefore$ Required area = Area of the shaded region

= Area of ΔABC

= Area ABD + Area BDC

= [Area APQB – Area APQD] + [Area BQRC – Area DQRC]

$$= \int_2^3 (y_{AB} - y_{AC}).dx + \int_3^5 (y_{BC} - y_{AC}).dx$$

$$= \int_2^3 \left[(3x-5) - \left(\frac{x+1}{3}\right)\right].dx + \int_3^5 \left[(7-x) - \left(\frac{x+1}{3}\right)\right].dx$$

$$= \frac{1}{3}\int_2^3 (9x - 15 - x - 1)\,dx + \frac{1}{3}\int_3^5 (21 - 3x - x - 1)\,dx$$

$$= \frac{1}{3}\int_2^3 (8x - 16).dx + \frac{1}{3}\int_3^5 (20 - 4x).dx$$

$$= \frac{8}{3}\int_2^3 (x-2).dx + \frac{4}{3}\int_3^5 (5-x).dx$$

$$= \frac{8}{3}\left[\frac{x^2}{2} - 2x\right]_2^3 + \frac{4}{3}\left[5x - \frac{x^2}{2}\right]_3^5$$

Equation of line AC is given by

$$(y-1)=\frac{2-1}{3+1}(x+1)$$

$$\Rightarrow \quad (y-1)=\frac{1}{4}(x+1)$$

$$\Rightarrow \quad 4y-4=x+1$$

$$\Rightarrow \quad x-4y+5=0 \qquad ...(3)$$

$\therefore$ Required area = Area of the shaded region

$$= \text{Area of } \Delta ABC$$

$$= \text{Area ABD + Area BDC}$$

$$= [\text{Area APOB} - \text{Area APOD}] + [\text{Area BOQC} - \text{Area DOQC}]$$

$$= \int_{-1}^{0}(y_{AB}-y_{AC})\,dx+\int_{0}^{3}(y_{BC}-y_{AC})\,dx$$

$$= \int_{-1}^{0}\left[(4x+5)-\left(\frac{x+5}{4}\right)\right]dx+\int_{0}^{3}\left[(5-x)-\left(\frac{x+5}{4}\right)\right]dx$$

$$= \frac{1}{4}\int_{-1}^{0}(16x+20-x-5)\,dx+\frac{1}{4}\int_{0}^{3}(20-4x-x-5)\,dx$$

$$= \frac{1}{4}\int_{-1}^{0}(15x+15)\,dx+\frac{1}{4}\int_{0}^{3}(15-5x)\,dx$$

$$= \frac{15}{4}\int_{-1}^{0}(x+1)\,dx+\frac{5}{4}\int_{0}^{3}(3-x)\,dx$$

$$= \frac{15}{4}\left[\frac{x^2}{2}+x\right]_{-1}^{0}+\frac{5}{4}\left[3x-\frac{x^2}{2}\right]_{0}^{3}$$

$$= \frac{15}{4}\left[0-\left\{\frac{(-1)^2}{2}+(-1)\right\}\right]+\frac{5}{4}\left[\left\{3(3)-\frac{(3)^2}{2}\right\}-0\right]$$

$$= \frac{15}{4}\left[-\left(\frac{1}{2}-1\right)\right]+\frac{5}{4}\left[9-\frac{9}{2}\right]=\frac{15}{8}+\frac{45}{8}=\frac{60}{8}$$

$$= \frac{15}{2} \text{ sq. units.}$$

Example 28. *Using integration, find the area of the triangle whose vertices are (2, 0), (4, 5) and (6, 3).*

Solution. Let A(2, 0), B(4, 5) and C(6, 3) be the vertices of a triangle. The rough sketch of the triangle is as shown in the figure.

Equation of line AB is given by

$$(y-0)=\frac{5-0}{4-2}(x-2)$$

$$\Rightarrow \quad y=\frac{5}{2}(x-2)$$

$$\Rightarrow \quad 2y=5x-10$$

$$= \frac{7}{4}[(8-8)-(2-4)] + \frac{7}{4}\left[\left(36-\frac{36}{2}\right)-(24-8)\right]$$

$$= \frac{7}{4}(0+2) + \frac{7}{4}\left[\frac{36}{2}-16\right] = \frac{7}{2} + \frac{7}{4}\left(\frac{36-32}{2}\right) = \frac{7}{2} + \frac{7}{2}$$

= 7 sq. units.

Example 29. *Using integration find the area of the region bounded by the following curves.*

$$y = 1 + |x+1|, \quad x = -2, \quad x = 3, \quad y = 0.$$

Solution. The given equations of the curves are

$$y = 1 + |x+1| \qquad \text{...(1)}$$

$$\Rightarrow \qquad y = \begin{cases} 1+(x+1) & ; \ x+1>0 \\ 1-(x+1) & ; \ x+1 \le 0 \end{cases}$$

$$\Rightarrow \qquad y = \begin{cases} x+2 & ; \ x>-1 \\ -x & ; \ x \le -0 \end{cases}$$

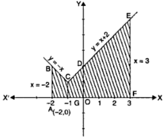
$x = -2$ is a straight line parallel to y-axis at a distance of 2 units to the left side of y-axis.
$x = 3$ is a straight line parallel to y-axis at a distance of 3 units to the right side of y-axis.
Table of values of x and y satisfying equation $y = x + 2$; $x > -1$ is given by

x	0	1	2	3
y	2	3	4	5

Table of values of x and y satisfying equation. $y = -x$; $x \le -1$ is given by

x	-1	-2
y	1	2

By joining these points we obtain a rough sketch of these curves is as shown in the figure.



$\therefore$ Required area

= Area of the shaded region.

= Area ABCEF

= Area ABCG + Area GCEF

$$= \int_{-2}^{-1}(-x)\,dx + \int_{-1}^{3}(x+2)\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{-2}^{-1} + \left[\frac{x^2}{2}+2x\right]_{-1}^{3}$$

$$= -\left[\frac{(-1)^2}{2} - \frac{(-2)^2}{2}\right] + \left[\left\{\frac{(3)^2}{2}+2(3)\right\} - \left\{\frac{(-1)^2}{2}+2(-1)\right\}\right]$$

$\therefore$ Required area = Area of the shaded region

$$= \text{Area ABCD}$$
$$= \text{Area DAC} + \text{Area CAB}$$
$$= [\text{Area DPAC} - \text{Area DPA}] + [\text{Area CAQB} - \text{Area AQB}]$$
$$= \int_{1/2}^{1} [x - (1 - x)]\, dx + \int_{1}^{\frac{3}{2}} [(2 - x) - (x - 1)]\, dx$$
$$= \int_{1/2}^{1} (2x - 1)\, dx + \int_{1}^{\frac{3}{2}} (3 - 2x)\, dx$$
$$= \left[\frac{2x^2}{2} - x\right]_{\frac{1}{2}}^{1} + \left[3x - \frac{2x^2}{2}\right]_{1}^{\frac{3}{2}}$$
$$= \left[\{(1)^2 - 1\} - \left\{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\right\}\right] + \left[\left\{3\left(\frac{3}{2}\right) - \left(\frac{3}{2}\right)^2\right\} - \{3(1) - (1)^2\}\right]$$
$$= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{9}{2} - \frac{9}{4}\right) - (3 - 1)\right]$$
$$= -\left(\frac{1-2}{4}\right) + \left(\frac{18-9}{4}\right) - 2 = \frac{1}{4} + \frac{9}{4} - 2 = \frac{10}{4} - 2 = \frac{10-8}{4}$$
$$= \frac{2}{4} = \frac{1}{2} \text{ sq. units.}$$

Example 31. *Find the area bounded by the curve $y^2 = 4a^2(x - 1)$ and the line $x = 1$ and $y = 4a$.*

Solution. The given equations of the curves are

$$y^2 = 4a^2(x - 1) \qquad \ldots(1)$$
$$\Rightarrow \quad (y - 0)^2 = 4a^2(x - 1)$$

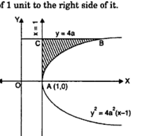
and $\quad x = 1, \quad y = 4a$

Equation (1) represents a parabola with vertex at (1, 0) opening right hand side of x-axis.

$x = 1$ is a straight line parallel to y-axis at a distance of 1 unit to the right side of it.

$y = 4a$ is a straight line parallel to the x-axis at a distance $4a$ units upwards to the x-axis.

The rough sketch of the curves is as shown in the figure.



$\therefore$ Required Area = Area of the shaded region

$$= \text{Area ABC}$$
$$= \int_{0}^{4a} (x - 1)\, dy$$
$$\left[\begin{array}{l} \because y^2 = 4a^2(x - 1) \\ \Rightarrow \dfrac{y^2}{4a^2} = (x - 1) \end{array}\right]$$

$$\int_0^{4a} \frac{y^2}{4a^2}.dy = \frac{1}{4a^2}\left[\frac{y^3}{3}\right]_0^{4a}$$

$$= \frac{1}{12a^2}[(4a)^3 - 0] = \frac{1}{12a^2}.64a^3$$

$$= \frac{16a}{3} \text{ sq. units.}$$

Example 32. *Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$ and the x-axis.*

Solution. The given equations of the curves are

$$x^2 = y \quad ...(1)$$

$$y = x + 2 \quad ...(2)$$

Equation (1) represents a parabola with vertex at the origin (0, 0) and symmetric along positive direction of y-axis.

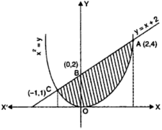
$y = x + 2$ represents a straight line.

Table of values of x and y satisfying equation (1) is given by

x	0	– 1	1	– 2	2
y	0	1	1	4	4

Table of values of x and y satisfying equation (2) is given by

x	0	– 1	1	2
y	2	1	3	4



By joining these points with a free hand, we obtain a rough sketch of the two curves as shown in the figure.

Now, to find the points of intersection of the two curves, let us solve equations (1) and (2) simultaneously

$\therefore$ From (1) and (2), we have

$$y = x^2$$

$$\Rightarrow \quad x + 2 = x^2$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad x^2 - 2x + x - 2 = 0$$

$$\Rightarrow \quad x(x-2) + 1(x-2) = 0$$

$$\Rightarrow \quad (x-2)(x+1) = 0$$

$$\Rightarrow \quad x = -1, \quad x = 2$$

For $x = -1$, $y = (-1)^2 = 1$

For $x = 2$, $y = (2)^2 = 4$

$\therefore$ The points of intersection of the two curves are $(-1, 1)$ and $(2, 4)$.

$\therefore$ Required area = Area of the shaded region

$$= \text{Area OABCO.}$$

$$= \int_{-1}^{2} (y_{\text{upper curve}} - y_{\text{lower curve}}).dx$$

$$= \int_{-1}^{2} [(x+2) - x^2]\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[\left\{\frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3}\right\} - \left\{\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right\}\right]$$

$$= \left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right] = 6 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= 8 - \frac{9}{3} - \frac{1}{2} = 8 - 3 - \frac{1}{2} = 5 - \frac{1}{2} = \frac{9}{2} \text{ sq. units.}$$

Example 33. *Sketch the curves and identify the region bounded by the curves* $x = \frac{1}{2}$, $x = 2$, $y = \log x$ *and* $y = 2^x$. *Find the area of this region.*

Solution. The given equations of the curves are

$$y = 2^x \qquad \ldots(1)$$

$$y = \log x \qquad \ldots(2)$$

$$x = \frac{1}{2} \qquad \ldots(3)$$

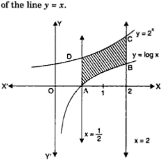
$$x = 2 \qquad \ldots(4)$$

Equation (1) is an exponential curve and equation (2) is a logarithmic curve.

Since the inverse of an exponential function is a logarithmic function and vice-versa. Therefore, these two curves are on the opposite sides of the line $y = x$.

Thus, the curves $y = 2^x$ and $y = \log x$ do not intersect.

The rough sketch of these curves is as shown in the figure.



$\therefore$ Required Area

$$= \text{Area of the shaded region.}$$

$$= \int_{1/2}^{2} (y_{\text{upper curve}} - y_{\text{lower curve}}).dx$$

$$= \int_{1/2}^{2} (2^x - \log x).dx$$

$$= \left[\frac{2^x}{\log 2} - x\log x + x\right]_{1/2}^{2}$$

$$= \left[\left\{ \frac{2^2}{\log 2} - 2\log 2 + 2 \right\} - \left\{ \frac{2^{1/2}}{\log 2} - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2} \right\} \right]$$

$$\left[\begin{aligned} &\because \int \underset{\text{II}}{1} . \underset{\text{I}}{\log x}\, dx \text{ (Integrating by parts)} \\ &= \log x . x - \int \frac{d}{dx}(\log x) . x\, dx \\ &= x\log x - \int \frac{1}{x} . x\, dx = x\log x - \int 1 . dx = (x\log x - x) \end{aligned} \right]$$

$$= \left[\frac{4}{\log 2} - 2\log 2 + 2 - \frac{\sqrt{2}}{\log 2} + \frac{1}{2}\log\frac{1}{2} - \frac{1}{2} \right] = \left[\frac{4-\sqrt{2}}{\log 2} + \frac{3}{2} - 2\log 2 - \frac{1}{2}\log 2 \right]$$

$$= \left[\frac{4-\sqrt{2}}{\log 2} - \frac{3}{2} - \frac{5}{2}\log 2 \right] \text{ sq. units.}$$

Example 34. *Find the area of the parabola $x^2 = 2 - y$ cut off by the line $x + y = 0$.*

Solution. The given equations of the curves are

$$x^2 = 2 - y$$

$$\Rightarrow \quad x^2 = -(y - 2) \quad \ldots(1)$$

$$x + y = 0$$

$$\Rightarrow \quad y = -x \quad \ldots(2)$$

Equation (1) represents a parabola opening downwards to y-axis with vertex at (0, 2) and meets the x-axis at $x = \pm\sqrt{2}$.

Equation (2), $x + y = 0$ represents a straight line.

Now, to find the points of intersection of two curves, let us solve equations (1) and (2).

$\therefore$ From (1) and (2), we have

$$x^2 = 2 - y \quad \text{and} \quad y = -x$$

$$\Rightarrow \quad x^2 = 2 - (-x)$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad x^2 - 2x + x - 2 = 0$$

$$\Rightarrow \quad x(x - 2) + 1(x - 2) = 0$$

$$\Rightarrow \quad (x + 1)(x - 2) = 0$$

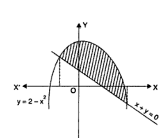
$$\Rightarrow \quad x = -1, x = 2$$

For $x = -1$, $y = -(-1) = 1$

For $x = 2$, $y = -(2) = -2$

$\therefore$ The points of intersection of the two curves are $(-1, 1)$ and $(2, -2)$.

The rough sketch of these curves are as shown in the figure.



$\therefore$ Required area = Area of the shaded region

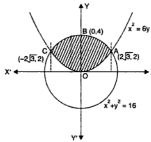
$$= \int_{-1}^{2} (y_{\text{upper curve}} - y_{\text{lower curve}})\, dx$$

Now, equation (1) *i.e.*, $x^2 + y^2 = 16$ is symmetrical about both x-axis and y-axis as the equation contains only even powers of x and y.

Equation (2) *i.e.* $x^2 = 6y$ is symmetrical about y-axis as it contains only even powers of x.

$\therefore$ Area bounded between the curves is symmetrical about y-axis.

The rough sketch of the two curves is as shown in the figure.



$\therefore$ Required area = Area of the shaded region.

$$= \int_{-2\sqrt{3}}^{2\sqrt{3}} (y_{\text{circle}} - y_{\text{parabola}})\, dx$$

$$= \int_{-2\sqrt{3}}^{2\sqrt{3}} \left(\sqrt{16-x^2} - \frac{x^2}{6}\right) dx$$

$$= 2\int_{0}^{2\sqrt{3}} \left(\sqrt{16-x^2} - \frac{x^2}{6}\right) dx \qquad [\because \text{ Integrand is an even function}]$$

$$= 2\left[\frac{x\sqrt{16-x^2}}{2} + \frac{16}{2}\sin^{-1}\frac{x}{4} - \frac{x^3}{18}\right]_0^{2\sqrt{3}}$$

$$\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 2\left[\left\{\frac{2\sqrt{3}\sqrt{16-12}}{2} + 8\sin^{-1}\left(\frac{2\sqrt{3}}{4}\right) - \frac{(2\sqrt{3})^3}{18}\right\} - 0\right]$$

$$= 2\left[2\sqrt{3} + 8\sin^{-1}\frac{\sqrt{3}}{2} - \frac{24\sqrt{3}}{18}\right]$$

$$= 4\sqrt{3} + 16\frac{\pi}{3} - \frac{8}{\sqrt{3}} = \frac{12-8}{\sqrt{3}} + \frac{16\pi}{3}$$

$$= \left(\frac{4}{\sqrt{3}} + \frac{16\pi}{3}\right) \text{ sq. units.}$$

Example 36. *Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $y^2 = 4x$.*

Solution. The given equations of the parabola are :

$$4x^2 + 4y^2 = 9$$

$$\Rightarrow \quad x^2 + y^2 = \frac{9}{4} \qquad \ldots(1)$$

$$y^2 = 4x \qquad \ldots(2)$$

Equation (1) represents a circle with centre at the origin (0, 0) and radius equal to $\frac{3}{2}$.

Equation (2), $y^2 = 4x$ represents a parabola whose vertex is at the origin and symmetrical to x-axis as it contains only even powers of y.

Now, to find the points of intersection of two curves, let us solve equations (1) and (2) simultaneously.

$\therefore$ From (1) and (2), we have

$$x^2 + y^2 = \frac{9}{4} \quad \text{and} \quad y^2 = 4x$$

$$\Rightarrow \quad x^2 + 4x = \frac{9}{4}$$

$$\Rightarrow \quad 4x^2 + 16x - 9 = 0$$

$$\Rightarrow \quad 4x^2 + 18x - 2x - 9 = 0$$

$$\Rightarrow \quad 2x(2x + 9) - 1(2x + 9) = 0$$

$$\Rightarrow \quad (2x + 9)(2x - 1) = 0$$

$$\Rightarrow \quad 2x + 9 = 0, \quad 2x - 1 = 0$$

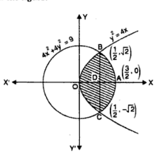
$$\Rightarrow \quad x = -\frac{9}{2}, \quad x = \frac{1}{2}$$

For $x = \frac{1}{2}$, $\quad y^2 = 4\left(\frac{1}{2}\right) \Rightarrow y^2 = 2 \Rightarrow y = \pm\sqrt{2}$

For $x = -\frac{9}{2}$, $\quad y^2 = 4\left(-\frac{9}{2}\right) \Rightarrow y^2 = -18 \Rightarrow y$ has imaginary value.

$\therefore$ The points of intersection of the two curves are $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$.

The rough sketch of these curves are as shown in the figure.



$\therefore$ Required area

$= \text{Area of the shaded region.}$

$= \text{Area OBAC}$

$= 2(\text{Area OBA})$

$= 2[\text{Area OBD} + \text{Area BDA}]$

$$= 2\int_0^{1/2} (y_{\text{parabola}})\, dx + 2\int_{1/2}^{3/2} (y_{\text{circle}})\, dx$$

$$= 2\int_0^{1/2} \sqrt{4x}\, dx + 2\int_{1/2}^{3/2} \left(\sqrt{\frac{9}{4} - x^2}\right) dx$$

$$= 4\int_0^{1/2} x^{1/2}\, dx + 2\int_{1/2}^{3/2} \sqrt{\left(\frac{3}{2}\right)^2 - x^2}\, dx$$

$$= 4\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{\frac{1}{2}} + 2\left[\frac{x\sqrt{\frac{9}{4} - x^2}}{2} + \frac{\frac{9}{4}}{2}\sin^{-1}\frac{x}{\frac{3}{2}}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 2\left[-\frac{\sqrt{3}}{8} - \frac{1}{2}\left(\frac{\pi}{6}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right)\right] + 2\left[\frac{\pi}{4} - \frac{\sqrt{3}}{8} - \frac{1}{2}\left(\frac{\pi}{6}\right)\right] = -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6}$$

$$= -\frac{\sqrt{3}}{2} + \pi - \frac{\pi}{3} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units.}$$

Example 38. *Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.*

Solution. The given equations of the curves are

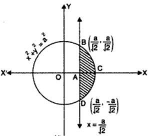
$$x^2 + y^2 = a^2 \quad \text{...(1)}$$

$$x = \frac{a}{\sqrt{2}} \quad \text{...(2)}$$

Equation (1) represents a circle with centre at the origin (0, 0) and radius a.

Equation (2), $x = \frac{a}{\sqrt{2}}$ is a straight line parallel to y-axis at a distance of $\frac{a}{\sqrt{2}}$ in the positive directon of x-axis.

The rough sketch of the curves are as shown in the figure.



Now, to find the points of intersection of the two curves, let us solve equations (1) and (2) simultaneously.

$\therefore$ From (1) and (2), we have

$$x = \frac{a}{\sqrt{2}}$$

$$\Rightarrow \quad x^2 + y^2 = a^2$$

$$\Rightarrow \quad \left(\frac{a}{\sqrt{2}}\right)^2 + y^2 = a^2$$

$$\Rightarrow \quad y^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}$$

$$\Rightarrow \quad y = \pm\frac{a}{\sqrt{2}}$$

$\therefore$ The points of intersection of the two curves are $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ and $\left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$.

$\therefore$ Required area = Area of the shaded region

= Area ABCD

$$= \int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\, .\, dx$$

$$= \left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a} \quad \left[\because \int \sqrt{a^2 - x^2}\, .\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left[\left\{ \frac{a\sqrt{a^2 - a^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{a}{a} \right\} - \left\{ \frac{\frac{a}{\sqrt{2}} \sqrt{a^2 - \frac{a^2}{2}}}{2} + \frac{a^2}{2} \sin^{-1} \frac{\frac{a}{\sqrt{2}}}{a} \right\} \right]$$

$$= \left(0 + \frac{a^2}{2} \sin^{-1} 1 \right) - \left(\frac{a}{2\sqrt{2}} \cdot \frac{a}{\sqrt{2}} + \frac{a^2}{2} \sin^{-1} \frac{1}{\sqrt{2}} \right)$$

$$= \frac{a^2}{2} \cdot \frac{\pi}{2} - \frac{a^2}{4} - \frac{a^2}{2} \left(\frac{\pi}{4} \right) = \frac{a^2\pi}{4} - \frac{a^2}{4} - \frac{a^2\pi}{8} = \frac{a^2\pi}{8} - \frac{a^2}{4}$$

$$= \frac{a^2}{4} \left(\frac{\pi}{2} - 1 \right) \text{ sq. units.}$$

Example 39. *Find the area of the region between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.*

Solution. The given equations of the curves are

$$x^2 + y^2 = 4 \quad \ldots(1)$$

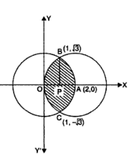
$$(x - 2)^2 + y^2 = 4 \quad \ldots(2)$$

Equation (1) represents a circle with centre at the origin (0, 0) and radius 2.

Equation (2) also represents a circle with centre (2, 0) and radius 2.

Since both the curves are symmetrical about x-axis, as both the equations have only even powers of y.

The rough sketch of the curves are as shown in the figure.



Now, to find the points of intersection of the two curves, let us solve equations (1) and (2) simultaneously.

$\therefore$ From (1) and (2), we have

$$x^2 + y^2 = 4$$

$$\Rightarrow \quad y^2 = 4 - x^2$$

$$(x - 2)^2 + y^2 = 4$$

$$\Rightarrow \quad x^2 + 4 - 4x + 4 - x^2 = 4$$

$$\Rightarrow \quad 4x = 4 \Rightarrow x = 1$$

For $x = 1$, $y^2 = 4 - (1)^2 = 4 - 1 = 3$

$$\Rightarrow \quad y = \pm\sqrt{3}$$

$\therefore$ The points of intersection of the two curves are $(1, \sqrt{3})$ and $(1, -\sqrt{3})$.

$\therefore$ Required area = Area of the shaded region.

= Area OBACO = 2 [Area OBA]

= 2 [Area OBP + Area BAP]

$$= 2\int_0^1 y_{\text{circle (2)}} \cdot dx + 2\int_1^2 y_{\text{circle (1)}} \cdot dx$$

$$= 2\int_0^1 \sqrt{4 - (x - 2)^2}\, dx + 2\int_1^2 \sqrt{4 - x^2}\, dx$$

$$= 2\left[\frac{(x-2)\sqrt{4-(x-2)^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 + 2\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2$$

$$\left[\because \int \sqrt{a^2-x^2}\,.dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 2\left[\left\{\frac{(1-2)\sqrt{4-(1-2)^2}}{2} + 2\sin^{-1}\left(\frac{1-2}{2}\right)\right\} - \left\{\frac{(0-2)\sqrt{4-(0-2)^2}}{2} + 2\sin^{-1}\left(\frac{-2}{2}\right)\right\}\right]$$

$$+ 2\left[\left\{\frac{2\sqrt{4-4}}{2} + 2\sin^{-1}\frac{2}{2}\right\} - \left\{\frac{\sqrt{4-1}}{2} + 2\sin^{-1}\frac{1}{2}\right\}\right]$$

$$= 2\left[\left\{\frac{-\sqrt{3}}{2} + 2\sin^{-1}\left(-\frac{1}{2}\right)\right\} - \left\{0 + 2\sin^{-1}(-1)\right\}\right] + 2\left[(0 + 2\sin^{-1}1) - \left(\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{1}{2}\right)\right]$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$= -\sqrt{3} - 4\sin^{-1}\left(\frac{1}{2}\right) + 4\sin^{-1}1 + 4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\left(\frac{1}{2}\right)$$

$$= -2\sqrt{3} - 8\sin^{-1}\left(\frac{1}{2}\right) + 8\sin^{-1}1$$

$$= -2\sqrt{3} - 8\left(\frac{\pi}{6}\right) + 8\left(\frac{\pi}{2}\right) = -2\sqrt{3} - \frac{4\pi}{3} + 4\pi$$

$$= \left(\frac{8\pi}{3} - 2\sqrt{3}\right) \text{ sq. units.}$$

Example 40. *Calculate the area of the region enclosed between the circles $x^2 + y^2 = 1$ and $\left(x - \frac{1}{2}\right)^2 + y^2 = 1$.*

Solution. The given equations of the curves are

$$x^2 + y^2 = 1 \qquad \text{...(1)}$$

$$\left(x - \frac{1}{2}\right)^2 + y^2 = 1 \qquad \text{...(2)}$$

Equation (1) represents a circle with centre at the origin (0, 0) and radius unity.

Equation (2) also represents a circle with centre at $\left(\frac{1}{2}, 0\right)$ and radius unity.

Since, both the curves are symmetrical about x-axis, as both the equation contain only even powers of y.

The rough sketch of the curves is as shown in the figure.

$$= 2\left[\left\{\frac{\left(\frac{1}{4}-\frac{1}{2}\right)\sqrt{1-\left(\frac{1}{4}-\frac{1}{2}\right)^2}}{2}+\frac{1}{2}\sin^{-1}\left(\frac{1}{4}-\frac{1}{2}\right)\right\} - \left\{\frac{\left(-\frac{1}{2}-\frac{1}{2}\right)\sqrt{1-\left(-\frac{1}{2}-\frac{1}{2}\right)^2}}{2}+\frac{1}{2}\sin^{-1}\left(-\frac{1}{2}-\frac{1}{2}\right)\right\}\right]$$

$$+ 2\left[\left\{\frac{\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}1\right\} - \left\{\frac{\frac{1}{4}\sqrt{1-\left(\frac{1}{4}\right)^2}}{2}+\frac{1}{2}\sin^{-1}\left(\frac{1}{4}\right)\right\}\right]$$

$$= \left[\left(-\frac{1}{4}\right)\sqrt{1-\frac{1}{4}}+\sin^{-1}\left(-\frac{1}{4}\right)\right] - \left[(-1)\sqrt{0}+\sin^{-1}(-1)\right]$$

$$+ [1\sqrt{0}+\sin^{-1}(1)] - \left[\frac{1}{4}\cdot\frac{\sqrt{15}}{4}+\sin^{-1}\frac{1}{4}\right] \quad [\because \ \sin(-\theta) = -\sin\theta]$$

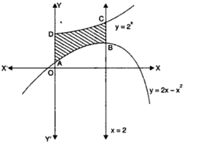
$$= -\frac{\sqrt{3}}{8}-\sin^{-1}\left(\frac{1}{4}\right)+\frac{\pi}{2}+\frac{\pi}{2}-\frac{\sqrt{15}}{16}-\sin^{-1}\frac{1}{4} = -\frac{\sqrt{3}}{8}-\frac{\sqrt{15}}{16}-2\sin^{-1}\frac{1}{4}+\pi$$

$$= \left(\frac{-(2\sqrt{3}+\sqrt{15})}{16}-2\sin^{-1}\frac{1}{4}+\pi\right) \text{ sq. units.}$$

Example 41. *Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$, $y = 2x - x^2$.*

Solution. The given equations of the curves are

$$y = 2x - x^2 \qquad \ldots(1)$$
$$y = 2^x \qquad \ldots(2)$$
$$x = 0 \qquad \ldots(3)$$
$$x = 2 \qquad \ldots(4)$$



Equation (1) represents a parabola opening downwards having vertex at (1, 1) and cuts the x-axis at points (0, 0) and (2, 0).

Equation (2), $y = 2^x$ shows an exponential curve.

$x = 0$ is the y-axis and $x = 2$ is a straight line parallel to the y-axis at a distance 2 units from it.

The rough sketch of the curves is as shown in the figure.

$\therefore$ Required area

$$= \text{Area of the shaded region}$$

$$= \int_0^2 (y_{\text{upper curve}} - y_{\text{lower curve}}) \, . \, dx$$

$$= \int_0^2 [2^x - (2x - x^2)] . dx = \int_0^2 (2^x - 2x + x^2) . dx$$

$$= \left[\frac{2^x}{\log 2} - \frac{2x^2}{2} + \frac{x^3}{3} \right]_0^2$$

$$= \left[\left\{ \frac{2^2}{\log 2} - (2)^2 + \frac{(2)^3}{3} \right\} - \left\{ \frac{2^0}{\log 2} - 0 + 0 \right\} \right] = \frac{4}{\log 2} - 4 + \frac{8}{3} - \frac{1}{\log 2}$$

$$= \left(\frac{3}{\log 2} - \frac{4}{3} \right) \text{ sq. units.}$$

Example 42. *Find the area of the smaller region bounded by the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and the straight line* $\frac{x}{a} + \frac{y}{b} = 1$.

Solution. The given equations of the curves are

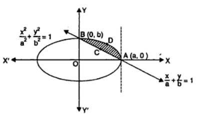
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...(1)}$$

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{...(2)}$$

Equation (1) represents an ellipse with centre at the origin, length of major and minor axis are $2a$ and $2b$ respectively.

The ellipse intersects x-axis and y-axis at $(\pm a, 0)$ and $(0, \pm b)$ respectively.

Equation (2) represents a straight line in the intercept form having intercepts a and b on axis.



$\therefore$ Equation (2) intersects x-axis at $(a, 0)$ and y-axis at $(0, b)$.

$\therefore$ Points of intersection of the two curves are $(a, 0)$ and $(0, b)$.

The rough sketch of the curves are as shown in the figure.

$\therefore$ Required area = Area of the shaded region

= Area ACBDA

$$= \int_0^a (y_{\text{upper curve}} - y_{\text{lower curve}}) \, . \, dx$$

$\therefore$ Equation (2) intersects x-axis at (3, 0) and y-axis at (0, 2).

$\therefore$ Points of intersection of the two curves are (3, 0) and (0, 2).

The rough sketch of the two curves is as shown in the figure.

$\therefore$ Required area = Area of the shaded region = Area BCAD

$$= \int_0^3 (y_{\text{upper curve}} - y_{\text{lower curve}}).dx$$

$$= \int_0^3 \left[\frac{2}{3}\sqrt{9-x^2} - \frac{6-2x}{3}\right] dx \qquad \left[\begin{array}{l} \because \text{ From (1), } \dfrac{x^2}{9}+\dfrac{y^2}{4}=1 \\ \Rightarrow \dfrac{y^2}{4} = 1-\dfrac{x^2}{9} \\ \Rightarrow y^2 = \dfrac{4}{9}(9-x^2) \\ \Rightarrow y = \pm\dfrac{2}{3}\sqrt{9-x^2} \end{array}\right]$$

$$= \frac{2}{3}\int_0^3 \left[\sqrt{(3)^2-x^2} - (3-x)\right] dx$$

$$= \frac{2}{3}\left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}\frac{x}{3} - 3x + \frac{x^2}{2}\right]_0^3$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{2}{3}\left[\left\{\frac{3\sqrt{9-9}}{2} + \frac{9}{2}\sin^{-1}1 - 9 + \frac{9}{2}\right\} - \left\{0 + \frac{9}{2}\sin^{-1}0 - 0\right\}\right]$$

$$= \frac{2}{3}\left[0 + \frac{9}{2}\left(\frac{\pi}{2}\right) - \frac{9}{2} - 0\right] = \left(\frac{3\pi}{2} - 3\right) \text{ sq. units.}$$

Example 44. Find the area of the region bounded by the curves $y = 4x - x^2$ and $y = x^2 - x$ above x-axis.

Solution. The given equations of the curves are

$$y = 4x - x^2 \qquad ...(1)$$

$$y = x^2 - x \qquad (2)$$

From equation (1), we have

$$x^2 = 4x - y$$

$\Rightarrow$ $x^2 - 4x = -y$

$\Rightarrow$ $x^2 - 4x + 4 = -y + 4$ [Adding 4 on both sides]

$\Rightarrow$ $(x-2)^2 = -(y-4)$ ...(3)

$\therefore$ Equation (3) represents a parabola opening downward and having vertex at (2, 4).

Now, from equation (2), we have

$$x^2 - x = y$$

Example 46. *Make a sketch of the region given below and find its area using integration.*

$\{(x, y) : 0 \le y \le x^2 + 3,\ 0 \le y \le 2x + 3,\ 0 \le x \le 3\}.$

Solution. Let $R = \{(x, y) : 0 \le y \le x^2 + 3,\ 0 \le y \le 2x + 3,\ 0 \le x \le 3\}$

$\Rightarrow \quad R = \{(x, y) : 0 \le y \le x^2 + 3\} \cap \{(x, y) : 0 \le y \le 2x + 3\} \cap \{(x, y) : 0 \le x \le 3\}$

$\Rightarrow \quad R = R_1 \cap R_2 \cap R_3$ (say).

Now, to find the points of intersection of these curves, let us solve equations $y = x^2 + 3$ and $y = 2x + 3$.

$\therefore \quad x^2 + 3 = 2x + 3$

$\Rightarrow \quad x^2 - 2x = 0$

$\Rightarrow \quad x(x - 2) = 0$

$\Rightarrow \quad x = 0,\ x = 2$

For $x = 0$, $y = 2(0) + 3 = 3$

For $x = 2$, $y = 2(2) + 3 = 7$

$\therefore$ The points of intersection of these curves are (0, 3) and (2, 7).

Region R_1 :

We have $\quad y = x^2 + 3$

$\Rightarrow \quad x^2 = y - 3$

$\Rightarrow \quad (x - 0)^2 = y - 3 \quad \ldots(1)$

Equation (1) represents a parabola opening upwards and having vertex at (0, 3), axis along the positive direction of y-axis.

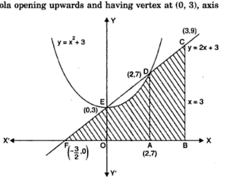
$\therefore$ Region R_1 is the region lying on or above the x-axis and outside the parabola.

Region R_2 :

We have $\quad y = 2x + 3 \quad \ldots(2)$

Equation (2) represents a straight line intersecting x-axis at $\left(-\frac{3}{2}, 0\right)$ and y-axis at (0, 3).

$\therefore$ Region R_2 is lying on or above x-axis and below the line.



Region R_3 : Region R_3 is lying between the lines $x = 0$ *i.e.*, y-axis and $x = 3$. The rough sketch of these curves is as shown in the figure.

$\therefore$ Required area = Area of the shaded region

= Area OADE + Area ABCD

$$= \int_0^2 (y_{\text{parabola}}).dx + \int_2^3 (y_{\text{line}}).dx$$

$$= \int_0^2 (x^2 + 3).dx + \int_2^3 (2x + 3)\,dx = \left[\frac{x^3}{3} + 3x\right]_0^2 + \left[\frac{2x^2}{2} + 3x\right]_2^3$$

$$= \left[\left\{\frac{(2)^3}{3} + 3(2)\right\} - \{0+0\}\right] + [\{(3)^2 + 3(3)\} - \{(2)^2 + 3(2)\}]$$

$$= \left(\frac{8}{3} + 6\right) + (9+9) - (4+6) = \frac{8}{3} + 6 + 18 - 10 = \frac{8}{3} + 14 = \frac{8+42}{3}$$

$$= \frac{50}{3} \text{ sq. units.}$$

Example 47. *Find the area of the region :*

$$\{(x, y) : x^2 + y^2 \le 1 \le x + y\}.$$

Solution. Let $R = \{(x, y) : x^2 + y^2 \le 1 \le x + y\}$

$\Rightarrow$ $R = \{(x, y) : x^2 + y^2 \le 1\} \cap \{(x, y) : 1 \le x + y\}$

$\Rightarrow$ $R = R_1 \cap R_2$

where $R_1 = \{(x, y) : x^2 + y^2 \le 1\}$

$R_2 = \{(x, y) : 1 \le x + y\}$.

Now, to find the points of intersection of the two curves let us solve the equations $x^2 + y^2 = 1$ and $x + y = 1$.

$\therefore$ $y = 1 - x$

$\Rightarrow$ $x^2 + (1-x)^2 = 1$

$\Rightarrow$ $x^2 + 1 + x^2 - 2x = 1$

$\Rightarrow$ $2x^2 - 2x = 0$

$\Rightarrow$ $2x(x-1) = 0$

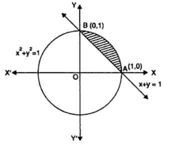
$\Rightarrow$ $x = 0, x = 1$

For $x = 0$, $y = 1 - 0 = 1$

For $x = 1$, $y = 1 - 1 = 0$

$\therefore$ The points of intersection of the two curves are (0, 1) and (1, 0).

The rough sketch of the two curves is as shown in the figure.



Region R_1 :

We have $x^2 + y^2 = 1$ ...(1)

Equation (1), represents a circle with centre at the origin (0, 0) and radius 1.

Region R_1 represents the interior of the circle $x^2 + y^2 = 1$.

Region R_2 :

We have $x + y = 1$ ...(2)

Equation (2), represents a straight line passes through the points (0, 1) and (1, 0). R_2 is the region lying above the line $x + y = 1$.

$\therefore$ Required area = Area of the shaded region.

$$= \int_0^1 (y_{\text{circle}} - y_{\text{line}}).dx$$

$$= \int_0^1 [(\sqrt{1-x^2}) - (1-x)]\, dx = \int_0^1 \left(\sqrt{1-x^2} - 1 + x\right) dx$$

Equation (3) represents a parabola with vertex at (0, 0) *i.e.*, at the origin and its axis along x-axis.

$\therefore$ Region R_2 is lying outside the parabola.

Region R_3 :

It is given that $x \geq 0$ and $y \geq 0$.

$\therefore$ Region R_3 is the region in the first quadrant.

$\therefore$ Required area = Area of shaded region

$$= \int_0^a (y_{\text{upper curve}} - y_{\text{lower curve}}).dx$$

$$= \int_0^a (y_{\text{circle}} - y_{\text{parabola}})\, dx$$

$$= \int_0^a \left(\sqrt{2ax - x^2} - \sqrt{ax}\right) dx = \int_0^a \sqrt{2ax - x^2}\,.dx - \sqrt{a}\int_0^a \sqrt{x}\,.dx$$

$$= \int_0^a \sqrt{a^2 - (x-a)^2}\,.dx - \sqrt{a}\int_0^a x^{1/2}\, dx$$

$$= \left[\frac{(x-a)\sqrt{a^2-(x-a)^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a - \sqrt{a}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^a$$

$$\left[\because \int \sqrt{a^2 - x^2}\,.dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left[\left\{\frac{(a-a)\sqrt{a^2-(a-a)^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a-a}{a}\right)\right\} - \left\{\frac{(0-a)\sqrt{a^2-(0-a)^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{0-a}{a}\right)\right\}\right] - \frac{2\sqrt{a}}{3}[a^{3/2} - 0]$$

$$= \left[\left\{0 + \frac{a^2}{2}\sin^{-1}(0)\right\} - \left\{0 + \frac{a^2}{2}\sin^{-1}(-1)\right\}\right] - \frac{2a^2}{3}$$

$$= \frac{a^2}{2}(0) + \frac{a^2}{2}\sin^{-1}(1) - \frac{2a^2}{3} = \frac{a^2}{2}\left(\frac{\pi}{2}\right) - \frac{2a^2}{3} = \frac{a^2\pi}{4} - \frac{2a^2}{3}$$

$$= a^2\left(\frac{\pi}{4} - \frac{2}{3}\right) \text{ sq. units.}$$

Example 49. *Find the area enclosed by the region :*

$\{(x, y) : y^2 \leq 5x,\ 5x^2 + 5y^2 \leq 36\}$.

Solution. Let $R = \{(x, y) : y^2 \leq 5x,\ 5x^2 + 5y^2 \leq 36\}$

$\Rightarrow$ $R = \{(x, y) : y^2 \leq 5x\} \cap \{(x, y) : 5x^2 + 5y^2 \leq 36\}$

$\Rightarrow$ $R = R_1 \cap R_2$

where $R_1 = \{(x, y) : y^2 \leq 5x\}$

$R_2 = \{(x, y) : 5x^2 + 5y^2 \leq 36\}$

Region R_1 :

We have $\quad y^2 = 5x$ ...(1)

Equation (1) represents a parabola with vertex at origin (0, 0) and x-axis as its axis.

The curve is symmetrical about x-axis. Region R_1 is lying in xy-plane and inside this parabola.

Region R_2 :

We have $\quad 5x^2 + 5y^2 = 36$ ...(2)

Equation (2) represents a circle with centre at origin (0, 0) and radius $\frac{6\sqrt{5}}{5}$.

Region R_2 is the region in xy-plane and is inside this circle.

Now, to find the points of intersection of two curves, let us solve the equations (1) and (2) simultaneously.

$$y^2 = 5x$$

$$\Rightarrow \quad 5x^2 + 5(5x) = 36$$

$$\Rightarrow \quad 5x^2 + 25x - 36 = 0$$

$$\Rightarrow \quad x = \frac{-25 \pm \sqrt{625 + 720}}{10}$$

$$\Rightarrow \quad x = \frac{-25 \pm \sqrt{1345}}{10} \qquad \left[\begin{array}{l} \because \text{ For a quadratic equation :} \\ ax^2 + bx + c = 0. \\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{array}\right]$$

$$\Rightarrow \quad x = \frac{-25 \pm 36.6}{10}$$

$$\Rightarrow \quad x = -\frac{61.6}{10}, \quad x = \frac{11.6}{10}$$

$$\Rightarrow \quad x = -6.16, \quad x = 1.16$$

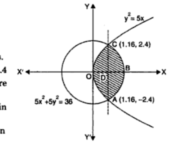
For $x = -6.16$; $y^2 = 5(-6.16)$

$\Rightarrow$ $y^2 = -30.18$ $\Rightarrow$ y has imaginary values.

For $x = 1.16$; $y^2 = 5(1.16) = 5.8$ $\Rightarrow$ $y = \pm 2.4$

$\therefore$ The points of intersection of two curves are (1.16, 2.4) and (1.16, – 2.4).

The rough sketch of these curves is as shown in the figure :



$\therefore$ Required area = Area of the shaded region

= Area OABC

= 2(Area OBC)

= 2[Area OCD + Area DCB]

$$= 2\int_0^{1.16} y_{\text{parabola}} \, .dx + 2\int_{1.16}^{\frac{6\sqrt{5}}{5}} y_{\text{circle}} \, .dx$$

$$= 2\sqrt{3}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{1.2} + 2\left[\frac{x\sqrt{\frac{16}{3} - x^2}}{2} + \frac{\frac{16}{3}}{2}\sin^{-1}\left(\frac{x}{\frac{4}{\sqrt{3}}}\right)\right]_{1.2}^{\frac{4\sqrt{3}}{3}}$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{4\sqrt{3}}{3}[(1.2)^{3/2} - 0] + \left[\left\{\frac{4\sqrt{3}}{3}\sqrt{\frac{16}{3} - \left(\frac{4\sqrt{3}}{3}\right)^2} + \frac{16}{3}\sin^{-1}\frac{\sqrt{3}}{4}\frac{4\sqrt{3}}{3}\right\} - \left\{(1.2)\sqrt{\frac{16}{3} - (1.2)^2} + \frac{16}{3}\sin^{-1}\frac{\sqrt{3}\,(1.2)}{4}\right\}\right]$$

$$= \frac{4\sqrt{3}}{3}(1.31) + \left[\left\{\frac{4\sqrt{3}}{3}\sqrt{\frac{16}{3} - \frac{16}{3}} + \frac{16}{3}\sin^{-1} 1\right\} - \left\{1.2\sqrt{\frac{16}{3} - 1.44} + \frac{16}{3}\sin^{-1}\frac{3\sqrt{3}}{10}\right\}\right]$$

$$= 3.02 + 0 + \frac{16}{3}\left(\frac{\pi}{2}\right) - 2.36 + \frac{16}{3}\sin^{-1}\frac{3\sqrt{3}}{10}$$

$$= \left[0.66 + \frac{8\pi}{3} + \frac{16}{3}\sin^{-1}\frac{3\sqrt{3}}{10}\right] \text{ sq. units.}$$

EXERCISE FOR PRACTICE

1. Find the area bounded by the curve $y = x^3$, x-axis and the ordinates $x = 2$, $x = 4$.
2. Using integration, find the area of the region bounded by the curves $y = 1 + |x + 1|$, $x = -3$, $x = 3$, $y = 0$.
3. Draw a rough sketch of the curve $y = x^2 - 9$ and find the area bounded by the curve, the lines $x = 0$, $x = 2$ and the x-axis.
4. Sketch the region bounded by $y = 2x - x^2$ and x-axis, and find its area using integration.
5. Find the area under the curve $y = (x^2 + 2)^2 + 2x$, between the ordinates $x = 0$, $x = 2$ and the x-axis.
6. Using integration, find the area of the region bounded by the line $3y = 2x + 4$, x-axis and the lines $x = 1$ and $x = 3$.
7. Find the area lying above the x-axis and under the parabola $y = 4x - x^2$.
8. Draw a rough sketch of the graph of the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and evaluate the area of the region under the curve and above the x-axis.
9. Find the area bounded by the curve $y = \cos x$, x-axis and the ordinates $x = 0$ and $x = 2\pi$.
10. Sketch the graph of $y = |x - 5|$. Evaluate $\int_0^1 |x - 5|\, dx$, what does this value of the integral represent on the graph.

11. Draw the rough sketch of $y^2 + 1 = x, x \le 2$. Find the area enclosed by the curve and the line $x = 2$.

12. Using integration, find the area of the region bounded between the line $x = 2$ and the parabola $y^2 = 8x$.

13. Find the area of the region common to the parabolas $4y^2 = 9x$ and $3x^2 = 16y$.

14. Find the area of the region enclosed between the circles $x^2 + y^2 = 1$, $(x - 1)^2 + y^2 = 1$.

15. Using integration, find the area of the region bounded by the triangle whose vertices are (2, 1), (3, 4) and (5, 2).

16. Find the area enclosed between two curves $y^2 = 9x$ and $x^2 = 9y$.

17. Find the area common to the circle $x^2 + y^2 = 16a^2$ and the parabola $y^2 = 6ax$.

18. Find the area bounded by the lines
$x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

19. Compare the areas under the curves
$y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.

20. Find the area of the region :
$\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$.

21. Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$.

22. Find the area bounded by the curve $y^2 = 4a^2(x - 1)$ and the line $x = 1$ and $y = 4a$.

23. Find the area enclosed by the parabolas
$y = 5x^2$ and $y = 2x^2 + 9$.

24. Find the area of the region $\{(x, y) : y^2 \le 5x, 5x^2 + 5y^2 \le 36\}$.

Answers

1. 60 sq. units **2.** 16 sq. units **3.** $\frac{46}{3}$ sq. units

4. $\frac{4}{3}$ sq. units **5.** $\frac{436}{15}$ sq. units **6.** $\frac{16}{3}$ sq. units

7. $\frac{32}{3}$ sq. units **8.** 3π sq. units **9.** 4 sq. units

10. $\frac{9}{2}$ sq. units **11.** $\frac{4}{3}$ sq. units **12.** $\frac{32}{3}$ sq. units

13. 4 sq. units **14.** $\frac{4}{3}$ sq. units **15.** 4 sq. units

16. 27 sq. units **17.** $\frac{4a^2}{3}(4\pi + \sqrt{3})$ sq. units **18.** 6 sq. units

19. $\frac{\pi}{2}$ sq. units [Each] **20.** $\left(\frac{\pi - 2}{4}\right)$ sq. units **21.** $\frac{50}{3}$ sq. units

22. $\frac{16a}{3}$ sq. units **23.** $12\sqrt{3}$ sq. units

24. $\left[\frac{4\sqrt{5}}{3} z^{3/2} + \frac{18\pi}{5} - z\sqrt{\frac{36}{5} - z^2} - \frac{36}{5}\sin^{-1}\frac{z\sqrt{15}}{6}\right]$ sq. units
where $z = 1.16$.

e.g.,
$$x.\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$
$$x.\frac{\partial x}{\partial y} + y.\frac{\partial y}{\partial z} = pz \text{ etc.}$$

In our present course, we shall be considering only ordinary differential equations.

10.3 ORDER AND DEGREE OF DIFFERENTIAL EQUATIONS

10.3.1 Order. The order of a differential equation is the order of the highest order derivative occuring in the equation.

The order of a differential equation is a positive integer.

10.3.2 Degree. The degree of a differential equation is the degree of the highest order differential co-efficient appearing in it, provided the differential co-efficients are made free from radicals and fractions.

e.g., (*i*) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$. Its order is 2 and degree is 1.

(*ii*) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = 2\left(\frac{d^2y}{dx^2}\right)^2$. Its order is 2 and degree is 2.

(*iii*) $\frac{dy}{dx} = \sin x + \cos x$. Its order is 1 and degree is 1.

(*iv*) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = p\left(\frac{d^2y}{dx^2}\right)$

The order of highest order differential co-efficient is 2. So, its order is 2.

Now, to find its degree, first we have to express it as a polynomial in derivatives.

i.e., on squaring both sides, we have

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 - p^2\left(\frac{d^2y}{dx^2}\right)^2 = 0$$

Here, the power of the highest order differential co-efficient is 2.

Therefore, its degree is 2.

10.4 LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear, if the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

i.e., if it is expressible in the form

$$A_0\frac{d^ny}{dx^n} + A_1\frac{d^{n-1}y}{dx^{n-1}} + A_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots A_{n-1}\frac{dy}{dx} + A_ny = B$$

where $A_0, A_1, A_2, \ldots\ldots, A_{n-1}, A_n$ and B are either constants or functions of independent variable x.

In particular, a linear differential equation of order one is of the form

$$A_0 \frac{dy}{dx} + A_1 y = B$$

Note. The degree of a linear differential equation is always one. But, the converse is not true.

e.g., $$x\frac{dy}{dx} + y\cos x = 1$$

and $$\frac{d^2y}{dx^2} + \frac{5y}{x} = x^2 \sin x$$ are linear differential equations.

The differential equation $y\frac{dy}{dx} - p = x$ is not a linear differential equation, because the dependent variable y and its derivative $\frac{dy}{dx}$ are multiplied together.

A differential equation which is not linear is called a Non-linear Differential Equation.

i.e., A differential equation will be non-linear differential equation if :

(*i*) its degree is more than one.

(*ii*) any of the differential co-efficient has exponent more than one.

(*iii*) exponent of the dependent variable is more than one.

(*iv*) products containing dependent variable and its differential co-efficients are present.

e.g., $$\left(\frac{d^2y}{dx^2}\right)^2 + 3\left(\frac{dy}{dx}\right)^2 + 5 = 0.$$

Order of highest order derivative *i.e.,* of $\left(\frac{d^2y}{dx^2}\right)^2$ is 2. Degree of $\left(\frac{d^2y}{dx^2}\right)$ is 2.

$\therefore$ Order of the equation is 2 and degree of the equation is 2.

This is an example of a non-linear differential equation, because $\left(\frac{d^2y}{dx^2}\right)$ is multiplied by itself.

10.5 FORMATION OF A DIFFERENTIAL EQUATION

Let there be an equation involving independent variable y, the dependent variable x and 'n' arbitrary constants, to form the differential equation of such family of curves we are to eliminate the 'n' arbitrary constants from the given equation.

This can be achieved by differentiating given equation n times and, we get a differential equation of n^{th} order corresponding to the given equation.

If n be an arbitrary constant and

$$y = n \sin 6x \quad \text{...(1)}$$

be an equation.

For different values of n, we get different equations.

If we eliminate the constant n by differentiating the equation (1), then the equation, which we get is called the differential equation of the given equation.

Differentiating (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\text{A}\cos x - \text{B}\sin x$$

$$= -(\text{A}\cos x + \text{B}\sin x) = -y \qquad \text{[Using (1)]}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} + y = 0 \qquad ...(3)$$

This shows that $y = \text{A}\cos x + \text{B}\sin x$ satisfies the given differential equation and hence is a solution of given equation.

10.6.1 Types of Solutions. There are three types of solutions :

(*a*) **General Solution.** A solution of a differential equation is called the general solution (or complete solution), if it contains as many arbitrary constants as the order of differential equation.

e.g., $y = \text{A}\cos x + \text{B}\sin x$, is a general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

because the numebr of arbitrary constants A and B *i.e.,* 2 is same as the order of differential equation.

(*b*) **Particular Solution.** A solution obtained by giving particular values to arbitrary constants in the general solution of a differential equation is called a particular solution of the differential equation, under consideration.

e.g., $y = 2\cos x + 3\sin x$ is a particular solution of the given differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

e.g., $y = 7x^4$ is a particular solution of the differential equation $x\frac{dy}{dx} - 4y = 0$.

(*c*) **Singular Solution.** The solution containing no arbitrary constant and which cannot be obtained by giving particular values to arbitrary constants in the general solution is called a singular solution.

e.g., (*i*) $y^2 = 4ax$ is the singular solution of the differential equation $y = x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}}$.

10.6.2 Note. 1. The finding of a solution of a given differential equation is generally referred to as solving the differential equation or integrating the differential equation. The process involves two steps :

(*a*) First to find the solution of the given differential equation *i.e.,* To find a relation, connecting the independent variable (say) x, and the dependent variable (say) y such that the relation (called the solution or primitive) does not contain any derivative.

(*b*) Second, to derive from the solution, if possible, y in terms of x or x in terms of y.

2. Suppose a solution of a differential equation contains n-arbitrary constants. To eliminate n arbitrary constants, we need $(n + 1)$ equations. The given relation along with n more relations obtained by successively differentiating it n-times provide us with $(n + 1)$ equations. The differential equation thus obtained is clearly of the n^{th} order.

   Hence, a solution of n^{th} order differential equation contain n arbitrary constants.

10.7 INITIAL VALUE PROBLEM

We have seen that a first order differential equation represents a one-parameter family of curves, a second-order differential equation represents a two-parameters family of curves, and so on. In order to specify a particular member of such a family, we need to assign particular value (s) to the parameter (s) involved.

So we require, besides the differential equation, some other conditions (known as subsidiary conditions) for the specification of the parameter. Usually, these subsidiary conditions are prescribed by assigning values to the unknown function and the requisite number (depending on the number of parameters to be determined) of the derivatives of the unknown function at some point of the domain of its definition. *e.g.*,

(*i*) Let the function $f(x) = 3x + 2$. Satisfies $f'(x) = 3$ and $f(1) = 5$.

Here, the function $f(x) = 3x + A$ is the solution of the differential equation

$$f'(x) = 3$$

where A is a parameter.

(*ii*) Let the function $f(x) = 3x^2 + 3$ satisfies $f'(x) = 6x$ and $f(1) = 6$.

Here, the function $f(x) = 3x + B$ is the solution of the differential equation

$$f'(x) = 6x$$

where B is the parameter.

These subsidiary conditions are generally prescribed of only one point of the domain of definition of the unknown function, these conditions are referred to as the initial conditions.

The differential equation with these initial values or initial conditions is generally known as an initial value problem.

10.8 DIFFERTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

10.8.1 Definition. A differential equation of first order and first degree is an equation of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \mathrm{M}dx + \mathrm{N}dy = 0$$

where M, N are functions of x and y.

10.9 METHODS OF SOLVING A FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATION

Differential equations with variable separable :

10.9.1 Definition. If in a differential equation, it is possible to get all the functions of x and dx to one side, and all the functions of y and dy to the other, the variables are said to be separable.

10.9.2 Solution of a Differential Equation by the Method of Variables Separable. Let us consider the differential equation :

4. **Remember :**

(*i*) $\log x + \log y = \log xy$ (*ii*) $\log x - \log y = \log \frac{x}{y}$

(*iii*) $n \log x = \log x^n$ (*iv*) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$

(*v*) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$.

10.10 EQUATIONS REDUCIBLE TO THE FORM IN WHICH VARIABLES CAN BE SEPARATED

Equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to the form in which the variables are separable.

Put $ax + by + c = z$

$$\Rightarrow \quad a + b\frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

$\therefore$ The given equation becomes

$$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z) \quad \Rightarrow \quad \frac{dz}{dx} - a = bf(z)$$

$$\Rightarrow \quad \frac{dz}{dx} = a + bf(z) \quad \Rightarrow \quad \frac{dz}{dx} = a + bf(z)$$

$$\Rightarrow \quad \frac{dz}{a + bf(z)} = dx$$ [Variables are separated]

On integrating both sides, we have

$$\int \frac{1}{a + bf(z)}\, dz = \int dx + C \quad \Rightarrow \quad \int \frac{1}{a + bf(z)}\, dz = x + C$$

where C is an arbitrary constant, and $z = ax + by + C$ which is the required solution.

SOME SOLVED EXAMPLES

Example 1. *Determine the order and degree of the following differential equations. State also, if these are linear or non-linear.*

(*i*) $\frac{d^2y}{dx^2} = \sqrt{1 + \frac{dy}{dx}}$ (*ii*) $xy\frac{dy}{dx} = \frac{(1+y^2)(1+x+x^2)}{1+x^2}$

(*iii*) $\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 8 = 0$ (*iv*) $y = \frac{dy}{dx} + \frac{C}{\frac{dy}{dx}}$

(v) $\frac{d^2y}{dx^2} = 1 + \sqrt{\frac{dy}{dx}}$ (vi) $\frac{d^3y}{dx^3} + y = 0$

(vii) $y = \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3}$ (viii) $y = x\frac{dy}{dx} + \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$

(ix) $\frac{1}{x}\frac{d^2y}{dx^2} = e^x$ (x) $(xy^2 + x)dx + (y - x^2y)dy = 0.$

Solution. (i) The given differential equation is

$$\frac{d^2y}{dx^2} = \sqrt{1 + \frac{dy}{dx}}$$

On squaring both sides, we get

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left(1 + \frac{dy}{dx}\right)$$

In this equation, the order of highest order derivative, *i.e.*, of $\frac{d^2y}{dx^2}$ is 2.

Degree of highest order derivative is 2.

∴ Order and degree of the given differential equation are 2 each.

The given differential equation is Non-linear, because $\frac{d^2y}{dx^2}$ is multiplied by itself.

(ii) The given differential equation is

$$xy\frac{dy}{dx} = \frac{(1+y^2)(1+x+x^2)}{(1+x^2)}$$

The order of highest order derivative, *i.e.*, of $\frac{dy}{dx}$ is 1.

Degree of highest order derivative is 1.

∴ Order and degree of the given differential equation are 1 each.

The given differential equation is non-linear, because y and $\frac{dy}{dx}$ are multiplied together.

(iii) The given differential equation is

$$\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 8 = 0$$

The order of highest order derivative *i.e.*, of $\left(\frac{d^2s}{dt^2}\right)$ is 2.

Degree of highest order derivative is 2.

∴ Order and degree of given differential equation are 2 each.

The given differential equation is Non-linear, because $\frac{d^2s}{dt^2}$ is multiplied by itself.

Degree of highest order derivative is 2.

$\therefore$ The order of given differential equation is 1 and degree is 2.

The given differential equation is Non-linear, because $\left(\frac{dy}{dx}\right)$ is multiplied by itself.

(*ix*) The given differential equation is $\frac{1}{x}\frac{d^2y}{dx^2} = e^x$

Order of highest order derivative *i.e.*, of $\frac{d^2y}{dx^2}$ is 2. Degree of highest order derivative is 1.

$\therefore$ Order of given equation is 2 and degree is 1.

The given differential equation is Linear, because dependent variable and its derivative occur only in the first degree and are not multiplied together.

(*x*) The given differential equation is

$$(xy^2 + x)dx + (y - x^2y)dy = 0$$

$$\Rightarrow \qquad (xy^2 + x) + (y - x^2y)\frac{dy}{dx} = 0$$

Order of highest order derivative *i.e.*, of $\frac{dy}{dx}$ is 1. Degree of highest order derivative is 1.

$\therefore$ Order of given differential equation is 1 and degree is 1.

The given differential equation is Non-linear, because y and $\frac{dy}{dx}$ are multiplied together.

Example 2. *Find the order and degree of the following differential equations. State also, if they are linear or non-linear.*

(*i*) $\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$ (*ii*) $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$

(*iii*) $y = x\frac{dy}{dx} + \frac{1}{\frac{dy}{dx}}$ (*iv*) $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = p$

(*v*) $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{4/3} = K\frac{d^2y}{dx^2}$ (*vi*) $t^2\frac{d^2s}{dt^2} + st\frac{ds}{dt} = s$

(*vii*) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} - x^2 = 0$ (*viii*) $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

(*ix*) $y + \frac{dy}{dx} = \frac{1}{6}\int y.dx$ (*x*) $x\frac{d^3y}{dx^3} + 6\cos x\frac{d^2y}{dx^2} + y\sin x = \sqrt{1-x^2}$.

Solution. (*i*) The given differential equation is

$$\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$$

$$\Rightarrow \qquad \sqrt{1-y^2} + y\sqrt{1-x^2}\,\frac{dy}{dx} = 0$$

Order of the highest order derivative *i.e.,* of $\frac{dy}{dx}$ is 1.

Degree of highest order derivative is 1.

$\therefore$ Order and degree of the given differential equation are 1 each.

The given differential equation is Non-linear, because y and $\frac{dy}{dx}$ are multiplied together.

(*ii*) The given differential equation is

$$\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$$

The order of highest order derivative *i.e.,* of $\frac{d^2y}{dx^2}$ is 2.

Degree of highest order derivative is 1.

$\therefore$ The order of given equation is 2 and degree is 1. The given differential equation is Linear.

(*iii*) The given differential equation is

$$y = x\frac{dy}{dx} + \frac{1}{\frac{dy}{dx}}$$

$$\Rightarrow \qquad y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + 1$$

Order of highest order derivative *i.e.,* of $\frac{dy}{dx}$ is 1.

Degree of highest order derivative is 2.

$\therefore$ Order of the given differential equation is 1 and degree is 2.

It is a Non-linear equation, because y and $\frac{dy}{dx}$ are multiplied together.

(*iv*) The given differential equation is

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = p$$

$$\Rightarrow \qquad \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = p\,\frac{d^2y}{dx^2}$$

On squaring both sides, we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = p^2\left(\frac{d^2y}{dx^2}\right)^2$$

Order of highest order derivative *i.e.*, of $\frac{d^2y}{dx^2}$ is 2. Degree of highest order derivative is 2.

$\therefore$ Order and degree of the given differential equation are 2 each.

It is a Non-linear differential equation, because $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are multiplied by itself.

(*v*) The given differential equation is

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{4}{3}} = \mathrm{K}\,\frac{d^2y}{dx^2}$$

Cubing both sides, we get

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^4 = \mathrm{K}^3\left(\frac{d^2y}{dx^2}\right)^3$$

Order of highest order derivative *i.e.*, $\frac{d^2y}{dx^2}$ is 2. Degree of highest order derivative is 3.

$\therefore$ Order of the given differential equation is 2 and degree is 3.

It is a Non-linear differential equation as $\frac{dy}{dx}$ is multiplied by itself.

(*vi*) The given differential equation is

$$t^2\frac{d^2s}{dt^2} + st\frac{ds}{dt} = s$$

Order of highest order derivative *i.e.*, of $\frac{d^2s}{dt^2}$ is 2. Degree of highest order derivative is 1.

$\therefore$ Order of the given differential equation is 2 and degree is 1.

The given differential equation is Non-linear, because s and $\frac{ds}{dt}$ are multiplied together.

(*vii*) The given differential equation is

$$\frac{d^2y}{dx^2} + y\frac{ds}{dx} - x^2 = 0$$

Order of the highest order derivative *i.e.*, of $\frac{d^2y}{dx^2}$ is 2. Degree of highest order derivative is 1.

$\therefore$ The order of the given differential equation is 2 and degree is 1.

It is a Non-linear differential equation, because y and $\frac{dy}{dx}$ are multiplied together.

(*viii*) The given differential equation is

$$\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$$

Order of the highest order derivative *i.e.*, of $\frac{d^2s}{dt^2}$ is 2. Degree of highest order derivative is 1.

$\therefore$ Order of the given differential equation is 2 and degree is 1.

It is a Non-linear differential equation, because $\left(\frac{ds}{dt}\right)$ is multiplied by itself.

(*ix*) The given differential equation is

$$y + \frac{dy}{dx} = \frac{1}{6}\int y.dx$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} + \frac{d^2y}{dx^2} = \frac{1}{6}y$$

$$\Rightarrow \qquad 6\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - y = 0$$

In this equation, order of highest order derivative *i.e.*, of $\frac{d^2y}{dx^2}$ is 2.

Degree of highest order derivative is 1.

$\therefore$ Order of the given differential equation is 2 and degree is 1.

It is a Linear differential equation as the dependent variable and its derivative occur only in first degree.

(*x*) The given differential equation is

$$x\frac{d^3y}{dx^3} + 6\cos x\frac{d^2y}{dx^2} + y\sin x = \sqrt{1-x^2}$$

Order of the highest order derivative *i.e.*, of $\frac{d^3y}{dx^3}$ is 3. Degree of the highest order derivative is 1.

$\therefore$ The order of the given differential equation is 3 and degree is 1.

It is a Linear differential equation, because the dependent variable and its derivative occur only in first degree.

10.11 SOME IMPORTANT RESULTS OF CO-ORDINATE GEOMETRY

(*i*) Equation of any straight line is $y = mx + C$, where m and C are arbitrary constants.

(*ii*) Equation of any straight line passing through the origin is $y = mx$ where m and C are arbitrary constants.

(*iii*) Equation of the circle having centre (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

(*iv*) Equation of the circle in general form is

$$x^2 + y^2 + 2gx + 2fy + C = 0$$

where $\qquad$ Centre $= (-g, -f)$ and $\quad$ Radius $= \sqrt{g^2 + f^2 - C}$

(*v*) Equation of the circle passing through the origin is

$$x^2 + y^2 + 2gx + 2fy = 0$$

where f and g are arbitrary constants.

(*vi*) Equation of circle passing through the origin and having centre on the x-axis is

$$(x-a)^2 + y^2 = a^2$$

where a is an arbitrary constant.

$$= \left(\frac{3y}{2\frac{dy}{dx}}\right)^3 = \frac{27y^3}{8\left(\frac{dy}{dx}\right)^3}$$

$$\Rightarrow \quad 8a\left(\frac{dy}{dx}\right)^3 = 27y$$

which is the required differential equation.

(*ii*) The given equation is

$$y = A\cos mx + B\sin mx \quad ...(1)$$

Since the given equation has two arbitrary constants, we shall differentiate it twice and we shall get a differential equation of second order.

Differentiating equation (1) w.r.t. x, we get

$$\frac{dy}{dx} = -mA\sin mx + mB\cos mx$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -m^2\cos mx - m^2 B\sin mx$$

$$= -m^2[A\cos mx + B\sin mx] \quad \text{[Using (1)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -m^2y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + m^2y = 0$$

which is the required differential equation.

(*iii*) The given equation is

$$C(y + C)^2 = x^3 \quad ...(1)$$

Since only one parameter is there, so we shall get a differential equation of first order.

Differentiating equation (1) w.r.t. x, we get

$$2C(y + C)\frac{dy}{dx} = 3x^2 \quad ...(2)$$

Dividing (1) by (2), we get

$$\Rightarrow \quad \frac{C(y+C)^2}{2C(y+C)\frac{dy}{dx}} = \frac{x^3}{3x^2} \quad \Rightarrow \quad \frac{(y+C)}{2\frac{dy}{dx}} = \frac{x}{3}$$

$$\Rightarrow \quad y + C = \frac{2}{3}x\frac{dy}{dx}$$

$$\Rightarrow \quad C = \left(\frac{2}{3}\frac{x\,dy}{dx} - y\right) \quad ...(3)$$

Substituting this value of C in equation (1), we have

$$\left[\frac{2}{3}x\frac{dy}{dx} - y\right]\left[y + \frac{2}{3}x\frac{dy}{dx} - y\right]^2 = x^3$$

$$\Rightarrow \quad \left(\frac{2}{3}x\frac{dy}{dx} - y\right)\left(\frac{2}{3}x\frac{dy}{dx}\right)^2 = x^3$$

Example 4. *Find the differential equations of the following families of curves :*

(*i*) $y = C(x - C)^2$; where C is an arbitrary constant.

(*ii*) $y = Ae^{mx} + Be^{nx}$; *where m, n are fixed and A, B are arbitrary constants.*

(*iii*) $y^2 = m(a^2 - x^2)$; *where m and a are arbitrary constants.*

(*iv*) $y^2 - 2ay + x^2 = a^2$; *where a is an arbitrary constant.*

(*v*) $e^x + Ce^y = 1$; *where C is an arbitrary constant.*

(*vi*) $y = e^x (A \cos x + B \sin x)$; *A and B are arbitrary constants.*

Solution. (*i*) The given equation is

$$y = C(x - C)^2 \qquad \text{...(1)}$$

Since there is one arbitrary constant in the given equation, so we differentiate it once and we shall get a differential equation of first order.

Differentiating equation (1) w.r.t. x, we get

$$\frac{dy}{dx} = 2C(x - C) \qquad \text{...(2)}$$

Dividing (1) by (2), we get

$$\frac{y}{\frac{dy}{dx}} = \frac{C(x-C)^2}{2C(x-C)}$$

$$\Rightarrow \quad \frac{y}{\frac{dy}{dx}} = \frac{x-C}{2} \qquad \Rightarrow \quad x - C = \frac{2y}{\frac{dy}{dx}}$$

$$\Rightarrow \quad C = \left(x - \frac{2y}{\frac{dy}{dx}}\right) \qquad \text{...(3)}$$

Substituting this value of C in equation (1), we have

$$y = \left(x - \frac{2y}{\frac{dy}{dx}}\right)\left[x - \left(x - \frac{2y}{\frac{dy}{dx}}\right)\right]^2 = \left(x - \frac{2y}{\frac{dy}{dx}}\right)\left[x - x + \frac{2y}{\frac{dy}{dx}}\right]^2$$

$$= \left(x - \frac{2y}{\frac{dy}{dx}}\right)\left(\frac{4y^2}{\left(\frac{dy}{dx}\right)^2}\right) = \frac{4xy^2}{\left(\frac{dy}{dx}\right)^2} - \frac{8y^3}{\left(\frac{dy}{dx}\right)^3}$$

$$\Rightarrow \quad y\left(\frac{dy}{dx}\right)^3 = 4xy^2\left(\frac{dy}{dx}\right) - 8y^3$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 = 4y\left(x\frac{dy}{dx} - 2y\right) \qquad \text{[Cancelling } y \text{ throughout]}$$

which is the required differential equation.

(*ii*) The given equation is

$$y = Ae^{mx} + Be^{nx} \qquad \text{...(1)}$$

Since the given equation contains two arbitrary constants A and B ; we shall differentiate it twice and we shall get a differential equation of second order.

$$\Rightarrow \qquad y\frac{dy}{dx} = -mx \qquad ...(2)$$

Differentiating again w.r.t. x, we have

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -m \qquad ...(3)$$

Substituting this value of $(-m)$ in equation (2), we have

$$y\frac{dy}{dx} = \left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right].x = xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \qquad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is the required differential equation.

(*iv*) The given equation is

$$y^2 - 2ay + x^2 = a^2 \qquad ...(1)$$

Since the given equation has only one arbitrary constant, so we shall get a differential equation of first order.

Differentiating equation (1), w.r.t. x, we get

$$2y\frac{dy}{dx} - 2a\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \qquad y\frac{dy}{dx} - a\frac{dy}{dx} + x = 0 \qquad \text{[Cancelling 2 throughout]}$$

$$\Rightarrow \qquad a\frac{dy}{dx} = x + y\frac{dy}{dx}$$

$$\Rightarrow \qquad a = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}} \qquad ...(2)$$

Substituting this value of a in equation (1), we have

$$y^2 - 2y\left(\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}\right) + x^2 = \left(\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}\right)^2.$$

$$\Rightarrow \qquad y^2 - 2y\left(\frac{x + yy_1}{y_1}\right) + x^2 = \left(\frac{x + yy_1}{y_1}\right)^2 \qquad \left[\text{Writing } \frac{dy}{dx} = y_1\right]$$

$$\Rightarrow \qquad y^2 - \frac{2xy}{y_1} - 2y^2 + x^2 = \frac{x^2 + y^2y_1^2 + 2xyy_1}{y_1^2}$$

$$\Rightarrow \qquad x^2 - y^2 - \frac{2xy}{y_1} = \frac{x^2 + y^2y_1^2 + 2xyy_1}{y_1^2}$$

$$\Rightarrow \qquad \frac{x^2y_1 - y^2y_1 - 2xy}{y_1} = \frac{x^2 + y^2y_1^2 + 2xyy_1}{y_1^2}$$

$$\Rightarrow \qquad y_1(x^2y_1 - y^2y_1 - 2xy) = x^2 + y^2y_1^2 + 2xyy_1$$

$$\Rightarrow \qquad x^2y_1^2 - y^2y_1^2 - 2xyy_1 = x^2 + y^2y_1^2 + 2xyy_1$$

Example 5. *Find the differential equation of the family of all straight lines passing through the origin.*

Solution. The general equation of the family of all straight lines passing through the origin is given by

$$y = mx$$

Please try yourself

[Hint : See Example 3. Part (*iv*)]

Example 6. *Find the differential equation of the following families of curves :*

(*i*) $y = Ae^{Bx}$ *; where A and B are arbitrary constants.*

(*ii*) $y = A \cos (x + B)$ *; where A and B are arbitrary constants.*

(*iii*) $y = Ax + \frac{B}{x}$ *; where A and B are arbitrary constants.*

(*iv*) $y = Ae^{3x} + Be^{5x}$ *; where A and B are arbitrary constants.*

(*v*) $y = ae^{x} + be^{2x} + ce^{3x}$ *; where a, b and c are arbitrary constants.*

(*vi*) $y = a \sin (bx + c)$ *; where b is fixed, and a, c are arbitrary constants.*

Solution. (*i*) The given equation is

$$y = Ae^{Bx} \qquad ...(1)$$

Since the given equation contains two arbitrary constants A and B, so we shall differentiate it twice and we shall get a differential equation of second order.

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = ABe^{Bx} = B(Ae^{Bx})$$

$$\Rightarrow \quad \frac{dy}{dx} = By \qquad ...(2) \quad \text{[Using equation (1)]}$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = B\frac{dy}{dx} \qquad ...(3)$$

From equation (2), we have

$$B = \frac{1}{y}\frac{dy}{dx}$$

Substituting this value of B in equation (3), we get

$$\frac{d^2y}{dx^2} = \left(\frac{1}{y}\frac{dy}{dx}\right)\frac{dy}{dx} \quad \Rightarrow \quad y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

which is the required differential equation.

(*ii*) The given equation is

$$y = A \cos (x + B) \qquad ...(1)$$

Since the given equation has two arbitrary constants A and B, so we shall differentiate it twice and we shall get a differential equation of order two.

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = -A \sin (x + B) \qquad ...(2)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = -A \cos (x + B)$$

$$\Rightarrow \quad 1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad (y-b) = \frac{-\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \qquad \text{...(3)}$$

Substituting this value of $(y-b)$ in equation (2), we have

$$(x-a) - \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x-a) = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}.\frac{dy}{dx} \qquad \text{...(4)}$$

Now, substituting the values of $(y-b)$ and $(x-a)$ from equations (3) and (4) in equation (1), we have

$$\left[\left(\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right).\frac{dy}{dx}\right]^2 + \left[-\left(\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right)\right]^2 = r^2$$

$$\Rightarrow \quad \left[\frac{(1+y_1^2)}{y_2}.y_1\right]^2 + \left[-\left(\frac{1+y_1^2}{y_2}\right)\right]^2 = r^2$$

$$\Rightarrow \quad \frac{(1+y_1^2)^2 y_1^2}{y_2^2} + \frac{(1+y_1^2)^2}{y_2^2} = r^2$$

$$\Rightarrow \quad (1+y_1^2)^2[y_1^2+1] = r^2y_2^2$$

$$\Rightarrow \quad (1+y_1^2)^3 = r^2y_2^2$$

$$\Rightarrow \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left[\frac{d^2y}{dx^2}\right]^2$$

which is the required differential equation.

Example 8. *Find the differential equation of all circles passing through origin and having their centres on the axis of x.*

Or

Find the differential equation of the system of circles touching the y-axis at the origin.

Solution. The general equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(1)}$$

Since, the circle passes through the origin, *i.e.*, (0, 0)

$$\therefore \quad (0)^2 + (0)^2 + 2g(0) + 2f(0) + c = 0$$

$$\Rightarrow \quad c = 0$$

$\therefore$ Equation of circle reduces to

$$x^2 + y^2 + 2gx + 2fy = 0 \qquad ...(2)$$

If the centre $(-g, -f)$ lies on x-axis.

$$\therefore \qquad f = 0$$

$\therefore$ Equation (2) becomes

$$x^2 + y^2 + 2gx = 0 \qquad ...(3)$$

Differentiating equation (3), w.r.t. x, we have

$$2x + 2y\frac{dy}{dx} + 2g = 0$$

$$\Rightarrow \qquad x + y\frac{dy}{dx} + g = 0 \qquad \text{[Cancelling 2 throughout]}$$

$$\Rightarrow \qquad g = -\left(x + y\frac{dy}{dx}\right) \qquad ...(4)$$

Substituting this value of g in equation (3), we have

$$x^2 + y^2 + 2\left[-\left(x + y\frac{dy}{dx}\right)\right]x = 0$$

$$\Rightarrow \qquad x^2 + y^2 - 2x^2 - 2xy\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad y^2 - x^2 - 2xy\frac{dy}{dx} = 0$$

which is the required differential equation.

Example 9. *Find the differential equation of lines in xy-plane.*

Solution. The general equation of a straight line in xy-plane is given by

$$y = mx + c \qquad ...(1)$$

where m and c are arbitrary constants.

Since the equation (1) has two arbitrary constants, so, we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = m \qquad ...(2)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = 0 \qquad ...(3)$$

which is the required differential equation.

Example 10. *Find the differential equation corresponding to* $y^2 = a(b - x)(b + x)$ *by eliminating parameters a and b.*

Solution. Please try yourself.

[**Hint :** $\qquad y^2 = a(b - x)(b + x)$ $\qquad \left[\textbf{Ans. } x\left(y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right) = y\frac{dy}{dx}\right]$

$\Rightarrow \qquad y^2 = a(b^2 - x^2)$

See Example 4 Part (*iii*).]

Differentiating equation (3), w.r.t. x, we have

$$\frac{d^3y}{dx^3} = 0$$

which is the required differential equation.

Example 12. *Formulate the differential equations of the family of curves :*

(i) $Ax^2 + By^2 = 1$ *; where A and B are arbitrary constants.*

(ii) $(x^2 - y^2) = c(x^2 + y^2)^2$ *; where C is an arbitrary constant.*

(iii) $\frac{x}{a} + \frac{y}{b} = 1$ *; where a and b are arbitrary constants.*

(iv) $y = Ae^{2x} + Be^{-3x}$ *; where A and B are arbitrary constants.*

Solution. *(i)* The given equation is

$$Ax^2 + By^2 = 1 \qquad \text{...(1)}$$

Since equation (1) contains two arbitrary constants A and B, so we shall differentiate it twice and we shall get a differential equation of order two.

Differentiating equation (1), w.r.t. x, we have

$$2Ax + 2By\frac{dy}{dx} = 0$$

$$\Rightarrow \quad Ax + By\frac{dy}{dx} = 0 \qquad \text{...(2)}$$

Differentiating again w.r.t. x, we have

$$A + By\frac{d^2y}{dx^2} + B\frac{dy}{dx}.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad A + By\frac{d^2y}{dx^2} + B\left(\frac{dy}{dx}\right)^2 = 0 \qquad \text{...(3)}$$

From equation (2), $\quad Ax = -By\frac{dy}{dx}$

$$\Rightarrow \quad A = -B\frac{y}{x}\frac{dy}{dx} \qquad \text{...(4)}$$

Substituting this value of A in equation (3), we have

$$-B\frac{y}{x}\frac{dy}{dx} + By\frac{d^2y}{dx^2} + B\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad -\frac{y}{x}\frac{dy}{dx} + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \text{[Cancelling B throughout]}$$

$$\Rightarrow \quad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is the required differential equation.

(ii) The given equation is

$$x^2 - y^2 = c(x^2 + y^2)^2 \qquad \text{...(1)}$$

Differentiating equation (1) w.r.t. x, we have

$$2x - 2y\frac{dy}{dx} = 2c(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left(x - y\frac{dy}{dx}\right) = c\,(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) \quad ...(2) \text{ [Cancelling 2 throughout]}$$

From equation (1), we have

$$c = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad ...(3)$$

Substituting this value of c in equation (2), we have

$$\left(x - y\frac{dy}{dx}\right) = \frac{2\,(x^2 - y^2)}{(x^2 + y^2)^2}.(x^2 + y^2)\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (x^2 + y^2)\left(x - y\frac{dy}{dx}\right) = 2\,(x^2 - y^2)\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad x^3 + xy^2 - x^2y\frac{dy}{dx} - y^3\frac{dy}{dx} = 2x^3 + 2x^2y\frac{dy}{dx} - 2xy^2 - 2y^3\frac{dy}{dx}$$

$$\Rightarrow \quad (3xy^2 - x^3) = \frac{dy}{dx}(3x^2y - y^3)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3xy^2 - x^3}{3x^2y - y^3}$$

which is the required differential equation.

(*iii*) The given equation is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad ...(1)$$

Differentiating equation (1) w.r.t. x, we have

$$\frac{1}{a} + \frac{1}{b}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{1}{b}\frac{dy}{dx} = -\frac{1}{a}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b}{a} \quad ...(2)$$

Differentiating equation (2), w.r.t. x, we have

$$\frac{d^2y}{dx^2} = 0$$

which is the required differential equation.

(*iv*) The given equation is

$$y = Ae^{2x} + Be^{-3x} \quad ...(1)$$

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = 2Ae^{2x} - 3Be^{-3x} \quad ...(2)$$

Differentiating equation (2), w.r.t. x again, we have

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 9Be^{-3x} \quad (3)$$

Adding equations (2) and (3), we have

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 6Ae^{2x} + 6Be^{-3x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} = 6(Ae^{2x} + Be^{-3x})$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} + \frac{dy}{dx} = 6y \qquad \text{[Using equation (1)]}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

which is the required differential equation.

Example 13. *Obtain a differential equation that should be satisfied by the family of concentric circles $x^2 + y^2 = a^2$.*

Solution. The given equation is

$$x^2 + y^2 = a^2 \qquad ...(1)$$

where a is an arbitrary constant.

Since the given equation (1) has only one arbitrary constant, so we shall differentiate it once, and we shall get a differential equation of first order.

Differentiating equation (1) w.r.t. x, we have

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad x + y\frac{dy}{dx} = 0$$

which is the required differential equation.

Example 14. *Find the differential equation of all conics whose axes coincides with the axes of co-ordinates.*

Solution. The equation of any conic whose axes coincide with the axes of co-ordinate is given by

$$Ax^2 + By^2 = 1 \qquad ...(1)$$

where A and B are arbitrary constants. [For solution See Example 12 Part (*i*)]

Example 15. *Find the differential equation of all the circles in the first quadrant which touch the co-ordinate axes.*

Solution. The equation of all the circles in the first quadrant which touch the co-ordinate axes is given by

$$(x - a)^2 + (y - a)^2 = a^2 \qquad ...(1)$$

$$\Rightarrow \qquad x^2 + a^2 - 2ax + y^2 + a^2 - 2ay = a^2$$

$$\Rightarrow \qquad x^2 + y^2 - 2ax - 2ay + a^2 = 0 \qquad ...(2)$$

Differentiating equation (2), w.r.t. x, we have

$$2x + 2y\frac{dy}{dx} - 2a - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad x + y\frac{dy}{dx} - a - a\frac{dy}{dx} = 0 \qquad \text{[Cancelling 2 throughout]}$$

$$\Rightarrow \qquad a\left(1 + \frac{dy}{dx}\right) = x + y\frac{dy}{dx}$$

$$\Rightarrow \qquad a = \left(\frac{x + y\dfrac{dy}{dx}}{1 + \dfrac{dy}{dx}}\right) \qquad ...(3)$$

Example 20. *Assume that a spherical rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop.*

Solution. Let r be the radius of the rain drop at any time t.

Let V be the volume of the rain drop and S be the surface area of the rain drop.

Now, according to the given condition

$$\frac{dv}{dt} \propto S$$

$$\Rightarrow \quad \frac{dv}{dt} = -kS \qquad ...(1)$$

where k is the constant of proportionality. (–ve) sign shows that volume is decreasing due to evaporation.

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k(4\pi r^2) \qquad \left[\begin{array}{l}\because \text{ Volume of sphere} = \frac{4}{3}\pi r^3 \\ \text{Surface area of sphere} = 4\pi r^2\end{array}\right]$$

$$\Rightarrow \quad \frac{4}{3}\pi.3r^2\frac{dr}{dt} = -4\pi r^2 k$$

$$\Rightarrow \quad \frac{dr}{dt} = -k$$

which is the required differential equation.

Example 21. *Find the differential equation of the family of curves $xy = Ae^x + Be^{-x} + x^2$.*

Solution. The given equation is

$$xy = Ae^x + Be^{-x} + x^2 \qquad ...(1)$$

where A and B are arbitrary constants.

Since the given equation contains two arbitrary constants, so we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating equation (1) w.r.t. x, we have

$$y + x\frac{dy}{dx} = Ae^x - Be^{-x} + 2x \qquad ...(2)$$

Differentiating again w.r.t. x, we have

$$\frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} = Ae^x + Be^x + 2$$

$$\Rightarrow \quad 2\frac{dy}{dx} + x\frac{d^2y}{dx^2} = xy - x^2 + 2 \qquad \text{[Using equation (1)]}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$$

which is the required differential equation.

Example 22. *Find the differential equation of all circles in the xy-plane.*

Solution. The general equation of any circle in the xy-plane is given by

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(1)$$

where g, f and c are arbitrary constants.

Example 2. *Show that y = 4 sin 3x is a solution of the differential equation* $\frac{d^2y}{dx^2} + 9y = 0$.

Solution. The given equation is

$$y = 4 \sin 3x \quad ...(1)$$

The given differential equation is

$$\frac{d^2y}{dx^2} + 9y = 0 \quad ...(2)$$

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = 12 \cos 3x \quad ...(3)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = -36 \sin 3x \quad ...(4)$$

Now, substituting the values of y and $\frac{d^2y}{dx^2}$ in the L.H.S. of equation (2), we have

$$\frac{d^2y}{dx^2} + 9y = -36 \sin 3x + 9(4 \sin 3x) = -36 \sin 3x + 36 \sin 3x = 0$$

$\therefore$ $y = 4 \sin 3x$ is a solution of the given differential equation.

Example 3. *Show that the equation y = A cos 2x – B sin 2x is a solution of the differential equation* $\frac{d^2y}{dx^2} + 4y = 0$.

Solution. The given function is

$$y = A \cos 2x - B \sin 2x \quad ...(1)$$

The given differential equation is

$$\frac{d^2y}{dx^2} + 4y = 0 \quad ...(2)$$

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = -2A \sin 2x - 2B \cos 2x \quad ...(3)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = -4A \cos 2x + 4B \sin 2x = -4[A \cos 2x - B \sin 2x]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -4y \quad \text{[Using equation (1)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 4y = 0$$

which is same as the given differential equation.

$\therefore$ $y = A \cos 2x - B \sin 2x$ is a solution of the given differential equation.

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = 2ae^{2x} - be^{-x} \quad ...(3)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x} \quad ...(4)$$

Now, substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the L.H.S. of equation (2), we have

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = (4ae^{2x} + be^{-x}) - (2ae^{2x} - be^{-x}) - 2(ae^{2x} + be^{-x})$$

$= 4ae^{2x} + be^{-x} - 2ae^{2x} + be^{-x} - 2ae^{2x} - 2be^{-x} = 0$ which is true.

$\therefore$ $y = ae^{2x} + be^{-x}$ is a solution of the given differential equation.

Example 7. *Show that* $y = \frac{c}{x} + d$ *is a solution of the differential equation*

$$\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0.$$

Solution. The given function is

$$y = \frac{c}{x} + d \quad ...(1)$$

The given differential equation is

$$\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0 \quad ...(2)$$

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = -\frac{c}{x^2} \quad ...(3)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{2c}{x^3} \quad ...(4)$$

Now, substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the L.H.S. of equation (2), we have

$$\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = \frac{2c}{x^3} + \frac{2}{x}\left(-\frac{c}{x^2}\right)$$

$$= \frac{2c}{x^3} - \frac{2c}{x^3} = 0$$

which is true.

$\therefore$ $y = \frac{c}{x} + d$ is a solution of the given differential equation.

Example 8. *Verify that* $y + x + 1 = 0$ *is a solution of the differential equation* $(y - x)dy - (y^2 - x^2)dx = 0.$

Example 10. *Show that $y = Ce^{-x}$ is a solution of the differential equation $\frac{dy}{dx} + y = 0$.*

Solution. The given function is

$$y = Ce^{-x} \quad ...(1)$$

The given differential equation is

$$\frac{dy}{dx} + y = 0 \quad ...(2)$$

Differentiating equation (1) w.r.t. x, we have

$$\Rightarrow \quad \frac{dy}{dx} = -Ce^{-x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -y \quad \text{[Using equation (1)]}$$

$$\Rightarrow \quad \frac{dy}{dx} + y = 0$$

which is same as the given differential equation

$\therefore$ $y = Ce^{-x}$ is a solution of the differential equation $\frac{dy}{dx} + y = 0$.

Example 11. *Show that $y = a \cos (\log x) + b \sin (\log x)$ is a solution of the differential equation*

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

Solution. The given function is

$$y = a \cos (\log x) + b \sin (\log x) \quad ...(1)$$

The given differential equation is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \quad ...(2)$$

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = -\frac{a \sin (\log x)}{x} + \frac{b \cos (\log x)}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -a \sin (\log x) + b \cos (\log x) \quad ...(3)$$

Differentiating again w.r.t. x, we have

$$x\frac{d^2y}{dx^2} + 1.\frac{dy}{dx} = -\frac{a \cos (\log x)}{x} - \frac{b \sin (\log x)}{x}$$

$$\Rightarrow \quad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[a \cos (\log x) + b \sin (\log x)]$$

$$\Rightarrow \quad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \quad \text{[Using equation (1)]}$$

$$\Rightarrow \quad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

which is same as equation (2).

$\therefore$ $y = a \cos (\log x) + b \sin (\log x)$ is a solution of the given differential equation.

Solution. The given equation is

$$Ax^2 + By^2 = 1 \qquad \text{...(1)}$$

The given differential equation is

$$x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y.\frac{dy}{dx} \qquad \text{...(2)}$$

Differentiating equation (1) w.r.t. x, we have

$$2Ax + 2By\frac{dy}{dx} = 0$$

$$\Rightarrow \quad Ax + By\frac{dy}{dx} = 0 \qquad \text{...(3) [Cancelling '2' throughout]}$$

Differentiating again w.r.t. x, we have

$$A + B\left[\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right] = 0$$

$$\Rightarrow \quad A = -B\left[\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right]$$

$$\Rightarrow \quad -\frac{A}{B} = \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} \qquad \text{...(4)}$$

From equation (3), we have

$$Ax = -By\frac{dy}{dx}$$

$$\Rightarrow \quad -\frac{A}{B} = \frac{y}{x}\frac{dy}{dx} \qquad \text{...(5)}$$

Equating equations (4) and (5), we have

$$\frac{y}{x}\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}$$

$$\Rightarrow \quad y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2}$$

$$\Rightarrow \quad xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is same as given equation (2)

$\therefore$ $Ax^2 + By^2 = 1$ is the solution of the given differential equation.

Example 15. *Show that $y = be^x + Ce^{2x}$ is a solution of the differential equation*

$$y_2 - 3y_1 + 2y = 0.$$

Solution. The given function is

$$y = be^x + Ce^{2x} \qquad \text{...(1)}$$

The given differential equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \qquad \text{...(2)}$$

Solution. The given equation is

$$y = x^3 + ax^2 + bx + C \quad ...(1)$$

The given differential equation is

$$\frac{d^3y}{dx^3} = 6 \quad ...(2)$$

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = 3x^2 + 2ax + b \quad ...(3)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = 6x + 2a \quad ...(4)$$

Differentiating again w.r.t. x, we have

$$\frac{d^3y}{dx^3} = 6$$

which is same as the given equation (2).

$\therefore$ $y = x^3 + ax^2 + bx + C$ is a solution of the given differential equation $\frac{d^3y}{dx^3} = 6$.

Example 19. *Solve the following differential equations :*

(i) $\frac{dy}{dx} = \frac{x}{x^2+1}$ (ii) $\frac{dy}{dx} = \frac{1}{x^2+2x+2}$

(iii) $(x^2+1)dy = dx$ (iv) $\frac{dy}{dx} + 2x = e^{3x}$

(v) $(x+2)\frac{dy}{dx} = x^2 + 4x - 9$ (vi) $\frac{dy}{dx} - x\sin^2 x = \frac{1}{x\log x}$.

Solution. (i) The given equation is

$$\frac{dy}{dx} = \frac{x}{x^2+1} \quad ...(1)$$

Separating the variables

$$dy = \frac{x}{x^2+1}.dx$$

Integrating both sides, we get

$$\int dy = \int \frac{x}{x^2+1}\,dx$$

$$\Rightarrow \quad y = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx \quad \text{[Multiply and divided by 2]}$$

$$\Rightarrow \quad y = \frac{1}{2}\log|x^2+1| + C \quad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$$

which is the required solution.

(ii) the given equation is

$$\frac{dy}{dx} = \frac{1}{x^2+2x+2} \quad ...(1)$$

$$= \log|\log x| + \frac{1}{2}.\frac{x^2}{2} - \frac{1}{2}\int \underset{\text{I}}{x} \underset{\text{II}}{\cos 2x}\, dx \qquad \text{[Integrating by parts]}$$

$$= \log|\log x| + \frac{x^2}{4} - \frac{1}{2}\left[x.\int \cos 2x\, dx - \int\left[\frac{d}{dx}(x)\int \cos 2x.dx\right]dx\right]$$

$$= \log|\log x| + \frac{x^2}{4} - \frac{1}{2}\left[x\frac{\sin 2x}{2} - \int \frac{1.\sin 2x}{2}dx\right]$$

$$\Rightarrow \qquad y = \log|\log x| + \frac{x^2}{4} - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$$

which is the required solution.

Example 20. *Solve the following differential equations :*

(*i*) $\frac{dy}{dx} = e^{x+y} + x^2e^y$ (*ii*) $\frac{dy}{dx} = e^{x-y} + e^{2\log x - y}$

(*iii*) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$ (*iv*) $\frac{dy}{dx} = \tan^2 x$

(*v*) $\frac{dy}{dx} = x^2 + \sin 3x$ (*vi*) $\frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x$.

Solution. (*i*) The given equation is

$$\frac{dy}{dx} = e^{x+y} + x^2e^y = e^x.e^y + x^2e^y$$

$$\Rightarrow \qquad \frac{dy}{dx} = e^y(e^x + x^2)$$

Separating the variables,

$$\frac{1}{e^y}dy = (e^x + x^2)dx$$

Integrating both sides, we get

$$\int e^{-y}\, dy = \int (e^x + x^2)\, dx$$

$$\Rightarrow \qquad \frac{e^{-y}}{-1} = e^x + \frac{x^3}{3} + C$$

$$\Rightarrow \qquad e^x + e^{-y} + \frac{x^3}{3} + C = 0$$

which is the required solution.

(*ii*) The given equation is

$$\frac{dy}{dx} = e^{x-y} + e^{2\log x - y} \qquad ...(1)$$

$$\Rightarrow \qquad \frac{dy}{dx} = e^x.e^{-y} + e^{\log x^2}.e^{-y}$$

$$\Rightarrow \qquad \frac{dy}{dx} = e^x.e^{-y} + x^2.e^{-y} \qquad [\because\ e^{\log f(x)} = f(x)]$$

$$\Rightarrow \qquad \frac{dy}{dx} = e^{-y}(e^x + x^2)$$

Integrating both sides, we get

$$\int dy = \int \tan^2 x . dx \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$$\Rightarrow \qquad y = \int (\sec^2 x - 1) dx$$

$$\Rightarrow \qquad y = \tan x - x + C$$

which is the required solution.

(*v*) The given equation is

$$\frac{dy}{dx} = x^2 + \sin 3x \qquad ...(1)$$

Separating the variables

$$dy = (x^2 + \sin 3x)dx$$

Integrating both sides, we get

$$\int dy = \int (x^2 + \sin 3x) dx$$

$$\Rightarrow \qquad y = \frac{x^3}{3} - \frac{\cos 3x}{3} + C$$

which is the required solution.

(*vi*) The given equation is

$$\frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x \qquad ...(1)$$

Separating the variables

$$dy = (\sin^3 x \cos^2 x + xe^x)dx$$

Integrating both sides, we get

$$\int dy = \int (\sin^3 x \cos^2 x + xe^x) dx$$

$$\Rightarrow \qquad y = \int \sin^3 x \cos^2 x \, dx + \int xe^x dx$$

$$\Rightarrow \qquad y = \int \sin^2 x \sin x \cos^2 x \, dx + \int \underset{\text{I}}{x} \underset{\text{II}}{e^x} dx$$

$$\Rightarrow \qquad y = \int (1 - \cos^2 x) \cos^2 x . \sin x \, dx + x . \int e^x dx - \int 1 . e^x dx$$

$$[\because \sin^2 A + \cos^2 A = 1]$$

Put $\cos x = z \Rightarrow -\sin x \, dx = dz \Rightarrow \sin x \, dx = -dz$

$$\therefore \qquad y = -\int (1 - z^2) z^2 dz + xe^x - e^x + C$$

$$= \int (z^4 - z^2) dz + xe^x - e^x + C$$

$$= \frac{z^5}{5} - \frac{z^3}{3} + xe^x - e^x + C$$

$$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + xe^x - e^x + C \qquad [\because z = \cos x]$$

which is the required solution.

Example 21. *Solve the following differential equations*

(i) $(e^x + e^{-x})dy = (e^x - e^{-x})dx$ *(ii)* $(1 + \cos x)dy = (1 - \cos x)dx$

(iii) $(\sin x + \cos x)dy + (\cos x - \sin x)dx = 0$

(iv) $\frac{1}{x}\frac{dy}{dx} = \tan^{-1} x$

(v) $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$ *(vi)* $(x - y^2x)dx - (y - x^2y)dy = 0.$

Solution. *(i)* The given equation is

$$(e^x + e^{-x})dy = (e^x - e^{-x})dx \quad ...(1)$$

Separating the variables

$$dy = \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

Integrating both sides, we get

$$\int dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

$$\Rightarrow \quad y = \log|e^x + e^{-x}| + C \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)|\right]$$

which is the required solution.

(ii) The given equation is

$$(1 + \cos x)dy = (1 - \cos x)dx \quad ...(1)$$

Separating the variables,

$$dy = \frac{1-\cos x}{1+\cos x}dx \qquad \left[\begin{array}{l}\because 1-\cos 2A = 2\sin^2 A \\ \Rightarrow 1-\cos A = 2\sin^2 \frac{A}{2} \\ \quad 1+\cos 2A = 2\cos^2 A \\ \Rightarrow 1+\cos A = 2\cos^2 \frac{A}{2}\end{array}\right]$$

$$\Rightarrow \quad dy = \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}}dx$$

$$\Rightarrow \quad dy = \tan^2 \frac{x}{2}dx$$

Integrating both sides, we get

$$\int dy = \int \tan^2 \frac{x}{2}dx$$

$$\Rightarrow \quad y = \int\left(\sec^2 \frac{x}{2} - 1\right)dx \qquad [\because \sec^2 A - \tan^2 A = 1]$$

$$\Rightarrow \quad y = \frac{\tan \frac{x}{2}}{\frac{1}{2}} - x + C$$

$\Rightarrow \qquad y = 2\tan\frac{x}{2} - x + C$

which is the required solution.

(*iii*) The given equation is

$(\sin x + \cos x)dy + (\cos x - \sin x)dx = 0 \qquad ...(1)$

$\Rightarrow \qquad (\sin x + \cos x)dy = -(\cos x - \sin x)dx$

Separating the variables,

$$dy = \frac{(\sin x - \cos x)}{(\sin x + \cos x)}dx$$

Integrating both sides, we get

$$\int dy = \int \frac{-(\cos x - \sin x)}{\sin x + \cos x}dx \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)|\right]$$

$\Rightarrow \qquad y = -\log|\sin x + \cos x| + C$

$\Rightarrow \quad y + \log|\sin x + \cos x| + C_1 = 0 \quad$ where : $C_1 = -C$

which is the required solution.

(*iv*) The given equation is

$$\frac{1}{x}\frac{dy}{dx} = \tan^{-1}x \qquad ...(1)$$

Separating the variables

$$dy = x\tan^{-1}x.dx$$

Integrating both sides, we get

$$\int dy = \int \underset{\text{II}}{x}\,\underset{\text{I}}{\tan^{-1}x}\,dx \qquad \text{[Integrating by parts]}$$

$$\Rightarrow \qquad y = \tan^{-1}x.\int x.dx - \int\left\{\frac{d}{dx}(\tan^{-1}x).\int x\,dx\right\}dx$$

$$\Rightarrow \qquad y = \tan^{-1}x.\frac{x^2}{2} - \int\frac{1}{1+x^2}.\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\frac{1+x^2-1}{1+x^2}dx$$

[Add and subtract 1 to the numerator]

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int 1.dx + \frac{1}{2}\int\frac{1}{1+x^2}dx$$

$$\therefore \qquad y = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C \qquad \left[\because \int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

which is the required solution.

(*v*) The given equation is

$(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0 \qquad ...(1)$

$\therefore$ We have
$$y = \int \frac{1}{y^2 + (\sqrt{2})^2}\, dt$$
$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + C \qquad \left[\because \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}\right]$$
$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{z - \frac{1}{z}}{\sqrt{2}} \right) + C \qquad \left[\because t = \left(z - \frac{1}{z}\right)\right]$$
$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{z^2 - 1}{\sqrt{2}\, z} \right) + C$$
$$\Rightarrow \qquad y = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan^2 x - 1}{\sqrt{2} \tan x} \right) + C \qquad [\because \; z = \tan x]$$

which is the required solution.

(*ii*) The given equation is
$$\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}} \qquad \text{...(1)}$$

Separating the variables,
$$dy = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}\, dx$$
$$= \frac{3e^{2x}(1 + e^{2x})}{e^x + \frac{1}{e^x}}\, dx = \frac{3e^{2x}(1 + e^{2x})}{\left(\frac{e^{2x} + 1}{e^x}\right)}\, dx$$
$$= \frac{3e^{2x}(1 + e^{2x}).e^x}{(1 + e^{2x})}\, dx$$
$$\Rightarrow \qquad dy = 3e^{3x}\, dx$$

Integrating both sides, we get
$$\int dy = 3 \int e^{3x}\, dx$$
$$\Rightarrow \qquad y = 3.\left(\frac{e^{3x}}{3}\right) + C$$
$$\Rightarrow \qquad y = e^{3x} + C$$

which is the required solution.

(*iii*) The given equation is
$$\sin^4 x \frac{dy}{dx} = \cos x$$

Separating the variables,
$$dy = \frac{\cos x}{\sin^4 x}\, dx$$

Integrating both sides, we get

$$\int dy = -\int\left(\sqrt{a+x} - \frac{a}{\sqrt{a+x}}\right)dx$$

$$\Rightarrow \quad y = -\int \sqrt{a+x}\, dx + a\int \frac{1}{\sqrt{a+x}}\, dx$$

$$\Rightarrow \quad y = -\frac{(a+x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + a\,\frac{(a+x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$\Rightarrow \quad y = -\frac{(a+x)^{3/2}}{3/2} + \frac{a(a+x)^{1/2}}{1/2} + C$$

$$\Rightarrow \quad y = -\frac{2}{3}(a+x)^{3/2} + 2a\sqrt{a+x} + C$$

which is the required solution.

(*vi*) The given equation is

$$\cos x\frac{dy}{dx} + \cos 2x = \cos 3x \qquad \ldots(1)$$

$$\Rightarrow \quad \cos x\frac{dy}{dx} = \cos 3x - \cos 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos 3x - \cos 2x}{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4\cos^3 x - 3\cos x}{\cos x} - \frac{2\cos^2 x - 1}{\cos x}$$

$$[\because \ \cos 3A = 4\cos^3 A - 3\cos A,\ \cos 2A = 2\cos^2 A - 1]$$

$$\Rightarrow \quad \frac{dy}{dx} = 4\cos^2 x - 3 - 2\cos x + \frac{1}{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx} = 4\left(\frac{1+\cos 2x}{2}\right) - 3 - 2\cos x + \sec x$$

$$\Rightarrow \quad \frac{dy}{dx} = (2\cos 2x - 1 - 2\cos x + \sec x)$$

Separating the variables,

$$dy = (2\cos 2x - 1 - 2\cos x + \sec x)dx$$

Integrating both sides, we get

$$\int dy = \int (2\cos 2x - 1 - 2\cos x + \sec x)\, dx$$

$$\Rightarrow \quad y = \sin 2x - x - 2\sin x + \log|\sec x + \tan x| + C$$

which is the required solution.

Example 23. *Solve the following differential equations :*

(*i*) $xy(y+1)dy = (x^2+1)dx$ (*ii*) $(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$

(*iii*) $\frac{dy}{dx} = x^5 \tan^{-1}(x^3)$ (*iv*) $\sqrt{1-x^6}\, dy = x^2 dx$

(*v*) $\frac{dy}{dx} = \cos^3 x \sin^4 x + x\sqrt{2x+1}$ (*vi*) $\frac{dy}{dx} = \log x.$

Solution. (*i*) The given equation is

$$xy(y+1)dy = (x^2+1)dx \qquad ...(1)$$

Separating the variables,

$$y(y+1)dy = \left(\frac{x^2+1}{x}\right)dx$$

Integrating both sides, we get

$$\int (y^2+y)dy = \int\left(\frac{x^2}{x}+\frac{1}{x}\right)dx$$

$$\Rightarrow \quad \int (y^2+y)dy = \int\left(x+\frac{1}{x}\right)dx$$

$$\Rightarrow \quad \frac{y^3}{3}+\frac{y^2}{2} = \frac{x^2}{2} + \log|x| + C$$

which is the required solution.

(*ii*) The given equation is

$$(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0 \qquad ...(1)$$

$$\Rightarrow \quad x^2(1-y)dy = -y^2(1+x^2)dx$$

Separating the variables,

$$\frac{1-y}{y^2}dy = -\frac{1+x^2}{x^2}dx$$

Integrating both sides, we get

$$\int\left(\frac{1-y}{y^2}\right)dy = -\int\left(\frac{1+x^2}{x^2}\right)dx$$

$$\Rightarrow \quad \int\left(\frac{1}{y^2}-\frac{1}{y}\right)dy = -\int\left(\frac{1}{x^2}+1\right)dx$$

$$\Rightarrow \quad \frac{y^{-2+1}}{-2+1} - \log|y| = -\left[\frac{x^{-2+1}}{-2+1}+x\right]+C$$

$$\Rightarrow \quad -\frac{1}{y} - \log|y| = -\left(-\frac{1}{x}+x\right)+C$$

$$\Rightarrow \quad -\frac{1}{y} - \log|y| = \frac{1}{x} - x + C$$

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y} + \log|y| - x + C = 0$$

which is the required solution.

(*iii*) The given equation is

$$\frac{dy}{dx} = x^5 \tan^{-1}(x^3) \qquad ...(1)$$

Separating the variables

$$dy = x^5 \tan^{-1}(x^3).dx$$

Integrating both sides, we get

$$\int dy = \int x^5 \tan^{-1}(x^3)\,dx$$

$$\Rightarrow \qquad (1 - x^2)dy = xy(y - 1)dx$$

Separating the variables,

$$\frac{1}{y(y-1)}\,dy = \frac{x}{1-x^2}\,dx$$

Integrating both sides, we get

$$\int \frac{1}{y(y-1)}\,dy = \int \frac{x}{1-x^2}\,dx$$

$$\Rightarrow \qquad \int\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = -\frac{1}{2}\int \frac{-2x}{1-x^2}\,dx \qquad \text{[By partial fractions]}$$

$$\Rightarrow \quad \log|y-1| - \log|y| = -\frac{1}{2}\log|1-x^2| + \log C \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$$

$$\Rightarrow \qquad \log\left|\frac{y-1}{y}\right| = \frac{1}{2}(2\log C - \log|1-x^2|)$$

$$\Rightarrow \qquad 2\log\left|\frac{y-1}{y}\right| = \log\left|\frac{C^2}{1-x^2}\right| \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$\Rightarrow \qquad \log\left(\frac{y-1}{y}\right)^2 = \log\left(\frac{C^2}{|1-x^2|}\right) \qquad [\because 2\log C = \log C^2]$$

$$\Rightarrow \qquad \frac{(y-1)^2}{y^2} = \frac{C_1}{|1-x^2|} \qquad [\because C^2 = C_1]$$

$$\Rightarrow \qquad C_1 y^2 = (y-1)^2\,|1-x^2|$$

which is the required solution.

(*iii*) The given equation is

$$\frac{dy}{dx} = e^{x+y} \qquad \text{...(1)}$$

$$\Rightarrow \qquad \frac{dy}{dx} = e^x . e^y$$

Separating the variables

$$\frac{1}{e^y}\,dy = e^x\,dx$$

Integrating both sides, we get

$$\int e^{-y}\,dy = \int e^x\,dx$$

$$\Rightarrow \qquad -e^{-y} = e^x + C$$

$$\Rightarrow \qquad e^x + e^{-y} + C = 0$$

which is the required solution.

(*iv*) The given equation is

$$(1 + x^2)dy = (1 + y^2)dx \qquad \text{...(1)}$$

Separating the variables,

$$\frac{1}{1+y^2}\,dy = \frac{1}{1+x^2}\,dx$$

Integrating both sides, we get

$$\int \frac{1}{1+y^2}\,dy = \int \frac{1}{1+x^2}\,dx$$

$\Rightarrow$ $\tan^{-1} y = \tan^{-1} x + \tan^{-1} C$ $\left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$\Rightarrow$ $\tan^{-1} y - \tan^{-1} x = \tan^{-1} C$

$\Rightarrow$ $\tan^{-1}\left(\frac{y-x}{1+xy}\right) = \tan^{-1} C$ $\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$

$\Rightarrow$ $\frac{y-x}{1+xy} = C$

$\Rightarrow$ $(y-x) = C(1+xy)$

which is the required solution.

(*v*) The given equation is

$$\frac{dy}{dx} + y = 1 \qquad ...(1)$$

$\Rightarrow$ $\frac{dy}{dx} = 1 - y$

Separating the variables,

$$\frac{1}{1-y}\,dy = dx$$

Integrating both sides, we get

$$\int \frac{1}{1-y}\,dy = \int 1.dx$$

$\Rightarrow$ $-\log|1-y| = x + C$ $\Rightarrow$ $x + \log|1-y| + C$

which is the required solution.

(*vi*) The given equation is

$$(1-y)x\frac{dy}{dx} + (1+x)y = 0 \qquad ...(1)$$

$\Rightarrow$ $(1-y)x\frac{dy}{dx} = -(1+x)y$

Separating the variables,

$$\frac{(1-y)}{y}\,dy = -\frac{(1+x)}{x}\,dx$$

Integrating both sides, we get

$$\int \frac{1-y}{y}\,dy = -\int \frac{1+x}{x}\,dx$$

$\Rightarrow$ $\int\left(\frac{1}{y}-1\right)dy = -\int\left(\frac{1}{x}+1\right)dx$

$\Rightarrow$ $\log|y| - y = -[\log|x| + x] + C$

$\Rightarrow$ $\log|y| - y = -\log|x| - x + C$

$\Rightarrow$ $\log|x| + \log|y| + x - y = C$ $[\because \log m + \log n = \log mn]$

$\Rightarrow$ $\log|xy| + x - y = C$

which is the required solution.

Separating the variables,

$$\frac{1}{1+y^2}\,dy = \frac{x}{1+x^2}\,dx$$

Integrating both sides, we get

$$\int \frac{1}{1+y^2}\,dy = \int \frac{x}{1+x^2}\,dx$$

$$\Rightarrow \quad \int \frac{1}{1+y^2}\,dy = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx \qquad \text{[Multiply and divide the R.H.S. by 2]}$$

$$\Rightarrow \quad \tan^{-1} y = \frac{1}{2}\log|1+x^2| + C \qquad \left[\because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \quad \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$$

which is the required solution.

(*v*) The given equation is

$$\frac{dy}{dx} = \sin^2 y \qquad ...(1)$$

Separating the variables,

$$\frac{1}{\sin^2 y}\,dy = dx$$

Integrating both sides, we get

$$\int \frac{1}{\sin^2 y}\,dy = \int 1.dx$$

$$\Rightarrow \quad \int \operatorname{cosec}^2 y\,dy = \int dx$$

$$\Rightarrow \quad -\cot y = x + C$$

$$\Rightarrow \quad x + \cot y + C = 0$$

which is the required solution.

(*vi*) The given equation is

$$(x+1)\,\frac{dy}{dx} = 2xy \qquad ...(1)$$

Separating the variables,

$$\frac{1}{y}\,dy = \frac{2x}{x+1}\,dx$$

Integrating both sides, we get

$$\int \frac{1}{y}\,dy = \int \frac{2x}{x+1}\,dx$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = 2\int \frac{x}{x+1}\,dx$$

$$\Rightarrow \qquad \int \frac{1}{y}\,dy = 2\int \frac{x+1-1}{x+1}\,dx$$

[Add and subtract 1 to the numerator in R.H.S.]

$$\Rightarrow \qquad \log|y| = 2\int\left(1-\frac{1}{x+1}\right)dx$$

$$\Rightarrow \qquad \log|y| = 2[x - \log|x+1|] + C$$

which is the required solution.

(*vii*) The given equation is

$$\frac{dy}{dx} = 1 + x + y + xy \qquad \text{...(1)}$$

Please try yourself.

[**Hint :** See part (*iii*) of the same example.] $\left[\textbf{Ans. } \log|1+y| = x + \frac{x^2}{2} + C\right]$

Example 26. *Solve the following differential equations*

(*i*) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

(*ii*) $(1+x)(1+y^2)dx + (1+y)(1+x^2)dy = 0$

(*iii*) $(y+xy)dx + (x-xy^2)dy = 0$

(*iv*) $(y^3+1)(e^x + xe^x)dx - xe^x y^2 dy = 0$

(*v*) $\cos x\,(1+\cos y)dx - \sin y\,(1+\sin x)dy = 0$

(*vi*) $\dfrac{dy}{dx} = y^2 \tan ax$ (*vii*) $\dfrac{dy}{dx} = (1+x)(1+y^2)$

(*viii*) $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0.$

Solution. (*i*) The given equation is

$$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0 \qquad \text{...(1)}$$

Separating the variables,

$$\sec^2 x \tan y\, dx = -\sec^2 y \tan x\, dy$$

$$\Rightarrow \qquad \frac{\sec^2 x}{\tan x}\,dx = -\frac{\sec^2 y}{\tan y}\,dy$$

Integrating both sides, we get

$$\int \frac{\sec^2 x}{\tan x}\,dx = -\int \frac{\sec^2 y}{\tan y}\,dy \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$$

$$\Rightarrow \qquad \log|\tan x| = -\log|\tan y| + \log C$$

$$\Rightarrow \qquad \log|\tan x| + \log|\tan y| = \log C \qquad [\because \log m + \log n = \log mn]$$

$$\Rightarrow \qquad \log|\tan x \tan y| = \log C$$

$$\Rightarrow \qquad |\tan x \tan y| = C$$

which is the required solution.

(*ii*) The given equation is

$$(1+x)(1+y^2)dx + (1+y)(1+x^2)dy = 0 \qquad \text{...(1)}$$

$$\Rightarrow \qquad (1+x)(1+y^2)dx = -(1+y)(1+x^2)dy$$

(*v*) The given equation is

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y} \quad ...(1)$$

Separating the variables,

$$(\sin y + y\cos y)dy = x(2\log x + 1)dx$$

Integrating both sides, we get

$$\int(\sin y + y\cos y)\,dy = \int(2x\log x + x)\,dx$$

$$\Rightarrow \quad \int \sin y\,dy + \int \underset{\mathrm{I}}{y}\,\underset{\mathrm{II}}{\cos y}\,dy = 2\int \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{\log x}\,dx + \int x\,dx \qquad \text{[Integrating by parts]}$$

$$\Rightarrow \quad -\cos y + y\sin y - \int 1.\sin y\,dy = 2\left[\log x.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}\,dx\right] + \frac{x^2}{2} + C$$

$$\Rightarrow \quad -\cos y + y\sin y - (-\cos y) = x^2\log x - \int x\,dx + \frac{x^2}{2} + C$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y = x^2\log x - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = x^2\log x + C$$

which is the required solution.

(*vi*) The given equation is

$$x(1 + y^2)dx + y(1 + x^2)dy = 0$$

Separating the variables,

$$\frac{x}{1+x^2}\,dx + \frac{y}{1+y^2}\,dy = 0$$

Integrating both sides, we get

$$\int \frac{x}{1+x^2}\,dx + \int \frac{y}{1+y^2}\,dy = C$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{2x}{1+x^2}\,dx + \frac{1}{2}\int \frac{2y}{1+y^2}\,dy = C \qquad \text{[Multiply and divided by 2]}$$

$$\Rightarrow \quad \frac{1}{2}\log|1 + x^2| + \frac{1}{2}\log|1 + y^2| = C \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$$

$$\Rightarrow \quad \log|1 + x^2| + \log|1 + y^2| = 2C$$

$$\Rightarrow \quad \log|(1 + x^2)(1 + y^2)| = 2C \qquad [\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad \log|(1 + x^2)(1 + y^2)| = \log C_1 \qquad [\text{where } \log C_1 = 2C]$$

$$|(1 + x^2)(1 + y^2)| = C_1$$

which is the required solution.

Example 28. *Solve the following differential equations :*

(*i*) $(e^y + 1) \cos x\, dx + e^y \sin x\, dy = 0$ (*ii*) $x \cos^2 y\, dx = y \cos^2 x\, dy$

(*iii*) $(1 + x^2)dy = xy\, dx$ (*iv*) $\dfrac{dy}{dx} = \dfrac{xe^x \log x + e^x}{x \cos y}$

(*v*) $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$ (*vi*) $x \log x\, dy - y dx = 0$

(*vii*) $\log \dfrac{dy}{dx} = ax + by.$

Solution. (*i*) The given equation is

$$(e^y + 1) \cos x\, dx + e^y \sin x\, dy = 0 \qquad ...(1)$$

$$\Rightarrow \quad (e^y + 1) \cos x\, dx = - e^y \sin x\, dy$$

Separating the variables,

$$\frac{\cos x}{\sin x}\, dx = -\frac{e^y}{e^y+1}\, dy$$

Integrating both sides, we get

$$\int \frac{\cos x}{\sin x}\, dx = -\int \frac{e^y}{e^y+1}\, dy$$

$$\Rightarrow \quad \log|\sin x| = -\log|e^y + 1| + \log C \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)|\right]$$

$$\Rightarrow \quad \log|e^y + 1| + \log|\sin x| = \log C$$

$$\Rightarrow \quad \log|(e^y + 1) \sin x| = \log C \qquad [\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad |(e^y + 1) \sin x| = C$$

which is the required solution.

(*ii*) The given equation is

$$x \cos^2 y\, dx = y \cos^2 x\, dy \qquad ...(1)$$

Separating the variables,

$$\frac{x}{\cos^2 x}\, dx = \frac{y}{\cos^2 y}\, dy$$

$$\Rightarrow \quad x \sec^2 x\, dx = y \sec^2 y\, dy$$

Integrating both sides, we get

$$\int \underset{\text{I}}{x} \underset{\text{II}}{\sec^2 x}\, dx = \int \underset{\text{I}}{y} \underset{\text{II}}{\sec^2 y}\, dy \qquad \text{[Integrating by parts]}$$

$$\Rightarrow \quad x.\tan x - \int 1.\tan x.dx = y.\tan y - \int 1.\tan y\, dy + C$$

$$\Rightarrow \quad x \tan x - (-\log|\cos x|) = y \tan y - (-\log|\cos y|) + C$$

$$\Rightarrow \quad x \tan x + \log|\cos x| = y \tan y + \log|\cos y| + C$$

which is the required solution.

(*iii*) The given equation is

$$(1 + x^2)dy = xy\,dx \qquad ...(1)$$

Separating the variables,

$$\frac{1}{y}\,dy = \frac{x}{1+x^2}\,dx$$

Integrating both sides, we get

$$\int \frac{1}{y}\,dy = \int \frac{x}{1+x^2}\,dx$$

$$\Rightarrow \quad \log|y| = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx \qquad \text{[Multiply and divide the R.H.S. by 2]}$$

$$\Rightarrow \quad \log|y| = \frac{1}{2}\log|1+x^2| + C \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$$

$$\Rightarrow \quad 2\log|y| = \log|1+x^2| + 2C$$

$$\Rightarrow \quad \log y^2 - \log(1+x^2) = 2C$$

$$\Rightarrow \quad \log\frac{y^2}{1+x^2} = 2C$$

$$\Rightarrow \quad \frac{y^2}{1+x^2} = e^{2C}$$

$$\Rightarrow \quad \frac{y^2}{1+x^2} = C_1 \qquad [\because C_1 = e^{2C}]$$

$$\Rightarrow \quad y^2 = C_1(1+x^2)$$

which is the required solution.

(*iv*) The given equation is

$$\frac{dy}{dx} = \frac{xe^x \log x + e^x}{x\cos y} \qquad ...(1)$$

Separating the variables,

$$\cos y\,dy = \frac{e^x(x\log x + 1)}{x}\,dx$$

Integrating both sides, we get

$$\int \cos y\,dy = \int e^x\left(\log x + \frac{1}{x}\right)dx$$

$$\Rightarrow \quad \sin y = e^x \log x + C \qquad \left[\because \int e^x[f(x) + f'(x)]dx = e^x f(x)\right]$$

which is the required solution.

About the Book

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