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2nd EDITION

Integral Calculus for JEE Main & Advanced

Indefinite Integration

Definite Integration

Area Under the Curve

Differential Equations

Vinay Kumar

McGRAW HILL EDUCATION WINNING WE EDGE SERIES

Integral Calculus *for*

JEE Main & Advanced

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Second Edition

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PREFACE

This book is meant for students who aspire to join the Indian Institute of Technologies (IITs) and various other engineering institutes through the JEE Main and Advanced examinations. The content has been devised to cover the syllabi of JEE and other engineering entrance examinations on the topic *Integral Calculus*. The book will serve as a text book as well as practice problem book for these competitive examinations.

As a tutor with more than thirteen years of teaching this topic in the coaching institutes of Kota, I have realised the need for a comprehensive textbook in this subject.

I am grateful to McGraw-Hill Education for providing me an opportunity to translate my years of teaching experience into a comprehensive textbook on this subject.

This book will help to develop a deep understanding of Integral Calculus through concise theory and problem solving. The detailed table of contents will enable teachers and students to easily access their topics of interest.

Each chapter is divided into several segments. Each segment contains theory with illustrative examples. It is followed by Concept Problems and Practice Problems, which will help students assess the basic concepts. At the end of the theory portion, a collection of Target Problems have been given to develop mastery over the chapter.

The problems for JEE Advanced have been clearly indicated in each chapter.

The collection of objective type questions will help in a thorough revision of the chapter. The Review Exercises contain problems of a moderate level while the Target Exercises will assess the students' ability to solve tougher problems. For teachers, this book could be quite helpful as it provides numerous problems graded by difficulty level which can be given to students as assignments.

I am thankful to all teachers who have motivated me and have given their valuable recommendations. I thank my family for their whole-hearted support in writing this book. I specially thank Mr. Devendra Kumar and Mr. S. Suman for their co-operation in bringing this book.

Suggestions for improvement are always welcomed and shall be gratefully acknowledged.

Vinay Kumar

CONTENT

$$\int \cos x \cos 2x\,dx = \frac{1}{2}\int 2\cos x \cos 2x\,dx$$
$$= \frac{1}{2}\int (\cos 3x + \cos x)\,dx$$
$$= \frac{1}{2}\left(\frac{\sin 3x}{3} + \frac{\sin x}{1}\right) + C$$
$$\int \sin mx \sin nx\,dx$$
$$= \frac{1}{2}\left[\int \cos(m-n)x\,dx - \int \cos(m+n)x\,dx\right]$$
$$= \begin{cases} \dfrac{\sin(m-n)x}{2(m-n)} - \dfrac{\sin(m+n)x}{2(m+n)} & \text{if } m^2 \neq n^2 \\ \pm\left(\dfrac{x}{2} - \dfrac{\sin 2nx}{4n}\right) & \text{if } m = \pm n \end{cases}$$

CHAPTER

1

INDEFINITE INTEGRATION

1.1 INTRODUCTION

Integral calculus is to find a function of a single variable when its derivative f(x) and one of its values are known. The process of determining the function has two steps. The first is to find a formula that gives us all the functions that could possibly have f as a derivative. These functions are the so-called antiderivatives of f, and the formula that gives them all is called the indefinite integral of f. The second step is to use the known function value to select the particular antiderivative we want from the indefinite integral.

A physicist who knows the velocity of a particle might wish to know its position at a given time. Suppose an engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period. In each case, the problem is to find a function F whose derivative is a known function f. If such a function F exists, it is called an antiderivative of f.

There are two distinct ways in which we may approach the problem of integration. In the first way we regard integration as the reverse of differentiation; this is the approach via the indefinite integral. In the second way we regard integration as the limit of an algebraic summation; this is the approach via the definite integral. For the moment we shall consider only the first and we begin with the formal definition.

Indefinite integration is the process which is the inverse of differentiation, and the objective can be stated as follows : given a function y = f(x) of a single real variable x, there is a definite process whereby we can find (if it exists) the function F(x) such that

$$\frac{dF(x)}{dx} = f(x)$$

Definition

A function F(x) is called the **antiderivative (primitive)** of the function f(x) on the interval [a, b] if at all points of the interval F'(x) = f(x).

Find the antiderivative of the function $f(x) = x^2$.

From the definition of an antiderivative it follows that the function $f(x) = \frac{x^3}{3}$ is an antiderivative, since $\left(\frac{x^3}{3}\right)' = x^2$.

It is easy to see that if for the given function f(x) there exists an antiderivative, then this antiderivative is not the only one. In the foregoing example, we could take the following functions as antiderivatives:

$$F(x) = \frac{x^3}{3} + 1,\quad F(x) = \frac{x^3}{3} - 7 \text{ or,}$$

$$\text{generally, } F(x) = \frac{x^3}{3} + C$$

(where C is an arbitrary constant), since

$$\left(\frac{x^3}{3}+C\right)' = x^2$$

On the other hand, it may be proved that functions of the form $\frac{x^3}{3}+C$ exhaust all antiderivatives of the function x^2. This is a consequence of the following theorem. The example shows that a function has infinitely many antiderivatives. We are going to show how to find all antiderivatives of a given function, knowing one of them.

Constant Difference Theorem

If $F_1(x)$ and $F_2(x)$ are two antiderivatives of a function f(x) on an interval [a, b], then the difference between them is a constant.

Proof By virtue of the definition of an anti-derivative we have $\left.\begin{matrix} F_1'(x) = f(x) \\ F_2'(x) = f(x) \end{matrix}\right\}$ ...(1)

for any value of x on the interval [a, b].

Let us put $F_1(x) - F_2(x) = g(x)$ ...(2)

Then by (1) we have

$$F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

or $g'(x) = [F_1(x) - F_2(x)]' = 0$

for any value of x on the interval [a, b].

But from $g'(x) = 0$ it follows that g(x) is a constant. Indeed, let us apply the Lagrange's theorem to the function g(x), which, obviously, is continuous and differentiable on the interval [a, b]. No matter what the point x on the interval [a, b], we have, by virtue of the Lagrange's theorem, $g(x) - g(a) = (x - a)\, g'(c)$ where $a < c < x$.

Since $g'(c) = 0$,

$g(x) - g(a) = 0$

or, $g(x) = g(a)$ ...(3)

Thus, the function g(x) at any point x of the interval [a, b] retains the value g(a), and this means that the function g(x) is constant on [a, b]. Denoting the constant g(a) by C, we get from (2) and (3),

$$F_1(x) - F_2(x) = C.$$

Thus, if for the function F_1 and F_2 there exists an interval [a, b] such that $F_1'(x) = F_2'(x)$ in [a, b], then there exists a number C such that $F_1(x) = F_2(x) + C$ in [a, b].

From this theorem it follows that if for a given function f(x) some one antiderivative F(x) is found, then any other antiderivative of f(x) has the form F(x) + C, where C is an arbitrary constant.

For example, if $F'(x) = 6x^2 + 2x$, then

$F(x) = 2x^3 + x^2 + C,$

for some number C. There are no other antiderivatives of $6x^2 + 2x$.

If $F'(x) = 4x - 3$ and $F(1) = 3$,

then $F(x) = 2x^2 - 3x + C$, for some number C.

Since, $F(1) = 2 - 3 + C = 3$, we have $C = 4$.

Thus $F(x) = 2x^2 - 3x + 4$,

and there is just one function F satisfying the given conditions.

The equation $F'(x) = 4x - 3$ is an example of a differential equation, and the condition $F(1) = 3$ is called a boundary condition of F.

Given F' and a **boundary condition** on F, there is a unique antiderivative F of F' satisfying the given boundary condition. This function F is called the solution of the given **differential equation**.

Definition

If the function F(x) is an antiderivative of f(x), then the expression F(x) + C is the **indefinite integral** of the function f(x) and is denoted by the symbol $\int f(x)\,dx$. It is the set of all antiderivatives of f(x). Thus, by definition

$$\int f(x)\,dx = F(x) + C,$$

if $F'(x) = f(x)$.

Here, the function f(x) is called the **integrand**, f(x) dx is the **element of integration** (the expression under the integral sign), the variable x the **variable of integration**, and $\int$ is the **integral sign**.

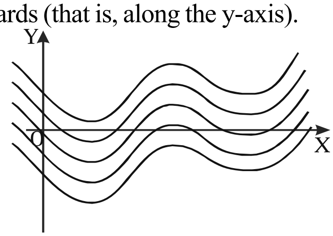
Thus, an indefinite integral is a family of functions $y = F(x) + C$

(one antiderivative for each value of the constant C).

The symbol $\int$ for the integral was introduced by Leibnitz. This elongated S stood for a "sum" in his notation.

The graph of an antiderivative of a function f(x) is called an **integral curve** of the function y = f(x). It is obvious that any integral curve can be obtained by a translation (parallel displacement) of any other integral curve in the vertical direction.

From the geometrical point of view, an indefinite integral is a collection (family) of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards (that is, along the y-axis).



Existence of Antiderivative

A natural question arises : do antiderivatives (and, hence, indefinite integrals) exist for every function f(x) ? The answer is no.

Let us find an antiderivative of a continuous function

$$f(x)=\begin{cases} x+1 & , \quad 0\le x<1 \\ 3-x^2 & , \quad 1\le x\le 2 \end{cases}, \text{ on the interval } [0, 2].$$

On integrating both the formulae we get

$$F(x)=\begin{cases} \dfrac{x^2}{2}+x+C_1 , & 0\le x<1 \\ 3x-\dfrac{x^3}{3}+C_2 , & 1\le x\le 2 \end{cases}$$

To ensure that F '(1) = f(1) = 2, we first make F(x) continuous :

$$F(1^-) = F(1^+)$$

$$\Rightarrow \frac{1}{2}+1+C_1=3-\frac{1}{3}+C_2$$

$$\Rightarrow C_1+\frac{3}{2}=\frac{8}{3}+C_2$$

$$\Rightarrow C_1=\frac{7}{6}+C_2$$

$$\text{Now, } F(x)=\begin{cases} \dfrac{x^2}{2}+x+\dfrac{7}{6}+C_2 , & 0\le x<1 \\ 3x-\dfrac{x^3}{3}+C_2 , & 1\le x\le 2 \end{cases}$$

We further observe that $F'(1^+) = F'(1^-)$.

Thus, we obtain the antiderivative F(x) of the function f(x).

Let us note, however, without proof, that if a function f(x) is continuous on an interval [a, b], then this function has an antiderivative (and, hence, there is also an indefinite integral).

Now, let us find an antiderivative of a discontinuous function $f(x)=\dfrac{1}{x^2}$.

$F(x)=-\dfrac{1}{x}$ is an antiderivative of $f(x)=\dfrac{1}{x^2}$ on $(-\infty, 0)$ and on $(0, \infty)$. However, it is not an an antiderivative on [–1, 1] since the interval includes 0 where F′(x) does not exist.

We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval. Thus, we write

$$\int \frac{1}{x^2}dx=-\frac{1}{x}+C$$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$. This is true despite the fact that the general antiderivative of the function $f(x) = 1/x^2, x \ne 0$, is

$$F(x)=\begin{cases} -\dfrac{1}{x}+C_1 & \text{if } x<0 \\ -\dfrac{1}{x}+C_2 & \text{if } x>0 \end{cases}.$$

Example 1. Prove that y = sgn x does not have an antiderivative on any interval which contains 0.

Solution $y=\text{sgn}(x)=\begin{cases} -1 & , \quad x<0 \\ 0 & , \quad x=0 \\ 1 & , \quad x>0 \end{cases}$

Here we present two antiderivatives :

$$f(x)=\begin{cases} -x+c_1 , & x<0 \\ x+c_2 , & x>0 \end{cases}$$

$$g(x)=\begin{cases} -x+c & , \quad x<0 \\ x+c & , \quad x\ge 0 \end{cases}$$

f(x) is discontinuous at x = 0, if $c_1 \ne c_2$. g(x) is continuous at x = 0. None of these functions are differentiable at x = 0 i.e. we are unable to ensure that f'(0) or g'(0) is 0. Hence, y = sgn x does not have an antiderivative on any interval which contains 0. However, the function has an antiderivative (either f or g) on any interval which does not contain 0.

Indefinite Integration

The finding of an antiderivative of a given function f(x) is called indefinite integration of the function f(x). Thus, the problem of indefinite integration is to find the function F(x) whose derivative is the given function f(x) i.e. given the equation $\dfrac{dF(x)}{dx}=f(x)$, we have to find the function F(x).

The process of integration is not so simple. Although rules may be given which cover this operation with various types of simple functions, indefinite integration is a tentative process, and indefinite integrals are found by trial.

This chapter is devoted to working out methods by means of which we can find antiderivatives (and indefinite integrals) for certain classes of elementary functions.

Note: Though the derivative of an elementary function is always an elementary function, the antiderivative of the elementary function may not prove to be representable by a finite number of elementary functions. We shall return to this question at the end of the chapter.

1.2 ELEMENTARY INTEGRALS

Rules of integration

Assume that f and g are functions with antiderivatives $\int f(x)$ and $\int g(x)\,dx$. Then the following hold :

(a) $\int kf(x)\,dx = k\int f(x)\,dx$ for any constant k.

(b) $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$,

(c) $\int (f(x)-g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$.

Proof

(a) It is only required to show that the derivative of $k\int f(x)dx$ is cf(x). The differentiation shows :

$$\frac{d}{dx}\left(k\int f(x)\,dx\right) = k\frac{d}{dx}\left(\int f(x)\,dx\right) = kf(x)$$

A constant moves past the derivative symbol.

(b) We show that the derivative of $\int f(x) + \int g(x)\,dx$ is f(x) + g(x) : $\frac{d}{dx}\left(\int f(x)\,dx + \int g(x)\,dx\right)$

$$= \frac{d}{dx}\left(\int f(x)\,dx\right) + \frac{d}{dx}\left(\int g(x)\,dx\right) = f(x) + g(x).$$

(c) The proof of property (c) is similar to that of property (b).

The last two parts of theorem extend to any finite number of functions. For instance,

$$\int (f(x) - g(x) + h(x))\,dx$$

$$= \int f(x)\,dx - \int g(x)\,dx + \int h(x)\,dx.$$

Note:

(i) We have $\int (1+x)dx = \int 1dx + \int xdx = (x + C_1) + \left(\frac{x^2}{2} + C_2\right) = x + \frac{x^2}{2} + C_1 + C_2$

Here, we have two arbitrary constants when one will suffice. This kind of problem is caused by introducing constants of integration too soon and can be avoided by inserting the constant of integration in the final result, rather than in intermediate computations.

(ii) $\int x^2e^x\,dx = x^2e^x - 2(xe^x - e^x + C)$

$= x^2e^x - 2xe^x + 2e^x - 2C = (x^2 - 2x + 2)\,e^x + C'$

where $C' = 2C$

When arbitrary constants are algebraically combined with other numbers, the final algebraic expression is just as arbitrary.

(iii) Note the following representations :

(a) $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

(b) $d\left(\int f(x)dx\right) = f(x)\,dx$

(c) $\int f'(x)dx = f(x) + C$

(d) $\int df(x) = f(x) + C$

Elementary formulae

We begin by listing a number of standard forms, that is to say formulae for integrals which we shall be free to quote once we have listed them. Each formula is of the the type

$$\int f(x)dx = F(x)$$

and its validity can be etablished by showing that

$$\frac{d}{dx}F(x) = f(x)$$

These formulae should be known and quoted, without proof, whenever needed.

(i) $\int dx = kx + C$, where k is a constant

(ii) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where n ¹ 1

(iii) $\int \frac{1}{x}dx = \ln|x| + C$

(iv) $\int a^x dx = \frac{a^x}{\ell n\ a} + C$, where $a > 0$

(v) $\int e^x dx = e^x + C$

(vi) $\int \sin x\,dx = -\cos x + C$

(vii) $\int \cos x\,dx = \sin x + C$

(viii) $\int \sec^2 x\,dx = \tan x + C$

(ix) $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$

(x) $\int \sec x \tan x\, dx = \sec x + C$

(xi) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$

(xi) $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$

(xii) $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$

(xiii) $\int \frac{1}{|x|\sqrt{x^2-1}}\, dx = \sec^{-1} x + C$

We have some additional results which will be established later :

(xiv) $\int \tan x\, dx = \frac{1}{a} \ln \sec|x| + C$

(xv) $\int \cot x\, dx = \frac{1}{a} \ln \sin|x| + C$

(xvi) $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

$= \ln \tan\left|\frac{\pi}{4} + \frac{x}{2}\right| + C$

$= -\ln|\sec x - \tan x| + C$

(xvii) $\int \operatorname{cosec} x\, dx = \ell n|\operatorname{cosec} x - \cot x| + c$

$= \ell n \left|\tan\frac{x}{2}\right| + c$

$= -\ell n|\operatorname{cosec} x + \cot x|$

Here, we must analyse carefully the formula

$$\int \frac{1}{x} dx = \ell n|x| + C$$

We have two cases :

(i) Let $x > 0$, then $|x| = x$ and the formula attains the form $\int \frac{dx}{x} = \ell n x + C$

Differentiating, we get $(\ell n x + C)' = \frac{1}{x}$.

(ii) Let $x < 0$, then $|x| = -x$ and the formula has the form

$$\int \frac{dx}{x} = \ell n(-x) + C$$

Differentiating, we have $(\ell n(-x) + C)' = \frac{-1}{-x} = \frac{1}{x}$

Since ln x is real when $x > 0$ and $\frac{d}{dx}(\ell n\, x) = \frac{1}{x}$,

so $\int \frac{1}{x} dx = \ln x + C$ is defined for $x > 0$.

When $x < 0$, i.e. $-x > 0$, $\frac{d}{dx}\ell n(-x) = \frac{-1}{-x} = \frac{1}{x}$.

Therefore when $x < 0$, $\int \frac{1}{x} dx = \ln(-x) + C$.

Hence, both these results will be included if we write

$$\int \frac{1}{x} dx = \ell n|x| + C.$$

In the formula and examples where integrals of this type occurs, i.e., where the value of an integral involves the logarithm of a function and the function may become negative for some values of the variable of the function, the absolute value sign enclosing the function should be given, but it has generally been omitted, though it is always understood to be present and it should be supplied by the students.

Direct integration

Direct integration is such a method of computing integrals in which they are reduced to the elementary formulae by applying to them the principal properties of indefinite integrals.

For example :

$$\int \frac{dx}{x+3} = \ln|x+3| + C$$

$$\int 3^x.4^{2x}\, dx = \int (3.16)^x\, dx = \int 48^x\, dx = \frac{48^x}{\ell n 48} + C$$

$$\int 4x^5\, dx = \frac{4}{6}x^6 + C = \frac{2}{3}x^6 + C.$$

Example 1. Evaluate

$$\int \left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right) dx$$

Solution $\int \left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right) dx$

$$= \int x^3\, dx + \int 5x^2\, dx - \int 4 dx + \int \frac{7}{x} dx + \int \frac{2}{\sqrt{x}} dx$$

$$= \int x^3\,dx + 5.\int x^2\,dx - 4.\int 1.dx$$

$$+ 7.\int \frac{1}{x}dx + 2.\int x^{-1/2}dx$$

$$= \frac{x^4}{4} + 5\frac{x^3}{3} - 4x + 7\ln|x| + 2\left(\frac{x^{1/2}}{1/2}\right) + C$$

$$= \frac{x^4}{4} + \frac{5}{3}x^3 - 4x + 7\ln|x| + 4\sqrt{x} + C$$

Example 2. Evaluate $\int \frac{2^x + 3^x}{5^x}\,dx$

Solution $\int \frac{2^x + 3^x}{5^x}\,dx$

$$= \int \left(\frac{2^x}{5^x} + \frac{3^x}{5^x}\right)dx = \int \left[\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right]dx$$

$$= \frac{(2/5)^x}{\ln 2/5} + \frac{(3/5)^x}{\ln 3/5} + C$$

Example 3. Evaluate

(i) $\int 5^{\log_e x}\,dx$

(ii) $\int 2^{\log_4 x}\,dx$

Solution

(i) $\int 5^{\log_e x}\,dx = \int x^{\log_e 5}dx = \frac{x^{\log_e 5 + 1}}{\log_e 5 + 1} + C$

(ii) $\int 2^{\log_4 x}\,dx = \int 2^{\log_{2^2} x}\,dx = \int 2^{1/2\,\log_2 x}\,dx$

$$= \int 2^{\log_2 \sqrt{x}}\,dx = \int \sqrt{x}\,dx = \frac{x^{3/2}}{3/2} + C$$

$$\therefore \int 2^{\log_4 x}\,dx = \frac{2}{3}x^{3/2} + C$$

Example 4. Evaluate $\int e^{x\ln a} + e^{a\ln x} + e^{a\ln a}\,dx$

Solution $\int e^{x\ln a} + e^{a\ln x} + e^{a\ln a}\,dx$

$$= \int e^{\ln a^x} + e^{\ln x^a} + e^{\ln a^a}\,dx$$

$$= \int (a^x + x^a + a^a)\,dx$$

$$= \int a^x\,dx + \int x^a\,dx + \int a^a\,dx$$

$$= \frac{a^x}{\ln a} + \frac{x^{a+1}}{a+1} + a^a.x + C.$$

Extension of elementary formulae

If $\int f(x)dx = F(x)$ and a, b are constants, then

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) \quad ...(1)$$

We prove (1) by observing that, when $y = ax + b$,

$$\frac{1}{a}\frac{d}{dx}F(y) = \frac{1}{a}\frac{d}{dy}F(y).\frac{dy}{dx} = \frac{1}{a}f(y).a = f(ax+b).$$

For example:

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \quad n \neq -1$

(ii) $\int \sin(5x+2)dx = -\frac{1}{5}\cos(5x+2) + C$

(iii) $\int \sec^2(3x+5)dx = \frac{1}{3}\tan(3x+5) + C$

(iv) $\int \sec(ax+b).\tan(ax+b)\,dx$

$$= \frac{1}{a}\sec(ax+b) + C$$

(v) $\int (\cos 7x + \sin(2x-6))\,dx$

$$= \frac{1}{7}\sin 7x - \frac{1}{2}\cos(2x-6) + C$$

(vi) $\int \frac{dx}{2x+1} = \frac{1}{2}\ln|2x+1| + C$

(vii) $\int \frac{x}{2x+1}dx = \frac{1}{2}\int \frac{2x}{2x+1}dx = \frac{1}{2}\int\left(1 - \frac{1}{2x+1}\right)dx$

$$= \frac{1}{2}\left\{x - \frac{1}{2}\ln|2x+1|\right\} + C$$

In this example, we break the function up into parts like 1 and $\frac{1}{2x+1}$ whose integrals we know from the list of elementary integrals.

(viii) $\int \frac{dx}{25+4x^2} = \int \frac{dx}{4\left(\frac{25}{4}+x^2\right)} = \frac{1}{4}\int \frac{dx}{\left(\frac{5}{2}\right)^2 + x^2}$

$$= \frac{1}{4}\cdot\frac{1}{\frac{5}{2}}\tan^{-1}\frac{x}{\frac{5}{2}}+C = \frac{1}{10}\tan^{-1}\frac{2x}{5}+C$$

(ix) $\int \frac{dx}{a^2+b^2x^2} = \int \frac{dx}{a^2+(bx)^2} = \frac{1}{ab}\tan^{-1}\frac{bx}{a}+C$

(x) $\int \frac{dx}{\sqrt{1-9x^2}} = \int \frac{dx}{\sqrt{9\left(\frac{1}{9}-x^2\right)}} = \int \frac{dx}{3\sqrt{\frac{1}{9}-x^2}}$

$$= \frac{1}{3}\frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2-x^2}} = \frac{1}{3}\sin^{-1}\frac{x}{\frac{1}{3}}+C = \frac{1}{3}\sin^{-1}3x+C$$

Note:

(i) $\int \frac{dx}{x^2}$ is a 'convenient form' for $\int \frac{1}{x^2}dx$ and such symbols are commonly used. Strictly, the first symbol has no meaning save as a shorthand for the second symbol; as the definition shows, there can be no question here of 'dividing dx by x^2'.

(ii) In the chapter of indefinite integration, the simplification of square root functions are done without much consideration to the sign of the expressions. However, this must be taken seriously in the chapter of definite integration. For example,

$$\int \sqrt{1+\sin 2x}\ dx$$

$$= \int \sqrt{(\sin x+\cos x)^2}\ dx$$

$$= \int (\sin x+\cos x)\,dx$$

$$= -\cos x + \sin x + C$$

A more elegant way of handling the situation is illustrated below :

$$\int \sqrt{1+\sin 2x}\ dx = \int \sqrt{(\sin x+\cos x)^2}\ dx$$

$$= \int |\sin x+\cos x|\ dx$$

$$= \text{sgn}(\sin x+\cos x)\cdot\int(\sin x+\cos x)\,dx$$

$$= \text{sgn}(\sin x+\cos x)\cdot\{-\cos x+\sin x\} + C.$$

(iii) We notice that the integral of $\frac{1}{|x|\sqrt{x^2-a^2}}$, where $a>0$ is $\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C$. But we would like to know the integral of $\frac{1}{x\sqrt{x^2-a^2}}$.

From the standard result we obtain,

$$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \begin{cases} \frac{1}{a}\sec^{-1}\frac{x}{a}+C & , x>a \\ -\frac{1}{a}\sec^{-1}\frac{x}{a}+C & , x<-a \end{cases}$$

CAUTION

$$\int_{-2}^{-2/\sqrt{3}} \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x\Big|_{-2}^{-2/\sqrt{3}}$$

$$= \left(\frac{5\pi}{6}-\frac{2\pi}{3}\right) = \frac{\pi}{6}$$

This is a wrong result since the integral of a negative function must be negative. This happened because the antiderivative used in the calculation is wrong.

$$\int_{-2}^{-2/\sqrt{3}} \frac{1}{x\sqrt{x^2-1}}dx = -\sec^{-1}x\Big|_{-2}^{-2/\sqrt{3}}$$

$$= -\left(\frac{5\pi}{6}-\frac{2\pi}{3}\right) = -\frac{\pi}{6}.$$

Alternatively, we have

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|+C$$

Thus,

$$\int_{-2/\sqrt{3}}^{-\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x|\Big|_{-2/\sqrt{3}}^{-\sqrt{2}} = \frac{\pi}{4}-\frac{\pi}{6} = \frac{\pi}{12}.$$

Determination of function

Example 5. Let f be a polynomial function such that for all real x, $f(x^2+1) = x^4+5x^2+3$ then find the primitive of $f(x)$ w.r.t. x.

Solution $f(x^2+1) = (x^2+1)^2+3x^2+2$

$= (x^2+1)^2+3(x^2+1)-1$

We replace x^2+1 by x

$\therefore\ f(x) = x^2+3x-1$

Now we integrate f(x) w.r.t. x :

$$\int f(x)dx = \int (x^2+3x-1)\,dx$$

$$= \frac{x^3}{3}+\frac{3x^2}{2}-x+C$$

Example 6. Given $f''(x) = \cos x$, $f'\left(\frac{3\pi}{2}\right) = e$ and $f(0) = 1$, then find $f(x)$.

Solution $f''(x) = \cos x$

Integrating w.r.t. x :

$f'(x) = \sin x + C_1$

Now, $f'\left(\frac{3\pi}{2}\right) = e$

$\Rightarrow \; e = -1 + C \qquad \Rightarrow C_1 = e + 1$

$f'(x) = \sin x + e + 1$

Integrating again w.r.t. x :

$f(x) = -\cos x + (e + 1)x + C_2$

$f(0) = 1 \quad \Rightarrow 1 = -1 + C_2 \quad \Rightarrow C_2 = 2$

$\therefore \; f(x) = (e + 1)x - \cos x + 2$

Example 7. A curve $y = f(x)$ such that $f''(x) = 4x$ at each point (x, y) on it and crosses the x-axis at (–2, 0) at an angle of 45°. Find the value of $f(1)$.

Solution $f''(x) = 4x \Rightarrow f'(x) = 2x^2 + c$

$f'(-2) = \tan 45° = 1 = 8 + c \Rightarrow c = -7$

Now $\quad f'(x) = 2x^2 - 7$

$\Rightarrow \; f(x) = \frac{2}{3}x^3 - 7x + d$

Since, $\quad f(-2) = 0, \; -\frac{16}{3} + 14 + d = 0$

$\Rightarrow \; d = -\frac{26}{3}$

$\therefore \; f(x) = \frac{1}{3}[2x^3 - 21x - 26]$

And, thus $\quad f(1) = -\frac{45}{3} = -15.$

Example 8. Find the antiderivatives of the function $y = x + 2$ which touch the curve $y = x^2$.

Solution Since the function $y = x + 2$ is a derivative of any of its antiderivatives, it follows, that the equation for finding the abscissa of the point of tangency has the form $2x = x + 2$.

The root of this equation is x = 2. The value of the function $y = x^2$ at the point x = 2 is equal to 4. Consequently, among all the antiderivatives of the functions $y = x + 2$,

i.e. the function $F(x) = \frac{1}{2}x^2 + 2x + C$,

we must find that whose graph passes through the point P(2, 4). The constant C can be found from the condition $F(2) = 4$:

$\Rightarrow \; \frac{1}{2}.4 + 2.2 + C = 4 \Rightarrow C = -2.$

$\Rightarrow \; F(x) = \frac{1}{2}x^2 + 2x - 2.$

Example 9. Deduce the expansion for $\tan^{-1}x$ from the formula $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^5 + \ldots$ when $x < 1$,

Solution We have

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^5 + \ldots..$

Integrating both sides w.r.t. x, we have

$$\tan^{-1}x = \int \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

No constant is added since $\tan^{-1}x$ vanishes with x.

Concept Problems A

1. Find an antiderivative of the function :
   (i) $f(x) = 1 - 4x + 9x^2$
   (ii) $f(x) = x\sqrt{x} + \sqrt{x} - 5$
   (iii) $f(x) = \sqrt[4]{x + 1}$
   (iv) $f(x) = (x/2 - 7)^3$.

2. (a) Show that $F(x) = \frac{1}{6}(3x + 4)^2$ and $G(x) = \frac{3}{2}x^2 + 4x$ differ by a constant by showing that they are antiderivatives of the same function.
   (b) Find the constant C such that $F(x) - G(x) = C$ by evaluating F(x) and G(x) at a particular value of x.
   (c) Check your answer in part (b) by simplifying the expression $F(x) - G(x)$ algebraically.

3. Let F and G be defined piecewise as

$$F(x) = \begin{cases} x, & x > 0 \\ -x & x < 0 \end{cases} \text{ and } G(x) = \begin{cases} x + 2, & x > 0 \\ -x + 3, & x < 0 \end{cases}$$

   (a) Show that F and G have the same derivative.
   (b) Show that $G(x) \neq F(x) + C$ for any constant C.
   (c) Do parts (a) and (b) violate the constant difference theorem ? Explain.

4. (a) Graph some representative integral curves of the function $f(x) = e^x/2$.
   (b) Find an equation for the integral curve that passes through the point (0, 1).

5. Prove that the following functions donot have an antiderivative on any interval which contains 0.

   (i) $y = \begin{cases} x + 1, & x > 0 \\ x, & x \le 0 \end{cases}$ (ii) $y = \begin{cases} x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$

6. Find the function satisfying the given equation and the boundary conditions.
(i) $F'(x)=3(x+2)^3, F(0)=0$
(ii) $s''(t)=8, s'(0)=7, s(-1)=-3$
(iii) $f'(x)=x^2+5, f(0)=-1$.

7. If $f''(x)=10$ and $f'(1)=6$ and $f(1)=4$ then find $f(-1)$.

8. Evaluate the following integrals :
(i) $\int 2^x . e^x\,dx$ (ii) $\int \frac{e^{2x}-1}{e^x}dx$
(iii) $\int e^{ax+b}\,dx$ (iv) $\int a^{px+q}\,dx$

9. Evaluate the following integrals :
(i) $\int e^{\ell n\sqrt{x}}\,dx$ (ii) $\int e^{-\ell n x^2}\,dx$
(iii) $\int \ell n\left(\frac{1}{e^x}\right)dx$ (iv) $\int e^{m\ln x}\,dx$

10. Evaluate the following integrals :
(i) $\int a^{mx}.b^{nx}\,dx$ (ii) $\int (2x+3x)2\,dx$
(iii) $\int \frac{e^{3x}+e^{5x}}{e^x+e^{-x}}dx$ (iv) $\int e^{\ell n 2+\ell n x}\,dx$

11. Evaluate the following integrals :
(i) $\int \frac{dx}{2\cdot\sqrt{x}}$ (ii) $\int (ax+b)^n\,dx$
(iii) $\int \frac{dx}{3-2x}$ (iv) $\int \frac{(1+x)^3}{x}dx$

12. Evaluate the following integrals :
(i) $\int \frac{x\ dx}{a+bx}dx$ (ii) $\int \frac{2x-1}{x-2}dx$
(iii) $\int \frac{x^2}{1+x^2}dx$ (iv) $\int \frac{x^4}{1+x^2}dx$

13. Evaluate the following integrals :
(i) $\int \cos x^{o}\,dx$ (ii) $\int \sec^2(ax+b)\,dx$
(iii) $\int \cot^2 x\,dx$ (iv) $\int \frac{1-\cos x}{1+\cos x}dx$

14. Evaluate the following integrals :
(i) $\int \frac{dx}{x\sqrt{25x^2-2}}$ (ii) $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$
(iii) $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}}$

15. Evaluate the following integrals :
(i) $\int \frac{\cos\left(1-\frac{x}{2}\right)}{\sin^2\left(1-\frac{x}{2}\right)}dx$ (ii) $\int \frac{1}{\sqrt{\left(3-\frac{x^2}{4}\right)}}dx$
(iii) $\int \frac{1}{3+(2-3x)^2}dx$

Practice Problems A

16. Integrate the following functions :
(i) $\frac{1}{3}\sin\frac{2-5x}{3}+\frac{2}{5}\cos\frac{3-2x}{5}+\frac{4}{1+2x}+\frac{1}{\sqrt{(3x+1}}$
(ii) $(7-4x)^3+\frac{7}{(3-7x)}-4\operatorname{cosec}^2(4x+3)+\frac{2}{16+9x^2}$

17. Show that
$$I=\int \frac{dx}{4\sin x-5\cos x}=\frac{1}{\sqrt{41}}\ell n\left|\tan\left(\frac{x}{2}-\frac{\alpha}{2}\right)\right|+C$$
where $\alpha=\tan^{-1}\frac{5}{4}$.

18. Prove that
$$\int \frac{1-x^{2m}}{1-x^2}dx=x+\frac{1}{3}x^3+\frac{1}{5}x^5+\ldots+\frac{1}{2m-1}x^{2m-1}.$$

19. Given the continuous periodic function f(x), $x\in R$. Can we assert that the antiderivative of that function is a periodic function ?

20. Show that $\int |x|dx = \frac{x|x|}{2}+C$

21. Find all the antiderivatives of the function $f(x)=3/x$ whose graphs touch the curve $y=x^3$.

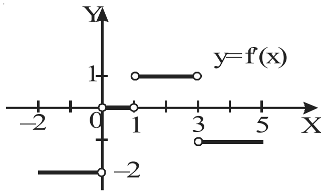
22. In each case, find a function f satisfying the given conditions.
(a) $f'(x^2)=1/x$ for $x>0$, $f(1)=1$.
(b) $f'(\sin^2 x)=\cos^2 x$ for all x, $f(1)=1$.
(c) $f'(\sin x)=\cos^2 x$ for all x, $f(1)=1$.
(d) $f'(\ln x)=\begin{cases}1 & \text{for } 0<x\le 1,\\ x & \text{for } x>1,\end{cases}$, $f(0)=0$.

23. Use the following information to graph the function f over the closed interval $[-2, 5]$.

(a) The graph of f is made of closed line segments joined end to end.

(b) The graph starts at the point $(-2, 1)$.

(c) The derivative of f is shown below:



24. Is there a function f such that $f(0) = -2$, $f(1) = 1$, and $f''(x) = 0$ for all x? If so, how many such functions are there?

25. Find all functions f(x) such that $f''(x) = 2 \sin 3x$.

26. A function g, defined for all positive real numbers, satisfies the following two conditions: $g(1) = 1$ and $g'(x^2) = x^3$ for all $x > 0$. Compute g(4).

27. Find a polynomial P of degree ≤ 5 with $P(0) = 1$, $P(1) = 2$, $P'(0) = P''(0) = P'(1) = P''(1) = 0$.

28. Find a function f such that $f''(x) = x + \cos x$ and such that $f(0) = 1$ and $f'(0) = 2$.

1.3 INTEGRATION BY TRANSFORMATION

Standard methods of integration

The different methods of integration all aim at reducing a given integral to one of the fundamental or known integrals. There are several methods of integration :

(i) Method of Transformation, i.e., it is useful to properly transform the integrand and then take advantage of the basic table of integration formulae.

(ii) Method of Substitution, i.e., a change of the independent variable helps in computing a large number of indefinite integrals.

(iii) Integration by parts

In some cases, when the integrand is a rational fraction it may be broken into **Partial Fractions** by the rules of algebra, and then each part may be integrated by one of the above methods.

In some cases of irrational functions, the method of **Integration by Rationalization** is adopted, which is a special case of (ii) above.

In some cases, integration by the method of **Successive Reduction** is resorted to, which really falls under case (iv) It may be noted that the classes of integrals which are reducible to one or other of the fundamental forms by the above processes are very limited, and that the large majority of the expressions, under proper restrictions, can only be integrated by the aid of infinite series, and in some cases when the integrand involves expressions under a radical sign containing powers of x beyond the second, the investigation of such integrals has necessitated the introduction of higher classes of transcedental function such as elliptic functions, etc.

In fact, integration is, on the whole, a more difficult operation than differentiation. We know that elementary functions are differentiated according to definite rules and formulae but integration involves, so to say, an "individual" approach to every function. Differential calculus gives general rules for differentiation, but integral calculus gives no such corresponding general rules for performing the inverse operation. In fact, so simple an integral in appearance as

$$\int \sqrt{x} \cos x dx, \text{ or} \int \frac{\sin x}{x} dx$$

cannot be worked out that is, there is no elementary function whose derivative is $\sqrt{x} \cos x$, or $(\sin x)/x$, though the integrals exist. There is quite a large number of integrals of these types.

Method of Transformation

In the method of transformation, we find the integrals by manipulation i.e. by simplifying and converting the given integrand into standard integrands. It requires experience to find an appropriate transformation of the integrand, thus reducing the given integral to a standard form.

The student must not, however, take for granted that whenever one or other of the transformations is applicable, it furnishes the simplest method of integration. The most suitable transformation in each case can only be arrived at after considerable practice and familiarity with the results introduced by such transformations.

Example 1. Evaluate $I = \int \frac{5(x-3)^2}{x\sqrt{x}}dx$

Solution $I = \int \frac{5x^2 - 30x + 45}{x\sqrt{x}}dx$

$$= 5\int x^{1/2}dx - 30\int \frac{dx}{\sqrt{x}} + 45\int x^{-3/2}dx$$

$$= 5.\frac{2}{3}x^{3/2} - 30.2\sqrt{x} + 45(-2x^{-1/2}) + C$$

$$= \frac{10}{3}x^{3/2} - 60\sqrt{x} - 90x^{-1/2} + C.$$

Example 2. Evaluate $\int \frac{x^4}{x^2+1}dx$

Solution $\int \frac{x^4}{x^2+1}dx = \int \frac{x^4-1+1}{x^2+1}dx$

$$= \int \frac{x^4-1}{x^2+1} + \frac{1}{x^2+1}\,dx$$

$$= \int (x^2-1)\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \frac{x^3}{3} - x + \tan^{-1}x + C$$

Example 3. Evaluate $\int \frac{x^2+3}{x^6(x^2+1)}dx$

Solution $\int \frac{(x^2+1)+2}{x^6(x^2+1)}dx$

$$= \int \frac{1}{x^6}dx + \int \frac{2}{x^6(x^2+1)}dx$$

$$= \int \frac{1}{x^6}dx + 2\int \frac{(x^6+1)-x^6}{x^6(x^2+1)}dx$$

$$= \int \frac{1}{x^6}dx + 2\int \frac{x^4-x^2+1}{x^6}dx - 2\int \frac{1}{x^2+1}dx$$

$$= \int \frac{1}{x^6}dx + 2\int\left(\frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6}\right)dx - 2\int \frac{1}{1+x^2}dx$$

$$= \frac{-1}{5x^5} + 2\left(-\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}\right) - 2\tan^{-1}x + C.$$

Example 4. Evaluate $\int \left(\frac{1}{x^{3/4}} + \frac{1-x^4}{1-x} + \sec x \tan x\right)dx.$

Solution Let $I = \int \left(\frac{1}{x^{3/4}} + \frac{1-x^4}{1-x} + \sec x \tan x\right)dx$

$I = \int x^{-3/4}dx + \int (1+x+x^2+x^3)\,dx + \int \sec x \tan x\,dx,$

$[\because (1-x^n)/(1-x) = 1+x+x^2+x^3+ .... + x^{n-1}]$

$I = 4x^{1/4} + x + (x^2/2) + (x^3/3) + (x^4/4) + \sec x + C.$

Example 5. Evaluate $I = \int \frac{1+\sin x}{1-\cos x}dx.$

Solution Here $I = \int \frac{1+\sin x}{1-\cos x}dx$

$$= \int \frac{1+2\sin\frac{1}{2}x\cos\frac{1}{2}x}{2\sin^2\frac{1}{2}x}dx$$

$$= \frac{1}{2}\int \operatorname{cosec}^2 \frac{1}{2}x\,dx + \int \cot\frac{1}{2}x\,dx$$

$$= -\cot\frac{1}{2}x + 2\ln\left(\sin\frac{1}{2}x\right) + C.$$

Example 6. Evaluate $I = \int \frac{dx}{\sqrt{3}\cos x + \sin x}$

Solution We have

$$I = \int \frac{dx}{\sqrt{3}\cos x + \sin x} = \int \frac{dx}{2\left[\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right]}$$

$$= \frac{1}{2}\int \frac{dx}{\sin\left(x + \frac{1}{3}\pi\right)} = \frac{1}{2}\int \operatorname{cosec}\left(x + \frac{1}{3}\pi\right)dx$$

$$= \frac{1}{2}\ln\left|\tan\left(\frac{1}{2}x + (\pi/6)\right)\right| + C.$$

Example 7. Evaluate $\int \frac{1-\sin\theta}{\cos\theta}d\theta$

Solution We break the integrand into two summands

$$\int \frac{1-\sin\theta}{\cos\theta}d\theta = \int\left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)d\theta$$

$$= \int (\sec\theta - \tan\theta)\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + \ln|\cos\theta| + C.$$

Since $\ln A + \ln B = \ln AB$, the answer can be simplified to $\ln(|\sec\theta + \tan\theta|\,|\cos\theta|) + C.$

But $\sec\theta\cos\theta = 1$ and $\tan\theta\cos\theta = \sin\theta$. The answer becomes even simpler :

$$\int \frac{1-\sin\theta}{\cos\theta}\,d\theta = \ln(1+\sin\theta)+C.$$

Example 8. Evaluate $\int \dfrac{dx}{\sec x + \operatorname{cosec} x}$

Solution $\int \dfrac{dx}{\sec x + \operatorname{cosec} x} = \dfrac{1}{2}\int \dfrac{2\sin x\, \cos x}{\sin x + \cos x}\,dx$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int (\sin x + \cos x)\,dx - \frac{1}{2\sqrt{2}}$$

$$\int \frac{1}{\sin\left(x+\frac{\pi}{4}\right)}\,dx$$

$$= \frac{1}{2}(-\cos x + \sin x) - \frac{1}{2\sqrt{2}}\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$$

Example 9. Evaluate $\int \dfrac{\sin 2x}{\sin 5x\, \sin 3x}\,dx$

Solution $\int \dfrac{\sin 2x}{\sin 5x\, \sin 3x}\,dx = \int \dfrac{\sin(5x-3x)}{\sin 5x\, \sin 3x}\,dx$

$$= \int \frac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\sin 5x\, \sin 3x}\,dx$$

$$= \int (\cot 3x - \cot 5x)\,dx$$

$= 1/3 \ln|\sin 3x| - 1/5 \ln|\sin 5x| + C.$

Example 10. Evaluate $\int \dfrac{\sin x}{\cos 3x}\,dx$

Solution $\int \dfrac{\sin x}{\cos^3 x}\,dx = \int \dfrac{2\sin x\, \cos x}{2\cos x\, \cos^3 x}\,dx$

$$= \int \frac{\sin(3x-x)}{2\cos x\, \cos 3x}\,dx$$

$$= \int \frac{\sin 3x\, \cos x - \cos 3x\, \sin x}{2\cos x\, \cos 3x}\,dx$$

$$= \frac{1}{2}\int (\tan 3x - \tan x)\,dx$$

$$= \frac{1}{6}\ln|\sec 3x| - \frac{1}{2}\ln|\sec x| + C.$$

Integrals of the form

$\int \cos ax \cos bx\, dx$, $\int \sin ax \sin bx\, dx$

$\int \sin ax \cos bx\, dx$, in which $a \neq b$.

We use the addition formulae to change products to sums or differences, which can be integrated easily. For example :

$$\int \cos x \cos 2x\, dx = \frac{1}{2}\int 2\cos x \cos 2x\, dx$$

$$= \frac{1}{2}\int (\cos 3x + \cos x)\,dx$$

$$= \frac{1}{2}\left(\frac{\sin 3x}{3} + \frac{\sin x}{1}\right) + C$$

$$\int \sin mx \sin nx dx$$

$$= \frac{1}{2}\left[\int \cos(m-n)xdx - \int \cos(m+n)xdx\right]$$

$$= \begin{cases} \dfrac{\sin(m-n)x}{2(m-n)} - \dfrac{\sin(m+n)x}{2(m+n)} & \text{if } m^2 \neq n^2 \\ \pm\left(\dfrac{x}{2} - \dfrac{\sin 2nx}{4n}\right) & \text{if } m = \pm n \end{cases}$$

$$\int \sin mx \cos nx dx = \frac{1}{2}\left[\int \sin(m-n)xdx + \int \sin(m+n)xdx\right]$$

$$= \begin{cases} -\dfrac{\cos(m-n)x}{2(m-n)} - \dfrac{\cos(m+n)x}{2(m+n)} & \text{if } m^2 \neq n^2 \\ \mp \dfrac{\cos 2nx}{4n} & \text{if } m = \pm n \end{cases}$$

$$\int \cos mx \cos nx dx = \frac{1}{2}\left[\int \cos(m-n)xdx + \int \cos(m+n)xdx\right]$$

$$= \begin{cases} \dfrac{\sin(m-n)x}{2(m-n)} + \dfrac{\sin(m+n)x}{2(m+n)} & \text{if } m^2 \neq n^2 \\ \dfrac{x}{2} + \dfrac{\sin 2nx}{4n} & \text{if } m^2 = n^2 \end{cases}$$

Example 11. Evaluate $\int \sin 8x \sin 3x\, dx$

Solution We have $\sin 8x \sin 3x = \dfrac{1}{2}(\cos 5x - \cos 11x)$,

and so $\int \sin 8x \sin 3x\, dx$

$$= \frac{1}{2}\int (\cos 5x - \cos 11x)\,dx$$

$$= \frac{1}{10}\sin 5x - \frac{1}{22}\sin 11x + C.$$

Note: If a question has one of the functions like $\sin^2x$, $\cos^2x$, $\sin^3x$ or $\cos^3x$, then we replace them by

$$\frac{1-\cos x}{2},\frac{1+\cos 2x}{2},\frac{3\sin x-\sin 3x}{4},\frac{3\cos x+\cos 3x}{4}$$

respectively. The idea is to first express the function in terms of multiple angles as above and then integrate it. Also $\tan^2x$ and $\cot^2x$ should be replaced by $\sec^2x - 1$ and $\text{cosec}^2x - 1$ respectively.

For example :

$$\int \cos2x\, dx = \int (1 + \cos 2x)\, dx$$

$$= \frac{x}{2}+\frac{1}{4}\sin 2x + C,$$

$$\int \sin2x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx$$

$$= \frac{x}{2}+\frac{1}{4}\sin 2x + C.$$

$$\int \sin3x dx = \int \frac{1}{4}(3\sin x - \sin x)dx$$

$$= \frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C.$$

Example 12. Evaluate $\int \cos4x\, dx$.

Solution $\cos^4x = \{\frac{1}{2}(1+\cos 2x)\}^2$

$$= \frac{1}{4}\{1+2\cos 2x+\cos^2 2x\}$$

$$= \frac{1}{4}[1+2\cos 2x+\frac{1}{2}(1+\cos 4x)]$$

$$= \frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$

$$\therefore\ \cos^4 x\, dx = \int (\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x)\, dx$$

$$= \frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x + C.$$

Example 13. Evaluate $\int \sin6x\, dx$.

Solution To evaluate $\int \sin6x\, dx$ we should first expressed $\sin^6 x$ as a sum of sines/cosines of multiples of x and then integrate each term of the sum to obtain the result.

Since $\sin^2 x = \frac{1}{2}(1-\cos 2x)$,

therefore $\sin^6 x = \frac{1}{8}(1-\cos 2x)^3$,

$$= \frac{1}{8}[1-3\cos 2x+3\cos^2 2x-\cos^3 2x] \quad ...(1)$$

Writing $\cos^2 2x = \frac{1}{2}(1+\cos 4x)$,

$$\cos^3 2x = \frac{1}{4}(\cos 6x+3\cos 2x),$$

(1) becomes

$$\sin^6x = \frac{1}{8}[1 - 3\cos 2x + \frac{3}{2}(1 + \cos 4x) - \frac{1}{4}(\cos 6x+3\cos 2x)\},$$

$$= \frac{1}{32}[10-15\cos 2x+6\cos 4x-\cos 6x] \quad ...(2)$$

From (2) we have $\int \sin^6 x dx$

$$= \frac{1}{32}[10x-\frac{15}{2}\sin 2x+\frac{3}{2}\sin 4x-\frac{1}{6}\sin 6x]+C \quad ...(3)$$

If the exponents are not too big, this method works well.

Example 14. Evaluate $I = \int \sin^3 x\cos^3 x\, dx$

Solution $I = \frac{1}{8}\int (2\sin x\cos x)^3\, dx$

$$= \frac{1}{8}\int \sin^3 2x\, dx$$

$$= \frac{1}{8}\int \frac{3\sin 2x-\sin 6x}{4}\, dx$$

$$= \frac{1}{32}\int (3\sin 2x-\sin 6x)\, dx$$

$$= \frac{1}{32}\left[-\frac{3}{2}\cos 2x+\frac{1}{6}\cos 6x\right]+C.$$

Example 15. Evaluate $\int \frac{dx}{\sin(x-\alpha)\sin(x-\beta)}$

Solution $I = \frac{1}{\sin(\beta-\alpha)}\int \frac{\sin[(x-\alpha)-(x-\beta)]}{\sin(x-\alpha)\sin(x-\beta)}dx$

$$= \frac{1}{\sin(\beta-\alpha)}\int \frac{\sin(x-\alpha)\cos(x-\beta)-\cos(x-\alpha)\sin(x-\beta)}{\sin(x-\alpha)\sin(x-\beta)}dx$$

$$= \frac{1}{\sin(\beta-\alpha)}\int [\cot(x-\beta)-\cot(x-\alpha)]dx$$

$$=\frac{1}{\sin(\beta-\alpha)}\int(\ln|\sin(x-\beta)|-\ln|\sin(x-\alpha)|]+C.$$

$$=\frac{1}{\sin(\beta-\alpha)}\ln\left|\frac{\sin(x-\beta)}{\sin(x-\alpha)}\right|+C.$$

Example 16. Evaluate $\int \tan(x-\alpha)\cdot\tan(x+\alpha)\cdot\tan 2x\,dx$

Solution $\tan 2x=\tan\,(\overline{x+\alpha}+\overline{x-\alpha})$

$$=\frac{\tan(x+\alpha)+\tan(x-\alpha)}{1-\tan(x+\alpha)\cdot(x-\alpha)}$$

or, $\tan 2x-\tan 2x\cdot\tan(x+\alpha)\tan(x-\alpha)$

$=\tan(x+\alpha)+\tan(x+\alpha)$

$\therefore$ $\tan(x+\alpha)\tan(x+\alpha)\tan 2x$

$=\tan 2x-\tan(x+\alpha)-\tan(x-\alpha)$

$\therefore$ $I=\int[\tan 2x-\tan(x+\alpha)-\tan(x-\alpha)]\,dx$

$=\frac{1}{2}\ln|\sec 2x|-\ln\sec|(x+\alpha)|$

$-\ln\sec|(x-\alpha)|+C$

$=\ln|\sqrt{\sec 2x}\cdot\cos(x+\alpha)\cdot\cos(x-\alpha)|+C.$

Example 17. Evaluate

$$\int\frac{dx}{\tan x+\cot x+\sec x+\operatorname{cosec} x}$$

Solution $I=\int\frac{dx}{\tan x+\cot x+\sec x+\operatorname{cosec} x}$

$$=\int\frac{\sin x\cos x\,dx}{1+\sin x+\cos x}$$

$$=\int\frac{\sin x dx}{\sec x+\tan x+1}$$

Multiplying and dividing by $(1+\tan x-\sec x)$, we get

$$=\int\frac{\sin x(1+\tan x-\sec x)}{(1+\tan x)^2-\sec^2 x}dk$$

$$=\int\frac{\sin x(1+\tan x-\sec x)}{2\tan x}dk$$

$$=\frac{1}{2}\int\cos x\,(1+\tan x-\sec x)\,dx$$

$$=\frac{1}{2}\int(\cos x+\sin x-1)\,dx$$

$$=\frac{1}{2}(\sin x-\cos x-x)+C.$$

Example 18. Integrate the function

$$\frac{5\cos^3 x+2\sin^3 x}{2\sin^2 x\cos^2 x}+\sqrt{(1+\sin 2x)}$$

$$+\frac{1+2\sin x}{\cos^2 x}+\frac{1-\cos 2x}{1+\cos 2x}\text{ w. r. t. x.}$$

Solution The given function may be written as

$$\frac{5\cos^3 x+2\sin^3 x}{2\sin^2 x\cos^2 x}+\sqrt{(\cos^2 x+\sin^2 x+2\sin x\cos x)}$$

$$+\frac{1}{\cos^2 x}+\frac{2\sin x}{\cos^2 x}+\frac{2\sin^2 x}{2\cos^2 x}$$

$=\frac{5}{2}\operatorname{cosec} x\cot x+\sec x\tan x+\cos x+\sin x$

$+\sec^2 x+2\sec x\tan x+2(\sec^2 x-1)$

$=\frac{5}{2}\operatorname{cosec} x\cot x+3\sec x\tan x+\cos x+\sin x$

$+3\sec^2 x-2.$

Now integrating, we get

$$I=\frac{5}{2}\int\operatorname{cosec} x\cot x\,dx+3\int\sec x\tan x\,dx$$

$$+\int\cos x dx+\int\sin x dx+3\int\sec^2 x\,dx-2\int dx$$

$=-\frac{5}{2}\operatorname{cosec} x+3\sec x+\sin x-\cos x$

$+3\tan x-2x+C.$

Example 19. Evaluate $\int\frac{\cos 5x+\cos 4x}{1-2\cos 3x}dx.$

Solution $\int\frac{\cos 5x+\cos 4x}{1-2\cos 3x}dx$

$$=\int\frac{\sin 3x(\cos 5x+\cos 4x)}{\sin 3x-2\cos 3x\sin 3x}dx$$

$$=\int\frac{2\sin\frac{3x}{2}\cos\frac{3x}{2}.2\cos\frac{9x}{2}\cos\frac{x}{2}}{\sin 3x-\sin 6x}dx$$

$$=4\int\frac{\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\cos\frac{9x}{2}\sin\frac{3x}{2}}dx$$

$$=-\int 2\cos\frac{3x}{2}\cos\frac{x}{2}dx=-\int(\cos 2x+\cos x)\,dx$$

$$=-\left(\frac{\sin 2x}{2}+\sin x\right)+C\cdot$$

Concept Problems B

1. Evaluate the following integrals :

(i) $\int \frac{(1+x)^2}{x\left(1+x^2\right)} dx$

(ii) $\int \frac{x^4+x^2+1}{2\left(1+x^2\right)} dx$

(iii) $\int \frac{x^3-4x^2+5x-2}{x^2-2x+1} dx$

(iv) $\int \frac{8^{1+x}+4^{1-x}}{2^x} dx$

2. Evaluate the following integrals :

(i) $\int \cos^3 x\, dx$

(ii) $\int \sin^4 x\, dx$

(iii) $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x \, . \cos^2 x} dx$

(iv) $\int \frac{a \sin^3 x + b\cos^3 x}{\sin^2 x \cos^2 x}$

3. Evaluate the following integrals :

(i) $\int \cos 2x \cos 3x \, dx$

(ii) $\int \cos^2 x \sin 4x \, dx$.

(iii) $\int \sin 2x \, . \cos^2 x \, dx$

(iv) $\int 4\cos\frac{x}{2}\cdot \cos x \cdot \sin\frac{21}{2}x \, dx$

4. Evaluate the following integrals :

(i) $\int \sqrt{1+\sin x}$

(ii) $\int \frac{\cos^4 x-\sin^4 x}{\sqrt{1+\cos 4x}} dx$

(iii) $\int \sin x \sin 2x \sin 3x \, dx$

(iv) $\int \sin x \cos x \cos 2x \cos 4x \, dx$

5. Evaluate the following integrals :

(i) $\int \sec^2 x \operatorname{cosec}^2 x \, dx$

(ii) $\int \cot^2 x \cos^2 x \, dx$

(iii) $\int \tan^2 x \sin^2 x \, dx$

(iv) $\int (\cot^2 x - \cos^2 x)\, dx$

6. Evaluate the following integrals :

(i) $\int \frac{dx}{1+\cos x}$ (ii) $\int \frac{dx}{1+\sin x}$

(iii) $\int \frac{dx}{1-\sin 3x}$ (iv) $\int \frac{\cos x - \cos 2x}{1-\cos x} dx$

7. Evaluate the following integrals :

(i) $\int \frac{1+\cos^2 x}{1+\cos 2x} dx$ (ii) $\int \frac{1-\tan^2 x}{1+\tan^2 x} dx$

(iii) $\int \frac{1+\tan^2 x}{1+\cot^2 x} dx$ (iv) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx$

8. Evaluate the following integrals :

(i) $\int \frac{\cos 5x + \cos 4x}{1-2\cos 3x} dx$

(ii) $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} dx$

(iii) $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

(iv) $\int \frac{\operatorname{cosec} x + \tan^2 x + \sin^2 x}{\sin x} dx$

9. Evaluate the following integrals :

(i) $\int (3 \sin x \cos^2 x - \sin^3 x)\, dx$

(ii) $\int \left[\sin\alpha \sin(x-\alpha) + \sin^2\left(\frac{x}{2}-\alpha\right)\right] dx$

(iii) $\int \left[\sin^2\left(\frac{9\pi}{8}+\frac{x}{4}\right) - \sin^2\left(\frac{7\pi}{8}+\frac{x}{4}\right)\right] dx$

(iv) $\int \cot\left(\frac{3}{4}\pi - 2x\right)\cos 4x \, dx.$

Practice Problems B

10. Evaluate the following integrals :

(i) $\int \frac{dx}{25+4x^2}$

(ii) $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$

(iii) $\int \frac{\left(x^2 + \sin^2 x\right) \sec^2 x}{1 + x^2} dx$

(iv) $\int \frac{\sin x}{\cos^2 x} (1 - 3\sin^3 x)\, dx$

11. Evaluate the following integrals :

(i) $\int \frac{\sqrt{x^4 + x^{-4} + 2}}{x^3} dx$

(ii) $\int \frac{dx}{\sqrt{2x+3} + \sqrt{2x-3}} dx$

(iii) $\int \frac{\left(\sqrt{x} + 1\right)\left(x^2 - \sqrt{x}\right)}{x\sqrt{x} + x + \sqrt{x}} dx$

(iv) $\int \left(\frac{1 - x^{-2}}{x^{1/2} - x^{-1/2}} - \frac{2}{x^{3/2}} + \frac{x^{-2} - x}{x^{1/2} - x^{-1/2}} \right) dx$

12. Evaluate the following integrals :

(i) $\int \frac{2x^3 + 3x^2 + 4x + 5}{2x + 1} dx$

(ii) $\int \left(\frac{x^{-6} - 64}{4 + 2x^{-1} + x^{-2}} \cdot \frac{x^2}{4 - 4x^{-1} + x^{-2}} - \frac{4x^2(2x+1)}{1 - 2x} \right) dx$

(iii) $\int \left(\frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}} \right)^2 \left(\frac{\sqrt{x}-1}{\sqrt{x}+1} - \frac{\sqrt{x}+1}{\sqrt{x}-1} \right) dx$

(iv) $\int \frac{\sqrt{1-x^2} + 1}{\sqrt{1-x} + 1/\sqrt{1+x}} dx.$

13. Evaluate the following integrals :

(i) $\int \frac{(\sqrt{x}+1)(x^2 - \sqrt{x})}{x\sqrt{x} + x + \sqrt{x}} dx.$

(ii) $\int \frac{\sqrt{1-x^2} - 1}{x} \left(\frac{1-x}{\sqrt{1-x^2} + x - 1} + \frac{\sqrt{1+x}}{\sqrt{1+x} - \sqrt{1-x}} \right) dx$

(iii) $\int \frac{\frac{x^4 + 5x^3 + 15x - 9}{x^6 + 3x^4} + \frac{9}{x^4}}{(x^3 - 4x + 3x^2 - 12)/x^5} dx$

(iv) $\int \frac{\sqrt[3]{x + \sqrt{2 - x^2}}\ \sqrt[6]{1 - x\sqrt{2 - x^2}}}{\sqrt[3]{1 - x^2}} dx$

14. Evaluate the following integrals :

(i) $\int \frac{5\cos^3 x + 3\sin^3 x}{\sin^2 x \cos^2 x} dx$

(ii) $\int (\cos^6 x + \sin^6 x)\, dx$

(iii) $\int \sin^3 x \cos\frac{x}{2} dx$

(iv) $\int \frac{dx}{\sqrt{3}\cos x + \sin x}$

15. Evaluate the following integrals :

(i) $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx$

(ii) $\int \frac{\sin 2x + \sin 5x - \sin 3x}{\cos x + 1 - 2\sin^2 2x} dx$

(iii) $\int \frac{\cos x - \sin x}{\cos x + \sin x}(2 + 2\sin 2x) dx$

(iv) $\int \left[\frac{\cot^2 2x - 1}{2\cot 2x} - \cos 8x \cot 4x \right] dx$

16. Evaluate the following integrals :

(i) $\int \frac{\cos x}{\sqrt{1 + \cos x}} dx$ (ii) $\int \frac{dx}{\sin x \sin(x + \alpha)}$

(iii) $\int \{1 + \cot(x - \alpha)\cot(x + \alpha\}\, dx$

17. Let $f(0) = 0$ and $f'(x) = \frac{1}{\sqrt{(1 - x^2)}}$ for $-1 < x < 1$,

show that $f(x) + f(a) = f\{x\sqrt{(1 - a^2)} + a\sqrt{(1 - x^2)}\}$.

1.4 INTEGRATION BY SUBSTITUTION

The method of substitution is one of the basic methods for calculating indefinite integrals. Even when we integrate by some other method, we often resort to substitution in the intermediate states of calculation. The success of integration depends largely on how appropriate the substitution is for simplifying the given integral. Essentially, the study of methods of integration reduces to finding out what kind of substitution has to be performed for a given element of integration.

Substitution – change of variable

Let $I = \int f(x)\,dx$, and let $x = \phi(u)$.

Then, by definition, $\frac{dI}{dx} = f(x)$ and $\frac{dx}{du} = \phi'(u)$.

Now, $\frac{dI}{du} = \frac{dI}{dx}\frac{dx}{du} = f'(x)\,\phi'(u) = f\{\phi(u)\}\,\phi'(u)$.

$\therefore \quad I = \int f\{\phi(u)\}\,\phi'(u)\,du.$

Note:

Thus, in the integral $\int f(x)dx$, we put $x = \phi(u)$ in the expression f(x) and also we replace dx by $\phi'(u)$ du and then we proceed with the integration with u as the new variable. After evaluating the integral we need to replace u by the equivalent expression in x.

Note that though from $x = \phi(u)$ we can write $\frac{dx}{du} = \phi'(u)$ in making our substitution in the given integral, we generally use it in the differential form $dx = \phi'(u)\,du$. It really means that when x and u are connected by the relation $x = \phi(u)$, I being the function of x whose differential coefficient with respect to x is f(x), it is, when expressed in terms of u, identical with the function whose differential coefficient with respect to u is $f\{\phi(u)\}\,\phi'(u)$ which later, by a proper choice of $\phi(u)$, may possibly be of a standard form, and therefore easy to find out.

Note:

Sometimes it is found convenient to make the substitution in the form $g(x) = u$ where corresponding differential form will be $g'(x)\,dx = du$.

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $\quad u = g(x)$, then $du = g'(x)$ and

$$\int f(u)du = F(u) + C.$$

Steps for substitution

(i) Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.

(ii) Compute $du = g'(x)\,dx$.

(iii) Rewrite the integral in terms of the variable u.

(iv) Evaluate the resulting integral in terms of u.

(v) Replace u by g(x) to obtain an antiderivative in terms of x.

The main challenge in using substitution is to think of an appropriate substitution. One should try to choose u to be some function in the integrand whose differential also occurs (except for a constant factor). If that is not possible, try choosing u to be some complicated part of the integrand. Finding the right substitution is a bit of an art. It is not unusual to guess a wrong substitution; if the first guess does not work, try another substitution.

Example 1. Find $\int x^3 \cos(x^4 + 2)\,dx$.

Solution We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\,dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using $x^3\,dx = du/4$, we have

$$\int x^3 \cos(x^4 + 2)\,dx$$

$$= \int \cos u \,.\, \frac{1}{4}\,du = \frac{1}{4}\int \cos u\,du$$

$$= \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4 + 2) + C$$

Example 2. Evaluate $\int \frac{(\ell n\,x)^2}{x}\,dx$

Solution $\int \frac{(\ell n\,x)^2}{x}\,dx$

Put $\ell nx = t$

$\Rightarrow \quad \frac{1}{x}\,dx = dt$

$$I = \int t^2 . \left(\frac{dx}{x}\right) = \int t^2 dt = \frac{t^3}{3} + C = \frac{(\ell n\,x)^3}{3} + C.$$

Example 3. Evaluate $\int (1 + \sin^2 x)\cos x\,dx$

Solution Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\int (1 + t^2)\,dt = t + \frac{t^3}{3} + C$$

$$= \sin x + \frac{\sin^3 x}{3} + C$$

Example 4. Evaluate $\int \frac{dx}{x\sqrt{4x^2-5}}$

Solution $\int \frac{dx}{x\sqrt{4x^2-5}} = \int \frac{\frac{du}{2}}{\frac{u}{2}\sqrt{u^2-a^2}}$

where $a = \sqrt{5}$

Put $u = 2x,\ x = u/2,\quad dx = du/2$,

$$= \int \frac{du}{u\sqrt{u^2-a^2}}$$

$$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

$$= \frac{1}{\sqrt{5}}\sec^{-1}\left(\frac{2|x|}{\sqrt{5}}\right) + C$$

Example 5. Evaluate $\int \sqrt{\frac{x}{4-x^3}}\,dx$.

Solution $I = \int \sqrt{\frac{x}{4-x^3}}\,dx = \int \frac{\sqrt{x}\,dx}{\sqrt{4-x^3}}$

Here integral of $\sqrt{x}$ is $\frac{2}{3}x^{3/2}$ and $4 - x^3 = 4 - (x^{3/2})^2$

Put $x^{3/2} = t \Rightarrow \sqrt{x}\,dx = \frac{2}{3}dt$

$$I = \frac{2}{3}\int \frac{dt}{\sqrt{4-t^2}} = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{2}\right) + C.$$

Example 6. Evaluate $\int \frac{10x^9 + 10^x \ln 10}{10^x + x^{10}}dx$

Solution Put $10^x + x^{10} = t$

$\Rightarrow (10^x \ln 10 + 10\,x^9)\,dx = dt$

$$I = \int \frac{1}{t}dt = \ln t = \ln|10^x + x^{10}| + C.$$

Example 7. Evaluate $\int \frac{x^x(1+(\ln x)dx}{(x^x+1)}$.

Solution Put $x^x = t$

$\Rightarrow x^2(1 + \ln x)\,dx = dt$

$$I = \int \frac{dt}{t+1} = \ln(t+1)$$

$= \ln(x^x + 1) + C.$

Example 8. Evaluate $\int \frac{d(x^2+1)}{\sqrt{x^2+2}}$

Solution $I = \int \frac{d(x^2+1)}{\sqrt{x^2+2}}$

We know $d(x^2 + 1) = 2x\,dx$

$$\therefore\quad I = \int \frac{2xdx}{\sqrt{x^2+2}}$$

Put $x^2 + 2 = t^2 \therefore 2x\,dx = 2t\,dt$

$$I = \int \frac{2t\,dt}{t} = 2t + C$$

$$\Rightarrow\quad I = 2\sqrt{x^2+2} + C.$$

Alternative :

$$I = \int \frac{d(x^2+1)}{\sqrt{x^2+2}} = \int \frac{d(x^2+1)}{\sqrt{(x^2+1)+1}}$$

$= 2\sqrt{(x^2+1)+1} + C$

[considering $x^2 + 1$ as the variable of integration]

$= 2\sqrt{x^2+2} + C.$

$$\int \frac{x^2dx}{4+3x^3} = \frac{1}{9}\int \frac{d(4+3x^3)}{4+3x^3} = \frac{1}{9}\ell n\left|4+3x^3\right| + C$$

Deduction 1

The General Poser Rule for integration

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C,\ n \neq -1.$$

In other words it means that if we are to integrate any function of x raised to the power n and multiplied by the derivative of that function, then we shall apply the above power formula on that function.

Proof

$$I = \int [f(x)]^n f'(x)dx$$

Put $f(x) = t \Rightarrow f(x)\,dx = dt$

$$I = \int t^n dt = \frac{t^{n+1}}{n+1} = \frac{[f(x)]^{n+1}}{n+1} + C$$

For example, $\int \tan^5 x \sec^2 x dx = \frac{\tan^6 x}{6} + C$

Example 9. Evaluate $\int (x^2+1)^2\,(2x)\,dx$.

Solution Letting $g(x) = x^2 + 1$, you obtain

$g'(x) = 2x$ and $f(g(x)) = [g(x)]^2$.

From this, we can recognize that the integrand follows the $f(g(x))\, g'(x)$ pattern. Thus, we write

$$\int \underbrace{(x^2+1)^2}_{[g(x)]^2} \underbrace{(2x)}_{g'(x)} dx = \frac{1}{3}(x^2+1)^3 + C.$$

Example 10. Evaluate $I = \int \frac{\cos x}{\sin^7 x} dx$

Solution Here power formula is applicable on sin x as its derivative i.e. cosx is present in the numerator. But you should note the form of power formula which

will be $\int \frac{1}{x^n} dx = \frac{1}{-(n-1)x^{n-1}}$,

i.e. when power of x is in the denominator, then decrease the power by one and multiply the denominator by the decreased power with sign changed.

Here, $I = \frac{1}{-6\sin^6 x} + C$

Example 11. Evaluate $\int \frac{-4x}{(1-2x^2)^2} dx$

Solution $\int \frac{-4x}{(1-2x^2)^2} dx = \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}} \overbrace{(-4x)dx}^{du}$

$$= -\overbrace{(1-2x^2)^{-1}}^{u^{-1}(-1)} + C = \frac{1}{2x^2-1} + C.$$

Example 12. Find $\int \frac{x^2 dx}{(x^3-2)^5}$.

Solution Let u be the value in the parenthesis, that is, let $u = x^3 - 2$. Then $du = 3x^2\, dx$. so by substitution :

$$\int \frac{x^2 dx}{(x^3-2)^5} = \int \frac{du/3}{u^5} = \frac{1}{3}\int u^{-5} du$$

$$= \frac{1}{3}\frac{u^{-4}}{-4} + C = -\frac{1}{12}(x^3-2)^{-4} + C.$$

Deduction 2

$$I = \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

We prove this as follows :

Put $f(x) = z$.

$\therefore$ $f'(x)\, dx = dz$.

$\therefore$ $I = \int \frac{dz}{z} = \ln|z| = \ln|f(x)| + C$

Hence, if the integrand be a fraction such that its numerator is the differential coefficient of the denominator, then the integral is equal to $\ln|(\text{denominator})| + C$.

For example, consider $I = \int \frac{x^2}{7+x^3} dx$.

Here the derivative of the denominator $(7 + x^3)$ is $3x^2$.

$\therefore$ $I = \frac{1}{3}\int \frac{3x^2 dx}{7+x^3} = \frac{1}{3}\ell n|7 + x^3| + C$

Also, $\int \frac{\cos x - \sin x}{\sin x + \cos x} dx = \ln|(\sin x + \cos x)| + C$

$$\int \frac{2ax+b}{ax^2+bx+c} dx = \ln|(ax^2+bx+c)| + C$$

Example 13. Find $\int \frac{dx}{1+e^x}$

Solution We write the integrand as follows :

$$\frac{1}{1+e^x} = \frac{e^{-x}}{e^{-x}}\left(\frac{1}{1+e^x}\right) = \frac{e^{-x}}{e^{-x}+1}$$

$u = e^{-x} + 1$, $du = -e^{-x}\, dx$

$$\int \frac{dx}{1+e^x} = \int \frac{e^{-x}dx}{e^{-x}+1} = \int \frac{-du}{u}$$

$= -ln|u| + C = -ln(e^{-x} + 1) + C$

Remember, $e^{-x} + 1 > 0$ for all x, so

$ln|e^{-x} + 1| = ln(e^{-x} + 1)$.

Example 14. Integrate $\int \frac{2\sin x}{5+3\cos x} dx$.

Solution $I = -\frac{2}{3}\int \frac{-3\sin x}{5+3\cos x} dx$

Now, since the numerator of the integrand is the differential coefficient of the denominator,

$$I = -\frac{2}{3}\ln|(5 + 3\cos x)| + C.$$

With the help of the deduction 2, we can prove some of the standard results mentioned earlier.

(i) $\int \tan x\, dx = \ln|\sec x| + C$

Proof :

Put $\cos x = z$, then $-\sin x\, dx = dz$.

$\therefore$ $I = \int \frac{\sin x}{\cos x} dx = -\int \frac{dz}{z} = -\ln z + C$

$= -\ln \cos x + C = \ln|\sec x| + C$.

Alternative :

$$\int \tan x\, dx = \int \frac{\sec x \tan x}{\sec x} dx = \ln|\sec x| + C$$

(ii) $\int \cot x\, dx = \ln|\sin x| + C$

Proof:

By substituting $\sin x = z$, this result follows.

Alternative :

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C$$

(iii) $\int \operatorname{cosec} x\, dx = \ln\left|\tan\frac{x}{2}\right| + C$

Proof:

$$\int \operatorname{cosec} x\, dx = \int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{1}{2}x\cos\frac{1}{2}x}$$

We multiply the numerator and denominator by $\sec^2\frac{1}{2}x$ to get numerator as the differential coefficient of the denominator.

$$= \int \frac{\frac{1}{2}\sec^2\frac{1}{2}x}{\tan\frac{1}{2}x} dx = \ln\left|\tan\frac{x}{2}\right| + C.$$

Alternative 1 :

$$\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx$$

$= \ln|(\operatorname{cosec} x - \cot x)| + C$

Alternative 2 :

$$\int \operatorname{cosec} x\, dx = \int \frac{dx}{\sin x} = \int \frac{\sin x}{\sin^2 x} dx$$

$$= -\int \frac{d(\cos x)}{1-\cos^2 x} = -\int \frac{dz}{1-z^2}, \text{ where } z = \cos x$$

$$= \frac{1}{2}\ln\frac{1-z}{1+z} = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$$

It should be noted that the different forms in which the integral of cosec x is obtained by different methods can be easily shown to be identical by elementary trigonometry.

We have,

$$\frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| = \frac{1}{2}\ln\left|\frac{2\sin^2\frac{1}{2}x}{2\cos^2\frac{1}{2}x}\right| = \frac{1}{2}\ln\left|\tan^2\frac{1}{2}x\right|$$

$$= \ln\left|\tan\frac{1}{2}x\right|.$$

(iv) $\int \sec x\, dx = \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

$= \ln|(\sec x + \tan x)| + C.$

Proof:

$$\int \sec x\, dx = \int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(\frac{1}{2}\pi + x\right)}$$

$$= \int \frac{dx}{2\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$

$$= \int \frac{\frac{1}{2}\sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right)dx}{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$

$$= \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

Alternative 1 :

$$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \ln|(\sec x + \tan x)| + C$$

since the numerator is the derivative of the denominator.

While this is the shortest method, it does seem artificial. The next method may seem a little less contrived.

Alternative 2 :

$$\int \sec x\, dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1-\sin^2 x} dx$$

The substitution $z = \sin x$ and $dz = \cos x\, dx$ transforms this last integral into the integral of a rational function :

$$\int \frac{dz}{1-z^2} = \frac{1}{2}\int\left(\frac{1}{1+z} + \frac{1}{1-z}\right) dz$$

$$= \frac{1}{2}[\ln|1+z| - \ln|1-z|] + C$$

$$= \frac{1}{2}\ln\left|\frac{1+z}{1-z}\right| + C.$$

Since $z = \sin x$, $\frac{1}{2}\ln\left|\frac{1+z}{1-z}\right| = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$.

The reader may check that this equals $\ln|\sec x + \tan x|$ by showing that $\frac{1+\sin x}{1-\sin x} = (\sec x + \tan x)^2$.

Neither method gives, ln |tan(x/2 + π/4)|. However, we can to show that tan(x/2 + π/4) = sec x + tan x.

Example 15. Evaluate $\int \frac{1+x}{1+x^2}dx$

Solution $\int \frac{1+x}{1+x^2}dx = \int \frac{1}{1+x^2}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx$

$$= \tan^{-1}x + \frac{1}{2}\ln(1+x^2) + C$$

Example 16. Integrate $\int \frac{x+9}{x^3+9x}dx$

Solution $I = \int \frac{x+9+x^2-x^2}{x(x^2+9)}dx$

$$= \int \frac{dx}{x^2+9} + \int \frac{dx}{x} - \int \frac{x}{x^2+9}dx$$

$$= \frac{1}{3}\tan^{-1}\frac{x}{3} + \ln|x| - \frac{1}{2}\ln(x^2+9) + C.$$

Example 17. Integrate

(i) $\int \frac{1}{x\cos^2(1+\ln x)}dx$

(ii) $\int \frac{1}{x(1+\ln x)^m}dx$, $m \neq 1$.

Solution

(i) Here $I = \int \frac{1}{x\cos^2(1+\ln x)}dx$

Putting 1 + ln x = t, so that (1/x) dx = dt, we have

$I = \int dt/\cos^2 \int = \sec^2\ t\ dt$

= tan t = tan (1 + ln x) + C.

(ii) Here $I = \int \frac{1}{x(1+\ln x)^m}dx$.

Putting 1 + ln x = t, so that (1/x) dx = dt, we have

$$I = \int \frac{dt}{t^m} = \frac{t^{-m+1}}{= m+1}$$

$$= \frac{(1+\ln x)^{-m+1}}{(1-m)} - = \frac{1}{(1-m)}(1+\ln x)^{1-m} + C.$$

Example 18. Evaluate $\int \frac{\cot x}{3+2\ln\sin x}dx$

Solution $\int \frac{\cot x}{3+2\ln\sin x}dx = \frac{1}{2}\int \frac{2\cot x}{(3+2\ln\sin x)}dx$

$$= \frac{1}{2}\ln|3+2\ln\sin x| + C$$

Example 19. Find $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$.

Solution Here integrand contains expression of the form cosθ, where θ = xe^x is a function of x, therefore, put z = xe^x.

Let z = xe^x, then dz = ($e^x + xe^x$) dx = $e^x(1 + x)$dx

Now $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx = \int \frac{dz}{\cos^2 z} = \int \sec^2 z\,dz$

= tan z + c = tan(xe^x) + C.

Example 20. Evaluate $I = \int \frac{dx}{(ax+b)+\sqrt{(ax+b)}}$

Solution $I = \int \frac{dx}{\sqrt{(ax+b)}\left[\sqrt{(ax+b)}+1\right]}$

Now, $\frac{d}{dx}\left[\sqrt{(ax+b)}+1\right] = \frac{a}{2\sqrt{(ax+b)}}$

$\therefore\quad I = \frac{2}{a}\int \frac{\frac{a}{2\sqrt{(ax+b)}}}{[\sqrt{(ax+b)}+1]}dx$

$$= \frac{2}{a}\ln\left[\sqrt{(ax+b)}+1\right] + C$$

Rationalizing Substitutions

Some irrational functions can be changed into rational functions by means of appropriate substitutions. Very often in integration processes emphasis is mainly laid upon finding a change of variable which reduces the given integral to an integral of a rational function; then the further course of integration becomes clear. We say that such a change rationalizes the integral.

In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective.

Example 21. Evaluate $\int \frac{dx}{2\sqrt{x}(x+1)}$.

Solution Put $x = t^2 \Rightarrow dx = 2t\,dt$

$$I = \int \frac{dx}{2\sqrt{x}(x+1)} = \int \frac{2t\ dt}{2t(t^2+1)}$$

$$= \int \frac{dt}{1+t^2} = \tan^{-1}t + C = \tan^{-1}\left(\sqrt{x}\right) + C$$

Example 22. Evaluate $\int \frac{\sqrt{x+4}}{x}dx.$

Solution Let $u = \sqrt{x+4}$. Then $u^2 = x+4$, so $x = u^2 - 4$ and $dx = 2u\,du$.

Therefore, $\int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2-4}\,2u\,du$

$$= 2\int \frac{u^2}{u^2-4}du = 2\int\left(1+\frac{4}{u^2-4}\right)du$$

$$= 2\int du + 8\int \frac{du}{u^2-4}$$

$$= 2u + 8\cdot\frac{1}{2\cdot 2}\, ln\left|\frac{u-2}{u+2}\right| + C$$

$$= 2\sqrt{x+4} + 2\, ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.$$

Example 23. Evaluate

$$\int \cos x\, \sqrt{(4-\sin^2 x)}dx = dt$$

Solution Put $\sin x = t$ so that $\cos x\, dx = dt$. Then the given integral

$$= \int \sqrt{(4-t^2)}\,dt = \int \sqrt{(2^2-t^2)}\,dt$$

$$= \frac{1}{2}t\sqrt{(2^2-t^2)} + \frac{2^2}{2}\sin^{-1}(t/2) + C$$

$$= \frac{1}{2}\sin x\,.\sqrt{(4-\sin^2 x)} + 2\sin^{-1}(1/2\sin x) + C.$$

Example 24. Integrate $\int \sqrt{2+\sqrt{4+\sqrt{x}}}\,dx$

Solution Let

$$\sqrt{2+\sqrt{4+\sqrt{x}}}\,dx = u,\quad 2+\sqrt{4+\sqrt{x}} = u^2$$

$$\Rightarrow \sqrt{4+\sqrt{x}} = u^2 - 2$$

$$\Rightarrow 4+\sqrt{x} = (u^2-2)^2$$

$$\sqrt{x} = (u^2-2)^2 - 4$$

$$\Rightarrow x = [(u^2-2)^2-4]^2$$

$$dx = 2[(u^2-2)^2-4]^2\cdot 2(u^2-2)\cdot 2u\,du$$

$$= 8u(u^2-2)(u^4-4u^2)\,du$$

$$= 8(u^3-2u)(u^4-4u^2)\,du$$

$$= 8(u^7-4u^5-2u^5-8u^3]\,du$$

$$\therefore\ I = 8\int u(u^7-6u^5-8u^3)du$$

$$= \int (u^8-6u^6-8u^4)du$$

$$= 8\left[\frac{u^9}{9}-\frac{6u^7}{7}-\frac{8u^5}{5}\right]+C,$$

where $u = \sqrt{2+\sqrt{4+\sqrt{x}}}$.

Example 25. Evaluate $I = \int \sqrt{(1+\sin x)}\,dx$

Solution We have $I = \int \sqrt{(1+\sin x)}\,dx$

$$= \int \sqrt{\left\{1-\cos\left(\frac{1}{2}\pi + x\right)\right\}}dx$$

$$= \int \sqrt{\left[2\sin^2\left(\frac{1}{4}\pi+\frac{1}{2}x\right)\right]}dx = 2\int \sin\left(\frac{1}{2}x+\frac{1}{4}\pi\right)dx.$$

Now putting $\frac{1}{2}x + \frac{1}{4}\pi = t$, so that

$$= 2\cos\left(\frac{1}{2}x+\frac{1}{4}\pi\right) + C.$$

Simplifying integrals step by step

If we do not know what substitution to make, we try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in the following integrals.

(i) $\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}dx$

(a) $u = \tan x$, followed by $v = u^3$, then by $w = 2 + v$

(b) $u = \tan^3 x$, followed by $v = 2 + u$

(c) $u = 2 + \tan^3 x$.

(ii) $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$

(a) $u = x - 1$, followed by $v = \sin u$, then by $w = 1 + v^2$

(b) $u = \sin(x-1)$, followed by $v = 1 + u^2$

(c) $u = 1 + \sin^2(x-1)$

Example 26. $\int \frac{\sec x\,dx}{\cos(2x+\alpha)+\cos\alpha}$

Solution $I = \int \frac{\sec x\,dx}{\sqrt{2\cos(x+\alpha)\cos x}}$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec x\,dx}{\sqrt{(\cos x\cos\alpha - \sin x\sin\alpha)\cos x}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec^2 x\ dx}{\sqrt{\cos\alpha - \tan x \sin\alpha}}$$

$$= \frac{1}{\sqrt{2\sin\alpha}}\int \frac{\sec^2 x\ dx}{\sqrt{\cot\alpha - \tan x}}$$

Put $\cot\alpha - \tan x = t^2$

$\Rightarrow$ $-\sec^2 x\, dx = 2t\, dt$

$$I = \frac{1}{\sqrt{2\sin\alpha}}\int \frac{-2t\, dt}{t} = \frac{-2}{\sqrt{2\sin\alpha}}\int dt$$

$$= -\frac{\sqrt{2}}{\sqrt{\sin\alpha}}\sqrt{\cot\alpha - \tan x} + C$$

$$= C - \sqrt{\frac{2(\cot\alpha - \tan x)}{\sin\alpha}}$$

Example 27. Evaluate

$$\int \frac{\sin^3 x\ dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}$$

Solution

$$I = \int \frac{\sin^3 x\ dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}$$

Put $\tan^{-1}(\sec x + \cos x) = t$

$$\Rightarrow \frac{1}{1 + (\sec x + \cos x)^2}(\sec x \tan x - \sin x)\, dx = dt$$

$$\Rightarrow \frac{\sin^3 x\, dx}{\cos^4 x + 3\cos^2 x + 1} = dt$$

$$\therefore\ I = \int \frac{dt}{t} = \ln|t| + C$$

$= \ln|\tan^{-1}(\sec x + \cos x)| + C.$

Special forms

Example 28. If $\int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x}dx$ $= ln|f(x)| + g(x) + C$ where C is the constant of integration and f (x) is positive, then find f (x) + g (x).

Solution $I = \int \frac{(e^x + \cos x + 1) - (e^x + \sin x + x)}{e^x + \sin x + x}dx$

$= ln|e^x + \sin x + x| - x + C$

$\therefore$ $f(x) = e^x + \sin x + x$ and $g(x) = -x$

$\Rightarrow$ $f(x) + g(x) = e^x + \sin x.$

Example 29. Evaluate

$$\int \frac{x + e^x(\sin x + \cos x) + \sin x \cos x}{(x^2 + 2e^x \sin x - \cos^2 x)^2}dx$$

Solution

$$I = \frac{1}{2}\int \frac{2x + 2e^x(\sin x + \cos x) + 2\sin x \cos x}{(x^2 + 2e^x \sin x - \cos^2 x)^2}dx$$

Put $x^2 + 2e^x \sin x - \cos^2 x = t$

$[2x + 2(e^x \cos x + e^x \sin x) + 2\sin x \cos x]dx = dt$

$[2x + 2e^x(\cos x + \sin x) + 2\sin x \cos x]\, dx = dt$

$$I = \frac{1}{2}\int \frac{dt}{t^2} = C - \frac{1}{2}\cdot\frac{1}{t}$$

$$= C - \frac{1}{2(x^2 + 2e^x \sin x - \cos^2 x)}.$$

Example 30. Evaluate

$$\int \frac{e^{2x} - e^x + 1}{(e^x \sin x + \cos x)(e^x \cos x - \sin x)}dx$$

Solution $f(x) = e^x \sin x + \cos x$

$f'(x) = e^x \cos x + \sin x\, e^x - \sin x$

$g(x) = e^x \cos x - \sin x$

$g'(x) = \cos x \cdot e^x - e^x \sin x - \cos x$

Now $f(x)\cdot g'(x)$

$= (e^x \sin x + \cos x)(\cos x \cdot e^x - e^x \sin x - \cos x)$

$= e^{2x} \sin x \cos x - e^{2x} \sin^2 x - e^x \sin x \cos x$

$+\ e^x \cos^2 x - e^x \sin x \cos x - \cos^2 x$ ...(1)

and $g(x)\cdot f'(x)$

$= (e^x \cos x - \sin x)(e^x \cos x + \sin x\, e^x - \sin x)$

$= e^{2x} \cos^2 x + e^{2x} \sin x \cos x - e^x \sin x \cos x$

$- e^x \sin x \cos x - e^x \sin^2 x + \sin^2 x$ ...(2)

$f(x)\cdot g'(x) - g(x)\cdot f'(x) = e^{2x} - e^x + 1$

$$I = \int \frac{f(x)g'(x) - g(x)f'(x)}{f(x)g(x)}dx = ln\left|\frac{g(x)}{f(x)}\right| + C$$

$$= \ln\left|\frac{e^x \cos x - \sin x}{e^x \sin x + \cos x}\right| + C.$$

Taking x^n common

Many integrals can be evaluated by taking x^n common from some bracketed expression and then using substitution. Some of the suggested forms are given below :

(i) $\int \frac{dx}{x(x^n+1)}$ $n \in N$

Take x^n common and put $1 + x^{-n} = t$.

(ii) $\int \frac{dx}{x^2\left(x^n+1\right)^{(n-1)/n}}$ $n \in N$

Take x^n common and put $1+x^{-n} = t^n$.

(iii) $\int \frac{dx}{x^n\left(1+x^n\right)^{1/n}}$

Take x^n common and put $1 + x^{-n} = t$.

Example 31. Evaluate $\int \frac{dx}{x(x^4-1)}$

Solution Let $I = \int \frac{dx}{x(x^4-1)} = \int \frac{dx}{x^5\left(1-\frac{1}{x^4}\right)}$

Put $1-\frac{1}{x^4} = t$

$$\Rightarrow \frac{4}{x^5}dx = dt \Rightarrow \frac{1}{x^5}dx = \frac{dt}{4}$$

$$\therefore\ I = \frac{1}{4}\int\frac{dt}{t} = \frac{1}{4}\ln|t| + C = \frac{1}{4}\ln\left|1-\frac{1}{x^4}\right| + C.$$

Example 32. Evaluate $\int \frac{dx}{x^4(x^3+1)^2}$

Solution Let $I = \int \frac{dx}{x^4(x^3+1)^2} = \int \frac{dx}{x^{10}\left(1+\frac{1}{x^3}\right)^2}$

Put $1+\frac{1}{x^3} = t$

$$\Rightarrow -\frac{3}{x^4}dx = dt \Rightarrow \frac{1}{x^6} = (t-1)^2 \Rightarrow \frac{1}{x^4}dx = -\frac{dt}{3}$$

$$I = \frac{-1}{3}\int\frac{(t-1)^2}{t^2}dt = \frac{-1}{3}\int\frac{t^2+1-2t}{t^2}dt$$

$$= \frac{-1}{3}\int 1+\frac{1}{t^2}-\frac{2}{t}dt = \frac{-1}{3}\left[t-\frac{1}{t}-2\ln|t|\right]+C$$

where $t = 1+\frac{1}{x^3}$.

Example 33. Evaluate $\int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx$.

Solution Let $I = \int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx$

$$= \int \frac{x^4(5+4x)dx}{x^{10}\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2}$$

$$= \int \frac{5/x^6+4/x^5}{\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2}dx$$

Put $1+\frac{1}{x^4}+\frac{1}{x^5} = t \Rightarrow \left(-\frac{4}{x^5}-\frac{5}{x^6}\right)dx = dt$

$$I = \int -\frac{dt}{t^2} = \frac{1}{t} + C = \frac{1}{1+\frac{1}{x^4}+\frac{1}{x^5}} + C$$

$$= \frac{x^5}{x^5+x+1} + C.$$

Example 34. Evaluate $\int \frac{dx}{x^2(x+\sqrt{1+x^2})}$

Solution Let

$$I = \int \frac{dx}{x^2(x+\sqrt{1+x^2})} = \int \frac{1}{x^3\left(1+\sqrt{\frac{1}{x^2}+1}\right)}$$

Put $1+\frac{1}{x^2} = t^2 \Rightarrow \frac{-2}{x^3}dx = 2t\,dt$

$$I = -\int\frac{t\,dt}{1+t} = -\int\frac{t+1-1}{1+t}dt = -\int\left(1-\frac{1}{t+1}\right)dt$$

$= -t+\ln|t+1|+C$, where $t = \sqrt{1+\frac{1}{x^2}}$.

Example 35. Evaluate $\int \frac{(x^4-1)dx}{x^2\sqrt{(x^4+x^2+1)}}$

Solution $I = \int \frac{(x^4-1)dx}{x^2.x\sqrt{x^2+\frac{1}{x^2}+1}} = \int \frac{\left(x-\frac{a}{x^3}\right)dx}{\sqrt{x^2+\frac{1}{x^2}+1}}$

Put $x^2+\frac{1}{x^2}=t \Rightarrow 2\left(x-\frac{1}{x^3}\right)dx = dt$

$$I=\frac{1}{2}\int\frac{dt}{\sqrt{(t+1)}}=\sqrt{(t+1)}+C=\frac{\sqrt{(x^4+x^2+1}}{x}+C.$$

Example 36. Given that $f(0)=f'(0)=0$ and $f''(x)=\sec^4 x+\sec^2 x\tan^2 x+4$, find f(x).

Solution $f''(x)=\sec^2 x+\sec^2 x\tan^2 x+4$

Integrating we get

$$f'(x)=\tan x+\frac{\tan^3 x}{3}+4x+C$$

But, $f'(0)=0$

$\therefore\ 0=0+0+0+C$

$\therefore\ C=0$

Then, $f'(x)=\tan x+\dfrac{\tan^3 x}{3}+4x$

$\Rightarrow\ f'(x)=\tan x+\dfrac{1}{3}(\sec^2 x-1)\tan x+4x$

$\Rightarrow\ f'(x)=\dfrac{1}{3}\tan x\sec^2 x+\dfrac{2}{3}\tan x+4x$

Integrating both sides w.r.t. x we get

$$f(x)=\frac{1}{6}\tan^2 x+\frac{2}{3}\log|\sec x|+2x^2+D$$

But, $f(0)=0$

$\therefore\ 0=0+0+0+D$

$\Rightarrow\ D=0$

Then, $f(x)=\dfrac{2}{3}\ln|\sec x|+\dfrac{1}{6}\tan^2 x+2x^2.$

Example 37. Let F(x) be the primitive of $\dfrac{3x+2}{\sqrt{x-9}}$ w.r.t. x. If $F(10)=60$ then find the value of F(13).

Solution $F(x)=\displaystyle\int\frac{3x+2}{\sqrt{x-9}}dx$

Let $x-9=t^2 \Rightarrow dx=2t\,dt$

$$\therefore\ F(x)=\int\left(\frac{3(t^2+9)+2}{t}\cdot 2t\right)dt$$

$$=2\int(29+3t^2)dt=2[29t+t^3]$$

$$F(x)=2[29\sqrt{x-9}+(x-9)^{3/2}]+C$$

Given $F(10)=60=2[29+1]+C \Rightarrow C=0$

$$\therefore\ F(x)=2[29\sqrt{x-9}+(x-9)^{3/2}]$$

$$F(13)=2[29\times 2+4\times 2]$$

$$=4\times 33=132.$$

Concept Problems C

1. (a) Evaluate $\int(5x-1)^2\,dx$ by two methods : first square and integrate, then let $u=5x-1$.
   (b) Explain why the two apparently different answers obtained in part (a) are really equivalent.
2. Avni (using the substitution $u=\cos\theta$) claims that $\int 2\cos\theta\sin\theta\,d\theta=-\cos^2\theta$, while Meet (using the substitution $u=\sin\theta$) claims that the answer is $\sin^2\theta$. Who is right?
3. Integrate $\dfrac{x^2}{(x^2+1)^2}$ by the substitutions
   (a) $x=\tan\theta$,
   (b) $u=x^2+1$,
   and verify the argeement of the results.
4. Evaluate (i) $\displaystyle\int\frac{dx}{x\ln x}$ (ii) $\displaystyle\int\frac{dx}{x\ln x\ln\ln x}$
5. Evaluate the following integrals :
   (i) $\displaystyle\int\frac{\tan\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}dx$
   (ii) $\displaystyle\int\frac{1}{2\sqrt{x}}\tan^4\sqrt{x}.\sec^2\sqrt{x}dx$
   (iii) $\int\cos x\cos(\sin x)dx$
   (iv) $\int(x^2+1)\cos(x^3+3x+2)dx$
6. Evaluate the following integrals :
   (i) $\displaystyle\int\frac{x^2}{\sqrt{1-x}}dx$
   (ii) $\displaystyle\int\sqrt{\frac{x}{1-x^3}}dx$
   (iii) $\displaystyle\int\frac{x^2+3x+1}{(x+1)^2}dx$
   (iv) $\int(27e^{9x}+e^{12x})^{1/3}dx$
7. Evaluate the following integrals :
   (i) $\displaystyle\int\frac{dx}{e^x+1}$ (ii) $\displaystyle\int\frac{e^x-1}{e^x+1}dx$
   (iii) $\displaystyle\int\left(1-\frac{1}{x^2}\right)e^{x+1/x}dx$ (iv) $\int x^{\log_e x^2}\cdot\log_{e^x}(x)dx$

8. Evaluate the following integrals :

(i) $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx$

(ii) $\int e^x \tan(e^x)\sec(e^x)\, dx$

(iii) $\int (3 \sin x \cos^2 x - \sin^3 x)\, dx$

(iv) $\int \frac{\sec x \operatorname{cosec} x}{\log(\tan x)} dx$

Practice Problems C

9. Evaluate the following integrals :

(i) $\int (3x + 2)\sqrt{2x + 1}\, dx$

(ii) $\int \frac{x^2 + 1}{\sqrt[3]{(x^3 + 3x + 6)}} dx$

(iii) $\int \frac{1}{\sqrt{x}(4+3\sqrt{x})^2} dx$

(iv) $\int \left(5^{5^{5^x}} \times 5^{5^x} \times 5^x\right) dx$

10. Evaluate the following integrals :

(i) $\int \frac{\cos 2x}{(\cos x - \sin x)^2}$

(ii) $\int \frac{dx}{x \sin^2(1+\log x)}$

(iii) $\int \frac{dx}{(1 + x^2)\sqrt{(\tan^{-1} x + 3)}}$

(iv) $\int \frac{(x^2 - 1)dx}{(x^4 + 3x^2 + 1)\tan^{-1}(x + 1/x)}$

11. Evaluate the following integrals :

(i) $\int \frac{\tan x\, dx}{a + b\tan^2 x}$

(ii) $\int \frac{\cot x}{\ln \sin x} dx$

(iii) $\int \frac{\sec x\, dx}{\ln(\sec x + \tan x)}$

(iv) $\int \frac{a\cos x - b\sin x}{a\sin x + b\cos x + c} dx$

12. Evaluate the following integrals :

(i) $\int (\tan^3 x - x\tan^2 x)dx$

(ii) $\int \frac{\tan x \sec^2 x}{(a^2 + b^2\tan^2 x)^2} dx$

(iii) $\int \frac{\tan x\, dx}{\log \sec x}$

(iv) $\int \frac{\cos(\tan^{-1} x)dx}{(1+x^2)\sqrt{\sin(\tan^{-1} x)}}$

13. Evaluate the following integrals :

(i) $\int \frac{e^{3x}}{(1+e^x)^3} dx$

(ii) $\int \frac{\sec^4 x\, dx}{\sqrt{(\tan x)}}$

(iii) $\int \frac{2x\sin^{-1} x^2}{\sqrt{(1-x^4)}} dx$

(iv) $\int \sin 2x \cos^{11/2} x(1+\cos^{5/2} x)^{1/2} dx$

14. Evaluate the following integrals :

(i) $\int \frac{(x+1)(x+\log x)^2}{2x} dx$

(ii) $\int \frac{x\, dx}{(2x+1)^3}$

(iii) $\int \frac{\cos^2 x}{2+\sin x} dx$

(iv) $\int \left[\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right] \ln x\, dx$

15. Evaluate the following integrals :

(i) $\int \frac{\sqrt{(x - x^2)}}{x^3} dx$

(ii) $\int \frac{dx}{x^2(1+x^4)^{3/4}}$

(iii) $\int \frac{dx}{x^2(x+\sqrt{1+x^2})}$

(iv) $\int \frac{(x^4+1)}{x^2\sqrt{x^4+x^2-1}} dx$

16. Evaluate the following integrals :

(i) $\int \frac{dx}{x^2(1+x^5)^{4/5}}$ (ii) $\int \frac{x^2-1}{x\sqrt{(1+x^4)}}dx$

17. Show that the integral

$$\int((x^2-1)(x+1))^{-2/3}dx$$

can be evaluated with any of the following substitutions.

(a) $u = 1/(x+1)$

(b) $u = ((x-1)/(x+1))^k$ for $k = 1, 1/2. 1/3, -1/3, -2/3$, and -1

(c) $u = \tan^{-1} x$

(d) $u = \tan^{-1}\sqrt{x}$

(e) $u = \cos^{-1} x$

1.5 INTEGRALS INVOLVING SINE AND COSINE

Positive integral powers of sine and cosine

1. Odd positive index

Any odd positive power of sines and cosines can be integrated immediately by substituting $\cos x = z$ and $\sin x = z$ respectively as shown below.

Example 1. Evaluate $\int \sin^3 x\, dx$

Solution $I = \int \sin^2 x \sin x\, dx$

$= -\int(1-z^2)\,dz$, putting $z = \cos x$

$$= -\left(z - \frac{1}{3}z^3\right) + C = -\left(\cos x - \frac{1}{3}\cos^3 x\right) + C$$

Example 2. Evaluate $\int \cos^5 x\, dx$

Solution $I = \int \cos^4 x \cos x\, dx$

$= \int(1-z^2)^2\,dx$, putting $z = \sin x$

$$= \int(1-2z^2+z^4)dz = z - \frac{2}{3}z^3 + \frac{1}{5}z^5 + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

Example 3. Integrate $\int \cos^3 4x\, dx.$

Solution $\int \cos^3 4x\, dx. = dx\int \cos^2 4x \cos 4x\, dx$

$$= \int(1-\sin^2 4x)\cos 4x\, dx$$

$$= \int \cos 4x\, dx - \int \sin^2 4x \cos 4x\, dx$$

$$= \frac{1}{4}\sin 4x - \frac{1}{12}\sin^3 4x + C.$$

2. Even positive index

In order to integrate any even positive power of sine and cosine, we should first express it in terms of multiple angles by means of trigonometry and then integrate it. The simplest examples are the two integrals

$$\int \cos^2 x\, dx \text{ and } \int \sin^2 x\, dx,$$

which can be integrated by means of the identities

$$\cos^2 x = \frac{1}{2}(1+\cos 2x),$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

We get $\int \cos^2 x\, dx = \frac{1}{2}\int(1+\cos 2x)\,dx$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + C,$$

$$\int \sin^2 x\, dx = \frac{1}{2}\int(1-\cos 2x)\,dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + C.$$

Going on to the higher powers, consider the integral

$$\int \cos^{2n} x\, dx,$$

where n is an arbitrary positive integer. We write

$$\cos^{2n} x = (\cos^2 x)^n = \left[\frac{1}{2}(1+\cos 2x)\right]^n$$

We expand using binomial theorem and integrate the terms using previous methods.

Similarly,

$$\sin^{2n} x = (\sin^2 x)^n = \left[\frac{1}{2}(1-\cos 2x)\right]^n.$$

Example 4. Evaluate $\int \sin^4 x\, dx$

Solution $\int \sin^4 x\, dx = \frac{1}{2^2}\int(1-\cos 2x)^2\,dx$

$$= \frac{1}{4}\int(1-2\cos 2x+\cos^2 2x)\,dx$$

$$= \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}\int(1+\cos 4x)dx\right]$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right] + C.$$

Note: It should be noted that when the index is large, it would be more convenient to express the powers of sines or cosines of angles in terms of multiple angles by the use of De Moivre's theorem, as shown below.

Let $z = \cos x + i \sin x$ ...(1)

$\Rightarrow \frac{1}{z} = \cos x - i \sin x$ ...(2)

From (1) and (2),

$2 \cos x = z + \frac{1}{z}$ ...(3)

$2 i \sin x = z - \frac{1}{z}$ ...(4)

From De Moivre's Theorem

$z^n = \cos nx + i \sin nx$ ...(5)

$\frac{1}{z^n} = \cos nx - i \sin nx$ ...(6)

From (5) and (6),

$z^n + \frac{1}{z^n} = 2 \cos nx$ ...(7)

$z^n - \frac{1}{z^n} = 2 i \sin nx$ ...(8)

Example 5. Evaluate $\int \sin^8 x\, dx$.

Solution Using (4), we have

$$2^8 i^8 \sin^8 x = \left(z - \frac{1}{z}\right)^8$$

$$= \left(z^8 + \frac{1}{z^8}\right) - 8\left(z^6 + \frac{1}{z^6}\right)$$

$$+ 28\left(z^4 + \frac{1}{z^4}\right) - 56\left(z^2 + \frac{1}{z^2}\right) + 70$$

$$- 56.2 \cos 2x + 70 \text{ [using (7)]}$$

$\therefore \sin^8 x = 2^{-7}(\cos 8x - 8 \cos 6x + 28 \cos 4x - 56 \cos 2x + 35)$

$\therefore \int \sin^8 x\, dx$

$$= 2^{-7}\int (\cos 8x - 8\cos 6x + 28\cos 4x - 56 \cos 2x + 35)dx$$

$$= \frac{1}{2^7}\left[\frac{\sin 8x}{8} - \frac{8\sin 6x}{6} + 28\frac{\sin 4x}{4} - 56\frac{\sin 2x}{2} + 35x\right] + C$$

$$= \frac{1}{2^7}\left[\frac{1}{8}\sin 8x - \frac{4}{3}\sin 6x + 7\sin 4x - 28\sin 2x + 35x\right] + C.$$

Example 6. Evaluate $\int \cos^8 x\, dx$.

Solution Using (3), we have

$$2^8 \cos^8 x = z^8 + {}^8C_1 z^6 + {}^8C_2 z^4 + {}^8C_3 z^2 + {}^8C_4$$

$$+ {}^8C_5 \frac{1}{z^2} + {}^8C_6 \frac{1}{z^4} + {}^8C_7 \frac{1}{z^6} + \frac{1}{z^8}$$

$$= \left(z^8 + \frac{1}{z^8}\right) + 8\left(z^6 + \frac{1}{z^6}\right) + 28\left(z^4 + \frac{1}{z^4}\right) + 56\left(z^2 + \frac{1}{z^2}\right) + 70$$

$$= \cos 8x + 8.2 \cos 6x + 28.2 \cos 4x + 56.2 \cos 2x + 70$$

[using (7)]

$$\therefore \int \cos^8 x\, dx = \frac{1}{128}\int (\cos 8x + 8 \cos 6x + 28 \cos 4x + 56 \cos 2x + 35)\, dx$$

$$= \frac{1}{128}\left(\frac{\sin 8x}{9} + 8\frac{\sin 6x}{6} + 28.\frac{\sin 4x}{4} + 56.\frac{\sin 2x}{2} + 35x\right) + C$$

Integral powers of tangent and cotangent

Any integral power of tangent and cotangent can be readily integrated. Thus,

(i) $\int \tan^3 x\, dx = \int \tan x \,.\, \tan^2 x\, dx$

$$= \int \tan x(\sec 2x - 1)dx$$

$$= \int \tan x\, d(\tan x) - \int \tan x dx = \frac{1}{2}\tan^2 x - \ell n\,|\sec x| + C.$$

(ii) $\int \cot^4 x\, dx = \int \cot^2 x\, (\operatorname{cosec}^2 x - 1)dx$

$$= \int \cot^2 x \operatorname{cosec}^2 x dx - \int \cot^2 x\, dx$$

$$= -\int \cot^2 x d(\cot x) - \int (\operatorname{cosec}^2 x - 1)dx$$

$$= -\frac{1}{3}\cot^3 x + \cot x + x + C.$$

Positive integral powers of secant and cosecant

1. Even positive index

Even positive powers of secant or cosecant admit of immediate integration in terms of tan x or cot x. Thus,

(i) $\int \sec^4 x\, dx = \int (1 + \tan^2 x)\sec^2 x dx$

$$= \int \sec^2 x\, dx + \int \tan^2 x\, d(\tan x)$$

$$= \tan x + \frac{1}{3}\tan^3 x + C.$$

(ii) $\int \operatorname{cosec}^6 x\, dx = \int \operatorname{cosec}^4 x. \operatorname{cosec}^2 x\, dx$

$$= \int (1+\cot^2 x)^2 \operatorname{cosec}^2 x\,dx$$

$$= -\int (1+2\cot^2 x+\cot^4 x)\,d(\cot x)$$

$$= -\cot x - \frac{2}{3}\cot 3x - \frac{1}{5}\cot^5 x + C$$

2. Odd positive index

Odd positive powers of secant and cosecant are to be integrated by the application of the rule of integration by parts (to be dealt later).

(iii) $\int \sec^3 x\,dx = \int \sec x\,.\,\sec^2 x\,dx$

$$= \sec x\tan x - \int \sec x\tan^2 x\,dx$$

$$= \sec x\tan x - \int \sec x(\sec^2 x - 1)dx$$

$$= \sec x\tan x + \int \sec x\,dx - \int \sec^3 x\,dx.$$

$\therefore$ Transposing $\int \sec^3 x\,dx$ to the left side, writing the value of $\int \sec x\,dx$ and dividing by 2, we get

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C.$$

(iv) $\int \sec^5 x\,dx = \int \sec^3 x\sec^2 x\,dx$

$$= \sec^3 x\tan x - \int 3\sec^3 x\tan^2 x\,dx$$

$$= \sec^3 x\tan x - 3\int \sec^3 x(\sec^2 x - 1)dx$$

$$= \sec^3 x\tan x + 3\int \sec^3 c\,x\,dx - \int \sec^5 x\,dx$$

Now, transposing $3\int \sec^5 x\,dx$ and writing the value of $\int \sec^3 x\,dx$ we get ultimately

$$\int \sec^5 x\,dx = \frac{\tan x\sec^3 x}{4}$$

$$+\frac{3}{4}\frac{\tan x\sec x}{2} + \frac{3}{4}\frac{1}{2}\ln\left|\tan\left(\frac{\pi}{4}+\frac{\pi}{2}\right)\right| + C.$$

(v) $\int \operatorname{cosec}^3 x\,dx = \int \operatorname{cosec} x\operatorname{cosec}^2 x\,dx$

$$= -\operatorname{cosec} x\cot x - \int \operatorname{cosec} x\cot^2 x\,dx$$

$$= -\operatorname{cosec} x\cot x - \int \operatorname{cosec} x(\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\operatorname{cosec} x\cot x + \int \operatorname{cosec} x\,dx - \int \operatorname{cosec}^3 x\,dx$$

$\therefore$ Transposing $\int \operatorname{cosec}^3 x\,dx$ and writing the value of $\int \operatorname{cosec} x\,dx$ $\int \operatorname{cosec}^3 x\,dx = -\frac{1}{2}\operatorname{cosec} x\cot x + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C.$

Example 7. Find $\int \tan^6 x\,dx$.

Solution Here power of tanx is even positive integer therefore change $\tan^2 x$ into $\sec^2 x - 1$ and then put $z = \tan x$.

Now $\int \tan^6 x\,dx = \int (\tan^2 x)^3\,dx = \int (\sec^2 x - 1)^2 dx$

$$= \int (\sec^6 x - 3\sec^4 x + 3\sec^2 x - 1)dx$$

$$= \int \sec^6 x\,dx - 3\int \sec^4 x\,dx + 3\int \sec^2 x\,dx - \int dx$$

$$= \int \sec^6 x\,dx - 3\int \sec^4 x\,dx + 3\tan x - x \quad ...(1)$$

Let $z = \tan x$ then $dz = \sec^2 x\,dx$

Now $\int \sec^6 x\,dx - 3\int \sec^4 x\,dx$

$$= \int \sec^4 x\sec^2 x\,dx - 3\int \sec^2 x\sec^2 x\,dx$$

$$= \int (1+\tan^2 x)^2\sec^2 x\,dx - 3\int (1+\tan^2 x)\sec^2 x\,dx$$

$$= \int (1+z^2)^2\,dz - 3\int (1+z^2)dz$$

$$= \int (1+2z^2+z^4)dz - 3\int (1+z^2)dz$$

$$= z + 2.\frac{z^8}{3} + \frac{z^5}{5} - 3z - 3.\frac{z^3}{3}$$

$$= \frac{z^5}{5} - \frac{z^3}{3} - 2z = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} - 2\tan x$$

Putting in (1) we get

$$\int \tan^6 x\,dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} - 2\tan x + 3\tan x - x + C$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C.$$

Integrals Involving Sine and Cosine Together

$$\int \sin^m x \cos^n x\,dx$$

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosine. Then expand and integrate.

$$\int \overbrace{\sin^{2k+1}}^{\text{Odd}} x \cos^n x dx = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosine}} \cos^n x \overbrace{\sin x dx}^{\text{Save for du}}$$

$$= \int (1-\cos^2 x)^k \cos^n x \sin x\, dx$$

Here, we put $\cos x = t$.

Example 8. Evaluate $\int \frac{\sin^3 x}{\sqrt[3]{\cos^2 x}} dx$

Solution Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\int \frac{\sin^3 x}{\sqrt[3]{\cos^2 x}} dx = -\int \frac{(1-t^2)}{t^{2/3}} dt = \int \frac{\cos^4 x}{\sin^2 x} dx$$

$$= \int \frac{(1-\sin^2 x)^2}{\sin^2 x} dx = \int (\operatorname{cosec}^2 x - 2 + \sin^2 x)\, dx$$

$$= \int \left(\operatorname{cosec}^2 x - 2 + \frac{1-\cos 2x}{2}\right) dx.$$

$$= -\cot x - (3/2)x - (1/4)\sin 2x + C.$$

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sine. Then, expand and integrate.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x dx$$

$$= \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to sine}} \overbrace{\cos x dx}^{\text{Save for du}}$$

$$= \int \sin^m x (1-\sin^2 x)^k \cos x\, dx$$

Here, we put $\sin x = t$.

Example 9. Evaluate $\int \frac{\cos^3 x}{\sin^4 x} dx$

Solution $\int \frac{\cos^3 x}{\sin^4 x} dx = \int \frac{\cos^2 x \cos x dx}{\sin^4 x}$

$$= \int \frac{(1-\sin^2 x)\cos x dx}{\sin^4 x}$$

Denoting $\sin x = t$, $\cos x\, dx = dt$, we get

$$\int \frac{\cos^3 x}{\sin^4 x} dx = \int \frac{(1-t^2)dt}{t^4}$$

$$= \int \frac{dt}{t^4} - \int \frac{dt}{t^2} = -\frac{1}{3t^3} + \frac{1}{t} + C$$

$$= -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + C.$$

3. If the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$.

Putting them into the integral we get

$$\int \sin^{2p} x \cos^{2q} x\, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^p \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^q dx$$

Powering and opening brackets, we get terms containing cos 2x to odd and even powers. The terms with odd powers are integrated as indicated in Case 1. We again reduce the even exponents by formulae given above. Continuing in this manner we arrive at terms of the form $\int \cos kx\, dx$, which can easily be integrated.

4. If both exponents are even, and atleast one of them is negative, then the preceding technique does not give the desired result. If in the expression $\sin^m x \cos^n x$, $m + n$ is a negative even integer, then one should make the substitution $\tan x = t$ (or $\cot x = t$).

We have $\cos\theta = \frac{1}{\sqrt{1+x^2}}$, $\sin\theta = \frac{x}{\sqrt{1+x^2}}$,

and $d\theta = \frac{dx}{1+x^2}$,

The integral transforms into

$$\int \frac{x^m dx}{(1+x^2)^{\frac{m+n}{2}+1}}$$

Hence, if $m + n = -2r$, this becomes

$$\int x^m (1+x^2)^{r-1} dx,$$

a form which is immediately integrable.

Take, for example, $\int \frac{\sin^2 \theta d\theta}{\cos^6 \theta}$.

Let $x = \tan\theta$, and we get

$$\int x^2(1+x^2)dx = \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + C.$$

Next, to find $\int \frac{d\theta}{\sin\theta\cos^5\theta}$
making the same substitution, we obtain

$$\int \frac{(1+x^2)^2 dx}{x}.$$

Hence, the value of the proposed integral is

$$\frac{\tan^4\theta}{4} + \tan^2\theta + \ln|\tan\theta| + C$$

Again, to find $\int \frac{d\theta}{\sin^{3/2}\theta\cos^{5/2}\theta}$

Here the transformed expression is $\int \frac{(1+x^2)dx}{x^{3/2}}$, and

accordingly the value of the proposed integral is

$$\frac{2}{3}\tan^{\frac{3}{2}}\theta - \frac{2}{\tan^{1/2}\theta} + C.$$

In many cases it is more convenient to assume $x = \cot\theta$.

For example, to find $\int \frac{d\theta}{\sin^4\theta}$.

Since $d(\cot\theta) = -\frac{d\theta}{\sin^2\theta}$, if $\cot\theta = x$, the transformed

integral is $-\int(1+x^2)\,dx = -\cot\theta - \frac{\cot^3\theta}{3} + C.$

Example 10. Evaluate $\int \frac{\sqrt{\tan x}}{\sin 2x}dx$

Solution Here $m = -1/2$, $n = -3/2$,
so that $m + n = -2$, a negative integer.
Hence, we put $\tan x = t$, $\sec^2 x\,dx = dt$

$$\int \frac{\sqrt{\tan x}}{\sin 2x}dx = \int \frac{\sqrt{\tan x}}{\frac{2\sin x\cos x}{\cos^2 x}}\cos^2 x\,dx$$

$$= \int \frac{\sqrt{t}}{2t}dt = \int \frac{1}{2\sqrt{t}}dt = \sqrt{t} + C = \sqrt{\tan x} + C.$$

5. If in the expression $\sin^m x\cos^n x$, $m + n$ is a negative odd integer, then one should multiply the integrand by suitable power of $(\sin^2 x + \cos^2 x)$ and expand it into simpler integrals.

Example 11. Evaluate $\int \frac{dx}{\sin x\cos^2 x}$

Solution Here sum of powers of sin x and cos x is odd and negative, therefore multiply by suitable power of $(\cos^2 x + \sin^2 x)$.

$$I = \int \frac{dx}{\sin x\cos^2 x} = \int \frac{\cos^2 x + \sin^2 x}{\sin x\cos^2 x}dx$$

$$= \int \frac{\cos^2 x}{\sin x\cos^2 x}dx + \int \frac{\sin^2 x}{\sin x\cos^2 x}dx$$

$$= \int \operatorname{cosec} x\,dx + \int \frac{\sin x}{\cos^2 x}dx$$

$$= \ln\left|\tan\frac{x}{2}\right| + \int -\frac{dz}{z^2} \qquad \left[\because \int \operatorname{cosec} x\,dx = \ln\tan\frac{x}{2}\right]$$

$[z = \cos x]$

$$= \ln\left|\tan\frac{x}{2}\right| + \frac{1}{z} + c = \ln\tan\frac{x}{2} + \sec x + c.$$

Consider some more illustrations.

Example 12. Evaluate $\int \sin^4 x\cos^2 x\,dx$.

Solution Here power of neither cosx nor sin x is odd and positive and sum of their powers is even and positive.

Let $z = \cos x + i\sin x$, then $\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right)$

and $\sin x = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

Also $z^n + \frac{1}{z^n} = 2\cos nx$ and $z^n - \frac{1}{z^n} = 2i\sin nx$

Now, $\sin^4 x\cos 2x = \left(\frac{1}{2i}\right)^4\left(\frac{1}{2}\right)^2\left(z - \frac{1}{z}\right)^4\left(z + \frac{1}{z}\right)^2$

$$= \frac{1}{64}\left(z - \frac{1}{z}\right)^2\left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^2$$

$$= \frac{1}{64}\left(z - \frac{1}{z}\right)^2\left(z^2 - \frac{1}{z^2}\right)^2$$

$$= \frac{1}{64}\left(z^2 - 2 + \frac{1}{z^2}\right)^2\left(z^4 - 2 + \frac{1}{z^4}\right)$$

$$= \frac{1}{64}\left(z^6 - 2z^2 + \frac{1}{z^2} - 2z^4 + 4 - \frac{2}{z^4} + z^2 - \frac{2}{z^2} + \frac{1}{z^6}\right)$$

$$= \frac{1}{64}\left[\left(z^6 + \frac{1}{z^6}\right) - 2\left(z^4 + \frac{1}{z^4}\right) + \left(z^2 + \frac{1}{z^2}\right) - 2\left(z^2 + \frac{1}{z^2}\right) + 4\right]$$

$$= \frac{1}{64}(2\cos 6x - 2\cos 4x - 2\cos 2x + 4)$$

$$= \frac{1}{32}(\cos 6x - \cos 4x - \cos 2x + 2)$$

$$\therefore \int \sin^4 x \cos^2 x dx$$

$$= \frac{1}{32}\left(\frac{\sin 6x}{6} - \frac{\sin 4x}{4} - \frac{\sin 2x}{2} + 2x\right) + C.$$

Example 13. Integrate $\int \sin^2 x \cos^5 x \, dx$.

Solution $I = \int \sin^2 x \cos^4 x \cos x \, dx$

$$= \int \sin^2 x(1 - \sin 2x)^2 d(\sin x)$$

$$= \int z^2(1 - z^2)^2 dz, \quad \text{Putting } z = \sin x$$

$$= \int (z^2 - 2z^4 + z^6) dz$$

$$= \frac{1}{3}z^3 - \frac{2}{5}z^5 + \frac{1}{7}z^7 + C$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$$

Example 14. Integrate $\int \frac{\sin^2 x}{\cos^6 x} dx$.

Solution Here $p + q = 2 - 6 = -4$.

$\therefore$ Put $\tan x = z$, then $\sec^2 x \, dx = dz$.

Now, $I = \int \tan^2 x . \sec^4 x \, dx$

$$= \int z^2(1 + z^2) dz = \frac{1}{3}z^3 + \frac{1}{5}z^5 + C$$

$$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C.$$

Example 15. Integrate $\int \frac{dx}{\sin^{1/2} x \cos^{7/2} x}$.

Solution Here, $p + q = -\frac{1}{2} - \frac{7}{2} = -4$.

$\therefore$ Put $\tan x = z$, then $\sec^2 x \, dx = dz$.

Now, $I = \int \frac{\sec^4 x \, dx}{\tan^{1/2} x} = \int \frac{1 + z^2}{z^{1/2}} dz$

$$= \int (z^{-1/2} + z^{3/2}) dz = 2z^{1/2} + \frac{2}{5}z^{5/2} + C$$

$$= 2\tan^{1/2} x + \frac{2}{5}\tan^{5/2} x + C$$

Example 16. Evaluate $\int \sin^5 x \cos^4 x \, dx$.

Solution It is the exponent of the sine which is an odd positive integer. Hence

$$\int \sin^5 x \cos^4 x \, dx = \int (\sin^2 x)^2 \cos^4 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x) \cos^4 x \sin x \, dx$$

$$= \int \cos^4 x \sin x \, dx - 2\int \cos^6 x \sin x \, dx + \int \cos^8 x \sin x \, dx$$

$$= \frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C.$$

Example 17. Evaluate $\int \frac{\sin^2 x}{\cos^6 x} dx$

Solution $\int \frac{\sin^2 x}{\cos^6 x} dx = \int \frac{\sin^2 x(\sin^2 x + \cos^2 x)^2}{\cos^6 x} dx$

$$= \int \tan^2 x (1 + \tan^2 x)^2 dx$$

Put $\tan x = t$, then $x = \tan^{-1} t$, $dx = \frac{dt}{1 + t^2}$ and we get

$$\int \frac{\sin^2 x}{\cos^6 x} dx = \int t^2(1 + t^2)^2 \frac{dt}{1 + t^2}$$

$$= \int t^2(1 + t^2) dt = \frac{t^3}{3} + \frac{t^5}{5} + C$$

$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

Example 18. Evaluate $\int \frac{\cos^5 x}{\sin^2 x}\,dx.$

Solution Let $I = \int \frac{\cos^5 x}{\sin^2 x}\,dx = \int \frac{\cos^4 x}{\sin^2 x}\cos x\,dx$

$$= \int \frac{(1-\sin^2 x)^2}{\sin^2 x}\cos x\,dx.$$

Put $\sin x = t$ so that $\cos x\,dx = dt$.

Then $I = \int \frac{(1-t^2)^2}{t^2}\,dt = \int \frac{1-2t^2+t^4}{t^2}\,dt$

$$= \int \left[\frac{1}{t^2} - 2 + t^2\right] dt = -\frac{1}{t} - 2t + \frac{t^2}{3}$$

$$= -\frac{1}{\sin x} - \sin x + \frac{\sin^3 x}{3}$$

$$= \operatorname{cosec} x - 2\sin x + \frac{1}{3}\sin^3 x + C.$$

Example 19. Evaluate $\int \frac{dx}{\sin^3 x \cos^5 x}.$

Solution Here the integrand is $\sin^{-3} x \cos^{-5} x$. It is of the type $\sin^m x \cos^n x$, where

$m + n = -3 - 5 = -8$

i.e. a negative even integer.

$$\therefore \quad I = \int \frac{dx}{(\sin^3 x/\cos^3 x)\cos^3 x.\cos^5 x}$$

$$= \int \frac{\sec^3 x dx}{\tan^3 x} = \int \frac{\sec^6 x.\sec^2 x dx}{\tan^3 x}$$

$$= \int \frac{(1+\tan^2 x)^3 \sec^2 x dx}{\tan^3 x}$$

Now put $\tan x = t$ so that $\sec^2 x\,dx = dt$.

$$\therefore \quad I = \int \frac{(1+t^2)^3 dt}{t^3} = \int \left(\frac{1}{t^3} + \frac{3}{t} + 3t + t^2\right) dt$$

$$= -\{1/2t^2\} + 3\ln|t| + \frac{3}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C.$$

Example 20. Evaluate $I = \int \sin^3 x \cos^2 2x.$

Solution Here we use the formulae

$\sin^3 x = 1/4\,(3\sin x - \sin 3x)$,

$\cos^2 2x = 1/2\,(1 + \cos 4x)$,

Multiplying these two expressions and replacing $\sin x \cos 4x$, for example by $1/2\,(\sin 5x - \sin 3x)$, we obtain

$$I = \frac{1}{16}\int (7\sin x - 5\sin 3x + 3\sin 5x - \sin 7x)\,dx$$

$$= -\frac{7}{16}\cos x + \frac{5}{18}\cos 3x - \frac{3}{80}\cos 5x + \frac{1}{112}\cos 7x + C$$

The integral may of course be obtained in different forms by different methods. For example

$$I = \int \sin^3 x \cos^2 2x\,dx$$

$$= \int (4\cos^4 x - 4\cos^2 x + 1)(1 - \cos^2 x)\sin x\,dx,$$

which reduces, on making the substitution $\cos x = t$, to

$$\int (4t^6 - 8t^4 + 5t^2 - 1)\,dt$$

$$= \frac{4}{7}\cos^7 x - \frac{8}{5}\cos^5 x + \frac{5}{3}\cos^3 x - \cos x.$$

It may be verified that this expression and that obtained above differ only be a constant.

Example 21. Evaluate $\int \sqrt{\frac{\cos^3 x}{\sin^{11} x}}\,dx.$

Solution $I = \int \cos^{3/2} x \cdot \sin^{-11/2} x\,dx.$

Here $\frac{3}{2} + \frac{-11}{2} = -4 =$ negative even integer.

So, we put $\tan x = z$, then $\sec^2 x\,dx = dz$

$$\therefore \quad I = \int \cos^{3/2} x \cdot \sin^{-11/2} x \frac{dz}{\sec^2 x}$$

$$= \int \cos^{7/2} x \cdot \sin^{-11/2} x\,dz$$

$$= \int \frac{\cos^{7/2} x}{\sin^{7/2} x} \cdot \operatorname{cosec}^2 x\,dz$$

$$= \int \cot^{7/2} x \cdot \operatorname{cosec}^2 x\,dz$$

$$= \int z^{-\frac{7}{2}} \cdot \left(1 + \frac{1}{z^2}\right) dz = \int \left(z^{-\frac{7}{2}} + z^{-\frac{11}{2}}\right) dz$$

$$= \frac{z^{-\frac{5}{2}}}{-\frac{5}{2}} + \frac{z^{-\frac{9}{2}}}{-\frac{9}{2}} + C.$$

$$= \frac{-2}{5}\cot^{5/2} x + \frac{-2}{9}\cot^{9/2} x + C.$$

Example 22. Evaluate $\int \frac{dx}{\sqrt{(\cos^3 x \sin^5 x)}}.$

Solution Here the integrand is of the type $\cos^m x \sin^n x$. We have $m = -3/2$, $n = -5/2$, $m + n = -4$ i.e., and even negative integer.

$$\therefore \int \frac{dx}{\sqrt{(\cos^3 x \sin^5 x)}} = \int \frac{dx}{\cos^{3/2} x \sin^{5/2} x}$$

$$= \int \frac{dx}{\cos^{3/2} x(\sin^{5/2} x / \cos^{5/2} x).\cos^{5/2} x}$$

$$= \int \frac{dx}{\cos^4 x \tan^{5/2} x} = \int \frac{\sec^4 x}{\tan^{5/2} x} dx$$

$$= \int \frac{\sec^2 x}{\tan^{5/2} x} \sec^2 x \, dx$$

$$= \int \frac{(1+\tan^2 x)}{\tan^{5/2} x} \sec^2 x \, dx = \int \frac{(1+t^2)}{t^{5/2}} dt,$$

putting $\tan x = t$ and $\sec^2 x \, dx = dt$

$$= \int (t^{-5/2} + t^{-1/2}) \, dt = -\frac{2}{3} t^{-3/2} + 2t^{1/2} + C$$

$$= -\frac{2}{3} (\tan x)^{-3/2} + (\tan x)^{1/2} + C$$

$$= 2\sqrt{(\tan x)} - \frac{2}{3} (\tan x)^{-3/2} + C.$$

Integrals Involving Secants and Tangents

$$\int \sec^m x \tan^n x dx$$

1. If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then, expand and integrate.

$$\int \sec^{\overbrace{2k}^{\text{Even}}} x \tan^n x dx = \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x dx}^{\text{Save for du}}$$

$$= \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x \, dx$$

Here, we put $\tan x = t$.

2. If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then, expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x dx$$

$$= \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x \, dx}^{\text{Save for du}}$$

$$= \int \sec^{m-1} x(\sec^2 x - 1)^k \sec x \tan x \, dx$$

Here, we put $\sec x = t$.

3. If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor; then expand and repeat if necessary.

$$\int \tan^n x dx = \int \tan^{n-2} x \overbrace{(\tan^2 x)}^{\text{Convert to secants}} dx$$

$$= \int \tan^{n-2} x(\sec^2 x - 1)dx$$

4. If the integral is of the form $\int \sec^m x dx$, where m is odd and positive, use integration by parts.
5. If none of the first four cases applies, try converting to sines and cosines.

A similar strategy is adopted for

$$\int \mathbf{cosec}^m \, \mathbf{x} \, \mathbf{cot}^n \, \mathbf{x} \mathbf{dx}.$$

Example 23. Find $\int \tan\theta \sec^4\theta d\theta$.

Solution Here power of $\sec\theta$ is an even positive integer, therefore, we put $z = \tan\theta$

Let $z = \tan\theta$, then $dz = \sec^2\theta d\theta$

Now $\int \tan\theta \sec^4\theta d\theta = \int \tan\theta \sec^2\theta d\theta$

$$= \int \tan\theta \, (1 + \tan^2\theta) \sec^2\theta d\theta$$

$$= \int z(1 + z^2) \, dz$$

$$= \int (z + z^5) \, dz = \frac{z^2}{2} + \frac{z^4}{4} + C.$$

Example 24. Evaluate $\int \tan^3 x dx$.

Solution Here power of tanx is an odd positive integer therefore, we put $z = \sec x$.

Let $z = \sec x$, then $dz = \sec x \tan x \, dx$

Now $\int \tan^3 x dx = \int \frac{\tan^2 x \sec x \tan x}{\sec x} dx$

$$= \int \frac{\sec^2 x - 1}{\sec x} \sec x \tan x \, dx = \int \frac{z^2 - 1}{z} dz$$

$$= \int \left(z - \frac{1}{z}\right) dz = \frac{z^2}{2} - \ln|z| + C$$

$$= \frac{\sec^2 x}{2} - \ln|\sec x| + C.$$

Example 25. Find $\int \tan^3 2x \sec 2x dx$.

Solution Here power of $\sec^2 x$ is not an even positive integer and power of $\tan^2 x$ is an odd positive integer therefore, we put $z = \sec^2 x$

Let $z = \sec^2 x$, then $dz = 2\sec^2 x \tan^2 x dx$

Now $\int \tan^3 2x \sec 2x dx = \int \tan^2 2x \sec 2x \tan 2x dx$

$$= \int \frac{(\sec^2 2x - 1) 2 \sec 2x \tan 2x}{2} dx$$

$$= \int \frac{(z^2 - 1)}{2} dz = \frac{1}{2} \int (z^2 - 1) dz$$

$$= \frac{1}{2}\left[\frac{z^3}{3} - z\right] + c = \frac{\sec^3 2x}{6} - \frac{\sec^2 x}{2} + C.$$

Example 26. Find $\int \tan^6 x dx$.

Solution Here power of sec x is not an even positive integer and power of tan x is an even positive integer therefore, we change $\tan^2 x$ into $\sec^2 x - 1$ and then put

$z = \tan x$.

Now $\int \tan^4 x dx = \int (\tan^2 x)^2 dx = \int (\sec^2 x - 1)^2 dx$

$$= \int (\sec^4 x - 2\sec^2 x + 1) dx$$

$$= \int \sec^4 x dx - 2 \int (\sec^2 x dx + \int dx$$

$$= \int \sec^4 x dx - 2 \tan x + x \quad ...(1)$$

Again $\int \sec^4 x dx = \tan x + \frac{\tan^3 x}{3}$ [from previous example]

$\therefore$ From (1), $\int \tan^4 x dx$

$$= \tan x + \frac{\tan^3 x}{3} - 2\tan x + x + C$$

$$= \frac{\tan^3 x}{3} - \tan x + x + C.$$

Example 27. Find $\int \cot^2 x \operatorname{cosec}^4 x \, dx$.

Solution Here power of cosecx is even positive integer, therefore, put $z = \cot x$.

Let $z = \cot x$, then $dz = -\operatorname{cosec}^2 x \, dx$

Now, $\int \cot^2 x \operatorname{cosec}^4 x \, dx = \int \cot^2 x \operatorname{cosec}^2 x . \operatorname{cosec}^2 x \, dx$

$$= \int \cot^2 x (1 + \cot^2 x) \operatorname{cosec}^2 x \, dx$$

$$= \int z^2 (1 + z^2)(-dz)$$

$$= -\int (z^2 + z^4) dz = -\left(\frac{z^3}{3} + \frac{z^6}{5}\right) + C$$

$$= -\frac{\cot^3 x}{3} - \frac{\cot^5 x}{5} + C.$$

Practice Problems D

1. Evaluate the following integrals :

(i) $\int \cos^2 x \sin^3 x \, dx$ (ii) $\int \frac{\sin^3 \theta d\theta}{\sqrt{\cos\theta}}$

(iii) $\int \frac{d\theta}{\sin\theta \cos^3\theta}$ (iv) $\int \frac{dx}{\sin^3 x \cos^5 x}$

2. Evaluate the following integrals :

(i) $\int \frac{d\theta}{\sin^{1/2}\theta \cos^{7/2}\theta}$ (ii) $\int \frac{\sqrt{(\tan x)}}{\sin x \cos x} dx$

(iii) $\int \cos^5 x \, dx$ (iv) $\int \sin^7 x \, dx$

3. Evaluate the following integrals :

(i) $\int \frac{\sec^4 x}{\sqrt{\tan x}} dx$

(ii) $\int \frac{dx}{\sin^6 x}$

(iii) $\int \frac{\sqrt{\sin^3 2x}}{\sin^5 x} dx$

(iv) $\int \sqrt{\cos x - \cos^3 x \, dx}$

4. Integrate the following functions :

(i) $\sin^5 x \sec^6 x$ (ii) $\tan^2 x \sec^4 x$

(iii) $\sec^6 x$ (iv) $\operatorname{cosec}^5 x$.

5. Evaluate the following integrals :

(i) $\int \cos^5 x \operatorname{cosec}^2 x \, dx$

(ii) $\int \sqrt{\frac{\cos x}{\sin^5 x}} dx$

(iii) $\int \frac{\sec^2 x}{\sin 2x} dx$

(iv) $\int \frac{dx}{\sin\frac{x}{2}\sqrt{\cos^3\frac{x}{2}}}$

6. Evaluate the following integrals :

(i) $\int \cot^3 x \operatorname{cosec}^3 x \, dx$

(ii) $\int \left(\frac{\sec x}{\tan x}\right)^4 dx$

(iii) $\int \frac{\sin^3 x}{\cos^{2/5} x} dx$

(iv) $\int \frac{dx}{\sin^4 x \cos^2 x}$

7. Evaluate the following integrals :

(i) $\int \cos^6 \frac{1}{2} x\, dx$ (ii) $\int \tan^3 3x \sec 3x\, dx$

(iii) $\int \tan^{3/2} x \sec^4 x\, dx$ (iv) $\int \tan^4 x \sec^4 x\, dx$

1.6 RATIONALIZATION BY TRIGONOMETRIC SUBSTITUTION

Consider the integrand $\sqrt{a^2 - x^2}$.

If we change the variable from x to θ by the substitution x = a sin θ, then the identity

$1 - \sin^2\theta = \cos^2\theta$

allows us to get rid of the square root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta}$$
$$= \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a|\cos\theta|$$

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution x = a sin θ (the old variable is a function of the new one).

In general we can make a substitution of the form x = g(t). To make our calculations simpler, we assume that g has an inverse function; that is, g is one-one. We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward.

Hence, we make the substitution x = a sin θ restricting θ to lie in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that it defines a one-one function. Thus, x = a sin θ leads to $\theta = \sin^{-1}\left(\frac{x}{a}\right)$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

To simplify the integration of $\sqrt{x^2 + a^2}$, we substitute x = a tan θ. Further, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If x = a sec θ is substituted to simplify the integration of $\sqrt{x^2 - a^2}$ we want to be able to set $\theta = \sec^{-1}(x/a)$ when we are done.

As we know, the function in these substitutions have inverses only for selected values of θ.

For reversibility, x = a tan θ requires $\theta = \tan^{-1}\left(\frac{x}{a}\right)$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

x = a sec θ requires $\theta = \sec^{-1}\left(\frac{x}{a}\right)$ with

$$\begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \ge 1 \\ \frac{\pi}{2} < \theta \le \pi & \text{if } \frac{x}{a} \le -1 \end{cases}$$

To simplify calculations with the substitution x = a sec θ, we will restrict its use to integrals in which x/a ≥ 1. This will place θ in [0, π/2) and make tan θ ≥ 0. We will then have

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \tan^2\theta} = |a \tan\theta| = a \tan\theta,$$

free of absolute values, provided a > 0.

Any algebraic expression in x which contains only one surd of a quadratic form, is capable of being rationalized by such trigonometric substitutions.

For example :

(i) $\int \frac{dx}{(a^2 - x^2)^{3/2}}$

Let x = a sin θ, and we get

$$\frac{1}{a^3}\int \frac{d\theta}{\cos^2\theta} = \frac{\tan\theta}{a^2} + C = \frac{x}{a^2\sqrt{a^2 - x^2}} + C$$

(ii) $\int \frac{dx}{x^2(1 + x^2)^{\frac{1}{2}}}$

Let x = tanθ, and the integral becomes

$$\int \frac{\cos\theta\, d\theta}{\sin^2\theta} = \int \frac{d(\sin\theta)}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = -\frac{\sqrt{1 + x^2}}{x} + C$$

(iii) $\int \frac{dx}{x^3(x^2 - 1)^{1/2}}$

Let x = sec θ, and the integral becomes

$$\int \cos^2\theta\, d\theta = \frac{\sin\theta\cos\theta}{2} + \frac{\theta}{2} + C$$
$$= \frac{\sqrt{x^2 - 1}}{2x^2} + \frac{1}{2}\sec^{-1} x + C$$

Suppose R denotes a rational function of the entities involved. The integral of the form

1. $\int R(x, \sqrt{b^2 - a^2x^2})dx$ is simplified by the substitution $x = \frac{b}{a}\sin\theta$

2. $\int R(x, \sqrt{a^2x^2 + b^2})dx$ is simplified by the substitution $x = \frac{b}{a}\tan\theta$

3. $\int R(x, \sqrt{a^2x^2 - b^2})dx$ is simplified by the substitution $x = \frac{b}{a}\sec\theta$

Example 1. Compute the integral $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}}$

Solution We use the substitution $x = a\sin\theta$, with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ then $dx = a\cos\theta\, d\theta$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \int \frac{a\cos\theta d\theta}{\sqrt{(a^2 - a^2\sin^2\theta)^3}}$$

$$= \int \frac{a\cos\theta d\theta}{a^3\cos^3\theta} = \frac{1}{a^2}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{a^2}\tan\theta + C$$

$$= \frac{1}{a^2}\frac{\sin\theta}{\cos\theta} + C = \frac{1}{a^2}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} + C.$$

The restriction on θ serves two purposes – it enables us to replace $|\cos\theta|$ by $\cos\theta$ to simplify the calculations, and it also ensures that the substitutions can be rewritten as $\theta = \sin^{-1}(x/a)$, if needed.

Example 2. Find $\int \frac{(16-9x^2)^{3/2}}{x^6}dx.$

Solution Let $x = \frac{4}{3}\sin\theta$. Then $dx = \frac{4}{3}\cos\theta\, d\theta$ and $\sqrt{16-9x^2} = 4\cos\theta$. Hence

$$\int \frac{(16-9x^2)^{3/2}}{x^6}dx = \int \frac{(64\cos^3\theta)\left(\frac{4}{3}\cos\theta\, d\theta\right)}{\frac{4096}{729}\sin^6\theta}$$

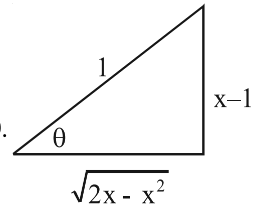
$$= \frac{243}{16}\int \cos^4\theta\,\mathrm{cosec}^2\theta\, d\theta = -\frac{243}{80}\cot^5\theta + C$$

$$= -\frac{1}{80}\frac{(16-9x^2)^{5/2}}{x^5} + C$$

Example 3. Find $\int \frac{x^2dx}{\sqrt{2x-x^2}}$

Solution We have $I = \int \frac{x^2dx}{\sqrt{1-(x-1)^2}}$

Let $x - 1 = \sin\theta$.



Then $dx = \cos\theta\, d\theta$ and $\sqrt{2x-x^2} = \cos\theta$. Hence,

$$I = \int \frac{(1+\sin\theta)^2}{\cos\theta}\cos\theta\, d\theta$$

$$= \int (1+\sin\theta)^2\, d\theta = \int\left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta + C$$

$$= \frac{3}{2}\sin^{-1}(x-1) - 2\sqrt{2x-x^2}$$

$$-\frac{1}{2}(x-1)\sqrt{2x-x^2} + C$$

$$= \frac{3}{2}\sin^{-1}(x-1) - \frac{1}{2}(x+3)\sqrt{2x-x^2} + C.$$

Example 4. Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$.

Solution We set

$$x = 2\tan\theta,\ dx = 2\sec2\theta\, d\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta.$$

Then, $\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta d\theta}{|\sec\theta|}$

$\sqrt{\sec^2\theta} = |\sec\theta|$ $\sec\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + C$$

$$= \ln |\sqrt{4+x^2}| + C'. \quad [\text{Taking } C' = C - \ln 2]$$

Note: A trigonometric substitution can sometimes help us to evaluate an integral containing an integer power of a quadratic binomial, as in the next example.

Example 5. Evaluate $I = \int \frac{dx}{(x^2+4)^2}$

Solution Put $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$,

Also, $\theta = \tan^{-1}\frac{x}{2}$.

$$I = \int \frac{2\sec^2\theta d\theta}{(4\sec^2\theta)^2}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta} = \frac{1}{16}\int 2\cos^2\theta\, d\theta$$

$$= \frac{1}{16}\int (1+\cos 2\theta)\, d\theta = \frac{1}{16}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{1}{8(4+x^2)} + C.$$

Example 6. Evaluate $\int \frac{\sqrt{x^2-25}}{x}\, dx$, assuming that $x \le -5$.

Solution The integrand involves a radical of the form $\sqrt{x^2-a^2}$ with $a = 5$, so we make the substitution

$$x = 5\sec\theta, \quad \frac{\pi}{2} < \theta \le \pi$$

$$\frac{dx}{d\theta} = 5\sec\theta\tan\theta \text{ or, } dx = 5\sec\theta\tan\theta\, d\theta$$

Thus, $\int \frac{\sqrt{x^2-25}}{x}\, dx$

$$= \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta)d\theta$$

$$= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\, d\theta$$

$$= -5\int \tan^2\theta\, d\theta$$

$$= -5\int (\sec^2\theta - 1)d\theta = -5\tan\theta + 5\theta + C$$

where $\tan\theta = -\frac{\sqrt{x^2-25}}{5}$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2-25}}{x}\, dx = -\sqrt{x^2-25} + 5\sec^{-1}\left(\frac{x}{5}\right) + C.$$

Example 7. Find $\int \frac{dx}{(4x^2-24x+27)^{3/2}}$

Solution

$$\int \frac{dx}{(4x^2-24x+27)^{3/2}} = \int \frac{dx}{(4(x-3)^2-9)^{3/2}}$$

Let $x - 3 = \frac{3}{2}\sec\theta$.

Then $dx = \frac{3}{2}\sec\theta\tan\theta\, d\theta$ and

$$\sqrt{4x^2-24x+27} = 3\tan\theta.$$

So, $\int \frac{dx}{(4x^2-24x+27)^{3/2}} = \int \frac{\frac{3}{2}\sec\theta\tan\theta d\theta}{27\tan^3\theta}$

$$= \frac{1}{18}\int \operatorname{cosec}\theta\cot\theta\, d\theta$$

$$= \frac{1}{18}\operatorname{cosec}\theta + C = -\frac{1}{9}\frac{x-3}{\sqrt{4x^2-24x+27}} + C$$

Example 8. Integrate $\int \frac{dx}{x\sqrt{x^4-1}}$.

Solution Here $\sqrt{x^4-1} = \sqrt{(x^2)^2-1}$ which is of the form $\sqrt{x^4-a^2}$. Hence, the substitution $x^2 = \sec\theta$ should be tried.

$$\text{Now } \int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{dx}{x\sqrt{(x^2)^2-1}} \qquad ...(1)$$

Let $x^2 = \sec\theta$ then $2x\, dx = \sec\theta\tan\theta\, d\theta$

$$dx = \frac{\sec\theta\tan\theta}{2x}d\theta = \frac{\sec\theta\tan\theta}{2\sqrt{\sec\theta}}d\theta$$

Now from (1),

$$\int \frac{dx}{x\sqrt{x^4-1}} = \int\left(\frac{1}{\sqrt{\sec\theta}\sqrt{\sec^2\theta-1}}\right)\frac{\sec\theta.\tan\theta}{2\sqrt{\sec\theta}}d\theta$$

$$= \int\left(\frac{1}{\sqrt{\sec\theta.\tan\theta}}\right)\frac{\sec\theta\tan\theta}{2\sqrt{\sec\theta}}d\theta$$

$$= \int \frac{1}{2}d\theta = \frac{1}{2}\theta + C = \frac{1}{2}\sec^{-1}(x^2) + C.$$

Note: When integrand involves expressions of the form :

(i) $\sqrt{\dfrac{a-x}{a+x}}$ put $x = a\cos 2\theta$.

(ii) $\sqrt{\dfrac{x}{a-x}}$ put $x = a\sin^2\theta$.

(iii) $\sqrt{\dfrac{x}{a+x}}$ put $x = a\tan^2\theta$.

(iv) $\sqrt{\dfrac{x-a}{b-x}}$ or $\sqrt{(x-a)(b-x)}$

put $x = a\cos^2\theta + b\sin^2\theta$

(v) $\sqrt{\dfrac{x-a}{x-b}}$ or $\sqrt{(x-a)(x-b)}$

put $x = a\sec^2\theta - b\tan^2\theta$

Example 9. Evaluate $I = \int \dfrac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

Solution Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

so that $dx = 2(\beta-\alpha)\sin\theta\cos\theta\, d\theta$.

Also $(x-\alpha) = (\beta-\alpha)\sin^2\theta$,

and $(\beta - x) = (\beta-\alpha)\cos^2\theta$.

Making these substitutions, the given integral

$$I = \int \frac{2(\beta-\alpha)\sin\theta\cos\theta\, d\theta}{\sqrt{(\beta-\alpha)\cos^2\theta.(\beta-\alpha)\sin^2\theta}}$$

$$= \int \frac{2(\beta-\alpha)\sin\theta\cos\theta}{(\beta-\alpha)\cos\theta\sin\theta} d\theta$$

$$= 2\int d\theta = 2\theta + C = \cos^{-1}(\cos 2\theta) + C \quad ...(1)$$

But $x = \alpha\cos^2\theta + \beta\sin^2\theta$;

$\therefore$ $2x = \alpha(1+\cos 2\theta) + \beta(1-\cos 2\theta)$

i.e. $(\beta-\alpha)\cos 2\theta = (\alpha+\beta-2x)$

or $\cos 2\theta = (\alpha+\beta-2x)/(\beta-\alpha)$.

$\therefore$ From (1), we get

$$I = \cos^{-1}\left(\frac{\alpha+\beta-2x}{\beta-\alpha}\right) + C.$$

Note that using $(x-\alpha) = (\beta-\alpha)\sin^2\theta$, the answer can also be written as $2\sin^{-1}\sqrt{\dfrac{x-\alpha}{\beta-\alpha}} + C$.

Example 10. Integrate $\int \sqrt{\dfrac{1-x}{1+x}}dx$.

Solution Let $x = \cos 2\theta$ then $dx = -2\sin 2\theta\, d\theta$

Now $\int \sqrt{\dfrac{1-x}{1+x}}dx = \int \sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}(-2\sin 2\theta)d\theta$

$$= -2\int \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\, 2\sin\theta\cos\theta\, d\theta$$

$$= -2\int \frac{\sin\theta}{\cos\theta} 2\sin\theta\cos\theta\, d\theta = -2\int 2\sin^2\theta\, d\theta$$

$$= -2\int (1-\cos 2\theta)d\theta = -2\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= -2\theta + \sin 2\theta + C \quad ...(1)$$

$\because$ $\cos 2\theta = x$, $2\theta = \cos^{-1}x$ and $\sin 2\theta$

$= \sqrt{1-\cos^2 2\theta} = \sqrt{1-x^2}$

Hence from (1),

$$\int \sqrt{\frac{1-x}{1+x}}dx = -\cos^{-1}x + \sqrt{1-x^2} + C.$$

Example 11. Evaluate $\int \dfrac{dx}{\sqrt{(x-a)(x-b)}}$

Solution Put $x = a\sec^2\theta - b\tan^2\theta$ ...(1)

$\therefore$ $dx = [a.2\sec^2\theta\tan\theta - 2b\tan\theta\sec^2\theta]\, d\theta$

$= 2(a-b)\sec^2\theta\tan\theta\, d\theta$.

Thus, $x - a = a(\sec^2\theta - 1) - b\tan^2\theta$

$= (a-b)\tan^2\theta$ ...(2)

and $x - b = a\sec^2\theta - b(1+\tan^2\theta$

$= (a-b)\sec^2\theta$ ...(3)

$$\therefore \int \frac{dx}{\sqrt{(x-a)(x-b)}} = \int \frac{2(a-b)\sec^2\theta\tan\theta}{(a-b)\tan\theta\sec\theta}d\theta$$

$$= 2\int \sec\theta\, d\theta = 2\ln|\sec\theta + \tan\theta| + C.$$

From (2) and (3),

$$\sec\theta = \sqrt{\left(\frac{x-b}{a-b}\right)} \text{ and } \tan\theta = \sqrt{\left(\frac{x-a}{a-b}\right)}.$$

$\therefore$ The given integral

$$= 2\ln\left[\frac{\sqrt{(x-b)}+\sqrt{(x-a)}}{\sqrt{(a-b)}}\right] + C$$

$$= 2\ln\left[\sqrt{(a-b)} + \sqrt{(x-a)}\right] + C_1,$$

omitting the constant term $-2\ln\sqrt{(a-b)}$ which may be added to the constant of integration C.

Example 12. Evaluate $I = \int \dfrac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$

Solution We make the substitution

$x = \sin^2 t \Rightarrow dx = 2\sin t \cos t\, dt$

Hence,

$$I = \int \frac{2\sin t \cos t dt}{(1+\sin t)\sqrt{\sin^2 t - \sin^4 t}} = \int \frac{2dt}{1+\sin t}$$

$$= 2\int \frac{1-\sin t}{\cos^2 t} dt = 2\tan t - \frac{2}{\cos t} + C$$

$$= \frac{2\sqrt{x}}{\sqrt{1-x}} - \frac{2}{\sqrt{1-x}} + C = \frac{2(\sqrt{x}-1)}{\sqrt{1-x}} + C.$$

Practice Problems E

Evaluate the following integrals :

1. $\int \frac{x^2 dx}{(4-x^2)^{5/2}}$
2. $\int \frac{dx}{(4x-x^2)^{3/2}}$
3. $\int \frac{x^3 dx}{\sqrt{a^8 - x^8}}$
4. $\int \frac{dx}{x\sqrt{2ax - a^2}}$
5. $\int \frac{dx}{(9+x^2)^2}$
6. $\int \frac{\sqrt{y^2 - 49}}{y} dy, y < -7$
7. $\int \frac{dx}{(1-x^2)\sqrt{(1-x^2)}}$
8. $\int \frac{\sqrt{x}}{\sqrt{(a^3 - x^3)}} dx$
9. $\int \frac{x^2}{\sqrt{(a^6 - x^6)}} dx$
10. $\int \frac{dx}{x\sqrt{(x^4 - 1)}}$
11. $\int \frac{x^2 dx}{(4-x^2)^{5/2}}$
12. $\int \frac{1-x+x^2}{(1-x^2)^{3/2}} dx$
13. $\int \frac{dx}{(4x-x^2)^{3/2}}$
14. $\int \frac{dx}{(9+x^2)^2}$
15. $\int \frac{dx}{(x-\beta)\sqrt{(x-\alpha)(\beta-x)}}$

1.7 INTEGRALS OF THE FORM

$$\int \frac{\mathbf{dx}}{\mathbf{ax^2 + bx + c}}, \int \frac{\mathbf{dx}}{\sqrt{\mathbf{ax^2 + bx + c}}}, \int \sqrt{\mathbf{ax^2 + bx + c}}\, \mathbf{dx}$$

Some Standard Integrals

(i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(iii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

(iv) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + C$

(v) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + C$

(vi) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(vii) $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(viii) $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2+a^2}\right) + C$

(ix) $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2-a^2}\right| + C$

Let us evaluate some of these standard integrals.

(a) $I = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

Proof : Put $x = a\tan\theta$, then $dx = a\sec^2\theta\, d\theta$

$$\therefore \quad I = \int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Note: Putting $x = a\cot\theta$, the above integral takes up the form $-(1/a)\cot^{-1}(x/a)$, which evidently differs from the previous form by a constant. Sometimes,

$$\int \frac{dx}{a^2+x^2} \text{ is written in the form } -\frac{1}{a}\cot^{-1}\frac{x}{a} + C$$

It is an established convention of the calculus that the integral of a real function shall be presented as a real function and not as a function involving complex numbers. For example (anticipating formulae from the chapter of differentiation, to illustrate the point)

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \quad (a, a > 0)$$

uses only real numbers and is the standard form for this integral.
The calculation

$$\int \frac{1}{x^2+a^2}dx = \int \frac{1}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)dx$$

$$= \frac{1}{2ai}\ell n \frac{x-ai}{x+ai}$$

though it can be fitted to the theory, is rejected because it does not give the integral in terms of real numbers.

(b) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

Proof : $\int \frac{dx}{x^2 - a^2} = \int \frac{1}{2a}\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\}dx$

$$= \frac{1}{2a}\left\{\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right\}$$

$$= \frac{1}{2a}\{\ln|(x-a)| - \ln|(x+a)|\} + C.$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Note: The above is an example of integration by breaking up the integrand into partial fractions.

(c) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

The proof is as before, noticing that

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right)$$

(d) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|(x + \sqrt{x^2 \pm a^2}\right| + C$

Proof : Put $\sqrt{x^2 \pm a^2} = t - x$,

or, $t = x + \sqrt{x^2 \pm a^2}$

$$\therefore \quad dt = \left(1 + \frac{2x}{2\sqrt{x^2 \pm a^2}}\right)dx = \frac{t}{\sqrt{x^2 \pm a^2}}dx$$

$$\therefore \quad \int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \int \frac{dx}{t} = \ln|t| + C$$

$$= \ln|(x + \sqrt{x^2 \pm a^2}| + C$$

The first of the integrals (d) can also be evaluated by putting $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$. Thus,

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a\sec^2\theta\, d\theta}{a\sec\theta} = \int \sec\theta\, d\theta$$

$$= \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}d\theta$$

$$= \ln|\sec\theta + \tan\theta|$$

$$= \ln|\sqrt{(1+\tan^2\theta)} + \tan\theta| + C'$$

$$= \ln\left|\sqrt{1 + \frac{x^2}{a^2}} + \frac{x}{a}\right| + C' = \ln\left|\frac{x + \sqrt{(x^2 + a^2)}}{a}\right| + C'$$

Similarly, put $x = a\sec\theta$, in the other integral.

(e) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C, \quad (|x| < |a|)$

Put $x = a\sin\theta$, then $dx = a\cos\theta\, d\theta$.

$$\therefore \quad I = \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta = \theta = \sin^{-1}\frac{x}{a} + C$$

Note: Putting $x = a\cos\theta$, the integral $\int \frac{dx}{\sqrt{(a^2 - x^2)}}$ can be put in the form $-\cos^{-1}\frac{x}{a} + C'$ instead of $\sin^{-1}\frac{x}{a} + C$.

(f) $\int \sqrt{x^2 - a^2}\,dx = \frac{x\sqrt{x^2 - a^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Proof : Putting $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$, we get

$$\int \sqrt{a^2 - x^2}\,dx = a^2\int \cos^2\theta\, d\theta$$

$$= a^2 . \frac{1}{2}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}a^2\left[\int \cos 2\theta\, d\theta + \frac{1}{2}\int d\theta\right]$$

$$= \frac{1}{2}a^2\left[\frac{1}{2}\sin 2\theta + \theta\right] + C$$

$$= \frac{1}{2}a^2 . \sin\theta\cos\theta + \frac{1}{2}a^2\theta + C$$

$$= \frac{1}{2}a^2\frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a} + C$$

$$= \frac{x\sqrt{x^2 - a^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

Note: For the evaluation of the integrals $\int \sqrt{x^2 + a^2}\,dx$ and $\int \sqrt{x^2 - a^2}\,dx$, refer the section of Integration by Parts.

Example 1. Evaluate $\int \frac{1}{4+9x^2}\,dx$

Solution We have $\int \frac{1}{4+9x^2}$

$$= \frac{1}{9}\int \frac{1}{\frac{4}{9}+x^2}\,dx = \frac{1}{9}\int \frac{1}{(2/3)^2+x^2}\,dx$$

$$= \frac{1}{9}\cdot\frac{1}{(2/3)}\tan^{-1}\left(\frac{x}{2/3}\right)+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)+C.$$

Example 2. Evaluate the following integrals :

(a) $\int\sqrt{16-x^2}\,dx$, (b) $\int\sqrt{x^2-16}\,dx$,

(c) $\int\sqrt{x^2+16}\,dx$,

Solution

(a) The integrand is of the form $\sqrt{a^2-x^2}$, where a=4.

$\therefore \int\sqrt{16-x^2}\,dx = 1/2\,x\sqrt{16-x^2}+8\sin^{-1}(x+4)+C$

(b) The integrand is of the form $\sqrt{x^2-a^2}$, where a = 4.

$\therefore \int\sqrt{x^2-16}\,dx = 1/2\,x\sqrt{x^2-16}-8\ln|\sqrt{x^2-16}+x|+C.$

(c) The integrand is of the form $\sqrt{x^2+a^2}$, where a = 4.

$\therefore \int\sqrt{x^2+16}\,dx = 1/2\,x\sqrt{x^2+16}$

$+8\ln(\sqrt{x^2+16}+x)+C.$

The integrals of the form $\int\frac{dx}{ax^2+bx+c}$,

$\int\frac{dx}{\sqrt{ax^2+bx+c}}$, and $\int\sqrt{ax^2+bx+c}\,dx$ can be evaluated by first expressing ax^2+bx+c in the form of a perfect square and then applying the standard results.

For example $x^2+2x+2=(x+1)^2+1$.

and hence the substitution $u = x + 1$, $du = dx$ yields

$$\int\frac{1}{x^2+2x+2}\,dx = \int\frac{1}{u^2+1}\,du$$

$= \tan^{-1}u + C = \tan^{-1}(x+1)+C.$

In general, the objective is to convert $ax^2 + bx + c$ into either a sum or difference of two squares–either $u^2 \pm a^2$ or $a^2 - u^2$.

Thus, in each case write

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$$

and put $x+\frac{b}{2a}=u$ and use the standard formulae.

Example 3. Evaluate $\int\frac{1}{4+(2-3x)^2}\,dx.$

Solution Here we shall apply the formula

$$\int\frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

But in place of x, we have (2 – 3x) and hence we shall divide by the derivative of 2 – 3x which is –3.

$$\therefore\ I = \frac{1}{-3}\cdot\frac{1}{2}\tan^{-1}\frac{2-3x}{2}+C = \frac{-1}{6}\tan^{-1}\frac{2-3x}{2}+C.$$

Example 4. Evaluate $\int\frac{dx}{(2x^2+x-1)}$.

Solution We have

$$\int\frac{dx}{(2x^2+x-1)} = \frac{1}{2}\int\frac{dx}{\left(x^2+\frac{1}{2}x-\frac{1}{2}\right)}$$

$$= \frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{4}\right)^2-\frac{1}{2}-\frac{1}{16}}$$

$$= \frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{4}\right)^2-\frac{9}{16}} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4}{3}\ln\left|\frac{x+\frac{1}{4}-\frac{3}{4}}{x+\frac{1}{4}+\frac{3}{4}}\right|+C$$

$$= \frac{1}{2}\ln\left|\frac{2x-1}{2(x+1)}\right|+C = \frac{1}{3}\ln\left|\frac{2x-1}{x+1}\right|-\frac{1}{3}\ln 2+C$$

$$= \frac{1}{3}\ln|(2x-1)/(x-1)|+C_1.$$

Example 5. Evaluate $\int\frac{1}{x^2-x+1}\,dx$

Solution $\int\frac{1}{x^2-x+1}\,dx$

$$= \int\frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1}\,dx$$

$$= \int\frac{1}{(x-1/2)^2+3/4}\,dx$$

$$= \int \frac{1}{(x-1/2)^2 + \left(\sqrt{3}/2\right)^2}\, dx$$

$$= \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

Example 6. Evaluate $\int \frac{dx}{\sqrt{x^2-4x+14}}$.

Solution $x^2-4x+14 = (x^2-4x+4)+10 = (x-2)^2+10$

$$I = \int \frac{dx}{\sqrt{(x-2)^2+10}}$$

Put $x-2=t \Rightarrow dx = dt$

$$I = \int \frac{dt}{\sqrt{t^2+10}} = \ln\left(t+\sqrt{t^2+10}\right) + C$$

$$= \ln\left(x-2+\sqrt{x^2-4x+14}\right) + C$$

Example 7. Evaluate $\int \frac{1}{\sqrt{x^2-8x-9}}\, dx$

Solution $\int \frac{1}{\sqrt{x^2-8x-9}}\, dx$

$$= \int \frac{1}{\sqrt{x^2-8x+16-25}}\, dx$$

$$= \int \frac{1}{\sqrt{(x-4)^2-5^2}}\, dx$$

$$= \ln\left|x-4+\sqrt{(x-4)^2-5^2}\right| + C$$

$$= \ln\left|x-4+\sqrt{x^2-8x-9}\right| + C.$$

Example 8. Evaluate $\int \frac{dx}{\sqrt{(4+3x-2x^2)}}$.

Solution $\int \frac{dx}{\sqrt{(4+3x-2x^2)}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\{2(3/2)x - x^2\}}}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left\{2-\left(x^2-\frac{3}{2}x\right)\right\}}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left\{2+\frac{9}{16}-\left(x^2-\frac{3}{2}x+\frac{9}{16}\right)\right\}}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\{(41/16)-(x-3/4)^2\}}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\{(\sqrt{41/4})^2-(x-3/4)^2\}}}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left\{\frac{x-\frac{3}{4}}{(\sqrt{41/4})}\right\} + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right) + C.$$

Example 9. Evaluate $\int \sqrt{x^2+2x+5}\, dx$

Solution $\int \sqrt{x^2+2x+5}$

$$= \int \sqrt{x^2+2x+1+4}\, dx$$

$$= \frac{1}{2}(x+1)\sqrt{(x-1)^2+2^2}$$

$$+ \frac{1}{2}.(2)^2 \ln\left((x+1) + \sqrt{(x+1)^2+2^2}\right) + C$$

$$= \frac{1}{2}(x+1)\sqrt{x^2-2x+5}$$

$$+ 2\ln\left((x+1) + \sqrt{x^2+2x+5}\right) + C$$

Example 10. Evaluate $\int \sqrt{2x^2+3x+4}\, dx$

Solution $\int \sqrt{2x^2+3x+4}\, dx$

$$= \sqrt{2} \int \sqrt{\left(x^2+\frac{3}{2}x+2\right)}\, dx$$

$$=\sqrt{2}\int\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\,dx$$

$$=\sqrt{2}\int\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\,dx$$

$$=\sqrt{2}\left[\frac{1}{2}\left(x+\frac{3}{4}\right)\sqrt{2x^2+3x+4}\right]$$

$$+\frac{1}{2}\frac{23}{16}\ln\left(x+\frac{3}{4}+2\sqrt{2x^2+3x+4}\right)+C$$

$$=\frac{1}{4\sqrt{2}}(4x+3)\sqrt{2x^2+3x+4}$$

$$+\frac{23}{16\sqrt{2}}\ln\left(x+\frac{3}{4}+\sqrt{2x^2+3x+4}\right)+C$$

Example 11. Evaluate $\int\sqrt{(x-1)(2-x)}\,dx.$

Solution We have $\int\sqrt{(x-1)(2-x)}\,dx$

$$=\int\sqrt{(-x^2+3x-2)}\,dx$$

$$=\int\sqrt{\left\{-2-\left(x-\frac{3}{2}\right)^2+\frac{9}{4}\right\}}\,dx$$

$$=\int\sqrt{\left\{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2\right\}}\,dx$$

$$=\frac{1}{2}\left(x-\frac{3}{2}\right)\sqrt{\left\{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2\right\}}$$

$$+\frac{1}{2}\cdot\frac{1}{4}\sin^{-1}\left\{\left(x-\frac{3}{2}\right)/(1/2)\right\}+C$$

$$=\frac{1}{4}(2x-3)\sqrt{(3x-x^2-2)}+\frac{1}{8}\sin^{-1}(2x-3)+C.$$

Example 12. Evaluate $\int\frac{x^2}{\sqrt{x^2+a^2}}\,dx$

Solution We evaluate $\int\frac{x^2}{\sqrt{x^2+a^2}}\,dx$

by letting $x=a\tan\theta$,

$dx=a\sec^2\theta\,d\theta,\ -\frac{\pi}{2}<x<\frac{\pi}{2}.$

Then, $\sqrt{x^2+a^2}=\sqrt{a^2(1+\tan^2\theta)}=a|\sec\theta|=a\sec\theta,$

and $\int\frac{x^2}{\sqrt{x^2+a^2}}dx=\int\frac{a^2\tan^2\theta}{a\sec\theta}a\sec^2\theta\,d\theta$

$$=a^2\int\tan^2\theta\sec\theta\,d\theta$$

$$=a^2\int(\sec^3\theta-1)\sec\theta\,d\theta$$

$$=a^2\left\{\int\sec^3\theta\,d\theta-\int\sec\theta\,d\theta\right\}$$

$$=a^2\left\{\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\int\sec\theta\,d\theta-\int\sec\theta\,d\theta\right\}$$

$$=\frac{a^2}{2}(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|)+C$$

$$=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|+C$$

$$=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\ln\left|\sqrt{x^2+a^2}+x\right|+C',$$

where $C'=C+(a^2\ln a)/2.$

Example 13. Evaluate $\int\frac{\cos\theta+\sin\theta}{\sqrt{5+\cos2\theta}}d\theta$

Solution $\int\frac{\cos\theta+\sin\theta}{\sqrt{5+\cos2\theta}}d\theta$

$$=\int\frac{\cos\theta}{\sqrt{5+\cos2\theta}}d\theta+\int\frac{\sin\theta}{\sqrt{5+\cos2\theta}}d\theta$$

$$=\int\frac{\cos\theta}{\sqrt{6-2\sin^2\theta}}d\theta+\int\frac{\sin\theta}{\sqrt{4+2\cos^2\theta}}d\theta$$

↓ Put $\sin\theta=u$ ↓ Put $\cos\theta=v$

$$=\int\frac{du}{\sqrt{6-2u^2}}-\int\frac{dv}{\sqrt{4+2v^2}}.$$

Now, the above integrals can be evaluated easily.

Practice Problems F

Evaluate the following integrals :

1. $\int \frac{dx}{9x^2-4}$

2. $\int \frac{1}{2x^2+x-1}\,dx$

3. $\int \frac{dx}{x^2+9x+20}$

4. $\int \frac{dx}{x^2+x+1}$

5. $\int \frac{dx}{\sqrt{x^2+x+1}}$

6. $\int \frac{dx}{\sqrt{2ax-x^2}}$

7. $\int \frac{1}{\sqrt{2x^2+3x-2}}\,dx$

8. $\int \sqrt{4-3x-2x^2}\,dx$

9. $\int \sqrt{5x^2+8x+4}\,dx$

10. $\int \sqrt{3x^2-6x+10}\,dx$

11. $\int \sqrt{2ax+x^2}\,dx$

12. $\int \frac{dx}{(x^2-2x\cos\theta+1)}$

13. $\int \frac{\cos x}{4-\sin^2 x}\,dx$

14. $\int \frac{\cos x\,dx}{\sin^2 x-6\sin x+12}\,dx$

15. $\int \frac{\sin x\,dx}{\sqrt{\cos^2 x+4\cos x+1}}$

16. $\int \frac{dx}{x[(\log x)^2+2\log x-3]}$

17. (a) Evaluate the integral $\int \frac{1}{\sqrt{2x-x^2}}\,dx$ in three ways: using the substitution $u = \sqrt{x}$, using the substitution $u = \sqrt{2-x}$, and completing the square.
(b) Show that the answers in part (a) are equivalent.

1.8 INTEGRALS OF THE FORM

$$\int \frac{\mathbf{px+q}}{\mathbf{ax^2+bx+c}}\,\mathbf{dx},\ \int \frac{\mathbf{px+q}}{\sqrt{\mathbf{ax^2+bx+c}}}\,\mathbf{dx},\ \text{and}$$

$$\int \mathbf{(px+q)\sqrt{ax^2+bx+c}\,dx}$$

1. $\int \frac{\mathbf{px+q}}{\mathbf{ax^2+bx+c}}\,\mathbf{dx}$. $(a\neq 0, p\neq 0)$

Here, noting that the differential of

$ax^2+bx+c=(2ax+b)dx$,

the given integral can be written as

$$\frac{p}{2a}\int \frac{2ax+\frac{2aq}{p}}{ax^2+bx+c}\,dx = \frac{p}{2a}\int \frac{(2ax+b)+\frac{2aq}{p}-b}{ax^2+bx+c}\,dx$$

$$= \frac{p}{2a}\left\{\int \frac{(2ax+b)}{ax^2+bx+c}\,dx + \frac{2aq-pb}{p}\int \frac{dx}{ax^2+bx+c}\right\}$$

The first integal is equal to $\ln|ax^2+bx+c|$, since the numerator of the integral is equal to the differential coefficient of the denominator. The second integral is evaluated as in the previous section.

Note:

(i) To express $px + q$ as the sum of two terms we might also proceed thus :
Let $px + q \equiv \ell$ (derivative of the denominator) $+ m$, where the constants ℓ, m are to be determined by comparing the coefficients.

(ii) If $ax^2 + bx + c$ breaks up into two real linear factors, then instead of proceeding as above, we may break up the integrand into the sum of partial fractions, and then integrate each separately.

2. $\int \frac{\mathbf{px+q}}{\sqrt{\mathbf{ax^2+bx+c}}}\,\mathbf{dx}$. **$(a\neq 0, p\neq 0)$**

Let $I=\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

Observe that the derivative of ax^2+bx+c is $2ax+b$.
We therefore write $(px+q)\equiv \ell(2ax+b)+m$.
Comparing coefficients of x on both sides of the above identity, we have $p = 2a\ell$ ...(1)
Also, comparing the constant terms on both sides, we have $q = b\ell + m$ ...(2)
From (1) and (2), $\ell = p/(2a)$, $m = q - bp/(2a)$ ...(3)
With these values of ℓ and m, we can write

$$I=\int \frac{\ell(2ax+b)+m}{\sqrt{ax^2+bx+c}}\,dx,$$

$$=\ell\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + m\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$=2\ell\sqrt{ax^2+bx+c}+mJ,$$

where $J=\int \frac{dx}{\sqrt{ax^2+bx+c}}$.

The method of integrating J has been discussed earlier, and therefore we can evaluate I.

3. $\int (px + q)\sqrt{ax^2 + bx + c}\,dx$. $(a \neq 0)$

To integrate this, transform

$$px+q=\frac{p}{2a}(2ax+b)+\left(q-\frac{bp}{2a}\right)$$

Then the integral reduces to the sum of two integrals,

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx$$

$$=\ell\int(2ax+b)\sqrt{ax^2+bx+c}\,dx+mJ,$$

$$=\frac{2\ell}{3}(ax^2+bx+c)^{3/2}+mJ,$$

where $J = \int\sqrt{ax^2+bx+c}\,dx$ can be integrated as before. Recall that $\ell = p/(2a)$, $m = q - bp/(2a)$.

Example 1. Evaluate $\int \frac{3x+1}{2x^2-2x+3}dx$.

Solution Here $(d/dx)(2x^2-2x+3)=4x-2$.

$$\therefore\ I=\int\frac{3x+1}{2x^2-2x+3}dx=\int\frac{\frac{3}{4}(4x-2)+1+\frac{3}{2}}{(2x^2-2x+3)}dx$$

$$=\frac{3}{4}\int\frac{4x-2}{2x^2-2x+3}dx+\frac{5}{2}\int\frac{1}{2x^2-2x+3}dx$$

$$=\frac{3}{4}\ln(2x^2-2x+3)+\frac{5}{2.2}\int\frac{dx}{x^2-x+(3/2)}$$

$$=\frac{3}{4}\ln(2x^2-2x+3)+\frac{5}{4}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)-\left(\frac{1}{4}\right)}$$

$$=\frac{3}{4}\ln(2x^2-2x+3)+\frac{5}{4}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+(\sqrt{5/2})^2}$$

$$=\frac{3}{4}\ln(2x^2-2x+3)$$

$$+\frac{5}{4}\frac{1}{(\sqrt{5/2})}\tan^{-1}\left\{\frac{x-\frac{1}{2}}{(\sqrt{5/2})}\right\}+C$$

$$=\frac{3}{4}\ln(2x^2-2x+3)+\frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+C.$$

Example 2. Evaluate $\int\frac{(x+2)dx}{\sqrt{(5-12x-9x^2)}}$

Solution We have $\int\frac{(x+2)dx}{\sqrt{(5-12x-9x^2)}}$

$$=\frac{1}{3}\int\frac{(x+2)dx}{\sqrt{\left(\frac{5}{9}-\frac{4}{3}x-x^2\right)}}$$

$$=\frac{1}{3}\int\frac{-\frac{1}{2}\left(-2x-\frac{4}{3}\right)+2-\frac{2}{3}}{\sqrt{\left(\frac{5}{9}-\frac{4}{3}x-x^2\right)}}dx$$

$$=-\frac{1}{6}\int\left(\frac{5}{9}-\frac{4}{3}x-x^2\right)^{-1/2}\left(-2x-\frac{4}{3}\right)dx$$

$$+\frac{4}{9}\int\frac{dx}{\sqrt{\left\{\frac{5}{9}-\left(x^2+\frac{4}{3}x\right)\right\}}}$$

$$=\frac{1}{6}\frac{\left(\frac{5}{9}-\frac{4}{3}x-x^2\right)^{1/2}}{\frac{1}{2}}$$

$$+\frac{4}{9}\int\frac{dx}{\sqrt{\left\{\frac{5}{9}\left(x+\frac{2}{3}\right)^2+\frac{4}{9}\right\}}}$$

$$=-\frac{1}{3}\sqrt{\left(\frac{5}{9}-\frac{4}{3}x-x^2\right)}+\frac{4}{9}\int\frac{dx}{\sqrt{\left\{1-\left(x+\frac{2}{3}\right)^2\right\}}}$$

$$=\frac{1}{9}\sqrt{(5-12x-9x^2)}+\frac{4}{9}\sin^{-1}\left(\frac{x+\frac{2}{3}}{1}\right)+C$$

$$=-\frac{1}{9}\sqrt{(5-12x-9x^2)}+\frac{4}{9}\sin^{-1}\left(\frac{3x+2}{3}\right)+C.$$

Example 3. Evaluate $\int \frac{3x+1}{\sqrt{x^2+4x+1}}dx$.

Solution The linear expression in the numerator can be expressed as

$$3x+1 = l\frac{d}{dx}(x^2+4x+1)+m$$

$\Rightarrow$ $3x+1 = l(2x+4)+m$

Comparing the coefficients of x and constants both sides.

$3 = 2l$ & $1 = 4l+m$

$\Rightarrow$ $l = 3/2$ & $m = -5$

$$\Rightarrow I = \int\frac{3x+1}{\sqrt{x^2+4x+1}} = \int\frac{3/2(2x+4)-5}{\sqrt{x^2+4x+1}}dx$$

$$= \frac{3}{2}\int\frac{2x+4}{\sqrt{x^2+4x+1}} - 5\int\frac{dx}{\sqrt{x^2+4x+1}}$$

Let $I_1 = \frac{3}{2}\int\frac{2x+4}{\sqrt{x^2+4x+1}} = \frac{3}{2}\int\frac{dt}{\sqrt{t}}$

(where $t = x^2+4x+1$)

$$= 3\sqrt{t}+C = 3\sqrt{x^2+4x+1}+C$$

Let $I_2 = 5\int\frac{dx}{\sqrt{x^2+4x+1}} = 5\int\frac{dx}{\sqrt{(x+2)^2-3}}$

$$= 5\ln|x+2+\sqrt{(x+2)^2-3}|+C$$

$\Rightarrow$ $I = I_1 - I_2$

$$= 3\sqrt{x^2+4x+1} - 5\ln|x+2+\sqrt{x^2+4x+1}|+C$$

Example 4. Evaluate $\int (2x-5)\sqrt{(2+3x-x^2)}\,dx$.

Solution We have $(2x-5) = -(3-2x)-2$.

$$\therefore \int (2x-5)\sqrt{(2+3x-x^2)}\,dx$$

$$= -\int(3-2x)\sqrt{(2+3x-x^2)}\,dx - 2\int\sqrt{(2+3x-x^2)}\,dx$$

$$= -\frac{(2+3x-x^2)^{3/2}}{3/2} - 2\int\sqrt{\left\{\frac{17}{4}-\left(x-\frac{3}{2}\right)^2\right\}}\,dx$$

$$= -\frac{2}{3}(2+3x-x^2)^{3/2} - 2\left[\frac{1}{2}\left(x-\frac{3}{2}\right)\sqrt{\left\{\frac{17}{4}-\left(x-\frac{3}{2}\right)^2\right\}}\right.$$

$$\left.+\frac{1}{2}\cdot\frac{17}{4}\sin^{-1}\left\{\frac{x-3/2}{\sqrt{(17/4)}}\right\}\right]+C$$

$$= -\frac{2}{3}(2+3x-x^2)^{3/2} - \frac{1}{2}(2x-3)\sqrt{(2x+3x-x^2)}$$

$$-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+C.$$

Example 5. Evaluate $\int (x-5)\sqrt{x^2+x}\,dx$

Solution Let $(x-5) = \lambda.\frac{d}{dx}(x^2+x)+\mu$. Then,

$x-5 = \lambda(2x+1)+\mu$.

Comparing coefficients of like powers of x, we get

$$1 = 2\lambda \text{ and } \lambda+\mu = -5 \Rightarrow \lambda = \frac{1}{2} \text{ and } \mu = -\frac{11}{2}$$

$$\int (x-5)\sqrt{x^2+x}\,dx$$

$$= \int\left(\frac{1}{2}(2x+1)-\frac{11}{2}\right)\sqrt{x^2+x}\,dx$$

$$= \int\frac{1}{2}(2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int\sqrt{x^2+x}\,dx$$

$$= \frac{1}{2}\int(2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int\sqrt{x^2+x}\,dx$$

$$= \frac{1}{2}\int\sqrt{t}\,dt - \frac{11}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx$$

where $t = x^2+x$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2} - \frac{11}{2}\left[\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right]\right.$$

$$\left.-\frac{1}{2}\left(\frac{1}{2}\right)^2\ln\left[\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right\}\right]+C$$

$$= \frac{1}{3}t^{3/2} - \frac{11}{2}$$

$$\left[\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\ell n\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right]+C$$

$$= \frac{1}{3}(x^2+x)^{3/2}$$

$$-\frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\ell n\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right]+C$$

Example 6. $\int \frac{(x\cos\theta - 1)\,dx}{x^2 - 2x\cos\theta + 1}$

Solution The integral becomes

$$\int \frac{\cos\theta(x-\cos\theta)\,dx}{x^2 - 2x\cos\theta + 1}$$

$$-\int \frac{\sin^2\theta\,dx}{(x-\cos\theta)^2 + \sin^2\theta}$$

Hence, $\int \frac{(x\cos\theta - 1)dx}{x^2 - 2x\cos\theta + 1} = \frac{\cos\theta}{2}\ln|x^2 - 2x\cos\theta + 1|$

$$-\sin\theta\tan^{-1}\frac{x-\cos\theta}{\sin\theta} + C.$$

Example 7. Evaluate $\int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x}\,dx$

Solution $I = \int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x}\,dx$

$$= \int \frac{(4\sin x - 1)\cos x}{6 - (1 - \sin^2 x) - 4\sin x}\,dx$$

$$= \int \frac{(4\sin x - 1)\cos x}{\sin^2 x - 4\sin x + 5}\,dx$$

Put $\sin x = t$, so that $\cos x\,dx = dt$.

$$\therefore\quad I = \int \frac{(4t-1)dt}{(t^2 - 4t + 5)} \qquad \text{....(1)}$$

Now, let $(4t - 1) = \lambda(2t - 4) + \mu$

Comparing coefficients of like powers of t, we get

$2\lambda = 4, -4\lambda + \mu = -1$

$\Rightarrow \lambda = 2, \mu = 7$ ...(2)

$$\therefore\quad I = \int \frac{2(2t-4)+7}{t^2 - 4t + 5}\,dt \qquad \text{[using (1) and (2)]}$$

$$= 2\int \frac{2(2t-4)+7}{t^2-4t+5}\,dt + 7\int \frac{dt}{t^2-4t+5}$$

$$= 2\ln(t^2 - 4t + 5) + 7\int \frac{dt}{t^2 - 4t + 4 - 4 + 5}$$

$$= 2\ln(t^2 - 4t + 5) + 7\int \frac{dt}{(t-2)^2 + (1)^2}$$

$= 2\ln(t^2 - 4t + 5) + 7\tan^{-1}(t-2) + C$

$= 2\ln(\sin^2 x - 4\sin x + 5) + 7\tan^{-1}(\sin x - 2) + C.$

Integrals of the form

$$\int \frac{px^2 + qx + r}{ax^2 + bx + c}\,dx,\ \int \frac{px^2 + qx + r}{\sqrt{ax^2 + bx + c}}\,dx, \text{ and}$$

$$\int (ax^2 + bx + c)\sqrt{(ex^2 + fx + g)}\,dx$$

In this case express the numerator as

$N^r = l(D^r) + m$ (derivative of D^r) $+ n$

i.e. put $ax^2 + bx + c$

$= l(px^2 + qx + r) + m(2ax + b) + n$

where l, m and n are to be so chosen by comparing the coefficients of x^2, x and constant term, that it becomes equal to the given numerator.

Example 8. Evaluate $I = \int \frac{2x^2 + 3x + 4}{x^2 + 6x + 10}dx$

Solution $I = \int \frac{l(x^2 + 6x + 10) + m(2x+6) + n}{x^2 + 6x + 10}\,dx$

$\therefore\quad 2x^2 + 3x + 4 = l(x^2 + 6x + 10) + m(2x + 6) + n$

On comparing the coefficients of x^2, x and constant term, we have

$l = 2, 6l + 2m = -3, 10l + 6m + n = 4$

$\therefore\quad l = 2, m = -\frac{9}{2}$ and $n = 11$.

$$\therefore\quad I = \int 2dx - \frac{9}{2}\int \frac{(2x+6)}{x^2 + 6x + 10}dx + 11\int \frac{dx}{(x+3)^2 + 1}$$

$= 2x - \frac{9}{2}\ell n|x^2 + 6x + 10| + 11\tan^{-1}(x+3) + C$

Example 9. Evaluate $\int \frac{x^2 - 3x + 1}{\sqrt{1-x^2}}dx$

Solution

Let $x^2 - 3x + 1 = l(1 - x^2) + m\frac{d}{dx}(1-x^2) + n$

Comparing the coefficients like powers of x

$l = -1, m = 3/2, n = 2$

$$\int \frac{x^2 - 3x + 1}{\sqrt{1-x^2}}dx = \int \frac{-(1-x^2) + \frac{3}{2}(-2x) + 2}{\sqrt{1-x^2}}dx$$

$$= -\int \sqrt{1-x^2}\,dx - \int \frac{3x}{\sqrt{1-x^2}}dx + 2\int \frac{1}{\sqrt{1-x^2}}dx$$

$$= -\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]$$

$$-3\left(-\frac{1}{2}\right)\int\frac{-2x}{\sqrt{1-x^2}}dx+2\int\frac{1}{\sqrt{1-x^2}}dx$$

$$=-\frac{x}{2}\sqrt{1-x^2}+\frac{3}{2}\sin^{-1}x+3\sqrt{1-x^2}+C$$

$$=\frac{6-x}{2}\sqrt{1-x^2}+\frac{3}{2}\sin^{-1}x+C.$$

Example 10. Evaluate $\int\frac{(x^2+4x+7)}{\sqrt{x^2+x+1}}$.

Solution

Let $x^2+4x+7=A(x^2+x+1)+B(2x+1)+C$

Comparing the coefficients of x^2, x and constant term,we get $A=1,\quad A+2B=4,\quad A+B+C=7$

$$\Rightarrow A=1, B=\frac{3}{2}, C=\frac{9}{2}$$

So $I=\int\sqrt{x^2+x+1}\,dx+\frac{3}{2}\int\frac{(2x+1)dx}{\sqrt{x^2+x+1}}$

$$+\frac{9}{2}\int\frac{dx}{\sqrt{x^2+x+1}}$$

Now $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$

$$I=\left(\frac{x+\frac{1}{2}}{2}\right)+\sqrt{x^2+x+1}+\frac{3}{8}\ell n\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)$$

$$+3\sqrt{x^2+x+1}+\frac{9}{2}\ell n\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+C.$$

Example 11. Evaluate $\int\frac{x^2-3x+1}{\sqrt{-1+4x-2x^2}}dx$.

Solution First of all, the quadratic trinomial under the radical $-1+4x-2x^2$ is brought to the form :

$$-1+4x-2x^2=-[1+2(x^2-2x)]=1-2(x-1)^2$$

Now we change of the variable of integration.

Put $u=x-1$ in the integral, which leads to

$$\int\frac{x^2-3x+1}{\sqrt{-1+4x-2x^2}}dx=\int\frac{(u+1)^2-3(u+1)+1}{\sqrt{1-2u^2}}du$$

$$=\int\frac{u^2}{\sqrt{1-2u^2}}du-\int\frac{u}{\sqrt{1-2u^2}}du-\int\frac{du}{\sqrt{1-2u^2}}$$

Each of the three integrals on the right hand side of the last equality can be evaluated using formulae.

$$\int\frac{x^2-3x+1}{\sqrt{-1+4x-2x^2}}dx$$

$$=\left[\frac{1}{4\sqrt{2}}(-\sqrt{2}u\sqrt{1-2u^2}+\sin^{-1}\sqrt{2}u)\right]$$

$$\left[\frac{1}{2}\sqrt{1-2u^2}\right]-\left[\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}u\right]+C$$

$$=\frac{1}{4}(2-u)\sqrt{1+2u^2}-\frac{3}{4\sqrt{2}}\sin^{-1}\sqrt{2}u+C$$

On returning to the original variable x we arrive at the final result : $\int\frac{x^2-3x+1}{\sqrt{-1+4x-2x^2}}dx$

$$=\frac{1}{4}(3-x)\sqrt{-1+4x-2x^2}-\frac{3}{4\sqrt{2}}\sin^{-1}\sqrt{2}(x-1)+C$$

Form $\int\frac{p_0x^n+p_1x^{n-1}+p_2x^{n-2}+....p^n}{ax^2+bx+c}dx$

In this case divide the N^r by D^r and express it as Quotient + $\frac{px+q}{ax^2+bx+c}$, where quotient will consist of certain terms which we shall integrate by power formula and $\frac{px+q}{ax^2+bx+c}$ will be integrated as explained before.

Example 12. Evaluate $I=\int\frac{x^4+x^3+2x+1}{x^2+x+1}dx$

Solution $I=\int\left[(x^2-1)+\frac{(3x+2)}{(x^2+x+1)}\right]dx$

$$=\int\left[x^2-1+\frac{3}{2}\frac{(2x+1)}{x^2+x+1}+\frac{1}{2\left\{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right\}}\right]dx$$

$$=\frac{x^3}{3}-x+\frac{3}{2}\ell n(x^2+x+1)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$$

Example 13. Evaluate . $\int\frac{(x^3+3)dx}{\sqrt{(x^2+1)}}$

Solution We have

$$\int \frac{(x^3+3)dx}{\sqrt{(x^2+1)}} = \int \frac{x(x^2+1)-x+3}{\sqrt{(x^2+1)}}dx$$

$$= \int \frac{x(x^2+1)}{\sqrt{(x^2+1)}}dx - \int \frac{x\,dx}{\sqrt{(x^2+1)}} + 3\int \frac{x\,dx}{\sqrt{(x^2+1)}}$$

$$= \frac{1}{2}\int (2x)\sqrt{(x^2+1)}\ dx$$

$$-\frac{1}{2}\int \frac{2x\,dx}{\sqrt{(x^2+1)}} + 3\int \frac{dx}{\sqrt{(x^2+1)}}$$

$$= \frac{1}{2}\left[\frac{2}{3}(x^2+1)^{3/2}\right] - \frac{1}{2}[2\sqrt{(x^2+1)}]$$

$$+3\,ln\,(x+\sqrt{x^2+1}\,)+C$$

$$= \frac{1}{3}(x^2+1)^{2/3} - \sqrt{(x^2+1)} + 3\,ln(x+\sqrt{x^2+1}\,)+C.$$

Practice Problems G

Evaluate the following integrals :

1. $\int \frac{x\ dx}{x^2+2x+1}$
2. $\int \frac{2x+3}{x^2+3x-10}dx$
3. $\int \frac{x}{x^2-5x+6}dx$
4. $\int \frac{x+1}{x^2+x+3}dx$
5. $\int \frac{3x+5}{x^2+2x-3}$
6. $\int \frac{2x+1}{\sqrt{3+2x-x^2}}dx$
7. $\int \frac{6x-5}{\sqrt{3x^2-5x+1}}dx$
8. $\int \frac{(3x+5)dx}{\sqrt{x^2+4x+3}}$
9. $\int (x-1)\sqrt{1+x+x^2}\ dx$
10. $\int (3x+5)\ \sqrt{x^2+2x-3}\ dx$
11. $\int (x+2)\sqrt{2x^2+2x+1}\ dx$
12. $\int \frac{\ln x\,dx}{x\sqrt{1-4\ln x-\ln^2 x}}$
13. $\int \frac{x^2dx}{\sqrt{x^2-16}}$
14. $\int \frac{x^2+2x-1}{2x^2+3x+1}dx$
15. $\int \frac{x^2\,dx}{\sqrt{1-2x-x^2}}$.
16. $\int \frac{(2x^2-3x)dx}{\sqrt{x^2-2x+5}}$
17. $\int \frac{x^3dx}{(x^2-2x+2)}$
18. $\int \frac{x^2+2x+3}{\sqrt{(x^2+x+1)}}dx$
19. $\int \frac{(x+1)\sqrt{x+2}}{\sqrt{x-2}}dx$
20. $\int \frac{3x^3-8x+5}{\sqrt{x^2-4x-7}}dx$
21. $\int \frac{x^3+1}{\sqrt{x^2+x}}dx$
22. $\int \frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{\sqrt{4-x^4}}dx.$

1.9 INTEGRALS OF THE FORM

$$\int \frac{\mathbf{ax^2+b}}{\mathbf{x^4+px^2+q}}\,\mathbf{dx},\quad \mathbf{q>0}$$

Here we have a very interesting method for evaluating the integral of a rational function in which the denominator is of degree 4 and the numerator is either of degree 2 or is constant. Moreover the odd powers of x occur neither in the numerator nor in the denominator.

Let us consider first the integral $\int \frac{ax^2+b}{x^4+px^2+1}dx$

We divide N^r and D^r by x^2 and then put $x \pm \frac{1}{x} = t$.

Example 1. Evaluate $I = \int \frac{1-x^2}{1+x^2+x^4}dx$

Solution We divide N^r and D^r by x^2.

$$I = \int \frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}+1}\quad \text{Put } x+\frac{1}{x}=t$$

$$= -\int \frac{dt}{t^2-1} = -\frac{1}{2}\,\ell n\left|\frac{t-1}{t+1}\right| + C$$

$$= -\frac{1}{2}\,\ell n\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$$

Example 2. Evaluate $\int \frac{x^2+1}{x^4+1}dx$.

Solution $I = \int \frac{x^2+1}{x^4+1}dx$

Divide above and below by x^2.

$$\therefore \quad I = \int \frac{\left(1+\frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2}} dx$$

Here $\left(1+\frac{1}{x^2}\right)$ is $\frac{d}{dx}\left(x - \frac{1}{x}\right)$ hence we express the Dr as a perfect square of $x - \frac{1}{x}$ by subtracting and adding 2.

$$\therefore \quad I = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2 + \frac{1}{x^2} - 2 + 2} = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x - \frac{1}{x}\right)^2 + \left(\sqrt{2}\right)^2}$$

Put $x - \frac{1}{x} = t \; \therefore \left(1 + \frac{1}{x^2}\right)dx = dt$

$$\therefore \quad I = \int \frac{dt}{t^2 + \left(\sqrt{2}\right)^2} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{\left(x - \frac{1}{x}\right)}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{x^2 - 1}{\sqrt{2}x} + C.$$

Example 3. Evaluate $\int \frac{1}{x^4 + 1 + 5x^2} dx$

Solution We express the numerator in terms of $(x^2 + 1)$ and $(x^2 - 1)$.

$1 = \ell(x^2 + 1) + m(x^2 - 1)$

$\Rightarrow \ell + m = 0, \ell - m = 1$

$\Rightarrow \ell = 1/2, \; m = -1/2$

Hence, $1 = \frac{1}{2}(x^2 + 1) - \frac{1}{2}(x^2 - 1)$

$$I = \frac{1}{2}\int \frac{2}{x^4 + 1 + 5x^2} dx$$

$$= \frac{1}{2}\int \frac{1 + x^2}{x^4 + 1 + 5x^2} dx - \frac{1}{2}\int \frac{x^2 - 1}{x^4 + 1 + 5x^2} dx$$

$$= \frac{1}{2}\int \frac{1 + (1/x^2)}{x^2 + (1/x^2) + 5} dx - \frac{1}{2}\int \frac{1 - (1/x^2)}{x^2 + (1/x^2) + 5} dx$$

$$= (I_1 - I_2)/2$$

For I_1, we write $x - \frac{1}{x} = u \Rightarrow \left(1 + \frac{1}{x^2}\right)dx = du$

$$\Rightarrow \quad I_1 = \int \frac{du}{u^2 + (\sqrt{7})^2} = \frac{1}{\sqrt{7}} \tan^{-1} \frac{u}{\sqrt{7}}$$

$$= \frac{1}{\sqrt{7}} \tan^{-1}\left(\frac{x - 1/x}{\sqrt{7}}\right)$$

For I_2, we write $x + \frac{1}{x} = v \Rightarrow \left(1 - \frac{1}{x^2}\right)dx = dv$

$$I_2 = \frac{dv}{v^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{v}{\sqrt{3}} = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x + 1/x}{\sqrt{3}}\right)$$

Combining the two results, we get $I = (I_1 - I_2)/2$

$$= \frac{1}{2\sqrt{7}} \tan^{-1}\left(\frac{x - 1/x}{\sqrt{7}}\right) - \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x + 1/x}{\sqrt{3}}\right) + C$$

Example 4. Evaluate $I = \int \frac{x^2}{x^4 + x^2 + 1} dx$

Solution Let $I = \int \frac{x^2}{x^4 + x^2 + 1} dx$

We express the numerator as

$$x^2 = \frac{1}{2}(x^2 + 1) + \frac{1}{2}(x^2 - 1)$$

$$\therefore \quad I = \frac{1}{2}\int \frac{(x^2 + 1) + (x^2 - 1)}{x^4 + x^2 + 1} dx$$

$$= \frac{1}{2}\int \frac{(1 + 1/x^2) + (1 - 1/x^2)}{x^2 + 1 + (1/x^2)} dx$$

$$= \frac{1}{2}\int \frac{(1 + 1/x^2)dx}{(x - 1/x)^2 + 3} + \frac{1}{2}\int \frac{(1 - 1/x^2)dx}{(x + 1/x)^2 - 1}$$

In the first integral put $x - \frac{1}{x} = t$ so that $\left(1 + \frac{1}{x^2}\right)dx = dt$, and in the second integral put $x + \frac{1}{x} = z$ so that $\left(1 - \frac{1}{x^2}\right)dx = dz$.

$$\therefore \quad I = \frac{1}{2}\int \frac{dt}{t^2 + (\sqrt{3})^2} + \frac{1}{2}\int \frac{dz}{z^2 - 1}$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + \frac{1}{2}\frac{1}{2 \times 1} \ln \frac{z - 1}{z + 1} + C$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left\{\frac{(x - 1/x)}{\sqrt{3}}\right\} + \frac{1}{4} \ln \frac{(x + 1/x) - 1}{(x + 1/x) + 1} + C$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left\{\frac{x^2-1}{(\sqrt{3})x}\right\} + \frac{1}{4}\ln\frac{x^2-x+1}{x^2+x+1} + C.$$

Example 5. Evaluate $\int \frac{1}{x^4+1}\,dx$

Solution We have,

$$I = \int \frac{1}{x^4+1}\,dx = \int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} - \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx$$

Putting $x - \frac{1}{x} = u$ in 1st integral and $x + \frac{1}{x} = v$ in 2nd integral, we get

$$I = \frac{1}{2}\int \frac{du}{u^2+\left(\sqrt{2}\right)^2} - \frac{1}{2}\int \frac{dv}{v^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\,\frac{1}{2\sqrt{2}}\ln\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\ln\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) - \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}\,x+1}{x^2+x\sqrt{2}+1}\right| + C$$

Example 6. Evaluate $I = \int \frac{dx}{x^4+8x^2+9}$

Solution We express the numerator in terms of (x^2+3) and (x^2-3). Note that 3 is the square root of the constant term 9 in the denominator.

$$1 = \ell(x^2+3) + m(x^2-3)$$

$\Rightarrow \ell + m = 0,\ 3\ell - 3m = 1$

$\Rightarrow \ell = 1/6,\ m = -1/6$

Hence $1 = 1/6\,(x^2+3) - 1/6\,(x^2-3)$

$$I = \frac{1}{6}\int \frac{x^2+3}{x^4+8x^2+9}\,dx - \frac{1}{6}\int \frac{x^2-3}{x^4+8x^2+9}\,dx$$

$$= \frac{1}{6}\int \frac{1+3/x^2}{x^4+3/x^2+8}\,dx - \frac{1}{6}\int \frac{1-3/x^2}{x^4+3/x^2+8}\,dx$$

$$= \frac{1}{6}\int \frac{1+3/x^2}{(x-3/x)^2+14}\,dx - \frac{1}{6}\int \frac{1-3/x^2}{(x+3/x)^2+2}\,dx$$

Put $x - 3/x = u$ Put $x + 3/x = v$

$(1+3/x^2)\,dx = du$ $(1-3/x^2)\,dx = dv$

$$I = \frac{1}{6}\int \frac{du}{u^2+14} - \frac{1}{6}\int \frac{dv}{v^2+2}$$

Now, the above integrals can be evaluated easily.

Example 7. Evaluate $\int \frac{x}{x^4+x^2+1}\,dx$

Solution We have,

$$I = \int \frac{x}{x^4+x^2+1}\,dx = \int \frac{x}{(x^2)^2+x^2+1}\,dx$$

Let $x^2 = t$, then $d(x^2) = dt$

$$\Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{dt}{2x}$$

$$I = \int \frac{x}{t^2+t+1}\cdot\frac{dt}{2x} = \frac{1}{2}\int \frac{1}{t^2+t+1}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C.$$

Example 8. Evaluate $I = \int \frac{x^4+1}{x^6+1}\,dx$

Solution $I = \int \frac{(x^2+1)^2-2x^2}{x^6+1}\,dx$

$$= \int \frac{(x^2+1)}{(x^4-x^2+1)}dx - \int \frac{x^2}{x^6+1}dx$$

In the second integral, put $x^3 = t \Rightarrow 3x^2\,dx = dt$

$$I = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}-1\right)}dx - \frac{2}{3}\int \frac{dt}{t^2+1}$$

In the first integral, put $x - 1/x = v \Rightarrow 1 + 1/x^2\,dx = dv$

$$I = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(\left(x-\frac{1}{x}\right)^2+1^2\right)}dx - \frac{2}{3}\tan^{-1}t$$

$$= \int \frac{dv}{v^2+1^2} - \frac{2}{3}\tan^{-1}x^3$$

$$= \tan^{-1}\left(x-\frac{1}{x}\right) - \frac{2}{3}\tan^{-1}x^3 + C$$

Example 9. Using substitution only to evaluate $\int \operatorname{cosec}^3 x\,dx$.

Solution $I = \int \frac{\sin x}{\sin^4 x}dx = \int \frac{\sin x}{(1-\cos^2 x)^2}dx$

Put $\cos x = t$

$$I = -\int \frac{dt}{(1-t^2)^2} = -\int \frac{dt}{t^4-2t^2+1}$$

$$= -\frac{1}{2}\int \frac{(t^2+1)-(t^2-1)}{t^4-2t^2+1}dt$$

$$I = -\frac{1}{2}\underbrace{\int \frac{t^2+1}{t^4-2t^2+1}dt}_{I_1} + \frac{1}{2}\underbrace{\int \frac{t^2-1}{t^4-2t^2+1}dt}_{I_2} \quad ...(1)$$

Hence, $I = \frac{1}{2}(I_2 - I_1)$

Consider, $I_1 = \int \frac{1+\frac{1}{t^2}}{t^2-2+\frac{1}{t^2}}dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2}dt$

Put $t - \frac{1}{t} = y$

Hence, $I_1 = \int \frac{dy}{y^2} = -\frac{1}{y} = -\frac{1}{t-\frac{1}{t}} = -\frac{t}{t^2-1}$

$$= \frac{\cos x}{1-\cos^2 x}$$

$$I_2 = \int \frac{dt}{t^2-1} = \frac{1}{2}ln\left|\frac{t-1}{t+1}\right| = \frac{1}{2}ln\left|\frac{\cos x-1}{\cos x+1}\right|$$

$$= \frac{1}{2}ln\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\therefore \quad I = \frac{1}{2}\left[\frac{1}{2}ln\left(\frac{1-\cos x}{1+\cos x}\right) - \frac{\cos x}{\sin^2 x}\right] + C$$

Example 10. Evaluate $\int \frac{(1+x^2)dx}{(1-x^2)\sqrt{1+x^2+x^4}}$

Solution Let, $I = \int \frac{(1+x^2)dx}{(1-x^2)\sqrt{1+x^2+x^4}}$

$$= \int \frac{x^2\left(1+\frac{1}{x^2}\right)dx}{x\left(\frac{1}{x}-x\right)x\sqrt{\frac{1}{x^2}+1+x^2}}$$

$$= -\int \frac{(1+1/x^2)dx}{(1-1/x)\sqrt{(x-1/x)^2+3}}$$

Put $x - \frac{1}{x} = t$,

$$= -\int \frac{dt}{t\sqrt{t^2+3}}$$

Again put $t^2 + 3 = s^2 \Rightarrow 2t\,dt = 2s\,ds$.

$$= -\int \frac{s\,ds}{s(s^2-3)} = -\int \frac{ds}{s^2-(\sqrt{3})^2}$$

$$= -\frac{1}{2\sqrt{3}}\ln\left|\frac{s-\sqrt{3}}{s+\sqrt{3}}\right| + C$$

$$= -\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{t^2+3}-\sqrt{3}}{\sqrt{t^2+3}+\sqrt{3}}\right| + C$$

$$= -\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{(x-1/x)^2+3}-\sqrt{3}}{\sqrt{(x-1/x)^2+3}+\sqrt{3}}\right| + C$$

$$= -\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{x^2+\frac{1}{x^2}+1}-\sqrt{3}}{\sqrt{x^2+\frac{1}{x^2}+1}+\sqrt{3}}\right| + C$$

Example 11. Evaluate $I=\int \frac{x-1}{x+1}\frac{dx}{\sqrt{x^3+x^2+x}}$

Solution $I=\int \frac{x-1}{x(x+1)}\frac{dx}{\sqrt{x+\frac{1}{x}+1}}$

$$=\int \frac{(x^2-1)}{x(x+1)^2}\frac{dx}{\sqrt{x+\frac{1}{x}+1}}$$

$$=\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}$$

Put $x+1/x+1=t^2 \Rightarrow (1-1/x^2)\,dx=2t\,dt$

$$I=\int \frac{2t\,dt}{(t^2+1)t}=2\int \frac{dt}{t^2+1}=2\tan^{-1}t+C$$

$$=2\tan^{-1}\sqrt{x+1/x+1}+C$$

Example 12. Evaluate $\int \frac{1}{x\sqrt{x^4+3x^2+1}}dx$

Solution $\int \frac{1}{x\sqrt{x^4+3x^2+1}}dx$

$$=\int \frac{1}{x^2\sqrt{x^2+1/x^2+3}}dx$$

Now, $1=\ell(x^2+1)+m(x^2-1)$

$\Rightarrow \ell+m=0,\ \ell-m=1$

$\Rightarrow \ell=1+m,\ 1+2m=0$

$\Rightarrow m=-1/2,\ \ell=1/2$

$$I=\int \frac{1/2(x^2+1)-1/2(x^2-1)}{x^2\sqrt{x^2+\frac{1}{x^2}+3}}dx$$

$$=\frac{1}{2}\int \frac{1+1/x^2}{\sqrt{\left(x-\frac{1}{x}\right)^2+5}}dx-\frac{1}{2}\int \frac{(1-1/x^2)dx}{\sqrt{(x+1/x)^2+1}}$$

Put $x-1/x=u$ Put $x+1/x=v$

$\left(1+\frac{1}{x^2}\right)dx=du$ $\left(1-\frac{1}{x^2}\right)dx=dv$

$$I=\frac{1}{2}\int \frac{du}{\sqrt{u^2+(\sqrt{5})^2}}-\frac{1}{2}\int \frac{dv}{\sqrt{v^2+1^2}}$$

$$=\frac{1}{2}\{\ln(u+\sqrt{u^2+5}\}-\frac{1}{2}\{\ln(v+\sqrt{v^2+1})\}+C$$

Form $\int \frac{f(x)}{\sqrt{(ax^4+2bx^3+cx^2+2bx+a)}}dx$

where f(x) is a rational function of x.
The denominator can be written as

$$x\sqrt{\left\{a\left(x^2+\frac{1}{x^2}\right)+2b\left(x+\frac{1}{x}\right)+c\right\}}$$

and hence the substitution is

$$x+\frac{1}{x}=z \quad \text{or,} \quad x-\frac{1}{x}=z,$$

according as f(x) is expressible in the form

$$\left(x-\frac{1}{x}\right)g\left(x+\frac{1}{x}\right) \quad \text{or,} \quad \left(x+\frac{1}{x}\right)g\left(x-\frac{1}{x}\right).$$

If $b=0$, the substitution $x^2+\frac{1}{x^2}=z$ or $x^2-\frac{1}{x^2}=z$ is sometimes useful.

Example 13. Integrate $\int \frac{1-x^2}{1+x^2}\frac{dx}{\sqrt{(1+x^2+x^4)}}$.

Solution $I=\int \frac{-x^2\left(1+\frac{1}{x^2}\right)dx}{x\left(x+\frac{1}{x}\right)\sqrt{x^2\left(x^2+\frac{1}{x^2}+1\right)}}$

$$=-\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-1}}$$

$$=-\int \frac{dz}{z\sqrt{(z^2-1)}} \qquad \text{putting } x+\frac{1}{x}=z$$

$$=\int \frac{\operatorname{cosec}\theta\cot\theta}{\operatorname{cosec}\theta\cot\theta}d\theta \qquad \text{putting } z=\operatorname{cosec}\theta$$

$$=\int d\theta=\theta+C=\operatorname{cosec}^{-1}z+C$$

$$=\operatorname{cosec}^{-1}\left(\frac{x^2+1}{x}\right)+C=\sin^{-1}\left(\frac{x}{1+x^2}\right)+C.$$

Example 14. Evaluate

$$\int \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}dx.$$

Solution Let $I = \int \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}}$

$$= \int \frac{(1-\tan^2 x)dx}{(1+\tan x)^2 \cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}}$$

$$I = \int \frac{-\left(1 - \frac{1}{\tan^2 x}\right)\sec^2 x dx}{\left(\tan x + 2 + \frac{1}{\tan x}\right)\sqrt{\tan x + 1 + \frac{1}{\tan x}}}$$

Put $y^2 = \tan x + 1 + \frac{1}{\tan x}$

$$\Rightarrow \quad 2ydy = \left(\sec^2 x - \frac{\sec^2 x}{\tan^2 x}\right)dx$$

$$\therefore \quad I = \int \frac{-2ydy}{(y^2+1)y} = -2\tan^{-1}y + C$$

$$= -2\tan^{-1}\sqrt{\tan x + 1 + \frac{1}{\tan x}} + C$$

Practice Problems H

Evaluate the following integrals :

1. $\int \frac{x^2 - 1}{x^4 - 7x^2 + 1}dx$
2. $\int \frac{dx}{x^4 + x^2 + 1}$
3. $\int \frac{dx}{x^4 - x^2 + 1}$
4. $\int \frac{x^2 dx}{x^4 - x^2 + 1}$
5. $\int \frac{x^2 - 1}{x^2 + 1} \cdot \frac{dx}{\sqrt{x^4 + 1}}$
6. $\int \frac{(1 + x^2)dx}{(1 - x^2)\sqrt{(1 - 3x^2 + x^4)}}$.
7. $\int \frac{dx}{x^4 + 18x^2 + 81}$
8. $\int \frac{(x^2 - 1)dx}{x\sqrt{x^4 + 3x^2 + 1}}$
9. $\int \frac{x^2 dx}{x^4 + a^4}$
10. $\int \frac{(x^2 - a^2)dx}{x^4 + a^2x^2 + a^4}$
11. $\int \sqrt{\tan x}\, dx$
12. $\int \frac{(x^2 - 1)dx}{x\sqrt{(x^2 + ax + 1)(x^2 + bx + 1)}}$
13. $\int \frac{(1 + x^2)}{(1 - x^2)} \frac{dx}{\sqrt{1 + x^2 + x^4}}$

1.10 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Integrals of the form

(i) $\int \frac{dx}{a\cos^2 x + 2b\sin x\cos x + b\sin^2 x}$

(ii) $\int \frac{dx}{a\cos^2 x + b}$ (iii) $\frac{dx}{a + b\sin^2 x}$

Here, we shall find the integral of the fractions whose N^r is a constant and D^r contains $\cos^2 x$, $\sin^2 x$ and sin x cos x or any of them and a constant.

In such cases we divide the N^r and D^r by $\cos^2 x$ or $\sin^2 x$ and then N^r becomes either $\sec^2 x$ or $\text{cosec}^2 x$ and D^r becomes something in terms of $\sec^2 x$ and $\tan^2 x$ or something in terms of $\text{cosec}^2 x$ and $\cot^2 x$.

Then in the new D^r we shall replace $\sec^2 x$ by $1 + \tan^2 x$ and $\text{cosec}^2 x$ by $1 + \cot^2 x$. Hence the N^r shall be either $\sec^2 x$ or $\text{cosec}^2 x$ and D^r in terms of tan x or cot x and we shall either put tan x = t or cot x = t as the case may be. The above shall be more clear by the following examples:

Example 1. Evaluate $\int \frac{dx}{2 - \sin^2 x}$

Solution Put tan x = t

$$\int \frac{dx}{2 - \sin^2 x} = \int \frac{dt}{\left(2 - \frac{t^2}{1+t^2}\right)(1+t^2)}$$

$$= \int \frac{dt}{2 + t^2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

Example 2. Evaluate $\int \frac{dx}{1 + 3\cos^2 x}$

Solution $\int \frac{dx}{1 + 3\cos^2 x} = \int \frac{\sec^2 x\, dx}{\tan^2 x + 4}$

$$=\frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right)+C.$$

Example 3. Evaluate $\int \frac{dx}{3\sin^2 x+4\cos^2 x}$

Solution $\int \frac{dx}{3\sin^2 x+4\cos^2 x}=\int \frac{\sec^2 x}{3\tan^2 x+4}dx$

$$=\int \frac{dt}{3t^2+4} \text{ where } t=\tan x$$

$$=\frac{1}{3}\int \frac{dt}{t^2+\left(2/\sqrt{3}\right)^2}=\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{t}{2/\sqrt{3}}\right)+C$$

$$=\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{2}\tan x\right)+C.$$

Example 4. Evaluate $I=\int \frac{dx}{1+3\sin^2 x}$

Solution Here divide the N^r and D^r by $\sin^2 x$

$$I=\int \frac{\operatorname{cosec}^2 x dx}{\operatorname{cosec}^2 x+3}=\int \frac{\operatorname{cosec}^2 x dx}{1+\cot^2 x+3}$$

Put $\cot x=t$

$\therefore$ $-\operatorname{cosec}^2 x\, dx=dt.$

$$I=-\int \frac{dt}{4+t^2}=-\frac{1}{2}\tan^{-1}\frac{t}{2}+C=-\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\cot x\right)+C$$

Note: We could have divided the N^r and D^r by $\cos^2 x$ as well.

$$I=\int \frac{dx}{1+3\sin^2 x}=\int \frac{\sec^2 x dx}{\sec^2 x+3\tan^2 x}$$

$$=\int \frac{\sec^2 x dx}{1+\tan^2 x+3\tan^2 x}.$$

Now put $\tan x=t$

$\therefore$ $\sec^2 x\, dx=dt$

$$I=\int \frac{dt}{1+4t^2}=\frac{1}{2}\tan^{-1}(2t)+C=\frac{1}{2}\tan^{-1}(2\tan x)+C.$$

The two answers obtained in different forms can be easily shown to be differing only by a constant.

Example 5. Evaluate $\int \frac{dx}{(2\sin x+3\cos x)^2}.$

Solution $\int \frac{dx}{(2\sin x+3\cos x)^2}=\int \frac{\sec^2 x dx}{(2\tan x+3)^2}$

Put $\tan x = t$, so that $\sec^2 x\, dx = dt$

$$=\int \frac{dt}{(2t+3)^2}=-\int \frac{1}{2(2t+3)}+C,$$

$$=-\frac{1}{2(2\tan x+3)}+C.$$

Example 6. Evaluate

$$\int \frac{d\theta}{5\cos^2\theta-4\cos\theta\sin\theta-2\sin^2\theta}.$$

Solution Divide the N^r and D^r by $\cos^2\theta$

$$I=\int \frac{\sec^2\theta d\theta}{5-4\tan\theta-2\tan^2\theta}.$$

Now, put $\tan\theta=t$

$\therefore$ $\sec^2\theta\, d\theta=dt.$

$$I=\int \frac{dt}{5-4t-2t^2}=\frac{1}{2}\int \frac{dt}{\left(\sqrt{\frac{7}{2}}\right)^2-(t+1)^2}$$

$$I=\frac{1}{2}\cdot\frac{1}{2\sqrt{\frac{7}{2}}}\ln\left|\frac{\sqrt{\frac{7}{2}}+(t+1)}{\sqrt{\frac{7}{2}}-(t+1)}\right|+C$$

$$=\frac{1}{2\sqrt{14}}\ln\left|\frac{\sqrt{\frac{7}{2}}+\tan\theta+1}{\sqrt{\frac{7}{2}}-\tan\theta-1}\right|+C$$

Example 7. Evaluate

$$I=\int \frac{dx}{\sin^4 x+\cos^4 x-\sin^2 x\cos^2 x}.$$

Solution $I=\int \frac{\sec^4 x\, dx}{\tan^4 x+1+\tan^2 x}$

Put $\tan x=t$, $\sec^2 x\, dx=dt$

$$I=\int \frac{(1+t^2)dt}{t^4+t^2+1}=\int \frac{\left(1+\frac{1}{t^2}\right)dt}{t^2+\frac{1}{t^2}+1}$$

$$=\int \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+3}$$

Now, put $t-\frac{1}{t}=v \Rightarrow \left(1+\frac{1}{t^2}\right)dt=dv$

$$I=\int \frac{dv}{v^2+\sqrt{3}^2}=\frac{1}{\sqrt{3}}\tan^{-1}\frac{v}{\sqrt{3}}+C$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x-\frac{1}{\tan x}}{\sqrt{3}}\right)+C.$$

Example 8. Evaluate $I=\int\frac{dx}{(\cos^2 x+4\sin^2 x)^2}$

Solution $I=\int\frac{dx}{\cos^4 x+16\sin^4 x+8\cos^2 x\sin^2 x}$

$$=\int\frac{\sec^4 x\,dx}{1+16\tan^4 x+8\tan^2 x}$$

$$=\int\frac{\sec^4 x\,dx}{1+(2\tan x)^4+2(2\tan x)^2}$$

Now, put $2\tan x=\tan\theta \Rightarrow 2\sec^2 x\,dx=\sec^2\theta\,d\theta$

$$=\frac{1}{2}\int\frac{\left(\frac{1+\tan^2\theta}{4}\right)\sec^2\theta\,d\theta}{\sec^4\theta}$$

$$=\frac{1}{8}\int(4\cos^2\theta+\sin^2\theta)\,d\theta$$

Now, the above integral can be evaluated easily.

$$\therefore\quad I=\frac{5}{2}\tan^{-1}(2x)+\frac{3\tan}{21+4\tan^2 x}+C$$

Example 9. Evaluate $I=\int\frac{dx}{(\sin x+2\sec x)^2}$

Solution $I=\frac{1}{4}\int\frac{dx}{\left(\frac{\sin x}{2}+\sec x\right)^2}$

$$=\frac{1}{4}\int\frac{\sec^2 x\,dx}{\left(\tan^2 x+\frac{\tan x}{2}+1\right)^2}$$

$$=\frac{1}{4}\int\frac{\sec^2 x\,dx}{\left(\left(\tan x+\frac{1}{4}\right)^2+\frac{15}{16}\right)^2}$$

Put $\tan x+\frac{1}{4}=\frac{\sqrt{15}}{4}\tan\theta$

$$\Rightarrow\quad \sec^2 x\,dx=\frac{\sqrt{15}}{4}\sec^2\theta\,d\theta$$

$$=\frac{1}{4}\int\frac{\frac{\sqrt{15}}{4}\sec^2\theta}{\left(\frac{15}{16}\right)^2\sec^4\theta}\,d\theta$$

$$=\frac{1}{4}\cdot\frac{\sqrt{15}}{4}\cdot\left(\frac{16}{15}\right)^2\int\cos^2\theta\,d\theta$$

Now, the above integral can be evaluated easily.

$$\therefore\quad \frac{8}{15^{3/2}}\tan^{-1}\left(\frac{4\tan x+1}{\sqrt{15}}\right)+\frac{4\tan x+1}{15(2\tan^2 x+\tan x+2)}+C$$

Integrals of the form

(i) $\int\frac{dx}{a+b\sin x}$ (ii) $\int\frac{dx}{a+b\cos x}$

(iii) $\int\frac{dx}{a+b\sin x+c\cos x}$

In general, the integrals of the form

$\int F(\cos x,\sin x)\,dx$,

where $F(\cos x, \sin x)$ is a rational function of $\cos x$ and $\sin x$, can often be evaluated by the substitution

$$\tan\frac{x}{2}=t$$

We then have $\frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$,

i.e. $dx=2(1+t^2)^{-1}\,dt$,

$\cos x=(1-t^2)/(1+t^2)$, $\sin x=2t/(1+t^2)$, and

consequently, $\int F(\cos x,\sin x)\,dx$

$$=\int F\left(\frac{1-t^2}{1+t^2},\frac{2t}{1+t^2}\right)\frac{2dt}{1+t^2}$$

This substitution enables us to integrate any function of the form $F(\cos x, \sin x)$. For this reason it is sometimes called a "universal trigonometric substitution". However, in particular it frequently leads to extremely complex rational functions. It is therefore convenient to know some other substitutions (in addition to the "universal" one) that sometimes lead more quickly to the desired end.

Example 10. Evaluate $\int\frac{1}{1+\sin x+\cos x}\,dx$

Solution Let $I=\int\frac{1}{1+\sin x+\cos x}\,dx$

$$=\int\frac{1}{1+\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}\,dx$$

$$=\int\frac{1+\tan^2 x/2}{1+\tan^2 x/2+2\tan x/2+1-\tan^2 x/2}\,dx$$

$$=\int\frac{\sec^2 x/2}{2+2\tan x/2}\,dx$$

Putting $\tan\frac{x}{2}=t$ and $\frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$, we get

$$I=\int\frac{1}{t+1}\,dt=\ln|t+1|+C=\ln\left|\tan\frac{x}{2}+1\right|+C$$

Example 11. Integrate $I = \int \frac{dx}{4+5\cos x}$

Solution

$$I = \int \frac{dx}{4(\cos^2 x/2 + \sin^2 x/2) + 5(\cos^2 x/2 - \sin^2 x/2)}$$

$$= \int \frac{dx}{9\cos^2(x/2) - \sin^2(x/2)}.$$

$$= \int \frac{\sec^2(x/2)\,dx}{9-\tan^2(x/2)} \quad ...(1)$$

Putting $\tan x/2 = t$, $1/2 \sec^2(x/2)\,dx = dt$ in (1), we have

$$I = \int \frac{dt}{9-t^2} = \frac{1}{3}\ln\left|\frac{3+t}{3-t}\right| + C,$$

$$= \frac{1}{3}\ln\left|\frac{3+\tan x/2}{3-\tan x/2}\right| + C.$$

Example 12. Integrate $\int \frac{dx}{5-13\sin x}$

Solution $$I = \frac{dx}{5\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right) - 13.2\sin\frac{x}{2}\cos\frac{x}{2}}$$

Multiplying the numerator and denominator by $\sec^2\frac{x}{2}$, we get

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{5\left(\tan^2\frac{x}{2}+1\right) - 26\tan\frac{x}{2}}$$

$$= \int \frac{2\,dz}{5z^2 - 26z + 5}, \quad \text{Putting } \tan\frac{x}{2} = z$$

$$= \frac{2}{5}\int \frac{dz}{\left(z-\frac{13}{5}\right)^2 - \left(\frac{12}{5}\right)^2}$$

$$= \frac{2}{5}\int \frac{du}{u^2-a^2}, \text{ where } u = z - \frac{13}{5} \text{ and } a = \frac{12}{5}$$

$$= \frac{2}{5}\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C = \frac{1}{12}\ln\left|\frac{z-5}{z-1/5}\right| + C$$

$$= \frac{1}{12}\ln\left|\frac{5\tan\frac{x}{2} - 25}{5\tan\frac{x}{2} - 1}\right| + C, \text{ on restoring the value of } z.$$

Example 13. Evaluate $\int \frac{dx}{3\cos x + 4\sin x + 5}$

Solution Putting $3 = r\cos\alpha$, $4 = r\sin\alpha$, so that $r = 5$, $\alpha = \tan^{-1}(4/3)$. We have

$$\int \frac{dx}{3\cos x + 4\sin x + 5}$$

$$= \int \frac{dx}{5\cos(x-\alpha)+5}$$

$$= \frac{1}{16}\int \sec^2\frac{1}{2}(x-\alpha)\,dx,$$

$$= \frac{1}{10}[2\tan(x-\alpha)/2] + C,$$

$$= \frac{1}{5}\tan[(x-\tan^{-1}(4/3))/2] + C.$$

Example 14. Integrate $\int \frac{dx}{13+3\cos x+4\sin x}$

Solution

$$I = \int \frac{dx}{13\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+3\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)+4.2\sin\frac{x}{2}\cos\frac{x}{2}}$$

Multiplying the numerator and denominator by

$\sec^2\frac{x}{2}$, then

$$I = \int \frac{\sec^2\frac{x}{2}\,dx}{10\tan^2\frac{x}{2} + 8\tan\frac{x}{2} + 16}$$

$$= \int \frac{2\,dx}{10z^2+8z+16}. \quad \text{Putting } z = \tan\frac{x}{2}$$

$$= \frac{1}{5}\int \frac{dz}{\left(z+\frac{2}{5}\right)^2 + \left(\frac{6}{5}\right)^2} = \frac{1}{5}\int \frac{du}{u^2+a^2}$$

where $u = z + \frac{2}{5}$, $a = \frac{6}{5}$

$$= \frac{1}{5}\frac{1}{a}\tan^{-1}\frac{u}{a} + C = \frac{1}{6}\tan^{-1}\frac{5z+2}{6} + C$$

$$= \frac{1}{6}\tan^{-1}\frac{5\tan\frac{1}{2}x + 2}{6} + C$$

Example 15. Evaluate $\int \frac{a+b\sin x}{(b+a\sin x)^2}\,dx$

Solution Here $I = \int \frac{a+b\sin x}{(b+a\sin x)^2}\,dx$

$$= \frac{b}{a}\int \frac{\frac{a^2}{b} - b + (b + a\sin x)}{(b + a\sin x)^2}dx$$

$$I = \frac{a^2 - b^2}{a}\int \frac{dx}{(b + a\sin x)^2} + \frac{b}{a}\int \frac{dx}{(b + a\sin x)} \qquad ...(1)$$

Now let, $A = \frac{\cos x}{b + a\sin x} \Rightarrow \frac{dA}{dx} = \frac{-b\sin x - a}{(b + a\sin x)^2}$

$$\Rightarrow \frac{dA}{dx} = -\frac{b}{a}\left\{\frac{a\sin x + b + \frac{a^2}{b} - b}{(b + a\sin x)^2}\right\}$$

$$\Rightarrow \frac{dA}{dx} = -\frac{b}{a}\left\{\frac{1}{b + a\sin x} + \frac{a^2 - b^2}{b(b + a\sin x)^2}\right\},$$

Integrating both sides w.r.t. 'x', we get

$$A = \frac{b}{a}\int \frac{dx}{b + a\sin x} - \frac{(a^2 - b^2)}{a}\int \frac{dx}{(b + a\sin x)^2}$$

$$\Rightarrow \frac{a^2 - b^2}{a}\int \frac{dx}{(b + a\sin x)^2} = -\frac{b}{a}\int \frac{dx}{b + a\sin x} - A \quad ...(2)$$

From (1) and (2),

$$I = -\frac{b}{a}\int \frac{dx}{(b + a\sin x)} - A + \frac{b}{a}\int \frac{dx}{(b + a\sin x)}$$

$$\Rightarrow I = -A + C$$

$$\Rightarrow I = -\left(\frac{\cos x}{b + a\sin x}\right) + C.$$

Example 16. Evaluate $\int \frac{dx}{(16 + 9\sin x)^2}$

Solution Let $A = \frac{\cos x}{16 + 9 + \sin x} \qquad ...(1)$

$$\Rightarrow \frac{dA}{dx} = \frac{(16 + 9\sin x)(-\sin x) - \cos x(9\cos x)}{(16 + 9\sin x)^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-16\sin x - 9}{(16 + 9\sin x)^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-\frac{16}{9}(9\sin x + 16) + \frac{256}{9} - 9}{(16 + 9\sin x)^2}$$

$$\Rightarrow \frac{dA}{dx} = -\frac{16}{9}\frac{1}{(16 + 9\sin x)} + \frac{175}{9(16 + 9\sin x)^2} \qquad ...(2)$$

Integrating both sides of (2) w.r.t. 'x', we get

$$A = -\frac{16}{9}\int \frac{dx}{16 + 9\sin x} + \frac{175}{9}\int \frac{dx}{(16 + 9\sin x)^2}$$

$$\Rightarrow \frac{175}{9}\int \frac{dx}{(16 + 9\sin x)^2}$$

$$= A + \frac{16}{9}\int \frac{(1 + \tan^2 x/2)dx}{16 + 16\tan^2 x/2 + 18\tan x/2}$$

$$\Rightarrow \frac{175}{9}\int \frac{dx}{(16 + 9\sin x)^2} = A + \frac{16}{9}\int \frac{2dt}{16t^2 + 18t + 16}$$

where $\tan x/2 = t$

$$\Rightarrow \frac{175}{9}\int \frac{dx}{(16 + 9\sin x)^2} = A + \frac{2}{9}\int \frac{dt}{t^2 + \frac{9}{8}t + 1}$$

$$= A + \frac{2}{9}\int \frac{dt}{\left(t + \frac{9}{16}\right)^2 + \left(\frac{\sqrt{175}}{16}\right)^2}$$

$$= A + \frac{2}{9} \times \frac{16}{\sqrt{175}}\tan^{-1}\left(\frac{16t + 9}{\sqrt{175}}\right) + C_1$$

$$\Rightarrow \int \frac{dx}{(16 + 9\sin x)^2} = \frac{9}{175}\frac{\cos x}{(16 + 9\sin x)^2} + \frac{2}{(175)^{3/2}}\tan^{-1}\left(\frac{16\tan x/2 + 9}{\sqrt{175}}\right) + C$$

Integrals of the form $\int \frac{a\cos x + b\sin x + c}{d\cos x + e\sin x + f}dx$

Since cos x, sin x, and 1 are linearly independent, therefore we can find real numbers p, q, and r such that

$a\cos x + b\sin x + c \equiv p(d\cos x + e\sin x + f)$
$+ q(-d\sin x + e\cos x) + r \qquad ...(1)$

Equating the coefficients of cos x and sin x on both sides, and also equating the constant terms on both sides, we have

$$a = pd + de,\ b = pe - qd,\ c = pf + r \qquad ....(2)$$

Solving the equations (2) we get the values of p, q and r. Substituting the expression the expression for a cos x + b sin x + c from (1) in the integrand, we have

$$I = \int \frac{a\cos x + b\sin x + c}{d\cos x + e\sin x + f}dx$$

$$= p + q\int\frac{-d\sin x + e\cos x}{d\cos x + e\sin x + f}dx + \int\frac{r\,dx}{d\cos x + e\sin x + f}$$

$= p + q\ \ln|d\cos x + e\sin x + f| + rI_1$ ...(3)

where $I_1 = \int\frac{dx}{d\cos x + e\sin x + f}$.

To evaluate I_1 we use the substitution $\tan x/2 = t$. Substituting the value of I_1 in (3), we get the expression for I from (3).

Note: We could have used the substitution $\tan x/2 = t$ from the very beginning.

Example 17. Evaluate $\int\frac{2\cos x + 3\sin x + 4}{3\cos x + 4\sin x + 5}dx$

Solution Let

$2\cos x + 3\sin x + 4 \equiv \ell\,(3\cos x + 4\sin x + 5)$
$+ m\,(-3\sin x + 4\cos x) + n$ ...(1)

Equating the coefficients of cos x and sin x on both sides of (1), and also equating the constant terms on both sides, we have

$2 = 3\ell + 4m,\ 3 = 4\ell - 3m,\ 4 = 5\ell + n$ ...(2)

Solving the system of equations (2), we have

$\ell = \frac{18}{25},\ m = -\frac{1}{25},\ n = \frac{2}{5}.$ ...(3)

Substituting the expression for 2 cos x + 3 sin x + 4 from (1), the given integrand and the values of ℓ, m, n from (3), we have

$$\int\frac{2\cos x + 3\sin x + 4}{3\cos x + 4\sin x + 5}dx$$

$$= \frac{18}{25}x - \frac{1}{25}\ln|3\cos x + 4\sin x + 5| + \frac{2}{5}\int\frac{dx}{3\cos x + 4\sin x + 5}$$

$$= \frac{18}{25}x - \frac{1}{25}\ln|3\cos x + 4\sin x + 5| + \frac{2}{25}\tan[(x - (\tan^{-1}4/3))/2] + C.$$

Example 18. Evaluate $\int\frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}dx$

Solution Let 2 sin x + 3 cos x

$= \ell$(denominator) + m(derivative of denominator)

$= \ell(3\sin x + 4\cos x) + m(3\cos x - 4\sin x)$

$= (3\ell - 4m)\sin x + (4\ell + 3m)\cos x$

Now comparing the coefficients of sin x and cos x of both sides, we get

$3\ell - 4m = 2$ and $4\ell + 3m = 3$,

$\Rightarrow \ell = \frac{18}{25},\ m = \frac{1}{25}.$

$\therefore$ 2 sin x + 3 cos x

$$= \frac{18}{25}(3\sin x + 4\cos x) + \frac{1}{25}(3\cos x - 4\sin x)$$

$$\therefore\ I = \frac{18}{25}\int dx + \frac{1}{25}\int\frac{3\cos x - 3\sin x}{3\sin x + 4\cos x}dx$$

$$= \frac{18}{25}x + \frac{1}{25}\ln|(3\sin x + 4\cos x)| + C$$

Example 19. Evaluate $\int\frac{3\cos x + 2}{\sin x + 2\cos x + 3}dx.$

Solution Let $I = \int\frac{3\cos x + 2}{\sin x + 2\cos x + 3}dx.$

Let $3\cos x + 2 = \lambda\,(\sin x + 2\cos x + 3)$
$+ \mu\,(\cos x - 2\sin x) + \nu.$

Comparing the coefficients of sin x, cos x and constant term on both sides, we get

$\lambda - 2\mu = 0,\ 2\lambda + \mu = 3,\ 3\lambda + \nu = 2$

$\Rightarrow \lambda = \frac{6}{5}, \mu\ \frac{3}{5}$ and $\nu = -\frac{8}{5}$

$$\therefore I = \int\frac{\lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + \nu}{\sin x + 2\cos x + 3}dx$$

$$= \lambda\int dx + \mu\int\frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}dx + \nu\int\frac{1}{\sin x + 2\cos x + 3}dx$$

$= \lambda x + \mu\log|\sin x + 2\cos x + 3| + \nu I_1$, where

$$I_1 = \int\frac{1}{\sin x + 2\cos x + 3}dx$$

Putting $\sin x = \frac{2\tan x/2}{1 + \tan^2 x/2}$, $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$

we get $I_1 = \int\frac{1}{\frac{2\tan x/2}{1 + \tan^2 x/2} + \frac{2(1 - \tan^2 x/2)}{1 + \tan^2 x/2} + 3}dx$

$$= \int\frac{1 + \tan^2 x/2}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3(1 + \tan^2 x/2)}dx$$

$$= \int\frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 5}dx$$

Putting $\tan\frac{x}{2} = t$ and $\frac{1}{2}\sec^2\frac{x}{2} = dt$

or $\sec^2\frac{x}{2}\,dx = 2\,dt$, we get

$$I_1 = \int \frac{2dt}{t^2+2t+5}$$

$$= 2\int \frac{dt}{(t+1)^2+2^2} = \frac{2}{2}\tan^{-1}\left(\frac{t+1}{2}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)$$

Hence, $I = \lambda x + \mu \log|\sin x + 2\cos x + 3|$

$$+ \nu\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) + C,$$

where $\lambda = \frac{6}{5}$, $\mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$.

Example 20. Evaluate $\int \frac{(5\sin x + 6)dx}{\sin x + 2\cos x + 3}$.

Solution Let $5\sin x + 6 = A(\sin x + 2\cos x + 3)$ $+ B(\cos x - 2\sin x) + C$

Equating the coefficients of sin x, cos x and constant term, we get

$$\left.\begin{array}{l} A-2B=5 \\ 2A+B=0 \\ 3A+C=6 \end{array}\right\} \Rightarrow A=1, B=-2, C=3$$

$$I = \int dx - 2\int \frac{(\cos x - 2\sin x)dx}{\sin x + 2\cos x + 3} + 3\int \frac{dx}{\sin x + \cos x + 3}$$

$$= x - 2\ell n|\sin x + 2\cos x + 3| + 3I_1$$

In I_1, put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}dx = 2dt$

So, $I_1 = \int \frac{2dt}{t^2+2t+5} = \int \frac{2dt}{(t+1)^2+4}$

$$= \tan^{-1}\left(\frac{t+1}{2}\right) + C = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{2}\right) + C.$$

Example 21. Evaluate $\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx$.

Solution To evaluate $I = \int \frac{P(x)}{Q(x)}dx$,

assume $P(x) = AQ(x) + BQ'(x)$ ...(1)

where A and B are constants.

Then $I = \int \frac{AQ(x)+BQ'(x)}{Q(x)}dx$

$$= A\int dx + B\int \frac{Q'(x)}{Q(x)}dx$$

$$= Ax + B\ln|Q(x)| + C$$

From (1), by comparing coefficients of same type of terms, one gets constants A and B.

In the present problem, $I = \int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}$

Denominator $Q(x) = 9e^x - 4e^{-x}$

Numerator $P(x) = 4e^x + 6e^{-x}$

As $Q'(x) = 9e^x + 4e^{-x}$, we take

$4e^x + 6e^{-x} = A(9e^x - 4e^{-x}) + B(9e^x + 4e^{-x})$

By comparing the coefficients of e^x and e^{-x}, we get

$4 = 9A + 9B$

$6 = -4A + 4B$

$\therefore\ A = -\frac{19}{36}$, $B = \frac{35}{36}$

$$\therefore\ I = \int \frac{A(9e^x-4e^{-x})+B(9e^x+4e^{-x})}{9e^x-4e^{-x}}dx$$

$$= A\int dx + B\int \frac{9e^x+4e^{-x}}{9e^x-4e^{-x}}dx$$

$$= Ax + B\ln|9e^x - 4e^{-x}| + C$$

$$= -\frac{19}{36}x + \frac{35}{36}\ln|9e^x-4e^{-x}| + C.$$

Note:

To evaluate $\int \frac{ae^x+be^{-x}+c}{pe^x+qe^{-x}+r}dx$, we can find real numbers l, m, and n such that $a\,e^x + b\,e^{-x} + c$ $\equiv l(a\,e^x + b\,e^{-x} + c) + m(a\,e^x - b\,e^{-x}) + n$ and integrate the three integrals.

Integrals of the forms $\int \frac{\cos x \pm \sin x}{f(\sin 2x)}dx$

We have $\frac{d}{dx}(\sin x \pm \cos x) = \cos x \mp \sin x$ and $(\sin x \pm \cos x)^2 = 1 \pm \sin 2x$.

Example 22. Evaluate

$$\int (\cos x - \sin x)(3 + 4\sin 2x)dx.$$

Solution $I = \int (\cos x - \sin x)(3 + 4\sin 2x)dx$

Here derivative of sin x + cos x is cos x – sin x
and 3 + 4 sin 2x = 3 + 4((sin x + cos x)^2–1)

$\therefore$ Put sin x + cos x = t

$\Rightarrow$ (cos x – sin x)dx = dt

So $I=\int(3+4(t^2-1)dt$

$$=\frac{t}{3}[4t^2-3]+C$$

$$=\left(\frac{\sin x+\cos x}{3}\right)[4(\sin x+\cos x)^2-3]+C$$

$$=\left(\frac{\sin x+\cos x}{3}\right)(1+4\sin 2x)+C.$$

Example 23. Evaluate $I=\int\frac{\cos x-\sin x}{\sqrt{\sin 2x}}dx$

Solution Put sin x + cos x = u

$\Rightarrow$ (cosx – sin x) dx = du

Also, u^2 = 1 + sin 2x

Hence, $I=\int\frac{du}{\sqrt{u^2-1}}$

$$=\ln\left|u+\sqrt{u^2-1}\right|+C, \text{ where } u=\sin x+\cos x.$$

Example 24. Evaluate $I=\int\frac{\sin x+2\cos x}{9+16\sin 2x}dx$

Solution Let sin x + 2 cos x

$= \ell$ (cos x – sin x) + m (sin x + cos x)

$\Rightarrow$ $-\ell+m=1$ and $\ell+m=2$

$\Rightarrow$ $\ell=1/2$ and $m=3/2$

$$I=\frac{1}{2}\int\frac{\cos x-\sin x}{9+16\sin 2x}dx+\frac{3}{2}\int\frac{\sin x-\cos x}{9+16\sin 2x}dx$$

Put sinx + cos x = u and sin x – cos x = v

(cos x – sin x) dx = du (cos x + sin x) dx = dv

$$I=\frac{1}{2}\int\frac{du}{9+16(u^2-1)}+\frac{3}{2}\int\frac{dv}{9+16(1-v^2)}$$

$$=\frac{1}{2}\int\frac{du}{16u^2-7}+\frac{3}{2}\int\frac{dv}{25-v^2}$$

Now, the above integrals can be evaluated easily.

Example 25. Evaluate $\int\sqrt{\tan x}\,dx$.

Solution $I=\int\sqrt{\tan x}\,dx$.

$$=\frac{1}{2}\left[\int(\sqrt{\tan x}+\sqrt{\cot x})dx+\int(\sqrt{\tan x}-\sqrt{\cot x})dx\right].$$

Now, $I_1=\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx$

$$=\int\frac{\sqrt{2}d(\sin x-\cos x)}{\sqrt{\sin 2x}}=\sqrt{2}\int\frac{d(\sin x-\cos x)}{\sqrt{1-(1-\sin 2x)}}$$

$$=\sqrt{2}\int\frac{d(\sin x-\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}$$

$$=\sqrt{2}\int\frac{dz}{\sqrt{1-z^2}}, \text{ where } z=\sin x-\cos x$$

$$=\sqrt{2}\sin^{-1}z+c_1=\sqrt{2}\sin^{-1}(\sin x-\cos x)+c_1.$$

$$I_2=\int(\sqrt{\tan x}-\sqrt{\cos x})dx=\int\frac{\sin x-\cos x}{\sqrt{\sin x\cos x}}dx$$

$$=\int\frac{-d(\sin x+\cos x)}{\sqrt{\sin x 2x}}=-\sqrt{2}\int\frac{d(\sin x+\cos x)}{\sqrt{(1+\sin 2x)-1}}$$

$$=-\sqrt{2}\int\frac{d(\sin x+\cos x)}{\sqrt{(\sin x+\cos x)^2-1}}$$

$$=-\sqrt{2}\int\frac{dz}{\sqrt{z^2-1}}, \text{ where } z=\sin x+\cos x$$

$$=-\sqrt{2}\ln(z+\sqrt{z^2-1})+c^2$$

$$=-\sqrt{2}\ln(\sin x+\cos x+\sqrt{\sin 2x})+c_2$$

$\therefore$ $I=\frac{1}{2}I_1+\frac{1}{2}I_2$

$$=\frac{1}{\sqrt{2}}\sin^{-1}(\sin x-\cos x)$$

$$-\frac{1}{\sqrt{2}}\ln(\sin x+\cos x+\sqrt{\sin 2x})+C.$$

Example 26. Evaluate $\int\frac{dx}{\cos x+\operatorname{cosec} x}$.

Solution $I=\int\frac{dx}{\cos x+\frac{1}{\sin x}}=\int\frac{\sin x}{\sin x\cdot\cos x+1}dx$

$$=\int\frac{2\sin x}{2+\sin 2x}dx=\int\frac{(\sin x+\cos x)+(\sin x-\cos x)}{2+\sin 2x}dx$$

$$=\int\frac{\sin x+\cos x}{2+\sin 2x}dx+\int\frac{\sin x-\cos x}{2+\sin 2x}$$

$$=\int\frac{\frac{d}{dx}(\sin x-\cos x)}{3-(1-\sin 2x)}dx-\int\frac{\frac{d}{dx}(\sin x+\cos x)}{1+(1+\sin 2x)}dx$$

$$=\int\frac{\frac{d}{dx}(\sin x-\cos x)}{3-(\sin x-\cos x)^2}dx-\int\frac{\frac{d}{dx}(\sin x+\cos x)}{1+(\sin x+\cos x)^2}dx$$

$= I_1 - I_2$.

For I_1, put sin x – cos x = z. Then

$$I_1=\int\frac{dz}{3-z^2}=\int\frac{dz}{(\sqrt{3})^2-z^2}$$

$$=\frac{1}{2\sqrt{3}}\ln\frac{\sqrt{3}+z}{\sqrt{3}-z}+c_1$$

$$= \frac{1}{2\sqrt{3}} \ln \frac{\sqrt{3} + \sin x - \cos x}{\sqrt{3} - \sin x + \cos x} + c_1$$

For I_2, put $\sin x + \cos x = t$. Then

$$I_2 = \int \frac{dt}{1+t^2} = \tan^{-1} t + c_2$$
$$= \tan^{-1}(\sin x + \cos x) + c_2$$

$\therefore \quad I = I_1 - I_2$

$$= \frac{1}{2\sqrt{3}} \ln \frac{\sqrt{3} + \sin x - \cos x}{\sqrt{3} - \sin x + \cos x} - \tan^{-1}(\sin x + \cos x) + C.$$

Example 27. Evaluate $\int \frac{dx}{2\sin x + \sec x}$

Solution Let $I = \int \frac{dx}{2\sin x + \sec x}$

$$= \int \frac{\cos x\,dx}{\sin 2x + 1} = \frac{1}{2}\int \frac{2\cos x\,dx}{1 + \sin 2x}$$

$$= \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{(\sin^2 + \cos^2 x + 2\sin x \cos x)} dx$$

$$= \frac{1}{2}\int \frac{\cos x + \sin x}{(\sin x + \cos x)^2} dx + \int \frac{(\cos x - \sin x)}{(\sin x + \cos x)^2} dx$$

$$= \frac{1}{2}\int \frac{dx}{\sin x + \cos x} + \frac{1}{2}\int \frac{dv}{v^2}, \text{ where } v = \sin x + \cos x$$

$$= \frac{1}{2\sqrt{2}} \int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x} - \frac{1}{2v}$$

$$= \frac{1}{2\sqrt{2}} \int \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} - \frac{1}{2(\sin x + \cos x)}$$

$$= \frac{1}{2\sqrt{2}} \ln \left|\operatorname{cosec}\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right)\right|$$
$$- \frac{1}{2(\sin x + \cos x)} + C$$

Integrals of the forms

$$\int \sqrt{\sec^2 x \pm a}\,dx,\ \int \sqrt{\operatorname{cosec}^2 x \pm a}\,dx,$$
$$\int \sqrt{\tan^2 x \pm a}\,dx,\ \int \sqrt{\cot^2 x \pm a}\,dx$$

Method :

(i) Write $\sqrt{\sec^2 x \pm a} = \frac{\sec^2 \pm a}{\sqrt{\sec^2 x \pm a}}$

$$= \frac{\sec^2 x}{\sqrt{\sec^2 x \pm a}} \pm a \frac{\cos x}{\sqrt{1 \pm a\cos^2 x}}$$

In the first part, put $u = \tan x$ and in the second part, put $v = \sin x$.

(ii) In case of $\sqrt{\operatorname{cosec}^2 x \pm a}$ proceed as in (i) and put put $u = \cot x$ in the first part, and $v = \sin x$ in the second part.

(iii) In case of $\sqrt{\tan^2 x \pm a}$ or $\sqrt{\cot^2 x \pm a}$, change $\tan^2 x$ into $\sec^2 x - 1$ and $\cot^2 x$ into $\operatorname{cosec}^2 x - 1$ and then proceed as in (i) and (ii).

Example 28. Evaluate $\int \sqrt{\tan^2 x - 3}\,dx$

Solution $\int \sqrt{\tan^2 x - 3}\,dx = \int \sqrt{\sec^2 x - 4}\,dx$

$$= \int \frac{\sec^2 x - 4}{\sqrt{\sec^2 x - 4}} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan^2 x - 3}} dx - \int \frac{4}{\sqrt{\sec^2 x - 4}} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan^2 x - 3}} dx - 4\int \frac{\cos x}{\sqrt{1 - 4\cos^2 x}} dx$$

Put $\tan x = u$ Put $\sin x = v$

$\sec^2 x\,dx = du$ $\cos x\,dx = dv$

Now, the above integrals can be evaluated easily.

Example 29. Evaluate $\int \frac{\sqrt{3 + \cos 2x}}{\cos x} dx$

Solution $\int \frac{\sqrt{3 + \cos 2x}}{\cos x} dx = \int \sqrt{\frac{2 + 2\cos^2 x}{\cos^2 x}}\,dx$

$$= \sqrt{2}\int \sqrt{\sec^2 x + 1}\,dx = \sqrt{2}\int \frac{\sec^2 x + 1}{\sqrt{\sec^2 x + 1}} dx$$

$$= \sqrt{2}\left\{\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 2}} dx + \int \frac{dx}{\sqrt{\sec^2 x + 1}}\right\}$$

$$= \sqrt{2}\left\{\int \frac{\sec^2 x\,dx}{\sqrt{\tan^2 x + 2}} + \int \frac{\cos x\,dx}{\sqrt{1 + \cos^2 x}}\right\}$$

Put $\tan x = u$ Put $\sin x = v$

$\sec^2 x\,dx = du$ $\cos x\,dx = dv$

Now, the above integrals can be evaluated easily.

Example 30. Evaluate $\int \sqrt{\operatorname{cosec}^2 x - 2}\,dx$.

Solution $I = \int \frac{\operatorname{cosec}^2 x - 2}{\sqrt{\operatorname{cosec}^2 x - 2}} dx$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\operatorname{cosec}^2 x - 2}} dx - 2\int \frac{dx}{\sqrt{\operatorname{cosec}^2 x - 2}}$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot^2 x - 1}} dx - 2\int \frac{\sin x}{\sqrt{1 - 2\sin^2 x}} dx \quad ...(1)$$

Let $I_1 = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot^2 x - 1}} dx$, put $z = \cot x$,

then $dz = -\operatorname{cosec}^3 x\, dx$

Now $I_1 = -\int \frac{dz}{\sqrt{z^2 - 1}} = -\ln(z + \sqrt{z^2 - 1})$

$$= -\ln(\cot x + \sqrt{\cot^2 x - 1})$$

Let $I_2 = -2\int \frac{\sin x}{\sqrt{1 - 2\sin^2 x}} dx = -2\int \frac{\sin x}{\sqrt{2\cos^2 x - 1}} dx$

Put $z = \sqrt{2}\cos x$, then $dz = -\sqrt{2}\sin x\, dx$

Now $I_2 = \sqrt{2}\int \frac{dz}{\sqrt{z^2 - 1}} = \sqrt{2}\ln(z + \sqrt{z^2 - 1})$

$$= \sqrt{2}\ln(\sqrt{2}\cos x + \sqrt{2\cos^2 x - 1})$$

$$= \sqrt{2}\ln(\sqrt{2}\cos x + \sqrt{\cos 2x})$$

From (1), $I = I_1 + I_2$

$$= -\ln(\cot x + \sqrt{\cot^2 x - 1})$$

$$+ \sqrt{2}\ln(\sqrt{2}\cos x + \sqrt{\cos 2x}) + C.$$

Practice Problems I

1. Evaluate the following integrals :

(i) $\int \frac{dx}{\cos^2 x - 3\sin^2 x}$ (ii) $\int \frac{dx}{4 - 5\sin^2 x}$

(iii) $\int \frac{dx}{(3\sin x - 4\cos x)^2}$

(iv) $\int \frac{1}{(\sin x - 2\cos x)(2\sin x + \cos x)} dx$

2. Evaluate the following integrals :

(i) $\int \frac{dx}{3 + \cos^2 x}$ (ii) $\int \frac{\sin x}{\sin 3x} dx$

(iii) $\int \frac{\cos x}{\cos 3x} dx$ (iv) $\int \frac{dx}{(\sin x + \cos x)^2}$

3. Evaluate the following integrals :

(i) $\int \frac{dx}{13 + 12\cos x}$ (ii) $\int \frac{1}{5 + 4\cos x} dx$

(iii) $\int \frac{dx}{1 - 2\sin x}$ (iv) $\int \frac{dx}{5 + 4\sin x}$

4. Evaluate the following integrals :

(i) $\int \frac{dx}{3 + 2\sin x + \cos x}$

(ii) $\int \frac{dx}{2\sin x + 3\cos x - 5}$

(iii) $\int \frac{dx}{2\sin x - \cos x + 3}$

(iv) $\int \frac{dx}{5 - 4\sin x + 3\cos x}$

5. Evaluate the following integrals :

(i) $\int \frac{d\theta}{2 - 3\cos 2\theta}$ (ii) $\int \frac{(1 - \sin\theta)}{2\cos\theta + 3} d\theta$

(iii) $\int \frac{\cos x\, dx}{5 - 3\cos x}$ (iv) $\int \frac{\cos x}{\cos 2x} dx$

6. Evaluate the following integrals :

(i) $\int \frac{d\theta}{1 - \sin^4\theta}$ (ii) $\int \frac{dx}{\sin^4 x + \cos^4 x}$

(iii) $\int \frac{dx}{1 - \cos^4 x}$

(iv) $\int \frac{d\theta}{(a\cos^2\theta + b\sin^2\theta)^2}$

7. Evaluate the following integrals :

(i) $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} dx$ (ii) $\int \frac{\cos x\, dx}{2\sin x + 3\cos x}$

(iii) $\int \frac{4\sin x + 5\cos x}{5\sin x + 4\cos x} dx$

(iv) $\int \frac{6 + 3\sin x + 14\cos x}{3 + 4\sin x + 5\cos x} dx$

8. Evaluate the following integrals :

(i) $\int \frac{\cos x - \sin x}{9\sin 2x - 6} dx$ (ii) $\int \frac{\cos\left(x + \frac{\pi}{4}\right)}{2 + \sin 2x} dx$

(iii) $\int \frac{\cos x - \sin x}{\cos x + \sin x}(2 + 2\sin 2x)\, dx$

(iv) $\int \frac{dx}{\sin x + \tan x}$

9. Evaluate the following integrals :

(i) $\int \frac{\cos x}{\cos x + \sin x} dx$ (ii) $\frac{1}{a + b\cot x}$

(iii) $\int \frac{\sqrt{1+\sin 2x}}{1+\cos 2x}\,dx$

10. Evaluate the following integrals :

(i) $\int \sqrt{2+\tan^2 x}\,dx$ (ii) $\int \sqrt{1+\sec x}\,dx$

(iii) $\int \sqrt{1+\operatorname{cosec} x}\,dx$

(iv) $\int \operatorname{cosec} x \sqrt{\cos 2x}\,dx$

11. Evaluate the following integrals :

(i) $\int \frac{1}{(\cos x + 2\sin x)^2}\,dx$

(ii) $\int \frac{dx}{(\sin^2 x + 2\cos^2 x)^2}\,dx$

(iii) $\int \frac{\cos\theta\, d\theta}{(5+4\cos\theta)^2}$

(iv) $\int \frac{dx}{\sin^6 x + \cos^6 x}$

1.11 INTEGRATION BY PARTS

Integration by parts is a method to compute integrals of the form $\int u(x).v(x)\,dx$ in which u can be differentiated easily and g can be integrated easily.

Formula for integration by parts

$$\int u.v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}.\int v\,dx\right]dx$$

The formula comes from the product rule of differentiation

$$\frac{d}{dx}(u.w) = \frac{du}{dx}.w + u.\frac{dw}{dx}$$

Integrating both sides w.r.t. x

$$u.w = \int\left(\frac{du}{dx}.w\right)dx + \int\left(u.\frac{dw}{dx}\right)dx$$

$$\int\left(u.\frac{dw}{dx}\right)dx = u.w - \int\left(\frac{du}{dx}.w\right)dx$$

Let $\frac{dw}{dx} = v$, then $w = \int v\,dx$

Hence, $\int u.v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}.\int v\,dx\right]dx$

It is most frequently used in the integration of expressions that may be represented in the form of a product of two factors u and v in such a way that the finding of the function v from its derivative, and the evaluation of the integral $\int v\,du$ should, taken together, be a simpler problem.

The rule for integration by parts is widely used for integrals of the form.

$\int x^k \sin ax\,dx, \int x^k \cos ax\,dx$

$\int x^k e^{ax}\,dx, \int x^k \ln x\,dx$

and certain integrals containing inverse trigonometric functions are evaluated by means of integration by parts.

Note:

While using integration by parts, we choose u and v such that

(a) $\int v\,dx$ is simple and

(b) $\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ is simple to integrate.

This is generally obtained, by keeping the order of u and v as per the order of the letters in **ILATE**, where I – Inverse function, L – Logarithmic function, A – Algebraic function, T – Trigonometric function and E – Exponential function. When the integrand of an integration by parts problem consists of the product of two different types of functions, we should let u designate the function that appears first in the acronym ILATE, and let v denote the rest.

However, to become skilled at breaking up a given integrand into the factors u and v, one has to solve problems; we shall show how this is done in a number of cases.

For example, since the integrand of $\int x\cos x\,dx$ is the product of the algebraic function x with the trigonometric function cos x, we should let u = x and v = cos x.

$$\int x\cos x\,dx = x\int \cos x\,dx - \int\left(\frac{d(x)}{dx}\int \cos x\,dx\right)dx$$

$$= x\sin x - \int 1.\sin x\,dx$$

$$= x\sin x - (-\cos x) + C$$

$$= x\sin x + \cos x + C$$

Example 1. Integrate $\int xe^x dx$

Solution $I = x\int e^x dx - \int\left\{\frac{dx}{dx}\int e^x\, dx\right\}dx$

$= xe^x - \int 1.e^x dx = xe^x - e^x + C$

Note:

In the above integral, instead of taking x as the first function and e^x as the second, if we take e^x as the first function and x as the second, then applying the rule for integration by parts we get

$$\int (e^x.x)dx = e^x.\frac{1}{2}x^2 - \int e^x.\frac{1}{2}x^2 dx.$$

The integral $\frac{1}{2}\int e^x x^2\, dx$ on the right side is more complicated than the one we started with, for it involves x^2 instead of x.

Thus, while applying the rule for integration by parts to the product of two functions, care should be taken to choose properly the first function, i.e., the function to be differentiated.

A little practice and experience will enable the student to make the right choice.

Note:

1. When determining the function v from its derivative, we can take any arbitrary constant, since it does not enter into the final result. This can be seen by adding the arbitrary constant C in the formula of integration by parts. It is however convenient to consider this constant equal to zero.

$$\int u.v\, dx$$

$$= u\left(\int v dx + C_1\right) - \int\left[\frac{du}{dx}.\left(\int v dx + C_1\right)\right]dx$$

$$= u\int v dx + u C_1 - \int\left[\frac{du}{dx}.\int v\, dx\right]dx - u C_1$$

$$= u\int v dx - \int\left[\frac{du}{dx}.\int v\, dx\right]dx$$

We see that the constant C_1 cancels out, thus giving the same solution obtained by omitting C_1, thereby justifying the omission of the constant of integration when calculating v in integration by parts. Also note that $\int v dx$ should be taken as same in both terms.

2. Sometimes, a constant can be used to simplify the calculations. For example,

$$\int \ln(1+x)\, dx = \ln(x+1).(x+1) - \int \frac{1}{x+1}.(x+1)dx$$

$= (x+1).\ln(x+1) - x + C.$

Thus, we have simplified the computation of the second term by introducing a constant of integration $C_1 = 1$.

3. The method sometimes fails because it leads back to the original integral. For example, let us try to integrate $\int x^{-1}dx$ by parts.

$$\int x^{-1}dx = \frac{1}{x}.x - \int\left(-\frac{1}{x^2}\right)x\, dx$$

$$\Rightarrow \int x^{-1}dx = -1 + \int x^{-1}dx + C. \qquad ...(1)$$

and we are back where we started.

This example is often used to illustrate the importance of paying attention to the arbitrary constant C. If (1) is written without C, it leads to the equation $\int x^{-1}dx = -1 + \int x^{-1} dx$, which is false.

Example 2. Evaluate $\int x\tan^{-1}x\, dx$

Solution $\int x\tan^{-1}x\, dx$

$$= (\tan^{-1}x)\frac{x^2}{2} - \int \frac{1}{1+x^2}.\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2+1-1}{x^2+1}dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int 1 - \frac{1}{x^2+1}dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}[x - \tan^{-1}x] + C.$$

Example 3. Evaluate $\int x\ln(1+x)\, dx$

Solution $\int x\ln(1+x)\, dx$

$$= \ln(x+1).\frac{x^2}{2} - \int\frac{1}{x+1}.\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int\frac{x^2}{x+1}dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int\frac{x^2-1+1}{x+1}dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2-1}{x+1} + \frac{1}{x+1}\,dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\left[\int\left((x-1)+\frac{1}{x+1}\right)dx\right]$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\left[\frac{x^2}{2} - x + \ln|x+1|\right] + C$$

Note:

Very often an integral involving a single logarithmic function or a single inverse circular function can be evaluated by the application of the rule for integration by parts, by considering the integral as the product of the given function and unity; and taking the given function as the first function and unity as the second. The principle is illustrated below.

Example 4. Evaluate $\int \ln x\,dx$

Solution $\int \ln x \,.\, 1\,dx$

$$= \ln x\int dx - \int\left\{\frac{d}{dx}(\ln x)\,.\int dx\right\}dx$$

$$= \ln x\,.\,x - \int \frac{1}{x}\,.\,x\,dx$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + C.$$

Example 5. Evaluate $\int \tan^{-1} x\,dx$

Solution $I = \int \tan^{-1} x\,.\,1\,dx$

$$= \tan^{-1} x\int dx - \int\left\{\frac{d}{dx}\left(\tan^{-1} x\right)\int dx\right\}dx$$

$$= \tan^{-1} x\,.\,x - \int \frac{1}{1+x^2}\,.\,x\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results :

$$\int f^{-1}(x)dx = \int y f'(y)dy$$

$$y = f^{-1}(x),\ x = f(y),\ dx = f'(y)dy$$

$$= yf(y) - \int f(y)dy = xf^{-1}(x) - \int f(y)dy$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\int \ln x\,dx = \int ye^y dy$$

$$y = \ln x,\ x = e^x,\quad dx = e^y dy$$

$$= ye^y - e^y + C$$

$$= x\ln x - x + C.$$

For the integral of $\cos^{-1} x$ we get

$$\int \cos^{-1} x\,dx = y\cos y - \int \cos y\,dy$$

where $y = \cos^{-1} x$, $x = \cos y$

$$= x\cos^{-1} x - \int \cos y\,dy$$

$$= x\cos^{-1} x - \sin y + C$$

$$= x\cos^{-1} x - \sin(\cos^{-1} x) + C.$$

Generally, we can integrate $f^{-1}(x)$, the inverse function of $f(x)$, if we can integrate $f(x)$.

Example 6. Evaluate $\int \ln(x + \sqrt{x^2 + a^2})dx$

Solution $I = \int \ln(x + \sqrt{x^2 + a^2}).1dx$

$$= \ln(x + \sqrt{x^2 + a^2})\int 1dx$$

$$- \int\left[\frac{d}{dx}\left\{\ln\left(x + \sqrt{x^2+a^2}\right)\right\}.\int 1dx\right]dx$$

$$= \ln(x + \sqrt{x^2 + a^2}).x - \int \frac{1}{\sqrt{x^2 + a^2}}.x\,dx$$

$$= x\ln(x + \sqrt{x^2 + a^2}) - \int \frac{x\,dx}{\sqrt{(x^2 + a^2)}}$$

To evaluate $\int \frac{x\,dx}{\sqrt{(x^2 + a^2)}}$, put $x^2 + a^2 = z^2$, so that

$x\,dx = z\,dz$.

$$\therefore\quad I = x\ln(x + \sqrt{x^2 + a^2}) - \sqrt{x^2 + a^2} + C.$$

Example 7. Evaluate $\int x^n \ln x$.

Solution We have $\int x^n \ln x\,dx = \int (\ln x).\,x^n\,dx$

$$= (\ln x)\,.\,\frac{x^{n+1}}{n+1} - \int \frac{1}{x}\,.\,\frac{x^{n+1}}{n+1}\,dx,$$

[Integrating by parts taking x^n as the second function]

$$= (\ln x).\,\frac{x^{n+1}}{n+1} - \int \frac{x}{n+1}\,dx$$

$$= (\ln x)\,.\,\frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C.$$

Example 8. Evaluate $\int \frac{\ln(\sec^{-1} x)dx}{x\sqrt{(x^2-1)}}$.

Solution Put $\sec^{-1} x = t$ so that $\frac{1}{x\sqrt{(x^2-1)}}\,dx = dt$.

Then the given integral

$$= \int \ln t\, dt = \int (\ln t).\, 1\, dt$$

$$= (\ln t).\, t - \int \frac{1}{t}\, t\, dt$$

$$= t \ln t - \int dt = t \ln t - t + C$$

$$= t(\ln t - 1) + C = t.\,(\ln t - \ln e) + C$$

$$= t \ln\left(\frac{t}{e}\right) + C = (\sec^{-1} x) \ln\left(\frac{\sec^{-1} x}{e}\right) + C.$$

Example 9. Evaluate $\int \tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}\, dx$.

Solution Put $x = \cos\theta$ so that $dx = -\sin\theta\, d\theta$.

$\therefore$ The given integral

$$= \int \left\{\tan^{-1}\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}\right\}(-\sin\theta)\, d\theta$$

$$= -\int \{\tan^{-1}(\tan 1/2\,\theta) \sin\theta\, d\theta$$

$$= -\int \frac{1}{2}\theta \sin\theta\, d\theta = -\frac{1}{2}\int \theta \sin\theta\, d\theta$$

$$= -\frac{1}{2}[\theta.(-\cos\theta) - \int(-\cos\theta)\, d\theta]$$

$$= \frac{1}{2}[\theta\cos\theta - \int \cos\theta\, d\theta]$$

$$= \frac{1}{2}[\theta\cos\theta - \sin\theta] + C$$

$$= \frac{1}{2}[x\cos^{-1} x - \sqrt{(1-x^2)}\; + C.$$

Example 10. Evaluate $\int x^2 \tan^{-1} x\, dx$

Solution We have $\int x^2 \tan^{-1} x\, dx$

$$= \frac{x^3}{3}\tan^{-1} x - \int \frac{x^3}{3}.\frac{1}{1+x^2}$$

Integrating by parts taking x^2 as the second function

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x(x^2+1)-x}{1+x^2}\, dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int x\, dx + \frac{1}{3}.\frac{1}{2}\int \frac{2x}{1+x^2}\, dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C.$$

Note: The operation of integration by parts can be used successively, i.e. we can repeat it several times.

Example 11. Find $\int x^2 e^{-x}\, dx$.

Solution $\int x^2 e^{-x}\, dx = x^2(-e^{-x}) - \int(-e^{-x})\, 2x\, dx)$

$$= -x^2 e^{-x} + 2\int x e^{-x}\, dx$$

$$= -x^2 e^{-x} + 2\,[x(-e^{-x}) - \int(-e^{-x})\, dx]$$

$$= x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$= -e^{-x}(x^2 + 2x + 2) + C.$$

Example 12. Evaluate $I = \int x^3 e^{2x} dx$

Solution $I = \int x^3 e^{2x} dx = x^3 \frac{e^{2x}}{2} - \int 3x^2.\frac{e^{2x}}{2} dx$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\left(x^2\frac{e^{2x}}{2} - \int 2x\frac{e^{2x}}{2}dx\right)$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\int x e^{2x} dx$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\left(x.\frac{e^{2x}}{2} - \int 1.\frac{e^{2y}}{2}dx\right)$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}\int e^{2x}\, dx$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}e^{2x} + C$$

$$= \frac{1}{8}e^{2x}\left(4x^3 - 6x^2 + 6x - 3\right) + C$$

Example 13. Evaluate $\int (p^3 + 6p)\sin p\, dp$

Solution $I = \int (p^3 + 6p)\sin p\, dp$

$$= -(p^3 + 6p)\cos p + \int (3p^2 + 6)\cos p\, dp$$

$$= -(p^3 + 6p)\cos p + (3p^2 + 6)\sin p - \int 6p \sin p\, dp$$

$$= -(p^3 + 6p)\cos p + (3p^2 + 6)\sin p$$

$$\left[-p\cos p + \int \cos p\, dp\right]$$

$$= -(p^3 + 6p)\cos p + (3p^2 + 6)\sin p + 6p\cos p$$

$$- 6\sin p + C$$

$$= 3p^2 \sin p - (p^3 + 6p)\cos p + 6p\cos p + C$$

$= 3p^2 \sin p - p^3 \cos p + C$
$= (3 \sin p - p \cos p)\, p^2 + C.$

Example 14. Evaluate $\int \sec^3 x\, dx$

Solution $I = \int \sec^3 x\, dx = \int \sec x \,.\, \sec^2 x\, dx$

Here both functions are trigonometric and integral of $\sec^2 x$ is $\tan x$ which is simpler than the integral of $\sec x$ which is $\ln(\sec x + \tan x)$. Therefore, we take $\sec^2 x$ as y.

$$I = \sec x \tan x - \int (\sec x \tan x) \tan x\, dx$$
$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$
$$= \sec x \tan x - \sec x(\sec^2 x - 1)dx$$
$$= \sec x \tan x - I + \int \sec x\, dx \quad [\because I = \int \sec^3 x\, dx]$$

or $2I = \sec x \tan x + \ln(\sec x + \tan x)$

$$\therefore \quad I = \frac{\sec x \tan x}{2} + \frac{1}{2}\ln(\sec x + \tan x) + C.$$

Note: While performing integration by parts, sometimes the original integral appears on the right hand side, which we name as I and transfer on the left hand side.

Example 15. Evaluate $I = \int \sqrt{a^2 - x^2}\, dx$

Solution Integrating by parts, taking the first function $\sqrt{a^2 - x^2}$ and second function 1, we have

$$I = \int \sqrt{a^2 - x^2}\,.1dx$$
$$= \sqrt{a^2 - x^2}\; x - \int \left[\frac{1}{2}\cdot\frac{(-2x)}{\sqrt{a^2 - x^2}}\right] x\, dx$$
$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}} dx$$
$$= x\sqrt{a^2 - x^2} - \int \frac{(a^2 - x^2) - a^2}{\sqrt{a^2 - x^2}} dx$$
$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$
$$= x\sqrt{a^2 - x^2} - I + a^2 \sin^{-1}(x/a) + C_1$$

or $2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}(x/a) + C_1$

or $I = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1}(x/a) + C,$

where we have written C for $1/2\, C_1$.

Example 16. Evaluate $I = \int \sqrt{x^2 - a^2}\, dx.$

Solution Integrating by parts, taking the first function $\sqrt{x^2 - a^2}$, and second function 1, we have

$$I = \sqrt{x^2 - a^2}\,.\,x - \int \frac{1}{2}\cdot\frac{2x}{\sqrt{x^2 - a^2}} x\, dx$$
$$= x\sqrt{x^2 - a^2} - \int \frac{(x^2 - a^2) + a^2}{\sqrt{x^2 - a^2}} dx,$$
$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}},$$
$$= x\sqrt{x^2 - a^2} - I - a^2 \ln|\sqrt{x^2 - a^2} + x|,$$

or $2I = x\sqrt{x^2 - a^2} - a^2 \ln|\sqrt{x^2 - a^2} + x| + C_1$

or $I = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{1}{2} a^2 \ln|\sqrt{x^2 - a^2} + x| + C$

where we have written C for $1/2\, C_1$.

Example 17. Evaluate $I = \int \sqrt{x^2 + a^2}\, dx.$

Solution Integrating by parts, taking the first function $\sqrt{x^2 + a^2}$, and second function 1, we have

$$I = \sqrt{x^2 + a^2}\,.x - \int \frac{1}{2}\frac{2x}{\sqrt{x^2 + a^2}} x\, dx$$
$$= x\sqrt{x^2 + a^2} - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}} dx$$
$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\, dx + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}$$
$$= x\sqrt{x^2 + a^2} - I + a^2 \ln|\sqrt{x^2 + a^2} + x| + C_1$$

or $2I = x\sqrt{x^2 + a^2} + a^2 \ln|\sqrt{x^2 + a^2} + x| + C_1$

or $I = \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{1}{2} a^2 \ln|\sqrt{x^2 + a^2} + x| + C,$

where we have written C for $1/2\, C_1$.

Note:

(i) $\int e^{ax}.\sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C$

(ii) $\int e^{ax}.\cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C$

Proof:

Let us evaluate $I = \int e^{ax} \sin bx\, dx.$

Integrating by parts taking sin x as the second function, we get

$$I = \frac{e^{ax}\cos bx}{b} - \int ae^{ax}\left(-\frac{\cos bx}{b}\right)dx$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx\,dx.$$

Again integrating by parts taking cos bx as the second function, we get

$$I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left[\frac{e^{ax}\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}dx\right]$$

or $\quad I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx$

$$-\frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

or $\quad I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx) - \frac{a^2}{b^2}I.$

Transposing the term $-\frac{a^2}{b^2}I$ to the left hand side, we get

$$\left(1 + \frac{a^2}{b^2}\right)I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$$

or $\quad \frac{1}{b^2}(a^2 + b^2)I = \frac{1}{b^2}e^{ax}(a\sin bx - b\cos bx).$

$\therefore\quad I = \int^{e^{ax}}(a\sin bx - b\cos bx).$

Thus, $\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C.$

We can proceed similarly to evaluate . $\int^{e^{ax}}\cos bxdx.$

Integration of a complex function of a real variable

We can define a complex function as

$f(x) = u(x) + iv(x)$

of a real variable x and also its derivative

$f'(x) = u'(x) + iv'(x).$

A function $F(x) = U(x) + iV(x)$ is called an antiderivative of a complex function of a real variable f(x) if

$F'(x) = f(x)$ ...(1)

that is, if $U'(x) + iV'(x) = u(x) + iv(x)$ ...(2)

From (2) it follows that $U'(x) = u(x)$, $V'(x) = v(x)$, that is, U(x) is an antiderivative of u(x) and V(x) is an antiderivative of v(x).

It follows, from this definition, that if

$F(x) = U(x) + iV(x)$

is an antiderivative of the function f(x), then any antiderivative of f(x) is of the form F(x) + C, where C is an arbitrary complex constant. We will call the expression F(x) + C the indefinite integral of a complex function of a real variable and we will write

$$\int f(x)\,dx = \int u(x)\,dx + i\int v(x)\,dx = F(x) + C$$

The definite integral of a complex function of a real variable, $f(x) = u(x) + iv(x)$, is defined as follows :

$$\int_a^b f(x)dx = \int_a^b u(x)dx + i\int_a^b v(x)dx$$

Now this provides an alternative method to evaluate the integrals $\int e^{ax}\cos bx\,dx$ and $Q = \int e^{ax}\sin bx\,dx.$

Let $P = \int e^{ax}\cos bx\,dx$

and $Q = \int e^{ax}\sin bx\,dx.$

$\therefore\quad P + iQ = \int e^{ax}(\cos bx + i\sin bx)\,dx$

$$= \int e^{ax}e^{ibx}\,dx = \int e^{(a+ib)x}\,dx$$

$$= \frac{e^{(a+ib)x}}{a+ib} + C + iD = \frac{a - ib}{a^2 + b^2}e^{ax}e^{ibx} + C + iD$$

$$= \frac{e^{ax}}{a^2 + b^2}(a - ib)(\cos bx + i\sin bx) + C + iD.$$

$$= \frac{(ae^{ax}\cos bx + be^{ax}\sin bx) + i(ae^{ax}\sin bx - be^{ax}\cos bx)}{a^2 + b^2}$$

$+ C + iD.$

Equating real and imaginary parts we get the values of P and Q as before.

$$P\frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

$$Q = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}.$$

Example 18. Integrate $\int\frac{\cos^3 x}{e^{3x}}dx.$

Solution $I = \int e^{-3x}\cos^3 x\,dx$

$$= \frac{1}{4}\int e^{-3x}(\cos 3x + 3\cos x)dx$$

$$= \frac{1}{4}\left[\int e^{-3x}\cos 3x\,dx + 3\int e^{-3x}\cos x\,dx\right]$$

$$= \frac{1}{4}\left[\frac{e^{-3x}}{18}(-3\cos 3x + 3\sin 3x) + 3\frac{e^{-3x}}{10}(-3\cos x + \sin x)\right] + C$$

$$= \frac{e^{-3x}}{8}\left\{\frac{1}{3}(\sin 3x - \cos 3x) + \frac{3}{5}(\sin x - 3\cos x)\right\} + C.$$

Example 19. Evaluate $I = \int e^{2x} \sin 3x \cos x\, dx$.

Solution $I = \frac{1}{2}\int e^{2x} . 2 \sin 3x \cos x\, dx$

$$= \frac{1}{2}\int e^{2x}(\sin 4x + \sin 2x)\, dx$$

$$= \frac{1}{2}[\int e^{2x} \sin 4x\, dx + \int e^{2x} \sin 2x\, dx]$$

$$= \frac{1}{2}\left[\frac{e^{2x}}{\sqrt{20}}\sin(4x - \tan^{-1} 2) + \frac{e^{2x}}{\sqrt{8}}\sin(2x - \tan^{-1} 1)\right] + C$$

$$= \frac{e^{2x}}{2}\left[\frac{1}{\sqrt{20}}\sin(4x - \tan^{-1} 2) + \frac{1}{\sqrt{8}}\sin\left(2x - \frac{\pi}{4}\right)\right] + C.$$

Example 20. Evaluate $\int xe^x \cos x\, dx$

Solution $\int xe^x \cos x\, dx$

$$= x\frac{e^x}{2}(\cos x + \sin x) - \int 1.\frac{e^x}{2}(\cos x + \sin x)\, dx$$

$$= x\frac{e^x}{2}(\cos x + \sin x)$$

$$-\frac{1}{2}\left[\frac{e^x}{2}(\cos x + \sin x) + \frac{e^x}{2}(\sin x - \cos x)\right] + C.$$

Alternative :

Let $I_2 = \int xe^x \cos x\, dx$ and $I_2 = \int xe^x \sin x\, dx$

Consider $I_1 + i I_2 = \int xe^x (\cos x + i \sin x)\, dx$

$= \int xe^x \cdot e^{ix}\, dx$

$$= \int xe^{(1+i)x} dx = \frac{xe^{(1+i)x}}{1+i} - \frac{e^{(1+i)x}}{(1+i)^2} + C + iD$$

$$I_1 = \int x\, e^x \cos x\, dx$$

$$= \text{real part of } \left\{x\frac{e^{(1+i)x}}{1+i} - \frac{e^{(1+i)x}}{2i} + C + iD\right\}$$

$$= \text{real part of } \frac{xe^x(\cos x + i\sin x)}{2}(1-i)$$

$$-\frac{e^x}{2}\frac{(\cos x + i\sin x)}{i} + C + iD$$

$$= \frac{xe^x}{2}(\cos x + \sin x) - \frac{e^x}{2}\sin x + C.$$

Example 21. Evaluate $I = \int \ln x . \sin^{-1}x\, dx$.

Solution $I = (x \ln x - x)\sin^{-1}x$

$$-\int \frac{x\ln x}{\sqrt{1-x^2}}dx + \int \frac{x}{\sqrt{1-x^2}}dx \qquad \text{...(1)}$$

[Taking $\sin^{-1}x$ as the first function]

Now in order to evaluate $\int \frac{x\ln x}{\sqrt{1-x^2}}dx$

Put $x = \sin\theta$, then $dx = \cos\theta\, d\theta$

$$\therefore \int \frac{x\ln x}{\sqrt{1-x^2}}dx = \int \sin\theta \ln\sin\theta\, d\theta$$

$$= -\cos\theta \ln \sin\theta - \int -\cos\theta . \cot\theta\, d\theta$$

$$= -\cos\theta \ln \sin\theta + \int \frac{\cos^2\theta}{\sin\theta}d\theta$$

$$= -\cos\theta \ln \sin\theta + \int \frac{1-\sin^2\theta}{\sin\theta}d\theta$$

$$= -\cos\theta \ln \sin\theta + \int (\text{cosec}\theta - \sin\theta)\, d\theta$$

$$= -\cos\theta \ln \sin\theta + \ln(\text{cosec}\theta - \cot\theta) + \cos\theta$$

$$= -\sqrt{1-x^2}\ln x + \ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$$

$\therefore$ From (1),

$$I = (x\ln x - x)\sin^{-1}x + \sqrt{1-x^2}\ln x - \ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) + C.$$

Example 22. Evaluate $\int \frac{x^3\sin^{-1}x}{(1-x^2)^{3/2}}$

Solution Let $I = \int \frac{x^3\sin^{-1}x}{(1-x^2)^{3/2}}\, dx$.

Put $x = \sin\theta$ or $\sin^{-1}x = \theta$.

Then $dx = \cos\theta\, d\theta$.

$$\therefore I = \int \frac{\sin^3\theta}{\cos^3\theta}\theta\cos\theta\, d\theta = \int \frac{\theta\sin\theta(1-\cos^2\theta)}{\cos^2\theta}d\theta.$$

$$= \int \theta[\sec\theta\tan\theta - \sin\theta]\, d\theta$$

$$= \theta(\sec\theta + \cos\theta) - \int 1.(\sec\theta + \cos\theta)\, d\theta,$$

(Integrating by parts taking $\sec\theta\tan\theta - \sin\theta$ as the second function)

$= \theta(\sec\theta + \cos\theta) - \ln|\sec\theta + \cos\theta| - \sin\theta + C$

$$= \theta\left(\frac{1}{\cos\theta} + \cos\theta\right) - \ln\left|\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right| - \sin\theta + C$$

$$= \theta\,\frac{(1+\cos^2\theta)}{\cos\theta} - \ln\left|\frac{1+\sin\theta}{\sqrt{(1-\sin^2\theta)}}\right| - \sin\theta + C$$

$$= \theta\,\frac{(1+1-\sin^2\theta)}{\sqrt{(1-\sin^2\theta)}} - \ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)^{1/2} - \sin\theta + C$$

$$= \theta\,\frac{2-\sin^2\theta}{\sqrt{(1-\sin^2\theta)}} - \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \sin\theta + C$$

$$= \frac{2-x^2}{\sqrt{(1-x^2)}}\sin^{-1}x - x - \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C.$$

Example 23. Evaluate $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx$.

Solution Putting $x = \sin^2\theta$ and $dx = 2\sin\theta\cos\theta\,d\theta$, we have

$$I = \int \frac{\theta - \left(\frac{\pi}{2} - \theta\right)}{\theta + \left(\frac{\pi}{2} - \theta\right)}\sin 2\theta\,d\theta$$

$[\because \sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \pi/2]$

$$= \frac{2}{\pi}\int\left(2\theta - \frac{\pi}{2}\right)\sin 2\theta\,d\theta$$

$$= \frac{4}{\pi}\int \theta\sin 2\theta\,d\theta - \int \sin 2\theta\,d\theta$$

$$= \frac{4}{\pi}\left[\theta\left(\frac{-\cos 2\theta}{2}\right) + \int\frac{\cos 2\theta}{2}d\theta\right] + \frac{\cos 2\theta}{2}$$

$$= \frac{4}{\pi}\left[\theta\left(\frac{-\cos 2\theta}{2}\right) + \frac{\sin 2\theta}{4}\right] + \frac{\cos 2\theta}{2} + C$$

$$= \frac{2}{\pi}[\theta(2\sin^2\theta - 1) + \sin\theta\cos\theta] + \frac{1-2\sin^2\theta}{2} + C$$

$$= \frac{2}{\pi}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x}] + \frac{1-2x}{2} + C$$

$$= \frac{2}{\pi}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x}] - x + C.$$

Concept Problems D

1. Evaluate $\int \ln(2x+3)dx$ using integration by parts. Similify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = \frac{3}{2}$ when going from dv to v.
2. Evaluate $\int x\tan^{-1}x\,dx$ using integration by parts. Simplify the computation of $\int v du$ by introducing a constant of integration $C_1 = \frac{1}{2}$ when going from dv to v.
3. What equation results if integration by parts is applied to the integral $\int \frac{1}{x\ln x}dx$ with the choices $u = \frac{1}{\ln x}$ and $dv = \frac{1}{x}\,dx$? In what sense is this equation true ? In what sense is it false ?
4. Evaluate the following integrals :

   (i) $\int x^3\ln^2 x\,dx$ (ii) $\int ln x\cdot\frac{1}{(x+1)^2}dx$

   (iii) $\int ln(1+x)^{1+x}dx$ (iv) $\int \frac{x\,dx}{1+\sin x}$
5. Evaluate the following integrals :

   (i) $\int \sec^{-1}x$ (ii) $\int x\sin^{-1}x$

   (iii) $\int \sin^{-1}\sqrt{x}$ (iv) $\int \sin^{-1}\frac{2x}{1+x^2}dx$
6. Prove that

   $$\int (\ln|u|)^2\,du = u(\ln|u|)^2 - 2u\ln|u| + 2u + C.$$
7. Prove that $\int e^{x^3}dx = \frac{e^{x^3}}{3x^2} + \frac{2}{3}\int\frac{e^{x^3}}{x^3}dx$.
8. Prove that

   $$\int f'(x)F(x)\,dx = f'(x)F(x) - f(x)F'(x) + \int f(x)F''(x)\,dx$$

   and generally $\int f^{(n)}(x)\,F(x)\,dx = f^{(n-1)}(x)\,F(x) - f^{(n-2)}(x)\,F'(x) + \ldots + (-1)^n\int f(x)F^{(n)}(x)\,dx$.

Practice Problems J

9. Evaluate the following integrals :

(i) $\int \frac{\ln \cos x}{\cos^2 x} dx$

(ii) $\int \sin x \,.\, \ln \tan x \; dx$

(iii) $\int \ln(1+2x^2+x^4) dx$

(iv) $\int e^x (1+x) \ln(xe^x) dx$

10. Evaluate the following integrals :

(i) $\int \operatorname{cosec}^2 x \ln \sec x \, dx.$

(ii) $\int \cos x \ln(\operatorname{cosec} x + \cot x) dx$

(iii) $\int \sin x \,.\, \ln (\sec x + \tan x) dx$

(iv) $\int \sec x \,.\, \ln(\sec x + \tan x) dx$

11. Evaluate the following integrals :

(i) $\int \ln(x + \sqrt{x^2 + a^2}) dx$

(ii) $\int \ln^2(x + \sqrt{1+x^2}) dx$

(iii) $\int x^2 \ln \frac{1+x}{1-x} dx$

(iv) $\int \frac{\ln x}{(x-1)^3} dx$

12. Evaluate the following integrals :

(i) $\int 2^x \sin x \, dx$ (ii) $\int 3^x \cos 3x \, dx$

(iii) $\int e^x \sin x \sin 2x \, dx$ (iv) $\int e^{2x} \sin^2 x \, dx$

13. Evaluate the following integrals :

(i) $\int \sin (\ln x) \; dx$ (ii) $\int e^x \sin x \sin 3x \, dx$

(iii) $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$ (iv) $\int x^3 \tan^{-1} x dx$

14. Evaluate the following integrals :

(i) $\int \cos^{-1} \frac{1}{x} dx$ (ii) $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$

(iii) $\int \frac{\cos^{-1} x}{x^3} dx$ (iv) $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$

15. Evaluate the following integrals :

(i) $\int x \sin x \cos^2 x \, dx$ (ii) $\int x \sec^2 x \tan x \, dx$

(iii) $\int x \cos x \; \cos 2x \; dx$

16. Evaluate the following integrals :

(i) $\int x^3 \; e^x \; dx$ (ii) $\int x^3 \cos x \, dx$

(iii) $\int x^3 \; \ln^2 x \, dx$

1.12 SPECIAL INTEGRALS

Using integration by parts we can establish the following integrals :

(i) $\int e^x [f(x) + f'(x)] \, dx = e^x . f(x) + C$

(ii) $\int [f(x) + xf'(x)] \, dx = x f(x) + C$

Proof:

(i) $\int e^x \{f(x) + f'(x)\} dx$

Integrating by parts $\int e^x f(x) \, dx$, we have

$$\int e^x f(x) dx = \int f(x) e^x dx$$

$$= f(x)e^x - \int f'(x) \, e^x \, dx$$

Transposing the second integral to the left hand side

$$\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C.$$

Alternatively, we may integrate by parts $\int e^x f'(x) \, dx$, and derive the same result.

Note:

$\int e^x \phi(x) dx$, when $\phi(x)$ can be broken up as the sum of two functions of x such that one is the differential coefficient of the other, can be easily integrated as above.

(ii) $\int [f(x) + xf'(x)] \, dx$

Integrating by parts $\int f(x). \, 1 \, dx$, we have

$$\int f(x) \, dx = f(x). \, x - \int f'(x). \, x \, dx$$

Transposing the second integral to the left hand side

$$\int [f(x) + xf'(x)] \, dx = x \, f(x) + C.$$

Example 1. Evaluate $\int \frac{xe^x}{(x+1)^2} dx.$

Solution We have $\int \frac{xe^x}{(x+1)^2} dx = \int (xe^x) \frac{1}{(x+1)^2} dx.$

Integrating by parts taking $\frac{1}{(x+1)^2}$ as the second function and xe^x as the first function, we have

$$\int \frac{xe^x}{(x+1)^2}\,dx$$

$$= (xe^x)\left(-\frac{1}{x+1}\right) - \int (e^x + xe^x)\left(-\frac{1}{x+1}\right) dx.$$

[Note that the integral of $\frac{1}{(x+1)^2}$ is $-\frac{1}{x+1}$]

$$= \frac{xe^x}{(x+1)^2} \int e^x (x+1)\frac{1}{x+1}\,dx$$

$$= \frac{xe^x}{x+1} + \int e^x\,dx = -\frac{xe^x}{x+1} + e^x + C$$

$$= e^x\left[1 - \frac{x}{x+1}\right] + C = e^x\frac{x+1-x}{x+1} + C = \frac{e}{x+1} + C.$$

Alternative :

We have $\int \frac{xe^x}{(x+1)^2}dx = \int e^x \frac{(x+1)-1}{(x+1)^2}\,dx$

$$= \int e^x\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right] dx$$

$\int e^x [f(x) + f'(x)]\,dx$, where

$$f(x) = \frac{1}{x+1} = e^x\frac{1}{x+1} + C.$$

Example 2. Evaluate $\int e^x \frac{2+\sin 2x}{1+\cos 2x}\,dx$

Solution We have $\int e^x \frac{2+\sin 2x}{1+\cos 2x}\,dx$

$$= \int e^x\left[\frac{2}{1+\cos 2x} + \frac{\sin 2x}{1+\cos 2x}\right] dx$$

$$= \int e^x\left[\frac{2}{2\cos^2 x} + \frac{2\sin x\cos x}{2\cos^2 x}\right] dx$$

$$= \int e^x [\sec^2 x + \tan x]\,dx$$

$\int e^x[f(x) + f'(x)]\,dx$, where $f(x) = \tan x = e^x \tan x + C$.

Example 3. Evaluate $\int e^{\cos^{-1}x}.\frac{(x+1)+\sqrt{1-x^2}}{(x+1)^2\sqrt{1-x^2}}dx.$

Solution Let $\cos^{-1}x = t \Rightarrow -\frac{dx}{\sqrt{1-x^2}} = dt$

$$I = -\int e^t \frac{(1+\cos t)+\sin t}{(1+\cos t)^2}dt$$

$$= -\int e^t\left(\frac{1}{1+\cos t} + \frac{\sin t}{(1+\cos t)^2}\right)dt$$

$$= C - \frac{e^t}{1+\cos t} = C - \frac{e^{\cos^{-1}x}}{1+x}.$$

Example 4. Evaluate $\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx$

Solution $I = \int e^x\left(\frac{(1+x^2)^2+1-2x^2}{(1+x^2)^{5/2}}\right)dx$

$$= \int e^x\left(\frac{1}{\sqrt{1+x^2}} - \frac{x}{(1+x^2)^{3/2}}\right.$$

$$\left. + \frac{x}{(1+x^2)^{3/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right)dx$$

$$= e^x\left(\frac{1}{\sqrt{1+x^2}} + \frac{x}{(1+x^2)^{3/2}}\right) + C.$$

Example 5. Evaluate $\int \sin 4x \cdot e^{\tan^2 x}\,dx$

Solution $\int \sin 4x \cdot e^{\tan^2 x}\,dx$

$$= \int 2\cos 2x \cdot 2\frac{\sin x}{\cos x}\cos^2 x\, e^{\tan^2 x}\,dx$$

$$= \int 2\cos 2x\cos^2 x \cdot 2\tan x \cdot e^{\tan^2 x}\,dx$$

$$= \int 2\cos 4x\cos^4 x \cdot 2\tan x\sec^2 x \cdot e^{\tan^2 x}\,dx$$

Put $\tan^2 x = t \Rightarrow 2\tan t\sec^2 t\,dx = dt$

$$= 2\int \frac{(1-t)e^t\,dt}{(1+t)^2(1+t)} = 2\int \frac{e^t(1-t)\,dt}{(1+t)^3}$$

$$= -2\int \frac{e^t[t+1-2]dt}{(1+t)^3}$$

$$= -2\int e^t\left[\frac{1}{(t+1)^2} - \frac{2}{(t+1)^3}\right]dt$$

$$= -2\left[e^t\frac{1}{(t+1)^2}\right] + C = C - 2e^{\tan^2 x}\cos^4 x.$$

Note:

$$\int e^x (f+g+f'+g')\,dx = e^x(f+g)+C$$

Example 6. Evaluate $I=\int e^x(2\sec^2 x-1)\tan x\,dx$

Solution $I=\int e^x(2\sec^2 x\tan x-\tan x)\,dx$

$$=\int e^x(2\sec^2 x\tan x+\sec^2 x-\sec^2 x-\tan x)\,dx$$

$$=e^x(\sec^2 x-\tan x)+C.$$

Example 7. Evaluate $I=\int \dfrac{e^{\sin x}(x\cos^3 x-\sin x)}{\cos^2 x}\,dx$

Solution $I=\int e^{\sin x}(x\cos x-\tan x\sec x)\,dx$

Put $\sin x=t \Rightarrow \cos x\,dx=dt$

$$dx=\frac{dt}{\cos x}=\frac{dt}{\sqrt{1-t^2}}$$

$$I=\int\left(e^t\sin^{-1}t-\frac{e^t t}{(1-t^2)^{3/2}}\right)dt$$

$$=\int e^t\left(\sin^{-1}t-\frac{t}{(1-t^2)^{3/2}}\right)dt$$

$$=\int e^t\left(\sin^{-1}t+\frac{1}{\sqrt{1-t^2}}-\frac{1}{\sqrt{1-t^2}}-\frac{1}{(1-t^2)^{3/2}}\right)dt$$

We have

$$\int e^x(f+g+f'+g')\,dx=e^x(f+g)+C$$

$$=e^t\left(\sin^{-1}t-\frac{1}{\sqrt{1-t^2}}\right)+C$$

Example 8. Evaluate

$$I=\int e^{\tan x}(\sin x-\sec x)\,dx$$

Solution $I=\int e^{\tan x}\sin x\,dx-\int e^{\tan x}\sec x\,dx$

$$=-e^{\tan x}\cdot\cos x$$

$$+\int e^{\tan x}\sec^2 x\cos x\,dx-\int e^{\tan x}\sec x\,dx$$

$$=-e^{\tan x}\cdot\cos x+C.$$

Note: We have $\int e^g(fg'+f')\,dx = e^g f+C$, because

$$\int e^g(fg'+f')\,dx=\int e^g g' f\,dx+\int e^g f'\,dx$$

Integrating the first integral $\int e^g g' f\,dx$ by parts,

$$=e^g f-\int e^g f'\,dx+\int e^g f'\,dx$$

$$=e^g f+C.$$

Example 9. Evaluate

$$I=\int e^{(x\sin x+\cos x)}\left(\frac{x^2\cos^2 x-x\sin x-\cos x}{x^2}\right)dx$$

Solution We have

$$I=\int e^{(x\sin x+\cos x)}\left(\frac{x^2\cos^2 x-x\sin x-\cos x}{x^2}\right)dx$$

$$=\int e^{(x\sin x+\cos x)}\left(\cos^2 x+\frac{-x\sin x-\cos x}{x^2}\right)dx$$

$$=\int e^{(x\sin x+\cos x)}\left\{x\cos x\left(\frac{\cos x}{x}\right)+\left(\frac{\cos x}{x}\right)'\right\}dx$$

$$=\int e^g(g'f+f')\,dx=e^g f+C$$

$$=e^{(x\sin x+\cos x)}\cdot\frac{\cos x}{x}+C.$$

Example 10. Evaluate $I=\int\dfrac{x+\sin x}{1+\cos x}\,dx$

Solution $I=\int\left(x\dfrac{1}{1+\cos x}+\dfrac{\sin x}{1+\cos x}\right)dx$

$$=\int\left(x\frac{\sec^2 x/2}{2}+\tan\frac{x}{2}\right)dx$$

$$=x\tan\frac{x}{2}+C.$$

Alternative :

$$I=\int\frac{x+\sin x}{1+\cos x}dx=\int\frac{x}{1+\cos x}dx+\int\frac{\sin x}{1+\cos x}dx$$

$$=\int\frac{x}{2\cos^2\frac{x}{2}}dx+\int\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx$$

$$=\frac{1}{2}\int x.\sec^2\frac{x}{2}dx+\int\tan\frac{x}{2}dx$$

$$=\frac{1}{2}\left[x.\frac{\tan\frac{x}{2}}{1/2}-\int 1.\frac{\tan\frac{x}{2}}{1/2}dx\right]+\int\tan\frac{x}{2}dx$$

$$=x\tan\frac{x}{2}-\int\tan\frac{x}{2}dx+\int\tan\frac{x}{2}dx=x\tan\frac{x}{2}+C$$

Example 11. Evaluate $I=\int\left(\ln(\ln x)+\dfrac{1}{(\ln x)^2}\right)dx$

Solution $I=\int\left(\ln(\ln x)+\dfrac{x}{x\ln x}-\dfrac{1}{\ln x}+\dfrac{x}{x(\ln x)^2}\right)dx$

$= x\ln(\ln x) - \dfrac{x}{\ln x} + C.$

Example 12. Integrate $\displaystyle\int\left[\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right]dx$.

Solution Let $z = \ln x$

$\therefore \quad dz = \dfrac{1}{x}dx = \dfrac{dx}{e^z}$

or $\quad dx = e^z dz$

Now $I = \displaystyle\int\left[\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right]dx = \int\left(\frac{1}{z} - \frac{1}{z^2}\right)e^z dz$

$= \displaystyle\int e^z\left[\frac{1}{z} + \left(-\frac{1}{z^2}\right)\right]dz$

$= \displaystyle\int e^z[f(z) + f'(z)]\,dz$, where $f(z) = \dfrac{1}{z}$

$= e^z f(z) + C = e^z.\dfrac{1}{z} + C = \dfrac{x}{\ln x} + C.$

Practice Problems K

1. Evaluate the following integrals :

(i) $\int e^x(\sin x - \cos x)\,dx$

(ii) $\int e^x(\tan x - \ln\cos x)\,dx$

(iii) $\int e^x \sec x.(1 + \tan x)\,dx.$

2. Evaluate the following integrals :

(i) $\displaystyle\int e^x\frac{1-\sin x}{1-\cos x}dx$

(ii) $\displaystyle\int e^x\frac{2+\sin 2x}{1+\cos 2x}dx$

(iii) $\displaystyle\int\frac{e^{2x}(\sin 4x - 2)}{1-\cos 4x}dx$

(iv) $\displaystyle\int\frac{e^x(1+x+x^3)}{(1+x^2)^{3/2}}dx$

3. Evaluate the following integrals :

(i) $\displaystyle\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ (ii) $\displaystyle\int e^x\frac{(x^3-x+2)}{(x^2+1)^2}dx$

(iii) $\displaystyle\int\frac{e^x(x-1)}{(x+1)^3}dx$ (iv) $\displaystyle\int e^x\left(\frac{1-x}{1+x}\right)^2 dx$

4. Evaluate the following integrals :

(i) $\displaystyle\int e^x\frac{(x^2-3x+3)}{(x+2)^2}dx$

(ii) $\displaystyle\int\frac{e^x(x^2+1)}{(x+1)^2}dx$

(iii) $\displaystyle\int e^x\frac{(1-x)^2}{(1+x^2)^2}dx$

(iv) $\displaystyle\int\frac{x^2e^x}{(x+2)^2}dx$

5. Evaluate the following integrals :

(i) $\int e^x[\ln(\sec x + \tan x) + \sec x]\,dx$

(ii) $\displaystyle\int e^x\left(\log x + \frac{1}{x^2}\right)dx$

6. Evaluate the following integrals :

(i) $\displaystyle\int\frac{2x+\sin 2x}{1+\cos 2x}dx$

(ii) $\int(\tan(\ln x) + \sec^2(\ln x)\}dx$

(iii) $\displaystyle\int\frac{x+\sqrt{(1-x^2)}\sin^{-1}x}{\sqrt{(1-x^2)}}dx$

1.13 MULTIPLE INTEGRATION BY PARTS

In calculating a number of integrals we had to use the method of integration by parts several times in succession. The result could be obtained more rapidly and in a more concise form by using the so-called generalized formula for integration by parts (or the formula for multiple integration by parts) :

Let u and v be two differentiable functions of x (differentiable n times), and let us denote

(u') by u' and v dx by v_1,

(u')' by u'' and v_1 dx by v_2, etc.

Now, $uv\,dx = uv_1 - u'v_1\,dx$ ...(1)

(by integrating by parts)

Again, $u'v_1\,dx = u'v_2 - u''v_2\,dx$, ...(2)

$$\int u''v_2\,dx = u''v_3 - \int u'''v_3\,dx, \quad ...(3)$$

$$\int u'''v_3\,dx = u'''v_4\,dx - \int u^{(4)}v_4\,dx, \quad ...(4)$$

where $u^{(4)}$ denotes fourth derivative of u.
Combining (1), (2), (3) and (4) we get

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4$$

$$+ (-1)^4 \int u^{(4)}v_4\,dx \quad ...(5)$$

And generally,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$$

$$+ (-1)^{n-1} u^{(n-1)} v_n + (-1)^n \int u^n v_n\,dx \quad ...(6)$$

where $u^{(n)}$ denotes n^{th} order derivative of u.
The use of the generalized formula for integration by parts is especially advantageous when calculating the integral $\int P_n(x)\phi(x)dx$ where $P_n(x)$ is a polynomial of degree n, and the factor $\phi(x)$ is such that it can be integrated successively n + 1 times.
For example,

$$I = \int x^4 \cos x\,dx.$$
$$= x^4 \sin x - 4x^3(-\cos x) + 12x^2(-\sin x) - 24x(\cos x) + 24\sin x + C$$
$$= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x\cos x + 24\sin x + C$$

Example 1. Applying the generalized formula for integration by parts, find the following integrals :

(i) $\int (x^3 - 2x^2 + 3x - 1)\cos 2x\,dx$

(ii) $\int (2x^3 + 3x^2 - 8x + 1)\sqrt{2x+6}\,dx$

Solution

(i) $\int (x^3 - 2x^2 + 3x - 1)\cos 2x\,dx$

$$=(x^3-2x^2+3x-1)\frac{\sin 2x}{2} - (3x^2-4x+3)\left(-\frac{\cos 2x}{4}\right)$$

$$+(6x-4)\left(-\frac{\sin 2x}{8}\right) - 6\frac{\cos 2x}{16} + C$$

$$= \frac{\sin 2x}{4}(2x^3-4x^2+3x) + \frac{\cos 2x}{8}(6x^2-8x+3) + C$$

(ii) $\int (2x^3 + 3x^2 - 8x + 1)\sqrt{2x+6}\,dx$

$$= (2x^3+3x^2-8x+1)\frac{(2x+6)^{3/2}}{3}$$

$$-(6x^2+6x-8)\frac{(2x+6)^{5/2}}{3.5}$$

$$+(12x+6)\frac{(2x+6)^{7/2}}{3.5.7} - 12\frac{(2x+6)^{9/2}}{3.5.7.9} + C$$

$$= \frac{\sqrt{2x+6}}{5.7.9}(2x+6)(70x^3 - 45x^2 - 396x + 897) + C.$$

Now, consider some more illustrations on integration by parts.

Example 2. Evaluate $\int \sin^{-1}\left\{\frac{2x+2}{\sqrt{4x^2+8x+13}}\right\} dx$

Solution $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$

$$= \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right) dx$$

Put, $2x + 2 = 3\tan\theta \Rightarrow 2\,dx = 3\sec^2\theta\,d\theta$

$$= \int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{3}{2}\int \theta\sec^2\theta\,d\theta = \frac{3}{2}\{\theta\tan\theta - \int \tan\theta\,d\theta\}$$

$$= \frac{3}{2}\{\theta\tan\theta - \ln|\sec\theta|\} + C$$

$$= \frac{3}{2}\left\{\frac{2x+2}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) - \ln\left(\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right)\right\} + C$$

$$= \frac{3}{2}\left\{\frac{2}{3}(x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \ln\sqrt{4x^2+8x+13}\right\} + C$$

$$= (x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \frac{3}{4}\ln(4x^2+8x+13) + C.$$

Example 3. Evaluate $\int \frac{x^2(x\sec^2 x + \tan x)dx}{(x\tan x + 1)^2}$

Solution $I = \int x^2 . \frac{x\sec^2 x + \tan x}{(x\tan x + 1)^2}\,dx$

$$= x^2\left(-\frac{1}{(x\tan x+1)}\right) - \int 2x . \left(-\frac{1}{(x\tan x+1)}\right) dx \quad ...(1)$$

$$\text{using, } \int \frac{x\sec^2 x + \tan x}{(x\tan x+1)^2}dx = \int \frac{dt}{t^2}$$

$$= -\frac{1}{t} = -\frac{1}{(x\tan x+1)}$$

$$\Rightarrow \quad I = -\left(\frac{x^2}{x\tan x + 1}\right) + \int \frac{2x(\cos x)dx}{x\sin x + \cos x}$$

Put, $x \sin x + \cos x = u$

$\Rightarrow$ $(x \cos x + \sin x - \cos x)\, dx = du$

$$\Rightarrow \; I = -\frac{x^2}{(x\tan x + 1)} + 2\int \frac{du}{u} = -\frac{x^2}{x\tan x + 1} + 2\ln|u| + C$$

$$= -\frac{x^2}{x\tan x + 1} + 2\ln|x\sin x + \cos x| + C.$$

Example 4. Evaluate $\int \frac{\sec x(2+\sec x)}{(1+2\sec x)^2}\, dx$

Solution Let

$$I = \int \frac{\sec x(2+\sec x)}{(1+2\sec x)^2} dx = \int \frac{2\cos x + 1}{(2+\cos x)^2}\, dx$$

$$= \int \frac{\cos x(\cos x + 2) + \sin^2 x}{(2+\cos x)^2} dx$$

$$= \int \frac{\cos x dx}{2+\cos x} + \int \frac{\sin^2 x}{(2+\cos x)^2} dx$$

In the first integral integrating by parts taking cos x as the second function and keeping the second integral unchanged, we have

$$I = \frac{\sin x}{2+\cos x} - \int \frac{\sin^2 x}{(2+\cos x)^2} dx + \int \frac{\sin^2 x}{(2+\cos x)^2} dx$$

$$\Rightarrow \; I = \frac{\sin x}{2+\cos x} + C.$$

Example 5. Evaluate $\int \frac{x^2 dx}{(x\sin x + \cos x)^2}$

Solution Let, $I = \int \frac{x^2 dx}{(x\sin x + \cos x)^2}\, dx$

We know,

$$\frac{d}{dx}\left(\frac{1}{x\sin x + \cos x}\right) = \frac{-\{\sin x + x\cos x - \sin x\}}{(x\sin x + \cos x)^2}$$

$$= \frac{-x\cos x}{(x\sin x + \cos x)^2}$$

$$\therefore \; \int \frac{-x\cos x}{(x\sin x + \cos x)^2} dx = \frac{1}{(x\sin x + \cos x)}$$

$$\Rightarrow \; I = \int \frac{-x \,.\, \cos x}{(x\sin x + \cos x)^2} \cdot \frac{-x}{\cos x}\, dx$$

$$vu = \left(\frac{-x}{\cos x}\right) \cdot \frac{1}{(x\sin x + \cos x)}$$

$$-\int \frac{-\cos x - x\sin x}{\cos^2 x} \cdot \frac{1}{x\sin x + \cos x} dx$$

$$= \frac{-x}{\cos x(x\sin x + \cos x)} + \int \sec^2 x\, dx$$

$$\therefore \; I = \frac{-x}{\cos x(x\sin x + \cos x)} + \tan x + C.$$

Example 6. Evaluate $\int \left(3x^2 \tan\frac{1}{x} - x\sec^2\frac{1}{x}\right) dx$

Solution $\int \left(3x^2 \tan\frac{1}{x} - x\sec^2\frac{1}{x}\right) dx$

$$= \int 3x^2 \tan\frac{1}{x}\, dx - \int x\sec^2\frac{1}{x}\, dx$$

$$= \tan\frac{1}{x} \,.\, x^3 - \int \left(\sec^2\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) x^3 dx$$

$$- \int x\sec^2\frac{1}{x}\, dx = x^3 \tan\frac{1}{x} + C.$$

Example 7. Evaluate $\int \cot^{-1}(x^2 + x + 1)\, dx$

Solution Let $I = \int \cot^{-1}(x^2 + x + 1)\, dx$

$$= \int \tan^{-1}\left(\frac{1}{x^2 + x + 1}\right) dx$$

$$= \int \tan^{-1}\left(\frac{(x+1)-(x)}{1 + x(x+1)}\right) dx$$

$$= \int \{\tan^{-1}(x+1) - \tan^{-1}(x)\}\, dx$$

$$= \int \tan^{-1}(x+1)\, dx - \int \tan^{-1}(x)\, dx$$

applying integration by parts, we get

$$= \{\tan^{-1}(x+1)\}\, x - \int x \,.\, \frac{1}{1+(x+1)^2}\, dx$$

$$- \left\{(\tan^{-1}x)x - \int x . \frac{1}{1+x^2} dx\right\}$$

$$= x\,\{\tan^{-1}(x+1) - \tan^{-1}x\} + \int \frac{x}{1+x^2} dx$$

$$- \int \frac{x}{1+(1+x^2)}\, dx$$

$$I = x\,\{\tan^{-1}(x+1) - \tan^{-1}(x)\} + \frac{1}{2}\ln(1+x^2) - I_1 \quad ...(1)$$

Let, $I_1 = \int \frac{x}{1+(1+x)^2}\,dx$

put $1 + x = t, dx = dt$

$$= \int \frac{t-1}{1+t^2}\,dt = \int \frac{t}{1+t^2}\,dt - \int \frac{1}{1+t^2}\,dt$$

$$= \frac{1}{2}\ln|1+t^2| - \tan^{-1}(t)$$

$$I_1 = \frac{1}{2}\ln|1+(1+x)^2| - \tan^{-1}(1+x) \qquad ...(2)$$

From (1) and (2),

$$I = x\{\tan^{-1}(x+1) - \tan^{-1}(x)\} + \frac{1}{2}\ln|1-x^2|$$

$$-\frac{1}{2}\ln|1+(1+x)^2\} + \tan^{-1}(1+x) + C.$$

Example 8. Evaluate $\int \frac{\ell n\left(1+x^2\right)}{\sqrt{1-x}}\,dx$

Solution $I = \int \frac{\ell n\left(1+x^2\right)}{\sqrt{1-x}}\,dx$

$1 - x = t^2 \Rightarrow dx = -2t\,dt$

$$I \;= -2\int \ln(1+(1-t^2)^2)\,dt$$

$$= -2\int \ln(t^4 - 2t^2 + 2).1\,dt$$

$$= -2\left[t\,\ell n\left(t^4 - 2t^2 + 2\right) - \int \frac{\left(4t^3 - 4t\right)t}{t^4 - 2t^2 + 2}\,dt\right]$$

$$= -2t\ln(t^4 - 2t^2 + 2) + 8\int \frac{t^4 - t^2}{t^4 - 2t^2 + 2}\,dt$$

$$= -2t\ln(t^4 - 2t^2 + 2)$$

$$+ 8\int \frac{\left(t^4 - 2t^2 + 2\right) + \left(t^2 - 2\right)}{t^4 - 2t^2 + 2}\,dt$$

$$I = -2t\ln(t^4 - 2t^2 + 2) + 8t + 8I_1 \qquad ...(1)$$

Now $I_1 = \int \frac{t^2}{t^4 - 2t^2 + 2}\,dt - \int \frac{2\,dt}{t^4 - 2t^2 + 2} =$

$$\underbrace{\int \frac{dt}{t^2 - 2 + \frac{2}{t^2}}}_{I_2\text{ (say)}} - \underbrace{\int \frac{\frac{2}{t^2}}{t^2 - 2 + \frac{2}{t^2}}\,dt}_{I_3\text{(say)}}$$

$$I_2 = \frac{1}{2}\int \frac{\left(1 - \frac{\sqrt{2}}{t^2}\right)dt}{\left(t + \frac{\sqrt{2}}{t}\right)^2 - 2\sqrt{2} - 2}$$

$$+ \frac{1}{2}\int \frac{\left(1 + \frac{\sqrt{2}}{t^2}\right)dt}{\left(t - \frac{\sqrt{2}}{t}\right)^2 + 2\sqrt{2} - 2}$$

$$I_2 = \frac{1}{2}\int \frac{\left(1 - \frac{\sqrt{2}}{t^2}\right)dt}{\left(t + \frac{\sqrt{2}}{t}\right)^2 - \left(\sqrt{2\left(\sqrt{2}+1\right)}\right)^2}$$

$$+\frac{1}{2}\int \frac{\left(1 + \frac{\sqrt{2}}{t^2}\right)dt}{\left(t - \frac{\sqrt{2}}{t}\right)^2 + \left(\sqrt{2\left(\sqrt{2}-1\right)}\right)^2}$$

$$I_3 = \frac{1}{\sqrt{2}}\left[\int \frac{\left(1 + \frac{\sqrt{2}}{t^2}\right) - \left(1 - \frac{\sqrt{2}}{t^2}\right)}{\left(t^2 - 2 - \frac{2}{t^2}\right)}\,dt\right]$$

$$= \frac{1}{\sqrt{2}}\left[\int \frac{\left(1 + \frac{\sqrt{2}}{t^2}\right)dt}{\left(t - \frac{\sqrt{2}}{t}\right)^2 + 2\left(\sqrt{2}-1\right)}\right.$$

$$\left. - \int \frac{\left(1 - \frac{\sqrt{2}}{t^2}\right)dt}{\left(t + \frac{\sqrt{2}}{t}\right)^2 - 2\left(\sqrt{2}+1\right)}\right]$$

$$= \frac{1}{\sqrt{2}}\int \frac{\left(1 + \frac{\sqrt{2}}{t^2}\right)dt}{\left(t - \frac{\sqrt{2}}{t}\right)^2 + \left(\sqrt{2\left(\sqrt{2}-1\right)}\right)^2}$$

$$-\frac{1}{\sqrt{2}}\int \frac{\left(1 - \frac{\sqrt{2}}{t^2}\right)dt}{\left(t + \frac{\sqrt{2}}{t}\right)^2 - \left(\sqrt{2\left(\sqrt{2}+1\right)}\right)^2}$$

$$I_1 = I_2 - I_3 = \frac{\sqrt{2}+1}{2}\int \frac{\left(1 - \frac{\sqrt{2}}{t^2}\right)dt}{\left(t + \frac{\sqrt{2}}{t}\right)^2 - \left(\sqrt{2\left(\sqrt{2}+1\right)}\right)^2}$$

$$+ \frac{\sqrt{2}+1}{2}\int \frac{\left(1 + \frac{\sqrt{2}}{t^2}\right)dt}{\left(t - \frac{\sqrt{2}}{t}\right)^2 + \left(\sqrt{2\left(\sqrt{2}-1\right)}\right)^2}$$

Now we can compute I from (1).

Example 9. Evaluate $\int \frac{x^2+n(n-1)}{(x\sin x+n\cos x)^2}\,dx$

Solution Here $I=\int \frac{x^2+n(n-1)}{(x\sin x+n\cos x)^2}\,dx$

Multiplying and dividing by x^{2n-2}, we get

$$I=\int \frac{(x^2+n(n-1))x^{2n-2}}{(x^n\sin x+nx^{n-1}\cos x)^2}\,dx$$

We put $x^n\sin x+n\,x^{n-1}\cos x=t$

$\Rightarrow$ $(n\,x^{n-1}\sin x)+(x^n\cos x)+n\,(n-1)\,x^{n-2}\cos x - (n\,x^{n-1}\sin x)\,dx=dt$

$\Rightarrow$ $x^{n-2}\cos x.\,(x^2+n\,(n-1))\,dx=dt$

Keeping this in mind we put,

$$I=\int \frac{(x^2+n(n-1)).x^{n-2}.\cos x}{(x^n\sin x+nx^{n-1}\cos x)^2}\,.\,x^n\,.\sec x\,dx$$

vu and applying integration by parts, we get

$$=x^n\sec x\,.\left(-\frac{1}{(x^n\sin x+nx^{n-1}\cos x)}\right)$$

$$+\int \frac{(x^n\sec x\tan x+nx^{n-1}.\sec x)}{(x^n\sin x+nx^{n-1}\cos x)dx}$$

$$=-\frac{(x\sec x)}{(x\sin x+n\cos x)}+\int \sec^2 x\,dx$$

$$=-\frac{(x\sec x)}{(x\sin x+n\cos x)}+\tan x+C.$$

Example 10. If $\cos\theta>\sin\theta>0$, then evaluate

$$I=\int\left\{\ln\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right)^{\cos^2\theta}+\ln\left(\frac{\cos 2\theta}{1+\sin 2\theta}\right)\right\}d\theta.$$

Solution Here,

$$I=\int\left\{2\cos^2\theta\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right\}d\theta$$

$$=\int (2\cos^2\theta-1)\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$$

$$=\int \cos 2\theta\,.\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$$

on applying integration by parts.

$$=\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right).\frac{\sin 2\theta}{2}\int\frac{2}{\cos 2\theta}.\frac{\sin 2\theta}{2}d\theta$$

$$=\frac{\sin 2\theta}{2}\ln\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right|+\frac{1}{2}\ln|\cos 2\theta|+C.$$

Example 11. Evaluate $\int\frac{dx}{(x^2+a^2)^2}$

Solution Here, we know

$$\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a} \quad ...(1)$$

Also, $\int\frac{1}{x^2+a^2}\,.\,1\,dx=\frac{1}{x^2+a^2}x-\int\frac{-2x}{(x^2+a^2)^2}\,x\,dx$

$$=\frac{x}{x^2+a^2}+2\int\frac{x^2+a^2-a^2}{(x^2+a^2)^2}dx$$

$$\int\frac{dx}{x^2+a^2}=\frac{x}{x^2+a^2}+2\int\frac{dx}{x^2+a^2}-2a^2\int\frac{dx}{(x^2+a^2)^2} \quad ...(2)$$

From (1) and (2),

$$\frac{1}{a}\tan^{-1}\frac{x}{a}=\frac{x}{x^2+a^2}+2\frac{1}{a}\tan^{-1}\frac{x}{a}-2a^2\int\frac{dx}{(x^2+a^2)^2}$$

$$\Rightarrow\ 2a^2\int\frac{dx}{(x^2+a^2)^2}=\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\Rightarrow\ \int\frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^2}\left\{\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}\right\}+C.$$

Example 12. Evaluate $\int\frac{dx}{(x^2+a^2)^3}$

Solution Let $I=\int\frac{dx}{(x^2+a^2)^3}$ ...(1)

and $I_1=\int\frac{1}{(x^2+a^2)^2}\,dx$ ...(2)

$$=\int\frac{1}{(x^2+a^2)^2}\,.\,1\,dx$$

$$uv=\frac{1}{(x^2+a^2)^2}\,.\,x-\int\frac{-2(2x)}{(x^2+a^2)^3}\,x\,dx$$

$$=\frac{x}{(x^2+a^2)^2}+4\int\frac{x^2+a^2-a^2}{(x^2+a^2)^3}dx$$

$$=\frac{x}{(x^2+a^2)^2}+4\int\frac{1}{(x^2+a^2)^2}dx-4a^2\int\frac{dx}{(x^2+a^2)^3}$$

$\Rightarrow$ $I_1=\frac{x}{(x^2+a^2)^2}+4I_1-4a^2\,.\,I$ [using (1) and (2)]

$$\Rightarrow \quad 4a^2 I = \frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2} I_1 \qquad ...(3)$$

{Using, previous example

$$I_1 = \int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + c\}$$

$$\Rightarrow \quad I = \frac{x}{4a^2(x^2+a^2)^2}$$

$$+\frac{3}{4a^2}\left\{\frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)\right\} + C.$$

Practice Problems L

Evaluate the following integrals using multiple integration by parts :

1. $\int x^2 e^{3x}\, dx$
2. $\int (x^3+3x+1)e^{3x}\, dx$
3. $\int x^3 \cos 2x\, dx$
4. $\int e^{-x}\cos^2 x dx$
5. $\int (x^3-2x^2+5)e^{3x} dx$
6. $\int (x^2-2x+3)\ell n x\, dx$
7. $\int (3x^2+x-2)\sin^2(3x+1)dx$
8. $\int \frac{x^2-7x+1}{\sqrt[3]{2x+1}}dx$

Evaluate the following integrals using integration by parts :

9. $\int \cos 2x\, \ln(1+\tan x)dx$
10. $\int \frac{\sqrt{x^2+1}}{x^4}\ln\left(1+\frac{1}{x^2}\right)dx$
11. $\int \frac{e^x(1+nx^{n-1}-x^{2n})}{(1-x^n)\sqrt{1-x^{2n}}}dx$.

1.14 INTEGRATION BY REDUCTION FORMULAE

In some cases of integration, we take recourse to the method of successive reduction of the integrand which mostly depends on the repeated application of integration by parts. This is specially the case when the integrands are complicated in nature and depend on certain parameter or parameters. These parameters may be positive, negative, or fractional indices, as for example, $x^n e^{ax}$, $\tan^n x$, $(x^2+a^2)^{n/2}$, $\sin^m x \cos^n x$, etc.

To obtain a complete integral of these trigonometric or algebraic functions, we first of all define these integrals by the letters I, J, U, etc., introducing the parameter or parameters as suffixes, and connect them with certain similar other integral or integrals whose suffixes are lower than that of the original integral. Then by repeatedly changing the value of the suffixes, the original integral can be made to rest on much simpler integrals. This last integral can be easily evaluated and knowing the value of this last integral, by the process of repeated substitution, the value of the original integral can be found out.

The formula in which a certain integral involving some parameters is connected with some integrals of lower order is called a **Reduction Formula**. In most of the cases the reduction formula is obtained by the process of integration by parts. In some cases the method of differentiation or other special devices are adopted.

Example 1. Find the reduction formulae for $\int \tan^n x dx$ and $\int \cot^n x dx$.

Solution $I_n = \int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx$

$$= \int \tan^{n-2} x(\sec^2 x - 1)dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$$

$$\Rightarrow \quad \int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$$

$$\therefore \quad I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}.$$

Similarly, $I_n = \int \cot^n x dx = \int \cot^{n-2} x \cot^2 x dx$

$$= \int \cot^{n-2} x(\operatorname{cosec}^2 x - 1)dx$$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx$$

$$\Rightarrow \quad \int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx$$

$$\therefore \quad I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}.$$

Above are the required reduction formulae which reduce the power of tanx or cotx by 2. By successive application of these formulae we shall ultimately depend on

$$\int \tan x dx \text{ or } \int \cot x dx$$

when n is odd and which are ln|sec x| or ln|sinx| respectively or will depend on

$$\int \tan^2 x dx \text{ or } \int \cot^2 x dx$$

i.e. $\int (\sec^2 x - 1)dx$ or $\int (\text{cosec}^2 x - 1)dx$

which are $(\tan x - x)$ or $(-\cot x - x)$ respectively.

For example,

$$\int \tan^7 x dx = \frac{\tan^6 x}{6} - \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} - \ln|\sec x| + C.$$

$$\int \tan^6 x dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \frac{\tan x}{1} - x + C.$$

Example 2. Obtain the reduction formula for $I_n = \int \sec^n x\,dx$.

Solution $I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \,.\, \sec^2 x\,dx$

Integrating by parts,

$I_n = \sec^{n-2} x \,.\, \tan x$

$- \int (n-2)\sec^{n-3} x \sec x \,.\, \tan x \,.\, \tan x\,dx$

In order to put this equation into the desired form, let us replace $\tan^2 x$ by $\sec^2 x - 1$ in the new integral to yield

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x (\sec^2 x - 1)dx$$

$$= \sec^{n-2} x \tan x - (n-2)\left[\int \sec^n x\,dx - \int \sec^{n-2} x\,dx\right]$$

$$= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

Transposing and simplifying,

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

This is the required reduction formula.

With the aid of this formula, using it several times, we can integrate any positive integral power of the secant. For example, assuming, $n = 3$, we get

$$\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \int \sec x\,dx\right)$$

$$= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C;$$

Assuming $n = 4$, we get

$$\int \sec^4 x\,dx = \frac{1}{3}\left(\sec^2 x \tan x + 2\int \sec^2 x\,dx\right)$$

$$= \frac{1}{3}(\sec^2 x \tan x + 2\tan x) + C, \text{ and so on.}$$

Example 3. Evaluate $\int \sec^7 x\,dx$.

Solution Using the above reduction formula,

$$I_7 = \int \sec^7 x\,dx = \frac{\sec^5 x \tan x}{6} + \frac{5}{6} I_5.$$

$$I_5 = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} I_3.$$

$$I_3 = \frac{\sec x \tan x}{2} + \frac{1}{2} I_1$$

$$I_1 = \int \sec x\,dx = \ln|\sec x + \tan x|.$$

$$\therefore \quad I_7 = \frac{\sec^5 x \tan x}{6} + \frac{5}{6}\frac{\sec^3 x \tan x}{4} + \frac{3.5}{4.6}\frac{\sec x \tan x}{2}$$

$$+ \frac{1.3.5}{2.4.6} \ln|\sec x + \tan x| + C.$$

Example 4. Find the reduction formula for $I_n = \int x \sin^n x\,dx$.

Solution $I_n = \int x \sin^n x\,dx = \int x \sin^{n-1} x \sin x\,dx$

Integrating by parts.

$I_n = (x \sin^{n-1} x)(-\cos x)$

$$+ \int \cos x\{\sin^{n-1} x + x(n-1)\sin^{n-2} x \cos x\}dx$$

$$I_n = -x\cos x \sin^{n-1} x + \int \sin^{n-1} x \cos x\,dx$$

$$+ (n-1)\int x \sin^{n-2} x (1 - \sin^2 x)dx$$

$$I_n = -x \cos x \sin^{n-1} x + \frac{\sin^n x}{n} + (n-1)(I_{n-2} - I_n)$$

$$\therefore \quad (1 + (n-1))I_n = -x\cos x \sin^{n-1} x + \frac{\sin^n x}{n} + (n-1)I_{n-2}$$

or, $$\int x \sin^n x\,dx = \frac{x \cos x \sin^{n-1} x}{n} + \frac{\sin^n x}{n^2}$$

$$+ \frac{n-1}{n}\int x \sin^{n-2} x\,dx$$

This is the required reduction formula.

Example 5. Find a reduction formula for the integral $\int \frac{\sin nx}{\sin x} dx$.

Solution Let $I_n = \int \frac{\sin nx}{\sin x} dx$

We have $\sin nx - \sin(n-2)x = 2\cos(n-1)x\sin x$

$$\therefore \quad \frac{\sin nx}{\sin x} = 2\cos(n-1)x + \frac{\sin(n-2)x}{\sin x}$$

Integrating both sides we get

$$\int \frac{\sin nx}{\sin x}dx = \frac{2\sin(n-1)x}{(n-1)} + \int \frac{\sin(n-2)x}{\sin x}dx$$

$$I_n = \frac{2\sin(n-1)x}{(n-1)} + I_{n-2}$$

Above is the required reduction formula.

Example 6. If $I_n = \int x^n\sqrt{a^2 - x^2}\,dx$, prove that

$$I_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{(n+2)} + \frac{(n+1)}{(n+2)}a^2 I_{n-2}.$$

Solution $I_n = \int x^n\sqrt{a^2-x^2}\,dx$

$$= \int x^{n-1}.\{x\sqrt{a^2-x^2}\}\,dx$$

Applying integration by parts we get

$$= x^{n-1}.\left\{-\frac{(a^2-x^2)^{3/2}}{3}\right\}$$

$$+\int (n-1)x^{n-2}.\left\{-\frac{(a^2-x^2)^{3/2}}{3}\right\}dx$$

$$= -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3}$$

$$+\frac{(n+1)}{3}\int x^{n-2}.(a^2-x^2)\sqrt{a^2-x^2}\,dx$$

$$\Rightarrow I_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3} + \frac{(n-1)a^2}{3}I_{n-2} - \frac{(n-1)}{3}I_n$$

$$\therefore I_n + \frac{(n-1)}{3}I_n$$

$$= -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3} + \frac{(n-1)a^2}{3}I_{n-2} \quad \Rightarrow$$

$$\left(\frac{n+2}{3}\right)I_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3} + \frac{(n-1)a^2}{3}I_{n-2}$$

$$\Rightarrow I_n = \frac{x^{n-1}(a^2-x^2)^{3/2}}{(n+2)} + \frac{(n-1)a^2}{(n+2)}I_{n-2}.$$

Example 7. Find a reduction formula for the integral

$$\int \frac{dx}{(x^2+k)^n}.$$

Solution To obtain a reduction formula, we integrate $1/(x^2+k)^{n-1}$ by parts, taking unity as the second function.

$$\text{Thus, } I_{n-1} = \int \frac{1}{(x^2+k)^{n-1}}\,1\,.\,dx$$

$$= \frac{x}{(x^2+k)^{n-1}} - \int x.\frac{-(n-1)}{(x^2+k)^n}.2x\,dx$$

$$\Rightarrow I_{n-1} = \frac{x}{(x^2+k)^{n-1}} + 2(n-1)\int \frac{(x^2+k)-k}{(x^3+k)^n}dx$$

$$\Rightarrow I_{n-1} = \frac{x}{(x^2+k)^{n-1}}$$

$$+2(n-1)\left[\int \frac{dx}{(x^2+k)^{n-1}} - k\int \frac{dx}{(x^2+k)^n}\right]$$

$$\Rightarrow I_{n-1} = \frac{x}{(x^2+k)^{n-1}} + 2(n-1)I_{n-1} - 2k(n-1)I_n$$

$$\therefore 2k(n-1)I_n = \frac{x}{(x^2+k)^{n-1}} + \{2(n-1)-1\}I_{n-1}$$

$$\Rightarrow 2k(n-1)I_n = \frac{x}{(x^2+k)^{n-1}} + (2n-3)I_{n-1}$$

$$\text{Hence, } \int \frac{dx}{(x^2+k)^{n-1}}$$

$$= \frac{x}{2k(n-1)(x^2+k)^{n-1}} + \frac{(2n-3)}{2k(n-1)}\int \frac{dx}{(x^2+k)^{n-1}}.$$

This is the required reduction formula. By repeated application of this formula the integral shall reduce to that of $\frac{1}{(x^2+k)}$ which is $\frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{x}{\sqrt{k}}\right)$.

Example 8. Evaluate $\int \frac{dx}{(x^2+3)^3}$.

Solution By the above reduction formula, we get

$$\int \frac{dx}{(x^2+3)^3} = \frac{x}{12(x^2+3)^2} + \frac{3}{12}\int \frac{dx}{(x^2+3)^2}$$

[Putting n = 3 and k = 3 in the formula]

$$= \frac{x}{12(x^2+3)^2} + \frac{1}{4}\left\{\frac{x}{6(x^2+3)} + \frac{1}{6}\int \frac{dx}{(x^2+3)}\right\}$$

[On applying the same reduction formula by putting n = 2 and k = 3]

$$= \frac{x}{12(x^2+3)^2} + \frac{x}{24(x^2+3)} + \frac{1}{24\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C.$$

Example 9. Evaluate $\int \frac{(x+2)dx}{(2x^2+4x+3)^2}$.

Solution Here $(d/dx)(2x^2+4x+3) = 4x+4$.

$$\therefore \int \frac{(x+2)dx}{(2x^2+4x+3)^2} = \int \frac{\frac{1}{4}(4x+4)+2-1}{2(x^2+4x+3)^2}dx$$

$$= \frac{1}{4}\int \frac{(4+4)dx}{(2x^2+4x+3)^2} = \int \frac{(2-1)dx}{\left(x^2+2x+\frac{3}{2}\right)^2}$$

$$= \frac{1}{4}\int (2x^2+4x+3)^{-2}(4x+4)\,dx$$

$$+ \frac{1}{4}\int \frac{dx}{\left(x^2+2x+\frac{3}{2}\right)^2}$$

$$= -\frac{1}{4(2x^2+4x+3)} + \frac{1}{4}\int \frac{dx}{\left\{(x+1)^2+\frac{1}{2}\right\}^2}.$$

Now put $x + 1 = t$ and then applying the reduction formula, we get

$$I = \frac{1}{4(2x^2+4x+3)}$$

$$+\frac{1}{4}\left[\frac{(x+1)}{(x+1)^2+\frac{1}{2}} + \sqrt{2}\tan^{-1}\{\sqrt{2}(x+1)\}\right] + C.$$

Example 10. If $I_m = \int (\sin x + \cos x)^m\,dx$, then show that $m\,I_m = (\sin x + \cos x)^{m-1}.(\sin x - \cos x) + 2(m-1)I_{m-2}$.

Solution $I_m = \int (\sin x + \cos x)^m\,dx$

$$= \int (\sin x + \cos x)^{m-1}.(\sin x + \cos x)\,dx,$$

(applying integration by parts.)

$$= (\sin x + \cos x)^{m-1}(\cos x + \sin x) - \int [(m-1)\sin x + \cos x)^{m-2}.(\cos x - \sin x).(\sin x - \cos x)]dx$$

$$= (\sin x + \cos x)^{m-1}(\sin x - \cos x) + (m-1)$$

$$\int (\sin x - \cos x)^{m-2}(\sin x + \cos x)^2\,dx$$

Since, $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$,

$$I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x)$$

$$+ (m-1)\int [(\sin x + \cos x)^{m-2}\{2 - (\sin x + \cos x)^2\}]\,dx$$

$$= (\sin x + \cos x)^{m-1}(\sin x - \cos x)$$

$$+ (m-1)\int 2(\sin x + \cos x)^{m-2}\,dx$$

$$-(m-1)\int (\sin x + \cos x)^m\,dx$$

$$I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + (m-1)I_{m-2} - (m-1)I_m$$

$$\Rightarrow (m-1)I_m + I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + 2(m-1)I_{m-2}$$

$$\Rightarrow m\,I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + 2(m-1)I_{m-2}.$$

Example 11. Obtain a reduction formula for

$\int x^m(\ln x)^n\,dx$ where m, n are positive integers).

Solution Here, since two parameters m, n are involved, we shall denote the integral by the symbol

$I_{m,n} = \int x^m(\ln x)^n\,dx$. Integrating by parts,

$$I_{m,n} = \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{1}{m+1}\int n(\ln x)^{n-1}\frac{1}{x}.x^{m+1}dx$$

$$= \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m(\ln x)^{n-1}dx$$

$$= \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}I_{m,n-1},$$

$$\Rightarrow I_{m,n} = \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}I_{m,n-1}.$$

Note:

1. Here we have connected $I_{m,n}$ with $I_{m,n-1}$ and by successive change the power of ln x can be reduced to zero, i.e., after n operations we shall get a term $I_{m,o}$, i.e., $\int x^m\,dx$, which is easily integrable. Thus, by step by step reduction, $I_{m,n}$ can be evaluated. It may be noted that when two parameters are involved this is the usual practice.
2. Students must be cautious in denoting these integrals. For example, $I_{m,n} \neq I_{n,m}$ in general.

Example 12. Obtain reduction formulae for

$\int x^m(1-x)^n\,dx.$

Solution Let $I_{m,n} = \int x^m(1-x)^n\,dx$

$$= \frac{x^{m+1}}{m+1}.(1-x)^n + \frac{n}{m+1}\int x^{m+1}.(1-x)^{n-1}dx$$

$$= \frac{x^{m+1}(1-x)^n}{m+1} + \frac{n}{m+1}\int x^m(1-x)^{n-1}\{1-(1-x)\}dx$$

$$= \frac{x^{m+1}(1-x)^n}{m+1} + \frac{n}{m+1}\left[I_{m,n-1} - I_{m,n}\right].$$

Transposing and simplifying,

$$I_{m,n} = \frac{x^{m+1}(1-x)^n}{m+n+1} + \frac{n}{m+n+1} I_{m,n-1}.$$

Example 13. Find the reduction formula for

$I_{m,n} = \int \cos^m x \sin nx dx$

Solution Integrating by parts, $I_{m,n}$

$$= \cos^m x\left(-\frac{\cos nx}{n}\right) + \int \frac{\cos nx}{n}.m.\cos^{m-1} x(-\sin x)dx$$

$$= -\cos^m x \frac{\cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x(\cos nx \sin x)dx \quad ...(1)$$

Now $\sin(nx-x) = \sin nx \cos x - \cos nx \sin x$.

$\therefore \quad \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$.

Subtituting in (1)

$$\therefore \quad \int \cos^m x \sin nx dx = -\frac{\cos^m x \cos nx}{n}$$

$$-\frac{m}{n}\int \cos^{m-1} x[\sin nx \cos x - \sin(n-1)x]dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx dx$$

$$+\frac{m}{n}\int \cos^{m-1} x \sin(n-1)x dx$$

Transposing the middle term to the left, we get

$$\int \cos^m x \sin nx dx = -\frac{\cos^m x \cos nx}{m+n}$$

$$+\frac{m}{m+n}\int \cos^{m-1} x \sin(n-1)x dx$$

Reduction formulae for $\int \sin^m\theta \cos^n\theta \, d\theta$

$$\int \sin^m\theta \cos^n\theta \, d\theta = \int \cos^{n-1}\theta \sin^m\theta d(\sin\theta)$$

Consequently, if we assume

$$u = \cos^{n-1}\theta, \; v = \frac{\sin^{m+1}\theta}{m+1},$$

The formula for integration by parts gives

$$\int \sin^m\theta \cos^n\theta \, d\theta = \frac{\cos^{n-1}\theta \sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}$$

$$\int \sin^{m+2}\theta \cos^{n-2}\theta \, d\theta \quad ...(1)$$

In like manner, if the integral be written in the form

$$-\int \sin^{m-1}\theta \cos^n\theta \, d(\cos\theta),$$

we obtain

$$\int \sin^m\theta \cos^n\theta \, d\theta = \frac{m-1}{n+1}\int \sin^{m-2}\theta \cos^{n+2}\theta \, d\theta$$

$$-\frac{\sin^{m-1}\theta \cos^{n+1}\theta}{n+1} \quad ...(2)$$

It may be observed that formula (2) can be derived from (1) by substituting $(\pi/2 - \phi)$ for θ, and interchanging the letters m and n in it.

Case of one Positive and one Negative Index

The results in (1) and (2) hold whether m or n be positive or negative. Accordingly, let one of them be negative (n suppose), and on changing n into – n, formula (2) becomes

$$\int \frac{\sin^m\theta}{\cos^n\theta} d\theta = \frac{\sin^{m-1}\theta}{(n-1)\cos^{n-1}\theta} -$$

$$\frac{m-1}{m-1}\int \frac{\sin^{m-2}\theta}{\cos^{n-2}\theta} d\theta \quad ...(3)$$

in which m and n are supposed to have positive signs.

By this formula the integral of $\frac{\sin^m\theta}{\cos^n\theta}$ is made to depend on another in which the indices of $\sin\theta$ and $\cos\theta$ are each diminished by two. The same method is applicable to the new integral and so on.

If m be an odd integer, the expression is integrable immediately.

If m be even, and n even and greater than m, the substitution of $\tan\theta = x$ is applicable.

If $m = n$, the expression becomes $\int \tan^m\theta \, d\theta$, which can be treated as before.

If $n < m$, the integral reduces to that of $\int \sin^{m-n}\theta \, d\theta$.

Again, if n be odd, and $n > m$, the integral reduces to

$\int \frac{d\theta}{\cos^{n-m}\theta}$, and if $n < m$ reduces to $\int \frac{\sin^{m-n+1}\theta d\theta}{\cos\theta}$.

The mode of finding these latter integrals will be considered subsequently.

Again, if the index of $\sin\theta$ be negative, we get, by changing the sign of m in (1),

$$\int \frac{\cos^n\theta}{\sin^m\theta}d\theta = \frac{\cos^{n-1}\theta}{(m-1)\sin^{m-1}\theta}$$

$$-\frac{n-1}{m-1}\int \frac{\cos^{n-2}\theta}{\sin^{m-2}\theta}d\theta \quad ...(4)$$

We shall next consider the case where the indices are both positive.

Case of both indices being positive

If $\sin^m\theta\ (1 - \cos^2\theta)$ be written instead of $\sin^{m+2}\theta$ in formula (1), it becomes

$$\int \sin^m\theta \cos^n\theta\, d\theta = \frac{\cos^{n-1}\theta \sin^{m+1}\theta}{m+1}$$

$$+\frac{n-1}{m+1}\int \sin^m\theta\, (\cos^{n-2}\theta - \cos^n\theta)\, d\theta$$

$$= \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}\int \sin^m\theta \cos^{n-2}\theta\, d\theta$$

$$-\frac{n-1}{m+1}\int \sin^m\theta \cos^n\theta\, d\theta.$$

Now, transposing the latter integral to the other side, and dividing by $\frac{m+n}{m+1}$, we get,

$$\int \sin^m\theta \cos^n\theta\, d\theta = \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+1}$$

$$+\frac{n-1}{m+1}\int \sin^m\theta \cos^{n-2}\theta\, d\theta \quad ...(5)$$

In like manner, from (2), we get

$$\int \sin^m\theta \cos^n\theta\, d\theta = \frac{n-1}{m+1}\int \sin^{m-2}\theta \cos^n\theta\, d\theta$$

$$-\frac{\sin^{m-1}\theta\cos^{n+1}\theta}{m+n}. \quad ...(6)$$

By aid of these formulae the integral of $\sin^m\theta \cos^n\theta$ is made to depend on another in which the index of either $\sin\theta$, or of $\cos\theta$, is reduced by two. By successive application of these formulae the complete integral can always be found when the indices are integers.

Case of both indices being negative

It remains to consider the case where the indices of $\sin\theta$ and $\cos\theta$ are both negative.

Writing $-m$ and $-n$ instead of m and n, in formula (1), it becomes

$$\int \frac{d\theta}{\sin^m\theta\cos^n\theta} = \frac{-1}{(m+n)\cos^{n+1}\theta\sin^{m-1}\theta}$$

$$+\frac{n+1}{m+n}\int \frac{d\theta}{\sin^m\theta\cos^{n+2}\theta}$$

or, transposing and multiplying by $\frac{m+n}{n+1}$,

$$\int \frac{d\theta}{\sin^m\theta\cos^{n+2}\theta} = \frac{1}{(n+1)\sin^{m-1}\theta\cos^{n+1}\theta}$$

$$+\frac{m+n}{n+1}\int \frac{d\theta}{\sin^m\theta\cos^n\theta} \quad ...(7)$$

Again, if we substitute n for n + 2 in this, it becomes

$$\int \frac{d\theta}{\sin^m\theta\cos^n\theta} = \frac{1}{(n-1)\sin^{m-1}\theta\cos^{n-1}\theta}$$

$$+\frac{m+n-2}{n-1}\int \frac{d\theta}{\sin^m\theta\cos^n\theta} \quad ...(8)$$

Making a like transformation in formula (8), it becomes

$$\int \frac{d\theta}{\sin^m\theta\cos^n\theta} = \frac{-1}{(m-1)\sin^{m-1}\theta\cos^{n-1}\theta}$$

$$+\frac{m+n-2}{m-1}\int \frac{d\theta}{\sin^{m-2}\theta\cos^n\theta}. \quad ...(9)$$

In each of these, one of the indices is reduced by two degrees, and consequently, by successive applications of the formulae, the integrals are reducible ultimately to those of one or other of the forms $\int \frac{d\theta}{\cos\theta}$ and $\int \frac{d\theta}{\sin\theta}$ and these can be easily integrated.

The formulae of reduction for $\int \frac{d\theta}{\sin^n\theta}$ and $\int \frac{d\theta}{\cos^n\theta}$ are given as follows :

$$\int \frac{d\theta}{\cos^n\theta} = \frac{\sin\theta}{(n-1)\cos^{n-1}\theta} + \frac{n-2}{n-1}\int \frac{d\theta}{\cos^{n-2}\theta} \quad ...(10)$$

$$\int \frac{d\theta}{\sin^n\theta} = \frac{-\cos\theta}{(n-1)\sin^{n-1}\theta} + \frac{n-2}{n-1}\int \frac{d\theta}{\sin^{n-2}\theta} \quad ...(11)$$

It may be here observed that, since $\sin^2\theta + \cos^2\theta = 1$, we have immediately.

$$\int \frac{d\theta}{\sin^m\theta\cos^n\theta} = \int \frac{d\theta}{\sin^{m-2}\theta\cos^n\theta} + \int \frac{d\theta}{\sin^m\theta\cos^{n-2}\theta} \quad ...(12)$$

and a similar process is applicable to the latter integrals. This method is often useful in elementary cases.

Application of Method of Differentiation

The formulae of reduction given above can also be readily arrived at by direct differentiation. Thus, for example, we have

$$\frac{d}{d\theta}\left(\frac{\sin^m\theta}{\cos^n\theta}\right) = \frac{m\sin^{m-1}\theta}{\cos^{n-1}\theta} + \frac{n\sin^{m+1}\theta}{\cos^{n+1}\theta};$$

and, consequently,

$$\int \frac{\sin^{m+1}\theta}{\cos^{n+1}\theta} d\theta = \frac{1}{n}\frac{\sin^m\theta}{\cos^n\theta} - \frac{m}{n}\int \frac{\sin^{m-1}\theta}{\cos^{n-1}\theta}$$

Again,

$$\frac{d}{d\theta}(\sin^m\theta\cos^n\theta) = m\sin^{m-1}\theta\cos^{n+1}\theta - n\sin^{m+1}\theta\cos^{n-1}\theta.$$

If we substitute for $\cos^{n+1}\theta$ its equivalent $\cos^{n-1}\theta\,(1-\sin^2\theta)$, we get

$$\frac{d}{d\theta}(\sin^m\theta\cos^n\theta) = m\sin^{m-1}\theta\cos^{n-1}\theta - (m+n)\sin^{m+1}\theta$$

$\cos^{n-1}\theta$, hence we get $\int \sin^{m+1}\theta\cos^{n-1}\theta d\theta$

$$= -\frac{\sin^m\theta\cos^n\theta}{m+n} + \frac{m}{m+n}\int \sin^{m-1}\theta\cos^{n-1}\theta d\theta.$$

We can simplify the following forms using trigonometric substitutions :

(i) $\int \frac{x^m dx}{(a^2+x^2)^{n/2}}$

We put $x = a\tan\theta$,

the integral $\int \frac{x^m dx}{(a^2+x^2)^{n/2}}$ transforms into

$\int \sin^m\theta\cos^{n-m-2}\theta d\theta$ (neglecting a constant multiplier).

(ii) $\int \frac{x^m dx}{(a^2-x^2)^{n/2}}$

In a like manner, the substitution $x = a\sin\theta$ transforms

$$\int \frac{x^m dx}{(a^2-x^2)^{n/2}} \text{ into } \int \frac{a^{m-n+1}\sin^m\theta d\theta}{\cos^{n-1}\theta}$$

and, if $x = a\sec\theta$, the integral $\int \frac{x^m dx}{(x^2-a^2)^{n/2}}$

transforms into $\int \frac{\cos^{n-m-2}\theta d\theta}{\sin^{n-1}\theta}$

(neglecting the constant multiplier).

A similar substitution may be applied in other cases.

For example, to find the integral $\int \frac{x^n dx}{(2ax-x^2)^{1/2}}$, let $x = 2a\sin^2\theta$, then $dx = 4a\sin\theta\cos\theta\, d\theta$, and the transformed integral is $2^{n+1}a^n\int \sin^{2n}\theta\, d\theta$, which can be solved easily.

Practice Problems

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1. Derive the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

2. Obtain a reduction formula for the following integrals

(i) $\int x^n e^x dx\ (n \ge 1)$ (ii) $\int (\ln x)^n dx\,(n \ge 1)$

3. Evaluate the following integrals :

(i) $\int \tan^4\theta d\theta$ (ii) $\int \frac{d\theta}{\tan^5\theta}$

(iii) $\int \frac{d\theta}{\sin^3\theta}$ (iv) $\int \cos^6\theta d\theta$

4. Evaluate the following integrals :

(i) $\int \frac{d\theta}{\sin\theta\cos^2\theta}$

(ii) $\int \cos^2\theta\sin^4\theta d\theta$

(iii) $\int \frac{d\theta}{\sin^3\theta\cos^2\theta}$

5. Evaluate the following integrals :

(i) $\int \frac{dx}{x^3\sqrt{1-x^2}}$ (ii) $\int \frac{x^4 dx}{(a^2+x^2)^2}$

(iii) $\int \frac{x^2 dx}{(a+cx^2)^{7/2}}$ (iv) $\int \frac{x^3 dx}{(a^2+x^2)^{3/2}}$

6. Show that

$$\int \cot^9 x\,dx = -\frac{\cot^8 x}{8} + \frac{\cot^6 x}{6} - \frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \ln|\sin x| + C$$

7. Show that $\int \cot^8 x\,dx = -\frac{\cot^7 x}{7} + \frac{\cot^5 x}{5} - \frac{\cot^3 x}{3} + \cot x + x + C.$

8. Derive the formula

$$\int \frac{\sin^{n+1}x}{\cos^{m+1}x}dx = \frac{1}{m}\frac{\sin^n x}{\cos^m x} - \frac{n}{m}\int \frac{\sin^{n-1}x}{\cos^{m-1}x}dx.$$

9. If $I_{m,n} = \int x^m \cos nx\,dx\ (n \ne 0)$, then show that,

$$I_{m,n} = \frac{x^m \sin nx}{n} + \frac{mx^{m-1}\cos nx}{n^2} - \frac{m(m-1)}{n^2}I_{m-2,n}.$$

10. If $I_n = \int \cos nx\ \mathrm{cosec}\ x\ dx$, then show that,

$$I_n - I_{n-2} = \frac{2\cos(n-1)x}{n-1}$$

11. If $I_n = \int \frac{1-\cos nx}{1-\cos x}\,dx$, then show that
$I_n + I_{n-2} = 2I_{n-1} + \frac{2\sin(n-1)x}{n-1}$.

12. If $I_n = \int \frac{x^n}{\sqrt{x^2+a^2}}\,dx\ (n \ge 2)$, then show that
$I_n = \frac{x^{n-1}\sqrt{x^2+a^2}}{n} - \frac{a^2(n-1)}{n} I_{n-2}$.

13. Prove that $\int \frac{dx}{x^n\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{(n-1)bx^{n-1}}$
$- \frac{(2n-3)a}{(2n-2)b}\int \frac{dx}{x^{n-1}\sqrt{ax+b}} + C\ (n \ne 1)$.

14. Deduce the reduction formula for
$I_n = \int \frac{dx}{(1+x^4)^n}$ and hence evaluate $I_2 = \int \frac{dx}{(1+x^4)^2}$.

1.15 INTEGRATION OF RATIONAL FUNCTIONS USING PARTIAL FRACTIONS

A rational function (a ratio of polynomials) is found by combining two or more rational expressions into one rational expression. Here, the reverse process is considered : given one rational expression, we express it as the sum of two or more rational expressions. A special type of sum of simpler fractions is called the partial fraction decomposition ; each term in the sum is a partial fraction. In this section we show how to integrate any rational function by expressing it as a sum of partial fractions, that we already know how to integrate.

To illustrate the method, observe that by taking the fractions 2/(x – 1) and 1/(x + 2) to a common denominator we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation :

$$\int \frac{x+5}{x^2+x-2}dx = \int\left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx$$
$$= -2\,ln\,|x-1| - ln\,|x+2| + C.$$

There are four types of partial fractions :

(i) $\frac{A}{x-a}$

(ii) $\frac{A}{(x-a)^k}$ (k a positive integer ≥ 2),

(iii) $\frac{Ax+B}{x^2+px+q}$ (the roots of the denominator are imaginary, that is, $p^2 - 4q < 0$,

(iv) $\frac{Ax+B}{(x^2+px+q)^k}$ (k a positive integer ≥ 2; the roots of the denominator are imaginary).

These are called partial fractions of the type (i), (ii), (iii) and (iv).

It is known that every proper rational fraction may be represented as a sum of partial fractions. We shall therefore first consider integrals of partial fractions. The integration of partial fractions of type (i), (ii) and (iii) does not present any particular difficulty.

(i) $\int \frac{A}{x-a}dx = A\ln|x-a| + C$

(ii) $\int \frac{A}{(x-a)^k}dx = A\int (x-a)^{-k}dx = A\frac{(x-a)^{-k+1}}{-k+1} + C$
$= \frac{A}{(1-k)(x-a)^{k-1}} + C$.

(iii) $\int \frac{Ax+B}{x^2+px+q} = \int \frac{\frac{A}{2}(2x+p) + \left(B - \frac{Ap}{2}\right)}{x^2+px+q}dx$

$= \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx + \left(B - \frac{Ap}{2}\right)\int \frac{dx}{x^2+px+q}$

$= \frac{A}{2}\ln|x^2+px+q|$

$+ \left(B - \frac{Ap}{2}\right)\int \frac{dx}{\left(x+\frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)}$

$= \frac{A}{2}\ln|x^2+px+q| + \frac{2B-Ap}{\sqrt{4q-p^2}}\tan^{-1}\frac{2x-p}{\sqrt{4q-p^2}} + C$

The integration of partial fractions of type (iv) requires more involved computations. Suppose we have an integral of this type :

(iv) $\int \frac{Ax+B}{(x^2+px+q)^k}dx$

We perform the transformations :

$$\int \frac{Ax+B}{(x^2+px+q)^k}dx = \int \frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{(x^2+px+q)^k}dx$$

$$= \frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{(x^2+px+q)^k}$$

The first integral is considered via the substitution, $x^2+px+q=t$, $(2x+p)\,dx=dt$:

$$\int \frac{2x+p}{(x^2+px+q)^k}dx = \int \frac{dt}{t^k} = \int t^{-k}dt = \frac{t^{-k+1}}{1-k}+C$$

$$= \frac{1}{(1-k)(x^2+px+q)^{k-1}}+C$$

We write the second integral (let us denote it by I_k) in the form

$$I_k = \int \frac{dx}{(x^2+px+q)^k} = \int \frac{dx}{\left[\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right]^k}$$

$$= \int \frac{dt}{(t^2+m^2)^k}$$

assuming $x+\frac{p}{2}=t$, $dx=dt$, $q-\frac{p^2}{4}=m^2$

We then do as follows :

$$I_k = \int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2}\int \frac{(t^2+m^2)-t^2}{(t^2+m^2)^k}dt$$

$$= \frac{1}{m^2}\int \frac{dt}{(t^2+m^2)^{k-1}} - \frac{1}{m^2}\int \frac{t^2}{(t^2+m^2)^k}dt \qquad ...(1)$$

Integrating by parts we have $\int \frac{t^2dt}{(t^2+m^2)^k}$

$$= -\frac{1}{2(k-1)}\left[t\frac{1}{(t^2+m^2)^k} - \int \frac{dt}{(t^2+m^2)^{k-1}}\right]$$

Putting this expression into (1), we have

$$I_k = \int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2}\int \frac{dt}{(t^2+m^2)^{k-1}}$$

$$+ \frac{1}{m^2}\frac{1}{2(k-1)}\left[\frac{1}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}}\right]$$

$$= \frac{t}{2m^2(k-1)(t^2+m^2)^{k-1}} - \frac{2k-3}{2m^2(k-1)}$$

$$\int \frac{dt}{(t^2+m^2)^{k-1}}$$

On the right side is an integral of the same type as I_k, but the exponent of the denominator of the integrand is less by unity $(k-1)$; we have thus expressed I_k in terms of I_{k-1}. Continuing in the same manner we will arrive at the familiar integral.

$$I_1 = \int \frac{dt}{t^2+m^2} = \frac{1}{m}\tan^{-1}\frac{t}{m}+C$$

Then substituting everywhere in place of t and m their values, we get the expression of integral (iv) in terms of x and the given number A, B, p, q.

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It is possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called proper.

To form a partial fraction decomposition of a rational expression, we use the following steps:

Step 1 : If f is improper, that is, $\deg(P) \geq \deg(Q)$, then we must take the preliminary step of dividing Q into P (by division) until a remainder R(x) is obtained such that $\deg(R) < \deg(Q)$. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials. For example,

$$\frac{x^4-3x^3+x^2+5x}{x^3+3} = x^2-3x-2+\frac{14x+6}{x^2+3}.$$

Then, apply the following steps to the remainder, which is a proper fraction.

Step 2 : Factor Q(x) completely into factors of the form $(ax+b)^m$ or $(cx^2+dx+e)^n$, where cx^2+dx+e is irreducible and m and n are integers.

It can be shown that any polynomial Q can be factored as a product of linear factors (of the form $ax+b$) and irreducible quadratic factors (of the form ax^2+bx+c, where $b^2-4ac<0$). For instance, if $Q(x)=x^4-16$, we could factor it as

$$Q(x) = (x^2-4)(x^2+4) = (x-2)(x+2)(x^2+4)$$

Note:

If the equation $Q(x)=0$ cannot be solved algebraically, then the method of partial fractions naturally fails and recourse must be had to other methods.

Step 3 :

(I) For each distinct linear factor $(ax + b)$, the decomposition must include the term $\frac{A}{ax+b}$.

(II) For each repeated linear factor $(ax + b)^m$, the decomposition must include the terms

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+....+\frac{A_m}{(ax+b)^m}$$

(III) For each distinct quadratic factor $(cx^2 + dx + e)$, the decomposition must include the term

$$\frac{Bx+C}{cx^2+dx+e}.$$

(IV) For each repeated quadratic factor $(cx^2 + dx + e)^n$, the decomposition must include the terms

$$\frac{B_1x+C_1}{cx^2+dx+e}+\frac{B_2x+C_2}{(cx^2+dx+e)^2}$$

$$+.....+\frac{B_nx+C_n}{(cx^2+dx+e)^n}$$

Step 4 : Use techniques to solve for the constants in the numerators of the decomposition.
We explain the details for the four cases that occur.

Case I : The denominator Q(x) is a product of distinct linear factors

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) .... (a_kx + b_k)$$

where no factor is repeated. In this case the partial fraction theorem states that there exist constants $A_1, A_2, ..., A_k$ such that

$$\frac{R(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+.....+\frac{A_k}{a_kx+b_k}$$

These constants can be determined as in the following example.

Example 1. Evaluate $\int\frac{x^2+2x-1}{2x^3+3x^2-2x}dx.$

Solution Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as

$$2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand has the form

$$\frac{x^2+2x-1}{x(2x-1)(x+2)}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2} \quad ...(1)$$

Method of Comparision of Coefficients

To determinator the values of A, B, and C, we multiply both sides of this equation by the product of the denominators, $x(2x – 1)(x + 2)$, obtaining

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2) + Cx(2x-1) \quad ...(2)$$

Expanding the right side of Equation 2 and writing it in the standard form for polynomials, we get

$$x^2+2x-1 = (2A+B+2C)x^2 + (3A+2B-C)x-2A \quad ...(3)$$

The polyomials in Equation 3 are identical, so their coefficients must be equal. The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side – namely, 1. Likewise, the coefficients of x are equal and the constant terms are equal. Ths gives the following system of equation for A, B and C.

$$2A+B+2C=1$$
$$3A+2B-C=2$$
$$-2A=-1$$

Solving, we get $A=\frac{1}{2}$, $B=\frac{1}{5}$, and $C=-\frac{1}{10}$, and so

$$\int\frac{x^2+2x-1}{2x^3+3x^2-2x}dx=\int\left(\frac{1}{2}\frac{1}{x}+\frac{1}{5}\frac{1}{2x-1}-\frac{1}{10}\frac{1}{x+2}\right)dx$$

$$=\frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+k$$

Method of Particular Values

Example 2. Evaluate $\int\frac{dx}{(x^2-1)(3+2x)}$

Solution $I=\int\frac{dx}{(x^2-1)(3+2x)}$

$$=\int\frac{dx}{(x+1)(x-1)(2x+3)}$$

Let $\frac{1}{(x+1)(x-1)(2x+3)}\equiv\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{2x+3}$

or $1\equiv A(x-1)(2x+3)+B(x+1)(2x+3) + C(x+1)(x-1)$

Putting $x = 1, -1, -\frac{3}{2}$ successively, we get

$$1 = B\cdot 2\cdot 5,$$
$$1 = A(-2)\cdot 1,$$
$$1 = C\cdot\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right)$$

$\therefore\quad B=\frac{1}{10}, A=-\frac{1}{2}, C=\frac{4}{5}.$

$$\therefore\quad I=\int\left(\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{2x+3}\right)dx$$

$$=-\frac{1}{2}\ln|x+1|+\frac{1}{10}\ln|x-1|+\frac{2}{5}\ln|2x+3|+k.$$

Heaviside Cover-up Method for Linear Factors

When the degree of the polynomial P(x) is less than the degree of Q(x), and Q(x) is a product of n distinct linear factors, each raised to the first power, there is a quick way to express P(x)/Q(x) into partial fractions.

Example 3. Find A, B, and C in the partial fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3} \quad ...(1)$$

Solution If we multiply both sides of equation (1) by $(x-1)$ to get

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}=A+\frac{B(x-1)}{x-2}+\frac{C(x-1)}{x-3}$$

and set x = 1, the resulting equation gives the value of A :

$$\frac{(1)^2+1}{(1-2)(1-3)}=A+0+0,\ A=1.$$

Thus, the value of A is the number we would have obtained if we had covered the factor (x – 1) in the denominator of the original fraction

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}$$

and evaluate the rest at x = 1:

$$A=\frac{(1)^2+1}{\cancel{(x-1)}(1-2)(1-3)}=\frac{2}{(-1)(-2)}=1.$$

Similarly, we find the value of B in (1) by covering the factor (x – 2) and evaluating the rest at x = 2 :

$$B=\frac{(2)^2+1}{(2-1)\cancel{(x-2)}(2-3)}=\frac{5}{(1)(-1)}=-5.$$

Shortcut Method :

Consider $x-a_1=0$, then $x=a_1$, put this value of x in all the expressions other than $x-a_1$ and so on , e.g.

$$\frac{x^2+1}{x(x-1)(x+1)}=\frac{0+1}{x(0-1)(0+1)}+\frac{1+1}{1(x-1)(1+1)}+\frac{1+1}{-1(-1-1)(x+1)}.$$

Case II : Q(x) is a product of linear factors, some of which are repeated.

Suppose the first linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of Q(x). Then instead of the single term $A_1/(a_1x+b_1)$, we would use

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+....+\frac{A_r}{(a_1x+b_1)^r}$$

For example, we write

$$\frac{x^3-x+1}{x^2(x-1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}$$

Example 4. Find $\int\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$

Solution The first step is to divide. The result of long division is

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}=x+1+\frac{4x}{x^3-x^2-x+1}$$

The second step is to factor the denominator $Q(x)=x^3-x^2-x+1$. Since Q(1) = 0, we know that x – 1 is a factor and we obtain

$$x^3-x^2-x+1=(x-1)^2(x+1)$$

Since the linear factor x – 1 occurs twice, the partial fraction decompostion is

$$\frac{4x}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$

Multiplying by the least common denominator, $(x-1)^2(x+1)$, we get

$$4x=A(x-1)(x+1)+B(x+1)+C(x-1)^2$$
$$=(A+C)x^2+(B-2x)x+(-A+B+C)$$

Now we equate the coefficients :

$A+C=0$

$B-2C=4$

$-A+B+C=0$

Solving, we obtain A = 1, B = 2, and C = –1,

$$\text{so}\quad\int\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$$

$$=\int\left[x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\right]dx$$

$$=\frac{x^2}{2}+x+\ln|x-1|-\frac{2}{x-1}-\ln|x+1|+k$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + k.$$

Note: The Heaviside cover-up method applies recursively to handle the more general case of repeated linear factors, as the following example illustrates.

$$\frac{5x-3}{(x-1)(x-2)^2} = \frac{1}{x-2}\cdot\frac{5x-3}{(x-1)(x-2)}$$

$$= \frac{1}{x-2}\left(\frac{-2}{(x-1)} + \frac{7}{x-2}\right)$$

$$= \frac{-2}{(x-1)(x-2)} + \frac{7}{(x-1)^2}$$

$$= \frac{(-2)/(1-2)}{x-1} + \frac{(-2)/(2-1)}{x-2} + \frac{7}{(x-2)^2}$$

$$= \frac{2}{x-1} + \frac{-2}{x-2} + \frac{7}{(x-2)^2}.$$

Application of limit

$$\frac{P(x)}{(x-a)^r Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + ... + \frac{A_r}{(x-a)^r} + ... \quad ..(1)$$

Now in order to obtain $A_1, A_2, ... A_r$, following steps are useful.

(i) To obtain the constant of the partial fraction corresponding to (x – a), multiply both sides of the identity (1) by (x – a) and then let $x \to \infty$.

(ii) To obtain the constant of the partial fraction containing $(x-a)^r$ in the denominator put x = a in the given fraction except for $(x-a)^r$ present in the denominator. This gives $A_r = \frac{P(a)}{Q(a)}$.

(iii) To obtain the other constants, we can multiply both sides of the identity again and again by (x – a) and then let $x \to \infty$, or put some particular values of x (say 0, 1, –1 etc.) except the roots of denominator in the identity thus forming a system of linear equations and solving it.

Example 5. Evaluate $\int \frac{x^3+2}{(x-1)(x-2)^3}dx$.

Solution

$$\frac{x^3+2}{(x-1)(x-2)^3} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

$$= \frac{1^3+2}{(x-1)(1-2)^3} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{2^3+2}{(2-1)(x-2)^3}$$

$$= \frac{-3}{(x-1)} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{10}{(x-2)^3} \quad ...(1)$$

To get B we multiply (1) by (x – 2) and let $x \to \infty$

So, $\lim_{x\to\infty} \frac{x^3+2}{(x-1)(x-2)^3}$

$$= \lim_{x\to\infty}\left[\frac{-3(x-2)}{(x-1)} + B + \frac{C}{(x-2)} + \frac{10}{(x-2)^2}\right]$$

$\Rightarrow$ 1 = –3 + B + 0 + 0 $\Rightarrow$ B = 4.

To get C let us put x = 0 in (1)

$$\therefore \frac{2}{(-1)(-2)^3} = \frac{-3}{-1} + \frac{B}{-2} + \frac{C}{(-2)^2} + \frac{10}{(-2)^3} \Rightarrow C = 2$$

$$\therefore \int \frac{x^3+2}{(x-1)(x-2)^3}dx$$

$$= \int\left(\frac{-3}{x-1} + \frac{4}{x-2} + \frac{2}{(x-2)^2} + \frac{10}{(x-2)^3}\right)dx$$

$$= -3\ln|x-1| + 4\ln|x-2| - \frac{2}{x-2} - \frac{5}{(x-2)^2} + C.$$

Example 6. Evaluate $\int \frac{x^2-x+2}{(x+1)(x-1)^3}dx$

Solution $\int \frac{x^2-x+2}{(x+1)(x-1)^3}dx$

$$\frac{x^2-x+2}{(x+1)(x-1)^3} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} \quad ...(1)$$

$x^2 - x + 2 = A(x-1)^3 + B(x+1)(x-1)^2$
$\quad + C(x+1)(x-1) + D(x+1)$

$x = -1 \Rightarrow 4 = -8A \Rightarrow A = -1/2$

$x = 1 \Rightarrow 2 = 2D \Rightarrow D = 1.$

Multiply both sides of (1) by (x – 1) and taking limit,

$$\lim_{x\to\infty} \frac{x^2-x+2}{(x+1)(x-1)^2}$$

$$= \lim_{x\to\infty}\left(\frac{A(x-1)}{(x+1)} + B + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}\right)$$

$$0 = -\frac{1}{2} + B \Rightarrow B = \frac{1}{2}.$$

Multiply by (x – 1) again

$$\lim_{x\to\infty} \frac{x^2 - x + 2}{(x+1)(x-1)}$$

$$= \lim_{x\to\infty} \left(\frac{A(x-1)^2}{(x+1)} + B(x-1) + C + \frac{D}{(x-1)} \right)$$

$$= \lim_{x\to\infty} \left(\frac{-\frac{1}{2}(x^2+1-2x) + \frac{1}{2}(x^2-1)}{(x+1)} + C + \frac{D}{(x-1)} \right)$$

$$= \lim_{x\to\infty} \left(\frac{-\frac{x^2}{2} - \frac{1}{2} + x + \frac{1}{2}x^2 - \frac{1}{2}}{(x+1)} + C + \frac{D}{(x-1)} \right)$$

$$= \lim_{x\to\infty} \left(\frac{x-1}{x+1} + C + \frac{D}{(x-1)} \right)$$

$1 = 1 + C + O \Rightarrow C = 0$

Now, the integral can be evaluated easily.

Example 7. Evaluate $\int \frac{2x+1}{(x+2(x-3)^2} dx$.

Solution Assume

$$\frac{2x+1}{(x+2)(x-3)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} \qquad ...(1)$$

The Heaviside method of determining coefficients gives

$$A = \frac{-4+1}{(-2-3)^2} = -\frac{3}{25},\ C = \frac{6+1}{3+2} = \frac{7}{5}.$$

Also, multiplying by x, and then making $x \to \infty$, we find $A + B = 0$ or $B = \frac{3}{25}$.

The integral is therefore

$$-\frac{3}{25}\ln(x+2) + \frac{3}{25}\ln(x-3) - \frac{7}{5(x-3)} + C.$$

Method of differentiation

Example 8. Find A, B and C, in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Solution We first clear of fractions :

$x - 1 = A(x+1)^2 + B(x+1) + C.$

Substituting $x = -1$ shows $C = -2$.

We then differentiate both sides with respect to x, obtaining

$1 = 2A(x+1) + B.$

Substituting $x = -1$ shows $B = 1$.

We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} + \frac{2}{(x+1)^3}.$$

Case III : Q(x) contains irreducible quadratic factors, none of which is repeated.

If Q(x) has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions taken earlier, the expression will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

where A and B are constants to be detemined. The term given above can be integrated by completing the square and using the formula

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

For instance, the function given by

$$f(x) = \frac{x}{(x-2)(x^2+1)(x^2+4)}$$ has a partial fraction decomposition of the form :

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}.$$

Example 9. Evaluate $\int \frac{2x^2-x+4}{x^3+4x} dx$.

Solution Since $x^3 + 4x = x(x^2+4)$ cannot be factored further, we write

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

Multiplying by $x(x^2+4)$, we have

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x \qquad ...(1)$$
$$= (A+B)x^2 + Cx + 4A$$

Equating coefficients, we obtain

$A + B = 2,\qquad C = -1,\ 4A = 4$

Thus $A = 1$, $B = 1$ and $C = -1$.

Alternatively, we can put $x = 0$ in (1) to obtain $A = 1$, and put $x = 2i$ to get

$$-4 - 2i = -4B + 2iC \qquad ...(2)$$

Separating real and imaginary parts in (2),

we get $B = 1$ and $C = -1$.

Now, $$\int \frac{2x^2-x+4}{x^3+4x} dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4} \right) dx$$

In order to integrate the second term we split it into two parts :

$$\int \frac{x-1}{x^2+4}dx = \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

We make the substitution $u = x^2 + 4$ in the first of these integrals so that $du = 2x\,dx$. We evaluate the second integral by means of the formula (1) :

$$\int \frac{2x^2-x+4}{x(x^2+4)}dx = \int \frac{1}{x}dx + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}(x/2) + K.$$

Example 10. Evaluate $\int \frac{3x^2+5x+8}{(x^2+2x+5)(x-1)}dx$.

Solution

$$\frac{3x^2+5x+8}{(x^2+2x+5)(x-1)} = \frac{A}{x-1} + \frac{B(2x+2)+C}{(x^2+2x+5)}$$

Note : Here we prefer to write $B(2x + 2) + C$ instead of the usual expression $Bx + C$, looking at the needs of the integration process.

Multiplication of both sides by $(x^2 + 2x + 5)^2 (x - 1)$ leads to $3x^2 + 5x + 8 = A(x^2 + 2x + 5)$
$+ [B(2x + 2) + C](x - 1)$ ...(1)

If $x = 1$, equation (1) reduces to $16 = 8A$, or $A = 2$.
If $x = -1$, $6 = 4A - 2C \Rightarrow C = 1$
If $x = 0$, $8 = 5A - 2B - C \Rightarrow B = 1/2$.

$$\therefore \int \frac{3x^2+5x+8}{(x^2+2x+5)(x-1)}dx$$

$$= \int\left(\frac{2}{x-1} + \frac{(1/2)(2x+2)+1}{(x^2+2x+5)}\right)dx$$

$$= \int \frac{2}{x-1}dx + \frac{1}{2}\int \frac{(2x+2)}{(x^2+2x+5)}dx + \int \frac{1}{(x^2+2x+5)}dx$$

$$= 2\ell n|x-1| + \frac{1}{2}\ell n\,(x^2+2x+4)$$

$$+ \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + k.$$

Example 11. Evaluate $\int \frac{f(x)}{x^3-1}$ dx, where f(x) is a polynomial of degree 2 in x such that $f(0) = f(1) = 3f(2) = -3$.

Solution Let $f(x) = ax^2 + bx + c$

Given $f(0) = f(1) = 3f(2) = -3$

$\therefore$ $f(0) = f(1) = 3f(2) = -3$

$f(0) = c = -3$
$f(1) = a + b + c = -3$
$3f(2) = 3(4a + 2b + c) = -3$

On solving we get $a = 1, b = -1, c = -3$

$\therefore$ $f(x) = x^2 - x - 3$.

$$\Rightarrow I = \int \frac{f(x)}{x^3-1}dx = \int \frac{x^2-x-3}{(x-1)(x^2+x+1)}dx$$

Using partial fractions, we get,

$$\frac{(x^2-x-3)}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$$

where, $A = -1, B = 2, C = 2$

$$\therefore I = \int -\frac{1}{x-1}dx + \int \frac{(2x+2)}{(x^2+x+1)}dx$$

$$= -\ln|x-1| + \int \frac{(2x+2)}{(x^2+x+1)} + \int \frac{1-dx}{x^2+x+1}$$

$$= -\ln|x-1| + \ln(x^2+x+1)$$

$$+ \int \frac{dx}{(x+1/2)^2 + (\sqrt{3}/2)^2}$$

$$= \ln|x-1| + \ln(x^2+x+1) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

Case IV : Q(x) contains a repeated irreducible quadratic factor

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction, the sum

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

occurs in the partial fraction decomposition.

Example 12. Find $\int \frac{2x^2+3}{(x^2+1)^2}dx$.

Solution Let $\frac{2x^2+3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$.

Then

$2x^2 + 3 = (Ax + B)(x^2 + 1) + Cx + D$
$= Ax^3 + Bx^2 + (A + C)x + (B + D)$

Compare coefficients : $A = 0$, $B = 2$, $A + C = 0$, $B + D = 0$. Hence, $C = 0$, $D = 1$. Thus,

$$\int \frac{2x^2+3}{(x^2+1)^2}dx = \int \frac{2}{x^2+1}dx + \int \frac{1}{(x^2+1)^2}dx$$

$$= 2\tan^{-1}x + \int \frac{1}{(x^2+1)^2}dx$$

In the second integral, let $x = \tan\theta$. Then

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{\sec^2\theta d\theta}{\sec^4\theta} = \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{1}{2}\left(\theta + \frac{\tan\theta}{\tan^2\theta+1}\right) = \frac{1}{2}\left(\tan^{-1}x + \frac{x}{x^2+1}\right)$$

Thus, $\displaystyle\int \frac{2x^2+3}{(x^2+1)^2}\,dx = \frac{5}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{x^2+1} + C$

Example 13. Evaluate $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx.$

Solution The form of the partial fraction decomposition is

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have $-x^3+2x^2-x+1$

$= A(x^2\ 1)^2 + (Bx+C)\,x\,(x^2+1) + (Dx+E)x$

$= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex$

$= (A+B)\,x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$

If we equate coefficients, we get the system

$A+B=0 \quad C=-1 \quad 2A+B+D=2$

$C+E=-1 \qquad A=1$

The solution is $A=1, B=-1, C=-1, D=1$, and $E=0$. Thus

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx = \int\left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx$$

$$= \int\frac{dx}{x} - \int\frac{x}{x^2+1}\,dx - \int\frac{dx}{x^2+1} + \int\frac{xdx}{(x^2+1)^2}$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + k.$$

Example 14. $\displaystyle\int \frac{x-1}{(x^2+2x+3)^2}\,dx$

Solution

$$\int \frac{x-1}{(x^2+2x+3)^2}\,dx = \int \frac{\frac{1}{2}(2x+2)+(-1-1)}{(x^2+2x+3)^2}\,dx$$

$$= \frac{1}{2}\int\frac{2x+2}{(x^2+2x+3)^2}\,dx - 2\int\frac{dx}{(x^2+2x+3)^2}$$

$$= -\frac{1}{2}\frac{1}{(x^2+2x+3)} - 2\int\frac{dx}{(x^2+2x+3)^2}$$

We apply the substitution $x+1=t$ to the last integral:

$$\int\frac{dx}{(x^2+2x+3)^2} = \int\frac{dx}{((x+1)^2+2)^2}$$

$$= \int\frac{dx}{(t^2+2)^2} = \frac{1}{2}\int\frac{(t^2+2)-t^2}{(t^2+2)^2}\,dt$$

$$= \frac{1}{2}\int\frac{dt}{t^2+2} - \frac{1}{2}\int\frac{t^2}{(t^2+2)^2}\,dt$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} - \frac{1}{2}\int\frac{t^2dt}{(t^2+2)^2}$$

Let us consider the last integral

$$\int\frac{t^2dt}{(t^2+2)^2} = \int\frac{t.tdt}{(t^2+2)^2}$$

Integrating by parts

$$= -\frac{1}{2}\frac{t}{t^2+2} + \frac{1}{2}\int\frac{dt}{t^2+2}$$

$$= -\frac{t}{2(t^2+2)} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}$$

We do not yet write the arbitary constant but will take it into account in the final result.

Consequently,

$$\int\frac{dx}{(x^2+2x+3)} = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}$$

$$-\frac{1}{2}\left[-\frac{x+1}{2(x^2+2x+3)} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}\right]$$

Finally we get,

$$\int\frac{x-1}{(x^2+2x+3)^2}\,dx = -\frac{x+2}{2(x^2+2x+3)}$$

$$-\frac{\sqrt{2}}{4}\tan^{-1}\frac{x+1}{\sqrt{2}} + C.$$

Example 15. Evaluate $\displaystyle\int\frac{(2x+3)dx}{(x^2+2x+3)^2}$.

Solution Here $(d/dx)\,(x^2+2x+3) = 2x+2$.

$$\therefore \quad I = \int\frac{(2x+3)dx}{(x^2+2x+3)^2} = \int\frac{(2x+2+1)dx}{(x^2+2x+3)^2}$$

$$= \int \frac{(2x+2)dx}{(x^2+2x+3)^2} = \int \frac{dx}{(x^2+2x+3)^2}$$

$$= -\frac{1}{(x^2+2x+3)} + \int \frac{dx}{(x^2+2x+3)^2} \quad ...(1)$$

Now let $I_1 = \int \frac{dx}{[(x^2+1)^2+2]^2}$.

Put $x + 1 = \sqrt{2}\tan t$, so that $dx = \sqrt{2}\sec^2 t\,dt$.

$$\therefore \quad I_1 = \int \frac{\sqrt{2}\sec^2 t\,dt}{(2\tan^2 t+2)^2} = \frac{\sqrt{2}}{4}\int \cos^2 t\,dt$$

$$= \frac{\sqrt{2}}{4}\int \frac{1}{2}(1+\cos 2t)dt = \frac{\sqrt{2}}{8}\left[t + \frac{1}{2}\sin 2t\right]$$

$$= \frac{\sqrt{2}}{8}[t + \sin t\cos t] + C$$

Now $\tan t = \frac{x+1}{\sqrt{2}}$

$$\Rightarrow \quad \sin t = \frac{x+1}{\sqrt{\{(x+1)^2+2\}}} = \frac{x+1}{\sqrt{(x^2+2x+3)}}, \text{ and}$$

$$\cos t = \frac{\sqrt{2}}{\sqrt{(x^2+2x+3)}}$$

Also $t = \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$.

$$\text{Hence, } I_1 = \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

$$+ \frac{\sqrt{2}}{8}\cdot\frac{x+1}{\sqrt{\{(x^2+2x+3)\}}}\cdot\frac{\sqrt{2}}{\sqrt{(x^2+2x+3)}}$$

$$= \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{4}\frac{x+1}{(x^2+2x+3)}.$$

$$\therefore \quad I = -\frac{1}{x^2+2x+3}$$

$$+ \frac{1}{4}\frac{x+1}{x^2+2x+3} + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C, \text{ from (1)}$$

$$= \frac{x+1-4}{4(x^2+2x+3)} + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

$$= \frac{x-3}{4(x^2+2x+3)} + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C.$$

The Ostrogradsky method

If $Q(x)$ has multiple roots, then

$$\int \frac{P(x)}{Q(x)}dx = \frac{X(x)}{Q_1(x)} + \int \frac{Y(x)}{Q_2(x)}dx \quad ...(1)$$

where $Q_1(x)$ is the greatest common divisor of the polynomial $Q(x)$ and its derivative $Q'(x)$, and

$$Q_2(x) = \frac{Q(x)}{Q_1(x)}.$$

$X(x)$ and $Y(x)$ are polynomials with undetermined coefficients, whose degrees are, respectively, less by unity than those of $Q_1(x)$ and $Q_2(x)$.
The undetermined coefficients of the polynomials $X(x)$ and $Y(x)$ are computed by differentiating the identity (1).

Example 16. Find $\int \frac{dx}{(x^3-1)^2}$

Solution

$$\int \frac{1}{(x^3-1)^2} = \frac{Ax^2+Bx+C}{x^3-1} + \int \frac{Dx^2+Ex+F}{x^3-1}dx.$$

Differentiating this identity, we get

$$\frac{1}{(x^3-1)^2} = \frac{(2Ax+B)(x^3-1) - 3x^2(Ax^2+Bx+C)}{(x^3-1)^2}$$

$$+ \frac{Dx^2+Ex+F}{x^3-1}$$

or, $1 = (2Ax+B)(x^2-1) - 3x^2(Ax^2+Bx+C)$
$+ (Dx^2+Ex+F)(x^3-1)$.

Equating the coefficients of the respective degrees of x, we will have

$D = 0,\ E - A = 0,\ F - 2B = 0,\ D + 3C = 0,$
$E + 2A = 0,\ B + F = -1,$

whence $A = 0, B = -\frac{1}{3}, C = 0, D = 0, E = 0, F = -\frac{2}{3}$

and, consequently,

$$\int \frac{dx}{(x^3-1)^2} = -\frac{1}{3}\frac{x}{x^3-1} - \frac{2}{3}\int \frac{dx}{x^3-1} \quad ...(1)$$

To compute the integral on the right of (1), we decompose the fraction $\frac{1}{x^3-1}$ into partial fractions :

$$\frac{1}{x^3-1}=\frac{L}{x-1}+\frac{Mx+N}{x^2+x+1},$$

That is, $1 = L(x^2+x+1)+Mx(x-1)+N(x-1)$.

Putting $x = 1$, we get $L=\frac{1}{3}$.

Equating the coefficients of identical degrees of x on the right and left, we find

$$L+M=0;\ \ L-N=1,\ \ \text{or } M=-\frac{1}{3};N=-\frac{2}{3}.$$

Therefore,

$$\int\frac{dx}{x^3-1}=\frac{1}{3}\int\frac{dx}{x-1}-\frac{1}{3}\int\frac{x+2}{x^2+x+1}dx$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2+x+1)$$

$$-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$$

and $$\int\frac{dx}{(x^3-1)^2}=-\frac{1}{3}\frac{x}{x^3-1}+\frac{1}{9}\ln\frac{x^2+x+1}{(x-1)^2}$$

$$-\frac{2}{3\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$$

Example 17. Find $\int\frac{x^3-3x^2+2x-3}{(x^2+1)^2}dx$.

Solution There is a repeated quadratic polynomial in the denominator. Hence,

$$\frac{x^3-3x^2+2x-3}{(x^2+1)^2}=\frac{A_1x+B_1}{x^2+1}+\frac{A_2x+B_2}{(x^2+1)^2}$$

for some constants A_1, B_1, A_2 and B_2.

An easy way to determine these constants is as follows. By long division,

$$\frac{x^3-3x^2+2x-3}{x^2+1}=x-3+\frac{x}{x^2+1},$$

and therefore

$$\frac{x^3-3x^2+2x-3}{(x^2+1)^2}=\frac{x-3}{x^2+1}+\frac{x}{(x^2+1)^2}.$$

Thus $A_1 = 1$, $B_1 = -3$, $A_2 = 1$, and $B_2 = 0$.

We now have

$$\int\frac{x^3-3x^2+2x-3}{(x^2+1)^2}dx$$

$$=\int\frac{x}{x^2+1}dx-\int\frac{3}{x^2+1}dx+\int\frac{x}{(x^2+1)^2}dx$$

$$=\frac{1}{2}\ln(x^2+1)-3\tan^{-1}x-\frac{1}{2(x^2+1)}+C.$$

Example 18. Show that $\int\frac{4x^9+21x^6+2x^3-3x^2-3}{(x^7-x+1)^2}$ dx is a rational function.

Solution This is of the type A/Q^2, where

$Q\equiv x^7-x+1$, $Q'=7x^6-1$

and Q, Q' have no common factor [Q = 0, Q' = 0 have no common root].

Hence there are polynomials C and D for which

$$A\equiv(x^7-x+1)C+(7x^6-1)D \qquad ...(1)$$

We choose the degree of C and D to be as low as the general theory permits and take C to be degree 5 [one less than that of $7x^6-1$], D to be of degree 6[one less than that of x^7-x+1].

If we take $C\equiv c_0+c_1x+....+c_5x^5$,

$D\equiv d_0+d_1x+...+d_6x^6$,

and equate coefficients in (1), we see that our first needs are $c_5+7d_6=c_4+7d_5=c_3+7d_4=0$

and that there is probably a solution with

$c_3=c_4=c_5=d_4=d_5=d_6=0$;

a little work then leads to

$C(x)\equiv-3x^2$, $\quad D(x)\equiv x^3+3$

We therefore consider

$$\int\frac{A}{Q^2}dx=\int\frac{-3x^2}{Q}dx+\int(x^3+3)\frac{Q'}{Q^2}dx$$

$$=-\int\frac{3x^2}{Q}dx-\frac{x^3+3}{Q}+\int\frac{3x^2}{Q}dx$$

since $(Q'/Q^2)\,dx=-d(Q^{-1})$.

Hence the integral is $-(x^2+3)/Q$ and is rational.

Substitutions leading to integral of rational functions

(i) If an integral is of the form $\int R(\sin x)\cos x\,dx$, the substitution $\sin x = t$, $\cos x\,dx = dt$ reduces this integral to the form $\int R(t)\,dt$, where R(t) is a rational function.

(ii) If the integral has the form $\int R(\cos x)\sin x\,dx$, it is reduced to an integral of a rational function by the substitution $\cos x = t$, $\sin x\,dx = -dt$.

(iii) If the integrand is dependent only on tan x, then the substitution $\tan x = t$, i.e. $x=\tan^{-1}t$, $dx=\frac{dt}{1+t^2}$ reduces this integral to an integral of a rational function :

$$\int R(\tan x)\,dx=\int R(t)\frac{dt}{1+t^2}$$

(iv) If the integrand has the form R (sin x, cosx), but sin x and cos x are involved only in even powers, then the same substitution is applied : tan x = t because $\sin^2 x$ and $\cos^2 x$ can be expressed rationally in terms of tan x :

$$\cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2}$$

$$\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2}$$

$$dx = \frac{dt}{1+t^2}$$

After the substitution we obtain an integral of a rational function.

Example 19. Compute the integral $\int \frac{\sin^3 x}{2+\cos x}dx$.

Solution This integral is readily reduced to the form $\int R(\cos x)\sin x\,dx$. Indeed,

$$\int \frac{\sin^3 x}{2+\cos x}dx = \int \frac{\sin^2 x \sin x dx}{2+\cos x}$$

$$= \int \frac{1-\cos^2 x}{2+\cos x}\sin x\,dx$$

We substitute cos x = z. Then sinx dx = – dx

$$\int \frac{\sin^3 x}{2+\cos x}dx = \int \frac{1-z^2}{2+z}(-dz) = \int \frac{z^2-1}{z+2}dz$$

$$= \int\left(z-2+\frac{3}{z+2}\right)dz = \frac{z^2}{2} - 2z + 3\ln(z+2) + C$$

$$= \frac{\cos^2 x}{2} - 2\cos x + 3\ln(\cos x + 2) + C$$

Example 20. Evaluate $\int \frac{\sin x}{\sin 4x}dx$.

Solution $\int \frac{\sin x}{\sin 4x}dx = \int \frac{\sin x dx}{2\sin 2x \cos 2x}$

$$= \int \frac{\sin x dx}{4\sin x\cos x\cos 2x} = \frac{1}{4}\int \frac{dx}{\cos x\cos 2x}$$

$$= \frac{1}{4}\int \frac{\cos dx}{\cos 2x \cos^2 x}$$

$$= \frac{1}{4}\int \frac{\cos x\,dx}{(1-\sin^2 x)(1-2\sin^2 x)}$$

Now put sin x = t so that cos x dx = dt.

$$\therefore \quad I = \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)} = \frac{1}{4}\int \frac{dt}{(t^2-1)(2t^2-1)}$$

$$= \frac{1}{4}\int\left[\frac{dt}{(t^2-1)(2t^2-1)}\right]dt,$$

resolving into partial fractions

$$= \frac{1}{4}\int \frac{dt}{(t^2-1)} - \frac{1}{4}\int \frac{dt}{t^2-(1/2)}$$

$$= \frac{1}{4}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| - \frac{1}{4}\cdot\frac{1}{2.(1/\sqrt{2})}\ln\left|\frac{t-(1/\sqrt{2})}{t+(1/\sqrt{2})}\right|$$

$$= \frac{1}{8}\ln\left|\frac{t-1}{t+1}\right| - \frac{1}{4\sqrt{2}}\ln\left|\frac{t\sqrt{2}-1}{t\sqrt{2}+1}\right| + C$$

$$= \frac{1}{8}\ln\left|\frac{\sin x-1}{\sin x+1}\right| - \frac{1}{4\sqrt{2}}\ln\left|\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1}\right| + C.$$

Example 21. Evaluate $\int \frac{dx}{\sin x+\sin 2x}$

Solution $\int \frac{dx}{\sin x+\sin 2x} = \int \frac{dx}{\sin x + 2\sin x\cos x}$

$$= \int \frac{dx}{\sin x(1+2\cos x)}$$

$$= \int \frac{\sin x\,dx}{\sin^2 x(1+2\cos x)} = \int \frac{\sin x dx}{(1-\cos^2 x)(1+2\cos x)}$$

Now putting cos x = t, so that – sin x dx = dt, we get

$$= -\int \frac{dt}{(1-t^2)(1+2t)} = -\int \frac{dt}{(1-t)(1+t)(1+2t)}$$

$$= -\int\left[\frac{1}{6(1-t)} - \frac{1}{2(1-t)} + \frac{4}{3(1+2t)}\right]dt$$

$$= \frac{1}{6}\ln|1-t| + \frac{1}{2}\ln|1+t| - \frac{2}{3}\ln|1+2t| + C$$

$$= \frac{1}{6}\ln(1-\cos x) + \frac{1}{2}\ln(1+\cos x)$$

$$- \frac{2}{3}\ln|1+2\cos x| + C.$$

Example 22. Evaluate $\int \frac{(1+\sin x)dx}{\sin x(1+\cos x)}$

Solution We have $I = \int \frac{(1+\sin x)dx}{\sin x(1+\cos x)}$

$$= \int \frac{1}{\sin x(1+\cos x)}dx + \int \frac{dx}{1+\cos x}$$

$$= \int \frac{\sin x dx}{(1-\cos x)(1+\cos x)^2} + \int \frac{1}{2}\sec^2\frac{x}{2}dx$$

For first integral put $\cos x = t$

$$I = -\int \frac{dt}{(1-t)(1+t)^2} + \tan\frac{x}{2}$$

on integration by partial fractions, we get

$$= \frac{1}{4}\ell n\frac{1-\cos x}{1+\cos x} + \frac{1}{2(1+\cos x)} + \tan\frac{x}{2} + C$$

$$= \frac{1}{4}\ell n\tan^2\frac{x}{2} + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2} + C$$

Example 23. Integrate $\int \sqrt[3]{\tan x}\, dx$

Solution $I = \int (\tan x)^{1/3} dx$

Put $\tan x = t^3 \Rightarrow \sec^2 x\, dx = 3t^2\, dt$

$$= \int \frac{3t^3\, dt}{1+t^6}$$

Put $t^2 = y \Rightarrow 2t\, dt = dy$

$$= \frac{3}{2}\int \frac{y}{1+y^3}dy$$

$$I_1 = \int \frac{y+1-1}{y^3+1}dy$$

$$= \int \frac{dy}{y^2-y+1} - \int \frac{dy}{y^3+1}$$

$$= \int \frac{dy}{\left(y-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \int \frac{y^2-(y^2-1)}{y^3+1}dy$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y-1}{\sqrt{3}}\right) - \frac{1}{3}ln(y^3+1)$$

$$+ \int \frac{y-1}{y^2-y+1}.$$

Now $\int \frac{y-1}{y^2-y+1}dy = \frac{1}{2}\int \frac{(2y-1)-1}{y^2-y+1}$

$$= \frac{1}{2}ln(y^2-y+1) - \frac{1}{2}\int \frac{dy}{y^2-y+1}$$

$$\Rightarrow I_1 = \frac{1}{\sqrt{3}}\tan^{-1}\frac{2y-1}{\sqrt{3}} - \frac{1}{3}ln(y^3+1)$$

$$+ \frac{1}{2}ln(y^2-y+1) + C.$$

$\therefore$ $I = \frac{3}{2}I_1$ where $y = (\tan x)^{2/3}$.

Example 24. Evaluate $\int \frac{1}{(e^x-1)^2}dx$.

Solution We have $\int \frac{1}{(e^x-1)^2}dx = \int \frac{e^x}{e^x(e^x-1)^2}$

dx, $= \int \frac{dt}{t(t-1)^2}$, putting $e^x = t$ so that $e^x\, dx = dt$.

Now $\frac{1}{t(t-1)} \equiv \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t-1)^2}$,

$\therefore$ $1 \equiv A(t-1)^2 + Bt(t-1) + Ct$, ...(1)

To find A, putting $t = 0$ on both sides of (1), we get $A = 1$.

To find C, put $t = 1$ and we get $C = 1$.

Thus $1 \equiv (t-1)^2 + Bt(t-1) + t$.

Comparing the coefficients of t^2 on both sides, we get $0 = 1 + B$ or, $B = -1$.

$$\therefore \frac{1}{t(t-1)^2} = \frac{1}{t} - \frac{1}{t-1} + \frac{1}{(t-1)^2}.$$

Hence $\int \frac{dt}{t(t-1)^2} = \int \frac{1}{t}dt - \int \frac{dt}{t-1} + \int \frac{dt}{(t-1)^2}$

$= \ln|t| - \ln|t-1| - \{1/(t-1)\} + C$

$= \ln e^x - \ln|e^x - 1| - \{1/(e^x-1)\} + C$

$= x - \ln|e^x - 1| - \{1/(e^x-1)\} + C.$

Example 25. Evaluate $\int \frac{\tan^{-1}x}{x^4}dx$.

Solution $I = \int \frac{\tan^{-1}}{x^4}dx = \int \tan^{-1}x.\frac{1}{x^4}dx$

$$= (\tan^{-1}x)\left(-\frac{1}{3x^2}\right) - \int \frac{1}{1+x^2}.\frac{1}{(-3x^3)}dx$$

$$= -\frac{\tan^{-1}x}{3x^3} + \frac{1}{3}\int \frac{dx}{x^3(1+x^2)},$$

Put $1 + x^2 = t$

$2x\, dx = dt$

$$= -\frac{\tan^{-1}x}{3x^3} + \frac{1}{6}\int \frac{dt}{(t-1)^2.t}$$

$$I = -\frac{\tan^{-1}x}{3x^3} + \frac{1}{6}I_1 \quad \text{...(1)}$$

where, $I_1 = \int \frac{1}{(1-t)^2.t}dt = \int\left\{\frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t}\right\}dt$

Comparing coefficients we get,

$A=-1, B=1, C=1$

$$\therefore \quad I_1=\int\left\{-\frac{1}{(t-1)}+\frac{1}{(t-1)^2}+\frac{1}{t}\right\}dt$$

$$=-\ln|t-1|-\frac{1}{(t-1)}+\ln|t| \quad \text{...(2)}$$

$\therefore$ From (1) and (2), we get

$$I=\frac{\tan^{-1}x}{3x^3}+\frac{1}{6}\left\{-\ln x^2-\frac{1}{x^2}+\ln(1+x^2)\right\}+C$$

$$I=-\frac{\tan^{-1}x}{3x^3}+\frac{1}{6}\ln\left|\frac{x^2+1}{x^2}\right|-\frac{1}{6x^2}+C.$$

Practice Problems N

1. Evaluate the following integrals :

(i) $\int \frac{x^3-1}{4x^3-x}\,dx$

(ii) $\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}\,dx$

2. Evaluate the following integrals :

(i) $\int \frac{x^4}{(1-x)^3}\,dx$

(ii) $\int \frac{6x^2-12x+4}{x^2(x-2)^2}\,dx$

3. Evaluate the following integrals :

(i) $\int \frac{x}{(x-1)(x^2+4)}\,dx$

(ii) $\int \frac{x^3dx}{x^4+3x^2+2}$ (iii) $\int \frac{x^3-1}{x^3+x}\,dx$

(iv) $\int \frac{x^4-2x^3+3x^2-x+3}{x^3-2x^2+3x}\,dx$

4. Evaluate the following integrals :

(i) $\int \frac{dx}{x^3+1}$ (ii) $\int \frac{dx}{x(x^2+1)}$

(iii) $\int \frac{x+2}{(2x^2+4x+3)^2}\,dx$ (iv) $\int \frac{1+x^{-2/3}}{1+x}\,dx$

5. Evaluate the following integrals :

(i) $\int \frac{dx}{\sin x(3+2\cos x)}$ (ii) $\int \frac{dx}{\sin 2x-2\sin x}$

(iii) $\int \frac{\sin\frac{\theta}{2}\tan\frac{\theta}{2}\,d\theta}{\cos\theta}$

(iv) $\int \frac{dx}{\ln x^x\left[(\ln x)^2-3\ln x-10\right]}$

6. Evaluate the following integrals :

(i) $\int \frac{dx}{\sin x(3+\cos^2 x)}$

(ii) $\int \sec x.\sec 2x\,dx$

(iii) $\int \frac{\cos^2 x+\sin 2x}{(2\cos x-\sin x)^2}\,dx$

(iv) $\int \frac{dx}{e^x+3+2e^{-x}}$

7. Applying Ostrogradsky's method, find the following integrals :

(i) $\int \frac{dx}{(x+1)^2(x^2+1)^2}$ (ii) $\int \frac{dx}{(x^4+1)^2}$

(iii) $\int \frac{dx}{(x^2+1)^4}$

(iv) $\int \frac{x^4-2x^2+2}{(x^2-2x+2)^2}\,dx$

8. Evaluate the following integrals :

(i) $\int \frac{5x^2-12}{\left(x^2-6x+13\right)^2}\,dx$

(ii) $\int \frac{x^3+x-1}{(x^2+2)^2}\,dx$

(iii) $\int \frac{x^6+x^4-4x^2-2}{x^3(x^2+1)^2}\,dx$

(iv) $\int \frac{dx}{x^4\left(x^3+1\right)^2}$

9. Evaluate the following integrals :

(i) $\int \frac{x^3+x^2+x+3}{(x^2+1)(x^2+3)}\,dx$ (ii) $\int \frac{dx}{x^4(x^3+1)^2}$

(iii) $\int \frac{x^7+2}{(x^2+x+1)^2}\,dx$ (iv) $\int \frac{3x^4+4}{x^2(x^2+1)^3}\,dx$

10. Evaluate the following integrals :

(i) $\int \frac{2x^3 + x^2 + 4}{(x^2+4)^2}\, dx$

(ii) $\int \frac{x^3 + x^2 - 5x + 15}{(x^2+5)(x^2+2x+3)}\, dx$

(iii) $\int \frac{dx}{(x^4 + 2x + 10)^3}$

(iv) $\int \frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3} dx$

1.16 SPECIAL METHODS FOR INTEGRATION OF RATIONAL FUNCTIONS

Note that sometimes partial fractions can be avoided when integrating a rational function. For instance, although the integral

$$\int \frac{x^2+1}{x(x^2+3)}\, dx$$

could be evaluated by the method of partial fractions, it is much easier, if we put $u = x(x^2+3) = x^3 + 3x$, then $du = (3x^2 + 3)\, dx$ and so

$$\int \frac{x^2+1}{x(x^2+3)}\, dx = \frac{1}{3}\, ln\, |x^3 + 3x| + C.$$

1. Integrals of the form $\int \frac{px^2+q}{(x^2+a)(x^2+b)} dx$

Assume x^2 as t for finding partial fractions.

Suppose that $\frac{pt+q}{(t+a)(t+b)} = \frac{A}{(t+a)} + \frac{B}{(x+b)}$,

then the integral becomes

$$\int \frac{A}{(x^2+a)} dx + \int \frac{B}{x^2+b} dx.$$

In general, if the numerator and the denominator of a given fraction contains even powers of x only, we can first write the fraction in a simpler form by putting t for x^2 and then break it up into partial fractions involving t, i.e., x^2, and then integrate it.

Example 1. Evaluate $\int \frac{dx}{x^2(1+x^2)}$

Solution Treating $1/x^2(1+x^2)$ as a function of x^2, we have

$$\frac{1}{x^2(1+x^2)} = \frac{1}{x^2} - \frac{1}{1+x^2},$$

Hence, $\int \frac{dx}{x^2(1+x^2)} = \frac{1}{x} - \tan^{-1} x + C.$

Example 2. Integrate $\int \frac{x^2}{x^4 + x^2 - 2} dx$.

Solution Putting $x^2 = z$, we have

$$\frac{x^2}{x^4 + x^2 - 2} = \frac{z}{z^2 + z - 2}$$

$$= \frac{z}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1}, \text{ say.}$$

$\therefore\ z = A(z-1) + B(z+2).$

Putting $z = -2$ and 1, we get respectively

$A = \frac{2}{3}$, $B = \frac{1}{3}$

$$\therefore\ \frac{x^2}{x^4 + x^2 - 2} = \frac{2}{3}\frac{1}{x^2+2} + \frac{1}{3}\frac{1}{x^2-1}$$

$$\therefore\ I = \frac{2}{3}\int \frac{dx}{x^2+2} + \frac{1}{3}\int \frac{dx}{x^2-1}$$

$$= \frac{2}{3}\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{3}\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$$

Example 3. Evaluate $\int \frac{x^3\, dx}{(x^2+1)^2}$.

Solution Regarding $x^2/(x^2+1)^2$ as a function of x^2, we find (by inspection)

$$\frac{x^2}{(x^2+1)^2} = \frac{(x^2+1)-1}{(x^2+1)^2} = \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}$$

Hence, $\int \frac{x^3\, dx}{(x^2+1)^2} = \int \frac{x\, dx}{x^2+1} - \int \frac{x\, dx}{(x^2+1)^2}$

$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C.$$

2. Integrals of the form $\int \frac{P(x)}{Q(x)} dx$ where Q(x) has a linear factor with high index

Substitute the linear factor as 1/t.

Example 4. Evaluate $\int \frac{3x+1}{(x-1)^3(x+1)}dx$

Solution Put $x-1 = 1/t \Rightarrow x = 1/t+1$

$\Rightarrow \; dx = -\frac{dt}{t^2}$. Thus, the integral

$$\int \frac{\left(\frac{3}{t}+4\right)dt}{\left(\frac{1}{t}\right)^3\left(\frac{1}{t}+2\right)t^2} = \int \frac{\left(\frac{3+4t}{t}\right)dt}{\frac{1}{t^3}\left(\frac{1+2t}{t}\right)t^2} = -\int \frac{3t+4t^2}{(2t+1)}dt$$

$$= -\int \frac{(2t+1)^2 - t - 1}{(2t+1)} = \int \left(2t + \frac{1}{2} - \frac{1}{2(2t+1)}\right)dt$$

Now, the integral can be evaluated easily.

3. Substitution

(i) Integrals of the form $\int \frac{x^m}{(ax+b)^n}dx$, $m, n \in N$

Put $ax + b = t$.

The integral becomes $\frac{1}{a^{m+1}} \int \frac{(t-b)^m dt}{t^n}$.

Expanding by the binomial theorem and integrating each term separately the required integral can be immediately obtained.

Example 5. Evaluate $I = \int \frac{x^2}{(x+2)^3}dx$

Solution Put $x + 2 = t$

$$I = \int \frac{(t-2)^2}{t^3}dt = \int \frac{t^2+4-4t}{t^3}dt$$

$$= \int \frac{1}{t} + \frac{4}{t^3} - \frac{4}{t^2}\, dt = \ln|t| - \frac{4}{3t^2} + \frac{4}{t} + C.$$

where $t = x + 2$.

Example 6. Integrate $\int \frac{(a+bx)^2}{(a'+b'x)^3}dx$

Solution Put $a' + b'x = z$ or $x = \frac{z-a'}{b'}$

$\therefore \; dx = \frac{1}{b'}dz$.

Now the given integral becomes

$$I = \int \frac{\left\{a + \frac{b}{b'}(z-a')\right\}^2}{z^3}\frac{dz}{b'}$$

$$= \frac{1}{b'^3}\int \frac{(bz + ab' - a'b)^2}{x^3}dx$$

$$= \frac{1}{b'^3}\left[\int \frac{dz}{z} + 2b(ab'-a'b)\int \frac{dz}{z^3} + (ab'-a'b)^2\int \frac{dz}{z^3}\right]$$

$$= \frac{b^2}{b'^3}\ln|z| - \frac{2b(ab'-a'b)}{b'^3}\frac{1}{z} - \frac{(ab'-a'b)^2}{2b'^3}\frac{1}{z^2} + C$$

$$= \frac{b^2}{b'^3}\ln|a'+b'x| - \frac{2b(ab'-a'b)}{b'^3(a'+b'x)} - \frac{(ab'-a'b)^2}{2b'^3(a'+b'x)^2} + C$$

Note: By the same process we can integrate $\int \frac{(a+bx)^m}{(a'+b'x)^n}dx$, where m is a positive integer, n being a rational number.

(ii) Integrals of the form $\int \frac{dx}{x^m(ax+b)^n}$, $m, n \in N$

Put $\frac{ax+b}{x} = t$

Example 7. Evaluate $I = \int \frac{dx}{x^2(x+2)^3}$

Solution $I = \int \frac{dx}{x^5\left(\frac{x+2}{x}\right)^3}$

Put $\frac{x+2}{x} = t \Rightarrow 1 + \frac{2}{x} = t \;\; \Rightarrow \frac{-2}{x^2}dx = dt$

$$I = -\frac{1}{2}\int \frac{(t-1)^3}{8t^3}dt$$

Now, the integral can be evaluated easily.

Example 8. Integrate $\int \frac{dx}{x^3(a+bx)^2}$

Solution Put $a + bx = zx$, or $\frac{a}{x} + b = z$.

Then $-\frac{a}{x^2}dx = dz$

The given integral

$$= -\frac{1}{a}\int \frac{dz}{x.z^2x^2} = -\frac{1}{a}\int \left(\frac{z-b}{a}\right)^3 \frac{dz}{z^2}$$

$$= -\frac{1}{a^4}\int \left(z - 3b + \frac{3b^2}{z} - \frac{b^3}{z^2}\right)dx$$

$$= -\frac{1}{a^4}\left[\frac{z^2}{2} - 3bz + 3b^2 \ln|z| + \frac{b^3}{z}\right] + C$$

$$= -\frac{1}{a^4}\left[\frac{1}{2}\left(\frac{a+bx}{x}\right)^2 - 3b\left(\frac{a+bx}{x}\right) + 3b^2 \ln\left|\frac{a+bx}{x}\right| + b^3\left(\frac{x}{a+bx}\right)\right] + C.$$

(iii) Integrals of the form $\int \frac{dx}{(x-a)^m (x-b)^n}$

This integral can be easily transformed into a form which is immediately integrable, by the substitution

$$\frac{x-a}{x-b} = t \text{ if } m < n$$

$$\Rightarrow \frac{(x-b)(1)-(x-a)(1)}{(x-b)^2} dx = dt \Rightarrow \frac{a-b}{(x-b)^2} dx = dt$$

Also, $x = \frac{a-bt}{1-t}$

$$\therefore \quad x - a = \frac{(a-b)t}{1-t}, \quad x - b = \frac{a-b}{1-t},$$

The integral transforms into $\int \frac{(1-t)^{m+n-2} dt}{(a-b)^{m+n-1} t^m}$

Expand the numerator by the Binomial Theorem, and the integral can be immediately obtained.

For example, take the integral

$$\int \frac{dx}{(x-a)^2 (x-b)^2}$$

Put $\frac{x-a}{x-b} = t$

$$= \frac{1}{(a-b)^4} \int \left(\frac{1}{t^2} - \frac{3}{t} + 3 - t\right) dt$$

$$= \frac{-1}{(a-b)^4}\left\{\frac{t^2}{2} - 3t + 3\ln t + \frac{1}{t}\right\} + C.$$

where $t = \frac{x-a}{x-b}$.

Example 9. Integrate $\int \frac{dx}{(x-1)^2 (x-2)^3}$

Solution Put $\frac{x-1}{x-2} = z$

$$\therefore \quad x = \frac{1-2z}{1-z}$$

$$\therefore \quad dx = -\frac{dz}{(1-z)^2}.$$

Hence, the integral transforms into

$$\int \frac{(1-z)^3}{z^2} dz = \int \frac{1 - 3z + 3z^2 - z^3}{z^2} dz$$

$$= -\frac{1}{z} - 3\ln|z| + 3z - \frac{1}{2}z^2 + C$$

$$= -\left(\frac{x-2}{x-1}\right) - 3\ln\left|\frac{x-1}{x-2}\right| + 3\left(\frac{x-1}{x-2}\right) - \frac{1}{2}\left(\frac{x-1}{x-2}\right)^2 + C$$

(iv) Integrals of the form $\int \frac{dx}{x(a+bx^n)}$

Here, the substitution of $x^n = \frac{1}{t}$ gives

$$-\frac{1}{n}\int \frac{dt}{at+b}$$

the value of which is obviously

$$-\frac{1}{na}\ln|at+b| + C = \frac{1}{na}\ln\left(\frac{x^n}{a+bx^n}\right) + C.$$

(v) Integrals of the form $\int \frac{x^{2m+1}}{(ax^2+b)^n} dx$, where m and n are integers

Put $ax^2 + b = t$.

The integral becomes $\int \frac{(t-b)^m dt}{2a^{m+1} t^n}$

a form which is immediately integrable by aid of the Binomial Theorem. It is evident that the expression is made integrable by the same transformation when n is either a fractional or a negative index.

To evaluate $I = \int \frac{x^5}{(x^2+1)^3} dx$

Put $x^2 + 1 = t \Rightarrow 2x\, dx = dt$

$$I = \frac{1}{2}\int \frac{(t-1)^2}{t^3} dt$$

$$= \frac{1}{x^2+1} - \frac{1}{4(x^2+1)} + \frac{1}{2}\ln|x^2+1| + C.$$

In general, if in a fraction, the numerator contains only odd powers of x and the denominator only even powers, then it is found more convenient to change the variable first by putting $x^2 = z$ and then break it up into partial fractions as usual.

It may be also observed that the more general expression $\frac{f(x^2)x}{(a+cx^2)^n}$ can be integrated by the same transformation, where $f(x^2)$ denotes an integral algebraic function of x^2.

Example 10. Integrate $\int \frac{x^3\,dx}{x^4+3x^2+2}$.

Solution Put $x^2 = z$.

$\therefore$ $2x\,dx = dz$. $\therefore$ $x^3\,dx = \frac{1}{2}z\,dz$.

$$\therefore \quad I = \frac{1}{2}\int \frac{z\,dx}{x^2+3z+2}.$$

Now, $\frac{z}{z^2+3z+2} = \frac{z}{(z+1)(z+2)}$

$= \frac{A}{z+1} + \frac{B}{z+2}$, say.

We determine $A = -1$, $B = 2$ as usual.

$$\therefore \quad I = \frac{1}{2}\left[2\int\frac{dz}{z+2} - \int\frac{dz}{z+1}\right]$$

$$= \frac{1}{2}[2\ln(z+2) - \ln(z+1)] + C$$

$$= \ln(x^2+2) - \frac{1}{2}\ln(x^2+1) + C$$

Example 11. Evaluate $\int \frac{dx}{x(1+x^3)^2}$

Solution Put $1 + x^3 = t$

$\therefore$ $3x^2\,dx = dt$.

$$I = \int\frac{dt}{3x^3t^2} = \frac{1}{3}\int\frac{dt}{(t-1)t^2} \text{ split into partial fractions}$$

$$= \frac{1}{3}\int\left(\frac{1}{t-1} - \frac{1}{t} - \frac{1}{t^2}\right)dt$$

$$= \frac{1}{2}\left[\ln|t-1| - \ln|t| + \frac{1}{t}\right] + C$$

$$= \frac{1}{3}\left[\ln\left|\frac{x^3}{1+x^3}\right| + \frac{1}{1+x^3}\right] + C.$$

Example 12. Evaluate $\int \frac{1-x^2}{x(1+x^2+x^4)}dx$

Solution Multiply above and below by x and put $x^2 = t$.

$$\therefore \quad I = \frac{1}{2}\int\frac{1-t}{t(1+t+t^2)}dt \text{ split into partial fractions.}$$

$$I = \frac{1}{2}\int\left(\frac{1}{t} - \frac{t+2}{(t^2+t+1)}\right)dt$$

$$= \frac{1}{2}\ln|t| - \frac{1}{4}\int\frac{2t+1+3}{(1+t+t^2)}dt$$

$$= \frac{1}{2}\ln|t| - \frac{1}{4}\ln(t^2+t+1) - \frac{3}{4}\int\frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2}\ln x^2 - \frac{1}{4}\ln(x^4+x^2+1) - \frac{3}{4}.\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{x^2+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

$$= \ln|x| - \frac{1}{4}\ln(x^4+x^2+1) - \frac{\sqrt{3}}{2}\tan^{-1}\frac{2x^2+1}{\sqrt{3}} + C$$

4. Integration by parts

Example 13. Evaluate $\int\frac{dx}{(x^2+1)^2}$.

Solution In order to evaluate it we integrate $\frac{1}{(x^2+1)^{2-1}}$ by parts choosing unity as the other function.

$$\int\frac{1}{(x^2+1)}dx = \frac{1}{(x^2+1)}x - \int x\frac{(-2x)}{(x^2+1)^2}dx$$

$$= \frac{x}{(x^2+1)} + 2\int\frac{x^2+1-1}{(x^2+1)^2}dx$$

$$= \frac{x}{(x^2+1)} + 2\int\frac{dx}{(x^2+1)} - 2\int\frac{dx}{(x^2+1)^2}$$

$$\therefore \quad 2\int\frac{dx}{(x^2+1)^2} = \frac{1}{2}.\frac{x}{x^2+1} + \frac{1}{2}\tan^{-1}x$$

or $$\int\frac{dx}{(x^2+1)^2} = \frac{1}{2}.\frac{x}{x^2+1} + \frac{1}{2}\tan^{-1}x + C.$$

Example 14. Evaluate $I = \int\frac{dx}{(x^4-1)^2}$

Solution We have $d(x^4-1) = 4x^3\,dx$

$$I = \int\frac{1}{4x^3}\cdot\frac{4x^3}{(x^4-1)^2}dx$$

$$= \frac{1}{4x^3}\cdot\frac{-1}{(x^4-1)} - \int\frac{3}{4x^4}\cdot\frac{1}{(x^4-1)}dx$$

$$= -\frac{1}{4x^3(x^4-1)} - \frac{3}{4}\int\left(\frac{1}{x^4-1}-\frac{1}{x^4}\right)dx$$

Now $\int \frac{1}{x^4-1}dx = \int \frac{1}{(x^2-1)(x^2+1)}dx$

$$= \frac{1}{2}\int\left(\frac{1}{x^2-1}-\frac{1}{x^2+1}\right)dx$$

Now, the integral can be evaluated easily.

Here are some more examples on integration of rational functions.

Example 15. Evaluate

$$\int \frac{2e^{5x}+e^{4x}-4e^{3x}+4e^{2x}+2e^x}{(e^{2x}+4)(e^{2x}-1)^2}dx$$

Solution Put $e^x = y$

$$\Rightarrow I = \int \frac{2y^4+y^3-4y^2+4y+2}{(y^2+4)(y^2-1)^2}dy$$

$$= \int \frac{y(y^2+4)+(y^4-2y^2+1)}{(y^2+4)\ (y^2-1)^2}dy$$

$$= -\frac{1}{2\,(e^{2x}-1)} + \tan^{-1}\left(\frac{e^x}{2}\right) + C.$$

Example 16. Evaluate $\int \frac{dy}{y^2(1+y^2)^3}$

Solution Put $y = \tan\theta$

$$\Rightarrow \int \frac{dy}{y^2(1+y^2)^3} = \int \frac{\cos^6\theta}{\sin^2\theta}d\theta = \int \frac{(1-\sin^2\theta)^3 d\theta}{\sin^2\theta}$$

$$= \int \frac{(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta)d\theta}{\sin^2\theta}$$

$$= \int (\operatorname{cosec}^2\theta - 3 + 3\sin^2\theta - \sin^4\theta)\,d\theta$$

$$= -\frac{1}{y} - \frac{15}{8}\tan^{-1}y - \frac{1}{2}\sin(2\tan^{-1}y)$$

$$-\frac{1}{32}\sin(4\tan^{-1}y) + C.$$

Example 17. Evaluate $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx$.

Solution We have

$$\frac{(x^2+1)(x^2+2)}{(x^3+3)(x^2+4)} = \frac{(y+1)(y+2)}{(y+3)(y+4)},\ \text{where } y = x^2.$$

Now let $\frac{(y+1)(y+2)}{(y+3)(y+4)} = 1 + \frac{A}{y+3} + \frac{B}{y+4}$,

resolving into partial fractions.

We have $A = \frac{(-3+1)(-3+2)}{(-3+4)} = 2$,

$$B = \frac{(-4+1)(-4+2)}{(-4+3)} = -6,$$

$$\therefore \frac{(y+1)(y+2)}{(y+3)(y+4)} = 1 + \frac{1}{y+3} - \frac{6}{y+4}.$$

$\therefore$ The given integral

$$I = \int\left[1+\frac{2}{x^2+3}-\frac{6}{x^2+4}\right]dx$$

$$= \int dx + 2\int\frac{dx}{x^2+3} - 6\int\frac{dx}{x^2+4}$$

$$= x + 2.\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 6.\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Example 18. Evaluate $\int (3x+1)/\{(x-1)^3(x+1)\}dx$.

Solution Putting $x - 1 = y$ so that $x = 1 + y$, we get

$$\frac{3x+1}{(x-1)^3(x+1)} = \frac{3(1+y)+1}{y^3(2+y)} = \frac{4+3y}{y^3(2+y)}$$

arranging the Nr. and the Dr. in ascending powers of y

$$= \frac{1}{y^3}\left[2+\frac{1}{2}y-\frac{1}{4}y^2+\frac{1}{4}\frac{y^3}{2+y}\right],$$

by actual division

$$= \frac{2}{y^3}+\frac{1}{2y^2}-\frac{1}{4y}+\frac{1}{4}.\frac{1}{(2+y)}$$

$$= \frac{2}{(x-1)^3}+\frac{1}{2(x-1)^2}-\frac{1}{4(x-1)}+\frac{1}{4(x+1)}$$

Hence the required integral

$$= \int\left[\frac{2}{(x-1)^3}+\frac{1}{2(x-1)^2}-\frac{1}{4(x-1)}+\frac{1}{4(x+1)}\right]dx$$

$$= \frac{-1}{(x-1)^2} - \frac{1}{2(x-1)} - \frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+1| + C$$

$$= \frac{-1}{(x-1)^2} - \frac{1}{2(x-1)} - \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| + C.$$

Example 19. Evaluate the integral $\int \frac{1}{x^3(x-1)}dx$.

Solution Let $x = \sec^2\theta$

$dx = 2\sec^2\theta\tan\theta\, d\theta$

$$\Rightarrow \quad I = \int \frac{2\sec^2\theta\tan\theta\, d\theta}{\sec^6\theta\tan\theta} \Rightarrow 2\int \cos^4\theta\, d\theta$$

$$I = 2\int \cos^4\theta\, d\theta = 2\int [(\cos^2 2\theta)^2]d\theta$$

$$= 2\int \left[\frac{1+\cos 2\theta}{2}\right]^2 d\theta = \frac{2}{4}\int (\cos^2 2\theta + 2\cos 2\theta)d\theta$$

$$= \frac{1}{2}\left[\int d\theta + \int \cos^2 2\theta\, d\theta + 2\int \cos 2\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\theta + \int \left(\frac{1+\cos 4\theta}{2}\right)d\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{12}{2}\left[\theta + \frac{\theta}{2} + \frac{\sin 4\theta}{8} + \sin 2\theta\right] + C$$

$$= \frac{\theta}{2} + \frac{\theta}{4} + \frac{\sin 4\theta}{16} + \frac{\sin 2\theta}{2} + C$$

$$= \frac{3\theta}{4} + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{16} + C \text{ where } x = \sec^2\theta.$$

Practice Problems O

1. Evaluate the following integrals :

 (i) $\int \frac{(3x^2-2)\,dx}{x^4-3x^2-4}$

 (ii) $\int \frac{x^2\,dx}{(x^2+1)(2x^2+1)}$

 (iii) $\int \frac{x^2dx}{(a^2-x^2)^2}$

 (iv) $\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}$

2. Evaluate the following integrals :

 (i) $\int \frac{x\,dx}{x^4-x^2-2}$ (ii) $\int \frac{dx}{x(a+bx^2)^2}$

 (iii) $\int \frac{x}{(x^2+2)(x^2+1)}dx$

 (iv) $\int \frac{(1-x^2)dx}{x(1+x^2+x^4)}$

3. Evaluate the following integrals :

 (i) $\int \frac{dx}{x^2(a-bx)^2}$ (ii) $\int \frac{dx}{x(a+bx^n)^2}$

 (iii) $\int \frac{dx}{x^4+4x^3+5x^2+4x+4}$

4. Evaluate the following integrals :

 (i) $\int \frac{dx}{(x^4+1)^2}$ (ii) $\int \frac{dx}{(x^2+9)^3}$

 (iii) $\int \frac{x^2dx}{(1-x^2)^3}$ (iv) $\int \frac{dx}{x^2(1+x^2)^2}$

5. Evaluate the following integrals :

 (i) $\int \frac{x^7}{(x^{12}-1)}dx$ (ii) $\int \frac{x^9\,dx}{(x^4-1)^2}$

1.17 INTEGRATION OF IRRATIONAL FUNCTIONS

Integration by Rationalization

Certain types of integrals of algebraic irrational expressions can be reduced to integrals of rational functions by a appropriate change of the variable. Such transformation of an integral is called its rationalization.

1. Let n be a positive integer. Any rational function of x and $\sqrt[n]{ax+b}$ can be transformed into a rational function by the substitution $t^n = ax + b$ and thus can be integrated by partial fractions.

For example, in $\int \frac{x^n dx}{(a+bx)^{1/2}}$, where n is a positive integer, we put $a + bx = t^2$, then $dx = \frac{2tdt}{b}$, and $x = \frac{t^2-a}{b}$. Making these substitutions, the integral becomes $\int \frac{2(t^2-a)^n dt}{b^{n+1}}$, which can be easily integrated.

Example 1. Evaluate $\int \sqrt[3]{x}\sqrt[7]{1+\sqrt[3]{x^4}}\,dx.$

Solution Let $x = t^3$

$\Rightarrow$ $dx = 3t^2$

$$I = \int t(1+t^4)^{1/7} \cdot 3t^2\,dt = 3\int t^3(1+t^4)^{1/7}\,dt$$

Put $1 + t^4 = X^7$

$\Rightarrow$ $4t^3\,dt = 7X^6\,dX$

$$I = \frac{3}{4}\cdot\int 7X^7 dX = \frac{21}{32}X^6 + C$$

Therefore, $I = \frac{21}{32}(1+x^{4/3})^{8/7} + C$.

Example 2. Integrate $\int \sqrt{\frac{1+x}{1-x}}\,dx$

Solution Rationalizing the numerator, we have

$$I = \int \frac{1+x}{\sqrt{(1-x^2)}}dx = \int \frac{dx}{\sqrt{(1-x^2)}} + \int \frac{x\,dx}{\sqrt{(1-x^2)}}$$

Put $1 - x^2 = z^2$ in the second integral, so that $-2x\,dx = 2z\,dz$

The second integral $= -\int dz = -z = -\sqrt{(1-x^2)}$

$\therefore$ $I = \sin^{-1}x - \sqrt{(1-x^2)} + C.$

Note: Integrals of the type $\int \sqrt{\frac{ax+b}{cx+d}}\,dx$ ($a \neq 0, c \neq 0$) of which the above is a particular case can be evaluated exactly in the same way.

Example 3. Evaluate $\int \frac{x\,dx}{(2x^2+3)\sqrt{x^2-1}}$.

Solution Put $x^2 - 1 = t^2 \Rightarrow x\,dx = t\,dt$

So $$I = \int \frac{t\,dt}{(2t^2+5)t} = \int \frac{dt}{2t^2+5} = \frac{1}{2}\int \frac{dt}{t^2+\frac{5}{2}}$$

$$= \frac{1}{\sqrt{10}}\tan^{-1}\left(\sqrt{\frac{2}{5}}\sqrt{x^2-1}\right) + C$$

2. Integrals of the form $\int \frac{dx}{x\sqrt{ax^n+b}}$

Here we put $ax^n + b = t^2$.

Example 4. Evaluate $\int \frac{dx}{x\sqrt{(1+x^n)}}$.

Solution Put $1 + x^n = t^2$.

$nx^{n-1}\,dx = 2t\,dt.$

$\therefore$ $$I = \int \frac{2t\,dt}{nx^{n-1}}\cdot\frac{1}{x.t} = \frac{2}{n}\int \frac{dt}{t^2-1} \quad \text{since } x^n = t^2 - 1$$

$$= \frac{2}{n}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{n}\ln\left|\frac{\sqrt{(1+x^n)}-1}{\sqrt{(1+x^n)}+1}\right| + C.$$

3. Integrals of the form $\int \frac{dx}{(a+cx^2)^{3/2}}$

Here, we put $x = \frac{1}{t}$ and the integral becomes

$$\int -\frac{t\,dt}{(at^2+c)^{3/2}} = \frac{1}{a(at^2+c)^{1/2}} + C$$

So, $$\int \frac{dx}{(a+cx^2)^{1/2}} = \frac{x}{a(a+cx^2)^{1/2}} + C.$$

Consider the integral $\int \frac{dx}{(a+2bx+cx^2)^{3/2}}$

This can be written in the form

$$\int \frac{c^{3/2}dx}{\{ac-b^2+(cx+b)^2\}^{3/2}}$$

which is reduced to the preceding form by substituting $cx + b = z$.

Hence, we get

$$\int \frac{dx}{(a+2bx+cx^2)^{3/2}}$$

$$= \frac{b+cx}{(ac-b^2)(a+2bx+cx^2)^{1/2}} + C$$

4. Integrals of the form $\int \frac{x\,dx}{(a+2bx+cx^2)^{3/2}}$

We substitute $x = \frac{1}{t}$. The integral becomes

$$\int \frac{-dx}{(az^2+2bz+c)^{3/2}}$$

Hence, $$\int \frac{x\,dx}{(a+2bx+cx^2)^{3/2}}$$

$$= -\frac{a+bx}{(ac-b^2)(a+2bx+cx^2)^{1/2}} + C$$

Combining this with the previous result, we get

$$\int \frac{(p+qx)\,dx}{(a+2bx+cx^2)^{1/2}} = \frac{bp-aq+(cp-bq)x}{(ac-b^2)(a+2bx+cx^2)^{1/2}} + C'$$

Example 5. Evaluate $I = \int \frac{\sqrt{a^2-x^2}}{x^4}dx$

Solution Put $x = \frac{1}{t}$, $dx = -\frac{dt}{t^2}$

Hence, $I = -\int \frac{\sqrt{a^2-1/t^2}}{(1/t^4)t^2}dt = -\int t\sqrt{a^2t^2-1}\,dt$

Now make one more substitution $\sqrt{a^2t^2-1} = z$. Then $2a^2tdt = 2zdz$ and

$$I = -\frac{1}{a^2}\int z^2dz = -\frac{1}{3a^2}z^3 + C$$

Returning to t and then to x, we obtain

$$I = -\frac{(a^2-x^2)^{3/2}}{3a^2x^3} + C.$$

Example 6. Find $\int \frac{\sqrt{x-x^2}}{x^4}dx.$

Solution Put $x = \frac{1}{z}$

The substitution yields $dx = -\frac{dz}{z^2}$, $\sqrt{x-x^2} = \frac{\sqrt{z-1}}{z}$,

and $\int \frac{\sqrt{x-x^2}}{x^4}dx = \int \frac{\frac{\sqrt{z-1}}{z}\left(-\frac{dz}{z^2}\right)}{1/z^4} = -\int z\sqrt{z-1}\,dz$

Let $z-1 = s^2$.

Then $-\int z\sqrt{z-1}\,dz = -\int (s^2+1)(s)(2s\,ds)$

$$= -2\left(\frac{s^5}{5}+\frac{s^3}{3}\right) + C.$$

$$= -2\left[\frac{(z-1)^{5/2}}{5}+\frac{(z-1)^{3/2}}{3}\right] + C$$

$$= -2\left[\frac{(1-x)^{5/2}}{5x^{5/2}}+\frac{(1-x)^{3/2}}{3x^{3/2}}\right] + C.$$

5. If the integrand is a rational function of fractional powers of x, i.e. the function $R\left(x, x^{\frac{p_1}{q_1}}, \ldots, x^{\frac{p_k}{q_k}}\right)$, then the integral can be rationalized by the substitution $x = t^m$, where m is the L.C.M. of the denominators $q_1, q_2, \ldots, q_k$ of the several fractional powers. By this means the integration of such expressions is reduced to that of rational functions.

For example, to find

$$\int \frac{(1+x^{1/4})dx}{1+x^{1/2}}$$

Let $x = z^4$, where 4 is L.C.M. of 2 and 4, and the transformed integral is

$$4\int \frac{z^3(1+z)dz}{1+z^2}$$

Consequently the value of the integral is

$$\frac{4x^{3/4}}{3} + 2x^{1/2} - 4x^{1/4} + 4\tan^{-1}(x^{1/4}) - 2\ln(1+x^{1/2}) + C.$$

Example 7. Find $\int \frac{dx}{x^{1/3}+x^{1/2}}$

Solution Because 6 is the L.C.M. of 2 and 3, we set $u = x^{1/6}$, so that $u^6 = x$ and $6u^5\,du = dx$. We now have

$$\int \frac{dx}{x^{1/3}+x^{1/2}} = \int \frac{6u^5du}{(u^6)^{1/3}+(u^6)^{1/2}}$$

$$= \int \frac{6u^5du}{u^2+u^3} = \int \frac{6u^5du}{u^2(u+1)} = \int \frac{6u^3du}{1+u}$$

When the degree of the numerator is greater than or equal to the degree of the denominator, division is often helpful. By long division,

$$\frac{6u^3}{1+u} = 6u^2 - 6u + 6 + \frac{-6}{1+u}$$

$$\int \frac{6u^3du}{1+u} = \int\left(6u^2-6u+6+\frac{-6}{1+u}\right)du$$

$$= 2u^3 - 3u^2 + 6u - 6\,ln|1+u| + C$$

$$= 2(x^{1/6})^3 - 3(x^{1/6})^2 + 6x^{1/6} - 6\,ln|1+x^{1/6}| + C$$

$$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6ln(1+x^{1/6}) + C.$$

Example 8. Evaluate $I = \int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}$

Solution The least common multiple of the numbers 3 and 6 is 6, therefore we make the substitution :

$x = t^6$, $dx = 6t^5\,dt$,

$$\Rightarrow \quad I = 6\int \frac{(t^6+t^4+t)t^5}{t^6(1+t^2)}dt = 6\int \frac{t^5+t^3+1}{1+t^2}dt$$

$$= 6\int t^3\,dt + 6\int \frac{dt}{t^2+1} = \frac{3}{2}t^4 + 6\tan^{-1} t + C.$$

$$\therefore\quad I = \frac{3}{2}x^{\frac{2}{3}} + 6\tan^{-1}\sqrt[6]{x} + C.$$

6. Again, any algebraic expression containing integral powers of x along with fractional powers of an expression of the form a + bx is immediately reduced to rational functions, by the substitution $a + bx = t^m$, where m is the L.C.M. of the denominators of the several fractional powers.

Example 9. Evaluate $I = \int \frac{(2x-3)^{\frac{1}{2}}\,dx}{(2x-3)^{\frac{1}{3}}+1}$

Solution The integrand is a rational function of $\sqrt[6]{2x-3}$. Therefore we put $2x - 3 = t^6 \Rightarrow dx = 3t^5\,dt$

Also $(2x-3)^{\frac{1}{2}} = t^3$ and $(2x-3)^{\frac{1}{3}} = t^2$.

$$\Rightarrow\quad I = \int \frac{3t^8}{t^2+1}\,dt$$

$$= 3\int (t^6 - t^4 + t^2 - 1)\,dt + 3\int \frac{dt}{1+t^2}$$

$$= 3\frac{t^7}{7} - 3\frac{t^5}{5} + 3\frac{t^3}{3} - 3t + 3\tan^{-1} t + C.$$

$$= 3\left[\frac{1}{7}(2x-3)^{\frac{7}{6}} - \frac{1}{5}(2x-3)^{\frac{5}{6}} + \frac{1}{3}(2x-3)^{\frac{1}{2}}\right.$$

$$\left. -(2x-3)^{\frac{1}{6}} + \tan^{-1}(2x-3)^{\frac{1}{6}}\right] + C.$$

Example 10. Find $\int \frac{dx}{\sqrt{2x-1} - \sqrt[4]{2x-1}}$.

Solution The substitution $2x - 1 = z^4$ leads to an integral of the form

$$\int \frac{dx}{\sqrt{2x-1} - \sqrt[4]{2x-1}} = \int \frac{2z^3\,dz}{z^2 - z} = 2\int \frac{z^2 dz}{z-1}$$

$$= 2\int\left(z + 1 + \frac{1}{z-1}\right)dz = (z+1)^2 + 2\ln|z-1| + C$$

$$= \left(1 + \sqrt[4]{2x-1}\right)^2 + \ln\left(\sqrt[4]{2x-1} - 1\right)^2 + C.$$

Example 11. Evaluate $\int \frac{dx}{\sqrt[3]{x+1} + \sqrt{x+1}}$

Solution Let $I = \int \frac{dx}{\sqrt[3]{x+1} + \sqrt{x+1}}$

$$\Rightarrow\quad I = \int \frac{dx}{(x+1)^{1/3} + (x+1)^{1/2}}$$

The least common multiple of 2 and 3 is 6. So substitute $x + 1 = t^6$

$\Rightarrow dx = 6t^5\,dt$

$$\Rightarrow\quad I = \int \frac{6t^5 dt}{t^2 + t^3} = 6\int \frac{t^3 dt}{1+t}$$

$$\Rightarrow\quad I = 6\int t^2 - t + 1 - \frac{1}{1+t})\,dt$$

$$\Rightarrow\quad I = 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right) + C.$$

On substituting $t = (1 + x)^{1/6}$, we get

$$I = 6\left[\frac{(1+x)^{1/2}}{3} - \frac{(1+x)^{1/3}}{2}\right.$$

$$\left. + (1+x)^{1/6} - \ln\left((x+1)^{1/6} + 1\right)\right] + C.$$

Example 12. Evaluate $I = \int \frac{(x+1)^{1/2} + (x+1)^{3/4}}{x(x+1)^{1/4}}\,dx$

Solution The common denominator of 1/2 and 3/4 is 4.

Putting $x + 1 = t^4$ and $dx = 4t^3\,dt$, we have

$$I = \int \frac{t^2 + t^3}{(t^3 - 1)t}\cdot 4t^3\,dt$$

$$= 4\int \frac{1+t}{t^4 - 1}\,dt = 4\int \frac{dt}{(t-1)(t^2-1)}$$

$$= 2\int\left(\frac{1}{t-1} - \frac{t+1}{t^2+1}\right)dt$$

$$= 2\int \frac{dt}{t-1} - \int \frac{2t}{t^2-1}\,dt - \int \frac{dt}{t^2-1}$$

$$= 2\ln|t-1| - \ln(t^2-1) - \tan^{-1} t + C$$

$$= 2\ln|(x+1)^{1/4} - 1| - \ln[(x+1)^{1/2} + 1]$$

$$- 2\tan^{-1}(x+1)^{1/4} + C.$$

Example 13. $\int \frac{(x^{-7/6} - x^{5/6})\,dx}{x^{1/3}(x^2+x+1)^{1/2} - x^{1/2}(x^2+x+1)^{1/3}}$

Solution

$$I = \int \frac{(x^{-7/6} - x^{5/6})\,dx}{x^{7/6}.x^{1/3}(x^2+x+1)^{1/2} - x^{1/2}(x^2+x+1)^{1/3}}$$

$$=\int \frac{(1-x^2)}{x^{3/2}(x^2+x+1)^{1/2}-x^{5/3}(x^2+x+1)^{1/3}}$$

$$=\int \frac{-\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}+1\right)^{1/2}-\left(x+\frac{1}{x}+1\right)^{1/3}}$$

(putting $x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx=dt$)

$$=-\int \frac{dt}{(t+1)^{1/2}-(t+1)^{1/3}}$$

(putting $(t+1)=u^6$)

$$=-\int \frac{6u^5du}{u^3-u^2}$$

$$=-6\int \frac{u^3}{u-1}du, \text{ (putting } u-1=z)$$

$$=\int \frac{(z+1)^3}{z}dz=-6\int \frac{z^3+3z^2+3z+1}{z}dz$$

$$=6\int \left(z^2+3z+3+\frac{1}{z}\right)dz$$

$$=-6\left(\frac{z^3}{3}+\frac{3z^2}{2}+3z+\ln|z|\right)+C,$$

where $z=\left(x+\frac{1}{x}+1\right)^{1/6}-1$

7. If the integrand is a rational function of x and fractional powers of a linear fractional function of the form $\frac{ax+b}{cx+d}$, then rationalization of the integral is effected by the substitution $\frac{ax+b}{cx+d}=t^m$, where m is the L.C.M. of the denominators of the several fractional powers.

Example 14. Evaluate $\int \frac{dx}{\sqrt{(x+a)}+\sqrt{(x+b)}}$.

Solution Rationalizing the denominator, we have

$$\int \frac{dx}{\sqrt{(x+a)}+\sqrt{(x+b)}}\int \frac{\sqrt{(x+b)}-\sqrt{(x+a)}}{(x+b)-(x+a)}dx$$

$$=\int \frac{(x+b)^{1/2}-(x+a)^{1/2}}{b-a}dx$$

$$=\frac{1}{b-a}\left[\frac{2}{3}(x+b)^{3/2}-\frac{2}{3}(x+a)^{3/2}\right]$$

$$=\frac{2}{3}\frac{1}{(b-a)}[(x+b)^{3/2}-(x+a)^{3/2}]+C.$$

Example 15. Evaluate $\int \sqrt{\left(\frac{x+1}{x+2}\right)}\frac{dx}{x+3}$.

Solution Put $\frac{x+1}{x+2}=z^2$

$$\Rightarrow x=\frac{2z^2-1}{1-z^2}=-2+\frac{1}{1-z^2} \text{ and } x+3=\frac{2-z^2}{1-z^2}$$

$$\Rightarrow dx=\frac{2zdz}{(1-z^2)^2}$$

$$\therefore \quad I=\int z.\frac{2zdz}{(1-z^2)^2}\frac{(1-z^2)}{(2-z^2)}=\int \frac{2z^2dz}{(2-z^2)(1-z^2)}$$

$$=\int \left(\frac{2}{1-z^2}-\frac{4}{2-z^2}\right)dz$$

$$=\ln\left|\frac{1+z}{1-z}\right|-\sqrt{2}\ln\left|\frac{\sqrt{2}+z}{\sqrt{2}-z}\right|+C,$$

where $z=\sqrt{\left(\frac{x+1}{x+2}\right)}$.

Example 16. Evaluate $\int \sqrt{\frac{(1-\sin x)(2-\sin x)}{(1+\sin x)(2+\sin x)}}dx$

Solution We have $\sqrt{\frac{1-\sin x}{1+\sin x}}=\frac{\cos x}{1+\sin x}$

$$\therefore \quad I=\int \left(\frac{\cos x}{1+\sin x}\cdot\sqrt{\frac{2-\sin x}{2+\sin x}}\right)dx$$

Let $1+\sin x=y$

$\Rightarrow \cos x\, dx=dy$

$$\frac{2-\sin x}{2+\sin x}=\frac{2-(y-1)}{2+(y-1)}=\frac{3-y}{1+y}$$

$$I=\int \left(\frac{1}{y}\cdot\sqrt{\frac{3-y}{1+y}}\right)dy$$

Put $\frac{3-y}{1+y}=t^2 \Rightarrow 3-y=t^2+t^2y$

$$\Rightarrow\ y(1+t^2) = 3 - t^2 \ \Rightarrow\ y = \frac{3-t^2}{1+t^2}\ dy$$

$$= \frac{(t^2+1)(-2t)-(3-t^2)2t}{(1+t^2)^2} = \frac{-2t-6t}{(1+t^2)^2}\,dt$$

$$= \frac{-8t}{(1+t^2)^2}\,dt$$

$$\therefore\quad I = \int\left(\frac{1+t^2}{3-t^2}\cdot\frac{t(-8t)}{(1+t^2)^2}\right)dt = -8\int\frac{t^2}{(3-t^2)(1+t^2)}\,dt$$

$$= 8\int\frac{t^2}{(t^2-3)(t^2+1)}\,dt$$

Let $\dfrac{t^2}{(t^2-3)(t^2+1)} = \dfrac{A}{(t^2-3)} + \dfrac{B}{(t^2+1)}$

$t^2 = A(t^2+1) + B(t^2-3)$

$$A + B = 1,\ -4B = -1 \Rightarrow B = \frac{1}{4},\ A = \frac{3}{4}$$

$$\frac{t^2}{(t^2-3)(t^2+1)} = \frac{3}{4(t^2-3)} + \frac{1}{4(t^2+1)}$$

Now, $I = 8\cdot\dfrac{1}{4}\left[\displaystyle\int\frac{3}{t^2-3}\,dt + \int\frac{dt}{t^2+1}\right]$

$$= 2\left[3\cdot\frac{1}{2\sqrt{3}}\ln\frac{t-\sqrt{3}}{t+\sqrt{3}} + \tan^{-1}(t)\right]$$

$$= \sqrt{3}\ln\frac{t-\sqrt{3}}{t+\sqrt{3}} + 2\tan^{-1}(t) + C.$$

Example 17. Evaluate $I = \displaystyle\int\frac{2}{(2-x)^2}\sqrt[3]{\frac{2-x}{2+x}}\,dx.$

Solution The integrand is a rational function of x and the expression $\sqrt[3]{\dfrac{2-x}{2+x}}$, therefore let us introduce the substitution

$$\sqrt[3]{\frac{2-x}{2+x}} = t \ \Rightarrow\ \frac{2-x}{2+x} = t^3,$$

$$\Rightarrow\ x = \frac{2-2t^3}{1+t^3} \ \Rightarrow\ 2 - x = \frac{4t^3}{1+t^3}$$

$$\Rightarrow\ dx = \frac{-12t^2}{(1+t^3)^2}\,dt.$$

Hence, $I = -\displaystyle\int\frac{2(1+t^3)^2 t.12t^2}{16t^6(1+t^3)^2}\,dt$

$$= -\frac{3}{2}\int\frac{dt}{t^3} = \frac{3}{4t^2} + C.$$

Returning to x, we get $I = \dfrac{3}{4}\sqrt[3]{\left(\dfrac{2+x}{2-x}\right)^2} + C.$

Example 18. Evaluate $I = \displaystyle\int\frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$

Solution Since $\sqrt[4]{(x-1)^3(x+2)^5}$ $= (x-1)(x+2)\sqrt[4]{\dfrac{x+2}{x-1}}$, the integrand is a rational function of x and $\sqrt[4]{\dfrac{x+2}{x-1}}$; therefore let us introduce the substitution

$$\sqrt[4]{\frac{x+2}{x-1}} = t \ \Rightarrow\ \frac{x+2}{x-1} = t^4,$$

$$\Rightarrow\ x = \frac{t^4+2}{t^4-1} \ \Rightarrow\ x - 1 = \frac{3}{t^4-1}$$

$$\Rightarrow\ x + 2 = \frac{3t}{t^4-1} \Rightarrow dx = \frac{-12t^3}{(t^4-1)^2}\,dt.$$

Hence, $I = -\displaystyle\int\frac{(t^4-1)(t^4-1)12t^3dt}{3.3t^4t(t^4-1)^2}$

$$= -\frac{4}{3}\int\frac{dt}{t^2} = \frac{4}{3t} + C.$$

Returning to x, we obtain $I = \dfrac{4}{3}\sqrt[4]{\dfrac{x-1}{x+2}} + C.$

Alternative :

Here we put $\dfrac{x-1}{x+2} = t$

$$\therefore\quad \frac{3}{(x+2)^2}\,dx = dt \ \text{ or } \ \frac{dx}{(x+2)^2} = \frac{dt}{3}$$

$$I = \int\frac{dx}{\sqrt[4]{\left(\dfrac{x-1}{x+2}\right)^3(x+2)^8}}$$

$$= \int\frac{dx}{(x+2)^2\left(\dfrac{x-1}{x+2}\right)^{3/4}} = \int\frac{dt}{3(t)^{3/4}} = \frac{1}{3}\int(t)^{-3/4}\,dt$$

$$= \frac{1}{3}\int\frac{(t)^{1/4}}{(1/4)} + C = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$$

Note: The integrals of the form $\int \frac{dx}{\sqrt[n]{(x-a)^p(x-b)^q}}$, where $p+q=2n$ are solved more comfortably using the substitution $\frac{x-a}{x-b}=t$, as shown above.

Example 19. Evaluate $\int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}}$.

Solution $I=\int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}}$

$$= \int \frac{dx}{\left(\frac{x+3}{x-4}\right)^{15/16}(x-4)^2}$$

Put $\frac{x+3}{x-4}=t \Rightarrow \left(\frac{(x-4)-(x+3)}{(x-4)^2}\right)dx = dt$

$$\Rightarrow \frac{dx}{(x-4)^2} = \frac{dt}{-7}$$

So, $I = \frac{-1}{7}\int \frac{dt}{t^{15/16}} = \frac{-1}{7}\int t^{-15/16}dt = \frac{-16}{7}t^{1/16}+C$

$$= \frac{-16}{7}\left(\frac{x+3}{x-4}\right)^{1/16}+C.$$

Practice Problems

P

1. Evaluate the following integrals :

(i) $\int \frac{\sqrt{2x+1}}{x^2}dx$ (ii) $\int \frac{x\,dx}{(a+bx)^{1/2}}$

(iii) $\int \sqrt{\frac{x+a}{x+b}}\,dx$

2. Evaluate the following integrals :

(i) $\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x^5}-\sqrt[6]{x^7}}dx$

(ii) $\int \frac{x^{-2/3}}{2x^{1/3}+(x-1)^{1/3}}dx$

(iii) $\int \frac{dx}{x\left(2+\sqrt[3]{\frac{x-1}{x}}\right)}$

3. Evaluate the following integrals :

(i) $\int \frac{1}{x\sqrt{(1+x^6)}}dx$ (ii) $\int \frac{x\,dx}{(px+q)^{3/2}}$

(iii) $\int \frac{x^5}{\sqrt{(1+x^2)}}dx$ (iv) $\int \frac{dx}{x^3\sqrt{x^2+1}}$

4. Evaluate the following integrals :

(i) $\int \frac{dx}{(1+x)^{3/2}+(1+x)^{1/2}}$

(ii) $\int \frac{dx}{\sqrt[4]{5-x}+\sqrt{5-x}}$

(iii) $\int \frac{dx}{\sqrt{(x+2)}+\sqrt[4]{(x+2)}}$

(iv) $\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}dx$

1.18 INTEGRALS OF THE TYPE $\int \frac{dx}{P\sqrt{Q}}$

Where P and Q are linear or quadratic expressions

1. Integrals of the form

$$\int \frac{dx}{(ax+b)\sqrt{(cx+d)}},\text{ where } (a\neq 0, c\neq 0)$$

To find the integral $\int \frac{dx}{P\sqrt{Q}}$, where P and Q are linear algebraic expressions of x, we put $Q=t^2$.

Thus, to integrate $\int \frac{dx}{(ax+b)\sqrt{(cx+d)}}$

Put $cx+d=t^2$

$\therefore$ $c\,dx = 2t\,dt$.

The integral then reduces to

$$\frac{2}{c}\int \frac{t\,dt}{\left(a\frac{t^2-d}{c}+b\right).t} = 2\int \frac{dt}{at^2+(bc-ad)},$$

which can be easily evaluated.

Example 1. Evaluate $\int \frac{dx}{(2x+1)\sqrt{(4x+3)}}$.

Solution Put $4x+3=t^2$, so that $4dx = 2t\,dt$ and

$$(2x+1)=\frac{2(t^2-3)}{4}+1=\frac{t^2-3}{2}+1=\frac{t^2-1}{2}$$

$$\therefore \int \frac{dx}{(2x+1)\sqrt{(4x+3)}} = \int \frac{\frac{1}{2}tdt}{\frac{1}{2}(t^2-1)t} = \int \frac{dt}{(t^2-1)}$$

$$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\ln\frac{\sqrt{(4x+3)}-1}{\sqrt{(4x+3)}+1} + C.$$

Example 2. Evaluate $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$, $\beta > \alpha$.

Solution Put $x - \alpha = t^2$

$\Rightarrow dx = 2t\,dt$ and $\beta - x = \beta - \alpha - t^2$

$$\therefore \quad I = \int \frac{2tdt}{\sqrt{\{t^2(\beta-\alpha-t^2)\}}} = 2\int \frac{dt}{\sqrt{(k^2-t^2)}},$$

where $k^2 = \beta - \alpha$,

$$= 2\sin^{-1}\frac{t}{k} + C = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + C$$

Example 3. Evaluate $\int \frac{x^2dx}{(x-1)\sqrt{(x+2)}}$.

Solution Put $(x+2) = t^2$, so that $dx = 2t\,dt$. Also $x = t^2 - 2$.

$$\therefore \int \frac{x^2dx}{(x-1)\sqrt{(x+2)}} = \int \frac{(t^2-2)^2.2t\,dt}{(t^2-3).t}$$

$$= 2\int \frac{t^4-4t^2+4}{t^2-3}dt$$

$$= 2\int [(t^2-1) + \frac{1}{t^2-3}]\,dt,$$

dividing the numerator by the denominator

$$= 2\left[\frac{1}{3}t^3 - t + \frac{1}{2\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|\right] + C$$

$$= 2\left[\frac{(x+2)^{3/2}}{3} - \sqrt{(x+2)} + \frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{(x+2)}+\sqrt{3}}\right|\right] + C.$$

2. Integrals of the form $\int \frac{dx}{(px^2+qx+r)\sqrt{ax+b}}$

We now consider the integrals $\int \frac{dx}{P\sqrt{Q}}$, where P is a quadratic algebraic expression of x and Q is linear. To find the integral, we put $Q = t^2$.

The substitution $(ax + b) = t^2$ transforms the integral

$\int \frac{dx}{(px^2+qx+r)\sqrt{ax+b}}$ into an integral of the form

$\int \frac{dt}{\ell t^4 + mt^2 + n}$, which can be easily evaluated.

Example 4. Evaluate $\int \frac{dx}{x^2\sqrt{(x+1)}}$.

Solution Put $(x + 1) = t^2$, so that $dx = 2t\,dt$. Also $x = t^2 - 1$.

$$\therefore \int \frac{dx}{x^2\sqrt{(x+1)}} = \int \frac{2t\,dt}{(t^2-1)^2.t} = 2\int \frac{dt}{(t+1)^2(t-1)^2}$$

$$= \int \frac{1}{2}\left[\frac{1}{(t+1)^2} + \frac{1}{(t+1)} + \frac{1}{(t-1)^2} - \frac{1}{(t-1)}\right]dt,$$

by partial fractions

$$= \frac{1}{2}\int \frac{dt}{(t+1)^2} + \frac{1}{2}\int \frac{dt}{(t+1)} + \frac{1}{2}\int \frac{dt}{(t-1)^2} - \frac{1}{2}\int \frac{dt}{(t-1)}$$

$$= -\frac{1}{2(t+1)} + \frac{1}{2}\ln|t+1| - \frac{1}{2(t-1)} - \frac{1}{2}\ln|t-1| + C$$

$$= -\frac{1}{2}\left[\frac{1}{(t+1)} + \frac{1}{(t-1)}\right] + \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + C$$

$$= -\frac{\sqrt{(x+1)}}{x} + \frac{1}{2}\ln\left|\frac{\sqrt{(x+1)}+1}{\sqrt{(x+1)}-1}\right| + C.$$

Example 5. Evaluate $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx.$

Solution Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

Putting $x + 1 = t^2$, and $dx = 2t\,dt$, we get

$$I = \int \frac{(t^2+1)\,2t\,dt}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}$$

$$\Rightarrow \quad I = 2\int \frac{(t^2+1)}{t^4+t^2+1}\,dt = 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt$$

$$\Rightarrow \quad I = 2\int \frac{du}{u^2+\left(\sqrt{3}\right)^2} \quad \text{where } t - \frac{1}{t} = u.$$

$$\Rightarrow\ I=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+C=\frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{t-\frac{1}{t}}{\sqrt{3}}\right\}+C$$

$$\Rightarrow\ I=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{t\sqrt{3}}\right)+C$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{x}{\sqrt{3(x+1)}}\right\}+C.$$

3. Integrals of the form

$$\int\frac{dx}{(px+q)\sqrt{(ax^2+bx+c)}}$$

Here, we consider the integrals $\int\frac{dx}{P\sqrt{Q}}$, where Q is a quadratic algebraic expression of x and P is linear.

To evaluate $\int\frac{dx}{(px+q)\sqrt{(ax^2+bx+c)}}$,

we put $P=\frac{1}{t}$ i.e. $px+q=\frac{1}{t}$.

so that $p\,dx=-\frac{dt}{t^2}$ and $x=\frac{1}{p}\left(\frac{1}{t}-q\right)$

The given integral then reduces to

$$-\frac{1}{p}\int\frac{dt}{t^2\frac{1}{t}\sqrt{\left\{\frac{a}{p^2}\left(\frac{1}{t}-q\right)^2+\frac{b}{p}\left(\frac{1}{t}-q\right)+c\right\}}}$$

$$-\frac{1}{p}\int\frac{dt}{\sqrt{\left\{\frac{a}{p^2}(1-qt)^2+\frac{bt}{p}(1-qt)+ct^2\right\}}}$$

which when simplified takes up the form $-\int\frac{dt}{\sqrt{(At^2+Bt+C)}}$, which can be integrated easily.

Example 6. Evaluate $\int\frac{dx}{(1+x)\sqrt{(1-x^2)}}$.

Solution Put $(1+x)=1/t$, so that $dx=-(1/t^2)\,dx$.

Also $x=(1/t)-1$.

$$\therefore\ \int\frac{dx}{(1+x)\sqrt{(1-x^2)}}=\int\frac{-(1/t^2)dt}{(1/t)\sqrt{[1-\{(1/t)-1\}^2]}}$$

$$=-\int\frac{dt}{\sqrt{[t^2-(1-t)^2]}}=-\int\frac{dt}{\sqrt{(2t-1)}}$$

$$=-\frac{1}{2}\int(2t-1)^{-1/2}.\,2\,dt=-\sqrt{(2t-1)}\,)+C$$

$$=-\sqrt{\left[\frac{2}{1+x}-1\right]}+C=-\sqrt{\left(\frac{1-x}{1+x}\right)}+C.$$

Example 7. Integrate $\int\frac{dx}{(2x+3)\sqrt{(x^2+3x+2)}}$.

Solution Put $2x+3=\frac{1}{z}$. $\Rightarrow 2\,dx=-\frac{1}{z^2}\,dz$

$$\Rightarrow\ dx=-\frac{1}{2}\frac{dz}{z^2}$$

and $x=\frac{1}{2}\left(\frac{1}{z}-3\right)$, $\quad z=\frac{1}{2x+3}$

$$\therefore\ I=-\frac{1}{2}\int\frac{dz}{z^2\frac{1}{z}\sqrt{\left\{\frac{1}{2^2}\left(\frac{1}{z}-3\right)^2+\frac{3}{2}\left(\frac{1}{z}-3\right)+2\right\}}}$$

$$=-\int\frac{dz}{\sqrt{(1-z^2)}}$$

$$=-\sin^{-1}z+C=-\sin^{-1}\left(\frac{1}{2x+3}\right)+C.$$

Alternative :

$$I=\int\frac{dx}{(2x+3)\sqrt{\left\{\frac{1}{4}(4x^2+12x+8)\right\}}}$$

$$=\int\frac{dx}{(2x+3)\frac{1}{2}\sqrt{(2x+3)^2-1)}}$$

Put $2x+3=z$

$$\Rightarrow 2dx=dz\Rightarrow dx=\frac{1}{2}dz.$$

$$\therefore\ I=\frac{dz}{z\sqrt{(z^2-1)}}=\sec^{-1}z+C'=\sec^{-1}(2x+3)+C'$$

Although apparently the forms of the two results are different, it can be easily shown (by using the properties of inverse trigonometric functions) that one differs from the other by a constant.

Example 8. Evaluate $\int \frac{dx}{(x-a)^{3/2}(x+a)^{1/2}}$.

Solution $I = \int \frac{dx}{(x-a)\sqrt{(x^2-a^2)}}$

Put $x - a = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\, dt.$

Also, $x = \frac{1}{t} + a = \frac{1+at}{t}$

$\Rightarrow$ $x^2 - a^2 = \frac{(1+at)^2 - a^2t^2}{t^2} = \frac{1+2at}{t^2}$

$$\Rightarrow \quad I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\cdot\frac{\sqrt{(1+2at)}}{t}}$$

$$= -\int \frac{dt}{\sqrt{(1+2at)}} = -\frac{1}{2a}2\sqrt{(1+2at)} + C$$

$$= -\frac{1}{a}\sqrt{\left(1+\frac{2a}{x-a}\right)} + C = -\frac{1}{a}\sqrt{\left(\frac{x+a}{x-a}\right)} + C.$$

4. Integrals of the form

$$\int \frac{\mathbf{dx}}{(\mathbf{ax^2+b})\sqrt{(\mathbf{cx^2+d})}}$$

Here we have integrals of the form $\int \frac{dx}{P\sqrt{Q}}$, where P and Q are pure quadratic expressions. It means that they do not contain terms of x, i.e. they are of the type $ax^2 + b$ and $cx^2 + d$. In this case put $x = 1/t$ and then the expression under the radical should be put equal to z^2.

Example 9. Evaluate $\int \frac{dx}{(x^2-1)\sqrt{(x^2+1)}}$.

Solution Put $x = 1/t$, so that $dx = -(1/t^2)\, dt$.

$$\therefore \quad I = \int \frac{dx}{(x^2-1)\sqrt{(x^2+1)}}$$

$$= \int \frac{-(1/t^2)dt}{\{(1/t^2)-\}\sqrt{\{(1/t^2)+1\}}}$$

$$= -\int \frac{t\,dt}{(1-t^2)\sqrt{(1+t^2)}}$$

Now put $1 + t^2 = z^2$ so that $t\,dt = z\,dz$. Then

$$I = -\int \frac{d\,dz}{[1-(z^2-1)]z} = \int \frac{dz}{2-z^2} = \int \frac{dz}{z^2-2}$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| = \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{(1+t^2)}-\sqrt{2}}{\sqrt{(1+t^2)}+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{1+(1/x^2)}-\sqrt{2}}{\sqrt{1+(1/x^2)}+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{1+(1/x^2)}-x\sqrt{2}}{\sqrt{1+(1/x^2)}+x\sqrt{2}}\right| + C.$$

Example 10. Show that

$$\int \frac{dt}{(x^2+a^2)\sqrt{x^2+b^2}}$$

$$= \frac{1}{a\sqrt{b^2-a^2}}\cos^{-1}\left(\frac{a}{b}\sqrt{\frac{x^2+b^2}{x^2+a^2}}\right) - C,$$

if $b^2 > a^2$, C being the constant of integration.

Solution Put $\frac{x^2+b^2}{x^2+a^2} = t^2$, so that $x^2 = \frac{b^2-a^2t^2}{t^2-1}$

$\therefore$ $2x\,dx = -[2(b^2-a^2)t/(t^2-1)^2]\,dt.$

Also, $x^2 + a^2 = \frac{b^2-a^2t^2}{t^2-1} + a^2 = (b^2-a^2)/(t^2-1),$

$x^2 + b^2 = \frac{b^2-a^2t^2}{t^2-1} + b^2$

$= (b^2-a^2)\, t^2/(t^2-1).$

$$\therefore \int \frac{dx}{(x^2+a^2)\sqrt{x^2+b^2}}$$

$$= -\frac{1}{\sqrt{(b^2-a^2)}}\int \frac{dt}{\sqrt{b^2-a^2t^2}},$$

$$= \frac{1}{a\sqrt{(b^2-a^2)}}\cos^{-1}(at/b) + C,$$

$$= \frac{1}{a\sqrt{b^2-a^2}} \cos^{-1}\left(\frac{a}{b}\sqrt{\frac{x^2+b^2}{x^2+a^2}}\right) + C.$$

5. Integrals of the form

$$\int \frac{dx}{(px^2+qx+r)\sqrt{(ax^2+bx+c)}}.$$

Here we shall consider two cases only.

Case I: If px^2+qx+r breaks up into two linear factors of the forms $(mx+n)$ and $(m'x+n')$, then we resolve $1/\{(mx+n)(m'x+n)\}$ into two partial fractions and the integral then transforms into the sum (or difference) of two integrals.

Case II: If px^2+qx+r is a perfect square, say, $(lx+m)^2$, then the substitution is $lx+m=1/t$.

Example 11. Evaluate

$$I = \int \frac{dx}{2x\sqrt{1-x}\sqrt{(2-x)+\sqrt{1-x}}}.$$

Solution Here, $I = \int \frac{dx}{2x\sqrt{1-x}\sqrt{(2-x)+\sqrt{1-x}}}$

$$I = \int \frac{2t\,dt}{2(1-t^2).t\sqrt{1+t^2+t}}$$

$$= -\int \frac{dt}{(1-t^2)\sqrt{t^2+t+1}}$$

$$= \int \frac{dt}{(t-1)(t+1)\sqrt{t^2+t+1}}$$

$$= \frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)\frac{dt}{\sqrt{t^2+t+1}}$$

$$\because \quad \frac{1}{(t-1)(t+1)} = \frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)$$

$$= \frac{1}{2}\int \frac{1}{(t-1)\sqrt{t^2+t+1}}.dt - \frac{1}{2}\int \frac{1.dt}{(t+1)\sqrt{t^2+t+1}}$$

$$\text{Let, } I = \frac{1}{2}I_1 - \frac{1}{2}I_2 \qquad \text{...(1)}$$

where $I_1 = \int \frac{dt}{(t-1)\sqrt{t^2+t+1}}$

and $I_2 = \int \frac{dt}{(t+1)\sqrt{t^2+t+1}}$

Put $(t-1) = \frac{1}{z}$ for I_1,

$$I_1 = \int \frac{-1/z^2 dz}{\frac{1}{z}\sqrt{\left(1+\frac{1}{z}\right)^2+\left(1+\frac{1}{z}\right)+1}}$$

$$= -\int \frac{dz}{\sqrt{\left(z+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\ln\left|\left(z+\frac{3}{2}\right)+\sqrt{z^2+3z+3}\right| \qquad \text{...(2)}$$

For I_2, put $(t+1) = \frac{1}{u}$

$$I_2 = -\int \frac{du}{\sqrt{\left(u-\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$= -\ln\left|\left(u-\frac{1}{2}\right)+\sqrt{u^2-u+1}\right| \qquad \text{...(3)}$$

$$\therefore \quad I = -\frac{1}{2}\left\{\ln\left(z\frac{3}{2}+\sqrt{z^2+3z+3}\right)\right\}$$

$$+\frac{1}{2}\ln\left|\left(u-\frac{1}{2}\right)+\sqrt{u^2-u+1}\right|+C,$$

where $z = \frac{1}{\sqrt{1-x}-1}$ and $u = \frac{1}{\sqrt{1-x}+1}$.

6. Integrals of the form

$$\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+c}}, \text{ where } r \in N$$

Here, we substitute, $x-k = \frac{1}{t}$.

Example 12. Evaluate $\int \frac{dx}{(x-3)^3\sqrt{x^2-6x+10}}$.

Solution Put $x-3 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$

$$\int \frac{dx}{(x-3)^3\sqrt{x^2-6x+10}}$$

$$= \int \frac{-1/t^2 dt}{1/t^3\sqrt{(1/t+3)^2-6(1/t+3)+10}}$$

$$= -\int \frac{t^2 dt}{\sqrt{1+t^2}} = \int \frac{dt}{\sqrt{1+t^2}} - \int \sqrt{1+t^2}\, dt$$

$$= \ln|t+\sqrt{1+t^2}| - \frac{t}{2}\sqrt{1+t^2} - \frac{1}{2}\ln|t+\sqrt{1+t^2}| + C$$

$$= \frac{1}{2}\ln|t+\sqrt{1+t^2}| - \frac{t}{2}\sqrt{1+t^2} + C$$

$$= \frac{1}{2}\left[\ln\left|\frac{1+\sqrt{x^2-6x+10}}{|x-3|}\right| - \frac{\sqrt{x^2-6x+10}}{|x-3|^2}\right] + C.$$

Example 13. Integrate

$$\int \frac{dx}{(x^2-2x+1)\sqrt{(x^2-2x+3)}}.$$

Solution $I = \int \frac{dx}{(x-1)^2\sqrt{\{(x-1)^2+2\}}}$

$$= \int \frac{dz}{z^2\sqrt{(z^2+2)}}, \text{ putting } z = x-1.$$

$$= \int \frac{\sqrt{2}\sec^2\theta\, d\theta}{2\tan^2\theta \cdot \sqrt{2}\sec\theta}, \text{ putting } x = \sqrt{2}\tan\theta.$$

$$= \frac{1}{2}\int \operatorname{cosec}\theta \cot\theta\, d\theta = \frac{1}{2}\operatorname{cosec}\theta + C.$$

Since $\tan\theta = \frac{1}{\sqrt{2}} z$, $\operatorname{cosec}\theta = \frac{\sqrt{(z^2+2)}}{z}$

$$\therefore \quad I = \frac{1}{2}\frac{\sqrt{(z^2+2)}}{z} + C = \frac{1}{2}\frac{\sqrt{(x^2-2x+3)}}{x-1} + C.$$

7. Integrals of the form

$$\int \frac{(ax+b)dx}{(cx+d)\sqrt{px^2+qx+r}}.$$

Here we put $(ax + b) = A(cx + d) + B$, and find the values of A and B by comparing the coefficients of x and constant term.

Example 14. Evaluate $\int \frac{(4x+7)}{(x+2)\sqrt{x^2+4x+8}}$.

Solution Let $4x+7 = A(x+2)+B$

$\Rightarrow A = 4, B = -1$

So, $I = 4\int \frac{dx}{\sqrt{x^2+4x+8}} - \int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

$$= 4\ell n\left(x+2+\sqrt{x^2+4x+8}\right)$$

$$+\frac{1}{2}\ell n\left|\frac{1}{x+2}+\sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right| + C$$

8. Integrals of the form

$$\int \frac{ax^2+bx+c}{(dx+e)\sqrt{px^2+qx+r}}\, dx$$

Here, we write,

$ax^2+bx+c = A(dx+e)(2px+q)+B(dx+e)+C$

where A, B and C are constants which can be obtained by comparing the coefficients of like terms on both sides.

Example 15. Evaluate $\int \frac{2x^2+5x+9}{(x+1)\sqrt{x^2+x+1}}\, dx$

Solution Let

$2x^2+5x+9 = A(x+1)(2x+1)+B(x+1)+C$

$\Rightarrow 2x^2+5x+9 = x^2(2A)+x(3A+B)+(A+B+C)$

$\Rightarrow A = 1, B = 2, C = 6$

by comparing the coefficients of like terms on both sides.

Thus, $\int \frac{2x^2+5x+9}{(x+1)\sqrt{x^2+x+1}}\, dx$

$$= \int \frac{(x+1)(2x+1)}{(x+1)\sqrt{x^2+x+1}}\, dx$$

$$+2\int \frac{x+1}{(x+1)\sqrt{x^2+x+1}}\, dx + 6\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

$$= \int \frac{2x+1}{\sqrt{x^2+x+1}}\, dx + 2\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$+6\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

$$= \int \frac{du}{\sqrt{u}} + 2\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{3}{4}\right)}} + 6\int \frac{-dt}{\sqrt{t^2-t+1}}$$

where $u = x^2+x+1$ and $\frac{1}{t} = x+1$

$$= 2\sqrt{x^2+x+1} + 2\ln|(x+1/2)+\sqrt{x^2+x+1}|$$

$$-6\int \frac{dt}{\sqrt{(t-1/2)^2+3/4}}$$

$$= 2\sqrt{x^2+x+1} + |2\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|$$

$$-6\ln\left|\left(t+\frac{1}{2}\right)+\sqrt{t^2-t+1}\right|+C$$

$$= 2\sqrt{x^2+x+1} + 2\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|$$

$$-6\ln\left|\frac{1-x+\sqrt{x^2+x+1}}{2(x+1)}\right|+C.$$

Practice Problems Q

1. Evaluate the following integrals :

(i) $\displaystyle\int \frac{dx}{(2x+1)\sqrt{(4x+3)}}$

(ii) $\displaystyle\int \frac{1}{(x-3)\sqrt{x+1}}\,dx$

2. Evaluate the following integrals :

(i) $\displaystyle\int \frac{dx}{x\sqrt{(9x^2+4x+1)}}$

(ii) $\displaystyle\int \frac{dx}{(1+x)\sqrt{(1+x-x^2)}}$

(iii) $\displaystyle\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

(iv) $\displaystyle\int \frac{2x\,dx}{(1-x^2)\sqrt{(x^4-1)}}$

3. Evaluate the following integrals :

(i) $\displaystyle\int \frac{dx}{(x^2+1)\sqrt{x}}$

(ii) $\displaystyle\int \frac{dx}{(x^2+5x+6)\sqrt{x+1}}$

(iii) $\displaystyle\int \frac{dx}{(x^2-4)\sqrt{x+1}}$

4. Evaluate the following integrals :

(i) $\displaystyle\int \frac{dx}{(3+4x^2)(4-3x^2)^{1/2}}$

(ii) $\displaystyle\int \frac{dx}{(2x^2+1)\sqrt{1-x^2}}$ (iii) $\displaystyle\int \frac{\sqrt{1+x^2}\,dx}{2+x^2}$

5. Evaluate the following integrals :

(i) $\displaystyle\int \frac{dx}{(x^2+4x)\sqrt{4-x^2}}$

(ii) $\displaystyle\int \frac{dx}{(4x^2+4x+1)\sqrt{(4x^2+4x+5)}}$

(iii) $\displaystyle\int \frac{dx}{(x^2+2x+2)\sqrt{x^2+2x-4}}$

(iv) $\displaystyle\int \frac{dx}{(x+1)^3\sqrt{x^2+3x+2}}$

6. Evaluate the following integrals :

(i) $\displaystyle\int \frac{x\,dx}{(x^2-3x+2)\sqrt{x^2-4x+3}}$

(ii) $\displaystyle\int \frac{(x^2+1)dx}{(x^2+2x+2)\sqrt{(x+1)}}$

(iii) $\displaystyle\int \frac{(2x+3)dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}$

1.19 INTEGRATION OF A BINOMIAL DIFFERENTIAL

Integrals of the form $\int x^m(a+bx^n)^p\,dx$

Expressions of the form $x^m(a+bx^n)^p$, in which where m, n, p are rational numbers, are called Binomial Differentials.

The integral $\int x^m(a+bx^n)^p\,dx$ is expressed through elementary functions only in the following situations :

1. p is a positive integer. Then, the integrand is expanded by the formula of the Newton binomial.

2. p is a negative integer. Then we put $x = t^k$, where k is the L.C.M of the denominators of the fractions m and n.

3. $\dfrac{m+1}{n}$ is an integer. We put $a+bx^n = t^\alpha$, where α is the denominator of the fraction p.

4. $\dfrac{m+1}{n}+p$ is an integer. We put $a+bx^n = t^\alpha x^n$, where α is the denominator of the fraction p.

Example 1. Evaluate $I=\int \sqrt[3]{x}(2+\sqrt{x})^2\,dx.$

Solution $I=\int x^{\frac{1}{3}}(2+x^{\frac{1}{2}})^2\,dx.$

Here p = 2, i.e. an integer, hence we have

$$I=\int x^{\frac{1}{3}}(x+4x^{\frac{1}{2}}+4)\,dx=\int (x^{\frac{4}{3}}+4x^{\frac{5}{6}}+4x^{\frac{1}{3}})\,dx$$

$$=\frac{3}{7}x^{\frac{7}{3}}+\frac{24}{11}x^{\frac{11}{6}}+3x^{\frac{4}{3}}+C.$$

Example 2. Evaluate $\int x^{-\frac{2}{3}}\left(1+x^{\frac{2}{3}}\right)^{-1}.$

Solution Here, p = – 1, is a negative integer and m and n are rational numbers.

Put $x = t^3$

$\Rightarrow$ dx $3t^2$ dt

$$\text{So}\quad I=\int t^{-2}(1+t^2)^{-1}3t^2dt=\int\frac{3\,dt}{1+t^2}$$

$$=3\tan^{-1}(x^{1/3})+C.$$

Example 3. Evaluate $\int\frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx.$

Solution $I=\int x^{-\frac{2}{3}}(1+x^{\frac{1}{3}})^{\frac{1}{2}}dx.$

Here $m=-\frac{2}{3},\ n=\frac{1}{3},\ p=\frac{1}{2}$

$$\Rightarrow\quad \frac{m+1}{n}=\frac{\left(-\frac{2}{3}+1\right)}{\frac{1}{3}}=1,\ \text{i.e. an integer.}$$

Let us make the substitution $1+x^{\frac{1}{3}}=t^2$

$$\Rightarrow\quad \frac{1}{3}x^{-\frac{2}{3}}\,dx=2t\,dt.$$

Hence, $I=6\int t^2\,dt=2t^3+C=2\,(1+x^{\frac{1}{3}})^{\frac{3}{2}}+C.$

Example 4. Evaluate $\int x^{-11}(1+x^4)^{-\frac{1}{2}}\,dx.$

Solution Here $p=-\frac{1}{2}$ is a fraction,

$$\frac{m+1}{n}=\frac{-11+1}{4}$$

$$=-\frac{5}{2}\ \text{also a fraction,}$$

but $\frac{m+1}{n}+p=-\frac{5}{2}-\frac{1}{2}=-3$ is an integer.

We put $1+x^4=x^4t^2$. Hence,

$$x=\frac{1}{(t^2-1)^{\frac{1}{4}}}\ \Rightarrow\ dx=-\frac{t\,dt}{2(t^2-1)^{\frac{5}{4}}}$$

Substituting these expression into the integral, we obtain

$$I=-\frac{1}{2}\int(t^2-1)^{\frac{11}{4}}\left(\frac{t^2}{t^2-1}\right)^{-\frac{1}{2}}\frac{t\,dt}{2(t^2-1)^{\frac{5}{4}}}$$

$$=-\frac{1}{2}\int(t^2-1)^2dt=-\frac{t^5}{10}+\frac{t^3}{3}-\frac{t}{2}+C.$$

Returning to x, we get

$$I=-\frac{1}{10x^{10}}\sqrt{(1+x^4)^5}+\frac{1}{3x^6}\sqrt{(1+x^4)^3}$$

$$-\frac{1}{2x^3}\sqrt{1+x^4}+C.$$

Example 5. Evaluate $\int\frac{dx}{x^3\sqrt[3]{(1+x^3)}}.$

Solution We have m = –3, n = 3, p = –1/3.

Here, $\frac{m+1}{n}\neq$ integer, but

$\frac{m+1}{n}+p=-1$, (an integer).

$\therefore$ We put $1+x^3=z^3x^3$.

$$\Rightarrow\ x^3(z^3-1)=1,\quad \Rightarrow x=\frac{1}{(z^3-1)^{1/3}}.$$

$$\Rightarrow\ dx=-\frac{z^2}{(z^3-1)^{4/3}}dz.$$

The denominator $=x^4z=\frac{z}{(z^3-1)^{4/3}}.$

$$\therefore\ I=-\int z\,dz=-\frac{1}{2}z^2+C=-\frac{1}{2}\frac{(1+x^2)^{2/3}}{x^2}+C.$$

Practice Problems R

Evaluate the following integrals :

1. $\int \frac{x^5\,dx}{(1+x^3)^{1/2}}$
2. $\int \frac{(1-x^2)dx}{x^{1/2}(1+x^2)^{3/2}}$
3. $\int x^{-1}(1+x^{1/3})^{-3}\,dx$
4. $\int x^5\sqrt[3]{(1+x^3)^2}\,dx$
5. $\int \frac{dx}{x\sqrt[3]{1+x^5}}$
6. $\int \frac{\sqrt[3]{1+x^3}}{x^2}dx$
7. $\int \frac{dx}{x^{11}\sqrt{1+x^4}}$
8. $\int \sqrt[3]{1+\sqrt[4]{x}}\,dx$
9. $\int x^{1/4}(2+3x^2)^3 dx$
10. $\int x^{1/3}(1-2\sqrt{x})^3 dx$
11. $\int x^{3/4}(1+x^{7/8})^{1/2}dx$
12. $\int x(1+8x^3)^{1/3}dx$

1.20 EULER'S SUBSTITUTION

Let $R(x, \sqrt{a+2bx+cx^2})$, denote a rational algebraic function of x and $\sqrt{a+2bx+cx^2}$.

Integrals of the form $\int R(x, \sqrt{ax^2+bx+c})\,dx$ are calculated with the aid of one of the three Euler substitutions :

1. $\sqrt{ax^2+bx+c} = t \pm \sqrt{a}$ is $a > 0$,
2. $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$ if $c > 0$,
3. $\sqrt{ax^2+bx+c} = (x-\alpha)t$

if $ax^2+bx+c = a(x-\alpha)(x-\beta)$ i.e. if α is real root of the trinomial ax^2+bx+c.

The integrand in $\int R(x, \sqrt{ax^2+bx+c})\,dx$ can be made rational in several ways, which we consider in order :

1. Assume $\sqrt{ax^2+bx+c} = t - x\sqrt{a}$ ...(1)

Then $bx + c = t^2 - 2xt\sqrt{a}$

$\therefore$ $bdx = 2tdt - 2\sqrt{a}\,(xdt + tdx)$,

or $dx(b + 2t\sqrt{a}) = 2\,dt\,(t - x\sqrt{a}) = 2dt\sqrt{ax^2+bx+c}$

$$\therefore \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{2dt}{b+2t\sqrt{a}} \quad ...(2)$$

$$\text{Also} \quad x = \frac{t^2-c}{b+2t\sqrt{a}} \quad ...(3)$$

This substitution obviously renders the proposed expression rational.

When $b = 0$, we get $\frac{dx}{\sqrt{ax^2+c}} = \frac{dt}{t\sqrt{a}}$, and $x = \frac{t^2-c}{2t\sqrt{a}}$.

For example, to find $\int \frac{dx}{(p+qx)\sqrt{1+x^2}}$

Here $x = \frac{t^2-1}{2t}$, and $\frac{dx}{(p+qx)\sqrt{1+x^2}} = \frac{2dt}{qt^2+2pt-q}$

$$\therefore \int \frac{dx}{(p+qx)\sqrt{1+x^2}} = \frac{1}{\sqrt{p^2+q^2}}\ln\left(\frac{qz+p-\sqrt{p^2+q^2}}{qz+p+\sqrt{p^2+q^2}}\right)+C$$

When the coefficient a is negative the preceding method introduces imaginary numbers : we proceed to other substitution in which they are avoided.

2. Assume $\sqrt{ax^2+bx+c} = \sqrt{c} + tx$ ...(4)

Squaring both sides, we get immediately.

$$ax + b = 2t\sqrt{c} + xt^2$$

$\therefore$ $dx(a - t^2) = 2dt\,(\sqrt{c} + xt) = 2dt\sqrt{ax^2+bx+c}$.

$$\text{Hence} \quad \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{2dt}{a-t^2} \quad ...(5)$$

$$\text{And} \quad x = \frac{2t\sqrt{c}-b}{a-t^2} \quad ...(6)$$

This substitution also evidently renders the proposed expression rational, provided c be positive.

For example, to find

$$\int \frac{dx}{x\sqrt{1-x^2}}$$

Assume $\sqrt{1-x^2} = 1 - xt$, and we get

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{dt}{t} = \ln|t| + C = \ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)+C$$

3. Again, when the roots of $ax^2 + bx + c$ are real, there is another substitution.

For, let α and β be the roots, and the radical becomes of the form

$\sqrt{a(x-\alpha)(x-\beta)}$, or $\sqrt{a(x-\alpha)(\beta-x)}$,

according as the coefficient of x^2 is positive or negative.

In the former case, assume $\sqrt{x-\alpha} = t\sqrt{x-\beta}$, and we get $x = \dfrac{\alpha-\beta t^2}{1-t^2}$, hence $x-\beta = \dfrac{\alpha-\beta}{1-t^2}$,

$$\therefore \quad \frac{dx}{x-\beta} = \frac{2tdt}{1-t^2}.$$

Accordingly,

$$\frac{dx}{\sqrt{a(x-\alpha)(x-\beta)}} = \frac{dx}{t(x-\beta)\sqrt{a}} = \frac{2}{\sqrt{a}}\frac{dt}{1-t^2} \quad ...(7)$$

In the latter case, let $\sqrt{x-\alpha} = t\sqrt{\beta-x}$, and we get

$$x = \frac{\alpha+\beta t^2}{1+t^2}$$

and $$\frac{dx}{\sqrt{a(x-\alpha)(\beta-x)}} = \frac{2}{\sqrt{a}}\frac{dt}{1+t^2} \quad ...(8)$$

For example, the integral $\displaystyle\int \frac{dx}{(p+qx)\sqrt{1-x^2}}$

transforms into $\displaystyle\int \frac{2dt}{(p+q)t^2+p-q}$,

on putting $x = \dfrac{t^2-1}{t^2+1}$.

It may be observed that in the application of the foregoing methods it is advisable that the student should in each case select whichever method avoids the introduction of imaginary numbers.

Thus, as already observed, the first should be employed only when a is positive: in like manner, the second requires c to be positive; and the third, that the roots be real.

It is easily seen that when a and c are both negative, the roots must be real; for the expression

$$\sqrt{-ax^2+bx-c}$$

is imaginary for all real values of x unless $b^2 - 4ac$ is positive i.e., unless the roots are real.

Accordingly, the third method is always applicable when the other two fail.

From the preceding investigation it follows that the expression $R(x, \sqrt{ax^2+bx+c})\,dx$ can be always rationalized.

Example 1. Evaluate $I = \displaystyle\int \frac{dx}{1+\sqrt{x^2+2x+2}}$

Solution Here $a = 1 > 0$, therefore we make the substitution $\sqrt{x^2+2x+2} = t - x$.

Squaring both sides of this equality and reducing the similar terms, we get

$$2x + 2tx = t^2 - 2,$$

$$\Rightarrow \quad x = \frac{t^2-2}{2(1+t)} \Rightarrow dx = \frac{t^2+2t+2}{2(1+t)^2}dt$$

$$1 + \sqrt{x^2+2x+2} = 1 + t - \frac{t^2-2}{2(1+t)} = \frac{t^2+4t+4}{2(1+t)^2}.$$

Substituting into the integral, we obtain

$$I = \int \frac{2(1+t)(t^2+2t+2)}{(t^2+4t+4)2(1+t)^2}\,dt = \int \frac{(t^2+2t+2)dt}{(1+t)(1+2)^2}.$$

Now let us expand the obtained proper rational fraction into partial fractions :

$$\frac{t^2+2t+2}{(t+1)(t+2)^2} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{D}{(t+2)^2}.$$

We find : A = 1, B = 0, D = –2.

Hence,

$$\int \frac{t^2+2t+2}{(1+t)(1+2)^2}dt = \int \frac{dt}{t+1} - 2\int \frac{dt}{(t+2)^2}$$

$$= \ln|t+1| + \frac{2}{t+2} + C.$$

Returning to x, we get $I = \ln(x+1+\sqrt{x^2+2x+2})$

$$+ \frac{2}{x+2+\sqrt{x^2+2x+2}} + C.$$

Example 2. Evaluate $I = \dfrac{dx}{x+\sqrt{x^2-x+1}}$

Solution Since here $c = 1 > 0$, we can apply the second Euler substitution $\sqrt{x^2-x+1} = tx - 1$,

$$\Rightarrow \quad (2t-1)x = (t^2-1)x^2 \Rightarrow x = \frac{2t-1}{t^2-1}$$

$$\Rightarrow \quad dx = -2\frac{t^2-t+1}{(t^2-1)^2}\,dt$$

$$\Rightarrow \quad x + \sqrt{x^2-x+1} = \frac{t}{t-1}.$$

Substituting into I, we obtain an integral of a rational fraction $\int \frac{dx}{x+\sqrt{x^2-x+1}} = \int \frac{-2t^2-2t-2}{t(t-1)(t+1)^2}\,dt,$

$$\frac{-2t^2-2t-2}{t(t-1)(t+1)^2} = \frac{A}{t}+\frac{B}{t-1}+\frac{D}{(t+1)^2}+\frac{E}{t+1}.$$

We find $A = 2;\ B = -\frac{1}{2};\ D = -3;\ E = -\frac{3}{2}.$

Hence, $I = 2\int \frac{dt}{t} - \frac{1}{2}\int\frac{dt}{t-1} - 3\int\frac{dt}{(t+1)^3} - \frac{3}{2}\int\frac{dt}{t+1}$

$$= 2\ ln\,|t| - \frac{1}{2}\,ln|t-1| + \frac{3}{t+1} - \frac{3}{2}\,ln\,|t+1| + C,$$

where $t = \frac{\sqrt{x^2-x+1}+1}{x}.$

Example 3. Find $\int \frac{dx}{x\sqrt{x^2+x+2}}$

Solution Let $x^2+x+2 = (z-x)^2$. Then

$$x = \frac{z^2-2}{1+2z} \Rightarrow dx = \frac{2(z^2+z+2)dz}{(1+2z)^2},$$

$$\sqrt{x^2+x+2} = \frac{z^2+z+2}{1+2z}$$

and $\int \frac{dx}{x\sqrt{x^2+x+2}} = \int \frac{\frac{2(z^2+z+2)dz}{(1+2z)^2}}{\frac{z^2-2}{1+2z}\ \frac{z^2+z+2}{1+2z}}\,dz$

$$= 2\int\frac{dz}{z^2-2} = \frac{1}{\sqrt{2}}\ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{x^2+x+2}+x-\sqrt{2}}{\sqrt{x^2+x+2}+x+\sqrt{2}}\right| + C$$

Example 4. Find $\int \frac{xdx}{(5-4x-x^2)^{3/2}}.$

Solution Let $5-4x-x^2 = (5+x)(1-x) = (1-x)^2 z^2$.

Then $x = \frac{z^2-5}{1+z^2} \Rightarrow dx = \frac{12z\,dz}{(1+z^2)^2},$

$$\sqrt{5-4x-x^2} = (1-x)z = \frac{6z}{1+z^2}$$

and $\int \frac{x\,dx}{(5-4x-x^2)^{3/2}} = \int \frac{\frac{z^2-5}{1+z^2}\ \frac{12z}{(1+z^2)^2}}{\frac{216z^2}{(1+z^2)^3}}\,dz$

$$= \frac{1}{18}\int\left(1-\frac{5}{z^2}\right)dz$$

$$= \frac{1}{18}\left(z+\frac{5}{z}\right) + C = \frac{5-2x}{9\sqrt{5-4x-x^2}} + C.$$

Example 5. Evaluate $I = \int \frac{x\,dx}{(\sqrt{7x-10-x^2})}.$

Solution In this case a < 0 and c < 0 therefore neither the first, nor the second Euler substitution is applicable. But the quadratic trinomial $7x-10-x^2$ has real roots $\alpha = 2$, $\beta = 5$, therefore we use the third Euler substitution :

$$\sqrt{7x-10-x^2} = \sqrt{(x-2)(5-x)} = (x-2)\,t.$$

$$\Rightarrow\ 5-x = (x-2)\,t^2 \ \Rightarrow\ x = \frac{5+2t^2}{1+t^3}$$

$$\Rightarrow\ dx = -\frac{6t\,dt}{(1+t^2)^2}$$

Also $\quad (x-2)\,t = \left(\frac{5+2t^2}{1+t^2}-2\right)t = \frac{3t}{1+t^2}.$

Hence, $I = -\frac{6}{27}\int\frac{5+2t^2}{t^2}dt$

$$= -\frac{2}{9}\int\left(\frac{5}{t^2}+2\right)dt = -\frac{2}{9}\left(-\frac{5}{t}+2t\right)+C,$$

where $t = \frac{\sqrt{7x-10-x^2}}{x-2}.$

Example 6. Evaluate $\int \frac{dx}{\left(x+\sqrt{x^2-4}\right)^{5/3}}.$

Solution $I = \int \frac{dx}{\left(x+\sqrt{x^2-4}\right)^{5/3}}$

Put $x+\sqrt{x^2-4} = t \Rightarrow \left(1+\frac{x}{\sqrt{x^2-4}}\right)dx = dt$

Also, $x+\sqrt{x^2-4} = t \ \Rightarrow\ \sqrt{x^2-4} = t-x$

$$\Rightarrow\quad x = \frac{t^2+4}{2t}$$

$$\Rightarrow\ x^2-4 = \left(\frac{t^2+4}{2t}\right)^2 - 4$$

$$= \frac{t^4+16+8t^2-16t^2}{4t^2} = \left(\frac{t^2-4}{2t}\right)^2$$

So, $I = \int\left(\frac{t^2-4}{2t^2}\right)\frac{1}{t^{5/3}}dt$

$$= \frac{1}{2}\int t^{-5/3}dt - 2\int t^{-11/3}dt$$

$$= \frac{1}{2}\frac{t^{-2/3}}{-2/3} - 2\frac{t^{-8/3}}{-8/3} + C = \frac{3}{4}t^{-8/3}[1-t^2]+C,$$

where $t = \left(x+\sqrt{x^2-4}\right)$.

Example 7. Evaluate $\int \frac{dx}{x^2\left(x+\sqrt{1+x^2}\right)}$

Solution $\int \frac{x^{-3}\,dx}{\left(1+\sqrt{x^{-2}+1}\right)}$

Put $1+x^{-2}=t^2 \Rightarrow -2x^{-3}\,dx = 2t\,dt$

$$\Rightarrow \int \frac{t\,dt}{1+t} = \int\left(1-\frac{1}{1+t}\right)$$

$$= t - \ln(1+t) + C$$

$$= -\sqrt{1+\frac{1}{x^2}} - \ln\left(1+\sqrt{1+\frac{1}{x^2}}\right) + C$$

$$= \ln\left(\frac{x+\sqrt{1+x^2}}{x}\right) - \frac{\sqrt{x^2+1}}{x} + C.$$

Example 8. Evaluate $\int (x+\sqrt{1+x^2})^n\,dx$.

Solution Let $I = \int (x+\sqrt{1+x^2})^n\,dx$

Put $x+\sqrt{1+x^2} = t$ ...(1)

$$\Rightarrow \left(1+\frac{1}{2\sqrt{1+x^2}}.2x\right)dx = dt$$

$$\Rightarrow \left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)dx = dt \qquad ...(2)$$

We have $t = x+\sqrt{1+x^2}$

$$= x+\sqrt{1+x^2} \times \frac{x-\sqrt{1+x^2}}{x-\sqrt{1+x^2}} = \frac{-1}{x-\sqrt{1+x^2}}$$

$$\therefore\ -\frac{1}{t} = x - \sqrt{1+x^2}$$

Subtracting we get,

$$2\sqrt{1+x^2} = t+\frac{1}{t} \qquad ...(3)$$

or, $\frac{1}{\sqrt{1+x^2}} = \frac{2t}{t^2+1}$

From (1), (2) and (3) we get $dx = \frac{t^2+1}{2t^2}dt$

$$\therefore\ I = \int t^n.\frac{t^2+1}{(2t^2)}dt = \frac{1}{2}\int (t^n+t^{n-2})\,dt$$

$$= \frac{1}{2}\left[\frac{t^{n+1}}{n+1}+\frac{t^{n-1}}{n-1}\right]+C$$

$$\Rightarrow I = \frac{1}{2(n+1)}[x+\sqrt{(1+x^2)}]^{n+1}$$

$$+\frac{1}{2(n-1)}(x+\sqrt{(1+x^2)})^{n-1}+C.$$

Example 9. Evaluate $\int \frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{1+x+x^2}}dx$

Solution Let $\sqrt{1+x+x^2} = xt+1$, then

$$1+x+x^2 = x^2t^2+2xt+1,\ x = \frac{2t-1}{1-t^2},$$

$$dx = \frac{2t^2-2t+2}{(1-t^2)^2}dt$$

$$\sqrt{1+x+x^2} = xt+1 = \frac{t^2-t+1}{1-t^2}$$

$$1-\sqrt{1+x+x^2} = \frac{-2t^2+t}{1-t^2}$$

Putting the expressions obtained into the original integral, we find $\int \frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{1+x+x^2}}dx$

$$= \int \frac{(-2t^2+t)^2(1-t^2)^2(1-t^2)(2t^2-2t+2)}{(1-t^2)^2(2t-1)^2(t^2-t+1)(1-t^2)^2}dt$$

$$= 2\int \frac{t^2}{1-t^2}dt = -2t + \ln\left|\frac{1+t}{1-t}\right| + C$$

$$= -\frac{2(\sqrt{1+x+x^2}-1)}{x} + \ln\left|\frac{x+\sqrt{1+x+x^2}-1}{x-\sqrt{1+x+x^2}+1}\right| + C$$

$$= -\frac{2(\sqrt{1+x+x^2}-1)}{x} + \ln\left|2x+2\sqrt{1+x+x^2}+1\right| + C$$

Practice Problems S

Evaluate the following integrals :

1. $\int \frac{dx}{x-\sqrt{x^2+2x+4}}$

2. $\int \frac{dx}{\sqrt{1-x^2}-1}$

3. $\int \frac{dx}{\sqrt{(2x-x^2)^3}}$

4. $\int \frac{(x+\sqrt{1+x^2})^{15}}{\sqrt{1+x^2}}dx$

5. $\int \frac{dx}{x\sqrt{(x^2-x+2)}}$

6. $\int \frac{x\,dx}{x-\sqrt{x^2-1}}$

7. $\int \frac{dx}{x-\sqrt{x^2-1}}$

8. $\int \sqrt{x-\sqrt{x^2-4}}\,dx$

1.21 METHOD OF UNDETERMINED COEFFICIENTS

To find $\int P_n(x)e^{kx}dx$ where $P_n(x)$ is a polynomial of degree n, we have to perform integration by parts n times. We then get $Q_n(x)e^{kx}$, where $Q_n(x)$ is a polynomial of degree n.

Knowing this, we need not perform integration by parts n times. We calculate the integral using the method of undetermined coefficients, the essence of which is explained by the following example.

Let us find $\int x^2e^x dx$.

Let $\int x^2e^x dx = (a_2x^2 + a_1x + a_0)e^x + C$

On differentiating both sides, we get

$$x^2e^x = [x^2a_2 + x(2a_2 + a_1) + a_1 + a_0]\, e^x$$

Equating the coefficients of identical powers of x in the polynomials on the right and left, we get

$a_2 = 1$

$2a_2 + a_1 = 0 \Rightarrow a_1 = -2$

$a_1 + a_0 = 0 \Rightarrow a_0 = 2$

Finally, we get

$$\int x^2e^x dx = (x^2 - 2x + 2)e^x + C$$

Example 1. Applying the method of undetermined coefficients, evaluate $I = \int (3x^3-17)e^{2x}dx$.

Solution Let $\int (3x^3-17)e^{2x}dx$

$= (Ax^3 + Bx^2 + Dx + E)\, e^{2x} + C.$

Differentiating the right and the left sides, we obtain

$$(3x^3 - 17)e^{2x} = 2(Ax^3 + Bx^2 + Dx + E)e^{2x} + (3Ax^2 + 2Bx + D)e^{2x}$$

Cancelling e^{2x} we have

$3x^3 - 17 = 2Ax^3 + (2B + 3A)x^2 + (2D + 2B)x + (2E + D).$

Equating the coefficients of equal powers of x in the left and right sides of this identity, we get

$3 = 2A,\ 0 = 2B + 3A,$

$0 = 2D + 2B,\ -17 = 2E + D$

Solving the system, we obtain

$$A = \frac{3}{2}, B = -\frac{9}{4}, D = \frac{9}{4}, E = -\frac{77}{8}$$

Hence,

$$\int (3x^3-17)e^{2x}dx = \left(\frac{3}{2}x^3 - \frac{9}{4}x^2 + \frac{9}{4}x - \frac{77}{8}\right)e^{2x} + C.$$

Note: The method of undetermined coefficients may also be applied to integrals of the form

$$\int P_n(x)\sin ax\,dx,\ \int P_n(x)\cos ax\,dx,$$

where $P_n(x)$ is a polynomial. In both cases the answer is of the form $Q_n(x)\cos kx + R_n(x)\sin kx$, where $Q_n(x)$ and $R_n(x)$ are polynomials of degree n (or less than n).

Example 2. Evaluate $I = \int (x^2+3x+5)\cos 2x\,dx$.

Solution Let us put $\int (x^2+3x+5)\cos 2x\,dx$

$= (A_0x^2 + A_1x + A_2)\cos 2x$

$$+(B_0x^2+B_1x+B_2)\sin 2x+C$$

Differentiating both sides :

$$(x^2+3x+5)\cos 2x$$
$$=-2(A_0x^2+A_1x+A_2)\sin 2x+(2A_0x+A_1)\cos 2x$$
$$+2(B_0x^2+B_1x+B_2)\cos 2x+(2B_0x+B_1)\sin 2x$$
$$=[2B_0x^2+(2B_1+2A_0)x+(A_1+2B_2)]\cos 2x+$$
$$[-2A_0x^2+(2B_0-2A_1)x+(B_1-2A_2)]\sin 2x$$

Equating the coefficients at equal powers of x in the multipliers cos2x and sin2x, we get a system of equations:

$$2B_0=1,\quad 2(B_1+A_0)=3,\quad A_1+2B_2=5,$$
$$-2A_0=0,\quad 2(B_0-A_1)=0,\quad B_1-2A_2=0$$

Solving the system, we find

$$A_0=0,\ B_0=\frac{1}{2},\ A_1=\frac{1}{2},\ B_1=\frac{3}{2},\ A_2=\frac{3}{4},\ B_2=\frac{9}{4}$$

Thus, $\int (x^2+3x+5)\cos 2x\,dx$

$$=\left(\frac{x}{2}+\frac{3}{4}\right)\cos 2x+\left(\frac{1}{2}x^2+\frac{3}{2}x+\frac{9}{4}\right)\sin 2x+C$$

Integrals of the form $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx$

where $P_n(x)$ is a polynomial of degree n.

Put $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx$

$$=Q_{n-1}(x)\sqrt{ax^2+bx+c}+K\int \frac{dx}{\sqrt{ax^2+bx+c}},\quad ...(1)$$

where $Q_{n-1}(x)$ is a polynomial of degree (n – 1) with undetermined coefficients and K is a number.

The coefficients of the polynomial $Q_{n-1}(x)$ and the number K are found by differentiating identity (1).

Example 3. Evaluate $I=\int \frac{x^3-x-1}{\sqrt{x^2+2x+2}}dx$.

Solution Here $P_m(x)=x^3-x-1$.

$P_{m-1}(x)=Ax^2+Bx+D$

We seek the integral in the form.

$$I=(Ax^2+Bx+D)\sqrt{x^2+2x+2}+K\int \frac{dx}{\sqrt{x^2+2x+2}}$$

Differentiating this equality, we obtain

$$I'=\frac{x^3-x-1}{\sqrt{x^2+2x+2}}$$

$$=(2Ax+B)\sqrt{x^2+2x+2}$$
$$+(Ax^2+Bx+D)\frac{x+1}{\sqrt{x^2+2x+2}}+\frac{K}{\sqrt{x^2+2x+2}}$$

Reduce to a common denominator and equate the numerators

$$x^3-x-1=(2Ax+B)(x^2+2x+2)$$
$$+(Ax^2+Bx+D)(x+1)+K.$$

Equating the coefficients at equal powers of x, we get the following system of equations :

$$2A+A=1,\qquad B+4A+B+A=0,$$
$$2B+4A+D+B=-1,\qquad 2B+D+K=-1$$

Solving the system, we obtain

$$A=\frac{1}{3},\ B=-\frac{5}{6},\ D=\frac{1}{6},\ K=\frac{1}{2}$$

Thus,

$$I=\left(\frac{1}{3}x^2-\frac{5}{6}x+\frac{1}{6}\right)\sqrt{x^2+2x+2}+\frac{1}{2}\int \frac{dx}{\sqrt{x^2+2x+2}}$$

where $\int \frac{dx}{\sqrt{x^2+2x+2}}$

$$=\int \frac{dx}{\sqrt{(x+1)^2+1}}=\ell n(x+1+\sqrt{x^2+2x+2})+C.$$

Example 4. $\int x^2\sqrt{x^2+4}\,dx$

Solution Let $\int x^2\sqrt{x^2+4}\,dx=\int \frac{x^4+4x^2}{\sqrt{x^2+4}}dx$

$$=(Ax^2+Bx^2+Cx+D)\sqrt{x^2+4}+\lambda\int \frac{dx}{\sqrt{x^2+4}}.$$

Differentiating both sides,

$$\frac{x^4+4x^2}{\sqrt{x^2+4}}=(3Ax^2+2Bx+C)\sqrt{x^2+4}$$
$$+\frac{(Ax^3+Bx^2+Cx+D)x}{\sqrt{x^2+4}}+\frac{\lambda}{\sqrt{x^2+4}}.$$

Multiplying by $\sqrt{x^2+4}$ and equating the coefficients of identical degree of x, we obtain

$$A=\frac{1}{4},\ B=0,\ C=\frac{1}{2},\ D=0,\ \lambda=-2.$$

Hence,

$$\int x^2\sqrt{x^2+4}\,dx$$

$$=\frac{x^2+2x}{4}\sqrt{x^2+4}-2\ln(x+\sqrt{x^2+4})+C$$

Differentiation under the sign of Integration

The integral of any expression of the form f(x, a), where a is independent of x, is obviously a function of a as well as of x. Suppose the integral to be denoted by F(x, a), i.e. let $F(x, a)=\int f(x, a)\,dx$, then $\frac{d}{dx}\{F(x, a)\}=f(x, a)$. Again, differentiating both sides with respect to a, we have, since x and a are independent,

$$\frac{d^2F(x,a)}{dadx}=\frac{df(x,a)}{da},\quad \frac{d}{dx}\left(\frac{dF(x,a)}{da}\right)=\frac{df(x,a)}{da}$$

Consequently, integrating with respect to x, we get

$$\frac{dF(x,a)}{da}=\int\frac{df(x,a)}{da}dx,$$

i.e. $$\frac{d}{dx}\int f(x, a)\,dx=\int\frac{df(x,a)}{da}dx. \quad ...(1)$$

In other words, if

$$u=\int f(x, a)dx,\text{ then }\frac{du}{da}=\int\frac{df}{da}dx,$$

provided a be independent of x; in which case, accordingly, it is permitted to differentiate under the sign of integration. By continuing the same process of reasoning we obviously get

$$\frac{d^nu}{da^n}=\int\frac{d^nf(x,a)}{da^n}dx, \quad ...(2)$$

where $u=\int f(x, a)dx$, a being independent of x.

Consider the formula $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$.

On differentiating w.r.t. the parameter a,

$$\int\frac{2adx}{(x^2+a^2)^2}=\frac{1}{a^2}\tan^{-1}\frac{x}{a}+\frac{x}{a(a^2+x^2)}$$

and so, on dividing by 2a, we get

$$\int\frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^3}\tan^{-1}\frac{x}{a}+\frac{x}{2a^2(a^2+x^2)}$$

We now proceed to consider the inverse process, namely, the method of integration under the sign of integration.

Integration under the sign of Integration

If in the last section we suppose f(x, a) to be derived function with respect to a of another function v, i.e. if

$f(x, a)=\frac{dv}{da}$, then $v=\int f(x, a)\,da$. Also, we have

$$\frac{d}{da}\left(\int vdx\right)=\int\frac{dv}{da}dx=\int f(x, a)\,dx=F(x, a).$$

Hence $\int vdx=\int F(x, a)da$. In other words, if

$$F(x, a)=\int f(x, a)\,dx,\text{ then}$$

$$\int F(x, a)\,da=\int\left(\int f(x, a)\,da\right)dx. \quad ...(3)$$

Integration of Implicit Functions

Consider the integral $\int R(x, y)\,dx$, where R(x, y) is a rational function of x and y.

Assume that we can find a variable t such that x and y are both rational functions of t, say

$$x=\phi(t),\ y=\psi(t).$$

Then, $\int R(x, y)\,dx=\int R\{\phi(t), \psi(t)\}\,\phi'(t)\,dt$,

and the latter integral being that of a rational function of t, can be easily evaluated.

Example 5. If $y(x-y)^2=x$ then prove that $\int\frac{dx}{x-3y}$.

Solution $\int\frac{dx}{x-3y}$

$$=\frac{1}{2}\ln((x-y)^2-1)+C$$

Put $x-y=t\Rightarrow x=t+y$

$$yt^2=x\Rightarrow yt^2=t+y\Rightarrow y=\frac{t}{t^2-1}$$

$$x=t+\frac{t}{t^2-1}=\frac{t^3-t+t}{t^2-1}=\frac{t^3}{t^2-1}$$

$$dx=\frac{(t^2-1)(3t^2)-t^3(2t)}{(t^2-1)^2}=\frac{3t^4-3t^2-2t^4}{(t^2-1)^2}dt$$

$$\Rightarrow dx=\frac{t^4-3t^2}{(t^2-1)^2}dt \qquad x-3y=\frac{t^3}{t^2-1}-\frac{3t}{t^2-1}$$

$$\int\frac{t^4-3t^2}{(t^2-1)}dt\times\frac{(t^2-1)}{t^3-3t}=\frac{t^3-3t}{t^2-1}$$

$$\Rightarrow \int \frac{(t^3-3t)}{(t^2-1)(t^2-3)} = \int \frac{t(t^2-3)\,dt}{(t^2-1)(t^2-3)}$$

$$= \frac{1}{2}\int \frac{2t\,dt}{(t^2-1)}$$

$= t^2 - 1 = u \Rightarrow 2t\,dt = du$

$$\Rightarrow \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(t^2-1) + C$$

$$= \frac{1}{2}\ln((x-y)^2-1) + C$$

Practice Problems

T

1. Find polynomials P and Q such that $\int \{(3x-1)\cos x + (1-2x)\sin x\}\,dx = P\cos x + Q\sin x + C.$

2. Evaluate $\displaystyle\int \frac{9x^3-3x^2+2}{\sqrt{3x^2-2x+1}}dx$

3. Evaluate $\displaystyle\int \frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}dx$

4. Prove that, when $x > a > b$,
$$\int \frac{dx}{(x-a)^2(x-b)} = \frac{1}{(a-b)^2}\ell n\frac{x-b}{x-a} - \frac{1}{(a-b)(x-a)} + C$$

5. Use the formula $\int e^{ax}dx = a^{-1}e^{ax}$ to prove that
   (i) $\int xe^{ax}dx = e^{ax}(xa^{-1} - a^{-2}) + C$
   (ii) $\int x^2e^{ax}dx = e^{ax}(x^2a^{-1} - 2xa^{-2} + 2a^{-3}) + C$
   (iii) $\int xe^{x}dx = e^{x}(x-1) + C$

6. Use the integral $\int (x^2+a^2)^{-1/2}dx$ to prove that
$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}} + C$$

7. If $y^2(x-y) = x^2$, then prove that
$$\int \frac{dx}{3y-2x} = \ln\left(\frac{y}{y-x}\right) + C.$$

8. Find a substitution to reduce the integral $\int R(x, y)\,dx$ when $(x^2+y^2)^2 = a^2(x^2-y^2)$.

1.22 NON-ELEMENTARY INTEGRALS

Given a function f(x), has it necessarily an indefinite integral F(x)? If some functions have indefinite integrals and others not, what property of f(x) will ensure that the function has an indefinite integral?

In our elementary work we were not concerned with such generalities; we concentrated on finding an explicit formula for the integral, if we could. The one useful general answer that is readily available to us comes from the theory of the definite integral and is : every continuous function f(x) has an indefinite integral F(x).

Theorem Any function f(x) continuous on an interval (a, b) has an antiderivative on that interval. In other words, there exists a function F(x) such that $F'(x) = f(x)$.

Knowing that a given function f(x) has an indefinite integral F(x), can we find an explicit formula for that integral?

Consider $f(x) = e^{-x^2}$. Since f is continuous, its integral exists, and if we define the function F by $F(x) = \int_0^x e^{-t^2}dt$ then we know (to be dealt in the next chapter) that $F'(x) = e^{-x^2}$. Thus, $f(x) = e^{-x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{-x^2}dx$ in terms of the functions we know.

The functions that we have been dealing with are called **elementary functions**. These are the polynomials, rational functions, power functions, exponential function, logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition. For instance, the function

$$f(x) = \sqrt{\frac{x^2-1}{x^3+2x-1}} + \ln(\cos x) - xe^{\sin 2x}$$ is an elementary function.

Note that if f is an elementary function, then f′ is an elementary function but $\int f(x)\,dx$ need not be an elementary function.

For instance, $(\sqrt{x})' = 1/(2\sqrt{x})$, $(x^3 - 3x^2)' = 3x^2 - 6x$, and $(\tan x)' = \sec^2 x$. But if you start with an elementary function f and search for an elementary function F whose derivative is to be f, you may be frustrated – not because it may be hard to find F – but because no such F exists.
It is not easy to tell by glancing at f whether the desired F is elementary. After all, x tan x looks no more complicated that x cos x, yet it is not the derivative of an elementary function, while x cos x is.

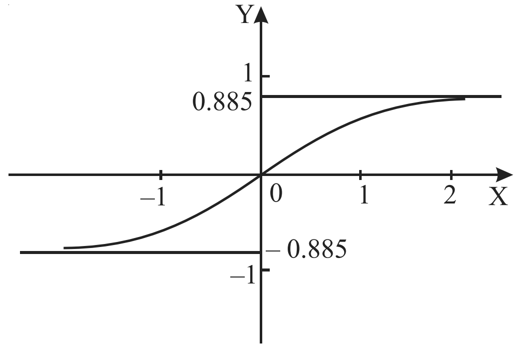
As another example, $\cos\sqrt{x}$ looks more complicated than $\cos x^2$. Yet it turns out that $\cos\sqrt{x}$ is the derivative of an elementary function, while $\cos x^2$ is not. [It is not hard to check that $(2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x})'$ is $\cos\sqrt{x}$]

Some integrals can be expressed in the form of an infinite series. The integral $\int \frac{e^{mx}dx}{x}$ cannot be obtained in terms of a finite number of elementary functions; it however may be exhibited in the shape of an infinite series. By expanding e^{mx} and integrating each term separately, we have

$$\int \frac{e^{mx}dx}{x} = \ln x + \frac{mx}{1} + \frac{m^2x^2}{1.2^2} + \frac{m^2x^2}{1.2.3^2} + ...$$

Even though an integral is non-elementary, we can study its properties. Consider $F(x) = \int e^{-x^2}dx$.

Since $\frac{dF(x)}{dx} = e^{-x^2} > 0$ for arbitrary x, it follows that F(x) is an increasing function. The derivative has a maximum at x = 0; hence, at x = 0, F(x) has a maximum angle of the tangent line with the x-axis. For large absolute values of x (positive or negative), the derivative $\frac{dF}{dx}$ is very small. This means that the function is almost constant. The graph of the function $F(x) = \int_0^x e^{-x^2}dx$ is shown in the figure (for the sake of definiteness, the lower limit has been chosen equal to zero).
Extensive tables have been compiled for this function and so computations involving this integral are not much complicated.



Non-elementary Integrals

The integrals which cannot be expressed using elementary functions in finite form are called non-elementary integrals.
In all such cases, the antiderivative is obviously some new function which does not reduce to a combination of a finite number of elementary functions.
Such functions are said to be non-integrable in elementary functions (non-integrable in finite form).
It is known that the following integrals are non-elementary :

$$\int \frac{e^x}{x}dx \qquad \int \sin(x^2)dx$$

$$\int \cos(e^x)dx \qquad \int \sqrt{x^3+1}\,dx$$

$$\int \frac{1}{\ln x}dx \qquad \int \frac{\sin x}{x}dx$$

$$\int \frac{\cos x}{x}dx \qquad \int \sqrt{1-k^2\sin^2 x}\,dx,\ (0<k^2<1)$$

We have seen that for some functions f(x) it is easy to discover an indefinite integral F(x), for others a difficult and lengthy affair; and for certain functions f(x) we shall find that an explicit formula for F(x) can be obtained only if we invent new types of function in addition to the types of function used in the definition of f(x).
Let us discuss this issue in more detail. Suppose we limit ourselves to using all the basic elementary functions except logarithmic. Then many integrals whose expressions we know should be regarded as "inexpressible in elementary functions".

For instance, the integrals $\int \frac{dx}{x}, \int \frac{dx}{x^2-1}$, cannot now be expressed "in finite form". For among the remaining basic elementary functions there is no function whose derivative is equal to $\frac{1}{x}$ or $\frac{1}{x^2-1}$ (since we cannot use the logarithmic function). But if the logarithmic function is included it becomes possible to find these integrals in finite form.

But there is nothing surprising in the fact that some other integrals remain non-integrable in elementary functions. To make them integrable it is necessary to extend the class of basic functions which we agree to use. This is exactly what is done in mathematical analysis: the non-elementary functions determined by the most important integrals inexpressible in terms of elementary functions are thoroughly investigated and tabulated. These new functions extend the variety of our techniques and make it possible to express the integrals of a number of formerly non-integrable functions.

Consider the function $f(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$

One of the antiderivatives of this function,

$$\frac{2}{\sqrt{\pi}} \int e^{-x^2} dx + C$$

which vanishes for $x = 0$ is called the Laplace function and is denoted by $\Phi(x)$. Thus,

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int e^{-x^2} dx + C_1 \text{ if } \Phi(0) = 0.$$

This function has been studied in detail. Table of its values for various values of x have been complied.

We note that $f(x) = \sqrt{(1-k^2 \sin^2 x)}$, $(0 < k^2 < 1)$ involves in its definition only trigonometric functions and a square root sign; being continuous it has an integral F(x); but there is no formula for F(x) in terms of root signs and trigonometric functions. In order to write down a formula for F(x), we must use a new sort of function, namely, an elliptic function.

One of the antiderivatives of f(x) i.e.

$$\int \sqrt{1-k^2 \sin^2 x}\, dx + C$$

which vanishes for $x = 0$ is called an elliptic integral and is denoted by E(x).

$$E(x) = \int \sqrt{1-k^2 \sin^2 x}\, dx + C' \text{ if } E(0) = 0$$

Table of the values of this function have also been complied for various values of x.

Chebyshev's theorem

(i) Consider the integral $\int x^p (1-x)^q dx$, where p and q are rational numbers. Chebyshev proved that there are only three cases for which the antiderivative in question is elementary, which are:

(A) if p is an integer,

(B) if q is an integer,

(C) if p + q is an integer.

In particular, $\int \sqrt{x}\sqrt[3]{1-x}\, dx$ and $\int \sqrt[3]{x-x^2}\, dx$ are not elementary. Chebyshev's theorem also holds for $\int x^p (1+x)^q dx$.

Concept Problems E

1. Two of these antiderivatives are elementary functions; find them.

(A) $\int \ln x\, dx$ (B) $\int \frac{\ln x\, dx}{x}$

(C) $\int \frac{dx}{\ln x}$

2. Two of these three antiderivatives are elementary. Find them.

(A) $\int \sqrt{1-4\sin^2\theta}\, d\theta$ (B) $\int \sqrt{4-4\sin^2\theta}\, d\theta$

(C) $\int \sqrt{1+\cos\theta}\, d\theta$

3. Two of these three integrals are elementary; evaluate them

(A) $\int \sin^2 x\, dx$ (B) $\int \sin\sqrt{x}\, dx$

(C) $\int \sin x^2\, dx$

4. Only one of the functions $\sqrt{x}\sqrt[3]{1-x}$ and $\sqrt{1-x}\sqrt[3]{1-x}$ has an elementary antiderivative. Find the function.

5. Three of these six antiderivatives are elementary. Find them.

(A) $\int x\cos x\, dx$ (B) $\int \frac{\cos x}{x} dx$

(C) $\int \frac{x\, dx}{\ln x}$ (D) $\int \frac{\ln x^2}{x} dx$

(E) $\int \sqrt{x-1}\sqrt{x}\sqrt{x+1}\, dx$

(F) $\int \sqrt{x-1}\sqrt{x+1}\, x\, dx$

Practice Problems U

6. Assuming that $\int (e^x/x)\,dx$ is not elementary (a theorem of Liouville), prove that $\int 1/\ln x\,dx$ is not elementary.

7. From the fact that $\int x\tan x\,dx$ is not elementary, deduce that the following are not elementary.

(A) $\int x^2\sec^2 x\,dx$ (B) $\int x^2\tan^2 x\,dx$

(C) $\int \dfrac{x^2dx}{1+\cos x}$

8. From the fact that $\int (\sin x)/x\,dx$ is not elementary, deduce that the following are not elementary :

(A) $\int (\cos^2x)/x^2\,dx$ (B) $\int (\sin^2x)/x^2\,dx$

(C) $\int \sin e^x\,dx$ (D) $\int \cos x\ln x\,dx$

9. Chebyshev's theorem : the integral $\int x^p(1-x)^q\,dx$, where p and q are rational numbers is elementary,
(A) if p is an integer,
(B) if q is an integer, or
(C) if p + q is an integer. ...(1)

(i) Deduce from (1) that $\int \sqrt{1-x^3}\,dx$ is not elementary.

(ii) Deduce from (1) that $\int (1-x^n)^{1/m}\,dx$, where m and n are positive integers, is elementary if and only if $m = 1, n = 1$ or $m = 2 = n$.

(iii) Deduce from (1) that $\int \sqrt{\sin x}\,dx$ is not elementary.

(iv) Deduce from (1) that $\int \sin^a x\,dx$, where a is rational, is elementary if and only if a is an integer.

(v) Deduce from (1) $\int \sin^p x\cos^q x\,dx$, where p and q are rational, is elementary if and only if p or q is an odd integer or p + q is an even integer.

(vi) Deduce from (v) that $\int \sec^p x\tan^q x\,dx$, where p and q are rational, is elementary if only p + q or q is odd, or if p is even.

(vii) (A) Deduce from (1) that $\int (x/\sqrt{1+x^n})\,dx$, where n is a positive integer, is elementary only when n = 1, 2 or 4.
(B) Evaluate the integral for n = 1, 2 and 4.

(viii)(A) Deduce from (1) that $\int (x^2/\sqrt{1+x^n})\,dx$, where n is a positive integer, is elementary only when n = 1, 2, 3 or 6.
(B) Evaluate the integral for n = 1, 2, 3 and 6.

(ix) (A) Using (1) determine for which positive integers n the integral $\int (x^n/\sqrt{1+x^4})\,dx$ is elementary
(B) Evaluate the integral for n = 3 and n = 5.

(x) Let $\theta = \int \dfrac{c\,dr}{r\sqrt{r^6-c^2}}$ where c is a constant. The integral is easily evaluated by the substitution..
(A) What substitution should we recommend?
(B) Using (1), determine for which positive integers,n, $\int (c/(r^n\sqrt{r^6-c^2}))\,dr$ is elementary.

10. (i) There are two values of a for which $\int \sqrt{1+a\sin^2\theta}\,d\theta$ is elementary. What are they?
(ii) From (1) deduce that there are two values of a for which $\int \dfrac{\sqrt{1+ax^2}}{\sqrt{1-x^2}}\,dx$ is elementary.

Target Problems for JEE Advanced

Problem 1. Evaluate $\int \dfrac{1}{\sin(x-a)\cos(x-b)}\,dx$.

Solution $I = \int \dfrac{1}{\sin(x-a)\cos(x-b)}\,dx$

$$= \frac{\cos(a-b)}{\cos(a-b)}\cdot\int\frac{dx}{\sin(x-a)\cos(x-b)}$$

$$= \frac{1}{\cos(a-b)}\cdot\int\frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}\,dx$$

$$= \frac{1}{\cos(a-b)}\int\left\{\frac{\cos(x-b).\cos(x-a)}{\sin(x-a)\cos(x-b)} + \frac{\sin(x-b).\sin(x-a)}{\sin(x-a)\cos(x-b)}\right\}dx$$

$$= \frac{1}{\cos(a-b)}\int\{\cot(x-a)+\tan(x-b)\}\,dx$$

$$= \frac{1}{\cos(a-b)}\{\ln|\sin(x-a)| - \ln|\cos(x-b)|\} + C$$

$$= \frac{1}{\cos(a-b)} \ln\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C.$$

Problem 2. Evaluate $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx.$

Solution $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx$

$$= \int \frac{\sin 3x(\cos 5x + \cos 4x)}{\sin 3x - \sin 6x} dx$$

$$= \int \frac{\left(2\sin\frac{3x}{2}\cos\frac{3x}{2}\right)\left(2\cos\frac{9x}{2}\cos\frac{x}{2}\right)}{-2\cos\frac{9x}{2}\cos\frac{3x}{2}} dx$$

$$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2} dx = -\int (\cos 2x + \cos x) dx$$

$$= -\left(\frac{\sin 2x}{2} + \sin x\right) + C.$$

Problem 3. Evaluate $\int \frac{x^2 + x + 1}{\sqrt{(x^2+2x+3)}} dx.$

Solution $I = \int \frac{(x^2 + 2x + 3) - (x-2)}{\sqrt{(x^2+2x+3)}} dx$

$$= \int \frac{x^2 + 2x + 3}{\sqrt{(x^2+2x+3)}} dx - \int \frac{x+2}{\sqrt{(x^2+2x+3)}} dx$$

$$= \int \sqrt{(x^2+2x+3)}\, dx - \int \frac{1/2(2x+2)+1}{\sqrt{(x^2+2x+3)}} dx$$

$$= \int \sqrt{(x+1)^2 + 2}\, dx$$

$$-\frac{1}{2}\int \frac{(2x+2)dx}{\sqrt{(x^2+2x+3)}} - \int \frac{dx}{\sqrt{\{(x+1)^2+2\}}} dx$$

Denoting the integrals by I_1, I_2, I_3 respectively,

$$I_1 - I_3 = \int \sqrt{x^2 + a^2} dz - \int \frac{dz}{\sqrt{(z^2+a^2)}},$$

(where $z = x + 1$, $a^2 = 2$)

$$= \frac{1}{2} z\sqrt{z^2+a^2} + \frac{1}{2} a^2 \ln(z + \sqrt{z^2+a^2})$$

$$- \ln(z + \sqrt{z^2+a^2})$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+3},$$

on restoring the value of z and a^2.

For I_2, we put $x^2 + 2x + 3 = z$, so that $(2x + 2)\, dx = dz$,

$$I_2 = \int \frac{dz}{\sqrt{z}} = 2\sqrt{z} = 2\sqrt{x^2+2x+3}$$

$$\therefore \quad I = \frac{1}{2}(x+1)\sqrt{x^2+2x+3} - \sqrt{x^2+2x+3} + C$$

$$= \frac{1}{2}(x-1)\sqrt{x^2+2x+3} + C.$$

Problem 4. Evaluate $\int \frac{dx}{\sqrt{\sin(x+\alpha)\cos^3(x-\beta)}}.$

Solution Substitute $x - \beta \Rightarrow dx = dy$. Given integral

$$I = \int \frac{dy}{\sqrt{\cos^3 y \sin(y + \beta + \alpha)}}$$

$$\Rightarrow \quad I = \int \frac{dy}{\sqrt{\cos^3 y \sin(y+\theta)}} \quad (\theta = \alpha + \beta)$$

$$= \int \frac{dy}{\sqrt{\cos^3 y(\sin y \cos\theta + \cos y \sin\theta)}}$$

$$= \int \frac{dy}{\sqrt{\cos^4 y(\cos\theta \tan y + \sin\theta)}}$$

$$= \int \frac{\sec^2 y dy}{\sqrt{(\cos\theta \tan y + \sin\theta)}}$$

Now write $\sin\theta + \cos\theta \tan y = z^2$

$\Rightarrow \quad \cos\theta \sec^2 y\, dy = 2z\, dz$

$$= \int \frac{2z\sec\theta dz}{z} = 2z\sec\theta + C$$

$$= \sec\theta\sqrt{\sec\theta + \cos\theta\tan y} + C$$

$$= 2\sec\theta\sqrt{\frac{\sin(y+\theta)}{\cos y}} + C$$

$$= 2\sec(\alpha+\beta) \cdot \sqrt{\frac{\sin(x+\alpha)}{\cos(x-\beta)}} + C.$$

Problem 5. Let a matrix A be denoted as $A = \text{diag.}\left(5^x, 5^{5^x}, 5^{5^{5^x}}\right)$ then compute the value of the integral $\int (\det A) dx$.

Solution $A=\begin{vmatrix} 5^x & 0 & 0 \\ 0 & 5^{5^x} & 0 \\ 0 & 0 & 5^{5^{5^x}} \end{vmatrix} = 5^x \cdot 5^{5^x} \cdot 5^{5^{5^x}}$

$\therefore \quad I=\int 5^x \cdot 5^{5^x} \cdot 5^{5^{5^x}} dx$. Put $5^{5^{5^x}} = t$

$\therefore \quad (5^{5^{5^x}} \cdot \ln 5 \cdot 5^{5^x} \cdot \ln 5 \cdot 5^x \cdot \ln 5)\, dx = dt$

$\therefore \quad I=\frac{1}{\ln^3 5}\int dt = \frac{t}{\ln^3 5} + c = \frac{5^{5^{5^x}}}{\ln^3 5} + C.$

Problem 6. For any natural number m, evaluate

$$I=\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m} dx,\ x>0.$$

Solution $I=\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m} dx$

$$=\int (x^{3m}+x^{2m}+x^m)\frac{(2x^{3m}+3x^{2m}+6x^m)^{1/m}}{x} dx$$

$$=\int (x^{3m-1}+x^{2m-1}+x^{m-1}).(2x^{3m}+3x^{2m}+6x^m)^{1/m} dx$$

Put $2x^{3m}+3x^{2m}+6x^m=t$

$\Rightarrow \quad 6m\,(x^{3m-1}+x^{2m-1}+x^{m-1})\,dx=dt$

$\therefore \quad I=\int t^{1/m}\frac{dt}{6m}=\frac{1}{6m}\frac{t^{(1/m)+1}}{(1/m)+1}+C$

$$=\frac{1}{6(m+1)}\{2x^{3m}+3x^{2m}+6x^m\}^{\frac{m+1}{m}}+C.$$

Problem 7. Evaluate $\int \sqrt{\frac{\cos x-\cos^3 x}{(1-\cos^3 x)}}\, dx.$

Solution Let $I=\int\sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}\, dx$

$$=\int\frac{\sqrt{\cos x}(\sqrt{1-\cos^2 x})}{\sqrt{1-(\cos^{3/2} x)^2}} dx = \int\frac{\sqrt{\cos x}.\sin x dx}{\sqrt{1-(\cos^{3/2} x)^2}}$$

[Put $\cos^{3/2} x = t \Rightarrow \frac{3}{2}\cos^{1/2} x.(-\sin x)\, dx = dt$]

$\therefore \quad I=-\frac{2}{3}\int\frac{dt}{\sqrt{1-t^2}}=-\frac{2}{3}\sin^{-1}t+C$

$I=-\frac{2}{3}\sin^{-1}(\cos^{3/2} x)+C.$

Problem 8. Evaluate $\int\frac{dx}{\cos x+\operatorname{cosec} x}$.

Solution $I=\int\frac{dx}{\cos x+\frac{1}{\sin x}}=\int\frac{\sin x dx}{\cos x \,.\, \sin x+1}$

$$=\int\frac{2\sin x dx}{2+2\sin x\cos x}=\int\frac{2\sin x dx}{2+\sin 2x}$$

$$=\int\frac{[(\sin x+\cos x)+(\sin x-\cos x)]dx}{2+\sin 2x}$$

$$=\int\frac{\sin x+\cos x}{2+\sin 2x}dx+\int\frac{\sin x-\cos x}{2+\sin 2x}dx$$

$$=\int\frac{\sin x+\cos x}{3-(1-\sin 2x)}dx+\int\frac{\sin x-\cos x}{1+(1+\sin 2x)}$$

$$=\int\frac{\sin x+\cos x}{3-(\sin x-\cos x)^2}.dx+\int\frac{\sin x-\cos x}{1+(\sin x+\cos x)^2}.dx$$

Put $\sin x - \cos x = s$ and $\sin x + \cos x = t$

$\Rightarrow \quad (\cos x + \sin x)\,dx = ds$ and $(\cos x - \sin x)\,dx = dt$

$$I=\int\frac{ds}{3-s^2}-\int\frac{dt}{1+t^2}$$

$$=\int\frac{ds}{(\sqrt{3})^2-(s)^2}-\int\frac{dt}{1+t^2}$$

$$=\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+s}{\sqrt{3}-s}\right|-\tan^{-1}t+C$$

$$=\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}\right|$$

$-\tan^{-1}(\sin x+\cos x)+C.$

Problem 9. Evaluate $I=\int\frac{dx}{(a+dx^2)\sqrt{b-ax^2}}$.

Solution Substituting $ax^2 = b\sin^2\theta$

$\Rightarrow \quad dx=\sqrt{\frac{b}{a}}\cos\theta\, d\theta$

$$\therefore \quad I=\int\frac{\sqrt{\frac{b}{a}}\cos\theta\, d\theta}{\left(a+\frac{b^2}{a}\sin^2\theta\right)\sqrt{b-b\sin^2\theta}}$$

$$=\sqrt{a}\int\frac{\cos\theta d\theta}{(a^2+b^2\sin^2\theta).\cos\theta}$$

$$=\sqrt{a}\int\frac{d\theta}{a^2+b^2\sin^2\theta},$$

dividing N^r and D^r by $\cos^2\theta$, we get

$$= \sqrt{a}\int \frac{\sec^2\theta\, d\theta}{a^2\sec^2\theta + b^2\tan^2\theta} \qquad \text{Put } \tan\theta = t$$

$$= \sqrt{a}\int \frac{dt}{a^2(1+t^2)+b^2t^2} = \sqrt{a}\int \frac{dt}{(a^2+b^2)t^2+a^2}$$

$$= \frac{\sqrt{a}}{a^2+b^2}\int \frac{dt}{t^2+\frac{a^2}{a^2+b^2}}$$

$$= \frac{\sqrt{a}}{a^2+b^2}\cdot\left(\frac{\sqrt{a^2+b^2}}{a}\right)\tan^{-1}\left(\frac{t\sqrt{a^2+b^2}}{a}\right)+C$$

$$= \frac{1}{\sqrt{a(a^2+b^2)}}\cdot\tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a\sqrt{b-ax^2}}\right)+C$$

$\{\text{since, } t = \tan\theta = \frac{x}{\sqrt{b-ax^2}}\}$.

Problem 10. Evaluate

$$I = \int \frac{x^2}{(x\cos x - \sin x)(x\sin x + \cos x)}dx$$

Solution Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$I = \int \frac{\tan^2\theta\sec^2\theta\, d\theta}{[(\tan\theta\cos(\tan\theta) - \sin(\tan\theta)][\tan\theta\sin(\tan\theta)+\cos(\tan\theta)]}$$

$$= \int \frac{\tan^2\theta\sec^2\theta\cos^2\theta\, d\theta}{[\sin\theta\cos(\tan\theta) - \cos\theta\sin(\tan\theta)][\sin\theta\sin(\tan\theta)+\cos\theta\cos(\tan\theta)]}$$

$$= \int \frac{(\sec^2\theta - 1)\, d\theta}{[(\sec(\theta-\tan\theta)]\cos(\theta-\tan\theta)}$$

$$= -\int \frac{(\sec^2\theta - 1)\, d\theta}{\sec(\tan\theta-\theta)\cdot\cos(\tan\theta-\theta)}$$

Put $\tan\theta - \theta = y$

$$= -2\int \frac{dy}{2\sin y\cos y} = -2\int \operatorname{cosec} 2y\, dy$$

Put $2y = t$

$$= -\frac{2}{2}\int \operatorname{cosec} t\, dt = -ln(\operatorname{cosec} t - \cot t) + C$$

$$= -ln\left(\frac{1-\cos t}{\sin t}\right) + C = -ln\left(\tan\frac{t}{2}\right)+C$$

$$= -ln(\tan y) + C = -ln[\tan(\tan\theta - \theta)] + C$$
$$= -ln[\tan(x - \tan^{-1}x)] + C$$

$$= -ln\left[\frac{\tan x - x}{1+\tan x - x}\right] + C = ln\left|\frac{1+x\tan x}{\tan x - x}\right| + C$$

$$= ln\left|\frac{x\sin x + \cos x}{x\cos x - \sin x}\right| + C.$$

Alternative :

$f(x) = x\cos x - \sin x \Rightarrow f'(x) = -x\sin x$
$f'(x) = -x\sin x + \cos x - \sin x$
$g(x) = x\sin x + \cos x \Rightarrow g'(x) = x\cos x$
$g'(x) = x\cos x + \sin x - \sin x$
$f(x)\cdot g'(x) + g(x)\cdot f'(x) = -x^2$

$$I = -\int \frac{\frac{d}{dx}[f(x)\cdot g(x)]}{f(x)g(x)}dx = \int \frac{F'(x)}{F(x)}dx$$

$[f(x)g(x) = F(x)]$

$= ln(F(x)) + C = ln(f(x)\cdot g(x)) + C.$

Problem 11. Evaluate $\int \frac{\sin x + \sin 2x}{\sqrt{\cos x + \cos 2x}}dx.$

Solution $I = \int \frac{\sin x(1+2\cos x)}{\sqrt{\cos x + 2\cos^2 x - 1}}dx.$

Put $\cos x = t$ then $-\sin x\, dx = dt$

$$\therefore\quad I = -\int \frac{1+2t}{\sqrt{2t^2+t-1}}dt = -\frac{1}{2}\int \frac{(4t+1)+1}{\sqrt{2t^2+t-1}}dt$$

$$= -\frac{1}{2}\int \frac{d(2t^2+t-1)}{(2t^2+t-1)} - \frac{1}{2}\int \frac{dt}{\sqrt{2\left(t^2+\frac{t}{2}-\frac{1}{2}\right)}}$$

$$= -\frac{1}{2}\cdot\frac{(2t^2+t-1)^{-1/2+1}}{-\frac{1}{2}+1}$$

$$-\frac{1}{2\sqrt{2}}\int \frac{dt}{\left(t+\frac{1}{4}\right)^2 - \frac{1}{16} - \frac{1}{2}}$$

$$= -\sqrt{2t^2+t-1} - \frac{1}{2\sqrt{2}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2}}$$

$$= -\sqrt{2t^2+t-1}$$

$$-\frac{1}{2\sqrt{2}}\ln\left(t+\frac{1}{4}+\sqrt{\left(t+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2}\right)+C$$

$$= -\sqrt{\cos x + \cos 2x}$$

$$-\frac{1}{2\sqrt{2}}\ln\left(\cos x+\frac{1}{4}+\sqrt{\cos^2 x+\frac{\cos x}{2}-\frac{1}{2}}\right)+C.$$

Problem 12. Evaluate $I=\int\frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}dx$

Solution $I=\int\frac{(1-\tan x)dx}{\sqrt{\tan x}(1+6\sin x\cos x)}$

$$=\int\frac{(1-\tan x)dx}{\sqrt{\tan x}(\sec^2 x+6\tan x)}\sec^2 x dx$$

Putting $\tan x = t^2$ and $\sec^2 x\, dx = 2t\, dt$, we have

$$I=\int\frac{(1-t^2)2t\,dt}{t(1+t^4+6t^2)}=2\int\frac{\left(\frac{1}{t^2}-1\right)dt}{t^2+\frac{1}{t^2}+6}$$

$$=-2\int\frac{d\left(t+\frac{1}{t}\right)}{\left(t+\frac{1}{t}\right)^2+4}=-2\int\frac{dy}{y^2+4}$$

[putting $t+1/t=y$]

$$=-2\cdot\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)+C=-\tan^{-1}\left(\frac{t^2+1}{2t}\right)+C$$

$$=-\tan^{-1}\left(\frac{\tan x+1}{2\sqrt{\tan x}}\right)+C$$

$$=-\tan^{-1}\left(\frac{\sqrt{\tan x}+\sqrt{\cos x}}{2}\right)+C.$$

Problem 13. Evaluate $I=\int\frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx$

Solution $I=\int\frac{(x^2-1)(x^2+1)}{x^3\sqrt{x^2+\frac{1}{x^2}+1}}dx$

$$=\int\frac{\left(1+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)}{\sqrt{\left(x+\frac{1}{x}\right)^2-1}}dx$$

Putting $\left(x+\frac{1}{x}\right)^2-1=t\Rightarrow 2\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x^2}\right)dx=dt$

$$\Rightarrow\ I=\int\frac{dt}{2\sqrt{t}}=\sqrt{t}+C$$

$$=\sqrt{x^2+\frac{1}{x^2}+1}+C=\sqrt{\frac{x^4+x^2+1}{x}}+C.$$

Problem 14. Evaluate $\int\frac{(1-x\sin x).dx}{x(1-x^3e^{3\cos x})}$.

Solution Let $I=\int\frac{(1-x\sin x)dx}{(1-(xe^{\cos x})^3)}$.

Put $xe^{\cos x}=t$ so that $(xe^{\cos x}.(-\sin x)+e^{\cos x})\,dx=dt$

$$\therefore\ I=\int\frac{dt}{t(1-t^3)}=\int\frac{dt}{t(1-t)(1+t+t^2)}$$

$$=\int\left(\frac{A}{t}+\frac{B}{1-t}+\frac{Ct+D}{1+t+t^2}\right)dt$$

Comparing coefficients we get :

$$A=1, B=\frac{1}{3}, C=-\frac{2}{3}, d=-\frac{1}{3}$$

$$\therefore\ I=\int\frac{dt}{t}+\frac{1}{3}\int\frac{dt}{1-t}+\int\frac{\left(-\frac{2}{3}t-\frac{1}{3}\right)}{1+t+t^2}dt$$

$$=\ln|t|-\frac{1}{3}\ln|1-t|-\frac{1}{3}\ln|1+t+t^2|+C,$$

where $t=xe^{\cos x}$.

Problem 15. Evaluate $\int\sin 4x.e^{\tan^2 x}dx$

Solution Let $I=\int 4\sin x.\cos x.\cos 2x.\ e^{\tan^2 x}dx$

$$=4\int\tan x.\cos^2 x(\cos^2 x-\sin^2 x).\ e^{\tan^2 x}dx$$

$$=4\int\tan x.\cos^4 x(1-\tan^2 x)e^{\tan^2 x}dx$$

$$=4\int\frac{\tan x}{(\sec^2 x)^2}(1-\tan^2 x)e^{\tan^2 x}dx$$

Put $\tan^2 x=t\Rightarrow 2\tan x.\sec^2 x\,dx=dt$

$$\Rightarrow\ I=4\int\frac{(1-t)e^t}{(1+t)^3}.\frac{dt}{2}=2\int\frac{(1-t)e^t}{(1+t)^3}dt$$

$$=2\int\frac{2e^t-(1+t)e^t}{(1+t)^3}dt$$

$$= 2\int e^t\left(\frac{2}{(1+t)^3} - \frac{1}{(1+t)^2}\right)dt = -2\frac{1}{(1+t)^2}e^t + C$$

{using $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C$}

$$= -\frac{2e^{\tan^2 x}}{(1+\tan^2 x)^2} + C$$

$$= -2\cos^4 x \cdot e^{\tan^2 x} + C.$$

Problem 16. Evaluate $\int \frac{1-\cos x}{\cos x(1+\cos x)}dx.$

Solution Let $I = \int \frac{1-\cos x}{\cos x(1+\cos x)}dx.$

Let $\cos x = y$.

Then $\frac{1-\cos x}{\cos x(1+\cos x)} = \frac{1-y}{y(1+y)} = \frac{1}{y} - \frac{2}{1+y}$

$$= \frac{1}{\cos x} - \frac{2}{1+\cos x}$$

$$\therefore \quad I = \int \frac{1-\cos x}{\cos x(1+\cos x)}dx$$

$$= \int \frac{1}{\cos x}dx - \int \frac{2}{1+\cos x}dx$$

$$\Rightarrow \quad I = \int \sec x\, dx - \int \frac{2}{2\cos^2 x/2}dx$$

$$= \int \sec x - \int \sec^2 x/2\, dx$$

$$\Rightarrow \quad I = \ln|\sec x + \tan x| - 2\tan x/2 + C.$$

Problem 17. If the primitive of the function

$f(x) = \frac{x^{2009}}{(1+x^2)^{1006}}$ w.r.t. x is equal to $\frac{1}{n}\left(\frac{x^2}{1+x^2}\right)^m + C$

then find $(m+n)$ (where $m, n \in N$).

Solution $f(x) = \int \frac{x^{2009}}{(1+x^2)^{1006}}dx$

Put $1 + x^2 = t \Rightarrow 2x\,dx = dt$

$$I = \frac{1}{2}\int \frac{(t-1)^{1004}dt}{t^{1006}} = \frac{1}{2}\int\left(1-\frac{1}{t}\right)^{1004} \cdot \frac{1}{t^2}dt$$

Put $1 - \frac{1}{t} = y \Rightarrow \frac{1}{t^2}\,dt = dy$

$$\therefore \quad I = \frac{1}{2}\int y^{1004}dy = \frac{1}{2}\cdot\frac{y^{1005}}{1005} + C$$

$$= \frac{1}{2010}\cdot\left(\frac{t-1}{t}\right)^{1005} + C = \frac{1}{2010}\cdot\left(\frac{x^2}{1+x^2}\right)^{1005} + C$$

$\Rightarrow$ $m = 1005, n = 2010 \Rightarrow m + n = 3015.$

Problem 18. Suppose f(x) is a quadratic function such that $f(0) = 1$ and $f(-1) = 4$. If $\int \frac{f(x)dx}{x^2(x+1)^2}$ is a rational function, find the value of f(10).

Solution $g(x) = \int \frac{f(x)dx}{x^2(x+1)^2}$

Now $\quad g(x) = \int\left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}\right)dx$

$$g(x) = A\,ln|x| - \frac{B}{x} + C\,ln(x+1) - \frac{D}{x+1} + E$$

Since g (x) is a rational function, $A = C = 0$.

$$g(x) = \int \frac{f(x)dx}{x^2(x+1)^2} = \int\left(\frac{B}{x^2} + \frac{D}{(x+1)^2}\right)dx$$

Hence f (x) must of the form $f(x) = B(x+1)^2 + Dx^2$

$f(0) = 1 \Rightarrow B = 1$

$f(-1) = 4, D = 4$

$\therefore$ $f(x) = (x+1)^2 + 4x^2$

$f(x) = 5x^2 + 2x + 1$

$\therefore$ $f(10) = 500 + 20 + 1 = 521$

Problem 19. Evaluate

$$I = \int \tan\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}}\right)dx$$

Solution Let $\sqrt{x} = \tan^2\theta$

$\because$ $x \in (0,1), \quad \theta \in (0, \pi/4)$

$x = \tan^4\theta \Rightarrow \tan\theta = x^{1/4}$

$\therefore$ $dx = 4\tan^3\theta\sec^2\theta\, d\theta$

Also, $\sqrt{1+\sqrt{x}} = \sec\theta$

$$\Rightarrow \quad \tan\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}}\right)$$

$$= \tan\left(2\tan^{-1}\sqrt{\frac{\sec\theta-1}{\sec\theta+1}}\right)$$

$$= \tan\left(2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right) = \tan\theta$$

$$\therefore\quad I = \int \tan\theta \cdot 4\tan^3\theta\sec^2\theta\, d\theta$$

$$= \frac{4}{5}\tan^5\theta + C = \frac{4}{5}(x^{5/4}) + C.$$

Problem 20. Evaluate

$$I = \int \frac{(x-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^2(x+1)}\,dx.$$

Solution $I = \int \frac{(x^2-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^2(x+1)^2}\,dx$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)\sqrt{x^2\left(x^2+2x-1+\frac{2}{x}+\frac{1}{x^2}\right)}dx}{\frac{x^2(x^2+2x+1)}{x^2}}$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)\sqrt{\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)-1}\, dx}{\left(x+\frac{1}{x}+2\right)}$$

put $x+\frac{1}{x} = t \Rightarrow \left(x-\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{\sqrt{(t^2-2)+2t-1}}{(t+2)}\,dt = \int \frac{\sqrt{t^2+2t-3}}{(t+2)}\,dt$$

$$= \int \frac{t^2+2t-3}{(t+2)\sqrt{t^2+2t-3}}\,dt$$

$$= \int \frac{t(t+2)dt}{(t+2)\sqrt{t^2+2t-3}} - 3\int \frac{dt}{(t+2)\sqrt{t^2+2t-3}}$$

$$I = I_1 - 3I_2 \qquad \text{...(1)}$$

where, $I_1 = \int \frac{t\,dt}{\sqrt{t^2+2t-3}}$ and $I_2 = \int \frac{t\,dt}{(t+2)\sqrt{t^2+2t-3}}$

$$\therefore\quad I_1 = \int \frac{t\,dt}{\sqrt{(t+1)^2-4}}$$

put $t+1 = z$

$$= \int \frac{(z-1)dz}{\sqrt{z^2-2^2}} = \int \frac{zdz}{\sqrt{z^2-2^2}} - \int \frac{dz}{\sqrt{z^2-2^2}}$$

$$= \sqrt{z^2-2^2} - \ln|z+\sqrt{z^2-4}|$$

$$= \sqrt{t^2+2t-3} - \ln|(t+1)+\sqrt{t^2+2t+3}| \qquad \text{...(2)}$$

$$I_2 = \int \frac{dy}{y^2.\frac{1}{y}\sqrt{\left(\frac{1}{y}-2\right)^2+2\left(\frac{1}{y}-2\right)-3}},\ \text{put } t+2 = \frac{1}{y}$$

$$= \int \frac{dy}{\sqrt{1-2y-3y^2}} = \frac{1}{\sqrt{3}}\int \frac{dy}{\sqrt{\left(\frac{2}{3}\right)^2-\left(y+\frac{1}{3}\right)^2}}$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{y+\frac{1}{3}}{\frac{2}{3}}\right) = \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{5+t}{2+t}\right) \qquad \text{...(3)}$$

$$\therefore\quad I = \sqrt{t^2+2t-3} - \ln(t+1+\sqrt{t^2+2t-3})$$

$$-\sqrt{3}\sin^{-1}\left(\frac{t+5}{t+2}\right) + C,\ \text{where } t = x+\frac{1}{x}$$

Problem 21. Evaluate $\int \frac{x^x(x^{2x}+1)(\ln x+1)}{x^{4x}+1}dx$

Solution $I = \int \frac{x^x(x^{2x}+1)(\ln x+1)}{x^{4x}+1}dx$

Put $x^x = y \Rightarrow x^x(\ln x+1)\,dx = dy$

$$I = \int \frac{y^2+1}{y^4+1}dy$$

$$= \int \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy = \int \frac{1+\frac{1}{y^2}}{\left(y-\frac{1}{y}\right)^2+2}dy$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y-\frac{1}{y}}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^x-\frac{1}{x^x}}{\sqrt{2}}\right) + C.$$

Problem 22. Evaluate $\int \frac{e^x\,dx}{(\sin e^x + e^{-x}\cos e^x)^2}$

Solution $I = \int \frac{dx}{e^{-x}(\sin e^x + e^{-x}\cos e^x)^2}$

$$= \int \frac{dx}{e^{-3x}(e^x\sin e^x + \cos e^x)^2} = \int \frac{e^{2x}\cdot e^x\,dx}{(e^x\sin e^x + \cos e^x)^2}$$

Put $e^x = t \Rightarrow e^x\,dx = dt$

$$= \int \frac{t^2\,dt}{(t\sin t + \cos t)^2} \quad \text{Put } t = \tan\theta$$

$$= \int \frac{\tan^2\theta \sec^2\theta\, d\theta}{(\tan\theta \sin(\tan\theta) + \cos(\tan\theta))^2}$$

$$= \int \frac{\tan^2\theta \cos^2\theta \sec^2\theta\, d\theta}{(\cos(\tan\theta)\cos\theta + \sin(\tan\theta)\sin\theta)^2}$$

$$= \int \frac{(\sec^2\theta - 1)\,d\theta}{\cos^2(\tan\theta - \theta)}$$

Put $\tan\theta - \theta = y \Rightarrow (\sec^2\theta - 1)d\theta = dy$

$$= \int \frac{dy}{\cos^2 y} = \int \sec^2 y\, dy$$

$= \tan y + C = \tan(\tan\theta - \theta) + C$

$= \tan(t - \tan^{-1}t) + C = \tan(e^x - \tan^{-1} e^x) + C$.

Problem 23. For $x > 0$, evaluate

$$\int \frac{e^{\tan^{-1}x}}{(1+x^2)}\left[\left(\sec^{-1}\sqrt{1+x^2}\right)^2 + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right] dx.$$

Solution Note that $\sec^{-1}\sqrt{1+x^2} = \tan^{-1}x$

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x, \text{ for } x > 0$$

$$I = \int \frac{e^{\tan^{-1}x}}{1+x^2}\left((\tan^{-1}x)^2 + 2\tan^{-1}x\right) dx$$

Put $\tan^{-1}x = t$

$$= \int e^t (t^2 + 2t)\,dt$$

$$= e^t \cdot t^2 = e^{\tan^{-1}x}\left(\tan^{-1}x\right)^2 + C.$$

Problem 24. Evaluate $\int \left(\sin(101x)\cdot \sin^{99}x\right)dx$

Solution $I = \int \left(\sin(100x + x)\cdot(\sin x)^{99}\right) dx$

$$= \int \left((\sin(100x)\cos x + \cos 100x \cdot \sin x)(\sin x)^{99}\right)dx$$

$$= \int \underbrace{\sin(100x)}_{u}\underbrace{\cos x \cdot (\sin x)^{99}}_{v}\, dx$$

$$+ \int \cos(100x)\cdot(\sin x)^{100}\, dx$$

$$= \frac{\sin(100x)(\sin x)^{100}}{100} - \frac{100}{100}\int \cos(100x)(\sin x)^{100}\,dx$$

$$+ \int \cos(100x)(\sin x)^{100}\,dx$$

$$= \frac{\sin(100x)(\sin x)^{100}}{100} + C.$$

Problem 25. A function $f(x)$ continuous on R and periodic with period 2π satisfies $f(x) + \sin x \cdot f(x+\pi) = \sin^2 x$. Find $f(x)$ and evaluate $\int f(x)\,dx$.

Solution $f(x) + \sin x \cdot f(x+\pi) = \sin^2 x$ ...(1)

$x \to x + \pi$

$f(x+\pi) + \sin(\pi + x)\cdot f(x + 2\pi) = \sin^2(\pi + x)$

$f(x+\pi) - \sin x\, f(x) = \sin^2 x$ ...(2)

From (1), $-f(x+\pi) = \dfrac{\sin^2 x - f(x)}{\sin x}$

From (2), $f(x+\pi) = \sin^2 x + \sin x \cdot f(x)$

$\therefore \quad \sin^2 x + \sin x \cdot f(x) = \dfrac{\sin^2 x - f(x)}{\sin x}$

$\sin^3 x + \sin^2 x \cdot f(x) = \sin^2 x - f(x)$

$f(x)\,[1 + \sin^2 x] = \sin^2 x\,(1 - \sin x)$

$\therefore \quad f(x) = \dfrac{\sin^2 x(1 - \sin x)}{1 + \sin^2 x}$.

Now,

$$\int \frac{\sin^2 x(1-\sin x)}{1+\sin^2 x}dx = \int \frac{(1+\sin^2 x) - (1 - \sin x)}{1 + \sin^2 x}dx$$

$$= x - \int \frac{(1-\sin x) + \sin x + \sin^3 x}{1+\sin^2 x}dx$$

$$= x - \int \frac{1}{1+\sin^2 x}dx + \int \frac{\sin x}{2 - \cos^2 x}dx + \int \sin x\,dx$$

$$I_1 = \int \frac{\sec^2 x}{1 + \tan^2 x + \tan^2 x}dx = \frac{1}{2}\int \frac{\sec^2 x}{\tan^2 x + \frac{1}{2}}dx$$

Put $\tan x = t$

$$= \frac{1}{2}\int \frac{dt}{t^2 + \frac{1}{2}} = \frac{1}{2}\cdot\sqrt{2}\cdot\tan^{-1}(\sqrt{2}t)$$

$$I_1 = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\cdot\tan x) = I_2 = \int \frac{\sin x}{2 - \cos^2 x}dx$$

Put $\cos x = t$

$$= -\int \frac{dt}{2 - t^2} = -\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+t}{\sqrt{2}-t}$$

$$= -\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+\cos x}{\sqrt{2}-\cos x}$$

$$\therefore \quad I = x + \cos x - \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right)$$

$$+\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+\cos x}{\sqrt{2}-\cos x}.$$

Problem 26. Evaluate $\int \frac{x^{1/4}+5}{x-16}dx$.

Solution Put $x = t^4 \Rightarrow dx = 4t^3\,dt$

$$I = 4\int \frac{(t+5)}{t^4-16}t^3\,dt = 4\int \frac{t^4+5t^3}{t^4-16}dt$$

$$= 4\int \frac{t^4-16+5t^3+16}{t^4-16}dt$$

$$= 4\int dt + 5\int \frac{4t^3}{t^4-16}dt + 64\int \frac{dt}{t^4-16}$$

$$= 4t + 5\,ln(t^4-16) + 64\int \frac{dt}{(t^2-4)(t^2+4)}$$

$$= 4t + 5\,ln\,(t^4-16) + 8\int \frac{(t^2+4)-(t^2-4)}{(t^2-4)(t^2+4)}dt$$

$$= 4t + 5\,ln\,(t^4-16) + 8\left[\int \frac{dt}{(t^2-4)}dt - \int \frac{dt}{(t^2+4)}dt\right]$$

$$= 4t + 5\,ln\,(t^4-16) + 8\cdot\frac{1}{2\cdot 2}\,ln\frac{t-2}{t+2}$$

$$-8\cdot\frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$$= 4x^{1/4} + 5\,ln\,(x-16) + 2\,ln\left(\frac{x^{1/4}-2}{x^{1/4}+2}\right)$$

$$-4\tan^{-1}\left(\frac{x^{1/4}}{2}\right) + C.$$

Problem 27.

Evaluate $\int \frac{dx}{\left(a^2-\tan^2 x\right)\sqrt{b^2-\tan^2 x}}$, $(a > b)$.

Solution

Put $\tan x = b\sin\theta$

$$\Rightarrow \sec^2 x\,dx = b\cos\theta\,d\theta \Rightarrow dx = \frac{b\cos\theta}{1+b^2\sin^2\theta}d\theta$$

$$\text{Let } I = \int \frac{b\cos\theta \quad d\theta}{(1+b^2\sin^2\theta)(a^2-b^2\sin^2\theta)\cdot b\cos\theta}$$

$$I = \int \frac{d\theta}{(a^2-b^2\sin^2\theta)(1+b^2\sin^2\theta)}$$

$$= \frac{1}{(1+a^2)}\int \frac{(a^2-b^2\sin^2 d\theta) + (1+b^2\sin^2\theta)}{(a^2-b^2\sin^2\theta)\,(1+b^2\sin^2\theta)}d\theta$$

$$= \frac{1}{(1+a^2)}\left[\int \frac{d\theta}{1+b^2\sin^2\theta} + \int \frac{d\theta}{a^2-b^2\sin^2\theta}\right]$$

$$= \frac{1}{(1+a^2)}\left[\int \frac{\sec^2\theta\,d\theta}{1+(b^2+1)\tan^2\theta} + \int \frac{\sec^2\theta\,d\theta}{a^2+(a^2-b^2)\tan^2\theta}\right]$$

Put $\tan\theta = t$

$$= \frac{1}{(1+a^2)}\left[\int \frac{dt}{1+(b^2+1)t^2} + \int \frac{dt}{a^2+(a^2-b^2)t^2}\right]$$

$$= \frac{1}{(1+a^2)}\left[\frac{1}{\sqrt{1+b^2}}\tan^{-1}\left(t\sqrt{1+b^2}\right)\right.$$

$$\left.+\frac{1}{(a^2-b^2)}\frac{\sqrt{a^2-b^2}}{a}\tan^{-1}\frac{t\sqrt{a^2-b^2}}{a}\right]$$

$$= \frac{1}{(1+a^2)}\left[\frac{1}{\sqrt{1+b^2}}\tan^{-1}\left(\sqrt{1+b^2}\tan\theta\right)\right.$$

$$\left.+\frac{1}{a\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\tan\theta\right)\right]+C,$$

where $\tan x = b\sin\theta$.

Problem 28. Evaluate $\int \frac{x\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}dx$

Solution $\int \frac{x\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}dx$.

Integrating by parts taking 'x' as the first function and $\frac{\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}$ as second function. For the second integral put $a^2\cos^2 x + b^2\sin^2 x = t$

$$\Rightarrow (b^2-a^2)\sin x\cos x\,dx = \frac{1}{2}dt$$

$$\text{So, } I = x\cdot\left(-\frac{1}{a^2\cos^2 x+b^2\sin^2 x}\right)\frac{1}{2(b^2-a^2)}$$

$$+\frac{1}{2(b^2-a^2)}\int 1.\frac{1}{a^2\cos^2 x+b^2\sin^2 x}dx$$

$$=\frac{-x}{a^2\cos^2 x+b^2\sin^2 x\cdot 2(b^2-a^2)}$$

$$+\frac{1}{2(b^2-a^2)}\cdot\int\frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x}$$

$$=\frac{-x}{a^2\cos^2 x+b^2\sin^2 x\cdot 2(b^2-a^2)}$$

$$+\frac{1}{2b^2(b^2-a^2)}\int\frac{\sec^2 x\,dx}{\left(\frac{a}{b}\right)^2+\tan^2 x}$$

$$=-\frac{1}{2}\frac{x}{(b^2-a^2)(a^2\cos^2 x+b^2\sin^2 x)}$$

$$+\frac{1}{2b^2(b^2-a^2)}\frac{b}{a}\cdot\tan^{-1}\left(\frac{a\tan x}{a}\right)$$

$$=\frac{1}{2(b^2-a^2)}\left[\frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right)\right.$$

$$\left.-\frac{x}{(a^2\cos^2 x+b^2\sin^2 x)}\right]+C.$$

Problem 29. Evaluate $\int\frac{\sqrt{1+x^2}}{2+x^2}dx$.

Solution $I=\int\frac{\sqrt{1+x^2}}{2+x^2}dx$. Put $x=\tan\theta$

$$\Rightarrow\ I=\int\frac{\sec^3\theta\,d\theta}{2+\tan^2\theta}=\int\frac{d\theta}{\cos^3\theta\left(2+\frac{\sin^2\theta}{\cos^2\theta}\right)}$$

$$=\int\frac{d\theta}{\cos\theta(2\cos^2\theta+\sin^2\theta)}$$

$$\Rightarrow\ I=\int\frac{\cos\theta\,d\theta}{\cos^2\theta(2\cos^2\theta+\sin^2\theta)}$$

$$=\int\frac{\cos\theta\,d\theta}{(1-\sin^2\theta)(2-\sin^2\theta)}\quad\text{Put }\sin\theta=y$$

$$\Rightarrow\ I=\int\frac{dy}{(1-y)(2-y)}=\int\frac{dy}{y-2}-\int\frac{dy}{y-1}$$

$$=ln\left|\frac{y-2}{y-1}\right|+C=ln\left|\frac{\sin\theta-2}{\sin\theta-1}\right|+C$$

$$=ln\left|\frac{x-2\sqrt{1+x^2}}{x-\sqrt{1+x^2}}\right|+C.$$

Problem 30. Let $\int\frac{f'(x)g(x)-g'(x)f(x)}{(f(x)+g(x))\sqrt{f(x)g(x)-g^2(x)}}dx$

$=\sqrt{m}\tan^{-1}\left(\sqrt{\frac{f(x)-g(x)}{ng(x)}}\right)+C$, where $m, n\in N$ and C is constant of integration $(g(x)>0)$. Find the value of (m^2+n^2).

Solution Let

$$I=\int\frac{\{f'(x)g(x)-g'(x)f(x)\}dx}{\{f(x)+g(x)\}\sqrt{g(x)(f(x)-g^2(x))}}$$

$$I=\int\frac{\left\{\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}\right\}}{\left(\frac{f(x)}{g(x)}+1\right)\sqrt{\frac{f(x)}{g(x)}-1}}dx$$

Put $\frac{f(x)}{g(x)}-1=t^2$

$$\Rightarrow\ \frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}dx=2t\,dt$$

$$\therefore\ I=\int\frac{2t\,dt}{(t^2+2)\cdot t}=\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+C$$

$$=\sqrt{2}\tan^{-1}\left\{\sqrt{\frac{f(x)}{2g(x)}-\frac{1}{2}}\right\}+C$$

$$=\sqrt{2}\tan^{-1}\left\{\sqrt{\frac{f(x)-g(x)}{2g(x)}}\right\}+C$$

Hence $m=2, n=2$.

$\therefore\ m^2+n^2=8$.

Problem 31. If $\int\left(\frac{1}{\ln x\cdot\ln(\ln x)}+\ln(\ln(\ln x))\right)dx$ $=g(x)+C$ then find the value of $g(e^e)$.

Solution $f(x)=\ln(\ln(\ln x))$

$$f'(x)=\frac{1}{\ln(\ln x)}\cdot\frac{1}{\ln x}\cdot\frac{1}{x}$$

$$x f'(x) = \frac{1}{\ln x \cdot \ln(\ln x)}$$

$$\therefore \quad I = \underbrace{x \ln(\ln(\ln x))}_{g(x)} + C$$

$$\therefore \quad g(e^e) = 0.$$

Problem 32. Evaluate

$$\int \operatorname{cosec}^2 x \; \ell n \,(\cos x - \sqrt{\cos 2x})dx$$

Solution Let $I = \int \operatorname{cosec}^2 x \; \ell n \,(\cos x - \sqrt{\cos 2x})dx$

$$= -\cot x \;\ell n(\cos x - \sqrt{\cos 2x})$$

$$+ \int \frac{\cot x\left(-\sin x - \frac{-\sin 2x}{\sqrt{\cos 2x}}\right)}{(\cos x - \sqrt{\cos 2x})} dx$$

$$= -\cot x \;\ell n(\cos x - \sqrt{\cos 2x})$$

$$+ \int \frac{\cos x(2\cos x - \sqrt{\cos 2x})}{\sqrt{\cos 2x}(\cos x - \sqrt{\cos 2x})} dx$$

$$= -\cot x \;\ell n(\cos x - \sqrt{\cos 2x}) + I_1$$

$$\text{where } I_1 = \int \frac{\cos x(2\cos x - \sqrt{\cos 2x})}{\sqrt{\cos 2x}(\cos x - \sqrt{\cos 2x})} dx$$

$$I_1 = \int \frac{\cos x(2\cos x - \sqrt{\cos 2x})}{\sqrt{\cos 2x}(\cos x - \sqrt{\cos 2x})} \times \frac{(\cos x + \sqrt{\cos 2x})}{(\cos x + \sqrt{\cos 2x})} dx$$

$$= \int \frac{\cos x(1 + \cos x\sqrt{\cos 2x})}{\sqrt{\cos 2x}\sin^2 x} dx$$

$$= \int \frac{\cos x}{\sqrt{(1 - 2\sin^2 x)\sin^2 x}} dx + \int \cot^2 x \, dx$$

Put $\sin x = \frac{1}{t} \Rightarrow \cos x \, dx = -\frac{1}{t^2} dt$

$$= \int \frac{-\frac{dt}{t^2}}{\sqrt{\frac{(t^2-2)}{t^2} \times \frac{1}{t^2}}} - \cot x - x + C$$

$$= -\int \frac{t\,dt}{\sqrt{t^2 - 2}} - \cot x - x + C$$

$$= -\sqrt{t^2 - 2} - \cot x - x + C$$

$$\Rightarrow \quad I = -\cot x \;\ell n(\cos x - \sqrt{\cos 2x}) - \sqrt{\operatorname{cosec}^2 x - 2}$$
$$- \cot x - x + C.$$

Problem 33. If $I_{m,n} = \int \cos^m x \,.\, \cos nx \,.\, dx$, show that $(m + n)\, I_{m,n} = \cos^m x \,.\, \sin nx + m\, I_{(m-1,\, n-1)}$.

Solution We have

$$I_{m,n} = \int \underbrace{\cos^m x}_{u} . \underbrace{\cos nx}_{v} \; dx = (\cos^m x)\left[\frac{\sin nx}{n}\right]$$

$$- \int m \cos^{m-1} x(-\sin x) . \frac{\sin nx}{n} dx$$

$$= \frac{1}{n} \cos^m x \,.\, \sin nx + \frac{m}{n} \int \cos^{m-1} x$$
$$\{\sin x \,.\, \sin nx\}\, dx$$

We have

$$\cos(n-1)x = \cos nx \cos x + \sin nx \,.\, \sin x$$

$$\therefore \quad I_{m,n} = \frac{1}{n} \cos_m x \,.\, \sin x + \frac{m}{n} \int \cos^{m-1} x$$
$$\{\cos(n-1)x - \cos nx \,.\, \cos x\}\, dx$$

$$= \frac{1}{n} \cos_m x \,.\, \sin x + \frac{m}{n} \int \cos^{m-1} x \,.\, \cos(n-1)x \,.\, dx$$

$$- \frac{m}{n} \int \cos^m x \,.\, \cos nx \, dx$$

$$= \frac{1}{n} \cos^m x \,.\, \sin nx + \frac{m}{n} I_{m-1,\, n-1} - \frac{m}{n} I_{m,n}$$

$$\Rightarrow \quad I_{m,n} + \frac{m}{n} I_{m,n}$$

$$= \frac{1}{n}[\cos^m x \,.\, \sin nx + m\, I_{m-1,\, n-1}]$$

$$\Rightarrow \quad \left(\frac{m+n}{n}\right) I_{m,n}$$

$$= \frac{1}{n}[\cos^m x \,.\, \sin nx + m\, I_{m-1,\, n-1}]$$

$$\Rightarrow \quad (m+n)\, I_{m,n} = \cos^m x \,.\, \sin nx + m\, I_{m-1,\, n-1}.$$

Problem 34. If I_n denotes $\int z^n e^{1/z} dz$, then show that $(n+1)!\, I_n = I_0 + e^{1/z}(1\,!z^2 + 2!z^3 + ... + n\,!\, z^{n+1})$.

Solution $I_n = \int z^n e^{1/z} dz$, applying integration by parts taking $e^{1/z}$ as first function and z^n as second function. We get,

$$I_n = \frac{e^{1/z}.z^{n+1}}{(n+1)} - \int e^{1/z}\left(-\frac{1}{z^2}\right).\frac{z^{n+1}}{n+1}\, dz$$

$$= \frac{e^{1/z}.z^{n+1}}{(n+1)} + \frac{1}{(n+1)}\int e^{1/z}.z^{n-1}\, dz$$

$$= \frac{e^{1/z}.z^{n+1}}{(n+1)} + \frac{I_{n-1}}{(n+1)}$$

$$= \frac{e^{1/z}.z^{n+1}}{(n+1)} + \frac{I_{n-1}}{(n+1)}\left[\frac{e^{1/z}.z^n}{n} + \frac{1}{n}I_{n-2}\right]$$

$$= \frac{e^{1/z}(z)^{n+1}}{(n+1)} + \frac{e^{1/z}.(z)^n}{(n+1)n} + \frac{1}{(n+1)n}I_{n-2}.$$

$$= \frac{e^{1/z}.(z)^{n+1}}{(n+1)} + \frac{e^{1/z}.(z)^n}{(n+1)n} + \frac{e^{1/z}.(z)^{n-1}}{(n+1)n.(n-1)}$$

$$+ \frac{1}{(n+1)n(n-1)}I_{n-3}$$

... ...

... ...

$$= \frac{e^{1/z}(z)^{n+1}}{(n+1)} + \frac{e^{1/z}.(z)^n}{(n+1)n} + ... + \frac{e^{1/z}.(z)^{n-1}}{(n+1)n....3\cdot 2}$$

$$+ \frac{1}{(n+1)n(n-1)...3\cdot 2} I_0$$

Multiplying both sides by (n + 1)! We get,

$$(n+1)!I_n = (e^{1/z}.z^{n+1}.n! + e^{1/z}.z^2(n-1)! + ...$$
$$+ e^{1/z}.z^3.(2)! + e^{1/z}.z^2.1!) + I_0$$

$\Rightarrow$ $I_n(n+1)! = I_0 + e^{1/z}(1\,z^2 + 2!z^3 + ... + n!\,z^{n+1}).$

Problem 35. Evaluate

$$I = \int \frac{dx}{(x-2)(1+\sqrt{7x-10-x^2})}$$

Solution Here, we have $7x - 10 - x^2 = (x-2)(5-x)$, therefore put $\sqrt{7x-10-x^2} = (x-2)\,t$

$\Rightarrow$ $(5-x)(x-2) = (x-2)^2t^2$

$\Rightarrow$ $(5-x) = (x-2)t^2$

$$\Rightarrow \quad x = \frac{2t^2+5}{t^2+1}$$

$$\Rightarrow \quad dx = \frac{(t^2+1)4t-(2t^2+5)2t}{(t^2+1)^2}\, dt = \frac{-6t}{(t^2+1)^2}$$

and $x - 2 = \dfrac{2t^2+5}{t^2+1} - 2 = \dfrac{3}{t^2+1}$

and $1 + \sqrt{7x-10-x^2} = 1 + (x-2)\,t$

$$= 1 + \frac{3t}{t^2+1} = \frac{t^2+3t+1}{t^2+1}$$

Hence, we have

$$I = \int \frac{t^2+1}{3}.\frac{t^2+1}{t^2+3t+1}.\frac{-6t}{(t^2+1)^2}dt$$

$$= -\int \frac{2t\, dt}{t^2+3t+1} = -\int \frac{2t+3-3}{t^2+3t+1}\, dt$$

$$= -\int \frac{2t+3}{t^2+3t+1}\, dt + 3\int \frac{dt}{\left(t+\frac{3}{2}\right)^2 - \frac{5}{4}}$$

$= \ln|t^2 + 3t + 1|$

$$+ \frac{3}{\sqrt{5}}\int\left[\frac{dt}{\left(t+\frac{3}{2}\right)-\frac{\sqrt{5}}{3}} - \frac{dt}{\left(t+\frac{3}{2}\right)+\frac{5}{2}}\right] dt$$

$$= \ln|t^2+3t+1| + \frac{3}{\sqrt{5}}\ln\left|\frac{2t+3-\sqrt{5}}{2t+3+\sqrt{5}}\right| + C,$$

where $t = \sqrt{\dfrac{5-x}{x-2}}$.

Problem 36. Let f(x) and g(x) are differentiable functions satisfying the conditions :

(i) $f(0) = 2, g(0) = 1$

(ii) $f'(x) = g(x)$ and

(iii) $g'(x) = f(x)$. Find the functions f(x) and g(x).

Solution $f'(x) = g(x)$

$g'(x) = f(x)$

adding, $f'(x) + g'(x) = f(x) + g(x)$

$$\Rightarrow \quad \frac{f'(x)+g'(x)}{f(x)+g(x)} = 1 \Rightarrow \ln(f(x)+g(x)) = x + c$$

Since, $f(0) = 2$, $g(0) = 1$, put $x = 0$ to get $c = \ln 3$.

Hence $f(x) + g(x) = 3e^x$ ...(1)

Similarly subtraction gives

$$\frac{f'(x)+g'(x)}{f(x)+g(x)}=-x$$

Integrating $\ln(f(x)-g(x))=-x+c$

Since, $c=ln(1) \Rightarrow c=0$

$\Rightarrow f(x)-g(x)=e^{-x}$ ...(2)

On adding (1) and (2) we get f(x) and on subtracting we get g(x) :

$$f(x)=\frac{3e^{x}+e^{-x}}{2}, \text{and } g(x)=\frac{3e^{x}-e^{-x}}{2}.$$

Problem 37. Let f be an injective function such that $f(x)\,f(y)+2=f(x)+f(y)+f(xy)$ for all non negative real x and y with $f(0)=1$ and $f'(1)=2$. Find f(x) and show that $3\int f(x)\,dx - x\,(f(x)+2)$ is a constant.

Solution

We have $f(x)\,f(y)+2=f(x)+f(y)+f(xy)$ ...(1)

Putting x = 1 and y = 1 then $f(1)\,f(1)+2=3f(1)$ we get

$f(1)=1, 2$

$f(1)\neq 1$ ($\because$ f(0) = 1 and function is injective)

Thus $f(1)=2$

Replacing y by $\frac{1}{x}$ in (1)

$$f(x)\,f\left(\frac{1}{x}\right)+2=f(x)+f\left(\frac{1}{x}\right)+f(1)$$

$$\Rightarrow f(x)\,f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right) \quad (\because f(1)=2)$$

Hence f(x) is of the type $f(x)=1\pm x^n$

$f(1)=1\pm 1=2$ (given)

$\therefore$ $f(x)=1+x^n$ and $f'(x)=nx^{n-1} \Rightarrow f'(1)=n=2$

$\therefore$ $f(x)=1+x^2$

$$\therefore \; 3\int f(x)\,dx - x(f(x)+2) = 3\int(1+x^2)\,dx - x(1+x^2+2)$$

$$=3\left(x+\frac{x^3}{3}\right)-x(3+x^2)+C$$

$= C =$ constant.

Things to Remember

1. **Rules of integration**

(A) $\int cf(x)\,dx = c\int f(x)\,dx$ for any constant c.

(B) $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$,

(C) $\int (f(x)-g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$.

2. **Integrand f(x) Integral $\int f(x)dx$**

(i) $\int dx = kx + C$, where k is a constant

(ii) $\int x^n dx = \frac{x^{n+1}}{n+1}+C$, where $n \neq 1$

(iii) $\int \frac{1}{x}dx = \ln|x| + C$

(iv) $\int a^x dx = \frac{a^x}{\ell n\, a}+C$, where $a>0$

(v) $\int e^x dx = e^x + C$

(vi) $\int \sin x\, dx = -\cos x + C$

(vii) $\int \cos x\, dx = \sin x + C$

(viii) $\int \sec^2 x\, dx = \tan x + C$

(ix) $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$

(x) $\int \sec x \tan x\, dx = \sec x + C$

(xi) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$

(xii) $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + C$

(xiii) $\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$

(xiv) $\int \frac{1}{|x|\sqrt{x^2-1}}dx = \sec^{-1}x + C$

We have some additional results:

(xv) $\int \tan x\, dx = \frac{1}{a}\ell n \sec|x| + C$

(xvi) $\int \cot x\, dx = \frac{1}{a}\ell n \sin|x| + C$

(xvii) $\int \sec x\, dx = \ell n\,|\sec x + \tan x| + C$

$= \ell n \tan\left|\frac{\pi}{4}+\frac{x}{2}\right| + C$

$= -\ell n\,|\sec x - \tan x| + C$

3. If $\int f(x)dx = F(x)$, then

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b)$$

4. $\int f(g(x))g'(x)\,dx = F(g(x)) + C.$

5. $\int [f(x)]^n f'(x)dx = \dfrac{[f(x)]^{n+1}}{n+1} + C, n \neq -1$

6. $\int \dfrac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

7. $\int \dfrac{f(x)g'(x) - g(x)f'(x)}{f(x)g(x)}dx = \ln\left(\dfrac{g(x)}{f(x)}\right) + C$

8. **Taking x^n common** :

For $\int \dfrac{dx}{x(x^n+1)}$, $n \in N$, take x^n common and put $1 + x^{-n} = t$.

9. **Positive integral powers of sine and cosine**

(i) Any odd positive power of sines and cosines can be integrated immediately by substituting $\cos x = z$ and $\sin x = z$ respectively.
(ii) In order to integrate any even positive power of sine and cosine, we should first express it in terms of multiple angles by means of trigonometry and then integrate it.

10. **Positive integral powers of secant and cosecant**

(i) Even positive powers of secant or cosecant admit of immediate integration in terms of tanx or cotx.
(ii) Odd positive powers of secant and cosecant are to be integrated by the application of the rule of integration by parts

11. **Integrals of the form $\int \sin^m x \cos^n x\,dx$**

(i) If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosine. Then expand and integrate.

$$\int \sin^{\overbrace{2k+1}^{\text{Odd}}} x \cos^n x\,dx =$$

$$\int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosine}} \cos^n x \overbrace{\sin x dx}^{\text{Save for du}}$$

(ii) If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sine. Then, expand and integrate.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x dx$$

$$= \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to sine}} \overbrace{\cos x dx}^{\text{Save for du}}$$

(iii) If the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \text{ and } \cos^2 x = \frac{1+\cos 2x}{2}.$$

(iv) If in the expression $\sin^m x \cos^n x$, $m + n$ is a negative even integer, then one should make the substitution $\tan x = t$ (or $\cot x = t$).

(v) If in the expression $\sin^m x \cos^n x$, $m + n$ is a negative odd integer, then one should multiply the integrand by suitable power of $(\sin^2 x + \cos^2 x)$ and expand it into simpler integrals.

12. **Integrals of the form $\int \sec^m x \tan^n x\,dx$**

(i) If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then, expand and integrate.

$$\int \sec^{\overbrace{2k}^{\text{Even}}} x \tan^n x dx$$

$$= \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x dx}^{\text{Save for du}}$$

Here, we put $\tan x = t$.

(ii) If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then, expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x dx$$

$$= \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x dx}^{\text{Save for du}}$$

Here, we put $\sec x = t$.

(iii) If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor; then expand and repeat if necessary.

$$\int \tan^n x dx = \int \tan^{n-2} x \overbrace{(\tan^2 x)}^{\text{Convert to secants}} dx$$

$$= \int \tan^{n-2} x(\sec^2 x - 1)dx$$

(iv) If the integral is of the form $\int \sec^m x dx$, where m is odd and positive, use integration by parts.

(v) If none of the first four cases applies, try converting to sines and cosines.

A similar strategy is adopted for $\int \operatorname{cosec}^m x \cot^n x dx$.

13. Suppose R denotes a rational function of the entities involved. The integral of the form

(i) $\int R(x, \sqrt{b^2 - a^2x^2})dx$

is simplified by the substitution $x = \frac{b}{a}\sin\theta$

(ii) $\int R(x, \sqrt{a^2x^2 + b^2})dx$

is simplified by the substitution $x = \frac{b}{a}\tan\theta$

(iii) $\int R(x, \sqrt{a^2x^2 - b^2})dx$

is simplified by the substitution $x = \frac{b}{a}\sec\theta$

14. When integrand involves expressions of the form:

(i) $\sqrt{\frac{a-x}{a+x}}$ put $x = a\cos 2\theta$.

(ii) $\sqrt{\frac{x}{a-x}}$ put $x = a\sin^2\theta$.

(iii) $\sqrt{\frac{x}{a+x}}$ put $x = a\tan^2\theta$.

(iv) $\sqrt{\frac{x-a}{b-x}}$ or $\sqrt{(x-a)(b-x)}$

put $x = a\cos^2\theta + b\sin^2\theta$

(v) $\sqrt{\frac{x-a}{x-b}}$ or $\sqrt{(x-a)(x-b)}$

put $x = a\sec^2\theta - b\tan^2\theta$

15. Some Standard Integrals

(i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(iii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

(iv) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + C$

(v) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + C$

(vi) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(vii) $\int \sqrt{a^2-x^2}\,dx$

$= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(viii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}$

$+ \frac{a^2}{2}\ln\left(x + \sqrt{x^2+a^2}\right) + C$

(ix) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}$

$- \frac{a^2}{2}\ln\left|x + \sqrt{x^2-a^2}\right| + C$

16. For integrals of the form $\int \frac{px + q}{ax^2 + bx + c}dx$,

$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$, **and**

$\int (px + q)\sqrt{ax^2 + bx + c}\,dx$

we write $(px + q) \equiv \ell(2ax + b) + m$

17. For integrals of the form

$\int \frac{px^2+qx+r}{ax^2+bx+c}dx$, $\int \frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}dx$, **and**

$\int (px^2+qx+r)\sqrt{(ax^2+bx+c)}\,dx$

we express $px^2 + qx + r$

$= l(ax^2 + bx + c) + m(2ax + b) + n$

18. For integrals of the form $\int \frac{ax^2+b}{x^4+px^2+1}dx$

we express the numerator in terms of $(x^2 + 1)$ and $(x^2 - 1)$, then divide N^r and D^r by x^2,

and then put $x \pm \frac{1}{x} = t$.

19. For integrals of the form

(i) $\displaystyle\int \frac{dx}{a\cos^2 x + 2b\sin x\cos x + b\sin^2 x}$

(ii) $\displaystyle\int \frac{dx}{a\cos^2 x + b}$ (iii) $\displaystyle\int \frac{dx}{a + b\sin^2 x}$

we divide the N^r and D^r by $\cos^2 x$ or $\sin^2 x$ and then put $\tan x = t$ or $\cot x = t$.

20. For integrals of the form

(i) $\displaystyle\int \frac{dx}{a + b\sin x}$ (ii) $\displaystyle\int \frac{dx}{a + b\cos x}$

(iii) $\displaystyle\int \frac{dx}{a + b\sin x + c\cos x}$

use the substitution $\tan\dfrac{x}{2} = t$

21. For integrals of the form $\displaystyle\int \frac{a\cos x + b\sin x + c}{d\cos x + e\sin x + f}dx$

we write $a\cos x + b\sin x + c$
$\equiv p\,(d\cos x + e\sin x + f)$
$+ q(-d\sin x + e\cos x) + r$

22. For integrals of the form $\displaystyle\int \frac{ae^x + be^{-x} + c}{pe^x + qe^{-x} + r}dx$,

we write $a e^x + b e^{-x} + c$
$\equiv l\,(a e^x + b e^{-x} + c) + m(a e^x - b e^{-x}) + n$

23. For integrals of the form $\displaystyle\int \frac{\cos x \pm \sin x}{f(\sin 2x)}dx$

put $\sin x \mp \cos x = t$

24. For integrals of the form $\displaystyle\int\sqrt{\sec^2 x \pm a}\,dx$,

$\displaystyle\int\sqrt{\operatorname{cosec}^2 x \pm a}\,dx$, $\displaystyle\int\sqrt{\tan^2 x \pm a}\,dx$,

$\displaystyle\int\sqrt{\cot^2 x \pm a}\,dx$

For the first integral write $\displaystyle\sqrt{\sec^2 x \pm a} = \frac{\sec^2 \pm a}{\sqrt{\sec^2 x \pm a}}$

$$= \frac{\sec^2 x}{\sqrt{\sec^2 x \pm a}} \pm a\frac{\cos x}{\sqrt{1 \pm a\cos^2 x}}$$

In the first part, put $u = \tan x$ and in the second part, put $v = \sin x$.
For others proceed as above.

25. Formula for integration by parts

$$\int u.v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}.\int v\,dx\right]dx$$

The order of u and v is taken as per the order of the letters in **ILATE.**

26. $\displaystyle\int \ln x\,dx = x\ln x - x + C.$

27. (i) $\displaystyle\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + C$

(ii) $\displaystyle\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$

28. (i) $\displaystyle\int e^x[f(x) + f'(x)]\,dx = e^x.\,f(x) + C$

(ii) $\displaystyle\int [f(x) + xf'(x)]\,dx = x\,f(x) + C.$

29. Multiple integration by parts

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots..$$
$$+ (-1)^{n-1}u^{(n-1)}v_n + (-1)^n\int u^n v_n\,dx.$$

30. Reduction formula for $I_n = \displaystyle\int \frac{dx}{(x^2 + k)^{n-1}}$

$$I_n = \frac{x}{2k(n-1)(x^2 + k)^{n-1}} + \frac{(2n-3)}{2k(n-1)}I_{n-1}.$$

31. Reduction formula for $I_n = \displaystyle\int \sin^m\theta\cos^n\theta\,d\theta$

$$I_n = \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}\int \sin^m\theta\cos^{n-2}\theta\,d\theta.$$

32. Consider a rational function $f(x) = \dfrac{P(x)}{Q(x)}$.

Factor Q(x) completely into factors of the form $(ax + b)^m$ or $(cx^2 + dx + e)^n$, where $cx^2 + dx + e$ is irreducible and m and n are integers.

(i) For each distinct linear factor (ax + b), the decomposition must include the term $\dfrac{A}{ax + b}$.

(ii) For each repeated linear factor $(ax + b)^m$, the decomposition must include the terms

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \ldots. + \frac{A_m}{(ax + b)^m}.$$

(iii) For each distinct quadratic factor $(cx^2 + dx + e)$, the decomposition must include the term

$$\frac{Bx + C}{cx^2 + dx + e}.$$

(iv) For each repeated quadratic factor $(cx^2 + dx + e)^n$, the decomposition must include the terms

$$\frac{B_1x + C_1}{cx^2 + dx + e} + \frac{B_2x + C_2}{(cx^2 + dx + e)^2}$$

$$+ \ldots\ldots + \frac{B_n x + C_n}{(cx^2 + dx + e)^n}.$$

Use one of the following techniques to solve for the constants in the numerators of the decomposition.
(A) Heaviside Cover-up Method for linear factors
(B) Method of Comparision of Coefficients
(C) Method of Particular Values
(D) Application of Limit
(e) Method of Differentiation

33. Special methods for integrating a rational function

(i) For integrals of the form $\int \frac{px^2+q}{(x^2+a)(x^2+b)}dx$ assume x^2 as t for finding partial fractions

(ii) For integrals of the form $\int \frac{P(x)}{Q(x)}dx$ where Q(x) has a linear factor with high index, substitute the linear factor as 1/t.

(iii) **Substitutions**

(A) $\int \frac{x^m}{(ax+b)^n}dx$ Put $ax+b=t$.

(B) $\int \frac{dx}{x^m(ax+b)^n}$ Put $\frac{ax+b}{x}=t$

(C) $\int \frac{dx}{(x-a)^m(x-b)^n}$ Put $\frac{x-a}{x-b}=t$ if $m<n$

(D) $\int \frac{dx}{x(a+bx^n)}$ Put $x^n=\frac{1}{t}$

(E) $\int \frac{x^{2m+1}}{(ax^2+b)^n}dx$ Put $ax^2+b=t$

(iv) Integration by parts

34. Integration by irrational functions Substitution :

(i) $\int \frac{dx}{x\sqrt{ax^n+b}}$ Put $ax^n+b=t^2$

(ii) $\int \frac{dx}{(a+cx^2)^{3/2}}$ Put $x=\frac{1}{t}$

(iii) $\int \frac{xdx}{(a+2bx+cx^2)^{3/2}}$ Put $x=\frac{1}{t}$

(iv) If the integrand is of the form $R\left(x, x^{\frac{p_1}{q_1}}, \ldots, x^{\frac{p_k}{q_k}}\right)$, then put $x = t^m$, where m is the L.C.M. of the denominators $q_1, q_2, \ldots, q_k$ of the several fractional powers.

(v) If the integrand is of the form $R\left(x, (ax+b)^{\frac{p_1}{q_1}}, \ldots, (ax+b)^{\frac{p_k}{q_k}}\right)$, then put $ax+b = t^m$, where m is the L.C.M. of the denominators $q_1, q_2, \ldots, q_k$ of the several fractional powers.

(vi) If the integrand is of the form $R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_k}{q_k}}\right)$, put $\frac{ax+b}{cx+d} = t^m$, where m is the L.C.M. of the denominators $q_1, q_2, \ldots, q_k$ of the several fractional powers.

(vii) Integrals of the form $\int \frac{dx}{\sqrt[n]{(x-a)^p(x-b)^q}}$, where $p+q=2n$ are solved using the substitution $\frac{x-a}{x-b}=t$.

35. Integrals of the form $\int \frac{dx}{P\sqrt{Q}}$

(i) $\int \frac{dx}{(ax+b)\sqrt{(cx+d)}}$ Put $cx+d=t^2$

(ii) $\int \frac{dx}{(px^2+qx+r)\sqrt{ax+b}}$ Put $ax+b=t^2$

(iii) $\int \frac{dx}{(px+q)\sqrt{(ax^2+bx+c)}}$ Put $px+q=\frac{1}{t}$

(iv) $\int \frac{dx}{(ax^2+b)\sqrt{(cx^2+d)}}$ Put $x=1/t$ and then the expression under the radical should be put equal to z^2.

(v) $\int \frac{dx}{(px^2+qx+r)\sqrt{(ax^2+bx+c)}}$.

Case I: If px^2+qx+r breaks up into two linear factors then we resolve $1/(px^2+qx+r)$ into partial fractions and the integral then transforms into the sum (or difference) of two integrals.

Case II: If px^2+qx+r is a perfect square, say, $(lx+m)^2$, then put $lx+m=1/t$.

(vi) $\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+c}}$, where $r \in N$ Put $x-k=\frac{1}{t}$

(vii) $\int \frac{(ax+b)dx}{(cx+d)\sqrt{px^2+qx+r}}$

Write $(ax+b) = A(cx+d)+B$

(viii) $\int \frac{ax^2+bx+c}{(dx+e)\sqrt{px^2+qx+r}}\,dx$

Write $ax^2 + bx + c = A(dx + e)(2px + q) + B(dx+e)+C$

36. Integrals of the form $\int x^m(a+bx^n)^p\,dx$

(i) p is a positive integer. Then, the integrand is expanded by the formula of the Newton binomial.

(ii) p is a negative integer. Then we put $x = t^k$, where k is the L.C.M of the denominators of the fractions m and n.

(iii) $\frac{m+1}{n}$ is an integer. We put $a + bx^n = t^a$, where a is the denominator of the fraction p.

(iv) $\frac{m+1}{n} + p$ is an integer. We put $a + bx^n = t^a x^n$, where a is the denominator of the fraction p.

37. Integrals of the form $\int R(x, \sqrt{ax^2+bx+c})\,dx$ are calculated with the aid of one of the three Euler substitutions :

1. $\sqrt{ax^2+bx+c} = t \pm \sqrt{a}$ is $a > 0$,
2. $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$ if $c > 0$,
3. $\sqrt{ax^2+bx+c} = (x-a)\,t$ if $ax^2 + bx + c = a(x-a)(x-b)$ i.e. if a is real root of the trinomial ax^2+bx+c.

38. The method of undetermined coefficients can be applied to integrals of the form :

(i) $\int P_n(x)e^{kx}dx = Q_n(x)e^{kx}+C$, where $P_n(x)$ and $Q_n(x)$ are polynomials of degree n.

(ii) $\int P_n(x)\sin ax\,dx$ or $\int P_n(x)\cos ax\,dx = Q_n(x)\cos kx + R_n(x)\sin kx + C$ where $P_n(x)$ is a polynomial of degree n and $Q_n(x)$ and $R_n(x)$ are polynomials of degree n (or less than n).

(iii) $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx$

$= Q_{n-1}(x)\sqrt{ax^2+bx+c} + K\int \frac{dx}{\sqrt{ax^2+bx+c}}$,

where $P_n(x)$ is a polynomial of degree n and $Q_{n-1}(x)$ is a polynomial of degree $(n-1)$ with undetermined coefficients and K is a number.

39. Differentiation under the sign of integration

$$\frac{d}{dx}\int f(x,a)\,dx = \int \frac{d\,f(x,a)}{da}\,dx.$$

40. Integration under the sign of integration If $F(x,a) = \int f(x,a)\,dx$,

then $\int F(x,a)\,da = \int\left(\int f(x,a)\,da\right)dx.$

41. Integration of Implicit Functions If R(x, y) is a rational function of x and y, then, $\int R(x,y)\,dx = \int R\{f(t), y(t)\}\,f'(t)\,dt$, assuming that $x = f(t)$, $y = y(t)$.

42. It is known that the following integrals are non-elementary :

$\int \frac{e^x}{x}\,dx$ $\int \sin(x^2)dx$

$\int \cos(e^x)dx$ $\int \sqrt{x^3+1}\,dx$

$\int \frac{1}{\ln x}\,dx$ $\int \frac{\sin x}{x}\,dx$

$\int \frac{\cos x}{x}\,dx$ $\int \sqrt{1-k^2\sin^2 x}\,dx$, $(0 < k^2 < 1)$

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. If f be a continuous function satisfying $f'(\ln x) = \begin{cases} 1 & \text{for } 0 < x \le 1 \\ x & \text{for } x > 1 \end{cases}$ and $f(0) = 0$, then f(x) can be defined as

(A) $f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 1-e^x & \text{if } x > 0 \end{cases}$

(B) $f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ e^x-1 & \text{if } x > 0 \end{cases}$

(C) $f(x) = \begin{cases} x & \text{if } x < 0 \\ e^x & \text{if } x > 0 \end{cases}$

(D) $f(x) = \begin{cases} x & \text{if } x \le 0 \\ e^x-1 & \text{if } x > 0 \end{cases}$

2. The value of $\int \frac{dx}{x^{13}\left(x^6-1\right)}$ is

(A) $\ln\left(\left|\frac{x^6-1}{x^6}\right|\right) + x^{-6} + x^{-12} + C.$

(B) $\frac{1}{6}\ln\left(\left|\frac{x^6-1}{x^6}\right|\right)+\frac{1}{6}x^{-6}+\frac{1}{4}x^{-12}+C$

(C) $\frac{1}{6}\left|\left(\ln\left|\frac{x^6-1}{x^6}\right|\right)+x^{-6}+\frac{1}{2}x^{-12}\right|+C$

(D) None of these

3. $\int\sqrt{(1+\cos^2 x)}.\sin 2x.\cos 2x\,dx$ is

(A) $\frac{4}{10}\sqrt{(1+\cos^2 x)^3}\,(3-2\cos^2 x)+C$

(B) $\frac{2}{5}\sqrt{(1+\cos^2 x)^3}\,(3+2\cos^2 x)+C$

(C) $\frac{2}{5}\sqrt{(1+\cos^2 x)}\,(3-2\cos^2 x)+C$

(D) None of these

4. $\int\frac{\sqrt{\sin^3 2x}}{\sin^5 x}dx$ is

(A) $C-\frac{4\sqrt{2}}{5}\sqrt{\cot^7 x}$ (B) $C+\frac{4\sqrt{2}}{5}\sqrt{\cot^7 x}$

(C) $C-\frac{4\sqrt{2}}{5}\sqrt{\cot^5 x}$ (D) None of these

5. $\int\frac{x^3-x}{1+x^6}dx$ is

(A) $\frac{1}{6}\ln\frac{x^4-x^2+1}{(1+x^2)^2}+C$

(B) $\frac{1}{6}\tan^{-1}\frac{(x^2+1)^2}{2}+C$

(C) $\ln\frac{x^4-x^2+1}{(1+x^2)}+C$

(D) None of these

6. If $\int\frac{dx}{3\sin x+\sin^3 x}=\frac{1}{6}\ln\frac{t-1}{t+1}+\frac{1}{12}\ln\frac{2+t}{2-t}+C$, then

(A) $t=\sin x$ (B) $t=\tan x/2$

(C) $t=2\cos x$ (D) $t=\cos x$

7. If $\int\frac{dx}{\sin x.(2\cos^2 x-1)}$ is $-\frac{1}{2}f(x)+\frac{1}{\sqrt{2}}\ln\left|\frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}\right|+C$ Then f(x) is

(A) $\ln\left|\frac{1-\sin x}{1+\sin x}\right|$ (B) $\ln\left|\frac{1+\sin x}{1-\sin x}\right|$

(C) $\frac{e^x}{1+\sin x}$ (D) $\frac{e^x}{1+\cos x}$

8. $\int\frac{\sqrt[3]{x}}{x(\sqrt{x}+\sqrt[3]{x})}dx$ is

(A) $\tan^{-1}\left[\frac{x}{(\sqrt[6]{x}+1)}\right]+C$ (B) $\ln|x+(\sqrt[6]{x}+1)|+C$

(C) $\ln\frac{x}{(\sqrt[6]{x}+1)^6}+C$ (D) none of these

9. Let f(x) be a function satisfying $f'(x)=f(x)$ and $f(0)=2$. Then $\int\frac{f(x)}{3+4f(x)}dx$ is

(A) $1/4\ln(3+8e^x)+C$ (B) $1/8\ln(3+8e^x)+C$

(C) $1/2\ln(3+8e^x)+C$ (D) none of these

10. $\int\frac{\ln x}{x\sqrt{1+\ln x}}dx$ is equal to

(A) $1/3(1+\ln x)^{3/2}-\sqrt{1+\ln x}+C$

(B) $2/3(1+\ln x)^{3/2}-\sqrt{1+\ln x}+C$

(C) $2/3(1+\ln x)^{3/2}-2\sqrt{1+\ln x}+C$

(D) none of these

11. $\int\frac{\cos x+x\sin x}{x(x+\cos x)}dx$ is equal to

(A) $\ln\left|\frac{x}{x+\cos x}\right|+C$ (B) $\ln\left|\frac{x+\cos x}{x}\right|+C$

(C) $\ln\left|\frac{x\sin x}{x+\cos x}\right|+C$ (D) none of these

12. $\int\frac{e^{2x}dx}{\sqrt[4]{e^x+1}}$ is equal to

(A) $4/21(e^x+1)^{3/4}+C$

(B) $4/21(e^x+1)^{3/2}(3e^x-4)+C$

(C) $2/21(e^x+1)^{3/4}+C$

(D) none of these

13. If $f(x)=\lim_{n\to\infty}\frac{x^n-x^{-n}}{x^n+x^{-n}}$, $x>1$, then

$\int\frac{xf(x)\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$ is equal to

(A) $\ln(x+\sqrt{1+x^2})-x+C$

(B) $\frac{1}{2}\left[x^2\ln\left(x+\sqrt{1+x^2}\right)-x^2\right]+C$

(C) $\left(x\ln\left(x+\sqrt{1+x^2}\right)-\ln\left(x+\sqrt{1+x^2}\right)\right)+C$

(D) None of these

14. $\int \sin^2 (\ln x)\, dx$ is equal to

(A) $(x/10)(5 + 2\sin(2\ln x) + \cos(2\ln x)) + C$
(B) $(x/10)(5 + 2\sin(2\ln x) - \cos(2\ln x)) + C$
(C) $(x/10)(5 - 2\sin(2\ln x) - \cos(2\ln x)) + C$
(D) $(x/10)(5 - 2\sin(2\ln x) + \cos(2\ln x)) + C$

15. $\int e^x \dfrac{(1 + n.x^{n-1} - x^{2n})}{(1 - x^n)\sqrt{1 - x^{2n}}}\, dx$ is equal to

(A) $e^x\sqrt{\dfrac{1 - x^n}{1 + x^n}} + C$ (B) $e^x\sqrt{\dfrac{1 + x^n}{1 - x^n}} + C$

(C) $-e^x\sqrt{\dfrac{1 - x^n}{1 + x^n}} + C$ (D) $-e^x\sqrt{\dfrac{1 + x^n}{1 - x^n}} + C$

16. The value of the integral $\int \dfrac{dx}{x^n(1 + x^n)^{1/n}}$, $n \in N$ is

(A) $\dfrac{1}{(1-n)}\left(1 + \dfrac{1}{x^n}\right)^{1-1/n} + C$

(B) $\dfrac{1}{(1+n)}\left(1 - \dfrac{1}{x^n}\right)^{1+1/n} + C$

(C) $-\dfrac{1}{(1-n)}\left(1 - \dfrac{1}{x^n}\right)^{1-1/n} + C$

(D) $\dfrac{1}{(1-n)}\left(1 + \dfrac{1}{x^n}\right)^{1-1/n} + C$

17. Integral of $\sqrt{1 + 2\cot x\,(\cot x + \operatorname{cosec} x)}$ w.r.t. x is

(A) $2\, ln \cos\dfrac{x}{2} + C$

(B) $2\, ln \sin\dfrac{x}{2} + C$

(C) $\dfrac{1}{2}\, ln \cos\dfrac{x}{2} + C$

(D) $ln \sin x - ln(\operatorname{cosec} x - \cot x) + C$

18. $\int x . \dfrac{\ell n\left(x + \sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\, dx$ equals

(A) $ln(x + \sqrt{1+x^2}) - x + C$

(B) $\dfrac{x}{2} . ln^2(x + \sqrt{1+x^2}) - \dfrac{x}{\sqrt{1+x^2}} + C$

(C) $\dfrac{x}{2} . ln^2(x + \sqrt{1+x^2}) + \dfrac{x}{\sqrt{1+x^2}} + C$

(D) $ln(x + \sqrt{1+x^2}) + x + C$

19. The value of $\int \dfrac{(1 + \ln x)}{\sqrt{(x^x)^2 - 1}}\, dx$ is

(A) $\sec^{-1}(x^x) + C$
(B) $\tan^{-1}(x^x) + C$
(C) $\ln\left(x + \sqrt{(x^x)^2 - 1}\right) + C$
(D) None of these

20. If $\int \dfrac{(\sqrt{x})^5}{(\sqrt{x})^7 + x^6}\, dx = a \ln\left(\dfrac{x^k}{1 + x^k}\right) + c$, then the values of a and k are

(A) 2/5, 5/2 (B) 1/5, 2/5
(C) 5/2, 1/2 (D) 2/5, 1/2

21. The value of $\int \dfrac{\cos 7x - \cos 8x}{1 + 2\cos 5x}\, dx$ is equal to

(A) $\dfrac{\sin 2x}{2} + \dfrac{\cos 3x}{3} + C$
(B) $\sin x - \cos x + C$
(C) $\dfrac{\sin 2x}{2} - \dfrac{\cos 3x}{3} + C$
(D) none of these

22. The value of $\int \left(x + \dfrac{1}{x}\right)^{n+5} . \left(\dfrac{x^2 - 1}{x^2}\right) dx$, is equal to

(A) $\left(x + \dfrac{1}{x}\right)^{n+6} + C$ (B) $\dfrac{\left(x^2 - \dfrac{1}{x^2}\right)^{n+6}}{n+6} + C$

(C) $\dfrac{\left(x + \dfrac{1}{x}\right)^{n+6}}{n+6} + C$ (D) none of these

23. The value of $\int \dfrac{(1 - \cot^{n-2} x)dx}{\tan x + \cot x . \cot^{n-2} x}$ is equal to

(A) $\dfrac{1}{n} \ln|\sin^n x + \cos^n x| + C$

(B) $\dfrac{1}{2n} \ln|\sin^{n-1} x + \cos^{n-1} x| + C$

(C) $\dfrac{1}{(n-2)} \ln|\sin^{n-1} x + \cos^{n-1} x| + C$

(D) $\frac{2}{n-2}\ln|\sin^{n-1}x+\cos^{n-1}x|+C$

24. The value of $\int\frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$ is equal to

(A) $\frac{2(\sqrt{x}+1)}{\sqrt{1-x}}+C$ (B) $\frac{2(\sqrt{x}-1)}{\sqrt{1-x}}+C$

(C) $\frac{-2(\sqrt{x}-1)}{\sqrt{1-x}}+C$ (D) none of these

25. The value of $\int\frac{\sec x(2+\sec x)}{(1+2\sec x)^2}dx$, is equal to

(A) $\frac{\sin x}{2+\cos x}+c$ (B) $\frac{\cos x}{2+\cos x}+c$

(C) $\frac{-\sin x}{2+\sin x}+c$ (D) $-\frac{\cos x}{2+\sin x}+c$

26. The value of $\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx$ is equal to

(A) $\frac{e^x(x+1)}{(1+x^2)^{3/2}}+C$ (B) $\frac{e^x(1+x+x^2)}{(1+x^2)^{3/2}}+C$

(C) $\frac{e^x(1+x)}{(1+x^2)^{3/2}}+C$ (D) none of these

27. If $\int(\sin 3\theta+\sin\theta)\ e^{\sin Q}\cos Q\,d\,Q=(A\sin^3 Q$ $+B\cos^2 Q+C\sin Q+D\cos Q+E)\,e^{\sin Q}+F$, then

(A) $A=-4, B=12$ (B) $A=-4, B=-12$
(C) $A=4, B=12$ (D) $A=4, B=-12$

28. The value of

$$\int e^{(x\sin x+\cos x)}.\left(\frac{x^4\cos^3x-x\sin x+\cos x}{x^2\cos^2 x}\right)dx$$

is equal to

(A) $e^{x\sin x+\cos x}.\left(x+\frac{1}{x\cos x}\right)+C$

(B) $e^{x\sin x+\cos x}.\left(x\cos x+\frac{1}{x}\right)+C$

(C) $e^{x\sin x+\cos x}.\left(x-\frac{1}{x\cos x}\right)+C$

(D) none of these

29. The value of $\int\frac{dx}{\sec x+\operatorname{cosec}x}$ is equal to

(A) $\left\{(\sin x+\cos x)+\frac{1}{\sqrt{2}}\ln\left|\frac{\tan x/2-1-\sqrt{2}}{\tan x/2-1+\sqrt{2}}\right|\right\}+C$

(B) $2\left\{(\sin x+\cos x)+\frac{1}{\sqrt{2}}\ln\left|\frac{\tan x/2-1-\sqrt{2}}{\tan x/2-1+\sqrt{2}}\right|\right\}+C$

(C) $\frac{1}{2}\left\{(\sin x-\cos x)+\frac{1}{\sqrt{2}}\ln\left|\frac{\tan x/2-1-\sqrt{2}}{\tan x-1+\sqrt{2}}\right|\right\}+C$

(D) none of these

30. The value of $\int\frac{(1+x)}{x(1+xe^x)^2}dx$ is equal to

(A) $\ln\left|\frac{x}{1+xe^x}\right|+\frac{1}{(1+xe^x)}+C$

(B) $\ln\left|\frac{xe^x}{1+xe^x}\right|+\frac{1}{1+xe^x}+C$

(C) $\ln\left|\frac{xe^x}{1+e^x}\right|+\frac{1}{1+xe^x}+C$

(D) none of these

31. The value of $\int\frac{dx}{x+\sqrt{a^2-x^2}}$ is equal to

(A) $\frac{1}{2}\sin^{-1}\frac{x}{a}+\frac{1}{2}\ln|x+\sqrt{a^2-x^2}|+C$

(B) $\frac{1}{2}\sin^{-1}\frac{x}{a}-\frac{1}{2}\ln|x+\sqrt{a^2-x^2}|+C$

(C) $\frac{1}{2}\sin^{-1}\frac{x}{a}-\ln|x+\sqrt{a^2-x^2}|+C$

(D) $\frac{1}{2}\cos^{-1}\frac{x}{a}+\frac{1}{2}\ln|x+\sqrt{a^2-x^2}|+C$

32. The value of $\int\frac{dx}{(x+a)^{8/7}(x-b)^{6/7}}$ is equal to

(A) $\frac{3}{2(a+b)}\left(\frac{x+a}{x-b}\right)^{2/3}+c$

(B) $\frac{3}{(a+b)}\left(\frac{x-b}{x+a}\right)^{1/3}+c$

(C) $\frac{7}{(a+b)}\left(\frac{x-b}{x+a}\right)^{1/7}+c$

(D) none of these

33. The value of $\int\frac{\sec x\,dx}{\sqrt{\sin(2x+A)+\sin A}}$ is equal to

(A) $\sec A(\sqrt{\tan x+\sin A\cos A})+C$

(B) $\sqrt{2}$ sec A $\sqrt{\tan x.\sin A + \cos A)}$ + C

(C) $\frac{1}{\sqrt{2}}$ sec A (tan x. sin A + cos A)$^{3/2}$ + C

(D) none of these

34. If $I = \int \frac{1}{2p}\sqrt{\frac{p-1}{p+1}}\, dp = f(p) + c$, then f(p) is equal to

(A) $\frac{1}{2}\ln (p - \sqrt{p^2+1}) + C$

(B) $\left(\frac{1}{2}\cos^{-1} p + \frac{1}{2}\sec^{-1} p\right) + C$

(C) $\ln\sqrt{p+\sqrt{p^2+1}} - \frac{1}{2}\sec^{-1}p + C$

(D) none of these

35. $\int \frac{(2x^{12}+5x^9)}{(x^5+x^3+1)^3}\, dx$ is equal to

(A) $\frac{x^{10}}{2(x^5+x^3+1)^2} + C$

(B) $\frac{x^2+2x}{(x^5+x^3+1)} + C$

(C) $\ln (x^5+x^3+1+\sqrt{2x^7+5x^4}) + C$

(D) None of these

36. $\int e^{\tan x} (x \sec^2 x + \sin 2x)\, dx$ is equal to

(A) $e^{\tan x} (x + \sec^2 x) + C$

(B) $e^{\tan x}(x - \sec^2 x) + C$

(C) $e^{\tan x}(x \sec x + \sin x) + C$

(D) $e^{\tan x}(x - \cos^2 x) + C$

37. Let $x = f''(t) \cos t + f'(t) \sin t$ and $y = -f''(t) \sin t + f'(t) \cos t$. $\sqrt{\text{Then}}$ $\int \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2}$ dt equals $\sqrt{\text{Then}}$

(A) $f'(t) + f''(t) + C$ (B) $f''(t) + f'''(t) + C$

(C) $f(t) + f''(t) + C$ (D) $f'(t) - f''(t) + C$

38. $\int \frac{px^{p+2q-1} - qx^{q-1}}{x^{2p+2q} + 2x^{p+q} + 1} dx$

(A) $-\frac{x^p}{x^{p+q}+1} + C$ (B) $\frac{x^q}{x^{p+q}+1} + C$

(C) $-\frac{x^q}{x^{p+q}+1} + C$ (D) $\frac{x^p}{x^{p+q}+1} + C$

39. The value of the integral $\int \frac{(1-\cos\theta)^{2/7}}{(1+\cos\theta)^{9/7}}$ dq is

(A) $\frac{7}{11}\left(\tan\frac{\theta}{2}\right)^{\frac{11}{7}} + C$ (B) $\frac{7}{11}\left(\cos\frac{\theta}{2}\right)^{\frac{11}{7}} + C$

(C) $\frac{7}{11}\left(\sin\frac{\theta}{2}\right)^{\frac{11}{7}} + C$ (D) None of these

40. $\int e^{\tan^{-1} x} (1 + x + x^2)\, d(\cot^{-1} x)$ is equal to

(A) $-e^{\tan^{-1} x} + C$ (B) $e^{\tan^{-1} x} + C$

(C) $-x e^{\tan^{-1} x} + C$ (D) $x e^{\tan^{-1} x} + C$

41. If $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}} = -[f(x)]^{1/n} + c$, then f(x) is

(A) $(1 + x^n)$ (B) $1 + x^{-n}$

(C) $x^n + x^{-n}$ (D) None of these

42. $\int e^x \left(\frac{2\tan x}{1+\tan x} + \cot^2\left(x + \frac{\pi}{4}\right)\right) dx$ is equal to

(A) $e^x \tan\left(\frac{\pi}{4} - x\right) + C$ (B) $e^x \tan\left(x - \frac{\pi}{4}\right) + C$

(C) $e^x \tan\left(\frac{3\pi}{4} - x\right) + C$ (D) None of these

43. The value of the integral $\int (x^2 + x)(x^{-8} + 2x^{-9})^{1/10}\, dx$ is

(A) $\frac{5}{11}(x^2 + 2x)^{11/10} + c$

(B) $\frac{5}{6}(x + 1)^{11/10} + c$

(C) $\frac{6}{7}(x + 1)^{11/10} + c$

(D) None of these

44. $\int e^{x^4} (x + x^3 + 2x^5) e^{x^2}\, dx$ is equal to

(A) $\frac{1}{2} x e^{x^2} e^{x^4} + c$ (B) $\frac{1}{2} x^2 e^{x^4} + c$

(C) $\frac{1}{2} e^{x^2} e^{x^4} + c$ (D) $\frac{1}{2} x^2 e^{x^2} e^{x^4} + c$

45. $\int x\left(\frac{\ln a^{a^{x/2}}}{3a^{5x/2}b^{3x}} + \frac{\ln b^{b^x}}{2a^{2x}b^{4x}}\right) dx$ (where $a, b \in R^+$) equal to

(A) $\frac{1}{6 \ln a^2b^3}\, a^{2x}b^{3x} \ln \frac{a^{2x}b^{3x}}{e} + k$

(B) $\frac{1}{6\ln a^2b^3}\ \frac{1}{a^{2x}b^{3x}}\ln\frac{1}{ea^{2x}b^{3x}}+k$

(C) $\frac{1}{6\ln a^2b^3}\ \frac{1}{a^{2x}b^{3x}}\ln(a^{2x}b^{3x})+k$

(D) $-\frac{1}{6\ln a^2b^3}\frac{1}{a^{2x}b^{3x}}\ln(a^{2x}b^{3x})+k$

46. $\int\frac{\operatorname{cosec}^2x-2005}{\cos^{2005}x}dx$ is equal to

(A) $\frac{\cot}{(\cos x)^{2005}}+C$ (B) $\frac{\tan x}{(\cos x)^{2005}}+C$

(C) $\frac{-(\tan x)}{(\cos x)^{2005}}+C$ (D) None of these

47. If $xf(x)=3f^2(x)+2$, then

$\int\frac{2x^2-12xf(x)+f(x)}{(6f(x)-x)(x^2-f(x))^2}dx$ equals

(A) $\frac{1}{x^2-f(x)}+C$ (B) $\frac{1}{x^2+f(x)}+C$

(C) $\frac{1}{x-f(x)}+C$ (D) $\frac{1}{x+f(x)}+C$

48. If $\int f(x)\sin x\cos x\,dx=\frac{1}{2(b^2-a^2)}\ln f(x)+C$, then $f(x)$ is equal to

(A) $\frac{1}{a^2\sin^2x+b^2\cos^2x}$

(B) $\frac{1}{a^2\sin^2x-b^2\cos^2}$

(C) $\frac{1}{a^2\cos^2x+b^2\sin^2x}$

(D) $\frac{1}{a^2\cos^2x-b^2\sin^2x}$

49. The value of integral

$\int e^x\left(\frac{1}{\sqrt{1+x^2}}+\frac{1-2x^2}{\sqrt{(1+x^2)^5}}\right)dx$ is equal to

(A) $e^x\left(\frac{1}{\sqrt{1+x^2}}+\frac{x}{\sqrt{(1+x^2)^3}}\right)+C$

(B) $e^x\left(\frac{1}{\sqrt{1+x^2}}-\frac{x}{\sqrt{(1+x^2)^3}}\right)+C$

(C) $e^x\left(\frac{1}{\sqrt{1+x^2}}+\frac{x}{\sqrt{(1+x^2)^5}}\right)+C$

(D) None of these

50. If $I=\int\frac{\sin 2x}{(3+4\cos x)^3}dx$, then I equals

(A) $\frac{3\cos x+8}{(3+4\cos x)^2}+C$

(B) $\frac{3+8\cos x}{16(3+4\cos x)^2}+C$

(C) $\frac{3+\cos x}{(3+4\cos x)^2}+C$

(D) $\frac{3-8\cos x}{16(3+4\cos x)^2}+C$

51. $\int x\sin x\sec^3x\,dx$ is equal to

(A) $\frac{1}{2}[\sec^2x-\tan x]+C$

(B) $\frac{1}{2}[x\sec^2x-\tan x]+C$

(C) $\frac{1}{2}[x\sec^2x+\tan x]+C$

(D) $\frac{1}{2}[\sec^2x+\tan x]+C$

52. $\int\frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$ is equal to

(A) $2\log\left(\frac{1+\sin x}{2+\sin x}\right)+C$

(B) $\log\left(\frac{2+\sin x}{2+\sin x}\right)+C$

(C) $\log\left(\frac{1+\sin x}{2+\sin x}\right)+C$

(D) None of these

53. $\int\frac{\cos^2x.dx}{\sin x.\cos 3x}$ is equal to

(A) $\ln \frac{|C\sin x|}{\sqrt{1-4\sin^2 x}}$ (B) $\ln \frac{|C\cos x|}{\sqrt{\cos 2x}}$

(C) $\ln \frac{|C\sin x|}{\sqrt{1-4\sin^2 x}}$ (D) None of these

54. $\int \frac{x\,\ell nx}{(x^2-1)^{3/2}}\,dx$ equals

(A) $\sec^{-1}x - \frac{\ell nx}{\sqrt{x^2-1}} + C$

(B) $\sec^{-1}x + \frac{\ell nx}{\sqrt{x^2-1}} + C$

(C) $\cos^{-1}x - \frac{\ell nx}{\sqrt{x^2-1}} + C$

(D) none of these

55. $\int \frac{a+b\cos x}{(b+a\cos x)^2}\,dx$ is equal to

(A) $\frac{\sin x}{(b+a\cos x)}+C$ (B) $\frac{\cos x}{(b+a\sin x)}+C$

(C) $\frac{\sin x}{(b+a\sin x)}+C$ (D) $\frac{\cos x}{(b+a\cos x)}+C$

56. $\int \frac{\sqrt{x-1}}{x\sqrt{x+1}}\,dx$ is equal to

(A) $\ln|x-\sqrt{x^2-1}| - \tan^{-1}x + C$

(B) $\ln|x+\sqrt{x^2-1}| - \tan^{-1}x + C$

(C) $\ln|x-\sqrt{x^2-1}| - \sec^{-1}x + C$

(D) $\ln|x+\sqrt{x^2-1}| - \sec^{-1}x + C$

57. $\int \sin^2(\ln x)\,dx$ is equal to

(A) $x/10\,(5+2\sin(2\ln x)+\cos(2\ln x))+C$
(B) $x/10\,(5+2\sin(2\ln x)-\cos(2\ln x))+C$
(C) $x/10\,(5-2\sin(2\ln x)-\cos(2\ln x))+C$
(D) $x/10\,(5-2\sin(2\ln x)+\cos(2\ln x))+C$

58. $\int \frac{x.e^x}{\sqrt{1+e^x}}\,dx$ is equal to

(A) $\ln\sqrt{\frac{1+\sqrt{1+e^x}}{\sqrt{1+e^x}-1}} + (2x+1)\sqrt{1+e^x} + C$

(B) $(2x+1)\sqrt{1+e^x} - \ln\sqrt{\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1}} + C$

(C) $\ln\sqrt{\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}} - (2x+1)\sqrt{1+e^x} + C$

(D) none of these

59. If $\int \frac{\sin 2x}{\sin^4 x+\cos^4 x} = dx = a\cot^{-1}(b\tan^2 x)+c$, then

(A) $a=1, b=-1$ (B) $a=-1, b=1$
(C) $a=-1, b=-1$ (D) none of these

60. If $\int f(x)dx = g(x)$, then $\int f^{-1}(x)\,dx$ is equal to

(A) $g^{-1}(x)$ (B) $x\,f^{-1}(x)-g(f^{-1}(x))$
(C) $xf^{-1}(x)-g^{-1}(x)$ (D) $f^{-1}(x)$

61. The value of $\int \frac{f(x)\varphi'(x)+\varphi(x)f'(x)}{(f(x)\varphi(x)+1)\sqrt{f(x)\varphi(x)-1}}\,dx$ is

(A) $\sin^{-1}\sqrt{\frac{f(x)}{\varphi(x)}}$

(B) $\cos^{-1}\sqrt{f^2(x)-\varphi^2(x)}$

(C) $\tan^{-1}[f(x).f(x)]$

(D) none of these

62. $\int \frac{(1+x\cos x)}{x(1-x^2e^{2\sin x})}\,dx$ is equal to

(A) $\log|x\,e^{\sin x}| + 1/2\log|1-x^2e^{2\sin x}|+C$
(B) $\log|x\,e^{\sin x}| - 1/2\log|1-x^2e^{2\sin x}|+C$
(C) $\log|x\,e^{\sin x}| - 1/2\log|1+x^2e^{2\sin x}|+C$
(D) none of these

63. $\int \frac{dx}{\tan x.\cos 2x}$ is equal to

(A) $\ln\frac{|C\sin x|}{\sqrt{\cos 2x}}$ (B) $\ln\frac{|C\cos x|}{\sqrt{\cos 2x}}$

(C) $\ln\frac{|C\sin x|}{\sqrt{1-4\sin^2 x}}$ (D) none of these

64. If the anti-derivative of $\frac{x^3}{\sqrt{1+2x^2}}$ which passes through (1, 2) is $\frac{1}{m}(1+2x^2)^{1/2}(x^2-1)+2$. Then the value of m is

(A) 1 (B) 3
(C) 5 (D) 6

65. If $I_n = \int \cot^n x\,dx$ and $I_0+I_1+2(I_2+I_3+.........+I_8)$

$+ I_9 + I_{10}$ is $l\left(u+\frac{u^2}{2}+\frac{u^3}{3}+.....+\frac{u^9}{9}\right)$

(where $u = \cot x$) Then the value of l is

(A) 1 (B) –1

(C) 2 (D) –2

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

66. If $\int \frac{\left(x^{-7/6}-x^{5/6}\right)}{x^{1/3}\left(x^2+x+1\right)^{1/2}-x^{1/2}\left(x^2+x+1\right)^{1/3}}dx$

$= -A\left\{\frac{z^3}{3}+\frac{3z^B}{2}+\frac{3z^C}{D}+\log|z|\right\}+c$ where

$z=\left(x+\frac{1}{x}+1\right)^{1/6}-1$, then

(A) $A=6$ (B) $B+C=4$

(C) $B+C=3$ (D) $B+C+D=5$

67. $\int \frac{1-\sin x}{\cos x}dx$ equals

(A) $ln(1+\sin x)+C$

(B) $2\,ln\left(\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)+C$

(C) $2\,ln\left(\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)+C$

(D) $2\,ln\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+C$

68. $\int \frac{x^3\,dx}{\sqrt{1+x^2}}$ equals

(A) $\frac{\left(1+x^2\right)^{3/2}}{3}-\sqrt{1+x^2}+C$

(B) $x^2\sqrt{1+x^2}-\frac{1}{3}\sqrt{\left(1+x^2\right)^3}+C$

(C) $x^2\sqrt{1+x^2}-\frac{2}{3}\sqrt{\left(1+x^2\right)^3}+C$

(D) none of these

69. $\int \frac{\ell n(\tan x)}{\sin x\,\cos x}dx$ equal

(A) $\frac{1}{2}\ln^2(\cot x)+C$

(B) $\frac{1}{2}\ln^2(\sec x)+C$

(C) $\frac{1}{2}\ln^2(\sin x\sec x)+C$

(D) $\frac{1}{2}\ln^2(\cos x\,\mathrm{cosec}\,x)+C$

70. $\int \sqrt{\frac{x-1}{x+1}}\cdot\frac{1}{x^2}dx$ equals

(A) $\sin^{-1}\frac{1}{x}+\frac{\sqrt{x^2-1}}{x}+C$

(B) $\frac{\sqrt{x^2-1}}{x}+\cos^{-1}\frac{1}{x}+C$

(C) $\sec^{-1}x-\frac{\sqrt{x^2-1}}{x}+C$

(D) $\tan^{-1}\sqrt{x^2+1}-\frac{\sqrt{x^2-1}}{x}+C$

71. A curve $g(x)=\int x^{27}(1+x+x^2)^6(6x^2+5x+4)\,dx$ is passing through origin, then

(A) $g(1)=\frac{3^7}{7}$ (B) $g(1)=\frac{2^7}{7}$

(C) $g(-1)=\frac{1}{7}$ (D) $g(-1)=\frac{2}{7}$

72. If $I=\int\frac{\sin x+\sin^3 x}{\cos 2x}dx = P\cos x + Q\ln|f(x)| + R$ then

(A) $P=1/2, Q=-\frac{3}{4\sqrt{2}}$

(B) $P=1/4, Q=-\frac{1}{\sqrt{2}}, f(x)=\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}$

(C) $P=1/2, Q=-\frac{3}{4\sqrt{2}}, f(x)=\frac{\sqrt{2}\cos x+1}{\sqrt{2}\cos x-1}$

(D) $P=-1/2, Q=-\frac{3}{4\sqrt{2}}, f(x)=\frac{\sqrt{2}\cos x+1}{\sqrt{2}\cos x-1}$

73. If $\int\frac{e^{x-1}}{(x^2-5x+4)}2x\,dx = AF(x-1)+BF(x-4)$

$+C$ and $F(x)=\int \frac{e^x}{x}dx$, then

(A) $A=-2/3$
(B) $B=4/3\,e^3$
(C) $A=2/3$
(D) $B=8/3\,e^3$

74. If $\int \frac{x^2-x+1}{(x^2+1)^2}e^x\,dx=e^x f(x)+c$, then

(A) f(x) is an even function
(B) f(x) is a bounded function
(C) The range of f(x) is (0, 1]
(D) f(x) has two points of extrema

75. $\int \sin^{-1}x\cos^{-1}x\,dx=f^{-1}(x)\,[Ax-xf^{-1}(x)-\sqrt{1-x^2}\,]$ $+2x+C$, then

(A) $f(x)=\sin x$
(B) $f(x)=\cos x$
(C) $A=\frac{\pi}{4}$
(D) $A=\frac{\pi}{2}$

76. $\int \frac{dx}{x^2+ax+1}=f(g(x))+c$, then

(A) f(x) is inverse trigonometric function for $|a|>2$
(B) f(x) is logarithmic function for $|a|<2$
(C) g(x) is quadratic function for $|a|>2$
(D) g(x) is rational function for $|a|<2$

77. $\int \frac{\sin x\cos x}{\sqrt{1-\sin^4 x}}dx$ is equal to

(A) $\frac{1}{2}\sin^{-1}(\sin^2 x)+C$

(B) $-\frac{1}{2}\cos^{-1}(\sin^2 x)+C$

(C) $\tan^{-1}(\sin^2 x)+C$

(D) $\cos^{-1}\left(\frac{\cos x}{\sqrt{2}}\right)+C$

78. If $\int xe^{-5x^2}\sin 4x^2\,dx=Ke^{-5x^2}(A\sin 4x^2$ $+B\cos 4x^2)+C$. Then

(A) $K=-1/82$
(B) $K=1/82$
(C) $A=5$
(D) None of these

79. $\int 2^{mx}.\,3^{nx}\,dx$ when $m, n\in N$ is equal to

(A) $\frac{2^{mx}+3^{mx}}{m\ell n2+n\ell n3}+C$
(B) $\frac{e^{(m\ell n2+n\ell n3)x}}{m\ell n2+n\ell n3}+C$
(C) $\frac{2^{mx}.3^{mx}}{\ell n(2^m.3^n)}+C$
(D) $\frac{(mn).2^x.3^x}{m\ell n2+n\ell n3}+C$

80. $\int \frac{\ell n(\tan x)}{\sin x\cos x}dx$ equal

(A) $1/2\ln^2(\cot x)+C$
(B) $1/2\ln^2(\sec x)+C$
(C) $1/2\ln^2(\sin x\sec x)+C$
(D) $1/2\ln^2(\cos x\,\mathrm{cosec}\,x)+C$

81. $\int \sec^2\left(2x-\frac{\pi}{4}\right)dx$ equals

(A) $C-1/2\cot(2x+\pi/4)$
(B) $1/2\tan(2x-\pi/4)+C$
(C) $1/2(\tan 4x-\sec 4x)+C$
(D) None of these

82. $\int \frac{x^2+\cos^2 x}{1+x^2}\mathrm{cosec}^2 x\,dx$ is equal to

(A) $\cot x-\cot^{-1}x+C$
(B) $C-\cot x+\cot^{-1}x$
(C) $-\tan^{-1}x-\frac{\mathrm{cosec}\,x}{\sec x}+C$
(D) $e^{\ell n\tan^{-1}x}-\cot x+C$

83. If $\int \frac{dx}{x^4(1+x^3)^2}=a\ln\left|\frac{1+x^3}{x^3}\right|+\frac{b}{x^3}+\frac{c}{(1+x^3)}+d$, then

(A) $a=1/3, c=1/3$
(B) $b=-1/3, c=-1/3$
(C) $a=2/3, b=-1/3$
(D) $a=2/3, b=1/3$

84. If $\int \frac{1-x^7}{x(1+x^7)}dx=a\ln|x|+b\ln|x^7+1|+c$, then

(A) $a=1$
(B) $a=-1$
(C) $b=-2/7$
(D) $b=1/7$

85. If $\int e^x\left\{b\ln(x^2+1)+\frac{cx}{x^2+1}\right\}dx$ $=\frac{c}{2}e^x\ln(x^2+1)+K$, then the values of b and c can be

(A) $b=1, c=2$
(B) $b=1/3, c=1/2$
(C) $b=1/2, c=1$
(D) $b=2, c=3$

Comprehension - 1

In a certain problem the differentiation of product (f(x).g(x)) appears. One student commits mistake and differentiates as $\frac{df}{dx}\cdot\frac{dg}{dx}$ but he gets correct result if $f(x) = x^3$ and $g(4) = 9$, $g(2) = -9$ and $g(0) = -\frac{1}{3}$.

86. The function g(x) is

(A) $\frac{3}{(x-3)^3}$ (B) $\frac{4}{(x-3)^3}$

(C) $\frac{9}{(x-3)^3}$ (D) $\frac{27}{(x-3)^3}$

87. The derivative of $f(x-3)\,.\,g(x)$ with respect to x at $x = 100$ is

(A) 0 (B) 1

(C) –1 (D) 2

88. $\lim\limits_{x\to 0}\frac{f(x).g(x)}{x(1+g(x))}$ will be

(A) 0 (B) –1

(C) 1 (D) 2

Comprehension - 2

Consider a differentiable function $f : R \to R$ which satisfies $f^2\left(\frac{1}{\sqrt{2}}x\right) = f(x)$ for all $x \in R$ and $f(1) = 2$, $f(0) \neq 0$.

89. If $\alpha, \beta \in R$ satisfying $\alpha^2 + \beta^2 = 1$, then for all x, $f(\alpha x)\,.\,f(\beta x) =$

(A) 2f(x) (B) f(x)

(C) $abf^2(x)$ (D) None of these

90. The value of $\lim\limits_{x\to\infty}\frac{f(x)-x^2}{f(x)+2^x}$ is

(A) 1 (B) $\frac{1}{2}$

(C) 0 (D) None of these

91. $\int f(x)\,.\,ln\,2^{4x}$ is equal to

(A) $2^{x^2+1} + C$ (B) $2^{2x} + C$

(C) $2^{2x^2-1} + C$ (D) $2^{2x^2} + C$

Comprehension - 3

If A is square matrix and e^A is defined as $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \frac{1}{2}\begin{bmatrix} f(x) & g(x) \\ g(x) & f(x)\end{bmatrix}$, where $A = \begin{bmatrix} x & x \\ x & x\end{bmatrix}$ and $0 < x < 1$, I is an identity matrix.

92. $\int\frac{g(x)}{f(x)}dx$ is equal to

(A) $\ln(e^x + e^{-x}) + C$ (B) $\ln(e^x - e^{-x}) + C$

(C) $\ln(e^{2x} - 1) + C$ (D) None of these

93. $\int (g(x) + 1)\sin x\,dx$ is equal to

A) $\frac{e^x}{2}(\sin x - \cos x)$

(B) $\frac{e^{2x}}{5}(2\sin x - \cos x)$

(C) $\frac{e^x}{5}(\sin 2x - \cos 2x)$

(D) None of these

94. dx is equal to

(A) $-\operatorname{cosec}^{-1}(e^x) + C$

(B) $-\sec^{-1}(e^x) + C$

(C) $+\sec^{-1}(e^x) + C$

(D) None of these

Comprehension - 4

Let n be a positive integer or zero and let $I_n =$ dx (a > 0) We can find the reduction formula as $I_n = -+ a^2B\,I_{n-2}$, where A and B are constants. Also $I_1 = -(a^2 - x^2)^{3/2}$.

95. A must be equal to
(A) $n+1$ (B) $n-1$
(C) $n+2$ (D) n

96. B must be equal to
(A) $\frac{n+1}{n+2}$ (B) $\frac{n}{n+2}$
(C) $\frac{n+2}{n+1}$ (D) $\frac{n-1}{n+2}$

97. The value of integral $\int_0^a x^4\sqrt{a^2-x^2}\,dx$ must be equal to
(A) $\frac{\pi a^4}{32}$ (B) $\frac{\pi a^4}{16}$
(C) $\frac{\pi a^4}{64}$ (D) none

Comprehension - 5

Let $\int \frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx = fog(x)+C$, where f and g are some functions and C is an arbitrary constant.

98. The nature of the function $y = f(x)+(g(x))^2$ is
(A) even
(B) odd
(C) neither even nor odd
(D) one-one

99. If $\int f(x)\ g(x)\ dx = ax^3\ g(x) + b(1+x^2)^{3/2} + c(1+x^2)^{1/2} + d$, then $a+c$ is equal to
(A) $\frac{1}{2}$ (B) $\frac{1}{3}$
(C) 0 (D) 1

100. If $\int e^{g(x)}\ dx = ax\left(x+\sqrt{1+x^2}+ag(x)+x\right)$ then a is equal to
(A) $\frac{1}{2}$ (B) $\frac{1}{3}$
(C) 2 (D) 3

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

101. Assertion (A): $\int_1^2 \frac{9x+4}{x^5+3x^2+x}dx = \ell n\frac{80}{23}$

Reason (R): The function $\frac{9x+4}{x^5+3x^2+x}$ has a non-elementary antiderivative.

102. Assertion (A): $\int 5^x(\sec^2 x+\tan x\ln 5)\ dx = 5^x\tan x + C$

Reason (R): If $a>0$, then
$\int a^x(f(x)+f'(x)\log_a e)dx = a^x f(x)+C$.

103. Assertion (A): $\int \sin x\ .\ \cos x\ .\ \cos 2x\ .\ \cos 4x\ .\ \cos 8x\ .\ \cos 16x\ dx = -\frac{\cos 32x}{1024}+C.$

Reason (R) : $\int \sin 2^n x\ dx = -\frac{\cos 2^n x}{2^n}+C$.

104. Assertion (A): $\int \sqrt{1+\operatorname{cosec} x}\ dx$
$= \sin^{-1}\sqrt{\sin x} + \frac{1}{2}\cos^{-1}(1-2\sin x)+C.$

Reason (R): $\sqrt{1+\operatorname{cosec} x} =$
$$\frac{\sqrt{(1+\sin x)(1-\sin x)}}{\sqrt{\sin x(1-\sin x)}} = \frac{\cos x}{\sqrt{\sin x(1-\sin x)}}$$
$= \frac{\cos x}{\sqrt{\frac{1}{4}-\left(\frac{1}{2}-\sin x\right)^2}}$ for all x in the domain.

105. Assertion (A): $\int \frac{\tan(\ln x)\tan\left(\ln\frac{x}{2}\right)\tan(\ln 2)}{x}dx$
is equal to $\ln\left(\frac{\sec(\ln x)}{\sec\left(\ln\frac{x}{2}\right)x^{\tan(\ln 2)}}\right)+C$

Reason (R) : We have $\tan(A+B)\tan A\ \tan B = \tan(A+B)-\tan A-\tan B$ and $\int\frac{\tan(\ln x)}{x}dx = \ln\sec(\ln x)+C.$

106. **Assertion (A):** If $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7 + x^6} dx = a \ln \frac{x^k}{x^k + 1} + c$, the value of $(a + k)$ is 9/2.

Reason (R) : The given integral reduces to the form $\int \frac{f'(x)}{f(x)} dx$ where $f(x) = (x^{-5/2} + 1)$.

107. **Assertion (A):** If $\int \frac{1}{f(x)} dx = \ln(f(x))^2 + c$ then $f(x) = \frac{1}{2} x$.

Reason (R): When $f(x) = \frac{x}{2}$, $\int \frac{1}{f(x)} dx = \int \frac{2}{x} dx = 2 \ln|x| + c$

108. **Assertion (A):** For $-1 < a < 4$,

$$\int \frac{dx}{x^2 + 2(a-1)x + a + 5} = \lambda \ln |g(x)| + c,$$ where λ and c are constants.

Reason (R): For $-1 < a < 4$, $\frac{1}{x^2 + 2(a-1)x + a + 5}$ is a continuous function.

109. **Assertion (A):** If the primitive of $f(x) = \pi \sin \pi x + 2x - 4$, has the value 3 for $x = 1$, then there are exactly two values of x for which the primitive of f(x) vanishes.

Reason (R): $\cos \pi x$ has period 2.

110. **Assertion (A):** $\int \frac{\{f(x)\varphi'(x) - f'(x)\varphi(x)\}}{f(x)\varphi(x)} \{\ln \phi(x) - \ln f(x)\}\, dx = \frac{1}{2}\left\{\ln \frac{\varphi(x)}{f(x)}\right\}^2 + C.$

Reason (R): $\int (h(x))^n h'(x)\, dx = \frac{(h(x))^{n+1}}{n+1} + C.$

MATCH THE COLUMNS FOR JEE ADVANCED

111.

Column-I	**Column-II**
(A) If $\int \frac{2^x}{\sqrt{1-4^x}} dx = k \sin^{-1}(f(x)) + C$, then k is greater than	(P) 0
(B) If $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7 + x^6} dx = a \ln \frac{x^k}{x^k + 1} + c$, then ak is less than	(Q) 1
(C) $\int \frac{x^4 + 1}{x(x^2+1)^2} dx = k \ln|x| + \frac{m}{1+x^2} + n$, where n is the constant of integration, then mk is greater than	(R) 3
(D) $\int \frac{dx}{5 + 4\cos x} = k \tan^{-1}\left(m \tan \frac{x}{2}\right) + C$, then k/m is greater than	(S) 4

112.

Column-I	**Column-II**
(A) Let $f(x) = \int x^{\sin x} (1 + x \cos \cdot \ell n\, x + \sin x)\, dx$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$ of $f(\pi)$ is	(P) rational then the value
(B) Let $g(x) = \int \frac{1 + 2\cos x}{(\cos x + 2)^2} dx$ and $g(0) = 0$ then the value of $g(\pi/2)$ is	(Q) irrational
(C) If real numbers x and y satisfy $(x + 5)^2 + (y - 12)^2$	(R) integral the minimum

$= (14)^2$ then value of $\sqrt{(x^2+y^2)}$ is

(D) Let $k(x) = \int \frac{(x^2+1)dx}{\sqrt[3]{x^3+3x+6}}$ and $k(-1) = \frac{1}{\sqrt[3]{2}}$ then the value of $k(-2)$ is

(S) prime

113. Column-I **Column-II**

(A) $\int \frac{dx}{\sin x + \sin 2x}$

(B) $\int \frac{\sin x + \sin 2x}{\sqrt{\cos x + \cos 2x}} dx$

(C) $\int \frac{2\tan x + 3}{\sin^2 x + 2\cos^2 x}$

(D) $\int \frac{\tan x}{\sqrt{\sin^4 x + \cos^4 x}} dx$

(P) $-\sqrt{2\cos^2 x + \cos x - 1} - \frac{1}{2\sqrt{2}} \log|4 \cos x + 1 + \sqrt[4]{\cos^2 x + \frac{1}{2}\cos x - \frac{1}{2}}| + C$

(Q) $\frac{1}{6} \log(1 - \cos x\_ + \frac{1}{2} \log (1+\cos x) - \frac{2}{3} \log|1 + 2\cos x| + C$

(R) $\frac{1}{2} \log\left(\tan^2 x + \sqrt{1+\tan^4 x}\right) + C$

(S) $\log(2 + \tan^2 x \frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x}{\sqrt{2}}\right) + C$

114. Column-I **Column-II**

(A) If $\int x^2 d(\tan^{-1} x) = x f(x) + c$, then f(1) is equal to

(B) If $\int \sqrt{1 + 3\tan x(\tan x + \sec x)}\, dx = a \log\left|\cos\frac{x}{2} - \sin\frac{x}{2}\right| + C$

(C) If $\int x^2 e^{2x} dx = e^{2x} f(x) + c$, then the minimum value of

(D) If $\int \frac{x^4+1}{x(x^2+1)^2} dx = a \log|x| + \frac{b}{x^2+1} + c$ then a – b is equal to

(P) 0

(Q) –2 then a is equal to $(0 < x < \frac{\pi}{2})$

(R) $\frac{\pi}{4}$

(S) 1 f(x) is equal to

(T) $\frac{1}{8}$

115. If $x \in (0, 1)$ then match the entries of column-A with column-B considering 'c' as an arbitrary constant of integration.

Column-I	Column-II
(A) $\int \tan\left(2\tan^{-1}\sqrt{\dfrac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}}\right)dx$	(P) $\dfrac{4}{3}x^{3/4}+C$
(B) $\int \cot\left(2\tan^{-1}\sqrt{\dfrac{\sqrt{1+\sqrt{x}}-\sqrt[4]{x}}{\sqrt{1+\sqrt{x}}+\sqrt[4]{x}}}\right)dx$	(Q) $\dfrac{4}{5}x^{5/4}+C$
(C) $\int\left(\dfrac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}\right)dx$	(R) $\dfrac{2}{3}x^{3/4}+C$
(D) $\int\sqrt{x}\tan\left(2\tan^{-1}\left(\dfrac{\sqrt{\sqrt{1+\sqrt{x}}+1}-\sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1}+\sqrt{\sqrt{1+\sqrt{x}}-1}}\right)\right)dx$	(S) $\dfrac{2}{5}x^{5/4}+C$

Review Exercises for JEE Advanced

1. Evaluate $\displaystyle\int\frac{d\theta}{\tan\theta+\cot\theta+\sec\theta+\operatorname{cosec}\theta}$

2. Evaluate $\int x^{x+1}\ln x(1+\ln x)\,dx$.

3. Evaluate $\displaystyle\int\frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{(4x^2+1)(x-2)^2}dx$

4. Evaluate $\displaystyle\int\frac{1+x^2}{1-x^2}\frac{dx}{\sqrt{1+x^2+x^4}}$.

5. Evaluate $\displaystyle\int\frac{dx}{(1+x^4)\{(1+x^4)^{1/2}-x^2\}^{1/2}}$.

6. Evaluate the following integrals :

(i) $\displaystyle\int\frac{x^7}{(1-x^2)^5}dx$ (ii) $\displaystyle\int\frac{x^7dx}{(1-x^4)^2}$

7. Evaluate $\displaystyle\int\sqrt{\frac{a^2-x^2}{x^2-b^2}}\frac{dx}{x}$.

8. Prove that $\displaystyle\int\frac{e^x\,dx}{x^4}$

$\displaystyle=-\frac{e^2}{3}\left\{\frac{1}{x^3}+\frac{1}{2x^2}+\frac{1}{2x}\right\}+\frac{1}{3.2.1}\int\frac{e^xdx}{x}$.

9. Evaluate $\displaystyle\int\frac{\sin x\,dx}{\sqrt{(a^2\cos^2x+b^2\sin^2x)}}$.

10. Integrate $\displaystyle\int\frac{dx}{x\sqrt{(x^2+2x-1)}}$, by the substitution $z=x+\sqrt{(x^2+2x-1)}$ and show that the value is $2\tan^{-1}\{x+\sqrt{(x^2+2x-1)}\}+C$.

11. Evaluate

$$\int\frac{\sin^3x\,dx}{(\cos^4x+3\cos^2x+1)\tan^{-1}(\sec x+\cos x)}$$

12. Let f(x) be a polynomial of degree three such that $f(0)=1$, $f(1)=2$, $x=0$ is a critical point but f(x) does not have local extremum at $x=0$. Then evaluate

$$\int\frac{f(x)}{\sqrt{x^2+7}}dx.$$

13. Evaluate $\displaystyle\int\frac{3x^3+5x^2-7x+9}{\sqrt{2x^2+5x+7}}dx$

14. Evaluate $\displaystyle\int\frac{2e^{5x}+e^{4x}-4e^{3x}+4e^{2x}+2e^x}{(e^{2x}+4)(e^{2x}-1)^2}$

$\displaystyle=\tan^{-1}\left(\frac{e^x}{2}\right)-\frac{1}{2(e^{2x}-1)}+C$

15. Determine the coefficients A, B so that

$$\int \frac{dx}{(a+b\cos x)^2} = \frac{A\sin x}{a+b\cos x} + B\int \frac{dx}{a+b\cos x}.$$

16. Evaluate the following integrals :

(i) $\int \frac{(x+1)^4 dx}{(x^2+2x+2)^3}$ (ii) $\int \frac{dx}{(1+x^2)^4}$

(iii) $\int \frac{5-3x+6x^2+5x^3-x^4}{x^5-x^4-2x^3+2x^2+x-1}dx$

(iv) $\int \frac{x^4+8x^3-x^2+2x+1}{(x^2+x)(x^3+1)}dx.$

17. Evaluate the following integrals :

(i) $\int \frac{e^x(x^2+5x+7)}{(x+3)^2}dx$

(ii) $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

(iii) $\int \frac{\sqrt{1+\sin 2x}}{(1+\cos 2x)e^{-x}}dx$

(iv) $\int \frac{\ell n\, x}{(1+\ell n\, x)^2}dx$

18. Evaluate the following integrals :

(i) $\int \frac{1}{\sin x\sqrt{\sin x(1+\sin x)}}dx$

(ii) $\int \frac{a\sin x}{\cos x\sqrt{\cos^2 x-a^2\sin^2 x}}dx.$

19. Evaluate the following integrals :

(i) $\int \frac{x^2-4}{x^6-2x^4+x^2}dx$

(ii) $\int \frac{(x^2-1)^2 dx}{(1+x)(1+x^2)^3}.$

20. Evaluate the following integrals :

(i) $\int \frac{7\cos x+3\sin x+5}{3\cos x+4\sin x+5}dx$

(ii) $\int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}dx.$

21. Evaluate the following integrals :

(i) $\int \frac{\sin 2x\, dx}{(a+b\cos x)^2}$

(ii) $\int \frac{a+b\sin x}{(b+a\sin x)^2}dx.$

22. Evaluate the following integrals :

(i) $\int \frac{m.x^{m+2n-1}-n.x^{n-1}}{x^{2m+2n}+2x^{m+n}+1}dx$

(ii) $\int \frac{m.x^{m+n}-n}{x^{m+n+1}+x}dx$

23 Evaluate the following integrals:

(i) $\int \frac{x^3 dx}{(a+cx^2)^4}$ (ii) $\int \frac{dx}{x(a+bx^n)^2}$

(iii) $\int \frac{x^2 dx}{(1-x^2)^3}$ (iv) $\int \frac{(1-x^2)dx}{x(1+x^2+x^4)}.$

24. Evaluate the following integrals:

(i) $\int \frac{x^2-3}{x^4+2x^2+9}dx$ (ii) $\int \frac{x^2+2x+2}{x^4-2x^2+4}dx$

(iii) $\int \frac{1}{x^4-a^2x^2+a^4}dx$ (iv) $\int \frac{x^2+5}{x^4-2x^2+4}dx.$

25. Evaluate the following integrals:

(i) $\int \frac{dx}{(x-2)^{7/8}(x+3)^{9/8}}$

(ii) $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

(iii) $\int \frac{2x^2+7x+11}{(x+2)\sqrt{x^2+4x+8}}$

(iv) $\int \frac{1}{\sqrt{(x+1)}-\sqrt[4]{(x+1)}}dx.$

26. Evaluate the following integrals :

(i) $\int x(\cos^2 x)e^{-x}dx$ (ii) $\int \frac{xe^{a\tan^{-1}x}dx}{(1+x^2)^{3/2}}$

27. Evaluate the following integrals :

(i) $\int \frac{\cos^4 x\, dx}{\sin^3 x(\sin^5 x+\cos^5 x)^{3/5}}dx$

(ii) $\int \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}dx$

28. Evaluate the following integrals :

(i) $\int \frac{x^3 dx}{(a+bx^2)^{3/2}}$ (ii) $\int \frac{(1+x^4)dx}{(1-x^4)^{1/2}}$

29. Evaluate the following integrals :

(i) $\int \frac{\sin x}{\sqrt{(1+\sin x)}}dx$

(ii) $\int \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

30. Evaluate the following integrals :

(i) $\int \frac{2x+1}{\sqrt{(4x^2-2x+1)^3}}dx$

(ii) $\int \frac{\sqrt{2-x-x^2}}{x^2}\,dx$

(iii) $\int \frac{\sqrt{x}\sqrt{(1-2x)}}{x^4}\,dx$

(iv) $\int \frac{\sqrt{1+x^2}}{1-x^2}dx$

31. Evaluate the following integrals :

(i) $\int \sqrt{\left(\frac{\operatorname{cosec} x-\cot x}{\operatorname{cosec} x+\cot x}\right)\frac{\sec x}{\sqrt{(1+2\sec x)}}}\,dx$

(ii) $\int \cos^{-1}(x+\sqrt{x^2+1})\,dx$

32. Evaluate the following integrals :

(i) $\int \frac{dx}{[x+\sqrt{x(1+x)}]^2}$

(ii) $\int \frac{x^3-x+1}{\sqrt{x^2+2x+2}}\,dx$

(iii) $\int \frac{dx}{x^{11}\sqrt{1+x^4}}$

(iv) $\int \frac{dx}{(1+x^4)\sqrt{\sqrt{1+x^4}-x^2}}$

33. Evaluate the following integrals :

(i) $\int \frac{(x+1)dx}{(x^2+x+1)\sqrt{x^2+x+1}}$

(ii) $\int \frac{x^2dx}{(4-2x+x^2)\sqrt{2+2x-x^2}}$

34. Evaluate the following integrals :

(i) $\int \tan^{-1}x\,.\ln(1+x^2)dx$

(ii) $\int \frac{\tan^{-1}x}{x^4}dx$

35. Evaluate the following integrals :

(i) $\int \frac{\cos 2x.\sin 4x}{\cos^4 x(1+\cos^2 2x)}$

(ii) $\int \frac{3\cot 3x-\cot x}{\tan x-3\tan 3x}dx$

Target Exercises for JEE Advanced

1. Evaluate $\int \frac{(x^{4r-1}+x^{2r-1})}{x^{6r}-1}\,dx$

2. Evaluate $\int \frac{x^3(x^2-1)}{x^{10}-1}\,dx$

3. Evaluate $\int \frac{1}{(x-a)^n(x-b)}dx$

4. Evaluate $\int \frac{x^3+2x^2+x-7}{\sqrt{(x^2+2x+3)}}\,dx$

5. Evaluate $\int \frac{m.x^{m+3n-1}-n.x^{2n-1}}{x^{3m+3n}+3x^{2m+2n}+3x^{m+n}+1}\,dx$

6. Evaluate $\int \tan^{-1}\left(\frac{2\cos^2\theta}{2-\sin 2\theta}\right).\ \sec^2\theta\,.d\theta$

7. Prove that $\int \frac{x}{n}\left\{x+\sqrt{x^2+a^2}\right\}dx$

$= \frac{1}{4}(2x^2+a^2)\ell n\left\{x+\sqrt{x^2+a^2}\right\}$

$-\frac{1}{4}x\sqrt{a^2+x^2}+C$

8. Evaluate $\int \frac{dx}{(1+x^{2n})\{(1+x^{2n})^{1/n}-x^2\}^{1/2}}.$

9. Evaluate $\int \frac{x^2dx}{(1+x^4)(1+x^4)^{1/2}}.$

10. Evaluate

$$\int \frac{\sin\theta d\theta}{26\cos\theta+4\sin\theta+4\sin\theta\cos\theta+12\sin^2\theta+26}.$$

11. Evaluate $\int \frac{(\sin^{3/2}\theta+\cos^{3/2}\theta)d\theta}{\sqrt{\sin^3\theta.\cos^3\theta.\sin(\theta+\alpha)}}$

12. Evaluate

$$\int \left[9ax^2+(x^2+24a^2)\sqrt{x^2-3a^2}\right]^{1/3}\,dx$$

13. Use an appropriate substitution to obtain an integrand that is a rational function of a single variable or of trigonometric functions. Do not evaluate

(i) $\displaystyle\int \frac{(\sqrt{4-x^2})^3+1}{[(4-x^2)^3+5]\sqrt{4-x^2}}\,dx$

(ii) $\displaystyle\int \frac{(x^2-5)^7}{x^2+3+\sqrt{x^2-5}}\,dx$

14. Evaluate $\displaystyle\int \frac{dx}{\sqrt{\sin(x+\alpha)\cos^3(x-\beta)}}$

15. Evaluate

$$\int \frac{\cos x\,dx}{(1-\cos\alpha\sin x)\sqrt{(1+\cos 2\alpha\sin^2 x-2\cos\alpha\sin x)}}$$

16. Evaluate $\displaystyle\int \frac{m.x^{m+3n-1}-n.x^{2n-1}}{x^{3m+3n}+3x^{2m+2n}+3x^{m+n}+1}\,dx$

17. Evaluate $\displaystyle\int \sin^{-1}(x+\sqrt{x^2+a^2})\,dx$

18. Evaluate

$$\int \frac{\cos x\,dx}{(1-\cos\alpha\sin x)\sqrt{(1+\cos 2\alpha\sin^2 x-2\cos\alpha\sin x)}}$$

19. Evaluate

$$\int \frac{(\sin x-\cos x)dx}{(\sin x+\cos x)\sqrt{\sin x\cos x+\sin^2 x\cos^2 x}}$$

20. Evaluate $\displaystyle\int \sqrt{\frac{1-\cos\theta}{\cos\theta(1+\cos\theta)(2+\cos\theta)}}\,d\theta$

21. Evaluate $\displaystyle\int \frac{(1+x^4)^{1/2}\,dx}{1-x^4}.$

22. Evaluate $\displaystyle\int \frac{1-ax^2}{1+ax^2}\frac{dx}{\sqrt{1+2cx^2+a^2x^4}}.$

23. Evaluate $\displaystyle\int x\cdot\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}\,dx$

24. Evaluate $\displaystyle\int \sqrt{\frac{\cos x-\cos 3x}{4-3\cos x-\cos 3x}}\,dx$

25. Prove that

$$\int \frac{x^6+7x^5+15x^4+32x^3+23x^2+25x-3}{(x^2+x+2)^2(x^2+1)^2}\,dx$$

$$=\frac{1}{x^2+x+2}-\frac{3}{x^2+1}+\ln\frac{x^2+1}{x^2+x+2}+C$$

26 Evaluate $\displaystyle\int \frac{x^5-x^3+1}{(x^6-1)}dx$

27. Evaluate $\displaystyle\int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}dx$

28. Prove that $\displaystyle\int e^{f(x)}\cdot\left[xf'(x)+\frac{f''(x)}{\{f'(x)\}^2}\right]dx$ $\displaystyle= e^{f(x)}\cdot\left(x-\frac{1}{f'(x)}\right)+c.$ Hence or otherwise evaluate

$$\int e^{x\sin x+\cos x}\cdot\left(\frac{x^4\cos^3 x-x\sin x+\cos x}{x^2\cdot\cos^2 x}\right)dx.$$

29. Explain the following apparent difficulties :

(a) $\displaystyle\int \frac{dx}{(a+cx^2)^{3/2}}=\frac{x}{a(a+cx^2)^{1/2}}$

yet, when a becomes nearly zero, the denominator on the right-hand side becomes nearly zero, while that on the left hand remains finite.

(b) $\displaystyle\int \frac{dx}{\sqrt{(x-a)(x-b)}}=2\ln(\sqrt{x-a}+\sqrt{x-b})$

yet if a and b are positive, and x is less than either of them, the square root on the left hand side is real, but those on the right hand side are imaginary.

30. If $R=(x^2+ax)^2+bx$, and $u=\ln\dfrac{x^2+ax+\sqrt{R}}{x^2+ax-\sqrt{R}}$, find the relation between the integrals $\displaystyle\int\frac{dx}{\sqrt{R}},\int\frac{xdx}{\sqrt{R}}.$

31. Show that if the roots of $Q(x)=0$ are all real and distinct, and $P(x)$ is of lower degree than $Q(x)$, then

$\displaystyle\int R(x)\,dx=\Sigma\frac{P(\alpha)}{Q'(\alpha)}\ln|x-\alpha|,$ the summation applying to all the roots a of $Q(x)=0$.

Hint: The form of the partial fraction corresponding to a may be deduced from the facts that

$$\frac{Q(x)}{x-\alpha}\;Q'(\alpha),\;(x-\alpha)R(x)\;\frac{P(\alpha)}{Q'(\alpha)}$$

32. If $(x^2+y^2)^2=2c^2(x^2-y^2)$, then prove that

$$\int\frac{dx}{y(x^2+y^2+c^2)}=-\frac{1}{c^2}\ln\left|\frac{x^2+y^2}{x-y}\right|.$$

33. Show that $\displaystyle\int\frac{dx}{(x-x_0)y}$, where $y^2=ax^2+2bx+c$, may be expressed in one or other of the forms

$$-\frac{1}{y_0}\ln\left|\frac{axx_0+b(x+x_0)+c+yy_0}{x-x_0}\right|,$$

$$\frac{1}{z_0}\tan^{-1}\left\{\frac{axx_0+b(x+x_0)+c}{yz_0}\right\},$$

according as $ax_0^2+2bx_0+c$ is positive and equal to y_0^2 or negative and equal to $-z_0^2$.

34. Evaluate

$$\int\frac{(x-\alpha)dx}{\{p(x-\alpha)^2+q(x-\beta)^2\}\{r(x-\alpha)^2+s(x-\beta)^2\}^{1/2}}$$

by means of the substitution $t=(x-\alpha)^2/(x-\beta)^2$, showing that the integral becomes $\frac{1}{2(\alpha-\beta)}\int\frac{dt}{(pt+q)\sqrt{(rt+s)}}$ Show how to obtain the value of the integral

$$\int\frac{(x-2)dx}{(7x^2-36x+48)\sqrt{(x^2-2x-1)}}.$$

35. (i) By means of the substitution $\frac{x}{a}+\frac{a}{x}=u$ find the integral of $(a^4-x^4)/(a^4+a^2x^2+x^4)^{3/2}$

(ii) By means of the substitution $1-x=xy^2$

Previous Year's Questions (JEE Advanced)

A. Fill in the blanks :

1. If $\int\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx=Ax+B\log(9e^{2x}-4)+C$, then $A=......, B=.........$ and $C=........$ **[IIT - 1990]**

B. Multiple Choice Questions with ONE correct answer :

2. $\int\frac{(3x+1)}{(x-1)^3(x+1)}dx$ equal to **[IIT - 1992]**

(A) $\frac{1}{4}\log|x+1|-\frac{1}{4}\log|x-1|-\frac{1}{2(x-1)}-\frac{1}{(x-1)^2}+c$

(B) $\frac{1}{4}\log|x-1|-\frac{1}{4}\log|x+1|-\frac{1}{2(x-1)}-\frac{1}{(x-1)}+c$

(C) $\frac{1}{4}\log|x+1|+\frac{1}{4}\log|x-1|-\frac{1}{2(x-1)}-\frac{1}{(x-1)^2}+c$

(D) None of these

3. The value of the integral $\int\frac{\cos^3x+\cos^5x}{\sin^2x+\sin^4x}dx$ is **[IIT - 1995]**

(A) $\sin x-6\tan^{-1}(\sin x)+c$

(B) $\sin x-2(\sin x)^{-1}+c$

(C) $\sin x-2(\sin x)^{-1}-6\tan^{-1}(\sin x)+c$

(D) $\sin x-2(\sin x)^{-1}5\tan^{-1}(\sin x)+c$

4. $\int\frac{dx}{(x-p)\sqrt{(x-p)(x-q)}}$ is equal to **[IIT - 1996]**

(A) $\frac{2}{p-q}\sqrt{\frac{x-p}{x-q}}+c$

(B) $-\frac{2}{p-q}\sqrt{\frac{x-q}{x-p}}+c$

(C) $\frac{1}{\sqrt{(x-p)(x-q)}}+c$

(D) None of these

5. If $\int\frac{dx}{(\sin x+4)(\sin x-1)}=\frac{A}{\tan\frac{x}{2}-1}+B\tan^{-1}f(x)+C$ then **[IIT - 1997]**

(A) $A=\frac{1}{5}, B=-\frac{2}{5\sqrt{15}}, f(x)=\frac{4\tan x+1}{5}$

(B) $A=-\frac{1}{5}, B=-\frac{2}{5\sqrt{15}}, f(x)=\frac{4\tan\frac{x}{2}+1}{\sqrt{15}}$

(C) $A=\frac{2}{5}, B=-\frac{2}{5\sqrt{15}}, f(x)=\frac{4\tan x+1}{5}$

(D) $A=\frac{2}{5}, B=-\frac{2}{5\sqrt{15}}, f(x)=\frac{4\tan\frac{x}{2}+1}{\sqrt{15}}$

6. $\int\frac{\cos x-\sin x}{\cos x+\sin x}(2+2\sin 2x)\,dx$ is equal to **[IIT - 1997]**

(A) $\sin 2x + c$ (B) $\cos 2x + c$
(C) $\tan 2x + c$ (D) None of these

7. $\int \frac{dx}{(2x-7)\sqrt{x^2-7x+12}}$ is equal to **[IIT - 1997]**

(A) $2\sec^{-1}(2x-7)+c$ (B) $\sec^{-1}(2x-7)+c$
(C) $1/2\sec^{-1}(2x-7)+2$ (D) None of these

8. $\int \operatorname{cosec} x \log\left(\tan\frac{x}{2}\right) dx$ is equal to **[IIT - 1998]**

(A) $\sin x \log\left(\tan\frac{x}{2}\right)+c$

(B) $\sin x \log \tan\frac{x}{2} - x + c$

(C) $\sin x \log\left(\tan\frac{x}{2}\right)+x+c$

(D) None of these

9. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}} dx =$ **[IIT - 2006]**

(A) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}+c$

(B) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+c$

(C) $\frac{\sqrt{2x^4-2x^2+1}}{x}+c$

(D) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$

10. Let $f(x) = \frac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = \underbrace{(f \circ f \circ \ldots \circ f)}_{\text{f occur n times}}(x)$. Then $\int x^{n-2} g(x)\, dx$ equals **[IIT - 2007]**

(A) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+K$

(B) $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}}+K$

(C) $\frac{1}{n(n-1)}(1+nx^n)^{1+\frac{1}{n}}+K$

(D) $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}}+K$

11. Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1} dx$, $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1} dx$. Then, for an arbitrary constant C, the value of J – I equals **[IIT - 2008]**

(A) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+C$

(B) $\frac{1}{2}\log\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+C$

(C) $\frac{1}{2}\log\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+C$

(D) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+C$

12. The integral $\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}} dx$ equals (for some arbitrary constant K) **[IIT - 2012]**

(A) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(B) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(C) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(D) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

C. Subjective Problems :

13. Evaluate $\int \frac{\sin x}{\sin x - \cos x} dx$ **[IIT - 1978]**

14. Evaluate $\int \frac{x^2 dx}{(a+bx)^2}$ **[IIT - 1979]**

15. Evaluate the following integrals : **[IIT - 1980]**

(A) $\int \sqrt{1+\sin\left(\frac{1}{2}x\right)}\, dx$ (B) $\int \frac{x^2 dx}{\sqrt{1-x}}$

16. Evaluate $\int (e^{\log x}+\sin x)\cos x\, dx$. **[IIT - 1981]**

17. Evaluate $\int \frac{(x-1)e^x}{(x+1)^3} dx$ **[IIT - 1983]**

18. Evaluate the following $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ **[IIT - 1984]**

19. Evaluate the following $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$ **[IIT - 1985]**

20. Evaluate $\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx$ **[IIT - 1986]**

21. Evaluate $\int \left[\frac{\cos 2x)^{1/2}}{\sin x}\right] dx$ **[IIT - 1987]**

22. Evaluate $\int (\sqrt{\tan x}+\sqrt{\cot x})\,dx$ **[IIT - 1989]**

23. Find the indefinite integral

$$\int \left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}+\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$$ **[IIT - 1992]**

24. Find the indefinite integral

$$\int \cos 2\theta.\ \ln\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\,d\theta$$ **[IIT - 1994]**

25. Evaluate $\int \frac{(x+1)}{x(1+xe^x)^2}\,dx$ **[IIT - 1996]**

26. (A) $\int \frac{5x^4+4x^5}{(x^5+x+1)^2}\,dx$ **[IIT - 1996]**

(B) $\int \frac{3x+1}{(x-1)^3.(x+1)}\,dx.$

(C) $\int \frac{x^4}{(x-1)^3(x^2+1)}\,dx$

27. Integrate the following : $\int \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2}\frac{dx}{x}$

[IIT - 1997]

28. Integrate $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}\,dx$ **[IIT - 1999]**

29. Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

[IIT - 2001]

30. For any natural number m, evaluate

$$\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}\,dx, x>0.$$

[IIT - 2002]

D. Assertion & Reasoning :

31. Let F(x) be an idenfinite integral of $\sin^2 x$.

Assertion (A): The function F(x) satisfiesf $F(x+\pi)=F(x)$ for all real x.

Reasons (R): $\sin^2(x+\pi)=\sin^2 x$ for all real x.

[IIT - 2007]

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

E. Comprehension

Let $f(x) = (1-x)^2 \sin^2 x + x^2$ for all $x \in \mathbb{R}$, and let

$$g(x)=\int_1^x\left(\frac{2(t-1)}{t+1}-\ln t\right)f(t)\,dt \text{ for all } x\in(1,\infty)$$

32. Consider the statements: **[IIT - 2012]**

P : There exists some $x\in\mathbb{R}$ such that $f(x)+2x=2(1+x^2)$

Q : There exists some $x\in\mathbb{R}$ such that $2f(x)+1=2x(1+x)$

Then:

(a) both **P** and **Q** are true

(b) **P** is true and **Q** is false

(c) **P** is true and **Q** is true

(d) both **P** and **Q** are false

33. Which of the following is true? **[IIT - 2012]**

(a) g is increasing on $(1,\infty)$

(b) g is decreasing on $(1,\infty)$

(c) g is increasing on (1, 2) and decreasing on $(2,\infty)$

(d) g is decreasing on (1, 2) and increasing on $(2,\infty)$

ANSWERS

CONCEPT PROBLEMS—A

1. (i) $x-2x^2+3x^3$

(ii) $\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}-5x$

(iii) $\frac{4}{5}(x+1)^{5/4}$

(iv) $\frac{1}{2}(x/2-7)^4$

2. (B) $F(0)-G(0)=\frac{8}{3}$

4. (B) $\frac{1}{2}(e^x+1)$

6. (i) $\frac{3}{4}(x+2)^4-12$ (ii) $4t^2-9t-16$

(iii) $\frac{1}{3}x^3+5x-1.$

7. 12

8. (i) $\frac{2^x \cdot e^x}{1+\ell n 2}+C$ (ii) $e^x+e^{-x}+c$

(iii) $\frac{e^{ax+b}}{a}+C$ (iv) $\frac{a^{px+q}}{p.\ln a}\ a>0$

9. (i) $\frac{2}{3}x^{3/2}+C$ (ii) $-\frac{1}{x}+C$

(iii) $-\frac{x^2}{2}+C$ (iv) $\frac{x^{m+1}}{m+1}+C,\ m\neq 1$

10. (i) $\frac{a^{mx}\cdot b^{nx}}{m\,\ell n a+n\ell n b}+C$

(ii) $\frac{2^{2x}}{2\,\ell n 2}+\frac{3^{2x}}{2\,\ell n 3}+2\cdot\frac{6^x}{\ell n 6}+C$

(iii) $\frac{e^{4x}}{4}+C$ (iv) x^2+C

11. (i) $\sqrt{x}+C$ (ii) $\frac{(ax+b)^{n+1}}{a(n+1)}+C$

(iii) $\frac{\ell n(3-2x)}{-2}+C$

(iv) $\frac{x^3}{3}+\frac{3}{x}x^2+3x+\ell n x+C$

12. (i) $\frac{1}{b}\left[x-\frac{a}{b}\ell n(a+bx)\right]+C$

(ii) $2x+3\,ln\,(x-2)+C$

(iii) $x-\tan^{-1}x+C$

(iv) $\frac{x^3}{3}-x+\tan^{-1}x+C$

13. (i) $\frac{180}{\pi}\sin x^{\circ}+C$ (ii) $\frac{1}{a}\tan(ax+b)+C$

(iii) $-\cot x-x+C$ (iv) $2\tan\frac{x}{2}-x+C$

14. (i) $\frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{5x}{\sqrt{2}}\right|+C$ (ii) $\sec^{-1}|x+1|+C$

(iii) $\frac{1}{4}\sec^{-1}\left|\frac{2x-1}{2}\right|+C$

15. (i) $2\text{cosec}\left(1-\frac{x}{2}\right)+C$ (ii) $2\sin^{-1}\frac{x}{2\sqrt{3}}+C$

(iii) $\frac{-1}{3\sqrt{3}}\tan^{-1}\frac{2-3x}{\sqrt{3}}+C$

PRACTICE PROBLEMS—A

16. (i) $\frac{1}{5}\cos\frac{2-5x}{3}-\sin\frac{3-2x}{5}+2\log|1+2x|$

$+\frac{2}{3}\sqrt{3x+1}+C$

(ii) $-\frac{1}{16}(7-4x)^4-\log|3-7x|+\cot(4x+3)$

$+\frac{1}{6}\tan^{-1}\frac{3x}{4}+C$

19. No, we cannot. for instance, $f(x) = 1 + \cos x$ is a periodic function and $F(x)=\int(1+\cos x)dx=x+\sin x+C$ is a non-periodic function.

21. $y=3\ln x+1$

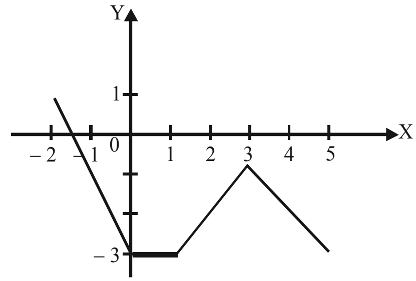
22. (a) $f(t)=2\sqrt{t}-1$ if $t>0$

(b) $f(t)=t-\frac{1}{2}t^2+\frac{1}{2}$ if $0\le t\le 1$

(c) $f(t)=t-\frac{1}{3}t^3+\frac{1}{3}$ if $|t|\le 1$

(d) $f(t)=t$ if $t\le 0$; $f(t)=e^t-1$ if $t>0$

23.



24. Yes, one

25. $f(x)=cx+d-2/9\sin 3x$, for constants c and d.

26. $\frac{67}{5}$

27. $6x^5-15x^4+10x^3+1$

CONCEPT PROBLEMS—B

1. (i) $\ln x + 2\tan^{-1}x + C$

(ii) $\frac{1}{2}\left[\frac{x^3}{3} + \tan^{-1}x\right] + C$

(iii) $\frac{1}{2}x^2 - 2x + C$

(iv) $\frac{4}{\ln 2}\left[2^{2x} - \frac{1}{3}.2^{-3x}\right] + C$

2. (i) $\frac{1}{4}\left(\frac{\sin 3x}{3} + 3\sin x\right) + C$

(ii) $\frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$

(iii) $\tan x - \cot x - 3x + c$

(iv) $a\sec x - b\operatorname{cosec} x + C$

3. (i) $\frac{1}{2}\left(\frac{\sin 5x}{5} + \sin x\right) + C$

(ii) $-\frac{\cos^2 x\cos 4x}{6} - \frac{\cos x.\cos 3x}{12} - \frac{\cos 2x}{24} + C$

(iii) $-\frac{1}{2}\left(\frac{\cos 2x}{2} + \frac{\cos 4x}{8}\right) + C$

(iv)

$-\left[\frac{1}{9}\cos 9x + \frac{1}{10}\cos 10x + \frac{1}{11}\cos 11x + \frac{1}{12}\cos 12x\right] + C$

4. (i) $2\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) + C$ (ii) $\frac{x}{\sqrt{2}} + C$

(iii) $-\frac{1}{8}\cos 2x - \frac{1}{16}\cos 4x + \frac{1}{24}\cos 6x + C$

(iv) $-\frac{1}{64}\cos 8x + C$

5. (i) $\tan x - \cot x + C$

(ii) $-\cot x - \frac{3}{2}x - \frac{1}{4}\sin 2x + C$

(iii) $\tan x - \frac{3}{2}x + \frac{1}{4}\sin 2x + C$

(iv) $-\cot x - \frac{3}{2}x - \frac{1}{4}\sin 2x + C$

6. (i) $\tan\frac{x}{2} + C$

(ii) $\tan x - \sec x + C$

(iii) $\frac{1}{3}(\tan 3x + \sec 3x) + C$

(iv) $(2\sin x + x) + C$

7. (i) $\frac{1}{2}(\tan x + x) + C$ (ii) $\frac{1}{2}\sin 2x + C$

(iii) $\tan x - x + C$ (iv) $-(\cot x + \tan x) + C$

8. (i) $[(-\sin x - \frac{1}{2}\sin 2x)] + C$

(ii) $x + \frac{\cos 2x}{4} + C$

(iii) $\tan x + C$

(iv) $-\cot x + \sec x - \cos x + C$

9. (i) $-\frac{\cos 3x}{3} + C$ (ii) $\frac{1}{2}(x - \sin x) + C$

(iii) $-\sqrt{2}\cos\frac{x}{2} + C$ (iv) $-\left(x + \frac{1}{\pi}\cos 4x\right) + C$

PRACTICE PROBLEMS—B

10. (i) $\frac{1}{10}\tan^{-1}\frac{2x}{5} + C$

(ii) $\frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + C$

(iii) $\tan x - \tan^{-1}x + C$

(iv) $\sec x - 3\tan x + \frac{9}{2}x - \frac{3}{4}\sin 2x + C$

11. (i) $\ln x - \frac{1}{4x^4} + C$

(ii) $\frac{1}{18}[(2x+3)]^{3/2} - (2x-3)^{3/2}] + C$

(iii) $\frac{x^2}{2} - x + C$

(iv) $\frac{4}{\sqrt{x}} - \frac{2}{3}x\sqrt{x} + C$

12. (i) $\frac{x^3}{3} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}\ln(2x+1) + C$

(ii) $x^2 + x + C$

(iii) $2x^{1/2} - \frac{2}{3}x^{3/2} + C$

(iv) $\frac{2(1+x)^{3/2}}{3} + C$

13. (i) $\frac{x^2}{2} - x + C$ (ii) $-x + C$

(iii) $x + 2\ln|x-2| + C$ (iv) $\sqrt[6]{2x} + C$

14. (i) $-5\operatorname{cosec} x + 3\sec x + C$

(ii) $\frac{1}{8}\left(5x + \frac{3\sin 4x}{4}\right) + C$

(iii) $-\frac{1}{4}\cos 3\frac{x}{2} - \frac{3}{4}\cos\frac{x}{2} + \frac{1}{28}\cos\frac{1}{28}$

$+\frac{1}{20}\cos\frac{5x}{2}+C$

(iv) $\frac{1}{2}\log\left|\tan\left(\frac{\pi}{4}+\frac{x-\frac{\pi}{6}}{2}\right)\right|+C$

15. (i) $\sec x - \operatorname{cosec} x + C$ (ii) $-2\cos x + C$

(iii) $\sin 2x + C$ (iv) $-\frac{\cos 8x}{8}+C$

16. (i) $2\sqrt{2}\sin\frac{x}{2} - \sqrt{2}\log\left|\sec\frac{x}{2}+\tan\frac{x}{2}\right|+C$

(ii) $\operatorname{cosec} a\,[\log\sin x - \log\sin(x+a)]+C$

(iii) $-\cot 2a\log[|\operatorname{cosec}(x-a)| + \log|\sin(x+a)|]+C$

CONCEPT PROBLEMS—C

2. They are both right.

4. (i) $\ln(\ln x)+C$ (ii) $\ln\ln\ln x + C$

5. (i) $(\tan\sqrt{x})^2 + C$ (ii) $\frac{1}{5}\tan^5\sqrt{x}$

(iii) $\sin(\sin x)+C$ (iv) $\frac{1}{3}\sin(x^2+3x+2)+C$

6. (i) $\frac{2}{15}\sqrt{1-x}\,(3x^2+4x+8)+C$

(ii) $\frac{2}{3}\sin^{-1}(x^{3/2})+C$

(iii) $x+\log|x+1|+\frac{1}{x+1}+C$

(iv) $\frac{(27+e^{3x})^{4/3}}{4}+C$

7. (i) $-\ln(1+e^{-x})+C$ (ii) $2\ln(e^{x/2}+e^{-x/2})+C$

(iii) $e^{x+1/x}+C$ (iv) $\frac{1}{4}x^{\log_e x^2}+C$

8. (i) $(3/2)\sqrt[3]{1-\sin 2x}+C$ (ii) $\sec e^x + C$

(iii) $-1/3\cos 3x + C$ (iv) $\log[\log(\tan x)]+C$

PRACTICE PROBLEMS—C

9. (i) $\frac{3}{10}(2x+1)^{5/2}+\frac{1}{6}(2x+1)^{3/2}+C$

(ii) $\frac{1}{2}(x^3+3x+6)^{2/3}+C$

(iii) $-\frac{2}{3(4+3\sqrt{x})}+C$

(iv) $\frac{5^{5^{5^x}}}{(\log 5)^3}+C$

10. (i) $-\frac{1}{2}\log|1-\sin 2x|+C$

(ii) $-\cot(1+\log x)+C$

(iii) $2\sqrt{(\tan^{-1}x+3)}+C$

(iv) $\ln\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right|+C$

11. (i) $\frac{1}{2(b-a)}\ln(a\cos^2 x + b\cos^2 x)+C$

(ii) $\ln|\ln\sin x|+C$

(iii) $\ln|\ln(\sec x+\tan x)|+C$

(iv) $\ln|a\sin x + b\cos x + c|+C$

12. (i) $\frac{(\tan x - x)^2}{2}+C$

(ii) $-\frac{1}{2b^2}\left\{\frac{1}{a^2+b^2\tan^2 x}\right\}+C$

(iii) $\ln|\ln\sec x|+c$

(iv) $2\sqrt{\sin(\tan^{-1}x)}+C$

13. (i) $x+2e^{-x}-1/2\,e^{-2x}+C$

(ii) $2\tan^{1/2}x+2/5\tan^{5/2}x+C$

(iii) $1/2(\sin^{-1}x^2)^2+C$

(iv) $1+\cos^{5/2}x+C$

14. (i) $1/6(x+\log x)^3+C$

(ii) $-(4x+1)/[8(2x+1)^2]+C$

(iii) $-3\ln|2+\sin x|-\frac{(2+\sin x)^2}{2}+4(2+\sin x)+C$

(iv) $\left(\frac{x}{e}\right)^x-\left(\frac{e}{x}\right)^x+C$

15. (i) $-\frac{2}{3}\frac{(x-x^2)^{3/2}}{x^3}+C$

(ii) $-\left(1+\frac{1}{x^4}\right)^{1/4}+C$

(iii) $-\ln\left(1+\sqrt{1+\frac{1}{x^2}}\right)+C$

(iv) $\frac{\sqrt{(x^4+x^2-1}}{x}+C$

16. (i) $-\left(1+\frac{1}{x^5}\right)^{1/5}+C$

(ii) $\log|(1+x^2+\sqrt{1+x^4})/x|+C$

PRACTICE PROBLEMS—D

1. (i) $\frac{1}{2}\cos^5 x-\frac{1}{3}\cos^3 x+C$

(ii) $\frac{2\cos^{3/2}\theta}{5}-2\cos^{1/2}\theta+\mathbf{C}$

(iii) $\frac{\tan^2\theta}{2}+\ln|\tan\theta|+C$

(iv) $-\frac{\cot^2 x}{2}+3\log\tan x+\frac{3}{2}\tan^2 x+\frac{\tan^4 x}{4}+C$

2. (i) $2\tan^{1/2}Q\left(1+\frac{\tan^2\theta}{5}\right)+C$

(ii) $2\sqrt{\tan x}+C$

(iii) $\sqrt{\sin x}\,dx\left(\frac{2}{3}-\frac{4}{7}\sin^2 x+\frac{2}{11}\sin^4 x\right)\sqrt{\sin^3 x}+C$

(iv) $-\cos x+\cos^3 x-\frac{3}{5}\cos^5 x+\frac{1}{7}\cos^7 x+C$

3. (i) $\frac{2}{5}\sqrt{\tan x}(5+\tan^2 x)+C$

(ii) $C-\cot x\frac{2}{3}\cot^3 x-\frac{1}{5}\cot^5 x$

(iii) $C-\frac{4\sqrt{2}}{5}\sqrt{\cot^5 x}$

(iv) $\frac{2}{3}\cos x\sqrt{\cos x}+C$

4. (i) $\sec x-\frac{2}{3}\sec^3 x+\frac{1}{5}\sec^5 x+C$

(ii) $\tan^3 x\left(\frac{1}{3}+\frac{1}{5}\tan^2 x\right)+C$

(iii) $\tan x\,(1+\frac{2}{3}\tan^2 x-\frac{1}{5}\tan^4 x)+C$

(iv) $-\frac{1}{4}\cot x\,\mathrm{cosec}^3 x-\frac{3}{8}\cot x\,\mathrm{cosec}\,x$

$+\frac{3}{8}\ln|\tan\frac{x}{2}|+C$

5. (i) $\frac{\sin^3 x}{3}-2\sin x-\mathrm{cosec}\,x+C$

(ii) $\frac{-2}{3}(\cot x)^{3/2}+C$

(iii) $\ln|\tan x|+C$

(iv) $\ln\left|\frac{1-\sqrt{\cos(x/2)}}{1+\sqrt{\cos(x/2)}}\right|+\frac{4}{\sqrt{\cos(x/2)}}$

$+2\tan^{-1}\sqrt{\cos(x/2)}+C$

6. (i) $-\frac{1}{5}\mathrm{cosec}^5 x+\frac{1}{3}\mathrm{cosec}^3 x+C$

(ii) $-\frac{1}{3\tan^3 x}-\frac{1}{\tan x}+C$

(iii) $-\frac{5}{3}\cos^{3/5}x+\frac{5}{13}\cos^{13/5}x+C$

(iv) $\tan x-2\cot x-\frac{1}{3}\cot^3 x+C.$

7. (i) $\frac{5}{16}x+\frac{1}{2}\sin x+\frac{3}{32}\sin 2x-\frac{1}{24}\sin^3 x+C$

(ii) $\frac{1}{9}\sec^3 3x-\frac{1}{3}\sec 3x+C$

(iii) $\frac{2}{5}\tan^{5/2}x+\frac{2}{9}\tan^{9/2}x+C$

(iv) $\frac{1}{7}\tan^7 x+\frac{1}{5}\tan^5 x+C$

PRACTICE PROBLEMS—E

1. $\frac{x^3}{12(4-x^2)^{3/2}}+C$

2. $\frac{x-2}{4\sqrt{4x-x^2}}+C$

3. $\frac{1}{4}\sin^{-1}\left(\frac{x}{a}\right)^4+C$

4. $\frac{2}{a}\tan^{-1}\sqrt{\frac{2x-a}{a}}+C$

5. $\frac{1}{54}\tan^{-1}\left(\frac{x}{3}\right)+\frac{x}{18(9+x^2)}+C$

6. $7\left[\frac{\sqrt{y^2-49}}{7}-\sec^{-1}\left(-\frac{y}{7}\right)\right]+C$

7. $\dfrac{x}{\sqrt{1-x^2}}+C$

8. $\dfrac{2}{3}\sin^{-1}(x/a)^{3/2}+C$

9. $\dfrac{1}{3}\sin^{-1}(x/a)^3+C$

10. $\dfrac{1}{2}\sec^{-1}x^2+C$

11. $\dfrac{x^3}{12(4-x^2)^{3/2}}+C$

12. $2\tan Q-\sec Q+Q+c$ where $\sin Q=x$

13. $\dfrac{x-2}{4\sqrt{4x-x^2}}+C$

14. $\dfrac{1}{54}\tan^{-1}\left(\dfrac{x}{3}\right)+\dfrac{x}{18(9+x^2)}+C$

15. $\dfrac{2}{(\alpha-\beta)}\sqrt{\dfrac{x-\alpha}{\beta-x}}+C$

PRACTICE PROBLEMS—F

1. $\dfrac{1}{12}\ln\dfrac{3x-2}{3x+2}+C$

2. $\dfrac{1}{3}\ln\left|\dfrac{2x-1}{2x+2}\right|+C$

3. $\ln\left|\dfrac{x+4}{x+5}\right|+C$

4. $\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+C$

5. $\ln\left|x+\dfrac{1}{2}+\sqrt{x^2+x+1}\right|+C$

6. $\sin^{-1}\left(\dfrac{x-a}{a}\right)+C$

7. $\dfrac{1}{\sqrt{2}}\ln\left|\left(x+\dfrac{3}{4}\right)+\sqrt{x^2+\dfrac{3}{2}x-1}\right|+C$

8. $\dfrac{1}{8}(4x+3)\sqrt{4-3x-2x^2}+\dfrac{41\sqrt{2}}{32}\sin^{-1}\dfrac{4x+3}{\sqrt{41}}+C$

9. $\dfrac{1}{10}(5x+4)\sqrt{5x^2+8x+4}$

$+\dfrac{2}{5\sqrt{5}}\ln\{(5x+4)+\sqrt{5(5x^2+8x+4)}\}+C$

10. $\sqrt{3}\left[\dfrac{t}{2}\sqrt{t^2+\dfrac{7}{3}}+\dfrac{7}{6}\ell n\left|t+\sqrt{t^2+\dfrac{7}{3}}\right|\right]+C,$

where $t=x-1$

11. $\dfrac{1}{2}(x+a)\sqrt{2ax+x^2}-\dfrac{a^2}{2}\ln\left|x+a+\sqrt{2ax+x^2}\right|+C$

12. $\dfrac{1}{\sin\theta}\tan^{-1}\dfrac{x-\cos\theta}{\sin\theta}+C$

13. $\dfrac{1}{4}\ln\dfrac{2+\sin x}{2-\sin x}+C$

14. $-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{3-\sin x}{\sqrt{3}}+C.$

15. $-\ln\left|\cos x+2+\sqrt{\cos^2 x+4\cos x+1}\right|+C.$

16. $\dfrac{1}{4}\ln\left|\dfrac{\ln x-1}{\ln x+3}\right|+C$

17. (a) $2\sin^{-1}(\sqrt{x/2})+C;-2\sin^{-1}(\sqrt{x/2}/\sqrt{2})$
$+C;\sin^{-1}(x-1)+C$

PRACTICE PROBLEMS—G

1. $ln(x+1)+\dfrac{1}{x+1}+C$

2. $\ln|x^2+3x-10|+C$

3 $-2\ln|x-2|+3\ln|x-3|+C$

4. $\dfrac{1}{2}\ln|x^2+x+3|+\dfrac{1}{\sqrt{11}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{11}}\right)+C$

5. $\dfrac{3}{2}\ln|x^2+2x-3|+\dfrac{1}{2}\ln\left|\dfrac{x-1}{x+3}\right|+C$

6. $-2\sqrt{3+2x-x^2}+3\sin^{-1}\left(\dfrac{x-1}{2}\right)+C$

7. $2\sqrt{3x^2-5x+1}+C$

8. $3\sqrt{x^2+4x+3}-\ell n\left|(x+2)+\sqrt{x^2+4x+3}\right|+C$

9. $\dfrac{1}{3}(x^2+x+1)^{3/2}-\dfrac{3}{8}(2x+1)\sqrt{1+x+x^2}$

$-\dfrac{9}{16}\ln(2x+1+2\sqrt{x^2+x+1})+C$

10. $(x^2+2x-3)^{3/2}+(x+1)(x^2+3x-2)-4$

$\ln\left|x+1+\sqrt{x^2+2x-3}\right|+C$

11. $\dfrac{1}{6}(2x^2+2x+1)^{3/2}+\dfrac{3}{8}(2x+1)\sqrt{2x^2+2x+1}$

$+\frac{3}{8\sqrt{2}}\ln\{(2x+1)+\sqrt{2(2x^2+2x+1)}\}+C$

12. $-\sqrt{1-4\ln x-\ln^2 x}-2\sin^{-1}\frac{2+\ln x}{\sqrt{5}}+C.$

13. $\frac{1}{2}x\sqrt{x^2-16}+8\ln|x+\sqrt{x^2-16}|+C$

14. $\frac{x}{2}+\frac{1}{8}\ln|2x^2+3x+1|-\frac{15}{8}\ln\left|\frac{x+\frac{1}{2}}{x+1}\right|+C$

15. $\frac{1}{2}(3-x)\sqrt{1-2x-x^2}+2\sin^{-1}\frac{x+1}{\sqrt{2}}+C.$

16. $x\sqrt{x^2-2x+5}-5\ln\left(x-1+\sqrt{x^2-2x+5}\right)+C$

17. $\frac{x^2}{2}-2x+\ln|x^2-2x+2|-2\tan^{-1}(x-1)+C$

18. $\frac{1}{4}(2x+5)\sqrt{x^2+x+1}+\frac{15}{8}\ln\{(x+\frac{1}{2})$

$+\sqrt{x^2+x+1}\}+C$

19. $\frac{1}{2}(x+6)\sqrt{x^2-4}+4\ln(x+\sqrt{x^2-4})+C$

20. $(x^2+5x+36)\sqrt{x^2-4x-7}$

$+112\ln\left|x-2+\sqrt{x^2-4x-7}\right|+C$

21. $\frac{1}{3}(3x^2+3x+1)\sqrt{x^2+x}$

$-\frac{3}{2}\left\{\frac{2x+1}{4}\sqrt{x^2+x}-\frac{1}{8}\ln(2x+1+2\sqrt{x^2+x})\right\}+C$

22. $\sin^{-1}\frac{x}{\sqrt{2}}-\ln\left(x+\sqrt{x^2+2}\right)+C.$

Hint. Divide the numerator by the denominator term by term.

PRACTICE PROBLEMS—H

1. $\frac{1}{6}\ln\left|\frac{x+\frac{1}{x}-3}{x+\frac{1}{x}+3}\right|+C$

2. $\frac{1}{4}\ln\frac{1+x+x^2}{1-x+x^2}+\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x\sqrt{3}}{1-x^2}\right)+C$

3. $\frac{1}{2}\tan^{-1}\frac{x^2-1}{x}-\frac{1}{4\sqrt{3}}\ell n\left|\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\right|+C$

4. $\frac{1}{2}\tan^{-1}\frac{x^2-1}{x}+\frac{1}{4\sqrt{3}}+\ln\left|\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\right|+C$

5. $\frac{1}{\sqrt{2}}\sec^{-1}\frac{t}{\sqrt{2}}+C,$ where $t=x+1/x$

6. $\sin^{-1}\left(\frac{x}{x^2-1}\right)+C$

7. $\frac{1}{54}\left\{\tan^{-1}x+\frac{3x}{x^2+9}\right\}+C.$

8. $\ln\left(x+\frac{1}{x}+\sqrt{x^2+\frac{1}{x^2}+3}\right)+C$

9. $\frac{1}{2\sqrt{2}a}\tan^{-1}\frac{x^2-a^2}{\sqrt{2}ax}+$

$\frac{1}{4\sqrt{2}a}\ell n\left|\frac{x^2-\sqrt{2}ax+a^2}{x^2+\sqrt{2}ax+a^2}\right|+C$

10. $\frac{1}{2a}\ell n\left|\frac{x^2-ax+a^2}{x^2+ax+a^2}\right|+C$

11. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\ln\left|\frac{y-\sqrt{2}}{y+\sqrt{2}}\right|+C$

where $y=\tan x-\frac{1}{\tan x}$

12. $2\ln\frac{\sqrt{x^2+ax+1}+\sqrt{x^2+bx+1}}{\sqrt{x}}+C$

13. $\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{1+x^2+x^4}+x\sqrt{3}}{1-x^2}\right|+C$

PRACTICE PROBLEMS—I

1. (i) $\frac{1}{2\sqrt{3}}\ln\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right|+C$

(ii) $-\frac{1}{2}\ell n\left|\frac{\cot x-\frac{1}{2}}{\cot x+\frac{1}{2}}\right|+C$

(iii) $-\dfrac{1}{3(3\tan x-4)}+C$

(iv) $\dfrac{1}{5}\ell n\left[\dfrac{\tan x-2}{2\tan x+1}\right]+C$

2. (i) $\dfrac{1}{2\sqrt{3}}\tan^{-1}\left(\dfrac{\sqrt{3}\tan x}{2}\right)+C$

(ii) $\dfrac{1}{2\sqrt{3}}\ln\left|\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|+C$

(iii) $\dfrac{1}{2\sqrt{3}}\ln\left|\dfrac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right|+C$

(iv) $-\dfrac{1}{1+\tan x}+C$

3. (i) $\dfrac{2}{5}\tan^{-1}\left\{\dfrac{1}{5}\tan\left(\dfrac{1}{2}x\right)\right\}+C$

(ii) $\dfrac{2}{3}\tan^{-1}\left(\dfrac{\tan x/2}{3}\right)+C$

(iii) $\dfrac{1}{\sqrt{3}}\ln\left|\dfrac{\tan(x/2)-2-\sqrt{3}}{\tan(x/2)-2+\sqrt{3}}\right|+C.$

(iv) $\dfrac{2}{3}\tan^{-1}\left(\dfrac{5\tan\frac{x}{2}}{3}\right)+C$

4. (i) $\tan^{-1}\left(\tan\dfrac{x}{2}+1\right)+C$

(ii) $-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{4\tan\frac{x}{2}-1}{\sqrt{3}}+C$

(iii) $\tan^{-1}\left(1+2\tan\dfrac{x}{2}\right)+C$

(iv) $\dfrac{1}{5}\tan\left(\dfrac{x}{2}+\dfrac{1}{2}\cos^{-1}\dfrac{3}{5}\right)+C$

5. (i) $\dfrac{1}{2\sqrt{5}}\ln\left|\dfrac{\sqrt{5}\tan\theta-1}{\sqrt{5}\tan\theta+1}\right|+C$

(ii) $\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\tan\dfrac{\theta}{2}\right)+\ln(\cos\theta+2)+C$

(iii) $\dfrac{1}{3}\ln\left|\dfrac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}+4}\right|+C$

(iv) $\dfrac{1}{2\sqrt{2}}\ln\left|\dfrac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|+C$

6. (i) $\dfrac{1}{2}\tan\theta+\dfrac{1}{2\sqrt{2}}\tan^{-1}(\tan\theta\sqrt{2})+C$

(ii) $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{\tan^2 x+1}{\sqrt{2}\tan x}\right)+C$

(iii) $\dfrac{-1}{2\tan x}+\dfrac{1}{2\sqrt{2}}\tan^{-1}\left(\dfrac{\tan x}{\sqrt{2}}\right)+C$

(iv) $\dfrac{(a+b)\varphi}{2(ab)^{3/2}}-\dfrac{(a-b)\sin 2\varphi}{4(ab)^{3/2}}+C$

where $\tan\phi=\sqrt{\dfrac{b}{a}}\tan\theta$

7. (i) $\dfrac{-5}{13}x+\dfrac{12}{13}\ln|3\cos x+2\sin x|+C$

(ii) $\dfrac{3}{13}x+\dfrac{2}{13}\ln|2\sin x+3\cos x|+C$

(iii) $\dfrac{40}{41}x+\dfrac{9}{41}\ln|5\sin x+4\cos x|+C$

(iv) $2x+\ln|3+4\sin x+5\cos x|+C$

8. (i) $\dfrac{1}{24}\ln\left|\dfrac{\sin x+\cos x-4/3}{\sin x+\cos x+4/3}\right|+C$

(ii) $\dfrac{1}{\sqrt{2}}\tan^{-1}(\sin x+\cos x)+C$

(iii) $\sin 2x+c$

(iv) $\dfrac{1}{2}\ln\tan\dfrac{x}{2}-\dfrac{1}{4}\tan^2\dfrac{x}{2}+C.$

9. (i) $\dfrac{1}{2}x+\dfrac{1}{2}\ln|\cos x+\sin x|+C$

(ii) $\dfrac{a}{a^2+b^2}x-\dfrac{b}{a^2+b^2}\ln|b\cos x+a\sin x|+C$

(iii) $(1/2)\{\sec x+\ln|\sec+\tan x|\}+C$

10. (i) $\tan^{-1}\left(\dfrac{\tan x}{2+\tan^2 x}\right)$

$+\dfrac{1}{2}\ln\left|\dfrac{\tan^2 x+\tan x+2}{\tan^2 x-\tan x+2}\right|+C$

(ii) $2\tan^{-1}\sqrt{\sec x-1}+C$

(iii) $-2\tan^{-1}\sqrt{\operatorname{cosec} x-1}+C$

(iv) $-\ln|\cot x+\sqrt{\cot^2 x-1}|+\sqrt{2}\ln|\sqrt{2}\cos x+\sqrt{2\cos^2 x-1}|+C$

11. (i) $\dfrac{-1}{2(1+2\tan x)}+C$

(ii) $\dfrac{3}{8\sqrt{2}}\tan^{-1}\left(\dfrac{\tan^2 x-2}{2\sqrt{2}\tan x}\right)-\dfrac{\tan x}{4(\tan^2 x+2)}+C$

(iii) $\dfrac{5}{9}\cdot\dfrac{\sin\theta}{5+4\cos\theta}-\dfrac{8}{27}\tan^{-1}\left(\dfrac{\tan\frac{\theta}{2}}{3}\right)+C$

(iv) $\tan^{-1}(\tan x-\cot x)+C$

CONCEPT PROBLEMS—D

2. $\dfrac{1}{2}(x^2+1)\tan^{-1}x-\dfrac{1}{2}+C$

4. (i) $\dfrac{x^4}{4}\left\{(\ln x)^2-\dfrac{\ln x}{2}+\dfrac{1}{8}\right\}+C$

(ii) $(\dfrac{x\ln x}{x+1}-\ln(x+1)+C$

(iii) $\dfrac{(1+x)^2}{4}[\ell n(1+x)-2]+C$

(iv) $-x\tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)-2\ell n\sec\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)+C$

5. (i) $x\sec^{-1}x-\ln(x+\sqrt{x^2-1})+C$

(ii) $\dfrac{1}{2}x^2\sin^{-1}x-\dfrac{1}{4}\sin^{-1}x+\dfrac{1}{4}x\sqrt{1-x^2}+C$

(iii) $\left(x-\dfrac{1}{2}\right)\sin^{-1}\sqrt{x}+\dfrac{1}{2}\sqrt{x(1-x)}+C$

(iv) $2x\tan^{-1}x-\ln(1+x^2)+C$

PRACTICE PROBLEMS—J

9. (i) $\tan x\,.\,\ln\cos x+\tan x-x+C$

(ii) $-\cos x\,.\,\ln\tan x+\ln\tan\dfrac{x}{2}+C$

(iii) $2\{x\ln(1+x^2)-2x+2\tan^{-1}x\}+C$

(iv) $xe^x\ln(xe^{x-1})+C$

10. (i) $-\cot x\ln(\sec x)+x+C$

(ii) $\sin x\ln(\operatorname{cosec} x+\cot x)+x+C$

(iii) $-\cos x\ln(\sec x+\tan x)+x+C$

(iv) $\dfrac{1}{2}\{\ln|\sec x+\tan x|\}^2+C$

11. (i) $x\,\ell n\left(x+\sqrt{x^2+a^2}\right)-\sqrt{x^2+a^2}+C$

(ii) $x\ln^2(x+\sqrt{1+x^2})-2\sqrt{1+x^2}\times\ln(x+\sqrt{1+x^2})+2x+C$

(iii) $\dfrac{1}{3}\left[x^2\ln\dfrac{1+x}{1-x}+\ln(1-x^2)+x^2\right]+C$

(iv) $-\dfrac{1}{2(x-1)^2}\ln x+\ln\dfrac{x}{x-1}-\dfrac{1}{x-1}+C$

12. (i) $\dfrac{2^x\sin\left(x-\cot^{-1}(\ln 2)\right)}{\sqrt{1+(\ln 2)^2}}+C$

(ii) $\dfrac{3^x\sin(3\sin 3x+(\ln 3)\cos 3x)}{9+(\ln 3)^2}+C$

(iii) $\dfrac{1}{4}e^x\{(\cos x+\sin x\}-\dfrac{1}{5}(\cos 3x+3\sin 3x)\}+C$

(iv) $\dfrac{e^{2x}}{8}(2-\sin 2x-\cos 2x)+C$

13. (i) $\dfrac{x}{2}[\sin(\ln x)-\cos(\ln x)]+C$

(ii) $\dfrac{e^x}{2}\left(\dfrac{2\sin 2x+\cos 2x}{5}-\dfrac{4\sin 4x+\cos 4x}{17}\right)+C$

(iii) $x\tan^{-1}\dfrac{x}{a}-\sqrt{ax}+a\tan^{-1}\dfrac{x}{a}+C$

(iv) $\dfrac{1}{4}\left\{(x^4-1)\tan^{-1}x-\dfrac{x^2}{3}+x\right\}+C$

14. (i) $x\sec^{-1}x-\ln(x+\sqrt{x^2-1})+C$

(ii) $\dfrac{x}{\sqrt{1-x^2}}\sin^{-1}x+\dfrac{1}{2}\ln(1-x^2)+C$

(iii) $\left(\dfrac{-1}{2x^2}\right)\cos^{-1}x+\dfrac{\sqrt{1-x^2}}{2x}+C$

(iv) $\dfrac{1}{2}\left[x\cos^{-1}x-\sqrt{1-x^2}\right]+C$

15. (i) $-\dfrac{x\cos 3x}{3}+\dfrac{1}{36}\sin 3x+\dfrac{1}{4}\sin x+C$

(ii) $\dfrac{1}{4}(x^2-x\sin 2x-\dfrac{1}{2}\cos 2x)+C$

(iii) $\frac{1}{18}(3x\sin 3x+\cos 3x)+\frac{1}{2}(x\sin x+\cos x)+C$

16. (i) $(x^3-3x^2+6x-6)e^x+C$
(ii) $x^3\sin x+3x^2\cos x-3.2.x\sin x-3.2.1.\cos x+C$
(iii) $\frac{1}{4}(\ell n\ x)^4\left[(\ell n x)-\frac{1}{2}\ell nx+\frac{1}{8}\right]+C$

PRACTICE PROBLEMS—K

1. (i) $-e^x\cos x+C$ (ii) $e^x\tan x+C$
(iii) $e^x\sec x+C$

2. (i) $-e^x\cot\frac{x}{2}+C$ (ii) $e^x\tan x+C$
(iii) $\frac{1}{2}e^{2x}\cot 2x+C$ (iv) $e^x\frac{x}{\sqrt{1+x^2}}+C$

3. (i) $e^x\cdot\frac{1}{1+x^2}+C$ (ii) $e^x\left(\frac{x+1}{x^2+1}\right)+C$
(iii) $\frac{e^x}{(1+x)^2}+C$ (iv) $e^x\left(\frac{x-3}{x+1}\right)+C$

4. (i) $e^x\left(\frac{x+1}{x+2}\right)+C$ (ii) $\frac{e^x(x-1)}{x+1})+C$
(iii) $\frac{e^x}{1+x^2}+C$ (iv) $e^x\frac{x-2}{x+2}+C$

5. (i) $e^x\ ln(\sec x+\tan x)+C$
(ii) $e^x\left(\log x-\frac{1}{x}\right)+C$

6. (i) $x\tan x+C$ (ii) $x\tan(\ln x)+C$
(iii) $x\sin^{-1}x+C$

PRACTICE PROBLEMS—L

1. $\frac{x^2}{3}e^{3x}-\frac{2}{9}xe^{3x}-\frac{2}{27}e^{3x}+C$

2. $\frac{e^{3x}}{3}\left(x^3-x^2+\frac{11}{3}x-\frac{2}{9}\right)+C$

3. $\frac{1}{4}x(2x^2-3)\sin 2x+\frac{3}{8}(2x^2-1)\cos 2x+C$

4. $-\frac{e^{-x}}{5}(\cos^2x-\sin 2x+2)+C$

5. $\left(\frac{1}{3}x^3-x^2+\frac{2}{3}x+\frac{13}{9}\right)e^{3x}+C.$

6. $\left(\frac{x^3}{3}-x^2+3x\right)\ell nx-\frac{x^3}{9}+\frac{x^2}{2}-3x+C$

7. $-\frac{18x^2+6x-13}{72}\sin(6x+2)$
$-\frac{6x+1}{72}\cos(6x+2)+\frac{1}{2}x^3+\frac{1}{4}x^2-x+C$

8. $\frac{3}{4}(x^2-7x+1)(2x+1)^{2/3}$
$-\frac{9}{40}(2x-7)(2x+1)^{5/3}+\frac{27}{320}(2x+1)^{8/3}+C$

9. $\frac{1}{2}\sin 2x\ln(1+\tan x)-\frac{x}{2}+\frac{1}{2}\ln(\cos x+\sin x)+C$

10. $1/9\left(1-\frac{1}{x^2}\right)^{3/2}\left[2-3\ln\left(1+\frac{1}{x^2}\right)\right]+C$

11. $e^x\sqrt{\frac{1+x^n}{1-x^n}}+C$

PRACTICE PROBLEMS—M

2. (i) $I_n=x^ne^x-nI_{n-1}$ (ii) $I_n=x(\ln x)^n-nI_{n-1}$

3. (i) $\frac{\tan^3\theta}{3}-\tan Q+Q+C$
(ii) $\frac{-1}{4\tan^4\theta}+\frac{1}{2\tan^2\theta}+\ln|\sin Q|+C$
(iii) $-\frac{\cos\theta}{2\sin^2\theta}+\frac{1}{2}\ln\left|\tan\frac{\theta}{2}\right|+C$
(iv) $\frac{\sin\theta\cos^3\theta}{6}\left(\cos^2\theta+\frac{5}{4}\right)$
$+\frac{5}{16}(\sin\theta\cos\theta+\theta)+C$

4. (i) $\frac{1}{\cos\theta}+\ln\left|\tan\frac{\theta}{2}\right|+C$
(ii) $\frac{\sin\theta\cos\theta}{2}\left(\frac{\sin^4\theta}{3}-\frac{\sin^2\theta}{12}-\frac{1}{8}\right)+\frac{\theta}{16}+C$
(iii) $\frac{1}{\cos\theta}-\frac{\cos\theta}{2\sin^2\theta}+\frac{3}{2}\ln\left|\tan\frac{\theta}{2}\right|+C$

5. (i) $\frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|-\frac{\sqrt{1-x^2}}{2x^2}+C$
(ii) $\frac{-x^3}{2(a^2+x^2)}+\frac{3}{2}\left(x-a\tan^{-1}\frac{x}{a}\right)+C$

(iii) $\dfrac{x^3}{a^2(a+cx^2)}\left\{\dfrac{1}{3}-\dfrac{cx^2}{5(a+cx^2)}\right\}+C$

(iv) $\dfrac{-(2a^2+3x^2)}{3(a^2+x^2)^{3/2}}+C$

14. $I_n=\dfrac{x}{4(n-1)(1+x^4)^{n-1}}+\dfrac{4n-5}{4(n-1)}I_{n-1}$;

$I_2=\dfrac{x}{4(1+x^4)}$

$+\dfrac{3}{4}\left(\dfrac{1}{2\sqrt{2}}\tan^{-1}\left(\dfrac{x-\frac{1}{x}}{\sqrt{2}}\right)-\dfrac{1}{4\sqrt{2}}\ell n\left(\dfrac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)\right)+C$

PRACTICE PROBLEMS—N

1. (i) $\dfrac{1}{4}x-\dfrac{1}{x}+\dfrac{1}{2x-1}+\dfrac{1}{2x+1}+C$

(ii) $4\ln|x-1|-7\ln|x+3|+5\ln|x-4|+C$

2. (i) $-\dfrac{1}{2}x^2-3x-\ln(1-x)^6-\dfrac{4}{1-x}+\dfrac{1}{2(1-x)^2}+C$

(ii) $-2\ln|x|+2\ln|x-2|-\dfrac{1}{x-2}-\dfrac{1}{x}+C$

3. (i) $\dfrac{1}{5}\ln|x-1|-\dfrac{1}{10}\ln|x^2+4|+\dfrac{2}{5}\tan^{-1}\dfrac{x}{2}+C$ (ii)

$\sqrt{2}\tan^{-1}\dfrac{x}{\sqrt{2}}-\tan^{-1}x+C$

(iii) $x-\tan^{-1}x+\ln\left|\dfrac{\sqrt{1+x^2}}{x}\right|+C$

(iv) $\dfrac{1}{2}x^2+\ln\left|\dfrac{x}{\sqrt{x^2-2x+3}}\right|+C$

4. (i) $\dfrac{1}{3}\ln|x^3+1|-\dfrac{1}{2}\ln|x^2-x+1|$

$+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}+C$

(ii) $\ln|x|-\dfrac{1}{2}\ln(x^2+1)+C$

(iii) $-\dfrac{1}{4}\dfrac{1}{(2x^2+4x+3)}+\dfrac{1}{4}\left[\dfrac{x+1}{x+1)^2+\frac{1}{2}}\right]$

$+\sqrt{2}\tan^{-1}\sqrt{2}(x+1)+C$

(iv) $\ln(1+x)+\dfrac{1}{3}\ln(1+x^{1/3})-\dfrac{1}{6}\ln(x^{2/3}-x^{1/3}+1)$

$+\dfrac{1}{\sqrt{3}}\tan^{-1}\left[\dfrac{2x^{1/3}-1}{\sqrt{3}}\right]+C.$

5. (i) $-\dfrac{1}{10}\ln|1-\cos x|+\dfrac{1}{2}\ln|1+\cos x|+C$

(ii) $-\dfrac{2}{5}\ln|3+2\cos x|+C$

(iii) $\ln\left|\dfrac{1-\sin\frac{\theta}{2}}{1+\sin\frac{\theta}{2}}\right|+\dfrac{1}{\sqrt{2}}\ln\left|\dfrac{\sqrt{2}\sin\frac{\theta}{2}+1}{\sqrt{2}\sin\frac{\theta}{2}-1}\right|+C$

(iv) $-\dfrac{1}{10}\ln|\ln x|+\dfrac{1}{35}\ln|\ln x-5|$

$+\dfrac{1}{14}\ln|\ln x+2|+C$

6. (i) $\dfrac{1}{8}\ln\left|\dfrac{1-\cos x}{1+\cos x}\right|-\dfrac{1}{4\sqrt{3}}\tan^{-1}\left(\dfrac{\cos x}{\sqrt{3}}\right)+C$

(ii) $\dfrac{1}{\sqrt{2}}\ln\left|\dfrac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|-\dfrac{1}{2}\ln\left|\dfrac{1+\sin x}{1-\sin x}\right|+C$

(iii) $\dfrac{2}{3}\ln\left|\dfrac{\sec x}{\tan x-2}\right|-\dfrac{1}{\tan x-2}-\dfrac{x}{5}+C$

(iv) $\ln\left(\dfrac{e^x+1}{e^x+2}\right)+C$

7. (i) $\dfrac{-x^2+x}{4(x+1)(x^2+1)}+\dfrac{1}{2}\ln|x+1|$

$-\dfrac{1}{4}\ln(x^2+1)+\dfrac{1}{4}\tan^{-1}x+C.$

(ii) $-\dfrac{3}{8}\tan^{-1}x-\dfrac{x}{4(x^4-1)}+\dfrac{3}{16}\ln\left|\dfrac{x-1}{x+1}\right|+C.$

(iii) $\dfrac{15x^5+40x^2+33x}{48(1+x^2)^3}+\dfrac{15}{48}\tan^{-1}x+C.$

(iv) $x-\dfrac{x-1}{x^2-2x+2}+2\ln(x^2-2x+2)$

$+3\tan^{-1}(x-1)+C$

8. (i) $\dfrac{13x-159}{8(x^2-6x+13)}+\dfrac{53}{16}\tan^{-1}\dfrac{x-3}{2}+C$ (ii)

$$\frac{x^2}{2}-\frac{1}{2}\ell n(x^2+2)-\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$$

(iii) $\frac{1}{x^2(x^2+1)}+\ln\sqrt{x^2+1}+C$

(iv) $\frac{2}{3}\ln\left(\frac{x^3+1}{x^3}\right)-\frac{1}{3x^3}-\frac{1}{3(x^3+1)}+C$

9. (i) $\ln\sqrt{x^2+3}+\tan^{-1}x+C$

(ii) $\frac{2}{3}\ln\left|\frac{x^3+1}{x^3}\right|-\frac{1}{3x^3}-\frac{1}{3(x^3+1)}+C$

(iii) $\frac{x}{x^2+x+1}+\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}-2\ln(x^2+x+1)$

$+\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}+2x+C$

(iv) $C-\frac{57x^4+103x^2+32}{8x(x^2+1)^2}-\frac{57}{8}\tan^{-1}x$

10. (i) $\ln(x^2+4)+\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{4}{x^2+4}+C$

(ii) $\ln\sqrt{x^2+2x+3}+\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$

$-\sqrt{5}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$

(iii) $\frac{1}{648}\left[\tan^{-1}\frac{x+1}{3}+\frac{3(x+1)}{x^2+2x+10}+\frac{18(x+1)}{(x^2+2x+10)^2}\right]+C$

(iv) $\frac{1}{2}\ln(x^2+2)-\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)-\frac{1}{(x^2+2)^2}+C$

PRACTICE PROBLEMS—O

1. (i) $\frac{1}{2}\ln\left|\frac{x-2}{x+2}\right|+\tan^{-1}x+C$

(ii) $\tan^{-1}x-\frac{1}{\sqrt{2}}\tan^{-1}(x\sqrt{2})+C$

(iii) $\frac{a^4}{2(a^2-x^2)}+\frac{x^2}{2}+a^2\ln|a^2-x^2|+C$

(iv) $-\frac{1}{x-2}-\tan^{-1}(x-2)+C$

2. (i) $\frac{1}{6}\{\ln|x^2-2|-\ln(x^2+1)\}+C$

(ii) $\frac{I}{2a(a+bx^2)}+\frac{1}{2a^2}\ln\left(\frac{x^2}{a+bx^2}\right)+C$

(iii) $\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}-\tan^{-1}x+C$

(iv) $\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2+x^2}{x^2\sqrt{3}}\right)-\frac{1}{4}\ln\left(\frac{x^4+x^2+1}{x^4}\right)+C$

3. (i) $\frac{2b}{a^2}\ln\left|\frac{x}{a-bx}\right|-\frac{(a-2bx)}{a^2x(a-bx)}+C$ (ii)

$\frac{1}{na^2}\ln\left|\frac{x^n}{a+bx^n}\right|+\frac{1}{na(a+bx^n)}+C$

(iii) $\frac{2}{25}\ln\left|\frac{(x+2)^2}{x^2+1}\right|+\frac{3}{25}\tan^{-1}x-\frac{1}{5(x+2)}+C$

4. (i) $-\frac{1}{4x^3(x^4+1)}+\frac{1}{4x^3}$

$+\frac{3}{4}\left(\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x^2-1}{\sqrt{2}x}-\frac{1}{4\sqrt{2}}\ell n\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|\right)+C$

(ii) $\frac{x}{216(x^2+9)}+\frac{x}{36(x^2+9)^2}+\frac{1}{648}\tan^{-1}\frac{x}{3}+C$

(iii) $\frac{1}{8}\frac{x(1+x^2)}{(1-x^2)^2}-\frac{1}{16}\ln\left|\frac{1+x}{1-x}\right|+C$

(iv) $-\frac{2+3x^2}{2x(1+x^2)}-\frac{3}{2}\tan^{-1}x+C$

5. (i) $\frac{1}{12}\ell n(x^4-1)-\frac{1}{14}\ell n(x^8+x^4+1)$

$+\frac{1}{4\sqrt{3}}\tan^{-1}\frac{2x^4+1}{\sqrt{3}}+C$

(ii) $\frac{1}{4}\left(\frac{2x^6-3x^2}{x^4-1}+\frac{3}{2}\ln\left|\frac{x^2-1}{x^2+1}\right|\right)+C$

PRACTICE PROBLEMS—P

1. (i) $C-\frac{\sqrt{2x+1}}{x}+\ln\left|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right|$

(ii) $\frac{2(a+bx)^{3/2}}{3b^2}-\frac{2a(a+bx)^{1/2}}{b^2}+C$

(iii) $\sqrt{(x+a)(x+b)}+(a-b)$

$\ln(\sqrt{x+a}+\sqrt{x+b})+C$

2. (i) $4x^{1/4}+6x^{1/6}+24x^{1/12}+2\ln|x^{1/12}-1|+C$

(ii) $\dfrac{-1}{2}\ln(t^2+t+1)-\dfrac{5}{\sqrt{3}}\tan^{-1}\left(\dfrac{2t+1}{\sqrt{3}}\right)$

$+\ln\left|\dfrac{(t+2)^{4/3}}{(t-1)^{1/3}}\right|+C$ where $t=\left(\dfrac{x-1}{x}\right)^{1/3}$

(iii) $\left(-\dfrac{2}{5}\right)\log(x^2+4)+\left(\dfrac{9}{10}\right)\log(x^4+9)+C$

3. (i) $\dfrac{1}{6}\ln\left|\dfrac{\sqrt{1+x^6}-1}{\sqrt{(1+x^6)}+1}\right|+C$

(ii) $\dfrac{2}{p^2}\left\{\sqrt{(px+q)}+\dfrac{q}{\sqrt{(px+q)}}\right\}+C$

(iii) $\dfrac{1}{5}(1+x^2)^{5/2}-\dfrac{2}{3}(1+x^2)^{3/2}+\sqrt{(1+x^2)}+C$

(iv) $-\dfrac{1}{2x^2}\sqrt{x^2+1}+\dfrac{1}{2}\ln\dfrac{1+\sqrt{x^2+1}}{|x|}+C$

4. (i) $2\tan^{-1}\sqrt{x+1}+C$

(ii) $-2(\sqrt[4]{5-x}-1)^2-4\ln(1+\sqrt[4]{5-x})+C$

(iii) $2(x+2)^{1/2}-4(x+2)^{1/4}+4\ln\{1+(x+2)^{1/4}\}+C$

(iv) $\ln\left|\dfrac{(\sqrt{x+1}-1)^2}{x+2+\sqrt{x+1}}\right|-\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2\sqrt{x+1}+1}{\sqrt{3}}+C$

PRACTICE PROBLEMS—Q

1. (i) $\dfrac{1}{2}\ln\left|\dfrac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}\right|+C$

(ii) $\dfrac{1}{2}\ln\left|\dfrac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+C$

2. (i) $\ln|x|-\ln|1+2x|+\sqrt{(9x^2+4x+1)}+C$

(ii) $\sin^{-1}\left(\dfrac{3x+1}{(1+x)\sqrt{5}}\right)+C$

(iii) $\dfrac{1}{\sqrt{2}}\sin^{-1}\left(\dfrac{x\sqrt{2}}{1+x}\right)+C$

(iv) $\dfrac{\sqrt{(x^2+1)}}{\sqrt{(x^2-1)}}+C$

3. (i) $\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x-1}{\sqrt{(2x)}}-\dfrac{1}{2\sqrt{2}}\ln\left|\dfrac{x-\sqrt{(2x)}+1}{x+\sqrt{(2x)}+1}\right|+C$

(ii) $2\tan^{-1}\left(\sqrt{x+1}\right)\sqrt{2}\tan^{-1}\left(\dfrac{\sqrt{x+1}}{\sqrt{2}}\right)+C$

(iii) $\dfrac{1}{4\sqrt{3}}\ln\left|\dfrac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}\right|-\dfrac{1}{2}\tan^{-1}\sqrt{x+1}+C$

4. (i) $\dfrac{1}{5\sqrt{3}}\tan^{-1}\left(\dfrac{5x}{\sqrt{12-9x^2}}\right)+C$

(ii) $-\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{1-x^2}{\sqrt{3}\,x^2}\right)+C$

(iii) $\dfrac{1}{2\sqrt{2}}\ln\dfrac{\sqrt{2+2x^2}-x}{\sqrt{2+2x^2}+x}+\ln(x+\sqrt{x^2+1})+C$

5. (i) $\dfrac{1}{8}\ln\left|\dfrac{\sqrt{4-x^2}-2}{x}\right|-\dfrac{1}{8\sqrt{3}}\sin^{-1}\dfrac{2(x+1)}{x+4}+C$

(ii) $-\dfrac{1}{8}\dfrac{\sqrt{(4x^2+4x+5)}}{2x+1}+C$

(iii) $-\dfrac{1}{2\sqrt{6}}\ln\left(\dfrac{\sqrt{x^2+2x-4}-\sqrt{6}\,(x+1)}{\sqrt{x^2+2x-4}+\sqrt{6}\,(x+1)}\right)+C$

(iv) $-\dfrac{2}{15}\sqrt{\dfrac{x+2}{x+1}}\,\dfrac{8x^2+12x+7}{(x+1)^2}+C$

6. (i) $-\dfrac{\sqrt{x^2-4x+3}}{x-1}-2\sin^{-1}\dfrac{1}{(x+1)}+C$

(ii) $2\sqrt{(x+1)}-\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2(x+1)}}-\dfrac{3}{2\sqrt{2}}$

$\ln\left|\dfrac{(x+2)-\sqrt{2(x+1)}}{(x+2)+\sqrt{2(x+1)}}\right|+C$

(iii) $\ln\dfrac{\sqrt{x^2+2x+4}-1}{\sqrt{x^2+2x+4}+1}$

$-\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{\sqrt{2(x^2+2x+4)}}{x+1}+C$

PRACTICE PROBLEMS—R

1. $\dfrac{2(1+x^3)^{1/2}(x^3-2)}{9}+C$

2. $\dfrac{2x^{1/2}}{(1+x^2)^{1/2}}+C$

3. $3\left[\ln\left|\dfrac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right|+\dfrac{2\sqrt[3]{x}+3}{2(1+\sqrt[3]{x})^2}\right]+C$

4. $\dfrac{1}{8}\sqrt[3]{(1+x^3)^8}-\dfrac{1}{5}\sqrt[3]{(1+x^3)^5}+C$

5. $\dfrac{1}{5}\ln\dfrac{|u-1|}{\sqrt{u^2+u+1}}+\dfrac{\sqrt{3}}{5}\tan^{-1}\dfrac{1+2u}{\sqrt{3}}+C,$

where $u=\sqrt[3]{1+x^5}$

6. $C-\dfrac{\sqrt[3]{1+x^3}}{x}+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2\sqrt[3]{1+x^3}+x}{x\sqrt{3}}$

$-\dfrac{1}{3}\ln\left|\dfrac{\sqrt[3]{1+x^3}+x}{\sqrt{\sqrt[3]{(1+x^3)^2}+x\sqrt[3]{1+x^3}+x^2}}\right|$

7. $C-\dfrac{1}{10}\sqrt{\left(\dfrac{1+x^4}{x^4}\right)^5}+\dfrac{1}{3}\sqrt{\left(\dfrac{1+x^4}{x^4}\right)^3}-\dfrac{1}{2}\sqrt{\dfrac{1+x^4}{x^4}}$

8. $12\left[\dfrac{\sqrt[3]{u^{13}}}{13}-\dfrac{3\sqrt[3]{u^{10}}}{10}+\dfrac{3\sqrt[3]{u^7}}{7}-\dfrac{\sqrt[3]{u^4}}{4}\right]+C,$

where $u=1+\sqrt[4]{x}$

9. $\dfrac{32}{5}x^{5/4}+\dfrac{144}{13}x^{13/4}+7\dfrac{72}{7}x^{21/4}+\dfrac{108}{29}x^{29/4}+C$

10. $3x^{4/3}\left(\dfrac{1}{4}-\dfrac{2}{11}x^{1/2}+\dfrac{12}{7}x-\dfrac{16}{17}x^{3/2}\right)+C.$

11. $\dfrac{16}{105}(1+x^{7/8})^{3/2}(3x^{7/8}-2)+C.$

12. $\dfrac{1}{3}\dfrac{t}{t^3-8}-\dfrac{1}{36}\ln\dfrac{|t-2|}{\sqrt{(t^2+2t+4)}}+\dfrac{1}{12\sqrt{3}}$

$\tan^{-1}\dfrac{t+1}{\sqrt{3}}+C$, where $t=x^{-1}(1+8x^3)^{1/3}$.

PRACTICE PROBLEMS—S

1. $2\ell n\left|\sqrt{x^2+2x+4}-x\right|-\dfrac{3}{2\left(\sqrt{x^2+2x+4}-x-1\right)}$

$-\dfrac{3}{2}\ell n\left|\sqrt{x^2+2x+4}-x-1\right|+C$

2. $\dfrac{1+\sqrt{1-x^2}}{x}+2\tan^{-1}\sqrt{\dfrac{1+x}{1-x}}+C$

3. $\dfrac{x-1}{\sqrt{2x-x^2}}+C$

4. $\dfrac{(x+\sqrt{1+x^2})^{15}}{15}+C$

5. $\dfrac{1}{\sqrt{2}}\ln\left|\dfrac{\sqrt{(x^2-x+2)}+x-\sqrt{2}}{\sqrt{(x^2-x+2)}+x+\sqrt{2}}\right|+C.$

6. $\dfrac{1}{3}\left[x^3+\sqrt{(x^2-1)^3}\right]+C$

7. $\dfrac{x^2}{2}+\dfrac{x}{2}\sqrt{x^2-1}-\dfrac{1}{3}\ln\left|x+\sqrt{x^2-1}\right|+C$

8. $\dfrac{\sqrt{2}}{3}[(x+2)^{3/2}-(x-2)^{3/2}]+C$

PRACTICE PROBLEMS—T

1. $P(x)=2x+2$; $Q(x)=3x-3$.

2. $\dfrac{3x^2+x-1}{3}\sqrt{3x^2-2x+1}+C$

3. $\dfrac{1}{3}(x^2-14x+111)\sqrt{x^2+4x+3}$

$-66\ln\left|2x+1+2\sqrt{x^2+4x+3}\right|+C$

7. **Hint:** Putting $y=tx$, we obtain $x=1/\{t^2(1-t)\}$, $y=1/\{t(1-t)\}$

8. Hint : Put $x^2+y^2=t(x-y)$, from where we obtain $x=a^2t(t^2+a^2)/(t^4+a^4)$, $y=a^2t(t^2-a^2)/(t^4+a^4)$.

CONCEPT PROBLEMS—E

1. (a) and (b)
2. (a) Not elementary (b) 2 sin q + C (c) $2\sqrt{1-\cos\theta}+C$
3. (a) and (b)
4. $\sqrt{1-x}\sqrt[3]{1-x}$
5. (a), (d) and (f)

PRACTICE PROBLEMS—U

9. (iii) Hint : let $u=\sin^2 x$

(vii) (b) $n=1:\dfrac{2}{3}(1+x)^{3/2}-2(1+x)^{1/2}+C$;

$n=2:\sqrt{1+x^2}+C$; $n=4:\dfrac{1}{2}\ln(x^2+\sqrt{1+x^4})+C$

(ix) (a) n is odd. (b) $n = 3 : \frac{1}{2}\sqrt{1+x^4} + C$;

$n = 5 : \frac{x^2}{4}\sqrt{1+x^4} - \frac{1}{4}\ln(x^2 + \sqrt{1+x^4}) + C$

10. (ii) $a = 0, -1$

OBJECTIVE EXERCISES

1. D	**2.** C	**3.** B
4. C	**5.** A	**6.** D
7. B	**8.** C	**9.** A
10. C	**11.** A	**12.** B
13. D	**14.** C	**15.** B
16. B	**17.** B	**18.** A
19. A	**20.** A	**21.** C
22. C	**23.** A	**24.** B
25. A	**26.** B	**27.** B
28. C	**29.** C	**30.** B
31. A	**32.** C	**33.** B
34. C	**35.** A	**36.** D
37. C	**38.** C	**39.** A
40. C	**41.** B	**42.** B
43. A	**44.** D	**45.** B
46. D	**47.** A	**48.** A
49. A	**50.** B	**51.** B
52. B	**53.** A	**54.** A
55. A	**56.** D	**57.** C
58. D	**59.** B	**60.** B
61. D	**62.** B	**63.** A
64. D	**65.** B	**66.** AC
67. ABCD	**68.** AC	**69.** ACD
70. C	**71.** AC	**72.** AC
73. AD	**74.** ABC	**75.** AD
76. ABD	**77.** ABD	**78.** AC
79. BC	**80.** ACD	**81.** ABC
82. BCD	**83.** BC	**84.** AC
85. AC	**86.** C	**87.** A
88. A	**89.** B	**90.** A
91. D	**92.** A	**93.** B
94. C	**95.** C	**96.** D
97. A	**98.** A	**99.** B
100. A	**101.** C	**102.** C
103. B	**104.** C	**105.** A
106. D	**107.** A	**108.** D
109. B	**110.** A	

111. (A) – (P,Q), (B) – (P), (C) – (RS), (D) – (PQ)
112. (A) – (Q), (B) – (P), (C) – (P,R), (D) – (PRS)
113. (A) – (Q), (B) – (P), (C) – (S), (D) – (R)
114. (A) – (R), (B) – (Q), (C) – (T), (D) – (P)
115. (A) – (Q), (B) – (Q), (C) – (Q), (D) – (Q)

REVIEW EXERCISES for JEE ADVANCED

1. 1/2 (sinq – lnq - q) + C **2.** $x^x(x \ln x - 1) + C$

3. $\frac{(\tan^{-1}2x)^2}{4} - 3\ln|x-2| + \frac{6}{x-2} + C$

4. $\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{1+x^2+x^4}+x\sqrt{3}}{1-x^2}\right| + C$

5. $\sin^{-1}\left(\frac{x}{(1+x^4)^{1/2}}\right) + C$

6. (i) $\frac{1}{8(1-x^2)^4} + \frac{1}{2(1-x^2)}$

$- \frac{1}{2(1-x^2)^3} + \frac{3}{4(1-x^2)^2} + C$

(ii) $\frac{1}{4}\left\{\ln|1-x^4| + \frac{1}{1-x^4}\right\} + C$

7. $\tan^{-1}\sqrt{\frac{a^2-x^2}{x^2-b^2}} + \frac{a}{b}\tan^{-1}\frac{b}{a}\sqrt{\frac{a^2-x^2}{x^2-b^2}} + C$

9. $\frac{-1}{\sqrt{(a^2-b^2)}}\ln\left|\sqrt{a^2-b^2}\cos x + \sqrt{a^2\cos^2 x + b^2\sin^2 x}\right|$
$+ C$, if $a > b$,

$\frac{-1}{\sqrt{b^2-a^2}}\sin^{-1}\left(\frac{\sqrt{b^2-a^2}}{b}\cos x\right) + C$, if $a < b$.

11. $\ln|\tan^{-1}(\sec x + \cos x)| + C$

12. $\frac{\sqrt{x^3+7}}{3}(x^2 - 14) + \ln|x + \sqrt{x^2+7}| + C$

13. $\frac{1}{64}(32x^2 - 20x - 373)\sqrt{2x^2+5x+7} +$

$\frac{3297}{128\sqrt{2}}\ln\left|4x+5+2\sqrt{4x^2+10x+14}\right| + C$

15. $A = -b/(a^2 - b^2)$, $B = a/(a^2 - b^2)$

16. (i) $\frac{3}{8}\tan^{-1}(x+1) - \frac{5x^3+15x^2+18x+8}{8(x^2+2x+2)^2} + C$ (ii)

$\frac{15x^5+40x^3+33x}{48(1+x^2)^3} + \frac{15}{48}\tan^{-1}x + C$

(iii) $\dfrac{3-7x-2x^2}{2(x^3-x^2-x+1)}+\ln\dfrac{|x-1|}{(x+1)^2}+C$

(iv) $\ln\left|\dfrac{x^3-x^2+x}{(x+1)^2}\right|-\dfrac{3}{x+1}+\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right)+C$

17. (i) $\dfrac{e^x(x+2)}{x+3})+C$ (ii) $xe^{\tan^{-1}x}+C$

(iii) $\dfrac{e^x\sec x}{2}+C$ (iv) $\dfrac{x}{1+\ell nx}+C$

18. (i) $-2\sqrt{\dfrac{1-\sin x}{\sin x}}+\sqrt{2}\tan^{-1}\sqrt{\dfrac{1-\sin x}{2\sin x}}+C$

(put $u = 1/\sin x$)

(ii) $\tan^{-1}\left(\dfrac{\sqrt{\cos^2 x-a^2\sin^2 x}}{a}\right)+C$

19. (i) $\dfrac{4}{x}+\dfrac{3x}{2(x^2-1)}+\dfrac{11}{4}\log\left|\dfrac{x-1}{x+1}\right|+C$ (ii)

$\dfrac{1}{2}\cdot\dfrac{1+x}{(1+x^2)^2}+\dfrac{1}{4}\cdot\dfrac{x-2}{x^2+1}+\dfrac{1}{4}\tan^{-1}x+C$

20. (i) $x+\ln|3\cos x+4\sin x+5|-(1/5)\tan(x-a)/2+C$ where $a=\tan^{-1}(4/3)$

21. (i) $-\dfrac{2}{b^2}\left[\ln|a+b\cos x|+\dfrac{a}{a+b\cos x}\right]+C$ (ii)

$C-\dfrac{\cos x}{b+a\sin x}$

22. (i) $-\dfrac{x^n}{x^{m+n}+1}+C$

(ii) $x^m+\dfrac{1}{x^n}=v$;

$\ln|v|+c=\ln|x^{m+n}+1|-n\ln|x|+C.$

23. (i) $-\dfrac{1}{4c^2(a+cx^2)^2}+\dfrac{a}{6c^2(a+cx^2)^3}+C$

(ii) $\dfrac{1}{na^2}\ln\left|\dfrac{x^n}{a+bx^n}\right|+\dfrac{1}{na(a+bx^n)}+C$

(iii) $\dfrac{1}{8}x\dfrac{(1+x^2)}{(1-x^2)^2}-\dfrac{1}{16}\ln\left|\dfrac{1+x}{1-x}\right|+C$

(iv) $\dfrac{\sqrt{3}}{2}\tan^{-1}\left(\dfrac{2+x^2}{x^2\sqrt{3}}\right)-\dfrac{1}{4}\tan\left(\dfrac{x^4+x^2+1}{x^4}\right)+C$

24. (i) $\dfrac{1}{4}\ln\left|\dfrac{x^2-2x+3}{x^4+2x+3}\right|+C$

(ii) $\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x^2-1}{\sqrt{3}}\right)+C$

(iii) $\dfrac{1}{2a^3}$

$\left[\tan^{-1}\left(\dfrac{x^2-a^2}{ax}\right)-\dfrac{1}{2\sqrt{3}}\ln\left|\dfrac{x^2-\sqrt{3}ax+a^2}{x^2+\sqrt{3}ax+a^2}\right|\right]+C$

(iv) $\dfrac{7}{4\sqrt{2}}\tan^{-1}\left(\dfrac{x^2-2}{\sqrt{2}x}\right)-\dfrac{\sqrt{3}}{8\sqrt{2}}$

$\ln\left|\dfrac{x^2-\sqrt{6}x+2}{x^2+\sqrt{6}x+2}\right|+C$

25. (i) $\dfrac{8}{5}\left(\dfrac{x+3}{x-2}\right)^{1/8}+C$

(ii) $-\ln\left(t-\dfrac{1}{2}+\sqrt{\left(t-\dfrac{1}{2}\right)^2+\dfrac{3}{4}}\right)+C,$

where $t=1/(x+1)$

(iii) $2\sqrt{x^2+4x+8}-\ell n\left|(x+2)+\sqrt{x^2+4x+8}\right|$

$-\dfrac{5}{2}\ell n\left|\dfrac{1}{(x+2)}+\sqrt{\dfrac{1}{(x+2)^2}+\dfrac{1}{4}}\right|+c$

(iv) $2y^2+4y+4\ln(y-1)$, where $x+1=y^4$

26. (i) $\dfrac{e^{-x}}{50}((3-5x)\cos 2x+(4+10x)\sin 2x$

$-25(x+1))+C$

(ii) $\dfrac{e^{a\tan^{-1}x}(ax-1)}{(1+a^2)(1+x^2)^{1/2}}+C$

27. (i) $-1/2\,(1+\cot^5 x)^{2/5}+C$

(ii) $-2\tan^{-1}\sqrt{1+\tan x+\cot x}+C$

28. (i) $\dfrac{2a+bx^2}{b^2(a+bx^2)^{1/2}}+C$ (ii) $\dfrac{x}{(1-x^4)^{1/2}}+C$

29. (i) $2\sqrt{1-\sin x}-\sqrt{2}\ln\left|\tan\left(\dfrac{x}{4}+\dfrac{\pi}{8}\right)\right|+C$

(ii) $\ln(1+t) - 1/4 \ln(1+t^2) + \frac{1}{2\sqrt{2}} \ln \frac{t^2 - \sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1}$

$- 1/2 \tan^{-1} t^2 + C$ where $t = \sqrt{\cot x}$

30. (i) $\frac{2x-1}{\sqrt{4x^2 - 2x + 1}} + C$

(ii) $-\frac{\sqrt{2 - x - x^2}}{x} + \frac{\sqrt{2}}{4}$

$\ln \frac{4 - x + 2\sqrt{2}\sqrt{2 - x - x^2}}{x} - \sin^{-1}\left(\frac{2x+1}{3}\right) + C$

(iii) $-2/5 \frac{(1-2x)^{5/2}}{x^{5/2}} - \frac{4}{3} \frac{(1-2x)^{3/2}}{x^{3/2}} + C$

(iv) $-\frac{1}{\sqrt{2}} \ln \left|\frac{\sqrt{1+x^2} - \sqrt{2}x}{\sqrt{1+x^2} + \sqrt{2}x}\right|$

$- \ln |x + \sqrt{1+x^2}| + C$

31. (i) $\sin^{-1}(1/2 \sec^2 1/2\, x) + C$

(ii) $1/2\, t \cos t - 1/2 \sin t - \frac{t}{\cos t}$

$+ \ln|\sec t + \tan t| + C$

32. (i) $\frac{2(3-4x)}{5(1-x-x^2)} + \frac{2}{5\sqrt{5}} \ln \left|\frac{\sqrt{5} + 1 + 2x}{\sqrt{5} - 1 - 2x}\right| + C$

(ii) $\left(\frac{1}{3}x^2 - \frac{5x}{6} + \frac{1}{6}\right)\sqrt{x^2 + 2x + 2} + \frac{5}{2}$

$\ln(x + 1 + \sqrt{x^2 + 2x + 2}) + C$

(iii) $-\frac{1}{10}\sqrt{\left(\frac{1+x^4}{x^4}\right)^5} + \frac{1}{3}\sqrt{\left(\frac{1+x^4}{x^4}\right)^3}\ C$

(iv) $\sin^{-1}\left(\frac{x}{\sqrt[4]{1+x^4}}\right) + C$

33. (i) $\frac{2(x-1)}{3\sqrt{x^2+x+1}} + C$

(ii) $\sin^{-1}\frac{x-1}{\sqrt{3}}\ \frac{\sqrt{2}}{3} \tan^{-1} \frac{\sqrt{2+2x-x^2}}{(1-x)\sqrt{2}}$

$-\frac{1}{\sqrt{6}} \ln \frac{\sqrt{6} + \sqrt{2+2x-x^2}}{\sqrt{6} - \sqrt{2+2x-x^2}} + C$

34. (i) $x \tan^{-1}x \,.\, \ln(1+x^2) + (\tan^{-1}x)^2 - 2x \tan^{-1}x$

$+ \ln(1+x^2) - \left(\ell n\sqrt{1+x^2}\right)^2 + C$

(ii) $\frac{-\tan^{-1}x}{3x^3} - \frac{1}{6x^2} - \frac{1}{6}\ln\left(\frac{x^2+1}{x^2}\right) + C$

35. (i) $\ln \frac{4\cos^4 x}{1 + \cos^2 2x} + \sec^2 x + C$

(ii) $\frac{1}{\sqrt{3}} \ln \left|\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right| + C$

TARGET EXERCISES for JEE ADVANCED

1. $\frac{1}{3r} \ln |x^{2n} - 1| - \frac{1}{6r} \ln |x^{4r} + x^{2r} + 1| + C)$

2. $\frac{1}{5} \ln \left|x + \frac{1}{x}\right| - \frac{1}{20} \ln \left|\left(x + \frac{1}{x}\right)^4 - 5\left(x + \frac{1}{x}\right)^2 + 5\right|$

$+ \frac{1}{4\sqrt{5}} \ln \left|\frac{\left(x + \frac{1}{x}\right)^2 - \left(\frac{5+\sqrt{5}}{2}\right)}{\left(x + \frac{1}{x}\right)^2 - \left(\frac{5-\sqrt{5}}{2}\right)}\right| + C$

3. $\frac{1}{(b-a)^n} \log \left|\frac{x-b}{x-a}\right|$

$+ \sum_{k=1}^{n-1} \frac{1}{k(b-a)^{n-k}(x-a)^k}$, $n^3 2$.

4. $\frac{1}{3}(x^2 + 2x + 3)^{3/2} - \frac{1}{2}(x+5)\sqrt{x^2 + 2x + 3}$

$- 6 \ln(x + 1 + \sqrt{x^2 + 2x + 3}) + C$

5. $C - \frac{1}{2}\left(\frac{x^n}{x^{m+n} + 1}\right)^2$

6. $x \cot^{-1}(x^2 - x + 1) + 1/2 \ln(x^2 - 2x + 2)$
$- 1/2 \ln(x^2 + 1) + C$

8. $\sin^{-1}\left(\frac{x}{(1 + x^{2n})^{1/2n}}\right) + C$

9. $\frac{1}{4\sqrt{2}} \ln \left|\frac{(1+x^4)^{1/2} + x\sqrt{2}}{1 - x^2}\right|$

$+ \frac{1}{4\sqrt{2}} \tan^{-1} \frac{(1+x^4)^{1/2}}{x\sqrt{2}} + C$

10. $\frac{1}{50} \ln [25 \tan^2(Q/2) + 4 \tan(Q/2) + 13]$

$-\dfrac{2}{25\sqrt{321}}\tan^{-1}\left[\dfrac{25\tan(\theta/2)+2}{\sqrt{321}}\right]+C$

11. $\dfrac{2}{\cos\alpha}\sqrt{\cos\alpha.\tan\theta+\sin\alpha}$

$-\dfrac{2}{\sin\alpha}\sqrt{\cos\alpha+\cot\theta.\sin\alpha}+C$

12. $3a\,x+\dfrac{x\sqrt{x^2-3a^2}}{2}-\dfrac{3a^2}{2}\ln|x+\sqrt{x^2+3a^2}|+C$

13. (i) $\displaystyle\int\frac{8\cos^3\theta+1}{64\cos^6\theta+5}d\theta$

(ii) $5^7\sqrt{5}\displaystyle\int\frac{\sec\theta\tan^{15}\theta\, d\theta}{5\sec^2\theta+3+\sqrt{5}\tan\theta}$

14. $\dfrac{2}{(\cos\beta\sqrt{\cos\alpha\sin\beta}+\sin\beta\sqrt{\sin\alpha\cos\beta})}+C$

15. $\dfrac{1}{\sin\alpha}\sin^{-1}\left[\dfrac{\sin\alpha\sin x}{1-\cos\alpha\sin x}\right]+C$

16. $C-\dfrac{1}{2}\left(\dfrac{x^n}{x^{m+n}+1}\right)^2$

17. $\dfrac{1}{2}\left(t-\dfrac{a^2}{t}\right)\sin^{-1}t+\dfrac{1}{2}\sqrt{1-t^2}-\dfrac{a^2}{2}$

$\ln\left|\dfrac{1}{t}+\sqrt{\dfrac{1}{t^2}-1}\right|+C$ where $t=x+\sqrt{x^2+a^2}$

18. $\dfrac{1}{\sin\alpha}\sin^{-1}\left[\dfrac{\sin\alpha\sin x}{1-\cos\alpha\sin x}\right]+C$

19. $\operatorname{cosec}^{-1}(1+\sin 2x)+C$

20. $\operatorname{cosec}^{-1}\left(2\cos^2\dfrac{\theta}{2}\right)+C$

21. $\dfrac{1}{2\sqrt{2}}\ln\left|\dfrac{\sqrt{1+x^4}+x\sqrt{2}}{1-x^2}\right|+\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{x\sqrt{2}}{\sqrt{1+x^4}}+C$

22. $\dfrac{1}{\sqrt{2(c-a)}}\ln\left|\dfrac{x\sqrt{2(c-a)}+\sqrt{1+2cx^2+a^2x^4}}{1+ax^2}\right|+C$

when $c>a$. $\dfrac{1}{\sqrt{2(a-c)}}\sin^{-1}$

$\left(\dfrac{x\sqrt{2(a-c)}}{1+ax^2}\right)$, when $a>c$.

23. $\ln\left|\sec\dfrac{x^2+1}{2}\right|+C$

24. $-\dfrac{2}{3}\sin^{-1}(\cos^{3/2}x)+C$

26 $\dfrac{1}{6}\ell n\left|\dfrac{x-1}{x+1}\right|+\dfrac{1}{6}\ell n(x^2+x+1)$

$+\dfrac{1}{3}\ell n(x^2-x+1)-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}+C$

27. $\dfrac{x+2}{2(x^2+2x+3)}-\dfrac{\sqrt{2}}{4}\tan^{-1}\dfrac{x+1}{\sqrt{2}}+\ln|x+1|+C$

28. $e^{x\sin x+\cos x}\cdot\left(x-\dfrac{1}{x\cos x}\right)+C$

29. If $R=(x^2+ax)^2+bx$, and $u=\ln\dfrac{x^2+ax+\sqrt{R}}{x^2+ax-\sqrt{R}}$, find the relation between the integrals $\displaystyle\int\frac{dx}{\sqrt{R}}, \int\frac{xdx}{\sqrt{R}}$.

31. The form of the partial fraction corresponding to a may be deduced from the facts that

$\dfrac{Q(x)}{x-\alpha}\to Q'(a), (x-a)R(x)\to\dfrac{P(\alpha)}{Q'(\alpha)}$

34. Last part, $a=2$ and $7x^2-36x+48-p(x^2-4x+4)$ must be a square : $p=3$ and from this $q=4, b=3$. Also $r=2, s=-1$ from $x^2-2x-1\equiv r(x^2-4x+4)+s(x^2-6x+9)$.

35. (i) $a^{-1}(u^2-1)^{-1/2}$; the integral reduces to $-\dfrac{1}{a}\displaystyle\int\frac{udu}{(u^2-1)^{1/2}}$

(ii) The integral reduces $-2\displaystyle\int\frac{2y^2+3}{1+3y^2+y^4}dy$

$$-2\int\left\{\frac{1}{y^2+\frac{1}{2}(3+\sqrt{5})}+\frac{1}{y^2+\frac{1}{2}(3-\sqrt{5})}\right\}dy$$

Moreover, $\left\{\dfrac{1}{2}(1+\sqrt{5})\right\}^2=\dfrac{1}{2}(3+\sqrt{5})$; integrals in inverse tangent form.

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. $\frac{-3}{2}, \frac{35}{36}$, any real value

2. A **3.** C

4. B **5.** D

6. A **7.** B

8. D **9.** D

10. A **11.** C

12. C

13. $\frac{1}{2}\log|\sin x - \cos x| + \frac{x}{2} + C$

14. $\frac{1}{b^3}\left[a + bx - 2a\log|a+bx| - \frac{a^2}{a+bx}\right] + C$

15. (a) $\pm 4\left[\sin\frac{x}{4} - \cos\frac{x}{4}\right] + C$

(b) $-2\left[\frac{(1-x)^{5/2}}{5} - \frac{2(1-x)^{3/2}}{3} + \sqrt{1-x}\right] + C$

16. $x\sin x + \cos x - \frac{1}{4}\cos 2x + C$

17. $\frac{e^x}{(x+1)^2} + C$ **18.** $-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$

19. $2\sqrt{1-x} - \cos^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x} + C$

20. $\frac{2}{\pi}[\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}] - x + C$

21. $\sqrt{2}\log\left[\frac{\sqrt{2}\cot x + \sqrt{\cos^2 x - 1}}{\cot x}\right]$

$-\log(\cot x + \sqrt{\cos^2 x - 1}) + C$

22. $\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + C$

23. $\frac{2}{3}x^{2/3} - \frac{12}{7}x^{7/12} + 2x^{1/2}$

$-\frac{12}{5}x^{5/12} + 3x^{1/3} + 6x^{1/6} - 12x^{1/12}$

24. $\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - \frac{1}{2}\ln\sec 2\theta + C$

$+ 12\log|x^{1/2} + 1|$

$+6\left[\left\{\frac{(1+x^{1/6})^3}{3} - \frac{3}{2}(1+x^{1/6})^2 + 3(1+x^{1/6})\right\}\right]$

25. $-\log\left|\frac{1+xe^x}{xe^x}\right| + \frac{1}{1+xe^x} + C$

26. (a) $\frac{3}{8}\tan^{-1}x - \frac{x}{4x(x^4-1)} - \frac{3}{16}\ln\left(\frac{x-1}{x+1}\right) + C$

(b) $\frac{1}{4}\log\frac{x+1}{x-1} - \frac{1}{2(x-1)} - \frac{1}{2(x-1)^2} + C$

(c) $\frac{x^2}{2} + x + 2\ln(x-1) - \left(\frac{1}{4}\right)$

$\ln(x^2+1) - \left(\frac{1}{2}\right)\tan^{-1}x + C$

(d) $\frac{1}{1+xe^x} + \log\frac{xe^x}{1+xe^x} + C$

27. $2\cos^{-1}\sqrt{x} - 2\log\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right) + C$

28. $-\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+1)$

$+\frac{3}{2}\tan^{-1}x + \frac{x}{1+x^2} + C$

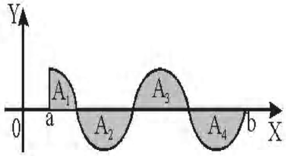
29. $(x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2+8x+13) + C$

30. $\frac{1}{6}\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{m+1} + C$

31. D

32. C

33. B



$$\therefore \quad \int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4$$

where A_1, A_2, A_3, A_4 are the areas of the shaded region.

Hence, the integral $\int_a^b f(x)dx$ represents the "net signed area" of the region bounded by the curve $y = f(x)$, x–axis and the lines $x = a$, $x = b$.

CHAPTER

2

DEFINITE INTEGRATION

2.1 INTRODUCTION

The definite integral is one of the basic concepts of mathematical analysis and is a powerful research tool in mathematics, physics, mechanics, and other disciplines. Calculation of areas bounded by curves, of arc length, volumes, work, velocity, path length, moment of inertia, and so forth reduce to the evaluation of a definite integral.

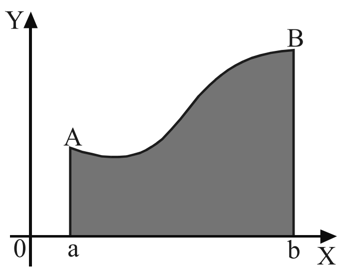
There are various problems leading to the notion of the definite integral : determining the area of a plane figure, computing the work of a variable force, finding the distance travelled by a body with a given velocity, and many other problems.

In the previous chapter we dealt with integration as the inverse process of differentiation. The concept of integration first arose in connection with determination of areas of plane regions bounded by curves and an integral was recognized as the limit of a certain sum. It was only later that Newton and Leibnitz established an intimate relationship between the processes of integration and differentiation, known now as the fundamental theorem of integral calculus which we shall discuss in this chapter. A definite integral will be defined as the limit of a sum and it will be shown how a definite integral can be used to define the area of some special region.

A definite integral may be described as an analytical substitute for an area. In the usual elementary treatment of the definite integral, defined as the limit of a sum, it is assumed that the function of x considered may be represented graphically, and the limit in question is the area between the curve, the axis of x, and the two bounding ordinates, say at $x = a$ and $x = b$.

The Area Problem

Let us understand the problem of finding the area of a curvilinear trapezoid. Consider a nonnegative continuous function $y = f(x)$, $x \in [a, b]$.



The figure AabB bounded by a segment of the axis of abscissas, segments of vertical lines $x = a$ and $x = b$, and the graph of the given function is called a curvilinear trapezoid.

In other words, a curvilinear trapezoid is the set of points in the plane whose coordinates x, y satisfy the following conditions: $a \le x \le b$, $0 \le y \le f(x)$.

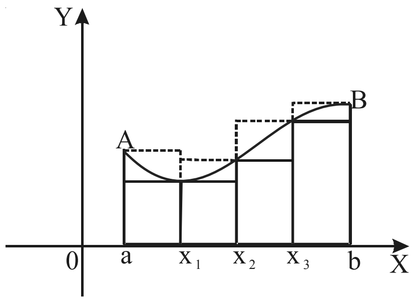
Let us find the area of this curvilinear trapezoid. To this end we partition the closed interval [a, b] into n subintervals of equal length

$$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b],$$

using for this purpose the points

$$x_i = a + \frac{b-a}{n} i, \quad i = 0, 1, \ldots, n.$$

We then denote by m_i and M_i the least and greatest values (respectively) of the function f(x) on the interval $[x_{i-1}, x_i]$, where i = 1,....., n.



The curvilinear trapezoid AabB is thus separated into n parts. Obviously, the area of the ith part is not less than $m_i(x_i - x_{i-1})$ and is not greater than $M_i(x_i - x_{i-1})$. Consequently, the area of the entire curvilinear trapezoid AabB is not less than the sum

$$m_1\Delta x_1 + ... + m_n\Delta x_n = \sum_{i=1}^{n} m_i \Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ and is not greater than the sum

$$M_1\Delta x_1 + ... + M_n\Delta x_n = \sum_{i=1}^{n} M_i \Delta x_i$$

Denoting respectively these sums by s_n and S_n, we see that S_{AabB} satisfy the inequalities

$$s_n \le S_{AabB} \le S_n,$$

Here s_n represents the area of the stepped figure contained in the given curvilinear trapezoid, and S_n the area of the steplike figure containing the given curvilinear trapezoid.

If the interval [a, b] is divided into sufficiently small subintervals, i.e. if n is sufficiently large, then the areas of these figures differ but slightly from each other as also from the area of the curvilinear trapezoid. Consequently, we may assume that the sequences (s_n) and (S_n) have one and the same limit and this limit is equal to the area of the figure AabB.

This assertion is obtained in the assumption that the curvilinear trapezoid under consideration has an area, but the latter notion is not yet defined. And so, the above reasoning leads us to the following definition.

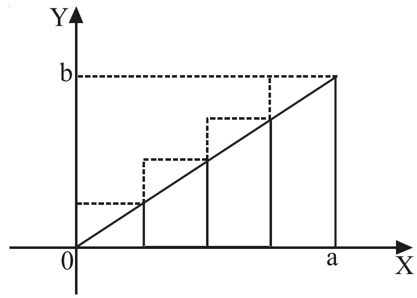
Definition Let there be given a continuous nonnegative function f(x), $x \in [a, b]$. Then, if the limits of the sequences (s_n) and (S_n) exist and are equal to each other, their common value is called the area of the cuvilinear trapezoid.

$$\{(x, y) \mid a \le x \le b, 0 \le y \le f(x)\}.$$

Example 1. Let us show that, according to the given definition, the area of the right angled triangle with the vertices at points (0, 0), (a, 0) and (a, b) is equal to $\frac{1}{2}ab$, that is, it is computed by the known formula.

Solution The given triangle is a curvilinear trapezoid for the function

$$f(x) = \frac{b}{a}x, \; x \in [0, a]$$



Using the points $x_i = \frac{a}{n}i$, i = 0, 1,.....,n, we divide the interval [0, a] into n equal parts of length $\frac{a}{n}$. Then

$$m_i = f(x_{i-1}) = \frac{b}{n}(i-1)$$

$$M_i = f(x_i) = \frac{b}{n}i$$

and therefore

$$s_n = \sum_{i=1}^{n} \frac{n}{b}(i-1)\frac{a}{n} = \frac{ab}{n^2}\cdot\frac{(n-1)n}{2} = \frac{ab}{2}\left(1 - \frac{1}{n}\right)$$

$$S_n = \sum_{i=1}^{n} \frac{n}{b}(i)\frac{a}{n} = \frac{ab}{n^2}\cdot\frac{(n+1)n}{2} = \frac{ab}{2}\left(1 + \frac{1}{n}\right)$$

Hence, it is obvious that

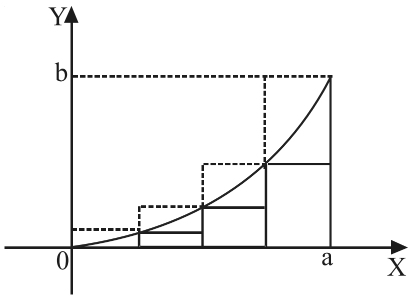
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = \frac{ab}{2}$$

Thus, it has been proved that the area of the given triangle is equal to $\frac{1}{2}ab$.

Example 2. Find the area of the figure bounded by a portion of the parabola $y = x^2$ and segments of the straight lines x = 0 and x = a, where a > 0.

Solution Proceeding as in previous example, we partition the interval [0, a] into n subintervals each of length $\frac{a}{n}$, using points $x_i = \frac{a}{n}i$, i = 0, 1,...., n.

Then



$$s_n = \frac{a}{n}\sum_{i=1}^{n} x_{i-1}^2 = \frac{a^3}{n^3}\sum_{i=1}^{n}(i-1)^2\ ,$$

$$S_n = \frac{a}{n}\sum_{i=1}^{n} x_i^2 = \frac{a^3}{n^3}\sum_{i=1}^{n} i^2$$

We have $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Hence,

$$S_n = \frac{a^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{a^3}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)$$

$$s_n = \frac{a^3}{n^3}\cdot\frac{(n-1)n(2n-1)}{6} = \frac{a^3}{3}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{2n}\right)$$

and therefore

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n = \frac{a^3}{3}$$

Thus, the area of the given figure is equal to $\frac{1}{3}a^3$.

Here we would like to note several points.

Note:

1. Let us return to the curvilinear trapezoid AabB, and proceeding in the usual manner, partition the interval [a, b] into n parts of equal length, making use of points x_i, $i = 0, 1, ..., n$. On each subinterval $[x_{i-1}, x_i]$ we choose arbitrarily some point and denote it by ξ_i.

If m_i and M_i are respectively, the least and the greatest values of the function f on the subinterval $[x_{i-1}, x_i]$, then, obviously, $m_i \le f(\xi_i) \le M_i$, where $i = 1, ..., n$. Let us now multiply either of these inequalities by $\Delta x_i = x_i - x_{i-1}$ and add termwise the inequlities thus obtained.

Then we get the following inequality :

$$\sum_{i=1}^{n} m_i\Delta x_i \le \sum_{i=1}^{n} f(\xi_i)\Delta x_i \le \sum_{i=1}^{n} M_i\Delta x_i$$

Hence, it follows that the limit $\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ exists, does not depend on the choice of points x_i and is always equal to the area of the figure AabB.

Thus, $\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = S_{AabB}$ ...(1)

2. In the previous examples we divided the interval [a, b] into n equal subintervals. It can be proved that formula (1) remains true also for the case when [a, b] is separated into n parts of arbitrary lengths but such that the greatest of these lengths tends to zero as $n \to \infty$.

The Definite Integral

Consider the function f(x) defined on the interval [a, b]. As before, we divide the interval [a, b] into n equal subintervals by means of points

$$x_i = a + \frac{b-a}{n}i\ ,\ i = 0, 1, ..., n.$$

On each of these subintervals $[x_{i-1}, x_i]$, $i = 1, ..., n$ we choose one point denoting it by ξ_i where $\xi_i \in [x_{i-1}, x_i]$

Then the sum

$$f(\xi_1)\Delta x_1 + ...+ f(\xi_n)\Delta x_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$, is called an **integral sum** of the function f.

Obviously, this sum depends both on the manner the interval [a, b] is subdivided and on the choice of points ξ_i.

Definition. If the limit $\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ exists and is independent of the coice of points ξ_i, then the function f is said to be **integrable** on the interval [a, b] and the limit is called the **definite integral** or simply the integral of the function f(x) with respect to x over the interval [a, b] and is denoted as $\int_a^b f(x)dx$

(read as " the integral of f(x)dx from a to b").

The symbol $\int$ is called the integral sign, the function f(x) the integrand, x the variable of integration, the expression f(x)dx the element of integration. The numbers a and b are called the lower and upper limits of integration.

Thus, according to the definition,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

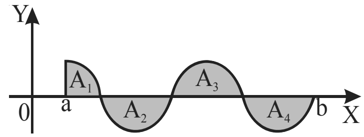
The interval [a, b] is called the interval of integration. The word 'limit' here has nothing to do with the word limit as used in differential calculus. It only signifies the 'end points' of the interval of integration.
We have a theorem that every continuous function is integrable, but integrability extends to a class of functions wider than the class of continuous functions.

Note:

1. The above definition of a definite integral is a special case of the more generalized definition as given below.
   Let f(x) be a bounded function defined in the interval (a, b), and let the interval (a, b) be divided in any manner into n sub-intervals (equal or unequal) of lengths $\delta_1, \delta_2, ..., \delta_n$. In each sub-interval choose a perfectly arbitrary point (which may be within or at either end points of the interval), and let these points be $x = \zeta_1, \zeta_2, ... \zeta_n$.
   Let $S_n = \sum_{r=1}^{n} \delta_r f(\zeta_r)$.
   Now, let n increase indefinitely in such a way that the greatest of the lengths $\delta_1, \delta_2, ... , \delta_n$ tends to zero. If, in this case, S_n tends to a definite limit which is independent of the way in which the interval (a, b) is sub-divided and the intermediate points $\zeta_1, \zeta_2, ..., \zeta_n$ are chosen, then this limit, when it exists, is called the definite integral of f(x) from a to b.
2. The process of forming the definite integral shows that the symbol $\int_a^b f(x)\,dx$ is a certain number.
   Its value only depends on the properties of the integrand and on the numbers a and b determining the interval of integration.

Geometrical interpretation of definite integral

In general, $\int_a^b f(x)\,dx$ represents an algebraic sum of areas of the region bounded by the curve y = f(x), the x-axis and the ordinates x = a and x = b. Here algebraic sum means that area which is above the x–axis will be added in this sum with + sign and area which is below the x–axis will be added in this sum with – sign. So, value of the definite integral may be positive, zero or negative.
This is because the value of f(x) in the integral sum is considered without modulus sign. The area above the x-axis enter into this sum with a positive sign, while those below the x-axis enter it with a negative sign.



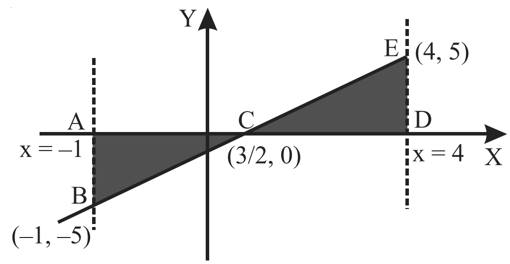
$\therefore \int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4$

where A_1, A_2, A_3, A_4 are the areas of the shaded region.

Hence, the integral $\int_a^b f(x)dx$ represents the "net signed area" of the region bounded by the curve y = f(x), x–axis and the lines x = a, x = b.

Example 3. Evaluate $\int_{-1}^{4} (2x-3)dx$.

Solution y = 2x – 3 is a straight line, which lie below the x–axis in $\left[-1, \frac{3}{2}\right)$ and above in $\left(\frac{3}{2}, 4\right]$



Now area of $\Delta ABC = \frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4}$

Area of $\Delta CDE = \frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4}$

(using formula from geometry)

So, $\int_{-1}^{4} (2x-3)dx = -\frac{25}{4} + \frac{25}{4} = 0$.

Example 4. Evaluate the following integrals by interpreting each in terms of areas :

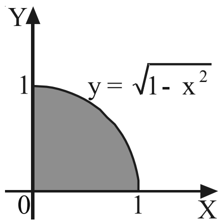
(a) $\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\int_{-1}^{2} (x+2)dx$

(c) $\int_0^3 (x-1)dx$

Solution

(a) We sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

Since $f(x)=\sqrt{1-x^2}\geq 0$, we can interpret this integral as the area under the cure $y=\sqrt{1-x^2}$ from $x = 0$ to $x = 1$. We have $y^2 = 1 - x^2$, that is $x^2 + y^2 = 1$, which shows that the graph of f is the quarter-circle with radius 1.



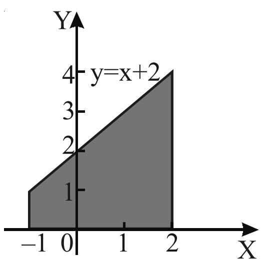
Therefore, $\int_0^1 \sqrt{1-x^2}\,dx =$ area of quarter-circle

$$=\frac{1}{4}\pi(1)^2=\frac{\pi}{4}.$$

(b) The graphs of the integrand is the line $y = x + 2$, so the region is a trapezoid whose base extends from $x = -1$ to $x = 2$. Thus,

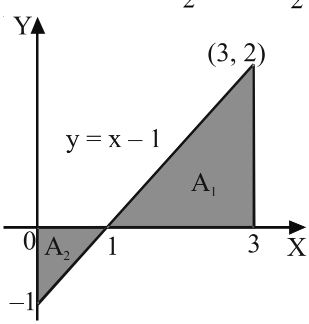
$$\int_{-1}^{2}(x+2)dx = \text{(area of trapezoid)}$$

$$=\frac{1}{2}(1+4)(3)=\frac{15}{2}$$



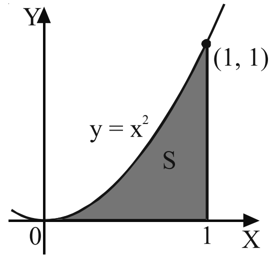
(c) The graph of $y = x - 1$ is a line with slope 1 as shown in the figure. We compute the integral as the difference of the areas of the two triangles :

$$\int_0^3 (x-1)\,dx = A_1 - A_2 = \frac{1}{2}(2\,.\,2)-\frac{1}{2}(1\,.\,1)=1.5.$$

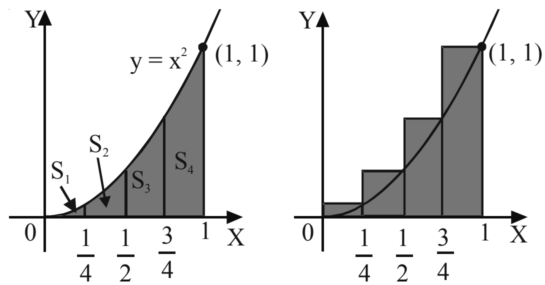


2.2 DEFINITE INTEGRAL AS A LIMIT OF SUM

Let us use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.



We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x=\frac{1}{4}$, $x=\frac{1}{2}$ and $x=\frac{3}{4}$ as in Figure (a)



(a) (b)

We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip [see Figure (b)]. In other words, the heights of these rectangle are the values of the function $f(x) = x^2$ at the right end points of the subintervals

$$\left[0,\frac{1}{4}\right],\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{1}{2},\frac{3}{4}\right] \text{ and } \left[\frac{3}{4},1\right].$$

Each rectangle has width $\frac{1}{4}$ and the heights are

$$\left(\frac{1}{4}\right)^2,\left(\frac{1}{2}\right)^2,\left(\frac{3}{4}\right)^2, \text{ and } 1^2.$$

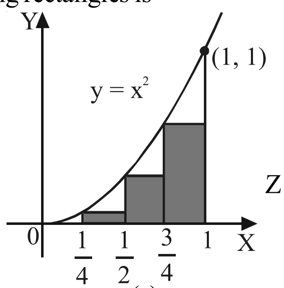
If we let R_4 be the sum of the areas of these approximating rectangles, we get R_4

$$= \frac{1}{4}, \left(\frac{1}{4}\right)^2 + \frac{1}{4}.\left(\frac{1}{2}\right)^2 + \frac{1}{4}.\left(\frac{3}{4}\right)^2 + \frac{1}{4}.1^2 = \frac{15}{32}$$

$= 0.46875$

From the Figure (b) we see that the area A of S is less than R_4, so $A < 0.46875$

Instead of using the rectangles in Figure (b) we could use the smaller rectangles in Figure (c) whose heights are the values of f at the left endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0). The sum of the areas of these approximating rectangles is



(c)

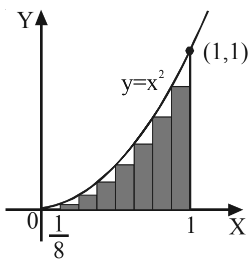
$$L_4 = \frac{1}{4}.0^2 + \frac{1}{4}.\left(\frac{1}{4}\right)^2 + \frac{1}{4}.\left(\frac{1}{2}\right)^2 + \frac{1}{4}.\left(\frac{3}{4}\right)^2 = \frac{7}{32}$$

$= 0.21875$

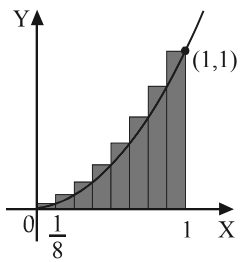
We see that the area of S is larger than L_4, so we have lower and upper estimates for A :

$0.21875 < A < 0.46875$

We can repeat this procedure with a larger number of strips. Figure (d), (e) shows what happens when we divide the region S into eight strips of equal width.



(d) Using left endpoint



(e) Using right endpoint

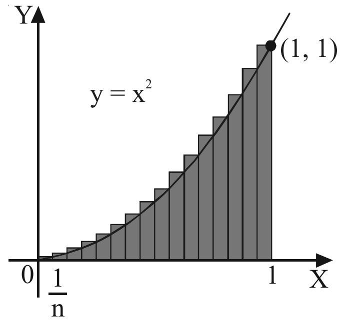
By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A :

$0.2734375 < A < 0.3984375$

So, we can say that the true area of S lies somewhere between 0.2734375 and 0.3984375. We could obtain better estimates by increasing the number of strips.

For the region S in previous example, we now show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$.

R_n is the sum of the areas of the n rectangles in Figure. Each rectangle has width 1/n and the heights are the values of the function $f(x) = x^2$ at the points 1/n, 2/n, 3/n, ..., n/n; that is, the heights are $(1/n)^2$, $(2/n)^2$, $(3/n)^2$, ..., $(n/n)^2$.



Thus, R_n

$$= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n}\cdot\frac{1}{n^2}(1^2 + 2^2 + 3^2 + \ldots. + n^2)$$

$$= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)$$

$$= \frac{(n+1)(2n+1)}{6n^2}$$

Thus, we have

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6}.1.2 = \frac{1}{3}$$

It can be shown that the lower approximating sums also approach 1/3, that is,

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{1}{n}(0)^2 + \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \ldots..$$

$$+\frac{1}{n}\left(\frac{n-1}{n}\right)^2$$

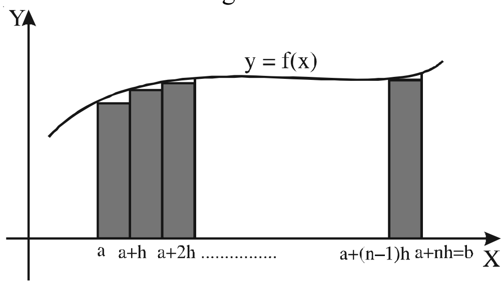
$$= \lim_{n\to\infty}\frac{(n-1(2n-1)}{6n^2} = \frac{1}{3}.$$

From Figures it apears that, as n increases, both L_n and R_n become better and better approximations to the area of S. Therefore, we define the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}.$$

Definite integral as a limit of sum

Let f(x) be a continuous real valued function defined on the closed interval [a, b] which is divided into n parts as shown in the figure.



The points of division on the x-axis are $a, a + h, a + 2h \,..........\, a + (n - 1)h, a + nh$,

where $\frac{b-a}{n} = h.$

Left end estimation

Let L_n denotes the area of these n rectangles.
Then, $L_n = hf(a) + hf(a + h) + hf(a + 2h) + ... + h\,f(a + (n - 1)h)$

Clearly, L_n represents an area very close to the area of the region bounded by curve y = f(x), x–axis and the ordinates x = a, x = b.

Hence, $\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h\,[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)]$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n-1} h\,f(a+rh), \text{ where } nh = b - a.$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n-1}\left(\frac{b-a}{n}\right) f\left(a + \frac{(b-a)\,r}{n}\right).$$

If for a function f(x) the limit exists, then we say the function is integrable on the interval [a, b].

It can be shown that, when f(x) is a continuous function, the above limit always exists. Hence, if a function f(x) is continuous on an interval [a, b], then it is integrable on that interval.

Right end estimation

Considering the sum of areas of rectangles using the heights at the right end points of the subintervals, we have

$R_n = hf(a + h) + hf(a + 2h) + ...+ hf(a + nh)$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{b-a}{n}\right) f\left(a + \left(\frac{b-a}{n}\right) r\right)$$

It follows from theorems that for a continuous function f,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \int_a^b f(x)dx.$$

That is, we may compute the integral using either the left end estimation or the right end estimation.

Note:

1. If a = 0, b = 1, then

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n} f\left(\frac{r}{n}\right)$$

2. From the definition of definite integral, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=\varphi(x)}^{\psi(x)} f\left(\frac{r}{n}\right) = \int_a^b f(x)dx, \text{ where}$$

(i) Σ is replaced by $\int$ sign,

(ii) $\frac{r}{n}$ is replaced by x,

(iii) $\frac{1}{n}$ is replaced by dx,

(iv) To obtain the limits of integration, we use

$$a = \lim_{n\to\infty}\frac{\phi(x)}{n} \text{ and } b = \lim_{n\to\infty}\frac{\psi(x)}{n}.$$

For example, $\lim_{n\to\infty}\sum_{r=1}^{pn}\frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^p f(x)\,dx$

since $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0, \quad \lim_{n\to\infty}\left(\frac{pn}{n}\right) = p.$

Example 1. Calculate $I = \int_{-1}^{4}(1+x)dx$ as the limit of sums.

Solution We divide the interval [–1, 4] into n equal parts. On each subinterval

$$[x_{i-1}, x_i] = \left[-1+\frac{5(i-1)}{n}, -1+\frac{5i}{n}\right]$$

the continuous function $1+x$ attains the least value at the left endpoint of the interval and the greatest value at the right end point.
Therefore

$$L_n = \sum_{i=1}^{n} f\left(-1+\frac{5(i-1)}{n}\right).\frac{5}{n}$$

$$= \sum_{i=1}^{n} \frac{5}{n}(i-1)\frac{5}{n} = \frac{25}{n^2}\sum_{i=1}^{n}(i-1)$$

$$R_n = \sum_{i=1}^{n} f\left(-1+\frac{5i}{n}\right).\frac{5}{n} = \sum_{i=1}^{n}\frac{5}{n}.i.\frac{5}{n} = \frac{25}{n^2}\sum_{i=1}^{n} i$$

Hence

$$R_n - L_n = \frac{25}{n^2}\left(\sum_{i=1}^{n} i - \sum_{i=1}^{n}(i-1)\right)$$

$$= \frac{25}{n^2}n = \frac{25}{n} \to 0 \text{ as } n \to \infty.$$

This means that the integral $I = \int_{-1}^{4}(1+x)dx$ exists.

To calculate it as the limit of sums, we can consider any of the sequence of sums.
Here, we use right end estimation.

$$I = \int_{-1}^{4}(1+x)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{25}{n^2}i = \lim_{n\to\infty}\frac{25n(n+1)}{2n^2} = \frac{25}{2}$$

Thus, $\int_{-1}^{4}(1+x)dx = \frac{25}{2}$.

Example 2. Evaluate $\int_0^2 x^2\,dx$.

Solution The function f to be integrated is defined by $f(x) = x^2$, and the interval of integration is [0, 2].

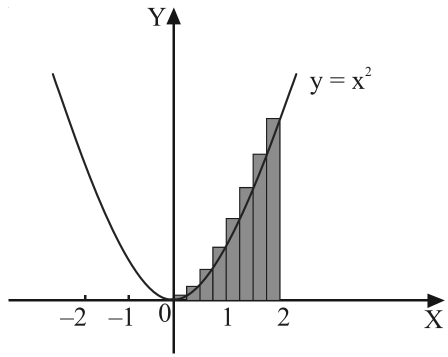
$$\int_0^2 x^2\,dx = \lim_{n\to\infty} R_n$$

The partition $\{x_0, \ldots, x_n\}$ which subdivides [0, 2] into n subintervals of equal length is given by

$$x_i = a + \frac{b-a}{n}i = 0 + \frac{2}{n}i = \frac{2i}{n},$$

for each $i = 0, \ldots, n$.

Moreover, $x_i - x_{i-1} = \frac{b-a}{n} = \frac{2}{n}, i = 1, \ldots, n$.



Since $f(x_i) = x_i^2$ and $x_i = \frac{2i}{n}$, it follows that

$$R_n = \sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\frac{8i^2}{n^3} = \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

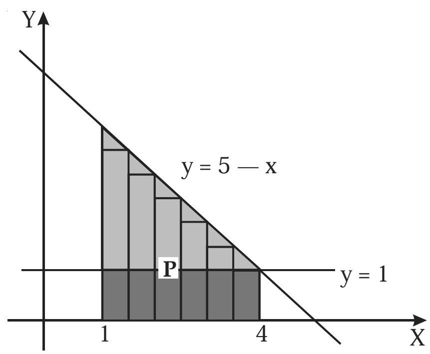
$$= \frac{8}{n^3}\frac{2n^3+3n^2+n}{6} = \frac{4}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right),$$

and so $\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{4}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) = \frac{4}{3}.2 = \frac{8}{3}$.

We conclude that $\int_0^2 x^2 dx = \frac{8}{3}$.

Example 3. Evaluate $\int_1^4 (5-x)\,dx$.

Solution The function f, defined by $f(x) = 5 - x$, is linear and decreasing on the interval [1, 4]. Its graph is shown in the figure.



The partition $\{x_0, \ldots, x_n\}$ subdivides the interval [1, 4] into subintervals of length $\frac{4-1}{n} = \frac{3}{n}$, and the points are given by

$$x_i = 1 + \left(\frac{3}{n}\right)i, \qquad i = 0, \ldots, n.$$

In addition, $x_i - x_{i-1} = \frac{3}{n}$, $\quad i = 1, \ldots\ldots, n$.

We shall compute the integral as

$$\int_1^4 (5 - x)\,dx = \lim_{n\to\infty} L_n.$$

We have $x_i = 1 + \frac{3i}{n}$ and $f(x_i) = 5 - x_i$, and so

$$f(x_i) = 5 - \left(1 + \frac{3i}{n}\right) = 4 - \frac{3i}{n}.$$

Since $x_i - x_{i-1} = \frac{3}{n}$, we get

$$L_n = \sum_{i=1}^{n}\left(4 - \frac{3i}{n}\right)\frac{3}{n}.$$

$$= \sum_{i=1}^{n}\left(4 - \frac{3i}{n}\right)\frac{3}{n} = \sum_{i=1}^{n}\left(\frac{12}{n} - \frac{9i}{n^2}\right)$$

$$= \sum_{i=1}^{n}\frac{12}{n} - \sum_{i=1}^{n}\frac{9i}{n^2} \qquad = \frac{12}{n}\sum_{i=1}^{n}1 - \frac{9}{n^2}\sum_{i=1}^{n} i$$

$$= \frac{12}{n}n - \frac{9}{n^2}\frac{n(n+1)}{2} = 12 - \frac{9}{2}\left(1 + \frac{1}{n}\right).$$

But it is easy to see that

$$\lim_{n\to\infty}\left[12 - \frac{9}{2}\left(1 + \frac{1}{n}\right)\right] = 12 - \frac{9}{2} = 7\frac{1}{2},$$

and we finally conclude that

$$\int_1^4 (5 - x)\,dx = \lim_{n\to\infty} L_n = 7\,\frac{1}{2}.$$

This answer can be checked by looking at the figure. The value of the integral is equal to the area of the shaded region P, which is divided by the horizontal line $y = 1$ into two pieces : a right triangle sitting on top of a rectangle.

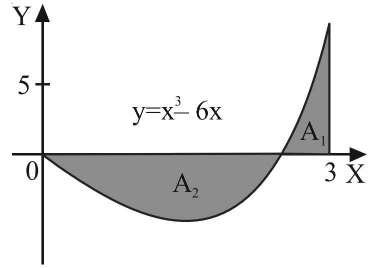
The area of the triangle is $\frac{1}{2}(3 \,.\, 3) = \frac{9}{2}$, and that of the rectangle is $3 \,.\, 1 = 3$. Hence

$$\int_1^4 (5 - x)\,dx = \text{area}(P) = \frac{9}{2} + 3 = 7\frac{1}{2}.$$

The excessive lengths of the computations in the above examples make it obvious that some powerful technique is needed to streamline the process of evaluating definite integrals.

Example 4. Evaluate $\int_0^3 (x^3 - 6x)\,dx$ using limit of sum and interpret the result.

Solution



$$\int_0^3 (x^3 - 6x)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x - \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\frac{27}{n^3}i^3 - \frac{18}{n}i\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right\}$$

$$= \lim_{n\to\infty}\left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75$$

This integral can be interpreted as net signed area because f takes on both positive and negative values. It is the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in the figure.

$$\int_0^3 (x^3 - 6x)dx = A_1 - A_2 = -6.75$$

Example 5. Express the following limit as a definite integral : $\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{6n}\right)$.

Solution $\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{6n}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+5n}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{5n}\left(\frac{1}{n+r}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\left(\frac{1}{1+\frac{r}{n}}\right)$$

Since the lower limit of r is 1, the lower limit of integration $= \lim_{n\to\infty} \frac{1}{n} = 0$

Since the upper limit of r is 5n, the upper limit of integration is $\lim_{n\to\infty}\frac{5n}{n}=5$.

Hence, the given limit is equivalent to $\int_0^5 \frac{1}{1+x}\,dx$.

Example 6. Use the definition of the integral as the limit of a sum to evaluate $\int_0^\alpha \sin x\,dx$

Solution Here $a=0, b=\alpha, \alpha=nh, f(x)=\sin x$.

$$\therefore \int_0^\alpha \sin x\,dx = \lim_{n\to\infty}\sum_{r=1}^{n}(\alpha/n)\sin(r\alpha/n)$$

$$= \lim_{n\to\infty}\frac{\alpha}{n}\left[\sin\frac{\alpha}{n}+\sin\frac{2\alpha}{n}+\ldots+\frac{\sin n\alpha}{n}\right]$$

$$= \lim_{n\to\infty}\frac{\alpha}{n}\frac{\sin\left(\frac{\alpha}{n}+\frac{n-1}{2}\cdot\frac{\alpha}{n}\right)\sin\frac{n\alpha}{2n}}{\sin\frac{\alpha}{2n}}$$

$$= \lim_{n\to\infty}\frac{2(\alpha/2n)}{\sin(\alpha/2n)}\cdot\sin\left(\frac{1}{2}\alpha\left(1+\frac{1}{n}\right)\right)\sin\left(\frac{\alpha}{2}\right)$$

$$= 2.\,1\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right) = 2\sin^2\left(\frac{\alpha}{2}\right) = 1-\cos\alpha.$$

Example 7. Show that $\int_a^b \frac{1}{x}dx = \ln\frac{b}{a}$ $(0<a<b)$ using limit of sum where the interval (a, b) is divided into n parts by the points of division a, ar, ar^2, ..., ar^{n-1}, ar^n.

Solution We have $ar^n = b$, i.e. $r = (b/a)^{1/n}$.
Evidently as $n\to\infty$, $r=(b/a)^{1/n}\to 1$, so that each of the intervals $a(r-1)$, $ar(r-1)$ ..... $\to 0$. Now, by definition,

$$\int_a^b \frac{1}{x}dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{ar^{k-1}}(ar^k - ar^{k-1})$$

$$\lim_{n\to\infty}\sum(r-1) = \lim_{n\to\infty} n(r-1)$$

$$= \lim_{n\to\infty} n[(b/a)^{1/n}-1]$$

$$= \lim_{n\to\infty}\frac{(b/a)^{1/n}-1}{1/n} = \ln\frac{b}{a}.$$

Example 8. Evaluate the integral $\int_1^2 \frac{dx}{x}$.

Solution We subdivide the interval [1, 2] into n parts so that the points of division x_i $(i = 0, 1, 2, ..., n)$ form the geometric progression :

$x_0 = 1; x_1 = q; x_2 = q^2; x_3 = q^3; ....; x_n = q^n = 2.$

Here $q = \sqrt[n]{2}$.

The length of the ith subinterval is equal to

$\Delta x_i = q^{i+1} - q^i = q^i(q-1),$

and $q^{n-1}(q-1)\to 0$ as $n\to\infty$, i.e. as $q\to 1$.

Now let us choose the right hand endpoints of the subintervals as the points $x_{i+1} = q^{i+1}$.

Forming an integral sum :

$$R_n = \sum_{i=0}^{n-1}\frac{1}{q^{i+1}}q^i(q-1) = \frac{n}{q}(q-1) = \frac{1}{2^{\frac{1}{n}}}n(2^{\frac{1}{n}}-1)$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{n(2^{\frac{1}{n}}-1)}{2^{\frac{1}{n}}} = \ell n\,2$$

and so, $\int_1^2 \frac{dx}{x} = \ell n\,2.$

Example 9. Evaluate the integral $\int_a^b \frac{dx}{x^2}$ $(a<b)$.

Solution

$$S_n = h\left[\frac{1}{a^2}+\frac{1}{(a+h)^2}+\frac{1}{(a+2h)^2}+....+\frac{1}{(a+\overline{n-1}h)^2}\right]$$

$$< h\left[\frac{1}{(a-h)a}+\frac{1}{a(a+h)}+\frac{1}{(a+h)(a+2h)}\right.$$

$$\left. +....+\frac{1}{(a+\overline{n-2}h)(a+\overline{n-1}h)}\right]$$

$$= \left[\left[\frac{1}{(a-h)}-\frac{1}{a}\right]+\left[\frac{1}{a}-\frac{1}{(a+h)}\right]+\left[\frac{1}{(a+h)}-\frac{1}{(a+2h)}\right] + ... + \frac{1}{(a+\overline{n-2}h)} - \frac{1}{(a+\overline{n-1}h)}\right]$$

Thus $S_n < \frac{1}{a-h} - \frac{1}{a+nh-h}$ ...(1)

Similarly, $S_n >$

$$h\left[\frac{1}{a(a+h)}+\frac{1}{(a+h)(a+2h)}+.......\right]$$

$$= \left[\left[\frac{1}{a}-\frac{1}{(a+h)}\right]+\left[\frac{1}{(a+h)}-\frac{1}{(a+2h)}\right] + ... + \frac{1}{(a+\overline{n-1}h)} - \frac{1}{(a+nh)}\right]$$

Thus $S_n > \frac{1}{a} - \frac{1}{a+nh}$ ...(2)

We have $b = a + nh$

From (1) and (2), using Sandwich Theorem, we get

$$\int_a^b \frac{dx}{x^2} = \lim_{n\to\infty} S_n = \lim_{h\to 0} \frac{1}{a-h} - \frac{1}{b-h} = \frac{1}{a} - \frac{1}{b}$$

Example 10. Evaluate the integral $\int_a^b \frac{dx}{\sqrt{x}}$, $a>0$, $b>0$

Solution By definition

$$S_n = h\left[\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a+2h}}+......+\frac{1}{\sqrt{a+(n-1)h}}\right]$$

We know that $2\sqrt{r} < \sqrt{r+h} + \sqrt{r}$ for sufficiently small $h>0$.

$$\frac{1}{2\sqrt{r}} > \frac{1}{\sqrt{r+h}+\sqrt{r}} = \frac{\sqrt{r+h}-\sqrt{r}}{h}$$

$$\therefore \frac{h}{\sqrt{r}} > 2[\sqrt{r+h} - \sqrt{r}]$$

Substituting $r = a$, $a+h$, $a+2h$, ... we have

$$\frac{h}{\sqrt{a}} > 2\left(\sqrt{a+h} - \sqrt{a}\right)$$

$$\frac{h}{\sqrt{a+h}} > 2\left(\sqrt{a+2h} - \sqrt{a+h}\right)$$

$$\frac{h}{\sqrt{a+2h}} > 2\left(\sqrt{a+3h} - \sqrt{a+2h}\right)$$

$$\frac{h}{\sqrt{a+(n-1)h}} > 2\left(\sqrt{a+nh} - \sqrt{a+(n-1)h}\right)$$

On addition,

$$h\left[\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{a+h}}+....+\frac{1}{\sqrt{a+\overline{n-1}h}}\right]$$

$$> 2\left(\sqrt{a+nh} - \sqrt{a}\right)$$

$$= 2\left|\sqrt{a+b-a} - \sqrt{a}\right| = 2\left|\sqrt{b} - \sqrt{a}\right|$$

$$\therefore \lim_{n\to\infty} S_n \ge 2\left(\sqrt{b} - \sqrt{a}\right)$$

Similarly, by considering $2\sqrt{r} > \sqrt{r} + \sqrt{r-h}$, we can prove that $\lim_{n\to\infty} S_n \le 2\left(\sqrt{b} - \sqrt{a}\right)$.

Hence, $\int_a^b \frac{dx}{\sqrt{x}} = \lim_{n\to\infty} S_n = 2\left(\sqrt{b} - \sqrt{a}\right)$.

Concept Problems A

1. Compute the area of the figure bounded by a portion of the straight line $y = x$ and segments of the straight lines $y = 0$ and $x = 3$.
2. Evaluate $\int_{-2}^{0} \sqrt{4-x^2}\,dx$,
3. Find the area of the curvilinear trapezoid defined by the graph of the function $y = e^x$ on the interval $0 \le x \le 1$.
4. First show that $\int_0^{\pi} \sin x dx = \lim_{n\to\infty} \frac{\pi}{n}\sum_{k=1}^{n} \sin\frac{k\pi}{n}$

   Use the fact $\sum_{k=1}^{n} \sin\frac{k\pi}{n} = \cot\frac{\pi}{2n}$ and L'Hospital's rule to show that finally that $\int_0^{\pi} \sin x dx = 2$.
5. Evaluate the following integrals as limit of sums:

   (i) $\int_1^4 (x^2 - x)\,dx$ (ii) $\int_0^3 (x^2 + 1)\,dx$

   (iii) $\int_0^2 (2x^3 + 5)\,dx$ (iv) $\int_0^1 (x + e^{2x})\,dx$

   (v) $\int_0^{\pi/2} \cos x\,dx$ (vi) $\int_a^b \sin x\,dx$
6. Let f(x) denote a linear function that is nonnegative on the interval [a, b]. For each value of x in [a, b], define A(x) to be the area between the graph of f and the interval [a, x].

   (a) Prove that $A(x) = \frac{1}{2}[f(a) + f(x)]\,(x - a)$.

   (b) Use part (a) to verify that $A'(x) = f(x)$.

Practice Problems A

7. Use appropriate formulas from geometry to evaluate the integrals.

   (a) $\int_0^1 (x+2\sqrt{1-x^2})dx$ (b) $\int_{-1}^3 (4-5x)dx$

   (c) $\int_{-2}^2 (1-3|x|)dx$

8. Evaluate the integral $\int_0^{10} \sqrt{10x-x^2}\,dx$ by completing the square and applying appropriate formulas from geometry.

9. Prove that the area of the curvilinear trapezoid defined by a portion of the parabola $y = 1 - x^2$ $(0 \le x \le 1)$ is equal to $\frac{2}{3}$.

10. Determine a region whose area is equal to the limit $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\tan\frac{i\pi}{4n}$. Do not evaluate the limit.

11. Express the integral $\int_2^6 \frac{x}{1+x^5}dx$ as a limit of integral sums. Do not evaluate the limit.

12. Evaluate $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(5+\frac{2i}{n}\right)^2$

13. Explain why $\frac{1}{100}\sum_{i=1}^{100} f\left(\frac{i}{100}\right)$ is an estimate of $\int_0^1 f(x)\,dx$.

14. Prove using the concept of definite integral as limit of sum that

$$\frac{a}{n}\sum_{k=1}^{n}\cos\frac{ka}{n} < \sin a < \frac{a}{n}\sum_{k=0}^{n-1}\cos\frac{ka}{n},\ 0 < a < \frac{\pi}{2}$$

15. Form the integral sum s_n by dividing the interval $[a, b]$ into n parts by the points $x_i = aq^i\,(i = 0, 1, 2, .., n)$, where $q = \sqrt[n]{\frac{b}{a}}$ and pass to the limit to compute the following definite integrals:

    (i) $\int_a^b x^2\,dx$

    (ii) $\int_a^b \frac{dx}{x}$, where $0 < a < b$

    (iii) $\int_a^b \sqrt{x}\,dx$.

16. Let A denote the area between the graph of $f(x) = \sqrt{x}$ and the interval $[0, 1]$, and let B denote the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$. Explain geometrically why $A + B = 1$.

17. Let A denote the area between the graph of $f(x) = 1/x$ and the interval $[1, 2]$, and let B denote the area between the graph of f and the interval $\left[\frac{1}{2}, 1\right]$. Explain geometrically why $A = B$.

2.3 RULES OF DEFINITE INTEGRATION

We have a number of rules regarding the definite integral $\int_a^b f(x)dx$.

1. If $a = b$, then $\Delta x = 0$ and so $\int_a^a f(x)dx = 0$. This is natural also from the geometric standpoint. Indeed, the base of a curvilinear trapezoid has length equal to zero. Consequently, its area is zero too.

2. **Order of integration**

$$\int_b^a f(x)\,dx = -\int_b^a f(x)\,dx$$

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But the definition as a limit of sum makes sense even if $a > b$.

Notice that if we reverse a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$.

Therefore, $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

Example 1. Evaluate the definite integrals :

(a) $\int_{\pi}^{\pi} \sin x \, dx$ (b) $\int_{4}^{1} (5-x) \, dx$

Solution

(a) Because the sine function is defined at $x = \pi$, and the upper and lower limits of integration are equal, we can write $\int_{\pi}^{\pi} \sin x \, dx = 0$.

(b) This integral is the same as $\int_{1}^{4} (5-x) \, dx$ except that the upper and lower limits are interchanged. Because the integral has a value of 7.5, we can write $\int_{4}^{1} (5-x) \, dx = -\int_{1}^{4} (5-x) \, dx = -7.5$.

3. Dummy variable

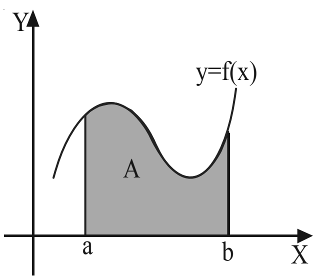
The definite integral $\int_{a}^{b} f(x) \, dx$ is a number which depends only on the form of the function f(x) and the limits of integration, and not on the variable of integration, which may be denoted by any letter. In fact, we could use any letter in place of x without changing the value of the integral

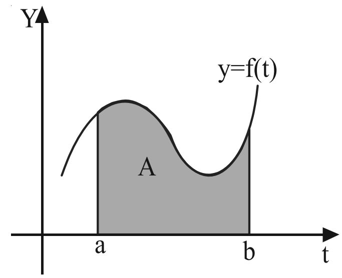
$$\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(t) \, f(t) \, dt = \int_{a}^{b} f(u) \, du.$$

Because the variable of integration in a definite integral plays no role in the end result, it is often referred to as a dummy variable.

Whenever you find it convenient to change the letter used for the variable of integration in a definite integral, you can do so without changing the value of the interval.

This result should not be surprising, since the area under the graph of the curve y = f(x) over an interval [a, b] on the x-axis is the same as the area under the graph of the curve y = f(t) over the interval [a, b] on the t-axis (See figure).



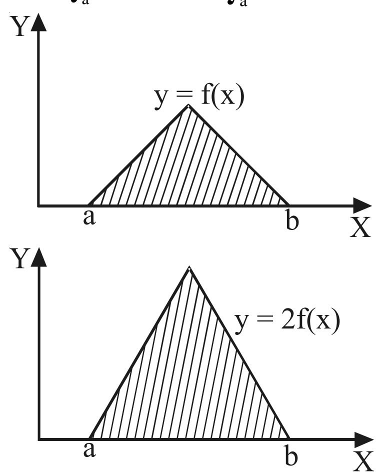


4. Homogeneous Property

$$\int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx,$$ where c is any constant.

This property says that the integral of a constant times a function is the constant times the integral of the function. In other words, a constant (independent of x) can be taken in front of an integral sign.

For example, $\int_{a}^{b} 2f(x) \, dx = 2\int_{a}^{b} f(x) \, dx$

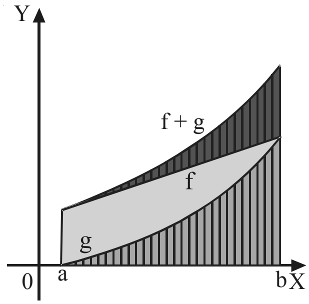


5. Additivity Property

$$\int_{a}^{b} [f(x)+g(x)] \, dx = \int_{a}^{b} f(x) dx + \int_{a}^{b} g(x) \, dx$$

and $$\int_{a}^{b} [f(x)-g(x)] \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx$$

This property says that the integral of a sum is the sum of the integrals. It says that the net signed area under f + g is the area under f plus the area under g. The figure helps us understand why this is true in view of how graphical addition works.



$$\int_a^b [f(x)+g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

In general, this property follows from the fact that the limit of a sum is the sum of the limits :

$$\int_a^b [f(x)+g(x)]dx = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i)+g(x_i)]\Delta x$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\Delta x + \sum_{i=1}^{n} g(x_i)\Delta x\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x + \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\Delta x$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

The above two properties can be combined into one formula known as the linearity property.

6. Linearity Property

For every real c_1 and c_2, we have

$$\int_a^b [c_1 f(x) + c_2 g(x)]\,dx = c_1\int_a^b f(x)\,dx + c_2\int_a^b g(x)\,dx.$$

Example 2. Use the properties of integrals to evaluate $\int_0^1 (4+3x^2)\,dx$.

Solution Using linearity property of integrals, we have

$$\int_0^1 (4+3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx$$

$$= \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

We have $\int_0^1 4\,dx = 4(1-0) = 4$ and we found in one of the previous examples that $\int_0^1 x^2\,dx = \frac{1}{3}$. So,

$$\int_0^1 (4+3x^2)\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

$$= 4 + 3\,.\,\frac{1}{3} = 5.$$

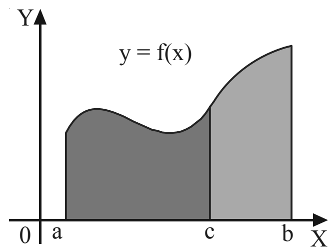
The next property tells us how to combine integrals of the same function over adjacent intervals :

7. Additivity with respect to the interval of integration

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

This theorem reflects the additive property of area. If an interval is decomposed into two intervals, the sum of the areas of the two parts is equal to the area of the whole.

Note that c may or maynot lie between a and b, Property 2 allows us to conclude that this property is valid not only when c is between a and b but for any arrangement of the points a, b, c, but the function must be integrable in the desired intervals. It is easy to prove for the case where $f(x) \ge 0$ and $a < c < b$. This can be seen from the geometric interpretation of the following figure



The area under $y = f(x)$ from a to c plus the area from c to b is equal to the total area from a to b.

Example 3. Given that $\int_0^{10} f(x)dx = 17$ and $\int_0^8 f(x)dx = 12$, find $\int_8^{10} f(x)dx$.

Solution By the rule of additivity, we have

$$\int_8^{10} f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$$

$$\text{So,} \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx$$

$$= 17 - 12 = 5$$

Example 4. If for every integer n, $\int_n^{n+1} f(x)\,dx = n^2$, then find the value of $\int_{-2}^{4} f(x)\,dx$.

Solution We have $\int_n^{n+1} f(x)\quad dx = n^2$

Putting $n = -2, -1, 0, 1, 2, 3$ we get

$$\int_{-2}^{-1} f(x) = 4\,,\quad \int_{-1}^{0} f(x)\ dx = 1\,,\quad \int_{0}^{1} f(x) = 0,$$

$$\int_{1}^{2} f(x)\ dx = 1,\ \int_{2}^{3} f(x)\ dx = 4\,,\quad \int_{3}^{4} f(x)\ dx = 9$$

Hence, $\int_{-1}^{4} f(x)\ dx = 4 + 1 + 0 + 1 + 4 + 9 = 39.$

Comparision properties of integral

Next, we have a comparison theorem which tells us that if one function has larger values than another throughout [a, b], its integral over this interval is also larger.

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

2. Domination Law

If $f(x) \leq g(x)$ for for every x in [a, b], then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

3. If $f(x) < g(x)$ for every x in [a, b], then

$$\int_a^b f(x)dx < \int_a^b g(x)dx.$$

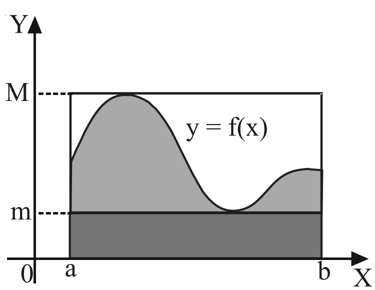
4. Max-Min Inequality

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m\,(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

If $f(x) \geq 0$, then $\int_a^b f(x)\,dx$ represents the area under the graph of f, so the geometric interpretation of Property 1 is simply that areas above the x-axis are positive.

Property 2 says that a bigger function has a bigger integral. It follows from Property 1 because $g - f \geq 0$.



Max-Min Inequality is illustrated in the figure for the case where $f(x) \geq 0$. If f is continuous we could take m and M to be the absolute minimum and maximum values of f on the interval [a, b]. In this case the inequality says that the area under the graph of f is greater than the area of the rectangle with height m and less than the area of the rectangle with height M. Since $m \leq f(x) \leq M$, Property 2 gives

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

Evaluating the integrals on the left and right sides, we obtain

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

This inequality is useful when we want to find a rough estimate of the value of an integral.

Example 5. Use Max-Min Inequality to estimate

$$\int_0^1 e^{-x^2}\ dx.$$

Solution Because $f(x) = e^{-x^2}$ is a decreasing function on [0, 1], its absolute maximum value is $M = f(0) = 1$ and its absolute minimum value is $m = f(1) = e^{-1}$. Thus by Max-Min Inequality,

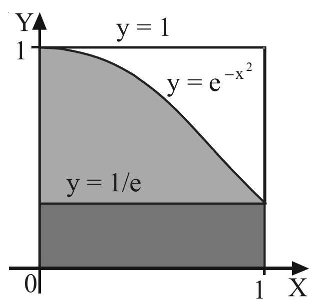
$$e^{-1}(1-0) \leq \int_0^1 e^{-x^2}\ dx \leq 1(1-0)$$

$$\text{or } e^{-1} \leq \int_0^1 e^{-x^2}\ dx \leq 1$$

Since $e^{-1} \approx 0.3679$, we can write

$$0.367 \leq \int_0^1 e^{-x^2}\ dx \leq 1$$

The result is illustrated in the figure.



The integral is greater than the area of the lower rectangle and less than the area of the square.

Example 6. Prove that $4 \leq \int_1^3 \sqrt{3 + x^3}\ dx \leq 2\sqrt{30}$

Solution Since the function $f(x) = \sqrt{3 + x^3}$ increases on the interval [1, 3].

$\therefore$ M = maximum value of $\sqrt{3 + x^3}$

$$= \sqrt{3 + 3^3} = \sqrt{30}$$

m = minimum value of $\sqrt{3+x^3}$

$= \sqrt{3+1^3} = 2$

$\therefore$ m = 2, M = $\sqrt{30}$, b – a = 2

Hence, $2.2 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$

$\Rightarrow 4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$

Mean Value Theorem for Integrals

If f is continuous on the interval [a, b], there is atleast one number c between a and b such that

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

Proof : Suppose M and m are the largest and smallest values of f, respectively, on [a, b]. This means that

$m \le f(x) \le M$ when $a \le x \le b$

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

[Domination Law]

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

Because f is continuous on the closed interval [a, b] and because the number

$$I = \frac{1}{b-a}\int_a^b f(x)\,dx$$

lies between m and M, the Intermediate Value Theorem says there exists a number c between a and b for which f(c) = I, that is,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

$$\Rightarrow \int_a^b f(x)\,dx = f(c)\,(b-a).$$

The Mean Value Theorem for Integrals does not specify how to determine c. It simply guarantees the existence of atleast one number c in the interval.

Since $f(x) = 1 + x^2$ is continuous on the interval [– 1, 2], the Mean Value Theorem for Integrals says there is a number c in [–1, 2] such that

$$\int_{-1}^{2} (1+x^2)\,dx = f(c)[2-(-1)] \qquad ...(1)$$

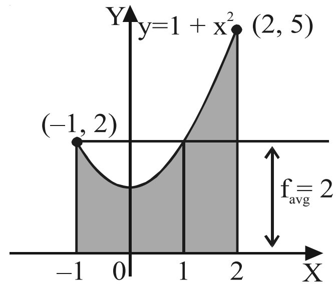
In this particular case we can find c explicitly.

From definite integral as limit of sum, we can evaluate $\int_{-1}^{2} (1+x^2)\,dx$ to be equal to 6.

Placing this value in (1), we get f(c) = 2.

Therefore, $1 + c^2 = 2$ so $c^2 = 1$.

Thus, in this case there happen to be two numbers c = ± 1 in the interval [–1, 2] that work in the Mean Value Theorem for Integrals.



For a nonnegative function the Mean Value Theorem for Integrals has a simple geometrical interpretation. It asserts that the area of the curvilinear trapezoid corresponding to the function f is equal to the area of the rectangle whose base is equal to the base of the trapezoid, and the altitude to one of the values of the integrable function.

Note: The formula in the theorem holds true not only for integrals in which the lower limit of integration is less than the upper one, but also for those in which the lower limit exceeds the upper one.

Example 7. Let f be the function defined by

$$f(x) = \begin{cases} 1 + x, & \text{when } 0 \le x < \frac{1}{2},\ \frac{1}{2} < x \le 1 \\ 0, & \text{when } x = \frac{1}{2} \end{cases}$$

Show that $\int_0^1 f(x)dx = \frac{3}{2}$ but there is no point c in [0, 1] such that $\int_0^1 f(x)dx = f(c)$.

Solution Here f is bounded and has only one point of discontinuity, i.e. $x = \frac{1}{2}$.

To find $\int_0^1 f(x)dx$, we consider the partition $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ obtained by dissecting [0, 1] into n equal parts.

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\sum_{r-1}^{n}\frac{1}{n}\left(1+\frac{r}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\left\{\left(1+\frac{1}{n}\right)+....+\left(1+\frac{n}{n}\right)\right\}\right]$$

$$= \lim_{n\to\infty}\left[n+\frac{1}{n}\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\frac{3n+1}{2n}=\frac{3}{2}.$$

But there is no point in [0, 1] at which f takes this value. The only likely candidate for the point c is the point $x = 1/2$ but $f(1/2) = 0$. This does not contradict the Mean Value Theorem for Integrals as the given function is discontinuous.

Average value of a function

If f is integrable on the interval [a, b], the average value of f on this interval is given by the integral

$$f_{avg} = \mu = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

From the Mean Value Theorem for Integrals we conclude that this average value is definitely attained by continuous functions at some $c \in [a, b]$. Hence, $f(c) = \mu$.

Example 8. Find the average value of the function $f(x) = 1 + x^2$ on the interval [–1, 2].

Solution With $a = -1$ and $b = 2$ we have

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)dx$$

$$= \frac{1}{3}\times 6 = 2.$$

Another frequently seen averaging process is the **root mean square (r.m.s.)** of f over [a, b], defined as follows :

$$f_{r.m.s} = \sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}}$$

One can show that $f_{avg} \le f_{r.m.s.}$, the equality holding only for constant functions.

Concept Problems B

1. Given that $\int_0^1 x^2\,dx = \frac{1}{3}$. Use this fact and the properties of integrals to evaluate $\int_0^1 (5-6x^2)dx$

2. Write as a single integral in the form

$$\int_a^b f(x) = \int_{-2}^{2} f(x)dx + \int_2^5 f(x)dx - \int_{-2}^{-1} f(x)dx$$

3. Find $\int_{-1}^{2}[f(x)+2g(x)]dx$ if

$$\int_{-1}^{2} f(x)dx = 5 \text{ and } \int_{-1}^{2} g(x)dx = -3$$

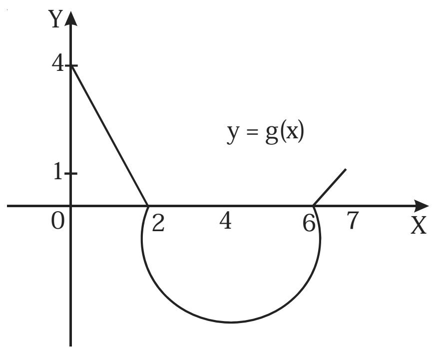
4. If the function f is integrable in a closed interval containing a, b, c and d, prove that

$$\int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^d f(x)dx = \int_a^d f(x)dx.$$

5. The graph of g consists of two straight lines and a semicircle. Use it to evaluate each integral.

(a) $\int_0^2 g(x)dx$ (b) $\int_2^6 g(x)dx$

(c) $\int_0^7 g(x)dx$



6. Replace the symbol * by either ≤ or ≥ so that the resulting expressions are correct. Give your reasons.

(a) $\int_0^1 x^2\,dx * \int_0^1 x^3\,dx$

(b) $\int_{-1}^{1} x^2\,dx * \int_{-1}^{1} x^3\,dx$

(c) $\int_1^3 x^2\,dx * \int_1^3 x^3\,dx.$

7. Find out which integral is greater :

(i) $\int_0^1 2^{x^2}\,dx$ or $\int_0^1 2^{x^3}\,dx$?

(ii) $\int_1^2 2^{x^2}\,dx$ or $\int_1^2 2^{x^3}\,dx$?

(iii) $\int_1^2 \ln x\,dx$ or $\int_1^2 (\ln x)^2\,dx$?

(iv) $\int_3^4 \ln x\,dx$ or $\int_3^4 (\ln x)^2\,dx$?

8. (a) If $\int_0^1 7f(x)\,dx = 7$, does $\int_0^1 f(x)\,dx = 1$?

(b) If $\int_0^1 f(x)\,dx = 4$ and $f(x) \geq 0$, does

$\int_0^1 \sqrt{f(x)}dx = \sqrt{4} = 2$?

9. If f is continuous on [a, b], $f(x) \geq 0$ on [a, b], and $f(x_0) > 0$ for some x_0 in [a, b], prove that

$\int_a^b f(x)dx > 0$.

10. Assume that f is integrable and nonnegative on [a, b]. If $\int_a^b f(x)dx = 0$, prove that f(c) = 0 at each point of continuity of f.

11. If f_{avg} really is a typical value of the integrable function f(x) on [a, b], then the number f_{avg} should have the same integral over [a, b] that f does. Does it ? That is, does

$\int_a^b f_{avg}dx = \int_a^b f(x)dx$?

12. It would be nice if average values of integrable functions obeyed the following rules on an interval [a, b] :

(a) $(f+g)_{avg} = f_{avg} + g_{avg}$

(b) $(kf)_{avg} = k\,f_{avg}$, (any number k)

(c) $f_{avg} \leq g_{avg}$ if $f(x) \leq g(x)$ on [a, b].

Do these rules ever hold ? Give reasons for your answers.

Practice Problems B

13. If $\int_0^9 f(x)dx = 37$ and $\int_0^9 g(x)dx = 16$, find

$\int_0^9 [2f(x)+3g(x)]dx$

14. Suppose $\int_{-2}^2 f(x)dx = 4$, $\int_2^5 f(x)dx = 3$, $\int_{-2}^5 g(x)dx = 2$. Which, if any, of the following statements are true ?

(a) $\int_5^2 f(x)dx = -3$

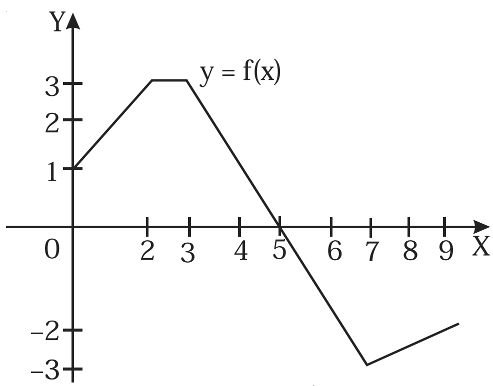
(b) $\int_{-2}^5 (f(x)+g(x))dx = 9$

(c) $f(x) \leq g(x)$ on the interval $-2 \leq x \leq 5$

15. The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

(a) $\int_0^2 f(x)dx$ (b) $\int_0^5 f(x)dx$

(c) $\int_5^7 f(x)dx$ (d) $\int_0^9 f(x)dx$



16. Evaluate the integral $\int_{-3}^0 (1+\sqrt{9-x^2})dx$ by interpreting it in terms of areas.

17. Let $f(x) = \begin{cases} -x-1 & \text{if } -3 \leq x < 0 \\ -\sqrt{1-x^2} & \text{if } 0 \leq x \leq 1 \end{cases}$ Evaluate

$\int_{-3}^1 f(x)dx$ by interpreting the integral as a difference of areas.

18. Determine whether the value of the integral is positive or negative.

(i) $\int_{-3}^{-1} \frac{x^4}{\sqrt{3-x}}dx$ (ii) $\int_{-2}^{4} \frac{x^3}{|x|+1}dx$

19. Draw the graph of the function $f(x) = x(x-2)(x-4) = x^3 - 6x^2 + 8x$, and indicate the region P^+ defined by the inequalities $0 \le x \le 3$ and $0 \le y \le f(x)$, and the region P^- defined by $0 \le x \le 3$ and $f(x) \le y \le 0$. Let $P = P^+ \cup P^-$, and suppose that $\int_0^2 f(x)\,dx = 4$ and $\int_0^3 f(x)\,dx = 2\frac{1}{4}$. Find area (P^+), area (P^-), and area (P).

20. Use the properties of integrals to verify the inequality without evaluating the integrals.
$$\int_1^2 \sqrt{5-x}\,dx \ge \int_1^2 \sqrt{x+1}\,dx$$

21. Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

22. Given that, when $x > 0$, the function f(x) is positive and is strictly decreasing, prove that
$$f(n+1) \le \int_n^{n+1} f(x)dx \le f(n)$$

23. Find the maximum and minimum values of $\sqrt{x^3+2}$ for $0 \le x \le 3$, and use these values to find bounds on the value of the integrals
$$\int_0^3 \sqrt{x^3+2}\,dx$$

24. Find the average value of
$$f(x) = \begin{cases} x+4, & -4 \le x \le -1 \\ -x+2, & -1 < x \le 2 \end{cases}$$ on $[-4, 2]$, using graph of f (without integrating).

25. Suppose that f and g are continuous on [a, b], $a \ne b$, and that $\int_a^b (f(x) - g(x))\,dx = 0$. Show that $f(x) = g(x)$ atleast once in [a, b].

26. The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\,dx$.

27. Let f be a function that is differentiable on [a, b]. In the chapter of derivatives, we defined the average rate of change of f over [a, b] to be $\frac{f(b)-f(a)}{b-a}$ and the instantaneous rate of change of f at x to be $f'(x)$. Here we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have $\frac{f(b)-f(a)}{b-a}$ = average value of f' on [a, b] Is this the case ?

28. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval ?

2.4 FIRST FUNDAMENTAL THEOREM OF CALCULUS

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus : differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibnitz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums as we did before.

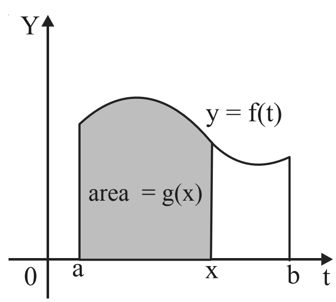
Area function

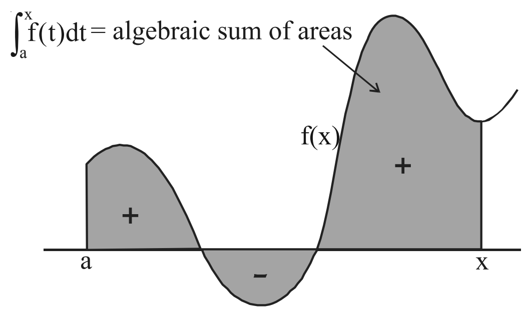
The First Fundamental Theorem deals with functions defined by an equation of the form
$$g(x) = \int_a^x f(t)\,dt \qquad ...(1)$$

where f is a function defined on [a, b] and x varies between a and b. Observe that g depends only on x, which appears as the variable upper limit in the integral.

If x is a fixed number, then the integral $\int_a^x f(t)\,dt$ is a definite number.

If we then let x vary, the number $\int_a^x f(t)\,dt$ also varies and defines a function of x denoted by g(x). If happens to be a positive function then g(x) can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. Think of g as the "area so far" function (See figure).





Let us discuss the question of notation. the independent variable in the upper limit is usually denoted by the same letter (say x) as the variable of integration. thus, we write

$$g(x) = \int_a^x f(x)dx$$

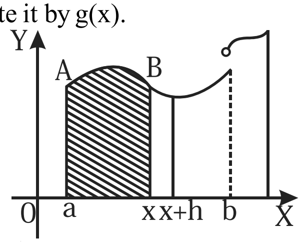
however the letter x in the element of integration only serves to designate the auxiliary variable (the variable of integration) which runs over the values ranging from the lower limit a to the upper limit x as the integral is formed. If it is necessary to evaluate a particular value of the function g(x), for instance, for x = b, i.e. g(b), we substitute b for x in the upper limit of the integral but do not replace by b the variable of integration. Therefore, it is more convenient to write

$$g(x) = \int_a^x f(t)dt$$

denoting the variable of integration by some other letter (in this case by t).

However, for simplicity, we shall often denote by the same letter both the variable of integration and the independent variable in the upper limit bearing in mind that in the upper limit and under the integral sign they have different meanings.

Now consider the graph of a bounded piecewise continuous function f with a point of discontinuity c. Let us take an arbitrary value $x \in [a, b]$. We shall be again interested in the definite integral of f on [a, x]. Let us denote it by g(x).



Hence, $g(x) = \int_a^x f(t)dt$

The number g(x) for the given x is represented in the figure by the area of the figure ABxa. g(x) changes as x varies on [a,b].

Theorem. If a function f is integrable on a closed interval [a, b], then the function $g(x) = \int_a^x f(t)dt$ is continuous at any point $x \in [a, b]$.

Proof : Let us take an arbitrary point x and assign an increment h to it (shown in figure is a positive h). We have

$$|g(x+h) - g(x)| = \left| \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right|$$

$$= \int_x^{x+h} f(t)\,dt \le M\,|\,h\,|$$

($M \ge |f(t)|, \forall\, t \in [a, b]$).

We have obtained the inequality

$|g(x+h) - g(x)| \le M\,|h|$,

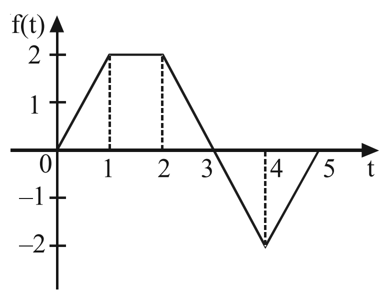
it follows that

$$\lim_{h \to 0} [g(x+h) - g(x)] = 0,$$

i.e. g is continuous at the point x.

It should be underlined that x may turn out to be either a point of continuity, or a point of discontinuity of f, but all the same, the function g(x) is continuous at this point.

Example 1. If f is the function whose graph is shown in the figure and $g(x) = \int_a^x f(t)\,dt$, find the values of g(0), g(1), g(2), g(3), g(4) and g(5). Then sketch a rough graph of g.



Solution First we notice that $g(0) = \int_0^0 f(t)\,dt = 0$.

From Figure we see that g(1) is the area of a triangle :

$$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2}\,(1\,.\,2) = 1$$

To find g(2) we add to g(1) the area of a rectangle :

$$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)dt + \int_1^2 f(t)\,dt$$
$$= 1 + (1\,.\,2) = 3$$

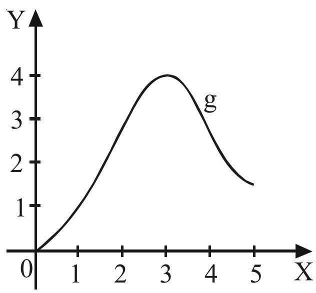
$$g(3) = g(2) + \int_2^3 f(t)\,dt = 3 + \frac{1}{2}(1\,.\,2) = 4$$

For $t > 3$, f(t) is negative and so we start subtracting areas :

$$g(4) = g(3) + \int_3^4 f(t) = 4 - \frac{1}{2}(2 \times 1) = 3$$

$$g(5) = g(4) + \int_4^5 f(t)\,dt = 3 - \frac{1}{2} \times 2 \times 1 = 2$$

We use these values to sketch the graph of g.



$$g(x) = \int_0^x f(t)dx$$

Notice that, because f(t) is positive for $t < 3$, we keep adding area for $t < 3$ and so g is increasing up to $x = 3$, where it attains a maximum value.
For $x > 3$, g decreases because f(t) is negative.
If we take $f(t) = t$ and $a = 0$, then, we have

$$g(x) = \int_0^x t\,dt = \frac{x^2}{2}$$

Notice that $g'(x) = x$, that is , $g' = f$. In other words, if g is defined as the integral of f, then g turns out to be an antiderivative of f, atleast in this case.
And if we sketch the derivative of the function g shown in the figure by estimating slopes of tangents, we get a graph like that of f.

First Fundamental Theorem of Calculus

If f is continuous on [a, b], then the function g defined

by $g(x) = \int_a^x f(t)\,dt \quad a \le x \le b$ ...(1)

is continuous on [a, b] and differentiable on (a, b), and $g'(x) = f(x)$.

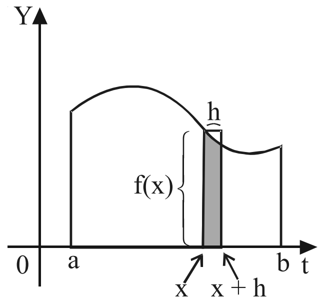
Proof : If x and $x + h$ are in (a, b), then

$$g(x+h) - g(x) = \int_x^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^{x+h} f(t)dt + \int_x^a f(t)dt = \int_x^{x+h} f(t)\,dt$$

and so, for $h \ne 0$,

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \quad ...(2)$$



For now let us assume that $h > 0$. Since f is continuous in $[x, x + h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x + h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x+ h]$. By Max-Min Inequality, we have

$$mh \le \int_x^{x+h} f(t)\,dt \le Mh$$

that is, $f(u)h \le \int_x^{x+h} f(t)\,dt \le f(v)\,h$

Since $h > 0$, we can divide this inequality by h:

$$f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v)$$

Now we use (2) to replace the middle part of this inequality :

$$f(u) \le \frac{g(x+h) - g(x)}{h} \le f(v) \quad ...(3)$$

Inequality 3 can be proved in similar manner for the case $h < 0$.
Now we let $h \to 0$. Then $u \to x$ and $v \to x$, since u and v lie between x and $x + h$.
Therefore

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x) \text{ and}$$

$$\lim_{h \to 0} f(v) = \lim_{v \to x} f(u) = f(x)$$

because f is continuous at x. We conclude, from (3) and the Sandwich Theorem, that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x) \quad ...(4)$$

If x = a or b, then equation 4 can be interpreted as a one-sided limit.
Using Leibnitz notation for derivatives, we can write the First Fundamental Theorem (FTC1) as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad ...(5)$$

when f is continuous. Roughly speaking, equation 5 says that if we first integrate f and then differentiate the result, we get back to the original function f.
This theorem is called the theorem on differentiating the definite integral with respect to its upper limit.
For a continuous function the derivative of the integral with respect to its upper limit is equal to the function itself.

The antiderivative of a continuous function

It follows from the First Fundamental Theorem that any continuous function has an antiderivative (primitive) which is definite integral with variable upper limit of the given function.

Theorem The function f(x) continuous on the closed interval [a, b] has an antiderivative on this interval. One of the antiderivatives is a function

$$F(x) = \int_a^x f(t)dt \qquad ...(1)$$

For example,

$F(x) = \int_0^x e^{t^2}\,dt$ is an antiderivative of $f(x) = e^{x^2}$,

since f(x) is continuous and FTC1 ensures that F′(x) = f(x).

Note: An integral with a variable upper limit is defined for any function f(x) integrable on [a, b]. However, for the function F(x) of form (1) to be an antiderivative for f(x), it is essential that the function f(x) be continuous. Thus, out of the definite integral we have constructed a function, which we call an indefinite integral of the integrand. Loosely speaking, an indefinite integral is the definite integral with a varying upper end point. The theorem is stated generally for indefinite integrals based at an arbitrary point x_0 in [a, b], namely $\int_{x0}^x f(t)\,dt$. Of course, for a given x_0 the two indefinite integrals $\int_{x0}^x f(t)\,dt$ and $\int_a^x f(t)\,dt$ just differ by a constant, namely $\pm\int_a^{x0} f(t)\,dt$.

Example 2. Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

Solution Since $f(t) = \sqrt{1+t^2}$ is continuous, the First Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1+x^2}$$

Example 3. If $F(t) = \int_0^t \frac{1}{x^2+1}\,dx$, find F′(1), F′(2), and F′(x).

Solution The integrand in this example is a continuous function f defined by $f(x) = \frac{1}{x^2+1}$.

By the First Fundamental Theorem,

$$F'(x) = f(x) = \frac{1}{x^2+1}.$$

In particular, $F'(1) = \frac{1}{1^2+1} = \frac{1}{2}$,

$$F'(2) = \frac{1}{2^2+1} = \frac{1}{5}.$$

Example 4. Find $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$.

Solution Here we neet to use the Chain Rule in conjunction with FTC1.
Let u = x^4. Then

$$\frac{d}{dx}\int_1^{x^4} \sec t\,dt = \frac{d}{dx}\int_1^{u} \sec t\,dt$$

$$= \frac{d}{du}\left(\int_1^u \sec t\,dt\right)\frac{du}{dx} \quad \text{(by the Chain Rule)}$$

$$= \sec u\,\frac{du}{dx} \quad \text{(by FTC1)}$$

$$= \sec(x^4).4x^3.$$

Example 5. Find the derivative $\frac{dy}{dx}$ of the implicit function $\int_{\pi/2}^x \sqrt{3-2\sin^2 z}\,dz + \int_0^y \cos t\,dt = 0$.

Solution Differentiate the left side of the equation with respect to x,

$$\frac{d}{dx}\left[\int_{\pi/2}^x \sqrt{3-2\sin^2 z}\,dz\right] + \frac{d}{dy}\left[\int_0^y \cos t\,dt\right]\frac{dy}{dx} = 0$$

$$\Rightarrow \sqrt{3-2\sin^2 x} + \cos y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{3-2\sin^2 x}}{\cos y}.$$

Example 6. Let

$$f(x) = \int_0^x \{(a-1)(t^2+t+1)^2 - (a+1)(t^4+t^2+1)\}dt$$

Find the value of 'a' for which $f'(x) = 0$ has two distinct real roots.

Solution Differentiating the given equation, we get

$$f'(x) = (a-1)(x^2+x+1)^2 - (a+1)(x^2+x+1)(x^2-x+1).$$

Now, $f'(x) = 0$

$\Rightarrow (a-1)(x^2+x+1) - (a+1)(x^2-x+1) = 0$

$\Rightarrow x^2 - ax + 1 = 0.$

For distinct real roots $D > 0$ i.e. $a^2 - 4 > 0$

$a^2 > 4 \Rightarrow a \in (-\infty, -2) \cup (2, \infty)$

Example 7. If $\int_{\sin x}^{1} t^2 f(t)\,dt = 1 - \sin x$, where $x \in \left(0, \frac{\pi}{2}\right)$, then find the value of $f\left(\frac{1}{\sqrt{3}}\right)$.

Solution We have $\int_1^{\sin x} t^2 f(t)\,dt = 1 - \sin x$

Differentiating both sides, we get

$-\sin^2 x\, f(\sin x)\cos x = -\cos x$

$$\Rightarrow f(\sin x) = \operatorname{cosec}^2 x = \frac{1}{\sin^2 x},\ x \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow f(z) = \frac{1}{z^2},\ z \in (0, 1).$$

$$\therefore\ f\left(\frac{1}{\sqrt{3}}\right) = 3.$$

Example 8. Find $\lim\limits_{x\to 0} \frac{\int_0^x \cos(t^2)dt}{x}$

Solution The limit is in indeterminate form 0/0. The integral with a variable upper limit

$\int_0^x \cos(t^2)dt$ has derivative

$$\left(\int_0^x \cos(t^2)dt\right)' = \cos(x^2).$$

Therefore, applying L'Hospital's rule, we obtain

$$\lim_{x\to 0} \frac{\int_0^x \cos(t)^2 dt}{x} = \lim_{x\to 0} \frac{\cos(x^2)}{1} = 1$$

Note that an antiderivative of $\cos(x^2)$ is not an elementary function, i.e. $\int_0^x \cos(t)^2 dt$ cannot be expressed in terms of elementary functions. This however, has not prevented us from calculating the required limit.

Example 9. Evaluate $\lim\limits_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}dt\right)$.

Solution $\lim\limits_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}dt\right)$

$$= 3\lim_{x\to 3} \frac{\int_3^x \frac{\sin t}{t}dt}{x-3} = 3\lim_{x\to 3}\frac{F(x)-F(3)}{x-3}$$

[applying L'Hospital's Rule and FTC1]

$$= 3F'(3) = 3\,\frac{\sin 3}{3} = \sin 3.$$

Example 10. Find

$$\lim_{x\to\infty} \frac{\int_0^x (\ln(t+\sqrt{1+t^2}) - \ln(1+t)dt}{x+1}$$

Solution The limit is in (∞/∞) form. Therefore, applying L'Hospital's Rule and FTC1, we get

$$= \lim_{x\to\infty}\left[\ln\left(x+\sqrt{1+x^2}\right) - \ln(1+x)\right]$$

$$= \lim_{x\to\infty} \ln\left(\frac{x+\sqrt{1+x^2}}{1+x}\right) = \lim_{x\to\infty}\ln\left(\frac{1+\sqrt{\frac{1}{x^2}+1}}{\frac{1}{x}+1}\right) = \ln 2.$$

Example 11. Find the critical points of the function f(x) if

(i) $f(x) = 1 + x + \int_1^x (ln^2 z + 2\,ln\, z)\,dz$

(ii) $f(x) = x - ln\, x + \int_1^x \left(\frac{1}{t} - 2 - 2\cos 4t\right) dt$

Solution

(i) $f(x) = 1 + x + \int_2^x (\ln^2 z + 2\ln z)\,dz$

$\therefore$ $f'(x) = 0 + 1 + \ln^2 x + 2\ln x$

For critical points $f'(x) = 0$

$\Rightarrow$ $1 + \ln^2 x + 2\ln x = 0$

$\Rightarrow$ $(\ln x + 1)^2 = 0 \Rightarrow \ln x = -1$

$\therefore$ $x = e^{-1} = \dfrac{1}{e}$.

(ii) $f(x) = x - \ln x + \int_1^x \left(\dfrac{1}{t} - 2 - 2\cos 4t\right) dt$

$\therefore$ $f'(x) = 1 - \dfrac{1}{x} + \dfrac{1}{x} - 2 - 2\cos 4x - 0$

$= -1 - 2\cos 4x$

For critical points $f'(x) = 0$

$\Rightarrow$ $\cos 4x = -\dfrac{1}{2} = \cos\dfrac{2\pi}{3}$

$\therefore$ $4x = 2n\pi \pm \dfrac{2\pi}{3}$ or $x = \dfrac{n\pi}{2} \pm \dfrac{\pi}{6}$

$\therefore$ $x = \dfrac{\pi}{6}, \dfrac{n\pi}{2} \pm \dfrac{\pi}{6}, n \in N$ $\qquad (\because x > 0)$

Example 12. If $f(x) = \int_0^x (t+1)(e^t - 1)(t-2)(t+4)\,dt$, then find the points of local minima of $f(x)$.

Solution Here $f'(x) = (x+1)(e^x - 1)(x-2)(x+4)$

Sign scheme for $f'(x)$:

$$\begin{array}{ccccccccc} + & & - & & + & & - & & + \\ & -4 & & -1 & & 0 & & 2 & \end{array}$$

Clearly $x = -1$ and $x = 2$ are the points of local minima.

Example 13. Find the points of maxima/minima of $\int_0^{x^2} \dfrac{t^2 - 5t + 4}{2 + e^t}\,dt$.

Solution Let $f(x) = \int_0^{x^2} \dfrac{t^2 - 5t + 4}{2 + e^t}\,dt$

$$f'(x) = \frac{x^4 - 5x^2 + 4}{2 + e^{x^2}}.2x - 0$$

$$= \frac{(x-1)(x+1)(x-2)(x+2).2x}{2 + e^x}$$

$$\begin{array}{ccccccccccc} - & & + & & - & & + & & - & & + \\ & -2 & & -1 & & 0 & & 1 & & 2 & \end{array}$$

From the sign scheme of $f'(x)$, it is clear that $f(x)$ has points of maxima at $x = -1, 1$ (as sign changes from +ve to –ve) and points of minima at –2, 0, 2 (as sign changes from –ve to + ve) at $x = 1$.

Example 14. Let $f(x) = \int_3^x \dfrac{dt}{\sqrt{t^4 + 3t^2 + 13}}$ If $g(x)$ is the inverse of $f(x)$ then find $g'(0)$.

Solution $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^4 + 3x^2 + 13}}$ when $y = f(x)$

$\therefore$ $g'(y) = \dfrac{1}{dy/dx} = \sqrt{x^4 + 3x^2 + 13}$

When $y = 0$ then $x = 3$.

Hence, $g'(0) = \sqrt{3^4 + 27 + 13} = \sqrt{121} = 11$.

Example 15. If $f(x)$ is a continuous function such that $\int_0^x f(t)\,dt \to \infty$ as $x \to \infty$, show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)\,dt = a$ where $a \in R^+$.

Solution We have to show that there exists some x such that $m^2x^2 + \int_0^x f(t)\,dt = a \ (a \in R^+)$ ...(1)

Consider the function

$$g(x) = m^2x^2 + \int_0^x f(t)\,dt$$

Since f is a continuous function, therefore g is also a continuous function. Also, $g(0) = 0$ and $g(x) \to \infty$ as $x \to \infty$. Thus, by intermediate value theorem, there must be some $x \in (0, \infty)$, such that $g(x) = a \ (a \in R^+)$. Hence, for every real m, there exists some a $(a \in R^+)$ that satisfies equation (1).

Example 16. Consider the function

$$f(x) = \cos x - \int_0^x (x - t)\,f(t)\,dt.$$

Show that $f''(x) + f(x) = -\cos x$.

Solution We have $f(x) = \cos x - \int_0^x (x-t)\ f(t)\,dt$

$= \cos x - x\int_0^x f(t)\,dt + \int_0^x t\,f(t)\,dt$

Differentiating w.r.t. x, we have

$f'(x) = -\sin x - xf(x) - \int_0^x f(t)\,dt + xf(x)$

$= -\sin x - \int_0^x f(t)\,dt$

Differentiating again w.r.t. x, we have

$f''(x) = -\cos x - f(x)$

i.e. $f''(x) + f(x) = -\cos x$. Hence proved.

Example 17. Show [assuming (f(t) to be continuous for all values of t considered in the problem] that, when p and k are constants,

$y = \dfrac{1}{p}\int_k^x f(t)\sin p(x-t)dt$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + p^2y = f(x)$.

Solution $y = \dfrac{1}{p}\int_k^x f(t)(\sin px\cos pt - \cos px\sin pt)\,dt$

$= \dfrac{1}{p}\left\{\sin px\int_k^x f(t)\cos pt\,dt - \cos px\int_k^x f(t)\sin pt\,dt\right\}$

$\dfrac{dy}{dx} = \dfrac{1}{p}\left[\sin px\,.\,f(x)\,.\,\cos px + \cos px\int_k^x f(t)\cos pt\,dt\right]$

$- \dfrac{1}{p}\left[\cos px\,.\,f(x)\sin px + \sin px\int_k^x f(t)\sin pt\,dt\right]$

$= \cos px\,.\int_k^x f(t)\cos pt\,dt + \sin px\int_k^x f(t)\sin pt\,dt$

On differentiating again,

$\dfrac{d^2y}{dx^2} = \cos^2 px\,.\,f(x) - p\sin px\int_k^x f(t)\cos pt\,dt$

$+ \sin^2 px\,f(x) + p\cos px\int_k^x f(t)\sin pt\,dt$

$= f(x) - p\int_k^x f(t)\ (\sin px\cos pt - \sin pt\cos px)\,dt$

$= f(x) - p\int_k^x f(t)\ \sin p(x-t)\,dt = f(x) - p^2 y.$

Concept Problems C

1. Prove that $\dfrac{d}{dx}\int_a^{g(x)} f(t)\,dt = f[g(x)]g'(x)$.

2. Suppose $\int_1^x f(t)dt = x^2 - 2x + 1$. Find f(x).

3. Let f be continuous on [a, b] and $\int_a^x f(t)\,dt = 0$ for all $x \in [a, b]$, then prove that $f(x) = 0$ for all $x \in [a, b]$.

4. Find the following derivatives :

(i) $\dfrac{d}{dx}\int_a^b \sin(x^2)dx$ (ii) $\dfrac{d}{da}\int_a^{x^2} \sin(x^2)dx$

(iii) $\dfrac{d}{dx}\int_0^{x^2} \sqrt{1+x^2}\,dx$ (iv) $\dfrac{d}{dx}\int_{x^2}^{x^3} \dfrac{dt}{\sqrt{1+t^2}}$

(v) $\dfrac{d}{dx}\int_{x^2}^{x^3} \dfrac{dx}{\sqrt{1+x^2}}$ (vi) $\dfrac{d}{dx}\int_{t^2}^{x^3} \dfrac{dt}{\sqrt{x^2+t^4}}$

5. Find the derivative of $y = \int_{1-3x}^{1} \dfrac{u^3}{1+u^2}du$.

6. If $F(x) = \int_1^x f(t)dt$ where $f(t) = \int_1^{t^2} \dfrac{\sqrt{1+u^4}}{u}du$ find F'' (2).

7. Find $\dfrac{d^2y}{dx^2}$ if $y = \int_x^{13} \dfrac{t^3\sin 2t}{\sqrt{1+3t}}\,dt$.

8. Let f ' be continuous in [a, b]. State under what conditions $\dfrac{d}{dx}\left(\int_a^x f(t)dt\right) = \int_a^x \dfrac{d}{dt}f(t)dt$.

9. Find an antiderivative F of $f(x) = x^2\sin(x^2)$ such that $F(1) = 0$.

10. Let f(t) be a function that is continuous and satisfies $f(t) \geq 0$ on the interval $\left[0, \dfrac{\pi}{2}\right)$. Suppose it is known that for any number x between 0 and $\dfrac{\pi}{2}$, the region under the graph of f on [0, x] has area $A(x) = \tan x$.

(a) Explain why $\int_0^x f(t)\,dt = \tan x$ for $0 \leq x < \dfrac{\pi}{2}$

(b) Differentiate both sides of the equation in part (a) and deduce the formula of f.

11. (a) Over what open interval does the formula $F(x)=\int_1^x \frac{dt}{t}$ represent an antiderivative of $f(x)=1/x$?

(b) Find a point where the graph of F crosses the x-axis.

12. (a) Over what open interval does the formula $F(x)=\int_1^x \frac{1}{t^2-9}dt$ represents an antiderivative of $f(x)=\frac{1}{x^2-9}$?

(b) Find a point where the graph of F crosses the x-axis.

13. Suppose that f has a positive derivative for all values of x and that f(1) = 0. Which of the following statements must be true of the function $g(x)=\int_0^x f(t)dt$?

(a) g is a differentiable function fx.
(b) g is a continuous function of x.
(c) The graph of g has a horizontal tangent at x = 1.
(d) g has a local maximum at x = 1.
(e) g has a local minimum at x = 1.
(f) The graph of g has an inflection point at x = 1.
(g) The graph of dg/dx crosses the x-axis at x = 1

14. Let $F(x)=\int_0^x \frac{t-3}{t^2+7}dt$ for $-\infty<x<\infty$.

(a) Find the value of x where F attains its minimum value.
(b) Find intervals over which F is over increasing or only decreasing.

Practice Problems C

15. Find $\frac{d^2}{dx^2}\int_0^x\left(\int_1^{\sin t}\sqrt{1+u^4}\,du\right)dt$

16. If $f(x)=\int_0^{g(x)}\frac{1}{\sqrt{1+t^3}}dt$, where $g(x)=\int_0^{\cos x}[1+\sin(t^2)]dt$, find $f'(\pi/2)$.

17. Find $f'(2)$ if $f(x)=e^{g(x)}$ and $g(x)=\int_2^x \frac{t}{1+t^4}dt$.

18. The function $f(x)=\int_4^{x^2}\sqrt{9+t^2}\,dt$ has inverse for $x\ge 0$. Find the value of $(f^{-1})'(0)$.

19. Suppose x and y are related by the equation $x=\int_0^y \frac{1}{\sqrt{1+4t^2}}dt$.

Show that $\frac{d^2y}{dx^2}$ is proportional to y and find the constant of proportionality.

20. Let f be a function such that $f(x)>0$. Assume that f has derivatives of all orders and that $\ln f(x)=f(x)\int_0^x f(t)dt$. Find

(i) $f(0)$, (ii) $f^{(1)}(0)$,
(iii) $f^{(2)}(0)$.

21. If $y=\int_0^x f(t)\sin\{K(x-t)\}\,dt$, then prove that

$$\frac{d^2y}{dx^2}+K^2y=K\,f(x).$$

22. Find f(4) if

(a) $\int_0^{x^2} f(t)dt = x\cos\pi x$,

(b) $\int_0^{f(x)} t^2dt = x\cos\pi x$.

23. If f is a continuous function such that

$$\int_0^x f(t)dt = xe^{2x}+\int_0^x e^{-t}f(t)dt$$

for all x, find an explicit formula for f(x).

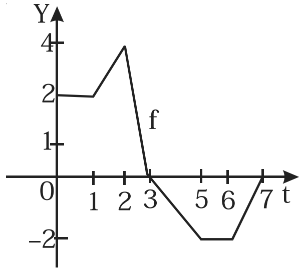
24. If $x\sin\pi x=\int_0^{x^2} f(t)$, where f is a continuous function, find f(4).

25. Find the critical points of the function

(i) $f(x)=\frac{2}{3}\sqrt{x^3}-\frac{x}{2}+\int_1^x\left(\frac{1}{2}+\frac{1}{2}\cos 2t-t^{\frac{1}{2}}\right)dt$

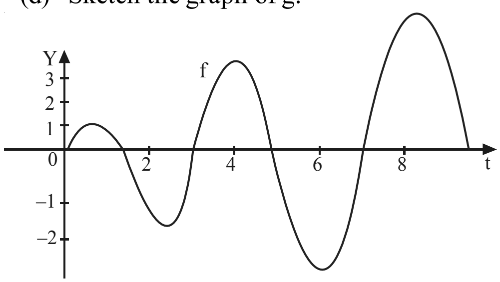
(ii) $f(x)=\int_0^x(\sin^2 2t-2\cos^2 2t+a)dt$

26. Let $g(x)=\int_0^x f(t)dt$, where f is the function whose graph is shown.

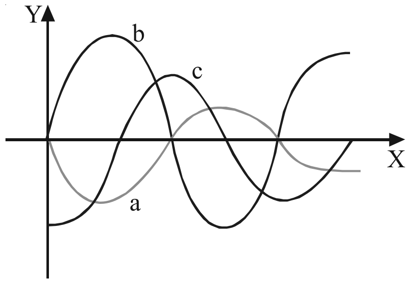


(a) Evaluate g(0), g(1), g(2), g(3), and g(6).
(b) On what interval is g increasing ?
(c) Where does g have a maximum value ?
(d) Sketch a rough graph of g.

27. Let $g(x)=\int_0^x f(t)dt$ where f is the function whose graph is shown.
(a) At what values of x do the local maximum and minimum values of g occur ?
(b) Where does g attain its absolute maximum value?
(c) On what intervals is g concave down?
(d) Sketch the graph of g.



28. The figure shows the graphs of f, f′, and $\int_0^x f(t)dt$. Identify each graph and explain your choices.



29. Find f(π/2) from the following information.
(i) f is positive and continuous.
(ii) The area under the curve y = f(x) from x = 0 to x = a is $\frac{a^2}{2}+\frac{a}{2}\sin a+\frac{\pi}{2}\cos a$.

30. Find the interval [a, b] for which the value of the integral $\int_a^b (2+x-x^2)dx$ is a maximum.

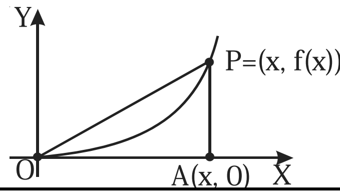
31. If f is a differentiable function such that $\int_0^x f(t)dt=[f(x)]^2$ for all x, find f.

32. Evaluate $\lim_{x\to\infty} xe^{-x^2}\int_0^x e^{t^2}dt$.

33. If $0<a<b$, find $\lim_{t\to 0}\left\{\int_0^1 [bx+a(1-x)]^t dx\right\}^{1/t}$.

34. Let f be a function such that f‴(x) is continuous, $f(x)\ge 0$, $f(0)=0$, $f'(0)=0$, and $f''(0)>0$. The graph of f is shown below. Find the limit as $x\to 0^+$ of the quotient

$$\frac{\text{Area under curve and above }[0,x]}{\text{Area of triangle OAP}}$$



2.5 SECOND FUNDAMENTAL THEOREM OF CALCULUS

The examples in the section of definite integral as limit of sum show that the direct evaluation of definite integral as limit of sum involves great difficulties. Even when the integrands are very simple (kx, x^2, e^x), this method involves cumbersome computations. The finding of definite integrals of more complicated functions leads to still greater difficulties. The natural problem that arises is to find some practically convenient way of evaluating definite integrals. This method, which was discovered by Newton and Leibnitz, utilizes the relationship that exists between integration and differentiation. The Newton-Leibnitz Formula yields a convenient method for computing definite integrals in cases where the antiderivative of the integrand is known.

The Newton-Leibnitz Formula

If f is continuous on [a, b] and F is any antiderivative of f on the interval [a, b], that is a function F exists such that $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
This formula is known as the Second Fundamental Theorem of Calculus (FTC2).

Proof : Let $F(x) = \int_a^x f(t)\,dt$. ...(1)

We know from the First Fundamental Theorem that $F'(x) = f(x)$, that is F is an antiderivative of f.
If G is any other antiderivative of f on [a, b], then we know that F and G differ by a constant:

$G(x) = F(x) + C$, for $a \le x \le b$. ...(2)

If we put $x = a$ in the formula for F(x), we get

$F(a) = \int_a^a f(t)\,dt = 0$ ...(3)

So, using (2) with $x = b$ and $x = a$, we have

$$G(b) - G(a) = [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a)$$
$$= \int_a^b f(t)\,dt \quad \text{using (1) and (3)}$$

Thus, the difference $F(b) - F(a)$ is independent of the choice of antiderivative F, since all antiderivatives differ by a constant quantity, which disappears upon subtraction anyway.
In short, the method to evaluate

$\int_a^b f(x)\,dx$ is as follows:

First we evaluate the indefinite integral $\int f(x)dx$ by the usual methods, and suppose the result is F(x).
Next we substitute for x in F(x) first the upper limit and then the lower limit, and subtract the second result from the first.

Thus, $\int_a^b f(x)dx = F(b) - F(a)$.

Note:

1. We also have two notations :

   $F(b) - F(a) = [F(x)]_a^b$ or $F(b) - F(a) = F(x)\Big|_a^b$

   The expression $\Big|_a^b$ is called the sign of double substitution.
   We can use any notation. It indicates that the value of the function corresponding to the lower index must be subtracted from the one corresponding to the upper index. We also have

   (a) $[cF(x)]_a^b = c[F(x)]_a^b$

   (b) $[F(x) + G(x)]_a^b = F(x)]_a^b + G(x)]_a^b$

   (c) $[F(x) - G(x)]_a^b = F(x)]_a^b - G(x)]_a^b$.

2. Since F(x) is an antiderivative of F'(x), we have

   $\int_a^x F'(x)dx = F(x) - F(a)$

   which can also be written as

   $\int_0^x dF(x) = F(x) - F(a)$

   We have thus come to a modification of the Newton-Leibnitz formula which makes it possible to state the basic theorem in the following way: The increment of a function on an interval is equal to the definite integral of the differential of the function over that interval.

3. With the help of the Second Fundamental Theorem, the value of a definite integral can be obtained much more easily than by the tedious process of summation. This also establishes the existence of the limit of the sum.

4. The Fundamental Theorems establish a connection between the integration as a particular kind of summation, and the integration as an operation inverse to differentiation. The connection between antiderivatives and definite integrals is as follows : FTC1 says that if f is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of f. FTC2 says that $\int_a^b f(x)\,dx$ can be found by evaluating $F(b) - F(a)$, where F is an antiderivative of f.

5. While evaluating a definite integral, arbitrary constant need not be added in the expression of the corresponding indefinite integral.

6. The indefinite integral $\int f(x)\,dx$ is a function of x, where as definite integral $\int_a^b f(x)\,dx$ is a number. Given $\int f(x)\,dx$ we can find $\int_a^b f(x)\,dx$, but given $\int_a^b f(x)\,dx$ we cannot find $\int f(x)\,dx$.

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions.

7. From the above theorem it is clear that the definite integral is a function of its upper and lower limits and not of the independent variable x.
It should be noted that if the upper limit is the independent variable, the integral is not a definite integral but simply another form of indefinite integral.
Thus, suppose

$$\int f(x)\,dx = F(x) \text{ then } \int_a^x f(x)\,dx = F(x) - F(a)$$

$$= F(x) + \text{constant} = \int f(x)\,dx.$$

Assume f is continuous on an open interval I and let F be any primitive of f on I. Then, for each a and each x in I, we have

$$F(x) = F(a) + \int_a^x f(x)\,dx.$$

8. Properties of a function deduced from properties of its derivative:
If a function f has a continuous derivative f′ on an open interval I, the Second Fundamental Theorem states that

$$f(x) = f(a) + \int_a^x f'(t)dt$$

For every choice of points x and a in I. This formula, which express f in terms of its derivative f′, enables us to deduce properties of a function from properties of its derivative.
Suppose f' is continuous and nonnegative on I. If $x > a$, then $\int_a^x f'(t)dt \geq 0$, and hence $f(x) \geq f(a)$.
In other words, if the derivative is continuous and nonnegative on I, the function is increasing on I.
Also, the indefinite integral of an increasing function is concave up. That is, if f′ is continuous and increasing on I, f is concave up on I. Similarly, f is concave down on those intervals where f′ is continuous and decreasing.

9. The Newton-Leibnitz Formula may be applied to all indefinite integrals, though care is necessary to ensure that (i) f(x) is continuous in [a, b], and (ii) F(x) is continuous in [a, b].
For example,

$$\int_{-1}^{2} \frac{dx}{x} \neq [\ln|x|]_{-1}^{2} \text{ i.e. } \ln 2$$

because the integrand, x^{-1}, is not continuous at $x = 0$; neither is $\ln|x|$.
Although we stated the Newton-Leibnitz Formula specifically for continuous functions, many discontinuous functions are integrable as well. We treat the integration of bounded piecewise continuous functions later and we shall explore the integration of unbounded functions in the section of improper integrals.

10. Sign of definite integral : Let us consider
$\int \frac{dx}{x^2} = -\frac{1}{x}$. This equation follows from the earlier found value of the derivative
$\frac{d\left(\frac{1}{x}\right)}{dx} = -\frac{1}{x^2}$ Is the sign of the integral correct here ? Can the integral of a positive function, $\frac{1}{x^2}$, be negative ? Any doubt is due to the fact that formula is not written in a proper fashion. If we write it as $\int \frac{dx}{x^2} = -\frac{1}{x} + C$ then we cannot say that the sign of the integral is alway negative since this also depends on the sign and value of the quantity C. Actually, all statements concerning the sign of the integral are referred to the definite integral. Let us take

$$\int_a^b \frac{1}{x^2}dx = -\frac{1}{x}\Big|_a^b = \left(-\frac{1}{b}\right) - \left(-\frac{1}{a}\right) = \frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}.$$

When $0 < a < b$ or $a < b < 0$, the integral is positive, as it should be.

Example 1. Evaluate the integral $\int_1^3 e^x\,dx$.

Solution The function $f(x) = e^x$ is continuous everywhere and we know that an antiderivative is $F(x) = e^x$, so the Second Fundamental Theorem gives

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e.$$

Notice that FTC2 says we can use any antiderivative F of f. So we may as well use the simplest one, namely $F(x) = e^x$, instead of $e^x + 7$ or $e^x + C$.

Example 2. Evaluate $\int_1^3 (3-2x+x^2)\,dx$.

Solution The function f defined by $f(x)=3-2x+x^2$ is continuous and has antiderivative F given by

$$F(x)=3x-x^2+\frac{1}{3}x^3.$$

Therefore, by the Fundamental Theorem

$$\int_1^3 (3-2x+x^2)\,dx = F(3)-F(1)$$

$$=(9-9+9)-(3-1+\frac{1}{3})=\frac{20}{3}.$$

Example 3. Evaluate $\int_0^{\pi/2} \cos^2 x\,dx$.

Solution Now $\int \cos^2 x\,dx = \frac{1}{2}\int 2\cos^2 x\,dx$

$$=\frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{2}x+\frac{1}{4}\sin 2x.$$

$$\therefore\quad \int_0^{\pi/2}\cos^2 x\,dx = \left[\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$=\frac{\pi}{4}+\frac{1}{4}\sin\pi=\frac{\pi}{4}.$$

Note: One should remember the values of the following definite integrals:

(a) $\int_0^{\pi/2}\sin x\,dx = \int_0^{\pi/2}\cos x\,dx = 1$

(b) $\int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}\cos^2 x\,dx = \frac{\pi}{4}$

(c) $\int_0^{\pi/2}\sin^3 x\,dx = \int_0^{\pi/2}\cos^3 x\,dx = \frac{2}{3}$

(d) $\int_0^{\pi/2}\sin^4 x\,dx = \int_0^{\pi/2}\cos^4 x\,dx = \frac{3\pi}{16}$.

Fourier Integrals

We have $\int \sin mx \sin nx\,dx$

$$=\frac{1}{2}\left[\int \cos(m-n)x\,dx - \int\cos(m+n)x\,dx\right]$$

$$=\begin{cases}\dfrac{\sin(m-n)x}{2(m-n)}-\dfrac{\sin(m+n)x}{2(m+n)} & \text{if } m^2\neq n^2\\ \pm\left(\dfrac{x}{2}-\dfrac{\sin 2nx}{4n}\right) & \text{if } m=\pm n\end{cases}$$

and $\int \sin mx\cos nx\,dx = \frac{1}{2}\left[\int \sin(m-n)x\,dx+\int\sin(m+n)x\,dx\right]$

$$=\begin{cases}-\dfrac{\cos(m-n)x}{2(m-n)}-\dfrac{\cos(m+n)x}{2(m+n)} & \text{if } m^2\neq n^2\\ \mp\dfrac{\cos 2nx}{4n} & \text{if } m=\pm n\end{cases}$$

Also,

$$\int \cos mx\ \cos nx\,dx = \frac{1}{2}\left[\int\cos(m-n)x\,dx+\int\cos(m+n)x\,dx\right]$$

$$=\begin{cases}\dfrac{\sin(m-n)x}{2(m-n)}+\dfrac{\sin(m+n)x}{2(m+n)} & \text{if } m^2\neq n^2\\ \dfrac{x}{2}+\dfrac{\sin 2nx}{4n} & \text{if } m^2=n^2\end{cases}.$$

An extremely fundamental consequence of the above results are the following so called "orthogonality relations" :

$$\int_{-\pi}^{\pi}\sin mx\sin nx\,dx=\begin{cases}0 & \text{if } m\neq n\\ \pi & \text{if } m=n\end{cases}$$

$$\int_{-\pi}^{\pi}\sin mx\cos nx\,dx = 0 \text{ for all m and n}$$

$$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx=\begin{cases}0 & \text{if } m\neq n\\ \pi & \text{if } m=n\end{cases}$$

where we assume that both m and n are positive. Also, when m and n are unequal integers, we have

$$\int_0^{\pi}\sin mx\sin nx\,dx = 0, \text{ and}$$

$$\int_0^{\pi}\cos mx\cos nx\,dx = 0,$$

When $m=n$, we have

$$\int_0^{\pi}\sin^2 nx\,dx = \frac{\pi}{2}, \text{ when n is an integer.}$$

Similarly, with the same condition, we have

$$\int_0^{\pi}\cos^2 nx\,dx = \frac{\pi}{2}.$$

Example 4. Evaluate $\int_0^2 |2x-1|\,dx$.

Solution Using the definition of absolute value, we rewrite the integrand as follows.

$$|2x-1|=\begin{cases}-(2x-1), & x<\dfrac{1}{2}\\ 2x-1, & x\geq\dfrac{1}{2}\end{cases}$$

Using this, we can rewrite the integral in two parts.

$$\int_0^2 |2x-1|\, dx = \int_0^{1/2} -(2x-1)dx + \int_{1/2}^{2} (2x-1)dx$$

$$= [-x^2+x]_0^{1/2} + [x^2-x]_{1/2}^{2}$$

$$= \left(-\frac{1}{4}+\frac{1}{2}\right) -1(0+0) + (4-2) - \left(\frac{1}{4}-\frac{1}{2}\right) = \frac{5}{2}.$$

Example 5. Calculate $\int_1^5 (|x-3|+|1-x|)dx$.

Solution We can represent the integrand as

$$f(x) = \begin{cases} 4-2x, & x \le 1, \\ 2, & 1 < x < 3, \\ 2x-4, & x \ge 3. \end{cases}$$ We get

$$\int_1^3 (|x-3| + |1-x|\, dx + \int_3^5 (|x-3| + |1-x|)\, dx$$

$$= \int_1^3 2\, dx + \int_3^5 (2x-4)\, dx$$

$$= 2x|_1^3 + (x^2-4x)|_3^5$$
$$= 4 + 8 = 12.$$

Example 6. Compute the integral $\int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}dx$

Solution $\sqrt{\frac{1+\cos 2x}{2}} = \sqrt{\frac{2\cos^2 x}{2}} = |\cos x|$

$$= \begin{cases} \cos x, & 0 \le x \le \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

Therefore,

$$\int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}\, dx = \int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} (-\cos x)dx$$

$$= \sin x\big|_0^{\pi/2} + (-\sin x)\big|_{\pi/2}^{\pi} = (1-0) + (0-(-1)) = 2.$$

Note: If we ignore the fact that cosx is negative in $\left[\frac{\pi}{2}, \pi\right]$ we get a wrong result:

$$\int_0^{\pi} \cos x dx = \sin x|_0^{\pi} = 0.$$

Existence of an integral

From the Fundamental Theorems we obtain the important result that whenever f(x) is a continuous function it possesses an antiderivative, and knowledge of the antiderivative is equivalent to ability to evaluate $\int_a^b f(x)\, dx$, for if F(x) is the antiderivative in question, the value of the integral is F(b) – F(a).

The question as to whether an antiderivative exists, and the question of the existence of an integral of the function f(x) in (a, b) are entirely independent questions. The First Fundamental Theorem however shows that when f(x) is a continuous function in the interval (a, x), then the function $F(x) = \int_a^x f(x)dx$, and the function F(x) which satisfies the differential equation $\frac{dy}{dx} = f(x)$, are identical, except for an arbitrary additive constant.

Thus, whenever f(x) possesses an antiderivative, which it certainly does when it is continuous, the Second Fundamental Theorem enables us to evaluate $\int_a^b f(x)\, dx$ if it exists.

The definite integral of f(x) may however exist, if f(x) possesses no antiderivative at all in the interval (a, b). The definite integral depends only upon the difference between two particular values of F(x), which may often be found by some special device when the form of F(x) is unknown.

CAUTION

Let us evaluate $\int_0^{\pi/4} x \tan x\, dx$

To apply the Fundamental Theorem, it is necessary to find a function F such that F '(x) = x tan x. The mathematicians have proved that there is such a function F but it is not an elementary function. That is, F is not expressible in terms of polynomials, logarithms, exponential, trigonometric functions, or any composition of these functions. We are therefore blocked, since the fundamental theorem of calculus is of use in computing $\int_a^b f(x)dx$ only if f is "nice" enough to be the derivative of an elementary function.

Note: The fact that a given function (theoretically) possesses an antiderivative does not mean that rules for obtaining its actual value are necessarily known.

Isolation of roots

If f is a continuous function in [a , b] and $\int_a^b f(x)\,dx = 0$ then the equation $f(x) = 0$ has atleast one root lying in (a , b).

Example 7. Let a, b, c be non-zero real numbers such that

$$\int_0^1 (e^{-x}+e^x)(ax^2+bx+c)\,dx$$

$$= \int_0^2 (e^{-x}+e^x)(ax^2+bx+c)\,dx$$

Then show that the quadratic equation $ax^2 + bx + c = 0$ has at least one root in (1, 2).

Solution Let $f(x) = (e^{-x}+e^x)(ax^2+bx+c)$

We have $\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$$\Rightarrow \int_1^2 f(x)\,dx = 0$$

If $f(x) > 0\ (< 0)\ \forall\ x \in [1, 2]$, then $\int_1^2 f(x)\,dx > 0\ (< 0)$. Since, the integral is zero, $f(x) = (e^{-x}+e^x)(ax^2+bx+c)$ must be positive for some values of x in [1, 2] and must be negative for some values of x in [1, 2]. As $e^{-x} + e^x \ge 2$, it follows that if $g(x) = ax^2 + bx + c$, then there exists some $\alpha, \beta \in [1, 2]$ such that $g(\alpha) > 0$ and $g(\beta) < 0$. Since g is continuous on R, there exists some γ between α and β such that $g(\gamma) = 0$. Thus $ax^2 + bx + c = 0$ has at least one root in (1, 2).

Finding displacement using velocity

If an object moves along a straight line with position function s(t), then its velocity is $v(t) = s'(t)$, so

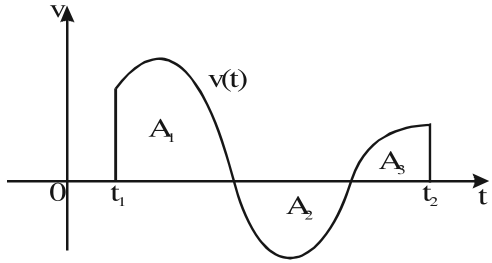
$$\int_{t2}^{t1} v(t)dt = s(t_2) - s(t_1)$$

is the net change of position, or displacement of the particle during the time period from t_1 to t_2.

If we want to calculate the distance travelled during the time interval, we have to consider the interval when $v(t) \le 0$ (the particle moves to the left). In both cases the distance is computed by integrating |v(t)|, the speed, Therefore

$$\int_{t_2}^{t_1} |v(t)|\,dt = \text{total distance travelled}$$

The figure below shows how both displacement and distance travelled can be interpreted in terms of areas under a velocity curve.



$$\text{Displacement} = \int_{t_1}^{t2} v(t)dt = A_1 - A_2 + A_3$$

$$\text{Distance} = \int_{t_1}^{t2} |v(t)|\,dt = A_1 + A_2 + A_3$$

The acceleration of the object is $a(t) = v'(t)$, so

$\int_{t1}^{t2} a(t)dt = v(t_2) - v(t_1)$ is the change in velocity from time t_1 to time t_2.

Example 8. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ m/s.

(a) Find the displacement of the particle during the time period $1 \le t \le 4$.

(b) Find the distance travelled during this time period.

Solution

(a) Displacement = s(4) – s(1)

$$= \int_1^4 v(t)dt = \int_1^4 (t^2 - t - 6)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}.$$

This means that the particle moved 4.5 m towards the left.

(b) Note that $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \le 0$ on the interval [1, 3] and $v(t) \ge 0$ on [3, 4]. Thus, the distance travelled is

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]dt + \int_1^3 v(t)\,dt$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \frac{61}{6} \approx 10.17 \text{ m}$$

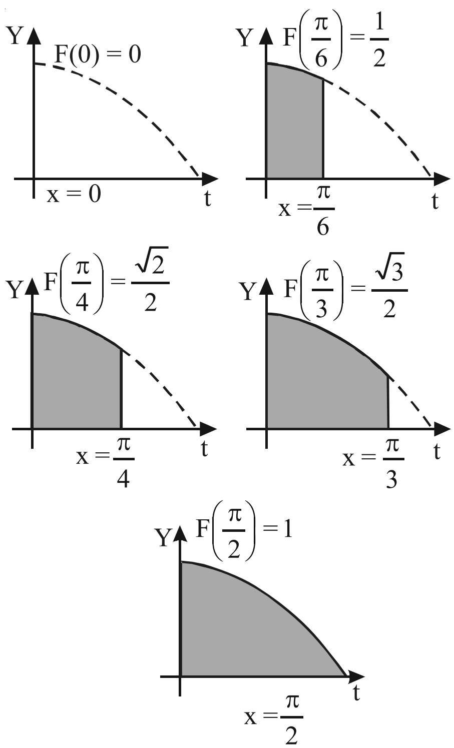
Now, consider some more examples.

Example 9. Evaluate the function $F(x)=\int_0^x \cos t\, dt$ at $x = 0, \pi/6, \pi/4, \pi/3$ and $\pi//2$.

Solution We could evaluate five different definite integrals, one for each of the given upper limits. However, it is much simpler to fix x(as a constant) temporarily and apply the Fundamental Theorem once, to obtain

$$\int_0^x \cos t\, dt = [\sin t]_0^x = \sin x - \sin 0 = \sin x.$$

Now using $F(x) = \sin x$, we obtain the results shown in the figure.



Example 10. Find the average of sin x for x in
(a) $[0, \pi/2]$ (b) $[0, 2\pi]$.

Solution
(a) The average for the interval $[0, \pi/2]$ is

$$\frac{\int_a^b \sin x dx}{(\pi/2)-0} = \frac{-\cos\Big|_0^{\pi/2}}{\pi/2} = \frac{1}{\pi/2} = \frac{2}{\pi} \approx 0.64$$

(b) The average for the interval $[0, 2\pi]$ is

$$\frac{\int_0^{2\pi} \sin x dx}{2\pi - 0} = \frac{-\cos x\Big|_0^{2\pi}}{2\pi} = \frac{0}{2\pi} = 0$$

Example 11. Find the mean value of $f(x) = \left(\frac{1}{x+1}\right)$ over the interval $[0, 2]$.

Solution $f(x) = \frac{1}{x+1}$

Mean value of f(x) over [0, 2]

$$= \frac{1}{(2-0)}\int_0^2 \frac{1}{x+1}dx = \frac{1}{2}[\ln(x+1)]_0^2 = \frac{1}{2}\ln 3.$$

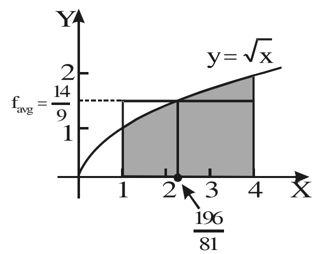
Example 12. Find the average value of the function $f(x) = \sqrt{x}$ over the interval [1, 4], and find all numbers in the interval at which the value of f is the same as the average.

Solution $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$

$$= \frac{1}{4-1}\int_1^4 \sqrt{x}dx = \frac{1}{3}\left[\frac{2x^{3/2}}{3}\right]_1^4$$

$$= \frac{1}{3}\left[\frac{16}{3} - \frac{2}{3}\right] = \frac{14}{9} \approx 1.6$$

The x-values at which $f(x) = \sqrt{x}$ is the same as the average satisfy $\sqrt{x} = 14/9$, from which we obtain $x = 196/81 \approx 2.4$ (Figure).



Example 13. Let the voltage e(t) in an electrical circuit be given by $e(t) = E\sin\omega t$ volts, where E and ω are constants.
(a) What is the average voltage over the interval of time $[0, p/\omega]$?
(b) What is the root mean square of the voltage over $[0, p/\omega]$?

Solution

(a) $e_{avg} = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin\omega t dt = \frac{\omega}{\pi}\left(-\frac{E}{\omega}\cos\omega t\right)\Bigg|_0^{\pi/\omega} = \frac{2E}{\pi}$

(b) $e_{r.m.s} = \sqrt{\dfrac{\omega}{\pi}\displaystyle\int_0^{\pi/\omega} E^2 \sin^2 \omega t dt} = \sqrt{\dfrac{\omega}{\pi}.\dfrac{E^2\pi}{2\omega}} = \dfrac{\sqrt{2}E}{\pi}$

since

$$\int_0^{\pi/\omega} E^2 \sin^2 \omega t dt = E^2\left(\frac{t}{2}+\frac{\sin 2\omega t}{4\omega}\right)\Bigg|_0^{\pi/\omega} = \frac{E^2\pi}{2\omega}$$

Example 14. If a positive function f satisfies $f(x) = \int_a^x \frac{1}{f(x)} dx$ and $\int_a^1 \frac{1}{f(x)} dx = \sqrt{2}$ then find f(x).

Solution Since $f(x) = \int_a^x \frac{1}{f(x)} dx$

Differentiating both sides w.r.t.x.

$$f'(x) = \frac{1}{f(x)} \Rightarrow 2f(x)\, f'(x) = 2$$

Integrating both sides,
$\{f(x)\}^2 = 2x + c$

$\therefore \quad f(x) = \sqrt{(2x+c)}$ ...(1)

But $\int_a^1 \frac{1}{f(x)} dx = \sqrt{2}$ ...(2)

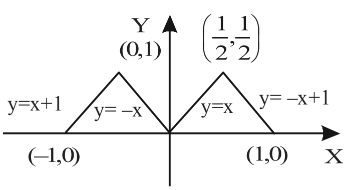
and $f(1) = \int_a^1 \frac{1}{f(x)} dx = \sqrt{2}$

$\Rightarrow \quad \sqrt{2+c} = \sqrt{2}$

$\therefore \quad c = 0$ then $f(x) = \sqrt{2x}$.

Example 15. If $f(x) = \min\{|x-1|, |x|, |x+1|\}$, then evaluate $\int_{-1}^1 f(x)dx$.

Solution Here, $f(x) = \min\{|x-1|, |x|, |x+1|\}$, Graphically it can be shown as :



$\therefore \quad \int_{-1}^1 f(x)dx$ = area of the triangles

$$= 2\int_0^1 f(x)dx = 2\left(\frac{1}{2}\times 1\times\frac{1}{2}\right) = \frac{1}{2}.$$

Example 16. Evaluate : $\int_{-3/2}^2 f(x)\,dx$, where f(x) is given by $f(x) = \max\limits_{-3/2 \le t \le x} (|t-1| - |t| + t + 1|)$.

Solution Let $g(t) = |t-1| - |t| + |t+1|$

$$= \begin{cases} -t & , \quad t < -1 \\ t+2 & , \quad -1 \le t < 0 \\ 2-t & , \quad 0 \le t < 1 \\ t & , \quad t \ge 1 \end{cases}$$

Hence, $f(x) = \begin{cases} 3/2 & , \quad -3/2 \le x < -1/2 \\ 2+x & , \quad -1/2 < x \le 0 \\ 2 & , \quad 0 < x \le 2 \end{cases}$

$$\therefore \quad \int_{-3/2}^2 f(x) = \int_{-3/2}^{-1/2} \frac{3}{2} dx + \int_{-1/2}^0 (2+x)\,dx + \int_0^2 2\,dx$$

$$= \frac{3}{2}\left(-\frac{1}{2}+\frac{3}{2}\right) + 0 - \left(-1+\frac{1}{8}\right) + 2\,(2-0)$$

$$= \frac{3}{2} + \frac{7}{8} + 4 = \frac{51}{8}.$$

Example 17. If $f(x) = ae^{2x} + be^x + cx$ satisfies the conditions $f(0) = -1$, $f'(\log 2) = 31$, and

$$\int_0^{\log 4} [f(x) - cx]\; dx = \frac{39}{2},$$ then find a, b and c.

Solution We have $f(x) = ae^{2x} + be^x + cx$.

$f(0) = -1 \Rightarrow a + b = -1$ ...(1)

$f'(x) = 2ae^{2x} + be^x + c$

$\therefore \quad f'(\log 2) = 2ae^{2\log 2} + be^{\log 2} + c = 8a + 2b + c$

$\therefore \quad f'(\log 2) = 31 \Rightarrow 8a + 2b + c = 31$ ...(2)

$$\int_0^{\log 4} [f(x) - cx]\; dx = \frac{39}{2}$$

$$\Rightarrow \int_0^{\log 4} (ae^{2x} + bc^x)\; dx = \frac{39}{2}$$

$$\Rightarrow \left(\frac{a}{2}e^{2x} + be^x\right)\Bigg|_0^{\log 4} = \frac{39}{2}$$

$$\Rightarrow \left(\frac{a}{2}e^{2\log 4} + be^{\log 4}\right) - \left(\frac{a}{2} + b\right) = \frac{39}{2}$$

$$\Rightarrow \frac{a}{2}(16) + b\,(4) - \frac{a}{2} - b = \frac{39}{2}$$

$$\Rightarrow \frac{15}{2}a + 3\,b = \frac{39}{2} \Rightarrow 15\,a + 6\,b = 39$$

$\Rightarrow$ $5a+2b=13$ ...(3)

Solving (1), (2) and (3) we get,

$a=5,\ b=-6,\ c=3$.

Example 18. Find the value of

$$\int_0^{\pi/4}(\tan^n x+\tan^{n-2}x)\, d(x-[x]),$$ where $[x]$ denotes the greatest integer function.

Solution Let $I=\int_0^{\pi/4}(\tan^n x+\tan^{n-2}x)\, d(x-[x])$

Now, $0<x\le\frac{\pi}{4} \Rightarrow [x]=0$.

Then $I=\int_0^{\pi/4}(\tan^{n-2}x(\sec^2 x-1)\, dx$

$$=\int_0^{\pi/4}\tan^{n-2}x\sec^2 x\, dx=\left[\frac{\tan^{n-1}x}{(n-1)}\right]_0^{\pi/4}$$

$$=\frac{1}{(n-1)}-0=\frac{1}{(n-1)}.$$

Example 19. Find the value of α which satisfies the equation $\int_{\pi/2}^{a}\sin x\, dx=\sin 2\alpha$, $\alpha\in[0,2\pi]$.

Solution We have $\int_{\pi/2}^{a}\sin x\, dx=\cos 2\alpha$

$\therefore$ $-\cos\alpha=\sin 2\alpha$

$\Rightarrow$ $\cos\alpha(2\sin\alpha+1)=0$

$\Rightarrow$ $\cos\alpha=0$ or $\sin\alpha=-\frac{1}{2}$

$\Rightarrow$ $\alpha=\frac{\pi}{2},\frac{3\pi}{2}$ or $\alpha=\frac{7\pi}{6},\frac{11\pi}{6}$.

These are the value of d belonging to $[0, 2\pi]$.

Example 20. Find the range of values of 'a' for which $\int_0^a(3x^2+4x-5)\, dx\le a^3-2$.

Solution $\int_0^a(3x^2+4x-5)\, dx\le a^3-2$

$\Rightarrow$ $a^3+2a^2-5a\le a^3-2$

$\Rightarrow$ $2a^2-5a+2\le 0$

$\Rightarrow$ $2a^2-4a-a+2\le 0$

$\Rightarrow$ $(2a-1)(a-2)\le 0$

$\Rightarrow$ $\frac{1}{2}\le a\le 2$.

Example 21. Find the range of the function

$f(x)=\int_0^x|t-1|\, dt$, where $0\le x\le 2$.

Solution $f(x)=\int_0^x|t-1|\, dt$

$$=\begin{cases}\int_0^x(1-t)dt, & 0\le x\le 1\\ \int_0^1(1-t)dt+\int_1^x(t-1)dt, & 1<x\le 2\end{cases}$$

$$=\begin{cases}x-\frac{x^2}{2}, & 0\le x\le 1\\ \frac{x^2}{2}-x+1, & 1<x\le 2\end{cases}$$

$\Rightarrow$ The range of the function f(x) is [0,1].

Example 22. Let $f(x)=\int_{-2}^{x}|t+1|\, dt$. Define f(x) in the interval $[-2,1]$.

Solution If $-2\le x\le -1$ then

$$f(x)=\int_{-2}^{x}-(t+1)\, dt=-\left[\frac{t^2}{2}+t\right]_{-2}^{x}$$

$$=-\frac{x^2}{2}-x+\left(\frac{4}{2}-2\right)=-\frac{x^2}{2}-x.$$

If $-1<x\le 1$ then

$$f(x)=\int_{-2}^{x}|t+1|\, dt=\int_{-2}^{-1}|t+1|\, dt+\int_{-1}^{x}|t+1|\, dt$$

$$=\int_{-2}^{-1}-(t+1)\, dt+\int_{-1}^{x}(t+1)\, dt$$

$$=\left[-\frac{t^2}{2}-t\right]_{-2}^{-1}+\left[\frac{t^2}{2}+t\right]_{-1}^{x}$$

$$=\frac{1}{2}+\frac{x^2}{2}+x-\left(\frac{1}{2}-1\right)$$

$$=\frac{x^2}{2}+x+1.$$

Thus, the definition of f(x) is as follows :

$$f(x)=\begin{cases}-\frac{x^2}{2}-x, & -2\le x\le -1\\ -\frac{x^2}{2}-x+1, & -1<x\le 1\end{cases}$$

Example 23.

Let $f(x)=\begin{cases}\displaystyle\int_0^x (1+|1-t|)\,dt & , x>2\\ 5x+1 & , x\le 2\end{cases}$.

Then show that
(i) f(x) is not continuous at $x=2$,
(ii) the right derivative of f(x) does not exist at $x=2$.

Solution

(i) $f(x)=\int_0^x (1+|1-t|)\,dt,\ x>2$

$$=\int_0^1 (2-t)\,dt+\int_1^x t\,dt$$

$$=\int_0^1 (2-t)\,dt+\int_1^x t\,dt$$

$$=\left[2t-\frac{t^2}{2}\right]_0^1+\left[\frac{t^2}{2}\right]_1^x$$

$$=\left(2-\frac{1}{2}-0\right)+\left(\frac{x^2}{2}-\frac{1}{2}\right)=\frac{1}{2}x^2+1.$$

We have $f(x)=\begin{cases}\frac{1}{2}x^2+1 & , x>2\\ 5x+1 & , x\le 2\end{cases}$

$$\lim_{x\to 2^+} f(x)=\lim_{h\to 0} f(2+h)$$

$$=\lim_{h\to 0}\frac{1}{2}(2+h)^2+1=3,$$

$$\lim_{x\to 2^-} f(x)=\lim_{h\to 0} f(2-h)$$

$$=\lim_{h\to 0} 5(2-h)+1=11$$

Also $f(2)=11$

$$\therefore\ \lim_{x\to 2^+} f(x)\ne\lim_{x\to 2^-} f(x)$$

Hence f(x) is not continuous at $x=2$.

(ii) $f'(2^+)=\lim_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$

$$=\lim_{h\to 0}\frac{\frac{1}{2}(2+h)^2+1-11}{h}$$

$$=\lim_{h\to 0}\frac{(2+h)^2-20}{2h}=\lim_{h\to 0}\frac{h^2+4h-16}{2h}=-\infty$$

Hence the right hand derivative of f(x) at $x=2$ does not exist.

Alternative : Since, f(x) is discontinuous from the right, it cannot have right hand derivative at $x=2$.

Example 24. Find the minimum value of the function

$$f(x)=\int_0^2 |x-t|\,dt.$$

Solution We have $f(x)=\int_0^2 |x-t|\,dt$

For $x\ge 2$, we have

$$f(x)=\int_0^2 (x-t)\,dt=\left[xt-\frac{t^2}{2}\right]_0^2=2x-2.$$

For $x<0$, we have

$$f(x)=\int_0^2 (t-x)\,dt=\left[\frac{t^2}{2}-xt\right]_0^2=2-2x.$$

For $0\le x<2$, we have

$$f(x)=\int_0^x (x-t)\,dt+\int_x^2 (t-x)\,dt$$

$$=\left[\frac{(x-t)^2}{2}\right]_x^0+\left[\frac{(t-x)^2}{2}\right]_x^2$$

$$=\frac{x^2}{2}+\frac{(2-x)^2}{2}=x^2-2x+2.$$

Thus, we have

$$f(x)=\begin{cases}2-2x, & x<0\\ x^2-2x+2, & 0\le x<2\\ 2x-2, & x\ge 2\end{cases}\quad\text{and}$$

$$f'(x)=\begin{cases}-2, & x>0\\ 2x-2, & 0<x<2\\ 2, & x>2\end{cases}$$

Now, we have $f'(x)=0$ at $x=1$, where $f'(x)$ changes sign.

$\Rightarrow$ f(x) strictly decreases in (0, 1) and strictly increases in (1, 2)

Hence, f(x) attains minima at $x=1$, and its minimum value, is $f(1)=1-2+2=1$.

Example 25. Find the range of the function

$$f(x)=\int_{-1}^1 \frac{\sin x\,dt}{(1-2t\cos x+t^2)}.$$

Solution We have $f(x)=\displaystyle\int_{-1}^1 \frac{\sin x\,dt}{\sin^2 x+(t-\cos x)^2}$

$$=\frac{\sin x}{\sin x}\tan^{-1}\left(\frac{t-\cos x}{\sin x}\right)\Bigg|_{-1}^{1}$$

$$=\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)-\tan^{-1}\left(\frac{-1-\cos x}{\sin x}\right)$$

$$=\tan^{-1}(\tan x/2)+\tan^{-1}(\cot x/2).$$

Case I : When $0<x<\pi$

$$0<\frac{x}{2}<\frac{\pi}{2} \text{ and } 0<\frac{\pi}{2}-\frac{x}{2}<\frac{\pi}{2}$$

$$\therefore \quad f(x)=\tan^{-1}(\tan x/2)+\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)$$

$$=x/2+\pi/2-x/2$$

$$=\pi/2$$

Case II : When $\pi<x<2\pi$

$$\frac{\pi}{2}<\frac{x}{2}<\pi \qquad \text{and} -\frac{\pi}{2}<\frac{\pi}{2}-\frac{x}{2}<0$$

$$\therefore \quad f(x)=x/2-\pi+\pi/2-x/2=-\pi/2$$

Hence, the range of f(x) is $\left\{-\frac{\pi}{2},\frac{\pi}{2}\right\}$.

Example 26. A cubic function f(x) vanishes at $x=-2$ and has relative minimum / maximum at $x=-1$ and $x=\frac{1}{3}$. If $\int_{-1}^{1} f(x)dx=\frac{14}{3}$, find f(x).

Solution Given f(x) is a cubic polynomial. Therefore, f′(x) is a quadratic polynomial. Also f(x) has relative minimum / maximum at

$$x=-1 \text{ and } x=\frac{1}{3}.$$

Hence, -1 and $\frac{1}{3}$ are roots of $f'(x)=0$.

$$\Rightarrow \quad f'(x)=a(x+1)\left(x-\frac{1}{3}\right)=a\left(x^2+\frac{2}{3}x-\frac{1}{3}\right)$$

Now integrating w.r.t x we get,

$$f(x)=a\left(\frac{x^3}{3}+\frac{x^2}{3}-\frac{x}{3}\right)+c$$

Since $f(-2)=0$, we have $a\left(\frac{-8}{3}+\frac{4}{3}+\frac{2}{3}\right)+c=0$

$$\Rightarrow \quad \frac{-2a}{3}+c=0 \Rightarrow c=\frac{2a}{3}$$

Thus, $f(x)=a\left(\frac{x^3}{3}+\frac{x^2}{3}-\frac{x}{3}\right)+\frac{2a}{3}$

$$\Rightarrow \quad f(x)=a\left(\frac{x^3}{3}+\frac{x^2}{3}-\frac{x}{3}+\frac{2}{3}\right)$$

We also have $\int_{-1}^{1} f(x)dx=\frac{14}{3}$

$$\Rightarrow \quad \frac{a}{3}\int_{-1}^{1}(x^3+x^2-x+2)\,dx=\frac{14}{3}$$

$$\Rightarrow \quad \frac{a}{3}\left[\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^2}{2}+2x\right]_{-1}^{1}=\frac{14}{3}$$

$$\Rightarrow \quad \frac{a}{3}\left[\frac{2}{3}+4\right]=\frac{14}{3} \Rightarrow a=3$$

$$\therefore \quad f(x)=x^3+x^2-x+2.$$

Example 27. Let α, β be the distinct positive roots of the equation $\tan x=2x$ then evaluate

$\int_0^1 2\sin\alpha x\cdot\sin\beta x\,dx$, independent of α and β.

Solution $I=\frac{1}{2}\int_0^1 2\sin\alpha x\cdot\sin\beta x\,dx$

$$=\frac{1}{2}\int_0^1 2\sin\alpha x\cdot\sin\beta x\,dx$$

$$=\frac{1}{2}\left[\frac{\sin(\alpha-\beta)x}{\alpha-\beta}-\frac{\sin(\alpha+\beta)x}{\alpha+\beta}\right]_0^1$$

$$=\frac{1}{2}\left[\frac{\sin(\alpha-\beta)}{\alpha-\beta}-\frac{\sin(\alpha+\beta)}{\alpha+\beta}\right] \qquad \text{...(1)}$$

Now $\quad 2\alpha=\tan\alpha$ and $2\beta=\tan\beta$

$$\Rightarrow \quad \begin{aligned}2(\alpha-\beta)&=\tan\alpha-\tan\beta\\ 2(\alpha+\beta)&=\tan\alpha+\tan\beta\end{aligned}$$

$$\therefore \quad 2(\alpha-\beta)=\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$$

and $2(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$

Substituting the value of $\frac{\sin(\alpha-\beta)}{2(\alpha-\beta)}$ and $\frac{\sin(\alpha+\beta)}{2(\alpha+\beta)}$

we get $I=(\cos\alpha\cdot\cos\beta)-(\cos\alpha\cdot\cos\beta)=0$.

Example 28. Prove that

$$\int\frac{x^{n-1}((n-2)x^2+(n-1)(a+b)x+nab}{(x+a)^2(x+b)^2}=\frac{b^{n-1}-a^{n-1}}{2(a+b)},$$

$a>0, b>0$.

Solution

$$\int\frac{x^{n-1}((n-2)x^2+(n-1)(a+b)x+nab}{(x+a)^2(x+b)^2}$$

$$=\int_a^b\frac{x^{n-1}[n(x+a)(x+b)-x(2x+a+b)]}{(x+a)^2(x+b)^2}dx$$

$$=\int_a^b\left[\frac{nx^{n-1}}{(x+a)(x+b)}-\frac{x^n(2x+a+b)}{(x+a)^2(x+b)^2}\right]dx$$

$$=\int_a^b \frac{d}{dx}\left[\frac{x^n}{(x+a)(x+b)}\right]dx = \left[\frac{x^n}{(x+a)(x+b)}\right]_a^b$$

$$=\frac{b^n}{(b+a)2b}-\frac{a^n}{2a(a+b)}=\frac{b^{n-1}-a^{n-1}}{2(a+b)}.$$

Example 29. Let $I=\int_0^{\pi/2}\frac{\cos x}{a\cos x+b\sin x}dx$ and

$J=\int_0^{\pi/2}\frac{\sin x}{a\cos x+b\sin x}dx$, where $a>0$ and $b>0$.

Compute the values of I and J.

Solution $aI+bJ=\frac{\pi}{2}$ ...(1)

and $bI-aJ=\int_0^{\pi/2}\frac{b\cos x-a\sin x}{a\cos x+b\sin x}dx$

$\therefore\ bI-aJ=ln\left[a\cos x+b\sin x\right]_0^{\pi/2}$

$\Rightarrow\ bI-aJ=ln\left(\frac{b}{a}\right)$ ...(2)

From (1) and (2)

$$a^2I+abJ=\frac{a\pi}{2}\Rightarrow b^2I-abJ=b\ln(b/a)$$

Adding, we get

$$I=\frac{1}{a^2+b^2}\left(\frac{a\pi}{2}+b\ \ln\left(\frac{b}{a}\right)\right)$$

Again $abI+b^2I=\frac{b\pi}{2}$

and $abI-a^2J=a\ln(b/a)$

Subtracting, we get

$$J=\frac{1}{a^2+b^2}\left(\frac{b\pi}{2}-a\ \ln\left(\frac{b}{a}\right)\right).$$

Alternative: We can convert $a\cos x+b\sin x$ into a single cosine say $\cos(x+\phi)$ and put $x+\phi=t$.

Concept Problems D

1. Prove the results

$\int_0^1 x^n\,dx=\frac{1}{n+1}$ and $\int_1^2 x^n\,dx=\frac{2^{n+1}-1}{n+1}$,

$n\in W$, and use them to evaluate the following:

(a) $\int_1^2 3x^2-2x+1\ dx$ (b) $\int_0^2 (t^3+t^2+t)\,dt$.

2. For the function $f(x)=1+3^x\ \ell n\ 3$ find the antiderivative F(x), which assumes the value 7 for $x=2$. At what values of x does the curve F(x) cut the x-axis ?

3. If $f(1)=12$, f′ is continuous, and

$\int_1^4 f'(x)dx=17$, what is the value of f(4) ?

4. For f(x) find such an antiderivative which attains the given magnitude $y=y_0$ at $x=x_0$.

5. Suppose that the function f is defined for all x such that $|x|>1$ and has the property that

$f'(x)=\frac{1}{x\sqrt{x^2-1}}$ for all such x.

(a) Explain why there exists two constants A and B such that
$f(x)=\sec^{-1}x+A$ if $x>1$;
$f(x)=-\sec^{-1}x+B$ if $x<-1$.

(b) Determine the values of A and B so that $f(2)=1=f(-2)$. Then sketch the graph of $y=f(x)$.

6. (a) Compute the area under the parabola $y=2x^2$ on the interval [1, 2] as the limit of a sum.

(b) Let $f(x)=2x^2$ and note that $g(x)=\frac{2}{3}x^3$ defines a function that satisfies $g'(x)=f(x)$ on the interval [1, 2]. Verify that the area computed in part (a) satisfies $A=g(2)-g(1)$.

(c) The function defined by $h(x)=\frac{2}{3}x^3+C$ for any constant C also satisfies $h'(x)=f(x)$. Is it true that the area in part (a) satisfies $A=h(2)-h(1)$?

7. (a) Prove that if f is continuous on [a, b], then $\int_a^b[f(x)-f_{avg}]dx=0$

(b) Does there exist a constant $c\neq f_{avg}$ such that $\int_a^b[f(x)-c]dx=0$?

8. Find the mean value of the function on each of the indicated closed intervals:

(a) $f(x)=\sqrt{x}$ on [0, 1], [0, 10], [0, 100],

(b) $f(x)=10+2\sin x+3\cos x$ on $[-\pi,\pi]$

(c) $f(x)=\sin(x+\alpha)$ on $[0,2\pi]$

9. (a) Find f_{avg} of $f(x)=x^2$ over [0, 2].

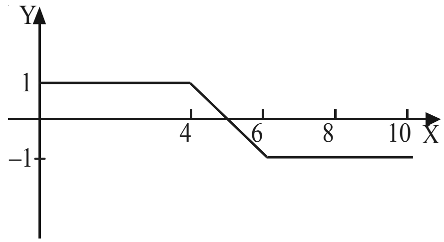
(b) Find a number c in [0, 2] such that $f(c) = f_{avg}$.

(c) Sketch the graph of $f(x) = x^2$ over [0, 2] and construct a rectangle over the interval whose area is the same as the area under the graph of f over the interval.

10. Construct the graph of the function

$F(x) = \int_a^x f(t)dt$ for a = 0, a = 4, a = 8. The function

f(x) is shown below:



11. Find all values of c for which

(a) $\int_0^c x(1-x)dx = 0$

(b) $\int_0^c |x(1-x)|\, dx = 0$

12. Find a cubic polynomial P for which

$P(0) = P(-2) = 0, P(1) = 15$, and $3\int_{-2}^{0} P(x)\,dx = 4$.

13. Find the number K, L and M such that the function

of the form $f(x) = \dfrac{Kx^2 + L}{x-1} + Mx$ satisfies the conditions $f(2) = 23$, $f'(0) = 4$ and

$\int_{-1}^{0} (x-1)f(x)dx = \dfrac{37}{6}$

14. Given the function $f(x) = \begin{cases} x^2 & \text{for } 0 \le x < 1 \\ \sqrt{x} & \text{for } 1 \le x \le 2 \end{cases}$

Compute $\int_0^2 f(x)dx$.

15. Find the abscissas of the points of intersection of the graph of the functions

$F_1(x) = \int_3^x (2t-5)\,dt$, $F_2(x) = x^2 - 5x + 6$.

16. If w'(t) is the rate of growth of a child in kg per year, what does $\int_5^{10} w'(t)dt$ represent ?

Practice Problems D

17. Evaluate the following integrals :

(i) $\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\, dx$

(ii) $\int_0^{\frac{\pi}{3}} \dfrac{x}{1+\sec x} dx$

(iii) $\int_0^2 |x^2 + 2x - 3|\, dx$

(iv) $\int_{-\sqrt{2}}^{-2} \dfrac{\sec^2(\sec^{-1} x)dx}{x\sqrt{x^2-1}}$

18. If $f(x) = |2^x - 1| + |x - 1|$ then evaluate $\int_{-2}^{2} f(x)\, dx$.

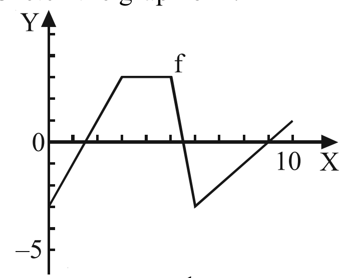
19. Let $F(x) = \int_0^x f(t)dt$, where f is the function whose graph is shown in the accompanying figure.

(a) Find F(0), F(3), F(5), F(7), and F(10).

(b) On what subintervals of the interval [0, 10] is F increasing ? decreasing?

(c) Where does F have its maximum value? Its minimum value ?

(d) Sketch the graph of F.



20. Show that $\int_0^x |t|\, dt = \frac{1}{2}x|x|$ for all real x and express $F(x) = \int_{-1}^{x} |t|\, dt$ in a piecewise form that does not involve an integral.

21. Write the equation of the tangent lines to the graph of the function $F(x) = \int_2^x (2t-5)\, dt$ at the points where the graph cuts the x-axis.

22. Find the critical points of the function

$$f(x) = x - \ell nx + \int_2^x \left(\frac{1}{z} - 2 - 2\cos 4z\right) dz$$

23. Find the greatest and the least value of the function

$F(x) = \int_1^x |t|\, dt$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

24. A function f is defined by

$$f(x) = \int_0^\pi \cos t \cos(x-t)dt \qquad 0 \le x \le 2\pi$$

Find the minimum value of f.

25. On the interval [5π/3, 7π/4] find the greatest value of the function $F(x) = \int_{5\pi/3}^x (6\cos u - 2\sin u)du$

26. Find a function f and a number a such that

$6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}$ for all $x > 0$.

27. If f(x) is a non-negative continuous function such that f(x) + f (x + 1/2) = 1, then find the value of

$$\int_0^2 f(x)dx.$$

28. Solve the inequality

$$\sqrt{x^2 - x - 12} - \int_0^x dz < x\int_0^{\pi/2} \cos 2x dx.$$

29. Find all the numbers a (a > 0) for each of which

$$\int_0^a (2 - 4x + 3x^2)\, dx \le a.$$

30. Find all solutions of the equation

$\int_0^\alpha \cos(x + \alpha^2)\, dx = \sin \alpha$ belonging to the interval [2, 3].

31. Find the values of A, B and C for which the function of the form $f(x) = Ax^2 + Bx + C$ satisfies the conditions

$$f'(1) = 8,\ f(2) + f''(2) = 33,\ \int_0^1 f(x)\,dx = \frac{7}{3}.$$

32. Find all the values of α from the interval [–π, 0] which satisfy the equation

$$\sin\alpha + \int_\alpha^{2\alpha} \cos 2x\, dx = 0$$

33. Evaluate the integrals $\int_{-\pi}^{x} \left|\frac{1}{2} + \cos t\right| dt$, if $0 \le x \le \pi$.

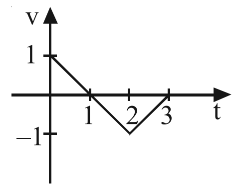
34. Find all real values of x such that

$$\int_0^x (t^2 - t)dt = \frac{1}{3}\int_{\sqrt{2}}^x (t - t^3)dt$$

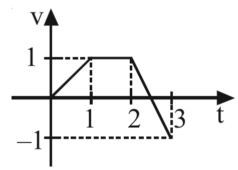
Draw a suitable figure and interpret the equation geometrically.

35. In each part, the velocity versus time curve is given for a particle moving along a line. Use the curve to find the displacement and the distance traveled by the particle over the time interval $0 \le t \le 3$.

(a)



(b)



36. Find the mean value of the velocity of a body falling freely from the altitude h with the initial velocity v_0.

37. Suppose that the velocity function of a particle moving along a line is $v(t) = 3t^3 + 2$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ by integrating.

38. The cross section of a trough has the form of a parabolic segment. Its base a = 1m, depth h = 1.5 m. Find the mean depth of the trough.

39. If a is positive and $I = \int_{-1}^{1} \frac{dx}{\sqrt{1 - 2ax + a^2}}$

then show that $I = 2$ if $a < 1$ and $I = \frac{2}{a}$ if $a > 1$.

40. If p, q are positive integers, show that

$$\int_0^\pi \cos px \sin qx\, dx = \begin{cases} 2q/(q^2 - p^2), & \text{if } (q-p) \text{ is odd,} \\ 0, & \text{if } (q-p) \text{ is even.} \end{cases}$$

41. Solve the following equations :

(i) $\int_{\sqrt{2}}^{x} \frac{dx}{x\sqrt{x^2 - 1}} = \frac{\pi}{12}$

(ii) $\int_{\ln 2}^{x} \frac{dx}{\sqrt{e^x - 1}} = \frac{\pi}{6}$

(iii) $\int_{-1}^{x} \left(8t^2 + \frac{28}{3}t + 4\right) dt = \frac{1.5x + 1}{\log_{x+1} \sqrt{x+1}}$

42. If oil leaks from a tank at a rate of r(t) litres per minute at time t, what does $\int_0^{120} r(t)dt$ represent?

43. A honeybee population starts with 100 bees and increases at a rate of n'(t) bees per week. What does $100 + \int_0^{15} n'(t)dt$ represent ?

44. The linear density of a rod of length 4m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per metre, where x is measured in metres from one end of the rod. Find the total mass of the rod.

2.6 INTEGRABILITY

Theorem. If a function f is continuous at every point of an interval [a, b], then we know that f is integrable over [a, b].

Note: If a function is integrable in a closed interval I, then it is integrable in every closed subinterval of I. However, it is important to realize that a function does not have to be continuous to be integrable and that there are many simple discontinuous functions which can be integrated.

A piecewise-continuous function does not have an antiderivative on any interval containing a point of discontinuity. Hence, we need to modify the Newton - Leibnitz Formula and the definition of antiderivative to allow integration of piecewise-continuous functions.

$$\text{Let } f(x) = \operatorname{sgn} x = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0, \quad x \in [-1, 1] \\ -1 \text{ for } x < 0 \end{cases}$$

This function is a piecewise-continuous on the closed interval [–1,1], but has no antiderivative.

Indeed, any function of the form

$$F(x) = \begin{cases} -x + C_1 & \text{for } x < 0 \\ x + C_2 & \text{for } x \geq 0 \end{cases}$$

where C_1 and C_2 are arbitrary numbers, has an antiderivative equal to sgn x for all $x \neq 0$. But even "the best" of these functions, i.e. the continuous function $F(x) = |x| + C$ (if $C_1 = C_2 = C$), does not have an antiderivative for $x = 0$.

Therefore, the function sgn x (and, in general, every piecewise-continuous function) does not have an antiderivative on any interval containing a point of discontinuity.

Here is an extended definition of an antiderivative which is suitable for integration of piecewise-continuous functions.

Extended definition of antiderivative

Definition The function F(x) is an antiderivative of the function f(x) on the closed interval [a, b] if :
(i) F(x) is continuous on [a, b], (ii) F '(x) = f(x) at the points of continuity of f(x).

Remark The function f(x) continuous on [a, b] is a special case of a piecewise-continuous function. Therefore for a continuous function the extended definition of an antiderivative coincides with the old definition since $F'(x) = f(x)\ \forall\ x \in [a, b]$ and the continuity of F(x) follows from its differentiability.

Here is an example of a function which has an antiderivative in the "new" sense and has no antiderivative in the "old" sense.

The function f(x) = sgnx had no antiderivative in the "old" sense on [–1, 1] whereas in the "new" sense the function F(x) = |x| is its antiderivative since it is continuous on [–1,1] and F'(x) = f(x) for $x \neq 0$, i.e. everywhere except for the point of discontinuity x = 0.

Hence, $\int_0^x \operatorname{sgn} t\, dt = |x|$.

Under the integral sign, there stands a bounded function discontinuous at the point x = 0. The integral as a function of the upper limit F(x) = |x| is a continuous function, at the point x = 0. But the derivative $F'(0)$ is not existent, and this does not contradict FTC1 which guarantees the existence of the derivative $F'(x)$ only if f is continuous at the point x.

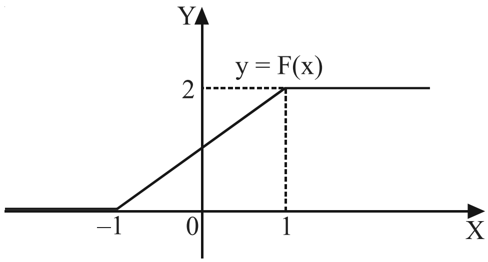
It is now clear that a function f(x), piecewise-continuous on the closed interval [a, b], has an antiderivative on this interval in the sense of the extended definition. The function $F(x) = \int_a^x f(t)dt$ is one of the antiderivatives.

Example 1. Find an antiderivative of the piecewise-continuous function

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}, \quad x \in R.$$

Solution One of the antiderivatives is an integral with a variable upper limit, and we can take any number, say, x = –2, as the lower limit of integration. Thus

$$F(x)=\int_{-2}^{x} f(t)dt = \begin{cases} 0 & \text{for} \quad x \le -1, \\ x+1 & \text{for} \quad -1<x<1, \\ 2 & \text{for} \quad x \ge 1, \end{cases}$$



The significance of the extended definition of the antiderivative is clear if we consider the following result which retains the Newton -Leibnitz Formula with the "new" definition of the antiderivative for piecewise-continuous functions.

Newton-Leibnitz Formula

For piecewise-continuous function the Newton-Leibnitz formula

$$\int_a^b f(x)dx = F(b) - F(a),$$

holds true, where F(x) is an antiderivative of the function f(x) on [a, b] in the sense of extended definition.

For example, $\int_{-1}^{2} \operatorname{sgn} x dx = |x|_{-1}^{2} = 2 - 1 = 1.$

Example 2. Calculate $I=\int_0^{\pi} \frac{\cos x}{\sqrt{1-\sin^2 x}} dx$

Solution The integrand f(x) is not defined at the point x = π/2. We divide the closed interval [0, π] in two: [0, π/2] and [π/2, π]. Setting f(π/2) = 1 on the first interval, we obtain an integral of the continuous function f(x) = 1;

$$I_1 = \int_0^{\pi/2} 1dx = \pi/2$$

On the second interval we set f(π/2) = –1 and again obtain an integral of the continuous function f(x) = –1:

$$I_2 = \int_{\pi/2}^{\pi} (-1)dx = -x\Big|_{\pi/2}^{\pi} = -\pi/2$$

The final result is $I_1 + I_2 = 0$.

Alternative : We use the extended definition of the antiderivative. The function F(x) which satisfies this definition has the form

$$F(x)=\begin{cases} x & \text{for} \quad 0 \le x \le \pi/2 \\ \pi - x & \text{for} \quad \pi/2 \le x \le \pi \end{cases}$$

Indeed, F(x) is continuous on [0,π] and F'(x) = f(x) ∀ x ∈ [0, π], x ≠ π/2, i.e. F'(x) = f(x) at the points of continuity of f(x). (Recall that x = π/2 is a point of discontinuity of f(x).)

In accordance with the Newton-Leibnitz formula valid for piecewise-continuous functions and for the extended definition of the antiderivative, we obtain

$$I=\int_0^{\pi} f(x)dx = F(x)\Big|_0^{\pi} = (\pi - x)\Big|_{x=\pi} - x\Big|_{x=0} = 0$$

Theorem Let f(x) be bounded in a ≤ x ≤ b and continuous in a < x < b. Let there be finite limits $f(a^+)$, $f(b^-)$, and let F(x) be a function for which

$$\frac{d}{dx}F(x) = f(x) \quad \text{when } a<x<b$$

and let there be finite limits $F(a^+)$, $F(b^-)$. Then

$$\int_a^b f(x)dx = F(b^-) - F(a^+).$$

Theorem If f is bounded on [a, b] and is continuous at every point of [a, b] except possibly at the endpoints, then f is integrable over [a, b].

Bounded on an interval I means that for some finite constant M, |f(x)| ≤ M for all x in I.

Example 3. Let f be the function defined by

$$f(x) = \begin{cases} \sin\frac{\pi}{x}, & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Show that f is integrable over [0, 2].

Solution This function is continuous everywhere except at 0, and its values oscillate wildly as x approaches 0. Since |f(x)| ≤ 1 for every x, the function is bounded on every interval. It therefore follows that f is integrable over [0, 2].

CAUTION

The following two examples show that a formal application of the Newton-Leibniz formula (i.e. the use of this formula without due account of the conditions of its applicability) may lead to errors in the result.

(i) Consider an integral $\int_0^1 \frac{dx}{2\sqrt{x}}$.

Taking the function $F(x) = \sqrt{x}$ as an antiderivative of the function $f(x) = 1/(2\sqrt{x})$ and using formally the Newton-Leibniz formula, we obtain

$$\int_0^1 \frac{dx}{2\sqrt{x}} = \sqrt{x}\Big|_0^1 = 1.$$

However, this result is incorrect since the function $f(x) = 1/(2\sqrt{x})$ is unbounded on [0, 1] and, consequently, the integral $\int_0^1 \frac{dx}{2\sqrt{x}}$ does not exist.

(ii) Consider an integral $I = \int_0^1 \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx$

At first sight the function $\tan^{-1}\frac{1}{x}$ may seem to be an antiderivative of the integrand function $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)$ and then, from the Newton-Leibniz formula, we obtain

$$I = \tan^{-1}\frac{1}{x}\Big|_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

However, this result is incorrect since the function $\tan^{-1}(1/x)$ is not an antiderivative of $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)$ on the interval [–1, 1]. It can be seen that the function has a discontinuity of the first kind at the point x = 0 whereas the derivative must be continuous at all points according to the definition itself.

To calculate the integral I, we note that the integrand function is as follows :

$$\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = \begin{cases} -\dfrac{1}{1+x^2} & \text{for } x \neq 0, \\ \text{is not defined} & \text{for } x = 0 \end{cases}$$

Extending the definition of this function to the point x = 0 by continuity, we get a continuous function

$$f(x) = -\frac{1}{1+x^2},\ x \in [-1, 1]$$

The function $F(x) = -\tan^{-1}x$ is an antiderivative of f(x) and therefore, from the Newton-Leibniz formula, we have

$$I = -\tan^{-1}x\Big|_{-1}^{1} = -\frac{\pi}{4} + \left(-\frac{\pi}{4}\right) = -\frac{\pi}{2}$$

Note that we can also construct an antiderivative for f(x) using the function $\tan^{-1}(1/x)$, namely,

$$G(x) = \begin{cases} \tan^{-1}\left(\dfrac{1}{x}\right) & \text{for } -1 \le x < 0 \\ -\dfrac{\pi}{2} & \text{for } x = 0 \\ \tan^{-1}\left(\dfrac{1}{x}\right) - \pi & \text{for } 0 < x \le 1 \end{cases}$$

From the Newton Leibniz formula we obtain

$$I = G(x)\Big|_{-1}^{1} = -\frac{3\pi}{4} - \left(-\frac{\pi}{4}\right) = -\frac{\pi}{2}$$

Note: Comparatively few functions have antiderivatives, a great many more have integrals.
A function may have an antiderivative but not an integral and vice-versa.
For example the function f defined by

$f(x) = 2x\sin(1/x^2) - (2/x)\cos(1/x^2),\ x \neq 0$

and $f(0) = 0$ has an antiderivative $F(x) = x^2\sin(1/x^2)$ but in the interval [–1, 1] there is no integral since f is unbounded in the neighbourhood of zero.
Further, the function ϕ defined by $\phi(x) = 1$ for $x \neq 0$ and $\phi(0) = 0$ is integrable being bounded and having only one point of discontinuity at x = 0 but has no antiderivative in the actual sence.
The next theorem shows that we can, without changing the value of the integral, assign arbitrary (bounded) values to the integrand at any finite number of points.

Theorem Let g(x) be zero except at a finite number of points in (a, b) and at these points let g(x) take finite values. Then g(x) is integrable over (a, b) and

$$\int_a^b g(x)dx = 0.$$

Corollary. Let f(x) be bounded and integrable in (a, b). Let $g(x) \equiv f(x)$ except at a finite number of points $x_1, \ldots, x_k$, at which the (finite) values of g(x) are $X_1, \ldots, X_k$. then g(x) is also integrable in (a, b) and

$$\int_a^b g(x)dx = \int_a^b f(x)dx.$$

Theorem Let f(x) be continuous in (a, b) except for a discontinuity of first kind at x = c, where a < c < b. then f(x) is integrable over each of the intervals (a, c), (c, b), (a, b) and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Note: The theorem can be extended to cover any finite number of discontinuities of first kind in the range of integration, including discontinuities of first kind at the end points a and b.

Theorem

(a) If f has a countable number of discontinuities of the first kind in [a, b], then f is integrable on [a, b].

(b) If f is not bounded on [a, b], then f is not integrable on [a, b].

Thus, a piecewise-continuous function (a function which has a countable number of points of discontinuity of the first kind on the closed interval [a, b]) is integrable on this interval.

If the conditions of the theorem are fulfilled then the value of the integral $\int_a^b f(x)dx$ does not depend on the values of f(x) at the points of discontinuity.

We therefore often raise and solve the problem of calculation of the integral of a function which is not defined either at a finite number of points of the interval [a, b] or on the set of points which can be covered by a finite number of intervals of an arbitrarily small length. In that case we assume that the definition of the function f(x) is completed arbitrarily at these points but the function remains bounded on the interval [a,b] and consequently, integrable.

For example, strictly speaking, the integral

$$\int_0^1 \frac{\sin x}{x}dx \qquad \ldots(1)$$

does not exist since at the point x = 0 the function $\frac{\sin x}{x}$ is not defined.

However, the integral $\int_0^1 f(x)dx$, where

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ m & \text{for } x = 0 \end{cases},$$ (m is an arbitrary number), exists and is independent of the choice of m. We therefore assume that integral (1) also exists and is equal to $\int_0^1 f(x)dx$.

It often happens that the integrand has a discontinuity which is simply due to a failure in its definition at a particular point in the range of integration, and can be removed by attaching a particular value to it at that point. In this case it is usual to suppose the definition of the integrand completed in this way. Thus, the integrals

$$\int_0^{1/2\pi} \frac{\sin mx}{x}dx, \quad \int_0^{1/2\pi} \frac{\sin mx}{\sin x}dx$$

can be treated as ordinary definite integrals, if the integrands are regarded as having the value m when x = 0.

Let us now evaluate $\int_0^{\pi} \frac{dx}{\cos^2 x(1+\tan^2 x)}$.

Here $f(x) = \frac{1}{\cos^2 x(1+\tan^2 x)}$

$$= \begin{cases} 1 & \text{for } 0 \le x < \pi/2,\ \pi/2 < x \le \pi, \\ \text{is not defined for } & x = \pi/2 \end{cases}$$

Extending the definition of this function to the point π/ 2, say, by continuity, i.e. setting f(π/2) = 1, we get f(x) ≡ 1 ∀ x ∈ [0, π], and, consequently, the required integral is equal to π.

Now, consider the function

f(x) = 0 where x is an integer,

f(x) = 1

otherwise, in the interval (0, m), where m is a positive integer. This function is integrable, since it is bounded and its discontinuities are finite in number, being situated at the points x = 1, 2, ..., m.

Note that the function $f(x) = 1/x^2$ is not integrable on any interval containing x = 0. Indeed, even if we extend f to be defined at 0, say by setting

$$f(x) = \begin{cases} 1/x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f would still be unbounded on any interval containing x = 0, so the theorem tells us that f is not integrable across any such interval.

Example 4. Which of the following integrals exist?

(a) $\int_0^1 \sin\frac{1}{x}dx$ (b) $\int_1^2 \frac{\ln x}{1-x}dx$

(c) $\int_0^{\pi/2} \tan x\, dx$

Solution This is the same as asking whether or not each function is integrable over its proposed interval of integration.

Let us suppose therefore, that f is bounded and continuous on the open interval (a, b). We may choose values f(a) and f(b) completely arbitrarily, and the resulting function will be integrable over [a, b]. Further more, the integral $\int_a^b f\,dx$ is independent of the choice of f(a) and f(b). Hence, if f is bounded and continuous on (a, b), we shall certainly adopt the point of view that f is integrable over [a, b] and, equivalently, that

$\int_a^b f\,dx$ is defined.

(a) Following this convention, we see that the function $\sin\frac{1}{x}$ is bounded and continuous on (0, 1), and so $\int_0^1 \sin\frac{1}{x}dx$ exists.

(b) Using L'Hospital's rule , one can easily show that

$$\lim_{x\to 1^+}\frac{\ln x}{1-x} = -1.$$

Hence, $\frac{\ln x}{1-x}$ is bounded and continuous on (1, 2), and so $\int_1^2 \frac{\ln x}{1-x}dx$ exists.

(c) Here $\lim_{x\to(\pi/2)^-}\tan x = \infty$, and we therefore conclude that tan x is not integrable over the interval $\left[0, \frac{\pi}{2}\right]$. We now give an example of a nonintegrable function.

Consider the Dirichlet function on the interval [0,1]. It is equal to 1 at rational points and to zero at irrational points. Therefore, if we take irrational points as points ξ_i in integral sums, then

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} 0.\Delta x_i = 0$$

and consequently, the limit of these sums is equal to zero as $n \to \infty$. And if rational points are chosen as ξ_i, then

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} 1.\Delta x_i = \sum_{i=1}^{n}\Delta x_i = 1$$

and, hence, the limit of these sums is equal to 1. Thus, in the case of Dirichlet function the limit of integral sums on the interval [0, 1] depends on the choice of points ξ_i and this means that the Dirichlet function is not integrable on [0, 1].

Theorem Let each of the functions f(x) and g(x) be bounded and integrable over [a, b]; and let c be a given constant. Then each of the functions

(i) cf(x) (ii) f(x) + g(x)

(iii) f(x)g(x)

is bounded and integrable over [a, b].

Example 5. Using examples, prove that the sum, the product and the quotient of two non-integrable functions can be integrable.

Solution Consider the Dirichlet function,

$$D(x) = \begin{cases} 0 \text{ if } x \text{ is an irrational number,} \\ 1 \text{ if } x \text{ is a rational number} \end{cases}$$

The function D(x) is non-integrable. The function f(x) = 2 + D(x) is not integrable either.

$$f(x) = \begin{cases} 2 \text{ if } x \text{ is an irrational number,} \\ 3 \text{ if } x \text{ is a rational number} \end{cases}$$

Indeed, if we assume that f(x) is integrable, then the difference of two integrable functions f(x) – 2 = D(x) must be integrable, but this contradicts the fact that D(x) is non-integrable.

Now, let g(x) = f(x).

Since g(x) = f(x), it follows that g(x) is non-integrable.

Let us consider the function

$$h(x) = \frac{1}{g(x)} = \begin{cases} 1/2 \text{ if } x \text{ is an irrational number} \\ 1/3 \text{ if } x \text{ is a rational number} \end{cases}$$

This function is not integrable either. The reason is similar to the reason for the non-integrability of the Dirichlet function.

We set up the sum, the product and the quotient of non-integrable functions:

$F_1(x) = f(x) + (-g(x)) \equiv 0,$

$F_2(x) = f(x)\,h(x) \equiv 1,$

$F_3(x) = f(x)/g(x) \equiv 1.$

Being constant, the functions F_1, F_2 and F_3 are integrable on any closed interval [a, b]. However, the integrability of a sum or a product does not imply the integrability of the summands or factors.

Example 6. Using examples, prove that the product of the integrable function f(x) by the non-integrable function g(x) may be (a) an integrable, (b) a nonintegrable function.

Solution

(a) Let us consider, for example, an integrable function $f(x) \equiv 0$ and a nonintegrable Dirichlet function D(x) on [a, b]. Since $f(x)\,D(x) \equiv 0$, it follows that f(x)D(x) is an integrable function on [a, b].

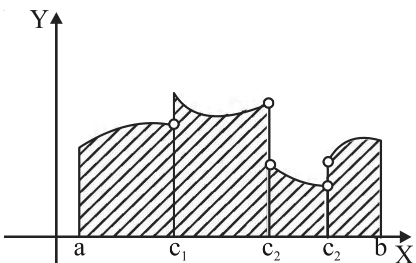
(b) Let $f(x) \equiv 2$ and $g(x) = D(x)$ on [a, b]. Then f(x) g(x) = 2D(x) is a function nonintegrable on [a, b].

Integrals of piecewise continuous functions

If a function f(x) has a countable number of points of discontinuity of the first kind in an interval [a, b], the integral of this function is defined as the sum of the ordinary integrals taken over the subintervals into which the interval [a, b] is broken up by all the points of discontinuity of the function. Denoting these points as $c_1, c_2, \ldots c_k$ $(a < c_1 < c_2 < \ldots < c_k < b)$ we write

$$\int_a^b f(x)dx = \int_a^{c1} f(x)dx + \int_{c1}^{c2} f(x)dx + \ldots.. + \int_{ck}^{b} f(x)dx$$

In the first integral on the right hand side the value of the function f(x) at the point c_1 is understood as its left hand limit $f(c_1^-)$, and in the second integral as the right hand limit $f(c_1^+)$. The values of the function f(x) at the other points of discontinuity are understood similarly. Under the conditions assumed, in every closed subinterval of integration the integrand is continuous. The geometrical meaning of the integral under consideration is clear from the figure.



The integral is equal to the sum of the areas of the trapezoids having as bases the subintervals $[a, c_1]$, $[c_1, c_2]$, ..., $[c_k, b]$ lying between the subsequent points of discontinuity.

Let us find the integral of the function

$$f(x) = \begin{cases} 1-x, & -1 \le x < 0 \\ x^2, & 0 \le x < 2 \\ -1, & 2 \le x \le 3, \end{cases} \quad \text{over } [-1, 3].$$

We have

$$\int_{-1}^{3} f(x)dx = \int_{-1}^{0}(1-x)\,dx + \int_0^2 x^2dx + \int_2^3(-1)dx$$

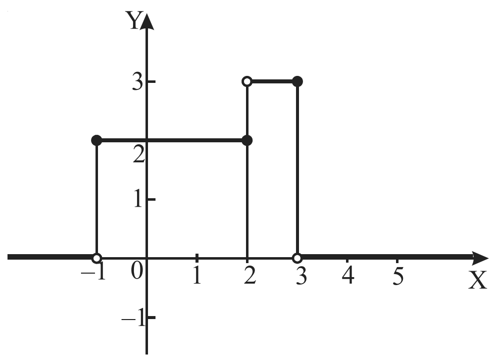
$$= \left[x - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_0^2 + [-x]_2^3$$

$$= \frac{3}{2} + \frac{8}{3} - 1 = \frac{19}{6}.$$

Thus, the Newton-Leibnitz Formula applies to bounded piecewise continuous functions with the restriction that $\left(\frac{d}{dx}\right)\int_a^x f(t)dt$ is expected to equal f(x) only at values of x at which f is continuous.

Example 7. If $f(x) = \begin{cases} 0 & -\infty < x < -1 \\ 2 & -1 \le x \le 2, \\ 3 & 2 < x \le 3, \\ -1 & 3 < x < \infty \end{cases}$, find $\int_0^4 f(x)dx$.

Solution The graph of the function f is shown in the figure.



Since f is a bounded piecewise continuous function, it is integrable over any closed interval. In particular,

$$\int_0^4 f\,dx = \int_0^2 f\,dx + \int_2^3 f\,dx + \int_3^4 f\,dx$$

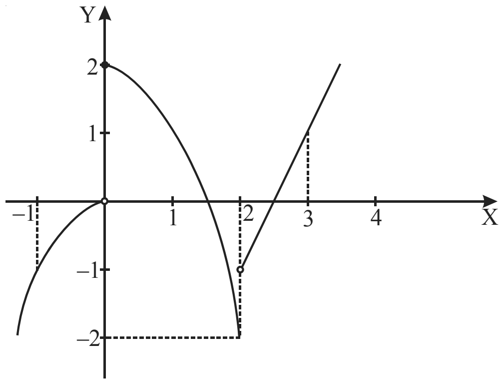
$$= 2\,.\,(2-0) + 3\,.\,(3-2) + (-1)\,(4-3)$$

$$= 4 + 3 - 1 = 6.$$

Example 8. Let f be the function defined by

$$f(x)=\begin{cases} x^3 & -\infty < x \le 0, \\ 2-x^2 & 0 < x \le 2, \\ 2x-5 & 2 < x < \infty. \end{cases}$$

The graph of f is drawn in the figure.



Solution The function is continuous except at 0 and at 2, and is bounded on any bounded interval. Thus f is integrable over the interval [–1, 3] and that

$$\int_{-1}^{3} f\,dx = \int_{-1}^{0} f\,dx + \int_{0}^{2} f\,dx + \int_{2}^{3} f\,dx .$$

For all x in [–1, 0], we have $f(x) = x^2$, and so

$$\int_{-1}^{0} f\,dx = \int_{-1}^{0} x^3 dx = \left.\frac{x^4}{4}\right|_{-1}^{0} = -\frac{1}{4}.$$

For all x in [0, 2], we have $f(x) = 2 - x^2$ except that $f(0) = 0$. Hence

$$\int_{0}^{2} f\,dx = \int_{0}^{2} (2-x^2)dx = \left.\left(2x - \frac{x^3}{3}\right)\right|_{0}^{2} = \frac{4}{3}.$$

Similarly, $f(x) = 2x - 5$ for all x in [2, 3] except that $f(2) = -2$

$$\int_{2}^{3} f\,dx = \int_{2}^{3} (2x-5)dx = \left.(x^2 - 5x)\right|_{2}^{3} = 0.$$

Hence, $\displaystyle\int_{-1}^{3} f = -\frac{1}{4} + \frac{4}{3} + 0 = \frac{13}{12}$.

Example 9. Evaluate $\displaystyle\int_{1}^{2} [2x^2 - 3]\,dx$

Solution At x = 1, value of $2x^2 - 3 = 2(1)^2 - 3 = -1$ and at x = 2, value of $(2x^2 - 3) = 2(2)^2 - 3 = 5$. The integers between – 1 to 5 are 0, 1, 2, 3, 4

$$\therefore \quad 2x^2 - 3 = 0 \Rightarrow x = \frac{\sqrt{3}}{\sqrt{2}}$$

$$2x^2 - 3 = 1 \Rightarrow x = \sqrt{2}$$

$$2x^2 - 3 = 2 \Rightarrow x = \frac{\sqrt{5}}{\sqrt{2}}$$

$$2x^2 - 3 = 3 \Rightarrow x = \sqrt{3}$$

$$2x^2 - 3 = 4 \Rightarrow x = \frac{\sqrt{7}}{\sqrt{2}}$$

$$\therefore \quad \int_{1}^{2} [2x^2 - 3]\,dx$$

$$= \int_{1}^{\sqrt{3}/\sqrt{2}} [2x^2-3]\,dx + \int_{\sqrt{3}/\sqrt{2}}^{\sqrt{2}} [2x^2-3]\,dx$$

$$+ \int_{\sqrt{2}}^{\sqrt{5}/\sqrt{2}} [2x^2-3]\,dx + \int_{\sqrt{5}/\sqrt{2}}^{\sqrt{3}} [2x^2-3]\,dx$$

$$+ \int_{\sqrt{3}}^{\sqrt{7}/\sqrt{2}} [2x^2-3]\,dx + \int_{\sqrt{7}/\sqrt{2}}^{2} [2x^2-3]\,dx$$

$$= (-1)\int_{1}^{\sqrt{3}/\sqrt{2}} dx + (0)\int_{\sqrt{3}/\sqrt{2}}^{\sqrt{2}} dx + (1)\int_{\sqrt{2}}^{\sqrt{5}/\sqrt{2}} dx$$

$$+ (2)\int_{\sqrt{5}/\sqrt{2}}^{\sqrt{3}} dx + (3)\int_{\sqrt{3}}^{\sqrt{7}/\sqrt{2}} dx + (4)\int_{\sqrt{7}/\sqrt{2}}^{2} dx$$

$$= (-1)\left(\frac{\sqrt{3}}{\sqrt{2}} - 1\right) + 0 + \left(\frac{\sqrt{5}}{\sqrt{2}} - \sqrt{2}\right)$$

$$+ 2\left(\sqrt{3} - \frac{\sqrt{5}}{\sqrt{2}}\right) + 3\left(\frac{\sqrt{7}}{\sqrt{2}} - \sqrt{3}\right) + 4\left(2 - \frac{\sqrt{7}}{\sqrt{2}}\right)$$

$$= 9 - \left\{\frac{\sqrt{3}}{\sqrt{2}} + \sqrt{2} + \frac{\sqrt{5}}{\sqrt{2}} + \sqrt{3} + \frac{\sqrt{7}}{\sqrt{2}}\right\}.$$

Example 10. Evaluate $\displaystyle\int_{0}^{2} [x^2 - x + 1]\,dx$, [.] is the greatest integer function.

Solution Let $I = \displaystyle\int_{0}^{2} [x^2 - x + 1]\,dx$

Let $f(x) = x^2 - x + 1 \Rightarrow f'(x) = 2x - 1$
For $x > 1/2$, $f'(x) > 0$ and for $x < 1/2$, $f'(x) < 0$.
The values of f(x) at x = 1/2 and 2 are 3/4 and 3.
Between them we have integers 1, 2.

Hence we solve $x^2 - x + 1 = 1, 2$

We get $x = 1,\ x = \dfrac{1+\sqrt{5}}{2}$

Also the values of f(x) at x = 0 and 1/2 are 1 and 3/4. There is no integer between them.

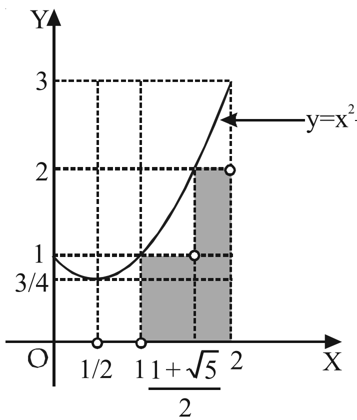
$$\therefore\quad I = \int_0^{1/2} [x^2 - x + 1]\,dx + \int_{1/2}^{1} [x^2 - x + 1]\,dx$$

$$+ \int_1^{\frac{1+\sqrt{5}}{2}} [x^2 - x + 1]\,dx + \int_{\frac{1+\sqrt{5}}{2}}^{2} [x^2 - x + 1]\,dx$$

$$= 0 + 0 + 1\int_1^{\frac{1+\sqrt{5}}{2}} 1.\,dx + 2\int_{\frac{1+\sqrt{5}}{2}}^{2} 1\,dx$$

$$= \left(\frac{1+\sqrt{5}}{2} - 1\right) + 2\left(2 - \frac{1+\sqrt{5}}{2}\right) = \left(\frac{5-\sqrt{5}}{2}\right).$$

Alternative: Graphical Method



It is clear from the figure that

$$\int_0^2 [x^2 - x + 1]\,dx = \text{Area of the bounded region}$$

$$= 0 + \left(\frac{1+\sqrt{5}}{2} - 1\right) \times 1 + \left(2 - \frac{1+\sqrt{5}}{2}\right) \times 2$$

$$= 3 - \left(\frac{1+\sqrt{5}}{2}\right) = \left(\frac{5-\sqrt{5}}{2}\right).$$

Example 11. Evaluate $\int_1^2 [x^3 - 1]\,dx$

where [.] denotes the greatest integer function.

Solution $1 \le x \le 2 \Rightarrow 1 \le x^3 \le 8 \Rightarrow 0 \le x^3 - 1 \le 7$

$$\text{So } I = \int_1^2 [x^3 - 1]\,dx$$

$$= \int_1^{2^{1/3}} [x^3 - 1]\,dx + \int_{2^{1/3}}^{3^{1/3}} [x^3 - 1]\,dx + \ldots + \int_{7^{1/3}}^{2} [x^3 - 1]\,dx$$

Now if $x \in \left[1, 2^{\frac{1}{3}}\right)$, then $x^3 \in [1, 2)$ or $[x^3 - 1] = 0$ and so on.

$$\text{Therefore } I = \int_1^{2^{1/3}} 0.dx + \int_{2^{1/3}}^{3^{1/3}} 1.dx + \ldots\ldots\ldots \int_{7^{1/3}}^{2} 6.dx$$

$$= [3^{1/3} - 2^{1/3}] + 2[4^{1/3} - 3^{1/3}] + 3[5^{1/3} - 4^{1/3}]$$
$$+ 4[6^{1/3} - 4^{1/3}] + 4[6^{1/3} - 5^{1/3}] + 6[2 - 7^{1/3}]$$
$$= 12 - [7^{1/3} + 6^{1/3} + 5^{1/3} + 4^{1/3} + 3^{1/3} + 2^{1/3}].$$

Example 12. Evaluate $\int_{\pi/4}^{\pi/2} \left[\sin x + \left[\frac{2x}{\pi}\right]\right] dx,$

where [.] denotes the greatest integer function.

Solution Here, $I = \int_{\pi/4}^{\pi/2} \left[\sin x + \left[\frac{2x}{\pi}\right]\right] dx.$

Also $\dfrac{\pi}{4} < x < \dfrac{\pi}{2} \Rightarrow \dfrac{1}{2} < \dfrac{2x}{\pi} < 1 \Rightarrow \left[\dfrac{2x}{\pi}\right] = 0.$

so that $I = \int_{\pi/4}^{\pi/2} [\sin x]\,dx = 0$.

Example 13. Find the mean value of the function on each of the indicated closed intervals:

(a) $f(x) = \cos x$ on $[0, 3\pi/2]$,

(b) $f(x) = \text{sgn } x$ on $[-1, 2]$.

Solution

(a) $f_{avg} = \dfrac{2}{3\pi}\int_0^{3\pi/2} \cos x\,dx = -\dfrac{2}{3\pi}$

Note that the continuous function cos x assumes the value $f_{avg} = -2/(3\pi)$, namely, $\cos\xi = -2/(3\pi)$, at the point $\xi = \cos^{-1}\left(-\dfrac{2}{3\pi}\right)$ of the closed interval $[0, 3\pi/2]$

(b) $f_{avg} = \dfrac{1}{3}\int_{-1}^{2} \text{sgn}\,x\,dx = \dfrac{1}{3}.$

In this case the discontinuous function sgnx does not assume the value $f_{avg} = 1/3$ on the closed interval $[-1, 3]$.

Example 14. Find the average value of $f(x) = x[2x]$ sgn $(x - 2)$ on $[1, 3]$, where [.] denotes the greatest integer function.

Solution The average value of f(x) over [1, 3]

$$= \frac{1}{(3-1)}\int_1^3 f(x)\,dx$$

$$= \frac{1}{2}\int_1^3 x[2x]\,\text{sgn}(x-2)\,dx$$

$$= \frac{1}{2}\left\{\int_1^{15} x[2x]\text{sgn}(x-2)dx + \int_{1.5}^{2} x[2x]\,\text{sgn}(x-2)\,dx\right.$$

$$\left. + \int_2^{2.5} x[2x]\text{sgn}(x-2)dx + \int_{2.5}^{3} x[2x]\text{sgn}(x-2)dx\right\}$$

$$= \frac{1}{2}\left\{\int_1^{1.5} -2x\,dx + \int_{1.5}^{2} -3x\,dx + \int_3^{2.5} 4x\,dx\right.$$

$$\left. + \int_{2.5}^{3} 5x\,dx\right\} = \frac{15}{4}.$$

Example 15. Find the mean value of the function

$f(x) = \{x\}$ on $\left[\frac{-1}{2}, 1\right]$.

Solution By definition, the mean value of function is

$$f_{avg} = \frac{1}{1+1/2}\int_{-1/2}^{1} \{x\}\,dx$$

$$= \frac{2}{3}\int_{-1/2}^{1} (x-[x])\,dx$$

$$= \frac{2}{3}\left[\int_{-1/2}^{0}(x+1)\,dx + \int_0^1 x\,dx\right]$$

$$= \frac{2}{3}\left(\left[\frac{x^2}{2}+x\right]_{-1/2}^{0} + \left[\frac{x^2}{2}\right]_0^1\right)$$

$$= \frac{2}{3}\left(\frac{-1}{8}+\frac{1}{2}+\frac{1}{2}\right) = \frac{7}{12}.$$

Example 16. Prove that $\int_0^x [x]dx = x[x] - \frac{1}{2}[x]([x]+1)$.

Solution Let n be an integer such that $n \le x < n+1$ i.e. $[x] = n$.

Then, $\int_0^x [x]dx = \int_0^n [x]dx + \int_n^x [x]dx$

$$= \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + \ldots$$

$$\ldots + \int_{n-1}^{n} [x]dx + \int_n^x [x]dx$$

$$= 0 + 1(2-1) + 2(3-2) + \ldots.$$

$$\ldots. + (n-1)[n-(n-1)] + n(x-n)$$

$$= [1+2+3 \ldots. + (n-1)] + nx - n^2$$

$$= nx + \frac{n(n-1)}{2} - n^2$$

$$= nx - \frac{n(n+1)}{2} \quad \text{(replacing } n = [x])$$

$$= [x]x - \frac{[x]([x]+1)}{2}.$$

Example 17. Let $f(x) = \int_0^x e^{t-[t]}\,dt$ $(x>0)$, where $[x]$ denotes greatest integer less than or equal to x. Show that f is continuous but not differentiable in $[0, 3)$ and $f(2) = 2(e-1)$.

Solution We have $f(x) = \int_0^x e^{t-[t]}dt = \int_0^x e^{\{t\}}dt$, so

$$f(x) = \begin{cases} \int_0^x e^t\,dt \text{ if } x \in [0,1] \\ \int_0^1 e^t\,dt + \int_1^x e^{t-1}\,dt \text{ if } x \in [1,2] \\ \int_0^1 e^t\,dt + \int_1^2 e^{t-1}\,dx + \int_2^x e^{t-2}\,dt \text{ if } x \in [2,3] \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} e^x - 1 & \text{if } x \in [0,1] \\ (e-1)+(e^{x-1}-1) & \text{if } x \in [1,2] \\ 2(e-1)+(e^{x-2}-1) & \text{if } x \in [2,3] \end{cases}$$

Clearly $f(x)$ is continuous $\forall$ but not differentiable at $x = 1$ and 2. Thus, f is continuous but not differentiable in $[0, 3)$.
Also $f(2) = 2(e-1) + 0 = 2(e-1)$.

Example 18. If the value of definite integral $\int_1^a x \cdot a^{-[\log_a x]}dx$ where $a > 1$, and $[\ .\ ]$ denotes the greatest integer, is $\frac{e-1}{2}$ then find the value of $[a]$.

Solution $\int_1^a x \cdot a^{-[\log_a x]}dx$

Put $\log_a x = t \Rightarrow a^t = x$

$$I = \ln a \cdot \int_0^1 (a^t \cdot a^{-[t]} \cdot a^t)\,dt = \ln a \cdot \int_0^1 (a^{t-[t]} \cdot a^t)\,dt$$

$$= \ln a \cdot \int_0^1 (a^{\{t\}} \cdot a^t)\,dt = \ln a \cdot \int_0^1 a^{2t}\,dt$$

$$= \left.\frac{\ln a}{2} \cdot \frac{a^{2t}}{\ln a}\right]_0^1 = \frac{1}{2}(a^2-1)$$

(as $\{t\} = t$ if $t \in (0, 1)$)

$$\therefore \quad \frac{1}{2}(a^2-1) = \frac{e-1}{2} \quad \Rightarrow \quad a = \sqrt{e}$$

$$\therefore \quad [a] = 1.$$

Alternative : $x \in (1, a) \Rightarrow \log_a x \in (0, 1)$

$\Rightarrow [\log_a x] = 0$

$$I = \int_1^a x\,dx = \frac{1}{2}(a^2-1) = \frac{e-1}{2} \Rightarrow a = \sqrt{e}.$$

$\therefore$ $[a] = 1.$

Example 19. Prove that for any positive integer K,

$$\frac{\sin 2Kx}{\sin x} = 2[\cos x + \cos 3x + ... + \cos(2K-1)x]$$

Hence, prove that $\int_0^{\pi/2} \sin 2Kx \,.\, \cot x\,dx = \frac{\pi}{2}.$

Solution

To prove: $\sin 2Kx$

$= 2\sin x[\cos x + \cos 3x + \cos 5x + ... + \cos(2K-1)x]$

R.H.S.

$= (\sin 2x) + (\sin 4x - \sin 2x) + (\sin 6x - \sin 4x) +$

$....... + (\sin 2Kx - \sin(2K-2)x) = \sin 2Kx$

L.H.S.

Now, $\int_0^{\pi/2} \sin 2Kx \,.\, \cot x\,dx$

$$= \int_0^{\pi/2} \left(\frac{\sin 2Kx}{\sin x}\right).\cos x\,dx$$

$$= \int_0^{\pi/2} 2\cos x[\cos x + \cos 3x + ... + \cos(2K-1)x]dx$$

$$= \int_0^{\pi/2} [(1+\cos 2x) + (\cos 4x + \cos 2x) + ... + (\cos 2Kx + \cos(2K-2)x)]\,dx$$

But we know that,

$$\int_0^{\pi/2} (\cos 2nx)\,dx = 0 \ \forall\, n \in I \text{ and } n \neq 0$$

$$\Rightarrow \int_0^{\pi/2} \sin 2Kx\,.\cot x\,dx = \int_0^{\pi/2} 1\,dx + 0 = \frac{\pi}{2}.$$

Example 20. A function f is defined on [0, 1] by $f(x) = 1/2^n$ for $1/2^{n+1} < x \le 1/2^n$, $n \in W$, and $f(0) = 0$.

Calculate $\int_0^x f(t)dt$, where x lies between $1/2^m$ and $1\ 2^{m-1}$.

Solution It is easy to see that f is discontinuous at $x = 1/2^n$, since we have

$$f\left(\frac{1}{2^n}^+\right) = \frac{1}{2^{n-1}} \text{ and } f\left(\frac{1}{2^n}^-\right) = \frac{1}{2^n}.$$

Thus, the points of discontinuity are

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...$$

Now

$$\int_0^x f(t)\,dt = \int_{1/2^m}^x f dx + \int_{1/2^{m+1}}^{1/2^m} f dx + \int_{1/2^{m+2}}^{1/2^{m+1}} f dx + ...$$

$$= \int_{1/2^m}^x \frac{1}{2^{m-1}}dx + \int_{1/2^{m+1}}^{1/2^m} \frac{1}{2^m}dx + \int_{1/2^{m+2}}^{1/2^{m+1}} \frac{1}{2^{m+1}}dx + ...$$

$$= \frac{1}{2^{m-1}}\left[x - \frac{1}{2^m}\right] + \frac{1}{2^m}\left(\frac{1}{2^m} - \frac{1}{2^{m+1}}\right) + \frac{1}{2^{m+1}}\left(\frac{1}{2^{m+1}} - \frac{1}{2^{m+2}}\right) + ...$$

$$= \frac{1}{2^{m-1}}\left(x - \frac{1}{2^m}\right) + \frac{1}{2^{2m-1}} + \frac{1}{2^{2m+3}} + \frac{1}{2^{2m+5}} + ...$$

$$= \frac{x}{2^{m-1}} - \frac{1}{2^{2m-1}} + \frac{1/2^{2m+1}}{1-\frac{1}{4}}$$

$$= \frac{x}{2^{m-1}} - \frac{1}{2^{2m-1}} + \frac{1}{3.2^{2m-1}}$$

$$= \frac{x}{2^{m-1}} - \frac{1}{3.2^{2m-2}}.$$

Concept Problems — E

1. Is the function $f(x) = 1/x$ integrable on the closed intervals (i) [1, 2] and (ii) [–1, 1] ?

2. Is the function $f(x) = \tan x.\cot x$ integrable on the closed intervals (i)$[\pi/6, \pi/4]$ and (ii) [–1, 1] ?

3. Is the function $f(x) = e^{-1/x}$ integrable on the closed intervals (i) [–3, –2] , (ii) [–1, 0] and (iii) [–1, 1]?

4. In each part, determine whether the function f is integrable on the interval [–1, 1].

(a) $f(x) = \cos x$

(b) $f(x) = \begin{cases} x/|x|, & x \neq 0 \\ x, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 1/x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(d) $f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

5. Is each of the following integrals defined ? Give a reason for your answer.

(a) $\int_0^1 \frac{\sin x}{x} dx$ (b) $\int_0^{1/2} \frac{\tan 2x}{x} dx$

(c) $\int_0^1 \frac{1}{x} dx$ (d) $\int_0^{1/e} \frac{1}{\ln x} dx$

(e) $\int_0^e \ln x\, dx$

6. State whether or not each of the following functions is integrable in the given interval. Give reasons for each answer.

(i) $f(x) = |x-1|$, [0, 3].

(ii) $F(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ -1 & \text{if } x \text{ irrational}, [0,1]. \end{cases}$

7. If $|f|$ is integrable in an interval [a, b], show that f need not be integrable in [a, b].

8. Use the Second Fundamental Theorem of Calculus to evaluate the integral, or explain why it does not exist.

(i) $\int_{-5}^{5} \frac{2}{x^3} dx$ (ii) $\int_{\pi}^{2\pi} \operatorname{cosec}^2 \theta\, d\theta$

(iii) $\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}} dt$

9. Is this computation correct?

$$\int_{-2}^{1} \frac{dx}{2x+1} = \frac{1}{2} \ln|2x+1| \Big|_{-2}^{1} = \frac{1}{2}\ln 3 - \frac{1}{2}\ln 3 = 0$$

10. Draw the graph of f, and evaluate $\int_a^b f(x)dx$ in each of the following examples.

(a) $f(x) = \begin{cases} 1 & \text{if } -\infty < x \le 0, \\ 5 & \text{if } 0 < x < 2, \\ 3 & \text{if } 2 \le x < \infty, \end{cases}$ and [a, b]=[–3, 3].

(b) $f(x) = \begin{cases} x^2 & \text{if } -\infty < x < 0, \\ 2 - x^2 & \text{if } 0 \le x < \infty, \end{cases}$

and [a, b] = [–2, 2].

(c) f(x) = n if $n \le x < n+1$ where n is any integer, and [a, b] = [0, 5].

11. Prove that $\int_a^b \frac{|x|}{x} dx = |b| - |a|$.

12. Evaluate the following integrals, where [x] denotes greatest integer less than or equal to x.

(i) $\int_0^3 [x]dx$

(ii) $\int_0^9 [\sqrt{t}]dt$

(iii) $\int_0^3 x\ d([x]-x)$

(iv) $\int_{-1}^{3} \left([x] + \left[x + \frac{1}{2}\right]\right) dx$

13. Show that $\int_a^b [x]dx + \int_a^b [-x]dx = a - b$.

14. Show that $\int_0^x \left(x - [x] - \frac{1}{2}\right) dx = \frac{1}{2}\{x\}(\{x\}-1)$

where [x] and {x} are integral and fractional parts of x, respectively.

Practice Problems E

15. Determine whether or not each of the following functions is integrable over the proposed interval. Give a reason for your answer.

(a) $\cos\frac{1}{x}$, [0, 1] (b) $\frac{x^2+x-2}{x-1}$, [0, 1]

(c) $\frac{x^2+x-2}{x-1}$, [0, 2] (d) $\frac{x^2+x+2}{x-1}$, [0, 2]

16. Prove that the function $f(x) = \frac{1}{x} - \left[\frac{1}{x}\right]$ for $x \ne 0$, $f(0) = 0$, is integrable on the closed interval [0, 1].

17. Prove that the function $f(x) = \operatorname{sgn}\left(\sin\frac{\pi}{x}\right)$ is integrable on the closed interval [0, 1].

18. A number is **dyadic** if it can be expressed as the quotient of two integers m/n, where n is a power of 2. (These are the fractions into which an inch is usually divided.)

Let $f(x) = \begin{cases} 0 & \text{if } x \text{ is dyadic} \\ 3 & \text{if } x \text{ is not dyadic} \end{cases}$

Why does f not have a definite integral over the interval [0, 1]?

19. Prove that the sum of an integrable and a non integrable function is a nonintegrable function.

20. Find out whether the following functions are integrable on the closed interval [0, 1]

(a) $f_1(x) = x$, (b) $g_1(x) = 1/x$
(c) $f_1(x) + g_1(x)$, (d) $f_1(x)\,g_1(x)$
(e) $f_2(x) = \sqrt{x}$, (f) $f_2(x)\,g_1(x)$.

21. $f(x) = \begin{cases} 1 & \text{for } -2 \le x \le 0, \\ D(x) & \text{for } 0 < x \le 2 \end{cases}$
where D(x) is the Dirichlet function,
$D(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$, Is the function f(x) integrable on the closed intervals [–2, 2], [–2, –1], [–1, 1], [1, 2] ?

22. Let there exist $\int_a^b |f(x)|\,dx$. Does it follow that the function f(x) is integrable on the closed interval [a, b] ?

23. Show that the function
$$f(x) = \begin{cases} \dfrac{x\ell nx}{1-x}, & 0 < x < 1 \\ 0, & x = 0 \\ -1, & x = 1 \end{cases}$$
is integrable on the interval [0, 1].

24. Can one assert that if a function is absolutely integrable on the interval [a, b], then it is integrable on this interval ?

25. Find the mean value of the function on each of the indicated closed intervals.
(a) f(x) = sinx on [0, π], [0, 2π], [α, α + 2π], [α, α + π],
(b) f(x) = sgn x on [–2, –1] [–2, 1], [–1, 3], [–2, 2], [1, 2].
(c) Is the mean value of the function on each intervalone of the values of the function on that interval ? Explain why in some cases the answer in positive and in the other negative.

26. Calculate $\int_0^2 f(x)dx$, where
$f(x) = \begin{cases} x^2 & \text{for } 0 \le x \le 1 \\ 2-x & \text{for } 1 < x \le 2 \end{cases}$ employing two techniques :
(i) using the antiderivative of f(x), constructed on the whole closed interval [0,2],
(ii) Dividing the interval [0, 2] into the intervals [0, 1] and [1, 2].

27. (a) If n is a positive integer, prove that
$$\int_0^n [t]dt = n(n-1)/2.$$
(b) If $f(x) = \int_0^x [t]\,dt$ for $x \ge 0$, draw the graph of f over the interval [0, 4].

28. (a) Prove that $\int_0^2 [t^2]dt = 5 - \sqrt{2} - \sqrt{3}$.
(b) Compute $\int_{-3}^3 [t^2]dx$.

29. (a) If n is a positive integer, prove that
$$\int_0^n [t]^2\,dx = n(n-1)(2n-1)/6.$$
(b) If $f(x) = \int_0^x [t]^2\,dx$ for $x \ge 0$, draw the graph of f over the interval [0, 3].
(c) Find all $x > 0$ for which $\int_0^x [t]^2\,dx = 2(x-1)$.

30. Let $f(x) = \dfrac{1}{x^2}$ and $F(x) = -\dfrac{1}{x}$. Find F(1) – F(–1).
Does $\int_{-1}^1 f(x)dx = F(1) - F(-1)$? Explain

2.7 IMPROPER INTEGRAL

Up to now, when speaking of definite integrals we assumed that the interval of integration was finite and closed and that the integrand was continuous or piecewise continuous. It is this particular case to which the theorems for existence the definite integral stated in the previous section applies. However, it often becomes necessary to extend the definition of the definite integral to an infinite interval of integration or to an unbounded integrand function.
We define integrals with infinite intervals of integration or infinite discontinuities within the interval of integration as improper integrals.

Integrals with infinite limits

Let y = f(x) be a continuous function in an infinite interval [a, ∞), i.e. for x ≥ a . Then we can take the integral of the function f(x) over any finite interval
$$[a, b]\ (b > a) : I(b) = \int_a^b f(x)dx$$
Now let us make b grow indefinitely. Then, there are two possibilities, namely, I(b) either has a limit as $b \to \infty$ or has no limit, which justifies the following definition :

1. The improper integral $\int_a^{\infty} f(x)dx$ of the function f(x) over the interval $[a, \infty)$ is the limit of the integral $\int_a^b f(x)dx$ as $b \to \infty$ provided that this limit exists :

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx \qquad ...(1)$$

If the limit exists the improper integral $\int_a^{\infty} f(x)dx$ is said to be convergent. If the limit does not exist equality (1) is meaningless, and the improper integral $\int_a^{\infty} f(x)dx$ is said to be divergent. In this case $\int_a^b f(x)dx$ either tends to infinity or has no limit at all.

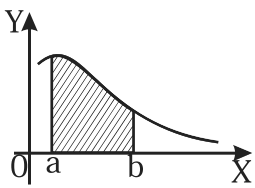
If an antiderivative F(x) of the integrand f(x) is known, it can easily be checked whether the given improper integral is convergent or divergent:

If $\lim_{x\to\infty} F(x) = F(\infty)$ exists, then, by the Newton-Leibnitz formula,

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}[F(b) - F(a)] = F(\infty) - F(a)$$

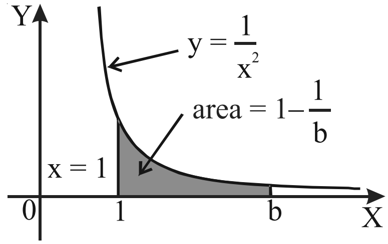
and the integral is convergent; if this limit does not exist the integral is divergent.

It is easy to see the geometric meaning of an improper integral :



If the integral $\int_a^b f(x)dx$ expresses the area of a region bounded by the curve y = f(x), the x-axis and the ordinates x = a, x = b, it is natural to consider that the improper integral $\int_a^{\infty} f(x)dx$ expresses the area of an unbounded (infinite) region lying between the curves y = f(x), x = a, and the x–axis.

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line x = 1. One might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look.



The area of the part of S that lies to the left of the line x = b (shaded in Figure) is

$$A(b) = \int_1^b \frac{1}{x^2}dx = -\frac{1}{x}\Big]_1^b = 1 - \frac{1}{b}$$

Notice that A(b) < 1 no matter how large b is chosen.

We also observe that $\lim_{b\to\infty} A(b) = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$

The area of the shaded region approaches 1 as $b \to \infty$, so we say that the area of the infinite region S is equal to 1 and we write

$$\int_1^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = 1.$$

Example 1. Evaluate $\int_0^{\infty} (1 - x)e^{-x}dx$

Solution Integrating by parts with $u = 1 - x$ and $v = e^{-x}$ yields

$$\int (1-x)\, e^{-x}\, dx = -e^{-x}(1-x) - \int e^{-x}\, dx$$
$$= -e^{-x} + xe^{-x} + e^{-x} + C = xe^{-x} + C$$

Thus, $\int_0^{\infty} (1 - x)e^{-x}dx = \lim_{b\to\infty}[xe^{-x}]_0^b = \lim_{b\to\infty}\frac{b}{e^b}$

The limit is an indeterminate form of type ∞/∞, so we will apply L'Hospital's rule by differentiating the numerator and denominator with respect to b. This yields

$$\int_0^{\infty} (1 - x)e^{-x}dx = \lim_{b\to\infty}\frac{1}{e^b} = 0$$

An explanation of why this integral is zero can be obtained by interpreting the integral as the net signed area between the graph of $y = (1 - x)e^{-x}$ and the interval $[0, \infty)$.

Example 2. Evaluate $\int_0^\infty [2e^{-x}]\, dx$, where [.] denotes the greatest integer function.

Solution Let $I = \int_0^\infty [2e^{-x}]dx$

Let $y = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} < 0 \ \forall\ x \in [0, \infty]$

$\therefore$ $2e^{-x}$ is decreasing function $\forall\ x \in [0, \infty]$

$\Rightarrow$ $0 < 2e^{-x} \le 2 \ \forall\ x \in [0, \infty)$

For $x > \ln 2 \Rightarrow e^x > 2 \Rightarrow e^{-x} < \frac{1}{2}$

$\Rightarrow$ $2e^{-x} < 1 \Rightarrow 0 \le 2e^{-x} < 1$

$\Rightarrow$ $[2e^{-x}] = 0$

$$\therefore\ I = \int_0^{\ln 2} [2e^{-x}]\, dx + \int_{\ln 2}^{\infty} [2e^{-x}]\, dx$$

$$= \int_0^{\ln 2} 1 . dx + \int_{\ln 2}^{\infty} 0 . dx$$

$= (\ln 2 - 0) + 0 = \ln 2.$

Example 3. Evaluate $\int_0^\infty \frac{x^2}{(1+x^2)^2} dx$.

Solution Put $x = \tan\theta$. $\therefore$ $dx = \sec^2\theta\, d\theta$

As x increases from 0 to ∞, θ increases from 0 to $\frac{\pi}{2}$.

$$I = \int_0^{\frac{1}{2}\pi} \frac{\tan^2\theta \sec^2\theta\, d\theta}{\sec^4\theta} = \int_0^{\frac{1}{2}\pi} \sin^2\theta\, d\theta = \frac{1}{4}\pi.$$

Note: Thus, sometimes an infinite integral can be tranformed into an ordinary definite integral by a suitable substitution, we must see that the transformation is legitimate.

Example 4. Show that $\int_0^\infty e^{-x} x^n dx = n!$, n being a positive integer.

Solution Let I_n denote the given integral.

$$I_n = \lim_{\varepsilon \to \infty} \int_0^{\varepsilon} e^{-x} x^n dx$$

$$= \lim_{\varepsilon \to \infty} \left\{ \left[-e^{-x} x^n\right]_0^{\varepsilon} + n\int_0^{\varepsilon} e^{-x} x^{n-1} dx \right\}$$

[integrating by parts]

$$= n \lim_{\varepsilon \to \infty} \int_0^{\varepsilon} e^{-x} x^{n-1} dx, \text{ since } \lim_{\varepsilon \to \infty} e^{-\varepsilon} . \varepsilon^n = 0$$

$= n I_{n-1} = n(n-1) I_{n-2}$ (as before)

$= n(n-1)(n-2) I_{n-3}$, ...

$$= n(n-1)(n-2) .... 2.1 \int_0^\infty e^{-x} dx.$$

$$= n!, \quad \text{since } \int_0^\infty e^{-x} dx = 1.$$

Example 5. Evaluate $\int_2^\infty \frac{x+3}{(x-1)(x^2+1)} dx.$

Solution

$$\int_2^\infty \frac{x+3}{(x+1)(x^2+1)} dx$$

$$= \lim_{b\to\infty} \int_2^b \frac{x+3}{(x+1)(x^2+1)} dx$$

$$= \lim_{b\to\infty} \int_2^b \left(\frac{2}{x-1} - \frac{2x+1}{x^2+1}\right) dx$$

$$= \lim_{b\to\infty} \left[2\ln(x-1) - \ln(x^2+1) - \tan^{-1}x\right]_2^b$$

$$= \lim_{b\to\infty} \left[\ln\frac{(x-1)^2}{x^2+1} - \tan^{-1}x\right]_2^b$$

$$= \lim_{b\to\infty} \left[\ln\left(\frac{(b-1)^2}{b^2+1}\right) - \tan^{-1}b\right] - \ln\left(\frac{1}{5}\right) + \tan^{-1}2$$

$$= 0 - \frac{\pi}{2} + \ln 5 + \tan^{-1}2 \approx 1.14.$$

Notice that we combined the logarithms in the antiderivative before we calculated the limits as $b \to \infty$. Had we not done so, we would have encountered the indeterminated form

$$\lim_{b\to\infty} (2\ln(b-1) - \ln(b^2+1)) = \infty - \infty.$$

The way to evaluate the indeterminate form, of course, is to combine the logarithms, so we would have arrived at the same answer in the end. But our original route was shorter.

2. The improper integral over the interval $(-\infty, b]$ is defined similarly

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx = F(b) - F(-\infty)$$

where $F(-\infty)$ is the limit (if it exists) of the antiderivative F(x) as $x \to -\infty$.

Example 6. Evaluate $\int_{-\infty}^0 xe^x\, dx.$

Solution $\int_{-\infty}^0 xe^x\, dx = \lim_{a\to-\infty} \int_a^0 xe^x\, dx$

We integrate by parts with u = x, dv = e^x dx so that du = dx, v = e^x :

$$\int_a^0 xe^x\,dx = xe^x]_a^0 - \int_a^0 e^x\,dx = -te^a - 1 + e^a$$

We know that $e^a \to 0$ as $a \to -\infty$, and by L'Hospital's Rule we have

$$\lim_{a\to-\infty} ae^a = \lim_{a\to-\infty} \frac{a}{e^{-a}} = \lim_{a\to-\infty} \frac{a}{-e^{-a}} = \lim_{a\to-\infty} (-e^a) = 0$$

Therefore

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} (-ae^a - 1 + e^a)$$

$$= -0 - 1 + 0 = -1.$$

3. If f(x) is a function continuous in $(0, \infty)$, the improper integral can be defined for the whole interval $(-\infty, \infty)$ by definition,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

If both integrals on the right hand side are convergent, the integral $\int_{-\infty}^{\infty} f(x)dx$ is said to be convergent.

If atleast one of the integrals on the right hand side is divergent the equality has no sense, and the integral on the left is called divergent.

Here we have a simpler notation. If an antiderivative F(x) is known, then

$$\int_{-\infty}^{\infty} f(x)dx = F(\infty) - F(-\infty)$$

where $F(\infty)$ and $F(-\infty)$ are respectively, the limits (if they exist) of F(x) as $x \to \infty$ and $x \to -\infty$. If atleast one of these limits does not exist the improper integral is divergent.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \tan^{-1} x\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}dx = \frac{1}{2}\ell n(1+x^2)\Big|_{-\infty}^{\infty}.$$

Here both $F(-\infty)$ and $F(\infty)$ are equal to infinity and the integral is divergent.

It should be noted that all the simplest properties of definite integrals enumerated are extended without any changes to improper integrals provided all the integrals on the right hand sides of the equalities are convergent.

Example 7. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

Solution $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_{0}^{\infty} \frac{1}{1+x^2}dx$

We must now evaluate the integrals on the right side separately :

$$\int_0^{\infty} \frac{1}{1+x^2}dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2} = \lim_{t\to\infty} \tan^{-1} x]_0^t$$

$$= \lim_{t\to\infty} (\tan^{-1} t - \tan^{-1} 0) = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$$

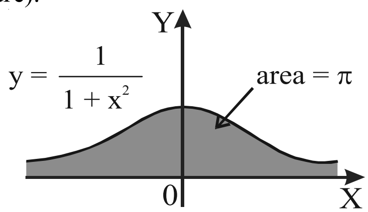
$$\int_{-\infty}^0 \frac{1}{1+x^2}dx = \lim_{t\to-\infty} \int_t^0 \frac{dx}{1+x^2} = \lim_{t\to-\infty} \tan^{-1} x]_t^0$$

$$= \lim_{t\to-\infty} (\tan^{-1} 0 - \tan^{-1} t) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Since $1/(1+x^2) > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the curve $y = 1/(1+x^2)$ and above the x-axis (see figure).



Example 8. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$.

Solution $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \int_{-\infty}^{\infty} \frac{e^x dx}{e^{2x} + 1}$

$$\int_0^{\infty} \frac{e^x dx}{e^{2x}+1} = \lim_{c\to\infty} \int_0^c \frac{e^x dx}{e^{2x}+1}$$

$$= \lim_{c\to\infty} \int_0^{e^c} \frac{du}{u^2+1} \text{ (by the substitution } u = e^x)$$

$$= \lim_{c\to\infty} \tan^{-1} u\Big]_1^{e^c} = \lim_{c\to\infty} (\tan^{-1}(e^c) - \tan^{-1}(1))$$

$$= \lim_{c\to\infty}\left(\tan^{-1}(e^c) - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Similarly, $\int_{-\infty}^0 \frac{e^x dx}{e^{2x}+1} = \lim_{c\to-\infty} \int_e^0 \frac{e^x dx}{e^{2x}+1}$

$$= \lim_{c\to-\infty}\int_{e^c}^{1}\frac{du}{u^2+1} = \lim_{c\to-\infty}\tan^{-1}u\Big]_{e^c}^{1}$$

$$= \lim_{c\to-\infty}\left(\frac{\pi}{4}-\tan^{-1}(e^c)\right) = \frac{\pi}{4}-\lim_{c\to-\infty}\tan^{-1}(e^c)$$

$$= \frac{\pi}{4}-0=\frac{\pi}{4}$$

Thus, $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}} = \int_0^{\infty}\frac{e^x dx}{e^{2x}+1}+\int_{-\infty}^{0}\frac{e^x dx}{e^{2x}+1}$

$$= \frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}.$$

Integral of a function with infinite discontinuity

We now proceed to the definition of the improper integral for functions with infinite discontinuities. Let $y = f(x)$ be a continuous function for all $x \in [a, b)$ (i.e. $a \le x < b$) having an infinite discontinuity at the right end point $x = b$ of the interval $[a, b]$. It is clear that the ordinary definition of the definite integral is inapplicable here. Let us first take the ordinary integral.

$I(\varepsilon) = \displaystyle\int_a^{b-\varepsilon} f(x)dx$ where $\varepsilon > 0$ and then make ε tend to 0.

Then $I(\varepsilon)$ either tends to a finite limit or has no finite limit (in the latter case it either tends to infinity or has no limit at all).

The improper integral $\displaystyle\int_a^b f(x)dx$ of a function $f(x)$ continuous for $a \le x < b$ and unbounded as $x \to b$ is the limit of the integral $\displaystyle\int_a^{b-\varepsilon} f(x)dx$ for $\varepsilon \to 0$ $(\varepsilon > 0)$ provided this limit exists:

$$\int_a^b f(x)dx = \lim_{\varepsilon\to0}\int_a^{b-\varepsilon} f(x)dx,\quad \varepsilon > 0.$$

The integral $\displaystyle\int_a^b f(x)\,dx$ of the function can also be evaluated as follows :

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^{b} f(x)\,dx$$

Example 9. Evaluate $\displaystyle\int_0^1\frac{dx}{\sqrt{1-x}}$.

Solution $\displaystyle\int_0^1\frac{dx}{\sqrt{1-x}} = \lim_{b\to1^-}\int_0^b\frac{dx}{\sqrt{1-x}}$

$$= -\lim_{b\to1^-}2\sqrt{1-x}\,\Big|_0^b = -\lim_{b\to1^-}2(\sqrt{1-b}-1)=2.$$

Example 10. Evaluate $\displaystyle\int_0^{\pi/2}\frac{\cos x}{\sqrt{1-\sin x}}dx.$

Solution The integrand is discontinuous at $x = \dfrac{\pi}{2}$.

$$\int_0^{\pi/2}\frac{\cos x}{\sqrt{1-\sin x}}dx = \lim_{u\to\pi/2^-}\int_0^u\frac{\cos x}{\sqrt{1-\sin x}}dx$$

$$= \lim_{u\to\pi/2^-} -\int_0^u(1-\sin x)^{-1/2}(-\cos x)\,dx$$

$$= \lim_{u\to\pi/2^-} -2(1-\sin x)^{1/2}\Big]_0^u$$

$$= \lim_{u\to\pi/2^-} -2[(1-\sin u)^{1/2}-1]=2$$

Example 11. Prove that $\displaystyle\int_0^{\pi/2}\sin^3\theta\cos^{-1/2}\theta d\theta = \frac{8}{5}$.

Solution The integrand is continuous in $0 \le \theta < \dfrac{1}{2}\pi$, but tends to ∞ as $\theta \to \dfrac{1}{2}\pi$. The integral can only be defined as $\displaystyle\lim_{\epsilon\to0}\int_0^{\frac{1}{2}\pi-\epsilon}\sin^3\theta\cos^{-\frac{1}{2}}\theta d\theta$ ...(1)

Put $\cos\theta = \lambda$, $-\sin\theta\, d\theta = d\lambda$

when $\theta = 0, \lambda = 1$; $\theta \to \dfrac{1}{2}\pi, \lambda \to 0$

Thus (1) is given by

$$\lim_{\delta\to0}\int_\delta^1(1-\lambda^2)\lambda^{-\frac{1}{2}}d\lambda$$

$$\lim_{\delta\to0}\left[2\lambda^{\frac{1}{2}}-\frac{2}{5}\lambda^{5/2}\right]_\delta^1 = 2-\frac{2}{5}=\frac{8}{5}.$$

5. Similarly, if the function $f(x)$ has an infinite discontinuity only at the left end point $x = a$ of the interval $[a, b]$ we put

$\displaystyle\int_a^b f(x)dx = \lim_{\delta\to0}\int_{a+\delta}^b f(x)dx$ where $\delta > 0$ provided this limit exists.

If the limit does not exist, the integral is divergent.

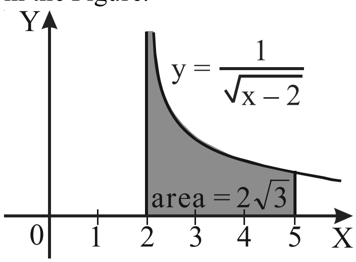
Example 12. Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

Solution We note first that the given integral is improper becuase $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$. Since the infinite discontinuity occurs at the left end point of [2, 5]

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+}\int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big]_t^5$$

$$= \lim_{t\to 2^+} 2(\sqrt{3}-\sqrt{t-2}) = 2\sqrt{3}.$$

Thus, the given improper integral is convergent and, since the integrand is positive, we can interpret the value of the integeral as the area of the shaded region as shown in the Figure.



Example 13. Evaluate $\int_0^1 \ln x\,dx$.

Solution We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since $\lim_{x\to 0^+} \ln x = -\infty$. Thus, the given integral is improper and we have

$$\int_t^1 \ln x\,dx = \lim_{t\to 0^+}\int_t^1 \ln x\,dx$$

Now we integrate by parts,

$$\int_t^1 \ln x\,dx = x\ln x\,\Big]_t^1 - \int_t^1 dx$$

$$= 1\ln - t\ln t - (1-t) = -t\ln t - 1 + t$$

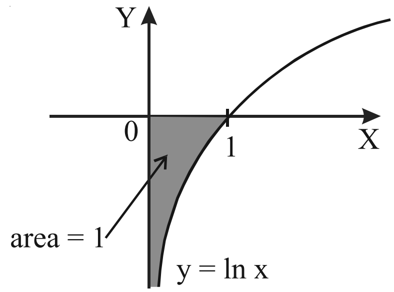
To find the limit of the first term we use L'Hospital's rule :

$$\lim_{t\to 0^+} t\,\ln t = \lim_{t\to 0^+}\frac{\ln t}{1/t} = \lim_{t\to 0^+}\frac{\ln t}{1/t^2} = \lim_{t\to 0^+}(-t) = 0.$$

Therefore, $\int_0^1 \ln x\,dx = \lim_{t\to 0^+}(-t\ln t - 1 + t)$

$$= -0 - 1 + 0 = -1$$

The figure shows that the area of the shaded region above $y = \ln x$ and below the x-axis is 1.



Example 14. Prove that $I = \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx = \pi$.

Solution The integrand $\to \infty$ both when $x \to 0$ and when $x \to 1$. We consider I as the limit, when δ and $\in \to 0$, of

$$I_1 = \int_\delta^{1-\in} x^{-1/2}(1-x)^{-1/2}\,dx$$

Put $x = \sin^2\theta$, where $0 < \theta < \frac{1}{2}\pi$, and let θ_1, θ_2 correspond to the values, δ, $1 - \in$ of x. Then

$dx = 2\sin\theta\cos\theta\,d\theta$ and

$$I_1 = \int_{\theta_1}^{\theta_2} \operatorname{cosec}\theta\sec\theta\;2\sin\theta\cos\theta\,d\theta = 2(\theta_2 - \theta_1)$$

Since $\theta_2 \to \frac{1}{2}\pi$ as $\in \to 0$ and $\theta_1 \to 0$ as $\delta \to 0$, $I = \pi$.

Note: The above is an example of how a convergent improper integral, such as

$$\int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx \qquad ...(1)$$

may, when we use a substitution, (here we use $x = \sin^2\theta$) become a proper integral, such as

$$\int_0^{\frac{1}{2}\pi} 2\,d\theta \qquad ...(2)$$

The solution that merely puts $x = \sin^2\theta$ in (1) and at once gets (2) gives the right answer; but since (1) is, in fact, an improper integral, this solution proves nothing.

6. If the function f(x) has an infinite discontinuity at an intermediate point $x = c$ of the interval [a, b] (i.e. $a < c < b$) then, by definition

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If both integrals on the right hand side of the equality are convergent, the integral $\int_a^b f(x)dx$ is also convergent.

If atleast one of these integrals on the right is divergent the integral is divergent. For example,

$$\int_{-1}^{2}\frac{dx}{\sqrt[3]{x^2}}=\int_{-1}^{0}\frac{dx}{\sqrt[3]{x^2}}+\int_{0}^{2}\frac{dx}{\sqrt[3]{x^2}}=3\sqrt[3]{x}\Big|_{-1}^{0}+3\sqrt[3]{x}\Big|_{0}^{2}=3+3\sqrt[3]{2}$$

Note : One should not be puzzled by the use of the term improper integral to denote something which has a definite value such as 2 or $\pi/2$. The distinction between an improper integral and a definite integral is similar to that between an infinite series and a finite series, and no one supposes that an infinite series is necessarily divergent.

Recall that the definite integral $\int_a^x f(t)\,dt$ we defined as a simple limit, i.e. as the limit of a certain finite sum. The improper integral is therefore the limit of a limit, or what is known as a repeated limit.

Example 15. Evaluate $\int_{-1}^{1}\frac{dx}{x^2}$.

Solution Since inside the interval of integration there exists a point $x = 0$ where the integrand is discontinuous, the integral must be represented as the sum of two terms :

$$\int_{-1}^{1}\frac{dx}{x^2}=\int_{-1}^{0}\frac{dx}{x^2}+\int_{0}^{1}\frac{dx}{x^2}$$

$$\int_{-1}^{1}\frac{dx}{x^2}=\lim_{\varepsilon_1\to 0^-}\int_{-1}^{\varepsilon_1}\frac{dx}{x^2}+\lim_{\varepsilon_2\to 0^+}\int_{\varepsilon_2}^{1}\frac{dx}{x^2}$$

We calculate each limit separately:

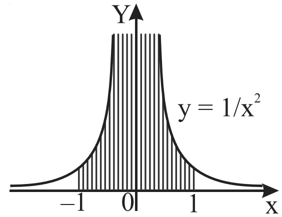
$$\lim_{\varepsilon_1\to 0^-}\int_{-1}^{\varepsilon_1}\frac{dx}{x^2}=-\lim_{\varepsilon_1\to 0^-}\frac{1}{x}\Big|_{-1}^{\varepsilon_1}$$

$$=-\lim_{\varepsilon_1\to 0^-}\left(\frac{1}{\varepsilon_1}-\frac{1}{-1}\right)=\infty$$

Thus, the integral diverges on the interval [–1, 0] :

$$\lim_{\varepsilon_2\to 0^+}\int_{\varepsilon_2}^{1}\frac{dx}{x^2}=-\lim_{\varepsilon_2\to 0^+}\left(1-\frac{1}{\varepsilon_2}\right)=\infty$$

And this means that the integral also diverges on the interval [0, 1]. Hence, the given integral diverges on the entire interval [–1, 1].



Here both integrals on the right hand side are divergent, and consequently, the given integral is also divergent. In other cases, an integral with the integrand function becoming infinite on the interval of integration can yield a finite result.

It should be noted that if we had begun to evaluate the given integral without paying attention to the discontinuity of the integrand at the point $x = 0$, the result would have been wrong :

$$\int_{-1}^{1}\frac{dx}{x^2}=-\frac{1}{x}\Big|_{-1}^{1}=-\left(\frac{1}{1}-\frac{1}{-1}\right)=-2,$$

which is impossible.

Example 16. Evaluate $\int_0^4\frac{dx}{(x-1)^2}$

Solution The integrand is discontinuous at $x = 1$, which is inside (0, 4).

$$\lim_{u\to 1^-}\int_0^4\frac{dx}{(x-1)^2}=\lim_{u\to 1^-}-\frac{1}{x-1}\Big]_0^u$$

$$=\lim_{u\to 1^-}-\left(\frac{1}{u-1}-(-1)\right)=\lim_{u\to 1^-}-\left(\frac{1}{u-1}+1\right)=\infty$$

Hence, $\int_0^4\frac{dx}{(x-1)^2}$ is divergent.

We do not have to consider $\lim_{u\to 1^+}\int_u^4\frac{dx}{(x-1)^2}$ at all.

For $\int_0^4\frac{dx}{(x-1)^2}$ to be convergent, both

$\lim_{u\to 1^-}\int_0^u\frac{dx}{(x-1)^2}$ and $\lim_{u\to 1^+}\int_u^4\frac{dx}{(x-1)^2}$ must exist.

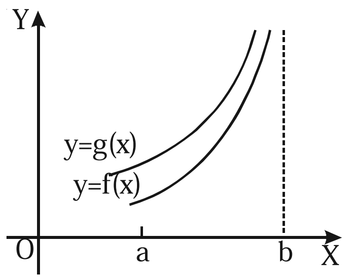
Note:

1. If the function f(x), defined on the interval [a, b], has within this interval, a finite number of points of discontinuity $a_1, a_2, \ldots, a_n$, then the integral of the functionf(x) on the interval [a, b] is defined as follows:

$$\int_a^b f(x)\,dx=\int_a^{a_1}f(x)dx+\int_{a_1}^{a_2}f(x)dx+\ldots+\int_{a_n}^{b}f(x)dx$$

if each of the improper integrals on the right side of the equation converges. But if even one of these integrals diverges, then $\int_a^b f(x)dx$ too is called divergent.

2. Let $0 \le f(x) \le g(x)$ for $a \le x < b$. Assume that $\lim_{x\to b^-} f(x) = \infty$ and $\lim_{x\to b^-} g(x) = \infty$. (See figure). It is not hard to show that, if $\int_a^b g(x)dx$ converges, then so does $\int_a^b f(x)dx$ and, equivalently, if $\int_a^b f(x)dx$ does not converge, then neither does $\int_a^b g(x)dx$. A similar result also holds for $a < x \le b$, with $\lim_{x\to a^+}$ replacing $\lim_{x\to b^-}$.



As an example, consider $\int_0^1 \frac{dx}{1-x^4}$. For $0 \le x < 1$, $1 - x^4 = (1 - x)(1 + x)(1 + x^2) < 4(1 - x)$ and $\frac{1}{4}\frac{1}{1-x} < \frac{1}{1-x^4}$.

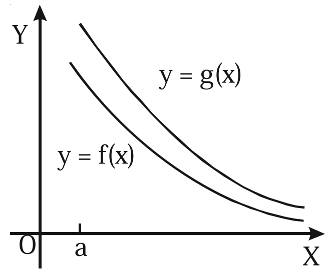
Since $\frac{1}{4}\int_0^1 \frac{dx}{1-x}$ does not converge, neither does $\int_0^1 \frac{dx}{1-x^4}$ converge.

Now consider $\int_0^1 \frac{dx}{x^2+\sqrt{x}}$.

For $0 < x \le 1$, $\frac{1}{x^2+\sqrt{x}} < \frac{1}{\sqrt{x}}$.

Since $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, so does $\int_0^1 \frac{dx}{x^2+\sqrt{x}}$.

3. Assume that $0 \le f(x) \le g(x)$ for $x \ge a$. Assume also that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$. (See figure). It is not hard to show that, if $\int_a^\infty g(x)\,dx$ coverges, so does $\int_a^\infty f(x)dx$ (and, equivalently, that, if $\int_a^\infty f(x)dx$ does not converge, then neither does $\int_a^\infty g(x)dx$.



As an example, consider $\int_1^\infty \frac{dx}{\sqrt{x^4+2x+6}}$.

For $x \ge 1$, $\frac{1}{\sqrt{x^4+2x+6}} < \frac{1}{x^2}$.

Since $\int_1^\infty \frac{dx}{x^2}$ converges, so does $\int_1^\infty \frac{dx}{\sqrt{x^4+2x+6}}$.

Example 17. Classify each of the following integrals as proper or improper. If improper, determine whether convergent or divergent, and, if convergent, evaluate it.

(a) $\int_a^1 \frac{1}{\sqrt{x}}dx$ (b) $\int_a^1 \frac{1}{x^2}dx$

(c) $\int_a^1 \sin\frac{1}{x}dx$ (d) $\int_{-1}^1 \frac{1}{x}dx$

Solution

(a) Since $\frac{1}{\sqrt{x}}$ takes on arbitrarily large values near 0, we know that $\int_0^1 \frac{1}{\sqrt{x}}dx$ is not a proper integral. For every t in [0, 1],

$$\int_t^1 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_t^1 = 2(1 - \sqrt{t}).$$

Since $\lim_{t\to 0^+} 2(1 - \sqrt{t})$ exists, we get

$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t\to 0^+} 2(1-\sqrt{t}) = 2$.

Hence, (a) is a convergent improper integral with value 2.

(b) The values of $\frac{1}{x^2}$ also increase without bound as x approaches zero, and (b) is therefore not a proper integral. For every t in (0, 1],

$$\int_t^1 \frac{1}{x^2} dx = -\frac{1}{x}\bigg|_t^1 = \frac{1}{t} - 1.$$

However, $\lim_{t\to 0^+} \int_t^1 \frac{1}{x^2} dx = \lim_{t\to 0^+} \frac{1}{t} - 1 = \infty$,

and, since the limit does not exist, the improper integral is divergent.

(c) Since $\left|\sin\frac{1}{x}\right| \le 1$ for all nonzero x, the function f defined by $f(x) = \sin\frac{1}{x}$ is bounded on [0, 1]. It is also continuous at every point of that interval. We now assign a value, say 0, to f(0), and it follows that f is integrable over [0,1], and hence, $\int_0^1 \sin\frac{1}{x} dx$ is a proper integral.

(d) $\int_{-1}^1 \frac{1}{x} dx$ is an improper integral.

It follows from the definition that

$$\int_{-1}^1 \frac{1}{x} dx = \int_{-1}^0 \frac{1}{x} dx + \int_0^1 \frac{1}{x} dx,$$

and that both integrals on the right should be convergent.

However,

$$\int_0^1 \frac{1}{x} dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x} dx = \lim_{t\to 0^+} (\ln 1 - \ln t) = \infty$$

and similarly $\int_{-1}^0 \frac{1}{x} dx$ is divergent.

We conclude that $\int_{-1}^1 \frac{1}{x} dx$ is divergent.

CAUTION

Failure to note the discontinuity of the function $\frac{1}{x}$ at 0 can result in the following error :

$$\int_{-1}^1 \frac{1}{x} dx = \ln|x|\Big|_{-1}^1 = 0 - 0 = 0.$$

Example 18. Find whether $\int_0^3 \frac{1}{(x-1)(x-3)} dx$ is convergent.

Solution Let $\int_0^3 \frac{1}{(x-1)(x-3)} dx$

$$= \int_0^1 \frac{1}{(x-1)(x-3)} dx + \int_1^3 \frac{1}{(x-1)(x-3)} dx$$

and both integrals on the right should be convergent. However, it easy to show that neither is convergent. A partial fractions decomposition yields

$$\frac{1}{(x-1)(x-3)} = -\frac{1}{2}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x-3}, \text{ and so}$$

$$\int \frac{1}{(x-1)(x-3)} dx = -\frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x-3| + c$$

$$= \frac{1}{2}\ln\left|\frac{x-3}{x-1}\right| + c.$$

In particular, therefore,

$$\int_0^1 \frac{1}{(x-1)(x-3)} dx = \lim_{t\to 1^-} \int_0^t \frac{1}{(x-1)(x-3)} dx$$

$$= \left(\lim_{t\to 1^-} \frac{1}{2}\ln\left|\frac{t-3}{t-1}\right|\right) - \frac{1}{2}\ln 3 = \infty,$$

which is sufficient to establish that the integral is divergent.

Example 19. If the value of the definite integral

$I = \int_1^\infty \frac{(2x^3-1)dx}{x^6+2x^3+9x^2+1}$ can be expressed in the form $\frac{A}{B}\cot^{-1}\frac{C}{D}$ where $\frac{A}{B}$ and $\frac{C}{D}$ are rationals in their lowest form, find the value of $(A + B^2 + C^3 + D^4)$.

Solution $I = \int_1^\infty \frac{(2x - x^{-2})dx}{x^4+2x+9+x^{-2}}$

(dividing N^r and D^r by x^2)

$$I = \int_1^\infty \frac{(2x-x^{-2})dx}{\left(x^2+\frac{1}{x}\right)^2+9}$$

Put $x^2 + \frac{1}{x} = t \Rightarrow \left(2x - \frac{1}{x^2}\right)dx = dt$

As $x \to 1$, $t \to 2$ and as $x \to \infty$, $t \to \infty$

$$\therefore \quad I = \int_2^\infty \frac{dt}{t^2+9} dx = \frac{1}{3}\tan^{-1}\frac{t}{3}\Big]_2^\infty$$

$$= \frac{1}{3}\left[\frac{\pi}{2} - \tan^{-1}\frac{2}{3}\right] = \frac{1}{3}\cot^{-1}\frac{2}{3}$$

Hence, $A = 1, B = 3, C = 2, D = 3$

$\therefore \quad (A + B^2 + C^3 + D^4) = 1 + 9 + 8 + 81 = 99.$

Concept Problems F

1. Which of following integrals are improper ? Why?

(a) $\int_1^2 \frac{1}{2x-1}dx$ (b) $\int_0^1 \frac{1}{2x-1}dx$

(c) $\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}dx$ (d) $\int_1^2 \ln(x-1)dx$

2. Explain why each of the following integrals is improper.

(a) $\int_1^{\infty} x^4 e^{-x^4} dx$ (b) $\int_0^{\pi/2} \sec x dx$

(c) $\int_0^2 \frac{x}{x^2-5x+6}dx$ (d) $\int_{-\infty}^0 \frac{1}{x^2+5}dx$

3. Find the error in the following steps:

$$\int_{-1}^1 \frac{1}{x^2}dx = \frac{-1}{x}\Big|_{-1}^1 = \frac{-1}{1} - \frac{-1}{-1} = -2.$$

4. Compute

(a) $\lim_{t\to 0+}\int_t^1 \frac{1}{x}dx$ (b) $\lim_{t\to 1-}\int_0^t \tan\frac{\pi}{2}x\, dx$.

How does the result give insight into the fact that neither integrand is integrable over the interval [0, 1] ?

5. Sketch the region whose area is $\int_0^{\infty}\frac{dx}{1+x^2}$, and use your sketch to show that

$$\int_0^{\infty}\frac{dx}{1+x^2} = \int_0^1 \sqrt{\frac{1-y}{y}}dy$$

6. Show that $\int_0^{\infty} x^2 e^{-x^2}dx = \frac{1}{2}\int_0^{\infty} e^{-x^2}dx$

7. Evaluate the following integrals :

(i) $\int_1^{\infty}\frac{dx}{x^2(x+1)}$ (ii) $\int_0^{\infty} x^3 e^{-x^2}dx$

(iii) $\int_0^{\frac{1}{e}}\frac{dx}{x\ln^2 x}$ (iv) $\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2}$

8. Evaluate the following integrals :

(i) $\int_{-1}^0 \frac{e^{\frac{1}{x}}}{x^3}dx$ (ii) $\int_{-\infty}^{\infty}\frac{1}{e^x+e^{-x}}dx$

(iii) $\int_3^5 \frac{x^2\, dx}{\sqrt{(x-3)(5-x)}}$ (iv) $\int_{-1}^1 \frac{dx}{(2-x)\sqrt{1-x^2}}$

9. Here is an argument that ln 3 equals $\infty - \infty$. Where does the argument go wrong ? Give reasons for your answer.

$$\ln 3 = \ln 1 - \ln\frac{1}{3}$$

$$= \lim_{b\to\infty}\ln\left(\frac{b-2}{b}\right) - \ln\frac{1}{3} \quad ...(1)$$

$$= \lim_{b\to\infty}\left[\ln\frac{x-2}{x}\right]_3^b \quad ...(2)$$

$$= \lim_{b\to\infty}[\ln(x-2) - \ln x]_3^b \quad ...(3)$$

$$= \lim_{b\to\infty}\int_3^b\left(\frac{1}{x-2} - \frac{1}{x}\right)dx \quad ...(4)$$

$$= \int_3^{\infty}\left(\frac{1}{x-2} - \frac{1}{x}\right)dx \quad ...(5)$$

$$= \int_3^{\infty}\frac{1}{x-2}dx - \int_3^{\infty}\frac{1}{x}dx \quad ...(6)$$

$$= \lim_{b\to\infty}[\ln(x-2)]_3^b - \lim_{b\to\infty}[\ln x]_3^b \quad ...(7)$$

$$= \infty - \infty.$$

10. Show that $\int_0^{\infty}\sin\theta\, d\theta$ and $\int_0^{\infty}\cos\theta\, d\theta$ are indeterminate.

11. $\int_{-\infty}^{\infty} f(x)dx$ may not equal $\lim_{b\to\infty}\int_{-b}^b f(x)dx$.

Show that $\int_0^{\infty}\frac{2xdx}{x^2+1}$ diverges and hence that $\int_{-\infty}^{\infty}\frac{2xdx}{x^2+1}$ diverges. Then show that $\lim_{b\to\infty}\int_{-b}^b \frac{2xdx}{x^2+1} = 0$.

12. (a) Show that if f is even and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$$

(b) Show that if f is odd and the necessary integrals exist, then $\int_{-\infty}^{\infty} f(x)dx = 0$

13. For each $x > 0$. let $G(x) = \int_0^{\infty} e^{-xt}\,dt$. Prove that $xG(x) = 1$ for each $x > 0$.

Practice Problems F

14. Given that $\int_0^1 \frac{\ln x}{(1+x)\sqrt{x}}dx$ is a convergent improper integral, prove that $\int_0^{\infty} \frac{\ln x\,dx}{(1+x)\sqrt{x}} = 0$.

15. Given that

$\int_0^{\pi/2} \ln\tan\theta d\theta, \int_0^{\pi/2} \sin^2\theta \ln\tan\theta\, d\theta$

are convergent improper integrals, prove that their values are 0, $\frac{\pi}{4}$ respectively.

16. Prove the following :

(i) $\int_0^4 \frac{dx}{(4-x)^{2/3}} = 3.4^{1/3}$

(ii) $\int_0^4 \frac{dx}{(x-2)^{2/3}} = 6\sqrt[3]{2}$

(iii) $\int_0^{\infty} \frac{dx}{a^2e^x + b^2e^{-x}} = \frac{1}{ab}\tan^{-1}\frac{b}{a}$

(iv) $\int_{1/2}^{1} \frac{dx}{x^4\sqrt{1-x^2}} = 2\sqrt{3}$

17. Evaluate the integrals

(i) $\int_0^b \frac{xdx}{(1+x)^3}$ (ii) $\int_0^b \frac{x^2dx}{(1+x)^4}$

and show that they converge to finite limits as $b \to \infty$.

18. Evaluate $\int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{\pi}{2a(a+b)}$, $a, b > 0$.

19. Prove that

$$\int_0^{2\lambda} \frac{\sin x}{x}dx = \int_0^{\lambda} \frac{\sin 2y}{y}dy = \frac{\sin^2\lambda}{\lambda} + \int_0^{\lambda} \frac{\sin^2 x}{x^2}dx.$$

Deduce that $\int_0^{\infty} \frac{\sin x}{x}dx = \int_0^{\infty} \frac{\sin^2 x}{x^2}dx$

(It may be assumed that the integrals are convergent)

20. Prove that, as $n \to \infty$, $\int_0^1 \cos nx \tan^{-1} x\,dx \to 0$.

21. Show that

(i) $\int_0^{\infty} \frac{x}{(1+x)^2}dx = \frac{1}{2}\int_0^{\infty} \frac{dx}{(1+x)^2} = \frac{1}{2}$.

(ii) $\int_0^{\infty} \frac{dx}{1+x^4} = \int_0^{\infty} \frac{x^2\,dx}{1+x^4} = -\frac{1}{25}$

22. Find $\int_0^2 f(x)dx$, where

$$f(x) = \begin{cases} \dfrac{1}{\sqrt[4]{x^3}} & \text{for } 0 \le x \le 1 \\ \dfrac{1}{\sqrt[4]{(x-1)^3}} & \text{for } 1 < x < 2 \end{cases}.$$

23. Prove that $\int_a^b \frac{dx}{\sqrt{\{(x-a)(b-x)\}}} = \pi$,

$\int_a^b \frac{xdx}{\sqrt{\{(x-a)(b-x)\}}} = \frac{1}{2}\pi(a+b)$,

(i) by means of the substitution $x = a + (b-a)\,t^2$,
(ii) by means of the substitution $(b-x)/(x-a) = t$, and
(iii) by means of the substitution $x = a\cos^2 t + b\sin^2 t$.

24. Prove that

(i) $\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}} = \frac{1}{\sqrt{3}}$.

(ii) $\int_0^1 \frac{dx}{(1+x)(2+x)\sqrt{x(1-x)}} = \pi\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right)$.

25. It can be proved that $\int_0^{\infty} \frac{x^{n-1}}{1+x}dx = \pi \operatorname{cosec} n\pi$ for $0 < n < 1$. Verify that this equation is correct for $n = 1/2$.

26. If a and b are positive, then prove that

(i) $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$,

(ii) $\displaystyle\int_0^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}.$

(iii) $\displaystyle\int_0^{\infty} \left(1+\frac{a^2}{x^2}\right) dx = \pi a.$

27. Prove that

(i) $\displaystyle\int_1^{\infty} \frac{dx}{\left(x+\sqrt{x^2+1}\right)^n} = \frac{n}{n^2-1}, \quad n>1$

(ii) $\displaystyle\int_1^{\infty} \frac{dx}{(1+e^x)(1+e^{-x})} = 1$

(iii) $\displaystyle\int_0^{\infty} \frac{x\ln x}{(1+x^2)^2} dx = 0$

(iv) $\displaystyle\int_0^{\infty} \frac{\sqrt{x}}{(1+x)^2} dx = \frac{1}{2} + \frac{1}{4}\pi.$

2.8 SUBSTITUTION IN DEFINITE INTEGRALS

When evaluating a definite integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Newton Leibnitz Formula.

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

While integrating an indefinite integral by the substitution of a new variable, it is sometimes rather troublesome to transform the result back into the original variable. In all such cases, while integrating the corrsponding integral between limits (i.e., corresonding definite integral), we can avoid the tedious process of restoring the original variable, by changing the limits of the definite integral to correspond with the change in the variable.

Substitution Theorem

If g' is continuous on [a, b] and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof : Let F be an antiderivative of f. Then, $F(g(x))$ is an antiderivative of $f(g(x))\, g'(x)$, so by the Second Fundamental Theorem, we have

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

But applying FTC2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)\, du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).$$

Therefore in a definite integral the substitution should be effected in three places (i) in the integrand, (ii) in the differential and (iii) in the limits.

Example 1. Evaluate $\displaystyle\int_0^4 \sqrt{2x+1}\, dx$

Solution Using the substitution, we have $u = 2x+1$ and $dx = du/2$. To find the new limits of integration we note that when $x = 0$, $u = 1$ and when $x = 4$, $u = 9$

Therefore, $\displaystyle\int_0^4 \sqrt{2x+1}\, dx = \int_1^9 \frac{1}{2}\sqrt{u}\, du$

$$= \frac{1}{2}\cdot\frac{2}{3} u^{3/2}\Big]_1^9 = \frac{1}{3}(9^{3/2} - 1^{3/2}) = \frac{26}{3}.$$

Example 2. Calculate $\displaystyle\int_1^e \frac{\ln x}{x} dx$

Solution Let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x} dx = \int_0^1 u\,du = \frac{u^2}{2}\Bigg]_0^1 = \frac{1}{2}.$$

Note: When computing the definite integral from this formula we do not return to the old variable.

Example 3. Let $f(0)=0$ and $\displaystyle\int_0^2 f'(2t)\, e^{f(2t)} dt = 5$, then find the value of $f(4)$.

Solution We have $\displaystyle\int_0^2 f'(2t)\, e^{f(2t)} dt = 5$

Put $e^{f(2t)} = y$

$\Rightarrow 2f'(2t)\, e^{f(2t)}\, dt = dy$

Now $\displaystyle\frac{1}{2}\int_{e^{f(0)}}^{e^{f(4)}} dy = 5 \quad \Rightarrow \int_{e^{f(0)}}^{e^{f(4)}} dy = 10$

$\Rightarrow e^{f(4)} - e^{f(0)} = 10 \quad \Rightarrow e^{f(4)} = 10 + e^0 = 11$

Hence, $f(4) = \ln 11$.

CAUTION

If we take $f(x) = \sqrt{x}$, then

$$\int_0^1 \sqrt{x}\, dx = \int_0^{\pi/2} \sqrt{\sin u}\ \cos u\, du \text{ is true.}$$

But $\int_0^1 \sqrt{x}\,dx = \int_0^{9\pi/2} \sqrt{(\sin u)}\ \cos u\,du$

is absurd since is $\sqrt{x}$ is not defined for negative x. On the other hand both the statements.

$\int_0^1 x^2\,dx = \int_0^{\pi/2} \sin^2 u \cos u\,du$ and

$\int_0^1 x^2\,dx = \int_\pi^{9\pi/2} \sin^2 u \cos u\,du$ are valid.

Example 4. Compute $\int_{-1}^{2} x \sin x^2 dx$

Solution We put $y = x^2$. Then

$$\int_{-1}^{2} x\sin x^2 dx = \frac{1}{2}\int_1^4 \sin y dy = \frac{1}{2}(-\cos y)\Big|_1^4$$

$$= \frac{1}{2}(\cos 1 - \cos 4)$$

Note that here we are not allowed to put $x = \sqrt{t}$, since $\sqrt{t} \geq 0$, and in the given integral x attains negative values as well.

Note: The derivative of a differentiable function need not be integrable. Examples to illustrate this are not however very easy to construct. if we assume that $g'(t)$ is continuous (the usual assumption), the integral

$\int_a^b f(g(x))\, g'(t)dt$ certainly exists.

Example 5. Evaluate $\int_0^1 \frac{\sin^{-1} x}{\sqrt{(1-x^2)}}dx$

Solution Put $\sin^{-1}x = \theta \Rightarrow d\theta = \frac{1}{\sqrt{(1-x^2)}}dx$

0 and 1 are the limits of x ; the corresponding limits of θ where $\theta = \sin^{-1}x$ are found as follows :

When $x = 0, \theta = \sin^{-1} 0 = 0$

When $x = 1, \theta = \sin^{-1} 1 = \frac{\pi}{2}$.

$$\therefore\quad I = \int_0^{\pi/2} \theta d\theta = \left[\frac{1}{2}\theta^2\right]_0^{\pi/2} = \frac{1}{8}\pi^2.$$

Note: Of course this example can be worked out by first finding the indefinite integral in terms of x and then substituting the limits.

Example 6. Compute the definite integral

$$\int_0^2 \frac{x^2dx}{x^3-17}.$$

Solution Let $u = x^3 - 17$.

Then $du = 3x^2dx$, or, equivalently, $x^2\,dx = \frac{du}{3}$.

$$\text{So, } \int \frac{x^2dx}{x^3-17} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + c$$

$$= \frac{1}{3}\ln|x^3 - 17| + c.$$

$$\text{Finally, } \int_0^2 \frac{x^2dx}{x^3-17} = \frac{1}{3}\ln|x^3-17|\Big|_0^2$$

$$= \frac{1}{3}\ln|8-17| - \frac{1}{3}\ln|-17|$$

$$= \frac{1}{3}\ln 9 - \frac{1}{3}\ln 17 = \frac{1}{3}\ln\frac{9}{17}.$$

Example 7. Compute $\int_0^{\pi/2} \frac{dx}{1+\cos x}$

Solution To find the indefinite integral of the function $f(x) = \frac{1}{1+\cos x}$ let us take advantage of the substitution $\tan\frac{x}{2} = t$. To be more precise, let us change the variable of integration in the following manner:

$x = 2\tan^{-1}t$.

From trigonometric formulae it follows that

$$\frac{1}{1+\cos x} = \frac{1}{1+\frac{1-t^2}{1+t^2}} = \frac{1+t^2}{1+t^2+1-t^2} = \frac{1+t^2}{2}$$

We then compute dx :

$$dx = \frac{2}{1+t^2}dt$$

Finally, since $\tan\frac{0}{2} = 0$, $\tan\frac{\pi}{2.2} = \tan\frac{\pi}{4} = 1$, we have to take 0 and 1 as the limits of integration in the new integral. Thus,

$$\int_0^{\pi/2} \frac{dx}{1+\cos x} = \int_0^1 \frac{1+t^2}{2}\,\frac{2}{1+t^2}dt = \int_0^1 dt = 1.$$

Example 8. Show that

$$\int_0^{1/2} \frac{dx}{(1-2x^2)\sqrt{1-x^2}} = \frac{1}{2}\ln(2+\sqrt{3}).$$

Solution Put $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$

Also when $x = 0$, $\theta = 0$, and

when $x = \frac{1}{2}, \theta = \frac{\pi}{6}$.

$$\therefore \quad I = \int_0^{\pi/6} \frac{\cos\theta\, d\theta}{\cos 2\theta \cos\theta} = \int_0^{\pi/6} \sec 2\theta\, d\theta$$

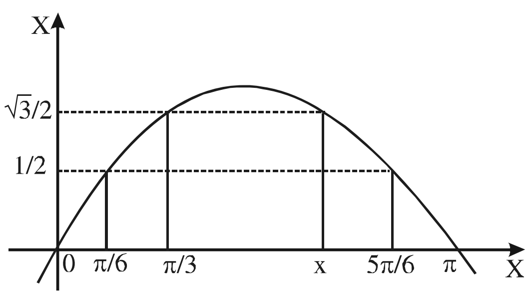
$$= \left[\frac{1}{2}\ln\tan\left(\frac{1}{4}\pi + \theta\right)\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\ln\tan\frac{5}{12}\pi - \ln\tan\frac{1}{4}\pi\right] = \frac{1}{2}\ln(2+\sqrt{3}).$$

Example 9. Compute the integral $\int_{1/2}^{\sqrt{3}/2} \frac{dx}{x\sqrt{1-x^2}}$

Solution Apply the substitution $x = \sin t$ (the given function is not monotonic), $dx = \cos t\, dt$. The new limits of integration t_1 and t_2 are found from the equations $\frac{1}{2} = \sin t$; $\frac{\sqrt{3}}{2} = \sin t$. We may put $t_1 = \frac{\pi}{6}$ and $t_2 = \frac{\pi}{3}$, but other values may also be chosen, for instance, $t_1 = \frac{5\pi}{6}$ and $t_2 = \frac{2\pi}{3}$.

In both cases, the variable $x = \sin t$ runs throughout the entire interval $\left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$.



Let us show that the results of the two integrations will coincide. Indeed,

$$\int_{1/2}^{\sqrt{3}/2} \frac{dx}{x\sqrt{1-x^2}} = \int_{\pi/6}^{\pi/3} \frac{\cot dt}{\sin t \cos t} = \int_{\pi/6}^{\pi/3} \frac{dt}{\sin t} = \ell n\left|\tan\frac{t}{2}\right|_{\pi/6}^{\pi/3}$$

$$= \ell n \tan\frac{\pi}{6} - \ell n \tan\frac{\pi}{12} = \ell n \frac{2+\sqrt{3}}{\sqrt{3}}.$$

On the other hand, taking into consideration that cost is negative on the interval $\left[\frac{2\pi}{3}, \frac{5\pi}{6}\right]$, we obtain

$$\int_{1/2}^{\sqrt{3}/2} \frac{dx}{x\sqrt{1-x^2}} = \int_{5\pi/6}^{2\pi/3} \frac{\cos t dt}{\sin t(-\cos t)} = \int_{2\pi/3}^{5\pi/6} \frac{dt}{\sin t}$$

$$= \ell n\left|\tan\frac{t}{2}\right|_{2\pi/3}^{5\pi/6} = \ell n \frac{\tan\frac{5}{12}\pi}{\tan\frac{\pi}{3}} = \ell n \frac{2+\sqrt{3}}{\sqrt{3}}.$$

Note: If we take $t_1 = \frac{5\pi}{6}$, $t_2 = \frac{\pi}{3}$, we need to split the interval of integration at $t = \frac{\pi}{2}$, since $\sqrt{1-x^2} = |\cos t|$ and cos t changes sign at $t = \frac{\pi}{2}$.

Example 10. Evaluate $\int_a^b \frac{dx}{x\sqrt{\{(x-a)(b-x)\}}}$

Solution Put $x = a\cos^2\theta + b\sin^2\theta$

$dx = 2(b-a)\sin\theta\cos\theta$

$$\therefore \quad I = \int_0^{\pi/2} \frac{2(b-a)\sin\theta\cos\theta d\theta}{(a\cos^2\theta + b\sin^2\theta)(b-a)\sin\theta\cos\theta}$$

$$= 2\int_0^{\pi/2} \frac{\sec^2\theta}{a + b\tan^2\theta}$$

$$= \frac{2}{\sqrt{b}}\int_0^{\pi/2} \frac{\sqrt{b}.\sec^2\theta d\theta}{(\sqrt{a})^2 + (\sqrt{b}\tan\theta)^2}$$

$$= \frac{2}{\sqrt{b}}.\frac{1}{\sqrt{a}}\left[\tan^{-1}\frac{\sqrt{b}\tan\theta}{\sqrt{a}}\right]_0^{\pi/2}$$

$$= \frac{2}{\sqrt{(ab)}}.\frac{\pi}{2} = \frac{\pi}{\sqrt{(ab)}}.$$

Example 11. Find the value of the definite integral

$$\int_e^{e^{2010}} \frac{1}{x}\left(1 + \frac{1 - \ln x}{\ln x \ln\left(\frac{x}{\ln x}\right)}\right) dx.$$

Solution Substituting $\ell n\, x = t$, we get

$$\int_1^{2010}\left(1 + \frac{1-t}{t(t-(\ln t))}\right) dt$$

$$= 2009 + \int_1^{2010 - \ln 2010} \frac{-1}{u} du \quad (u = t - \ell n\, t)$$

Example 12. Evaluate $\int_0^{\frac{\sqrt{2}-1}{2}} \frac{dx}{(2x+1)\sqrt{x^2+x}}$.

Solution Let $x^2 + x = t^2 \Rightarrow (2x+1)\,dx = 2t\,dt$

$$\Rightarrow \quad dx = \frac{2t\,dt}{(2x+1)}$$

When $x = \frac{\sqrt{2}-1}{2}$ then

$$t^2 = \frac{\sqrt{2}-1}{2}\left(\frac{\sqrt{2}-1}{2}+1\right)$$

$$= \frac{(\sqrt{2}-1)(\sqrt{2}+1)}{4} = \frac{1}{4} \Rightarrow t = \frac{1}{2}.$$

When $x = 0$ then $t = 0$

$$\text{Hence, } I = \int_0^{1/2} \frac{2t\,dt}{(2x+1)^2\,t} = \int_0^{1/2} \frac{2\,dt}{4x^2+4x+1}$$

$$= 2\int_0^{1/2} \frac{dt}{4(t^2)+1} = \frac{2}{4}\int_0^{1/2} \frac{dt}{t^2+(1/4)}$$

$$= \frac{1}{2}\cdot 2\left[\tan^{-1} 2t\right]_0^{1/2} = \tan^{-1} 1 = \frac{\pi}{4}.$$

Alternative:

Put $2x + 1 = \frac{1}{t}$

$$\Rightarrow \quad dx = -\frac{1}{2t^2}\,dt$$

If $x = 0$ then $t = 1$

If $x = \frac{1}{\sqrt{2}} - \frac{1}{2}$ then $\sqrt{2}-1+1 = \frac{1}{t}$

$$\Rightarrow \quad t = \frac{1}{\sqrt{2}}$$

Hence, the integral becomes

$$I = -\int_1^{1/\sqrt{2}} \frac{2t}{\sqrt{1-t^2}}\cdot\frac{t}{2t^2}\,dt = -\left[\sin t\right]_1^{1/\sqrt{2}} = \frac{\pi}{4}.$$

Example 13. Find $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$.

Solution Let $x = \frac{3}{2}\tan\theta$, which gives $dx = \frac{3}{2}\sec^2\theta\,d\theta$

and $\sqrt{4x^2+9} = \sqrt{9\tan^2\theta+9} = 3\sec\theta$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$;

when $x = 3\sqrt{3}/2$, $\tan\theta = \sqrt{3}$, so θ

$$= \pi\ 3.\ \int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$$

$$= \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\,\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\sec\theta}\sin\theta\,d\theta$$

Now we substitute $u = \cos\theta$ so that $du = -\sin\theta\,d\theta$.

When $\theta = 0$, $u = 1$; when $\theta = \pi/3$, $u = \frac{1}{2}$.

$$\text{Therefore, } \int_1^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\,du$$

$$\frac{3}{16}\int_1^{1/2} (1-u^2)\,du = \frac{3}{16}\left[u+\frac{1}{u}\right]_1^{1/2}$$

$$= \frac{3}{16}\left[\left(\frac{1}{2}+2\right)-(1+1)\right] = \frac{3}{32}.$$

Quadratic substitutions

A substitution of the type

$$y = x^2 - 2x + 10 \qquad \text{...(1)}$$

fails at the outset to give x uniquely in terms of y and must be replaced by one or other of

$$x = 1 + \sqrt{y-9},\ x = 1 - \sqrt{y-9}$$

at least for the purpose of substituting for x in terms of y. For $x \le 1$ we use the second form; for $x \le 1$ we use the first. In order to use (1) in an integral

$$\int_{-2}^{2} f(x)\,dx$$

we consider the integral in the form

$$\int_{-3}^{1} f(x)\,dx + \int_1^2 f(x)\,dx$$

and use in each integral the substitution appropriate to its range of values of x.

Suppose for example that $I = \int_0^7 (x^2 - 6x + 13)\,dx$.

We find by direct integration that $I = 48$.

Now let us apply the substitution $y = x^2 - 6x + 13$, which gives $x = 3 \pm \sqrt{(y-4)}$.

Since y = 8 when x = 1 and y = 20 when x = 7, we appear to be led to the result

$$I=\int_8^{20} y\frac{dx}{dy}dy=\pm\frac{1}{2}\int_8^{20}\frac{ydy}{\sqrt{y-4}}.$$

The indefinite integral is $(1/3)(y-4)^{3/2}+4(y-4)^{1/2}$,

and so we obtain the value $\pm\frac{80}{3}$, which is wrong, whichever sign we choose. The explanation is to be found in a closer consideration of the relation between x and y. The function $x^2-6x+13$ has a minimum for $x=3$, when $y=4$. As x increases from 1 to 3, y decreases from 8 to 4, and dx/dy is negative, so that

$$\frac{dx}{dy}=-\frac{1}{2\sqrt{y-4}}.$$

As x increases from 3 to 7, y increases from 4 to 20, and the other sign must be chosen. Thus

$$I=\int_1^7 ydx=\int_8^4-\frac{y}{2\sqrt{y-4}}dy+\int_4^{20}\frac{y}{2\sqrt{y-4}}dy,$$

a formula which will be found to lead to the correct result. Similarly, if we transform the integral

$\int_0^{\pi}dx=\pi$ by the substitution $x=\sin^{-1}y$, we must observe that dx/dy is $(1-y^2)^{-1/2}$ or $-(1-y^2)^{-1/2}$ according as $0\le x<\pi/2$ or $\pi/2<x\le\pi$.

Example 14. Transform the integral $\int_0^3 (x-2)^2dx$ by the substitution $(x-2)^2=t$.

Solution A formal application of the substitution throughout the interval [0, 3] would lead to wrong result, since the function x has two branches :

$x_1=2-\sqrt{t}$; $x_2=2+\sqrt{t}$. The former branch cannot attain values $x>2$, the latter values $x<2$.

To obtain a correct result we have to break up the given integral in the following way :

$$\int_0^3(x-2)^2dx=\int_0^2(x-2)^2dx+\int_2^3(x-2)^2dx$$

and to put $x=2-\sqrt{t}$ in the first integral, and

$x=2+\sqrt{t}$ in the second. Then we get

$$I_1=\int_0^2(x-2)^2dx=-\int_4^0 t\frac{dt}{2\sqrt{t}}=\frac{1}{2}\int_0^4\sqrt{t}dt=\frac{8}{3}$$

$$I_2=\int_2^3(x-2)^2dx=\int_0^1 t\frac{dt}{2\sqrt{t}}=\frac{1}{2}\int_0^1\sqrt{t}dt\frac{1}{3}$$

Hence, $I=\frac{8}{3}+\frac{1}{3}=3$ which is a correct result. It can be easily verified by directly computing the initial integral :

$$\int_0^3(x-2)^2dx=\left.\frac{(x-2)^3}{3}\right|_0^3=\frac{1}{3}+\frac{8}{3}=3.$$

Discontinuous substitutions

The example that follows shows that a formal application of the formula for a change of variable (without due account of the conditions for its applicability) may lead to an incorrect result.

Example 15. Since the integrand of the integral

$I=\int_{-1}^{1}\frac{1}{1+x^2}dx$ is positive, it follows that $I>0$.

However, if we make the change of variable $x=1/u$,

then $I=\int_{-1}^{1}\frac{1}{1+x^2}dx=-\int_{-1}^{1}\frac{1}{1+u^2}du=-I$

This implies $I=0$. Explain.

Solution The graph of $u=1/x$ has a discontinuity at $x=0$. Thus the indicated change of variable does not satisfy the requirements of the Substitution Theorem. To make the change of variables properly, we first break up the interval of integration, i.e.,

$$I=\int_{-1}^{0}\frac{1}{1+x^2}dx+\int_0^1\frac{1}{1+x^2}dx.$$

Now if we let $x=1/u$, we obtain

$$I=\int_{-1}^{-\infty}\frac{1}{1+u^2}du+\int_{\infty}^{1}\frac{1}{1+u^2}du$$

The integrals $\int_{-1}^{-\infty}\frac{1}{1+u^2}du$ and $\int_{\infty}^{1}\frac{1}{1+u^2}du$

can be evaluated to get the correct result.

However, by putting $x=\tan\theta$, in I

we get $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}d\theta=\frac{\pi}{2}$ easily.

Note: While applying Newton-Leibnitz Formula students are advised to check continuity of antiderivatives before putting the limits of integration. A discontinuous function used as an antiderivative may lead to wrong result.

Example 16. Find a mistake in the following evaluation of the integral :

$$\int_0^{\pi}\frac{dx}{1+2\sin^2 x}=\int_0^{\pi}\frac{dx}{\cos^2 x+3\sin^2 x}$$

$$=\int_0^{\pi}\frac{\sec^2 x dx}{1+3\tan^2 x}=\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan x)\Big|_0^{\pi}=0.$$

(The integral of a function positive everywhere turns out to be zero).

Solution The Newton-Leibnitz formula is not applicable here, since the antiderivative

$F(x)=\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan x)$ has a discontinuity at the point $x=\frac{\pi}{2}$ because :

$$\lim_{x\to\frac{\pi}{2}^-}F(x)=\lim_{x\to\frac{\pi}{2}^-}\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan x)$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}(\infty)=\frac{\pi}{2\sqrt{3}} \text{ and,}$$

$$\lim_{x\to\frac{\pi}{2}^+}F(x)=\lim_{x\to\frac{\pi}{2}^+}\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan x)$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}(-\infty)=-\frac{\pi}{2\sqrt{3}}$$

The correct result can be obtained in the following way:

$$\int_0^{\pi}\frac{dx}{\cos^2 x+3\sin^2 x}=\int_0^{\pi}\frac{1}{\cot^2 x+3}\frac{dx}{\sin^2 x}$$

$$=-\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\cot x)\Big|_{0^+}^{\pi^-}=\frac{-1}{\sqrt{3}}\left(-\frac{\pi}{2}-\frac{\pi}{2}\right)=\frac{\pi}{\sqrt{3}}.$$

It can also be found with the aid of the function $F(x)=\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan x)$. For this purpose divide the interval of integration $[0, \pi]$ into two subintervals, $\left[0,\frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2},\pi\right]$ and take into consideration the above indicated limit values of the function F(x) as $x\to\frac{\pi}{2}^{\mp}$. Now,

$$\int_0^{\pi}\frac{dx}{\cos^2 x+\sin^2 x}=\int_0^{\pi/2}\frac{dx}{\cos^2 x+\sin^2 x}+\int_{\pi/2}^{\pi}\frac{dx}{\cos^2 x+\sin^2 x}$$

$$=\frac{1}{\sqrt{3}}(\sqrt{3}\tan x)\Big|_0^{\pi/2^-}+\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan x)\Big|_{\pi/2^+}^{\pi}$$

$$=\frac{1}{\sqrt{3}}\left[\left(\frac{\pi}{2}-0\right)+\left(0-\left(-\frac{\pi}{2}\right)\right)\right]=\frac{\pi}{\sqrt{3}}.$$

Now, we have an alternative solution for above integral.

$$I=\int_0^{\pi}\frac{dx}{1+2\sin^2 x}$$

$$=\int_0^{\pi}\frac{\sec^2 x dx}{1+3\tan^2 x} \quad ..(1)$$

using $\int_0^{2a}f(x)dx=\begin{cases}0, & f(2a-x)=-f(x)\\ 2\int_0^{a}f(x)dx, & f(2a-x)=f(x)\end{cases}$

If $f(x)=\frac{\sec^2 x}{1+3\tan^2 x}$, we have $f(\pi-x)=f(x)$

$\therefore$ (1) reduces to $I=2\int_0^{\pi/2}\frac{\sec^2 x dx}{1+3\tan^2 x}$

$$=\frac{2}{\sqrt{3}}\int_0^{\infty}\frac{dt}{1+t^2}$$

Put $\sqrt{3}\tan x=t \Rightarrow \sqrt{3}\sec^2 x\,dx=dt$

$$=\frac{2}{\sqrt{3}}(\tan^{-1}(t))_0^{\infty}=\frac{2}{\sqrt{3}}\left(\frac{\pi}{2}-0\right)=\frac{\pi}{\sqrt{3}}.$$

$$\therefore \int_0^{\pi}\frac{dx}{1+2\sin^2 x}=\frac{\pi}{\sqrt{3}}.$$

Example 17. Calculate $I=\int_0^{2\pi}\frac{dx}{1+0.5\cos x}$

Solution The integrand function $f(x)=\frac{1}{1+0.5\cos x}$ is continuous on the closed interval $[0, 2\pi]$ and, consequently, has an antiderivative. An appropriate change of variable for finding an antiderivative of the function f(x) is $t = \tan(x/2)$. However, when we seek the integral I, such a change of variable does not satisfy the conditions of the Substitution Theorem. The change of variable $t = \tan(x/2)$ is permissible for each of the intervals $0 \le x < \pi$ and $\pi < x \le 2\pi$.

$\cos x=\frac{1-t^2}{1+t^2}$, $dx=\frac{2dt}{1+t^2}$ and we obtain an antiderivative

$$\phi(x)=\int\frac{dx}{1+0.5\cos x}=4\int\frac{dt}{3+t^2}\Big|_{t=\tan(x/2)}+C$$

$$= \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)+C$$

For any constant C, the function $\phi(x)$ is an antiderivative of $f(x) = \frac{1}{1+0.5\cos x}$ on the intervals $[0, \pi)$ and $(\pi, 2\pi]$.

Since it has a discontinuity of the first kind at the point $x = \pi$, i.e. $\phi(\pi^+) - \phi(\pi^-) = -\frac{4\pi}{\sqrt{3}}$, it follows that $\phi(x)$ is not an antiderivative of $f(x) = \frac{1}{1+0.5\cos x}$ on the whole closed interval $[0, 2\pi]$.

However, using $\phi(x)$, we can now easily construct an antiderivative for f(x) on the whole interval $[0, 2\pi]$ we set

$$F(x) = \begin{cases} \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) & \text{for } 0 \le x < \pi, \\ \frac{2\pi}{\sqrt{3}} & \text{for } x = \pi \\ \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)+\frac{4\pi}{\sqrt{3}} & \text{for } \pi < x \le 2\pi \end{cases}$$

We have thus taken $C = 0$ on $[0, \pi]$, extended the definition of $\phi(x)$ (for $C = 0$) to the point $x = \pi$ by continuity from the left and have taken $C = \frac{4\pi}{\sqrt{3}}$ on $(\pi, 2\pi]$.

We have got a function F(x) whose derivative is equal to the function f(x) at all points of the interval $[0, 2\pi]$, the point $x = \pi$ inclusive, i.e. F(x) is an antiderivative of f(x) on $[0, 2\pi]$.

From the Newton-Leibnitz formula we have

$$I = F(x)\Big|_0^{2\pi} = F(2\pi) - F(0) = \frac{4\pi}{\sqrt{3}} - 0 = \frac{4\pi}{\sqrt{3}}.$$

Note: We could have divided the integral I into two integrals $I = \int_0^{\pi} f(x)dx + \int_{\pi}^{2\pi} f(x)dx$ and use the fact that the antiderivative of f(x) on $[0, \pi]$ is the function

$$F_1(x) = \begin{cases} \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2}\right) & \text{for } 0 \le x < \pi \\ \frac{2\pi}{\sqrt{3}} & \text{for } x = \pi \end{cases}$$

and on $[\pi, 2\pi]$ the function

$$F_2(x) = \begin{cases} \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) & \text{for } 0 < x \le 2\pi \\ -\frac{2\pi}{\sqrt{3}} & \text{for } x = \pi \end{cases}$$

($F_1(x)$ results from $\phi(x)$ for $C = 0$ when the definition of $\phi(x)$ is extended to the point $x = \pi$ by continuity from the left and $F_2(x)$ from the right). In that case, applying the Newton-Leibnitz formula to each of the integrals, we obtain

$$I = F_1(x)\Big|_0^{\pi} + F_2(x)\Big|_{\pi}^{2\pi} = F_1(\pi) - F_1(0) + F_2(2\pi) - F_2(\pi)$$

$$= \frac{2\pi}{\sqrt{3}} - 0 + 0 - \left(-\frac{2\pi}{\sqrt{3}}\right) = \frac{4\pi}{\sqrt{3}}.$$

Example 18. Prove that $\int_0^{\pi/2} \frac{\cos^2\theta\, d\theta}{\cos^2\theta + 4\sin^2\theta} = \frac{\pi}{6}$.

Solution

$$I = \int_0^{\pi/2} \frac{d\theta}{1+4\tan^2\theta} = \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)(1+4\tan^2\theta)}$$

Now put $\tan\theta = x$

$$\therefore \quad I = \int_0^{\infty} \frac{dx}{(1+x^2)(1+4x^2)}$$

$$= \frac{1}{3}\int_0^{\infty}\left(\frac{-1}{1+x^2}+\frac{4}{1+4x^2}\right)dx = \frac{\pi}{6}.$$

Example 19. Evaluate $\int_0^{\pi} \frac{dx}{(5+4\cos x)^2}$

Solution $I = \int_0^{\pi} \frac{dx}{(5+4\cos x)^2}$

$$= \int_0^{\pi} \frac{\sec^4\frac{x}{2}dx}{\left\{5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)\right\}^2}$$

Put $\tan\frac{x}{2} = t \quad \therefore \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$\therefore \quad I = \int_0^{\infty} \frac{(1+t^2)2dt}{(9+t^2)^2} = 2\int_0^{\infty} \frac{9+t^2-8}{(9+t^2)^2}dt$$

$$= 2\int_0^{\infty} \frac{1}{9+t^2}dt - 16\int_0^{\infty}\frac{dt}{(9+t^2)^2}$$

$$= 2.\frac{1}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{\infty} - 16\left[\frac{t}{18(9+t^2)}+\frac{1}{54}\tan^{-1}\frac{t}{3}\right]_0^{\infty}$$

$$= \frac{2}{3}.\frac{\pi}{2} - \frac{16}{54}.\frac{\pi}{2} = \pi\left(\frac{1}{3}-\frac{4}{27}\right) = \frac{5\pi}{27}.$$

Example 20. Prove that

$$\int_0^{\pi/2} \frac{dx}{(\sqrt{\cos x}+\sqrt{\sin x})^4} = \frac{1}{3}$$

Solution $I = \int_0^{\pi/2} \frac{dx}{\cos^2 x(1+\sqrt{\tan x})^4}$

$$= \int_0^{\pi/2} \frac{\sec^2 x dx}{(1+\sqrt{\tan x})^4}.$$

Put $\tan x = z$ then $\sec^2 x\, dx = dz$ and

$$x = 0 \Rightarrow z = 0; \quad x = \frac{\pi}{2} \Rightarrow z = \infty.$$

$$\therefore \quad I = \int_0^{\infty} \frac{dz}{(1+\sqrt{z})^4}$$

Let $1+\sqrt{2} = t$; then $\frac{1}{2\sqrt{z}} dz = dt$ and

$z = 0 \Rightarrow \quad t = 1; \quad z = \infty \Rightarrow t = \infty.$

$$\therefore \quad I = \int_1^{\infty} \frac{2(t-1)}{t^4} dt = 2\int_1^{\infty}\left(\frac{1}{t^3}-\frac{1}{t^4}\right)dt$$

$$= 2\left[\frac{1}{-2t^2}+\frac{1}{3t^3}\right]_1^{\infty} = -2\left(-\frac{1}{2}+\frac{1}{3}\right) = 1-\frac{2}{3} = \frac{1}{3}.$$

Example 21.

If $\int_0^{\pi/4} \frac{\ln(\cot x)}{\left((\sin x)^{2009}+(\cos x)^{2009}\right)^2} \cdot (\sin 2x)^{2008} dx$

$= \frac{a^b \ln a}{c^2}$ (where a, b, c are in their lowest terms)

then find the value of $(a+b+c)$.

Solution

$$I = \int_0^{\pi/4} \frac{\ln(\cot x)}{\left((\sin x)^{2009}+(\cos x)^{2009}\right)^2} \cdot (\sin 2x)^{2008} dx$$

$$= \int_0^{\pi/4} \frac{\ln(\cot x)}{\left(1+(\cot x)^{2009}\right)^2} \cdot \frac{2^{2008}(\sin x)^{2008}(\cos x)^{2008}}{(\sin x)^{2009}(\sin x)^{2009}} dx$$

$$= -\frac{2^{2008}}{(2009)^2}\int_0^{\pi/4} \frac{\ln(\cot x)^{2009}}{(1+(\cot x)^{2009})^2} \cdot \frac{2009(\cot x)^{2008}}{(-\sin^2 x)} dx$$

Let $(\cot x)^{2009} = u, \quad \frac{-2009(\cot x)^{2008}}{\sin^2 x} dx = du$

$$I = \frac{2^{2008}}{(2009)^2}\int_{\infty}^{1} -\frac{\ln(u)}{(1+u)^2} du$$

$$= \frac{2^{2008}}{(2009)^2}\left[\frac{\ln u}{1+u}\right]_{\infty}^{1} - \frac{2^{2008}}{(2009)^2}\int_{\infty}^{1} \frac{du}{u(1+u)}$$

$$= -\frac{2^{2008}}{(2009)^2}\int_{\infty}^{1}\left[\frac{1}{u}-\frac{1}{u+1}\right]du$$

$$= -\frac{2^{2008}}{(2009)^2}\left[\ln\frac{u}{u+1}\right]_{\infty}^{1} = \frac{2^{2008}\ln 2}{(2009)^2}$$

$$= \frac{a^b \ln a}{c^2}$$

$\Rightarrow$ a = 2, b = 2008 and c = 2009

$\Rightarrow$ (a + b + c) = 4019.

Example 22. Show that the value of the definite integral $\int_{-\infty}^{a} \frac{(\sin^{-1} e^x + \sec^{-1} e^{-x})dx}{(\tan^{-1} e^a + \tan^{-1} e^x)(e^x + e^{-x})}$, $(a \in R)$ is independent of a.

Solution We have

$$I = \int_{-\infty}^{a}\left(\frac{\sin^{-1} e^x + \cos^{-1} e^x}{\tan^{-1} e^a + \tan^{-1} e^x}\right)\left(\frac{e^x}{e^{2x}+1}\right)dx$$

$$= \frac{\pi}{2}\int_{-\infty}^{a} \frac{1}{(\tan^{-1} e^a + \tan^{-1} e^x)}\left(\frac{e^x}{(e^{2x}+1)}\right)dx$$

Put $\tan^{-1} e^x = t \Rightarrow \frac{e^x}{e^{2x}+1} dx = dt$

$$I = \frac{\pi}{2}\int_0^{\tan^{-1} e^a} \frac{dt}{(t+\tan^{-1} e^a)}$$

$$= \frac{\pi}{2}\left[\ln(t+\tan^{-1} e^a)\right]_0^{\tan^{-1} e^a}$$

$$= \frac{\pi}{2}\left[\ln\left(2\tan^{-1} e^a\right) - \ln\left(\tan^{-1} e^a\right)\right] = \frac{\pi}{2}\ln 2.$$

Example 23. Evaluate $\int_{-1}^{1} x^2\, d(\ln x)$

Solution Here the limits are the values of ln x. Hence,

$\ln x = -1 \Rightarrow x = 1/e$

$\ln x = 1 \quad \Rightarrow x = e$

Thus the given integral is equal to

$$\int_{1/e}^{e} x^2 \cdot \frac{1}{x} dx = \int_{1/e}^{e} x\, dx$$

$$= \left.\frac{x^2}{2}\right|_{1/e}^{e} = \frac{e^2}{2} - \frac{1}{2e^2} = \frac{e^4-1}{2e^2}.$$

Concept Problems G

1. Applying substitution, calculate the following integrals :

(a) $\int_{-1}^{1}\frac{xdx}{\sqrt{5-4x}}$, (b) $\int_{0}^{0.75}\frac{dx}{(x+1)\sqrt{x^2+1}}$

(c) $\int_{0}^{\ln 2}\sqrt{e^x-1}dx$ (d) $\int_{0}^{1}\frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}dx$

2. Evaluate the following integrals :

(i) $\int_0^1 \frac{2x+3}{5x^2+1}$

(ii) $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\ dx$

(iii) $\int_0^{2\pi}\frac{\cos x}{\sqrt{4+3\sin x}}dx$

(iv) $\int_0^{\sqrt[3]{\pi^2}}\sqrt{x}\cos^2\left(x^{\frac{3}{2}}\right)dx$

3. Find out whether, when calculating the integral $\int_0^1\sqrt{1-x^2}dx$ by changing a variable x = sint, we can take, as the new limits of integration, the numbers
(a) π and $\pi/2$, (b) 2π and $5\pi/2$
(c) π and $5\pi/2$.
Calculate the integral in each case when this change of variable is permissible.

4. If f' is continuous on [a, b], show that
$2\int_a^b f(x)f'(x)dx = [f(b)]^2 - [f(a)]^2$.

5. Evaluate $\int_0^1 (1+5x-x^5)^4(x^2-1)(x^2+1)\,dx$.

6. If f is continuous and $\int_0^9 f(x)dx = 4$, find
$\int_0^3 xf(x^2)dx$

7. Meet says, " $\int_0^{\pi}\cos^2\theta\, d\theta$ is obviously positive. "Avni claims, "No, it's zero. Just make the substitution $u = \sin\theta$; hence $du = \cos\theta\, d\theta$. Then I get
$\int_0^{\pi}\cos^2\theta\ d\theta = \int_0^{\pi}\cos\theta\cos\theta\ d\theta$
$= \int_0^0\sqrt{1-u^2}du = 0$ Simple."
(a) Who is right? What is the mistake?
(b) Use the identity $\cos^2\theta = (1+\cos 2\theta)/2$ to evaluate the integral without substitution.

8. Avni asserts that $\int_{-2}^{1} 2x^2\,dx$ is obviously positive. "After all, the integrand is never negative and $-2 < 1$." "You are wrong again," Meet replies, "It's negative. Here are my computations. Let $u = x^2$; hence $du = 2x\,dx$. Then
$\int_{-2}^{1} 2x^2\,dx = \int_{-2}^{1} x\cdot 2x\,dx$
$= \int_4^1 \sqrt{u}\ du = -\int_4^1\sqrt{u}\ du,$
which is obviously negative." Who is right?

9. Why is it impossible to use the substitution
x= sin t in the integral $\int_2^3 x\sqrt[3]{1-x^2}\,dx$?

10. Verify the result of transforming the integrals
$\int_0^1(4x^2-x+1/16)dx$, and $\int_0^{\pi}\cos^2 xdx$
by the substitutions $4x^2 - x + 1/16 = y$, and $x = \sin^{-1}y$ respectively.

11. Make sure that a formal change of the variable $t = x^{2/5}$ leads to the wrong result in the integral
$\int_{-2}^{2}\sqrt[5]{x^2}\,dx$. Find the mistake and explain it.

12. Is it possible to make the substitution x = sec t in the integral $I = \int_0^1\sqrt{x^2+1}\,dx$.

13. Given the integral $\int_0^1\sqrt{1-x^2}\,dx$. Make the substitution x = sin t. Is it possible to take the numbers π and $\frac{\pi}{2}$ as the limits for t?

14. Given the integral $\int_0^{\pi}\frac{dx}{1+\cos^2 x}$. Make sure that the functions $F_1(x) = \frac{1}{\sqrt{2}}\cos^{-1}\frac{\sqrt{2}\cos x}{\sqrt{1+\cos^2 x}}$ and $F_2(x) = \frac{1}{\sqrt{2}}\tan^{-1}\frac{\tan x}{\sqrt{2}}$ are antiderivatives for the integrand. Is it possible to use both antiderivatives for computing the definite integral by the Newton-Leibnitz formula? If not, which of the antiderivatives can be used?

Practice Problems G

15. Evaluate the following definite integrals by finding antiderivatives:

(i) $\int_3^8 \frac{\sin\sqrt{x+1}}{\sqrt{x+1}}dx$

(ii) $\int_0^{\pi/4} \cos 2x\sqrt{4-\sin 2x}\ dx$

(iii) $\int_0^{1-e^{-2}} \frac{\ln(1-t)}{1-t}\ dt$

(iv) $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}\ dx$

16. Evaluate the following definite integrals by finding antiderivatives :

(i) $\int_0^{1/2} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$

(ii) $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}\ (b>a)$

(iii) $\int_0^{\pi/2} \frac{dx}{4+5\sin x}$

(iv) $\int_2^3 \frac{\sqrt{(x-2)^2}}{1+\sqrt{(x-2)^3}}dx$

17. Evaluate the following definite integrals by finding antiderivatives :

(i) $\int_0^{\pi/4} \frac{\sin\left(x-\frac{\pi}{4}\right)}{\sin 2x+2(1+\sin x+\cos x)}dx$

(ii) $\int_2^4 \frac{\sqrt{x^2-4}}{x^4}\ dx$

(iii) $\int_{\pi/4}^{\pi/2}\left(\sqrt{\frac{\sin x}{x}}+\sqrt{\frac{x}{\sin x}}\cos x\right)dx$

(iv) $\int_3^5 \frac{x^2\,dx}{\sqrt{(x-3)(5-x)}}$

18. Find the value of 'a' such that

$$\int_0^a \frac{dx}{e^x+4e^{-x}+5} = \ln \sqrt[3]{2}\,.$$

19. The integral $\int_0^{2\pi} \frac{dx}{5-3\cos x}$ is readily taken with the aid of the substitution $\tan\frac{x}{2}=z$. We have

$$\int_0^{2\pi} \frac{dx}{5-3\cos x}=\int_0^0 \frac{2\,dz}{(1+z^2)\left(5-3\frac{1-z^2}{1+z^2}\right)}=0.$$

Find the mistake.

20. Establish the following :

$$\int_0^1 \frac{f(x)dx}{\sqrt{1-x^2}}=\int_0^{\pi/2} f(\sin\theta)\,d\theta,$$

$$\int_a^b \frac{f(x)\,dx}{\sqrt{(x-a)(b-x)}}=2\int_0^{\pi/2} f(a\cos^2\theta+b\sin^2\theta)\,d\theta.$$

21. Evaluate $\int_1^{\frac{1+\sqrt{5}}{2}} \frac{x^2+1}{x^4-x^2+1}\ln\left(1+x-\frac{1}{x}\right)dx$

22. If $\int_0^1 \frac{2e^{2x}+xe^x+3e^x+1}{(e^x+1)^2(e^x+x+1)^2}dx$

$=K-\frac{1}{(e+1)(e+2)}$, where K is a constant. Find the value of K.

23. Suppose that the function f, g, f′ and g′ are continuous over [0, 1], $g(x)\neq 0$ for $x\in[0,1]$, $f(0)=0$, $g(0)=\pi$, $f(1)=\frac{2009}{2}$ and $g(1)=1$. Find the value of the definite integral,

$$\int_0^1 \frac{f(x)\cdot g'(x)\{g^2(x)-1\}+f'(x)\cdot g(x)\{g^2(x)+1\}}{g^2(x)}dx.$$

2.9 INTEGRATION BY PARTS FOR DEFINITE INTEGRALS

The formula for integration by parts in case of definite integrals is as follows :

$$\int_a^b u(x)v(x)dx$$

$$=\left(u(x)\int u(x)dx\right)\Big|_a^b-\int_a^b\left(u'(x)\int v(x)dx\right)dx$$

Example 1. Evaluate $\int_0^{\pi/4} \frac{x \cdot \sin x}{\cos^3 x} dx$

Solution Let $I = \int_0^{\pi/4} \frac{x \sin x}{\cos^3 x} dx$

$$= \int_0^{\pi/4} \underset{u}{x} \underbrace{\tan x \sec^2 x}_{v} dx$$

We have $\int \tan x \sec^2 x \, dx = \frac{\tan^2 x}{2}$

$$I = [x \int \tan x \sec^2 x \, dx] \Big|_0^{\pi/4}$$

$$- \int_0^{\pi/4} \left[\int \tan x \sec^2 x \, dx\right] dx$$

$$I = \frac{x \tan^2 x}{2}\Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \tan^2 x \, dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^{\pi/4} (\sec^2 x - 1)\, dx$$

$$= \frac{\pi}{8} + \frac{\pi}{8} - \frac{1}{2}\int_0^{\pi/4} \sec^2 x \, dx = \frac{\pi}{4} - \frac{1}{2}.$$

Example 2. Calculate $I = \int_{1/e}^{e} |\ln x| dx$

Solution Dividing the integral I into the sum of integrals over the closed intervals [1/e, 1] and [1, e] (to get rid of the absolute value and using, in each case, the formula of integration by parts, we obtain

$$I = -\int_{1/e}^{1} \ln x dx + \int_1^e \ln x dx$$

$$= -x \ln x\Big|_{1/e}^{1} + \int_{1/e}^{1} dx + x \ln x\Big|_1^e - \int_1^e dx$$

$$= -\frac{1}{e} + \left(1 - \frac{1}{e}\right) + e - (e-1) = 2\left(1 - \frac{1}{e}\right).$$

Example 3. Suppose f, f' and f'' are continuous on [0, e] and that $f'(e) = f(e) = f(1) = 1$ and $\int_1^e \frac{f(x)}{x^2} dx = \frac{1}{2}$, then find the value of $\int_1^e f''(x) \ln x dx$

Solution $I = \int_1^e \underbrace{f''(x)}_{v} \underbrace{\ln x}_{u} dx$

$$= \ln x \cdot f'(x)\Big|_1^e - \int_1^e \frac{f'(x)}{x} dx$$

$$I = 1 - I_1$$

$$I_1 = \int_1^e \frac{1}{x} f'(x) dx = \frac{1}{x} \cdot f(x)\Big|_1^e + \int_1^e \frac{f(x)}{x^2} dx$$

$$= \left(\frac{1}{e} - 1\right) + \frac{1}{2} = \frac{1}{e} - \frac{1}{2}$$

$$\therefore \quad I = 1 - \frac{1}{e} + \frac{1}{2} = \frac{3}{2} - \frac{1}{e}.$$

Example 4. Suppose that f, f ' and f '' are continuous on [0, ln 2] and that $f(0) = 0$, $f'(0) = 3$, $f(\ln 2) = 6$, $f'(\ln 2) = 4$ and $\int_0^{\ln 2} e^{-2x} \cdot f(x) dx = 3$. Find the value of $\int_0^{\ln 2} e^{-2x} \cdot f''(x) dx$.

Solution $I = \int_0^{\ln 2} \underbrace{e^{-2x}}_{u} \cdot \underbrace{f''(x)}_{v} dx$

$$= e^{-2x} \cdot f'(x)\Big|_0^{\ln 2} + 2\int_0^{\ln 2} e^{-2x} \cdot f'(x) dx$$

$$= \left(\frac{1}{4} f'(\ln 2) - 1 \cdot 3\right)$$

$$+ 2\left[e^{-2x} \cdot f(x)\Big|_0^{\ln 2} + 2\int_0^{\ln 2} e^{-2x} f(x) dx\right]$$

$$= (1-3) + 2\left[\frac{1}{4} f(\ln 2) + 2 \cdot 3\right] = -2 + 2\left[\frac{6}{4} + 6\right]$$

$$= -2 + 3 + 12 = 13.$$

Example 5. Evaluate $\int_0^{\pi/2} \frac{(1 + 2\cos x)}{(2 + \cos x)^2} dx.$

Solution Let $I = \int_0^{\pi/2} \frac{(1 + 2\cos x)}{(2 + \cos x)^2} dx$

$$= \int_0^{\pi/2} \frac{\cos x(2 + \cos x) + \sin^2 x}{(2 + \cos x)^2} dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{(2+\cos x)}\,dx + \int_0^{\pi/2} \frac{\sin^2 x\,dx}{(2+\cos x)^2}$$

In the first integral, integrating by parts taking cos x as second function and second integral unchanged, we have

$$I = \left[\frac{1}{(2+\cos x)}\sin x\right]_0^{\pi/2}$$

$$-\int_0^{\pi/2} \frac{\sin^2 x\,dx}{(2+\cos x)^2} + \int_0^{\pi/2} \frac{\sin^2 x\,dx}{(2+\cos x)^2}$$

$$= \frac{1}{(2+0)} - 0 = \frac{1}{2}.$$

Alternative :

$$I = \int_0^{\pi/2} \frac{(1+2\cos x)}{(2+\cos x)^2}\,dx$$

Dividing N^r and D^r by $\sin^2 x$, we get

$$I = \int_0^{\pi/2} \frac{(\operatorname{cosec}^2 x + 2\operatorname{cosec} x \cot x)dx}{(2\operatorname{cosec} x + \cot x)^2}$$

Put $2\operatorname{cosec} x + \cot x = t$

$\Rightarrow$ $(2\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x)\,dx = -\,dt$

When $x = 0$, $t = \infty$,

when $x = \frac{\pi}{2}$, $t = 2$.

$$\therefore\ I = \int_\infty^2 \frac{-dt}{t^2} = \int_2^\infty \frac{dt}{t^2} = -\left\{\frac{1}{t}\right\}_2^\infty = -\left\{0 - \frac{1}{2}\right\} = \frac{1}{2}.$$

Example 6. Evaluate $\displaystyle\int_0^{\pi/4} \frac{\sin^2 x dx}{e^{2mx}(\cos x - m\sin x)^2}$

Solution Let $I = \displaystyle\int_0^{\pi/4} \frac{\sin^2 x dx}{e^{2mx}(\cos x - m\sin x)^2}$

$$= \int_0^{\pi/4} \frac{\sin^2 x dx}{\{e^{mx}(\cos x - m\sin x)\}^2}$$

$$= -\frac{1}{(m^2+1)}\int_0^{\pi/4} \frac{\sin x}{e^{mx}}\,\frac{(-(m^2+1)e^{mx}\sin x)}{e^{mx}\{e^{mx}(\cos x - m\sin x)\}^2}\,dx$$

Integrating by parts taking $\frac{\sin x}{e^{mx}}$ as the first function,

$$I = -\frac{1}{(m^2+1)}\left\{\left[\frac{\sin x}{e^{mx}} \times \frac{1}{e^{mx}(\cos x - m\sin x)}\right]_0^{\pi/4}\right.$$

$$\left. + \int_0^{\pi/4} \frac{e^{mx}\cos x - \sin x\, m e^{mx}}{e^{2mx}} \cdot \frac{1}{e^{mx}(\cos x - m\sin x)}\,dx\right\}$$

$$= -\frac{1}{(m^2+1)}\left\{\frac{\frac{1}{\sqrt{2}}}{e^{m\pi/2}\left(\frac{1-m}{\sqrt{2}}\right)} - \left[\frac{e^{-2mx}}{2m}\right]_0^{\pi/4}\right\}$$

$$= -\frac{1}{(m^2+1)}\left\{\frac{1}{(m-1)e^{m\pi/2}} - \frac{1}{2m}(e^{-m\pi/2} - 1)\right\}$$

$$= -\frac{1}{(m^2+1)}\left\{\frac{(m+1)e^{-m\pi/2}}{2m(m-1)} + \frac{1}{2m}\right\}$$

Example 7. Show that

$$\lim_{n\to\infty}\left(\frac{(n^3+1)(n^3+2^3)(n^3+3^3)\ldots\ldots.(n^3+n^3)}{n^{3n}}\right)^{1/n}$$

$$= 4e^{\frac{\pi}{\sqrt{3}}}.e^{-3}.$$

Solution ln P

$$= \frac{1}{n}[\{\ln(n^3+1) + \ln(n^3+2^3) + \ldots..$$

$$\ldots. + \ln(n^3+n^3)\} - 3\ln 4]$$

Hence $T_r = \dfrac{1}{n}[\ln(n^3+r^3) - 3\ln n]$

$$= \frac{1}{n}\left[3\ln n + \ln\left(1+\left(\frac{r}{h}\right)^3\right) - 3.\ln n\right]$$

$$= \frac{\ln\left(1+\left(\frac{r}{h}\right)^3\right)}{n}$$

Let $S = \displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\left(\frac{r}{h}\right)^3\right)$

$$= \int_0^1 \ln(1+x^3)\,dx$$

$$= \int_0^1 \ln(1+x)\,dx + \int_0^1 \ln(x^2 - 1\,x + 1)\,dx$$

$$= \ln(1+x).x\,\Big]_0^1 - \int_0^1 \frac{1+x-1}{1+x}\,dx$$

$$+ \ln(1-x+x^2).x\,\Big]_0^1 - \int_0^1 \frac{x(2x-1)}{x^2-x+1}\,dx$$

$$= \ln 2 - (1 - \ln 2) - \int_0^1 \frac{2x^2 - x}{x^2 - x + 1}\,dx.$$

$$S = \ln 4 - 1 - I_1$$

$$I_1 = \int_0^1 \left(2 + \frac{x-2}{x^2 - x + 1}\right) dx = 2 - \frac{\pi}{\sqrt{3}}.$$

$$\Rightarrow \quad S = \ln 4 - 1 - 2 + \frac{\pi}{\sqrt{3}} = \ln 4 + \frac{\pi}{\sqrt{3}} - 3.$$

$$\therefore \quad \ln P = \ln 4 + \frac{\pi}{\sqrt{3}} - 3$$

$$\Rightarrow \quad P = 4e^{\frac{\pi}{\sqrt{3}}} \cdot e^{-3}.$$

Example 8. Show that for a differentiable function

$$f(x), \int_0^n f'(x)\left\{[x] - x + \frac{1}{2}\right\} dx$$

$$= \int_0^n f(x)dx + \frac{1}{2}f(0) + \frac{1}{2}f(n) - \sum_{r=0}^{n} f(r)\text{, where [.]}$$

denotes the greatest integer function and $n \in N$.

Solution

$$I = \int_0^n f'(x)[x]dx - \int_0^n xf'(x)dx + \frac{1}{2}\int_0^n f'(x)dx$$

$$\sum_{r=1}^{n} \int_{r-1}^{r} f'(x)[x]dx - \left\{(xf(x))_0^n - \int_0^n f(x)dx\right\} + \frac{1}{2}(f(x))_0^n$$

$$= \sum_{r=1}^{n} (r-1)\int_{r-1}^{r} f'(x)dx - nf(n)$$

$$+ \frac{1}{2}f(n) - \frac{1}{2}f(0) + \int_0^n f(x)dx$$

$$= \sum_{r=1}^{n} (r-1)\{f(r) - f(r-1)\}$$

$$-nf(n) + \frac{1}{2}f(n) + \int_0^n f(x)dx - \frac{1}{2}f(0)$$

$$= -f(1) - f(2) - \ldots\ldots - f(n-1) - f(n)$$

$$+ \frac{1}{2}f(n) + \frac{1}{2}f(0) + \int_0^n f(x)dx$$

$$= \sum_{r=1}^{n} f(r) + + \frac{1}{2}f(n) + \frac{1}{2}f(0) + \int_0^n f(x)dx.$$

Example 9. Prove that $\int \sin n\theta \sec\theta \, d\theta$

$$= -\frac{2\cos(n-1)\theta}{n-1} - \int \sin(n-2)\theta \sec\theta \, d\theta.$$

Hence or otherwise

$$\text{evaluate} \int_0^{\pi/2} \frac{\cos 5\theta \, \sin 3\theta}{\cos\theta} d\theta.$$

Solution Consider $\sin n\theta + \sin(n-2)\theta$

$= 2\sin(n-1)\theta\cos\theta$

$\Rightarrow \sin n\theta \sec\theta = 2\sin(n-1)\theta - \sin(n-2)\theta\sec\theta$

Hence $\int \sin n\theta \sec\theta \, d\theta$

$$= -\frac{2}{(n-1)} \cos(n-1)\theta - \int \sin(n-2)\theta \sec\theta \, d\theta$$

$$\text{Now, } \frac{1}{2}\int_0^{\pi/2} \frac{2\sin 3\theta \, \cos 5\theta}{\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin 8\theta - \sin 2\theta}{\cos\theta}$$

$$I = \frac{1}{2} I_8 - 1$$

$$I_8 = -\frac{2}{7}\cos 7\theta\,]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 6\theta}{\cos\theta} d\theta.$$

$$= \frac{2}{7} - \left[-\frac{2}{5}\cos 5\theta\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 4\theta}{\cos\theta} d\theta\Bigg]$$

$$= \frac{2}{7} - \left[\frac{2}{5} - \left\{\left(-\frac{2}{3}\cos 3\theta\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 2\theta}{\cos\theta} d\theta\right\}\right]$$

$$= \frac{2}{7} - \left[\frac{2}{5} - \frac{2}{3} + 2\right] = \frac{2}{7} - \frac{2}{5} + \frac{2}{3} - \frac{2}{1}$$

$$= \frac{30 - 42 + 70 - 210}{105} = -\frac{152}{105}$$

$$I = \frac{1}{2} I_8 - 1 = -\frac{76 + 105}{105} = -\frac{181}{105}.$$

Example 10. $\int_0^{\pi/2}\left(\frac{x}{x\sin x+\cos x}\right)^2 dx$

Solution $I=\int_0^{\pi/2}\frac{x\cos x}{(x\sin x+\cos x)^2}\cdot\frac{x}{\cos x}dx$

$$=-\frac{x}{\cos x}\left.\frac{1}{x\sin x+\cos x}\right|_0^{\pi/2}+\int_0^{\pi/2}\frac{\cos x+x\sin x}{(x\sin x+\cos x)}\cdot\sec^2 x\,dx$$

$$=\left[-\frac{x}{\cos x}\cdot\left(\frac{1}{x\sin x+\cos x}\right)+\frac{\sin x}{\cos x}\right]_0^{\frac{\pi}{2}}$$

$$=\left.\frac{\sin x(x\sin x+\cos x)-x}{\cos x(x\sin x+\cos x)}\right|_0^{\frac{\pi}{2}}$$

$$=\left.\frac{\sin x\cos x-x\cos^2 x}{\cos x(x\sin x+\cos x)}\right|_0^{\frac{\pi}{2}}$$

$$=\left.\frac{\sin x-x\cos x}{x\sin x+\cos x}\right|_0^{\pi/2}=\left(\frac{1-0}{\pi/2}\right)-0=\frac{2}{\pi}.$$

Example 11. Evaluate $I=\int_0^{\infty}\left(\frac{\tan^{-1}x}{x}\right)^3 dx$

Solution Put $x=\tan\theta$

$$I=\int_0^{\pi/2}\frac{\theta^3}{\tan^3\theta}\cdot\sec^2\theta\,d\theta=\int_0^{\pi/2}\underbrace{\frac{\sec^2\theta}{\tan^3\theta}}_{v}\cdot\underbrace{\theta^3}_{u}d\theta$$

$$=-\left.\frac{\theta^3}{2\tan^2\theta}\right|_0^{\pi/2}+\frac{3}{2}\int_0^{\pi/2}\frac{\theta^2}{\tan^2\theta}d\theta$$

$$=\frac{3}{2}\int_0^{\pi/2}\theta^2(\text{cosec}^2-1)\,d\theta$$

$$=\frac{3}{2}\left[\int_0^{\pi/2}(\underbrace{\theta^2}_{u}\cdot\underbrace{\text{cosec}^2\theta}_{v})\,d\theta-\frac{\pi^3}{24}\right]$$

$$=\frac{3}{2}\left[-\theta^2\cot\theta\Big|_0^{\pi/2}+2\int_0^{\pi/2}(\theta\cot\theta)d\theta-\frac{\pi^3}{24}\right]$$

$$=\frac{3}{2}\left[(0)+2\int_0^{\pi/2}-\frac{\pi^3}{24}\right]$$

$$=\frac{3\pi}{2}\ln 2-\frac{\pi^3}{16}.$$

Example 12. Show that

$$\int_0^{\pi/2}\sin x\cdot\ln\sin x\,dx=\ln 2-1.$$

Solution Let $I=\int_\theta^{\pi/2}\sin x\cdot\ln\sin x\,dx$.

Clearly, the required integral $=\lim\limits_{\theta\to 0} I$.

Now, $I=[\ln\sin x.(-\cos x)]_\theta^{\pi/2}$

$$-\int_\theta^{\pi/2}-\cos x\cdot\frac{\cos x}{\sin x}dx,\ \text{using by parts.}$$

$$=\cos\theta\cdot\ln\sin\theta+\int_\theta^{\pi/2}\frac{1-\sin^2 x}{\sin x}dx$$

$$=\cos\theta\cdot\ln\sin\theta+\int_\theta^{\pi/2}(\text{cosec}\,x-\sin x)\,dx$$

$$=\cos\theta.\ln\sin\theta+[-\ln(\text{cosec}\,x+\cot x)+\cos x]_\theta^{\pi/2}$$

$$=\cos\theta\cdot\ln\sin\theta+\ln(\text{cosec}\,\theta+\cot\theta)-\cos\theta$$

$$=\cos\theta\cdot\ln\sin\theta+\ln\frac{1+\cos\theta}{\sin\theta}-\cos\theta$$

$$=(\cos\theta-1)\ln\sin\theta+\ln(1+\cos\theta)-\cos\theta$$

$\therefore$ LHS $=\lim\limits_{\theta\to 0} I$

$$=\lim_{\theta\to 0}\{(\cos\theta-1)\ln\sin\theta\}+\ln 2-1$$

$$=\lim_{\theta\to 0}\frac{\ln\sin\theta}{\frac{1}{\cos\theta-1}}+\ln 2-1$$

[applying L' Hospital's Rule]

$$=\lim_{\theta\to 0}\frac{\frac{\cos\theta}{\sin\theta}}{\frac{-1}{(\cos\theta-1)^2}\cdot(-\sin\theta)}+\ln 2-1$$

$$=\lim_{\theta\to 0}\frac{\cos\theta(\cos\theta-1)^2}{\sin^2\theta}+\ln 2-1$$

$$=\lim_{\theta\to 0}\frac{\cos\theta(1-\cos\theta)^2}{1-\cos^2\theta}+\ln 2-1$$

$$=\lim_{\theta\to 0}\frac{\cos\theta(1-\cos\theta)}{1-\cos\theta}+\ln 2-1$$

$$=0+\ln 2-1=\ln 2-1.$$

Example 13. Prove that

$$\int_0^{\pi/2}\frac{x\sin x\cos x}{(a^2\cos^2 x+b^2\sin^2 x)^2}dx=\frac{\pi}{4ab^2(a+b)}.$$

Solution Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{(a^2\cos^2 x + b^2 \sin^2 x)^2}\,dx$

Integrating by parts taking x as the first function, we have

$$I = \left[x\left\{\frac{-1}{2(b^2-a^2)(a^2\cos^2 x + b^2\sin^2 x)}\right\}\right]_0^{\pi/2}$$

$$-\int_0^{\pi/2} \frac{-1}{2(b^2-a^2)(a^2\cos^2 x + b^2\sin^2 x)}\,dx$$

$$= -\frac{\pi}{4(b^2-a^2)b^2} + \frac{1}{2(b^2-a^2)}\int_0^{\pi/2}\frac{\sec^2 x\,dx}{a^2+(b\tan x)^2}$$

Put $b \tan x = t \Rightarrow \sec^2 x\, dx = \frac{dt}{b}$

$$= -\frac{\pi}{4(b^2-a^2)b^2} + \frac{1}{2(b^2-a^2)}\int_0^{\infty}\frac{dt}{b(a^2+t^2)}$$

$$= \frac{-\pi}{4(b^2-a^2)b^2} + \frac{1}{2ab(b^2-a^2)}\left[\tan^{-1}\left(\frac{t}{a}\right)\right]_0^{\infty}$$

$$= \frac{-\pi}{4(b^2-a^2)b^2} + \frac{1}{2ab(b^2-a^2)}\left(\frac{\pi}{2}-0\right)$$

$$= \frac{-\pi}{4(b^2-a^2)b^2} + \frac{1}{4ab(b^2-a^2)}$$

$$= \frac{\pi(b-a)}{4ab^2(b^2-a^2)}.$$

Example 14. Evaluate

$$\int_0^{3\pi/2} (\ln|\sin x|)\cos(2nx)\,dx,\ n \in N.$$

Solution Let $I_n = \int_0^{3\pi/2} (\ln|\sin x|)\cos(2nx)\,dx$

Integrating by parts, taking cos 2nx as the second function, we have

$$I_n = \left\{\ln|\sin x|\frac{\sin 2nx}{2n}\right\}_0^{3\pi/2} - \int_0^{3\pi/2}\frac{\cot x \sin 2nx}{2n}\,dx$$

$$= 0 - \frac{1}{2n}\int_0^{3\pi/2}\frac{\sin 2nx \cos x}{\sin x}\,dx$$

$$= -\frac{1}{2n} I_n' \qquad \text{...(1)}$$

$\therefore\ I_n' - I'_{n-1}$

$$= -\frac{1}{2n}\int_0^{3\pi/2}\frac{\cos x\{\sin 2nx - \sin(2n-2)x\}}{\sin x}\,dx$$

$$= -\frac{1}{2n}\int_0^{3\pi/2}\frac{\cos x\, 2\cos(2n-1)x \sin x}{\sin x}\,dx$$

$$= -\frac{1}{2n}\int_0^{3\pi/2}(2\cos(2n-1)x\cos x)\,dx$$

$$= -\frac{1}{2n}\int_0^{3\pi/2}(\cos 2nx + \cos(2n-2)x)\,dx$$

$$= -\frac{1}{2n}\left\{\frac{\sin 2nx}{2n} + \frac{\sin(2n-2)x}{2n-2}\right\}_0^{3\pi/2}$$

$$= -\frac{1}{2n}\{(0+0)-(0+0)\} = 0.$$

$\Rightarrow\ I_n' = I'_{n-1} = I'_{n-2} = \ldots\ldots = I_1'$

Now, $I_1' = \int_0^{3\pi/2}\frac{\sin 2x\cos x}{\sin x}\,dx$

$$= \int_0^{3\pi/2}(2\cos^2 x)\,dx = \int_0^{3\pi/2}(1+\cos 2x)\,dx$$

$$= \left\{x + \frac{\sin 2x}{2}\right\}_0^{3\pi/2} = \left\{\left(\frac{3\pi}{2}+0\right)-(0+0)\right\} = \frac{3\pi}{2}.$$

$\Rightarrow\ I_n' = \frac{3\pi}{2}$. From (1), $I_n = -\frac{1}{2n}I_n'$

Hence $I_n = -\frac{3\pi}{4n}$.

Concept Problems H

1. Applying the formula for integration by parts calculate the following integrals :

   (a) $\int_0^{\ell n 2} xe^{-x}\,dx$ (b) $\int_0^{2\pi} x^2\cos x\,dx$

   (c) $\int_0^1 \cos^{-1} x dx$

2. Let f and g be differentiable on [a, b] and suppose f′ and g′ are both continuous on [a, b], then prove that

$$\int_a^b f'(x)\,g(x)\,dx + \int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)\,g(a).$$

3. Suppose f and g are continuous and $f(a) = f(b) = 0$.

   Prove $\int_a^b f(x)\,g(x)\,dx = -\int_a^b f'(x)\,G(x)\,dx$,

   where $G(x) = \int_a^x g(t)\,dt$.

4. Suppose that f(1) = 2, f(4) = 7, f'(1) = 5, f'(4) = 3 and f" is continuous, Find the value of $\int_1^4 xf''(x)\,dx$

5. Show that

(a) $\int_0^x e^{-t}\, t\,dt = e^{-x}(e^x - 1 - x)$.

(b) $\int_0^x e^{-t}\, t^3 dt = 3!e^{-x}\left(e^x - 1 - x - \dfrac{x^2}{2!} - \dfrac{x^3}{3!}\right)$.

Practice Problems H

6. Calculate the following integrals :

(a) $\int_{-1}^{1} \dfrac{x dx}{x^2 + x + 1}$

(b) $\int_1^e (x\ell nx)^2 dx$

(c) $\int_0^3 \sin^{-1}\sqrt{\dfrac{x}{1+x}}\,dx$

(d) $\int_0^{2\pi} \dfrac{dx}{(2+\cos x)(3+\cos x)}$

(e) $\int_0^{\pi/2} \sin x \sin 2x \sin 3x\, dx$

(f) $\int_0^{\pi} (x \sin x)^2 dx$

7. Evaluate the following definite integrals by finding antiderivatives :

(i) $\int_0^{2\pi} [\cos x(1+x) + (1-x)\sin x]dx$

(ii) $\int_{\pi/2}^{\pi} x^{\sin x}(1 + x\cos x \cdot \ln x + \sin x)\,dx$

(iii) $\int_1^e \dfrac{dx}{\ln(x^x e^x)}$

(iv) $\int_0^1 x\,(\tan^{-1} x)^2\ dx$

8. Compute $\int_0^1 x f''(3x)\,dx$, given that f'(0) is defined, f(0) = 1, f(3) = 4, and f'(3) = –2.

9. Show that $\int_0^1 xf''(x)dx = 3$ for every function f(x) that satisfies the following conditions : (i) f(x) is defined for all x, (ii) f"(x) is continuous, (iii) f (0) = f (1), (iv) f' (1) = 3.

10. (a) Find an integer n such that $n\int_0^1 xf''(2x)dx = \int_0^2 tf''(t)dt$. (b) Compute $\int_0^1 xf''(2x)dx$, given that f(0) = 1, f(2) = 3, and f'(2) = 5.

11. Let f have derivatives of all orders.

(a) Explain why $f(b) = f(0) + \int_0^b f'(x)\,dx$.

(b) Using an integration by parts on the derivative integral in (a), with u = f'(x) and $\int v dx = x - b$, show that $f(b) = f(0) + f'(0)b + \int_0^b f^{(2)}(x)(b-x)dx$

(c) Similarly, show that

$$f(b) = f(0) + f'(0)b + \frac{f^{(2)}(0)}{2}b^2 + \frac{1}{2}\int_0^b f^{(3)}(x)\,(b-x)^2 dx$$

(d) Check that (c) is correct for any quadratic polynomial.

(e) Use another integration by parts on the formula in (c) to obtain the next formula.

2.10 REDUCTION FORMULA

Example 1. If $I_n = \int_0^{\pi/4} \tan^n x dx$ show that

$I_n + I_{n-2} = \dfrac{1}{n-1}$ and deduce the value of I_5.

Solution

$$\int_0^{\pi/4} \tan^n xdx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} xdx$$

$$\Rightarrow I_n = \frac{1}{n-1} - I_{n-2}.$$

$$I_n + I_{n-2} = \frac{1}{n-1}$$ Hence proved.

$$\text{Now } I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \left(\frac{1}{2} - I_1\right)$$

$$= \frac{1}{4} - \frac{1}{2} + \int_0^{\pi/4} \tan xdx = -\frac{1}{4} + [\ln|\sec x|]_0^{\pi/4}$$

$$= -\frac{1}{4} + \ln\sqrt{2} = \frac{1}{2}\ln 2 - \frac{1}{4}.$$

Example 2. If $u_n = \int_0^{\pi/2} x^n \sin x\,dx$, $(n > 0)$, then prove that $u_n + n(n-1)u_{n-2} = n\left(\frac{1}{2}\pi\right)^{n-1}$.

Solution Integrating by parts, $u_n = \int_0^{\pi/2} x^n \sin x\,dx$

$$u_n = \left[-x^n \cos x\right]_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1}\cos x\,dx$$

$$= n\left\{\left[x^{n-1}\sin x\right]_0^{\pi/2} - (n-1)\int_0^{\pi/2} x^{n-2}\sin x\,dx\right\}$$

$$= n\left(\frac{1}{2}\pi\right)^{n-1} - n(n-1)u_{n-2}$$

$$\therefore\ u_n + n(n-1)u_{n-2} = n\left(\frac{1}{2}\pi\right)^{n-1}.$$

Example 3. If $u_n = \int_0^{\pi/2} x(\sin x)^n\,dx$, $n > 0$, then prove that $u_n = \frac{n-1}{n}u_{n-2} + \frac{1}{n^2}$.

Solution $u_n = \int_0^{\pi/2} x(\sin x)^{n-1}.\sin x\ dx$

$$= \left.-x(\sin x)^{n-1}.\cos x\right]_0^{\pi/2}$$

$$+\left[\int_0^{\pi/2}(n-1)x(\sin x)^{n-2}.\cos x + (\sin x)^{n-1}\cos x\right]dx$$

$$= 0 + (n-1)\int_0^{\pi/2} x(\sin x)^{n-2}(1-\sin^2 x)\,dx$$

$$+\int_0^{\pi/2}(\sin x)^{n-1}.\cos x\,dx$$

$$\Rightarrow u_n = (n-1)u_{n-2} - (n-1)u_n + \frac{1}{n}$$

$$n u_n = (n-1)u_{n-2} + \frac{1}{n}$$

$$\Rightarrow u_n = \frac{n-1}{n}u_{n-2} + \frac{1}{n^2}.$$

Example 4. If $u_n = \int_0^1 x^n \tan^{-1} x\,dx$ then prove that

$$(n+1)u_n + (n-1)u_{n-2} = \frac{\pi}{2} - \frac{1}{n}.$$

Solution $u_n = \int_0^1 x^n \tan^{-1} x\,dx$

$$= \left.\frac{\tan^{-1}x \cdot x^{n+1}}{n+1}\right|_0^1 - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}$$

$$= \frac{\pi}{4(n+1)} - \frac{1}{n+1}\int_0^1 \frac{x^{n-1}[1+x^2-1]}{1+x^2}dx$$

$\Rightarrow (n+1)u_n$

$$= \frac{\pi}{4} - \left[\int_0^1 x^{n-1}dx - \int_0^1 x^{n-1}\cdot\frac{1}{1+x^2}dx\right]$$

$$= \frac{\pi}{4} - \frac{1}{n} + \left.x^{n-1}\tan^{-1}x\right|_0^1$$

$$- \int_0^1 (n-1)x^{n-2}\cdot\tan^{-1}x\,dx$$

$$\Rightarrow (n+1)u_n = \frac{\pi}{2} - \frac{1}{n} - (n-1)u_{n-2}$$

$$(n+1)u_n + (n-1)u_{n-2} = \frac{\pi}{2} - \frac{1}{n}.$$

Example 5. If $I_n = \int_0^1 \frac{t^n\,dt}{1+t^2}$, show that

$$I_{n+2} = \int_0^1 \frac{t^{n+1}}{n+1} - I_n.$$

Also evaluate $\int_0^1 \frac{t^6}{(1+t^2)}dt$.

Solution $I_{n+2} = \int_0^1 \frac{t^{n+2}}{1+t^2}dt = \int_0^1 \frac{t^n\{(1+t^2)-1\}}{1+t^2}dt$

$$= \int_0^1\left(t^n - \frac{t^n}{1+t^2}\right)dt \quad \left[\because \text{we require } \frac{t^{n+1}}{n+1} \text{ in result}\right]$$

$$= \frac{t^{n+1}}{n+1} - I_n.$$

Now $I_0 = \int \frac{dt}{1+t^2} = \tan^{-1} t$.

Using the reduction formula,

For $n = 0$, $I_2 = t - I_0 = t - \tan^{-1} t$,

For $n = 2$, $I_4 = \frac{t^3}{3} - I_2 = \frac{t^3}{3} - t + \tan^{-1} t$,

$$\therefore \int_0^1 \frac{t^6}{(1+t^2)}dt = \left[\frac{t^5}{5} - \frac{t^3}{3} + t - \tan^{-1} t\right]_0^1$$

$$= \left(\frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}\right) - 0 = \frac{13}{15} - \frac{\pi}{4}.$$

Example 6. Compute the integral

$I_n = \int_0^a (a^2 - x^2)^n\, dx$, where n is a natural number.

Solution The integral can be computed by expanding the integrand $(a^2 - x^2)^n$ according to Binomial theorem, but it involves cumbersome calculations. it is simpler to deduce a formula for reducing the integral I_n to the integral I_{n-1}. For this let us expand the integral I_n in the following way :

$$I_n = \int_0^a (a^2 - x^2)^{n-1}(a^2 - x^2)dx$$

$$= a^2 I_{n-1} - \int_0^a x(a^2 - x^2)^{n-1} x dx$$

and integrate the latter integral by parts.
We obtain

$$I_n = a^2 I_{n-1} + \frac{1}{2n} x\,(a^2 - x^2)^n \Big|_0^a - \frac{1}{2n}\int_0^a (a^2 - x^2)^n dx$$

$$= a^2 I_{n-1} - \frac{1}{2n} I_n$$

$$\Rightarrow I_n = a^2 \frac{2n}{2n+1} I_{n-1}$$

This formula is valid at any real n other than 0 and $-\frac{1}{2}$.

In particular, at natural n, taking into account that

$I_0 = \int_0^a dx = a$, we get

$$I_n = a^{2n+1}\frac{2n(2n-2)(2n-4)....6.4.2}{(2n+1)(2n-1)(2n-3)...5.3}$$

Example 7. Using the result of the preceding example obtain the following formula :

$$1 - \frac{{}^nC_1}{3} + \frac{{}^nC_2}{5} - \frac{{}^nC_3}{7} + ... + (-1)^n \frac{{}^nC_n}{2n+1}$$

$$= \frac{2n(2n-2)(2n-4)....6.4.2}{(2n+1)(2n-1)(2n-3)...5.3}.$$

Solution Consider the integral.

$$I_n = \int_0^1 (1-x^2)^n dx$$

$$= \frac{2n(2n-2)(2n-4)....6.4.2}{(2n+1)(2n-1)(2n-3)...5.3}$$

Expanding the integrand by the Binomial theorem and integrating within the limits from 0 to 1, we get :

$$I_n = \int_0^1 (1-x^2)^n dx$$

$$= \int_0^1 (1 - {}^nC_1 x^2 + {}^nC_2 x^4 - {}^nC_3 x^6 + ... + (-1)^n\, {}^nC_n x^{2n})dx$$

$$= \left[x - \frac{{}^nC_1 x^3}{3} + \frac{{}^nC_2 x^5}{5} - \frac{{}^nC_3 x^7}{7} + ... + \frac{(-1)^n\, {}^nC_n x^{2n+1}}{2n+1}dx\right]_0^1$$

$$= 1 - \frac{{}^nC_1}{3} + \frac{{}^nC_2}{5} - \frac{{}^nC_3}{7} + ... + (-1)^n \frac{{}^nC_n}{2n+1}.$$

Using the result of the previous example we can easily complete the proof.

Example 8. Find a reduction formula for the integral $\int \frac{\sin nx}{\sin x}dx$ and show that $\int_0^\pi \frac{\sin nx}{\sin x} = \pi$ or 0, according as n is odd or even.

Solution We have

$$\sin nx - \sin(n-2)x = 2\cos(n-1)x \sin x$$

$$\therefore \frac{\sin nx}{\sin x} = 2\cos(n-1)x + \frac{\sin(n-2)x}{\sin x}$$

Integrating both sides we get

$$\int \frac{\sin nx}{\sin x}dx = \frac{2\sin(n-1)x}{(n-1)} + \int \frac{\sin(n-2)x}{\sin x}dx$$

Above is the required reduction formula.
Taking the limits from 0 to π, we get

$$I_n = 0 + I_{n-2} = I_{n-4} ... = I_2 \text{ or } I_1$$

according as n is even or odd.

If n is even, $I_n = I_2 = \int_0^\pi \frac{\sin 2x}{\sin x}dx = \int_0^\pi 2\cos x dx$

$$= 2[\sin x]_0^\pi = 0.$$

If n is odd $= I_n = I_1 = \int_0^\pi \frac{\sin x}{\sin x}dx = [x]_0^\pi = \pi.$

Example 9. Prove that $\int_0^\pi \left(\frac{\sin n\theta}{\sin\theta}\right)^2 d\theta = n\pi$, $n \in W$.

Solution If the given integral is denoted by I_n then

$$I_n - I_{n-1} = \int_0^\pi \frac{\sin^2 n\theta - \sin^2(n-1)\theta}{\sin^2\theta}d\theta$$

$$= \int_0^{\pi} \frac{\sin(2n-1)\theta \sin\theta}{\sin^2\theta} d\theta$$

$$= \int_0^{\pi} \frac{\sin(2n-1)\theta}{\sin\theta} d\theta = \pi$$

[using the result of the previous question]

$\therefore$ $I_n - I_{n-1} = \pi$ or $I_n = \pi + I_{n-1} = \pi + (\pi + I_{n-2})$

or $I_n = 2\pi + I_{n-2} = 3\pi + I_{n-3} \ldots. = n\pi + I_0 = n\pi.$

$\because$ $I_0 = \int_0^{\pi} \frac{\sin^2 0}{\sin\theta} d\theta = \int_0^{\pi} 0\, d\theta = 0.$

Alternative :

We show that I_1, I_2, I_3,....constitute an arithmetic progression.

$$I_{n+1} - 2I_n + I_{n-1} = (I_{n+1} - I_n) - (I_n - I_{n-1})$$

$$= \int_0^{\pi/2} \frac{(\sin^2(n+1)x - \sin^2 nx) - (\sin^2 nx - \sin^2(n-1)x)}{\sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{(\sin(2n+1)x \sin x - \sin(2n-1)x \sin x)}{\sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{(\sin(2n+1)x - \sin(2n-1)x)}{\sin x}$$

$$= \int_0^{\pi/2} \frac{2\cos 2nx \sin x}{\sin x} dx = 2\int_0^{\pi/2} \cos 2nx\, dx$$

$$= 2 . \left.\frac{\sin 2nx}{2n}\right|_0^{\pi/2} = \frac{1}{n}(\sin n\pi - \sin 0) = 0 - 0 = 0.$$

$\therefore$ $I_{n-1} + I_{n+1} = 2I_n$

i.e., I_{n-1}, I_n, I_{n+1} from an A.P.

Now $I_0 = 0$ and $I_1 = \int_0^{\pi} d\theta = \pi$

Hence $I_2 = I_1 + (I_1 - I_0) = \pi + \pi - 0 = 2\pi$

Similarly, $I_n = nI$.

Example 10. If $S_n = \int_0^{\frac{1}{2}\pi} \frac{\sin(2n-1)x}{\sin x} dx$,

$$V_n = \int_0^{\frac{1}{2}\pi} \left(\frac{\sin nx}{\sin x}\right)^2 dx,$$

n being an integer, then show that

$$S_{n+1} = S_n = \frac{1}{2}\pi, \text{ and } V_{n+1} - V_n = S_{n+1}.$$

Also obtain the value of V_n.

Solution

$$S_{n+1} - S_n = \int_0^{\pi/2} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x} dx$$

$$= \int_0^{\pi/2} \frac{2.\cos 2nx.\sin x}{\sin x} dx = 2\int_0^{\pi/2} \cos 2nx\, dx$$

$$= 2\left[\frac{\sin 2nx}{2n}\right]_0^{\pi/2} = 0 \text{ for all integral values of } n.$$

$\therefore$ $S_{n+1} = S_n = S_{n-1} \ldots.. = S_1.$

Now, $S_1 = \int_0^{\pi/2} \frac{\sin x}{\sin x} dx = \int_0^{\pi/2} dx = \frac{\pi}{2}.$

$\therefore$ $S_{n+1} = S_n = \frac{\pi}{2}.$

Also, $V_{n+1} - V_n = \int_0^{\pi/2} \frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x} dx$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x.\sin x}{\sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x} dx = S_{n+1}$$

$\therefore$ $V_n - V_{n-1} = S_n = \frac{\pi}{2}$, $V_{n-1} - V_{n-2} = \frac{\pi}{2}$,

$\ldots.$, $V_2 - V_1 = \frac{\pi}{2}.$

$\therefore$ On adding, $V_n - V_1 = (n-1)\frac{\pi}{2}.$

Since $V_1 = \int_0^{\pi/2} dx = \frac{\pi}{2}$, we have $V_n = \frac{n\pi}{2}.$

Example 11. If $U_n = \int_0^{\pi}\left(\frac{1 - x\cos nx}{1 - \cos x}\right) dx$ where n is a positive integer or zero, then show that $U_{n+2} + U_n = 2U_{n+1}$. Hence show that $\int_0^{\pi} \frac{\sin^2 n\theta}{\sin^2\theta} = \frac{n\pi}{2}.$

Solution $U_n = \int_0^{\pi}\left(\frac{1 - \cos nx}{1 - \cos x}\right) dx$

$\therefore$ $U_{n+2} - U_{n+1}$

$$= \int_0^{\pi} \frac{(1 - \cos(n+2)x) - (1 \cos(n+1)x)}{(1 - \cos x)} dx$$

$$= \int_0^{\pi} \frac{\cos(n+1)x - \cos(n+2)x}{(1 - \cos x)} dx$$

$$= \int_0^{\pi} \frac{2\sin(n+3/2)x \sin x/2}{2\sin^2 x/2} dx$$

$$\Rightarrow U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\sin(n+3/2)x}{\sin x/2} dx \quad ...(1)$$

Similarly, $U_{n+1} - U_n = \int_0^{\pi} \frac{\sin(n+1/2)}{\sin x/2} dx \quad ...(2)$

From (1) and (2), we get

$(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n)$

$$= \int_0^{\pi} \frac{\sin(n+3/2)x - \sin(n+1/2)x}{\sin x/2} dx$$

$$= \int_0^{\pi} \frac{2\cos(n+1)x \sin x/2}{\sin x/2} dx$$

$$= 2\left\{\frac{\sin(n+1)x}{(n+1)}\right\}_0^{\pi}$$

Hence, $(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n) = 0$

$\therefore\ U_{n+2} + U_n = 2U_{n+1}$

$\Rightarrow\ U_{n+2} - U_{n+1} = U_{n+1} - U_n$

Similarly, $U_{n+2} - U_{n+1} = U_{n+1} - U_n$

$= U_n - U_{n-1} = ... = U_1 - U_0$

$\therefore\ U_n - U_{n-1} = U_1 - U_0 = \pi - 0$

$\Rightarrow\ U_n = \pi + U_{n-1} = \pi + \pi + U_{n-2} = 2\pi + U_{n-2}$

$U_n = n\pi + U_0$

$U_n = n\pi$

$[\because U_0 = 0]$

Hence, $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta} d\theta = \int_0^{\pi/2} \frac{1-\cos 2n\theta}{1-\cos 2\theta} d\theta$.

Put $2\theta = x$

$\therefore\ d\theta = \frac{dx}{2}$

Hence, $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta} d\theta = \frac{1}{2}\int_0^{\pi} \frac{1-\cos nx}{1-\cos x} dx$

$= \frac{1}{2} U_n = \frac{1}{2} n\pi$ from above.

Example 12. If $I_n = \int_{-\infty}^{0} e^x \sin^n x dx\ \forall\ n \geq 2 \in N$, then prove that I_{n-2}, I_n, I_{n+2} cannot be in G.P.

Solution $I_n = \int_{-\infty}^{0} e^x \sin^n x\, dx$

$$= \left[\sin^n x \int e^x dx\right]_{-\infty}^{0} - \int_{-\infty}^{0} n \sin^{n-1} x \cos x\, e^x dx$$

$$= -n \int_{-\infty}^{0} \sin^{n-1} x \cos x\, e^x dx$$

$$= -n\, [\sin^{n-1} x \cos x e^x]_{-\infty}^{0}$$

$$+ n\int_{-\infty}^{0} ((n-1)\sin^{n-2} x \cos^2 x - \sin^{n-1} x \sin x) e^x dx$$

$$= n(n-1)\int_{-\infty}^{0} (\sin^{n-2} x\,(1-\sin^2 x)) e^x dx$$

$$- n\int_{-\infty}^{0} \sin^n x\, e^x dx$$

$$= n(n-1)\int_{-\infty}^{0} \sin^{n-2} x\, e^x dx - n(n-1)$$

$$\int_{-\infty}^{0} \sin^n x\, e^x dx - n\int_{-\infty}^{0} \sin^n x\, e^x dx$$

$\Rightarrow\ I_n = n(n-1)I_{n-2} - n(n-1)I_n - nI_n$

$\Rightarrow\ I_n(1+n^2) = n(n-1)I_{n-2}$

$$\Rightarrow\ I_n = \frac{n(n-1)}{n^2+1} I_{n-2} \quad ...(1)$$

$$\text{Now } I_{n+2} = \frac{(n+1)(n+2)}{n^2+4n+5} I_n \quad ...(2)$$

From equation (1) and (2)

$$\frac{I_n}{I_{n-2}} = \frac{n(n-1)}{n^2+1} \text{ and } \frac{I_{n+2}}{I_n} = \frac{(n+1)(n+2)}{n^2+4n+5}$$

Let I_{n-2}, I_n and I_{n+2} are in G.P., then

$$\frac{I_n}{I_{n-2}} = \frac{I_{n+2}}{I_n} \Rightarrow \frac{n(n-1)}{n^2+1} = \frac{(n+1)(n+2)}{n^2+4n+5}$$

$\Rightarrow\ 2n^2 + 8n + 2 = 0$ which is not possible $\forall\ n \in N$.

$\Rightarrow\ I_{n-2}, I_n$ and I_{n+2} cannot be in G.P.

Practice Problems I

1. Derive a reduction formula and compute the integral $\int_{-1}^{0} x^n e^x dx$, (n is a positive integer).

2. Prove that if $J_m = \int_1^e \ln^m x\, dx$, then $J_m = e - mJ_{m-1}$, (m is a positive integer).

3. Evaluate $\int_0^{\frac{\pi}{4}} \frac{d}{dx}\left\{\int_1^x \sec^4\theta\, d\theta\right\} dx$.

4. If $I_n = \int_0^{\pi/2} x^n \sin x\, dx$, prove that $I_5 = \frac{5\pi^4}{16} - 15\pi^2 + 120$.

5. Evaluate $\int_0^a (a^2-x^2)^{5/2}dx$.

6. Evaluate

(i) $\int_0^a x^2\,(a^2-x^2)^{5/2}\,dx$,

(ii) $\int_0^1 \frac{x^4}{(1+x^2)^2}dx$ (iii) $\int_0^1 x^3 e^{x^2}dx$

7. Prove that $\int_0^{\pi/2} \cos^4 x \cos 3x\,dx = \frac{8}{35}$.

8. Prove that $\int_0^{\pi} \frac{\sin 7x}{\sin x}dx = \pi$.

9. Prove that $\int_0^{\pi} \frac{1-\cos 5x}{1-\cos x} = 5\pi$.

10. Prove that $\int \frac{dx}{(x^2+1)^4}$

11. Show that

(i) $I_n = \int_0^{\infty} \frac{dx}{(1+x^2)^n} = \frac{2n-3}{2n-2}I_{n-1}$

(ii) $\int_0^{\infty} \frac{dx}{(1+x^2)^5} = \frac{1.3.5.7}{2.4.6.8}\frac{\pi}{2}$.

12. Employing Euler's formula $e^{ix} = \cos x + i\sin x$, prove that $\int_0^{2\pi} e^{-inx}.e^{imx}dx = \begin{cases} 0 & \text{for } m \neq n, \\ 2\pi & \text{for } m = n \end{cases}$

13. Using Euler's formulae $\cos x = \frac{1}{2}(e^{ix}+e^{-ix})$, $\sin x = \frac{1}{2i}(e^{ix}+e^{-ix})$ calculate the integrals :

(a) $\int_0^{\pi/2} \sin^{2m} x \cos^{2n} x\,dx$

(b) $\int_0^{\pi} \frac{\sin nx}{\sin x}dx$

(c) $\int_0^{\pi} \cos^n x \cos nx\,dx$

(d) $\int_0^{\pi} \sin^n x \cos nx\,dx$

14. Find $\int_0^1 x^p(1-x)^q dx$ (p and q positive integers).

15. Show that $\int_0^1 (1-x^2)^n dx = \frac{2^{2n}(n!)^2}{(2n+1)!}$.

16. Utilizing the equation $\int_0^1 x^{n-1}dx = \frac{1}{n}$, compute the integral $\int_0^1 x^{n-1}(\ln x)^k\,dx$.

17. If $u_n = \int \cos n\theta \operatorname{cosec}\theta\,d\theta$, prove that $u_n - u_{n-2} = \frac{2\cos(n-1)\theta}{n-1}$. Hence or otherwise find the value of $\int_0^{\pi/2} \frac{\sin 3\theta \sin 5\theta}{\sin\theta}d\theta$.

18. Compute $\int_0^1 \frac{x^m\,dx}{\sqrt{1-x^2}}$ when m is (a) even, (b) odd (m>0).

19. Derive a formula for $I_n = \int_0^1 \frac{(1-x)^n}{\sqrt{x}}dx$, (n is a positive integer).

20. If $I_n = \int_0^1 (1-x^3)^n dx$, prove that $I_n = \frac{3n}{3n+1} I_{n-1}$.

Hence, evaluate $\frac{{}^nC_0}{1} - \frac{{}^nC_1}{4} + \frac{{}^nC_2}{7} - \ldots$

2.11 EVALUATION OF LIMIT OF SUM USING NEWTON-LEIBNITZ FORMULA

From the definition of definite integral, we have :

1. $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a+\frac{(b-a)\,r}{n}\right) = \int_a^b f(x)\,dx$

2. $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a+\left(\frac{b-a}{n}\right)r\right) = \int_a^b f(x)\,dx$

3. $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

4. $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{r=\phi(x)}^{\psi(x)} f\left(\frac{r}{n}\right) = \int_a^b f(x)dx$, where

(i) Σ is replaced by $\int$ sign

(ii) $\frac{r}{n}$ is replaced by x,

(iii) $\frac{1}{n}$ is replaced by dx,

(iv) To obtain the limits of integration, we use

$$a = \lim_{n\to\infty} \frac{\phi(x)}{n} \text{ and } b = \lim_{n\to\infty} \frac{\psi(x)}{n}$$

Note: that we have to see the following things before we can express the limit of its sum as a definite integral.

(i) Each term of the series must be multiplied by $1/n$ which $\to 0$ when $n \to \infty$.

(ii) All the terms should be some function of r/n which varies from term to term in A.P. with common difference $1/n$

Example 1. Evaluate

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\frac{1}{\sqrt{4n^2-9}}+\ldots..\frac{1}{\sqrt{3n^2}}\right].$$

Solution Let L

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\frac{1}{\sqrt{4n^2-9}}+\ldots..\frac{1}{\sqrt{3n^2}}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{(1-0)}{n}\frac{1}{\sqrt{4-\left(0+r\left(\frac{1-0}{n}\right)\right)^2}}$$

which is of the form

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{b-a}{n}f\left(a+r\left(\frac{b-a}{n}\right)\right)$$

Here $b=1$, $a=0$ and $f(x)=\frac{1}{\sqrt{4-x^2}}$.

So, $L=\int_0^1\frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2}\Big|_0^1 = \frac{\pi}{6}$.

Example 2. Evaluate $\lim_{n\to\infty}\sum_{k=1}^{k=n}\frac{1}{\sqrt{n^2+3kn}}$.

Solution $T_k = \frac{1}{n}\frac{1}{\sqrt{1+\frac{3k}{n}}}$

$$\text{Limit} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{\sqrt{1+\frac{3k}{n}}} = \int_0^1\frac{dx}{\sqrt{1+3x}}$$

$$= \int_0^1(1+3x)^{-1/2}dx = \left.\frac{(1+3x)^{1/2}}{3/2}\right|_0^1$$

$$= \frac{2}{3}[2-1] = \frac{2}{3}.$$

Example 3. Evaluate

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left[\left(1+\frac{2i}{n}\right)^2+1\right]$$

Solution $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left[\left(1+\frac{2i}{n}\right)^2+1\right]$

$$= 2\int_0^1\left((1+2x)^2+1\right)dx$$

$$= 2\left[\frac{(1+2x)^3}{2\cdot 3}+x\right]_0^1 = 2\left[\left(\frac{27}{6}+1\right)-\frac{1}{6}\right]$$

$$= 2\left[\frac{27}{6}+\frac{5}{6}\right] = \frac{32}{3}.$$

Example 4. Evaluate

$$\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+4^2}+\ldots..+\frac{n}{n^2+(2n-2)^2}\right]$$

Solution $\text{Limit} = \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{n}{n^2+(2r)^2} = \int_0^1\frac{1}{1+(2x)^2}dx$

$$= \frac{1}{2}\left[\tan^{-1}2x\right]_0^1 = \frac{1}{2}\tan^{-1}2.$$

Example 5. Evaluate

$$\lim_{n\to\infty}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+\dots\dots\dots+\frac{1}{2n}\right]$$

Solution $\lim_{n\to\infty}\left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+\dots\dots\dots+\frac{1}{2n}\right]$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r+n}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{1}{\left(\frac{r}{n}\right)+1}=\int_0^1\frac{dx}{x+1}$$

$$=\left[\ln(x+1)\right]_0^1=\ln 2.$$

Example 6. Evaluate $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2}\sin\frac{\pi i^2}{n^2}$

Solution $T_r=\frac{r}{n^2}\cdot\sin\frac{\pi r^2}{n^2}=\frac{1}{n}\cdot\frac{r}{n}\sin\pi\left(\frac{r}{n}\right)^2$

$$\text{Limit}=\lim_{n\to\infty}\sum_{r=1}^{n}T_r=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{n}\sin\pi\left(\frac{r}{n}\right)^2$$

$$=\int_0^1 x\sin\pi x^2\,dx$$

Put $\pi x^2=t\Rightarrow 2\pi x\,dx=dt$

$$S=\frac{1}{2\pi}\int_0^{\pi}\sin t\,dt=\frac{1}{\pi}.$$

Example 7. Evaluate

$$\lim_{n\to\infty}\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\dots.+\frac{1}{n}$$

Solution Limit

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n+r}{n^2+r^2}=\lim_{n\to\infty}\sum\frac{1}{n}\frac{(1+r/n)}{(1+r^2/n^2)}$$

$$=\int_0^1\frac{x+1}{x^2+1}dx=\left[\frac{1}{2}\ln(x^2+1)+\tan^{-1}x\right]_0^1$$

$$=\frac{1}{2}\ln 2+\frac{\pi}{4}.$$

Example 8. Prove that

$$\lim_{n\to\infty}\frac{1^m+2^m+3^m\dots.+n^m}{n^{m+1}}=\frac{1}{m+1},\quad m>-1.$$

Solution Limit

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^m+\left(\frac{2}{n}\right)^m+\dots+\left(\frac{n}{n}\right)^m\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^m$$

$$=\int_0^1 x^m\,dx=\left[\frac{x^{m+1}}{m+1}\right]_0^1=\frac{1}{m+1}.$$

Example 9. Evaluate

$$\lim_{n\to\infty}\frac{(1^2+2^2+3^2+\dots+n^2)(1^3+2^3+3^3+\dots+n^3)}{1^6+2^6+3^6+\dots+n^6}$$

Solution The given limit is $\lim_{n\to\infty}\frac{\sum_{r=1}^{n}r^2\times\sum_{r=1}^{n}r^3}{\sum_{r=1}^{n}r^6}$

$$=\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^2\times\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^3}{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^6}$$

$$=\frac{\int_0^1 x^2dx\int_0^1 x^3dx}{\int_0^1 x^6dx}$$

$$=\frac{\left[\frac{x^3}{3}\right]_0^1\left[\frac{x^4}{4}\right]_0^1}{\left[\frac{x^7}{7}\right]_0^1}=\frac{\frac{1}{3}\times\frac{1}{4}}{\frac{1}{7}}=\frac{7}{12}.$$

Example 10. Evaluate

$$\lim_{n\to\infty}\left[\frac{n}{(n+1)}\cdot\frac{1}{\sqrt{(2n+1}}+\frac{n}{(n+2)}\frac{1}{\sqrt{2(2n+2)}}+\dots.+\frac{n}{2n\sqrt{(n.3n)}}\right]$$

Solution General term $=\frac{n}{(n+r)\sqrt{r(2n+r)}}$

$$\therefore\quad\text{limit}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\left(1+\frac{r}{n}\right).n\sqrt{\left(2\frac{r}{n}+\frac{r^2}{n^2}\right)}}$$

$$= \int_0^1 \frac{dx}{(1+x)\sqrt{(x^2+2x)}} = \int_0^1 \frac{dx}{(x+1)\sqrt{\{(x+1)^2-1\}}}$$

$$= \left[\sec^{-1}(x+1)\right]_0^1 = \sec^{-1}2 - \sec^{-1}1 = \pi/3 - 0 = \pi/3$$

Example 11. Evaluate $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

Solution Let $L = \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

$$\ln L = \lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{n!}{n^n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{1.2.3\,........n}{n^n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(\frac{1}{n}\right)+\ln\left(\frac{2}{n}\right)+\ln\left(\frac{3}{n}\right)+.....+\ln\left(\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(\frac{r}{n}\right)$$

$$= \int_0^1 \ln x\,dx = x\ln x - x\Big]_0^1$$

$$= (0-1) - \lim_{x\to 0^+} x\ln x + 0 = -1-0 = -1$$

$$\Rightarrow\ L = \frac{1}{e}.$$

Example 12.

Show that $\lim_{n\to\infty}\prod_{r=1}^{n}\left[\phi\left(a+\frac{br}{n}\right)\right]^{1/n} = e^{\lambda}$

where $\lambda = \frac{1}{b}\int_a^{a+b}\ln\phi(x)\,dx.$

Solution We have

$$L = \lim_{n\to\infty}\prod_{r=1}^{n}\left[\phi\left(a+\frac{br}{n}\right)\right]^{1/n}$$

$$= \lim_{n\to\infty}\left[\phi\left(a+\frac{b}{n}\right)\cdot\phi\left(a+\frac{2b}{n}\right)...\phi\left(a+\frac{nb}{n}\right)\right]^{1/n}$$

Taking logarithm on both sides, we have

$$\ln L = \lim_{n\to\infty}\frac{1}{n}\left[\ln\phi\left(a+\frac{b}{n}\right)+\ln\phi\left(a+\frac{2b}{n}\right) +...+\ln\phi\left(a+\frac{nb}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\phi\left(a+\frac{br}{n}\right) = \int_0^1 \ln\phi\,(a+bx)\,dx$$

$$= \frac{1}{b}\int_a^{a+b}\ln\phi(z)\,dx \qquad [\text{Putting } a+bx=z]$$

which is the desired result.

Example 13. Evaluate

$$\lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)}\cdot(1^1\cdot 2^2\cdot 3^3.......n^n)^{\frac{1}{n^2}}.$$

Solution Let

$$L = \lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)}\cdot(1^1\cdot 2^2\cdot 3^3.......n^n)^{\frac{1}{n^2}}$$

$$\ln L = \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n}k\ln k$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n$$

$$+\frac{1}{n^2}\sum_{k=1}^{n}\left(k\ln k - k\ln n + k\ln n\right)$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n}k\ln\frac{k}{n} + \frac{\ln n}{n^2}\sum_{k=1}^{n}k$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\frac{k}{n} + \frac{\ln n}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \int_0^1 x\ln x\,dx + \frac{1}{2}\left(\frac{n+1}{n}\right)\ln n$$

$$= \int_0^1 x\ln x\,dx = -\frac{1}{4}.\ \text{(by integrating by parts).}$$

$$\therefore\ L = e^{-\frac{1}{4}}.$$

Example 14. Evaluate

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)....\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$

Solution If L be the required limit then ln

$$L=\int_0^1 \ln(1+x^2)dx=\left[x\ln(1+x^2)-2x+2\tan^{-1}x\right]_0^1$$

$$=\ln 2+\frac{\pi-4}{2}$$

$$\therefore\quad \ln L-\ln 2=\frac{\pi-4}{2}$$

$$\therefore\quad L=2e^{(\pi-4)/2}.$$

Example 15. Evaluate $\lim_{n\to\infty}({}^{2n}C_n)^{1/n}$.

Solution Let $L=\lim_{n\to\infty}\left(\frac{(2n)!}{n!n!}\right)^{1/n}$

$$L=\left[\frac{n!(n+1)(n+2).....(n+n)}{n!n!}\right]^{1/n}$$

$$=\left(\frac{n+1}{1}\cdot\frac{n+2}{2}......\frac{n+n}{n}\right)^{1/n}$$

$$\ln L=\frac{1}{n}\left(\ln\frac{n+1}{1}+\ln\frac{n+2}{2}+......+\ln\frac{n+n}{n}\right)$$

Here, $T_r=\frac{1}{n}\left(\ln\frac{n+r}{r}\right)=\frac{1}{n}\left(1+\frac{1}{r/n}\right)$

$$\therefore\quad S_n=\frac{1}{n}\sum_{r=1}^{n}\left(1+\frac{1}{r/n}\right)$$

Hence, $\ln L=\int_0^1 \ln\left(1+\frac{1}{x}\right)dx$

$$=\int_0^1(\ln(1+x)-\ln x)\,dx$$

$$=(1+x)\ln(1+x)-(1+x)-[x\,ln\,x-x]$$

$$=[(1+x)\ln(1+x)-1-x\ln x]_0^1$$

$$=(2\,ln\,2-1-0)-(0-1)$$

Thus, $\ln L=\ln 4\ \Rightarrow\ L=4$.

Example 16. Evaluate

$$\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+.........+\frac{3}{5n}\right]$$

Solution $\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{n+r}{n^2+r^2}=\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{1}{n}\frac{1+\frac{r}{n}}{1+\left(\frac{r}{n}\right)^2}$

$\because\ \lim_{n\to\infty}\left(\frac{r}{n}\right)=0$, when $r=1$, lower limit $=0$

and $\lim_{n\to\infty}\left(\frac{r}{n}\right)=\lim_{n\to\infty}\left(\frac{2n}{n}\right)=2$, when $r=2n$,

upper limit $=2$. The given limit

$$=\int_0^2\frac{1+x}{1+x^2}dx=\int_0^2\frac{1}{1+x^2}dx+\frac{1}{2}\int_0^2\frac{2x}{1+x^2}dx$$

$$=\tan^{-1}x]_0^2+\frac{1}{2}\ln(1+x^2)\Big]_0^2$$

$$=\tan^{-1}2+\frac{1}{2}\ln 5.$$

Example 17. Evaulate

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left\{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+.....\frac{1}{\sqrt{4n}}\right\}$$

Solution Let L

$$=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left\{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+.....+\frac{1}{\sqrt{4n}}\right\}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{4n}\frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{r}}=\lim_{n\to\infty}\sum_{r=1}^{4n}\frac{1}{n}\cdot\frac{1}{\sqrt{(r/n)}}$$

When $r=1$

$$\lim_{n\to\infty}\frac{1}{n}=0$$

and when $r=4n$ then $x=\lim_{n\to\infty}\frac{4n}{n}=4$

$$\therefore\quad L=\int_0^4\frac{dx}{\sqrt{x}}=2\sqrt{x}\Big|_0^4=2(2-0)=4.$$

Example 18. Evaluate

$$\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+.........+\frac{1}{64n}\right].$$

Solution Let

$$L=\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+.........+\frac{1}{64n}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{3n}\frac{n^2}{(n+r)^3}$$

Put $3n=m$, we get

$$L = \lim_{n\to\infty} \sum_{r=1}^{m} \frac{m^2/9}{\left(\frac{m}{3}+r\right)^3}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{m} \frac{3}{m} \left(\frac{1}{\left(1+\frac{3r}{m}\right)}\right)^3$$

$$= \int_0^3 \frac{dx}{(1+x)^3} = \left.\frac{-1}{2(1+x)^2}\right|_0^3 = \frac{15}{32}.$$

Example 19. Find the value of

$$S = \lim_{n\to\infty} \sum_{r=1}^{r=4n} \frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}.$$

Solution $T_r = \dfrac{1}{\sqrt{\frac{r}{n}}\cdot n\left(3\sqrt{\frac{r}{n}}+4\right)^2}$

$$S = \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{4n} \frac{1}{\left(3\sqrt{\frac{r}{n}}+4\right)^2 \cdot \sqrt{\frac{r}{n}}}$$

$$= \int_0^4 \frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$$

Put $3\sqrt{x}+4 = t \Rightarrow \dfrac{3}{2}\dfrac{1}{\sqrt{x}}dx = dt$

$$= \frac{2}{3}\int_4^{10} \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right]_{10}^{4} = \frac{2}{3}\left[\frac{1}{4}-\frac{1}{10}\right]$$

$$= \frac{2}{3}\cdot\frac{6}{40} = \frac{1}{10}.$$

Example 20. Compute

$$\lim_{n\to\infty} \frac{3}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\sqrt{\frac{n}{n+9}}+ \ldots\ldots + \sqrt{\frac{n}{n+3(n-1)}}\right]$$

Solution Transform the given expression in the following way :

$$\frac{3}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\ldots\ldots+\sqrt{\frac{n}{n+3(n-1)}}\right]$$

$$= \frac{3}{n}\left[\sqrt{\frac{1}{1+0}}+\sqrt{\frac{1}{1+\frac{3}{n}}}+\sqrt{\frac{1}{1+\frac{6}{n}}}+\ldots\ldots+\sqrt{\frac{1}{1+\frac{3(n-1)}{n}}}\right]$$

The given sum is the integral sum for the function

$f(x) = \sqrt{\dfrac{1}{1+x}}$ on the interval [0, 3].

Therefore,

$$\lim_{n\to\infty} \frac{3}{n}\left(1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\ldots\ldots\sqrt{\frac{n}{n+3(n-1)}}\right)$$

$$= \int_0^3 \sqrt{\frac{1}{1+x}}dx = \int_0^3 (1+x)^{-1/2}dx = 2\sqrt{1+x}\Big|_0^3$$

$$= 4-2 = 2.$$

Example 21. Compute

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{an}\right)$$

where a is a positive integer. Calculate approximately

$$\frac{1}{100}+\frac{1}{101}+\frac{1}{102}+\ldots\ldots+\frac{1}{300}.$$

Solution Let

$$P = \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots.+\frac{1}{an}\right) \qquad ...(1)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n+0}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots.+\frac{1}{n+n(a-1)}\right)$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n(a-1)}\frac{1}{(n+r)} = \lim_{n\to\infty}\sum_{r=0}^{n(a-1)}\frac{1}{n(1+r/n)}$$

$$= \int_0^{(a-1)}\frac{dx}{(1+x)} = [ln\,(1+x)]_0^{a-1}$$

Hence $P = ln\, a$.

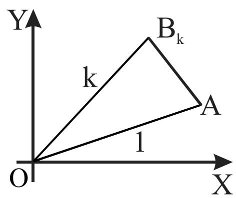
Put n = 100, and a = 3 in (1), we get

$$\frac{1}{100}+\frac{1}{101}+\frac{1}{102}+\ldots\ldots+\frac{1}{300} = ln\,3$$

$$= 2.303\log_{10}3 = (2.303)(0.4771) = 1.1 \text{ approx.}$$

Example 22. For positive integers $k = 1, 2, 3, \ldots, n$, let S_k denotes the area of ΔAOB_k (where 'O' is origin) such that $\angle AOB_k = \dfrac{k\pi}{2n}$, $OA = 1$ and $OB_k = k$. Find the value of $\lim\limits_{n\to\infty} \dfrac{1}{n^2}\sum\limits_{k=1}^{n} S_k$.

Solution



$$OB_k = k, \quad \angle AOB_k = \frac{k\pi}{2n}$$

$$S_k = \frac{1}{2}k\sin\frac{k\pi}{2n} \qquad \left(\text{using } \Delta = \frac{1}{2}ab\sin\theta\right)$$

$$\therefore \quad L = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} S_k = \frac{k}{2n^2}\sum_{n=1}^{\infty}\sin\frac{k\pi}{2n}$$

$$= \frac{1}{2n}\sum_{n=1}^{\infty}\frac{k}{n}\sin\frac{k\pi}{2n} = \frac{1}{2}\int_0^1 x\cdot\sin\frac{\pi x}{2}dx$$

$$= \frac{1}{2}\left[\frac{-2}{\pi}x\cos\frac{\pi x}{2}\Big|_0^1 + \frac{2}{\pi}\int_0^1\cos\frac{\pi x}{2}dx\right]$$

$$= \frac{1}{2}\left[0 + \frac{2}{\pi}\cdot\frac{2}{\pi}\left(\sin\frac{\pi x}{2}\right)_0^1\right] = \frac{2}{\pi^2}.$$

Example 23. Let

$$\lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+3}+\frac{1}{2n+5}+\ldots\ldots+\frac{1}{4n-1}\right)$$

$= \dfrac{A}{B}\,ln\,C$, where $A, B, C \in N$. Find the least value of $A + B + C$.

Solution Let given limit be L, then

$$L = \lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\frac{1}{2n+3}+\ldots\ldots+\frac{1}{4n}\right)$$

$$-\lim_{n\to\infty}\left(\frac{1}{2n+2}+\frac{1}{2n+4}+\frac{1}{2n+6}+\ldots\ldots+\frac{1}{4n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{2n}\frac{n}{2n+r} - \frac{1}{n}\sum_{r=1}^{n}\frac{n}{2n+2r}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{2n}\frac{1}{2+\frac{r}{n}} - \frac{1}{n}\sum_{r=1}^{n}\frac{1}{2+2\left(\frac{r}{n}\right)}\right]$$

$$= \int_0^2\frac{1}{2+x}dx - \int_0^1\frac{1}{2+2x}dx$$

$$= \left[\ln(2+x)\right]_0^2 - \frac{1}{2}\left[\ln(1+x)\right]_0^1$$

$$= \ln 4 - \ln 2 - \frac{1}{2}\ln 2 = \left(2 - \frac{3}{2}\right)\ln 2$$

$$= \frac{1}{2}\ln 2 = \frac{A}{B}\ln C.$$

Hence, least value $A + B + C = 1 + 2 + 2 = 5$.

Example 24. If H_n denote the harmonic mean of n positive integers $n+1, n+2, n+3, \ldots\ldots, n+n$, then find the value of $\lim\limits_{n\to\infty}\left(\dfrac{H_n}{n}\right)$.

Solution $$H_n = \frac{n}{\frac{1}{n+1}+\frac{1}{n+2}+\ldots..+\frac{1}{n+n}}$$

$$\Rightarrow \quad \frac{n}{H_n} = \frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{n+n}$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{n}{H_n}\right) = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\frac{1}{n}}{1+\frac{r}{n}}$$

$$= \int_0^1\frac{dx}{1+x} = ln\,2$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{H_n}{n}\right) = \frac{1}{\ln 2}.$$

Example 25. Let $C_n = \displaystyle\int_{\frac{1}{n+1}}^{\frac{1}{n}}\frac{\tan^{-1}(nx)}{\sin^{-1}(nx)}dx$ then find the value of $\lim\limits_{n\to\infty} n^2\cdot C_n$.

Solution $C_n = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{\tan^{-1}(nx)}{\sin^{-1}(nx)}dx$ (put $nx = t$)

$$\Rightarrow C_n = \frac{1}{n}\int_{\frac{n}{n+1}}^{1} \frac{\tan^{-1}(t)}{\sin^{-1}(t)}dt$$

$$L = \lim_{n\to\infty} n^2 \cdot C_n = \lim_{n\to\infty} n \cdot \int_{\frac{n}{n+1}}^{1} \frac{\tan^{-1} t}{\sin^{-1} t}dt \quad (\infty \times 0)$$

$$= \frac{\int_{\frac{n}{n+1}}^{1} \frac{\tan^{-1} t}{\sin^{-1} t}dt}{\frac{1}{n}} \quad \text{(applying L'Hospital's Rule)}$$

$$L = \lim_{n\to\infty} \frac{-\frac{\tan^{-1}\frac{n}{n+1}}{\sin^{-1}\frac{n}{n+1}}\left(\frac{1}{(n+1)^2}\right)}{-\frac{1}{n^2}} = \frac{\pi}{4}\cdot\frac{2}{\pi} = \frac{1}{2}.$$

Example 26. Evaluate

$$\lim_{n\to\infty}\left(\tan^{-1}\frac{1}{n}\right)\left(\sum_{k=1}^{n}\frac{1}{1+\tan(k/n)}\right).$$

Solution $\lim_{n\to\infty}\left(n\tan^{-1}\frac{1}{n}\right)\cdot \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\tan(k/n)}$

$$= 1.\int_0^1 \frac{dx}{1+\tan x} = \frac{1+\ln(\sin 1+\cos 1)}{2}.$$

Practice Problems J

1. Express the following limits in the form of an integral.

(i) $\lim_{n\to\infty}\frac{1}{n}\left(\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\ldots+\cos\frac{n\pi}{n}\right)$

(ii) $\lim_{n\to\infty}\frac{\pi}{6n}\left[\sec^2\left(\frac{\pi}{6n}\right)+\sec^2\left(2\frac{\pi}{6n}\right)+\ldots\right.$
$\left.\ldots+\sec^2\left((n-1)\frac{\pi}{6n}\right)+\frac{4}{3}\right]$

2. Evaluate the following limits :

(i) $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{(n+r)\sqrt{r(2n+r)}}$

(ii) $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$

(iii) $\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{1}{n+r}$

3. Evaluate the following limits :

(i) $\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\ldots.+\frac{n^2}{2n^3}\right]$

(ii) $\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots\ldots\ldots+2\sqrt{n}}{n\sqrt{n}}$

(iii) $\lim_{n\to\infty}\left[\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+\ldots..+\frac{1}{nb}\right]$

(iv) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\ldots\ldots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)$

4. Evaluate the following limits :

(i) $\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\tan\frac{3\pi}{2n}\ldots..\tan\frac{n\pi}{2n}\right]^{1/n}$

(ii) $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3}\ldots\left(1+\frac{n}{n}\right)^{1/n}\right]$

5. Prove that

(a) $\pi = \lim_{n\to\infty}\frac{4}{n^2}(\sqrt{n^2-1}+\sqrt{n^2-2^2}+\ldots+\sqrt{n^2-n^2})$.

(b) $\int_1^3 (x^2+1)dx = \lim_{n\to\infty}\frac{4}{n^3}\sum_{i=1}^{n}(n^2+2in+2i^2)$.

6. Evaluate the following limits :

(i) $\lim_{n\to\infty}\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{4n}$

(ii) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}\ldots..+\frac{1}{8n}\right]$

(iii) $\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+\ldots\ldots+\frac{3}{5n}\right]$

7. Evaluate the following limits :

(i) $\lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+.....+\frac{1}{n}\right)$

(ii) $\lim_{n\to\infty}\frac{2^k+4^k+6^k+..+(2n)^k}{n^{k+1}}$, $k\neq-1$

(iii) $\lim_{n\to\infty}\frac{3}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\sqrt{\frac{n}{n+9}}+.....\right.$ $\left.....+\sqrt{\frac{n}{n+3(n-1)}}\right]$

(iv) $\lim_{n\to\infty}\frac{n^2}{(n^2+1)^{3/2}}+\frac{n^2}{(n^2+2^2)^{3/2}}+$ $...+\frac{n^2}{\left[n^2+(n-1)^2\right]^{3/2}}$

8. Show that for each integer $m > 1$,

$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{m} < \ln m < 1+\frac{1}{2}+...+\frac{1}{m-1}.$$

9. Show that for each integer $m > 1$,

$$\ln 1+\ln 2+....+\ln(m-1) < m\ln m-m+1 < \ln 2+\ln 3+...+\ln m.$$

10. Find $\lim_{n\to\infty}\left(\frac{1^k+2^k+.....+n^k}{n^{k+1}}\right)$ for $k > 0$. Find the approximate value of $1^5+2^5+.....+100^5$.

11. Prove that when a is large the sum to infinity of the series $\frac{1}{a^2}+\frac{1}{a^2+1^2}+\frac{1}{a^2+2^2}+....$ is $\frac{1}{2}\pi/a$, approximately.

2.12 LEIBNITZ RULE FOR DIFFERENTIATION OF INTEGRALS

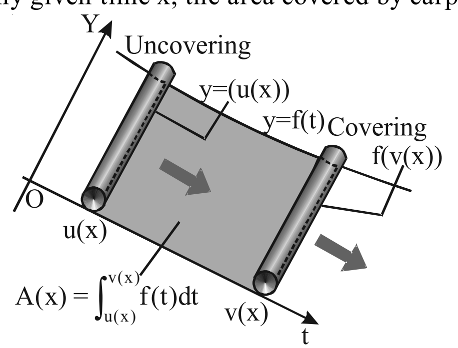
If f is continuous on [a, b], and u(x) and v(x) are differentiable functions of x whose values lie in [a, b], then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))\frac{dv}{dx}-f(u(x))\frac{du}{dx}$$

The following figure gives a geometric interpretation of Leibnitz rule. It shows a carpet of variable width f(t) that is being rolled up at the left at the same time x as it is being unrolled at the right. (In this interpretation time is x, not t.)

At time x, the floor is covered from u(x) to v(x). The rate $\frac{du}{dx}$ at which the carpet is being rolled up need not be the same as the rate $\frac{dv}{dx}$ at which the carpet is being laid down.

At any given time x, the area covered by carpet is



$$A(x)=\int_{u(x)}^{v(x)} f(t)dt.$$

The question is : at what rate is the covered area changing ?

At the instant x, A(x) is increasing by the width f(v(x)) of the unrolling carpet times the rate dv/dx at which the carpet is being unrolled. That is, A(x) is being increased at the rate

$$f(v(x))\frac{dv}{dx}.$$

At the same time, A is being decreased at the rate

$$f(u(x))\frac{du}{dx},$$

the width at the end that is being rolled up times the rate du/dx. The net rate of change in A is

$$\frac{dA}{dx}=f(v(x))\frac{dv}{dx}-f(u(x))\frac{du}{dx}.$$

which is precisely the Leibnitz rule.

Proof : To prove the rule, let F be an antiderivative of f on [a, b]. Then

$$\int_{u(x)}^{v(x)} f(t)\,dt = F(v(x))-F(u(x)). \quad ...(1)$$

Differentiating both sides of this equation with respect to x gives the equation we want :

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt=\frac{d}{dx}[F(v(x))-F(u(x))]$$

$$=F'(v(x))\frac{dv}{dx}-F'(u(x))\frac{du}{dx} \quad \text{[Chain Rule]}$$

$$= f(v(x))\,\frac{dv}{dx} - f(u(x))\,\frac{du}{dx}.$$

For example, if $F(x) = \int_{x}^{x^2} \sqrt{\sin t}\,dt$, then

$$F'(x) = 2x\,.\sqrt{\sin x^2} - 1\,.\,\sqrt{\sin x}\,.$$

Example 1. If $F(x) = \int_{e^{2x}}^{e^{3x}} \frac{t}{\ln t}\,dt$, then find first and second derivative of F(x) with respect to ln x at x = ln 2.

Solution $\frac{dF(x)}{d(\ln x)} = \frac{dF(x)}{dx}\cdot\frac{dx}{d(\ln x)}$

$$= \left[3\,.\,e^{3x}\,.\,\frac{e^{3x}}{\ln e^{3x}} - 2\,.\,e^{2x}\,\frac{e^{2x}}{\ln e^{2x}}\right]x = e^{6x} - e^{4x}.$$

Now, $\frac{d^2F(x)}{d(\ln x)^2} = \frac{d}{d(\ln x)}\,(e^{6x} - e^{4x})$

$$= \frac{d}{dx}\,(e^{6x} - e^{4x}) \times \frac{dx}{d(\ln x)} = (6\,e^{6x} - 4\,e^{4x})\,x$$

The first derivative of F(x) at x = ln 2 (i.e. $e^x = 2$) is $2^6 - 2^4 = 48$.
The second derivative of F(x) at x = ln 2 (i.e. $e^x = 2$) is $(6\,.\,2^6 - 4\,.\,2^4).\ln 2 = 5\,.\,2^6\,.\ln 2$.

Example 2. Which of the following functions are differentiable in (–1, 2)

(i) $\int_{x}^{2x} (\log x)^2\,dx$ (ii) $\int_{x}^{2x} \frac{\sin x}{x}\,dx$

(iii) $\int_{0}^{x} \frac{1 - t + t^2}{1 + t + t^2}\,dt$

Solution Since the functions $(\log x)^2$ and $\frac{\sin x}{x}$ are not well defined in (–1, 2), therefore the functions (i) and (ii) are not differentiable.

The function $f(t) = \frac{1 - t + t^2}{1 + t + t^2}$ is continuous on (–1, 2)

and $g(x) = \int_{0}^{x} \frac{1 - t + t^2}{1 + t + t^2}\,dt$ is the integral function of f(t), therefore g(x) is differentiable on (– 1, 2) such that, $g'(x) = f(x)$.

Example 3. Let $l = \lim_{x\to\infty} \int_{x}^{2x} \frac{dt}{t}$ and $m = \frac{1}{x \ln x}\int_{1}^{x} \ln t\,dt$ then prove that $l\,m = \ln 2$.

Solution $l = \lim_{x\to\infty} \ln 2x - \ln x = \ln 2$, and

$$m = \frac{\int_{1}^{x} \ln t}{x \ln x} = \lim_{x\to\infty} \frac{\ln x}{x\cdot\frac{1}{x} + \ln x} = \lim_{x\to\infty} \frac{\ln x}{1 + \ln x} = 1.$$

Hence $l \times m = \ln 2 \cdot 1 = \ln 2$.

Example 4. Evaluate $\lim_{x\to\infty} \frac{\left(\int_{0}^{x} e^{t^2}\,dt\right)^2}{\int_{0}^{x} e^{2t^2}\,dt}$

Solution $\lim_{x\to\infty} \frac{\left(\int_{0}^{x} e^{t^2}\,dt\right)^2}{\int_{0}^{x} e^{2t^2}\,dt} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \lim_{x\to\infty} \frac{2\,.\int_{0}^{x} e^{t^2}dt\,.\,e^{x^2}}{1\,.\,e^{2x^2}} \quad \text{Applying L' Hospital Rule}$$

$$= \lim_{x\to\infty} \frac{2\,.\int_{0}^{x} e^{t^2}dt}{e^{x^2}} = \lim_{x\to\infty} \frac{2\,.\,e^{x^2}}{2x\,.\,e^{x^2}} = 0.$$

Example 5. Let $f : (0, \infty) \to R$ be a continuous function such that $F(x) = \int_{0}^{x} t\,f(t)\,dt$. If $F(x^2) = x^4 + x^5$, then find $\sum_{r=1}^{12} f(r^2)$.

Solution We have

$$F(x^2) = \int_{0}^{x^2} t\,f(t)\,dt = x^4 + x^5 \qquad ...(1)$$

$\therefore$ On differentiating both the sides w.r.t. x, we get $2x\,(x^2)\,f(x^2) = 4x^3 + 5x^4$

$$\Rightarrow\ f(x^2) = 2 + \frac{5}{2}x \qquad ...(2)$$

$$\therefore\ \sum_{r=1}^{12} f(r^2) = \sum_{r=1}^{12}\left(2 + \frac{5}{2}r\right) = 24 + \left(\frac{5}{2}\right)\frac{(12)(13)}{2}$$

$$= 24 + (15)(13) = 24 + 195 = 216$$

Hence, $\sum_{r=1}^{12} f(r^2) = 219.$

Example 6. Let $f(x) = \int_{-1}^{x} e^{t^2}\,dt$ and $h(x) = f(1 + g(x))$, where g (x) is defined for all x, g'(x) exists for all x, and $g(x) \le 0$ for $x > 0$. If $h'(1) = e$ and $g'(1) = 1$, then find the possible value of g(1).

Solution Given $f(x) = \int_{-1}^{x} e^{t^2}\,dt$

$$h(x) = f(1+g(x)),\ g(x) \le 0 \text{ for } x > 0$$

Now $$h(x) = \int_{-1}^{1+g(x)} e^{t^2}\,dt$$

Differentiating w.r.t. x,

$$h'(x) = e^{(1+g(x))^2} \cdot g'(x)$$

$$h'(1) = e \qquad \text{(given)}$$

$$e^{(1+g(1))^2} \cdot g'(1) = e$$

$$\therefore\ (1 + g(1))^2 = 1$$

$$1 + g(1) = \pm 1$$

$$\Rightarrow\ g(1) = 0 \text{ or } g(1) = -2.$$

Example 7. Find the equation of the tangent to the curve $y = \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^2}}$ at $x = 1$.

Solution $$y = \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^2}}$$

$$\frac{dy}{dx} = \frac{3x^2}{\sqrt{1+x^6}} + \frac{2x}{\sqrt{1+x^2}}$$

$\Rightarrow$ Slope of tangent at x = 1:

$$m = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

Also, the value of y at x = 1 is :

$$y = \int_{1}^{1} \frac{dt}{\sqrt{1+t^2}} = 0$$

$\Rightarrow$ The equation of the tangent is

$$y = \frac{5}{\sqrt{2}}\,(x - 1).$$

Example 8. Let $F(x) = \int_{-1}^{x} \sqrt{4+t^2}\,dt$ and $G(x) = \int_{x}^{1} \sqrt{4+t^2}\,dt$ then compute the value of (FG)' (0).

Solution We have $F(x) = \int_{-1}^{x} f(t)\,dt$ and

$$G(x) = \int_{x}^{1} f(t)\,dt \quad \text{where } f(t) = \sqrt{4-t^2}$$

Let $H(x) = F(x) \cdot G(x)$

$$\therefore\ H'(x) = F(x) \cdot G'(x) + G(x) \cdot F'(x)$$

$$= \left(\int_{-1}^{x} f(t)\,dt\right)\left(-\sqrt{4+x^2}\right)$$

$$+ \left(\int_{-1}^{x} f(t)\,dt\right)\left(\sqrt{4+x^2}\right)$$

$$= (4 + x^2)\left[\int_{x}^{1} \sqrt{4+t^2}\,dt - \int_{-1}^{x} \sqrt{4+t^2}\,dt\right].$$

Now $$H'(0) = 4\left[\int_{0}^{1} \sqrt{4+t^2}\,dt - \int_{-1}^{0} \sqrt{4+t^2}\,dt\right]$$

We have $\int \sqrt{x^2 + a^2}\,dx$

$$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2+a^2}\right)$$

Hence, $\int_{0}^{1} \sqrt{4+t^2}\,dt$

$$= \left[\frac{t\sqrt{t^2+4}}{2} + 2\ln\left(t + \sqrt{t^2+4}\right)\right]_{0}^{1}$$

$$= \left[\frac{\sqrt{5}}{2} + 2\ln\left(\sqrt{5}+1\right)\right] - 2\ln 2$$

Also, $\int_{-1}^{0} \sqrt{4+t^2}\,dt = 2\ln 2 - \left\{-\frac{\sqrt{5}}{2} + 2\ln\left(\sqrt{5}-1\right)\right\}$

$$= 2\ln 2 + \frac{\sqrt{5}}{2} - 2\ln(\sqrt{5} - 1)$$

$\therefore$ $H'(0) = 4\,[2\ln(\sqrt{5}+1) + 2\ln(\sqrt{5}-1) - 4\ln 2]$
$= 4[2\ln(4) - 4\ln 2] = 4[4\ln(2) - 4\ln 2]$
$= 0.$

Example 9. Given a function g, continuous everywhere such that g(1)=5 and $\int_0^1 g(t)\,dt = 2.$

If $f(x) = \frac{1}{2}\int_0^x (x-t)^2 g(t)\,dt$, then compute the value of $f'''(1) - f''(1)$.

Solution $g(1) = 5$ and $\int_0^1 g(t)\,dt = 2.$

$$2f(x) = \int_0^x (x^2 - 2xt + t^2)\, g(t)\,dt$$

$$= x^2\int_0^x g(t)\,dt - 2x\int_0^x t\,g(t)\,dt + \int_0^x t^2 g(t)\,dt$$

Differentiating w.r.t. x,

$$2f'(x) = x^2 g(x) + \int_0^x g(t)dt\cdot 2x$$

$$-2\left[x^2 g(x) + \left(\int_0^x t\,g(t)\,dt\right)\right] + x^2 g(x)$$

$$2f'(x) = 2x\int_0^x g(t)dt - 2\int_0^x t\,g(t)\,dt$$

$$f''(x) = x\,g(x) + \int_0^x g(t)dt - x\,g(x)$$

$$= \int_0^x g(t)dt$$

Hence, $f''(1) = \int_0^1 g(t)\,dt = 2$ (given)

Also, $f'''(x) = g(x)$

$\Rightarrow$ $f'''(1) = g(1) = 5$

$\therefore$ $f'''(1) - f''(1) = 5 - 2 = 3.$

Modified Leibnitz Theorem

If $F(x) = \int_{g(x)}^{h(x)} f(x,t)\,dt$, then

$$F'(x) = \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}\,dt + f(x, h(x))\,h'(x) - f(x, g(x))\,.\,g'(x)$$

Example 10. If $f(x) = \int_{\ln x}^{x} \frac{dt}{x+t}$, then find $f'(x)$.

Solution $f'(x) = \int_{\ln x}^{x} \frac{-1}{(x+t)^2}\,dt + 1\,.\,\frac{1}{2x} - \frac{1}{x}\,\frac{1}{(x+\ln x)}$

$$= \left.\frac{1}{(x+t)}\right]_{\ln x}^{x} + \frac{1}{2x} - \frac{1}{x\,(x+\ln x)}$$

$$= \frac{1}{2x} - \frac{1}{x+\ln x} + \frac{1}{2x} - \frac{1}{x\,(x+\ln x)}$$

$$= \frac{1}{x} - \frac{x+1}{x\,(x+\ln x)} = \frac{\ln x - 1}{x\,(x+\ln x)}.$$

Alternative:

$f(x) = \int_{\ln x}^{x} \frac{dt}{x+t} = \left.\ln(x+t)\right]_{\ln x}^{x}$ (treating 't' as constant)

$\Rightarrow$ $f(x) = \ln x - \ln(x + \ln x)$

$\Rightarrow$ $f'(x) = \frac{1}{x} - \frac{1}{(x+\ln x)}\left(1 + \frac{1}{x}\right) = \frac{\ln x - 1}{x(x+\ln x)}.$

Concept Problems I

1. If $F(x) = \int_{x^2}^{e^x} \cos t\;dt$, find $F'(x)$.
2. If $f(x) = \int_0^{x^3} \sqrt{\cos t}\,dt$, find $f'(x)$.
3. Find the derivative of the function
   $$y = \int_0^{x} \frac{1 - t + t^2}{1 + t + t^2}\,dt \text{ at } x = 1.$$
4. If $f(x) = \int_x^{x^2} x^2 \sin t\,dt$ then find $f'(x)$.
5. If $x = \int_0^y \frac{dt}{\sqrt{1+4t^2}}$ and $\frac{d^2y}{dx^2} = ky$ then find k
6. Find the derivative with respect to x of the function y represented parametrically
   (i) $x = \int_0^t \sin t\;dt$, $y = \int_0^t \cos t\;dt$;
   (ii) $x = \int_1^{t^2} t\ln t\;dt$, $y = \int_{t^2}^{1} t^2 \ln t\;dt$.
7. If $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \frac{t}{1+t^4}\,dt$ then find the value of $f'(2)$.
8. Find the second derivative with respect to z of the function
   $$y = \int_0^{z^2} \frac{dx}{1+x^3} \text{ for } z = 1.$$

Practice Problems K

9. Find the derivative with respect to x of the function y specified implicitly by

$$\int_0^y e^t\,dt + \int_0^x \cos t\,dt = 0.$$

10. Find the value of the function $f(x) = 1 + x + \int_1^x ((\ln t)^2 + 2\ln t)\,dt$ where $f'(x)$ vanishes.

11. Find $\dfrac{d^2}{dx^2}\left(\int_0^{x^2} \dfrac{dt}{\sqrt{1-5t^3}}\right)^2$

12. If $x = \int_1^{t^2} z\ln z\,dz$ and $y = \int_{t^2}^1 z^2 \ln z\,dz$ find $\dfrac{dy}{dx}$.

13. If $\{F(x)\}^{101} = \int_0^x (F(t))^{100} \dfrac{dt}{1+\sin t}$, then find F(x).

14. If $\phi(x) = \cos x - \int_0^x (x-t)\,\varphi(t)\,dt$, then find the value of $\phi''(x) + \phi(x)$.

15. Find the interval in which

$$F(x) = \int_{-1}^x (e^t - 1)\,(2-t)dt,\ (x > -1) \text{ is}$$

increasing.

16. Find the point of maxima of $f(x) = \int_0^{x^2} \dfrac{t^2 - t}{e^t + 1}dt.$

17. At what value of x does the function $I(x) = \int_0^x xe^{-x^2}\,dx$ have an extremum? What is it equal to?

18. Let $g(x) = xe^{x^2}$ and let $f(x) = \int_1^x g(t)\left(t + \dfrac{1}{t}\right)dt.$ Compute the limit of $f''(x)/g''(x)$ as $x \to \infty$.

19. Let $g(x) = x^c e^{2x}$ and let $f(x) = \int_0^x e^{2t}(3t^2+1)^{1/2}\,dt.$ For a certain value of c, the limit of $f'(x)/g'(x)$ as $x \to \infty$ is finite and nonzero. Determine c and compute the value of the limit.

2.13 PROPERTIES OF DEFINITE INTEGRAL

Property P–1

Note that the definite integral is independent of what letter denotes the variable of integration. Thus,

$$\int_a^b f(x)\,dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

Property P–2

Order of integration : if we reverse a and b then sign of the integral is changed.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Property P–3

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

An interval can be decomposed into two intervals,

Example 1. Evaluate $\int_{-1}^{15} \text{sgn}(\{x\})\,dx$, where {.} denotes the fractional part function.

Solution We have

$$\text{sgn}(\{x\}) = \begin{cases} 1, & \text{if } x \text{ is not an integer} \\ 0, & \text{if } x \text{ is an integer} \end{cases}$$

$$\therefore \int_{-1}^{15} \text{sgn}(\{x\})\,dx$$

$$= \int_{-1}^{0} \text{sgn}(\{x\})\,dx + \int_0^{15} \text{sgn}(\{x\})\,dx \qquad [\text{P–3}]$$

$$= \int_{-1}^{0} 1.dx + 15\int_0^1 1.dx$$

$$= 1\,(0+1) + 15\,(1-0) = 16.$$

Example 2. Find the value of the definite integral

$$\int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{x^4 + x^2 + 2}{(x^2+1)^2}dx.$$

Solution

$$I = \int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{(x^2+1)^2 - (x^2-1)}{(x^2+1)^2}dx$$

$$= \int_{\sqrt{2}-1}^{\sqrt{2}+1} \left(1 - \frac{(x^2-1)}{(x^2+1)^2}\right)dx$$

$$= 2 - \underbrace{\int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{(x^2-1)}{(x^2+1)^2}dx}_{I_1}$$

$$I_1 = \int_{1/a}^{a} \frac{(x^2-1)}{(x^2+1)^2}dx \quad \text{where } (a = \sqrt{2}+1)$$

$$\text{put } x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$$

$$I_1 = \int_a^{1/a} \frac{\frac{1}{t^2}-1}{\left(\frac{1}{t^2}+1\right)^2}\cdot\left(-\frac{1}{t^2}\right)dt$$

$$= -\int_a^{1/a} \frac{(1-t^2)t^4}{t^4(1+t^2)^2}dt = -\int_a^{1/a} \frac{(1-t^2)}{(1+t^2)^2}dt$$

$$= \int_a^{1/a} \frac{t^2-1}{(t^2+1)^2}dt = -\int_{1/a}^{a} \frac{t^2-1}{(t^2+1)^2}dt = -I_1 \,\text{[P–1]}$$

$$\Rightarrow 2I_1 = 0 \Rightarrow I_1 = 0 \Rightarrow 2.$$

Example 3. Prove that $\int_0^{\infty} \frac{dx}{1+x^n} = \int_0^1 \frac{dx}{(1-x^n)^{1/n}} \ (n>1)$

Solution R.H.S. $= \int_0^1 \frac{dx}{(1-x^n)^{1/n}} = I \text{ (say)}$

$$\text{Put } x = \frac{1}{t} \Rightarrow dx = \frac{-1}{t^2}dt$$

$$\therefore \quad I = \int_{\infty}^{1} \frac{-\frac{1}{t^2}dt}{\left(1-\frac{1}{t^n}\right)^{1/n}} = \int_1^{\infty} \frac{dt}{t(t^n-1)^{1/n}}$$

Put $t^n - 1 = y^n \Rightarrow t^{n-1}\,dt = y^{n-1}\,dy$

$$\text{Now, R.H.S.} = \int_0^{\infty} \frac{y^{n-1}dy}{(1+y^n)y}$$

$$= \int_0^{\infty} \frac{y^n}{(1+y^n)}\frac{1}{y^2}dy = \int_0^{\infty} \frac{\frac{1}{y^2}dy}{1+\left(\frac{1}{y}\right)^n}$$

$$\text{Put } \frac{1}{y} = z \Rightarrow \frac{1}{y^2}\,dy = -dz$$

$$\therefore \quad I = \int_{\infty}^{0} \frac{-dz}{1+z^n} = \int_0^{\infty} \frac{dz}{1+z^n} \qquad \text{[P–2]}$$

$$= \int_0^{\infty} \frac{dx}{1+x^n} = \text{L.H.S.} \qquad \text{[P–1]}$$

Example 4. Prove that

$$\int_{1/e}^{\tan x} \frac{t}{1+t^2}dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)} = 1$$

Solution L.H.S. $= \int_{1/e}^{\tan x} \frac{t}{1+t^2}dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$

Put $t = \frac{1}{u}$ in the first integral

then $dt = \frac{-1}{u^2}\,du$

$$\therefore \quad \text{L.H.S.} = \int_e^{\cot x} \frac{\frac{1}{u}\left(\frac{-1}{u^2}\right)du}{\left(1+\frac{1}{u^2}\right)} + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$$

$$= \int_{\cot x}^{e} \frac{du}{u(1+u^2)} + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)} \qquad \text{[P–2]}$$

$$= \int_{\cot x}^{e} \frac{dt}{t(1+t^2)} + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)} \qquad \text{[P–1]}$$

$$= \int_{1/e}^{e} \frac{dt}{t(1+t^2)} = \int_{1/e}^{e} \left(\frac{1}{t} - \frac{t}{1+t^2}\right)dt \qquad \text{[P–3]}$$

$$= \left[\ln t - \frac{1}{2}\ln(1+t^2)\right]_{1/e}^{e}$$

$$= \left[1 - \frac{1}{2}\ln(1+e^2)\right] - \left[-1 - \frac{1}{2}\ln\left(1+\frac{1}{e^2}\right)\right]$$

$$= 2 - \frac{1}{2}\left\{\ln(1+e^2) - \ln\frac{(e^2+1)}{e^2}\right\}$$

$$= 2 - \frac{1}{2}\,[\ln e^2] = 2 - 1 = 1 = \text{R.H.S.}$$

Example 5. Find the value of

$$\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}dt$$

Solution Put $t = \sin^2 y$ in the first integral and $t = \cos^2 u$ in the second integral, we have

$$\int_0^{\sin^2 x} \sin^{-1}\ dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\ dt$$

$$= \int_0^{x} y \sin 2y\,dy - 1\int_{\pi/2}^{x} u \sin 2u\,du$$

$$= \int_0^x y \sin 2y\, dy + \int_x^{\pi/2} y \sin 2y\, dy \qquad \textbf{[P–2]}$$

$$= \int_0^{\pi/2} y \sin 2y\, dy \qquad \textbf{[P–3]}$$

$$= \left[y\left(-\frac{\cos 2y}{2}\right) + \frac{\sin 2y}{4} \right]_0^{\pi/2}$$

$$= \left(\frac{\pi}{4} + 0\right) - (0+0) = \frac{\pi}{4}.$$

Example 6. If $\int_0^1 \frac{e^t\, dt}{t+1} = a$, then show that

$\int_{b-1}^{b} \frac{e^{-t}\, dt}{t-b-1}$ is equal to $-ae^{-b}$.

Solution Given $\int_0^1 \frac{e^t\, dt}{(t+1)} = a$ ...(1)

Now let $I = \int_{b-1}^{b} \frac{e^{-t}\, dt}{t-b-1}$

Put $t = b - y \Rightarrow dt = -dy$

$$\Rightarrow\ I = \int_1^0 \frac{e^{-b+y}}{b-y-b-1}(-dy) = -e^{-b}\int_1^0 \frac{e^y\, dy}{-(y+1)}$$

$$= -e^{-b}\int_0^1 \frac{e^y\, dy}{(y+1)} \qquad \textbf{[P–2]}$$

$$= -e^{-b}\int_0^1 \frac{e^t\, dt}{t+1} \qquad \textbf{[P–1]}$$

$= -ae^{-b}$ from (1)

Property P–4

(i) $\int_{-a}^{a} f(x)dx = \int_0^a (f(x)+f(-x))\, dx$

(ii) $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$, if $f(x)$ is an even function i.e. $f(x) = f(-x)$.

(iii) $\int_{-a}^{a} f(x)\, dx = 0$, if $f(x)$ is an odd function i.e. $f(x) = -f(-x)$.

Proof : (i) $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$

$$= -\int_0^{-a} f(x)dx + \int_0^a f(x)dx$$

In the first integral on the far right side we make the substitution $u = -x$. Then $du = -dx$ and when $x = -a$, $u = a$. Therefore

$$-\int_0^{-a} f(x)dx = -\int_0^a f(-u)(-du) = \int_0^a f(-u)\, du$$

$$\int_{-a}^{a} f(x)dx = \int_0^a f(-u)\, du + \int_0^a f(x)dx$$

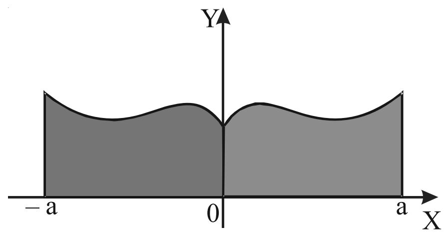
(ii) If f is even, then

$$\int_{-a}^{a} f(x)dx = \int_0^a f(u)du + \int_0^a f(x)dx = 2\int_0^a f(x)dx$$

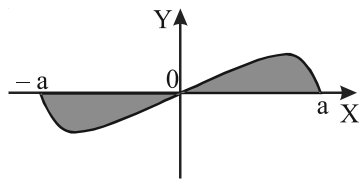
(iii) If f is odd, then

$$\int_{-a}^{a} f(x)dx = -\int_0^a f(u)du + \int_0^a f(x)dx = 0.$$

The above property is illustrated by Figures (a) and (b) For the case where f is even, part (ii) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry of the graph about y-axis. Thus, part (iii) says the integral of odd function is 0 because the areas cancel.



(a) f even, $\int_{?a}^{a} f(x)dx = 2\int_0^a f(x)dx$



(b) f odd, $\int_{-a}^{a} f(x)dx = 0$

Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6+1)\, dx = 2\int_0^2 (x^6+1)\, dx$$

$$= 2\left[\frac{1}{7}x^7 + x\right]_0^2 = 2\left(\frac{128}{7} + 2\right) = \frac{284}{7}.$$

Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx = 0.$$

This property of odd functions can save a lot of computation. For example,

$$\int_{-\pi/4}^{\pi/4} \sin^9 x\,dx = 0 \text{ by inspection.}$$

Example 7. Evaluate $\int_{-1}^{1} \frac{e^x+e^{-x}}{1+e^x}\,dx$

Solution $\int_{-1}^{1} \frac{e^x+e^{-x}}{1+e^x}\,dx$

$$= \int_0^1 \left(\frac{e^x+e^{-x}}{1+e^x} + \frac{e^{-x}+e^x}{1+e^{-x}}\right) dx \qquad \textbf{[P–4(i)]}$$

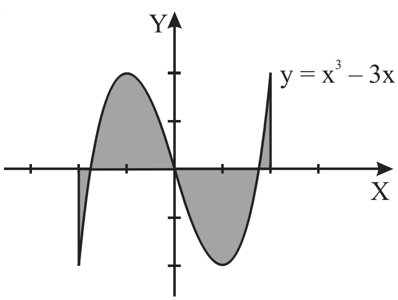
$$= \int_0^1 \left(\frac{e^x+e^{-x}}{1+e^x} + \frac{e^x(e^{-x}+e^x)}{e^x+1}\right) dx$$

$$= \int_0^1 (e^x+e^{-x})\,dx = e - 1 + \frac{(e^{-1}-1)}{-1} = \frac{e^2-1}{e}.$$

Example 8. Evaluate $\int_{-2}^{2} (x^3-3x)\,dx.$

Solution The integrand, $f(x) = x^3 - 3x$, is an an odd function, i.e., the equation $f(-x) = -f(x)$ is satisfied for every x. Its graph, drawn in the figure, is therefore symmetric about the origin. It follows that the region above the x-axis has the same areas as the region below it. We conclude that

$$\int_{-2}^{2} (x^3-3x)\,dx = 0$$



Example 9. Find the value of the integral

$$\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\,dx.$$

Solution $I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\,dx$

$$I = 2\int_0^{\pi/2} \sqrt{\cos x - \cos^3 x}\,dx \qquad \textbf{[P–4]}$$

$$I = 2\int_0^{\pi/2} \sin x\sqrt{\cos x}\,dx$$

$$I = -2\int_1^0 2t^2\,dt \qquad (\text{Putting } \cos x = t^2)$$

$$= 4\int_0^1 t^2\,dt \Rightarrow I = \frac{4}{3}.$$

Example 10. Prove $\int_{-1/2}^{1/2} \sqrt{(\cos x)}\ln\left(\frac{1-x}{1+x}\right)dx = 0$

Solution Let $I = \int_{-1/2}^{1/2} \sqrt{(\cos x)}\ln\left(\frac{1-x}{1+x}\right)dx$

Let $f(x) = \sqrt{\cos x}\,\ln\left(\frac{1-x}{1+x}\right)$

$\therefore$ $f(-x) = \sqrt{\cos(-x)}\ln\left(\frac{1+x}{1-x}\right)$

$$= \sqrt{(\cos x)}\,\ln\left(\frac{1+x}{1-x}\right)$$

$$= \sqrt{(\cos x)}\,\ln\left(\frac{1-x}{1+x}\right) = -f(x)$$

Hence f is odd function

$\therefore$ $I = 0.$ **[P–4]**

Example 11. If $f(x) = \frac{x^7-3x^5+7x^3-x+1}{\cos^2 x}$ then, evaluate $\int_{-\pi/4}^{\pi/4} f(x)\,dx.$

Solution $f(x) = \frac{x^7-3x^5+7x^3-x+1}{\cos^2 x}$

$$= \left[\frac{x^7-3x^5+7x^3-x}{\cos^2 x}\right] + \sec^2 x$$

$= $ (odd function) + (even function)

$$\int_{-\pi/4}^{\pi/4} f(x)\,dx = \int_{-\pi/4}^{\pi/4} \sec 2x\,dx$$

$$+ \int_{-\pi/4}^{+\pi/4} \left[\frac{x^7-3x^5+7x^3-x}{\cos^2 x}\right]dx$$

$= 2 + 0 = 2.$ **[P–4]**

Example 12. Evaluate $\int_{-n}^{n} (-1)^{[x]}\,dx,\ n \in N$, where [x] denotes the greatest integer function less than or equal to x.

Solution Let $I = \int_{-n}^{n} (-1)^{[x]}\,dx$

Suppose $f(x) = (-1)^{[x]}$

$\therefore\ f(-x) = (-1)^{[-x]} = (-1)^{-1-[x]},\ x \notin I$

$= -(-1)^{-[x]}$

$= -\dfrac{1}{(-1)^{[x]}} = -\dfrac{(-1)^{[x]}}{(-1)^{2[x]}}$

$= -(-1)^{[x]} = -f(x),\ x \notin I$

Note that the function is not odd, but the property can be applied since difference at few isolated points does not affect the integral.

$\therefore\ I = \int_{-n}^{n} (-1)^{[x]}\,dx = 0.$

The graph of an odd function is symmetric with respect to the origin. A function may be symmetric with respect to some other point of the x-axis.

Example 13. Find

$$\int_0^6 x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)\,dx.$$

Solution The point of symmetry is (3, 0). To exploit this we make the change of variable $x = u + 3$. Then

$x(x-1)\ldots(x-6) = g(u)$, where

$g(u) = (u+3)(u+2)(u+1)u(u-1)(u-2)(u-3)$

Thus, $\int_0^6 x(x-1)\ldots(x-6)\,dx = \int_{-3}^{3} g(u)\,du.$

But $g(-u) = -g(u)$, so the integral is 0.

Example 14. Let

$$I_n = \int_{-1}^{1} |x| \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots + \frac{x^{2n}}{2n}\right) dx.$$

Find $\lim\limits_{n\to\infty} I_n$.

Solution We have

$$I_n = 2\int_0^1 x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots + \frac{x^{2n}}{2n}\right)dx$$

$$\left(\int_{-1}^{1} (x^{odd})\,dx = 0\right) \qquad \textbf{[P–4]}$$

$$= 2\left[\frac{x^2}{1\cdot 2} + \frac{x^4}{2\cdot 4} + \frac{x^6}{4\cdot 6} + \ldots\ldots + \frac{x^{2n+2}}{2n(2n+2)}\right]_0^1$$

$$= 2\left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 4} + \frac{1}{4\cdot 6} + \ldots\ldots + \frac{1}{2n(2n+2)}\right]$$

$$= 1 + \frac{1}{2}\left[\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots\ldots + \left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

Hence, $\lim\limits_{n\to\infty}\left[1 + \frac{1}{2}\left(1 - \frac{1}{n+1}\right)\right] = \frac{3}{2}.$

Note:

1. If f is an odd function, then $g(x) = \int_a^x f(t)\,dt$ is an even function.

Proof: We have $g(-x) = \int_a^{-x} f(t)\,dt$

$\Rightarrow\ g(-x) = \int_a^{-a} f(t)\,dt + \int_{-a}^{-x} f(t)\,dt$

$\Rightarrow\ g(-x) = 0 + \int_{-a}^{-x} f(t)\,dt$

$[\because$ f is odd $\Rightarrow \int_{-a}^{a} f(t)\,dt = 0]$

$\Rightarrow\ g(-x) = -\int_a^x f(-y)\,dy$, where $t = -y$

$\Rightarrow\ g(-x) = \int_a^x f(y)\,dy$

$[\because$ f is odd]

$\Rightarrow\ g(-x) = \int_a^x f(t)\,dt \ \Rightarrow\ g(-x) = g(x)$

Hence, $g(x) = \int_a^x f(t)$ is even function, if f(t) is odd.

Example 15. If $g(x) = \int_0^x \ln\left(\frac{1-t}{1+t}\right)dt$, then find whether f is even or odd.

Solution Let $f(t) = \ln\left(\frac{1-t}{1+t}\right)$

$\therefore\ f(-t) = \ln\left(\frac{1+t}{1-t}\right) = -\ln\left(\frac{1-t}{1+t}\right) = -f(t)$

$\Rightarrow\ f(-t) = -f(t)$ i.e., f(t) is an odd function

$\therefore\ g(x) = \int_0^x \ln\left(\frac{1-t}{1+t}\right) dt$ is an even function, using the above property.

2. If f(t) is an even function, then $g(x) = \int_0^x f(t)\, dt$ is an odd function.

Proof: We have, $g(-x) = \int_0^{-x} f(t)\, dt$

$= -\int_0^x f(-y)\, dy$, where $t = -y$

$\Rightarrow g(-x) = -\int_0^x f(y)\, dy\ [\because f \text{ is even}]$

$\Rightarrow g(-x) = -\int_0^x f(t)\, dt$

$\Rightarrow g(-x) = -g(x)$

Hence, g(x) is an odd function.

3. If f(t) is an even function, then for non-zero 'a', $\int_a^x f(t)\, dt$ is not necessarily an odd function. It will be an odd function if $\int_0^a f(t)\, dt = 0$.

Because if $g(x) = \int_a^x f(t)\, dt$ is an odd function.

$g(-x) = -g(x)$

$\Rightarrow \int_a^{-x} f(t)\, dt = -\int_a^x f(t)\, dt$

$\Rightarrow \int_a^0 f(t)\, dt + \int_0^{-x} f(t)\, dt = -\int_a^0 f(t)\, dt - \int_0^x f(t)\, dt$

{Put $y = -t$ in the second integral of LHS}

$\{\because f(-y) = f(y)\}$

$\Rightarrow 2\int_a^0 f(t)\, dt = \int_0^x f(y)\, dy - \int_0^x f(t)\, dt$

$\Rightarrow 2\int_a^0 f(t)\, dt\, dt = 0$

$\Rightarrow -\int_0^a f(t)\, dt = 0 \qquad \Rightarrow \int_0^a f(t)\, dt = 0$

or $g(x) = \int_a^x f(t)\, dt$ is an odd function when f(t) is even and $\int_0^a f(t)\, dt = 0$.

Practice Problems L

1. $\int_0^\infty \ln\left(x + \frac{1}{x}\right)\frac{dx}{1+x^2} = \pi \ln 2$

2. Let $f(x) = \int_1^x \frac{\ln t}{(t+1)} dt$ if $x > 0$. Compute $f(x) + f\left(\frac{1}{x}\right)$. Also prove that $f(2) + \left(\frac{1}{2}\right) = \frac{1}{2}\ln^2 2$.

3. Prove that the function
$$F(x) = \int_0^x \frac{1}{1+t^2} dt + \int_0^{1/x} \frac{1}{1+t^2} dt$$
is constant on the interval $(0, \infty)$.

4. Prove that
$$\int_{-a}^a \varphi(x^2) dx = 2\int_0^a \varphi(x^2) dx,\ \int_{-a}^a \varphi(x^2) x\, dx = 0.$$

5. Evaluate the following integrals :

(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\, dx$

(ii) $\int_{-\pi/2}^{\pi/2} \ln\left(\frac{1-\sin x}{1+\sin x}\right) dx = 0$

(iii) $\int_{-10}^{10} [3 + 7x^{73} - 100x^{101}] dx$

6. Prove that

(i) $\int_{-1}^1 \left[\tan x + \frac{\sqrt[3]{x}}{(1+x^2)^7} - x^{17}\cos x\right] dx = 0$

(ii) $\int_{-5}^5 (3x^2 - x^{10}\sin x + x^5\sqrt{1+x^4}) dx = 250$

7. Evaluate the following integrals :

(i) $\int_{-1/2}^{1/2} \sec x \ln \frac{1-x}{1+x}\, dx$

(ii) $\int_{-1}^{3/2} |x \sin \pi x| dx$

(iii) $\int_{-1}^3 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right) dx$

(iv) $\int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right]^{1/2} dx$

8. Prove that one of the antiderivatives of an even function is an odd function and every antiderivative of an odd function is an even function.

9. Evaluate $\int_{-2}^{2}\left(x^3 f(x) + x\cdot f''(x) + 2\right)dx$, where f(x) is an even differentiable function

Property P–5

(i) $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

(ii) $\int_0^a f(x)\,dx = \int_0^a f(a+b-x)\,dx$

(i) Proof : $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

Put $x = a - t \Rightarrow dx = -dt$

Also, when $x = 0$, $t = a$ and when $x = a$, $t = 0$.

$$\therefore \quad \text{LHS} = \int_a^0 f(a-t)(-dt) = \int_0^a f(a-t)dt$$

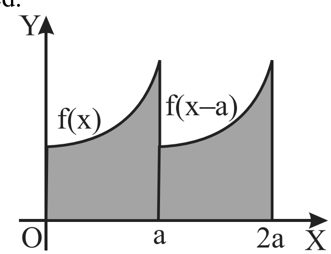
$$= \int_0^a f(a-x)dx = \text{RHS}$$

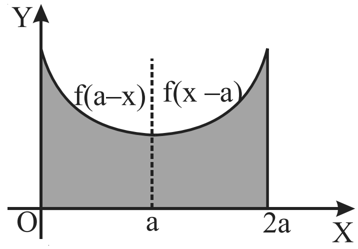
Graphical proof :

To draw $y = f(x - a)$, the graph of $y = f(x)$ is shifted right ward by 'a' units and to get $y = f(-x)$ we draw the image of $y = f(x)$ in the line $x = 0$.

Hence, the graph of $f(a - x)$ is the image of $f(x - a)$ in the line $x = a$ (see figure).

Finally, we observe that the graph of $f(a - x)$ is obtained when f(x) is inverted laterally in the region {0, a] i.e. the graph of f(x) is reflected about the line $x = a/2$. Thus, the graphs $y = f(x)$ and $y = f(a - x)$ form equal areas with the x-axis in $0 \le x \le a$. Hence, the formula is established.





(ii) Proof : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Put $x = a + b - t \;\Rightarrow dx = -dt$

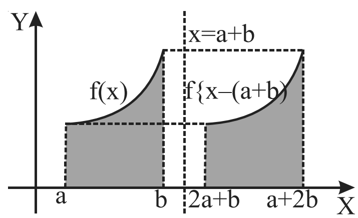
When $x = a$; $t = b$ and when $x = b$; $t = a$

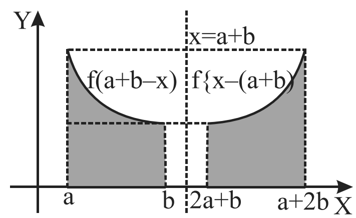
$$\therefore \quad \text{LHS} = \int_b^a f(a+b-t)\,(-dt) = \int_a^b f(a+b-t)dt$$

$$= \int_a^b f(a+b-x)dx = \text{RHS}.$$

Graphical proof :

We can get the graph $f(a + b - x)$ by shifting the graph of f(x) rightward by (a + b) units to get $f\{x - (a + b)\}$ and then reflecting the resulting graph about the line $x = a + b$





From the graph it is clear that

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

This property is the generalised form of the previous property. In this property too, f(x) inverts itself laterally in the region [a, b]. As a result graphs of f(x) and $f(a + b - x)$ form equal areas with x-axis in the interval [a, b]

Example 16. Evaluate $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$

Solution We have $I = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}} dx$

$$\left[\text{using} \int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx\right]$$

$$I = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx$$

Adding the above integrals, we have

$$2I = \int_{-\pi}^{\pi} \cos^2 x dx = -2\int_0^{\pi} \cos^2 x\, dx$$

$$[\because f(x) = \cos^2 x = f(-x)]$$

$$= \int_0^{\pi} (1+\cos 2x)\,dx = \left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = \pi$$

This gives $I = \pi/2$.

Example 17. Prove that $\int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\sin x)+g(\cos x)} dx = \frac{\pi}{4}$$

Solution Let $I = \int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)} dx$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} \frac{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)}{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)+g\left(\cos\left(\frac{\pi}{2}-x\right)\right)} \quad \textbf{[P–5]}$$

$$= \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\cos x)+g(\sin x)} dx$$

Adding the above integrals, we have

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{g(\sin x)}{g(\sin x)+g(\cos x)} + \frac{g(\cos x)}{g(\cos x)+g(\sin x)}\right) dx$$

$$= \int_0^{\pi/2} dx \Rightarrow I = \frac{\pi}{4}.$$

Note:

1. $\int_0^{\pi/2} \frac{g(\sin x)}{g(\sin x)+g(\cos x)} dx$

$$= \int_0^{\pi/2} \frac{g(\cos x)}{g(\sin x)+g(\cos x)} dx = \frac{\pi}{4}$$

2. $\int_0^{\pi/2} \frac{g(\tan x)}{g(\tan x)+g(\cot x)} dx$

$$= \int_0^{\pi/2} \frac{g(\cot x)}{g(\tan x)+g(\cot x)} dx = \frac{\pi}{4}$$

3. $\int_0^{\pi/2} \frac{g(\operatorname{cosec} x)}{g(\operatorname{cosec} x)+g(\sec x)} dx$

$$= \int_0^{\pi/2} \frac{g(\sec x)}{g(\operatorname{cosec} x)+g(\sec x)} dx = \frac{\pi}{4}$$

4. $\int_0^{a} \frac{g(x)}{g(x)+g(a-x)} dx = \frac{a}{2}$

Example 18. Evaluate $\int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x} dx$

Solution Let $f(x) = \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x} dx$

$$\therefore \quad f(\pi/2-x) = \frac{\cos(\pi/2-x)}{\cos(\pi/2-x)+\sin(\pi/2-x)}$$

$$= \frac{\sin x}{\sin x+\cos x}$$

Since $\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} f(\pi/2-x)\,dx$, **[P–5]**

therefore

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x} dx = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x} dx \quad ...(1)$$

$$\therefore \quad 2I = \int_0^{\pi/2} \frac{\cos x+\sin x}{\cos x+\sin x} dx = \int_0^{\pi/2} 1.dx = \pi/2.$$

$I = \pi/4$.

Note that we have added the two expressions for I as given in (1) to obtain a simple expression for I which could be easily evaluated.

Example 19. Find the value of the definite integral

$$\int_0^{\infty} \frac{dx}{(1+x^a)(1+x^2)} \quad (a>0).$$

Solution Put $x = \tan\theta$

$$I = \int_0^{\pi/2} \frac{d\theta}{1+(\tan\theta)^a} = \int_0^{\pi/2} \frac{(\cos\theta)^a}{(\sin\theta)^a+(\cos\theta)^a} d\theta$$

$$I = \int_0^{\pi/2} \frac{(\sin\theta)^a}{(\sin\theta)^a + (\cos\theta)^a} d\theta \quad \textbf{[P–5]}$$

Adding the integrals, we get $I = \frac{\pi}{4}$.

Example 20. If [.] stands for the greatest integer function, then evaluate

$$\int_4^{10} \frac{[x^2]dx}{[x^2 - 28x + 196] + [x^2]}.$$

Solution Let $I = \int_4^{10} \frac{[x^2].dx}{[(14-x)^2] + [x^2]}$ ...(1)

Then, $I = \int_4^{10} \frac{[(14-x)^2].dx}{[x^2] + [(14-x)^2]}$ ...(2)

[P–5]

Adding (1) and (2),

$$2I = \int_4^{10} 1.dx = 6$$

$\therefore \quad I = 3$

Example 21. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$

Solution

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$ ...(1)

Then $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ ...(2)

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right] \quad \textbf{[P–5]}$$

Adding (1) and (2), we get $2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx = \left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{6}$.

$\therefore \quad I = \frac{\pi}{12}$.

Example 22. Evaluate $\int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin x}+1}$.

Solution Let $I = \int_{-\pi/2}^{\pi/2} \frac{dx}{(e^{\sin x}+1)}$

$$= \int_{-\pi/2}^{0} \frac{dx}{e^{\sin x}+1} + \int_0^{\pi/2} \frac{dx}{e^{\sin x}+1}$$

In first integral put $x = -t$

$\therefore \quad dx = -dt$

$$= \int_{\pi/2}^{0} \frac{-dt}{e^{-\sin t}+1} + \int_0^{\pi/2} \frac{dx}{e^{\sin x}+1}$$

$$= \int_0^{\pi/2} \frac{dx}{e^{-\sin x}+1} + \int_0^{\pi/2} \frac{dx}{e^{\sin x}+1} \quad \textbf{[P–5]}$$

$$= \int_0^{\pi/2} 1\,.\,dx = [x]_0^{\pi/2} = \frac{\pi}{2}.$$

Example 23. Find $I = \int_{\cos^4 t}^{-\sin^4 t} \frac{\sqrt{f(z)}dz}{\sqrt{f(\cos 2t - z)} + \sqrt{f(z)}}$

Solution $I = \int_{\cos^4 t}^{-\sin^4 t} \frac{\sqrt{f(z)}dz}{\sqrt{f(\cos 2t - z)} + \sqrt{f(z)}}$ ...(1)

$$I = \int_{\cos^4 t}^{-\sin^4 t} \frac{\sqrt{f(\cos 2t - z)}dz}{\sqrt{f(\cos 2t - z)} + \sqrt{f(z)}} \quad ...(2)$$

[P–5]

Adding (1) and (2), we get

$$2I = \int_{\cos^4 t}^{-\sin^4 t} dz$$

$$2I = z\Big|_{\cos^4 t}^{-\sin^4 t}$$

$$I = -\frac{1}{2}(\sin^4 t + \cos^4 t)$$

$$= -\frac{1}{2}\left(1 - \frac{1}{2}\sin^2 2t\right)$$

$$I = \frac{1}{2} + \frac{1}{4}\sin^2 2t.$$

Example 24. Prove that $\int_0^{\pi/4} \ln(1+\tan\theta)\,d\theta = \frac{\pi}{8}\ln 2$

Solution Let $I = \int_0^{\pi/4} \ln(1+\tan\theta)\,d\theta$ ...(1)

$$= \int_0^{\pi/4} \ln\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta \quad \textbf{[P–5]}$$

$$= \int_0^{\pi/4} \ln\left(1 + \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \ln\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \ln 2\, d\theta - \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$$

$$= \ln 2 \int_0^{\pi/4} 1.d\theta - I \qquad [\text{from } (1)]$$

$$\Rightarrow 2I = \frac{\pi}{4}\ln 2.$$

Hence, $I = \frac{\pi}{8} \ln 2$.

Example 25. Evaluate $\int_{50}^{100} \frac{\ln x}{\ln x + \ln(150-x)} dx$.

Solution $I = \int_{50}^{100} \frac{\ln x}{\ln x + \ln(150-x)} dx \quad ...(1)$

Also $I = \int_{50}^{100} \frac{\ln(150-x)}{\ln(150-x) + \ln x} dx \quad$ **[P–5]**

$$I = \int_{50}^{100} \frac{\ln(150-x)}{\ln(150-x) + \ln x} dx \quad ...(2)$$

Adding (1) and (2) we get

$$2I = \int_{50}^{100} dx \Rightarrow 2I = 100 - 50 \Rightarrow I = 25.$$

Example 26. Suppose f is continuous and satisfies $f(x) + f(-x) = x^2$ then find the value of the integral $\int_{-1}^{1} f(x)dx$.

Solution $I = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} f(-x)dx \quad$ **[P–5]**

$$2I = \int_{-1}^{1} (f(x) + f(-x))dx = \int_{-1}^{1} x^2\, dx$$

$$2I = 2\int_0^1 x^2\, dx$$

$$\Rightarrow I = \int_0^1 x^2\, dx = \frac{1}{3}.$$

Note: Removal of x from the integral $\int_0^a xf(x)dx$.

Suppose you know the integral of f(x), then in order to evalutate xf(x) we try to remove the factor x. This is done by the help of property P–5, provided f(x) does not change when x is replaced by (a – x).

i.e. $f(a-x) = f(x) \quad ...(1)$

Let $I = \int_0^a xf(x)dx$

$$\therefore I = \int_0^a (a-x)f(a-x)dx \quad \textbf{[P–5]}$$

$$= \int_0^a (a-x)f(x)dx \quad \text{by } (1)$$

$$\therefore 2I = \int_0^a (x+a-x)f(x)dx = a\int_0^a f(x)dx$$

$$\therefore I = \frac{a}{2}\int_0^a f(x)dx \text{ provided } f(a-x) = f(x) \quad ...(2)$$

It is to be noted that x is eliminated only when f(x) remains unchanged when (a – x) is put in place of x. hence before applying the above method it should be observed that f(a – x) = f(x).

Example 27. Suppose $V = \int_0^{\pi/2} x\left|\sin^2 x - \frac{1}{2}\right|dx$, find the value of $\frac{96V}{\pi}$.

Solution $V = \int_0^{\pi/2} x\left|\sin^2 x - \frac{1}{2}\right|dx$

$$= \frac{1}{2}\int_0^{\pi/2} x\left|2\sin^2 x - 1\right|dx = \frac{1}{2}\int_0^{\pi/2} x|\cos 2x|dx$$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$

$$V = \frac{1}{8}\int_0^{\pi} t|\cos t|dt = \frac{1}{8}\int_0^{\pi} (\pi - t)|\cos t|dt \quad \textbf{[P–5]}$$

$$2V = \frac{\pi}{8}\int_0^{\pi} |\cos t|dt = \frac{2\pi}{8}$$

$$V = \frac{\pi}{8} \Rightarrow \frac{96}{\pi}\cdot\frac{\pi}{8} = 12.$$

Example 28. Show that $\int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n}} dx = \pi^2$.

Solution Let $I = \int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n}} \quad ...(1)$

$$I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}(2\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)} dx \quad \textbf{[P–5]}$$

$$= \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n} x}{(\sin^{2n} x + \cos^{2n} x)} dx \quad ...(2)$$

Adding (1) and (2) we get

$$2I=\int_0^{2\pi}\frac{2\pi\sin^2 nx}{\sin^{2n}x+\cos^{2n}x}\,dx$$

$$2I=2\pi\int_0^{2\pi}\frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}\,dx$$

$$\therefore\quad I=\pi\int_0^{2\pi}\frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}\,dx$$

$$=2\pi\int_0^{\pi}\frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}\,dx \qquad [P–6]$$

$$I=4\pi\int_0^{\pi/2}\frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}\,dx \qquad ...(3)$$

[P–6]

$$=4\pi\int_0^{\pi/2}\frac{\sin^{2n}\left(\frac{\pi}{2}-x\right)}{\sin^{2n}\left(\frac{\pi}{2}-x\right)+\cos^{2n}\left(\frac{\pi}{2}-x\right)}\,dx$$

[P–5]

$$I=4\pi\int_0^{\pi/2}\frac{\cos^{2n}x}{(\cos^{2n}x+\sin^{2n}x)}\,dx \qquad ...(4)$$

Adding (3) and (4) we get $2I=4\pi\int_0^{\pi/2}1.dx$

Hence, $I=\pi^2$.

Example 29. Let $I=\int_0^{\pi/4}(\pi x-4x^2)\ln(1+\tan x)dx$.

If the value of $I=\dfrac{\pi^3 \ln 2}{k}$ where $k\in N$, find k.

Solution $I=\int_0^{\pi/4}(\pi x-4x^2)\ln(1+\tan x)\,dx$

$$I=\int_0^{\pi/4}(p\,x-4\,x^2)\,[\ln 2-\ln(1+\tan x)]\,dx$$

[P–5]

$$2I=\ln 2\int_0^{\pi/4}(\pi x-4x^2)$$

$$I=\frac{\ln 2}{2}\left[\frac{\pi x^2}{2}-\frac{4}{3}x^3\right]_0^{\pi/4}=\frac{\ln 2}{2}\left[\frac{\pi^3}{32}-\frac{\pi^3}{48}\right]$$

$$=\frac{\pi^3}{192}\ln 2.$$

Example 30.

Evaluate $I=\displaystyle\int_0^{\pi}\frac{x\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$

Solution We have

$$I=\int_0^{\pi}\frac{(\pi-x)\sin 2(\pi-x)\sin\left(\frac{\pi}{2}\cos(\pi-x)\right)}{2(\pi-x)-\pi}$$

$$=\int_0^{\pi}\frac{(\pi-x)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{\pi-2x}dx$$

$$=\int_0^{\pi}\frac{(\pi-x)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$$

Adding the above integrals, we have

$$2I=\int_0^{\pi}\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx$$

$$\Rightarrow\quad I=\int_0^{\pi}\sin x\cos x\sin\left(\frac{\pi}{2}\cos x\right)dx$$

$$=\int_{-1}^{1}t\sin\left(\frac{\pi}{2}t\right)dt \quad [\text{putting } \cos x=t]$$

$$=\left[\frac{-t\cos\frac{\pi}{2}t}{\frac{\pi}{2}}\right]_{-1}^{1}+\frac{2}{\pi}\int_{-1}^{1}\cos\left(\frac{\pi}{2}t\right)dt$$

$$=0+\frac{2}{\pi}\left[\frac{\sin\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}}\right]_{-1}^{1}=\frac{8}{\pi^2}.$$

Example 31. Evaluate

$$I=\int_0^{\pi/4}\frac{x^2(\sin 2x-\cos 2x)}{(1+\sin 2x)\cos^2 x}dx$$

Solution We have $I=\displaystyle\int_0^{\pi/4}\frac{x^2(\sin 2x-\cos 2x)}{(1+\sin 2x)\cos^2 x}dx$

$$=\int_0^{\pi/4}\frac{2x^2(\sin 2x-\cos 2x)}{(1+\sin 2x)(1+\cos 2x)}dx \qquad ...(1)$$

Also,

$$I=\int_0^{\pi/4}\frac{\left(\frac{\pi}{4}-x\right)^2\left[\sin 2\left(\frac{\pi}{4}-x\right)-\cos 2\left(\frac{\pi}{4}-x\right)\right]}{\left[1+\sin 2\left(\frac{\pi}{4}-x\right)\right]\left[1+\cos 2\left(\frac{\pi}{4}-x\right)\right]}dx$$

$$= \int_0^{\pi/4} \frac{2\left(\frac{\pi}{4} - x\right)^2 (\cos 2x - \sin 2x)}{(1+\cos 2x)(1+\sin 2x)} \quad \ldots(2)$$

Adding the integrals (1) and (2), we have

$$2I = \int_0^{\pi/4} \frac{2\left[x^2 - \left(\frac{\pi}{4} - x\right)^2\right](\sin 2x - \cos 2x)}{(1+\cos 2x)(1+\sin 2x)} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\frac{\pi}{4}\left(2x - \frac{\pi}{4}\right)(\sin 2x - \cos 2x)}{(1+\sin 2x)(1+\cos 2x)}$$

$$= \frac{\pi}{4} \int_0^{\pi/4} \left(2x - \frac{\pi}{4}\right)\left(\frac{1}{1+\cos 2x} - \frac{1}{1+\sin 2x}\right) dx$$

$$= \frac{\pi}{4} \int_0^{\pi/4} \left[\frac{2x - \frac{\pi}{4}}{2\cos^2 x} - \frac{2x - \frac{\pi}{4}}{2\cos^2\left(\frac{\pi}{4} - x\right)}\right] dx$$

[using $\sin 2x = \cos(\pi/2 - x)$]

$$= \frac{\pi}{4} \int_0^{\pi/4} \left[\left(x - \frac{\pi}{8}\right)\sec^2 x - \left(x - \frac{\pi}{8}\right)\sec^2\left(x - \frac{\pi}{4}\right)\right] dx$$

$$= \frac{\pi}{4}(I_1 - I_2).$$

Now, we have

$$I_1 = \left[\left(x - \frac{\pi}{8}\right)\tan x\right]_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx$$

$$= \frac{\pi}{8} + [\ln|\cos x|]_0^{\pi/4} = \frac{\pi}{8} + \ln\frac{1}{\sqrt{2}}$$

and

$$I_2 = \left[\left(x - \frac{\pi}{8}\right)\tan\left(x - \frac{\pi}{4}\right)\right]_0^{\pi/4} - \int_0^{\pi/4} \tan\left(x - \frac{\pi}{4}\right) dx$$

$$= \frac{-\pi}{8} + \left[\ln\left|\cos\left(x - \frac{\pi}{4}\right)\right|\right]_0^{\pi/4} = \frac{-\pi}{8} - \ln\frac{1}{\sqrt{2}}$$

Hence, we have

$$I = \frac{\pi}{4}\left[\left(\frac{\pi}{8} + \ln\frac{1}{\sqrt{2}}\right) - \left(\frac{-\pi}{8} - \ln\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi}{2}\left(\frac{\pi}{8} + \ln\frac{1}{\sqrt{2}}\right) = \frac{\pi^2}{16} - \frac{\pi}{2}\ln 2.$$

Example 32. Evaluate $\int_0^1 \frac{x^2 \cos x}{(1+\sin x)^2} dx$

Solution Let $I = \int_0^1 \frac{x^2 \cos x}{(1+\sin x)^2} dx$

$$= \int_0^{\pi} x^2 \{(1+\sin x)^{-2} \cos x\} dx$$

(applying integration by parts.)

$$= \left(x^2 \frac{(1+\sin x)^{-1}}{-1}\right)_0^{\pi} - \int_0^{\pi} 2x . \frac{(1+\sin x)^{-1}}{-1} dx$$

$$I = (-\pi^2 + 0) + 2\int_0^{\pi} \frac{x}{1+\sin x} dx \quad \ldots(1)$$

[P–5]

$$I = -\pi^2 + 2\int_0^{\pi} \frac{(\pi - x)}{1+\sin x} dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = -2\pi^2 + 2\int_0^{\pi} \frac{\pi . dx}{1+\sin x}$$

$$2I = -2\pi^2 + 2\pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

$$= -2\pi^2 + 2\pi(\tan x - \sec x)_0^{\pi}$$

$$= -2\pi^2 + 2\pi\{0 - (-1-1)\}$$

$$2I = -2\pi^2 + 4\pi$$

$$I = -\pi^2 + 2\pi.$$

Example 33. If $\int_0^{\pi}\left(\frac{x}{1+\sin x}\right)^2 dx = \lambda$ then show that $\int_0^{\pi} \frac{2x^2 \cos^2(x/2)}{(1+\sin x)^2} dx = \lambda + 2\pi - \pi^2$.

Solution Let $I = \int_0^{\pi} \frac{2x^2 \cos^2(x/2)}{(1+\sin x)^2} dx$

$$= \int_0^{\pi} \frac{x^2(1+\cos x)}{(1+\sin x)^2} dx$$

$$= \int_0^{\pi} \frac{x^2}{(1+\sin x)^2} dx + \int_0^{\pi} \frac{x^2 \cos x}{(1+\sin x)^2} dx$$

$$= \lambda + \int_0^{\pi} x^2 . \frac{\cos x}{(1+\sin x)^2} dx$$

Integrating by parts taking x^2 as the first function, we get

$$I=\lambda+\left[x^2\left\{-\frac{1}{(1+\sin x)}\right\}\right]_0^{\pi}+2\int_0^{\pi}\left(\frac{x}{1+\sin x}\right)dx$$

$$I=\lambda-\pi^2+2\int_0^{\pi}\frac{x}{1+\sin x}dx \quad ...(1)$$

$$\text{and } I=\lambda-\pi^2+2\int_0^{\pi}\frac{(\pi-x)}{1+\sin(\pi-x)}dx \quad [P\text{–}5]$$

$$=\lambda-\pi^2+2\int_0^{\pi}\frac{(\pi-x)dx}{1+\sin x} \quad ...(2)$$

Adding (1) and (2) we get

$$2I=2\lambda-2\pi^2+2\pi\int_0^{\pi}\frac{dx}{(1+\sin x)}$$

$$\text{or, } I=\lambda-\pi^2+\pi\int_0^{\pi}\frac{(1-\sin x)}{1-\sin^2 x}dx$$

$$=\lambda-\pi^2+\pi\int_0^{\pi}(\sec^2 x-\sec x\tan x)\,dx$$

$$=\lambda-\pi^2+\pi\{\tan x-\sec x\}_0^{\pi}$$

$$=\lambda-\pi^2+\pi\{(0+1)-(0-1\}$$

$$=\lambda-\pi^2+2\pi$$

Hence, $I=\lambda-2\pi+\pi^2$.

Example 34. Show that $\int_0^{\frac{\pi}{2}} f(\sin 2x)\,\sin x\,dx$

$$=\sqrt{2}\int_0^{\frac{\pi}{4}} f(\cos 2x).\cos x\,dx.$$

Solution Let $I=\int_0^{\frac{\pi}{2}} f(\sin 2x)\sin x\,dx$ ...(1)

$$\text{Then } I=\int_0^{\frac{\pi}{2}} f[\sin(\pi-2x)]\sin\left(\frac{\pi}{2}-x\right)dx \quad [P\text{–}5]$$

$$\text{or, } I=\int_0^{\frac{\pi}{2}} f(\sin 2x)\cos x\,dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\frac{\pi}{2}} f(\sin 2x)(\sin x+\cos x)dx$$

$$=\sqrt{2}\int_0^{\frac{\pi}{2}} f(\sin 2x)\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)dx$$

$$=\sqrt{2}\int_0^{\frac{\pi}{2}} f(\sin 2x)\cos\left(x-\frac{\pi}{4}\right)dx$$

Put $x-\dfrac{\pi}{4}=t$

$$=\sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f\left(\sin 2\left(t+\frac{\pi}{4}\right)\right)\cos t\,dt \quad [P\text{–}4]$$

$$\therefore\ 2I=2\sqrt{2}\int_0^{\frac{\pi}{4}} f(\cos 2t).\cos t\,dt$$

$$\text{or, } I=\sqrt{2}\int_0^{\frac{\pi}{4}} f(\cos 2x).\cos x\,dx.$$

Example 35. Find the value of the definite integral

$$\int_2^4 (x(3-x)(4+x)(6-x)(10-x)+\sin x)dx.$$

Solution Let I

$$=\int_2^4 (x(3-x)(4+x)(6-x)(10-x)+\sin x)dx \quad ...(1)$$

$$\text{Now}\quad I=\int_2^4\big[(6-x)(3-(6-x))(4+(6-x))$$

$$(6-(6-x))(10-(6-x))+\sin(6-x)\big]dx \quad [P\text{–}5]$$

$$=\int_2^4\big((6-x)(x-3)(10-x)x(4+x)+\sin(6-x)\big)dx \quad ...(2)$$

∴ On adding (1) and (2), we get

$$2I=\int_2^4(\sin x+\sin(6-x))dx$$

$$=\left[-\cos x+\cos(6-x)\right]_2^4$$

$$=-\cos 4+\cos 2+\cos 2-\cos 4=2(\cos 2-\cos 4)$$

Hence, $I=\cos 2-\cos 4$.

Example 36. For any $t\in R$ and f being a continuous function,

$$\text{let } I_1=\int_{\sin^2 t}^{1+\cos^2 t} x\,f(x(2-x))\,dx \text{ and}$$

$$I_2=\int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))\,dx \text{ then find } \frac{I_1}{I_2}.$$

Solution $I_1=\int_{\sin^2 t}^{1+\cos^2 t} x\,f(x(2-x))\,dx$

$$=\int_{\sin^2 t}^{1+\cos^2 t}(1+\cos^2 t+\sin^2 t-x)\,f\{(1+\cos^2 t+\sin^2 t-x)$$

$$(2-(1+\cos^2 t+\sin^2 t-x)\}\,dx \quad [P\text{–}5]$$

$$=2\int_{\sin^2 t}^{1+\cos^2 t} f\{(2-x)\,x\}dx-\int_{\sin^2 t}^{1+\cos^2 t} x\,f\{\{2-x)x\}dx$$

$$\Rightarrow\ I_1=2I_2-I_1\ \Rightarrow\ 2I_1=2I_2$$

$$\Rightarrow\ \frac{I_1}{I_2}=I$$

Example 37. Suppose $I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x)dx$,

$I_2 = \int_0^{\pi/2} \cos(2\pi \sin^2 x)dx$ and

$I_3 = \int_0^{\pi/2} \cos(\pi \sin x)dx$, then show that

$I_1 = 0$ and $I_2 + I_3 = 0$.

Solution $I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x)dx$

$I_1 = \int_0^{\pi/2} \cos(\pi \cos^2 x)dx$ **[P–5]**

On adding

$$2I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x) + \cos(\pi \cos^2 x)dx$$

$$= \int_0^{\pi/2} 2\cos\left(\frac{\pi}{2}\right) \cdot \cos\left(\frac{\pi}{2}\cos 2x\right) dx = 0$$

$\therefore \quad I_1 = 0.$

$$I_2 = \int_0^{\pi/2} \cos(\pi(1-\cos 2x)dx = -\int_0^{\pi/2} \cos(\pi \cos 2x)dx$$

$$= -\frac{1}{2}\int_0^{\pi} \cos(\pi \cos t)dt \qquad [\text{Put } 2x = t]$$

$$= -\frac{2}{2}\int_0^{\pi/2} \cos(\pi \cos t)dt$$

$$I_2 = -\int_0^{\pi/2} \cos(\pi \sin t)dt = -I_3$$

$\therefore \quad I_2 + I_3 = 0$

Example 38. If the value of the definite integral $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2 - x + 1}dx = \frac{\pi^2}{\sqrt{n}}$ (where $n \in N$), then find the value of n.

Solution $I = \int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2 - x + 1}dx$ ...(1)

$$I = \int_0^1 \frac{\sin^{-1}\sqrt{1-x}}{x^2 - x + 1}dx = \int_0^1 \frac{\cos^{-1}\sqrt{x}}{x^2 - x + 1}dx \qquad ...(2)$$

[P–5]

On adding (1) and (2), we get

$$2I = \int_0^1 \frac{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}{x^2 - x + 1}dx = \frac{\pi}{2}\int_0^1 \frac{dx}{x^2 - x + 1}dx$$

$$= \frac{\pi}{2}\int_0^1 \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$2I = \frac{\pi}{2}\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\left[\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^1 = \frac{\pi^2}{3\sqrt{3}}$$

Hence, $I = \frac{\pi^2}{6\sqrt{3}} = \frac{\pi^2}{\sqrt{108}} = \frac{\pi^2}{\sqrt{n}}$.

$\Rightarrow \quad n = 108.$

Example 39. Let $f(x)$ be a continuous function on $[0, 4]$ satisfying $f(x)\ f(4-x) = 1$. Find the value of the definite integral $\int_0^4 \frac{1}{1+f(x)}dx$.

Solution Let $I = \int_0^4 \frac{1}{1+f(x)}dx$ ...(1)

[P–5]

$I = \int_0^4 \frac{1}{1+f(4-x)}dx$, put $f(4-x) = \frac{1}{f(x)}$

$\Rightarrow \quad I = \int_0^4 \frac{f(x)}{f(x)+1}dx$ ...(2)

On adding (1) and (2), we get

$$2I = \int_0^4 dx \Rightarrow I = 2.$$

Example 40. If f and g are continuous functions on $[0, a]$ satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$, then show that

$$\int_0^a f(x)\ g(x)dx = \int_0^a f(x)dx.$$

Solution Let $I = \int_0^a f(x)\ g(x)dx$ ...(1)

then $I = \int_0^a f(a-x)\ g(a-x)dx$ **[P–5]**

$$= \int_0^a f(x)[2-g(x)]dx$$

$[\because f(a-x) = f(x) \text{ and } g(a-x) + g(x) = 2]$

$$= 2\int_0^a f(x)dx - \int_0^a f(x)g(x)dx$$

$$= 2\int_0^a f(x)dx - I$$

$\therefore \quad 2I = 2\int_0^a f(x)dx$ or, $I = \int_0^a f(x)dx$.

Example 41. Let $f(x)=x^3-\frac{3x^2}{2}+x+\frac{1}{4}$. Find the value of $\int_{1/4}^{3/4} f(f(x))\,dx$.

Solution We have $f(x)=x^3-\frac{3x^2}{2}+x+\frac{1}{4}$

$$I=\int_{1/4}^{3/4} f(f(x))dx=\int_{1/4}^{3/4} f(f(1-x))\,dx \quad \textbf{[P–5]}$$

Now, $f(1-x)=(1-x)^3-\frac{3}{2}(1-x)^2+1-x+\frac{1}{4}$

$$=1-x^3-3x+3x^2-\frac{3}{2}(1+x^2-2x)+1-x+\frac{1}{4}$$

$$f(x)+f(1-x)=1 \quad ...(1)$$

Replace x by f(x)

$\Rightarrow$ $f(f(x))+f(1-f(x))=1$ ...(2)

Now, $I=\int_{1/4}^{3/4} f(f(x))dx$ ...(3)

Also $I=\int_{1/4}^{3/4} f(f(1-x))dx$

$$=a\int_{1/4}^{3/4} f(1-f(x))dx \quad ..(4)$$

[using (1)]

Adding $2I=\int_{1/4}^{3/4}[f(f(x))+f(1-f(x))]dx=\int_{1/4}^{3/4}dx$

$$2I=\frac{1}{2} \Rightarrow I=\frac{1}{4}.$$

Alternative :

Given $f(x)=x^3-\frac{3x^2}{2}+x+\frac{1}{4}$

$$=\frac{1}{4}(4x^3-6x^2+4x+1)$$

$$=\frac{1}{4}(4x^3-6x^2+4x-1+2)$$

$$f(x)=\frac{1}{4}[x^4-(1-x)^4]+\frac{2}{4}$$

$$\therefore \quad f(1-x)=\frac{1}{4}[(1-x)^4-x^4]+\frac{2}{4}$$

$$\therefore \quad f(x)+f(1-x)=\frac{2}{4}+\frac{2}{4}=1 \quad ...(1)$$

Replacing x by f(x) we have

$f\{f(x)\}+f[1-f(x)]=1$

$\therefore$ $f[f(x)]=1-f(1-f(x))$

From equation (1), $1-f(x)=f(1-x)=1-f(1-x)$

Again, $1-f(1-x)=f(x)$

$$\therefore \quad I=\int_{1/4}^{3/4} f\{f(x)\}dx=\int_{1/4}^{3/4} f(x)dx$$

$$=\int_{1/4}^{3/4} f(1-x)dx \quad \{\text{Applying } \textbf{P–5} \text{ and adding}\}$$

$$\Rightarrow \quad 2I=\int_{1/4}^{3/4}\{f(x)+f(1-x)\}dx=\int_{1/4}^{3/4}1\,dx$$

$$\therefore \quad I=\frac{1}{4}.$$

Example 42. Evaluate $\int_0^1 \cot^{-1}(1-x+x^2)\,dx$.

Solution Let $I=\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

$$=\int_0^1 \cot^{-1}(1-x(1-x))\,dx$$

$$=\int_0^1 \tan^{-1}\left(\frac{1}{1-x(1-x)}\right)dx$$

$$=\int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$$

$$=\int_0^1(\tan^{-1}x+\tan^{-1}(1-x))dx \quad (\because\ 0\le x<1)$$

$$=\int_0^1 \tan^{-1}x\,dx+\int_0^1 \tan^{-1}(1-x)\,dx$$

$$=\int_0^1 \tan^{-1}\,dx+\int_0^1 \tan^{-1}(1-(1-x))\,dx \quad \textbf{[P–5]}$$

$$=2\int_0^1 \tan^{-1}x\,dx$$

Integrating by parts taking unity as the second function, we have

$$I=2\left[[\tan^{-1}x.x]_0^1-\int_0^1\frac{x}{1+x^2}dx\right]$$

$$=2\left[\frac{\pi}{2}-\frac{1}{2}[\ln|1+x^2|]_0^1\right]=2\left[\frac{\pi}{2}-\frac{1}{2}\ln 2\right]$$

Hence, $I=\frac{\pi}{2}-\ln 2$.

Example 43. If $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}dx$

$= k\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx$ then find k.

Solution $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}dx$ ...(1)

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{-2x}{1-x^4}\right)dx \quad \text{[P–5]}$$

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\left(\pi - \cos^{-1}\frac{2x}{1-x^4}\right)dx \quad ...(2)$$

Adding (1) and (2),

$$\therefore \quad 2I = \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx$$

$$2I = 2\pi\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx \quad \text{[P–4]}$$

$\therefore \quad k = \pi.$

Example 44. Evaluate $\int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$

Solution Let $I = \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$ ...(1)

Also, $I = \int_0^1 \frac{dx}{[5+2(1-x)-2(1-x)^2][1+e^{2-4(1-x)}]}$

$$= \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{-2+4x})} \quad \text{[P–5]}$$

$$= \int_0^1 \frac{e^{2-4x}dx}{(5+2x-2x^2)(e^{2-4x}+1)} \quad ...(2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^1 \frac{(1+e^{2-4x})dx}{(5+2x-2x^2)(e^{2-4x}+1)}$$

$$= \int_0^1 \frac{dx}{5-2(x^2-x)}$$

$$= \int_0^1 \frac{dx}{\frac{1}{2}+5-2(x^2-x)} = \int_0^1 \frac{dx}{\frac{1}{2}+5-2\left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\int_0^1 \frac{dx}{\frac{11}{4}-\left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{4\sqrt{11}/2}\left|\ln\frac{\sqrt{11}/2+x-\frac{1}{2}}{\sqrt{11}/2-\left(x-\frac{1}{2}\right)}\right|_0^1$$

$$= \frac{1}{2\sqrt{11}}\left|\ln\frac{\frac{\sqrt{11}}{2}+\frac{1}{2}}{\frac{\sqrt{11}}{2}-\frac{1}{2}} - \ln\frac{\frac{\sqrt{11}}{2}-\frac{1}{2}}{\frac{\sqrt{11}}{2}+\frac{1}{2}}\right|$$

$$= \frac{1}{2\sqrt{11}}\left|2\ln\left(\frac{\sqrt{11}+1}{\sqrt{11}-1}\right)\right| = \frac{1}{\sqrt{11}}\ln\left(\frac{\sqrt{11}+1}{\sqrt{11}-1}\right)$$

$$= \frac{1}{\sqrt{11}}\ln\frac{\sqrt{11}+1}{\sqrt{11}-1}\,\frac{\sqrt{11}+1}{\sqrt{11}+1}$$

$$= \frac{1}{\sqrt{11}}\ln\frac{(\sqrt{11}+1)^2}{10}$$

$$\therefore \quad I = \frac{1}{2\sqrt{11}}\ln\frac{(\sqrt{11}+1)^2}{10}.$$

Concept Problems J

1. (a) Let $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}dx.$

Show that $I = a/2$.

(b) Use the result of part (a) to find

(i) $\int_0^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}dx$

(ii $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}dx$

2. Let f : [a, b] → R be a bounded integrable function for which $\forall\, x \in [a, b]$, $f(a+b-x) = f(x)$.

Prove that $\int_a^b xf(x)dx = \frac{a+b}{2}.\int_a^b f(x)dx$.

3. Prove that $\int_0^\pi x\varphi(\sin x)dx = \frac{1}{2}\pi\int_0^\pi \varphi(\sin x)dx.$

4. Prove that

(i) $\int_0^\pi \theta(\pi-\theta)\sin\theta\, d\theta = 4$,

(ii) $\int_0^\pi \theta(\pi-\theta)\cos\theta\, d\theta = 0$.

5. If a continuous function f on [0, a] satisfies f(x) f(a – x) = 1, a > 0 then find the value of $\int_0^a \frac{dx}{1+f(x)}$.

6. If $I_1 = \int_0^{\pi} xf(\sin^3 x + \cos^2 x)\,dx$ and $I_2 = \int_0^{\pi/2} f(\sin^3 x + \cos^2 x)\,dx$ then relate I_1 and I_2.

Practice Problems M

7. Evaluate the following integrals :

(i) $\int_0^1 \ln\left(\frac{1}{x}-1\right) dx$

(ii) $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$

(iii) $\int_0^{\pi/2} \sqrt{\sin 2\theta}\,\sin\theta\, d\theta$

(iv) $\int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x\cos x} dx$

8. Evaluate the following integrals :

(i) $\int_{\pi/6}^{\pi/3} \sin 2x \ln(\tan x)\, dx$

(ii) $\int_{\pi/8}^{3\pi/8} \ln\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$

(iii) $\int_{-\pi/4}^{\pi/4} \frac{\tan^2 x}{1+e^x} dx$

(iv) $\int_0^{\pi} \frac{dx}{1+2^{\tan x}}$

9. Evaluate the following integrals :

(i) $\int_{\pi/2}^{3\pi/2} |2\sin x|dx\ dx$

(ii) $\int_{\pi/2}^{3\pi/2} [2\sin x]dx$

(iii) $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$

10. Evaluate the following integrals :

(i) $\int_0^{\pi} \frac{x}{1+\sin x} dx$ (ii) $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x+\cos x} dx$

(iii) $\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x} dx$

11. Evaluate the following integrals :

(i) $\int_0^{\frac{1}{4}\pi} \frac{x\,dx}{1+\cos 2x+\sin 2x}$

(ii) $\int_{\pi/4}^{3\pi/4} \frac{x\sin x}{1+\sin x}$

(iii) $\int_0^{\pi} \frac{x^2\sin x\,dx}{(2x-\pi)(1+\cos^2 x)}$

(iv) $\int_0^{\pi} \frac{x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2} dx$

Property P-6

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$= \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \end{cases}$$

Proof: We have $\int_0^{2a} f(x)dx = \int_0^a f(x)(dx) + \int_a^{2a} f(x)dx$

Put x = 2a – t in the second integral ⇒ dt = – dx

Also when x = a; t = a and when x = 2a; t = 0

$$\therefore \int_0^{2a} f(x)dx = \int_0^a f(x)(dx) + \int_a^0 f(2a-t)(-dt)$$

$$= \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

Thus, $\int_0^{2a} f(x)dx$

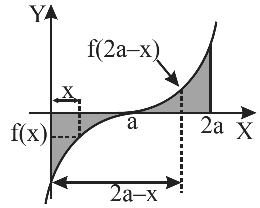
$$= \begin{cases} \int_0^a f(x)dx - \int_0^a f(x)(dx) & \text{if } f(2a-x) = -f(x) \\ \int_0^a f(x)dx + \int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \end{cases}$$

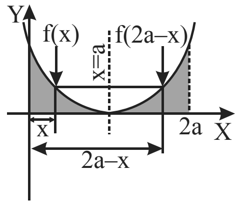
$$\int_0^{2a} f(x)dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \end{cases}$$

Graphical proof :

First of all consider the case when f(2a – x) = – f(x). Functions satisfying this condition are symmetric about the point (a, 0) and such functions are obtained simply by shifting an odd function horizontally by 'a' units.

Thus, $\int_0^{2a} f(x)dx = 0$ where f(x) = – f(2a – x).





Similarly, when $f(2a-x)=f(x)$, the graph is symmetric about the line $x = a$ and such functions are obtained by shifting an even function horizontally by 'a' units.

Thus, $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ when $f(x)=f(2a-x)$.

Example 45. Evaluate

$$I=\int_0^{\pi}\frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$$

Solution Here $f(\pi-x)=f(x)$

$$\therefore\quad I=\int_0^{\pi}\frac{(\pi-x)dx}{a^2\cos^2 x+b^2\sin^2 x}$$

Adding and dividing by 2, we get

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$$

$$=2.\frac{\pi}{2}\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}\qquad\textbf{[P–6]}$$

$$\therefore I=\pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x}=\frac{\pi}{b}.\frac{1}{a}\tan^{-1}\left[\frac{b\tan x}{a}\right]_0^{\pi/2}$$

$$=\frac{\pi}{ab}\left[\frac{\pi}{2}-0\right]=\frac{\pi^2}{2ab}.$$

Example 46. Evaluate $\int_{1/3}^{2/3}\left(x-\frac{1}{2}\right)^3 e^{\left(x-\frac{1}{2}\right)^2}dx$.

Solution Let $g(x)=\left(x-\frac{1}{2}\right)^3 e^{\left(x-\frac{1}{2}\right)^2}$. Then, we have

$$g\left(\frac{1}{3}+x\right)=\left(x-\frac{1}{6}\right)^3 e^{\left(x-\frac{1}{6}\right)^2}$$

and $g\left(\frac{2}{3}-x\right)=\left(\frac{1}{6}-x\right)^3 e^{\left(x-\frac{1}{6}\right)^2}=-g\left(\frac{1}{3}+x\right)$

$\Rightarrow$ g(x) is odd symmetric about $x=\frac{1/3+2/3}{2}=\frac{1}{2}$.

Hence, we have $\int_{1/3}^{2/3} g(x)dx=0$.

Example 47. Evaluate $I=\int_0^{\pi} x\sqrt{1+|\cos x|}\,dx$

Solution $I=\int_0^{\pi}(\pi-x)\sqrt{1+|\cos x|}\,dx$ **[P–5]**

$$2I=\pi\int_0^{\pi}(\sqrt{1+|\cos x|}\,dx$$

$$2I=2\pi\int_0^{\pi/2}\sqrt{1+\cos x}\,dx\qquad\textbf{[P–6]}$$

$$I=\pi\sqrt{2}\int_0^{\pi/2}\cos\frac{x}{2}dx=2\sqrt{2}\,\pi\sin\frac{x}{2}\Big|_0^{\pi/2}=2\pi.$$

Example 48. Show that the value of the definite integral

$\int_0^{2\pi} x\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$, is equal to

(i) $\pi\int_0^{2\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$

(ii) $2\pi\int_0^{\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$

(iii) $2\pi\int_0^{\pi}\ln\left(\frac{3-\cos x}{3+\cos x}\right)dx$

(iv) zero

Solution

(i) $I=\int_0^{2\pi} x\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$

$$I=\int_0^{2\pi}(2\pi-x)\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx\qquad\textbf{[P–5]}$$

$$\therefore\quad 2I=2\pi\int_0^{2\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$$

$$\therefore\quad I=\pi\int_0^{2\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx.$$

(ii) Using **P–6**, we get

$$I=2\pi\int_0^{\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$$

(iii) Now using **P–5,** $I=2\pi\int_0^{\pi}\ln\left(\frac{3-\cos x}{3+\cos x}\right)dx$

On adding (ii) and (iii) we get $I=0$.

Example 49. Without evaluating the integral at any stage, prove that ,

$$I=\int_0^{\pi} \cos^4 2x \, . \, \sin^2 4x \, dx=\frac{5}{16}\int_0^{\pi/2} \quad dy.$$

Solution $I=\int_0^{\pi} \cos^4 2x \sin^2 4x \, dx$

$$I=2\int_0^{\pi/2} \cos^4 2x \sin^2 4x \, dx \qquad \textbf{[P–6]}$$

$$I=2\int_0^{\pi/2} \cos^4 2x \, . \, 4\sin^2 2x \cos^2 2x \, dx$$

$$I=8\int_0^{\pi/2} \cos^6 2x \sin^2 2x \, dx$$

$$I=4\int_0^{\pi} \cos^6 t \sin^2 t \, dt \qquad \text{(Putting } 2x=t\text{)}$$

$$I=8\int_0^{\pi/2} \cos^6 t \sin^2 t \, dt \qquad \textbf{[P–6]}$$

$$I=8\int_0^{\pi/2} \sin^6 t \cos^2 t \, dt \qquad \textbf{[P–5]}$$

Adding, $2I=8\int_0^{\pi/2} \sin^2 t \cos^2 t \,(\sin^4 t+\cos^4 t)\, dt$

$$I=4\int_0^{\pi/2} \sin^2 t \cos^2 t \,(1-2\sin^2 t\cos^2 t)\, dt$$

$$I=\int_0^{\pi/2} \sin^2 2t\left(1-\frac{1}{2}\sin^2 2t\right) dt$$

$$I=\frac{1}{2}\int_0^{\pi} \sin^2 z\left(1-\frac{1}{2}\sin^2 z\right) dz$$

Putting $2t=z$

$$I=\frac{2}{2}\int_0^{\pi/2} \sin^2 z\left(1-\frac{1}{2}\sin^2 z\right) dz$$

$$I=\int_0^{\pi/2} \cos^2 z\left(1-\frac{1}{2}\cos^2 z\right) dz \qquad \textbf{[P–5]}$$

Adding, $2I=\int_0^{\pi/2} 1-\frac{1}{2}(\sin^4 z+\cos^4 z)\, dz$

$$=\int_0^{\pi/2} 1-\frac{1}{2}(1-2\sin^2 z\cos^2 z)\, dz$$

$$2I=\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{4}\sin^2 2z\right) dz$$

$$2I=\frac{1}{2}\int_0^{\pi}\left(\frac{1}{2}+\frac{1}{4}\sin^2 y\right) dz$$

(Putting $2z=y$)

$$I=\frac{1}{4}\times 2\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{4}\sin^2 y\right) dz$$

$$I=\frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{4}\cos^2 y\right) dz$$

Adding, $2I=\frac{1}{2}\int_0^{\pi/2}\left(1+\frac{1}{4}\right) dy$

$$I=\frac{5}{16}\int_0^{\pi/2} \quad dy.$$

Example 50. Prove that

$$\int_0^{\pi/2} \ln\sin x dx=\int_0^{\pi/2} \ln\cos x\, dx=-\frac{\pi}{2}\ln 2.$$

Solution Let $I=\int_0^{\pi/2} \ln\sin x dx$

$$I=\int_0^{\pi/2} \ln\sin\left(\frac{\pi}{2}-x\right) dx=\int_0^{\pi/2} \ln\cos x dx$$

$$\therefore \quad 2I=\int_0^{\pi/2} \ln\sin x dx+\int_0^{\pi/2} \ln\cos x dx$$

$$=\int_0^{\pi/2} (\ln\sin x+\ln\cos x)\, dx$$

$$=\int_0^{\pi/2} \ln\sin x\cos x\, dx=\int_0^{\pi/2} \ln\frac{\sin 2x}{2} dx$$

$$=\int_0^{\pi/2} \ln\sin 2x\, dx-\int_0^{\pi/2} \ln 2\, dx$$

In the first integral put $2x=t$ and adjust the limits.

$$2I=\frac{1}{2}\int_0^{\pi} \ln\sin t\, dt-[x\ln 2]_0^{\pi/2}$$

$$=\frac{1}{2}.2\int_0^{\pi/2} \ln\sin t\, dt-\frac{\pi}{2}\ln 2=I-\frac{\pi}{2}\ln 2$$

$\therefore$ Taking I to the L.H.S., we get

$$I=-\frac{\pi}{2}\ln 2=\frac{\pi}{2}\ln\frac{1}{2}.$$

Example 51. Evaluate $\int_{-\pi/4}^{\pi/4} \ln(\sin x+\cos x)\, dx.$

Solution $\int_{-\pi/4}^{\pi/4} \ln\left\{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right\} dx$

Putting $x+\frac{\pi}{4}=\theta$, $dx=d\theta$

$$=\int_0^{\pi/4}\ln\left(\sqrt{2}\sin\theta\right)d\theta$$

$$=\frac{1}{2}\int_0^{\pi/4}\ln 2\,d\theta+\int_0^{\pi/4}\ln\ \sin\theta\,d\theta$$

$$=\left(\frac{1}{4}\pi\ln 2\right)-\frac{1}{2}\pi\ln 2$$

$$=-\frac{1}{4}\pi\ln 2.$$

Example 52. Evaluate $I=\int_0^1\frac{\ln(1+x)}{1+x^2}dx$

Solution $I=\int_0^{\pi/4}\frac{\ln(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\,d\theta$

[putting $x=\tan\theta$]

$$=\int_0^{\pi/4}\ln\left(\frac{\sin\theta+\cos\theta}{\cos\theta}\right)d\theta$$

$$=\int_0^{\pi/4}\ln\left(\frac{\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)}{\cos\theta}\right)d\theta$$

$$=\int_0^{\pi/4}\ln\sqrt{2}\ d\theta+\int_0^{\pi/4}\ln\sin\left(\theta+\frac{\pi}{4}\right)d\theta$$

$$-\int_0^{\pi/4}\ln(\cos\theta)\,d\theta$$

$$=\ln\sqrt{2}\ [\theta]_0^{\pi/4}+I_1-I_2=\frac{\pi}{8}\ln 2+I_1-I_2$$

Let us put $\theta+\frac{\pi}{4}=\frac{\pi}{2}-t$, $d\theta=-dt$ in I_1.

Also, when $\theta=0$, then $t=\pi/4$ and when $\theta=\pi/4$, then $t=0$. Thus, we have

$$I_1=\int_{\pi/4}^{0}\ln\ \sin(\pi/2-t)(-dt)=\int_0^{\pi/4}\ln(\cos t)\,dt=I_2$$

Hence, we have

$$I=\frac{\pi}{8}\ln 2+I_2-I_2=\frac{\pi}{8}\ln 2.$$

Example 53. If $\int_0^{\pi}\ln\sin x\,dx=k$, then find the value of $\int_0^{\pi/4}\ln\left(1+\tan x\right)dx$ in terms of k.

Solution $I=\int_0^{\pi/4}\ln\left(1+\tan x\right)dx$

$$=\int_0^{\pi/4}\ln\left(1+\tan\left(\pi/4-x\right)\right)dx$$

$$=\int_0^{\pi/4}\ln\left(\frac{\sin(\pi/4-x)+\cos(\pi/4-x)}{\cos(\pi/4-x)}\right)dx$$

$$=\int_0^{\pi/4}\ln\left(\frac{2}{1+\tan x}\right)dx=(\ln 2)\,\pi/4-I$$

$$\Rightarrow I=\frac{\pi}{8}\cdot\ln 2$$

$(\because\ k=-\pi\ln 2)$

$$\therefore\ I=-\frac{k}{8}.$$

Example 54. Assume $\int_0^{\pi}\ln\ \sin\theta\ d\theta=-\pi\,ln\,2$ then rove that

$$\int_0^{\pi}\theta^3\ln\sin\theta\,d\theta=\frac{3\pi}{2}\int_0^{\pi}\theta^2\ln(\sqrt{2}\ \sin\theta)\,d\theta.$$

Solution Let $I=\int_0^{\pi}\theta^3\ \ln\sin\theta\,d\theta$ ...(1)

$$=\int_0^{\pi}(\pi-\theta)^3\ln\sin\theta\,d\theta \quad \textbf{[P–5]}$$

$$=\int_0^{\pi}(\pi^3-3\pi^2\theta+3\pi\theta^2-\theta^3)\ln\sin\theta\,d\theta$$

$$=\pi^3\int_0^{\pi}\ln\sin\theta\,d\theta-3\pi^2\int_0^{\pi}\theta\ln\sin\theta\,d\theta$$

$$+3\pi\int_0^{\pi}\theta^2\ln\sin\theta\,d\theta-\int_0^{\pi}\theta^3\ln\sin\theta\,d\theta$$

$$=\pi^3\int_0^{\pi}\ln\sin\theta\,d\theta-3\pi^2\int_0^{\pi}\theta\ln\sin\theta\,d\theta$$

$$+3\pi\int_0^{\pi}\theta^2\ln\sin\theta\,d\theta-I \quad \text{[From (1)]}$$

$$\therefore\ 2I=\pi^3I_1-3\pi I_2+3\pi\int_0^{\pi}\theta^2\ln\sin\theta\,d\theta.$$

Now $I_1=\int_0^{\pi}\ln\sin d\theta=-\pi\ln 2$ (given)

$$I_2=\int_0^{\pi}\theta\ln\sin\theta\,d\theta$$

$$=\int_0^{\pi}(\pi-\theta)\ln\sin(\pi-\theta)\,d\theta \quad \textbf{[P–5]}$$

$$=\int_0^{\pi}(\pi-\theta)\ln\sin\theta\,d\theta$$

$\therefore \quad 2I_2 = \pi\int_0^{\pi} \ln\sin\theta\, d\theta = -\pi^2 \ln 2$ (given)

$\therefore \quad I_2 = -\dfrac{\pi^2}{2}\ln 2.$

Now, $2I = -\pi^2\ln 2 + \dfrac{3\pi^4}{2}\ln 2 + 3\pi\int_0^{\pi} \theta^2 \ln\sin\theta\, d\theta$

$\Rightarrow \quad I = \dfrac{\pi^4}{2}\ln 2 + \dfrac{3\pi}{2}\int_0^{\pi} \theta^2\ln\sin\theta\, d\theta$

$= \dfrac{3\pi}{2}\int_0^{\pi} \theta^2 \ln\sqrt{2}\, d\theta + \dfrac{3\pi}{2}\int_0^{\pi} \theta^2\ln\sin\theta\, d\theta$

$= \dfrac{3\pi}{2}\int_0^{\pi} \theta^2 \ln(\sqrt{2}\sin\theta)\, d\theta$

Example 55. Evaluate

$$\int_0^{\pi} x(\sin^2(\sin x) + \cos^2(\cos x))\,dx$$

Solution $I = \int_0^{\pi} x(\sin^2(\sin x) + \cos^2(\cos x))\,dx$

$= \int_0^{\pi} (\pi - x)(\sin^2(\sin x) + \cos^2(\cos x))\,dx$ **[P–5]**

$2I = \pi\int_0^{\pi} (\sin^2(\sin x) + \cos^2(\cos x))\,dx$

$2I = 2\pi\int_0^{\pi/2} (\sin^2(\sin x) + \cos^2(\cos x))\,dx$ **[P–6]**

Also $I = \pi\int_0^{\pi/2} (\sin^2(\cos x) + \cos^2(\sin x))\,dx$

Adding, $2I = \pi\int_0^{\pi/2} 2\,dx$

$\Rightarrow \quad I = \pi\int_0^{\pi/2} dx = \dfrac{\pi^2}{2}.$

Example 56. Evaluate $\int_0^{\pi} x(\sin(\cos^2 x)\cos(\sin^2 x))\,dx$

Solution $I = \int_0^{\pi} x(\sin(\cos^2 x)\cos(\sin^2 x))\,dx$

$I = \int_0^{\pi} (\pi - x)\sin(\cos^2 x)\cos(\sin^2 x)\,dx$ **[P–5]**

$2I = \pi\int_0^{\pi} \sin(\cos^2 x)\cos(\sin^2 x)\,dx$

$2I = 2\pi\int_0^{\pi/2} \sin(\cos^2 x)\cos(\sin^2 x)\,dx$ **[P–6]**

$I = \pi\int_0^{\pi/2} \sin(\cos^2 x)\cos(\sin^2 x)\,dx$ ...(1)

$I = \pi\int_0^{\pi/2} \sin(\sin^2 x)\cos(\cos^2 x)\,dx$ ...(2)

[P–5]

Adding (1) and (2)

$2I = \dfrac{\pi}{2}\int_0^{\pi/2} \sin(\cos^2 x + \sin^2 x)\,dx$

$2I = \pi\int_0^{\pi/2} (\sin 1)\,dx \Rightarrow I = \dfrac{\pi^2}{4}(\sin 1).$

Example 57. Evaluate $\int_0^{2\pi} \dfrac{x^2 \sin x}{8 + \sin^2 x}\,dx.$

Solution Let $I = \int_0^{2\pi} \dfrac{x^2\sin x}{8 + \sin^2 x}$

$I = \int_0^{2\pi} \dfrac{-(2\pi - x)^2 \sin x}{8 + \sin^2 x}$ **[P–5]**

Adding the above integrals, we have

$2I = \int_0^{2\pi} \dfrac{2\pi(2x - 2\pi)\sin x}{8 + \sin^2 x}$

$\therefore \quad I = 2\pi\int_0^{2\pi} \dfrac{x\sin x}{9 - \cos^2 x} - 2\pi^2 \underbrace{\int_0^{2\pi} \dfrac{\sin x}{8 + \sin^2 x}\,dx}_{0}$ **[P–6]**

$$I = 2\pi\left[\left(\frac{-x}{6}\ln\frac{3 + \cos x}{3 - \cos x}\right)_0^{2\pi} + \frac{2}{6}\underbrace{\int_0^{2\pi} \ln\frac{3 + \cos x}{3 - \cos x}\,dx}_{0}\right]$$

$I = \dfrac{-\pi}{3}\left[2\pi\ln\dfrac{4}{2}\right] = \dfrac{-2\pi^2}{3}\ln 2.$

Example 58. Evaluate $I = \int_0^{\pi} x\dfrac{d(\cos x)}{1 + \cos^2 x}\,dx$

Solution We use integration by parts

$I = -\int_0^{\pi} x\dfrac{d(\cos x)}{1 + \cos^2 x} = -\int_0^{\pi} x\,d(\tan^{-1}(\cos x))$

$= -x\tan^{-1}(\cos x)\big|_0^{\pi} + \int_0^{\pi} \tan^{-1}(\cos x)\,dx$

$= -\pi\left(-\dfrac{\pi}{4}\right) - 0 + I_1 = \dfrac{\pi^2}{4} + I_1$

where $I_1 = \int_0^{\pi} \tan^{-1}(\cos x)\,dx$

To find I_1, we make note of the fact that the graph of the function $f(x) = \tan^{-1}(\cos x)$ is symmetric with respect to the point $(\pi/2, f(\pi/2)) = (\pi/2, 0)$. Therefore the integrals of this function over the closed intervals $[0, \pi/2]$ and $[\pi/2, \pi]$ are equal in absolute value and opposite in sign, and hence, the sum is zero, i.e.. $I_1 = 0$. We can also establish this fact as follows : we divide the integral I_1 into two itnegrals over the intervals $[0, \pi/2]$ and $[\pi/2, \pi]$ respectively and make a change of variable $x = \pi - t$ in the second integral. We obtain

$$I_1 = \int_0^{\pi/2} \tan^{-1}(\cos x)dx + \int_{\pi/2}^{\pi} \tan^{-1}(\cos x)dx$$

$$= \int_0^{\pi/2} \tan^{-1}(\cos x)dx + \int_{\pi/2}^{0} \tan^{-1}(-\cos t)(-dt)$$

$$= \int_0^{\pi/2} \tan^{-1}(\cos x)dx - \int_0^{\pi/2} \tan^{-1}(\cos t)dt = 0.$$

Thus $I_1 = 0$ and therefore $I = \pi^2/4$.

Example 59. Find the value of

$$2^{2010}\frac{\int\limits_0^1 x^{1004}(1-x)^{1004}dx}{\int\limits_0^1 x^{1004}(1-x^{2010})^{1004}dx}.$$

Solution Consider $I_2 = \int_0^1 x^{1004}(1-x^{2010})^{1004}dx$

Put $x^{1005} = t \Rightarrow 1005\, x^{1004}\, dx = dt$

So, $I_2 = \int_0^1 (1-t^2)^{1004}dt$ ...(1)

Also $I_2 = \frac{1}{1005}\int_0^1 \left[1-(1-t)^2\right]^{1004} dt$ ...(2)

[P–5]

$$\Rightarrow I_2 = \frac{1}{1005}\int_0^1 (t(2-t))^{1004} dt$$

$$= \frac{1}{1005}\int_0^1 t^{1004}(2-t)^{2004} dx$$

Put $t = 2y \Rightarrow dt = 2dy$

$$\text{So, } I_2 = \frac{1}{1005}\int_0^{1/2} (2y)^{1004}(2-2y)^{1004} dt$$

$$= \frac{1}{1005} 2\cdot 2^{1004}\cdot 2^{1004}\int_0^{1/2} y^{1004}(1-y)^{1004} dy$$

$$I_2 = \frac{1}{1005} 2^{2009}\int_0^{1/2} y^{1004}(1-y)^{1004} dy \quad ...(3)$$

$$\text{Now } I_1 = \int_0^1 x^{1004}(1-x)^{1004} dx$$

$$= 2\int_0^{1/2} x^{1004}(1-x)^{1004} dx \quad ...(4)$$

[P–6]

$\therefore$ From (3) and (4), we get

$$I_2 = \frac{1}{1005} 2^{2010}\frac{I_1}{4} \Rightarrow 2^{2010}\frac{I_1}{I_2} = 4020.$$

Practice Problems N

1. Let f be a continuous function. Show that $\int_0^2 f(x)\, dx = \int_0^1 [f(x) + f(x+1)]dx$.
2. Evaluate the following integrals :
   (i) $\int_0^1 \frac{\sin^{-1} x}{x} dx$ (ii) $\int_0^{\pi} \frac{x \tan x dx}{\sec x + \tan x}$
   (iii) $\int_0^{\infty} \left(\frac{\ln\left(x+\frac{1}{x}\right)}{1+x^2}\right) dx$ (iv) $\int_0^{\infty} \frac{\ln(1+x^2)}{1+x^2}dx$
3. Prove that
   (i) $\int_0^1 \ln\left(\frac{1}{x}-1\right) dx = 0$
   (ii) $\int_0^{\pi/2} \sin 2x \ln \tan x dx = 0$
   (iii) $\int_0^{\pi} \frac{xdx}{1+\cos\alpha \sin x} = \frac{\pi\alpha}{\sin\alpha}$
4. Evaluate the following integrals :
   (i) $\int_0^{2\pi} \frac{x(\sin x)^{2n}}{(\sin x)^{2n} + (\cos x)^{2n}}dx,\ n \in N$
   (ii) $\int_0^{\pi} x \ln (\sin x)dx$
   (iii) $\int_0^{\pi/2} (2\cos^2 x)\ln(\sin 2x)dx$
   (iv) $\int_0^{\infty} \left(\frac{\tan^{-1} x}{x}\right)^3 dx$
5. Evaluate the following integrals :
   (i) $\int_0^{\pi} x\cos^2 x \sin^4 x\, dx$
   (ii) $\int_0^{\pi} x \sin^6 x \cos^4 x dx$
   (iii) $\int_0^{\pi} \frac{xdx}{4\cos^2 x + 9\sin^2 x}$
   (iv) $\int_0^{\pi} x \sin^5 x dx$

6. Let A denote the value of the integral $\int_0^{\pi} \frac{\cos x}{(x+2)^2}\,dx$. Compute the integral $\int_0^{\pi/2} \frac{\sin x \cos x}{x+1}\,dx$ in terms of A.

Property P-7

If f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx,\ n \in I$

(ii) $\int_a^{a+nT} f(x)\,dx = n\int_0^{T} f(x)\,dx,\ n\in I,\ a \in R$

(iii) $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^{T} f(x)\,dx,\ m, n \in I$

(iv) $\int_{a+mT}^{b+nT} f(x)\,dx = (n-m)\int_0^{T} f(x)\,dx$
$+\int_a^{b} f(x)\,dx,\ n \in I,\ a, b\in R$

Let us prove $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx,\ n \in I.$

Proof : We have $\int_0^{nT} f(x)\,dx = \sum_{r=1}^{n}\int_{(r-1)T}^{rT} f(x)\,dx$

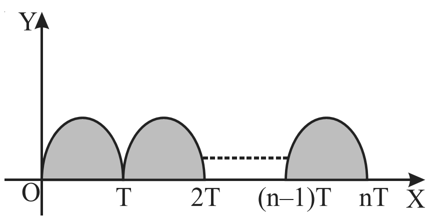
Put $x = (r-1)T + u$

$= \sum_{r=1}^{n}\int_0^{T} f((r-1)T+u)\,du,$

$= \sum_{r=1}^{n}\int_0^{T} f(u)\,du$, [since f(x) is periodic with period T, $f((r-1)T+y) = f(y)$]

$= n\int_0^{T} f(u)\,du = n\int_0^{T} f(x)\,dx.$

From the following graph the proof is obvious.



Example 60. Find the value of the definite integral $\int_0^{\pi}\left|\sqrt{2}\sin x + 2\cos x\right|dx$.

Solution We have

$$I = \sqrt{6}\int_0^{\pi}\left|\frac{1}{\sqrt{3}}\sin x + \frac{\sqrt{2}}{\sqrt{3}}\cos x\right|dx$$

$$= \sqrt{6}\int_0^{\pi}|\sin(x+\alpha)|\,dx$$

where $\sin\alpha = \frac{\sqrt{2}}{\sqrt{3}}$ and $\cos\alpha = \frac{1}{\sqrt{3}}$.

$\therefore\ \tan\alpha = \sqrt{2} \Rightarrow \alpha = \tan^{-1}(\sqrt{2})$

Put $x+\alpha = t$

$$= \sqrt{6}\int_{\alpha}^{\pi+\alpha}|\sin t|\,dt$$

$$= \sqrt{6}\left[\int_0^{\pi}\sin t\,dt\right] \qquad \textbf{[P–7]}$$

$$= 2\sqrt{6}.$$

Example 61. Evaluate $\int_{-3/2}^{10}\{2x\}dx$, where {.} denotes the fractional part of x.

Solution $f(x) = \{2x\}$ is a periodic function with period 1/2.

Let $I = \int_{-3/2}^{10}\{2x\}dx = \int_{-3(1/2)}^{20(1/2)}\{2x\}dx$

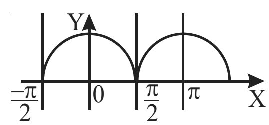
$= 23\int_0^{1/2} 2x\,dx$ (as $\{2x\} = 2x - [2x]$ and when $x \in [0, 1/2), [2x] = 0$)

$= 23x^2\Big|_0^{1/2} = \frac{23}{4}.$

Example 62. Evaluate $I = \int_{-\pi/4}^{32\pi/3}\sqrt{1+\cos 2x}\,dx$

Solution

$f(x) = \sqrt{1+\cos 2x} = \sqrt{2}\,|\cos x|$ is periodic with period π.



Hence,

$$I = \int_{-\pi/4}^{11\pi-\frac{\pi}{3}}|\cos x|\,dx$$

$$= \sqrt{2}\,.\,11\int_0^{\pi}\cos x\,dx + \int_{-\pi/4}^{-\pi/3}\cos x\,dx$$

$$= \sqrt{2}\,.\,11\int_{-\pi/2}^{\pi/2}\cos x\,dx + \int_{-\pi/4}^{-\pi/3}\cos x\,dx$$

$$= \sqrt{2}\left[11\times 2\times 1 + \sin\left(-\frac{\pi}{3}\right) - \sin\left(-\frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\left[22 - \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\right].$$

Example 63. Let $I = \int\limits_{k\pi}^{(k+1)\pi} \frac{|\sin 2x|dx}{|\sin x| + |\cos x|}$, ($k \in N$) and $J = \int_0^{\pi/4} \frac{dx}{\sin x + \cos x}$, then prove that

$$I = 2\int_0^{\pi/2} \frac{\sin 2x\,dx}{\sin x + \cos x} \text{ and } I = 4 - 4J.$$

Solution We have $I = \int_{k\pi}^{(k+1)\pi} \frac{|\sin 2x|dx}{|\sin x| + |\cos x|}$

Since the integrand is periodic with period π,

$$I = \int_0^{\pi} \frac{|\sin 2x|\,dx}{|\sin x| + |\cos x|} \qquad \textbf{[P–7]}$$

$$= 2\int_0^{\pi/2} \frac{\sin 2x\;dx}{\sin x + \cos x} \qquad \textbf{[P–6]}$$

$$= 2\int_0^{\pi/2} \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x}dx$$

$$= \int_0^{\pi/2} (\sin x + \cos x)dx - 2\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$= 4 - 4\int_0^{\pi/4} \frac{dx}{\sin x + \cos x} = 4 - 4J.$$

Example 64. Evaluate $I = \int_{-\frac{\pi}{3}}^{\frac{13\pi}{4}} \sqrt{\cot^{-1}(\cot x)}\;dx$

Solution The integrand is periodic with period π.

Hence,

$$I = \int_{-\pi+\frac{2\pi}{3}}^{3\pi+\frac{\pi}{4}} \sqrt{\cot^{-1}(\cot x)}$$

$$= 4\int_0^{\pi} \sqrt{x}\;dx + \int_{2\pi/3}^{\pi/4} \sqrt{x}\;dx$$

$$= \frac{2}{3}\left(\frac{33}{8} - \left(\frac{2}{3}\right)^{3/2}\right)\pi^{3/2}.$$

Example 65.

If $\int_a^b |\sin x|dx = 8$ and $\int_0^{a+b} |\cos x|dx = 9$, then find the value of $\int_a^b \sin x\;dx$.

Solution Here $\int_a^b |\sin x|\;x\;dx$ is the area under the curve from $x = a$ to $x = b$. Also the area from $x = a$ to $x = a + \pi$ is 2 square units. Hence $b - a = 4\pi$. Similarly, from the second integralm

$$a + b - 0 = \frac{9\pi}{2} \text{ i.e. } a + b = \frac{9\pi}{2}.$$

$$\Rightarrow a = \frac{\pi}{4}, b = \frac{17\pi}{4}.$$

Hence, $\int x\sin x\;dx = -x\cos x\,\Big|_{\pi/4}^{17\pi/4} + \int_{\pi/4}^{17\pi/4} \sin x\;dx$

$$= -\frac{17\pi}{4}\cos\frac{17\pi}{4} + \frac{\pi}{4}\cos\frac{\pi}{4} = \frac{4\pi}{\sqrt{2}} = -2\sqrt{2}\pi.$$

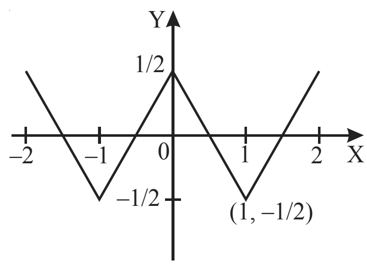
Example 66. Let the function f be defined by

$f(x) = |x - 1| - \frac{1}{2}, 0 \le x \le 2, f(x+2) = f(x)$ for all $x \in R$.

Evaluate

(i) $\int_0^{100} f(x)dx$ (ii) $\int_0^1 |f(2x)|\,dx$

Solution From the figure, we have $\int_0^2 f(x)dx = 0$.



(i) Using the property of periodic functions,

$$\int_0^{100} f(x)dx = 50\int_0^2 f(x)dx = 0.$$

(ii)

$$\int_0^1 |f(2x)|\,dx = \frac{1}{2}\int_0^2 |f(t)|\,dt = \frac{1}{2}.4\left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right) = \frac{1}{4}.$$

Example 67. Show that $\int_0^{p+q\pi} |\cos x|\;dx = 2q +$ $\sin p$ where $q \in N$ and $-\frac{\pi}{2} < p < \frac{\pi}{2}$.

Solution Let $I = \int_0^{p+q\pi} |\cos x|\;dx$

$$= \int_0^{q\pi} |\cos x|dx + \int_0^{p+q\pi} |\cos x|\;dx$$

$$= q\int_0^{\pi} |\cos x|\;dx + \int_0^{p} |\cos x|\;dx$$

$\{\because$ period of $|\cos x|$ is $\pi\}$

$$=q\left\{\int_0^{\pi/2}|\cos x|\,dx+\int_{\pi/2}^{\pi}|\cos x|\,dx\right\}+\int_0^{p}\cos x\,dx$$

$$=q\left\{\int_0^{\pi/2}\cos x\,dx-\int_{\pi/2}^{\pi}\cos x\,dx\right\}+\int_0^{p}\cos x\,dx$$

$$=q\,\{(\sin x)_0^{\pi/2}-(\sin x)_{\pi/2}^{\pi}\}+(\sin x)_0^{p}$$

$$=q\,\{(1-0)-(0-1)\}+\sin p-\sin 0$$

$$=2q+\sin p.$$

Example 68. Evaluate $\int_0^{2n\pi}[\sin x+\cos x]\,dx$, where[.] is the greatest integer function.

Solution Let $I=\int_0^{2n\pi}[\sin x+\cos x]\,dx$

$$[\sin x+\cos x]=\begin{cases}1\ , & 0\le x\le\dfrac{\pi}{2}\\[4pt] 0\ , & \dfrac{\pi}{2}<x\le\dfrac{3\pi}{4}\\[4pt] -1\ , & \dfrac{3\pi}{4}<x\le\pi\\[4pt] -2\ , & \pi<x<\dfrac{3\pi}{2}\\[4pt] -1\ , & \dfrac{3\pi}{2}<x<\dfrac{7\pi}{4}\\[4pt] 0\ , & \dfrac{7\pi}{4}\le x<2\pi\end{cases}$$

So, $\int_0^{2\pi}[\sin x+\cos x]dx$

$$=\int_0^{\pi/2}1.dx+\int_{\pi/2}^{3\pi/4}0.dx+\int_{3\pi/4}^{\pi}(-1)dx$$

$$+\int_{\pi}^{3\pi/2}(-2)dx+\int_{3\pi/2}^{7\pi/4}(-1)dx+\int_{7\pi/4}^{2\pi}0.dx$$

$$=\frac{\pi}{2}+0-\pi+\frac{3\pi}{4}-3\pi+2\pi-\frac{7\pi}{4}+\frac{3\pi}{2}+0=-\pi$$

Since $\sin x+\cos x$ is periodic function with period 2π,

so $I=\int_0^{2n\pi}[\sin x+\cos x]dx$

$$=n\int_0^{2\pi}[\sin x+\cos x]dx=-n\,\pi.$$

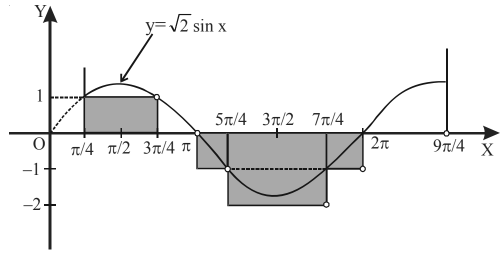
Alternative :

We have $\int_0^{2\pi}[\sin x+\cos x]\,dx$

$$=\int_0^{2\pi}\left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]dx$$

$$=\int_{\pi/4}^{9\pi/4}[\sqrt{2}\sin x]dx$$

= Area of the shaded region as shown in the figure.



$$=\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)\times1+1\times0+\left(\frac{5\pi}{4}-\pi\right)\times(-1)$$

$$+\left(\frac{7\pi}{4}-\frac{5\pi}{4}\right)\times(-2)+\left(2\pi-\frac{7\pi}{4}\right)\times(-1)=-\pi$$

Hence, $\int_0^{2\pi n}[\sin x+\cos x]dx=-n\,\pi.$

Example 69. Evaluate $\dfrac{\int_0^n[x]\,dx}{\int_0^n\{x\}\,dx}$, (where $[x]$ and $\{x\}$ are integral and fractional part of x and $n\in N$).

Solution We have, $\int_0^n[x]\,dx$

$$=\int_0^1 0.dx+\int_1^2 1.dx+\int_2^3 2.dx+\ldots+\int_{n-1}^{n}(n-1)\,dx$$

$$=0+1.(2-1)+2.(3-2)+\ldots.+(n-1)(n-(n-1))$$

$$=1+2+3+\ldots..+(n-1)=\frac{n(n-1)}{2} \qquad \ldots(1)$$

and $\int_0^n\{x\}\,dx=n\int_0^1\{x\}dx=n.\left(\dfrac{1}{2}.1.1\right)$ ...(2)

$$\therefore\ \frac{\int_0^n[x]dx}{\int_0^n\{x\}dx}=\frac{\frac{n(n-1)}{2}}{\frac{n}{2}}=(n-1)\ \text{[using (1) and (2)]}$$

Example 70. Evaluate $\int_0^5\dfrac{\tan^{-1}(x-[x])}{1+(x-[x])^2}dx$

Solution We have

$$I=\int_0^5\frac{\tan^{-1}\{x\}}{1+\{x\}^2}dx \qquad \text{[period 1]}$$

$$=5\int_0^1\frac{\tan^{-1}x}{1+x^2}\,dx=5\int_0^{\pi/4}u\,du$$

$$\text{[Putting }\tan^{-1}x=u]=5\left[\frac{u^2}{2}\right]_0^{\pi/2}=\frac{5\pi^2}{32}.$$

Example 71. Show that

$$\int_0^{2\pi}\sqrt{1+\sin^3 x}\,dx = \int_0^{2\pi}\sqrt{1+\cos^3 x}\,dx.$$

Solution In the integral on the left, replace sin x by cos $(\pi/2 - x)$. Then make the substitution $u = \pi/2 - x$:

$$\int_0^{2\pi}\sqrt{1+\sin^3 x}\,dx = \int_0^{2\pi}\sqrt{1+\cos^3\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_{\pi/2}^{-3\pi/2}\sqrt{1+\cos^3 u}\,(-du)$$

$$= \int_{-3\pi/2}^{\pi/2}\sqrt{1+\cos^3 u}\,du$$

$$= \int_0^{2\pi}\sqrt{1+\cos^3 u}\,du.$$

[since the integral is periodic with period 2π]

The next example uses the same idea.

Example 72. Suppose n is a positive integer and f(t) is any function defined for $-1 \le t \le 1$. Show that

$$\int_0^{2\pi} f(\sin nx)dx = \int_0^{2\pi} f(\cos nx)dx.$$

Solution Substitute $x = \pi/(2n) - u$ into the integral on the left.

Then $\sin nx = \cos(\pi/2 - nx) = \cos nu$, $dx = -du$

$$\Rightarrow \int_0^{2\pi} f(\sin nx)dx = -\int_{\pi/2n}^{(\pi/2n)-2\pi} f(\cos nu)du$$

$$= \int_{(\pi/2n)-2\pi}^{(\pi/2n} f(\cos nu)du = \int_0^{2\pi} f(\cos nu)\,du$$

We have used two facts :

(i) $f(\cos nu)$ has period 2π,

(ii) $\int_a^{a+2\pi} f(\cos nu)du = \int_0^{2\pi} f(\cos nu)du,$

where $a = \pi/2n - 2\pi$.

Example 73. Compute $\int_0^{2\pi} \sin^2(100x)\,dx$.

Solution Set $A = \int_0^{2\pi} \cos^2(100x)\,dx$ and

$B = \int_0^{2\pi} \sin^2(100x)\,dx$.

By the last example, A = B. On the other hand,

$$A+B = \int_0^{2\pi}\cos^2(100x)\,dx + \int_0^{2\pi}\sin^2(100x)\,dx$$

$$= \int_0^{2\pi}[\cos^2(100x)+\sin^2(100x)]\,dx$$

$$= \int_0^{2\pi} 1\cdot dx = 2\pi.$$

Hence, $A = \pi$.

Alternative :

Use the identity $\sin^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$.

$$\int_0^{2\pi}\sin^2(100x)\,dx$$

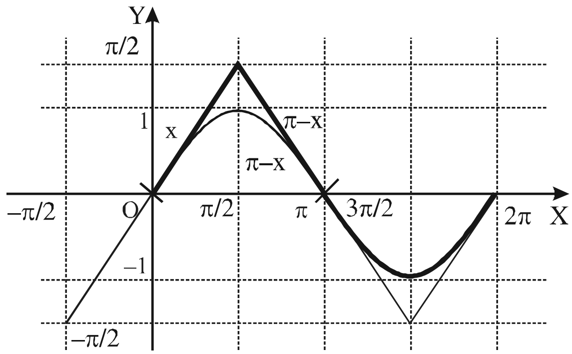
$$= \frac{1}{2}\int_0^{2\pi} dx + \frac{1}{2}\int_0^{2\pi}\cos(200x)\,dx = \pi.$$

The first integral on the right is π. The second is zero because cos 200 x makes a whole number of compute cycles from 0 to 2π.

Example 74. Find the value of the definite integral

$$\int_0^{2n\pi}\max.(\sin x, \sin^{-1}(\sin x))\,dx \quad (\text{where } n \in I).$$

Solution



$\therefore$ The period of the function is 2π.

$$I = \int_0^{2n\pi}\max.\left(\sin x, \sin^{-1}(\sin x)\right)dx$$

$$= n\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx + \int_{\pi}^{2\pi}(\sin x)\,dx\right]$$

$$= n\left[\frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{1}{2}\left(\pi^2 - \frac{\pi^2}{4}\right) - 2\right]$$

$$= n\left[\frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{3\pi^2}{8} - 2\right] = \frac{n(\pi^2 - 8)}{4}.$$

Example 75. It is known that f(x) is an odd function in the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$ and has a period equal to T. Prove that $\int_a^x f(t)\,dt$ is also perodic with period T.

Solution Given that $f(-x) = -f(x)$ and $f(x+T) = f(x)$

Let $F(x) = \int_a^x f(t)\,dt$

$\therefore \quad F(x+T) = \int_a^{x+T} f(t)\,dt$

$= \int_a^x f(t)\,dt + \int_x^{x+T} f(t)\,dt$

$= F(x) + \int_x^{T/2} f(t)\,dt + \int_{T/2}^{x+T} f(t)\,dt$

Put $t = u + T$

$= F(x) + \int_x^{T/2} f(t)\,dt + \int_{-T/2}^{x} f(u+T)\,du$

$= F(x) + \int_x^{T/2} f(t)\,dt + \int_{-T/2}^{x} f(u)\,du \quad [\because f(u+T) = f(u)]$

$= F(x) + \int_{-T/2}^{T/2} f(t)\,dt$

$= F(x) + 0 \quad \{\because f(t) \text{ is odd}\}$

$= F(x)$

$\therefore F(x+T) = F(x)$

$F(x)$ is periodic with period T.

Example 76. Let $\int_x^{x+p} f(t)dt$ be independent of x and

$I_1 = \int_0^p f(t)dt$ $I_2 = \int_{10}^{p^n+10} f(z)dz$ for some $p, n \in N$. Then evaluate $\frac{I_2}{I_1}$.

Solution Let $g(x) = \int_x^{x+p} f(t)dt$

Since $g(x)$ is independent of x, $g'(x) = 0$,

$\Rightarrow \quad f(x+p) - f(x) = 0$

$\Rightarrow \quad f(x)$ is periodic with period p.

Here, $I_1 = \int_0^p f(t)dt$ and

$I_2 = \int_{10}^{p^n+10} f(z)dz = \int_{10}^{p^{n-1}.p+10} f(z)dz$

$= \int_0^{p^{n-1}.p} f(z)dz = p^{n-1}\int_0^p f(z)dz \quad$ **[P–7]**

$\Rightarrow \frac{I_2}{I_1} = p^{n-1}.$

Example 77. Let $F(x)$ be a non-negative continuous function defined on R such that $F(x) + F\left(x+\frac{1}{2}\right) = 3.$

Find the value of $\int_0^{1500} F(x)dx$.

Solution We have $F(x) + F\left(x+\frac{1}{2}\right) = 3 \quad$ ...(1)

Replacing x by $x+\frac{1}{2}$ in (1), we get

$F\left(x+\frac{1}{2}\right) + F(x+1) = 3 \quad$ ...(2)

$\therefore$ From (1) and (2), we get $F(x) = F(x+1)$

$\Rightarrow \quad F(x)$ is a periodic function with period 1.

Now $I = \int_0^{1500} F(x)dx = 1500\int_0^1 F(x)dx \quad$ **[P–7]**

$= 1500\left[\int_0^{1/2} F(x)dx + \int_{1/2}^{1} F(x)dx\right]$

Putting $x = y + \frac{1}{2}$ in the second integral, we get

$I = 1500\left[\int_0^{1/2} F(x)dx + \int_0^{1/2} F\left(y+\frac{1}{2}\right)by\right]$

$= 1500\int_0^{1/2}\left(F(x) + F\left(x+\frac{1}{2}\right)\right)dx$

$= 1500\int_0^{1/2} 3\,dx \quad$ [using (1)]

Hence, $I = 1500(3)\left(\frac{1}{2}\right) = 750 \times 3 = 2250.$

Practice Problems O

1. Prove that

(i) $\int_0^{1000} e^{x-[x]}dx = 1000(e-1)$

(ii) $\int_0^{200\pi} \sqrt{1+\cos x}\;dx = 400\sqrt{2}$

(iii) $\int_0^{2000\pi} \frac{dx}{1+e^{\sin x}} = 1000\pi$

(iv) $\int_{10\pi+\frac{\pi}{6}}^{10\pi+\frac{\pi}{3}} (\sin x + \cos x)dx = (\sqrt{3}-1)$.

2. Evaluate the following integrals :

(i) $\int_{-1}^{2} e^{\{3x\}}\,dx$

(ii) $\int_0^{41\pi/2} \sin x\,dx$

(iii) $\int_{\pi}^{7\pi/2} |\cos x|\,dx$

(iv) $\int_{\pi}^{5\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\,dx$

3. Show that

(i) $\int_{-a}^{a} \varphi(x^2)\,dx = 2\int_0^a \varphi(x^2)\,dx$

$\int_{-a}^{a} x\varphi(x^2)\,dx = 0$

(ii) $\int_0^{\pi/2} \varphi(\cos x)\,dx = \int_0^{\pi/2} \varphi(\sin x)\,dx$

$= \frac{1}{2}\int_0^{\pi} \varphi(\sin x)\,dx$

(iii) $\int_0^{m\pi} \phi(\cos^2 x)\,dx = m\int_0^{\pi}\phi(\cos^2 x)\,dx$,

m being an integer.

4. A periodic function with period 1 is integrable over any finite interval. Also for two real numbers a, b and for two unequal non-zero postive integers m and n, $\int_a^{a+n} f(x)\,dx = \int_b^{b+m} f(x)\,dx$. Calculate the value of $\int_m^n f(x)\,dx$.

5. Suppose f is continuous on $(-\infty, \infty)$ with period p. Suppose f′ and f″ are also continuous. Prove that $\int_a^{a+p} f''(x)\,dx = \int_a^{a+p} f'(x)\,dx = 0$.

6. Given $\int_0^1 \frac{\sin t}{1+t}\,dt = \alpha$, find $\int_{4\pi-2}^{4\pi} \frac{\sin\frac{t}{2}}{4\pi+2-t}\,dt$ in terms of α.

7. Given an odd function f, defined everywhere, periodic with period 2, and integrable on every interval. Let $g(x) = \int_0^x f(t)\,dt$.

(a) Prove that $g(2n) = 0$ for every integer n.

(b) Prove that g is even and periodic with period 2.

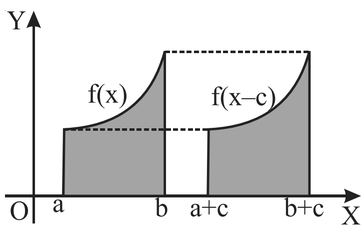
2.14 ADDITIONAL PROPERTIES

1. Shift property

The function $y = f(x - c)$ is obtained by shifting the graph of $y = f(x)$ 'c' units rightward.
Thus, it is obvious that

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx \text{ and also}$$

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx$$



We can prove the above results analytically, by substituting respectively $x - c = t$ and $x + c = t$ in the two formulae.
Also, we have for example,

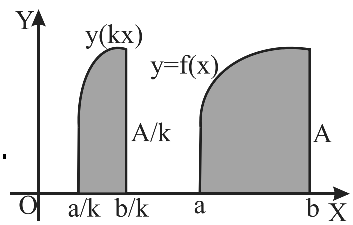
$$\int_{-1}^{0} (x+1)^3\,dx = \int_0^1 x^3\,dx = \int_1^2 (x-1)^3\,dx.$$

2. Expansion–Contraction property

$$K\int_{a/k}^{b/k} f(kx)\,dx = \int_a^b f(x)\,dx \text{ for every } k > 0.$$

Proof: Put $kx = t$ in the left hand side
$\therefore\ k\,dx = dt$
When $x = a/k$; $t = a$
when $x = b/k$; $t = b$

Thus, LHS $= k\int_a^b f(t)\frac{dt}{k} = \int_a^b f(x)\,dx$ = RHS



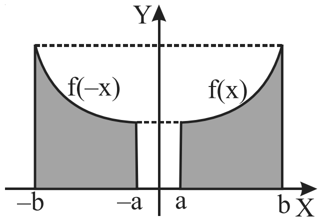
In order to sketch $y = f(kx)$ from $y = f(x)$ the later is compressed 'k' times along x-axis. Note that the graph of $y = f(kx)$ contracts w.r.t. $y = f(x)$ when $k > 1$ and expands when $0 < k < 1$. As a result the area formed by f(x) with x-axis in between the limits a and b contracts to the area formed by f(kx) with x-axis in between the limits a/k and b/k; and the latter area is (1/k) times the first area.

For example, $\int_0^{\pi} \sin x\,dx = 3\int_a^{\pi/3} \sin(3x)\,dx$

3. Reflection property

$$\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-x)\,dx$$

The curves $y = f(x)$ and $y = f(-x)$ are mirror images of each other in the line $x = 0$ i.e. y-axis.



4. Transformation of an integral into a new one with limits 0 and 1

For any given integral with finite limits a and b, one can always choose the linear substitution $x = pt + q$ (p, q constants) so as to transform this integral into a new one with limits 0 and 1.

Let $x = pt + q$

Since t must equal zero at $x = a$ and t must equal unity at $x = b$, we have for p and q the following system of equations :

$a = p.0 + q,$

$b = p.1 + q,$

$\Rightarrow\ p = b - a,\quad q = a.$

Hence, $\int_a^b f(x)dx = (b-a)\int_0^1 f[(b-a)t + a]dt$.

Example 1. Compute the sum of two integrals

$$\int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}dx.$$

Solution Let us tansform each of the given integrals into an integral with limits 0 and 1.

To this end apply the substitution

$x = (b-a)t + a$

$\Rightarrow\ x = -t - 4$ to the first integral. Then $dx = -dt$ and

$$I_1 = \int_{-4}^{-5} e^{(x+5)^2}dx - \int_0^0 e^{(-t+1)^2}dt = -\int_0^1 e^{(t-1)^2}dt$$

Similarly, we apply the substitution $x = \frac{t}{3} + \frac{1}{3}$ to the second integral. Then $dx = \frac{dt}{3}$ and

$$I_2 = 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}dx = \int_0^1 e^{(t-1)^2}dt$$

Hence, $I_1 + I_2 = -\int_0^1 e^{(t-1)^2}dt + \int_0^1 e^{(t-1)^2}dt = 0$.

Note that neither of the integrals $\int e^{(x+5)^2}dx$ nor $\int e^{9\left(x-\frac{2}{3}\right)^2}dx$ can be evaluated separately in elementary functions.

5. Integral of an inverse function

If f is invertible and f' is continuous, then a definite integral of f^{-1} can be expressed in terms of a definite integral of f.

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)dy.$$

Proof: Using integration by parts

$$\int_a^b f(x).1dx = xf(x)\Big|_a^b - \int_a^b f'(x)x\,dx$$

We have $y = f(x) \Rightarrow dy = f'(x)\,dx$.

Also, $x = f^{-1}(y)$

When $x = a$, $y = f(a)$ and when $x = b$, $y = f(b)$.

$$\Rightarrow\ \int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)dy.$$

This theorem can be given in a different form as :

If g (x) is the inverse of f (x) and f (x) has domain $x \in [a, b]$ where $f(a) = c$ and $f(b) = d$ then the value of

$$\int_a^b f(x)dx + \int_c^d g(y)dy = (bd - ac).$$

Proof: $y = f(x)\quad \Rightarrow x = f^{-1}(y) = g(y)$

$$I = \int_a^b f(x)dx + \int_c^d g(y)dy$$

$$= \int_a^b f(x)dx + \int_a^b xf'(x)dx\ [y = f(x) \Rightarrow f'(x)\,dx = dy)]$$

$$= xf(x)\Big]_a^b = b\,f(b) - a\,f(a) = bd - ac.$$

Example 2. Evaluate

$$I = \int_0^1 e^{\sqrt{e^x}}dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)dx.$$

Solution Let $y = f(x) = e^{\sqrt{e^x}}$.

Then $\ln y = \sqrt{e^x} \Rightarrow \ln(\ln y) = \frac{x}{2} \Rightarrow x = 2\ln(\ln y)$

$\Rightarrow\ f^{-1}(y) = 2\ln(\ln y)$.

Hence, $I = \int_{a=0}^{b=1} e^{\sqrt{e^x}}dx + 2\int_{c=e}^{d=e^{\sqrt{e}}} \ln(\ln y)dy$

where $f(0) = e$ and $f(1) = e^{\sqrt{e}}$

$\Rightarrow\ I = bd - ac = 1.e^{\sqrt{e}} - 0.e = e^{\sqrt{e}}$.

Example 3. If the value of the integral $\int_1^2 e^{x^2}dx$ is α, then find the value of $\int_e^{e^4}\sqrt{\ell n\,x}\,dx$.

Solution The functions e^{x^2} and $\sqrt{\ell n\,x}$ are inverses of each other. Assuming $f(x) = e^{x^2}$ and $a = 1$ and $b = 2$, we have $\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)dy$

$$\therefore\ \alpha = \int_1^2 e^{x^2}dx = 2.e^{2^2} - 1.e - \int_e^{e^4}\sqrt{\ell n\,y}\,dy$$

$$= 2e^4 - e - \int_e^{e^4} \sqrt{\ell n\, x}\, dx$$

Hence, $\int_e^{e^4} \sqrt{\ell n\, x}\, dx = 2e^4 - e - \alpha$.

Example 4. Let f(x) be a differentiable real valued function which is strictly monotonic and a, b two real numbers. Show that $\int_a^b (f(x)+f(a))\{f(x)-f(a)\}\,dx$

$$= 2\int_{f(a)}^{f(b)} x\{b - f^{-1}(x)\}dx.$$

Solution As f(x) is strictly monotonic, $f^{-1}(x)$ exists.
Also, if $x = f(y)$ then $f^{-1}(x) = y$
and $x = f(a) \Rightarrow y = f^{-1}\{f(a)\} = a$
$x = f(b) \Rightarrow y = f^{-1}(f(b)\} = b$.

$$\text{RHS} = 2\int_a^b f(y)\,(b-y)\,f'(y)\,dy$$

$\{\because x = f(y) \Rightarrow dx = f'(y)dy\}$

$$= \int_a^b d[\{f(y)\}^2]$$

$$= [(b-y)\{f(y)\}^2]_a^b - \int_a^b \{f(y)\}^2 \cdot (-1)\,dy,$$

(using integration by parts)

$$= -(b-a)\{f(a)\}^2 + \int_a^b \{f(y)\}^2\,dy$$

$$= -\int_a^b \{f(a)\}^2\,dy + \int_a^b \{f(y)\}^2\,dy$$

$$= \int_a^b [\{f(y)\}^2 - \{f(a)\}^2]dy$$

$$= \int_a^b [\{f(x)\}^2 - \{f(a)\}^2]\,dx,$$

$$= \int_a^b \{f(x)+f(a)\}\,(f(x)-f(a)\}\,dx.$$

Practice Problems P

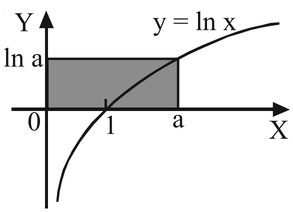
1. Prove that

(a) $\int_0^{1/2} \sin^{-1} x\,dx = \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \int_0^{\pi/6} \sin x dx$

(b) $\int_e^{e^2} \ln x dx = (2e^2 - e) - \int_1^2 e^x dx$.

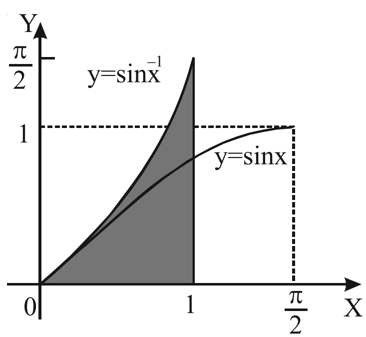
2. Show that for any number a > 1

$\int_1^a \ln x dx + \int_0^{\ln a} e^y dy = a \ln a$.



3. Use the accompanying figure to show that

$\int_0^{\pi/2} \sin x dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x dx$.



4. Evaluate $\int_0^1 (\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3})dx$

5. If g (x) is the inverse of *f* (x) and *f* (x) has domain x ∈ [1, 5], where f (1) = 2 and f (5) = 10 then find the value of $\int_1^5 f(x)dx + \int_2^{10} g(y)dy$.

6. Suppose f is continuous, f(0) = 0, f(1) = 1, f'(x) > 0, and $\int_0^1 f(x)dx = \frac{1}{3}$. Find the value of the integral $\int_0^1 f^{-1}(y)dy$

7. Show that $\int_0^\infty e^{-x^2}dx = \int_0^1 \sqrt{-\ell n y}dy$ by interpreting the integrals as areas.

8. Let f be an increasing function with f(0) = 0, and assume that it has an elementary antiderivative. Then f^{-1} is an increasing function, and $f^{-1}(0) = 0$. Prove that if f^{-1} is elementary, then it also has an elementary antiderivative.

9. Let a > 0, b > 0, and f a continuous strictly increasing function with f(0) = 0. Prove that

$ab \le \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx$.

Prove, moreover, that equality occurs if and only if b = f(a).

2.15 ESTIMATION OF DEFINITE INTEGRALS

Not all integrals can be evaluated using the methods discussed so far. So we try to find the approximate value of such integrals using some estimation techniques.

1. Domination Law

If $f(x) \le g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Actually $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$ unless $f(x) = g(x)$ for all x, in which case $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.

Example 1. Show that $\int_1^3 \frac{dx}{1+x^4} < \frac{1}{3}$.

Solution Since $\frac{1}{1+x^4} < \frac{1}{x^4}$, it follows from domination law that

$$\int_1^3 \frac{dx}{1+x^4} < \int_1^3 \frac{dx}{x^4} = -\left.\frac{1}{3x^3}\right|_1^3 = \frac{1}{3} - \frac{1}{81} < \frac{1}{3}.$$

This example illustrates an important technique : replacing the integrand f(x) by a slightly larger one, g(x), which can be easily integrated, gives the upper bound.

Note:

(i) If f(x) is an integrable function and $f(x) \ge 0$ for a $\le x \le b$, then actually $\int_a^b f(x)\,dx > 0$ unless f(x) = 0 for all x.

(ii) If at every point x of an interval [a, b], the inequalities $\psi(x) \le f(x) \le \phi(x)$ are fulfilled then

$$\int_a^b \psi(x)dx \le \int_a^b f(x)dx \le \int_a^b \phi(x)dx,\ a < b.$$

This means that an inequality between functions implies an inequality of the same sense between their definite integrals, or, briefly speaking, that it is allowable to integrate inequalities termwise.

On the other hand, simple geometrical considerations indicate that the differentiation of an inequality may lead to a senseless result. For instance, the inequality $f(x) < C$ (C is a constant) does not, of course, imply that $f'(x) < 0$.

(iii) Let f and g be integrable on [a, b]. Define M and m by $M(x) = \max\{f(x), g(x)\}$ and $m(x) = \min\{f(x), g(x)\}$. Then M(x) and m(x) are integrable and max

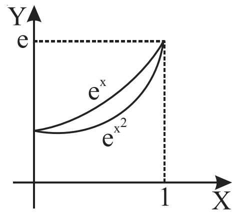
$$\left\{\int_a^b f(x)dx, \int_a^b g(x)dx\right\} \le \int_a^b \max\{f(x), g(x)\}dx,$$

$$\int_a^b \min\{f(x), g(x)\}dx \le \min\left\{\int_a^b f(x)dx, \int_a^b g(x)dx\right\}$$

Example 2. Estimate the value of

$$\int_0^1 e^{x^2}dx \text{ by using } \int_0^1 e^x dx$$

Solution For $x \in (0, 1)$, $e^{x^2} < e^x$



$$\Rightarrow \int_0^1 1\,dx < \int_0^1 e^{x^2}dx < \int_0^1 e^x dx$$

$$\Rightarrow 1 < \int_0^1 e^{x^2}dx < e - 1.$$

Example 3. If $I = \int_3^4 \frac{1}{\sqrt[3]{\ln x}}\,dx$, then prove that $0.92 < I < 1$.

Solution For $x > e$, we know that

$$1 < \ln x < \frac{x}{e}\left(\frac{e}{x}\right)^{1/3} < \frac{1}{\sqrt[3]{\log x}} < 1$$

$$\Rightarrow \int_3^4 e^{1/3}\ x^{-1/3}\ dx < I < \int_3^4 dx$$

$$\Rightarrow \frac{3}{2} e^{1/3}\ (4^{2/3} - 3^{2/3}) < I < 1$$

$$\Rightarrow 0.92 < I < 1.$$

Example 4. Consider the integrals,

$$I_1 = \int_0^1 e^{-x}\cos^2 x\ dx,\ I_2 = \int_0^1 e^{-x^2}\cos^2 x\ dx,$$

$$I_3 = \int_0^1 e^{-x^2}\ dx \text{ and } I_4 = \int_0^1 e^{-x^2/2}\ dx$$

Find the greatest integral.

Solution For $0 < x < 1$, $x^2 < x^2 < x$

$$\Rightarrow -x^2 > -x \Rightarrow e^{-x^2} > e^{-x}$$

$$\Rightarrow \int_0^1 e^{-x^2}\cos^2 x\,dx > \int_0^1 e^{-x}\cos^2 x\,dx$$

and $\cos^2 x \le 1$

$$\Rightarrow \int_0^1 e^{-x^2}\cos^2 x\,dx \le \int_0^1 e^{-x^2}\,dx$$

$$< \int_0^1 e^{-\frac{1}{2}x^2}\,dx = 1$$

Hence, I_4 is the greatest integral .

Example 5. Show that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 + x^5}} < \frac{\pi}{6}$.

Solution Let the given integral be I.

For $x \in (0, 1)$, $\sqrt{4 - x^2 + x^5} > \sqrt{4 - x^2}$

$$\therefore \int_0^1 \frac{1}{\sqrt{4 - x^2 + x^5}} < \int_0^1 \frac{1}{\sqrt{4 - x^2}}$$

$$\Rightarrow I < \left.\sin^{-1}\frac{x}{2}\right|_0^1 = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}.$$

Also, $\dfrac{1}{\sqrt{4-(x^2-x^5)}} > \dfrac{1}{\sqrt{4}}$

$$\int_0^1 \frac{1}{\sqrt{4-(x^2-x^5)}} > \int_0^1 \frac{1}{2}dx$$

$\Rightarrow \quad I > \dfrac{1}{2}.$

Example 6. Prove that

$$0.573 < \int_1^2 \frac{dx}{\sqrt{(4-3x+x^3)}} < 0.595.$$

Solution Let $I = \displaystyle\int_1^2 \frac{dx}{\sqrt{(4-3x+x^3)}}$

Put $x = 1 + u$, then

$$I = \int_0^1 \frac{du}{\sqrt{(2+3u+u^3)}}$$

Since, $2 + 3u^2 < 2 + 3u^2 + u^3 < 2 + 4u^2$, $(0 < u < 1)$

$$\int_0^1 \frac{du}{\sqrt{(2+4u^2)}} < \int_0^1 \frac{du}{\sqrt{(2+3u^2+u^3)}}$$

$$< \int_0^1 \frac{du}{\sqrt{(2+3u^2)}}.$$

Solving the two integrals, the inequality is established.

Example 7. Show that $\dfrac{\pi}{3\sqrt{3}} \le \displaystyle\int_0^1 \frac{dx}{1+x^2+2x^5} \le \frac{\pi}{4}$

Solution We have $1 + x^2 + 2x^5 > 1 + x^2$
and $1 + x^2 + 2x^5 < 1 + x^2 + 2x^2 = 1 + 3x^2$
[$\because x^5 < x^2$ on $(0, 1)$]

Hence, we have $\dfrac{1}{1+3x^2} < \dfrac{1}{1+x^2+2x^5} < \dfrac{1}{1+x^2}$

$$\Rightarrow \int_0^1 \frac{dx}{1+3x^2} < \int_0^1 \frac{dx}{1+x^2+2x^5} < \int_0^1 \frac{dx}{1+x^2}$$

$$\Rightarrow \left[\frac{\tan^{-1}\sqrt{3}x}{\sqrt{3}}\right]_0^1 < \int_0^1 \frac{dx}{1+x^2+2x^5} < \left[\tan^{-1}x\right]_0^1$$

$$\Rightarrow \frac{\pi}{3\sqrt{3}} \le \int_0^1 \frac{dx}{1+x^2+2x^5} \le \frac{\pi}{4},$$

which is the desired result.

Example 8. Show that

$$1 \le \int_0^{\pi/2} \sqrt{1-\sin^3 x}\,dx \le \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2})).$$

Solution $\sin^4 x \le \sin^3 x \le \sin^2 x$

$\Rightarrow \quad -\sin^2 x \le -\sin^3 x \le -\sin^4 x$

$\Rightarrow \quad 1 - \sin^2 x \le 1 - \sin^3 x \le 1 - \sin^4 x$

So, we get $\sqrt{1-\sin^2 x} \le \sqrt{1-\sin^3 x} \le \sqrt{1-\sin^4 x}$

$$\int_0^{\pi/2} \sqrt{1-\sin^2 x}\,dx = \int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1,$$

and, $\displaystyle\int_0^{\pi/2} \sqrt{1-\sin^4 x}\,dx = \int_0^{\pi/2} \sqrt{(1+\sin^2 x)\cos^2 x}\,dx$

$$= \int_0^{\pi/2} \cos x\sqrt{1+\sin^2 x}\,dx = \int_0^1 \sqrt{1+t^2}\,dt$$

$$= \left[\frac{1}{2}\left(t\sqrt{t^2+1} + \ln(t+\sqrt{t^2+1})\right)\right]_0^1$$

$$= \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2}))$$

$$\therefore \quad 1 \le \int_0^{\pi/2} \sqrt{1-\sin^3 x}\,dx \le \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2})).$$

Example 9. If $I_n = \displaystyle\int_0^{\pi/4} \tan^n x\,dx$, prove that

(i) $I_n + I_{n-2} = \dfrac{1}{n-1}$ (ii) $I_{n-1} + I_{n+1} = \dfrac{1}{n}$

(iii) $\dfrac{1}{n+1} < 2I_n < \dfrac{1}{n-1}$, where $n > 1$ is a natural number.

Solution $I_n = \displaystyle\int_0^{\pi/4} \tan^n x\,dx$

$$= \int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1)\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x\sec^2 x\,dx - \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$I_n = \left[\frac{\tan^{n-1} x}{(n-1)}\right]_0^{\pi/4} - I_{n-2}$$

$$\Rightarrow I_n + I_{n-2} = \frac{1}{n-1} \qquad ...(1)$$

Replace n by $(n + 1)$

$$\text{Then } I_{n+1} + I_{n-1} = \frac{1}{n} \qquad ...(2)$$

In the interval $\left[0, \dfrac{\pi}{4}\right]$, $\tan^n x < \tan^{n-2} x$

$$\therefore \quad \int_0^{\pi/4} \tan^n x\,dx < \int_0^{\pi/4} \tan^{n-2} x\,dx.$$

$\Rightarrow \quad I_n < I_{n-2}$

$\Rightarrow \quad I_n < \dfrac{1}{n-1} - I_n$ (from (1))

$$\Rightarrow \quad 2I_n < \frac{1}{(n-1)} \qquad \text{...(3)}$$

And similarly, $I_{n+2} < I_n$

$$\Rightarrow \quad \frac{1}{n+1} - I_n < I_n \quad \Rightarrow \quad \frac{1}{n+1} < 2I_n \qquad \text{...(4)}$$

From (1) and (2) we get $\dfrac{1}{(n+1)} < 2I_n < \dfrac{1}{(n-1)}$.

Example 10. Show that $\displaystyle\int_0^{10} \frac{x\,dx}{x^3+16} < \frac{144}{160}$ or less by breaking up the integration into two intervals [0, 1] and [1, 10] and using appropriate approximations for the integrand.

Solution $\displaystyle\int_0^{10} \frac{x\,dx}{x^3+16} < \int_0^{1} \frac{x\,dx}{x^3+16} + \int_1^{10} \frac{x\,dx}{x^3+16}.$

$$\int_0^{1} \frac{x\,dx}{x^3+16} < \int_0^{1} \frac{x\,dx}{16} = \frac{1}{32}; \int_1^{10} \frac{x\,dx}{x^3+16} < \int_1^{10} \frac{x\,dx}{x^3} = \frac{9}{10}$$

Thus, $\displaystyle\int_0^{10} \frac{x\,dx}{x^3+16} < \frac{1}{32} + \frac{9}{10} = \frac{149}{160}.$

2. Max-Min Inequality

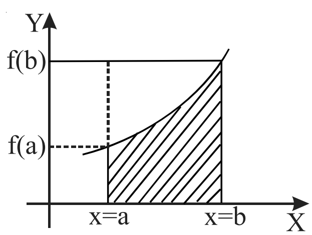
If $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Further,

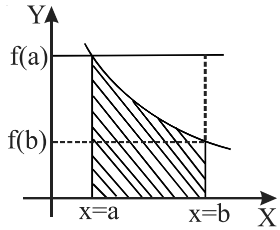
(i) For an increasing function in (a, b)

$$(b-a)\,f(a) < \int_a^b f(x)\,dx < (b-a)\,f(b)$$



(ii) For a decreasing function in (a, b)

$$(b-a)\,.\,f(b) < \int_a^b f(x)\,dx < (b-a)\,f(a)$$



Example 11. Estimate the value of the integrals :

(i) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$ (ii) $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\,dx$.

Solution

(i) Let $f(x) = \dfrac{\sin x}{x}$

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{(\cos x)(x - \tan x)}{x^2} < 0$$

$\Rightarrow$ f(x) is a strictly decreasing function.
f(0) is not defined, so we evaluate

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

$$f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}.$$

Hence, $\displaystyle\frac{2}{\pi}\,.\left(\frac{\pi}{2}-0\right) < \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx < 1\,.\left(\frac{\pi}{2}-0\right)$

$$\Rightarrow \quad 1 < \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx < \frac{\pi}{2}.$$

(ii) $f(x) = \dfrac{\sin x}{x}$ decreases on the interval $\left[\dfrac{\pi}{4}, \dfrac{\pi}{3}\right]$.

Hence, the least value of the function is

$$m = f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2\pi}, \text{ and}$$

its greatest value is $M = f\left(\dfrac{\pi}{4}\right) = \dfrac{2\sqrt{2}}{\pi}$.

Therefore,

$$\frac{3\sqrt{3}}{2\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) < \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\,dx < \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \frac{\sqrt{3}}{8} < \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\,dx < \frac{\sqrt{2}}{6}.$$

Example 12. Prove that

$$\frac{\pi}{\pi^3+10\pi+5}<\int_0^{\pi}\frac{dx}{x^3+10x+9\sin x+5}<\frac{\pi}{5}.$$

Solution Let $f(x) = x^3 + 10x + 9\sin x + 5$

$f'(x) = 3x^2 + 10 + 9\cos x > 0 \ \forall\ x \in R$

$\Rightarrow$ f(x) is strictly increasing

$\Rightarrow \dfrac{1}{f(x)}$ is strictly decreasing in $(0, \pi)$

$\Rightarrow$ Absolute maximum of f(x) in $[0, \pi]$ is $\dfrac{1}{5}$ and absolute minimum is $\dfrac{1}{\pi^3+10\pi+5}$.

So, $$\frac{\pi}{\pi^3+10\pi+5}<\int_0^{\pi}\frac{dx}{x^3+10x+9\sin x+5}<\frac{\pi}{5}.$$

Example 13. Prove that $1<\int_0^2\left(\dfrac{5-x}{9-x^2}\right)dx<\dfrac{6}{5}$.

Solution Let $f(x)=\dfrac{5-x}{(9-x^2)}$,

We first find the greatest and the least values of the integrand f(x) in the interval [0, 2].

$$\Rightarrow f'(x)=-\frac{(x-9)(x-1)}{(9-x^2)^2}$$

For $f'(x) = 0$, we have $x = 1$ as $x \in [0, 2]$.
Now $f(0) = 5/9$, $f(1) = 1/2$, $f(2) = 3/5$.

$\therefore$ The greatest and the least values of the integrand in the interval [0, 2] are, respectively, equal to $f(2) = 3/5$ and $f(1) = 1/2$.

Hence, $$(2-0)\frac{1}{2}<\int_0^2\left(\frac{5-x}{9-x^2}\right)dx<(2-0)\frac{3}{5}.$$

or, $$1<\int_0^2\left(\frac{5-x}{9-x^2}\right)dx<\frac{6}{5}.$$

3. $\left|\int_a^b f(x)dx\right| \le \left|\int_a^b |f(x)|\,dx\right|$

Proof For any x we have, $-|f(x)| \le f(x) \le |f(x)|$.
(If $f(x) > 0$ the right inequality turns into an equality and the left inequality is obvious; if $f(x) < 0$ these inequalities are proved similarly) It follows that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|\,dx$$

that is, $$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$$

Thus, the absolute value of the definite integral does not exceed the integral of the absolute value of the integrand function.

Note: Let K be a number such that $|f(x)| \le K$ $\forall\ x \in [a, b]$, then $\left|\int_a^b f(x)dx\right| \le K|b-a|$.

Proof If $a = b$, the result is trivial.
Let m, M be the bounds of f on [a, b].
Let $b > a$. Then for all x in [a, b], we have $f(x)| \le K$

$\Rightarrow -K \le f(x) \le K$

$\Rightarrow -K \le m \le f(x) \le M \le K$ ...(1)

It then follows from (1) that

$$-K(b-a) \le m(b-a) \le \int_a^b f(x)\,dx$$

$$\le M(b-a) \le K(b-a)$$

$$\Rightarrow \left|\int_a^b f(x)dx\right| \le K(b-a) \quad ...(2)$$

If $b < a$, then $a > b$.
Hence we get from (2) that

$$\left|\int_a^b f(x)dx\right| \le K(a-b)$$

$$\Rightarrow \left|-\int_a^b f(x)dx\right| \le K(a-b)$$

$$\Rightarrow \left|\int_a^b f(x)dx\right| \le K(a-b) \quad ...(3)$$

From (2) and (3), we get

$$\left|\int_a^b f(x)dx\right| \le K|b-a|.$$

Example 14. Estimate the absolute value of the integral

$$\int_{10}^{19}\frac{\sin x}{1+x^8}dx.$$

Solution $$I=\left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right| \le \int_{10}^{19}\left|\frac{\sin x}{1+x^8}\right|dx \quad ...(1)$$

Since $|\sin x| \le 1$ for $x \ge 10$, the inequality

$$\left|\frac{\sin x}{1+x^8}\right| \le \frac{1}{|1+x^6|} \text{ holds} \quad ...(2)$$

Also, since $10 \le x \le 19$, $1 + x^8 > 10^8$

$$\Rightarrow \frac{1}{1+x^8}<\frac{1}{10^8} \Rightarrow \frac{1}{|1+x^8|}<10^{-8} \quad ...(3)$$

From (2) and (3), we have

$$\left|\frac{\sin x}{1+x^8}\right| < 10^{-8}$$

$$\left|\int_{10}^{19}\frac{\sin x}{1+x^8}\,dx\right| < \int_{10}^{19} 10^{-8}\,dx$$

$$\therefore \quad \left|\int_{10}^{19}\frac{\sin x}{1+x^8}\,dx\right| < (19-10).10^{-8} < 10^{-7}.$$

4. Schwartz-Bunyakovsky inequality

For any functions f(x) and g(x), integrable on the interval (a, b), the following inequality holds :

$$\left|\int_a^b f(x)g(x)\,dx\right| \le \sqrt{\int_a^b f^2(x)dx\int_a^b g^2(x)dx}$$

Proof Consider the function $F(x) = [f(x) - \lambda g(x)]^2$ where λ is any real number. Since $F(x) \ge 0$, then

$$\int_a^b [f(x)-\lambda g(x)]^2\,dx \ge 0,$$

$$\Rightarrow \lambda^2\int_a^b g^2(x)dx - 2\lambda\int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx \ge 0$$

The expression in the left side of the latter inequality is a quadratic trinomial with respect to λ. It follows from the inequality that at any λ, this trinomial is non-negative. Hence, its discriminant is non-positive, i.e.

$$\left\{\int_a^b f(x)g(x)dx\right\}^2 - \int_a^b f^2(x)dx\int_a^b g^2(x)dx \le 0$$

Hence $\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\int_a^b f^2(x)dx\int_a^b g^2(x)dx}$,

which completes the proof.

Example 15. Prove that

$$\int_0^1 \sqrt{(1+x)(1+x^3)}\,dx \text{ cannot exceed } \sqrt{15/8}.$$

Solution $\int_0^1 \sqrt{(1+x)(1+x^3)}\,dx$

$$\le \sqrt{\left(\int_0^1 (1+x)dx\right)\left(\int_0^1 (1+x^3)dx\right)}$$

$$\le \sqrt{\left(x+\frac{x^2}{2}\right)_0^1\left(x+\frac{x^4}{4}\right)_0^1} \le \sqrt{\frac{3}{2}.\frac{5}{4}} \le \sqrt{\frac{15}{8}}.$$

Example 16. Prove that

$$1 > \int_0^1 \frac{1}{1+x^n}\,dx > \frac{n+1}{n+2} \quad (n > 0).$$

Solution $\int_0^1 \frac{1}{1+x^n}\,dx < \int_0^1 dx = 1.$

Using Schwartz-Bunyakovsky inequality,

$$\int_0^1 \frac{1}{1+x^n}\,dx \,.\int_0^1 (1+x^n)dx \ge \left[\int_0^1 dx\right]^2 = 1$$

$$\text{or,}\quad \int_0^1 \frac{1}{1+x^n}\,dx > \frac{1}{\int_0^1 (1+x^n)dx} = \frac{n+1}{n+2}.$$

5. If $m \le f(x) \le M$ and $g(x) > 0$ on [a, b], then

$$m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx.$$

Proof We have

$$\int_a^b m.g(x)dx \le \int_a^b f(x)g(x)dx \le \int_a^b M.g(x)dx$$

$$\Rightarrow m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx.$$

Example 17. Show that $\left|\int_0^{2\pi}\frac{\sin x dx}{1+x^2}\right| < \frac{\pi}{2}.$

Solution We regard the integral as

$$\int_0^{2\pi} f(x)\,g(x)\,dx, \quad f(x) = \sin x,\ g(x) = \frac{1}{1+x^2}.$$

Since $|f(x)| \le 1$ and $g(x) > 0$,

$$\left|\int_0^{2\pi}\frac{\sin x dx}{1+x^2}\right| < \int_0^{2\pi}\frac{dx}{1+x^2} = \tan^{-1}x\,\Big|_0^{2\pi} = \tan^{-1}2\pi.$$

But $\tan^{-1}x < \pi/2$ for all values of x, hence

$$\left|\int_0^{2\pi}\frac{\sin x dx}{1+x^2}\right| < \frac{\pi}{2}.$$

Example 18. If $f(a) = 0$ and $f(b) = 0$, then prove that

$$\int_a^b f(x)\,dx = -\frac{1}{2}\int_a^b (x-a)(b-x)\,f''(x)\,dx.$$

Further, if $|f''(x)| \le M$, $a \le x \le b$, then prove that

$$\left|\int_a^b f(x)dx\right| \le \frac{M}{12}(b-a)^3.$$

Solution We derive this result by two applications of integration by parts :

$$\int_a^b [(x-a)(b-x)]\,f''(x)\,dx$$

$$= -\int_a^b [(b-x)-(x-a)]\,f'(x)\,dx$$

$$= \int_a^b (-2)\,f(x)\,dx = -2\int_a^b f(x)\,dx.$$

Now, suppose $|f''(x)| \le M$, $a \le x \le b$.

From the preceding result,

$$\left|\int_a^b f(x)dx\right| = \frac{1}{2}\left|\int_a^b (x-a)(b-x)f''(x)\,dx\right|$$

but $(x-a)(b-x) \geq 0$ between a and b,

$$\Rightarrow \left|\int_a^b f(x)dx\right| \leq \frac{1}{2}M\int_a^b (x-a)(b-x)\,dx$$

$$\Rightarrow \left|\int_a^b f(x)dx\right| \leq \frac{M}{12}(b-a)^3.$$

Weighted Mean Value Theorem for integrals

Assume f and g are continuous on [a, b]. If g never changes sign in [a, b] then, for some c in [a, b], we have $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.

Proof Since g never changes sign in [a, b], g is always nonnegative or always nonpositive on [a, b]. Let us assume that g is nonnegative on [a, b]. Then we integrate the inequalities $mg(x) \leq f(x)\,g(x) \leq Mg(x)$ to obtain,

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx.$$

If $\int_a^b g(x)dx = 0$, this inequality shows that

$$\int_a^b f(x)g(x)dx = 0.$$

In this case, the theorem holds trivially for any choice of c since both members are zero. Otherwise, the integral of g is positive, and we may divide by this integral and apply the Intermediate Value Theorem to complete the proof. If g is nonpositive, we apply the same argument to –g.

The weighted mean value theorem sometimes leads to a useful estimate for the integral of a product of two functions, especially if the integral of one of the factors is easy to compute.

Generalized Mean Value Theorem for integrals

Assume g is continuous on [a, b], and assume f has a derivative which is continuous and never changes sign in [a, b]. Then, for some c in [a, b], we have

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx \quad ...(1)$$

Proof Let $G(x) = \int_a^x g(t)dt$. Since g is continuous, we have $G'(x) = g(x)$. Therefore, integration by parts gives us

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)G'(x)dx$$

$$= f(b)G(b) - \int_a^b f'(x)\,G(x)\,dx, \quad ...(2)$$

since $G(a) = 0$. By the weighted Mean Value Theorem, we have

$$\int_a^b f'(x)\,G(x)\,dx = G(c)\int_a^b f'(x)\,dx = G(c)[f(b) - f(a)]$$

for some c in [a, b]. Therefore, (2) becomes

$$\int_a^b f(x)\,g(x)dx = f(b)G(b) - G(c)[f(b) - f(a)]$$

$$= f(a)G(c) + f(b)[G(b) - G(c)].$$

This proves (1) since $G(c) = \int_a^c g(x)\,dx$

and $G(b) - G(c) = \int_c^b g(x)\,dx$.

Example 19. Estimate the integral $I = \int_0^1 \frac{\sin x}{1+x^2}dx$.

Solution By the generalized mean value theorem we have $\int_0^1 \frac{\sin x}{1+x^2}dx$

$$= \sin\mu\int_0^1 \frac{dx}{1+x^2} = \sin\mu\tan^{-1}x\Big|_0^1 = \frac{\pi}{4}\sin\mu, (0 < \mu < 1).$$

Since the function sinx increases on the interval [0, 1] then $\sin\mu < \sin 1$. Thus, we get an upper estimate of the integral :

$$\int_0^1 \frac{\sin x}{1+x^2}dx < \frac{\pi}{4}\sin 1 \approx 0.64.$$

It is possible to get a better estimation if we apply the same theorem in the form

$$\int_0^1 \frac{\sin x}{1+x^2}dx = \frac{1}{1+\xi^2}\int_0^1 \sin x dx = \frac{1}{1+\xi^2}(1-\cos 1)$$

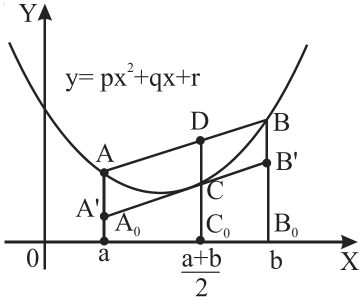
$$< 1 - \cos 1 \approx 0.46.$$

6. Concavity

By making the use of concavity of the graph we can make more appropriate approximation integrals as shown in the illustration below.

Let us estimate the integral $\int_a^b f(x)dx$, where

$f(x) = px^2 + qx + r$.



It is clear that this area is less than the area of a trapezoid with bases A_0A and B_0B. Let a midline C_0C be drawn. If we draw a tangent to the curve at point C, then this tangent will intersect the vertical lines at the points A' and B' and will form a trapezoid $A'B'B_0A_0$. The area of this trapezoid is obviously less than the area under the curve. Thus, in the case of the above function f(x) with $p>0$

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}.$$

Example 20. Using geometry, prove that :

(a) If the function f(x) increases and has a concave up graph in the interval [a, b], then

$$(b-a)\,f(a) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}$$

(b) If the function f(x) increases and has a concave down graph in the interval [a, b], then

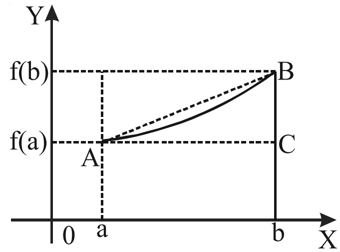
$$(b-a)\frac{f(a)+f(b)}{2} < \int_a^b f(x)dx < (b-a)f(b).$$

Solution

(a) Without loss of generality we may assume $f(x)>0$. If the graph of a function is concave up, it means that the curve lies below the chord through the points A(a, f(a)) and B(b, f(b)) (see figure). Therefore, the area of trapezoid aABb is greater than that of the curvilinear trapezoid bounded above by the graph of the function i.e.,

$$\int_a^b f(x)dx < \text{Area of aABb} = (b-a).\frac{f(a)+f(b)}{2}$$

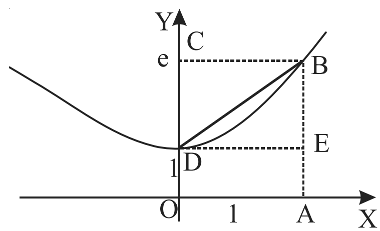
The inequality $(b-a)\,f(a) < \int_a^b f(x)dx$ is obvious because the area of rectangle aACb < the area under the curve.



(b) This can be proved as above.

Example 21. Estimate the value of $\int_0^1 e^{x^2}dx$.

Solution:



From the figure,

Area of rectangle OAED < $\int_0^1 e^{x^2}dx$ < Area of trapezium OABD

$$\Rightarrow \quad 1 < \int_0^1 e^{x^2}dx < \frac{1}{2}.1.(e+1)$$

$$\Rightarrow \quad 1 < \int_0^1 e^{x^2}dx < \frac{e+1}{2}.$$

Example 22. Estimate the integral $\int_0^1 \sqrt{1+x^4}\,dx$ using

(a) The mean value theorem for definite integral,

(b) Concavity and geometry,

(c) The inequality $\sqrt{1+x^4} < 1+\frac{x^4}{2}$,

(d) Schwartz-Bunyakovsky inequality.

Solution

(a) By the mean value theorem

$$I = \int_0^1 \sqrt{1+x^4}\,dx = \sqrt{1+\mu^4},\ \text{where } 0 \le x \le 1.$$

But $1 < \sqrt{1+\mu^4} < \sqrt{2}$.

$\Rightarrow \quad 1 < I < \sqrt{2} \approx 1.414.$

(b) The function $f(x) = \sqrt{1+x^4}$ is concave on the interval [0, 1], since

$$f''(x) = \frac{2x^2(x^4+3)}{(1+x^4)^{3/2}} > 0, \; 0 \le x \le 1.$$

Hence, $1 < \int_0^1 \sqrt{1+x^4}\,dx < \frac{1+\sqrt{2}}{2} \approx 1.207$.

(c) $1 < \int_0^1 \sqrt{1+x^4}\,dx < \int_0^1 \left(1+\frac{x^4}{2}\right) dx = 1 + \frac{1}{10} = 1.1$

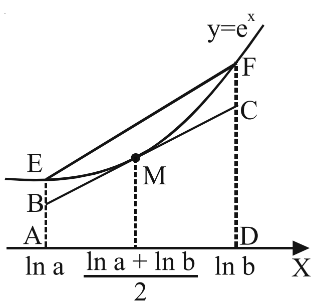
(d) Put $f(x) = \sqrt{1+x^4}$, $g(x) = 1$ and taking advantage of the Schwartz-Bunyakovsky inequality

$$\left|\int_0^1 \sqrt{1+x^4}\,dx\right| = \int_0^1 \sqrt{1+x^4}\,dx < \sqrt{\int_0^1 (1+x^4)dx . \int_0^1 1^2 dx}$$

$= \sqrt{1.2} \approx 1.095.$

Note: **Comparision of geometric, logarithmic, and arithmetic mean**

We knows that the graph of e^x is concave up over every interval of x-values.



With reference to the above figure we conclude that if $0 < a < b$ then

$$e^{(\ln a + \ln b)/2} . (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x dx$$

$$< \frac{e^{\ln a} + e^{\ln b}}{2} . (\ln b - \ln a).$$

From here we find that

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}.$$

This inequality says that the geometric mean of two positive numbers is less than their logarithmic mean, which is turn is less than their arithmetic mean.

7. We can begin with the inequality and generate other inequalities by successive integrations. For example :

$$\cos x \le 1 \Rightarrow \int_0^t \cos x dx \le \int_0^t 1\,dx$$

$\Rightarrow \sin x\Big|_0^t \le x\Big|_0^t$

$\Rightarrow \sin t \le t$

Reverting to the x notation, we have

$\sin x \le x$ ...(1)

$\Rightarrow \int_0^t \sin x dx \le \int_0^t x\,dx$

$\Rightarrow -\cos x\Big|_0^t \le \frac{x^2}{2}\Big|_0^t \Rightarrow 1 - \cos t \le \frac{t^2}{2}.$

Writing $1 - \cos t \le t^2/2$ as

$\cos x \ge 1 - x^2/2,$ ...(2)

we continue this process :

$\cos x \ge 1 - x^2/2 \Rightarrow \int_0^t \cos x dx \ge \int_0^t (1 - x^2/2)dx$

$\Rightarrow \sin t \ge t - t^3/3!$ ...(3)

$\sin x \ge x - x^3/3! \Rightarrow \int_0^t \sin x dx \ge \int_0^t (x - x^3/3!)dx$

$\Rightarrow 1 - \cos t \le t^2/2! - t^4/4!$ ...(4)

Clearly, this process can be continued.

Example 23. Let f (x) be a continuous function with continuous first derivative on (a, b), where $b > a$, and let $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to b^-} f(x) = -\infty$ and $f'(x) + f^2(x) \ge -1$, for all x in (a, b) then show that the minimum value of $(b - a)$ equals π.

Solution $f'(x) + f^2(x) \ge -1$

$\therefore f^2(x) + 1 \ge -f'(x)$ in (a, b)

$1 \ge -\frac{f'(x)}{1+f^2(x)}$ in (a, b)

$\therefore \int_0^b dx \ge -\int_a^b \frac{f'(x)}{1+f^2(x)} dx$

$b - a \ge -(\tan^{-1}(f(x))_a^b = -\left[\left(-\frac{\pi}{2}\right) - \frac{\pi}{2}\right]$

$\therefore (b - a) \ge \pi.$

Example 24. Suppose f is a differentiable real function such that $f(x)+f'(x)\le 1$ for all x, and $f(0)=0$, then find the largest possible value of $f(1)$

Solution Given $f'(x)+f(x)\le 1$

Multiplying by e^x

$$f'(x)e^x + f(x)e^x \le e^x$$

$$\frac{d}{dx}(e^x\cdot f(x)) \le e^x$$

Integrating between 0 and 1

$$\int_0^1 \frac{d}{dx}(e^x f(x))dx \le \int_0^1 e^x\,dx$$

$$e^x f(x)\Big|_0^1 \le e^x\Big|_0^1$$

$$e\cdot f(1) - e^0\cdot f(0) \le e-1$$

$$f(1) \le \frac{e-1}{e}.$$

Concept Problems K

1. It is known that $\int_a^b f(x)dx \ge 0$. Does it follow that $f(x)\ge 0\ \forall x\in[a,b]$? Give examples.

2. It is known that $\int_a^b f(x)dx > \int_a^b g(x)dx$. Does it follow that $f(x)\ge g(x)\ \forall x\in[a,b]$? Give examples.

3. Assume f is continuous on [a, b]. Assume also that $\int_a^b f(x)g(x)\,dx = 0$ for every function g that is continuous on [a, b]. Prove that $f(x)=0$ for all x in [a, b].

4. Prove that

 (i) $0 < \int_0^{\pi/2} \sin^{n+1}x\,dx < \int_0^{\pi/2}\sin^2 x\,dx,\ n>1$

 (ii) $1 < \int_0^{\pi/2}\sqrt{\sin x}\,dx < \sqrt{\frac{\pi}{2}}$

 (iii) $e^{-\frac{1}{4}} < \int_0^1 e^{x^2-x}\,dx < 1$

 (iv) $-\frac{1}{2} \le \int_0^1 \frac{x^3\cos x}{2+x^2}\,dx < \frac{1}{2}.$

5. Prove that

 (i) $0 < \int_0^2 \frac{x\,dx}{16+x^3} < \frac{1}{6}$ (ii) $\int_0^1 \frac{dx}{\sqrt{1+x^4}} \ge \frac{\pi}{4}$

 (iii) $\int_1^{100} e^{-x}\sin^2 x\,dx < 1$ (iv) $\int_0^1 \frac{dx}{4+x^3} > \ln\frac{5}{4}$

6. (a) Show that $1 \le \sqrt{1+x^3} \le 1+x^3$ for $x\ge 0$

 (b) Show that $1 \le \int_0^1 \sqrt{1+x^3}\,dx \le 1.25.$

7. Prove that

 (i) $\int_1^2 \frac{dx}{x^3+3x+1} < \frac{1}{5}$

 (ii) $3\sqrt{23} < \int_2^5 \sqrt{3x^3-1}\,dx < 10\sqrt{15} - 8\sqrt{6}/5$

 (iii) $2 < \int_0^4 \frac{dx}{1+\sin^2 x} < 4$

 (iv) $\frac{\pi}{2} < \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} < \frac{\pi}{2\sqrt{1-k^2}}$ $(0<k^2<1)$.

8. Show that $\left|\int_0^{2\pi} f(x)\sin 2x\,dx\right| \le \int_0^{2\pi} |f(x)|\,dx$.

9. Estimate the integral $\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1+\sin^2 x)\,dx$.

10. If $I = \int_0^1 \frac{dx}{1+x^{3/2}}$, prove that, $\ell n 2 < I < \frac{\pi}{4}$.

11. Show that $0.78 < \int_0^1 \frac{dx}{\sqrt{1+x^4}} < 0.93$.

12. Prove that, if $n>1$

 (i) $0 < \int_0^{\pi/2} \sin^{n+1}x\,dx < \int_0^{\pi/2}\sin^n x\,dx,$

 (ii) $0 < \int_0^{\pi/4} \tan^{n+1}x\,dx < \int_0^{\pi/4}\tan^n x\,dx.$

 (iii) $0.5 < \int_0^{1/2} \frac{dx}{\sqrt{(1-x^{2n})}} < 0.524.$

Practice Problems Q

13. Prove the inequalities :

 (i) $\int_1^3 \sqrt{x^4+1}\,dx \ge \frac{26}{3}$

 (ii) $\int_0^{\pi/2} x\sin x\,dx \le \frac{\pi^2}{8}$

(iii) $\frac{1}{17} \le \int_1^2 \frac{1}{1+x^4}dx \le \frac{7}{24}$.

14. Prove that

(i) $\frac{2\pi}{13} < \int_0^{2\pi} \frac{dx}{10+3\cos x} < \frac{2\pi}{7}$

(ii) $0 < \int_0^{\pi/4} x\sqrt{\tan x} < \frac{\pi^2}{32}$

(iii) $\frac{1}{2} < \int_{\pi/4}^{\pi/2} \frac{\sin x}{x}dx < \frac{1}{\sqrt{2}}$

(iv) $\left|\int_1^4 \frac{\sin x}{x}dx\right| \le \frac{3}{2}$.

15. Integrating by parts, prove that

$$0 < \int_{100\pi}^{200\pi} \frac{\cos x}{x}dx < \frac{1}{100\pi}.$$

16. Determine the signs of the integrals without evaluating them :

(a) $\int_{-1}^{2} x^3\,dx$

(b) $\int_0^{2\pi} \frac{\sin x}{x}dx$

(c) $\int_0^{\pi} x\cos x\,dx$.

17. Prove that

(i) $\frac{99\pi}{400} < \int_1^{100} \frac{\tan^{-1} x}{x^2}dx < \frac{99\pi}{200}$

(ii) $\frac{609(\ln 2)^2}{4} < \int_2^5 x^3(\ln x)^2 dx < \frac{609(\ln 5)^2}{4}$

(iii) $(1-e^{-1})\ln 10 < \int_1^{10} \frac{1-e^{-x}}{x}dx < \ln 10$

(iv) $\frac{1}{10\sqrt{2}} \le \int_0^1 \frac{x^9}{\sqrt{1+x}}dx \le \frac{1}{10}$.

18. Prove that $\int_0^1 \frac{dx}{\sqrt{2+x-x^2}}$ lies between $\frac{2}{3}$ and $\frac{1}{\sqrt{2}}$. Also find the exact value of the integral.

19. Using the Schwartz-Bunyakovsky inequality, prove that $\int_0^1 \sqrt{1+x^3}\,dx < \frac{\sqrt{5}}{2}$.

20. Using Schwartz-Bunyakovsky inequality with

$f^2(x) = \frac{1}{1+x^2}$, $g^2(x) = 1 + x^2$, show that

$\int_0^1 \frac{1}{1+x^2}dx > \frac{3}{4}$.

21. Show that the inequalities

$0.692 \le \int_0^1 x^x dx \le 1$ are valid.

22. If α and ϕ are positive acute angles then prove that

$$\phi < \int_0^{\varphi} \frac{dx}{\sqrt{(1-\sin^2\alpha\sin^2 x)}} < \frac{\varphi}{\sqrt{(1-\sin^2\alpha\sin^2\varphi)}}.$$

If $\alpha = \phi = 1/6\,\pi$, then prove that the integral lies between 0.523 and 0.541.

23. Let p be a polynomial of degree atmost 4 such that $p(-1) = p(1) = 0$ and $p(0) = 1$. If $p(x) \le 1$ for $x \in [-1, 1]$, find the largest value of $\int_{-1}^{1} p(x)dx$.

24. Proceeding from geometrical reasoning prove that:

(i) if the function f(x) increases in the interval [a, b] and has a concave down graph, then

$$(b-a)\frac{f(a)+f(b)}{2} < \int_a^b f(x)dx < (b-a)f(b).$$

(ii) Estimate the integral $\int_2^3 \frac{x^2\,dx}{1+x^2}$ using the above results.

25. Let f be twice continuously differentiable in $[0, 2\pi]$ and concave up. Prove that

$$\int_0^{2\pi} f(x)\cos x\,dx \ge 0.$$

26. Find the greatest and least values of the function

$I(x) = \int_0^x \frac{2t+1}{t^2-2t+2}dt$ on the interval $[-1, 1]$.

27. Given that f satisfies $|f(u)-f(v)| \le |u-v|$ for u and v in [a, b] then prove that

(i) f is continuous in [a, b] and

(ii) $\left|\int_a^b f(x)dx - (b-a)f(a)\right| \le \frac{(b-a)^2}{2}$.

2.16 DETERMINATION OF FUNCTION

Example 1. If f is continuous function such that $f(x + y) = f(x) + f(y)$ for all real numbers x and y, then prove that $f(x) = kx$ for some constant k.

Solution $f(x + y) = f(x) + f(y)$ ...(1)

Let us substitute t for y in (1) and then integrate from t = 0 to t = y with x held constant :

$$\int_{t=0}^{y} f(x+t)dt = \int_{t=0}^{y} f(x)dt + \int_{t=0}^{y} f(t)dt.$$

Then substituting u = x + t, du = dt on the left hand side yields the equation

$$\int_{u=x}^{x+y} f(u)du = y\,.\,f(x) + \int_{t=0}^{y} f(t)dt$$

from which we find that

$$y\,.\,f(x) = \int_{t=0}^{x+y} f(t)dt - \int_{t=0}^{x} f(t)dt - \int_{t=0}^{y} f(t)dt.$$

The right hand side is symmetric in the variables x and y, so interchanging them gives

$$y\,.\,f(x) = x\,.\,f(y), \text{ so that } \frac{f(x)}{x} = \frac{f(y)}{y},$$

for all x and y. Because x and y are independent, if follows that the function f(x)/x must be constant-valued, and therefore f(x) = kx for some constant k.

Example 2. If f is a positive-valued continuous functin such that f(x + y) = f(x) . f(y) for all real numbers x andy, then prove that $f(x) = e^{kx}$ for some constant k.

Solution f(x + y) = f(x) . f(y) ...(1)

Let g(x) = ln(f(x)). Taking the natural logarithm of both sides in (1) gives

$$g(x+y) = \ln(f(x+y)) = \ln(f(x)\,.\,f(y))$$
$$= \ln(f(x)) + \ln(f(y)) = g(x) + g(y).$$

The application of the previous example to g now yields ln(f(x)) = kx,

so that $f(x) = e^{kx}$.

Example 3. Find the real number a such that

$$6 + \int_a^x \frac{f(t)dt}{t^2} = 2\sqrt{x}.$$

Solution Differentiating both sides of the given expression, we get $\frac{f(x)}{x^2} = 2\cdot\frac{1}{2\sqrt{x}} \Rightarrow f(x) = x^{3/2}$.

Substituting this value in the given relation, we get,

$$6 + \int_a^x \frac{dt}{\sqrt{t}} = 2\sqrt{x}$$

$$6 + 2\sqrt{t}\Big]_a^x = 2\sqrt{x}$$

$$6 + 2[\sqrt{x} - \sqrt{a}] = 2\sqrt{x}$$

$$2\sqrt{a} = 6$$

$\Rightarrow$ a = 9

Hence, $f(x) = x^{3/2}$ and a = 9.

Example 4. A function f(x) satisfies

$f(x) = \sin x + \int_0^x f'(t)(2\sin t - \sin^2 t)\,dt$ then find $\sqrt{f(x)}$.

Solution Differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x + f'(x)(2\sin x - \sin^2 x)$$
$$\Rightarrow (1 + \sin^2 x - 2\sin x)\,f'(x) = \cos x$$

$$\Rightarrow f'(x) = \frac{\cos x}{1 + \sin^2 x - 2\sin x} = \frac{\cos x}{(1-\sin x)^2}$$

Integrating, $f(x) = \int \frac{\cos x\,dx}{(1-\sin x)^2}$

Put 1 – sin x = t

$$f(x) = -\int \frac{dt}{t^2} = \frac{1}{t} = \frac{1}{1-\sin x} + C.$$

Also , f(0) = 0, hence C = –1

$$f(x) = \frac{1}{1-\sin x} - 1 = \frac{1-1+\sin x}{1-\sin x} = \frac{\sin x}{1-\sin x}.$$

Example 5. Let f(x) be a continuous function such that f(x) > 0 for all x ≥ 0 and $(f(x))^{101} = 1 + \int_0^x f(t)\,dt$.

Solution Given $(f(x))^{101} = 1 + \int_0^x f(t)dt$

Differentiating,

$$101\cdot(f(x))^{100}\cdot f'(x) = f(x)$$
$$\therefore\ 101\cdot(f(x))^{99}\cdot f'(x) = 1 \text{ (as } f(x) > 0)$$

Integrating, $\frac{(101)(f(x))^{100}}{100} = x + C$

but f (0) = 1

$$\Rightarrow C = \frac{101}{100}$$

$$\therefore\ \frac{101}{100}(f(x))^{100} = x + \frac{101}{100}.$$

Putting x = 101,

$$\frac{101}{100}(f(101))^{100} = 101 + \frac{101}{100} = \frac{(101)(101)}{100}$$

$$\therefore\ (f(101))^{100} = 101.$$

Example 6. If $\int_x^{xy} f(t)\,dt$ is independent of x and f(2) = 2, find the value of $\int_1^x f(t)\,dt$.

Solution We have $\int_x^{xy} f(t)\, dt$.

Differentiating both sides with respect to x, treating y as constant, we get

$$\frac{d}{dx}\int_x^{xy} f(t)\, dt = 0$$

$\Rightarrow\ yf(xy) - f(x) = 0$ ...(1)

Putting $y = \frac{1}{x}$ in equation (1), we have

$$\frac{1}{x} f(1) - f(x) = 0$$

$\Rightarrow\ f(x) = f(1)\cdot\frac{1}{x}$

$\Rightarrow\ \int_1^x f(x)\, dx = f(1)\int_1^x \frac{1}{x}\, dx = f(1).\ln x$

Now, putting $y = \frac{1}{2}$ and $x = 2$ in equation (1),

we have $\frac{1}{2} f(1) - f(2) = 0$

$\Rightarrow\ f(1) = 2f(2) = 4.$

Hence, we have $\int_1^x f(x) f(x)\, dx = 4\ln x.$

Example 7. Consider a real valued continuous function f such that $f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t f(t))\, dt$.

Find the maximum and minimum values of the function f.

Solution We have $f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t f(t))\, dt$

$= \sin x + \pi \sin x + \int_{-\pi/2}^{\pi/2} t f(t)\, dt$

$\therefore\ f(x) = (\pi + 1)\sin x + A$ ...(1)

Now, $A = \int_{-\pi/2}^{\pi/2} t\,((\pi+1)\sin t + A)\, dt$

$= 2(\pi + 1)\int_0^{\pi/2} t \sin t\, dt$

$\Rightarrow\ A = 2(\pi + 1)$

Hence, $f(x) = (\pi + 1)\sin x + 2(\pi + 1)$

$\therefore\ f_{max.} = 3(\pi + 1)$ and $f_{min.} = (\pi + 1).$

Example 8. Let f(x) is a derivable function satisfying $f(x) = \int_0^x e^t \sin(x - t)\, dt$ and $g(x) = f''(x) - f(x)$.

Find the range of g (x).

Solution $f(x) = \int_0^x e^t \sin(x - t)\, dt$

$= \int_0^x e^{x-t} \sin(t)\, dt$ **[P–5]**

$\Rightarrow\ f(x) = e^x \int_0^x e^{-t} \sin t\, dt$

$\Rightarrow\ f'(x) = e^x \cdot e^{-x} \sin x + \left(\int_0^x e^{-t} \sin t\, dt\right) e^x$

$\Rightarrow\ f'(x) = \sin x + f(x)$ ...(1)

$\Rightarrow\ f''(x) = \cos x + f'(x)$

$= \cos x + \sin x + f(x)$ [using (1)]

$\Rightarrow\ f''(x) - f(x) = \sin x + \cos x$ ...(2)

$\Rightarrow\ g(x) = \sin x + \cos x$

$\Rightarrow$ The range of g (x) is $[-\sqrt{2}, \sqrt{2}]$.

Example 9. Let f(x) be a continuous function and c is a constant satisfying $\int_0^x f(t)\, dt = e^x - ce^{2x}\int_0^1 f(t)e^{-t}\, dt$ then find f(x) and the value of c.

Solution Put $x = 0$ in the given equation

$$0 = 1 - c\cdot\int_0^1 f(t)e^{-t}\, dt$$

Let $c = \frac{1}{k}$ where $k = \int_0^1 f(t)e^{-t}\, dt$.

Differentiating the given equation, we get

$$\int_0^x f(t)\, dt = e^x - e^{2x},$$

Integrating, we get

$f(x) = e^x - 2e^{2x}$

Now, $k = \int_0^1 (e^t - 2e^{2t})e^{-t}\, dt = \int_0^1 (1 - 2e^t)\, dt = 3 - 2e$

$\therefore\ c = \frac{1}{k} = \frac{1}{3 - 2e}.$

Example 10. If $f(x) = e^x + \int_0^1 (e^x + te^{-x}) f(t)\, dt$, find f(x).

Solution We can write $f(x) = Ae^x + Be^{-x}$, where

$$A = 1 + \int_0^1 f(t)\, dt \text{ and } B = \int_0^1 t\, f(t)\, dt.$$

$\therefore\ A = 1 + \int_0^1 (Ae^t + Be^{-t})dt = 1 + (Ae^t - Be^{-t})\Big|_0^t$

$\Rightarrow\ A = 1 + A(e^1 - 1) - B(e^{-1} - 1)$

$\Rightarrow\ (2 - e)A + (e^{-1} - 1)B = 1$ ...(1)

Now, $B = \int_0^1 t(Ae^t + Be^{-t})dt = A(te^t - e^t)_0^t$

$+\ B(-te^{-t} + e^{-t})\Big|_0^1$

$\Rightarrow$ $B = A + B(1 - 2e^{-1})$

$\Rightarrow$ $A - 2e^{-1}B = 0$ ...(2)

$\Rightarrow$ From (1) and (2), we get

$$A = \frac{2(e-1)}{4e - 2e^2},\ B = \frac{e-1}{4-2e}.$$

Hence, $f(x) = \frac{2(e-1)}{4-2e^2} \cdot e^x + \frac{e-1}{4-2e} \cdot e^{-x}$.

Example 11. If $f(x) = x + \int_0^1 (xy^2 + x^2y)(f(y))\,dy$, find f(x).

Solution $f(x) = x + x\int_0^1 y^2 f(y)\,dy + x^2\int_0^1 yf(y)\,dy$

$$= x\left(1 + \int_0^1 y^2 f(y)dy\right) + x^2\left(\int_0^1 yf(y)dy\right)$$

$\Rightarrow$ f(x) is a quadratic expression.

Let $f(x) = ax + bx^2$ ...(1)

then $f(y) = ay + by^2$,

where $a = 1 + \int_0^1 y^2 f(y)\,dy$

$$= 1 + \int_0^1 y^2(ay + by^2)\,dy$$

$$= 1 + \left(\frac{ay^4}{4} + \frac{by^5}{5}\right)_0^1$$

$\Rightarrow$ $a = 1 + \left(\frac{a}{4} + \frac{b}{5}\right)$

$\Rightarrow$ $20a = 20 + 5a + 4b$

$\Rightarrow$ $15a - 4b = 20.$ ...(2)

Also, $b = \int_0^1 y\,f(y)\,dy = \int_0^1 y\,(ay + by^2)\,dy$

$$= \left(\frac{ay^3}{3} + \frac{by^4}{4}\right)_0^1$$

$\Rightarrow$ $b = \frac{a}{3} + \frac{b}{4}$

$\Rightarrow$ $12b = 4a + 3b$

$\Rightarrow$ $9b - 4a = 0$ ...(3)

From (2) and (3),

$$a = \frac{180}{119} \text{ and } b = \frac{80}{119}$$

From (1),

$$f(x) = \frac{80x^2 + 180x}{119}.$$

Example 12. If $f(x) = x + \int_0^1 t(x+t)f(t)\,dt$, then find the value of the definite integral $\int_0^1 f(x)\,dx$.

Solution $f(x) = x + x\int_0^1 t\,f(t)\,dt + \int_0^1 t^2 f(t)\,dt$

$\therefore$ $f(x) = x(1 + A) + B$ where

$A = \int_0^1 t\,f(t)\,dt$ and $B = \int_0^1 t^2 f(t)\,dt$

Now $A = \int_0^1 t[t(1+A) + B]\,dt = \frac{t^3}{3}(1+A)\Big|_0^1 + \frac{B}{2}t^2\Big|_0^1$

$\Rightarrow$ $A = \frac{1+A}{3} + \frac{B}{2}$ $\Rightarrow$ $6A = 2(1+A) + 3B$

$\Rightarrow$ $4A - 3B = 2$ ...(1)

Again $B = \int_0^1 t^2[t(1+A) + B]\,dt = \left[\frac{t^4(1+A)}{4} + \frac{Bt^3}{3}\right]_0^1$

$$= \frac{1+A}{4} + \frac{B}{3}$$

$\Rightarrow$ $12B = 3 + 3A + 4B$

$\Rightarrow$ $8B - 3A = 3$ ...(2)

(1) $\times$ 3 gives $12A - 9B = 6$

(2) $\times$ 4 gives $-12A + 32B = 12$

Adding, we get $23B = 18$

$\Rightarrow$ $B = \frac{18}{23}$

$\therefore$ $4A = (3)\left(\frac{18}{23}\right) + 2$ $\Rightarrow 4A = \frac{54 + 46}{23} = \frac{100}{23}$

$\Rightarrow$ $A = \frac{25}{23}$.

$\therefore$ $\int_0^1 f(x)\,dx = \int_0^1 \left(\left(1 + \frac{25}{23}\right)x + \frac{18}{23}\right)dx$

$$= \left(1 + \frac{25}{23}\right)\frac{1}{2} + \frac{18}{23} = \frac{48}{(23)(2)} + \frac{18}{23}$$

$$= \frac{24}{23} + \frac{18}{23} = \frac{42}{23}.$$

Practice Problems R

1. If f is a continuous function such that $f(xy) = f(x) \cdot f(y)$ for all positive real numbers x and y, then prove that $f(x) = x^k$ for some constant k.

2. If F is a continuous function and

$F(x) = \int_0^x F(t)dt$, show that $F(x) = 0$ for every x.

3. A function f, defined for all positive real numbers, satisfies the equation $f(x^2) = x^3$ for every $x > 0$. Determine $f'(4)$.

4. Find f(x) if $[f(x)]^2 = 2\int_0^x f(t)\,dt$.

5. Find a function f and a value of the constant c such that $\int_c^x f(t)dt = \cos x - \frac{1}{2}$ for all real x.

6. Find a function f and a value of the constant c such that $\int_c^x tf(t)dt = \sin x - x\cos x - \frac{1}{2}x^2$ for all real x.

7. In each case, compute f(2) if f is continuous and satisfies the given formula for all $x \ge 0$.

(a) $\int_0^x f(t)dt = x^2(1+x)$.

(b) $\int_0^{x^2} f(t)dt = x^2(1+x)$.

(c) $\int_0^{f(x)} t^2dt = x^2(1+x)$.

(d) $\int_0^{x^2(1+x)} f(t)dt = x$.

8. A function f is defined for all real x by the formula $f(x) = 3 + \int_0^x \frac{1+\sin t}{2+t^2}dt$.
Without attempting to evaluate this integral, find a quadratic polynomial $p(x) = a + bx + cx^2$ such that $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$.

9. Find a function f and a number a such that $2 + \int_a^x f(t)dt = e^{3x}$.

10. If each case, give an example of a continuous function f satisfying the conditions stated for all real x, or else explain why there is no such function :

(a) $\int_0^x f(t)dt = e^x$

(b) $\int_0^{x^2} f(t)dt = 1 - 2^{x^2}$.

(c) $\int_0^x f(t)dt\, f^2(x) - 1$.

11. Let $F(x) = \int_0^x f(t)dt$. Determine a formula for computing F(x) for all real x if f is defined as follows :

(a) $f(t) = (t + |t|)^2$

(b) $f(t) = \begin{cases} 1-t^2 & \text{if } |t| \le 1, \\ 1-|t| & \text{if } |t| > 1. \end{cases}$

(c) $f(t) = e^{-|t|}$.

(d) f(t) = the maximum of 1 and t^2.

12. There is a function f, defined and continuous for all real x, which satisfies an equation of the form

$$\int_0^x f(t)dt = \int_0^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + c.$$

Find the function f and the value of the constant c.

13. A function f, continuous on the positive real axis, has the property that

$$\int_1^{xy} f(t)dt = y\int_1^x f(t)dt + x\int_1^y f(t)dt$$

for all $x > 0$ and all $y > 0$. If $f(1) = 3$, compute f(x) for each $x > 0$.

2.17 WALLIS' FORMULA

The formula to evaluate the integral

$I_{m,n} = \int_0^{\pi/2} \sin^m x \,.\, \cos^n x\,dx$, where m, n are any positive integers, is called Wallis' Formula.

First of all, we find a formula to evaluate a special case of the above integral i.e. $I_n = \int_0^{\pi/2} \sin^n x\,dx$.

To evaluate $I_n = \int_0^{\pi/2} \sin^n x\,dx$, we need to find a reduction formula. We have

$$I_n = \int_0^{\pi/2} \sin^n x\,dx$$

$$= [-\sin^{n-1} x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\pi/2} (n-1)\sin^{n-2} x \,.\, \cos^2 x\,dx$$

$= (n-1)\int_0^{\pi/2} \sin^{n-2} x \,.\, (1-\sin^2 x)\, dx$

$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\, dx - (n-1)$

$\int_0^{\pi/2} \sin^n x\, dx$

$\therefore\quad I_n + (n-1) I_n = (n-1) I_{n-2}$

$\Rightarrow\quad I_n = \left(\frac{n-1}{n}\right) I_{n-2}.$

Using the above formula repeatedly, we obtain

$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) \ldots\ldots I_0 \text{ or } I_1$

according as n is even or odd.

We have $I_0 = \frac{\pi}{2}, I_1 = 1.$

Hence, I_n

$$= \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\ldots\left(\frac{1}{2}\right).\frac{\pi}{2} & \text{if n is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\ldots\left(\frac{2}{3}\right).1 & \text{if n is odd} \end{cases}$$

Note:

1. $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$ **[P–5]**

2. In order to evaluate $\int_0^{\pi/2} \sin^n x dx$ or $\int_0^{\pi/2} \cos^n x dx$ we start with (n–1) in the numerator and go on diminishing by 2 till we get either 2 or 1. Similarly we start with n in the denominator and go on diminishing by 2 till we get either 2 or 1. Further, we multiply by $\frac{\pi}{2}$ in case n is even.

Example 1. Prove that $\int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta\, d\theta = \frac{3\pi}{16\sqrt{2}}.$

Solution L.H.S. $= \int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta\, d\theta$

$= \int_0^{\pi/4} (1-2\sin^2\theta)^{3/2} \cos\theta\, d\theta.$

Put $\sqrt{2}\sin\theta = \sin t$

$\therefore\quad \cos\theta\, d\theta = \frac{\cos t}{\sqrt{2}}\, dt$

When $\theta = 0$ then $t = 0$

When $\theta = \pi/4$ then $t = \pi/2$

$\therefore\quad$ L.H.S. $= \int_0^{\pi/2} \frac{\cos^4 t}{\sqrt{2}}\, dt = \frac{1}{\sqrt{2}}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$

$= \frac{3\pi}{16\sqrt{2}} =$ R.H.S.

Now, we develop a reduction formula for

$I_{m,n} = \int_0^{\pi/2} \sin^m x \,.\, \cos^n x\, dx.$

Then show that $I_{m,n} = \frac{m-1}{m+n} I_{m-2}, n$

We have $I_{m,n} = \int_0^{\pi/2} \sin^{m-1} x\, (\sin x \cos^n x)\, dx$

$= \left[-\frac{\sin^{m-1} x \,.\, \cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}}$

$+ \int_0^{\pi/2} \frac{\cos^{n+1} x}{n+1} (m-1) \sin^{m-2} x \cos x\, dx$

$= \left(\frac{m-1}{n+1}\right) \int_0^{\pi/2} \sin^{m-2} x \,.\, \cos^n x \,.\, \cos^2 x\, dx$

$= \left(\frac{m-1}{n+1}\right) \int_0^{\pi/2} (\sin^{m-2} x \,.\, \cos^n x - \sin^m x \,.\, \cos^n x)\, dx$

$= \left(\frac{m-1}{n+1}\right) I_{m-2,n} - \left(\frac{m-1}{n+1}\right) I_{m,n}$

$\Rightarrow\quad \left(1 + \frac{m-1}{n+1}\right) I_{m,n} = \left(\frac{m-1}{n+1}\right) I_{m-2,n}$

$\therefore\quad I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$

Using the above formula repeatedly, we obtain :

$I_{m-2,\,n} = \frac{m-3}{m+n-2} I_{m-4,n},$

$I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,\,n},$

....

$I_{2,n} = \frac{2}{3+n} \,.\, I_{1,n},$ if m is odd

$$I_{2,n} = \frac{1}{2+n},\ I_{0,n},\ \text{if m is even}$$

Thus,

$$I_{m,n} = \left\{\frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdot\frac{m-5}{m+n-4}\cdot\ \cdots\ \frac{2}{3+n}\cdot I_{1,n},\right\}$$

if m is even ...(1)

$$I_{m,n} = \left\{\frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdot\frac{m-5}{m+n-4}\cdot\ \cdots\ \frac{1}{2+n}\cdot I_{0,n},\right\}$$

if m is even ...(2)

$$\text{Now } I_{1,n} = \int_0^{\pi/2} \sin x \cos^n x dx = \left[\frac{-\cos^{n+1}}{n+1}\right]_0^{\pi/2} = \frac{1}{n+1}$$

...(3)

$$I_{0,n} = \int_0^{\pi/2} \sin^0 x \cos^n x dx = \int_0^{\pi/2} \cos^n x dx$$

$$= \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdot\ \cdots\ \dfrac{2}{3} & I_{0,1} \text{ if n is odd} \\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdot\ \cdots\ \dfrac{1}{2} & I_{0,0} \text{ if n is even} \end{cases}$$

...(4)

$$I_{0,1} = \int_0^{\pi/2} \sin^0 x\ \cos^1 x\, dx = [\sin x]_0^{\pi/2} = 1 \quad ...(5)$$

$$I_{0,0} = \int_0^{\pi/2} \sin^0 x\ \cos^0 x\, dx = [x]_0^{\pi/2} = \pi/2. \quad ...(6)$$

From (1) and (3) we find that if m is odd,

$$I_{m,n} = \frac{m-1}{m+n}\cdot\frac{m+3}{m+n-2}\cdot\frac{m-5}{m+n-4}\cdots\cdots\frac{2}{3+n}\cdot\frac{1}{1+n}, \quad ...(7)$$

whether n is odd or even.

From (2), (4) and (5) we find that if m is even and n is odd, then

$$I_{m,n} = \frac{m-1}{m+n}\cdot\frac{m+3}{m+n-2}\cdot\frac{m-5}{m+n-4}\cdots\cdots$$

$$\cdots\frac{1}{2+n}\cdot\frac{n-1}{n}\cdot\frac{1}{1+n}, \quad ...(8)$$

From (2), (4), and (6) we find that if m is even and n is also even, then

$$I_{m,n} = \frac{m-1}{m+n}\cdot\frac{m+3}{m+n-2}\cdots\cdots$$

$$\cdots\frac{1}{2+n}\cdot\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} \quad ...(9)$$

The above formulae (7), (8) and (9) can be combined to have the following simple formula covering all the cases :

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\, dx$$

$$= \frac{(m-1)(m-3)....(n-1)(n-3)....}{(m+n)(m+n-2).....} \times K,$$

the three sets of factors starting with m – 1, n – 1 and m + n and diminishing by 2 at a time, end up with either 1 or 2 according as the first factor of the set is odd or even, and K = π/2 if m and n are both even, and K = 1 if atleast one of m and n is odd.

Thus, $\int_0^{\pi/2} \sin^m x \,.\, \cos^n x\, dx$

$$= \frac{[(m-1)\ (m-3)..1 \text{ or } 2][(n-1)\ (n-3)\ (n-5)..1 \text{ or } 2]}{(m+n)\ (m+n-2)\ (m+n-4)....1 \text{ or } 2} K$$

where $K = \dfrac{\pi}{2}$ if both m and n are even $(m, n \in N)$;

= 1 otherwise.

The above formula is called Wallis formula.

Note: Wallis formula is applicable only when limits are 0 to $\dfrac{\pi}{2}$.

Example 2. Evaluate

(a) $\int_0^{\pi/2} \sin^5 x \cos^7 x\, dx$ (b) $\int_0^{\pi/2} \sin^6 x \cos^8 x\, dx$.

Solution $I_{5,7} = \int_0^{\pi/2} \sin^5 x \cos^7 x\, dx$

$$= \frac{4.2.6.4.2}{12.10.8.6.4.2} \times K$$

$= \dfrac{K}{120}$, where K is 1 since the exponents are not both even.

$$I_{6,8} = \int_0^{\pi/2} \sin^6 x \cos^8 x\, dx = \frac{5.3.1.7.5.3.1}{14.12.10.8.6.4.2} K,$$

$= \dfrac{5K}{2048}$, where K is π/2 since the exponents are both even $= \dfrac{5\pi}{4096}$.

Example 3. $\int_0^{2\pi} \sin^4 x \cos^3 x\, dx$.

Solution Since $\sin^4(2\pi - x)\cos^3(2\pi - x) = \sin^4 x \cos^3 x$,

$$\therefore \int_0^{2\pi} \sin^4 x \cos^3 x\, dx = 2\int_0^{\pi} \sin^4 x \cos^3 x\, dx. \quad ...(1)$$

Again, since $\sin^4(\pi - x)\cos^3(\pi - x) = -\sin^4 x \cos^3 x$,

$$\therefore \int_0^{\pi} \sin^4 x \cos^3 x\, dx = 0 \quad ...(2)$$

From (1) and (2), we find that $\int_0^{2\pi} \sin^4 x \cos^3 x\, dx = 0$.

Example 4. $\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx$

Solution $\sin^4(2\pi - x)\sin^6(2\pi - x) = \sin^4 x \cos^6 x$.

$$\therefore \int_0^{\pi/2} \sin^4 x \cos^6 x\, dx = 2\int_0^{\pi} \sin^4 x \cos^6 x\, dx \quad ...(3)$$

Again, since $\sin^4(\pi - x)\cos^6(\pi - x) = \sin^4 x \cos^6 x$, therefore,

$$\int_0^{\pi} \sin^4 x \cos^6 x\, dx = 2\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx \quad ...(4)$$

Also, $\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx = \dfrac{3.1.5.3.1}{10.8.6.4.2}.\dfrac{\pi}{2} = \dfrac{3\pi}{512}.$ ...(5)

From (3), (4) and (5), we have

$$\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx = 2.2\,\frac{3\pi}{512} = \frac{3\pi}{128}.$$

Example 5. Evaluate

$$\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^2 x(\sin x + \cos x)\, dx.$$

Solution The given integral

$$= \int_{-\pi/2}^{\pi/2} \sin^3 x \cos^2 x\, dx + \int_{-\pi/2}^{\pi/2} \sin^2 x \cos^3 x\, dx$$

$$= 0 + 2\int_0^{\pi/2} \sin^2 x \cos^3 x\, dx$$

($\because \sin^3 x \cos^2 x$ is odd and $\sin^2 x \cos^3 x$ is even)

$$= 2.\frac{1.2}{5.3.1} = \frac{4}{15}.$$

Example 6. Evaluate $\int_0^{\pi} x \sin^5 x \cos^6 x\, dx$.

Solution Let $I = \int_0^{\pi} x \sin^5 x \cos^6 x\, dx$

$$I = \int_0^{\pi} (\pi - x)\sin^5(\pi - x)\cos^6(\pi - x)\, dx \quad [P\text{–}5]$$

$$= \pi\int_0^{\pi} \sin^5 x . \cos^6 x\, dx - \int_0^{\pi} x \sin^5 x . \cos^6 x\, dx$$

$$\Rightarrow 2I = \pi . 2\int_0^{\pi/2} \sin^5 x . \cos^6 x\, dx$$

$$\Rightarrow I = \pi\,\frac{4.2.5.3.1}{11.9.7.5.3.1}$$

$$\Rightarrow I = \frac{8\pi}{693}.$$

Example 7. Evaluate $\int_0^{\pi} x \sin^6 x \cos^2 x dx$.

Solution Here if we replace x by $\pi - x$ in $\sin^6 x \cos^2 x$, it does not change.

$$\therefore I = \int_0^{\pi} (\pi - x)\sin^6 x \cos^2 x dx \quad [P\text{–}5]$$

$$\therefore 2I = \int_0^{\pi} (x + \pi - x)\sin^6 x \cos^2 x dx = \pi\int_0^{\pi} \sin^6 x \cos^2 x dx$$

$$2I = \pi.2\int_0^{\pi/2} \sin^6 x \cos^2 x dx \quad [P\text{–}6]$$

$$\therefore I = \pi.\frac{5.3.1.1}{8.6.4.2}.\frac{\pi}{2} = \frac{5\pi^2}{256}.$$

Example 8. Evaluate $\int_0^1 x^3(1-x)^5 dx$

Solution Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$x = 0$

$\Rightarrow \theta = 0;\ x = 1$

$$\Rightarrow \theta = \frac{\pi}{2}.$$

$$\therefore \int_0^1 x^3(1-x)^5 dx$$

$$= \int_0^{\pi/2} \sin^6\theta(\cos^2\theta)^5\, 2.\sin\theta.\cos\theta\, d\theta$$

$$= 2.\int_0^{\pi/2} \sin^7\theta \cos^{11}\theta\, d\theta$$

$$= 2.\frac{6.4.2.10.8.6.4.2}{18.16.14.12.10.8.6.4.2}$$

$$= \frac{6.4.2.10.8.6.4.2}{18.16.14.12.10.8.6.4.2}.$$

Example 9. Evaluate ${}^{100}C_{30}$. I if

$$I = \int_0^1 x^{70}(1-x)^{30} dx.$$

Solution Put $x = \sin^2\theta$

$$I = 2\int_0^{\pi/2} \sin^{141}\theta \cdot \cos^{61}\theta\, d\theta$$

$$I = \frac{2(140\cdot138......2)\,(60\cdot58.......2)}{202\cdot200........\cdot2}$$

$$I = \frac{2\cdot2^{70}\,(70\cdot69......1)\,2^{30}\,(30\cdot29.....1)}{2^{101}\,(101\cdot100.......1)}$$

$$= \frac{70!\,30!}{101\times100!} = \frac{1}{101}\cdot\frac{1}{{}^{100}C_{30}}$$

$$\therefore\ I\cdot{}^{100}C_{30} = \frac{1}{101}.$$

Finally, let us find the Wallis product, which expresses the number $\frac{\pi}{2}$ in the form of an infinite product.

Recall the formulae

$$I_{2m} = \int_0^{\pi/2}\sin^{2m}x\,dx = \frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} \quad ...(1)$$

$$I_{2m+1} = \int_0^{\pi/2}\sin^{2m+1}x\,dx = \frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\frac{6}{7}.\frac{4}{5}.\frac{2}{3} \quad ...(2)$$

We find, by means of term wise division,

$$\frac{\pi}{2} = \left(\frac{2\cdot4\cdot6...2m}{3.5...(2m-1)}\right)^2\frac{1}{2m+1}\frac{I_{2m}}{I_{2m+1}} \quad ...(3)$$

We shall now prove that $\lim\limits_{m\to\infty}\frac{I_{2m}}{I_{2m+1}} = 1$

For all x of the interval $\left(0,\frac{\pi}{2}\right)$ the inequalities $\sin^{2m-1}x > \sin^{2m}x > \sin^{2m+1}x$ hold.

Integrating from 0 to $\frac{\pi}{2}$, we get

$$I_{2m-1} \ge I_{2m} \ge I_{2m+1}$$

$$\Rightarrow \frac{I_{2m-1}}{I_{2m+1}} \ge \frac{I_{2m}}{I_{2m+1}} \ge 1 \quad ...(4)$$

We have $\frac{I_{2m-1}}{I_{2m+1}} = \frac{2m+1}{2m}$

Hence, $\lim\limits_{m\to\infty}\frac{I_{2m-1}}{I_{2m+1}} = \lim\limits_{m\to\infty}\frac{2m+1}{2m} = 1$

From inequality (4) we have $\lim\limits_{m\to\infty}\frac{I_{2m}}{I_{2m+1}} = 1$.

Passing to the limit in formula (3), we get the Wallis product for

$$\frac{\pi}{2} = \lim_{m\to\infty}\left[\left(\frac{2.4.6...2m}{3.5...(2m-1)}\right)^2\frac{1}{2m+1}\right]$$

This formula may be written in the form

$$\frac{\pi}{2} = \lim_{m\to\infty}\left(\frac{2}{1}.\frac{2}{3}.\frac{4}{3}.\frac{4}{5}.\frac{6}{5}....\frac{2m-2}{2m-1}.\frac{2m}{2m-1}.\frac{2m}{2m+1}\right).$$

Practice Problems S

1. Evaluate the following integrals :

(i) $\int_0^{\pi/2}\sin^5x\,dx$ (ii) $\int_0^{\frac{1}{2}\pi}\cos^6x\,dx$

2. Evaluate the following integrals :

(i) $\int_0^{\pi/2}\sin^5x\cos^4x\,dx$

(ii) $\int_0^{\frac{\pi}{2}}\sin^7x\cos^4x\,dx$

(iii) $\int_0^{\pi/2}\sin^3x\cos^5x\,dx$

(iv) $\int_0^{\pi}\sin^6\frac{x}{2}\cos^8\frac{x}{2}\,dx$

3. Evaluate the following integrals :

(i) $\int_0^a x\left(a^2-x^2\right)^{\frac{7}{2}}dx$

(ii) $\int_0^2 x^{3/2}\sqrt{2-x}\,dx$

(iii) $\int_0^1 x^3(1-x^2)^{5/2}dx$

(iv) $\int_0^{2a}x^5\sqrt{(2ax-x^2)}\,dx$

4. Evaluate the following integrals :

(i) $\int_0^{3\pi/2}\cos^4 3x\cdot\sin^2 6x\,dx$

(ii) $\int_0^1 x^6\sin^{-1}x\,dx$

(iii) $\int_0^1 x^3(1-x)^{9/2}dx$

(iv) $\int_0^1 x^4(1-x)^{1/4}dx$

5. Evaluate the following integrals :

(i) $\int_0^1(1-x^2)^n dx$

(ii) $\int_0^1 \frac{x^{2n}dx}{\sqrt{1-x^2}}$

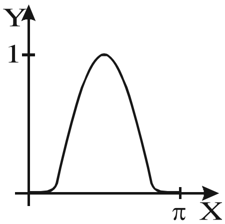
(iii) $\int_0^{2a} x^{9/2}(2a-x)^{-1/2}dx$

(iv) $\int_0^{\infty} \frac{x^4 dx}{(a^2+x^2)^2}$

6. Prove that

$$\int_0^{\pi/2} \cos^m x \sin^m x dx = 2^{-m} \int_0^{\pi/2} \cos^m x dx.$$

7. One of the numbers π, $\pi/2$, $35\pi/128$, $1-\pi$ is the correct value of the integral $\int_0^{\pi} \sin^8 x dx$. Use the graph of $y = \sin^8 x$ and a logical process of elimination to find the correct value.



2.18 LIMIT UNDER THE SIGN OF INTEGRAL

The value of limit of a definite integral can be determined by finding the limit of the integrand with respect to a quantity of which the limits of integration are independent.

For example, $\lim_{n\to\infty} \int_a^b f(x,n)\,dx = \int_a^b \lim_{n\to\infty} f(x,n)\,dx$, if a and b are independent of n.

Example 1. Evaluate $\lim_{n\to\infty} n \int_0^{\pi/2}\left(1-\sqrt[n]{\sin x}\right)dx$.

Solution $\lim_{n\to\infty} n\int_0^{\pi/2}\left(1-\sqrt[n]{\sin x}\right)dx$

Put $n = \frac{1}{t}$

$$= \lim_{t\to 0^+} \int_0^{\pi/2}\left(\frac{1-(\sin x)^t}{t}\right)dx$$

$$= \int_0^{\pi/2} \lim_{t\to 0^+}\left(\frac{1-(\sin x)^t}{t}\right)dx$$

$$= -\int_0^{\pi/2} \ln(\sin x)\,dx = \frac{\pi}{2}\ln 2.$$

Example 2. Evaluate $\lim_{t\to 0}\int_0^{2\pi}\frac{|\sin(x+t)-\sin x|}{|t|}dx$

Solution

$$= \int_0^{2\pi}\lim_{t\to 0}\left|\frac{\sin(x+t)-\sin x}{t}\right|dx$$

$$= \int_0^{2\pi}\left|\lim_{t\to 0}\frac{\sin(x+t)-\sin x}{t}\right|dx$$

(since modulus function is continuous)

$$= \int_0^{2\pi}|\cos x|\,dx = 4.$$

Example 3. If $\lim_{n\to\infty}\frac{3}{2}\int_{-(a)^{1/3}}^{a^{1/3}}\left(1-\frac{t^3}{n}\right)^n \cdot t^2\,dt = \sqrt{2}$, where $n \in N$, then find the value of 'a'.

Solution $\lim_{n\to\infty}\left(1-\frac{t^3}{n}\right)^n$ (1^∞ form)

$$= e^{\lim_{n\to\infty} n\left(1-\frac{t^3}{n}-1\right)} = e^{-t^3}$$

$$\Rightarrow \lim_{n\to\infty}\frac{3}{2}\int_{-(a)^{1/3}}^{a^{1/3}}\left(1-\frac{t^3}{n}\right)^n \cdot t^2\,dt$$

$$= \frac{3}{2}\int_{-(a)^{1/3}}^{a^{1/3}}\lim_{n\to\infty}\left(1-\frac{t^3}{n}\right)^n \cdot t^2\,dt$$

$$= \int_{-a^{1/3}}^{a^{1/3}} e^{-t^3}\cdot t^2\,dt.$$

Put $e^{-t^3} = y \Rightarrow -t^2\cdot e^{-t^3}dt = \frac{dy}{3}$

$$= \frac{1}{3}\int_{e^{-a}}^{e^a} dy = \frac{1}{3}(e^a - e^{-a}).$$

It is given that $\frac{1}{3}(e^a - e^{-a}) = \frac{2\sqrt{2}}{3}$.

$\Rightarrow e^a - e^{-a} = 2\sqrt{2} \Rightarrow e^{2a} - 2\sqrt{2}\,e^a - 1 = 0$

$\Rightarrow e^a = \sqrt{2}+\sqrt{3}$ ($\sqrt{2}-\sqrt{3}$ rejected)

$\Rightarrow a = \ln\left(\sqrt{2}+\sqrt{3}\right)$.

Example 4. Evaluate $\lim_{n\to\infty}\int_0^a \frac{e^x\,dx}{1+x^n}$.

Solution $\lim_{n\to\infty}\int_0^a \frac{e^x\,dx}{1+x^n}$

$$= \lim_{n\to\infty}\int_0^1 \frac{e^x\,dx}{1+x^n} + \lim_{n\to\infty}\int_1^a \frac{e^x\,dx}{1+x^n}$$

$$= \int_0^1 \lim_{n\to\infty}\frac{e^x\,dx}{1+x^n} + \int_1^a \lim_{n\to\infty}\frac{e^x\,dx}{1+x^n}$$

$n\to\infty$ and $x\in(0,1)$, $x^n\to 0$

and $n\to\infty$ and $x\in(1,a)$, $x^n\to\infty$

$$\therefore\quad I = \int_0^1 e^x\,dx + \int_1^a 0\,dx = e-1.$$

Example 5. Show that $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{{}^nC_k}{n^k(k+3)} = e-2$

Solution $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{{}^nC_k}{n^k(k+3)} = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k+3}\,{}^nC_k\cdot\frac{1}{n^k}$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}{}^nC_k\cdot\frac{1}{n^k}\int_0^1 x^{k+2}dx \quad \left\{\because \int_0^1 x^{k+2}dx = \frac{1}{k+3}\right\}$$

$$= \int_0^1\left(x^2\lim_{n\to\infty}\sum_{k=0}^{n}{}^nC_k\cdot\left(\frac{x}{n}\right)^k\right)dx$$

$$= \int_0^1 x^2\left\{\lim_{n\to\infty}\sum_{k=0}^{n}\left(1+\frac{x}{n}\right)^n\right\}dx = \int_0^1 x^2\,.\,e^x\,dx$$

$$\left\{\because\ \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x\right\}$$

$$= (x^2\,.\,e^x)_0^1 - \int_0^1 2x\,.\,e^x\,dx$$

$$= e-2$$

$$\left\{(x\,e^x)_0^1 - \int_0^1 e^x dx\right\}$$

$$= e-2\,\{e-e+1\}$$

$$= e-2.$$

Practice Problems T

1. Prove that $\lim_{\lambda\to\infty}\int_0^\infty \frac{1}{1+\lambda x^4}dx = 0$.
2. If f is continuous in [0, 1], show that

   $\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}dx = \frac{\pi}{2}f(0)$.
3. (a) Make a conjecture about the value of the limit

   $\lim_{k\to 0}\int_1^b t^{k-1}dt\ (b>0)$

   (b) Check your conjecture by evaluating the integral and finding the limit. [Hint : Interpret the limit as the definition of the derivative of an exponential function]
4. Let f have a continuous derivative for x in [a, b].

   Examine $\lim_{c\to\infty}\int_a^b f(x)\sin cx\ dx$.
5. Prove that

   $$\lim_{\omega\to\infty}\frac{e^{k\omega^2x^2}}{\int_a^b e^{k\omega^2x^2}\,dx} = \begin{cases}0 & \text{if } x<b,\\ \infty & \text{if } x=b\end{cases}$$

   $(\omega>0, k>0, b>a>0)$.
6. Show that $\int_0^\infty x^{-rx}$ sin ax dx equals $a/(a^2+r^2)$, where r > 0 and a are constant.

2.19 DIFFERENTIATION UNDER THE SIGN OF INTEGRAL

A definite integral can be differentiated with respect to a quantity of which the limits of integration are independent.

Let a function f(x, α) be continuous for $a\le x\le b$ and $c\le\alpha\le d$. Then for any $\alpha\in[c,d]$ if

$I(\alpha) = \int_a^b f(x,\alpha)dx$, then

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b\left(\frac{d}{d\alpha}f(x,\alpha)\right)dx.$$

For example, if $I(\alpha) = \int_0^\pi \frac{\ln(1+\sin\alpha\cos x)}{\cos x}dx$

then we have $\frac{dI}{d\alpha} = \int_0^\pi \frac{\cos\alpha dx}{1+\sin\alpha\cos x}$.

With the help of the above result firstly, new integrals can be deduced from certain known standard integrals. Secondly, the value of a given integral can be found by first differentiating the integral then evaluating the new integral thus obtained and finally integrating the result with respect to the same quantity with respect to which the integral was first differentiated.

Example 1. Evaluate $I(k) = \int_0^1 \frac{x^k-1}{\ln x}dx,\ (k\ge 0)$.

Solution $I(k)=\int_0^1 \frac{x^k-1}{\ln x}dx$

$$\Rightarrow \frac{d}{dk}(I(k))=\int_0^1 \frac{d}{dk}\left(\frac{x^k-1}{\ln x}\right)dx$$

$$=\int_0^1 \frac{x^k \ln x}{\ln x}dx$$

$$=\int_0^1 (x^k)dx = \left(\frac{x^{k+1}}{k+1}\right)_0^1$$

$$=\frac{1}{k+1}[1-0]=\frac{1}{k+1}.$$

$$\therefore \frac{d}{dx}(I(k))=\frac{1}{k+1}.$$

Integrating both sides w.r.t. 'k'. we get,

$$I(k)=\ln(k+1)+c \quad ...(1)$$

Given $I(k)=\int_0^1 \frac{x^k-1}{\ln x}dx$

$\therefore$ $I(0)=0$ (when $k=0$) ...(2)

also from (1), $I(0)=\ln(1)+c$

$\therefore$ $I(0)=c.$ ...(3)

From (2) and (3), $c=0$

$\Rightarrow$ $I(k)=\ln(k+1).$

Example 2. Evaluate $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx$

Solution Let $I=\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx$

Differentiating w.r.t. a, we have

$$\frac{dI}{da}=\int_0^\infty \frac{1}{(1-x^2)(1+a^2x^2)}dx$$

$$=\int_0^\infty\left[\frac{1}{(1-a^2)(1+x^2)}-\frac{a^2}{(1-a^2)(1+a^2x^2)}\right]dx$$

$$=\left[\frac{1}{1-a^2}\tan^{-1}x-\frac{a}{1-a^2}\tan^{-1}ax\right]_0^\infty$$

$$=\frac{\pi}{2}\left[\frac{1}{1-a^2}-\frac{a}{1-a^2}\right]=\frac{\pi}{2(1+a)}.$$

$$I=\frac{\pi}{2}\int\frac{da}{1+a}=\frac{\pi}{2}\ln(1+a)+A.$$

When $a=0, I=0 \Rightarrow A=0.$

Hence, $\int_0^\infty \frac{\tan^{-1}ax}{x(1+x^2)}dx=\frac{\pi}{2}\ln(1+a).$

Example 3. Evaluate $\int_0^\infty \frac{\ln(1+a^2x^2)}{1+b^2x^2}dx$

Solution Let $I=\int_0^\infty \frac{\ln(1+a^2x^2)}{1+b^2x^2}dx$

Then $\frac{dI}{da}=\int_0^\infty \frac{2ax^2}{(1+b^2x^2)(1+a^2x^2)}dx$

$$=2a\int_0^\infty\left(\frac{1}{(b^2-a^2)(1+a^2x^2)}-\frac{1}{(b^2-a^2)(1+b^2x^2)}\right)dx$$

$$=\frac{2a}{b^2-a^2}\left[\frac{1}{a}\tan^{-1}ax-\frac{1}{b}\tan^{-1}bx\right]_0^\infty$$

$$=\frac{2a}{b^2-a^2}\left(\frac{1}{a}-\frac{1}{b}\right).\frac{\pi}{2}=\frac{\pi}{b(a+b)}.$$

Therefore $I=\frac{\pi}{b}\ln(a+b)+A.$

Now when $a=0, I=0 \Rightarrow A=-\left(\frac{\pi}{b}\right)\ln b.$

Hence $\int_0^\infty \frac{\ln(1+a^2x^2)}{1+b^2x^2}dx=\frac{\pi}{b}\ln\frac{a+b}{b}.$

Example 4. Evaluate $\int_0^{\pi/2}\sec\theta.\tan^{-1}(a\cos\theta)\,d\theta.$

Solution The given definite integral is a function of 'a'. Let its value be I(a).

Then, $I(a)=\int_0^{\pi/2}\sec\theta\,.\tan^{-1}(\cos\theta)\,d\theta$

$$\Rightarrow I'(a)=\int_0^{\pi/2}\sec\theta\,.\frac{1}{1+a^2\cos^2\theta}.\cos\theta\,d\theta$$

$$=\int_0^{\pi/2}\frac{1}{1+a^2\cos^2\theta}d\theta$$

$$=\int_0^{\pi/2}\frac{\sec^2\theta\,d\theta}{1+\tan^2\theta+a^2},\ \text{put } \tan\theta=t$$

$$=\int_0^\infty\frac{dt}{t^2+(a^2+1)}=\frac{1}{\sqrt{a^2+1}}\left(\tan^{-1}\frac{t}{\sqrt{a^2+1}}\right)_0^\infty$$

$$=\frac{1}{\sqrt{a^2+1}}\left(\frac{\pi}{2}-0\right).$$

$$I'(a)=\frac{\pi}{2\sqrt{a^2+1}},$$

Integrating both sides w.r.t. 'a', we get

$\Rightarrow \quad I(a) = \frac{\pi}{2} \ln |a + \sqrt{a^2 + 1}| + C$

Since $I(0) = 0$, i.e. $0 + C = 0$, we have $C = 0$.

$\Rightarrow \quad I(a) = \frac{\pi}{2} \ln |a + \sqrt{1 + a^2}|$.

Example 5. Evaluate $\int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} x}{x} dx$.

Solution Let $I = \int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} x}{x} dx$

$$\frac{dI}{da} = \int_0^{\infty} \frac{1 \cdot x}{(1 + a^2 x^2) x} dx = \int_0^{\infty} \frac{dx}{1 + a^2 x^2}$$

$$= \frac{1}{a^2} \int_0^{\infty} \frac{dx}{x^2 + \frac{1}{a^2}} = \frac{1}{a^2} a \tan^{-1} x \Big]_0^{\infty} = \frac{\pi}{2a}.$$

$$I = \frac{\pi}{2} \int \frac{da}{a} \Rightarrow I = \frac{\pi}{2} \ln a + C.$$

When $a = 1, I = 0 \Rightarrow C = 0$

Hence, $I = \frac{\pi}{2} \ln a$.

Example 6. $\int_0^{\pi/2} \ln\left(\frac{1 + a \sin x}{1 - a \sin x}\right) \frac{dx}{\sin x}, (|a| < 1)$.

Solution $I = \int_0^{\pi/2} \ln\left(\frac{1 + a \sin x}{1 - a \sin x}\right) \frac{dx}{\sin x} \quad (|a| < 1)$

$$\frac{dI}{da} = \int_0^{\pi/2} \frac{2 \sin x}{(1 - a^2 \sin^2 x)} \cdot \frac{dx}{\sin x}$$

$$= \int_0^{\pi/2} \frac{2 \sec^2 x}{1 + \tan^2 x - a^2 \tan^2 x} dx$$

$$= \int_0^{\pi/2} \frac{2 \sec^2 x}{(1 - a^2) \tan^2 x + 1} dx \text{ (put } \tan x = t)$$

$$= \int_0^{\infty} \frac{2\,dt}{(1 - a^2) t^2 + 1}$$

$$= \frac{2}{(1 - a^2)} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{1}{\sqrt{1 - a^2}}\right)^2}$$

$$= \frac{2}{\sqrt{1 - a^2}} \left[\tan^{-1}\left(t\sqrt{1 - a^2}\right)\right]_0^{\infty} = \frac{\pi}{\sqrt{1 - a^2}}.$$

Now, $dI = \frac{\pi\, da}{\sqrt{1 - a^2}}$

$\Rightarrow \quad I = \pi \sin^{-1} a$, since at $a = 0, I = 0$.

Example 7. Evaluate $I = \int_0^{\pi} \frac{e^{-ax} \sin mx}{x} dx$.

Solution $\frac{dI}{dm} = \int_0^{\pi} e^{-ax} \cos mx dx = \frac{a}{a^2 + m^2}$.

$\therefore \quad I = a \int \frac{dm}{a^2 + m^2} = \tan^{-1}\left(\frac{m}{a}\right) + C$

Since the integral vanishes when $m = 0, C = 0$.

$\therefore \quad I = \tan^{-1}\left(\frac{m}{a}\right)$

Example 8. Evaluate the integral $\int_0^{\infty} \frac{\sin bx}{x} dx$.

Solution Let $I = \int_0^{\infty} \frac{e^{-ax} \sin bx}{x} dx, a > 0$.

Using differentiation under the integral sign, we have

$$\frac{dI}{db} = \int_0^{\infty} e^{-ax} \cos bx\, dx$$

$$= \frac{a}{a^2 + b^2}, a > 0.$$

Now, integrating with respect to b,

$$I = a \int \frac{db}{a^2 + b^2} = a \frac{1}{a} \tan^{-1} \frac{b}{a} + C$$

$$= \tan^{-1} \frac{b}{a} + C \quad \text{...(1)}$$

where C is the constant of integration.

From the given integral, we see that when $b = 0, I = 0$.

$\therefore$ From (1), we deduce $C = 0$.

$$\therefore \int_0^{\infty} \frac{e^{-ax} \sin bx}{x} dx = \tan^{-1} \frac{b}{a} \quad \text{...(2)}$$

Assuming I a continuous function of a, we deduce from (2), when $a \to 0$,

$$\int_0^{\infty} \frac{\sin bx}{x} dx = \frac{\pi}{2}, \text{ or } -\frac{\pi}{2}$$

according as $b >$ or < 0.

When $b = 1$ we have $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Application of integration under the sign of integral

Example 9. Find the value of $\int_0^{\infty} e^{-x^2}dx$.

Solution Denoting the proposed integral by k, and substituting ax for x, we get

$$\int_0^{\infty} e^{-a^2x^2}adx = k$$

Multiplying both sides by $e^{a^{-2}}$, we get

$$\therefore \quad \int_0^{\infty} e^{-a^2(1+x^2)}adx = ke^{a^{-2}}$$

$$\text{Hence } \int_0^{\infty}\int_0^{\infty} e^{-a^2(1+x^2)}adadx = k\int_0^{\infty} e^{-a^2}da \qquad ...(1)$$

Since, $\int_0^{\infty} e^{-a^2(1+x^2)}ada = \frac{1}{2}\frac{1}{1+x^2}$, we have from (1)

$$\frac{1}{2}\int_0^{\infty}\frac{dx}{1+x^2} = k^2$$

$$\therefore \quad \frac{\pi}{4} = k^2$$

Hence, $\int_0^{\infty} e^{-x^2}dx = k = \frac{1}{2}\sqrt{\pi}$.

Example 10. Using the equation $\int_0^1 x^{a-1}dx = \frac{1}{a}$ prove that $\int_0^{\infty}\frac{e^{-a_1x}-e^{-a_0x}}{z}dz = \ln\left(\frac{a_0}{a_1}\right)$.

Solution We have $\int_0^1\int_{a_0}^{a_1} x^{a-1}dadx = \int_{a_0}^{a_1}\frac{x^a}{a}\Big|_0^1 da$

$$= \int_{a_0}^{a_1}\frac{da}{a} = \ln\left(\frac{a_1}{a_0}\right)$$

Also, $\int_0^1\int_{a_0}^{a_1} x^{a-1}dadx = \int_0^1\frac{x^{a_1-1}-x^{a_0-1}}{\ln x}dx$

Hence $\int_0^1\frac{x^{a_1-1}-x^{a_0-1}}{\ln x}dx = \ln\left(\frac{a_1}{a_0}\right)$

Now, if we put $x = e^{-z}$ in this equation, we get

$$\int_0^{\infty}\frac{e^{-a_1x}-e^{-a_0x}}{z}dz = \ln\left(\frac{a_0}{a_1}\right)$$

Example 11. Given $\int_0^{\infty} e^{-ax}\cos mxdx = \frac{a}{a^2+m^2}$, prove that

$$\int_0^{\infty}\frac{e^{-a_1x}-e^{-a_0x}}{x}\cos mx\,dx = \frac{1}{2}\ln\left(\frac{a_1^2+m^2}{a_0^2+m^2}\right)$$

Solution We have

$$\int_0^{\infty}\left(\int_{a_0}^{a_1} e^{-ax}da\right)\cos mx\,dx = \int_{a_2}^{a_1}\frac{ada}{a^2+m^2}$$

Simplifying both sides, we get

$$\int_0^{\infty}\frac{e^{-a_1x}-e^{-a_0x}}{x}\cos mx\,dx = \frac{1}{2}\ln\left(\frac{a_1^2+m^2}{a_0^2+m^2}\right)$$

Example 12. Using the equation $\int_0^{\infty} e^{-ax}\sin mxdx = \frac{m}{a^2+m^2}$, prove that $\int_0^{\infty}\frac{\sin mx}{x}dx = \frac{\pi}{2}$.

Solution We get

$$\int_0^{\infty}\int_{a_0}^{a_1} e^{-ax}\sin mxdadx = \int_{a_0}^{a_1}\frac{mda}{a^2+m^2}$$

$\therefore$

$$\int_0^{\infty}\frac{e^{-a_0x}-e^{-a_1x}}{x}\sin mxdx = \tan^{-1}\left(\frac{a_1}{m}\right) - \tan^{-1}\left(\frac{a_0}{m}\right).$$

If we make $a_0 = 0$ and $a_1 = \infty$ in the latter result, we obtain

$$\int_0^{\infty}\frac{\sin mx}{x}dx = \frac{\pi}{2}.$$

Practice Problems U

1. If $F(t) = \int_2^3 \sin(x+t^2)dx$, find $F'(t)$.
2. Show that $\int_0^{\pi}\frac{\ell n\,(1+a\cos x)}{\cos x}dx = \pi\sin^{-1}a, (|a|<1)$
3. Evaluate $\int_0^{\pi}\ln(1+b\cos x)\,dx$
4. Evaluate $\int_0^1\frac{\tan^{-1}ax}{x\sqrt{1-x^2}}dx$
5. Evaluate $\int_0^{\pi/2}\ln(1+\cos\theta\cos x)\frac{dx}{\cos x}$
6. Evaluate $\int_0^a\sqrt{a^2-x^2}\cos^{-1}\frac{x}{a}dx$.
7. Show that $\int_0^1\frac{\ell n\,(1-a^2x^2)}{x^2\sqrt{(1-x^2)}}dx = \pi[\sqrt{1-a^2}-1], (a^2<1)$.

2.20 INTEGRATION OF INFINITE SERIES

The integral of the sum of a finite number of terms is equal to the sum of the integral of these terms. Now, the question arises whether this principle can be extended to the case when the number of terms is not finite.

In other words, is it always permissible to integrate an infinite series term by term ? It is beyond the scope of this book to investigate the conditions under which an infinite series can properly be integrated term by term. We should merely state the theorem that applies to most of the series that are ordinarily met with in elementary mathematics. For a complete discussion, students may consult any textbook on Mathematical Analysis.

Theorem. A power series can be integrated term by term throughout any interval of convergence, but not necessarily extending to the end points of the interval. Thus, if f(x) can be expanded in a convergent infinite power series for all values of x in certain continuous range, viz.,

$$f(x) = a_0 + a_1x + a_2x^2 + \text{.....} \text{ to } \infty,$$

then $\int_a^b f(x)\,dx = \int_a^b (a_0 + a_1x + a_2x^2 + \text{.....})\,dx$

$$= \Sigma \int_a^b a_r x^r\,dx,$$

Also, $\int_a^x f(x)dx = \int_a^x (a_0 + a_1x + a_2x^2 + \text{.....})\,dx$

$$= \Sigma \int_a^x a_r x^r\,dx,$$

provided the intervals (a, b) and (a, x) lie within the interval of convergence of the power series.

For example, If $|x| < 1$, we have

$$\frac{1}{\sqrt{(1-x^2)}} = 1 + \frac{1}{2}x^2 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^2 + \ldots$$

Hence, integrating term by term between the limits 0 and x,

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^2}{3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.5}\frac{x^7}{7} + \ldots$$

Note that this series is due to Newton.

If we put $x = \frac{1}{2}$ we get

$$\pi = 6\left\{\frac{1}{2} + \frac{1}{2.3.2^2} + \frac{1.3}{2.4.5.2^2} + \ldots\right\},$$

from which π can be calculated without much trouble.

Also, if $|x| < 1$, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^2}{3} - \ldots$

Integrating this between the limits 0 and x,

$$(1+x)\ln(1+x) - x = \frac{x^2}{1.2} - \frac{x^2}{2.3} + \frac{x^2}{3.4} - \ldots$$

Assuming that the function on the right-hand is continuous up to the limit x = 1. We infer that

$$\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \ldots = 2\ln 2 - 1$$

We can also apply the processes of differentiation to functions expressed by power-series.

Assuming that $f(x) = a_0 + a_1x + a_2x^2 + \text{.....}$ to ∞ is a continuous and differentiable function of x, then

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots$$

Here we mention some important expansions which are useful in certain problems :

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \text{.....}\infty = \ell n2$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{.....}\infty = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \text{.....}\infty = \frac{\pi^2}{12}$

(iv) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{.....}\infty = \frac{\pi^2}{8}$

(v) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \text{.....}\infty = \frac{\pi^2}{24}$

Example 1. Prove that $\int_0^1 \frac{1}{x}\ln(1+x^4)dx = \frac{\pi^2}{48}$.

Solution $I = \int_0^1 \frac{1}{x}\left(x^4 - \frac{1}{x}x^8 + \frac{1}{3}x^{12} - \frac{1}{4}x^{16}\text{....}\right)dx$

$$= \int_0^1\left(x^3 - \frac{1}{2}x^7 + \frac{1}{3}x^{11} - \frac{1}{4}x^{15} - \text{......}\right)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^8}{16} + \frac{x^{12}}{36} - \frac{x^{16}}{64}\text{.....}\right]_0^1$$

$$= \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{8^2} + \ldots.$$

$$= \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots\right] - 2\left[\frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{12^2} + \ldots\right]$$

$$= \frac{\pi^2}{24} - 2.\frac{1}{4^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right]$$

$$= \frac{\pi^2}{24} - \frac{1}{8}.\frac{\pi^2}{6} = \frac{2\pi^2 - \pi^2}{48} = \frac{\pi^2}{48}.$$

Example 2. Find by integration the expansion series for $\tan^{-1} x$.

Solution We have $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots\ldots + \ldots\infty$, if $x^2 < 1$.

Now, integrating both sides between the limits 0 and x

$$\int_0^x \frac{dx}{1+x^2} = \int_0^x (1 - x^2 + x^4 + \ldots)dx.$$

$$\therefore \quad \tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots, -1 < x < 1.$$

Example 3. Prove that $\sum_{n=0}^{\infty}\left(\frac{1}{3n+1} - \frac{1}{3n+2}\right) = \frac{\pi}{3\sqrt{3}}.$

Solution $\sum_{n=0}^{\infty}\int_0^1 (t^{3n} - t^{3n+1})dt = \sum_{n=0}^{\infty}\int_0^1 (t^{3n}(1-t)dt$

$$= \int_0^1\left((1-t)\sum_{n=0}^{\infty} t^{3n}\right)dt$$

$$= \int_0^1 (1-t)\cdot\{1 + t^3 + t^6 + t^8 + \ldots\ldots\infty\}\,dt$$

$$= \int_0^1 \frac{(1-t)}{1-t^3}dt = \int_0^1 \frac{dt}{1+t^2+t}$$

$$= \int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{(t+(1/2))^2}{\sqrt{3}}\Bigg]_0^1$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\sqrt{3} - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{3\sqrt{3}}.$$

Example 4. Evaluate $I = \int_0^{\pi/2} \tan x \ln \sin x dx$

Solution $\int_0^{\pi/2} \frac{\sin x}{\cos x} \ln\sqrt{(1-\cos^2 x)}dx$

Now put cosx = t and adjust the limits for t.

$$I = -\frac{1}{2}\int_1^0 \frac{1}{t}\ln(1-t^2)dt$$

$$= \frac{1}{2}\int_1^0 \frac{1}{t}\left(t^2 + \frac{1}{2}t^4 + \frac{1}{3}t^6 + \frac{1}{4}t^8\ldots\right)dt$$

$$= \frac{1}{2}\int_1^0\left(t + \frac{1}{2}t^3 + \frac{1}{3}t^5 + \frac{1}{4}t^7 + \ldots\right)$$

Integrating and putting the limits, we get

$$I = -\frac{1}{2}\left[\frac{1}{2} + \frac{1}{2.4} + \frac{1}{3.6} + \frac{1}{4.8} + \ldots\right]$$

$$= -\frac{1}{2}\left[\left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \right.$$

$$\left.\frac{1}{3}\left(\frac{1}{3} - \frac{1}{6}\right) + \frac{1}{4}\left(\frac{1}{4} - \frac{1}{8}\right) + \ldots\right]$$

$$= -\frac{1}{2}\left[\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right\}\right.$$

$$\left.-\frac{1}{2}\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\right\}\right]$$

$$= -\frac{1}{2}\left(\frac{\pi^2}{6} - \frac{1}{2}.\frac{\pi^2}{6}\right) = -\frac{\pi^2}{24}.$$

Example 5. Show that $\int_0^1 \frac{\ln x}{(1+x)}dx$

$$= -\int_0^1 \frac{\ln(1+x)}{x}dx = -\frac{\pi^2}{12}.$$

Solution Let $I = \int_0^1 \frac{\ln x}{(1+x)}dx$

Integrating by parts taking ln x as the first function, we have

$$I = [\ln x . \ln(1+x)]_0^1 \int_0^1 \frac{\ln(1+x)}{x}dx$$

$$\Rightarrow \quad I = 0 - \int_0^1 \frac{\ln(1+x)}{x}dx$$

$$= -\int_0^1 \left(\frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots}{x} \right) dx$$

$$= -\int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} - + \ldots \right) dx$$

$$= \left[1 - \frac{x^2}{2^2} + \frac{x^3}{3^3} - \frac{x^4}{4^2} + \ldots \right]_0^1$$

$$= \left(1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^2} + \ldots \right) = \frac{\pi^2}{12}.$$

Example 6. Show that sum of infinite series

$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \ldots$ can be expressed in the form $\int_0^1 \frac{z^{a-1}}{1+z^b} dz$ and hence prove that

$$\frac{1}{1} - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} + \ldots = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln 2 \right).$$

Solution Let $I = \int_0^1 z^{a-1}(1+z^b)^{-1} dz$

Expanding by binomial theorem,

$$I = \int_0^1 z^{a-1}(1 - z^b + z^{2b} - z^{3b} + z^{4b} \ldots) dz$$

$$= \int_0^1 (z^{a-1} - z^{a+b-1} + z^{a+2b-1} - z^{a+3b-1} \ldots) dz$$

$$= \left[\frac{z^a}{a} - \frac{z^{a+b}}{a+b} + \frac{z^{a+2b}}{a+2b} - \frac{z^{a+3b}}{a+3b} + \ldots \right]_0^1$$

$$= \frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \ldots$$

Again if we put a = 1 and b = 3, we get

$$I = \int_0^1 \frac{z^0}{(1+z^3)} dz = \frac{1}{1} - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} \ldots$$

L.H.S. on integrating using partial fractions

$$= \left[\frac{1}{3}\ln(z+1) - \frac{1}{6}\ln(z^2 - z + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2z-1}{\sqrt{3}} \right]_0^1$$

$$= \left[\frac{1}{3}\ln 2 - 0 + \frac{1}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}} \right] - \left[0 - 0 + \frac{1}{\sqrt{3}} \tan^{-1}\left(-\frac{1}{\sqrt{3}} \right) \right]$$

$$= \frac{1}{3}\ln 2 + \frac{1}{\sqrt{3}}\left(\frac{\pi}{6} + \frac{\pi}{6} \right) = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln 2 \right).$$

Example 7. If $|x| < 1$ then find the sum of the series

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots \infty$$

Solution Let

$$S = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots + \frac{2^n x^{2^n - 1}}{1+x^{2^n}}$$

Integrating both sides

$$\int S\, dx = \ln(1+x) + \ln(1\, x^2) + \ln(1+x^4)$$

$$+ \ln(1+x^8) + \ldots + \ln(1+x^{2^n}) + c$$

$$= \ln \{(1+x)(1+x^2)(1+x^4)(1+x^8) \ldots (1+x^{2^n})\} + c$$

$$= \ln \left\{ \frac{(1-x)(1+x)(1+x^2)(1+x^4)(1x+x^8)\ldots(1+x^{2^n})}{(1-x)} \right\} + c$$

$$= \ln \left\{ \frac{(1-x^2)(1+x^2)(1+x^4)(1x+x^8)\ldots(1+x^{2^n})}{(1-x)} \right\} + c$$

$$= \ln \left\{ \frac{(1-x^4)(1+x^4)(1+x^8)\ldots(1+x^{2^n})}{(1-x)} \right\} + c$$

....

$$= \ln \left\{ \frac{(1-x^{2^n})(1+x^{2^n})}{(1-x)} \right\} + c = \ln \left\{ \frac{1-x^{2^{n+1}}}{(1-x)} \right\} + c$$

When $n \to \infty$, $\int S\, dx = \ln \left\{ \frac{1-0}{1-x} \right\} + c = -\ln(1-x) + c$

Differentiating both sides, we get $S = \frac{1}{(1-x)}$.

Polynomial approximation to logarithm

Example 8. Prove that

$$-\ln(1-x) = x + \frac{x^2}{2} + \int_0^x \frac{u^2}{1-u}\,du.$$

Solution We have $\ln(1-x)$

$$= \int_1^{1-x} \frac{dt}{t}, \text{ which is valid if } x < 1.$$

The change of variable $t = 1 - u$ converts this to the form $-\ln(1-x)$

$$= \int_0^x \frac{du}{1-u}, \text{ valid for } x < 1.$$

From the algebraic identity $1 - u^2 = (1-u)(1+u)$, we obtain the formula

$$\frac{1}{1-u} = 1 + u + \frac{u^2}{1-u}, \text{ valid for any real } u \neq 1.$$

Integrating this from 0 to x, where $x < 1$, we have

$$-\ln(1-x) = x + \frac{x^2}{2} + \int_0^x \frac{u^2}{1-u}\,du.$$

Practice Problems V

1. Find the sum of the series

$$\frac{x^2}{1.2} - \frac{x^3}{2.3} + \frac{x^4}{3.4} - ... + (-1)^{n+1}\frac{x^{n+1}}{n(n+1)} + \cdots, |x| < 1.$$

2. If $|x| < 1$ then find the sum of the series

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + ...\infty.$$

3. Starting from

$$\frac{1}{1+x} - 1 + x - x^2 + ... + x^{2n-1} = \frac{x^{2n}}{1+x},$$

show that

$$t - \frac{t^2}{2} + \frac{t^2}{3} - ... - \frac{t^{2n}}{2n} \le \ln(1+t) \le t - \frac{t^2}{2} + \frac{t^3}{3} - + \frac{t^{2n+1}}{2n+1}$$

for $t \ge 0$.

4. Evaluate the following integrals :

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{x^4+1}$

(ii) $\displaystyle\int_0^1 \frac{\ln(1-x)}{x}\,dx$

(iii) $\displaystyle\int_0^{\infty} \frac{dx}{(x+1)(x+2)}$

(iv) $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$, $a, b > 0$.

5. Prove that $\displaystyle\int_0^1 x^n \ln x\,dx = \frac{1}{(n+1)^2}$, $n > -1$.

6. Prove that if $|x| < 1$

$$\frac{x^3}{1.3} - \frac{x^5}{3.5} + \frac{x^7}{5.7} - ... = \frac{1}{2}(1+x^2)\tan^{-1}x - \frac{1}{2}x.$$

7. Prove that

(i) $\displaystyle\int_0^1 \frac{x^{m-1}}{1+x^n}\,dx = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + ..$

(ii) $\displaystyle\int_0^x \frac{\sin x}{x}\,dx = x - \frac{x^3}{3.3!} + \frac{x^5}{5.5!} - ...$

(iii) $\displaystyle\int_a^b \frac{e^x}{x}\,dx = \ln\frac{b}{a} + (b-a) + \frac{b^2-a^2}{2.2!} + \frac{b^3-a^3}{3.3!} + ..$

(iv) $\displaystyle\int_0^1 \frac{\tan^{-1}x}{x}\,dx = \sum_0^{\infty} (-1)^n \frac{1}{(2n+1)^2}$.

8. Evaluate

$$\int_{\ell n2}^{\ell n3} f(x)dx, \text{ where } f(x) = e^{-x} + 2e^{-2x} + 3e^{-3x} + ..\infty.$$

9. Let P_n denote the polynomial of degree n given by $P_n(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^n}{n} = \sum_{k=1}^{n} \frac{x^k}{k}$.

Then, for every $x < 1$ and every $n \ge 1$, prove that

$$-\ln(1-x) = P_n(x) + \int_0^x \frac{u^n}{1-u}\,du.$$

2.21 APPROXIMATION OF DEFINITE INTEGRALS

This section presents some ways of approximating a definite integral $\int_a^b f(x)\,dx$. There are several reasons for wanting to make such an approximation. First, the integrand f(x) may not have an elementary antiderivative; thus the fundamental theorem of calculus could not be used (for example, $\int_0^1 e^{x^2}dx$). Second, even though the antiderivative is elementary, it may be tedious to compute [for example, $\int_0^1 1/(1+x^5)\,dx$]. Third, the values of the integrand f(x) may be known only at a few values of x.

The definite integral $\int_a^b f(x)\,dx$ is, by definition, a limit of sums of the form $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$ ...(1)

Any such sum consequently provides an estimate of $\int_a^b f(x)\,dx$. However, the two methods described in this section, the trapezoidal method and Simpson's method, generally provide much better estimates for the same amount of arithmetic.

Trapezoidal method

The sum (1) can be thought of as a sum of areas of rectangles. In the trapezoidal method, trapezoids are used instead of rectangles. Recall that the area of a trapezoid of height h and bases b_1 and b_2 is $(b_1 + b_2)\,h/2$. Let n be a positive integer. Divide the interval [a, b] into n sections of equal length h = (b – a)/n with

$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h, \dots x_n = b$

The sum

$$\frac{f(x_0)+f(x_1)}{2}\cdot h + \frac{f(x_1)+f(x_2)}{2}\cdot h + \dots + \frac{f(x_{n-1})+f(x_n)}{2}\cdot h$$

is the **trapezoidal estimate** of $\int_a^b f(x)\,dx$.

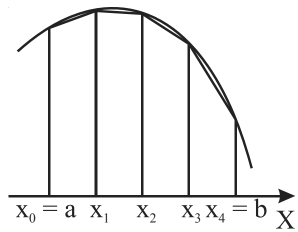
It is usually written

$$\frac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+\dots+2f(x_{n-1})+f(x_n)] \quad ...(2)$$

Note that $f(x_0)$ and $f(x_n)$ have coefficient 1, while all the other f(x)'s have coefficient 2. This is due to the double counting of the edges common to two trapezoids.

The diagram illustrates the trapezoidal approximation for the case n = 4. Note that if f is concave down, the trapezoidal approximation underestimates $\int_a^b f(x)\,dx$.

If f is a linear function, the trapezoidal method, of course, gives the integral exactly.



Example 1. Use the trapezoidal method with n = 4 to estimate $\int_0^1 \frac{dx}{1+x^2}$.

Solution In this case, a = 0, b = 1, and n = 4, so h = (1 – 0)/4 = 1/4. The trapezoidal estimate is

$$= \frac{h}{2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right].$$

The trapezoidal sum is therefore approximately

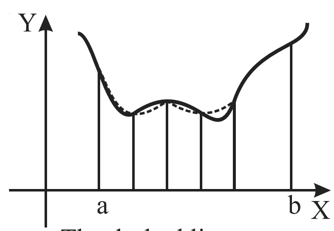
$$= \frac{1}{8}(1 + 1.882 + 1.6 + 1.28 + 0.5) = \frac{1}{8}(6.262) \approx 0.783.$$

Thus, $\int_0^1 \frac{dx}{1+x^2} \approx 0.783$.

The integral can be evaluated by the Fundamental Theorem of Calculus. It equals $\tan^{-1} 1 - \tan^{-1} 0 = \pi/4 \approx 0.782$.

Simpson's method

In the trapezoidal method a curve is approximated by lines. In Simpson's method a curve is approximated by parabolas (see figure). Simpson's method is exact if f(x) is a polynomial of degree atmost 3.



The dashed lines are parts of parabolas

In Simpson's method the interval [a, b] is divided into an even number of sections.

Divide the interval [a,b] into n sections of equal length h = (b – a)/n with

$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h, \dots, x_n = b.$

The sum

$$\frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

is the Simpson's estimate of $\int_a^b f(x)\,dx$.

Example 2. Use Simpson's method with n = 4 to estimate $\int_0^1 \frac{dx}{1+x^2}$.

Solution Here h = 1/4. Simpson's formula takes the form $\frac{\frac{1}{4}}{3}\left[f(0)+4f\left(\frac{1}{4}\right)+2f\left(\frac{2}{4}\right)+4f\left(\frac{3}{4}\right)+f(1)\right]$

The Simpson's approximation of $\int_0^1 \frac{dx}{1+x^2}$ is therefore

$$= \frac{1}{12}(1+3.765+1.6+2.56+0.5)$$

$$= \frac{1}{12}(9.425) \approx 0.785.$$

Thus $\int_0^1 \frac{dx}{1+x^2} \approx 0.785.$

Simpson's method usually provides a much more accurate estimate of an integral than the trapezoidal estimate for the same amount of arithmetic.

Example 3. Show that the approximation

$$\int_a^b F(x)\,dx = \frac{b-a}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right],$$

is exact for F(x) a cubic polynomial.

Solution Let $F(x) = A + 2Bx + 3Cx^2 + 4Dx^3$. It then follows that

$$\int_a^b F(x)\,dx = (Ax + Bx^2 + Cx^3 + Dx^4)\Big]_a^b$$

$$= (b-a)\,(A + B(b+a) + C(b^2+ba+a^2)$$

$$+ D(b^3 + b^2a + ba^2 + a^3))$$

$$= \frac{b-a}{6}\Bigg\{(A+2Ba+3Ca^2+4Da^3)$$

$$+ (A+2Bb+3Cb^2+4Db^3)$$

$$+ 4\left(A + 2B\left(\frac{a+b}{2}\right)+3C\left(\frac{a+b}{2}\right)^2+4D\left(\frac{a+b}{2}\right)^3\right)\Bigg\}.$$

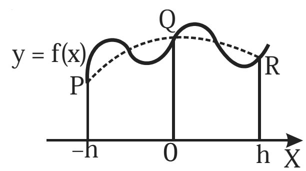
$$= \frac{b-a}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right].$$

Practice Problems W

1. Let $f(x) = Ax^2 + Bx + C$. Show that

$$\int_{-h}^{h} f(x)\,dx = \frac{h}{3}[f(-h)+4f(0)+f(h)].$$

2. Let f be a function. Show that there is a parabola $y = Ax^2 + Bx + C$ that passes through the three points (–h, f(–h)), (0, f(0)), and (h, f(h)).



3. Let $f(x) = Ax^2 + Bx + C$. Shows that

$$\int_{c-h}^{c+h} f(x)\,dx = \frac{h}{3}[f(c-h)+4f(c)+f(c+h)].$$

4. Show that if $f(x) = x^3$,

$$\int_{-h}^{h} f(x)\,dx = \frac{h}{3}[f(-h)+4f(0)+f(h)]$$

5. (a) Show that $\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)$ for $f(x) = 1, x, x^2$ and x^3.

   (b) Let a and b be two numbers, $-1 \le a < b \le 1$, such that $\int_{-1}^{1} f(x)\,dx = f(a)+f(b)$ for f(x) = 1, x, x^2, and x^3. Show that $a = -1/\sqrt{3}$ and $b = 1/\sqrt{3}$.

   (c) Show that the approximation $\int_{-1}^{1} f(x)\,dx \approx f(-1/\sqrt{3})+f(1/\sqrt{3})$ has no error when f is a polynomial of degree atmost 3.

6. Estimate $\int_0^3 f(x)\,dx$ if it is known that f(0) = 10, f(0.5) = 13, f(1) = 14, f(1.5) = 16, f(2) = 18, f(2.5) = 10, f(3) = 6 by (a) the trapezoidal method. (b) Simpson's method.

Target Problems for JEE Advanced

Problem 1. Find a mistake in the following evaluation:

$$\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}\bigg|_0^{\sqrt{3}}$$

$$= \frac{1}{2}\tan^{-1}(-\sqrt{3}) - \tan^{-1}0] = -\frac{\pi}{6},$$

where $\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}\right) = \frac{1}{1+x^2}(x \neq 1)$.

Solution The result is wrong since the integral of a function positive everywhere cannot be negative. The mistake is due to the fact that the function

$\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}$ has a discontinuity at the point x=1:

$$\lim_{x\to 1^-}\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{4};$$

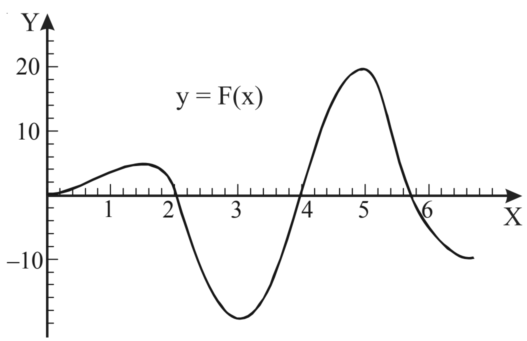
$$\lim_{x\to 1^+}\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = -\frac{\pi}{4}$$

The correct value of the integral under consideration is equal to

$$\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1}x\Big|_0^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}0 = \frac{\pi}{3}.$$

Here the Newton-Leibnitz formula is applicable, since the function $F(x) = \tan^{-1}x$ is continuous on the interval $\left[0, \frac{\pi}{3}\right]$ and the equality $F'(x) = f(x)$ is fulfilled on the whole interval.

Problem 2. Let $F(x) = \int_{x^2}^{4x^2} \sin(\sqrt{t})dt$.



Find all critical points of F between 0 and 2π.

Solution There are two natural ways to go at this. One would be to evaluate the integral, (by substitution $t = s^2$), then take the derivative, then work out what values of x give a derivative of zero. The other, which we choose here, would be to find the derivative without doing the integral, with the rest of the plan being the same.

Thus, $f'(x)\ \frac{d}{dx}\int_{x^2}^{4x^2} f(t)dt = 8xf(4x^2) - 2xf(x^2)$.

Since here, $f(x) = \sin\sqrt{x}$, we have $f'(x) = 8x\sin(2x) - 2x\sin(x)$.

Clearly this is zero at $x = 0, \pi$, and 2π. But from the graph, there must be other points as well.

With the trigonometric identity $\sin 2x = 2\sin x\cos x$, we have $F'(x) = 2x\sin x(8\cos x - 1)$. Now the other two critical points come into focus :

they are the places where $\cos x = 1/8$, and those are $\cos^{-1}(1/8)$ and $2\pi - \cos^{-1}(1/8)$.

Problem 3. Let a be a positive real number. Find the value of a such that the definite integral

$$\int_a^{a^2} \frac{dx}{x+\sqrt{x}}$$

achieves its smallest possible value.

Solution Let F(a) denote the given definite integral. Then

$$F'(a) = \frac{d}{da}\int_a^{a^2}\frac{dx}{x+\sqrt{x}} = 2a.\frac{1}{a^2+\sqrt{a^2}} - \frac{1}{a+\sqrt{a}}.$$

Setting $F'(a) = 0$, we find that

$$2a + 2\sqrt{a} = a + a \text{ or } (\sqrt{a}+1)^2 = 2.$$

We find $\sqrt{a} = \pm\sqrt{2} - 1$, and because

$$\sqrt{a} > 0, a = (\sqrt{2}-1)^2 = 3 - 2\sqrt{2}.$$

Problem 4. For $a > 0$, find the minimum value of the integral $\int_0^{1/a}(a^3 + 4x - a^5x^2)e^{ax}\,dx$.

Solution Let $\int(a^3 + 4x - a^5x^2)e^{ax}$

$= e^{ax}[Ax^2+Bx+C]+D$

Differentiating both sides $(a^3 + 4x - a^5x^2)e^{ax}$

$= e^{ax}[2Ax + B] + [Ax^2 + Bx + C]e^{ax}a$

$\Rightarrow A = -a^4,$

$$B = \frac{4+2a^4}{a},\ C = \frac{-a^4-4}{a^2}$$

$$\therefore \quad I = \left\{ e^{ax}\left[-a^4x^2 + \frac{4+2a^4}{a}x - \frac{a^4+4}{a^2} \right] \right\}_0^{\frac{1}{a}}$$

$$= e\left[-a^2 + \frac{4+2a^4}{a^2} - \frac{a^4+4}{a^2} \right] - \left[-\left(\frac{a^4+4}{a^2} \right) \right]$$

$$= e\left[-a^2 + \frac{a^4}{a^2} \right] + \frac{a^4+4}{a^2}$$

$$\therefore \quad I = a^2 + \frac{4}{a^2} = \left(a - \frac{2}{a} \right)^2 + 4.$$

At $a = \sqrt{2}$, I has a minimum value of 4.

Problem 5. If $f(x) = a|\cos x| + b|\sin x|$ $(a, b \in R)$ has a local minimum at $x = -\dfrac{\pi}{3}$ and satisfies $\int_{-\pi/2}^{\pi/2} (f(x))^2\,dx = 2$. Find the values of a and b and hence find b^2/a^2.

Solution $f(x) = \begin{cases} a\cos x + b\sin x & \text{if } 0 \le x \le \dfrac{\pi}{2} \\ a\cos x - b\sin x & \text{if } -\dfrac{\pi}{2} \le x < 0 \end{cases}$

For $-\pi/2 < x < 0$

$f'(x) = -a\sin x - b\cos x$ ...(1)

and $f''(x) = -a\cos x + b\sin$ ...(2)

Since $f(x)$ has a minima at $x = -\pi/3$

$f'(-\pi/3) = 0$ and $f''(-\pi/3) > 0$

Now $f'(-\pi/3) = a \cdot \dfrac{\sqrt{3}}{2} - \dfrac{b}{2} = 0 \Rightarrow \sqrt{3}\,a - b = 0.$

and $f''(-\pi/3) = -\dfrac{a}{2} - b \cdot \dfrac{\sqrt{3}}{2} = -\dfrac{1}{2}[a + b\sqrt{3}]$

$= -2a > 0.$

Hence, $a < 0$ and $b < 0$

$$\text{Now, } I = \int_{-\pi/2}^{\pi/2} (f(x))^2\,dx$$

$$= \int_{-\pi/2}^{0} f^2(x)\,dx + \int_0^{\pi/2} f^2(x)\,dx$$

$$= \int_{-\pi/2}^{0} (a^2\cos^2 x - 2ab\sin x\cos x + b^2\sin^2 x)\,dx$$

$$+ \int_0^{\pi/2} (a^2\cos^2 x + 2ab\sin x\cos x + b^2\sin^2 x)\,dx$$

On solving, $I = \dfrac{\pi a^2}{2} + \dfrac{\pi b^2}{2} + 2ab = 2$ (given)

$$\Rightarrow \quad 2(\sqrt{3} + \pi)a^2 = 2$$

$$\Rightarrow \quad a = -\frac{1}{\sqrt{\pi + \sqrt{3}}} \text{ and } b = -\frac{\sqrt{3}}{\sqrt{\pi + \sqrt{3}}}.$$

Problem 6. For a positive constant t, let α, β be the roots of the quadratic equation $x^2 + t^2x - 2t = 0$. If the minimum value of

$$\int_{-1}^{2} \left(\left(x + \frac{1}{\alpha^2} \right)\left(x + \frac{1}{\beta^2} \right) + \frac{1}{\alpha\beta} \right) dx \text{ is } \sqrt{\frac{a}{b}} + c,$$

where $a, b, c \in N$, then find the least value of $(a + b + c)$.

Solution If α and β are the roots of $x^2 + t^2x - 2t = 0$, then we have $\alpha + \beta = -t^2$ and $\alpha\beta = -2t$.

$$\text{So} \quad \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = \frac{t^2}{4} + \frac{1}{t}$$

$$\text{and} \quad \frac{1}{(\alpha\beta)^2} + \frac{1}{\alpha\beta} = \frac{1}{4t^2} - \frac{1}{2t}.$$

$$\text{Now } I = \int_{-1}^{2} \left(\left(x + \frac{1}{\alpha^2} \right)\left(x + \frac{1}{\beta^2} \right) + \frac{1}{\alpha\beta} \right) dx$$

$$= \int_{-1}^{2} \left(x^2 + \left(\frac{t^2}{4} + \frac{1}{t} \right) x + \left(\frac{1}{4t^2} - \frac{1}{2t} \right) \right) dx$$

$$= \frac{3t^2}{8} + \frac{3}{4t^2} + 3.$$

Differentiating I w.r.t. t,

$$\text{we get} \quad \frac{dI}{dt} = \frac{3t}{4} - \frac{3}{2t^2} = 0.$$

So we get $t = \pm\sqrt[4]{2}$, and since t is taken to be positive then

$$I_{min} = I(\sqrt[4]{2}) = \frac{3\sqrt{2}}{4} + 3 = \sqrt{\frac{18}{16}} + 3 = \sqrt{\frac{9}{8}} + 3$$

$$\equiv \sqrt{\frac{a}{b}} + c.$$

$\Rightarrow$ The least value of $a + b + c = 20$.

Problem 7. Consider the function

$f(x) = \int_0^x f(1 + t^3)^{-1/2}\,dt$. If g is the inverse of f, then find the value of $\dfrac{g''(y)}{g^2(y)}$.

Solution We have $f(x) = \int_0^x f(1 + t^3)^{-1/2}\,dt$

i.e. $f\{g(x)\} = \int_0^{g(x)} (1 + t^3)^{-1/2}\,dt$

i.e. $x = \int_0^{g(x)} (1 + t^3)^{-1/2}\,dt$

[$\because$ g is inverse of f, $f\{g(x)\} = x$]

Differentiating w.r.t. x, we have

$1 = (1 + g^3)^{-1/2} \cdot g'$

i.e. $(g')^2 = 1 + g^3$

Differentiating again w.r.t. x, we have

$2g'g'' = 3g^2g'$

$$\Rightarrow \frac{g''}{g^2} = \frac{3}{2}.$$

Problem 8. Let $a + b = 4$ where $a < 2$ and let $g(x)$ be a differentiable function of x. If $\frac{dg}{dx} > 0$ for all x, prove that $\int_0^a g(x) + dx + \int_0^b g(x)\,dx$ increases as $(b - a)$ increases.

Solution Let $y = \int_0^a g(x)\ dx + \int_0^b g(x)\ dx$ ...(1)

$z = b - a = (4 - a) - a = 4 - 2a$ ...(2)

We have to prove that $\frac{dy}{dx} > 0$.

Differentiating (1) w.r.t a, $\frac{dy}{da} = g(a) + g(b) \cdot \frac{db}{da}$

or $\frac{dy}{da} = g(a) + g(b)\frac{d}{da}(4 - a) = g(a) - g(b)$.

Differentiating (2) w.r.t. a, $\frac{dz}{da} = -2$.

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{da}}{\frac{dz}{da}} = \frac{g(a) - g(b)}{-2} = \frac{g(4-a) - g(a)}{2} \quad ...(3)$$

As $a < 2$, $4 - a > a$. Also $\frac{dg}{dx} > 0$ for all x, implies that g is an increasing function.

$\therefore$ $g(4 - a) > g(a)$.

Hence (3) $\Rightarrow \frac{dy}{dz} > 0$, i.e., y is an increasing function of z.

$\therefore \int_0^a g(x)\ dx + \int_0^b g(x)\ dx$ increases as $b - a$ increases.

Problem 9. Evaluate $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n-k}{n^2}\cos\frac{4k}{n}$.

Solution We have $S = \frac{1}{n}\sum_{k=1}^{n}\left(1 - \frac{k}{n}\right)\cos 4\left(\frac{k}{n}\right)$

$$= \int_0^1 \underbrace{(1-x)}_{u}\underbrace{\cos 4x}_{v}\,dx$$

$$= (1-x)\frac{\sin 4x}{4}\Big]_0^1 + \frac{1}{4}\int_0^1 \sin 4x\,dx$$

$$= 0 + \frac{1}{4}\int_0^1 \sin 4x\,dx = \frac{-1}{16}\cos 4x\Big]_0^1$$

$$= -\left(\frac{1}{16}\cos 4 - \frac{1}{16}\right) = \frac{1}{16}(1 - \cos 4).$$

Example 10. If $a, b, c \in R^+$ then show that $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{(k+an)(k+bn)}$ is equal to

(i) $\frac{1}{a-b}\ln\frac{a(b+1)}{b(a+1)}$ if $a \ne b$

(ii) $\frac{1}{a(1+a)}$ if $a = b$.

Solution

(i) We have $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2\left(a+\frac{k}{n}\right)\left(b+\frac{k}{n}\right)}$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\left(a+\frac{k}{n}\right)\left(b+\frac{k}{n}\right)}$$

$$= \int_0^1 \frac{1}{(a+x)(b+x)}dx$$

$$= \frac{1}{a-b}\int_0^1 \frac{(a+x)-(b+x)}{(a+x)(b+x)}dx$$

$$= \frac{1}{a-b}\int_0^1\left(\frac{1}{b+x} - \frac{1}{a+x}\right)dx$$

$$= \frac{1}{a-b}[\ln(b+x) - \ln(a+x)]_0^1$$

$$= \frac{1}{a-b}\left[\ln\frac{(b+x)}{(a+x)}\right]_0^1$$

$$= \frac{1}{a-b}\ln\frac{(b+1)a}{(a+1)b},\ \text{if } a \ne b.$$

(ii) Now if $a = b$ then the given limit

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{\left(a+\frac{k}{n}\right)^2} = \int_0^1\frac{dx}{(a+x)^2} = \frac{-1}{a+x}\Big]_0^1$$

$$= \frac{1}{a} - \frac{1}{a+1} = \frac{1}{a(a+1)}.$$

Problem 11. Evaluate

$$\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)^{\frac{2}{n^2}}\left(1+\frac{2^2}{n^2}\right)^{\frac{4}{n^2}}\left(1+\frac{3^2}{n^2}\right)^{\frac{6}{n^2}}\cdots\left(1+\frac{n^2}{n^2}\right)^{\frac{2n}{n^2}}\right\}$$

Solution Let A denote the given expression, then

$$\ln A=\sum_{r=1}^{n}\frac{2r}{n^2}\ln\left(1+\frac{r^2}{n^2}\right)$$

$$\therefore\quad \lim_{n\to\infty}\ln A=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}2\frac{r}{n}\ln\left(1+\frac{r^2}{n^2}\right)$$

$$=\int_0^1 2x\ln(1+x^2)dx$$

$$=\int_1^2\ln z\,dz,\quad \text{putting } 1+x^2=z$$

$$=[z\ln z-z]_1^2=2\ln 2-1=\ln\frac{4}{e}.$$

Since $\ln\lim_{n\to\infty}A=\lim_{n\to\infty}\ln A=\ln\frac{4}{e}$,

$\therefore$ The required limit $=\lim_{n\to\infty}A=\frac{4}{e}$.

Problem 12. Let f(x), $x\ge 0$, be a non-negative continuous function, and let $F(x)=\int_0^x f(t)\,dt,\ x\ge 0$. If for some $c>0$, $f(x)\le c\,F(x)$ for all $x\ge 0$, then show that $f(x)=0$ for all $x\ge 0$.

Solution Given that, for $x\ge 0$, $F(x)=\int_0^x f(t)\,dt$

$\Rightarrow F(0)=\int_0^0 f(t)\,dt=0$

As $f(x)\le c\,F(x)\ \forall\, x\ge 0$, we get

$f(0)\le c\,F(0)\Rightarrow f(0)\le 0.$

Since $f(x)\ge 0\ \forall\, x\ge 0$, we get

$f(0)\ge 0$

$\therefore\ f(0)=0$

Since, f is continuous on $[0,\infty]$, F is differentiable on $[0,\infty)$ and $F'(x)=f(x)\ \forall\, x\ge 0$.

Since, $f(x)\le cF(x)\le 0\ \forall\, x\ge 0$, multiplying both sides by e^{-cx} (the integrating factor) we get

$e^{-cx}F'(x)-ce^{-cx}F(x)\le 0\quad [\because e^{-cx}>0\ \forall\, x]$

$\Rightarrow \frac{d}{dx}[e^{-cx}F(x)]\le 0$

So, $g(x)=e^{-cx}F(x)$ is a decreasing function on $[0,\infty]$ i.e. $g(x)\le g(0)$ for each $x\ge 0$.

But we know that $g(0)=e^{-c(0)}F(0)=0$

$\therefore\ g(x)\le 0\ \forall\, x\ge 0$

$\Rightarrow\ e^{-cx}F(x)\le 0\ \forall\, x\ge 0$

$\Rightarrow\ F(x)\le 0\ \forall\, x\ge 0$

So, $f(x)\le c\,F(x)\le 0\ \forall\, x\ge 0$.

But it is given that $f(x)\ge 0\ \forall\, x\ge 0$.

hence, $f(x)=0\ \forall\, x\ge 0$.

Problem 13. Let f(x) be defined in the interval $0<x\le 1$ as follows :

$$f(x)=2,\ \frac{1}{2}<x\le 1,\quad f(x)=-3,\ \frac{1}{3}<x\le\frac{1}{2}$$

$$f(x)=4,\ \frac{1}{4}<x\le\frac{1}{3},\quad f(x)=-5,\ \frac{1}{5}<x\le\frac{1}{4},$$

and so on, the values being alternatively positive and negative. Show that the integral $\int_0^1 f(x)dx=\ln 2$.

Solution We have $\int_0^1 f(x)dx=\int_{1/2}^1 2dx-\int_{1/3}^{1/2}3dx$

$$+\int_{1/4}^{1/3}4dx-\int_{1/4}^{1/3}4dx-\int_{1/5}^{1/4}5dx+\ldots.$$

$$=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots.=\ln 2.$$

Problem 14. Evaluate the definite integral

$$\int_{-1}^{+1}\frac{2u^{332}+u^{998}+4u^{1664}\sin u^{691}}{1+u^{666}}du.$$

Solution The term is $\frac{4u^{1664}\sin u^{691}}{1+u^{666}}$ is odd in u, so its integral is 0. Now make the substitution

$u=v^{1/333}\Rightarrow du=\frac{1}{333}v^{-332/333}dv$ to find that

$$\int_{-1}^{1}\frac{2u^{332}+u^{998}}{1+u^{666}}du=\frac{1}{333}\int_{-1}^{+1}\frac{2+v^2}{1+v^2}dv$$

$$=\frac{1}{333}\int_{-1}^{1}\left(1+\frac{1}{1+v^2}\right)dv$$

$$=\frac{2}{333}\int_0^1\left(1+\frac{1}{1+v^2}\right)dv$$

$$=\frac{2}{333}\left(1+\int_0^1\frac{1}{1+v^2}dv\right)=\frac{2}{333}(1+\tan^{-1}1)$$

$$=\frac{2}{333}\left(1+\frac{\pi}{4}\right).$$

Problem 15. Show that $\int_0^{\pi}\frac{dx}{(a-\cos x)}=\frac{\pi}{\sqrt{(a^2-1)}}$.

Hence or otherwise evaluate $\int_0^{\pi} \frac{dx}{(\sqrt{5}-\cos x)^3}$.

Solution Let $I = \int_0^{\pi} \frac{dx}{(a-\cos x)}$ ...(1)

$$= \int_0^{\pi} \frac{dx}{a-\cos(\pi - x)} \quad \textbf{[P–5]}$$

$$= \int_0^{\pi} \frac{dx}{(a+\cos x)} \quad ...(2)$$

Adding (1) and (2), $2I = \int_0^{\pi} \frac{2a\,dx}{(a^2+\cos^2 x)}$

$$= 2a \,.\, 2\int_0^{\pi/2} \frac{dx}{(a^2-\cos^2 x)}$$

$$\Rightarrow \quad I = 2a\int_0^{\pi/2} \frac{dx}{(a^2-\cos^2 x)}$$

$$= 2a\int_0^{\pi/2} \frac{\sec^2 dx}{a^2(1+\tan^2 x)-1}$$

$$= 2a\int_0^{\pi/2} \frac{\sec^2 x\,dx}{(a^2-1)+(a\tan x)^2}$$

Put $a\tan x = t \Rightarrow a\sec^2 x\,dx = dt$ When $x = 0, t = 0$; $x = \pi/2, t = \infty$

$$= 2\int_0^{\infty} \frac{dt}{(\sqrt{a^2-1})^2+t^2}$$

$$= \frac{2}{\sqrt{(a^2-1)}}\left\{\tan^{-1}\left(\frac{t}{\sqrt{(a^2-1)}}\right)\right\}_0^{\infty}$$

$$= \frac{2}{\sqrt{(a^2-1)}}\{\tan^{-1}\infty - \tan^{-1}0\}$$

$$= \frac{2}{\sqrt{(a^2-1)}}\left\{\frac{\pi}{2}-0\right\}.$$

Hence, $I = \frac{\pi}{\sqrt{(a^2-1)}}$ or, $\int_0^{\pi} \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{(a^2-1)}}$

Differentiating both sides w.r.t. 'a', we get

$$-\int_0^{\pi} \frac{dx}{(a-\cos x)^2} = \frac{-\pi a}{(a^2-1)^{3/2}}$$

Again differentiating both sides w.r.t. 'a' we get

$$2\int_0^{\pi} \frac{dx}{(a-\cos x)^3} = \frac{\pi(2a^2+1)}{(a^2-1)^{3/2}}$$

Put $a = \sqrt{5}$ on both sides, we get

$$2\int_0^{\pi} \frac{dx}{(\sqrt{5}-\cos x)^3} = \frac{\pi(11)}{(4)^{3/2}}$$

or, $\int_0^{\pi} \frac{dx}{(\sqrt{5}-\cos x)^{3/2}} = \frac{11\pi}{16}$.

Problem 16. Evaluate $\int_1^{\infty} \frac{dx}{(x-\cos\alpha)\sqrt{(x^2-1)}}$, $0 < \alpha < 2\pi$.

Solution Let $I = \int_1^{\infty} \frac{dx}{(x-\cos\alpha)\sqrt{(x^2-1)}}$

Put $x - \cos\alpha = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}$

When $x = 1$ then $t = \frac{1}{1-\cos\alpha}$ and when $x \to \infty$ then $t = 0$

$$\therefore \quad I = \int_{\frac{1}{1-\cos\alpha}}^{0} \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+\cos\alpha\right)^2-1}}$$

$$= \int_0^{\frac{1}{1-\cos\alpha}} \frac{dt}{\sqrt{(1+t\cos\alpha)^2-t^2}}$$

$$= \int_0^{\frac{1}{1-\cos\alpha}} \frac{dt}{\sqrt{(-t^2\sin^2\alpha+2t\cos\alpha+1)}}$$

$$= \frac{1}{|\sin\alpha|}\int_0^{\frac{1}{1-\cos\alpha}} \frac{dt}{\sqrt{-\left(t^2-\frac{2t\cos\alpha}{\sin^2\alpha}-\frac{1}{\sin^2\alpha}\right)}}$$

$$= \frac{1}{|\sin\alpha|}\int_0^{\frac{1}{1-\cos\alpha}} \frac{dt}{\sqrt{-\left\{\left(t-\frac{\cos\alpha}{\sin^2\alpha}\right)^2-\frac{\cos^2\alpha}{\sin^4\alpha}-\frac{1}{\sin^2\alpha}\right\}}}$$

$$= \frac{1}{|\sin\alpha|}\int_0^{\frac{1}{1-\cos\alpha}} \frac{dt}{\sqrt{\left(\frac{1}{\sin^2\alpha}\right)^2-\left(t-\frac{\cos\alpha}{\sin^2\alpha}\right)^2}}$$

$$=\frac{1}{|\sin\alpha|}\sin^{-1}(t\sin^2\alpha-\cos\alpha)\Big|_0^{\frac{1}{1-\cos\alpha}}$$

$$=\frac{1}{|\sin\alpha|}\left\{\sin^{-1}\left(\frac{\sin^2\alpha}{1-\cos\alpha}-\cos\alpha\right)-\sin^{-1}(0-\cos\alpha)\right\}$$

$$=\frac{1}{|\sin\alpha|}\{\sin^{-1}(1)-\sin^{-1}(-\cos\alpha)\}$$

$$=\frac{1}{|\sin\alpha|}\left\{\frac{\pi}{2}-\sin^{-1}(-\cos\alpha)\right\}$$

$$=\frac{\cos^{-1}(-\cos\alpha)}{|\sin\alpha|}=\frac{\cos^{-1}\cos(\pi-\alpha)}{|\sin\alpha|}=\frac{|\pi-\alpha|}{|\sin\alpha|}$$

$$=\begin{cases}\dfrac{\pi-\alpha}{\sin\alpha}, 0<\alpha<\pi\\[2mm] \dfrac{\alpha-\pi}{-(\sin\alpha)}, \pi<\alpha<2\pi\end{cases}$$

Finally, $I=\dfrac{\pi-\alpha}{\sin\alpha}$.

Problem 17. If $n>1$, evaluate $\displaystyle\int_0^\infty\frac{1}{(x+\sqrt{1+x^2})^n}dx$.

Solution $\displaystyle\int_0^\infty\frac{dx}{(x+\sqrt{1+x^2})^n}=\int_0^{\pi/2}\frac{\sec^2\theta}{(\tan\theta+\sec\theta)^n}d\theta$

[Putting $x=\tan\theta$]

$$=\int_0^{\pi/2}\frac{\sec^2\theta\cos^n\theta}{(1+\sin\theta)^n}d\theta=\int_0^{\pi/2}\frac{\cos^{n-2}\theta}{(1+\sin\theta)^n}d\theta$$

$$=\int_0^{\pi/2}\frac{\cos^{n-2}\theta}{(1+\sin\theta)}d\theta$$

$$=\int_0^{\pi/2}\frac{\cos^{n-2}\left(\frac{\pi}{2}-\theta\right)}{\left[1+\sin\left(\frac{\pi}{2}-\theta\right)\right]^n}d\theta$$

$$=\int_0^{\pi/2}\frac{\sin^{n-2}\theta}{(1+\cos\theta)^n}d\theta$$

$$=\int_0^{\pi/2}\frac{\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{n-2}}{\left(2\cos^2\frac{\theta}{2}\right)^n}-d\theta$$

$$=\frac{1}{2^2}\int_0^{\pi/2}\frac{\sin^{n-2}\frac{\theta}{2}}{\cos^{n+2}\frac{\theta}{2}}d\theta=\frac{1}{4}\int_0^{\pi/2}\tan^{n-2}\frac{\theta}{2}\sec^4\frac{\theta}{2}d\theta$$

[Putting $z=\tan\frac{\theta}{2}$] $\displaystyle=\frac{1}{2}\int_0^1 z^{n-2}(1+z^2)dz$

$$=\frac{1}{2}\left[\frac{z^{n-1}}{n-1}+\frac{z^{n+1}}{n+1}\right]_0^1=\frac{1}{2}\left(\frac{1}{n-1}+\frac{1}{n+1}\right)=\frac{n}{n^2-1}.$$

Problem 18. Prove that

$$\int_{-\infty}^\infty\frac{dx}{(x^2+ax+a^2)(x^2+bx+b^2)}=\frac{2\pi(a+b)}{\sqrt{3}ab(a^2+ab+b^2)}.$$

Solution Let $\displaystyle I=\int_{-\infty}^\infty\frac{dx}{(x^2+ax+a^2)(x^2+bx+b^2)}$

$$=\frac{1}{(a^3-b^3)}\int_{-\infty}^\infty\frac{(x^3-b^3)-(x^3-a^3)}{(x^2+ax+a^2)(x^2+bx+b^2)}dx$$

$$=\frac{1}{(a^3-b^3)}\left[\int_{-\infty}^\infty\frac{x-b}{x^2+ax+a^2}dx-\int_{-\infty}^\infty\frac{x-a}{x^2+bx+b^2}dx\right]$$

$$=\frac{1}{(a^3-b^3)}\left[\int_{-\infty}^\infty\frac{\frac{1}{2}(2x+a)-(a/2+b)}{x^2+ax+a^2}dx\right.$$

$$\left.-\int_{-\infty}^\infty\frac{\frac{1}{2}(2x+b)-(a+b/2)}{x^2+bx+b^2}dx\right]$$

$$=\frac{1}{(a^3-b^3)}\frac{1}{2}\int_{-\infty}^\infty\frac{(2x+a)dx}{(x^2+ax+a^2)}$$

$$+\frac{(a/2+b)}{(a^3-b^3)}\int_{-\infty}^\infty\frac{dx}{(x+a/2)^2+(a\sqrt{3}/2)^2}$$

$$-\frac{1}{(a^3-b^3)}\frac{1}{2}\int_{-\infty}^\infty\frac{(2x+a)dx}{(x^2+bx+b^2)}$$

$$+\frac{(a+b/2)}{(a^3-b^3)}\int_{-\infty}^\infty\frac{dx}{(x+b/2)^2+(b\sqrt{3}/2)^2}$$

$$=\frac{1}{2}\cdot\frac{1}{(a^3-b^3)}\left[\ln\left(\frac{x^2+ax+a^2}{x^2+bx+b^2}\right)\right]_{-\infty}^\infty$$

$$-\frac{(a/2+b)}{(a^3-b^3)}\frac{1}{a\sqrt{3}/2}\left\{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right\}$$

$$+\frac{(a+b/2)}{(a^3-b^3)}\frac{1}{b\sqrt{3}/2}\left\{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right\}$$

$$=\frac{1}{2}\frac{1}{(a^3-b^3)}\left[\ln\left(\frac{1+\frac{a}{x}+\frac{a^2}{x^2}}{1+\frac{b}{x}+\frac{b^2}{x^2}}\right)\right]_{-\infty}^{\infty}$$

$$-\frac{(a+2b)\pi}{(a^3-b^3)a\sqrt{3}}+\frac{(b+2a)\pi}{(a^3-b^3)b\sqrt{3}}$$

$$=\frac{1}{2}\frac{1}{(a^3-b^3)}[\ln 1-\ln 1]$$

$$+\frac{\pi}{(a^3-b^3)\sqrt{3}}\left[\frac{b+2a}{b}-\frac{a+2b}{a}\right]$$

$$=0+\frac{\pi}{(a-b)(a^2+ab+b^2)\sqrt{3}}\left(\frac{ab+2a^2-ab-2b^2}{ab}\right)$$

$$=\frac{2\pi(a+b)}{\sqrt{3}ab(a^2+ab+b^2)}=\text{R.H.S.}$$

Problem 19. Evaluate $\int_0^1 (tx+1-x)^n\,dx$, $n \in N$ and is independent of x. Hence show that

$$\int_0^1 x^k(1-x)^{n-k}\,dx=\frac{1}{{}^nC_k(n+1)}$$

Solution Let $I=\int_0^1 (tx+1-x)^n\,dx$

$$=\int_0^1 ((t-1)x+1)^n\,dx$$

$$=\left\{\frac{((t-1)x+1)^{n+1}}{(n+1)(t-1)}\right\}_0^1$$

$$=\frac{1}{n+1}(t^n+t^{n-2}+\ldots+t+1\} \quad \ldots(1)$$

Again, $I=\int_0^1 \{(1-x)+tx\}^n\,dx$

$$=\int_0^1 \{{}^nC_0(1-x)^n+{}^nC_1(1-x)^{n-1}(tx)$$
$$+{}^nC_2(1-x)^{n-2}(tx)^2+\ldots+{}^nC_r(1-x)^{n-r}(tx)^r$$
$$+\ldots\ldots+{}^nC_n(tx)^n\}\,dx$$

$$=\int_0^1\left\{\sum_{r=0}^{n}{}^nC_r(1-x)^{n-r}(tx)^r\right\}dx$$

$$=\sum_{r=0}^{n}{}^nC_r t^r\left(\int_0^1(1-x)^{n-r}x^r dx\right) \quad \ldots(2)$$

From (1) and (2)

$$\sum_{r=0}^{n}{}^nC_r t^r\left(\int_0^1(1-x)^{n-r}x^r dx\right)$$

$$=\frac{1}{n+1}\{t^n+t^{n-1}+t^{n-2}+\ldots+t+1\}$$

Equating the coefficient of t^k on both sides,

$${}^nC_k\left(\int_0^1(1-x)^{n-r}x^r dx\right)=\frac{1}{n+1}$$

$$\Rightarrow \int_0^1(1-x)^{n-r}x^r dx=\frac{1}{{}^nC_k(n+1)}.$$

Problem 20. Prove the identity

$$\sum_{k=0}^{n}(-1)^k\frac{{}^nC_k}{(k+m+1)}=\sum_{k=0}^{m}(-1)^k\frac{{}^mC_k}{k+n+1}$$

Solution We have

$$\int_0^1 x^m(1-x)^n\,dx=\int_0^1 x^n(1-x)^m\,dx \quad \textbf{[P–5]}$$

$$\Rightarrow \int_0^1 x^m({}^nC_0-{}^nC_1x+{}^nC_2x^2-\ldots\ldots+(-1)^n\,{}^nC_nx^n)\,dx$$

$$=\int_0^1 x^n({}^mC_0-{}^mC_1x+{}^mC_2x^2-\ldots.+(-1)^m\,{}^mC_mx^m)\,dx$$

$$\Rightarrow \int_0^1 ({}^nC_0x^m-{}^nC_1x^{m+1}+{}^nC_2x^{m+2}-\ldots$$
$$+(-1)^n\,{}^nC_nx^{m+n})\,dx$$

$$=\int_0^1 ({}^mC_0x^n-{}^mC_1x^{n+1}+{}^mC_2x^{n+2}-\ldots.$$
$$+(-1)^m\,{}^mC_mx^{m+n})\,dx$$

$$\Rightarrow \frac{{}^nC_0}{(m+1)}-\frac{{}^nC_1}{m+2}+\frac{{}^nC_2}{(m+3)}-\ldots+\frac{(-1)^n\,{}^nC_n}{(m+n+1)}$$

$$=\frac{{}^mC_0}{n+1}-\frac{{}^mC_1}{n+2}+\frac{{}^mC_2}{n+3}-\ldots+(-1)^m\frac{{}^mC_m}{(m+n+1)}$$

$$\Rightarrow \sum_{k=0}^{n}(-1)^k\frac{{}^nC_k}{(k+m+1)}=\sum_{k=0}^{m}(-1)^k\frac{{}^mC_k}{(k+n+1)}$$

Hence proved.

Problem 21. Evaluate $I=\int_0^1 2\sin(\alpha t)\sin(\beta t)\,dt$ if

(a) $\tan\alpha=\alpha$ and $\tan\beta=\beta$
(b) $\tan\alpha=\alpha$, $\tan\beta=\beta$ and $\alpha=\beta$

Solution $I=\int_0^1 2\sin\alpha t\sin\beta t\,dt$

(a) Integrating by parts, taking $\sin\alpha t$ as the second function, we have

$$I=\left[2\sin\beta t\left(-\frac{\cos\alpha t}{\alpha}\right)\right]_0^1-\int_0^1 2\beta\cos\beta t\left(\frac{-\cos\alpha t}{\alpha}\right)dt$$

$$\therefore\quad I=-\frac{2}{\alpha}\cos\alpha\sin\beta+\frac{2\beta}{\alpha}\int_0^1\cos\beta t\cos\alpha t\,dt$$

$$=-\frac{2}{\alpha}\cos\alpha\sin\beta$$

$$+\frac{2\beta}{\alpha}\left[\left[\cos\beta t\frac{\sin\alpha t}{\alpha}\right]_0^1+\frac{\beta}{\alpha}\int_0^1\sin\beta t\sin\alpha t\,dt\right]$$

$$I=-\frac{2}{\alpha}\cos\alpha\sin\beta+\frac{2\beta}{\alpha}\sin\alpha\cos\beta+\frac{\beta^2}{\alpha^2}I$$

$$\Rightarrow\quad I\left(1-\frac{\beta^2}{\alpha^2}\right)=\frac{2\beta}{\alpha}\sin\alpha\cos\beta-\frac{2}{\alpha}\cos\alpha\sin\beta$$

$$=\cos\alpha\cos\beta\left[\frac{2\beta}{\alpha}\tan\alpha-\frac{2}{\alpha}\tan\beta\right]$$

$=0$, given $\tan\alpha=\alpha$ and $\tan\beta=\beta$.

(b) Given $\alpha=\beta$

$$I=\int_0^1 2\sin^2\alpha t\,dt=\int_0^1(1-\cos2\alpha t)\,dt$$

$$=\left[t-\frac{\sin2\alpha t}{2\alpha}\right]_0^1=1-\frac{\sin2\alpha}{2\alpha}$$

$$=1-\frac{2\tan\alpha}{2\alpha(1+\tan^2\alpha)}=1-\frac{2\alpha}{2\alpha(1+\alpha^2)}$$

$$=1-\frac{1}{1+\alpha^2}=\frac{\alpha^2}{1+\alpha^2}.$$

Problem 22. Evaluate $\int_0^\infty xe^{-2x}\,dx$.

Solution $$\int_0^\infty xe^{-2x}\,dx=\lim_{t\to\infty}\int_0^t \underbrace{x}_{u}\,\underbrace{e^{-2x}dx}_{dv}$$

$$=\lim_{t\to\infty}\left[\left(\frac{x}{-2}e^{-2x}\right)\Big|_0^t-\int_0^t\frac{1}{-2}e^{-2x}dx\right]$$

$$=\lim_{t\to\infty}\left[\frac{-xe^{-2x}}{-2}-\frac{e^{-2x}}{4}\right]_0^t$$

$$=\lim_{t\to\infty}\left[-\frac{1}{2}te^{-2t}-\frac{1}{4}e^{-2t}+0+\frac{1}{4}\right]$$

$$=-\frac{1}{2}\lim_{t\to\infty}\left(\frac{t}{e^{2t}}\right)+\frac{1}{4}$$

[We have $\lim_{t\to\infty}\left(-\frac{1}{4}e^{-2t}\right)=0$ and using L'Hospital's rule]

$$=-\frac{1}{2}\lim_{t\to\infty}\left(\frac{1}{2e^{2t}}\right)+\frac{1}{4}=\frac{1}{4}.$$

Problem 23. Prove that

$$\int_0^1 x^{n-1}(1-x)^{m-1}dx=\frac{1.2.3...(m-1)}{n(n+1)...(n+m-1)}$$

when m and n are positive, and m is an integer.

Solution Integrating by parts, we have

$$\int x^{n-1}(1-x)^{m-1}dx=\frac{x^n}{n}(1-x)^{m-1}$$
$$+\frac{m-1}{n}\int x^n(1-x)^{m-2}dx$$

Moreover, since n and m – 1 are positive, the term $x^n(1-x)^{m-1}$ vanishes for both limits.

$$\therefore\quad\int_0^1 x^{n-1}(1-x)^{m-1}dx=\frac{m-1}{n}\int_0^1 x^n(1-x)^{m-2}dx.$$

The repeated application of this formula reduces the integral to depend on $\int_0^1 x^{m+n-2}dx$, the value of which is $\frac{1}{m+n-1}$.

Hence we have

$$\int_0^1 x^{n-1}(1-x)^{m-1}dx=\frac{1.2.3...(m-1)}{n(n+1)....(m+m-1)}$$

This shows that when either m or n is an integer the definite integral $\int_0^1 x^{n-1}(1-x)^{m-1}dx$ can be easily evaluated.

Problem 24. If the value of the definite integral

$$\int_{-1}^1\cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)\cdot\left(\cot^{-1}\frac{x}{\sqrt{1-(x^2)^{|x|}}}\right)dx$$

$$=\frac{\pi^2(\sqrt{a}-\sqrt{b})}{\sqrt{c}}$$, where $a, b, c\in N$ in their lowest form, then find the value of $(a+b+c)$.

Solution Let

$$I=\int_{-1}^{1}\cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)\cdot\left(\cot^{-1}\frac{x}{\sqrt{1-(x^2)^{|x|}}}\right)dx \quad ...(1)$$

[P–5]

$$I=\int_{-1}^{1}\left(\cot^{-1}\frac{1}{\sqrt{1-x^2}}\right)\left(\cot^{-1}\frac{-x}{\sqrt{1-(x^2)^{|x|}}}\right)dx \quad ...(2)$$

On adding

$$2I=\int_{-1}^{1}\cot^{-1}\frac{1}{\sqrt{1-x^2}}\left\{\cot^{-1}\frac{x}{\sqrt{1-(x^2)^{|x|}}}+\pi-\cot^{-1}\frac{x}{\sqrt{1-(x^2)^{|x|}}}\right\}dx$$

$$2I=\int_{-1}^{1}\pi\cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=\int_{-1}^{1}\pi\tan^{-1}\sqrt{1-x^2}\,dx$$

(As $\tan^{-1}\sqrt{1-x^2}$ is even function)

$$=2\pi\int_0^1\tan^{-1}\sqrt{1-x^2}\,dx \quad ...(3)$$

$$\therefore\quad I=\pi\int_0^1\underbrace{1}_{v}\,\underbrace{\tan^{-1}(\sqrt{1-x^2})}_{u}dx \quad ...(4)$$

Integrating by parts

$$I=\pi\tan^{-1}(\sqrt{1-x^2})\cdot x\Big]_0^1-\int_0^1\frac{x}{(1+1-x^2)}\frac{(-x)}{\sqrt{1-x^2}}dx$$

$$=0+\pi\int_0^1\frac{x^2}{(2-x^2)\sqrt{1-x^2}}dx$$

Put $x=\sin\theta \Rightarrow dx=\cos\theta\,d\theta$

$$I=\pi\int_0^{\pi/2}\frac{\sin^2\theta}{(2-\sin^2\theta)}d\theta$$

$$=-\pi\int_0^{\pi/2}\frac{2-\sin^2\theta-2}{2-\sin^2\theta}d\theta$$

$$\therefore\quad I=2\pi\int_0^{\pi/2}\frac{d\theta}{2-\sin^2\theta}-\frac{\pi^2}{2}$$

$$I=2\pi\int_0^{\pi/2}\frac{\sec^2\theta d\theta}{2+2\tan^2\theta-\tan^2\theta}-\frac{\pi^2}{2}$$

$$=2\pi\int_0^{\pi/2}\frac{\sec^2\theta\,d\theta}{2+\tan^2\theta}-\frac{\pi^2}{2} \quad \text{Put } \tan\theta=t$$

$$I=2\pi\int_0^{\infty}\frac{dt}{2+t^2}-\frac{\pi^2}{2}$$

$$=2\pi\frac{1}{\sqrt2}\tan^{-1}\frac{t}{\sqrt2}\Big]_0^{\infty}-\frac{\pi^2}{2}=\frac{\pi^2}{\sqrt2}-\frac{\pi^2}{2}$$

$$=\frac{\pi^2(\sqrt2-1)}{2}=\frac{\pi^2(\sqrt a-\sqrt b)}{\sqrt c}$$

$\Rightarrow\ a=2, b=1$ and $c=4$

$\Rightarrow\ a+b+c=2+1+4=7.$

Problem 25. Let $<\varepsilon_n>$ be a sequence of positive real numbers, such that $\lim\limits_{n\to\infty}\varepsilon_n=0$. Find

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon_n\right).$$

Solution It is well known that

$$-1=\int_n^1\ln x\,dx=\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right)$$

Then, $\displaystyle\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon_n\right)\geq\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right)$

Given $\varepsilon>0$ there exists n_0 such that $0<\varepsilon_n\leq\varepsilon$ for all $n\geq n_0$.

Then $\displaystyle\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon_n\right)\leq\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon\right)$

Since $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon\right)=\int_0^1\ln(x+\varepsilon)dx$

$$=\int_{\varepsilon}^{1+\varepsilon}\ln x\,dx$$

we obtain the result when ε goes to 0 and so

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon_n\right)=-1$$

Problem 26. Evaluate the limit

$$\lim_{x\to0^+}\int_x^{2x}\frac{\sin^m t}{t^n}dt \quad (m, n\in N).$$

Solution We use the fact that $\dfrac{\sin t}{t}$ is decreasing in the interval $(0, \pi)$ and $\lim\limits_{t\to0^+}\dfrac{\sin t}{t}=1$.

For all $x \in \left(0, \frac{\pi}{2}\right)$ and $t \in [x, 2x]$ we have

$$\frac{\sin 2x}{2x} < \frac{\sin t}{t} < 1, \text{ thus}$$

$$\left(\frac{\sin 2x}{2x}\right)^m \int_x^{2x} \frac{t^m}{t^n} < \int_x^{2x} \frac{\sin^m t}{t^n} dt < \int_x^{2x} \frac{t^m}{t^n} dt,$$

$$\int_x^{2x} \frac{t^m}{t^n} dt = x^{m-n+1} \int_1^2 u^{m-n} du.$$

The factor $\left(\frac{\sin 2x}{2x}\right)^m$ tends to 1. If $m - n + 1 < 0$, the limit of x^{m-n+1} is infinity ; if $m - n + 1 > 0$ then 0. If $m - n + 1 = 0$ then $x^{m-n+1} \int_1^2 u^{m-n} du = \ln 2$. Hence,

$$\lim_{x \to 0^+} \int_x^{2x} \frac{\sin^m t}{t^n} dt = \begin{cases} 0, & m \ge n \\ \ln 2, & n - m = 1 \\ \infty, & n - m > 1 \end{cases}.$$

Problem 27. Let f be a continuous function on [0, 1] such that for every $x \in [0, 1]$ we have

$$\int_x^1 f(t)dt \ge \frac{1-x^2}{2}. \text{ Show that } \int_0^1 f^2(t)dt \ge \frac{1}{3}.$$

Solution From the inequality

$$0 \le \int_0^1 (f(x) - x)^2 dx = \int_0^1 f^2(x)dx$$

$$-2\int_0^1 xf(x)dx + \int_0^1 x^2 dx$$

We get

$$\int_0^1 f^2(x) \ge 2\int_0^1 xf(x)dx - \int_0^1 x^2 dx = 2\int_0^1 xf(x)dx - \frac{1}{3}.$$

From the hypotheses, we have

$$\int_0^1 \int_x^1 f(t)dt\,dx \ge \int_0^1 \frac{1-x^2}{2} dx \text{ or, } \int_0^1 tf(t)dt \ge \frac{1}{3}.$$

This completes the proof.

Problem 28. Let $F : (1, \infty) \to R$ be the function defined by $F(x) = \int_x^{x^2} \frac{dx}{\ln t}$. Show that F is one-to-one (i.e. injective) and find the range of F.

Solution From the definition we have

$$F'(x) = \frac{x-1}{\ln x}, x > 1.$$

Therefore $F'(x) > 0$ for $x \in (1, \infty)$. Thus F is strictly increasing and hence one-to-one. Since

$$F(x) \ge (x^2 - x).\min\left\{\frac{1}{\ln t} : x \le t \le x^2\right\} = \frac{x^2 - x}{\ln x^2} \to \infty$$

as $x \to \infty$, it follows that the range of F is $(F(1^+), \infty)$. In order to determine $F(1^+)$ we substitute $t = e^v$ in the definition of F and we get

$$F(x) = \int_{\ln x}^{2\ln x} \frac{e^v}{v} dv.$$

Hence, $F(x) < e^{2\ln x} \int_{\ln x}^{2\ln x} \frac{1}{v} dv = x^2 \ln 2$

and similarly $F(x) > x \ln 2$. Thus $F(1^+) = \ln 2$. Hence the range of F(x) is $(\ln 2, \infty)$.

Problem 29. If f is a continuous real function such that $f(x - 1) + f(x + 1) \ge x + f(x)$ for all x, what is the minimum possible value of $\int_1^{2005} f(x)dx$?

Solution Let $g(x) = f(x) - x$. Then $g(x - 1) + x - 1 + g(x + 1) + x + 1 \ge x + g(x) + x$, or, $g(x - 1) + g(x + 1) \ge g(x)$. But now, $g(x + 3) \ge g(x + 2) - g(x + 1) \ge -g(x)$. Therefore

$$\int_a^{a+6} g(x)dx = \int_a^{a+3} g(x)dx + \int_{a+3}^{a+6} g(x)dx$$

$$= \int_a^{a+3} (g(x) + g(x + 3))dx \ge 0$$

It follows that

$$\int_1^{2005} g(x) = \sum_{n=0}^{333} \int_{6n+1}^{6n+7} g(x)dx \ge 0,$$

so that

$$\int_1^{2005} f(x)dx = \int_1^{2005} (g(x) + x)dx \ge \int_1^{2005} x\,dx$$

$$= \left[\frac{x^2}{2}\right]_1^{2005} = \frac{2005^2 - 1}{2} = 2010012$$

The equality holds for $f(x) = x$.

Problem 30. For $\theta \in \left(0, \frac{\pi}{2}\right)$, find the value of

$$\int_0^\theta \ln(1 + \tan\theta \tan x)\, dx.$$

Solution Let $I = \int_0^\theta \ln(1 + \tan\theta \tan x)\, dx$

$$I = \int_0^\theta \ln(1 + \tan\theta \tan(\theta - x))\, dx \qquad \textbf{[P–5]}$$

$$I = \int_0^\theta \ln\left(1 + \frac{\tan\theta(\tan\theta - \tan x)}{1 + \tan\theta \tan x}\right) dx$$

$$= \int_0^{\theta} \ln\left(\frac{1+\tan^2\theta}{1+\tan\theta\tan x}\right) dx$$

$$I = \int_0^{\theta} \ln(1+\tan^2\theta)\,dx - \int_0^{\theta} \ln(1+\tan\theta\tan x)\,dx$$

$I = 2\theta \ln \sec\theta - I$

$2I = 2\theta \ln \sec\theta$

$\Rightarrow\ I = \theta \ln \sec\theta.$

Problem 31. $\displaystyle\int_0^{\pi/2} \ln(a^2\cos^2 x + b^2\sin^2 x)\,dx.$

Solution $I = \displaystyle\int_0^{\pi/2} \ln(a^2\cos^2 x + b^2\sin^2 x)\,dx \ (a>0, b>0)$

$$= \int_0^{\pi/2} \ln\{a^2(\cos^2 x + k^2\sin^2 x)\}\,dx$$

$$= \pi\ln a + \underbrace{\int_0^{\pi/2} \ln(\cos^2 x + k^2\sin^2 x)dx}_{I_1}$$

$$I_1 = \int_0^{\pi/2} \ln(\cos^2 x + k^2\sin^2 x)dx$$

$$\frac{dI_1}{dk} = \int_0^{\pi/2} \frac{2k\sin^2 x}{\cos^2 x + k^2\sin^2 x}dx$$

$$= 2k\int_0^{\pi/2} \frac{\tan^2 x\ \sec^2 x}{(1+k^2\tan^2 x)(1+\tan^2 x)}dx$$

(put $\tan x = t$)

$$= 2k\int_0^{\infty} \frac{t^2}{(1+k^2t^2)(1+t^2)}dx$$

$$= \frac{2k}{k^2-1}\int_0^{\infty} \frac{(1+k^2t^2)-(1+t^2)}{(1+k^2t^2)(1+t^2)}dx$$

$$= \frac{2k}{k^2-1}\left[\int_0^{\infty}\frac{dt}{1+t^2} - \int_0^{\infty}\frac{dt}{1+k^2t^2}\right]$$

$$= \frac{2k}{k^2-1}\left[\frac{\pi}{2} - \frac{1}{k}\tan^{-1} tk\Big]_0^{\infty}\right]$$

$$= \frac{2k}{k^2-1}\left[\frac{\pi}{2} - \frac{\pi}{2k}\right]$$

$$\frac{dI_1}{dk} = \frac{2k}{k^2-1}\frac{\pi}{2}\left(\frac{k-1}{k}\right) = \frac{\pi}{k+1}.$$

$I_1 = \pi\, ln(1+k) + C$

If $k = 1$, $I_1 = 0$

$\Rightarrow\ C = -\pi\, ln 2$

$$I_1 = \pi\ln\left(\frac{1+k}{2}\right)$$

$$= \pi\ln\left(\frac{1+(b/a)}{2}\right) = \pi\ln\left(\frac{a+b}{2a}\right) \quad (\because k = b/a)$$

$$\therefore\ I = \pi\ln a + \pi\ln\left(\frac{a+b}{2a}\right)$$

$$= \pi\ln\left(\frac{a.(a+b)}{2a}\right) = \pi\ln\left(\frac{a+b}{2}\right).$$

Problem 32. Using definite integral as a limit of sum for the integral $\int_0^{1000} x^{10}\,dx$, determine an approximate value for the sum $1^{10} + 2^{10} + 3^{10} + \ldots + 1000^{10}$.

Solution Let $I = \displaystyle\int_0^{1000} x^{10}\,dx = \frac{1000^{11}}{11}$

The graph of $y = x^{10}$ is increasing. Hence, Left end estimation < I < Right end estimation Dividing the interval [0, 1000] into [0, 1], [1, 2], [2, 3], ... , [999, 1000], we get using left end estimation

$$\sum_{r=0}^{999} r^{10} < \int_0^{1000} x^{10}dx$$

$$\Rightarrow\ S - 1000^{10} < \frac{1000^{11}}{11}$$

$$\Rightarrow\ S < 1000^{10} + \frac{1000^{11}}{11}$$

Using right end estimation

$$\sum_{r=1}^{1000} r^{10} > \int_0^{1000} x^{10}dx$$

$$\Rightarrow\ S > \frac{1000^{11}}{11}$$

Hence, $\dfrac{1000^{11}}{11} < S < 1000^{10} + \dfrac{1000^{11}}{11}$.

Problem 33. Find a step function S(x) on [1, 2.5] such that $1/x \le S(x)$ and $\int_1^{2.5} S(x)\ dx < 1$. Also conclude that $e > 2.5$.

Solution Define S(x) by $1 < \frac{5}{4} < \frac{6}{4} < \frac{7}{4} < \frac{8}{4} < \frac{9}{4} < \frac{10}{4}$

$= 2.5$ with the values $1, \frac{4}{5}, \frac{4}{6}, \ldots, \frac{4}{9}$.

Then $1/x \le S(x)$ and

$$\int_a^b S\,dx = \frac{1}{4}\left(1+\frac{4}{5}+\frac{4}{6}+\frac{4}{7}+\frac{4}{8}+\frac{4}{9}\right)$$

$$= \frac{1}{4}+\frac{1}{5}+...+\frac{1}{9} = 2509/25820 < 1.$$

Therefore $\int_1^{2.5} dx/x < 1$, $\ln 2.5 < 1, 2.5 < e$.

Problem 34. If the function $f : [0, 16] \to R$ is differentiable. If $0 < \alpha < 1$ and $1 < \beta < 2$, then prove that

$$\int_0^{16} f(t)\,dt = 4(\alpha^3 f(\alpha^4) + \beta^3 f(\beta^4))$$

Solution $I = \int_0^{16} f(t)\,dt$

Consider $g(x) = \int_0^{x^4} f(t)\,dt \Rightarrow g(0) = 0$

LMVT for g in [0, 1] gives, some $\alpha \in (0, 1)$ such that

$$\frac{g(1)-g(0)}{1-0} = g'(\alpha) \qquad ...(1)$$

Similarly LMVT in [1, 2] gives some $\beta \in (1, 2)$ such

that $\dfrac{g(2)-g(1)}{2-1} = g'(\beta) \qquad ...(2)$

Addind (1) and (2).

$$g'(\alpha) + g'(\beta) = g(2) - \underbrace{g(0)}_{\text{zero}};$$

But $g'(x) = f(x^4)\cdot 4x^3$ $4\left(\alpha^3 f(\alpha^4) + \beta^3 f(\beta^4)\right)$

$$= \int_0^{x^4} f(t)\,dt.$$

Problem 35. Let f be a continuous function on $[a, b]$. If $F(x) = \left(\int_a^x f(t)\,dt - \int_x^b f(t)\,dt\right)(2x - (a+b)$ then prove that there exist some $c \in (a, b)$ such that

$$\int_a^c f(t)\,dt - \int_c^b f(t)\,dt = f(c)(a+b-2c).$$

Solution Given F (x)

$$= \left(\int_a^x f(t)\,dt - \int_x^b f(t)\,dt\right)(2x-(a+b)) \qquad ...(1)$$

Since, f is continuous, F (x) is also continuous.

Also, put $x = a$:

$$F(a) = \left(-\int_a^b f(t)\,dt\right)(a-b) = (b-a)\int_a^b f(t)\,dt$$

and put $x = b$:

$$F(b) = \left(\int_a^b f(t)\,dt\right)(b-a)$$

Hence, $F(a) = F(b)$.

Thus, Rolle's Theorem is applicable to F (x).

$\therefore$ There exists some $c \in (a, b)$ such that $F'(c) = 0$.

Now, $F'(x) = 2\left(\int_a^x f(t)\,dt - \int_x^b f(t)\,dt\right)$

$$= +(2x-(a+b))[f(x)+f(x)] = 0.$$

$\therefore$ $F'(c) = \left(\int_a^c f(t)\,dt - \int_c^b f(t)\,dt\right) = f(c)[(a+b)-2c]$.

Problem 36. Comment upon the nature of roots of the quadratic equation $x^2 + 2x = k + \int_0^1 |t+k|\,dt$ depending on the value of $k \in R$.

Solution $D = 4 + 4\left(k + \int_0^1 |k+t|\,dt\right)$

$$= 4 + 4k + 4\int_0^1 |k+t|\,dt.$$

Let $k \geq 0$,

$$I = \int_0^1 |k+t|\,dt = \int_0^1 (k+t)dt = kt + \frac{t^2}{2}\Bigg|_0^1$$

$$= k + \frac{1}{2}$$

Hence, $D = 4 + 4k + 4\left(k + \frac{1}{2}\right)$

$$= 4 + 8k + 2 = 8k + 6 > 0 = 4\left(2k + \frac{3}{2}\right).$$

Let $k \leq -1$,

$$I = -\int_0^1 (k+t)dt = -\left[kt + \frac{t^2}{2}\right]_0^1 = -\left[k + \frac{1}{2}\right]$$

$$\therefore\ D = 4 + 4k - 4\left(k + \frac{1}{2}\right) = 2 > 0.$$

Let $-1 < k < 0$,

$$I = \int_0^1 |k+t|\,dt$$

Let $k = -y \Rightarrow 0 < y < 1$

$$I = \int_0^1 |t-y|\,dt = \int_0^y (y-t)\,dt + \int_y^1 (t-y)\,dt$$

$$= \left[ty - \frac{t^2}{2}\right]_0^y + \left[\frac{t^2}{2} - yt\right]_y^1$$

$$= \left(y^2 - \frac{y^2}{2}\right) + \left(\frac{1}{2} - y\right) - \left(\frac{y^2}{2} - y^2\right)$$

$$= y^2 - y + \frac{1}{2} = k^2 + k + \frac{1}{2}$$

$$D = 4 + 4k + 4(k^2 + k + \frac{1}{2})$$

$$= 4\left[1 + k + k^2 + k + \frac{1}{2}\right] = 4\left[k^2 + 2k + \frac{3}{2}\right]$$

$$= 4\left[(k+1)^2 + \frac{3}{2}\right] > 0.$$

Hence, $D > 0\ \forall\ k \in R \Rightarrow$ roots are real and distinct.

Problem 37. For $a > 0, b > 0$ verify that

$\int_0^\infty \frac{\ln x\ dx}{ax^2 + bx + a}$ reduces to zero by a substitution $x = 1/t$. Using this or otherwise, evaluate

$$\int_0^\infty \frac{\ln x\ dx}{x^2 + 2x + 4}.dx$$

Solution

$I = \int_0^\infty \frac{\ln x\ dx}{x^2 + 2x + 4} dx$ (put $x = 2t$ to make coefficient of x^2 and constant term same $\Rightarrow dx = 2dt$)

$$= 2\int_0^\infty \frac{\ln 2 + \ln t}{4(t^2 + t + 1)} dt$$

$$= \frac{\ln 2}{2}\underbrace{\int_0^\infty \frac{dt}{t^2 + t + 1}}_{I_1} + \frac{1}{2}\underbrace{\int_0^\infty \frac{\ln t\,dt}{t^2 + t + 1}}_{I_2}$$

$$I_2 = \int_0^\infty \frac{\ln t\,dt}{t^2 + t + 1} \quad \text{put } t = \frac{1}{y} \Rightarrow dt = -\frac{1}{y^2}dy$$

$$= \int_\infty^0 \frac{+\ln y \cdot (+1)}{\left(\frac{1}{y^2} + \frac{1}{y} + 1\right) y^2}\ dy.$$

$$I_2 = \int_\infty^0 \frac{\ln y\,dy}{y^2 + y + 1} = -\int_0^\infty \frac{\ln y\,dy}{y^2 + y + 1} = -I_2$$

$$\text{Now, } I_1 = \int_0^\infty \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{(t + (1/2))^2}{\sqrt{3}}\Bigg|_0^\infty$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{6}\right] = \frac{2\pi}{3\sqrt{3}}.$$

$$\therefore \quad I = \frac{\ln 2}{2} \cdot \frac{2\pi}{3\sqrt{3}} = \frac{2\pi}{3\sqrt{3}} = \frac{\pi \ln 2}{3\sqrt{3}}.$$

Problem 38. Find $\lim\limits_{a\to\infty} \int_0^1 e^{x^2} \cos ax\ dx$.

Solution Letting $u = e^{x^2}$ and $du = \cos ax\ dx$, we have

$$\int_0^1 e^{x^2} \cos ax\ dx = \frac{e^{x^2} \sin ax}{a}\Bigg|_0^1 - \int_0^1 \frac{2xe^{x^2} \sin ax\ dx}{a}$$

$$= \frac{e \sin a}{a} - \frac{2}{a}\int_0^1 xe^{x^2} \sin ax\ dx.$$

Now, for x in [0, 1], $-e \le xe^{x^2} \sin ax \le e$.

Hence, $-e = \int_0^1 -e\ dx \le \int_0^1 xe^{x^2} \sin ax\ dx \le \int_0^1 e\ dx = e$.

Consequently, $\left|\int_0^1 xe^{x^2} \sin ax\ dx\right| \le e$

and $\lim\limits_{a\to\infty}\left(\frac{e \sin a}{a} - \frac{1}{a}\int_0^1 2xe^{x^2} \sin ax\,dx\right) = 0$.

Thus, the limit of $\int_0^1 e^{x^2} \cos ax\ dx$ as $a \to \infty$ is 0.

Things to Remember

1. $\int_a^b f(x)\ dx$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \left(\frac{b-a}{n}\right) r\right)$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a + \frac{(b-a)\,r}{n}\right).$$

2. If $a = 0, b = 1$, then

$$\int_0^1 f(x)\ dx = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right)$$

3. $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{r=\varphi(x)}^{\psi(x)} f\left(\frac{r}{n}\right) = \int_a^b f(x)dx$, where

$$a = \lim_{n\to\infty} \frac{\phi(x)}{n} \text{ and } b = \lim_{n\to\infty} \frac{\psi(x)}{n}$$

4. Rules of definite integration

(i) $\int_a^a f(x)\,dx = 0$

(ii) $\int_b^a f(x)dx = -\int_a^b f(x)dx$

(iii) $\int_a^b f(x)dx = -\int_a^b f(t)dt$

(iv) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

(v) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(vi) $\int_a^b [c_1 f(x)+c_2 g(x)]dx = c_1\int_a^b f(x)dx + c_2\int_a^b g(x)dx$

(vii) $\int_a^c f(x)dx + \int_c^a f(x)dx = \int_a^b f(x)dx$

5. Mean Value Theorem for Integrals : If f is continuous on the interval [a, b], there is atleast one number c between a and b such that

$$\int_a^b f(x)dx = f(c)\,(b-a)$$

Average value of a function : If f is integrable on the interval [a, b], the average value of f on this interval is given by the integral

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx.$$

6 If a function f is integrable on a closed interval [a, b], then the function $g(x) = \int_a^x f(t)dt$ is continuous at any point $x \in [a, b]$.

7. First Fundamental Theorem of Calculus:
If f is continuous on [a, b], then the function g definedby $g(x) = \int_a^x f(t)dt$ $a \le x \le b$ is continuous on [a, b] and differentiable on (a, b), and $g'(x) = f(x)$.

8. The Newton-Leibnitz Formula : If f is continuous on [a, b] and F is any antiderivative of f on the interval [a, b], that is a function F exists such that $F'(x) = f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

9. Improper integrals

(i) $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$

(ii) $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$

(iii) $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx$

(iv) If f(x) has an infinite discontinuity only at the left end point x = a of the interval [a, b],

$$\int_a^b f(x)dx = \lim_{\delta\to0}\int_{a+\delta}^b f(x)dx \text{ where } \delta > 0.$$

(v) If f(x) has an infinite discontinuity only at the right end point x = b of the interval [a, b],

$$\int_a^b f(x)dx = \lim_{\varepsilon\to0}\int_a^{b-\varepsilon} f(x)dx, \text{ where } \varepsilon > 0.$$

(vi) If the function f(x) has an infinite discontinuity at an intermediate point x = c of the interval [a, b] (i.e. a < c < b) then, by definition

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

10. $\int_a^b u(x)v(x)dx$

$$= \left(u(x)\int v(x)dx\right)\Big|_a^b - \int_a^b\left(u'(x)\int v(x)dx\right)dx$$

11. Leibnitz rule for differentiation of integrals : If f is continuous on [a, b], and u(x) and v(x) are differentiable functions of x whose values lie in [a, b], then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$

12. Modified Leibnitz Rule : If $F(x) = \int_{g(x)}^{h(x)} f(x,t)\,dt$, then

$$F'(x) = \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}dt + f(x, h(x)))h'(x) - f(x, g(x)).g'(x)$$

13. Property P–1 : $\int_a^b f(x)dx = \int_a^b f(t)dt$

$$= \int_a^b f(u)du$$

Property P–2 : $\int_b^a f(x)dx = -\int_a^b f(x)dx$

Property P–3 : $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Property P–4 :

(i) $\int_{-a}^a f(x)\,dx = \int_0^a (f(x)+f(-x))\,dx$

(ii) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if f(x) is an even function

(iii) $\int_{-a}^{a} f(x)\,dx = 0$, if f(x) is an odd function

Property P-5

(i) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(ii) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Property P–6

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$= \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \end{cases}$$

Property P–7

If f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in I$

(ii) $\int_a^{a+nT} f(x) = n\int_0^T f(x)\,dx,\ n \in I,\ a \in R$

(iii) $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx,\ m, n \in I$

(iv) $\int_{a+mT}^{b+nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx$

$+\int_a^b f(x)\,dx,\ n \in I,\ a, b \in R$

14. Shift property : $\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx$

15. Expansion–Contraction property

$k\int_{a/k}^{b/k} f(kx)dx = \int_a^b f(x)dx$ for every $k > 0$.

16. Reflection property

$\int_a^b f(x)dx = \int_{-b}^{-a} f(-x)dx$

17. $\int_a^b f(x)dx = (b-a)\int_0^1 f[(b-a)t+a]dt$

18. $\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)dy$

19. Estimation of definite integral :

(i) If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)dx \ge 0$

(ii) If $f(x) \le g(x)$ for for every x in [a, b], then

$\int_a^b f(x)dx \le \int_a^b g(x)dx$

(iii) If $m \le f(x) \le M$ for $a \le x \le b$, then

$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

(iv) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$

(v) Schwartz-Bunyakovsky inequality

$$\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\int_a^b f^2(x)dx\int_a^b g^2(x)dx}$$

(vi) (a) if the function f(x) increases and has a concave up graph in the interval [a, b], then

$$(b-a)\,f(a) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}$$

(b) If the function f(x) increases and has a concave down graph in the interval [a, b], then

$$(b-a)\frac{f(a)+f(b)}{2} < \int_a^b f(x)dx < (b-a)f(b).$$

20. Weighted Mean Value Theorem for integrals : Assume f and g are continuous on [a, b]. If g never changes sign in [a, b] then, for some c in [a, b], we have $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$

21. Generalized Mean Value Theorem for integrals: Assume g is continuous on [a, b], and assume f has a derivative which is continuous and never changes sign in [a, b]. Then, for some c in [a, b], we have

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$$

22. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)...\left(\frac{1}{2}\right).\frac{\pi}{2} & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)...\left(\frac{2}{3}\right).1 & \text{if } n \text{ is odd} \end{cases}$$

23. Wallis formula : $\int_0^{\pi/2} \sin^m x \,.\, \cos^n x\,dx =$

$$\frac{[(m-1)\,(m-3)....1 \text{ or } 2]\,[(n-1)\,(n-3)\,(n-5)....1 \text{ or } 2]}{(m+n)\,(m+n-2)\,(m+n-4)....1 \text{ or } 2}K$$

where K

$= \frac{\pi}{2}$ if both m and n are even $(m, n \in N)$;

= 1 otherwise.

24. $\lim_{n\to\infty}\int_a^b f(x,n)\,dx = \int_a^b \lim_{n\to\infty} f(x,n)\,dx$, if a and b are independent of n.

25. Let a function $f(x, \alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. Then for any $\alpha \in [c, d]$ if

$I(\alpha) = \int_a^b f(x,\alpha)dx$, then

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b \left(\frac{d}{d\alpha} f(x,\alpha)\right) dx.$$

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. $\int \sqrt[3]{x}\sqrt[5]{1+\sqrt[3]{x^4}}\,dx$ is equal to

(A) $\frac{5}{8}(1+x^{4/3})^{6/5}+C$

(B) $\frac{5}{8}(1+x^{2/3})^{6/5}+C$

(C) $\frac{5}{8}(1+x^{6/5})^{4/3}+C$

(D) None of these

2. If m and n are positive integers and

$f(x) = \int_1^x (t-a)^{2n}((t-b)^{2m+1}\,dt,\ a \ne b$, then

(A) $x = b$ is a point of local minimum

(B) $x = b$ is a point of local maximum

(C) $x = a$ is a point of local minimum

(D) $x = a$ is a point of local maximum

3. If $f : [0, \pi] \to R$ is continuous and

$\int_0^\pi f(x)\sin x\,dx = \int_0^\pi f(x)\cos x\,dx = 0$, then the number of roots of f(x) in $(0, \pi)$ is

(A) zero

(B) exactly one

(C) exactly two

(D) atleast two

4. If $I = \int_\alpha^\beta \left[\log\log x + \frac{1}{(\log x)^2}\right]dx$, then I is equal

(A) $\alpha \log\log\alpha - \beta \log\log\beta$

(B) $\frac{1}{\alpha}-\frac{1}{\beta}+\log\log\alpha-\log\log\beta$

(C) $\frac{\beta-\alpha}{\alpha\beta}+\alpha\log\log\alpha-\beta\log\log\beta$

(D) None of these

5. Let $I_n = \int_{1/(n+1)}^{1/n} \frac{\tan^{-1}(nx)}{\sin^{-1}(nx)}dx$, then $\lim_{n\to\infty}(n^2 I_n)$ is equal to

(A) 1 (B) 0 (C) –1 (D) $\frac{1}{2}$

6. Given $\left|\int_b^a f(x)dx\right| = \int_a^b |f(x)|\,dx$ and $f'(x) \ne 0$ at any $x \in (a, b)$; if $g(x) = \int_0^x f(t)dt$, f(x) being continuous and differentiable in (a, b) then $\int_a^b f(x)g(x)dx = 0$ implies

(A) $g(x) = 0$ has atmost one root in (a, b)

(B) $g(x) = 0$ has atleast one root in (a, b)

(C) $g(x) = 0$ has exactly one root in (a, b)

(D) $g(x) = 0\ \forall\ x \in (a,b)$

7. Given

$\int_0^1 (x^5+x^4+x^2)\sqrt{4x^3+5x^2+10}\,dx = \alpha.19\sqrt{19}$

then α is equal to

(A) $\frac{1}{30}$ (B) $\frac{1}{20}$ (C) $\frac{1}{15}$ (D) $\frac{1}{10}$

8. If $\ell_1 = \int_0^{\pi/2}\frac{x}{\sin x}dx$ and $\ell_2 = \int_0^1 \frac{\tan^{-1}x}{x}dx$, then value of $\frac{\ell_1}{\ell_2}$ is

(A) 1 (B) 2 (C) $\frac{1}{2}$ (D) 3

9. The value of the integral $\int_{-10}^{0} \frac{\left|\frac{2[x]}{3x-[x]}\right|}{\left(\frac{2[x]}{3x-[x]}\right)}dx$, where [.] denotes the greatest integer function is,

(A) 0 (B) –10
(C) 10 (D) None of these

10. Let a function h(x) be defined as h(x) = 0, for all $x \neq 0$. Also $\int_{-\infty}^{\infty} h(x).f(x)dx = f(0)$ for every function f(x). Then the value of the definite integral $\int_{-\infty}^{\infty} h'(x).\sin x dx$ is

(A) equal to zero (B) equal to 1
(C) equal to –1 (D) non existent

11. If $f : R \to R$ is a continuous and differentiable function such that $\int_{-1}^{x} f(t)dt + f'''(3)\int_{x}^{0} dt = \int_{1}^{x} t^3 dt - f'(1)\int_{0}^{x} t^2 dt + f''(2)\int_{x}^{3} t\, dt$, then the value of $f'(4)$ is

(A) $48 - 8f'(1) + f''(2)$ (B) $48 - 8f'(1) - f''(2)$
(C) $48 + 8f'(1) + f''(2)$ (D) None of these

12. If $f(y) = e^y$, $g(y) = y$, $y > 0$ and $F(t) = \int_0^1 f(t-y)\, g(y)\, dt$, then

(A) $F(t) = e^t - (1 + t)$ (B) $F(t) = te^t$
(C) $F(t) = te^{-1}$ (D) $F(t) = 1 - e^t(1 + t)$

13. Suppose that F(x) is an antiderivative of $f(x) = \frac{\sin x}{x}$, where $x > 0$, then $\int_1^3 \frac{\sin 2x}{x} dx$ can be expressed as

(A) $F(6) - F(2)$ (B) $\frac{1}{2}(F(6) - F(2))$

(C) $\frac{1}{2}(F(3) - F(1))$ (D) $2(F(6) - F(2))$

14. The value of $\int_{-k}^{k} (-1)^{\mathrm{sgn}[x]} dx$ is

(A) $-2(k-1)$ (B) $-2k$
(C) k (D) none of these

15. If $f(x) = \int_0^x \frac{t \sin t\, dt}{\sqrt{1 + \tan^2 x \sin^2 t}}$ for $0 < x < \frac{\pi}{2}$, then

(A) $f(0^+) = -\pi$

(B) $f\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8}$

(C) f is continuous and differentiable in $\left(0, \frac{\pi}{2}\right)$

(D) f is continuous but not differentiable in $\left(0, \frac{\pi}{2}\right)$

16. If $\int_{-\pi/4}^{3\pi/4} \frac{e^{\pi/4} dx}{(e^x + e^{\pi/4})(\sin x + \cos x)} = k\int_{-\pi/2}^{\pi/2} \sec x\, dx$, then the value of k is

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{1}{2\sqrt{2}}$ (D) $-\frac{1}{\sqrt{2}}$

17. $\int_{-\pi/3}^{0} \left[\cot^{-1}\left(\frac{2}{2\cos x - 1}\right) + \cot^{-1}\left(\cos x - \frac{1}{2}\right)\right] dx$ is equal to

(A) $\frac{\pi^2}{6}$ (B) $\frac{\pi^2}{3}$ (C) $\frac{\pi^2}{8}$ (D) $\frac{3\pi^2}{8}$

18. $\int_0^{\infty} \left(\frac{\pi}{1 + \pi^2 x^2} - \frac{1}{1 + x^2}\right) \ln x\, dx$ is equal to

(A) $-\frac{\pi}{2} \ln \pi$ (B) 0

(C) $\frac{\pi}{2} \ln 2$ (D) None of these

19. $\int_0^4 \frac{(y^2 - 4y + 5)\sin(y-2)dy}{2y^2 - 8y + 1}$ is equal to

(A) 0 (B) 2
(C) –2 (D) None

20. If $f(x) = \int_0^1 \frac{dt}{1 + |x - t|}$, then $f'\left(\frac{1}{2}\right)$ is equal to

(A) 0 (B) $\frac{1}{2}$
(C) 1 (D) None

21. The value of $\int_0^1 \left(\prod_{r=1}^{\pi} (x + r)\right) \left(\sum_{k=1}^{n} \frac{1}{x + k}\right) dx$ equals

(A) n (B) n!
(C) $(n+1)!$ (D) $n \cdot n!$

22. f(x) is a continuous for all real values of x and satisfies $\int_n^{n+1} f(x)\, dx = \frac{n^2}{2}\ \forall\ n \in I$, then $\int_{-3}^{5} f(|x|)\, dx$ is equal to

(A) 19/2 (B) 35/2
(C) 17/2 (D) None

23. Let f be integrable over [0, a] for any real values of a. If $I_1 = \int_0^{\pi/2} \cos\theta\, f(\sin\theta + \cos^2\theta)\, d\theta$ and $I_2 = \int_0^{\pi/2} \sin 2\theta\, f(\sin\theta + \cos^2\theta)\, d\theta$, then

(A) $I_1 = -2I_2$ (B) $I_1 = I_2$
(C) $2I_1 = I_2$ (D) $I_1 = -I_2$

24. If $\alpha, \beta\ (\beta > \alpha)$ are the roots of $g(x) = ax^2 + bx + c = 0$ and f(x) is an even function, then

$$\int_\alpha^\beta \frac{e^{f\left(\frac{g(x)}{x-\alpha}\right)}dx}{e^{f\left(\frac{g(x)}{x-\alpha}\right)} + e^{f\left(\frac{g(x)}{x-\beta}\right)}} \text{ is equal to}$$

(A) $\left|\frac{b}{2a}\right|$ (B) $\frac{\sqrt{b^2-4ac}}{|2a|}$

(C) $\left|\frac{b}{a}\right|$ (D) None of these

25. If the function $f: [0, 8] \to R$ is differentiable, then for $0 < a, b < 2$, $\int_0^8 f(t)\,dt$ is equal to

(A) $3[\alpha^3 f(\alpha^2) + \beta^2 f(\beta^2)]$
(B) $3[\alpha^3 f(A) + \beta^2 f(\beta)]$
(C) $3[\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3)]$
(D) $3[\alpha^2 f(\alpha^2) + \beta^2 f(\beta^2)]$

26. Let g(x) be a continuous and differentiable function such that

$$\int_0^2 \left\{\int_{\sqrt{2}}^{\sqrt{5/2}} [2x^2 - 3]dx\right\}.\, g(x)\, dx = 0, \text{ then } g(x) = 0$$

when $x \in (0, 2)$ has (where [.] denotes the greatest integer function)

(A) exactly one real root
(B) atleast one real root
(C) no real root
(D) none of these

27. The values of x satisfying

$$\int_0^{2[x+14]} \left\{\frac{x}{2}\right\}dx = \int_0^{\{x\}} [x+14]dx, \text{ lie in the interval}$$

(where [.] and {.} denotes the greatest integer and fractional part of x)

(A) [–14, 13) (B) (0, 1)
(C) (–15, –14] (D) none of these

28. The value of $\int_0^2 \frac{dx}{(17+8x-4x^2)(e^{6(1-x)}+1)}$, is equal to

(A) $-\frac{1}{8\sqrt{21}}\ln\left|\frac{2-\sqrt{21}}{2+\sqrt{21}}\right|$

(B) $-\frac{1}{8\sqrt{21}}\ln\left|\frac{2+\sqrt{21}}{\sqrt{21}-2}\right|$

(C) $-\frac{1}{8\sqrt{21}}\left\{\ln\left|\frac{2+\sqrt{21}}{\sqrt{21}-2}\right| - \ln\left|\frac{2+\sqrt{21}}{\sqrt{21}-2}\right|\right\}$

(D) none of these

29. The value of $\int_{-1}^{10} \operatorname{sgn}(x - [x])\, dx$, is equal to (where [.] denotes the greatest integer function)

(A) 9 (B) 10
(C) 11 (D) none of these

30. The function

$$f(x) = \int_0^x \log_{|\sin t|}\left(\sin t + \frac{1}{2}\right)dt,\ x \in (0, 2\pi)$$

strictly increases in the interval

(A) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (B) $\left(\frac{5\pi}{6}, 2\pi\right)$

(C) $\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$ (D) $\left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)$

31. Let $f: (0, \infty) \to R$ and $F(x) = \int_0^x t\,f(t)\,dt$. If $F(x^2) = x^4 + x^5$, then $\sum_{r=1}^{12} f(r^2)$, is equal to

(A) 216 (B) 219
(C) 221 (D) 223

32. Let f(x) be a continuous functions for all x, such that $f(x))^2 = \int_0^x f(t).\frac{2\sec^2 t}{4+\tan t}\,dt$ and $f(0) = 0$, then

(A) $f\left(\frac{\pi}{4}\right) = \ln\frac{5}{4}$ (B) $f\left(\frac{\pi}{4}\right) = \frac{3}{4}$

(C) $f\left(\frac{\pi}{2}\right) = 2$ (D) none of these

33. The value of

$$\lim_{n\to\infty} \frac{1}{n^2}\left\{\sin^3\frac{\pi}{4n} + 2\sin^3\frac{2\pi}{4n} + \ldots + n\sin^3\frac{n\pi}{4n}\right\}, \text{ is}$$

equal to

(A) $\frac{\sqrt{2}}{9\pi^2}(52-15n)$

(B) $\frac{2}{9\pi^2}(52-15n)$

(C) $\frac{1}{9\pi^2}(15n-15)$

(D) none of these

34. The value of $\int_0^{\pi/2} \ln(\sin^2\theta + k^2\cos^2\theta)\, d\theta$, is equal to

(A) $\pi \ln(1+k) - \pi \ln 2$
(B) $\pi \ln 2 - \ln(1+k)$
(C) $\ln(1+k) - \pi \ln 2$
(D) none of these

35. If $I_n = \int_1^e (\ln x)^n\, dx$ then $I_n + nI_{n-1}$ is

(A) 1 (B) e
(C) e + 1 (D) e − 1

36. If f : R → R is continuous and differentiable function such that $\int_{-1}^{x} f(t)dt + f''(3)\int_x^0 dt$ $= \int_1^x t^3\, dt - f'(1)\int_0^x t^2\, dt + f'(2)\int_x^3 t\,.dt$, then the value of $f'(4)$ is

(A) $48 - 8f'(1) + f'(2)$
(B) $48 - 8f'(1) - f''(2)$
(C) $48 + 8f'(1) + f''(2)$
(D) none of these

37. Let $I_1 = \int_0^1 \frac{e^x dx}{1+x}$ and $I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)}$, then $\frac{I_1}{I_2}$ is equal to

(A) 3/e (B) e/3
(C) 3e (D) 1/3e

38. If $g(x) = \int_0^x (|\sin t| + |\cos t|)dt$, then $g\left(x + \frac{\pi n}{2}\right)$ is equal to, where $n \in N$

(A) $g(x) + g(\pi)$ (B) $g(x) + g\left(\frac{n\pi}{2}\right)$

(C) $g(x) + g\left(\frac{\pi}{2}\right)$ (D) none of these

39. Let $f(x) = \text{minimum}\left(|x|, 1-|x|, \frac{1}{4}\right), \forall x \in R$, then the value of $\int_{-1}^{1} f(x)dx$ is equal to

(A) $\frac{1}{32}$ (B) $\frac{3}{8}$

(C) $\frac{4}{32}$ (D) none of these

40. If the function $\int_0^x f(t)\, dt \to 5$ as $|x| \to 1$, then the value of 'a' so that the equation $2x + \int_0^x f(t)dt = a$ has atleast two roots of opposite signs in (–1, 1) is

(A) $a \in (0, 1)$ (B) $a \in (0, 3)$
(C) $a \in (-\infty, 1)$ (D) $a \in (3, \infty)$

41. Let $f(x) = \int_0^x (\sin t - \cot t)(e^t - 2)(t-1)^3(t-2)^5\, dt$ $(0 < x \le 4)$, then the number of points where f(x) assumes local maximum value is

(A) one (B) two
(C) three (D) none of these

42. Let $f: R \to R$ such that $f(x+2y) = f(x) + f(2y) + 4xy$ $\forall x, y \in R$ and $f'(0) = 0$. If $I_1 = \int_0^1 f(x)dx$, $I_2 = \int_{-1}^{0} f(x)dx$ and $I_3 = \int_{1/2}^{2} f(x)dx$, then

(A) $I_1 = I_2 > I_3$ (B) $I_1 > I_2 > I_3$
(C) $I_1 = I_2 < I_3$ (D) $I_1 < I_2 < I_3$

43. The value of $\int_0^1 e^{2x-[2x]} d(x-[x])$. (where [.] denotes the greatest integer function) is

(A) e + 1 (B) e
(C) e − 1 (D) none of these

44. The value of $\int_0^{\pi/4} \ln\cos\left(\frac{\pi}{4} + x\right)^{\cot(\pi/4 - x)} dx$ is

(A) 0 (B) 1
(C) 2 (D) not defined

45. $I_1 = \int_0^{\pi/2} \frac{\cos^2 x}{1+\cos^2 x} dx$, $I_2 = \int_0^{\pi/2} \frac{\sin^2 x}{1+\sin^2 x} dx$, $I_3 = \int_0^{\pi/2} \frac{1+2\cos^2 x.\sin^2 x}{4+2\cos^2 x \sin^2 x} dx$, then

(A) $I_1 = I_2 > I_3$ (B) $I_3 > I_1 = I$
(C) $I_1 = I_2 = I_3$ (D) none of these

46. The value of $\left[\int_0^{9\pi/4} (|\sin x - \cos x|)dx + \int_{-1}^{5} \{-x\}dx\right]$ is (where [.] and {.} represents greatest integer and fractional part).

(A) 3 (B) –4
(C) 2 (D) 4

47. The value of $\int_{-2}^{1}\left[x\left[1+\cos\left(\frac{\pi x}{2}\right)\right]+1\right]dx$, where [.] denotes greatest integer function, is
(A) 1 (B) 1/2
(C) 2 (D) none of these

48. If $m=\int_{-2}^{0}\frac{|\sin x|}{-2\left[\frac{x}{\pi}\right]+\frac{1}{2}}dx$, $n=\int_{0}^{2}\frac{|\sin x|}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}dx$ (where [ ] = G.I.F.), then
(A) m = n (B) m = –n
(C) m = 2n (D) None of these

49. If [.] is G.I.F. then $\int_{4}^{10}\frac{[x^2]dx}{[x^2-28x+196]+[x^2]}$ is
(A) 0 (B) 1
(C) 3 (D) 4

50. If $\int_{a}^{b}\frac{x^n}{x^n+(16-x)}dx = 6$ then
(A) a = 4, b = 12, $n \in R$
(B) a = 14, $n \in R$
(C) a = –4, b = 20, $n \in R$
(D) a = 2, b = 8, $n \in R$

51. The value of
$$\int_{0}^{\pi} e^{\sec x}\sec^3 x(\sin^2 x+\cos x+\sin x+\sin x\cos x)dx$$
equals
(A) 0 (B) e + 1/e
(C) –e – 1/e (D) e

52. If $I_n=\int_{0}^{\pi/2}x^n\cos x\,dx$, then the value of $2^8(I_8+56I_6)$ is
(A) π^8 (B) 8^π
(C) 5^π (D) π^5

53. Given $I_m=\int_{1}^{e}((\ln x)^m\,dx$

If $\frac{I_m}{K}+\frac{I_{m-2}}{L}=e$, then the value of K and L are
(A) 1 – m, 1/m
(B) (1/1 – m), m
(C) $\frac{1}{1-m},\frac{m(m-2)}{m-1}$
(D) $\frac{m}{m-1}, m-2$

54. Let f be integrable
$$I_1=\int_{0}^{\pi/2}\cos\theta f(\sin\theta+\cos^2\theta)d\theta \text{ and}$$
$$I_2=\int_{0}^{\pi/2}\sin 2\theta.\ f(\sin\theta+\cos^2\theta)d\theta\text{, then}$$
(A) $I_1 = I_2$ (B) $I_1 + I_2 = 0$
(C) $I_1 = 2I_2$ (D) none of these

55. If f''(x) = k in [0, a], then
$$\int_{0}^{a}f(x)dx-\left\{xf(x)-\frac{x^2}{2!}f'(x)+\frac{x^3}{3}f''(x)\right\}_0^a \text{ is}$$
(A) $-\frac{ka^4}{24}$ (B) ka^2
(C) $a^3/3$ (D) none of these

56. If x satisfies the equation
$$x^2\left(\int_{0}^{1}\frac{dt}{t^2+2t\cos\alpha+1}\right)-x\left(\int_{-3}^{3}\frac{t^2\sin 2t}{t^2+1}\right)-2=0$$
$(0<\alpha<\pi)$, then x is
(A) $2\sqrt{\frac{\sin\alpha}{\alpha}}$ (B) $4\sqrt{\frac{\sin\alpha}{\alpha}}$
(C) $2\sqrt{\frac{\cos\alpha}{\alpha}}$ (D) none of these

57. $\int_{a}^{b}\frac{\sin(x-a)-\cos(x-a)}{\sin(b-x)-\cos(b-x)}dx$
$$=\ell\int_{a}^{b}\frac{\sin(b-x)-\cos(b-x)}{\sin(x-a)-\cos(x-a)}dx.$$
Then the value of ℓ is
(A) 1 (B) 1/2
(C) 2 (D) a function of a and b

58. If $F(x)=\int_{0}^{x^3}f(x)g(t)dt$, $f(x)=\int_{1}^{x}\frac{dt}{t}$ and $g(t)=\frac{1}{1+t^2+\sin^2 t}$, and F'(x) is
$$\frac{3x^2\ln x}{1+x^6+\sin^2 x^3}+\varphi(x)\int_{0}^{x^3}g(t)dt \text{ then } \phi(x) \text{ is}$$
(A) x (B) x^2
(C) $1/x^2$ (D) 1/x

59. $f(x)=\lim_{n\to\infty}\left[\frac{1}{1.3}+\frac{2}{1.3.5}+...+\frac{n}{1.3.5..(2n+1)}\right]$

then $\int_0^1 [f(x)]d(x-[x])$ ([.] is G.I.F.) is

(A) 0 (B) 1
(C) 2 (D) None

60. If $f(x)=\int_0^x \frac{1}{(f(t))^2}dt$ and $\int_0^2 \frac{1}{(f(t))^2}dt=\sqrt[3]{6}$, then f(9) is equal to

(A) 2 (B) 0
(C) 3 (D) None

61. If $f_n(x)=\frac{n+1}{(n-1)!}.x^n$, then the value of

$\int_0^1\left(\sum_{n=1}^{\infty} f_n(x)\right)dx$ is

(A) 2 (B) 3
(C) 4 (D) None

62. If $\int_{a-n}^{n} e^{x(a-n)}dx=\lambda$, then the value of

$\int_{a-n}^{n} xe^{x(a-n)}dx$, $a\neq 2n$, is

(A) $\frac{a\lambda}{2}$ (B) $a\lambda$
(C) $2a\lambda$ (D) None of these

63. For $0<x<\frac{\pi}{2}$, $\int_{1/\sqrt{2}}^{1/2}\cos x.d(\cos x)$ is

(A) 1 (B) 2
(C) 3 (D) None of these

64. If f(x + y) = f(x) + f(y) for all x and y and

$a=\int_1^3 (x-1)^2 f(x-1)dx$ and

$b=\int_{-3}^1 (x+1)^2 f(x+1)dx$, then a is equal to

(A) b (B) 2b
(C) 3b (D) None of these

65. The value of $\int_{\pi/4}^{\pi/2}\sin 2x\tan^{-1}(\sin x)dx$ is equal to

(A) $\frac{\pi+2}{2}$ (B) $\frac{2-\pi}{2}$

(C) $\frac{\pi-2}{2}$ (D) $\frac{-\pi-2}{2}$

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

66. If $f(x)=[x]+\left[x+\frac{1}{3}\right]+\left[x+\frac{2}{3}\right]$, then

([.] denotes the greatest integer function)

(A) f(x) is discontinuous at x = 1, 10, 15

(B) f(x) is continuous at x = n/3, where n is any integer

(C) $\int_0^{2/3} f(x)\,dx=\frac{1}{3}$

(D) $\lim_{x\to\frac{2}{3}} f(x)=2$

67. Consider a real valued continuous function f(x) defined on the interval [a, b]. Which of the following statements does not hold(s) good?

(A) If $f(x)\geq 0$ on [a, b] then

$\int_a^b f(x)dx\leq\int_a^b f^2(x)dx$.

(B) If f(x) is increasing on [a, b], then $f^2(x)$ is increasing on [a, b].

(C) If f(x) is increasing on [a, b], then $f(x)\geq 0$ on (a, b).

(D) If f(x) attains a minimum at x = c where $a<c<b$, then $f'(c)=0$.

68. Let f(x) be a non constant twice derivable function defined on R such that $f(2+x)=f(2-x)$ and $f'\left(\frac{1}{2}\right)=0=f'(1)$. Then which of the following alternative(s) is/are correct?

(A) $f(-4)=f(8)$.

(B) Minimum number of roots of the equation $f''(x)=0$ in (0, 4) are 4.

(C) $\int_{-\pi/4}^{\pi/4} f(2+x)\sin x\,dx=0$.

(D) $\int_0^2 f(t)5^{\cos\pi t}dt=\int_2^4 f(4-t)5^{\cos\pi t}dt$.

69. Let [.] represent the greatest integer function.

$\int_0^x [t]dt=\int_0^{[x]} t\,dt$, if

(A) x = 5/2 (B) x = 6

(C) $x=-\frac{7}{2}$ (D) $x=\frac{101}{2}$

70. Let $f(x)=\int_{-2}^x |t+1|\,dt$, then

(A) f(x) is continuous in [–1, 1]

(B) f(x) is differentiable in [–1, 1]
(C) f'(x) is continuous in [–1, 1]
(D) f'(x) is differentiable in [–1, 1]

71. Which of the following function(s) is/are even?

(A) $f(x)=\int_0^x \ln\left(t+\sqrt{1+t^2}\right)dt$

(B) $g(x)=\int_0^x \frac{(2^t+1)t}{2^t-1}dt$

(C) $h(x)=\int_0^x \left(\sqrt{1+t+t^2}-\sqrt{1-t+t^2}\right)dt$

(D) $l(x)=\int_0^x \ln\left(\frac{1-t}{1+t}\right)dt$

72. Let $f:[1,\infty)\to R$ and $f(x)=x\int_1^x \frac{e^t}{t}dt-e^x$, then

(A) f(x) is an increasing function
(B) $\lim_{x\to\infty} f(x)\to\infty$
(C) f'(x) has a maxima at $x=e$
(D) f(x) is a decreasing function

73. If x satisfies the equation

$$x^2\left(\int_0^1 \frac{dt}{t^2+2t\cos\alpha+1}\right)-x\left(\int_{-3}^3 \frac{t^2\sin 2t}{t^2+1}dt\right)-2=0$$

$(0<\alpha<\pi)$, thet the value of x is

(A) $\sqrt[2]{\left(\frac{\sin\alpha}{\alpha}\right)}$ (B) $-\sqrt[2]{\left(\frac{\sin\alpha}{\alpha}\right)}$

(C) $\sqrt{\left(\frac{\sin\alpha}{\alpha}\right)}$ (D) none of these

74. If $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{3k}{n}\right)^2+2\right)\frac{3}{n}=\int_0^b f(x)dx$ then

(A) $b=1$ (B) $f(x)=(9x^2+2)$

(C) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{3k}{n}\right)^2+2\right)\frac{3}{n}=15$

(D) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{3k}{n}\right)^2+2\right)\frac{3}{n}=5$

75. $\lim_{x\to\infty} {}^nC_x\left(\frac{m}{n}\right)^x\left(1-\frac{m}{n}\right)\left(1-\frac{m}{n}\right)^{n-x}$ equals to

(A) $\frac{m^x}{x!}.e^{-m}$ (B) $\frac{m^x}{x!}.e^{m}$

(C) $.^0$ (D) $\frac{m^{x+1}}{me^m x!}$

76. $\lim_{n\to\infty}\left(\frac{n!}{(2n)^n}\right)^{1/n}$; $n\in N$ is equals to $(a^{-1}e^b)$, then

(A) $a=2$ (B) $a=1$
(C) $b=-1$ (D) $b=1$

77. Let f(x) be a periodic function with period 3 and $f\left(-\frac{2}{3}\right)=7$ and $g(x)=\int_0^x f(t+n)dt$ where $n=3k$, $k\in N$, then

(A) $g'(-2/3)=7$ (B) $g'(-2/3)=-7$

(C) $g'(7/3)=7$ (D) $g'\left(\frac{16}{3}\right)=7$

78. The value of the definite integral

$\int_0^{2\pi} x\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$, is

(A) $\pi\int_0^{2\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$

(B) $2\pi\int_0^{\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$

(C) zero

(D) $2\pi\int_0^{\pi}\ln\left(\frac{3-\cos x}{3+\cos x}\right)dx$

79. The function f(x) is defined for $x\ge 0$ and has its inverse g(x) which is differentiable. If f(x) satisfies

$\int_0^{g(x)} f(t)\,dt = x^2$ and $g(0)=0$ then

(A) f(x) is an odd linear polynomial
(B) f(x) is some quadratic polynomial
(C) $f(2)=1$
(D) $g(2)=4$

80. Let f(x) be a quadratic function with positive integral coefficients such that for every $\alpha, \beta \in R$, $\beta > \alpha$, $\int_{\alpha}^{\beta} f(x)\,dx > 0$. Let $g(t) = f''(t).\,f(t)$ and $g(0) = 12$, then

(A) 16 such quadratic functions are possible
(B) $f(x) = 0$ has either no real or distinct roots
(C) Minimum value of f(1) is 6
(D) Maximum value of f(1) is 11.

81. $\lim\limits_{x\to0^+} \int_x^{2x} \frac{\sin^m t}{t^n}$ (m, n, $\in$ N) equals

(A) 0 if $m \geq n$ (B) $\ln 2$ if $n - m = 1$
(C) ∞ if $n - m > 1$ (D) None of these

82. If $f(x) = \int_0^{\pi/2} \frac{\ln(1 + x\sin^2\theta)}{\sin^2\theta}\,d\theta$, $x \geq 0$ then

(A) $f(t) = \pi(\sqrt{t+1} - 1)$
(B) $f'(t) = \frac{\pi}{2\sqrt{t+1}}$
(C) f(x) cannot be determined
(D) None of these

83. Let $I_n = \int_0^{\sqrt{3}} \frac{dx}{1+x^n}$ (n = 1, 2, 3 ...) and $\lim\limits_{n\to\infty} I_n = I_0$ (say), then which of the following statement(s) is/are correct? (Given : e=2.71828)

(A) $I_1 > I_0$ (B) $I_2 < I_0$
(C) $I_0 + I_1 + I_2 > 3$ (D) $I_0 + I_1 > 2$

84. Let $J = \int_{-1}^{2} \left(\cot^{-1}\frac{1}{x} + \cot^{-1}x\right) dx$ and

$K = \int_{-2\pi}^{7\pi} \frac{\sin x}{|\sin x|}\,dx.$

Then which of the following alternative(s) is/are correct ?

(A) $2J + 3K = 8\pi$ (B) $4J^2 + K^2 = 26\pi^2$
(C) $2J - K = 3\pi$ (D) $\frac{J}{K} = \frac{2}{5}$

85. If $f : R \to R$ be a continuous function such that $f(x) = \int_1^x 2t\,f(t)dt$, then which of the following does not hold(s) good?

(A) $f(\pi) = e^{\pi^2}$ (B) $f(1) = e$
(C) $f(0) = 1$ (D) $f(2) = 2$

Comprehension - 1

A continuous function f satisfies f(2x) = 3f(x) for all x. Moreover $\int_0^1 f(x)dx = 1$ and let $\int_1^2 f(x)dx = S$.

86. $\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx$ is equal to

(A) $3^n S$ (B) $2^n S$
(C) $6^n S$ (D) None of these

87. $\int_{\frac{1}{8}}^{1} f(x)dx$ is equal to

(A) $\frac{13}{27}S$ (B) $\frac{7}{8}S$
(C) $\frac{43}{216}S$ (D) None of these

88. The value of S is

(A) 5 (B) 2
(C) 1 (D) None of these

Comprehension - 2

Let f be a continuous real function such that $f(x-1) + f(x+1) \geq x + f(x)$ for all x.
Let $g(x) = f(x) - x$. Then $g(x-1)+x-1+g(x+1)+(x+1) \geq x+g(x)+x$
$\Rightarrow g(x+1) + g(x-1) \geq g(x)$.

89. g(x) satisfies the inequality

(A) $g(x+2) \geq g(x+3) + g(x+1)$
(B) $g(x+1) \geq g(x+2) - g(x)$
(C) $g(x+3) + g(x) \geq 0$
(D) None of these

90. $\int_a^{a+6} g(x)\,dx$ is equal to

(A) $\int_a^{a+3} (g(x) + g(x+3))dx$
(B) $\int_a^{a+3} (g(x) - g(x+3))dx$
(C) $\int_a^{a+3} g(x+3)dx$
(D) None of these

91. The minimum value of $\int_1^{25} f(x)dx$ is

(A) 12 (B) 312
(C) 1248 (D) None of these

Comprehension - 3

Let $f : R \to R$ be a differentiable function such that $f(x) = x^2 + \int_0^x e^{-t} f(x-t)dt$.

92. f(x) increases for
(A) $x>1$ (B) $x<-2$
(C) $x>2$ (D) None of these

93. $y=f(x)$ is
(A) Injective but not surjective
(B) Surjective but not injective
(C) Bijective
(D) Neither injective nor surjective

94. The value of $\int_0^1 f(x)\,dx$ is
(A) 1/4 (B) – 1/12
(C) 5/12 (D) 12/7

Comprehension - 4

Let $f: R \to R$ be a continuous function with $\int_0^1 f(x)\, f'(x)\,dx = 0$ and $\int_0^1 f(x)^2\, f'(x)\,dx = 18.$

95. The value of $f(1)^3 - f(0)^3$ is
(A) 0 (B) 6
(C) 18 (D) 54

96. The value of f(0) is
(A) 2 (B) 3
(C) –3 (D) 0

97. $\int_0^1 f(x)^4\, f'(x)\,dx$ is equal to
(A) $\frac{486}{5}$ (B) $\frac{486}{25}$
(C) $\frac{243}{5}$ (D) $\frac{243}{35}$

Comprehension - 5

Let C be a curve defined by $y = e^{a+bx^2}$. The curve C passes through the point (P(1, 1) and the slope of the tangent at P is (–2). Also C_1 and C_2 are the circles $(x-a)^2 + (y-b)^2 = 3$, $(x-6)^2 + (y-11)^2 = 27$ respectively.

98. The value of $a^2 + b^2$ is equal to
(A) 2 (B) 8
(C) 18 (D) 32

99. The length of the shortest line segment AB which is tangent to C_1 at A and to C_2 at B is
(A) $9\sqrt{3}$ (B) $10\sqrt{3}$
(C) 11 (D) 12

100. If f is a real valued derivable function satisfying $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ with $f'(1)=2$. Then the value of the integral $\int_b^a f(x)\, d(\ell n\, x)$ is equal to
(A) 0 (B) $\frac{e^2-e^{-2}}{2}$
(C) $\frac{e^{-2}-e^2}{2}$ (D) 2

Assertion (a) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

101. Assertion (a) : $f: [0, \pi] \to R$ is continuous and $\int_0^\pi f(x)\sin x\,dx = 0$ then equation $f(x) = 0$ has at least one root between $(0, \pi)$.
Reason (R) : f(x) is continuous and $f(\alpha)f(\beta) < 0$, $\alpha < \beta$ then $f(x) = 0$ has atleast one root between (α, β).

102. Assertion (a) : $I = \int_0^\pi \frac{dx}{\cos^2 x + 3\sin^2 x}$ can be evaluated by substitution $\cot x = t$.
Reason (R) : by substitution $\tan x = t$,

$$I = \int_0^\pi \frac{dx}{\cos^2 x + 3\sin^2 x} \Rightarrow I = \int \frac{\sec^2 x\,dx}{1+3\tan^2 x}$$

$$\Rightarrow\quad I = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^\pi = 0$$

103. Assertion (a) : $\int_0^1 \left[\frac{\tan x}{x}\right] dx = 1$ (where [.] denotes greatest integer function)
Reason (R) : For $x > 0$ and wherever tan x is defined, $\tan x > x$.

104. Let f(x) is continuous and positive for $x \in [a,b]$, g(x) is continuous for $x \in [a,b]$ and $\int_a^b |g(x)|\,dx > \int_a^b g(x)dx$, then

Assertion (a) : The value of $\int_a^b f(x)\,g(x)\,dx$ can be zero.
Reason (R) : Equation $g(x) = 0$ has atleast one root in $x \in (a, b)$.

105. Assertion (a) : The value of $\int_{-4}^{-5} \sin(x^2-3)dx + \int_{-2}^{-1} \sin(x^2+12x+33)$ zero.

Reason (R): $\int_{-a}^{a} f(x)dx = 0$ if f(x) is an odd function.

106. Assertion (a) : The value of $\int_{-2}^{\frac{-2}{\sqrt{3}}} \sec^{-1} x\, dx$ is $\frac{4\pi}{3} - \frac{5\pi}{3\sqrt{3}} - \ln|2\sqrt{3} - 3|$

Reason (R) : The function $F(x) = x \sec^{-1} x - \ln|x + \sqrt{x^2 - 1}| + c$ is the indefinite integral of $\sec^{-1} x$ for all $|x| > 1$ and $\int_{-2}^{\frac{-2}{\sqrt{3}}} \sec^{-1} x dx = F\left(\frac{-2}{\sqrt{3}}\right) - F(-2)$.

107. Assertion (a) : If $\frac{d^2 f}{dx^2} = a(x)$ then $\int_0^1 \left(\int_0^x a(t)dt\right) dx = f(1) - f(0)$

Reason (R) : The fundamental theorem of integral calculus that $\int_a^b f'(x)dx = f(b) - f(a)$.

108. Assertion (a) : $\lim_{x \to 1^-} (1 - t) \sum_{n=1}^{\infty} \frac{t^n}{1 + t^n} = \ln 2$

Reason (R) : $\lim_{x \to 1^-} (-\ln t) \sum_{n=1}^{\infty} \frac{1}{1 + e^{-n \ln t}}$

$= \int_0^{\infty} \frac{dx}{1 + e^x} = -\ln(e^{-x} + 1)|_0^{\infty} = \ln 2$

109. Assertion (a) : Let $f : [0, 1] \to [0, \infty)$ be a continuously differentiable function with $M = \max_{0 \le x \le 1} |f'(x)|$ then

$$\left|\int_0^1 f^3(x)\,dx - f^2(0)\int_0^1 f(x)\,dx\right| \le M\left(\int_0^1 f(x)dx\right)^2$$

Reason (R) : Since $-M f(x) \le f(x) . f'(x) \le M f(x)$, $x \in [0, 1]$ by integration and then multiplying by f(x) we get $-Mf(x) \int_0^x f(t)dt \le \frac{1}{2} f^3(x) - \frac{1}{2} f^2(0)$ $f(x) \le M f(x) \int_0^x f(t)dt$. Integrating the last inequality on [0, 1], it follows that $-M\left(\int_0^1 f(x)dx\right)^2 \le \int_0^1 f^3(x)\,dx - f^2(0)\int_0^1 f(x)dx$ $\le M\left(\int_0^1 f(x)dx\right)^2$

110. Assertion (a) : $\int_{-1}^{1} \frac{\sin x - x^2}{3 - |x|} dx$ is same as $\int_0^1 \frac{-2x^2}{3 - |x|} dx$

Reason (R) : Since $\frac{\sin x}{3 - |x|}$ is an odd function so that $\int_{-1}^{1} \frac{\sin x - x^2}{3 - |x|} = \int_0^1 \frac{-2x^2}{3 - |x|} dx$

MATCH THE COLUMNS FOR JEE ADVANCED

111.

	Column-I	Column-II
(A)	The function $f(x) = \frac{e^{x\cos x} - 1 - x}{\sin x^2}$ is not defined at x = 0. The value of $f(0)$ so that f is continuous at x = 0 is	(P) – 1
(B)	The value of the definite integral $\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$ equals a + b $\ln$ 2 where a, b ∈ N then (a + b) equals	(Q) 0
(C)	Given $e^n \int_0^n \frac{\sec^2\theta - \tan\theta}{e^\theta} d\theta = 1$ then the value of tan (n) is equal to	(R) 1/2
(D)	Let $a_n = \int_{1/n+1}^{1/n} \tan^{-1}(nx)dx$ and $b_n = \int_{1/n+1}^{1/n} \sin^{-1}(nx)dx$ then $\lim_{n\to\infty} \frac{a_n}{b_n}$ has the value equal to	(S) 1

112. **Column-I** **Column-II**

(A) $\lim\limits_{n\to\infty}\left[\int_0^2 \frac{\left(1+\frac{t}{n+1}\right)^n}{n+1}dt\right]$ is equal to (P) $e-\frac{1}{2}e^2-\frac{3}{2}$

(B) Let f(x) be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and g be the function satisfying $f(x) + g(x) = x^2$

Then the value of the integral $\int_0^1 f(x)\,g(x)\,dx$ is (Q) e^2

(C) $\int_0^1 e^{e^x}(1+xe^x)$ is equal to (R) $e^2 - 1$

(D) $\lim\limits_{k\to 0}\frac{1}{k}\int_0^k (1+\sin 2x)^{1/x}dx$ is equal to (S) e^e

113. **Column-I** **Column-II**

(A) If f(x) is an integrable function for $x \in \left[\frac{\pi}{6},\frac{\pi}{3}\right]$ and (P) $\frac{3}{2}$

$I_1 = \int_{\pi/6}^{\pi/3} \sec^2\theta\, f(2\sin 2\theta)d\theta$ and $I_2 = \int_{\pi/6}^{\pi/3} \mathrm{cosec}^2\theta$ $f(2\sin 2\theta)d\theta$ then I_1/I_2

(B) If $f(x + 1) = f(3 + x)$ for $\forall$ x, and the value of $\int_a^{a+b} f(x)\,dx$ is indepedent of a then the value of b can be (Q) 1

(C) The value $\int_1^4 \frac{\tan^{-1}[x^2]}{\tan^{-1}[x^2]+\tan^{-1}[25+x^2-10x]}$ (where [·] denotes the greatest integer function) is (R) 2

(D) If $I = \int_0^2 \sqrt{x+\sqrt{x+\sqrt{x+\ldots\infty}}}\,dx$ (where $x > 0$) then [I] is equal to (where [·] denotes the greatest integer function) (S) 4

114. **Column-I** **Column-II**

(A) If f(x) and f'(x) are continuous functions on (a, b) $\lim\limits_{x\to a^+} f(x) \to \infty$, $\lim\limits_{x\to b^-} f(x) = -\infty$ and $f'(x) + f^2(x) \ge -1$ $\forall\, x \in (a, b)$, then minimum value of $\left(\frac{b-a}{\pi}\right)$ (P) –2

(B) If f(x) is differentiable function such that $f(x) + f'(x) \le 1$, $\forall x \in R$ and $f(0) = 0$, then the largest possible value of $f(1)+\frac{1}{e}$ is (Q) –1

(C) $f(x) = \int_{-1}^{x} e^{t^2}dt$, $h(x) = f(1 + g(x))$, $g(x) \le 0$, $\forall\, x > 0$, $h'(1) = e$ and $g'(1) = 1$, then the possible value g(1) can take is (R) 0

(D) If $g : [0, \infty)$ and is defined as $g(x) = \int_{-1}^{x} e^{t^2}dt$ and $f(x) \le g(x)$, $\forall\, x \ge 0$. Then the value of g(g(x)) is (S) 1

115.

Column I	Column II
(A) $\lim_{x\to\infty} \frac{\ln x}{x}\int_{e^3}^{x}\frac{dt}{\ln t}$ is	(P) 0
(B) $\lim_{x\to\infty} (e^{\sqrt{x^4+1}} - e^{x^2+1})$ is	(Q) $\frac{1}{2}$
(C) $\lim_{n\to\infty} (-1)^n \sin(\pi\sqrt{n^2+0.5n+1})\sin\frac{(n+1)\pi}{4n}$ is where $n \in N$	(R) 1
(D) The value of the integral $\int_0^1 \frac{\tan^{-1}\left(\frac{x}{x+1}\right)}{\tan^{-1}\left(\frac{1+2x-2x^2}{2}\right)}dx$ is	(S) Non existent

Review Exercises for JEE Advanced

1. Prove that

(i)

$$\lim_{n\to\infty} \frac{1}{n^3}+\frac{4}{8+n^3}+.....+\frac{r^2}{r^3+n^3}+......+\frac{1}{2n}=\frac{1}{3}\ln 2$$

(ii) $$\lim_{n\to\infty} \frac{n^{1/2}}{n^{3/2}}+\frac{n^{1/2}}{(n+3)^{3/2}}+\frac{n^{1/2}}{(n+6)^{3/2}}+......+\frac{n^{1/2}}{\{n+3(n-1)\}^{3/2}}=\frac{1}{3}$$

2. Verify the following and give a geometric interpretation using the concept of area bounded by graphs.

(i) $$\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1 (x^k - x^{k+1})dx = \frac{1}{2}$$

(ii) $$\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^1 (x^{1/k} - x^{1/(k+1)})dx = -\frac{1}{2}$$

3. When C and S are defined by

$$C=\int_0^{\pi}\frac{\cos^2\theta d\theta}{a^2\cos^2\theta+b^2\sin^2\theta},$$

$$S=\int_0^{\pi}\frac{\sin^2\theta d\theta}{a^2\cos^2\theta+b^2\sin^2\theta}$$

prove that $C=\frac{\pi}{a(a+b)}$, $S=\frac{\pi}{b(a+b)}$ $(a, b>0)$.

4. Prove that

(i) $$\int_0^{\pi/2}\frac{\sin^2\theta\cos\theta}{(\sin\theta+\cos\theta)^2}d\theta = \int_0^{\pi/2}\frac{\cos^2\theta\sin\theta}{(\sin\theta+\cos\theta)^2}d\theta$$

(ii) $$\int_0^{\pi}\frac{x\sin^3 x}{1+\cos^2 x}dx = \frac{1}{2}\pi^2 - \pi$$

(iii) $$\int_b^a \frac{x^{n-1}[(n-2)x^2+(n-1)(a+b)x+nab]}{(x+a)^2(x+b)^2}dx = \frac{b^{n-1}-a^{n-1}}{2(a+b)}$$

(iv) $$\int_0^{\pi}(\pi x - x^2)^3\cos 2x\,dx = \frac{3}{4}\pi(\pi^2-15)$$

5. Prove that $\int_0^{\pi} f(\sec\frac{x}{2}+\tan\frac{x}{2})\frac{dx}{\sqrt{\sin x}}$

$$= \int_0^{\pi} f(\operatorname{cosec} x)\frac{dx}{\sqrt{\sin x}}.$$

6. Prove, by the substitution $\sqrt{(1+x^4)}=(1+x^2)\cos\phi$, or otherwise, that

$$\int_0^1 \frac{1-x^2}{1+x^2}\frac{dx}{\sqrt{(1+x^4)}}=\frac{\pi}{4\sqrt{2}}.$$

7. (i) Prove that $\int_0^{\pi}\frac{\cos\theta d\theta}{a^2\cos^2\theta+b^2\sin^2\theta}=0$

(ii) Prove that, when a and b are positive,

$\int_0^{\pi}\frac{\sin\theta d\theta}{a^2\cos^2\theta+b^2\sin^2\theta}$ is equal to

$$\frac{2}{b\sqrt{(a^2-b^2)}}\tan^{-1}\frac{\sqrt{(a^2-b^2)}}{b},$$

$$\frac{1}{b\sqrt{(b^2-a^2)}}\ \ell n\frac{b+\sqrt{(b^2-a^2)}}{b-\sqrt{(b^2-a^2)}}$$
according as $a > b$, $a < b$.
Prove that, when $b = a\lambda$, the integral is a continuous function of λ at $\lambda = 1$.

8. Prove that $\int_0^a \frac{dx}{a^2+x^2} = \frac{\pi}{4a}$ and by differentiation w.r.t. a, prove also that $\int_0^a \frac{dx}{(a^2+x^2)^2} = \frac{\pi+2}{8a^3}$.

9. Show that, when $a > 0$,

(i) $\int_0^\infty \frac{\ln x}{a^2+x^2}dx = \frac{\pi}{2}\frac{\ln a}{a}$

(ii) $\int_0^\infty \frac{\ln x}{(a^2+x^2)^3}dx = \frac{\pi}{16a^5}(3\ln a - 4)$

10. Calculate $\int_a^b x dx$, where $0 < a < b$, by dividing (a,b) into n parts by the points of division a, ar, ar^2, ..., ar^{n-1}, ar^n, where $r^n = b/a$. Apply the same method to find the more general integral $\int_a^b x^m dx$.

11. Prove that

(i) $\int_0^\pi \frac{1+2\cos x}{(2+\cos x)^2} = \frac{1}{2}$

(ii) $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{1}{4}\pi$

(iii) $\int_0^1 \frac{\sqrt{(1-x^2)}}{1-x^3\sin^2\alpha}dx = \frac{\pi}{4\cos^2\frac{1}{2}\alpha}$

(iv) $\int_0^1 \frac{16(x-1)}{x^4-2x^3+4x-4}dx = \pi$

12. Find a function g(x) continuous in $(0, \infty)$ and positive in $(0, \infty)$ satisfying $g(0) = 0$ and

$$\int_0^x g^2(t)dt = \frac{2}{x}\left(\int_0^x g(t)dt\right)^2.$$

13. Consider the function

$$f(x) = \begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases},\ \text{where}$$

[x] denotes the greatest integer function and I is the set of integers. If $g(x) = \max\{x^2, f(x)|x|\}$, $-10 \le x \le 10$, then find the value of $\int_{-2}^2 g(x)dx$.

14. Let $f: R \to R$ be defined by $f(x) = \int_1^3 \frac{1}{|t-x|+1}dt$, then find the value of $\int_1^3 f(t)dt$.

15. Let $f: R^+ \to R$ be a differentiable function with $f(1) = 3$ and satisfying

$$\int_1^{xy} f(t)dt = y\int_1^x f(t)dt + x\int_1^y f(t)dt,\ \forall x, y \in R^+.$$

Find f(x).

16. Given a positive integer p. A step function s is defined on the interval [0, p] as follows : $s(x) = (-1)^n n$ if x lies in the interval $n \le x < n+1$, where $n = 0, 1, 2, ..., p-1$; $s(p) = 0$.

Let $f(p) = \int_0^p s(x)dx$.

(A) Calculate f(3), f(4), and f(f(3)).
(B) For what value (or values) of p is $|f(p)| = 7$?

17. Prove that $\frac{1}{b-a}\int_a^b f(x)dx = \frac{t}{c-a}\int_a^c f(x)dx + \frac{1-t}{b-c}\int_c^b f(x)dx$, $a < c < b$, $0 < t < 1$.

18. Given two functions f and g, integrable on every interval and having the following properties :
f is odd, g is even, $f(5) = 7$, $f(0) = 0$, $g(x) = f(x+5)$,

$f(x) = \int_0^x g(t)dt$ for all x. Prove that

(A) $f(x-5) = -g(x)$ for all x;

(B) $\int_0^5 f(t)dt = 7$;

(C) $\int_0^x f(t)dt = g(0) - g(x)$.

19. Prove that, as $n \to \infty$,

(i) $\int_0^1 \frac{n\cos x}{(1+nx)^2}dx \to 1$

(ii) $\int_0^1 \frac{n\cos x}{1+n^2x}dx \to 0$.

20. (A) If $I_n(x) = \int_0^x t^n(t^2+a^2)^{-1/2}$, use integration by parts to show that $nI_n(x) = x^{n-1}\sqrt{x^2+a^2} - (n-1)a^2 I_{n-2}(x)$ if $n \ge 2$.

(B) Use part (a) to show that $\int_0^2 x^5 (x^2+5)^{-12} dt$, $= 168/5 - 40\sqrt{5}/3$.

21. A function f, continuous on the positive real axis, has the property that for all choices of $x > 0$ and $y > 0$, the integral $\int_x^{xy} f(t)dt$ is independent of x (and therefore depends only on y). If $f(2) = 2$, compute the value of the integral $A(x) = \int_1^x f(t)dt$ for all $x > 0$.

22. Let $A = \int_0^1 \frac{e^t}{(t+1)} dt$. Express the values of the following integrals in terms of A :

(A) $\int_{a-1}^{a} \frac{e^{-t}}{t-a-1} dt.$ (B) $\int_0^1 \frac{te^{t^2}}{t^2+1} dt.$

(C) $\int_0^1 \frac{e^t}{(t+1)^2} dx.$ (D) $\int_0^1 e^t \ln(1+t)\, dt..$

23. Find the number a, $0 \le a \le 2\pi$ that maximizes the function $f(a) = \int_0^{2\pi} \sin x \sin(x+a)\, dx$

24. Proceeding from the equation

$\int_0^\infty \frac{dx}{x^2+a} = \frac{\pi}{2\sqrt{a}}$, evaluate the integral

$\int_0^\infty \frac{dx}{(x^2+a)^{n+1}}.$

25. Evaluate the integral $\int_0^\infty \frac{1-e^{-ax}}{xe^x} dx$.

26. Evaluate the integral $\int_{-1}^{1} \frac{\sin\alpha\, dx}{1-2x\cos\alpha+x^2}$. For what values of x is the integral a discontinuous function of α?

27. Prove that if $m \ge 1$ and

$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos nx dx,$

$J_{m,n} = \int_0^{\pi/2} \sin^m x \sin nx dx,$

then $(m+n) I_{m,n} = \sin n\pi/2 - mJ_{m-1,n-1}$ and express $I_{m,n}$ in terms of $I_{m-2,n-2}$ when $m \ge 2$.

28. Show that the value of the integral $\int_0^2 375x^5 (x^2+1)^{-4} dx$ is 2^n for some integer n.

29. By considering the value of $\int_0^1 (1-x^2)^n dx$, prove that if n be a positive integer,

$$1 - \frac{n}{1.3} + \frac{n(n-1)}{1.3.5} - \frac{n(n-1)(n-2)}{1.3.5.7} + \dots = \frac{2.4.6...2n}{3.5.7...(2n+1)}.$$

30. If $f_{ave}[a, b]$ denotes the average value of f on the interval [a, b] and $a < c < b$, show that

$$f_{avg}[a, b] = \frac{c-a}{b-a} f_{avg}[a, c] + \frac{b-c}{b-a} f_{avg}[c, b]$$

31. Prove that $\int_{-1}^{1} \frac{dx}{\sqrt{(1-2\alpha x+\alpha^2)}}$ is equal to 2 if $-1 \le \alpha \le 1$ and to $2/\alpha$ if $|\alpha| > 1$.

32. Suppose that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \text{ on } R$$

Show that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \quad (n = 0, 1, 2, ..., N)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \quad (n = 1, 2, ..., N)$$

33. Find all the values of a for which the inequality

$$\frac{1}{\sqrt{a}}\int_1^a \left(\frac{3}{2}\sqrt{x}+1-\frac{1}{\sqrt{x}}\right) dx < 4 \text{ is satisfied.}$$

34. Find the average value of the function

(i) $f(x) = \frac{\cos(\pi/x)}{x^2}$ over the interval [1, 3]

(ii) $f(x) = \frac{2}{e^x+1}$ on the interval [0, 2].

35. Determine a pair of numbers a and b for which $\int_0^1 (ax+b)(x^2+3x+2)^{-2} dx = 3/2.$

36. Prove that $\int_0^\pi \frac{\sin nx}{\sin x} dx$ is equal to π or to 0 according as n is odd or even.

37. Prove that

(i) $\int_0^\infty \frac{dx}{x^4\sqrt{a^2+x^2}} = \frac{2-\sqrt{2}}{3a^4},$

(ii) $\int_0^1 \frac{x^3 \sin^{-1} x}{\sqrt{1-x^2}} dx = \frac{7}{9},$

(iii) $\int_1^\infty \frac{\ln x}{x^n}dx = \frac{1}{(n-1)^2},\ n > 1$,

(iv) $\int_1^\infty \frac{dx}{(x+1)^2(x^2+1)} = \frac{1}{4}(1-\ln 2)$.

38. Prove that

$$I_n = \int_0^{\pi/2} x\sin^n x\,dx$$

$$= \frac{1}{n^2} + \frac{n-1}{n}\int_0^{\pi/2} x\sin^{n-2} x\,dx$$

and hence deduce that $I_5 = \frac{149}{225}$.

39. Evaluate $\int_0^\infty \frac{1}{(1+x^2)^{n+1/2}}dx$

40. Show that if n is a positive integer, then

$$\int_0^{2\pi} \frac{\cos(n-1)x - \cos nx}{1-\cos x}dx = 2\pi$$

and deduce that $\int_0^{2\pi}\left(\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right)^2 dx = 2n\pi$.

41. Show that if $I_n = \int_0^\pi \frac{x\sin nx}{\sin x}dx$,

then $I_n = \frac{2}{(n-1)^2}\,[(-1)^{n-1} - 1] + I_{n-2}$.

Deduce that $I_5 = \frac{\pi^2}{2}$, and $I_6 = \frac{1036}{225}$.

42. The function $f(x) = \sqrt{x}$ is continuous on [0, 4] and therefore integrable on this interval.

Evaluate $\int_0^4 \sqrt{x}dx$ using subintervals of unequal length given by the partitions $0 < 4(1)^2/n^2 < 4(2)^2/n^2 < \ldots < 4(n-1)^2/n^2 < 4$ and the right endpoint of the k^{th} subinterval.

43. Show that for any positive integer n

(A) $\left|\int_0^{\pi/6} \sin^n x\,dx\right| \le \frac{\pi}{3.2^{n+1}}$

(B) $\left|\int_{\pi/3}^{\pi/2} \cos^n x\,dx\right| \le \frac{\pi}{3.2^{n+1}}$

44. Let the function f(x) be positive on the interval [a, b]. Prove that the expression

$$\int_a^b f(x)dx\int_a^b \frac{dx}{f(x)}$$

reaches the least value only if f(x) is constant on this interval.

45. If $|x| < 1$, prove that

$$\frac{x^3}{1.2} - \frac{x^4}{3.4} + \frac{x^6}{5.6} - \ldots = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2).$$

Hence show that $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \ldots = 0.438\ldots$

46. Assuming that the function g defined by $g(x) = 2x$ is integrable over the interval [0, 2], use the partition $\left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$ to show that

$$3 \le \int_0^2 2x\,dx \le 5.$$

47. Find the limit, when n tends to infinity of

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{4n}.$$

48. Let f be a continuous function such that $f(0) = 2$ and $f(x) \to 3$ as $x \to \infty$. Find the limit of $(1/b)\int_0^b f(x)dx$ as (a) $b \to 0$, (b) $b \to \infty$.

49. Let $f(x) = x - [x] - \frac{1}{2}$, if x is not an integer, and let $f(x) = 0$, if x is an integer. ([x] denotes the greatest integer $\le x$.) Define a new function P as follows :

$P(x) = \int_0^x f(t)dt$ for every real x.

(A) Draw the graph of f over the interval [–3, 3] and prove that f is periodic with period 1.

(B) Prove that $P(x) = \frac{1}{2}(x^2 - x)$, if $0 \le x \le 1$ and that P is periodic with period 1.

(C) Express P(x) in terms of [x].

(D) Determine a constant c such that $\int_0^1 (P(t)+c)dt = 0$.

(E) For the constant c of part(d), let $Q(x) = \int_0^x (P(t)+c)dt$. Prove that Q is periodic with period 1 and that $Q(x) = \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12}$, if $0 \le x \le 1$.

50. Given an even function f, defined everywhere, periodic with period 2, and integrable on every interval. Let $g(x) = \int_0^x f(t)dt$, and let $A = g(1)$.

(A) Prove that g is odd and that $g(x+2) - g(x) = g(2)$.

(B) Compute g(2) and g(5) in terms of A.

(C) For what value of A will g be periodic with period 2 ?

Target Exercises for JEE Advanced

1. Evaluate the following limits :

(i) $\lim\limits_{n\to\infty} \frac{n}{(n+1)\sqrt{(2n+1)}} + \frac{n}{(n+2)\sqrt{2(2n+2)}} + \frac{n}{(n+3)\sqrt{3(2n+3)}}$ ..... up to n terms

(ii) $\lim\limits_{n\to\infty} \frac{1}{n^2}\left[\sin^3\frac{\pi}{4n} + 2\sin^3\frac{2\pi}{4n} + 3\sin^3\frac{3\pi}{4n} + \dots + n\sin^3\frac{n\pi}{4n}\right]$

$$\left[\sin^3\frac{\pi}{4n} + 2\sin^3\frac{2\pi}{4n} + 3\sin^3\frac{3\pi}{4n} + \dots + n\sin^3\frac{n\pi}{4n}\right]$$

2. Prove that, when a and b are positive,

$\int_0^{\pi/2} \frac{b - a\cos\theta}{b^2 - 2ab\cos\theta + a^2} d\theta$ is equal to $\frac{\pi}{2b} + \frac{1}{b}\tan^{-1}\frac{a}{b}$, or $-\frac{1}{b}\tan^{-1}\frac{b}{a}$ according as b is greater than or less than a

3. Show how it follows from the equality $\int_1^x \frac{dx}{x} = \ln x$ that the sum of n terms of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots$ lies between $\ln(n+1)$ and $1 + \ln n$.

4. Prove that

$$\int_0^{k\pi} \sin\left[\frac{2x}{\pi}\right]dx = \frac{\pi}{2}\cdot\frac{\sin k \sin(k+1/2)}{\sin(1/2)},$$ where [.] denotes the greatest integer function.

5. When $a > 0$ and $b > 0$, prove that

$$\int_0^\infty \frac{\ln(1+b^2x^2)}{1+a^2x^2}dx = \frac{\pi}{a}\ln\frac{a+b}{a}.$$

6. Prove that, if $I = \int_0^\infty e^{-x^2}\cos 2xy\,dx$ then $\frac{dI}{dy} = -2yI$, and assuming the formula $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$ prove that $I = \frac{1}{2}\sqrt{\pi}e^{-y^2}$.

7. Prove that, if $y_1 > 0$, $y_1 \le y \le y_2$, and

$$\phi(y) = \int_0^\infty \frac{\sin xy}{x(a^2+x^2)}dx$$

then $\phi''(y) - a^2\phi(y) = -\frac{1}{2}\pi$

Show further that, when $y > 0$ and $a > 0$,

$$\phi(y) = \frac{1}{2}\pi(1 - e^{-ay})/a^2$$

8. If $nh = 1$ always, then show that $\lim\limits_{n\to\infty}\prod_{r=1}^{n}[1 + (rh)^{2k}]^{1/r} = e^\lambda$, where $\lambda = \frac{1}{2k}\int_0^1 \frac{\ln(1+x)}{x}dx$.

9. Show that for a differentiable function f(x)

$$\int_0^n f'(x)\left\{[x] - x + \frac{1}{2}\right\}dx = \int_0^n f(x)dx + \frac{1}{2}f(0) + \frac{1}{2}f(n) - \sum_{r=0}^{n} f(r),$$

where [.] denotes the greatest integer function and $n \in N$.

10. Prove that $\int_0^1 \frac{\ln x}{1+x^2}dx = -\int_1^\infty \frac{\ln x}{1+x^2}dx$ and $\int_1^\infty \frac{\ln x}{1+x^2}dx = 0$ and deduce that if $a > 0$ then $\int_0^\infty \frac{\ln x}{a^2+x^2}dx = \frac{\pi}{2a}\ln a$.

11. A particle moves along a straight line. Its position at time t is f(t). When $0 \le t \le 1$, the position is given by the integral

$$f(t) = \int_0^t \frac{1 + 2\sin \pi x \cos \pi x}{1+x^2}\,dx.$$

For $t \ge 1$, the particle moves with constant acceleration (the acceleration it acquires at time $t = 1$). Compute the following :

(A) Its acceleration at time $t = 2$;
(B) Its velocity when $t = 1$
(C) Its velocity when $t > 1$
(D) The difference $f(t) - f(1)$ when $t > 1$.

12. Assume that the function f is defined for all x and has a continuous derivative. Assume that $f(0) = 0$ and that $0 < f'(x) \le 1$.

Prove that $\left[\int_0^1 f(x)\,dx\right]^2 \ge \int_0^1 [f(x)]^3 dx$

13. Let p(x) be a polynomial of degree at most 3.

(A) Show that there is a number c between 0 and 1 such that $\int_{-1}^{1} p(x)dx = p(c) + p(-c)$.

(B) Show that there is number c such that $\int_{-1/2}^{1/2} p(x)dx = \frac{1}{3}[p(-c) + p(0) + p(c)]$.

14. (A) Let $G(a) = \int_0^\infty \frac{dx}{(1+x^a)(1+x^2)}$. Evaluate G(0), G(1), G(2).

(B) Show, using the substitution $x = 1/y$, that $G(a) = \int_0^\infty \frac{x^a\,dx}{(1+x^a)(1+x^2)}$.

(C) From (b), show that $G(a) = \pi/4$, independent of a.

15. If $f_1(x) = \int_0^x f(t)dx$, $f_2(x) = \int_0^x f_1(t)dt$, ...,

$f_k(x) = \int_0^x f_{k-1}(t)dt$, then prove that

$$f_k(x) = \frac{1}{(k-1)!}\int_0^x f(t)\,(x-t)^{k-1}\,dt.$$

16. Evaluate the integral $\int_{-1}^{3} t^3(4+t^3)^{-1/2}dt$, given that $\int_{-1}^{3}(4+t^3)^{1/2}dt = 11.35$. Leave the answer in terms of $\sqrt{3}$ and $\sqrt{31}$.

17. A function F is defined by the indefinite integral $F(x) = \int_1^x \frac{e^t}{t}dt$, if $x > 0$.

(A) For what values of x is it true that $\ln x \le F(x)$?

(B) Prove that $\int_1^x \frac{e^t}{t+a}dt = e^{-a}[F(x)+a) - F(1+a)]$.

(C) In a similar way, express the following integrals in terms of F :

$$\int_1^x \frac{e^{at}}{t}dt,\ \int_1^x \frac{e^t}{t^2}dt,\ \int_1^x e^{1/t}dt.$$

18. The strength of an alternating current values according to the law

$$i = i_0 \sin\left(\frac{2\pi t}{T} + \phi\right)$$

where i_0 is the amplitude, t is time, T is the period and ϕ is the initial phase, Find the mean value of the square of the current strength :

(A) on the interval of time [0, T],
(B) on the interval [0, T/2] (the period of the function is $i^2(t)$)
(C) on the arbitrary interval $[0, t_0]$ and the limit of that mean value as $t_0 \to \infty$

19. Evaluate the following integrals :

(i) $\int_1^\infty \frac{\tan^{-1}(x-1)dx}{\sqrt[3]{(x-1)^4}}$ (ii) $\int_{-1}^{1} \ln\frac{1+x}{1-x}\frac{x^3\,dx}{\sqrt{1-x^2}}$

(iii) $\int_1^\infty \frac{x^2-2}{x^3\sqrt{x^2-1}}dx$.

20. (A) Show that $0 < \int_0^1 \frac{t^4(1-t)^4}{1+t^2}dt$ and that $\int_0^1 \frac{t^4(1-t)^4}{1+t^2}dt = \frac{22}{7} - \pi$.

(B) Evaluate $\int_0^1 t^4(1-t)^4 dt$ then apply the results of (a) to conclude that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

21. Use the identity $1 + x^6 = (1 + x^2)(1 - x^2 + x^4)$ to prove that for $a > 0$, we have

$$\frac{1}{1+a^6}\left(a - \frac{a^3}{3} + \frac{a^5}{5}\right) \le \int_0^a \frac{dx}{1+x^2} \le a - \frac{a^3}{3} + \frac{a^5}{5}$$

Taking a = 1/10, calculate the approximate value of the integral.

22. Prove that $y = 8\int_0^x t^2e^{-2t^2}\sin 2(x-t)dt$ is a solution of the differential equation

$$\frac{d^2y}{dx^2}+4y=16x^2e^{-2x^2}.$$

Given that every solution of the differential equation must be of the form e^{-2x^2} + A cos2x + B sin 2x, where A and B are constants, prove that

$$\int_0^x t^2e^{-2t^2}\sin 2(x-t)dt=\frac{1}{8}(e^{-2x^2}-\cos 2x).$$

23. When $I_n(\alpha)=\frac{1}{n!2^n}\int_0^1(1-t^2)^n\cos\alpha t dt$

and n is a positive integrer, prove that

(i) $\alpha^2 I_{n+1}=(2n+1)I_n-I_{n-1}$,

(ii) $\frac{dI_n}{d\alpha}=-\alpha I_{n+1}$

and deduce that $I_n(\alpha)$ is a solution of the differential equation $\frac{d^2y}{d\alpha^2}+\frac{2n+2}{\alpha}\frac{dy}{d\alpha}+y=0$.

24. Prove that

$$\int_0^{\pi/2}\frac{\cos^2\theta d\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^2}=\frac{\pi}{4a^3b},$$

$$\int_0^{\pi/2}\frac{\sin^2\theta d\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^2}=\frac{\pi}{4ab^3}$$

$$\int_0^{\pi/2}\frac{d\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^2}=\frac{\pi}{4ab}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

25. Prove that, when $0\le x<1$,

$$\int_0^{\pi/2}\tan^{-1}\left(\frac{1-x}{1+x}\tan\theta\right)d\theta=\frac{\pi^2}{8}-\frac{x}{1}-\frac{x^3}{3^2}-....$$

26. Prove, when a, b, α, β are constant, that

$$y=\frac{1}{a-b}\int_\alpha^x f(t)e^{a(x-t)}dt+\frac{1}{b-a}\int_\beta^x f(t)e^{b(x-t)}dt$$

is a solution of the differential equation

$$\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby=f(x)$$

27. (A) Give a geometric argument to show that

$$\frac{1}{x+1}<\int_x^{x+1}\frac{1}{t}dt<\frac{1}{x},\ x>0$$

(B) Use the result in part (a) to prove that

$$\frac{1}{x+1}<\ln\left(1+\frac{1}{x}\right)<\frac{1}{x},\ x>0$$

(C) Use the result in part (b) to prove that

$$e^{x/(x+1)}<\left(1+\frac{1}{x}\right)^x<e,\ x>0$$

and hence that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$.

(D) Use the inequality in part (c) to prove that

$$\left(1+\frac{1}{x}\right)^x<e<\left(1+\frac{1}{x}\right)^{x+1},\ x>0$$

28. Prove that, for any fixed value of λ,

$$\int_0^\lambda\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{\lambda}{a}$$

$$\int_0^\lambda\frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^3}\tan^{-1}\frac{\lambda}{a}+\frac{\lambda}{2a^2(\lambda^2+a^2)}$$

and deduce that $\int_0^\infty\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{4a^3}$.

29. Show from graphical considerations that if f(x) steadily diminishes, as x inereases from 0 to ∞, the series f(1) + f(2) + f(3) + ... is convergent, and that its sum lies between I and I + f(1), provided the integral $I=\int_1^\infty f(x)dx$, be finite.

Apply this to the series

$$\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+....$$

30. Show that

(A) $|\cos x-(1-x^2)/2!+x^4/4!-x^6/6!|\le x^8/8!$

(B) $|\sin x-(x-x^3)/3!+x^5/5!-x^7/7!|\le x^9/9!$

31. Assume that the function f defined by $f(x)=x^2+1$ is integrable over the interval [0, 1]. Using the partition $\left\{0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1\right\}$, show that

$$\frac{31}{25} \le \int_0^1 f(x)dx \le \frac{36}{25}.$$

32. Using the inequality $\sin x \ge x - \frac{x^3}{6}$ $(x \ge 0)$ and the Schwartz-Bunyakovsky inequality, show that

$$1.096 < \int_0^{\pi/2} \sqrt{x \sin x}\,dx < 1.111$$

33. The function f(x) is continuous in (0, 1) and

$$I_1 = \int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}dx,\quad I_2 = \int_{1/\sqrt{n}}^{1} \frac{nf(x)}{1+n^2x^2}dx$$

Use the mean value theorem for integrals to prove that for some ξ_n in $(1/\sqrt{n}, 1)$.

$I_1 = f(\xi_n)\tan^{-1}\sqrt{n}$, $I_2 = f(\zeta_n)\{\tan^{-1}n - \tan^{-1}\sqrt{n}\}$.
Deduce that, as $n \to \infty$, $I_2 \to 0$ and

$$I = \int_0^1 \frac{nf(x)}{1+n^2x^2}dx \to \frac{1}{2}\pi f(0)$$

34. When f(x) is bounded and is strictly decreasing in (0, 1) prove that $\int_0^{1-1/n} f(x)dx$

$$\ge n^{-1}\left\{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots\ldots+f\left(\frac{n-1}{n}\right)\right\}$$

$$\ge \int_{1/n}^{1} f(x)dx.$$

35. The function f(x) is differentiable in $a \le x \le b$, and $f(a) = f(b) = 0$; show, by dividing the range (a,b) into two equal parts and applying the mean-value theorem to each part that there is at least one point ξ in (a,b) for which

$$|f'(\xi)| > \frac{4}{(b-a)^2}\int_a^b f(x)dx.$$

Verify the theorem for the function $\sin^2 x$ in $(0, \pi)$.

36. When $I_n = \frac{1}{n}\int_0^1 \frac{\sin nx}{1+x^2}dx$,

By property, $|I_1| \le \frac{1}{n}\int_0^1 \frac{|\sin nx|}{1+x^2}dx$

and since $|\sin nx| \le 1$, we have

$$|I_n| \le \frac{1}{n}\int_0^1 \frac{dx}{1+x^2}$$

Deduce that $I_n \to 0$ as $n \to \infty$.

37. Given that f(x) is continuous in (a, b), prove that

as $n \to \infty$, $I_n = \frac{1}{\sqrt{n}}\int_a^b f(x)\cos nx\,dx \to 0$.

38. Arrange the functions

$$\frac{x}{\sqrt{(\ln x)}},\quad \frac{x\sqrt{(\ln x)}}{\ln\ln x},\quad \frac{x\ln\ln x}{\sqrt{(\ln x)}},\quad \frac{x\ln\ln\ln x}{\sqrt{(\ln\ln x)}}$$

according to their order of magnitude for large x.

39. Show that $\int_0^x \frac{dt}{1+t^4} = x - \frac{1}{5}x^5 + \frac{1}{9}x^9 - \ldots$ if $-1 \le x \le 1$. Also deduce that

$$1 - \frac{1}{5} + \frac{1}{9} - \ldots = \frac{\pi + 2\ln(\sqrt{2}+1)}{4\sqrt{2}}.$$

40. If $\phi(x) = \dfrac{3\int_0^x (1+\sec t)\ln \sec t\,dt}{\ln\sec x\ [x + \ln(\sec x + \tan x)]}$ then

(i) $\phi(x)$ is even,

(ii) $\phi(x) \to \frac{3}{2}$ when $x \to \frac{\pi}{2}$ through values less than $\frac{\pi}{2}$.

41. (A) If ϕ'' is continuous and nonzero on [a, b], and if there is a constant $m > 0$ such that $\phi'(t) \ge m$ for all t in [a, b], prove that

$$\left|\int_a^b \sin\varphi(t)dt\right| \le \frac{4}{m}.$$

(B) If $a > 0$, show that $\left|\int_a^x \sin(t^2)dt\right| \le \frac{2}{a}$ for all $x > a$.

42. A sequence of polynomials (called the Bernoulli polynomials) is defined inductively as follows :

$P_0(x) = 1$; $P'_n(x) = nP_{n-1}(x)$ and $\int_0^1 P_n(x)\,dx = 0$, if $n \ge 1$,

(A) Determine explicit formulas for $P_1(x)$, $P_2(x)$,...., $P_5(x)$.

(B) Prove, by induction, that $P_n(x)$ is a polynomial in x of degree n.

(C) Prove taht $P_n(0) = P_n(1)$ if $n \geq 2$.
(D) Prove that $P_n(x+1) - P_n(x) = nx^{n-1}$ if $n \geq 1$.
(E) Prove that for $n \geq 2$ we have

$$\sum_{r=1}^{k-1} r^n = \int_0^k P_n(x)dx = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1}.$$

(F) Prove that $P_n(1-x) = (-1)^n P_n(x)$ if $n \geq 1$.

(G) Prove that $P_{2n+1}(0) = 0$ and $P_{2n-1}\left(\frac{1}{2}\right) = 0$ if $n \geq 1$.

43. Prove that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+m+1}$

$$= \sum_{k=0}^{n} (-1)^k \binom{m}{k} \frac{1}{k+n+1}$$

44. Let $f(x) = \int_0^x (1+t^3)^{-1/2}\, dt$ if $x \geq 0$.

(A) Show that f is strictly increasing on the nonnegative real axis.

(B) Let g denote the inverse of f. Show that the second derivative of g is proportional to g^2 [that is, $g''(y) = cg^2(y)$ for each y in the domain of g] and find the constant of proportionality.

45. Let n be a fixed integer. Let $f: R \to R$ be given

by $g(x) = \begin{cases} x & \text{if } x \leq 0 \\ 2^n & \text{if } 2^n - 2^{n-2} < x \leq 2^{n+1} - 2^{n-1} \end{cases}$

Prove that $\int_0^{2^n} f(x)dx = \int_0^{2^n} xdx = 2^{2^n - 1}$.

46. A function, called the integral logarithm and denoted by Li, is defined as follows :

$$\text{Li}(x) = \int_2^x \frac{dt}{\ln t} \text{ if } x \geq 2.$$

(A) Prove that $\text{Li}(x) = \frac{x}{\ln x} + \int_2^x \frac{dt}{\ln^2 t} - \frac{2}{\ln 2}$.

(B) Show that there is a constant b such that $\int_b^{\ln x} \frac{e^t}{t} dt = \text{Li}(x)$ and find the value of b.

(C) Express $\int_0^x \frac{e^{2t}}{(t-1)} dt$ in terms of the integral logarithm, where $c = 1 + \frac{1}{2}\ln 2$.

(D) Let $f(x) = e^4\,\text{Li}(e^{2x-4}) - e^2\,\text{Li}(e^{2x-2})$ if $x > 3$. Show that $f'(x) = \frac{e^{2x}}{x^2 - 3x + 2}$

47. Assume that integrable functions $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ are given on the interval [a, b], the function $p_1(x)$ is non-negative, and the function $p_2(x)$, $p_3(x)$ satisfy the inequality $p_3(x) \leq p_2(x) \leq p_4(x)$. Prove that

$$\int_a^b p_3(x)p_1(x)dx \leq \int_a^b p_2(x)p_1(x)dx$$

$$\leq \int_a^b p_4(x)p_1(x)dx.$$

48. Given two functions f and g whose derivatives f′ and g′ satisfy the equations $f'(x) = g(x)$, $g'(x) = -f(x)$, $f(0) = 0$, $g(0) = 1$ for every x in some open interval J containing 0. For example, these equations are satisfies when $f(x) = \sin x$ and $g(x) = \cos x$.

(A) Prove that $f^2(x) + g^2(x) = 1$ for every x in J.

(B) Let F and G be another pair of functions satisfying the given conditions. Prove that $F(x) = f(x)$ and $G(x) = g(x)$ for every x in J.

49. Since $\sin x < x$ for $x > 0$, it follows that

$$\int_0^x \sin t\,dt \leq \int_0^x t\,dt \quad \text{or,} \quad 1 - \frac{x^2}{2!} \leq \cos x \leq 1.$$

In a similar fashion show that $x - (x^3/3!) \leq \sin x \leq x$.

50. Prove by repeated differentiation of the identity $\frac{1}{1-x} = 1 + x + x^2 + x^2 + \ldots$, where $|x| < 1$, that, if m be a positive integer $(1-x)^{-m} = 1 + mx$

$$+ \frac{m(m+1)}{1.2}x^2 + \frac{m(m+1)(m+2)}{1.2.3}x^3 + \ldots$$

Previous Year's Questions (JEE Advanced)

A. Fill in the blanks :

1. $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$.

Then $\int_0^{\pi/2} f(x)\,dx = \ldots\ldots$ **[IIT - 1987]**

2. The integral $\int_0^{1.5} [x^2]\,dx$, where [ ] denotes the greatest integer function, equals.... **[IIT - 1988]**

3. The value of $\int_{-2}^{2} |1-x^2|\,dx$ is......... **[IIT - 1989]**

4. The value of $\int_{\pi/4}^{3\pi/4} \frac{\varphi}{1+\sin\varphi}\,d\varphi$ is... **[IIT - 1993]**

5. The value of $\int_{2}^{3} \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$ is.. **[IIT - 1994]**

6. If for nonzero x, af(x) + bf(1/x) = 1/x – 5 where $a \neq b$, then $\int_{1}^{2} f(x)dx =$ .... **[IIT - 1996]**

7. For n> 0, $\int_{0}^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}\,dx =$........... **[IIT - 1996]**

8. The value of $\int_{1}^{e^{37}} \frac{\pi\sin(\pi \ln x)}{x}\,dx$ is.............. **[IIT - 1997]**

9. Let $\frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}$, x > 0. If $\int_{1}^{4} \frac{2e^{\sin x^2}}{x}\,dx$ = F(k) – F(1) then one of the possible values of k is.............. **[IIT - 1997]**

B. True/False :

10. The value of the integral

$$\int_{0}^{2\alpha}\left[\frac{f(x)}{\{f(x)+f(2a-x)\}}\right]dx \text{ is equal to a.}$$

[IIT - 1988]

C. Multiple Choice Questions with ONE correct answer :

11. The value of the definite integral $\int_{0}^{1}(1+e^{-x^2})\,dx$ is **[IIT - 1981]**
(A) –1 (B) 2
(C) $1+e^{-1}$ (D) None

12. Let a, b, c be non-zero real numbers such that $\int_{0}^{1}(1+\cos^8 x)(ax^2+bx+c)\,dx = \int_{0}^{2}(1+\cos^8 x)(ax^2+bx+c)\,dx$. Then the quadratic equation $ax^2+bx+c=0$ has **[IIT - 1981]**
(A) No root in (0, 2)
(B) At least one root in (0, 2)
(C) A double root in (0, 2)
(D) Two imaginary roots

13. The value of the integral $\int_{0}^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}\,dx$ is **[IIT - 1983]**
(A) $\pi/4$ (B) $\pi/2$
(C) π (D) None of these

14. For any integer n the integral
$$\int_{0}^{\pi} e^{\cos^2 x}\cos^3(2n+1)x\,dx \text{ has the value}$$
[IIT - 1985]
(A) π (B) 1
(C) 0 (D) None

15. f : R → R, g : R → R are continuous functions
The value of integral $\int_{-\pi/2}^{\pi/2}[f(x)+f(-x)][g(x)-g(-x)]\,dx$ is **[IIT - 1990]**
(A) π (B) 1
(C) –1 (D) 0

16. The value of $\int_{3}^{5} \frac{x^2}{x^2-4}\,dx$ is **[IIT - 1992]**
(A) $2-\log_e(15/7)$
(B) $2+\log_e(15/7)$
(C) $2+4\log_e 3-4\log_e 7+4\log_e 5$
(D) $2-\tan^{-1}(15/7)$

17. $\int_{-\pi}^{\pi}(\cos ax-\sin bx)^2\,dx$ where a and b are integers is equal to **[IIT - 1992]**
(A) $-\pi$ (B) 0
(C) π (D) 2π

18. The value of $\int_{-\pi}^{\pi}(1-x^2)\sin x\cos^2 x\,dx$ is **[IIT - 1992]**
(A) 0 (B) $\pi-\pi^3/3$
(C) $2\pi-\pi^3$ (D) $7/2-2\pi^3$

19. $\int_{-1}^{1}|(1-x)|\,dx$ equals **[IIT - 1992]**
(A) –2 (B) 0
(C) 2 (D) 4

20. Integral $\int_{0}^{1}|\sin 2\pi x|\,dx$ is equal to **[IIT - 1992]**
(A) 0 (B) $3/\pi$
(C) $\pi/4$ (D) $2/\pi$

21. The value of $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is **[IIT - 1993]**

(A) 0 (B) 1
(C) $\pi/2$ (D) $\pi/4$

22. The value of $\int_{-1}^{1} \frac{\sin x - x^2}{3-|x|}$ dx is **[IIT - 1994]**

(A) 0 (B) $2\int_0^1 \frac{\sin x}{3-|x|}$ dx
(C) $2\int_0^1 \frac{-x^2}{3-|x|}$ (D) $2\int_0^1 \frac{\sin x - x^2}{3-|x|}$ dx

23. If f(x) = A sin (πx/2) + B, f' $\left(\frac{1}{2}\right) = 2\sqrt{2}$ and $\int_0^1 f(x)\ dx = \frac{2A}{\pi}$, then the constant A and B are **[IIT - 1995]**

(A) $\pi/2$ and $\pi/2$ (B) $2/\pi$ and $3/\pi$
(C) 0 and $-4/\pi$ (D) $8/\pi$ and 0

24. The value of $\int_0^{2\pi} [2\sin x]\ dx$ where [ ] represents the greatest integer function is **[IIT - 1995]**

(A) $-5\pi/3$ (B) $-\pi$
(C) $5\pi/3$ (D) -2π

25. Let f be a positive function, It

$I_1 = \int_{1-k}^{k} x.f\ [x(1-x)]\ dx$ & $I_2 = \int_{1-k}^{k} f[x(1-x)]\ dx$, where $(2k-1) > 0$, then I_1/I_2 is **[IIT - 1997]**

(A) 2 (B) k
(C) 1/2 (D) 1

26. If $g(x) = \int_0^x \cos^4 t\ dt$, then $g(x+\pi)$ equals **[IIT - 1997]**

(A) $g(x)+g(\pi)$ (B) $g(x)-g(\pi)$
(C) $g(x)g(\pi)$ (D) $g(x)/g(\pi)$

27. If $\int_0^x f(t)\ dt = x + \int_x^1 tf(t)\ dt$, then the value of f(1) is **[IIT - 1998]**

(A) 1/2 (B) 0
(C) 1 (D) –1/2

28. Let f(x) x – [x], for every real number x, where [x] is the integral part of x. Then $\int_{-1}^{1} f(x)\ dx$ is **[IIT - 1998]**

(A) 1 (B) 2
(C) 0 (D) 1/2

29. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to **[IIT - 1999]**

(A) 2 (B) –2
(C) 1/2 (D) –1/2

30. If for a real number y, [y] is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\ dx$ is **[IIT - 1999]**

(A) $-\pi$ (B) 0
(C) $-\pi/2$ (D) $\pi/2$

31. Let $g(x) = \int_0^x f(t)\ dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$ for $t \in [0, 1]$ and $0 \le f(t) \le \frac{1}{2}$ for $t \in (1, 2]$. Then g(2) satisfies the inequality **[IIT - 2000]**

(A) $-3/2 \le g(2) < 1/2$ (B) $0 \le g(2) < 2$
(C) $3/2 < g(2) \le 5/2$ (D) $2 < g(2) < 4$

32. If f(x) = $\begin{cases} e^{\cos x} \sin x & \text{for } |x| \le 2 \\ 2 & \text{otherwise} \end{cases}$, then $\int_{-2}^{3} f(x)\ dx$ **[IIT - 2000]**

(A) 0 (B) 1
(C) 2 (D) 3

33. The value of the integral $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right|$ dx is **[IIT - 2000]**

(A) 3/2 (B) 5/2
(C) 3 (D) 5

34. $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}$ dx, a > 0 **[IIT - 2001]**

(A) π (B) πa
(C) $\pi/2$ (D) 2π

35. Let f : (0, ∞) $\to$ R and $F(x) = \int_0^x f(t)dt$. If $F(x^2) = x^2(1+x)$, then f(4) equals **[IIT - 2001]**

(A) 5/40 (B) 7
(C) 4 (D) 2

36. Let $f(x) = \int_1^x \sqrt{2-t^2}$ dt. Then the real roots of the equation $x^2 - f'(x) = 0$ are **[IIT - 2002]**

(A) ± 1 (B) $\pm 1/\sqrt{2}$
(C) $\pm 1/2$ (D) 0 and 1

37. Let $T > 0$ be a fixed real number. Suppose f is a continuous function such that for all $x \in R$, $f(x + T) = f(x)$. If $I = \int_0^T f(x)\,dx$, then the value of $\int_3^{3+3T} f(2x)\,dx$ is **[IIT - 2002]**

(A) (3/2) I (B) 2I
(C) 3I (D) 6I

38. The integral $\int_{-1/2}^{1/2}\left([x] + \ell n\left(\frac{1+x}{1-x}\right)\right)dx$ equals **[IIT - 2002]**

(A) –1/2 (B) 0
(C) 1 (D) 2 ℓn (1/2)

39. If $I(m, n) = \int_0^1 t^m\,(1+t)^n\,dt$, then the expression for I(m, n) in terms of I(m + 1, n – 1) is **[IIT - 2003]**

(A) $\frac{2^n}{m+1} - \frac{n}{m+1} I(m+1, n-1)$

(B) $\frac{n}{m+1} I(m+1, n-1)$

(C) $\frac{2^n}{m+1} + \frac{n}{m+1} I(m+1, n-1)$

(D) $\frac{m}{n+1} I(m+1, n-1)$

40. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\,dt$, then f(x) increases in **[IIT - 2001]**

(A) (–2, 2) (B) no value of x
(C) $(0, \infty)$ (D) $(-\infty, 0)$

41. If f(x) is differentiable and $\int_0^{t^2} x f(x)\,dx = \frac{2}{5}t^5$ then f (4/25) equals **[IIT - 2004]**

(A) 2/5 (B) –5/2
(C) 1 (D) 5/2

42. The value of the integral $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$ is **[IIT - 2004]**

(A) $\pi/2 + 1$ (B) $\pi/2 - 1$
(C) –1 (D) 1

43. $\int_{-2}^{0} \{x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)\}\,dx$ is equal to **[IIT - 2005]**

(A) –4 (B) 0
(C) 4 (D) 6

44. Let f be a non-negative function defined on the interval [0, 1].

If $\int_0^x \sqrt{1 - [f'(t)]^2}\,dt = \int_0^x f(t)dt$, $0 \le x \le 1$ and $f(0) = 0$, then **[IIT - 2009]**

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

45. Let f be a real-valued function defined on the interval (–1, 1) such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1}\,dt$, for all $x \in (-1, 1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to **[IIT - 2010]**

(A) 1 (B) $\frac{1}{3}$

(C) $\frac{1}{2}$ (D) $\frac{1}{e}$

46. The value of $I = \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)}\,dx$ is **[IIT - 2011]**

(A) $\frac{1}{4}\ln\frac{3}{2}$ (B) $\frac{1}{2}\ln\frac{3}{2}$

(C) $\ln\frac{3}{2}$ (D) $\frac{1}{6}\ln\frac{3}{2}$

D. One or More than ONE correct :

47. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ and

$T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$ for n = 1, 2, 3, ... Then **[IIT - 2008]**

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$

(C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$

48. Let f(x) be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'\left(\frac{1}{4}\right) = 0$. Then **[IIT - 2008]**

(A) $f''(x)$ vanishes at least twice on [0, 1]

(B) $f'\left(\frac{1}{2}\right) = 0$

(C) $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx = 0$

(D) $\int_0^{1/2} f(t)e^{\sin \pi t} dt = \int_{1/2}^{1} f(1-t)e^{\sin \pi t} dt$

49. If $\int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x} dx$, n = 0, 1, 2, ..., then **[IIT - 2009]**

(A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(C) $\sum_{m=1}^{10} I_{2m} = 0$ (D) $I_n = I_{n+1}$

50. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$ is (are) **[IIT - 2010]**

(A) $\frac{22}{7} - \pi$ (B) $\frac{2}{105}$

(C) 0 (D) $\frac{71}{15} - \frac{3\pi}{2}$

E. Subjective Problems:

51. Show that : $\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{6n}\right) = \log 6$ **[IIT - 1981]**

52. Evaluate $\int_0^1 (tx + 1 - x)^n dx$, where n is a positive integer and t is a parameter independent of x.

Hence show that $\int_0^1 x^k(1-x)^{n-k} dx = [{}^nC_k (n+1)]^{-1}$ for k = 0, 1,......., n. **[IIT - 1981]**

53. Show that $\int_0^{\pi} xf(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx.$ **[IIT - 1982]**

54. Find the value of $\int_{-1}^{3/2} |x \sin \pi x|\, dx$. **[IIT - 1982]**

55. Evaluate : $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$ **[IIT - 1983]**

56. Evaluate the following $\int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ **[IIT - 1984]**

57. Given a function f(x) such that

(i) It is integrable over every interval on the real line and

(ii) $f(t + x) = f(x)$, for every x and a real t, then show that the integral $\int_a^{a+t} f(x)\, dx$ is independent of a. **[IIT - 1984]**

58. Evaluate the following: $\int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx$ **[IIT - 1985]**

59. Evaluate $\int_0^{\pi} \frac{x\, dx}{1 + \cos\alpha \sin x}$, $< \alpha < \pi$ **[IIT - 1986]**

60. Evaluate $\int_0^1 \log[\sqrt{1-x} + \sqrt{1+x}]\, dx$ **[IIT - 1988]**

61. If f and g are continuous function on [0, a]satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$ then show that $\int_0^a f(x)g(x)dx = \int_0^a f(x)\, dx$ **[IIT - 1989]**

62. Show that $\int_0^{\pi/2} f(\sin 2x) \sin x\, dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x) \cos x\, dx$ **[IIT - 1990]**

63. Prove that for any positive integer k, $\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + ... + \cos (2k-1) x]$

Hence prove that $\int_0^{\pi/2} \sin 2kx \cot x\, dx = \frac{\pi}{2}$ **[IIT - 1990]**

64. If f is a continous function with $\int_0^x f(t)\, dt \to \infty$ as $|x| \to \infty$, then show that every line y = mx intersects the curve $y^2 + \int_0^x f(t)\, dt = 2$ **[IIT - 1991]**

65. Evaluate $\int_0^{\pi} \frac{x \sin(2x) \sin(\pi/2.\cos x)}{2x-\pi} dx$

[IIT - 1991]

66. Determine a positive integer $n \le 5$, such that

$\int_0^1 e^x (x-1)^n dx = 16 - 6e.$ **[IIT - 1992]**

67. Evaluate $\int_2^3 \frac{(2x^5 + x^4 - 2x^3 + 2x^2 + 1)}{(x^2+1)(x^4-1)} dx$

[IIT - 1993]

68. Show that $\int_0^{n\pi+v} |\sin x|\, dx = 2n + 1 - \cos v$

where n is a positive integer and $0 \le v \le \pi$.

[IIT - 1994]

69. Let $I_m = \int_0^{\pi} \frac{1-\cos mx}{1-\cos x} dx$. Use mathematical induction to prove that $I_m = m\pi$, m = 0, 1, 2,.........

[IIT -1995]

70. Evaluate $\int_{-(1/\sqrt{3})}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{2x}{1+x^2}\right) dx$

[IIT - 1995]

71. Let a + b = 4, where a < 2 and let g (x) be a differentiable function, If dg/dx > 0 for all x. Prove that $\int_0^a g(x)dx + \int_0^b g(x)dx$ increases as (b – a) increases **[IIT - 1997]**

72. Evaluate $\int_0^{\pi/4} \ell n(1 + \tan x)\, dx$ **[IIT - 1997]**

73. Determine the value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$

[IIT - 1997]

74. Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = 2\int_0^1 \tan^{-1} x\, dx.$

Hence or otherwise, evaluate the integral

$\int_0^1 \tan^{-1}(1-x+x^2)\, dx.$ **[IIT - 1998]**

75. Integrate $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$ **[IIT - 1999]**

76. For x > 0, let $f(x) = \int_1^x \frac{\ln t}{1+t} dt$. Find the function f(x) + f(1/x) and show that f(e) + f(1/e) = 1/2. Here $\ln t = \log_e t$. **[IIT - 2000]**

77. Let f(x), $x \ge 0$ be a non-negative continuous function and let $F(x) = \int_0^x f(t)\, dt,\ x \ge 0$. If for some $c > 0$, $f(x) \le cF(x)$ for all $x \ge 0$, show that f(x) = 0 for all $x \ge 0$. **[IIT - 2001]**

78. Let f(x) be an even function then prove that

$\int_0^{\pi/2} f(\cos 2x) \cos x\, dx = \sqrt{2} \int_0^{\pi/4} f(\sin 2x) \cos x\, dx.$

[IIT - 2003]

79. Let f(x) be a differentiable function defined as $f[0, 4] \to R$ show that

(i) $8f'(a) f(b) = \{f(4)^2 - f(0)\}^2$; when $a, b \in (0, 4)$.

(ii) $\int_0^4 f(x)\, dx = 2[\alpha f(\alpha^2) + \beta f(\beta^2)]\ \forall\ 0 < a, \beta < 2.$

[IIT - 2004]

80. Find the value of $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$

[IIT - 2004]

81. Evalutate

$\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right) \sin x\, dx$

[IIT - 2005]

G. Match the Columns

82.

	Column - I		**Column - II**
(A)	$\int_{-1}^{1} \frac{dx}{1+x^2}$	(P)	$\frac{1}{2}\log\left(\frac{2}{3}\right)$
(B)	$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$	(Q)	$2\log\left(\frac{2}{3}\right)$
(C)	$\int_2^3 \frac{dx}{1-x^2}$	(R)	$\frac{\pi}{3}$
(D)	$\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$	(S)	$\frac{\pi}{2}$

H. Comprehension Based Questions.

Let the definite integral be defined by the formula $\int_a^b f(x)\, dx = \frac{b-a}{2}(f(a) + f(b))$. For more accurate result for $c \in (a, b)$, we can use

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = F(c)$ so that for $c = \frac{a+b}{2}$, we get $\int_a^b f(x)\,dx$

$= \frac{b-a}{4}(f(a)+f(b)+2f(c))$.

83. $\int_0^{\pi/2} \sin x\,dx$ **[IIT -2006]**

(A) $\frac{\pi}{8}(1+\sqrt{2})$ (B) $\frac{\pi}{4}(1+\sqrt{2})$

(C) $\frac{\pi}{8\sqrt{2}}$ (D) $\frac{\pi}{4\sqrt{2}}$

84. If $\lim_{x\to a} \frac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$,

then f(x) is of maximum degree **[IIT - 2006]**

(A) 4 (B) 3

(C) 2 (D) 1

85. If $f''(x) < 0 \ \forall\ x \in (a, b)$ and c is a point such that $a < c < b$, and (c, f(c)) is the point lying on the curve for which F(c) is maximum, then f'(c) is equal to **[IIT - 2006]**

(A) $\frac{f(b)-f(a)}{b-a}$ (B) $\frac{2(f(b)-f(a))}{b-a}$

(C) $\frac{2f(b)-f(a)}{2b-a}$ (D) 0

I. Integer Answer Type:

86. The value of $5050 \frac{\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx}$ is

[IIT - 2006]

87. For any real number, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10]

by $f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd,} \\ 1+[x]-x & \text{if } [x] \text{ is even} \end{cases}$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\,dx$ is

[IIT - 2010]

88. Let $f: [1,\infty) \to [2,\infty)$ be a differentiable function such that $f(1) = 2$. if **[IIT - 2011]**

$$6\int_1^x f(t)dt = 3x\,f(x) - x^3$$

for all $x \ge 1$, then the value of f(2) is

ANSWERS

CONCEPT PROBLEMS—A

1. 4.5 **2.** π square unit

3. $e-1$.

5. (i) 13.5 (ii) 12

(iii) 18 (iv) $e^2/2$

(v) 1 (vi) cos a – cos b

15. (i) $\frac{b^3-a^3}{3}$ (ii) $\ln \frac{b}{a}$

(iii) $\frac{2}{3}(b^{3/2}-a^{3/2})$.

PRACTICE PROBLEMS—A

7. (A) $(1+\pi)/2$ (B) -4

(C) -8

8. $\frac{25}{2}\pi$

10. The region under the graph of $y = \tan x$ from 0 to $\pi/4$.

11. $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{2+4i/n}{1+(2+4i/n)^5}\cdot\frac{4}{n}$ **12.** 218/3

CONCEPT PROBLEMS—B

1. 3

2. $\int_{-1}^{5} f(x)\,dx$ **3.** –1

5. (i) 4

(ii) -2π

(iii) $4.5-2\pi$

6. (A) $\ge$

(B) $\ge$

(C) $\le$

7. (i) First (ii) Second

(iii) First (iv) Second

8. (A) True (B) False
11. Yes

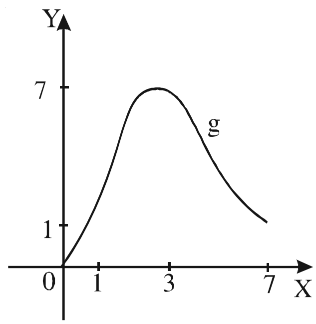
PRACTICE PROBLEMS—B

13. 122
14. (A) True
(B) True
(C) False
15. (A) 4 (B) 10
(C) –3 (D) 2
16. $3+\frac{9}{4}\pi$ **17.** $\frac{6-\pi}{4}$
18. both the integrals are positive
19. $4, \frac{7}{4}, \frac{23}{4}$.
23. $3\sqrt{2} \le \int_0^3 \sqrt{x^3+2}dx \le 3\sqrt{29}$
24. 3/2 **27.** Yes
28. Yes

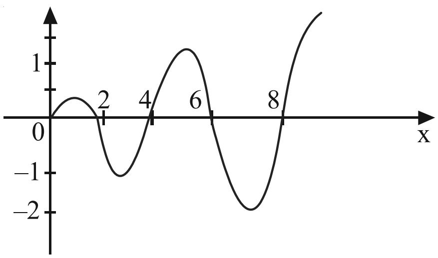
CONCEPT PROBLEMS—C

2. $2x-2$
4. (i) 0
(ii) $-\sin(a^2)$ (iii) $2x\sqrt{1+x^4}$
(iv) $\frac{3x^2}{\sqrt{1+x^6}}-\frac{2x}{\sqrt{1+x^4}}$ (v) $\frac{3x^2}{\sqrt{1+x^6}}-\frac{2x}{\sqrt{1+x^4}}$
(vi) $-\int_{t^2}^{x^2}\frac{xdx}{(x^2+t^4)^{3/2}}+\frac{3x^2}{\sqrt{x^2+x^{12}}}$
5. $y'=\frac{3(1-3x)^3}{1+(1-3x)^2}$ **6.** $\sqrt{257}$
7. $-\frac{x^2}{2}(1+3x)^{-3/2}(6\sin 2x+15x\sin 2x+4x\cos 2x + 12x^2\cos 2x)$
8. $f(a)=0$.
9. $F(x)=\int_1^x t^2\sin(t^2)dt$ **10.** (b) $\sec^2 x$.
11. (A) $(0,\infty)$ (b) $x=1$ **12.** (a) $(-3,3)$ (b) $x=1$
13. (A) True. Since f is continuous, g is differentiable by First Fundamental Theorem of Calculus.
(B) True, g is continuous because it is differentiable.
(C) True, since $g'(1)=f(1)=0$
(D) False, since $g''(1)=f'(1)>0$.
(E) True, since $g'(1)=0$ and $g''(1)=f'(1)>0$.
(F) False, $g''(x)=f'(x)>0$, so g'' never changes sign
(G) True, since $g'(1)=f(1)=0$ and $g'(x)=f(x)$ is an increasing function of x (because $f'(x)>0$).
14. (A) $x=3$
(B) Increasing on $[3,\infty)$, decreasing on $(-\infty,3]$

Practice Problems—C

15. $\sqrt{1+\sin^4 x}.\cos x$ **16.** –1
17. $\frac{2}{17}$ **18.** $\frac{1}{20}$
20. (i) 1
(ii) 1
(iii) 4
22. (A) 1/4 (b) $\sqrt[3]{12}$
23. $e^{2x}(1+2x)/(1-e^{-x})$ **24.** $\pi/2$
25. (i) $\left\{\frac{\pi n}{2}+\frac{\pi}{4}, n\in W\right\}$
(ii) $\{\pi n/2 \pm (1/4)\cos^{-1}(2a-1)/3, n\in I\}$ for $a \in (-1,2)$, for $a \notin [1,2]$ the function has no critical points.
26. (A) 0, 2, 5, 7, 3
(B) (0, 3)
(C) $x=3$
(D)


27. (A) local max. at 1 and 5; local min. at 3 and 7
(B) 9
(C) (1/2, 2), (4, 6), (8, 9)

(D)


28. $f = c, f' = b, \int_0^x f(t)dt = a$

29. 1/2

30. $[-1, 2]$

31. $f(x) = \frac{1}{2}x$ or $f(x) = 0$

32. 1/2.

33. $(b^b a^{-a})^{\frac{1}{b-a}} e^{-1}$

34. 2/3

CONCEPT PROBLEMS—D

1. (A) 5 (B) $\frac{26}{3}$

2. $F(x) = 3^x + x - 4, x = 1$

3. 29.

4. $F(x) = y_0 + \int_{x0}^{x} f(t)dt$

6. (A) $\frac{14}{3}$

(C) The statement is true.

7. (B) No

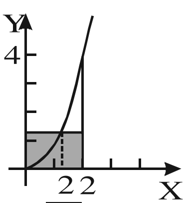
8. (A) $2/3, 10^{3/2}/15, 100^{3/2}/150$.

(B) 10

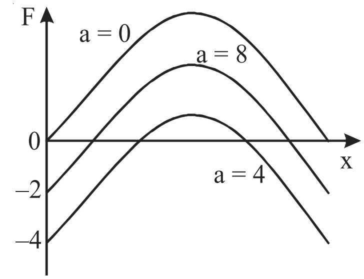
(C) 0

9. (A) $\frac{4}{3}$

(B) $\frac{2}{\sqrt{3}}$

(c)


$\frac{2}{\sqrt{3}}$ 2

10.


11. (A) $0, \frac{3}{2}$ (B) 0

12. $P(x) = 4x + 8x^2 + 3x^3$

13. $K = 3, L = 1, M = 5$

14. $\frac{1}{3}(4\sqrt{2} - 1)$.

15. The curves coincide for $x \in R$.

16. The increase in the child's weight in kg between the ages of 5 and 10.

PRACTICE PROBLEMS—D

17. (i) $\frac{\pi^4}{1024} + \frac{\pi}{2} + 2$

(ii) $\frac{\pi^2}{18} - \frac{\pi}{3\sqrt{3}} + 2\log_e\left(\frac{2}{\sqrt{3}}\right)$

(iii) 4

(iv) $\sqrt{3} - 1$

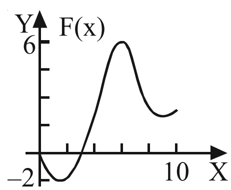
18. $5 + \frac{9}{4\log_e 2}$

19. (A) $-F(0) = 0, F(3) = 0, F(5) = 6, F(7) = 6, F(10) = 3$

(B) increasing on $\left[\frac{3}{2}, 6\right]$ and $\left[\frac{37}{2}, 10\right]$,

decreasing on $\left[0, \frac{3}{2}\right]$ and $\left[6, \frac{37}{4}\right]$

(C) maximum $\frac{15}{2}$ at $x = 6$, minimum $-\frac{9}{4}$ at $x = \frac{3}{2}$

(D)


20. $F(x) = \begin{cases} (1 - x^2)/2, & x < 0 \\ (1 + x^2)/2, & x \geq 0 \end{cases}$

21. $y = 2 - x; y = x - 3$

22. $\left\{\frac{\pi}{6}, \frac{\pi n}{2} \pm \frac{\pi}{6}, n \in N\right\}$

23. $\max\limits_{x\in[-1/2,\,1/2]} F(x) = F\left(\frac{1}{2}\right) = -\frac{3}{8}$,

$\min\limits_{x\in[-1/2,\,1/2]} F(x) = F\left(-\frac{1}{2}\right) = -\frac{5}{8}$.

24. $f(p) = -\pi/2$.
25. $3\sqrt{3} - 2\sqrt{2} - 1$
26. $f(x) = x^{3/2}, a = 9$.
27. 1
28. $[4, \infty)$
29. $a = 1$
30. $\sqrt{2\pi}, \frac{-1+\sqrt{8\pi+1}}{2}$
31. $A = 7, B = -6, C = 3$
32. $-\pi, -\frac{\pi}{3}, 0$
33. $\sqrt{3} + \frac{1}{2}x + \sin x + \pi/6$ if $0 \le x \le 2\pi/3$; $2\sqrt{3} - \frac{1}{2}x - \sin x + 5\pi/6$ if $2\pi/3 \le x \le \pi$
34. $0, \pm\sqrt{2}$
35. (A) Displacement $= -\frac{1}{2}$; distance $= \frac{1}{2}$

(B) Displacement $= \frac{3}{2}$; distance $= 2$

36. $(v_0 + 2gh)/2$
37. $\frac{263}{4}$
38. 1 m.
41. (i) $x = 2$
(ii) $x = \ln 4$
(iii) $\{-1, 4\}$
42. Number of litres of oil leaked in the first 2 hours.
43. Number of bees in the first 15 weeks.
44. $46\frac{2}{3}$ kg

CONCEPT PROBLEMS—E

1. (i) Yes (ii) No
2. (i) Yes (ii) Yes
3. (i) Yes
(ii) Yes
(iii) No
4. (A) Yes (B) Yes
(C) No (D) Yes
5. a, b, d are defined.
6. (i) Yes (ii) No
8. Consider $f = \begin{cases} 1 & \text{if } x \text{ rational} \\ -1 & \text{if } x \text{ irrational} \end{cases}$.
9. (i) Does not exist
(ii) Does not exist
(iii) π
10. No
11. (A) 16 (B) 4 (C) 10
13. (i) 3 (ii) 22 (iii) $-\frac{9}{2}$ (iv) 6

PRACTICE PROBLEMS—E

16. a, b, c are integrable.
21. (A) Yes (B) No
(C) No (D) Yes
(E) Yes (F) No
22. No, Yes, No, No
23. No, consider the example

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ -1 & \text{if } x \text{ is an irrational number} \end{cases}$$

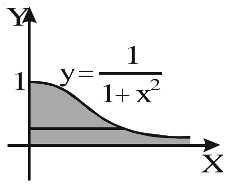
24. Make sure that the function f(x) is continuous both inside the interval (0, 1) and at the end-points $[\lim\limits_{x\to 0^+} f(x) = f(0)$ and $\lim\limits_{x\to 1^-} f(x) = f(1)]$.
25. No. Consider the function

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } x \text{ is irrational, on the interval } [0,1] \end{cases}$$

26. (A) $2/\pi, 0, 0, (2/\pi)\cos^{-1}a$
(B) $-1, -1/3, 1/2, 0, 1$
(C) Not necessarily because of discontinuity.
27. 5/6
29. (B) $2\sum_{k=1}^{8} k(\sqrt{k+1} - \sqrt{k}) = 2(21 - 3\sqrt{2} - \sqrt{3} - \sqrt{5} - \sqrt{6} - \sqrt{7})$
30. (C) $x = 1, x = \frac{5}{2}$
31. No, f is discontinuous.

CONCEPT PROBLEMS—F

1. (B), (C), (D)
2. (A) Infinite interval
(B) Infinite discontinuity
(C) Infinite discontinuity
(D) Infinite interval
3. The integral is improper; the integrand is undefined at 0.
4. both limits are ¥

5. 

7. (i) $1-\ln 2$ (ii) $1/2$
(iii) 1 (iv) π

8. (i) $-\dfrac{2}{e}$ (ii) $\pi/2$

(iii) $\dfrac{33\pi}{2}$ (iv) $\dfrac{\pi}{\sqrt{3}}$

9. Error when we move from step (3) to step(4)

PRACTICE PROBLEMS—F

15. (i) $\ln\tan\theta = -\ln\tan\left(\frac{1}{2}\pi - \theta\right)$

(ii) Use $2\sin^2\theta = 1-\cos 2\theta$, quote (i) and integrate $\cos 2\theta \ln\tan\theta$ by parts. Note $\sin 2\theta \ln\tan\theta \to 0$ as $\theta \to 0, \frac{1}{2}\pi$

17. (i) $\dfrac{1}{2}$ (ii) $\dfrac{1}{3}$

18. $t=\tan\theta$ gives I, on using partial fractions, as

$$\frac{1}{b^2-a^2}\int_0^{\infty}\left\{\frac{b^2}{a^2+b^2t^2}-\frac{1}{1+t^2}\right\}dt$$

22. 8

CONCEPT PROBLEMS—G

1. (A) 1/6 (B) $\dfrac{1}{\sqrt{2}}\ln\left(\dfrac{9+4\sqrt{2}}{7}\right)$

(C) $2-\pi/2$ (D) $\pi^2/4$

2. (i) $\dfrac{1}{5}\log 6+\dfrac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$

(ii) $\dfrac{4\sqrt{2}}{3}$ (iii) 0

(iv) $\dfrac{\pi}{3}$

3. Yes in each case ; $\pi/4$ **5.** $-4/5$

6. 2

8. Avni is right.

9. The limits of integration cannot be covered by sint.

11. The function $x=\pm\sqrt{t^5}$ is double-valued. To obtain the correct result it is necessary to divide the initial interval of integration into two parts :

$$\int_{-2}^{2}\sqrt[5]{x^2}\,dx=\int_{-2}^{0}\sqrt[5]{x^2}\,dx+\int_{0}^{2}\sqrt[5]{x^2}\,dx$$ and apply the substitutions $x=-\sqrt{t^5}$ in $-2<x<0$ and $x=\sqrt{t^5}$ in $0<x<2$.

12. It is impossible, since $\sec t \geq 1$ and the interval of integration is [0, 1]

13. Yes, it is possible

14. The antiderivative $F_1(x)$ will lead to the correct result and $F_2(x)$ to the wrong one, since this function is discontinuous in the interval $[0, \pi]$

PRACTICE PROBLEMS—G

15. (i) $2[\cos 2-\cos 3]$ (ii) $\dfrac{8-3\sqrt{3}}{3}$

(iii) -2 (iv) $\dfrac{\pi^2}{8}$

16. (i) $\dfrac{1}{2}\ln(2+\sqrt{3})$ (ii) π

(iii) $\dfrac{1}{3}\ell n2$ (iv) $2-\dfrac{2}{3}\ln 2-\dfrac{2\pi}{3\sqrt{3}}$

17. (i) $\dfrac{4-3\sqrt{2}}{4}$ (ii) $\dfrac{\sqrt{3}}{32}$

(iii) $\sqrt{2\pi}-\sqrt{\dfrac{\pi}{\sqrt{2}}}$ (iv) $\dfrac{33\pi}{2}$

18. ln 11.

19. The substitution $x=\tan\dfrac{x}{2}$ is not applicable, since the function $\tan\dfrac{x}{2}$ is discontinuous at $x=\pi$.

21. $\dfrac{\pi}{8}\ln 2$ **22.** $\dfrac{1}{4}$

23. 2009

CONCEPT PROBLEMS—H

1. (A) 0.5 ln(e/2)
(B) 4π
(C) 1

3. Apply integration by parts to f and G. Note that $G' = g$.

4. 2

PRACTICE PROBLEMS—H

6. (A) $0.5\ln 3-\dfrac{\pi\sqrt{3}}{2}$ (B) $\dfrac{5e^3-2}{27}$

(C) $\dfrac{4\pi}{3}-\sqrt{3}$ (D) $2\pi\left(\dfrac{1}{\sqrt{3}}-\dfrac{1}{2\sqrt{2}}\right)$

(E) 1/6 (f) $\dfrac{\pi^3}{6}-\dfrac{\pi}{4}$

7. (i) 2π (ii) $\left(\pi-\dfrac{\pi^2}{4}\right)$

(iii) $\ln 2$

(iv) $\dfrac{\pi}{4}\left(\dfrac{\pi}{4}-1\right)+\dfrac{1}{2}\ell n 2$

8. –1.

10. (A) $n=4$ (B) 2

11. (E) $f(b)=f(0)+f^{(1)}(0)b+\dfrac{f^{(2)}(0)b^2}{2}+\dfrac{f^{(3)}(0)b^3}{6}$

$+\dfrac{1}{6}\int_0^b f^{(4)}(x)(b-x)^3dx$

PRACTICE PROBLEMS—I

1. $(-1)^n n!\left[1-\dfrac{1}{e}\left(\dfrac{1}{n!}+\dfrac{1}{(n-1)!}+\ldots+\dfrac{1}{1!}+1\right)\right]$.

3. $\dfrac{4}{3}$ **5.** $(5\pi/32)a^6$

6. (i) $\pi a^4/32$; (ii) $5/4-3\pi/8$;

(iii) 1/2.

10. $\dfrac{x}{6(x^2+1)^3}+\dfrac{5x}{24(x^2+1)^2}+\dfrac{5x}{16(x^2+1)}+\dfrac{5}{16}\tan^{-1}x$

13. (A) $\dfrac{\pi(2m)!(2n)!}{2^{2m+2n+1}m!n!(m+n)!}$

(B) 0, if n is even, π, in n is odd

(C) $\pi/2^n$

(D) $\dfrac{\pi}{2^n}\sin\dfrac{n\pi}{2}$

14. $\dfrac{p!q!}{(p+q+1)!}$. **16.** $(-1)^k\dfrac{k!}{n^{k+1}}$.

17. $\dfrac{71}{105}$

18. (A) $\dfrac{(m-1)(m-3)\ldots 3\times 1}{m(m-2)\ldots 4\times 2}\dfrac{\pi}{2}$

(B) $\dfrac{(m-1)(m-3)\ldots 4\times 2}{m(m-2)\ldots 3\times 1}$. Put $x=\sin\varphi$.

19. $2\dfrac{2n(2n-2)\ldots 4\times 2}{(2n+1)(2n-1)\ldots 3\times 1}$. Put $x=\sin\varphi$.

20. $\dfrac{3^n\cdot n!}{1\cdot 4\cdot 7\cdot\ldots\cdot(3n+1)}$

PRACTICE PROBLEMS—J

1. (i) $\dfrac{1}{\pi}\int_0^\pi \cos x dx=0$

(ii) $\int_0^{\pi/6}\sec^2 x dx=\dfrac{\sqrt{3}}{3}$

2. (i) $\dfrac{1}{3}\pi$ (ii) $\dfrac{\pi}{2}$

(iii) $\ln 3$

3. (i) $\dfrac{1}{3}\ln 2$ (ii) $\dfrac{16}{3}$

(iii) $\ln\dfrac{b}{a}$ (iv) $2\left(\sqrt{2}-1\right)$

4. (i) 1 (ii) $e^{\frac{\pi^2}{12}}$

6. (i) ln 4] (ii) 3/8

(iii) $\tan^{-1}2+\dfrac{1}{2}\ln 5$

7. (i) $\dfrac{\pi}{4}+\dfrac{1}{2}\ell n2$ (ii) $\dfrac{2^k}{k+1}$

(iii) 2 (iv) $1/\sqrt{2}$

8. Consider upper and lower sums for the integral $\int_1^m \dfrac{1}{x}dx$ obtained by dividing the interval [1, m] into m – 1 equal parts.

10. $\dfrac{1}{k+1}$; $\approx 1.67\times 10^{11}$.

CONCEPT PROBLEMS—I

1. $\cos(e^x)\,e^x-\cos(x^2)2x$ **2.** $3x^2\sqrt{\cos x^3}$

3. $\frac{1}{3}$.
4. $x^2(2x \sin x^2 - \sin x) + 2(\cos x - \cos x^2)x$
5. 4
6. (i) $\frac{dy}{dx} = \cot t$ (ii) $\frac{dy}{dx} = -t^2$
7. $\frac{2}{17}$
8. –2.

PRACTICE PROBLEMS—K

9. $y' = -\frac{\cos x}{e^y}$.
10. $1 + \frac{2}{e}$
11. $\frac{8x^2}{1-5x^6} + \frac{40x^6+4}{(1-5x^6)^{3/2}} \int_0^{x^2} \frac{dt}{\sqrt{1-5t^3}}$
12. $-t^2$
13. $\frac{1}{101}(\tan x - \sec x + 1)$
14. $-\cos x$
15. $[0, 2]$
16. $x = 0$
17. Minimum at $x = 0$.
18. $\frac{1}{2}$
19. $c = 1$; limit $= \frac{1}{2}\sqrt{3}$

PRACTICE PROBLEMS—L

2. $f(x) + f(1/x) = \frac{1}{2}(\ln x)^2$
5. (i) 0 (ii) 0
   (iii) 60
7. (i) 0 (ii) $\frac{3\pi+1}{\pi^2}$
   (iii) π (iv) 4 ln 4/3
9. 8

CONCEPT PROBLEMS—J

1. (B) (i) $\frac{3}{2}$ (ii) $\frac{\pi}{4}$
6. $2l_1 = \pi l_2$

PRACTICE PROBLEMS—M

7. (i) 0 (ii) $\frac{\pi}{8} \cdot \ln 2$
   (iii) $\frac{\pi}{4}$ (iv) $\frac{\pi}{3\sqrt{3}}$
8. (i) 0 (ii) 0
   (iii) $\left(1 - \frac{\pi}{4}\right)$ (iv) $\pi/2$
9. (i) 4 (ii) $-\pi/2$
   (iii) 0
10. (i) π (ii) $\frac{\pi}{2\sqrt{2}} \ln(1 + \sqrt{2})$
    (iii) $\frac{\pi^2}{16}$
11. (i) $\frac{\pi}{16}\ln 2$ (ii) $\pi[\frac{\pi}{4} - (\sqrt{2} - 1)]$
    (iii) $\frac{\pi^2}{4}$ (iv) $\frac{\pi^2}{32}$

PRACTICE PROBLEMS—N

2. (i) $\frac{\pi}{2}\ln 2$ (ii) $\frac{\pi}{2}(\pi - 2)$
   (iii) $\pi \ln 2$ (iv) $\pi \ln 2$
4. (i) π^2 (ii) $-\frac{\pi^2}{2} \cdot \ln 2$
   (iii) $-\frac{\pi}{2}\ln 2$ (iv) $\frac{3\pi}{2}\ln 2 - \frac{\pi^3}{16}$
5. (i) $\frac{\pi^2}{32}$ (ii) $\frac{3\pi^2}{512}$
   (iii) $\frac{\pi^2}{12}$ (iv) $\frac{8}{15}\pi$
6. $\frac{1}{2}\left(\frac{1}{2} + \frac{1}{\pi+2} - A\right)$

PRACTICE PROBLEMS—O

2. (i) $3(e-1)$ (ii) 1
   (iii) 5 (iv) $\frac{\pi}{4}$
4. 0
6. $-\alpha$

PRACTICE PROBLEMS—P

3. $\tan^{-1}t + \frac{t}{1+t^2} - \frac{1}{2t}$ 4. 0
5. 48
6. $\frac{2}{3}$

CONCEPT PROBLEMS—K

1. No
2. No
9. $\pi < I < 2\pi$.
11. To estimate the integral from below use the inequality $1 + x^4 < (1 + x^2)^2$, the Schwartz-Bunyakovsky being used for estimating it from above.

PRACTICE PROBLEMS—Q

16. (A) Plus (B) Minus
(C) Plus
18. $2\sin^{-1}\frac{1}{3}$
21. The function $f(x) = \begin{cases} x^x & \text{at } 0 < x \le 1 \\ 1 & \text{at } x = 0 \end{cases}$ is continuous on the interval, it reaches the least value $m = e^{-\frac{1}{e}} \approx 0.692$ at $x = \frac{1}{e}$ and the greatest value $M = 1$ at $x = 0$ and at $x = 1$.
23 8/5
24. (ii) $0.85 < I < 0.90$.
26. $I(1) \approx 1.66$ the greatest value; $I\left(-\frac{1}{2}\right) \approx -0.11$ the least value.

PRACTICE PROBLEMS—R

3. 3
4. x or 0. 5. $f(t) = -\sin t;\ c = \pi/3$
6. $f(t) = \sin t - 1;\ c = 0$
7. (A) 16 (B) $1 + 3/\sqrt{2}$
(C) $(36)^{1/3}$ (D) 1/5
8. $p(x) = 3 + \frac{1}{2}x + \frac{1}{4}x^2$ 9. $f(x) = 3e^{3x}, a = \frac{1}{3}\ln 2$
10. (A) No such function
(B) $-2^x \ln 2$
(C) $\frac{1}{2}x \pm 1$
11. (A) $\frac{2}{3}x^2(x + |x|)$
(B) $x - \frac{1}{3}x^3$ if $|x| \le 1$; $x - \frac{1}{2}x|x| + \frac{1}{6}\frac{|x|}{x}$ if $|x| > 1$
(C) $1 - e^{-x}$ if $x \ge 0$; $e^x - 1$ if $x < 0$
(D) x if $|x| \le 1$; $\frac{1}{3}x^3 + \frac{2}{3}\frac{|x|}{x}$ if $|x| > 1$
12. $f(x) = 2x^{15};\ c = -\frac{1}{9}$ 13. $3 + 3\ln x$

PRACTICE PROBLEMS—S

1. (i) $\frac{8}{15}$ (ii) $\frac{5}{32}\pi$
2. (i) $\frac{8}{315}$ (ii) $\frac{16}{3.5.7.11}$
(iii) $\frac{1}{24}$ (iv) $\frac{5\pi}{2^{11}}$
3. (i) $\frac{a^9}{9}$ (ii) $\frac{\pi}{2}$
(iii) $\frac{2}{63}$ (iv) $\frac{33}{16}\pi a^7$
4. (i) $\frac{5\pi}{192}$ (ii) $\frac{\pi}{14} - \frac{16}{245}$
(iii) $\frac{2^5}{3.7.11.13}$ (iv) $\frac{2^{13}}{5.7.913.17}$
5. (i) $\frac{2.4.6...(2n)}{3.5.7....(2n+1)}$ (ii) $\frac{1.3.5...(2n-1)}{2.4.6....2n}.\frac{\pi}{2}$
(iii) $\frac{63}{8}\pi a^5$ (iv) $\frac{\pi}{32a^3}$
7. $\frac{35\pi}{128}$

PRACTICE PROBLEMS—T

4. 0
5. Change the variable by the formula $z = k\omega^2 x^2$, and then apply L'Hospital's rule.

PRACTICE PROBLEMS—U

1. $\int_2^3 \cos(x+t^2)2t\,dx$ 4. $\pi \ln\left(\frac{1+\sqrt{1-b^2}}{2}\right)$

5. $\frac{\pi}{2}\left\{\ell n\left(a+\sqrt{a^2+1}\right)\right\}$ 6. $\frac{1}{2}\left(\frac{\pi^2}{4}-\theta^2\right)$

7. $\frac{\pi^2 a^2}{16}+\frac{a^2}{4}$

PRACTICE PROBLEMS—V

1. $(x+1)\,ln\,(1+x)-x$ 2. $\frac{1}{1-x}$

3. Integrating both sides between 0 and t yields

$$\ln(1+t)-\left(t-\frac{t^2}{2}+...-\frac{t^{2n}}{2n}\right)=\int_0^t \frac{x^{2n}}{1+x}dx.$$

Now use

$$0\le\int_0^t \frac{x^{2n}}{1+x}dx\le\int_0^t x^{2n}dx=\frac{t^{2n+1}}{2n+1}.$$

4. (i) 0 (ii) $-\frac{\pi^2}{6}$

(iii) $\ln 2$ (iv) $\frac{\pi}{2(a+b)}$

8. $\frac{1}{2}$

OBJECTIVE EXERCISES

1. A 2. A 3. D
4. D 5. D 6. C
7. A 8. B 9. D
10. C 11. B 12. A
13. A 14. A 15. C
16. C 17. A 18. A
19. A 20. B 21. D
22. B 23. B 24. B
25. C 26. B 27. A
28. C 29. C 30 D
31. B 32. A 33. A
34. A 35. B 36. B
37. C 38. B 39. B
40. B 41. C 42. D
43. C 44. D 45. C
46. C 47. C 48. B
49. C 50. A 51. C
52. A 53. A 54. C
55. A 56. A 57. A
58. D 59. A 60. C
61. D 62. A 63. D
64. D 65. C 66. AC
67. ABCD 68. ABCD 69. ACD
70. ABCD 71. ACD 72. AB
73. AB 74. AC 75. AD
76. AC 77. ACD
78. ABCD 79. ACD
80. ABCD 81. ABC 82. AB
83. ACD 84. AB
85. ABCD 86. C 87. C
88. A 89. C 90. A
91. B 92. B 93. B
94. C 95. D 96. C
97. A 98. A 99. C
100. B 101. A 102. C
103. C 104. A 105. B
106. C 107. D 108. A
109. A 110. A
111. (A)–(R) ; (B)–(P) ; (C)–(S) ; (D)–(R)
112. (A)–(R) ; (B)–(P) ; (C)–(S) ; (D)–(Q)
113. (A)–(Q) ; (B)–(RS) ; (C)–(QS) ; (D)–(S)
114. (A)–(S) ; (B)–(S) ; (C)–(PR) ; (D)–(R)
115. (A)–(R) ; (B)–(S) ; (C)–(Q) ; (D)–(Q)

REVIEW EXERCISES for JEE ADVANCED

3. Having worked C one finds S by obsering that $a^2C+b^2S=\pi$.

12. $g(x)=cx^{\sqrt{2}+1}, c\in R$ 13. $\frac{275}{48}$

14. $6\ln 3-4$ 15. $3\ln(ex)$

21. $4\ln x$

22. (A) $-Ae^{-a}$ (B) $\frac{1}{2}A$

(C) $A+1-\frac{1}{2}e$ (D) $e\ln 2-A$

23. $a=0$ 24. $\frac{\pi}{2}\frac{1.3.5...(2n-1)}{2^n n!}$

25. $\ln(1+\alpha), (\alpha>-1)$.

26. The value of integral is $\pi/2$ if $2n\pi < \alpha < (2n+1)\pi$, and $-\pi/2$ if $(2n-1)\pi < \alpha < 2n\pi$, n being any integer; and 0 if α is a multiple of π.

33. $(0, 4)$

34. (i) $-\dfrac{\sqrt{3}}{4\pi}$ (ii) $2 + \ln\dfrac{2}{e^2+1}$.

35. $a = 9,\ b = \dfrac{27}{2}$

36. Use the identity $\dfrac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x} = 1 + 2\cos x + 2\cos 2x + \ldots + 2\cos nx$ to prove that $\dfrac{\sin nx}{\sin x} = 2\cos(n-1)x + 2\cos(n-3)x + \ldots$, where the last term is 1 or $2\cos x$.

39. $\dfrac{(2n-2)(2n-4)\ldots.4.2}{(2n-1)(2n-3)\ldots..5.3.1}$ **42.** $\dfrac{16}{3}$

44. Apply the Schwartz-Bunyakovsky inequality in the form

$$\left[\int_a^b \sqrt{f(x).\frac{1}{f(x)}}\,dx\right]^2 \le \int_a^b f(x)dx \int_a^1 \frac{1}{f(x)}dx$$

47. $\ln 4$

49. (C) $P(x) = \dfrac{1}{2}(x-[x])^2 - \dfrac{1}{2}(x-[x])$

(D) $\dfrac{1}{12}$

50. (B) $g(2) = 2A,\ g(5) = 5A$
(C) $A = 0$

TARGET EXERCISES for JEE ADVANCED

1. (i) $\dfrac{\pi}{3}$ (ii) $\dfrac{\sqrt{2}}{9\pi^2}(52 - 15\pi)$

2. The integral reduces, with $t = \tan\frac{1}{2}\theta$, to

$$\frac{1}{b}\int_0^1 \frac{dt}{1+t^2} + \frac{b^2-a^2}{b}\int_0^1 \frac{dt}{(a-b)^2+(a+b)^2t^2}$$

The difference between $a > b$ and $a < b$ comes in the answer to the second of these integrals.

11. (A) $\pi - \dfrac{1}{2}$ (B) $\dfrac{1}{2}$

(C) $\dfrac{1}{2} + (\pi - \dfrac{1}{2})(t-1)$ (D) $\dfrac{1}{2}(t-1) + (\pi - \dfrac{1}{2})(t-1)^2/2$

14. (A) $G(0) = G(1) = G(2) = \pi/4$

16. $\dfrac{2}{3}(3\sqrt{31} + \sqrt{3} - 11.35)$

17. (A) $x \ge 1$ (c) $F(ax) - F(a); F(x) - \dfrac{e^x}{x} + e; xe^{1/x} - e - F\left(\dfrac{1}{x}\right)$

18. (A) $(i_0)^2/2$

(B) $\dfrac{(i_0)^2}{2} - \dfrac{(i_0)^2 T}{8\pi t_0}\left[\sin(\dfrac{4\pi t_0}{T} + 2\varphi) - \sin 2\varphi\right]$

(C) $(i_0)^2/2$

19. (i) $\dfrac{3+2\sqrt{3}}{4}\pi - \dfrac{3}{2}\ln 2$

(ii) $\dfrac{5\pi}{3}$. Put $x = \cos\varphi$ and integrate by parts

(iii) 0

21. 0.099

32. Integrate the inequality

$$\sqrt{x\sin x} > \sqrt{x^2 - \frac{x^4}{6}} = x\sqrt{1 - \frac{x^2}{6}} \text{ on } 0 \le x \le \frac{\pi}{6}$$

and write Schwartz-Bunyakovskyn equality

$$\int_6^{\pi/2}\sqrt{x\sin x} \le \sqrt{\int_0^{\pi/2} x\,dx \int_0^{\pi/2}\sin x\,dx} = \sqrt{\frac{\pi^2}{8}} = \frac{\pi}{2\sqrt{2}}$$

33. In I_2 note that $f(\xi_n)$ is bounded , $f(x)$ being continuous.

37. $|f(x)| \le M$, say, since $f(x)$ is continuous; and so $|I_n| \le M(b-a)/\sqrt{n}$

42. $P_1(x) = x - \dfrac{1}{2}$; $P_2(x) = x^2 - x + \dfrac{1}{6}$; $P_3(x) = x^3 - \dfrac{3}{2}x^2 + \dfrac{1}{2}x$; $P_4(x) = x^4 - 2x^3 + x^2 - \dfrac{1}{30}$; $P_5(x) = x^5 - \dfrac{5}{2}x^4 + \dfrac{5}{3}x^3 - \dfrac{1}{6}x$

43. $1/(k+m+1) = \int_0^1 t^{k+m}dt$

44. (B) constant $= \frac{3}{2}$

46. (B) $b = \log 2$ (C) $e^2 Li(e^{2x-2})$

48. Consider $h(x) = [F(x) - f(x)]^2 + [G(x) - g(x)]^2$

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. $-\left(\frac{15\pi+32}{60}\right)$ **2.** $2-\sqrt{2}$

3. 4 **4.** $\pi(\sqrt{2}-1)$

5. 1/2

6. $\frac{1}{a^2-b^2}\left[a(\log 2-5)+\frac{7b}{2}\right]$ **7.** π^2

8. 2 **9.** 16 **10.** T

11. D **12.** B **13.** A

14. C **15.** D **16.** B

17. D **18.** A **19.** C

20. D **21.** D **22.** C

23. D **24.** B **25.** C

26. A **27.** A **28.** A

29. A **30.** C **31.** B

32. C **33.** B **34.** C

35. C **36.** A **37.** C

38. A **39.** A **40.** D

41. A **42.** B **43.** C

44. C **45.** B **46.** A

47. AD **48.** ABCD **49.** ABC

50. A

52. $\frac{t^{n+1}-1}{(t-1)(n+1)}$ **54.** $\frac{3}{\pi}+\frac{1}{\pi^2}$

55. $\frac{1}{20}\log 3$ **56.** $\frac{6-\pi\sqrt{3}}{12}$ **58.** $\frac{\pi^2}{16}$

59. $\frac{\pi\alpha}{\sin\alpha}$ **60.** $\frac{1}{2}\left[\log 2+\frac{\pi}{2}-1\right]$

65. $\frac{8}{\pi^2}$ **66.** $n=3$

67. $\frac{3}{2}\log 2-\frac{1}{10}$ **68.** $2n+1-\cos v$

70. $\frac{\pi}{2}\left[\frac{\pi}{6}+\frac{1}{2}\ell n\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}-\frac{2}{\sqrt{3}}\right|\right]$

72. $\frac{\pi}{8}\ell n\, 2$ **73.** π^2

74. $\log 2$ **75.** $\left(\frac{\pi}{2}\right)$

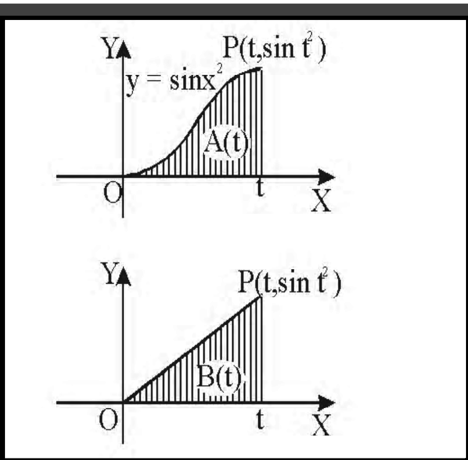
78. 2π **80.** $\frac{4\pi}{\sqrt{3}}\left[\tan^{-1}-\frac{\pi}{4}\right]$

81. $\frac{24}{5}\left[e\cos\left(\frac{1}{2}\right)+\frac{1}{2}e\sin\left(\frac{1}{2}\right)-1\right]$

82. (A)–(S) ; (B)–(S) ; (C)–(P) ; (D)–(R)

83. A] **84.** D **85.** A

86. 5051 **87.** 4 **88.** 6



CHAPTER

3

AREA UNDER THE CURVE

3.1 CURVE SKETCHING

One of the applications of definite integral is the calculation of areas bounded by curves. In the previous chapter we dealt with definite integration as the limit of a sum. The concept of integration arose in connection with determination of area. We shall now use definite integral to find the area of plane regions bounded by curves.

Let us first focus on the techniques of tracing the curves bounding the plane regions. The following procedure is to be applied in sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to evaluate the area under the curves.

1. Extent

Find the domain of definition of the function and the values of the function at the points of discontinuity (if possible) and the endpoints of the domain.

We should find out if there is any region of the plane such that no part of the curve lies in it. Such a region is easily obtained on solving the equation for one variable in terms of the other.

The curve will not exist for those values of one variable which make the other imaginary. For this, find the value of y in terms of x from the equation of the curve and find the value of x for which y is imaginary. Similarly, find the value of x in terms of y and determine the values of y for which x is imaginary. The curve does not exist for these values of x and y.

For example, the values of y obtained from $y^2 = 4ax$ are imaginary for negative values of x. So, the curve does not exist on the left side of y-axis.

Similarly, the curve $a^2y^2 = x^2(a - x)$ does not exist for $x > a$ as the values of y are imaginary for $x > a$.

In the curve, $a^2y^2 = x^2 (x - a) (2a - x)$, we find that for $0 < x < a$, y^2 is negative, i.e., y is imaginary. Therefore, the curve does not exist in the region bounded by the lines $x = 0$ and $x = a$. For $a < x < 2a$, y^2 is positive i.e., y is real. Therefore, the curve exist in the region bounded by the lines $x = a$ and $x = 2a$.

2. Intercepts

When we put $x = 0$ in the equation of the curve, we get the y-intercept and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (We can omit this step if the equation is difficult to solve.)

We should also see whether the curve passes through the origin or not. If the point (0, 0) satisfies the equation of the curve, it passes through the origin.

3. Sign Scheme of f(x)

When graphing a function f, the zeros of the function, i.e. solutions of the equation $f(x) = 0$ and the points of discontinuity of the function divide its domain of definition into intervals where the function is of constant sign. The sign scheme of the function helps in locating parts of the graph which lie above/below the x-axis.

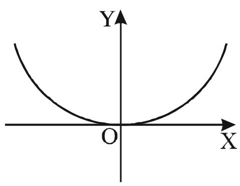
4. Symmetry

The aim of symmetry is to make the calculations as short as possible. Indeed, if a function is even or odd, then we may consider the part of the domain which belongs to the positive x-axis, rather than the whole domain. On this part of the domain we must carry out complete investigation of the behaviour of the function and construct its graph, then, resorting to symmetry, complete the construction on the whole of the domain. Consider the graph of an equation $F(x, y) = 0$ in the x-y plane.

(i) The graph of $F(x, y) = 0$ is symmetric about the y-axis if on replacing x by $-x$, the equation of the curve does not change. i.e. $F(x, y) = 0$ implies $F(-x, y) = 0$.

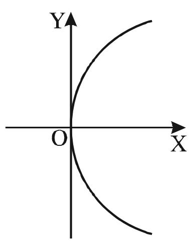
If F represents a function, then it is said to be even.

For example, the parabola $y = x^2$ is symmetric with respect to the y-axis.



(ii) The graph of $F(x, y) = 0$ is symmetric about the x-axis if on replacing y by $-y$, the equation of the curve does not change. i.e. $F(x, y) = 0$ implies $F(x, -y) = 0$.

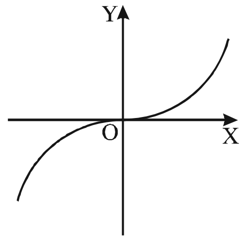
For example, the parabola $y^2 = x$ is symmetric with respect to the x-axis.



(iii) The graph of $F(x, y) = 0$ is symmetric about the origin if on replacing x by $-x$ and y by $-y$, the equation of the curve does not change.

i.e. $F(x, y) = 0$ implies $F(-x, -y) = 0$.

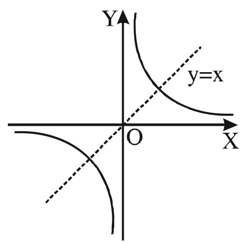
If F represents a function, then it is said to be odd. For example, the graph of $y = x^3$ is symmetric with respect to origin.



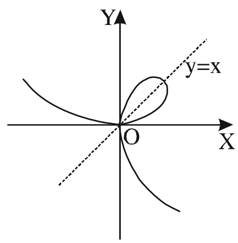
Further, the circle $x^2 + y^2 = r^2$, ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are symmetric with respect to the y-axis, the x-axis and the origin.

(iv) The graph of $F(x, y) = 0$ is symmetric about the line $y = x$ if on interchanging x and y, the equation of the curve does not change. i.e. $F(x, y) = 0$ implies $F(y, x) = 0$.

For example, the graph of $xy = c^2$ is symmetric with respect to the line $y = x$.



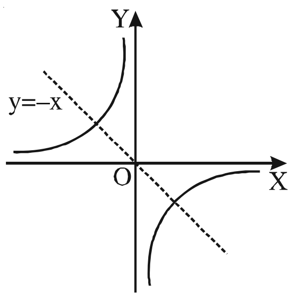
Also, the graph of $x^3 + y^3 = 3axy$, $a > 0$ is symmetric with respect to the line $y = x$.



(v) The graph of $F(x, y) = 0$ is symmetric about the line $y = -x$ if on replacing x by $-y$ and y by $-x$, the equation of the curve does not change.

i.e. $F(x, y) = 0$ implies $F(-y, -x) = 0$.

For example, the graph of $xy = -c^2$ is symmetric with respect to the line $y = -x$.



Note: For graphs of algebraic equations, the symmetry is judged as follows :

(a) If all the powers of y in the equation are even, the curve is symmetric about the x- axis.

(b) If all the powers of x are even, the curve is symmetric about the y-axis.

(c) If all powers of x and y are even, the curve is symmetric about the x-axis as well as y-axis.

Thus, the curve $x^2 + y^2 = 4ax$ is symmetrical about x-axis, The curve $a^2x^2 = y^3 (2a - y)$ is not symmetrical about x-axis, for besides containing a term in y^4, the equation also contains a term in y^3 so that even powers as well as odd powers of y occur in the equation.

5. Periodicity

Find out whether the function is periodic or not. If $f(x + T) = f(x)$ for all x in the domain, where T is a positive constant, then f is said to be a periodic function and the smallest such number T is called the fundamental period. If we know what the graph looks like in an interval of length T, then we can repeat the same segment of the graph to sketch the entire graph.

6. Monotonicity

The steps for determining intervals of monotonicity are as follows :

(i) Determine the derivative of the function f'(x) and find the critical points i.e. points where the derivative is zero or does not exist.

(ii) Determine the sign of f'(x) in different intervals formed by the critical points.

(iii) Determine monotonic nature of function in accordance with following categorization :

$f'(x) \geq 0$: equality holding for distinct points only

$\Rightarrow$ strictly increasing interval

$f'(x) \geq 0$: equality holding for sub-intervals

$\Rightarrow$ non-decreasing or increasing interval

$f'(x) \leq 0$: equality holding for distinct points only

$\Rightarrow$ strictly decreasing interval

$f'(x) \leq 0$: equality holding for subintervals

$\Rightarrow$ non-increasing or decreasing interval.

Note that at the points where $f'(x) = 0$, the curve has horizontal tangents.

7. Local Maximum and Minimum Values

Find the critical points of f and use the tests for discrimination of local maximum and minimum.

The First Derivative Test

Suppose that $x = a$ is a critical point of a continuous function $y = f(x)$.

(i) If $f'(x)$ changes from positive to negative at $x = a$, then f has a local maximum at $x = a$.

(ii) If $f'(x)$ changes from negative to positive at $x = a$, then f has a local minimum at $x = a$.

(iii) If $f'(x)$ does not change sign at $x = a$ (that is, if $f'(x)$ is positive on both sides of $x = a$ or negative on both sides), then f has no local maximum or minimum at $x = a$.

The Second Derivative Test

Let $x = a$ be a stationary point of a function f (i.e. $f'(a) = 0$). The function f has a local maximum at a if $f''(a)$ is negative, and a local minimum if $f''(a)$ is positive.

8. Concavity and Points of Inflection

If the second derivative $f''(x)$ is everywhere positive within an interval, the arc of the curve $y = f(x)$ corresponding to that interval is concave up. If the second derivative $f''(x)$ is everywhere negative in an interval, the corresponding arc of the curve $y = f(x)$ is concave down.

The concavity of the graph of f will change only at points where $f''(x) = 0$ or $f''(x)$ does not exist. Let $x = c$ be such a point and the signs of $f''(c - h)$ and $f''(c + h)$ be opposite, then the point $x = c$ is called a point of inflection.

The following table contrasts the interpretations of the signs of f, f', and f''. (It is assumed that f, f', and f'' are continuous.)

	Where the ordinate f(x)	Where the slope f '(x)	Where f "(x)
Is positive	The graph is above the x axis	The graph slopes upward	The graph is concave Upward
Is negative	The graph is below the x axis	The graph slopes downward	The graph is concave downward
Changes sign	The graph crosses the x axis.	The graph has a horizontal tangent and a relative maximum or minimum.	The graph has an inflection point

9. Asymptotes

(i) If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$.

(ii) The line $x = a$ is a vertical asymptote if at least one of the following statements is true :

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(iii) If there are limits

$$\lim_{x\to\infty} \frac{f(x)}{x} = m_1 \text{ and } \lim_{x\to\infty} [f(x) - m_1x] = c_1,$$

then the straight line $y = m_1x + c_1$ will be an asymptote (a right inclined asymptote or, when $m_1 = 0$, a right horizontal asymptote).

If there are limits

$$\lim_{x\to-\infty} \frac{f(x)}{x} = m_2 \text{ and } \lim_{x\to-\infty} [f(x) - m_2x] = c_2,$$

then the straight line $y = m_2x + c_2$ is an asymptote (a left inclined asymptote or, when $m_2 = 0$, a left horizontal asymptote).

Example 1. Find the asymptotes of $y = x + \dfrac{1}{x}$ and sketch the curve.

Solution We have $\lim\limits_{x\to 0^+} y = \lim\limits_{x\to 0^+} \left(x + \dfrac{1}{x}\right) = \infty$

$$\lim_{x\to 0^-} y = \lim_{x\to 0^-} \left(x + \frac{1}{x}\right) = -\infty$$

$\Rightarrow$ $x = 0$ is a vertical asymptote.

Now, $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} \left(x + \dfrac{1}{x}\right) = \infty$

$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} \left(x + \frac{1}{x}\right) = -\infty$$

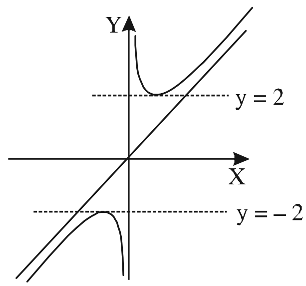
$\Rightarrow$ There is no horizontal asymptote.

Further, $\lim\limits_{x\to\infty} \dfrac{y}{x} = \lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x^2}\right) = 1.$

$$\lim_{x\to\infty} (y - x) = \lim_{x\to\infty} \left(x + \frac{1}{x} - x\right) = \lim_{x\to\infty} \frac{1}{x} = 0.$$

$\therefore$ $y = x + 0$ i. e. $y = x$ is an oblique asymptote.

A rough sketch of the curve is as follows :



Example 2. Analyse the equation $(y - x)^2 = x^3$ and trace its graph.

Solution Solving for y, we have

$$y = x \pm x^{3/2}. \qquad ...(1)$$

(i) Intercepts. The given equation shows that the curve contains the points (0, 0) and (1, 0) on the coordinate axes.

(ii) Symmetry. There is no symmetry with respect to the axes or the origin.

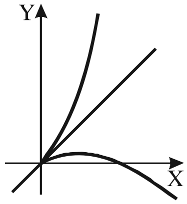
(iii) Extent. Equation (1) shows that we must exclude $x < 0$. It also shows that for each value of x (except $x = 0$), there are two values of y, and that the curve consists of two branches which start at the origin and lie on opposite sides of the line $y = x$.

For the upper branch, $\lim\limits_{x\to\infty} y = \infty$ and for the lower branch $\lim\limits_{x\to\infty} y = -\infty$. Hence, the curve extends indefinitely up and down.

(iv) Asymptotes. Equation (1) shows that there is no vertical asymptote and the discussion of the extent in y shows that there is no horizontal asymptote.

(v) Maximum, minimum, and inflectional points. Taking the positive sign in (1) and differentiating, we get $y' = 1 + 3/2x^{1/2}$, $y'' = 3/4\, x^{-1/2}$.

Since $y' \neq 0$, the upper branch has no extreme point except the origin, which is a minimum point since it is a left end point and the slope there is 1. Since $y'' > 0$ for $x > 0$, the upper branch is concave up and has no point of inflection.



For the lower branch we have

$$y' = 1 - 3/2x^{1/2},\ y'' = -3/4\, x^{-1/2}.$$

Hence again the origin is a minimum point as in the case of the upper branch. Setting $y' = 0$, we get $x = 4/9$ and $y = 4/27$. This point is a maximum point since $y'' < 0$ for $x > 0$. The last statement also tells us that the lower branch is concave down and has no point of inflection.

Example 3. Applying the derivative, construct the graphs of the following functions:

(i) $f(x) = \dfrac{(2-x)^3}{(x-3)^2}$ (ii) $f(x) = x + e^{-x}$

(iii) $f(x) = \dfrac{6\sin x}{2 + \cos x}$

Solution

(i) The function $f(x) = \dfrac{(2-x)^3}{(x-3)^2}$ is defined for all $x \neq 3$. It is neither even nor odd, nor periodic. Its graph cuts the x-axis at the point $x = 2$ and the y-axis at the point $y = 8/9$. Since $f(x) \to -\infty$ as $x \to 3$, the straight line $x = 3$ is a vertical asymptote to the graph of the function f(x). Note that

$$f(x) = \frac{8 - 12x + 6x^2 - x^3}{x^2 - 6x + 9} = -x + \frac{8 - 3x}{x^2 - 6x + 9},$$

As $x \to \infty$ and as $x \to -\infty$, $f(x) \to -x$. Therefore, the line $y = -x$ is an inclined asymptote to the graph of the function. Let us find the derivative

$$f'(x) = \frac{3(2-x)^2(-1)(x-3)^2 - (2-x)^3 2(x-3)}{(x-3)^4}$$

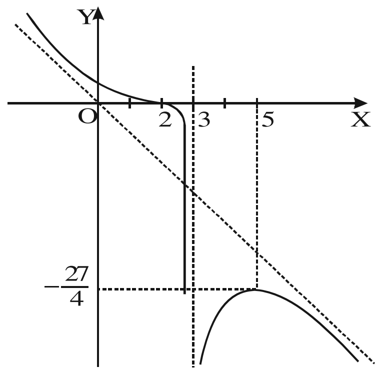
$$= \frac{(2-x)^2(5-x)}{(x-3)^3}.$$

The points $x = 2$ and $x = 5$ are the critical points. On the intervals $(-\infty, 2)$, $(2, 3)$ and $(5, \infty)$, $f'(x)$ is negative and, consequently, the function f(x) decreases. On the interval (3, 5) the derivative is positive and the function f(x) increases.

We tabulate the various information about the function as follows:

x	$(-\infty, 2)$	$(2, 3)$	$(3, 5)$	$(5, \infty)$
y:	∞ to 0	0 to $-\infty$	$-\infty$ to $-\dfrac{27}{4}$	$-\dfrac{27}{4}$ to $-\infty$
y′	–	–	+	–
	↘	↘	↗	↘

The graph of the function f(x) is shown below.



(ii) The function $f(x) = x + e^{-x}$ is defined for all $x \in R$. It is neither even nor odd. It is not periodic. The graph cuts the y-axis at $y = 1$. It does not intersect the x-axis.

Since $\lim\limits_{x\to\infty} e^{-x} = 0$, the line $y = x$ is an inclined asymptote as $x \to \infty$.

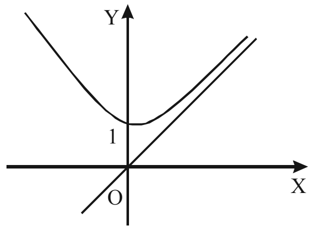
As $x \to -\infty$ there is no inclined asymptote because $\dfrac{f(x)}{x} \to -\infty$ as $x \to -\infty$.

Let us find the derivative

$$f'(x) = 1 - e^{-x}.$$

The point $x = 0$ is a critical point.

It is easy to see that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. Therefore, the function f(x) decreases on the infinite interval $(-\infty, 0)$ and increases on $(0, \infty)$. It follows that at the point $x = 0$ the function has a relative minimum. The graph of the function f(x) is shown below.



(iii) The function f(x) = $\dfrac{6\sin x}{2+\cos x}$ is defined for all $x \in R$. It is an odd function, and periodic with a period 2π. It is, therefore, sufficient to construct its graph on the interval $[0, \pi]$. Since $2 + \cos x \geq 1$, the function possesses no vertical asymptotes.

We find the derivative

$$f'(x) = 6\frac{\cos x(2+\cos x) - \sin x(-\sin x)}{(2+\cos x)^2} = 6$$
$$= \frac{2\cos x + 1}{(2+\cos x)^2}.$$

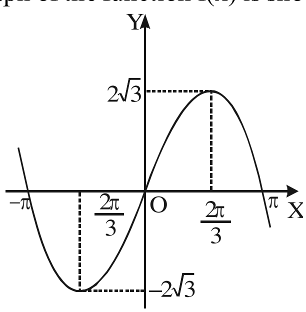
The points $x = 2\pi n \pm \frac{2}{3}\pi$, $n \in I$, are critical.

On the interval $(0, 2\pi/3)$ the derivative is positive and, consequently, the function f(x) increases.

On the interval $(2\pi/3, \pi)$ the derivative is negative and the function f(x) decreases.

The point $x = 2\pi/3$ is a point of maximum. In fact at the points $x = 2\pi n + \frac{2\pi}{3}$, $n \in I$, the function f(x) possesses maxima and at the points $x = 2\pi n - \frac{2\pi}{3}$, $n \in I$, it possesses minima.

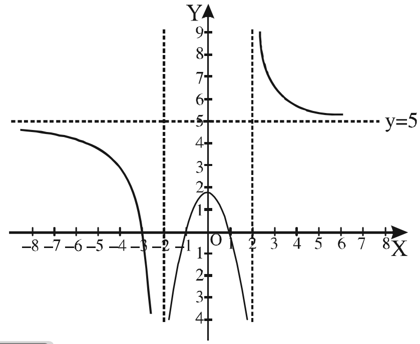
The graph of the function f(x) is shown below.



Example 4. Which one of the following functions resembles the graph of a rational function as shown?

(A) $\dfrac{5(x^2-1)(x+3)}{(x^2-4)(x+2)}$ (B) $\dfrac{(x^2-1)(x+3)}{(x^2-4)(x+2)}$

(C) $\dfrac{5(x^2-1)(x+3)}{(x^2-4)}$ (D) $\dfrac{5(x^2-4)(x+2)}{(x^2-1)(x+3)}$



Solution The function seems to have a zero (i.e. crosses the x-axis) at – 3, – 1 and 1. Of all the choices given only (A), (B) and (C) satisfy this.

The function in (C) would change sign around the vertical asymptote x = – 2 and the graph does not, so (C) is not the correct answer either.

The horizontal asymptote is the limit of the function at positive or negative infinity, and from the graph, this limit should be 5. Answer (B) does not satisfy this, so (A) must be the correct answer, and indeed, it satisfies all other requirements.

Example 5. Construct the graph of $f(x) = \dfrac{x^2-1}{x^2+1}$ and find the area bounded by y = f(x) and x-axis.

Solution Here, $f(x) = \dfrac{x^2-1}{x^2+1} = 1 - \dfrac{2}{x^2+1}$

(i) The function f(x) is well defined for all real x.
$\Rightarrow$ Domain of f(x) is R.

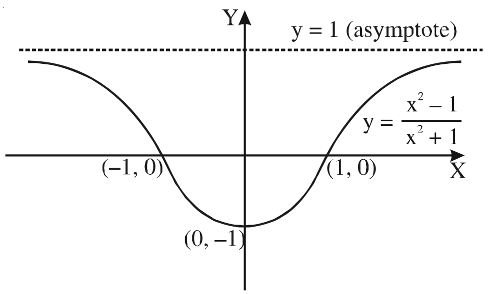
(ii) f (–x) = f(x), so it is an even function and hence graph is symmetrical about y-axis.

(iii) Obviously the function is non-periodic.

(iv) $f(x) \to 1^-$ for $x \to \infty$ (we are considering $x > 0$ only as curve is symmetrical about y-axis). Hence y = 1 is an asymptote of the curves. It may be observed that f(x) < 1 for any $x \in R$ and consequently its graph lies below the line y = 1 which is the asymptote to the graph of the given function.

(v) Again $\dfrac{2}{x^2+1}$ decreases in $(0, \infty)$, thus f(x) increases in $(0, \infty)$.

(vii) The greatest value $\to$ 1 for $x \to \infty$ and the least value is –1 for x = 0. Thus, its graph is as shown in figure.



Example 6. Construct the graph of $f(x) = xe^x$. Find the area bounded by $y = f(x)$ and its asymptote.

Solution

(i) The function is well defined for all real x $\Rightarrow$ domain of f(x) is R.

(ii) There is no symmetry in the graph.

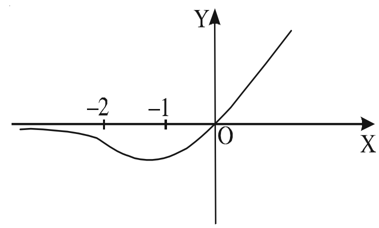
(iii) Obviously function is non-periodic.

(iv) $f(x) \to 0^-$ as $x \to -\infty$. Hence y = 0 is an asymptote of the curve.

(v) $f'(x) = (x+1)e^x \Rightarrow$ f(x) increases for $x \geq -1$ and decreases for $x \leq -1$.
Hence, x = –1 is the point of absolute minima.

Minimum value $= f(-1) = -\frac{1}{e}$.

(vi) $f''(x) = (x+2)e^x \Rightarrow$ f(x) is concave up for $x > -2$ and concave down for $x < -2$ and hence $x = -2$ is a point of inflection.



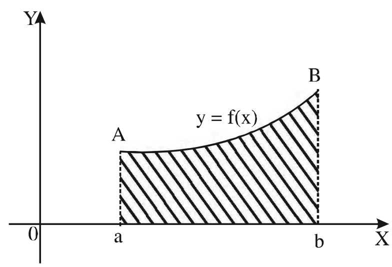
Practice Problems A

1. Plot the following curves :
 (i) $y = \pm\sqrt{x^2-1}$ (ii) $y = 2 \pm \sqrt{(x-1)^2-1}$
 (iii) $y = \pm\sqrt{x^2+1}$ (iv) $4y^2 + 4y - x^2 = 0$
2. Construct the graph of the following functions :
 (i) $y = 1 + x^2 - 0.5x^4$ (ii) $y = (x+1)(x-2)^2$.
3. Construct the graph of the following functions :
 (i) $y = (1-x^2)^{-1}$. (ii) $y = x^4(1+x)^{-3}$.
 (iii) $y = (1+x)^4(1-x)^{-4}$.
4. Construct the graph of the following functions :
 (i) $y = x(1-x^2)^{-2}$ (ii) $y = 2x - 1 + (x+1)^{-1}$
5. Construct the graph of the following functions :
 (i) $y = 0.5(\sqrt{x^2+x+1} - \sqrt{x^2-x+1})$
 (ii) $y = \sqrt{x^2+1} - \sqrt{x^2-1}$
 (iii) $y = (x+2)^{2/3} - (x-2)^{2/3}$.
6. Construct the graph of the following curves :
 (i) $y^2 = 8x^2 - x^4$ (ii) $y^2 = (x-1)(x+1)^{-1}$.
7. Plot the graph of the following functions :
 (i) $y = \frac{\cos x}{\cos 2x}$ (ii) $y = \frac{x^2+2x-3}{x^2+2x-8}$
8. Construct the following curves :
 (i) x = cost, y = sin2t
 (ii) x = cos3t, y = sin3t
 (iii) x = cos(5t + 1), y = sin(5t + 1)
 (iv) x = cost, $y = \cos\left(t + \frac{\pi}{4}\right)$

3.2 AREA OF A CURVILINEAR TRAPEZOID

It is known that the definite integral of a non-negative function is the area of the corresponding curvilinear trapezoid. This is the geometrical meaning of the definite integral, which is the basis of its application to computing the areas of plane figures.

Consider the curvilinear trapezoid aABb bounded by the graph of a nonnegative continuous function $y = f(x)$, $x \in [a, b]$, the interval [a, b] of the x-axis, and the line segments x = a and x = b (a < b).



If $f(x) \geq 0$ for $x \in [a, b]$, then the area bounded by curve $y = f(x)$, x-axis, $x = a$ and $x = b$ is $A = \int_a^b f(x)\,dx$

Example 1. Compute the area of the figure bounded by the curves $f(x) = x^2 - 2x + 2$, $x = -1$, $x = 2$, and the segment $[-1, 2]$ of the x-axis.

Solution The region described is a curvilinear trapezoid lying above the x-axis, therefore its area is computed as

$$A = \int_{-1}^{2} (x^2 - 2x + 2)dx = \left.\frac{x^3}{3}\right|_{-1}^{2} - \left.x^2\right|_{-1}^{2} + \left.2x\right|_{-1}^{2} = 6\cdot$$

Example 2. Find the area bounded by the curve $y = \ell n\, x + \tan^{-1}x$, x-axis and ordinates $x = 1$ and $x = 2$.

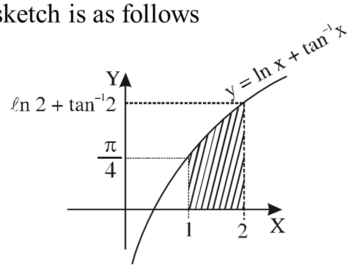
Solution Let $f(x) = \ell n\, x + \tan^{-1}x$

$$f'(x) = \frac{1}{x} + \frac{1}{1+x^2} > 0$$

$\Rightarrow$ f(x) is a strictly increasing function.

Since $f(1) = \frac{\pi}{4} > 0$, f(x) is positive for all $x \in [1, 2]$.

A rough sketch is as follows



$\therefore$ The required area $= \int_1^2 (\ell n\, x + \tan^{-1} x)\, dx$

$$= \left[x\, \ell n\, x - x + x \tan^{-1} x - \frac{1}{2}\, \ell n\,(1 + x^2)\right]_1^2$$

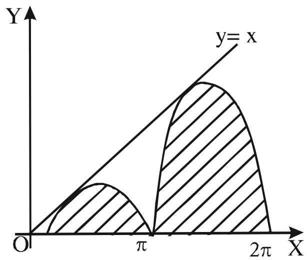
$$= 2\, \ell n\, 2 - 2 + 2 \tan^{-1}2 - \frac{1}{2}\, \ell n\, 5 - 0 + 1$$

$$- \tan^{-1} 1 + \frac{1}{2}\, \ell n\, 2$$

$$= \frac{5}{2}\, \ell n\, 2 - \frac{1}{2}\, \ell n\, 5 + 2 \tan^{-1}2 - \frac{\pi}{4} - 1.$$

Example 3. Find the area bounded by $y = x\,|\sin x|$ and x-axis between $x = 0$ and $x = 2\pi$

Solution $y = \begin{cases} x \sin x, & \text{if } \sin x \geq 0, \quad \text{i.e.,} \quad 0 \leq x \leq \pi \\ -x \sin x, & \text{if } \sin x < 0, \quad \text{i.e.,} \quad \pi < x \leq 2\pi \end{cases}$



The required area $= \int_0^{\pi} x \sin x\, dx + \int_{\pi}^{2\pi} (-x \sin x)\, dx$

$$= x\,(-\cos x)\Big|_0^{\pi} - \int_0^{\pi} (-\cos x)\, dx$$

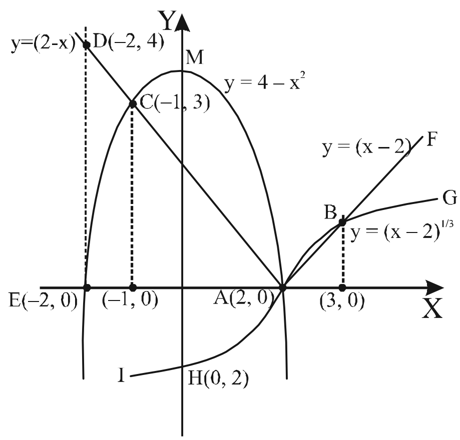
$$-(x(-\cos x))\Big|_{\pi}^{2\pi} + \int_{\pi}^{2\pi} (-\cos x)\, dx$$

$$= \pi + \sin x\,\Big|_0^{\pi} + (2\pi + \pi) - \sin x\,\Big|_{\pi}^{2\pi} = 4\pi.$$

Example 4. If a function f(x) is defined as $f(x) = \max\,\{4 - x^2, |x - 2|, (x - 2)^{1/3}\}$ for $x \in [-2, 4]$, then find the area bounded by the curve and x-axis

Solution $y = 4 - x^2$ is a parabola EMA. $y = |x - 2|$ is a pair of straight lines ABF and ACD $y = (x - 2)^{1/3}$ is the curve IHABG. Thus, from the graph $f(x) = \max\,\{4 - x^2, |x - 2|, (x - 2)^{1/3}\}$ is equivalent to

$$f(x) = \begin{cases} 2 - x & \text{if} \quad -2 \leq x \leq -1 \\ 4 - x^2 & \text{if} \quad -1 \leq x \leq 2 \\ (x - 2)^{1/3} & \text{if} \quad 2 \leq x \leq 3 \\ x - 2 & \text{if} \quad 3 \leq x \leq 4 \end{cases}$$



Now, the area bounded by the curve and the x-axis

$$= \int_{-2}^{-1} (2-x)dx + \int_{-1}^{2} (4-x^2)dx + \int_{2}^{3} (x-2)^{1/3} dx + \int_{3}^{4} (x-2)dx$$

$$= \left[2x - \frac{x^2}{2}\right]_{-2}^{-1} + \left[4x - \frac{x^3}{3}\right]_{-1}^{+2} + \left[\frac{3}{4}(x-2)^{4/3}\right]_{2}^{3} + \left[\frac{x^2}{2} - 2x\right]_{3}^{4}$$

$$= \frac{7}{2} + 9 + \frac{3}{4} + \frac{3}{2} = \frac{59}{4}.$$

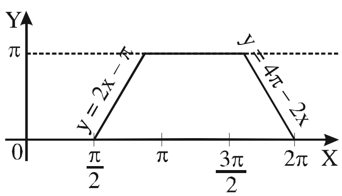
Example 5. Consider the function, $f(x) = \cos^{-1}(\cos x) - \sin^{-1}(\sin x)$ in $[0, 2\pi]$. Find the area bounded by the graph of the function and the $x-$axis .

Solution $\cos^{-1}(\cos x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi < x \le 2\pi \end{cases}$,

$$\sin^{-1}(\sin x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ x - 2\pi & \frac{3\pi}{2} < x \le 2\pi \end{cases}$$

$$\text{Hence, } f(x) = \begin{cases} 0 & \text{if } x \in \left(0, \frac{\pi}{2}\right) \\ 2x - \pi & \text{if } x \in \left(\frac{\pi}{2}, \pi\right) \\ \pi & \text{if } x \in \left(\pi, \frac{3\pi}{2}\right) \\ 4\pi - 2x & \text{if } x \in \left(\frac{3\pi}{2}, 2\pi\right) \end{cases}$$

The graph of $f(x)$ is :

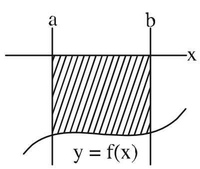


$\therefore$ The required area = area of the trapezium

$$= \frac{1}{2}\pi\left(\frac{3\pi}{2} + \frac{\pi}{2}\right) = \pi^2.$$

Note: If $f(x) \le 0$ for $x \in [a, b]$, then the area bounded by the curve $y = f(x)$, x-axis, $x = a$ and $x = b$ is

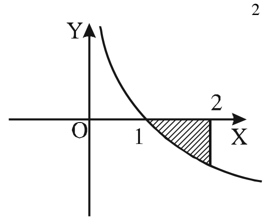
$$A = -\int_a^b f(x)\,dx$$



Further, if $y = f(x)$ does not change sign in $[a, b]$, then area bounded by $y = f(x)$, x-axis and the ordinates $x = a, x = b$ is $\left|\int_a^b f(x)\,dx\right|$.

Example 6. Find area bounded by $y = \log_{\frac{1}{2}} x$ and x-axis between $x = 1$ and $x = 2$.

Solution A rough sketch of $y = \log_{\frac{1}{2}} x$ is as follows



$$\text{The required area} = -\int_1^2 \log_{\frac{1}{2}} x\,dx$$

$$= -\int_1^2 \log_e x \,.\, \log_{\frac{1}{2}} e\,dx$$

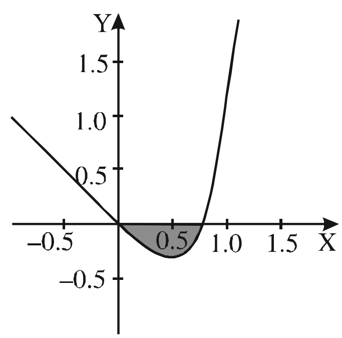
$$= -\log_{\frac{1}{2}} e \,.\, [x\log_e x - x]_1^2$$

$$= -\log_{\frac{1}{2}} e \,.\, (2\log_e 2 - 2 - 0 + 1)$$

$$= -\log_{\frac{1}{2}} e \,.\, (2\log_e 2 - 1).$$

Example 7. Find the area of the region bounded by the curve $y = e^{2x} - 3e^x + 2$ and the x-axis.

Solution We find the points of intersection of the curve and the x-axis : $e^{2x} - 3e^x + 2 = 0$. The curve intersects the x-axis at $x = 0$ and $x = ln\,2$. The graph of $f(x) = e^{2x} - 3e^x + 2$ is shown in the figure.

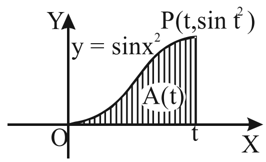


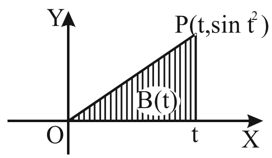
We see that on the interval $[0, ln\,2]$, $f(x) \le 0$. So the area of the given region is

$$A = \int_0^{\ln 2} -(e^{2x} - 3e^x + 2)dx = \left(-\frac{e^{2x}}{2} + 3e^x - 2x\right)\Bigg|_0^{\ln 2}$$

$$= \frac{3}{2} - 2\ln 2.$$

Example 8. The figure shows two regions in the first quadrant. A(t) is the area under the curve $y = \sin x^2$ from 0 to t and B(t) is the area of the triangle with vertices O, P and M(t, 0). Find $\lim_{t\to 0} \frac{A(t)}{B(t)}$.





Solution We have $A(t) = \int_0^t \sin x^2\, dx$, $B(t) = \frac{t \sin t^2}{2}$

$$\therefore \quad \lim_{t\to 0} \frac{A(t)}{B(t)} = \lim_{t\to 0} \frac{2\int_0^t \sin x^2\, dx}{t \sin t^2}$$

$$= \lim_{t\to 0} \frac{2\int_0^t \sin x^2\, dx}{t^3 \frac{\sin t^2}{t^2}}$$

$$= \lim_{t\to 0} \frac{2\int_0^t \sin x^2\, dx}{t^3}$$

Hence, $\lim_{t\to 0} \frac{A(t)}{B(t)} = \lim_{t\to 0} \frac{2\sin t^2}{3t^2} = \frac{2}{3}$.

Example 9. Let $f : [0, \infty) \to R$ be a continuous and strictly increasing function such that

$f^3(x) = \int_0^x t f^2(t)dt$, $\forall x > 0$. Find the area enclosed by $y = f(x)$, the x-axis and the ordinate at $x = 3$.

Solution We have $f^3(x) = \int_0^x t f^2(t)\,dt$

On differentiating w.r.t. x, we get

$\Rightarrow 3f^2(x)f'(x) = x f^2(x)$

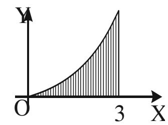
$\Rightarrow f^2(x)[3f'(x) - x] = 0$

As $f(x) \neq 0$

$\Rightarrow f'(x) = \frac{x}{3} \Rightarrow f(x) = \frac{x^2}{6} + C$

But $f(0) = 0 \Rightarrow C = 0 \Rightarrow f(x) = \frac{x^2}{6}$,

which is a non-negative function.



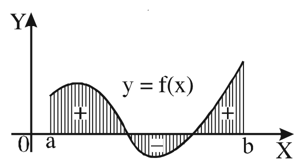
$$A = \frac{1}{6}\int_0^3 x^2\, dx = \left.\frac{x^3}{18}\right]_0^3 = \frac{3}{2}.$$

3.3 AREA BOUNDED BY A FUNCTION WHICH CHANGES SIGN

If on the interval [a, b] the function $f(x) \geq 0$, then, as we know from the area of a curvilinear trapezoid bounded by the curve $y = f(x)$, the x-axis and the straight lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x)dx$$

If f(x) changes sign on the interval [a, b] a finite number of times, then we break up the integral throughout [a, b] into the sum of integrals over the subintervals. The integral will be positive on those subintervals where $f(x) \geq 0$, and negative where $f(x) \leq 0$.



The integral over the entire interval will yield the difference of the areas above and below the x-axis. To find the sum of the areas in the ordinary sense, one has to find the sum of the absolute values of the integrals over the above indicated subintervals or compute the integral

$$A = \int_a^b |f(x)|dx$$

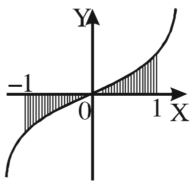
For example, if $f(x) \geq 0$ for $x \in [a,c]$ and $f(x) \leq 0$ for $x \in [c,b]$ $(a < c < b)$ then the area bounded by curve $y = f(x)$ and x–axis between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x)|dx = \int_a^c |f(x)|dx + \int_c^b |f(x)|dx$$

$$= \int_a^c f(x)dx - \int_c^b f(x)dx.$$

Example 1. Find the area bounded by $y = x^3$ and x–axis between the ordinates $x = -1$ and $x = 1$.

Solution The graph of $y = x^3$ is shown below :



The required area $= \int_{-1}^{0} -x^3dx + \int_0^1 x^3dx$

$$= -\left.\frac{x^4}{4}\right]_{-1}^{0} + \left.\frac{x^3}{4}\right]_0^1$$

$$= 0 - \left(-\frac{1}{4}\right) + \frac{1}{4} - 0 = \frac{1}{2}.$$

Example 2. Compute the area bounded by the curve $y = \sin x$ and the x-axis, for $0 \leq x \leq 2\pi$.

Solution Since $\sin x \geq 0$ when $0 \leq x \leq \pi$ and $\sin x \leq 0$ when $\pi < x \leq 2\pi$, we have

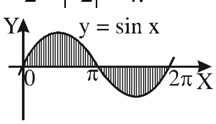
$$A = \int_0^{2\pi} |\sin x|dx = \int_0^{\pi} \sin xdx + \left|\int_{\pi}^{2\pi} \sin xdx\right|$$

$$\int_0^{\pi} \sin xdx = -\cos x\Big|_0^{\pi} = -(\cos\pi - \cos 0)$$

$$= -(-1-1) = 2.$$

$$\int_{\pi}^{2\pi} \sin xdx = -\cos x\Big|_{\pi}^{2\pi} = -(\cos 2\pi - \cos\pi) = -2.$$

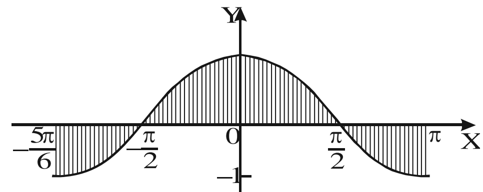
Consequently, $A = 2 + |-2| = 4$.



Example 3. Compute the area of the plane figure bounded by the interval $\left[-\frac{5\pi}{6}, \pi\right]$ of the x-axis, the graph of the function $f(x) = \cos x$, and segments of the straight lines $x = -\frac{5\pi}{6}$ and $x = \pi$.

Solution The graph of $f(x) = \cos x$ is shown below

:



Solving the equation $\cos x = 0$, we find that the graph of the function $y = \cos x$ on the interval $\left[-\frac{5\pi}{6}, \pi\right]$ intersects the x-axis at the points $x_1 = -\frac{\pi}{2}$, $x_2 = \frac{\pi}{2}$.

Consequently, the require area

$$= \int_{-5\pi/6}^{\pi} |\cos x|dx$$

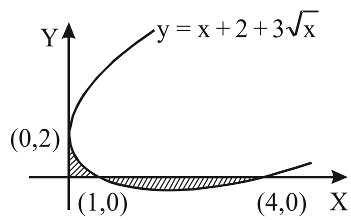
$$= -\int_{-5\pi/6}^{-\pi/2} \cos xdx + \int_{-\pi/2}^{\pi/2} \cos xdx - \int_{\pi/2}^{\pi} \cos xdx$$

$$= -\sin x\Big|_{-5\pi/6}^{-\pi/2} + \sin x\Big|_{-\pi/2}^{\pi/2} - \sin x\Big|_{\pi/2}^{\pi} = \frac{7}{2}.$$

Example 4. Compute the area of the figure bounded by the curve $(y - x - 2)^2 = 9x$ and the co-ordinate axes.

Solution The given equation represents a parabola touching y–axis at (0, 2) and cutting x–axis at (1, 0) and (4, 0).

Solving $(y - x - 2)^2 = 9x$ for y, we get $y = x + 2 + 3\sqrt{x}$ and $y = x + 2 - 3\sqrt{x}$. The curve below the line $y = x + 2$ is given by $y = x + 2 - 3\sqrt{x}$.



The required area

$$= \int_0^1 \left(x + 2 - 3\sqrt{x}\right) dx - \int_1^4 \left(x + 2 - 3\sqrt{x}\right) dx = 1.$$

Example 5. The area bounded by $y = x^2 + 2$ and $y = 2|x| - \cos\pi x$ is of the form of $\frac{p}{q}$ where p and q are relatively prime, then find $p - q$.

Solution Let us solve for points of intersection

$x^2 + 2 = 2|x| - \cos\pi x$

$\Rightarrow$ $(|x| - 1)^2 + 1 = -\cos\pi x$

$\Rightarrow$ $x = \pm 1.$

For $-1 < x < 1$, $x^2 + 2 > 2|x| - \cos\pi x$

The required area

$$= \int_{-1}^{1} \left(x^2 + 2 - 2|x| + \cos\pi x\right) dx = \frac{8}{3}.$$

Thus, $p - q = 8 - 3 = 5$.

Concept Problems A

1. Find the area enclosed between $y = \sin x$ and x-axis as x varies from 0 to $\frac{3\pi}{2}$.
2. Under what condition does the value of the integral $\int_a^b f(x)dx$ coincide with the value of the area of the curvilinear trapezoid bounded by the curves $y = f(x)$, $x = a$, $x = b$, $y = 0$?
3. Which of the following statements are true?
   (i) If $C > 0$ is a constant, the region under the line $y = C$ on the interval [a, b] has area $A = C(b - a)$.
   (ii) If $C > 0$ is a constant and $b > a \geq 0$, the region under the line $y = Cx$ on the interval [a, b] has area $A = \frac{1}{2}C(b - a)$
   (iii) The region under the parabola $y = x^2$ on the interval [a, b] has area less than $\frac{1}{2}(b^2 + a^2)(b - a)$.
   (iv) The region under the curve $y = \sqrt{1 - x^2}$ on the interval [–1, 1] has area $A = \frac{\pi}{2}$.
   (v) Let f be a function that satisfies $f(x) \geq 0$ for x in the interval [a, b]. Then the area under the curve $y = f^2(x)$ on the interval [a, b] must always be greater than the area under $y = f(x)$ on the same interval.
   (vi) If f is even and $f(x) \geq 0$ throughout the interval [–a, a], then the area under the curve $y = f(x)$ on this interval is twice the area under $y = f(x)$ on [0, a].
4. Express with the aid of an integral the area of a figure bounded by :
   (i) The coordinate axes, the straight line $x = 3$ and the parabola $y = x^2 + 1$.
   (ii) The x-axis, the straight lines $x = a$, $x = b$ and the curve $y = e^x + 2$ $(b > a)$.
5. Prove that the whole area (when finite) included between the axis of x and the curve $y = \frac{1}{a}\phi\left(\frac{x}{a}\right)$ is independent of the value of a, assuming $\phi(x) \geq 0$.
6. Compute the area of the figure bounded by the parabola $y = x^2 - 4x + 5$, the x-axis and the straight lines $x = 3$, $x = 5$.
7. Find the area between curve $y = x^2 - 3x + 2$ and x–axis (i) bounded between $x = 1$ and $x = 2$. (ii) bound between $x = 0$ and $x = 2$.
8. Find the area bounded by the curve $y = x(x - 1)(x - 2)$ and the x-axis.
9. After four seconds of motion the speed, which is proportional to the time, is equal to 1 cm/s. What is the distance travelled in the first ten seconds ?

Practice Problems B

10. Compute the area of the curvilinear trapezoid bounded by the x-axis and the curve $y = x - x^2\sqrt{x}$.
11. Find the area of the figure bounded by the curve $y = \sin^3 x + \cos^3 x$ and the segment of the x-axis joining two successive points of intersection of the curve with the x-axis.

12. Find the area of the region bounded by the graphs of $y = \dfrac{2x}{\sqrt{x^2+9}}$, $y = 0$, $x = 0$, and $x = 4$.

13. Show that the area under $y = \dfrac{1}{x}$ on the interval $[1, a]$ equals the area under the same curve on $[k, ka]$ for any number $k > 0$.

14. Compute the area of the curvilinear trapezoid bounded by the curve $y = e^{-x}(x^2 + 3x + 1) + e^2$, the x-axis and two straight lines parallel to the y-axis drawn through the points of extremum of the function y.

15. A rectangle with edges parallel to the coordinate axes has one vertex at the origin and the diagonally opposite vertex on the curve $y = kx^m$ at the point where $x = b$ ($b > 0$, $k > 0$, and $m \geq 0$). Show that the fraction of the area of the rectangle that lies between the curve and the x-axis depends on m but not on k or b.

16. (a) Find the area A(b) under the curve $y = e^{-x}\cos^2 x$ in the interval $[0, b]$.

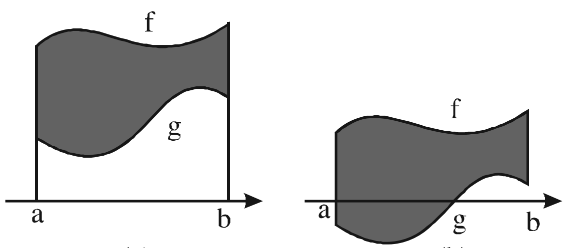
(b) Find $\lim_{b\to\infty} A(b)$.

17. Prove that the area between the curve $\left(\dfrac{x}{a}\right)^{2/3} + \dfrac{y}{b} = 1$ and the segment $(-a, a)$ of the axis of x is $\dfrac{4}{5}ab$.

3.4 AREA OF A REGION BETWEEN TWO NON-INTERSECTING GRAPHS

If f and g are continuous functions on the interval [a, b], and if $f(x) \geq g(x)$ for all x in [a, b], then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

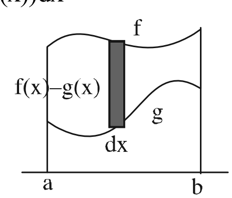
$$A = \int_a^b (f(x) - g(x))dx$$



(a) (b)

We explain this using vertical strips :

Note that the vertical strip has height $f(x) - g(x)$ if $y = f(x)$ is above $y = g(x)$ and the area of the vertical strip is $(f(x) - g(x))dx$

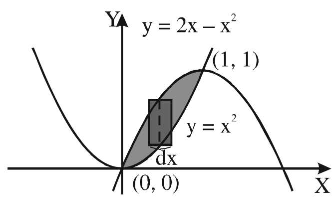


Thus, if $f(x) \geq g(x)$ for $x \in [a,b]$ then the bounded area $= \int_a^b (f(x) - g(x))dx$.

Remark : It is not necessary to make an extremely accurate sketch; the only purpose of the sketch is the determine which curve is the upper boundary and which is the lower boundary.

Example 1. Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution We first find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or, $2x^2 - 2x = 0$. Thus $2x(x - 1) = 0$, so $x = 0$ or 1. The points of intersection are (0, 0) and (1, 1).



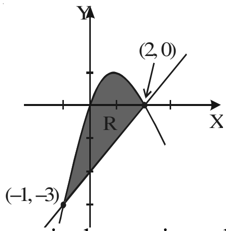
We see from the figure that the top and bottom boundaries are $y_1 = 2x - x^2$ and $y_2 = x^2$ respectively. The area of a typical rectangle is $(y_1 - y_2)dx = (2x - x^2 - x^2)$ dx and the region lies between $x = 0$ and $x = 1$. So the total area is

$$A = \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

Example 2. Find the area of the region between the curves $y = x - 2$ and $y = 2x - x^2$.

Solution



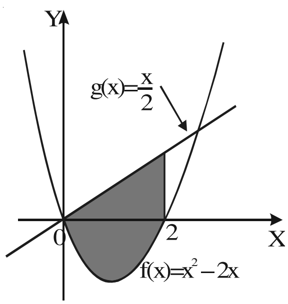
The graphs intersect in those points whose coordinates are the simultaneous solutions of the given equations. Eliminating y between these equations, we have $x - 2 = 2x - x^2$ or $x^2 - x - 2 = (x - 2)(x + 1) = 0$. Thus $x = 2$ or $x = -1$, and the common points of the two graphs are $(2, 0)$ and $(-1, -3)$. The graph of $y = x - 2$ is a straight line. The graph of $y = 2x - x^2$ is a parabola that is concave down and has its vertex at $(1, 1)$. The region whose area is sought is shaded in the figure. This area is given by

$$\int_{-1}^{2} [(2x - x^2) - (x - 2)]\,dx = \int_{-1}^{2} (-x^2 + x + 2)\,dx$$

$$= \left(-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right)\Bigg|_{-1}^{2} = \frac{9}{2}.$$

Example 3. Compute the area of the region between the graphs of f and g over the interval [0, 2] if $f(x) = x(x - 2)$ and $g(x) = x/2$.

Solution The two graphs are shown in the figure.



The shaded portion represents the required region. Since $f \le g$ over the interval [0, 2], the required area

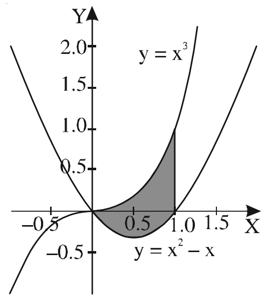
$$= \int_0^2 [g(x) - f(x)]\,dx = \int_0^2 \left(\frac{5}{2}x - x^2\right) dx$$

$$= \frac{5}{2}\frac{2^2}{2} - \frac{2^3}{3} = \frac{7}{3}.$$

Example 4. Find the area of the region between the curves $y = x^3$ and $y = x^2 - x$ on the interval [0, 1].

Solution We need to know which curve is the top curve in [0, 1]. Solving $x^3 = x^2 - x$, or $x(x^2 - x + 1) = 0$. The only real root is $x = 0$. To see which curve is on top, takes some representative value, such as $x = 0.5$, and note that because $(0.5)^3 > 0.5^2 - 0.5$, the curve $y = x^3$ must be above $y = x^2 - x$. Thus, the cubic curve is on top throughout the interval [0, 1].

The region is shown in the figure.

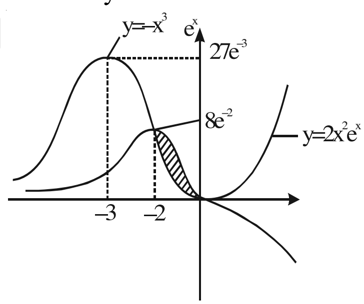


Thus, the required area is given by

$$[\underbrace{x^3}_{\text{Top curve}} - \underbrace{(x^2 - x)}_{\text{Bottom curve}}]\,dx = \left(\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Bigg|_0^1 = \frac{5}{12}$$

Example 5. Sketch the curves $y = 2x^2e^x$ and $y = -x^3e^x$ and compute the area of the finite portion of the figure bounded by these curves.

Solution



Solving the two curves $2x^2e^x = -x^3e^x$

$\Rightarrow \quad x = 0$ or -2.

Consider the curve $y = 2x^2e^x \quad \frac{dy}{dx} = 2xe^x(x + 2)$

$\Rightarrow$ the curve has a horizontal tangent at $x = 0, 2$

Also it increases in $x < -2$ and $x > 0$, and decreases in $(-2, 0)$. Similarly for $y = -x^3e^x$

we have $\frac{dy}{dx} = -x^2e^x(x + 3)$

It increases in $(-\infty, -3)$ and decreases in $(-3, \infty)$.

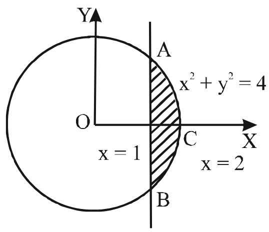
Hence, $A = \int_{-2}^{0} (2x^2e^x + x^3e^x)\,dx$.

Integrating by parts we get the result

$$A=\left[\left(x^3-x^2+2x-2\right)e^x\right]_{-2}^{0}=18e^{-2}-2.$$

Example 6. Find the area of smaller portion of the circle $x^2+y^2=4$ cut off by the line $x=1$.

Solution Equation of the circle is $x^2+y^2=4$ and equation of the line is $x=1$.



The required area = area ABCA

$$=2\int_1^2 y\,dx=2\int_1^2\sqrt{4-x^2}\,dx$$

$$=2\left[\frac{x\sqrt{2^2-x^2}}{2}+\frac{2^2}{2}\sin^{-1}\frac{x}{2}\right]_1^2$$

$$=0+2\sin^{-1}(1)-2\left\{\frac{\sqrt{3}}{2}+2\sin^{-1}\left(\frac{1}{2}\right)\right\}$$

$$=\frac{4\pi-3\sqrt{3}}{3}.$$

Example 7. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is divided into two parts by a line parallel to the $y-$axis. Find the equation of the line if the area of the smaller part is $ab\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$.

Solution Let the line be $x=h$ parallel to the $y-$axis. The smaller area

$$=2\int_h^a\frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=2\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_h^a$$

$$=\frac{2b}{a}\left[\frac{\pi a^2}{4}-\left(\frac{h}{2}\sqrt{a^2-h^2}+\frac{a^2}{2}\sin^{-1}\frac{h}{a}\right)\right]$$

Hence, $\frac{2b}{a}\left[a^2\left(\frac{\pi}{4}-\frac{1}{2}\sin^{-1}\left(\frac{h}{a}\right)\right)\right]-\frac{b}{a}h\sqrt{a^2-h^2}$

$$=ab\frac{\pi}{3}-\frac{\sqrt{3}}{4}ab\ \text{(given)}$$

Let $\frac{\pi}{2}-\sin^{-1}\frac{h}{a}=\frac{\pi}{3}$

$\Rightarrow\ \sin^{-1}\frac{h}{a}=\frac{\pi}{6}=\sin^{-1}\frac{1}{2}$

$\therefore\ h=\frac{a}{2}$

For this value of $h=\frac{a}{2}$,

$$\frac{bh}{a}\sqrt{a^2-h^2}$$

$$=\frac{b}{a}\cdot\frac{a}{2}\cdot\frac{\sqrt{3}\,a}{2}=\frac{\sqrt{3}\,ab}{4}$$

which agrees with the second term.

Example 8. For what value of the parameter $a>0$ is the area of the figure bounded by the curves $y=a\sqrt{x}$ and $y=\sqrt{1-x}$ and the $y-$axis equal to the number b? Also state the value of b for which the problem has a solution .

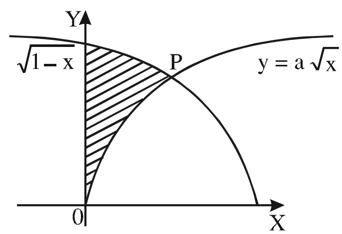
Solution Solving the curves : $P\left(\frac{1}{a^2+1},\frac{a}{\sqrt{a^2+1}}\right)$

Hence, $b=\int_0^{\frac{1}{a^2+1}}\left(\sqrt{1-x}-a\sqrt{x}\right)dx$

$$\Rightarrow\ b=-\frac{2}{3}(1-x)^{3/2}-\frac{2}{3}ax^{3/2}\Big|_0^{\frac{1}{a^2+1}}$$

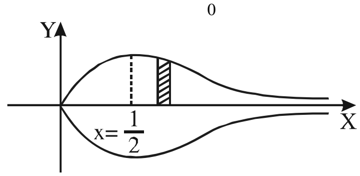
$$\Rightarrow\ a=\frac{|2-3b|}{3b\sqrt{4-3b}}.$$

The problem has a solution for $0<b<4/3$.



Example 9. Sketch the curve $y^2=xe^{-2x}$ and compute the area of the figure bounded by the curve. You may assume that the value of the integral $\int_0^{\infty}e^{-x^2}dx$ is $\frac{\sqrt{\pi}}{2}$.

Solution



$$A = 2\int_0^\infty y\,dx$$

$$A = 2\int_0^\infty \sqrt{x}\,e^{-x}\,dx$$

Consider

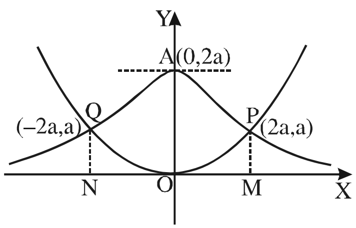
$$I = \int_0^\infty \sqrt{x}\,e^{-x}\,dx = \left.\frac{\sqrt{x}\,e^{-x}}{}\right]_0^\infty + \int_0^\infty \frac{e^{-x}}{2\sqrt{x}}\,dx$$

$$= 0 + \int_0^\infty e^{-t^2}\,dt \qquad \text{where } x = t^2$$

Hence $A = \sqrt{\pi}$.

Example 10. Find the area included between the parabola $x^2 = 4ay$ and the curve $y = 8a^3/(x^2 + 4a^2)$.

Solution The curve $(x^2 + 4a^3) = 8a^3$ is symmetrical about y-axis. Equating to zero the coefficient of the highest power of x in the curve parallel to x-axis. Also this curve cuts the y-axis at (0, 2a). Solving the two given equations $x^2 = 4ay$ and $y = 8a^3/(x^2 + 4a^2)$ we get their points of intersection as $(\pm 2a, a)$.
Also both the curves are symmetrical about y-axis.



Now the required area OPAQO
$= 2 \times$ area OPA (by symmetry)
$= 2 \times$ [area OAPM – area OPM]

$$= 2\int_0^{2a} y\,dx, \text{ for } y = \frac{8a^3}{(x^2 + 4a^2)} - \int_0^{2a} y\,dx, \text{ for } x^2 = 4ay$$

$$= 2\int_0^{2a} \frac{8a^3}{x^2 + 4a^2}\,dx - 2\int_0^{2a} \frac{x^2}{4a}\,dx$$

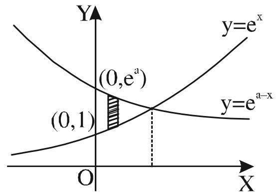
$$= 16a^3 \cdot \frac{1}{2a}\left[\tan^{-1}\frac{x}{2a}\right]_0^{2a} - \frac{1}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = 2\pi a^2 - \frac{4a^2}{3}$$

$$= 2\pi - \left[2\pi - \frac{4}{3}\right]a^2.$$

Example 11. Let 'a' be a positive constant. Consider two curves C_1: $y = e^x$, C_2 : $y = e^{a-x}$. Let S be the area of the part surrounding by C_1, C_2 and the y-axis, then find $\lim\limits_{a \to 0} \frac{S}{a^2}$.

Solution Solving $e^x = e^{a-x}$, we get $e^{2x} = e^a$

$$\Rightarrow x = \frac{a}{2}$$



$$\text{Area } S = \int_0^{a/2} (e^a \cdot e^{-x} - e^x)\,dx = \left[-(e^a \cdot e^{-x} + e^x)\right]_0^{a/2}$$

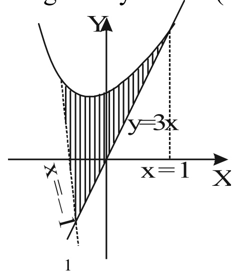
$$= (e^a + 1) - (e^{a/2} + e^{a/2}) = e^a - 2e^{a/2} + 1 = (e^{a/2} - 1)^2$$

$$\therefore \frac{S}{a^2} = \left(\frac{e^{a/2} - 1}{a}\right)^2 = \frac{1}{4}\left(\frac{e^{a/2} - 1}{a/2}\right)^2$$

$$\therefore \lim_{a \to 0} \frac{S}{a^2} = \frac{1}{4}.$$

Example 12. Find the area enclosed by curve $y = x^2 + x + 1$ and its tangent at (1,3) and between the ordinates $x = -1$ and $x = 1$.

Solution $\frac{dy}{dx} = 2x + 1$. The slope of tangent is $\frac{dy}{dx} = 3$ at $x = 1$.
The equation of tangent is $y - 3 = 3(x - 1) \Rightarrow y = 3x$



$$\text{The required area} = \int_{-1}^{1} (x^2 + x + 1 - 3x)\,dx$$

$$= \int_{-1}^{1} (x^2 - 2x + 1)\,dx = \left.\frac{x^3}{3} - x^2 + x\right]_{-1}^{1}$$

$$= \left(\frac{1}{3} - 1 + 1\right) - \left(-\frac{1}{3} - 1 - 1\right) = \frac{2}{3} + 2 = \frac{8}{3}.$$

Example 13. Suppose $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 5$ and $h(x) = (fog)(x)$. Find the area enclosed by the graph of the function $y = f(x)$ and the pair of tangents drawn to it from the origin

Solution Given

$g(x) = 2x + 1,\ h(x) = (2x+1)^2 + 4$

$h(x) = f[g(x)]$

$= (2x+1)^2 + 4 = f(2x+1)$

Let $2x + 1 = t$

$\Rightarrow\ f(t) = t^2 + 4$

$\therefore\ f(x) = x^2 + 4$ ...(1)

Solving, $y = mx$ and $y = x^2 + 4$, we get

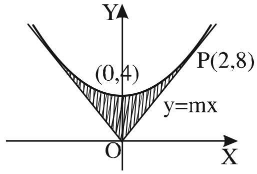
$x^2 - mx + 4 = 0$

For tangency, put $D = 0$

$\Rightarrow\ m^2 = 16$

$\Rightarrow\ m = \pm 4$

The tangents are $y = 4x$ and $y = -4x$

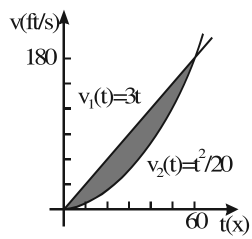


The required area $= 2\int_0^2 [(x^2+4) - 4x]dx$

$= 2\int_0^2 [(x-2)^2 dx = \frac{2}{3}(x-2)^3\Big]_0^2 = \frac{16}{3}$.

Practice Problems C

1. Find the area included between curves $y = 2x - x^2$ and $y + 3 = 0$.
2. Find area between curves $y = x^2$ and $y = 3x - 2$ from $x = 0$ to $x = 2$.
3. Calculate the areas of the figures bounded by $3y = -x^2 + 8x - 7$, $y + 1 = 4/(x-3)$
4. At what values of the parameter $a > 0$ is the area of the figure bounded by the curves $x = a$, $y = 2^x$, $y = 4^x$ larger or equal to the area bounded by the curves $y = 2^x$, $y = 0$, $x = 0$, $x = a$?
5. Find the area of the figure bounded by the straight line $y = -8x - 46$ and the parabola $y = 4x^2 + ax + 2$, if it is known that the tangent to the parabola at the point $x = -5$ makes an angle $\pi - \tan^{-1} 20$ with the positive x-axis.
6. Find the area of the figure bounded by the parabola $y = ax^2 + 12x - 14$ and the straight line $y = 9x - 32$, if the tangent drawn to the parabola at the point $x = 3$ is known to make the angle $\pi - \tan^{-1}6$ with the x-axis.
7. Find the area of the figure bounded by the curve $y - 15 = e^{2x}$ and the curve $y = 7\int e^x dx$ passing through the point A(0, 10).
8. Find the area of the region in the first quadrant below $y = -7x + 29$ and above the portion of $y = 8/(x^2 - 8)$ that lies in the first quadrant.
9.
10. Find the approximate area of the region that lies below the curve $y = \sin x$ and above the line $y = 0.2x$, where $x \geq 0$.
11. The accompanying figure shows velocity versus time curves for two cars that move along a straight track, accelerating from rest at a common starting line.
    (a) How far apart are the cars after 60 seconds ?
    (b) How far apart are the cars after T seconds, where $0 \leq T \leq 60$?



12. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the square bounded by $x = 0$, $x = 4$, $y = 0$, $y = 4$ into three equal areas.
13. Find the area of the region bounded by the parabola $(y-2)^2 = (x-1)$ and the tangent to it at ordinate $y = 3$ and x–axis.
14. Find the area of the region bounded by the curves $y = x^2$ and $y = \frac{2}{1+x^2}$.

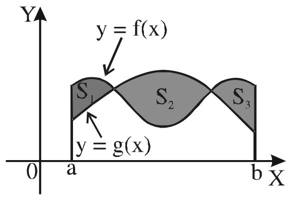
3.5 AREA OF A REGION BETWEEN TWO INTERSECTING GRAPHS

If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x, then we split the given region S into several regions S_1, S_2, ... with areas $A_1, A_2, \ldots$ as shown in the figure. We then define the area of the region S to be the sum of the areas of the smaller regions $S_1, S_2, \ldots$, that is, $A = A_1 + A_2 + \ldots$. The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

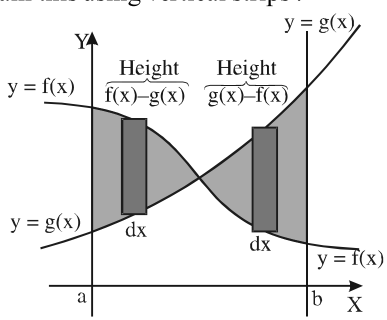
$$A = \int_a^b |f(x) - g(x)|\,dx$$

When evaluating the integral, however, we must split it into integrals corresponding to $A_1, A_2, \ldots$.

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \ge g(x) \\ g(x) - f(x) & \text{when } g(x) \ge f(x) \end{cases}$$



We explain this using vertical strips :



Area by vertical strips

Note that the vertical strip has height $f(x) - g(x)$ if $y = f(x)$ is above $y = g(x)$, and height $g(x) - f(x)$ if $y = g(x)$ is above $y = f(x)$. In either case, the height can be represented by $|f(x) - g(x)|$, and the area of the vertical strip is $dA = |f(x) - g(x)|dx$

Thus, we have $A = \int_a^b |f(x) - g(x)|\,dx$.

Example 1. Compute the area of the region between the graphs of f and g over the interval $[-1, 2]$ if $f(x) = x$ and $g(x) = x^3/4$.

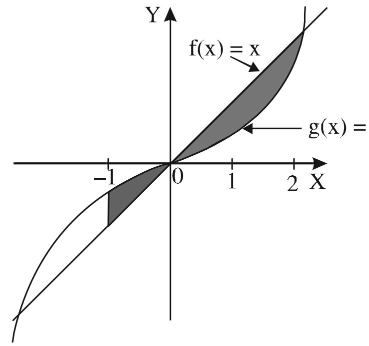
Solution The region is shown in the figure.

Here we do not have $f \le g$ throughout the interval $[-1, 2]$. However, we do have $f \le g$ over the subinterval $[-1, 0]$ and $g \le f$ over the subinterval $[0, 2]$.

$$A = \int_{-1}^{0} [g(x) - f(x)]\,dx + \int_0^2 [f(x) - g(x)]\,dx$$

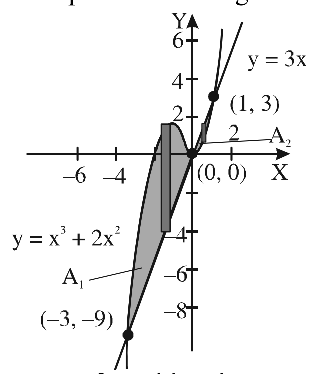
$$= \int_{-1}^{0} \left(\frac{x^2}{4} - x\right) dx + \int_0^2 \left(x - \frac{x^3}{4}\right) dx$$

$$= -\frac{1}{4}\frac{(-1)^4}{4} + \frac{(-1)^2}{2} + \frac{2^2}{2} - \frac{12^4}{44} = \frac{23}{16}.$$



Example 2. Find the area of the region bounded by the line $y = 3x$ and the curve $y = x^3 + 2x^2$.

Solution The region between the curve and the line is the shaded portion of the figure.



Part of the process of graphing these curves is to find which is the top curve and which is the bottom. To do this we need to find where the curves intersect :

$$x^3 + 2x^2 = 3x \text{ or } x(x + 3)(x - 1) = 0$$

The points of intersection occur at $x = -3$, 0 and 1. In the subinterval $[-3, 0]$, labeled A_1 in figure, the curve $y = x^3 + 2x^2$ is on top (test a typical point in the subinterval, such as $x = -1$), and on $[0, 1]$, the region labeled A_2, curve $y = 3x$ is on top. The representative vertical strips are shown in the figure and the area between the curve and the line is given by the sum

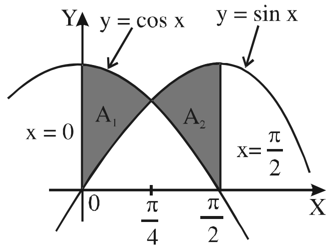
$$A = \int_{-3}^{0} [(x^3 + 2x^2) - (3x)]\,dx + \int_0^1 [(3x) - (x^3 + 2x^2)]dx$$

$$= \left(\frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2\right)\Bigg|_{-3}^{0} + \left(\frac{3}{2}x^2 - \frac{1}{4}x^4 - \frac{2}{3}x^3\right)\Bigg|_0^1$$

$$= 0 - \left(\frac{81}{4} - \frac{54}{3} - \frac{27}{2}\right) + \left(\frac{3}{2} - \frac{1}{4} - \frac{2}{3}\right) - 0 = \frac{71}{6}.$$

Example 3. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Solution The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \le x \le \pi/2$). The region is sketched in the figure.



Observe that $\cos x \ge \sin x$ when $0 \le x \le \pi/4$ but $\sin x \ge \cos x$ when $\pi/4 \le x \le \pi/2$. Therefore , the required area is

$$A = \int_0^{\pi/2} |\cos x - \sin x|\,dx = A_1\,A_2$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin\,]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

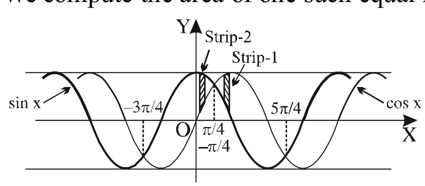
$$= 2\sqrt{2} - 2.$$

Note: In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$A = 2A_1 = 2\int_0^{\pi/4} (\cos x - \sin x)\,dx.$$

Example 4. Using the concept of area prove that $\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx = \int_{-3\pi/4}^{\pi/4} (\cos x - \sin x)\,dx$.

Solution The curves $y = \sin x$ and $y = \cos x$ intersect each other at several points enclosing regions of equal areas. We compute the area of one such equal region.



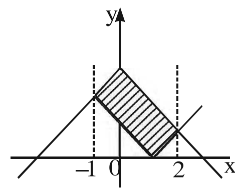
Using strip-1, $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$

Using strip-2, the same area $A = \int_{-3\pi/4}^{\pi/4} (\cos x - \sin x)\,dx$.

Hence, we get the desired result.

Example 5. Find the area of the figure bounded by the curves $y = |x - 1|$, $y = 3 - |x|$

Solution The curves meet at two points (see figure). Solving the equation $3 - |x| = |x - 1|$, we find the abscissa of these points : $x = -1$, $x = 2$.



Therefore, $A = \int_{-1}^{2} (3 - |x| - |x - 1|)\,dx$

We divide the integral into three integrals over the closed intervals $[-1, 0]$, $[0,1]$, $[1, 2]$ respectively. We obtain

$$A = \int_{-1}^{0} [(3 + x) - (1 - x)]dx + \int_0^1 [(3 - x) - (1 - x)]dx$$

$$+ \int_1^2 [(3 - x) - (x - 1)]dx = 1 + 2 + 1 = 4.$$

Example 6. Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

Solution To make an accurate sketch of the region, we need to know where the curves $x = y^2$ and $y = x - 2$ intersect. We find intersection points by equating the expressions for y.

$x = y^2$ and $x = y + 2$ ...(1)

This yields

$y^2 = y + 2$ or $y^2 - y - 2 = 0$ or $(y + 1)(y - 2) = 0$

from which we obtain $y = -1$, $y = 2$. Substituting these values in either equation in (1) we see that the corresponding x-values are $x = 1$ and $x = 4$, respectively, so the points of intersection are $(1, -1)$ and $(4, 2)$. (Fig. a)

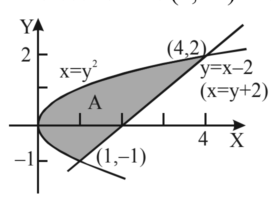


Fig: (a)

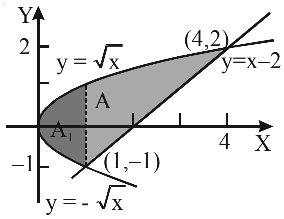


Fig: (b)

The equations of the boundaries must be written so that y is expressed explicitly as a function of x. The upper boundary can be written as $y = \sqrt{x}$ (rewrite $x = y^2$ as $y = \pm\sqrt{x}$ and choose the + for the upper portion of the curve). The lower portion of the boundary consists of two parts : $y = -\sqrt{x}$ for $0 \le x \le 1$ and $y = x - 2$ for $1 \le x \le 4$ (Figure b).

Because of this change in the formula for the lower boundary, it is necessary to divide the region into two parts and find the area of the each part separately.

With $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x}$, $a = 0$ and $b = 1$, we obtain

$$A_1 = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx$$

$$= 2\int_0^1 \sqrt{x}dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{4}{3} - 0 = \frac{4}{3}$$

With $f(x) = \sqrt{x}$, $g(x) = x - 2$, $a = 1$, and $b = 4$, we obtain

$$A_2 = \int_1^4 [\sqrt{x} - (x-2)]dx = \int_1^4 (\sqrt{x} - x + 2)dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4$$

$$= \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{19}{6}$$

Thus, the area of the entire region is

$$A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}.$$

Example 7. The line $y = mx$ bisects the area enclosed by the lines $x = 0$, $y = 0$, $x = \frac{3}{2}$ and the curve $y = 1 + 4x - x^2$. Find m.

Solution Here, the curve is $x^2 - 4x = 1 - y$,

i.e., $(x - 2)^2 = -(y - 5)$.

It is a parabola whose vertex is (2, 5), axis is $x - 2 = 0$.

The area enclosed by the lines $x = 0$, $y = 0$, $x = \frac{3}{2}$ and the curve

$$= \int_0^{3/2} ydx = \int_0^{3/2} (1 + 4x - x^2)dx$$

$$= \left[x + 2x^2 - \frac{x^3}{3}\right]_0^{3/2}$$

$$= \frac{3}{2} + 2\cdot\frac{9}{4} - \frac{1}{3}\cdot\frac{27}{8} = \frac{39}{8}.$$

The area bounded by the lines $y = mx$, $y = 0$ and $x = 3/2$

$$= \int_0^{3/2} ydx = \int_0^{3/2} mx\,dx = m\left[\frac{x^2}{2}\right]_0^{\frac{3}{2}} = \frac{9}{8}m.$$

According to the question,

$$\frac{9}{8}m = \frac{1}{2}\cdot\frac{39}{8}$$

$$\therefore \quad m = \frac{13}{6}.$$

Concept Problems B

1. Supose that f and g are integrable on [a, b], but neither $f(x) \ge g(x)$ nor $g(x) \ge f(x)$ holds for all x in [a, b] [i.e., the curves $y = f(x)$ and $y = g(x)$ are intertwined].
   (a) What is the geometric significance of the integral $\int_a^b [f(x) - g(x)]dx$?
   (b) What is the geometric significance of the integral $\int_a^b |f(x) - g(x)|\,dx$?
2. Find the area of the region R lying between the lines $x = -1$ and $x = 2$ and between the curves $y = x^2$ and $y = x^3$.
3. Find the area of the closed figure bounded by the following curves.
   (i) $y = 3x + 18 - x^2$, $y = 0$
   (ii) $y = x^2 - 2x + 2$, $y = 2 + 4x - x^2$
   (iii) $y = x^3 - 3x^2 - 9x + 1$, $x = 0$, $y = 6$ $(x < 0)$
   (iv) $y = \dfrac{6x^2 - x^4}{9}$, $y = 1$
4. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = \frac{x}{4}$, above left by the curve $y = 1 + \sqrt{x}$ and above right by the curve $y = \frac{2}{\sqrt{x}}$

5. Find the area included between the curves $y = \sin^{-1} x$, $y = \cos^{-1} x$ and the x-axis.

6. Find the area bounded by $f(x) = \max\{\sin x, \cos x\}$, $x = 0$, $x = 2\pi$ and the x-axis.

Practice Problems D

7. Find the area of the closed figure bounded by the following curves.
   (i) $y = \sin\frac{\pi}{2}x$, $y = x^2$
   (ii) $y = -2x^2 + 5x + 3$, $y + 1 = \frac{4}{x+1}$
   (iii) $y = \frac{-x^2 + 3x - 1}{x}$, $x = 1$, $x = 2$, $y = 0$
   (iv) $y = \sqrt{x}$, $y = \sqrt{4-3x}$, $y = 0$

8. Find the area of the closed figure bounded by the curves $y = 2 - |2 - x|$ and $y = \frac{3}{|x|}$

9. Compute the area of the figure bounded by the curves $y = \frac{\ln x}{4x}$ and $y = x \ln x$.

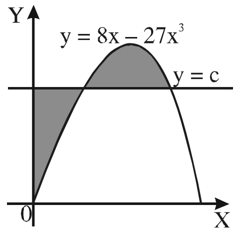
10. Find the area of the figure lying between the curve $y = xe^{-\frac{x^2}{2}}$ and its asymptote.

11. Find the area included between curves $y = \frac{4-x^2}{4+x^2}$ and $5y = 3|x| - 6$.

12. Find the area bounded by the curve $|y| + \frac{1}{2} = e^{-|x|}$.

13. For what values of m do the line $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region ? Find the area of the region.

14. The figure shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the areas of shaded regions are equal.



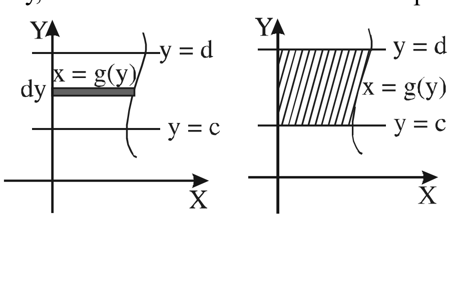
15. A circle with radius 1 and centre on the y-axis touches the curve $y = |2x|$ twice. Find the area of the region that lies between the two curves.

16. Find the area of the figure bounded by the curves $y = (x + 1)^2$, $x = \sin y$, $y = 0$ $(0 \le y \le 1)$

17. Find the area of the figure bounded by the curves $y = e^{-x}|\sin x|$, $y = 0$ $(x \ge 0)$ (assume that the area of this unbounded figure is the limit, as $A \to \infty$, of the areas of the curvilinear trapezoids corresponding to the variation of x from 0 to A).

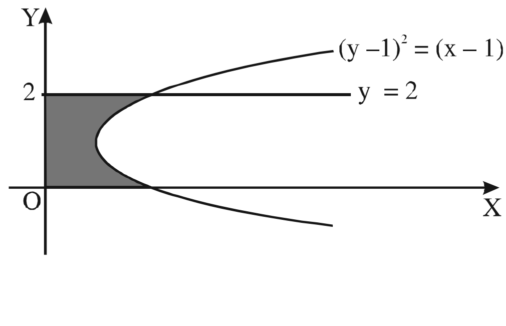
3.6 AREA BY HORIZONTAL STRIPS

For many regions it is easier to form horizontal strips rather than vertical strips. The procedure for horizontal strips duplicates the procedure for vertical strips. If we want to find the area between the curve $x = g(y)$ and the y-axis on the interval [c, d], we form horizontal strips. Such a region is shown in the figure, together with a typical horizontal approximating rectangle of width dy, which we refer to as horizontal strip.



Thus, if $g(y) \ge 0$ for $y \in [c, d]$ then the area bounded by the curve $x = g(y)$, y–axis and the abscissa $y = c$ and $y = d$ is $\int_{y=c}^{d} g(y)dy$.

Sometimes, a curve in the xy-plane can be defined as the graph of a function of y and not as a function of x. An example is the parabola defined by the equation $(y-1)^2 = x - 1$ and illustrated in the figure.



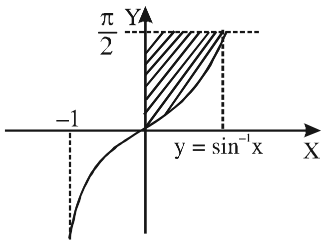
Although this equation cannot be solved uniquely for y in terms of x, it is easy to do the opposite. We get $x = (y-1)^2 + 1 = y^2 - 2y + 2$, and so the curve is the graph of the function f defined by $f(y) = y^2 - 2y + 2$. The area of the region bounded by the parabola, the two coordinate axes, and the horizontal line $y = 2$ is given by

$$\text{Area} = \int_0^2 f(y)\,dy = \int_0^2 (y^2 - 2y + 2)\,dy$$

$$= \left(\frac{y^3}{3} - y^2 + 2y\right)\Bigg|_0^2 = \frac{8}{3}.$$

Example 1. Find the area bounded between $y = \sin^{-1}x$ and y–axis between $y = 0$ and $y = \frac{\pi}{2}$.

Solution $y = \sin^{-1}x \Rightarrow x = \sin y$



The required area $= \int_0^{\frac{\pi}{2}} \sin y\,dy = -\cos y\Big]_0^{\frac{\pi}{2}} = -(0-1) = 1.$

Alternative :

The area in the above example can also evaluated by integration with respect to x. The required area = (area of rectangle formed by $x = 0, y = 0, x = 1, y = \frac{\pi}{2}$) – (area bounded by $y = \sin^{-1}x$, x–axis between $x = 0$ and $x = 1$)

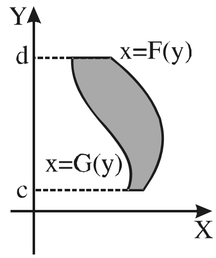
$$= \frac{\pi}{2} \times 1 - \int_0^1 \sin^{-1}x\,dx$$

$$= \frac{\pi}{2} - (x\sin^{-1}x + \sqrt{1-x^2})^1$$

$$= \frac{\pi}{2} - \left(\frac{\pi}{2} + 0 - 0 - 1\right) = 1.$$

Note: If F and G are continuous functions and if $F(y) \geq G(y)$ for all y in [c, d], then the area of the region bounded on the left by $x = G(y)$, on the right by $x = F(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d (F(y) - G(y))dy$$



Further, if we wish to find the area between the curves $x = F(y)$ and $x = G(y)$ where $F(y) \geq G(y)$ for some values of y but $G(y) \geq F(y)$ for other values of y, then we split the given region S into several regions.

Approximation by horizontal strips of width dy

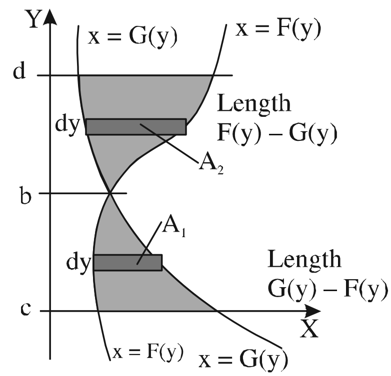


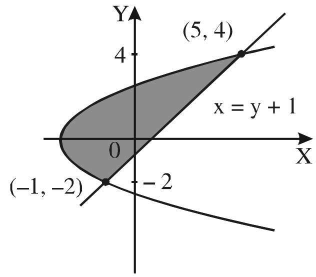
Figure Area by horizontal strips

Note that regardless of which curve is "ahead" or "behind," the horizontal strip has length $|F(y) - G(y)|$ and area $dA = |F(y) - G(y)|\,dy$. However, in practice, we must make sure to find the points of intersection of the curve and divide the intergrals so that in each region one curve is the leading curve ("right curve") and the other is the trailing curve ("left curve"). Suppose the curve intersect where $y = b$ for b on the interval [c, d], as shown in the figure, then

$$A = \int_c^b \underbrace{[G(y) - F(y)]}_{\text{G ahead of F}}dy + \int_b^d \underbrace{[F(y) - G(y)}_{\text{F ahead of G}}.$$

Example 2. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution By solving the two equations we find that the points of intersection are (–1, –2) and (5, 4).



We solve the equation of the parabola for x and notice from the figure that the left and the right boundary curves are $x_L = \frac{1}{2}y^2 - 3$ and $x_R = y + 1$.

We must integrate between the appropriate y-values, y = –2 and y = 4. Thus

$$A = \int_{-2}^{4}(x_R - x_L)\,dy$$

$$= \int_{-2}^{4}[(y+1) - (1/2\,y^2 - 3)]\,dy$$

$$= \int_{-2}^{4}(-1/2\,y^2 + y + 4)\,dy$$

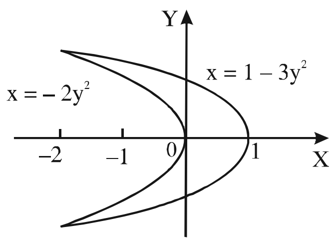
$$= \left.\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y\right]_{-2}^{4}$$

$$= -\frac{1}{6}(64) + 8 + 16 - \left(\frac{4}{3} + 2 - 8\right) = 18.$$

Example 3. Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$.

Solution We rewrite the equations as $y^2 = -\frac{x}{2}$ and $3y^2 = 1 - x \Rightarrow y^2 = \frac{1-x}{3}$.

The graph is shown below :



Solving the two equations :

$$\left(\frac{x-1}{3}\right) = \frac{x}{2} \Rightarrow 2x - 2 = 3x$$

$\Rightarrow\ x = -2 \Rightarrow y = 1, -1.$

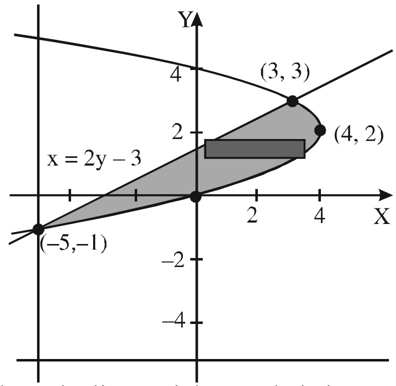
Since $1 - 3y^2 \geq -2y^2$ for $-1 \leq y \leq 1$, the required area

$$= \int_{-1}^{1}[(1-3y^2) - (-2y^2)]\,dy$$

$$= 2\left(y - \frac{y^3}{3}\right)_0^1 = \frac{4}{3}.$$

Example 4. Find the area of the region between the parabola $x = 4y - y^2$ and the line $x = 2y - 3$.

Solution The figure shows the region between the parabola and the line, together with a typical horizontal strip.



To find where the line and the parabola intersect, we solve $4y - y^2 = 2y - 3$ to obtain $y = -1$ and $y = 3$ Throughout the interval [–1, 3], the parabola is to the right of the line (test a typical point between –1 and 3, such as y = 0). Thus, the horizontal strip has area

$$dA = [\underbrace{(4y - y^2)}_{\text{Right curve}} - \underbrace{(2y-3)}_{\text{Left curve}}]dy$$

and the area between the parabola and the line is given by

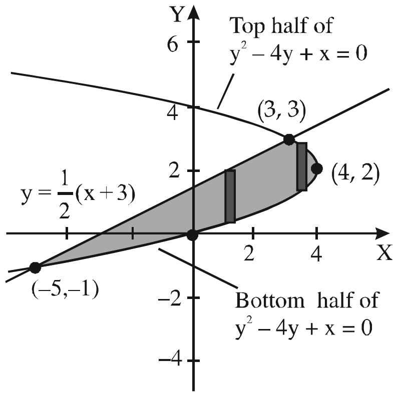
$$A = \int_{-1}^{3}[(4y - y^2) - (2y - 3)\,dy$$

$$= \int_{-1}^{3}(3 + 2y - y^2)\,dy = (3y + y^2 - \tfrac{1}{3}y^3)\Big|_{-1}^{3}$$

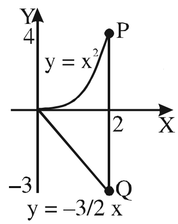
$$= (9 + 9 - 9) - (-3 + 1 + \tfrac{1}{3}) = 10\tfrac{2}{3}.$$

***Alternative* :**

In this example, the area can also be found by using vertical strips, but the procedure is more complicated. Note in the figure that on the interval [–5, 3], a representative vertical strip would extend from the bottom half of the parabola $y^2 - 4y + x = 0$ to the line $y = \frac{1}{2}(x + 3)$, whereas on the interval [3, 4], a typical vertical strip would extend from the bottom half of the parabola $y^2 - 4y + x = 0$ to the top half. Thus, the area is given by the sum of two integrals. It can be shown that the computation of area by vertical strips gives the same result as that found by horizontal strips.



Example 5. (i) Find the area of the region shown in the figure. The region is bounded by the curve $y = x^2$, the line $y = -3/2\,x$, and the line $x = 2$.



(ii) Find the area of the same region, but this time use cross sections parallel to the x axis.

Solution (i) Here, $f(x) = x^2$ and $g(x) = -3/2x$. For x in [0, 2], the cross-sectional length is $x^2 - (-3/2\,x)$. Thus the area of the region is

$$\int_0^2 \left[x^2 - \left(-\frac{3x}{2}\right)\right] dx = \int_0^2 \left(x^2 + \frac{3x}{2}\right) dx$$

$$= \left(\frac{x^3}{3} + \frac{3x^2}{4}\right)\Bigg|_0^2 = \left(\frac{2^3}{3} + \frac{3\cdot 2^2}{4}\right) - \left(\frac{0^3}{3} + \frac{3\cdot 0^2}{4}\right) = \frac{17}{3}.$$

(ii) Since the cross-sectional length is to be expressed in terms of y, first express the equations of the curves bounding the region in terms of y. The curve $y = x^2$ may be written as $x = \sqrt{y}$, since we are interested only in positive x. The curve $y = -2/3\,x$ can be expressed as $x = -2/3\,y$ by solving for x in terms of y. The line $x = 2$ also bounds the region. The point P in the figure lies on the parabola $y = x^2$ and has the x coordinate 2. Thus $P = (2, 2^2) = (2, 4)$. The point Q lies on the line $y = -\frac{3}{2}x$ and has x coordinate 2. Thus, $Q = (2, -\frac{3}{2}\cdot 2) = (2, -3)$. Consequently, a cross section of the region is determined for each number y in the interval [–3, 4]. The area of the region is therefore

$\int_{-3}^{4} x\,dy$. For $0 \le y \le 4$, the cross section is determined by the line $x = 2$ and the parabola $x = \sqrt{y}$. Thus for $0 \le y \le 4$, $x_1 = 2 - \sqrt{y}$, the large minus the smaller. For $-3 \le y \le 0$, the cross section is determined by the line $x = 2$ and the line $x_2 = -2/3y$. Thus for $-3 \le y \le 0$,

$$x = 2 - \left(-\frac{2}{3}y\right) = 2 + \frac{2y}{3}$$

The integral now breaks into two separate integrals,

$$\int_{-3}^{0} x_1\,dy + \int_0^4 x_2\,dy$$

$$= \int_{-3}^{0}\left(2 + \frac{2y}{3}\right) dy + \int_0^4 (2 - \sqrt{y})\,dy$$

$$= \left(2y + \frac{y^2}{3}\right)\Bigg|_{-3}^{0} + \left(2y - \frac{2}{3}y^{3/2}\right)\Bigg|_0^4$$

$$= 0 - (-3) + \frac{8}{3} - 0 = \frac{17}{3}.$$

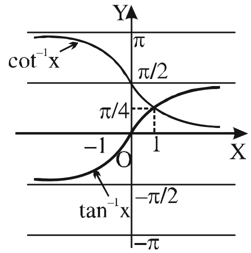
(i) Needed only one integral, but (ii) needed two. Moreover, in (ii) the formula for the cross-sectional length when $0 \le y \le 4$ involved $\sqrt{y}$, which is a little harder to integrate than x^2, which appeared in the corresponding formula in (i). Although both approaches to finding the area of the region in are valid, the one with cross sections parallel to the y axis is more convenient here.

Example 6. Compute the area enclosed between the curves $y = \tan^{-1}x$, $y = \cot^{-1}x$ and the y-axis.

Solution The required area

$$A = \int_0^1 (\cot^{-1}x - \tan^{-1}x)\,dx$$

$$= \int_0^1 \left(\frac{\pi}{2} - 2\tan^{-1}x\right) dx = \ln 2$$

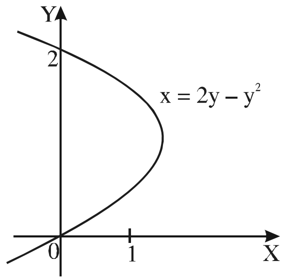


Alternative :

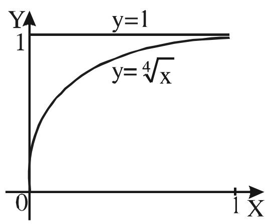
$$A = 2\int_0^{\pi/4} x\,dy = 2\cdot\int_0^{\pi/4} \tan y\,dy = \ln 2.$$

Practice Problems E

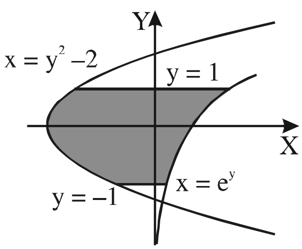
1. The area of the region that lies to the right of the y-axis and to the left of the parabola $x = 2y - y^2$ is given by integral $\int_0^2 (2y - y^2)dy$. (Turn your head clockwise and think of the region as lying below the curve $x = 2y - y^2$ from $y = 0$ to $y = 2$.) Find the area of the region.



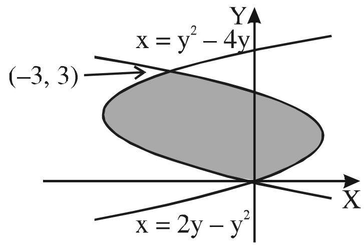
2. The boundaries of the shaded region are the y-axis, the line $y = 1$, and the curve $y = \sqrt[4]{x}$. Find the area of this region by writing x as a function of y and integrating with respect to y.



3. Find the area of the shaded region



4.



5. Find the area of the region in the 1st quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 3$.

6. Compute the area enclosed between the curves $y = \sec^{-1}x$, $y = \operatorname{cosec}^{-1}x$ and line $x - 1 = 0$.

7. Find the area of the region bounded by $x = y^2$ and $x = 3y - 2y^2$.

8. Find the area of the region given by

(i) y-axis, $x = y^3 - 3y^2 - 4y + 12$.

(ii) $y = \dfrac{1}{\sqrt{1-x^2}}$, $y = \dfrac{2}{x+1}$, y – axis.

9. (a) If $f(y) = -y^2 + y + 2$, sketch the region bounded by the curve $x = f(y)$, the y-axis, and the lines $y = 0$ and $y = 1$. Find its area.

(b) Find the area bounded by the curve $x = -y^2 + y + 2$ and the y-axis.

(c) The equation $x + y^2 = 4$ can be solved for x as a function of y, or for y as plus or minus a function of x. Sketch the region in the first quadrant bounded by the curve $x + y^2 = 4$, and find its area first by integrating a function of y and then by integrating a function of x.

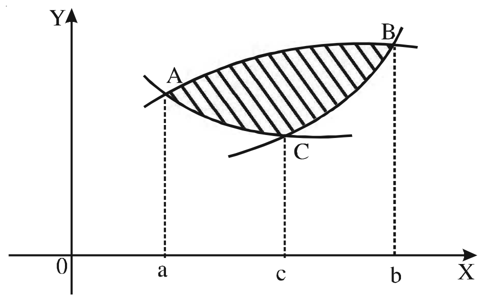
10. Find area common to circle $x^2 + y^2 = 2$ and the parabola $y^2 = x$.

11. Find the area of the region bounded by $y^2 + 4x = 0$ and $(y^2 + 4)x + 8 = 0$.

3.7 AREA OF A REGION BETWEEN SEVERAL GRAPHS

If it is required to find the area of a plane figure of more complicated form, then we try to express the sought for area in the form of an algebraic sum of the area of certain curvilinear trapezoids. For instance, the area of the figure represented in the figure below is computed by the formula

$$S = S_{aABb} - S_{aACc} - S_{cCBb}$$



Let the curves AB, BC and AC be the respective graphs of the following functions :

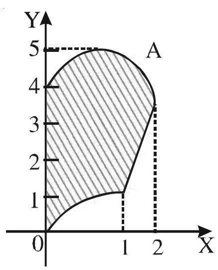
$y = f(x), x \in [a, b]$; $y = g(x), x \in [a, c]$

and $y = h(x), x \in [c, b]$. Then

$$S = \int_a^b f(x)dx - \int_a^c g(x)dx - \int_c^b h(x)dx$$

Example 1. Compute the area of the plane figure bounded by the curves $y = \sqrt{x}$, $x \in [0, 1]$, $y = x^2$, $x \in [1, 2]$ and $y = -x^2 + 2x + 4$, $x \in [0, 2]$.

Solution



The required area

$$S = \int_0^2 (-x^2 + 2x + 4)dx - \int_0^1 \sqrt{x}dx - \int_1^2 x^2 dx$$

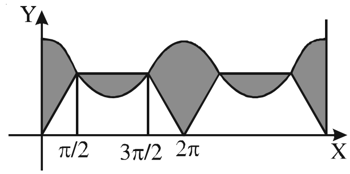
$$= -\frac{x^3}{3}\Big|_0^2 + x^2\Big|_0^2 + 4x\Big|_0^2 - \frac{2}{3}x^{3/2}\Big|_0^1 - \frac{x^3}{3}\Big|_1^2 = \frac{19}{3}.$$

Example 2. Find the area bounded by the curves $y = \sin^{-1}|\sin x| + \cos^{-1}\cos x$ and $y = \cos x + \pi$ and the lines $x = 0$ and $x = 4\pi$.

Solution Let $f(x) = \sin^{-1}|\sin x| + \cos^{-1}\cos x$

$\Rightarrow f(x) = \sin^{-1}(\sin x) + \cos^{-1}(\cos x)$ (for $0 \le x \le \pi$)

$$\Rightarrow f(x) = \begin{cases} 2x, & 0 \le x \le \frac{\pi}{2} \\ \pi, & \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ 4\pi - 2x, & \frac{3\pi}{2} < x \le 2\pi \end{cases}$$



$\Rightarrow$ The area bounded is shown in the figure

$$\Rightarrow A = 4\int_0^{\pi/2} (\pi + \cos x - 2x)\,dx$$

$$+ 2\int_{\pi/2}^{3\pi/2} (\pi - (\pi + \cos x))\,dx$$

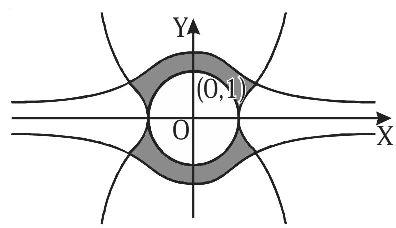
$$= 4[\pi x + \sin x - x^2]_0^{\pi/2} - 2[\sin x]_{\pi/2}^{3\pi/2}$$

$$= 4\left[\frac{\pi^2}{2} + 1 - \frac{\pi^2}{4}\right] - 2[-2]$$

$$= 2\pi^2 + 4 - \pi^2 + 4 = (\pi^2 + 8).$$

Example 3. Find the area of the region enclosed between the curves $4|y| = |4 - x^2|$ and $|y|(x^2 + 4) = 12$.

Solution



The required area

$$= 4\left[\int_0^{2\sqrt{2}} \frac{12}{x^2 + 4}dx - \left(\int_0^2 \frac{4 - x^2}{4}dx + \int_2^{2\sqrt{2}} \frac{x^2 - 4}{4}dx\right)\right]$$

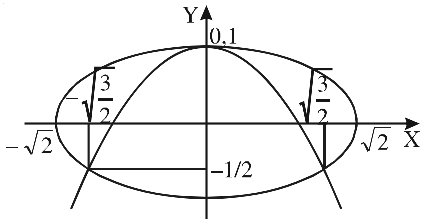
$$= 4\left[\frac{12}{2} \times \tan^{-1}\frac{x}{2}\Big|_0^{2\sqrt{2}} - \frac{1}{4} \times 4x - \frac{x^3}{3}\Big|_0^2 + \frac{1}{4} \times \frac{x^3}{4} - 4x\Big|_2^{2\sqrt{2}}\right]$$

$$= 4\left(6\tan^{-1}\sqrt{2}-\frac{4}{3}+\frac{4-2\sqrt{2}}{3}\right)$$

$$= \frac{4}{3}(18\tan^{-1}\sqrt{2}-2\sqrt{2}).$$

Example 4. Find the area bounded by the region inside the ellipse, $x^2+2y^2=2$ and outside the graph of the function, $f(x)=1-x^2$

Solution The graph is shown below :



$$A = 2\int_{-1/2}^{1}\left[\sqrt{2-2y^2}-\sqrt{1-y}\right]dy$$

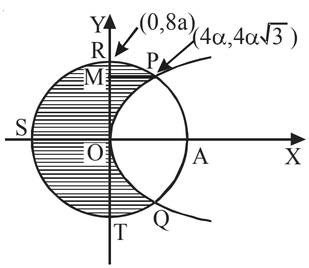
$$= 2\sqrt{2}\int_{-1/2}^{1}\sqrt{1-y^2}\,dy - 2\int_{-1/2}^{1}\sqrt{1-y}\,dy$$

$$= \frac{2\sqrt{2}\,\pi}{3}-\frac{3\sqrt{3}}{2\sqrt{2}}=\frac{8\pi-9\sqrt{3}}{6\sqrt{2}}.$$

Example 5. Show that the larger of the two areas into which the circle $x^3+y^2=64a^2$ is divided by the parabola

$y^2=12ax$ is $\frac{16}{3}a^2[8\pi-\sqrt{3}]$.

Solution $x^2+y^2=64a^2$ is a circle with centre (0, 0) and radius 8a and $y^2=12ax$ is a parabola whose vertex is at (0, 0) and latus rectum 12a. Both the curves are symmetrical about x-axis. Solving the two equations, the co-ordinates of the common point P are $(4a, 4a\sqrt{3})$.



Now the area of the larger portion of the circle (i.e. the shaded area) = the area PRSTQOP = the area of the semi-circle RST + 2 area OPR

$$= \frac{1}{2}\cdot\pi(8a)^2+[\text{area OPM}+\text{area MPR}]$$

$$= \frac{1}{2}\cdot\pi(8a)^2+2\left[\int_0^{4a\sqrt{3}}x\,dy,\ \text{for } y^2=12ax\right]$$

$$+2\left[\int_{4a\sqrt{3}}^{3a}x\,dy,\ \text{for } x^2+y^2=64a^2\right]$$

$$=32\pi a^2+2\int_0^{4a\sqrt{3}}\frac{y^2}{12a}dy+2\int_{4a\sqrt{3}}^{8a}\sqrt{(4a^2-y^2)}\,dy$$

$$=32\pi a^2+\frac{1}{6a}\left[\frac{y^3}{3}\right]_0^{4a\sqrt{3}}$$

$$+2\left[\frac{1}{2}y\sqrt{(64a^2-y^2)}+\frac{64a^2}{2}\sin^{-1}\frac{y}{8a}\right]_{4a\sqrt{3}}^{8a}$$

$$=32\pi a^2+\frac{1}{6a}\left[\frac{64\times3\sqrt{3}a^3}{3}\right]$$

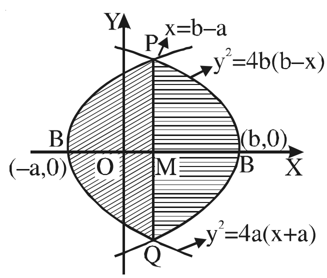
$$+2[\{0-8a^2\sqrt{3}\}+32a^2\{\sin^{-1}1-\sin^{-1}(\sqrt{3/2})\}]$$

$$=32\pi a^2+\frac{32\sqrt{3}a^2}{3}-16a^2\sqrt{3}+\frac{32}{3}a^2\pi$$

$$=\frac{128}{3}a^2\pi-\frac{16}{3}a^2\sqrt{3}=\frac{16}{3}a^2(8\pi-\sqrt{3}).$$

Example 6. Show that the area included between the parabolas $y^2=4a(x+a)$, $y^2=4b(b-x)$ is $\frac{8}{3}(a+b)\sqrt{ab}$.

Solution $y^2=4a(x+a)$ represents a parabola whose vertex is (–a, 0) and latus rectum is 4a. Also $y^2=4b(b-x)$ represents a parabola whose vertex is (b, 0) and latus rectum 4b. Both the curves have been shown in the figure.



Equating the values of y^2 from the two given equations of parabolas, we get $4a(x+a)=4b(b-x)$ or, $x=b-a$ i.e. the abscissa of the point of intersection P is $b-a$. Now both the curves are symmetrical about x-axis.

∴ The required area

= 2 [Area APM + Area PMB], by symmetry

$$= 2\left[\int_{-a}^{b-a} y\,dx, \text{ for the parabola } y^2 = 4a(x+a)\right.$$

$$\left. + \int_{b-a}^{b} y\,dx, \text{ for the parabola } y^2 = 4b(b-x)\right]$$

$$= 2\left[\int_{-a}^{b-a} \sqrt{\{4a(x+a)\}}\,dx + \int_{b-a}^{b} \sqrt{\{4b(b-x)\}}dx\right]$$

$$= 4\sqrt{a}\int_{-a}^{b-a} (x+a)^{1/2}\,dx + 4\sqrt{b}\int_{b-a}^{b} (b-x)^{1/2}dx$$

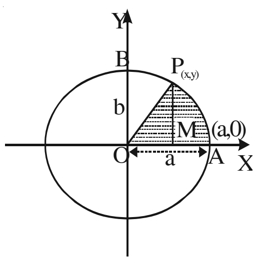
$$= 4\sqrt{a}\left[\frac{2}{3}(x-a)^{3/2}\right]_{-a}^{b-a} - 4\sqrt{b}\left[\frac{2}{3}(b-x)^{3/2}\right]_{b-a}^{b}$$

$$= \frac{1}{3}8\sqrt{a}.b^{3/2} + \frac{1}{3}8\sqrt{b}a^{3/2}$$

$$= \frac{8}{3}(a+b)\sqrt{(ab)}.$$

Example 7. If P(x, y) be any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and S be the sectorial area bounded by the curve, the x-axis and the line joining the origin to P, show that $x = a\cos(2S/xb)$, $y = b\sin(2S/ab)$.

Solution The given ellipse is shown in the figure. We have



S = sectorial area OAP (i.e. the shaded area)

= area of the ΔOMP + area PMA

$$= \frac{1}{2}\text{OM}.\text{MP} + \int_x^a y\,dx, \text{ for the ellipse}$$

$$= \frac{1}{2}xy + \int_x^a \frac{b}{a}\sqrt{(a^2-x^2)}dx,$$

$$= \frac{1}{2}x\frac{b}{a}\sqrt{(a^2-x^2)}$$

$$+ \frac{b}{a}\left[\frac{x}{2}\sqrt{(a^2-x^2)} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_x^a$$

$$= \frac{bx}{2a}\sqrt{(a^2-x^2)}$$

$$+ \frac{b}{a}\left[0 + \frac{a^2\pi}{4} - \frac{x}{2}\sqrt{(a^2-x^2)} - \frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{bx}{2a}\sqrt{(a^2-x^2)} + \frac{b}{a}.\frac{1}{2}a^2\left(\frac{\pi}{2} - \sin^{-1}\frac{x}{a}\right)$$

$$- \frac{bx}{2a}\sqrt{(a^2-x^3)}$$

$$= \frac{ab}{2}\left(\frac{\pi}{2} - \sin^{-1}\frac{x}{a}\right) = \frac{ab}{2}\cos^{-1}\frac{x}{a}$$

Thus, $S = \dfrac{ab}{2}\cos^{-1}\dfrac{x}{a}$.

$\therefore\ \cos^{-1}\dfrac{x}{a} = \dfrac{2S}{ab}$ or $\dfrac{x}{a} = \cos\dfrac{2S}{ab}$ or $x = a\cos\dfrac{2S}{ab}$.

Also $y = \dfrac{b}{a}\sqrt{a^2-x^2} = \dfrac{b}{a}\sqrt{a^2 - a^2\cos^2\dfrac{2S}{ab}} = b\sin\dfrac{2S}{ab}$.

Example 8. Find the area common to the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 4y^2 = 9$.

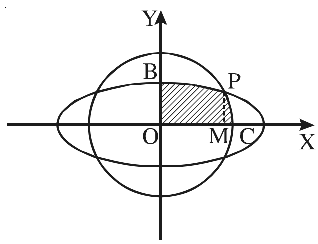
Solution The equation of the circle

$x^2 + y^2 = 4$, ...(1)

and the equation of the ellipse is

$x^2 + 4y^2 = 9$. ...(2)

Both the curves (1) and (2) are symmetrical about both the axes and have been shown in the figure.



Solving (1) and (2) for x, we have

$x^2 + 4(4 - x^2) = 9$ or $3x^2 = 7$ or $x^2 = 7/3$

∴ The x-coordinate of the point of intersection P lying in the first quadrant is $\sqrt{(7/3)}$. Also putting $y = 0$ in $x^2 + y^2 = 4$, we get $x = 2$ at C.

Now the required area is symmetrical about both the axes.

∴ The area common to the circle and the ellipse

= 4 × (common area lying in the first quadrant)

= 4 × area OCPB

= 4 [area OBPM + area CPM]

$$= 4\left[\int_0^{\sqrt{(7/3)}} y\,dx, \text{ for the ellipse}\right.$$

$$\left. + \int_{\sqrt{(7/3)}}^{2} y\,dx, \text{ for the circle}\right]$$

$$=4\left[\int_0^{\sqrt{(7/3)}}\frac{1}{2}\sqrt{(9-x^2)}dx+\int_{\sqrt{(7/3)}}^{2}\sqrt{(4-x^2)}dx\right]$$

$$=2\left[\frac{x\sqrt{(9-x^2)}}{2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^{\sqrt{(7/3)}}$$

$$+\left[\frac{x\sqrt{(4-x^2)}}{2}+2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{(7/3)}}^{2}$$

$$=2\left[\frac{1}{2}\cdot\sqrt{\frac{7}{3}}\sqrt{\frac{20}{3}}+\frac{9}{2}\sin^{-1}\frac{1}{3}\sqrt{\frac{7}{3}}\right]$$

$$+4\left[2\sin^{-1}(1)-\frac{1}{2}\sqrt{\frac{7}{3}}\sqrt{\frac{5}{3}}-\sin^{-1}\frac{1}{2}\sqrt{\frac{7}{3}}\right]$$

$$=\frac{2\sqrt{35}}{3}+9\sin^{-1}\frac{\sqrt{7}}{3\sqrt{3}}+4\pi-\frac{2}{3}\sqrt{35}-8\sin^{-1}\frac{\sqrt{7}}{2\sqrt{3}}$$

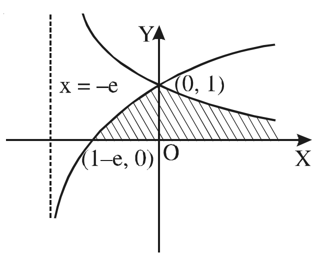
$$=4\pi+9\sin^{-1}\left(\frac{1}{3}\cdot\sqrt{\frac{7}{3}}\right)-8\sin^{-1}\left(\frac{1}{2}\cdot\sqrt{\frac{7}{3}}\right).$$

Example 9. Find the area enclosed between the curves $y=\ln(x+e)$, $x=\ln\left(\frac{1}{y}\right)$ and x-axis.

Solution The given curves are $y=\ln(x+e)$ and

$$x=\ln\left(\frac{1}{y}\right)\Rightarrow\frac{1}{y}=e^x\Rightarrow y=e^{-x}$$

Using transformation of graph we can sketch the curves.



Hence, the required area

$$=\int_{1-e}^{0}\ln(x+e)\,dx+\int_0^{\infty}e^{-x}dx$$

$$=\int_1^e \ln t\,dt+\int_0^{\infty}e^{-x}dx \quad (\text{putting } x+e=t)$$

$$=(t\ln t-t)_1^e-(e^{-x})_0^{\infty}=1+1=2.$$

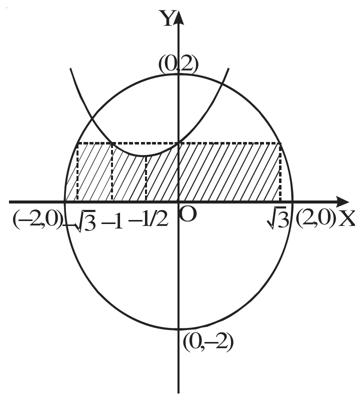
Example 10. Find the area enclosed by circle $x^2+y^2=4$, parabola $y=x^2+x+1$, the curve $y=\left[\sin^2\frac{x}{4}+\cos\frac{x}{4}\right]$ and x-axis (where [.] is the greatest integer function).

Solution $y=\left[\sin^2\frac{x}{4}+\cos\frac{x}{4}\right]$

$\because\ 1<\sin^2\frac{x}{4}+\cos\frac{x}{4}<2$, for $x\in(-2,2]$

$\therefore\ y=\left[\sin^2\frac{x}{4}+\cos\frac{x}{4}\right]=1$

Now we have to find out the area enclosed by the circle $x^2+y^2=4$, parabola $\left(y-\frac{3}{4}\right)=\left(x+\frac{1}{2}\right)^2$, line $y=1$ and x-axis. The required area is shaded in the figure.



Hence the required area

$$=\sqrt{3}\times1+(\sqrt{3}-1)\times1+\int_{-1}^{0}(x^2+x+1)\,dx$$

$$+2\int_{\sqrt{3}}^{2}(\sqrt{4-x^2})dx$$

$$=(2\sqrt{3}-1)+\left[\frac{x^3}{2}+\frac{x^2}{2}+x\right]_{-1}^{0}$$

$$+2\left[\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

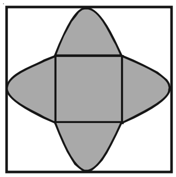
$$=(2\sqrt{3}-1)+\left[0-\left(-\frac{1}{3}+\frac{1}{2}-1\right)\right]$$

$$+2\left[(0+\pi)-\left(\frac{\sqrt{3}}{2}+\frac{2\pi}{3}\right)\right]$$

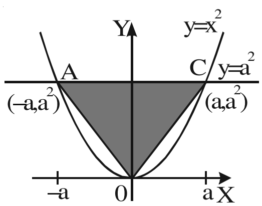
$$=(2\sqrt{3}-1)+\frac{5}{6}+\frac{2\pi}{3}-\sqrt{3}=\left(\frac{2\pi}{3}+\sqrt{3}-\frac{1}{6}\right).$$

Practice Problems F

1. Find the area included between the ellipses $x^2 + 2y^2 = a^2$ and $2x^2 + y^2 = a^2$.
2. The circle $x^2 + y^2 = a^2$ is divided into three parts by the hyperbola $x^2 - 2y^2 = \frac{a^2}{4}$. Determine the areas of these parts.
3. Find the area of the shaded region in the figure. The curves are parabolas. The inscribed square has area 4, and the circumscribed square has area 16.



4. A figure is bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, $x = 1$. At what point of the curve $y = x^2 + 1$, must a tangent be drawn for it to cut off a trapezoid of the greatest area from the figure ?
5. A figure is bounded by the curves $y = (x + 3)^2$, $y = 0$, $x = 0$. At what angles to the x-axis must straight lines be drawn through the point (0, 9) for them to partition the figure into three parts of the same size?
6. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

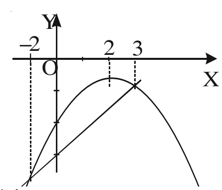


7. Let the sequence $a_1, a_2, a_3, \ldots$ be in G.P. If the area bounded by the parabolas $y^2 = 4a_n x$ and $y^2 + 4a_n(x - a_n) = 0$ be Δ_n, prove that the sequence $\Delta_1, \Delta_2, \Delta_3, \ldots$ is also in GP.
8. What part of the area of a square is cut off by the parabola passing through two adjacent vertices of the square and touching the midpoint of one of its sides?
9. For what value of the parameter $a > 0$ is the area of the figure bounded by the curves $y = a\sqrt{x}$, $y = \sqrt{2 - x}$ and the y-axis equal to the number b? For what values of b does the problem have a solution?
10. Show that the area bounded by the semi-cubical parabola $y^2 = ax^3$ and a double ordinate is 2/5 of the area of the rectangle formed by this ordinate and the abscissa.
11. Prove that the area common to the two ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ is $4ab \tan^{-1} b/a$.
12. The area between the parabola $2cy = x^2 + a^2$ and the two tangents drawn to it from the origin is $\frac{1}{3} a^2/c$.
13. Let A and B be the points of intersection of the parabola $y = x^2$ and the line $y = x + 2$, and let C be the point on the parabola where the tangent line is parallel to the graph of $y = x + 2$. Show that the area of the parabolic segment cut from the parabola by the line four-thirds the area of the triangle ABC.
14. Compute the areas of the curvilinear figures formed by intersection of the ellipse $\frac{x^2}{4} + y^2 = 1$ and the hyperbola $\frac{x^2}{2} - y^2 = 1$.
15. Find the value of c for which the area of the figure bounded by the curves $y = \frac{4}{x^2}$, $x = 1$ and $y = c$ is equal to $\frac{9}{4}$.

3.8. DETERMINATION OF PARAMETERS

Example 1. Find the area of the figure bounded by the parabola $y = ax^2 + 12x - 14$ and the straight line $y = 9x - 32$ if the tangent drawn to the parabola at the point $x = 3$ is known to make an angle $\pi - \tan^{-1}6$ with the x-axis.

Solution



$y = ax^2 + 12x - 14$

$$\frac{dy}{dx} = 2ax + 12 \Rightarrow \qquad \left.\frac{dy}{dx}\right|_{x=3} = 6a + 12.$$

Hence, $\tan(\pi - \tan^{-1}6) = 5a + 12 - 6 = 6a + 12$

$\Rightarrow \; a = -3$

Hence, $y = -3x^2 + 12x - 14$

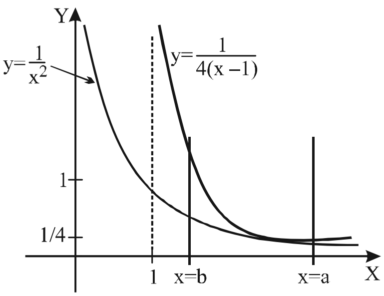
(note that $D < 0$, so $y < 0 \; \forall \; x \in R$)

The point of intersection of the line with parabola are $x = -2$ or 3.

$$\text{Hence } A = \int_{-2}^{3} [-3x^2 + 12x - 14] - (9x - 32)]dx = \frac{125}{2}.$$

Example 2. Find the value of 'a' ($a > 2$) for which the reciprocal of the area enclosed between $y = \frac{1}{x^2}$, $y = \frac{1}{4(x-1)}$, $x = 2$ and $x = a$ is 'a' itself and for what values of $b \in (1, 2)$, the area of the figure bounded by the lines $x = b$ and $x = 2$ is $1 - \frac{1}{b}$.

Solution



$x^2 = 4(x-1) \Rightarrow (x-2)^2 = 0$

$\Rightarrow$ The curves touch other.

$$\therefore \int_2^a \left(\frac{1}{4(x-1)} - \frac{1}{x^2}\right)dx = \frac{1}{a}$$

$a = e^2 + 1$.

$$\text{Also} \quad 1 - \frac{1}{b} = \int_b^2 \left(\frac{1}{4(x-1)} - \frac{1}{x^2}\right)dx$$

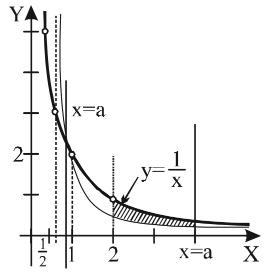
$\Rightarrow b = 1 + e^{-2}$.

Example 3. For what value of 'a' ($a > 2$) is the area of the figure bounded by the curves, $y = \frac{1}{x}$, $y = \frac{1}{2x-1}$, $x = 2$ and $x = a$ equal to $\ln \frac{4}{\sqrt{5}}$?

Solution

Solving the equations $y = \frac{1}{2x-1}$ and $y = \frac{1}{x}$

We get $x = 1$.



$$A = \int_2^a \left(\frac{1}{x} - \frac{1}{2x-1}\right)dx = \ln\frac{4}{\sqrt{5}}$$

$$\Rightarrow \left[\ln x - \frac{1}{2}\ln(2x-1)\right]_2^a = \ln\frac{4}{\sqrt{5}}$$

$$\Rightarrow \left[2\ln\frac{x^2}{2x-1}\right]_2^a = \ln\frac{16}{5}$$

$$\Rightarrow \ln\left(\frac{a^2}{2a-1}\right) - \ln\frac{4}{3} = \ln\frac{16}{5}$$

$$\Rightarrow \ln\frac{a^2}{2a-1} = \ln\frac{64}{15} \Rightarrow \frac{a^2}{2a-1} = \frac{64}{15}$$

$\Rightarrow 15a^2 - 128a + 64 = 0$

$$\Rightarrow a = 8 \; ; \; a = \frac{8}{15}.$$

Thus, $a = 2$ since a is greater than 2.

Example 4. If the line $y = mx$ divides the area enclosed by the lines $x = 0$, $y = 0$, $x = 3/2$ and the curve $y = 1 + 4x - x^2$ into two equal parts, then find the value of m.

Solution The given curve is $y - 5 = -(x - 2)^2$. Thus given curve is a parabola with vertex at (2, 5) and axis $x = 2$

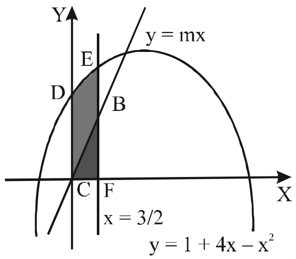
Given that area CBFC = Area CDEBC

So area CDEBFC = 2 Area CBFC

$$\text{Area CDEBFC} = \int_0^{3/2} (1 + 4x - x^2)dx$$

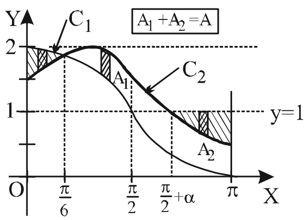
$$= \left| x + 2x^2 - \frac{x^3}{3} \right|_0^{3/2} = \frac{3}{2} + 2\left(\frac{9}{4}\right) - \frac{9}{8} = \frac{39}{8}.$$

$$\text{Area CBFC} = \int_0^{3/2} mx\,dx = \frac{9m}{8}$$



So we must have $\dfrac{39}{8}=\dfrac{18m}{8}$ or $m=\dfrac{13}{6}$

Example 5. Consider the two curves $C_1: y=1+\cos x$ & $C_2: y=1+\cos(x-\alpha)$ for $\alpha \in \left(0, \dfrac{\pi}{2}\right)$; $x \in [0, \pi]$. Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 and $x=0$ is same as that of the figure bounded by C_2, $y=1$ and $x=\pi$. For this value of α, find the ratio in which the line $y=1$ divides the area of the figure by the curves C_1, C_2 and $x=\pi$.



Solution $1+\cos x = 1+\cos(x-\alpha)$ $x = \alpha - x$

$\Rightarrow \quad x=\dfrac{\alpha}{2}$

Now $$\int_0^{\pi/2}(\cos x-\cos(x-\alpha))\,dx$$

$$= -\int_{\frac{\pi}{2}+\alpha}^{\pi}(\cos(x-\alpha))\,dx$$

or, $[\sin x-\sin(x-\alpha)]_0^{\alpha/2} = [\sin(x-\alpha)]_{\pi}^{\frac{\pi}{2}+\alpha}$

$$\left[\sin\frac{\alpha}{2}-\sin\left(-\frac{\alpha}{2}\right)\right]-[0-\sin(-\alpha)]$$

$$=\sin\left(\frac{\pi}{2}\right)-\sin(\pi-\alpha)$$

$$2\sin\frac{\alpha}{2}-\sin\alpha = 1-\sin\alpha,$$

Hence, $2\sin\dfrac{\alpha}{2}=1 \Rightarrow \alpha=\dfrac{\pi}{3}$.

Example 6. The possible values of $b>0$, so that the area of the bounded region enclosed between the parabola

$y=x-bx^2$ and $y=\dfrac{x^2}{b}$ is maximum.

Solution The given parabolas are

$$y=x-bx^2 \qquad ...(1)$$

and $$y=\frac{x^2}{b} \qquad ...(2)$$

The abscissae of their points of intersection are given by

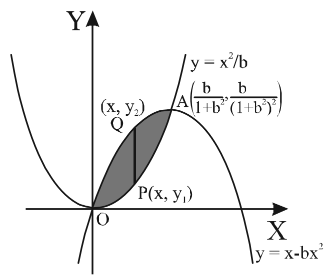
$$\frac{x^2}{b}=x-bx^2$$

$\Rightarrow \; x^2(1+b^2)-bx=0 \Rightarrow x=0,\, x=\dfrac{b}{1+b^2}$

Thus, the two parabolas intersect at O (0, 0) and

$$A\left(\frac{b}{1+b^2}, \frac{b}{(1+b^2)^2}\right)$$

The region enclosed by the two parabolas is shaded in the figure.



To find the area of this region, let us slice it into vertical strips. The approximating rectangle shown in the figure has length $= y_2 - y_1$, width $= \Delta x$ and it can move between $x=0$ and $x=\dfrac{b}{1+b^2}$. So, the area A enclosed by the two parabolas is given by

$$A=\int_0^{\frac{b}{1+b^2}}(y_2-y_1)\,dx$$

$$=\int_0^{\frac{b}{1+b^2}}\left(x-bx^2-\frac{x^2}{b}\right)dx=\left[\frac{x^2}{2}-\frac{bx^3}{3}-\frac{x^3}{3b}\right]_0^{\frac{b}{1+b^2}}$$

$$=\frac{1}{2}\left(\frac{b}{1+b^2}\right)^2-\frac{1}{3b}(1+b^2)\left(\frac{b}{1+b^2}\right)^3$$

$$=\frac{1}{2}\left(\frac{b}{1+b^2}\right)^2-\frac{1}{3}\left(\frac{b}{1+b^2}\right)^2=\frac{1}{6}\frac{b^2}{\left(1+b^2\right)^2}$$

Now, $A=\frac{1}{6}\frac{b^2}{\left(1+b^2\right)^2}$

$$\Rightarrow \frac{dA}{db}=\frac{1}{6}\left[\frac{\left(1+b^2\right)^2\cdot 2b-b^2\cdot 2\left(1+b^2\right)\cdot 2b}{\left(1+b^2\right)^4}\right]$$

$$\Rightarrow \frac{dA}{db}=\frac{b}{3\left(1+b^2\right)^3}\left[1+b^2-2b^2\right]$$

$$\Rightarrow \frac{dA}{db}=\frac{b(1-b)(1+b)}{3\left(1+b^2\right)^3}$$

For maximum value of A, we must have

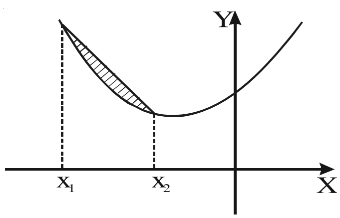
$$\frac{dA}{db}=0 \Rightarrow b(1-b)(1+b)=0$$

$\Rightarrow 1-b=0 \qquad \Rightarrow b=1 \qquad [\because b>0]$

Since $b>0$. Therefore, $1+b>0$. Thus, $\frac{dA}{db}$ changes its sign from positive to negative in the neighbourhood of $b=1$.

Example 7. Find the value(s) of the parameter "a' ($a>0$) for each of which the area of the figure bounded by the straight line, $y=\frac{a^2-ax}{1+a^4}$ and the parabola $y=\frac{x^2+2ax+3a^2}{1+a^4}$ is the greatest .

Solution



$$A=\int_{x_1}^{x_2}\frac{\left(a^2-ax\right)-\left(x^2+2ax+3a^2\right)}{1+a^4}dx$$

where x_1 and x_2 are the roots of,

$x^2+2ax+3a^2=a^2-ax$

$x=-2a$ or $x=-a$

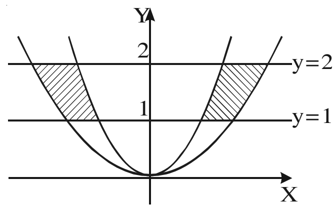
$$A=\int_{-2a}^{-a}\frac{\left(a^2-ax\right)-\left(x^2+2ax+3a^2\right)}{1+a^4}dx$$

$$=\frac{a^3}{1+a^4} \Rightarrow \frac{dA}{da}=0$$

gives $a=3^{1/4}$. The greatest area is bounded when $a=3^{1/4}$.

Example 8. Compute the area of the figure bounded by the straight lines, $y=1$, $y=2$ and the curve $y=ax^2$ and $y=\frac{1}{2}ax^2$. Find also the values of the parameter a $(a\geq 1)$ for which the area is greatest.

Solution



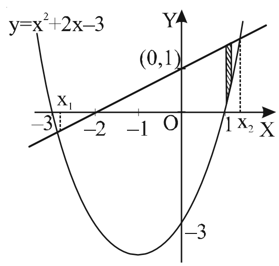
$$A=2\int_1^2\left(\sqrt{\frac{2y}{a}}-\sqrt{\frac{y}{a}}\right)dy$$

$$A=\frac{4\left(5-3\sqrt{2}\right)}{3\sqrt{a}}$$

Since $a\geq 1$, the greatest area is obtained when $a=1$, and the greatest area is $\frac{4\left(5-3\sqrt{2}\right)}{3}$.

Example 9. If the area bounded by $y=x^2+2x-3$ and the line $y=kx+1$ is the least, find k and also the least area.

Solution



Let x_1 and x_2 be the roots of the equation

$x^2+2x-3=kx+1$

$\Rightarrow x^2+(2-k)x-4=0$

$\Rightarrow x_1+x_2=k-2,\ x_1x_2=-4.$

Now, $A=\int_{x_1}^{x_2}[(kx+1)-(x^2+2x-3)]dx$

$$= \left[(k-2)\frac{x^2}{2} - \frac{x^3}{3} + 4x\right]_{x_1}^{x_2}$$

$$= \left[(k-2)\frac{x_2^2 - x_1^2}{2} - \frac{1}{3}\left(x_2^3 - x_1^3\right) + 4(x_2 - x_1)\right]$$

$$= (x_2 - x_1)\left[\frac{(k-2)^2}{2} - \frac{1}{3}\left((x_2+x_1)^2 - x_1x_2\right) + 4\right]$$

$$= \sqrt{(x_2+x_1)^2 - 4x_1x_2}\ \left[\frac{(k-2)^2}{2} - \frac{1}{3}\left((k-2)^2 + 4\right) + 4\right]$$

$$= \frac{\sqrt{(k-2)^2+16}}{6}\left[\frac{1}{6}(k-2)^2 + \frac{16}{3}\right]$$

$$= \frac{[(k-2)^2+16]^{3/2}}{6}.$$

$\Rightarrow$ A is minimum if $k = 2$ and $A_{min} = \frac{32}{3}$.

Least value of a variable area

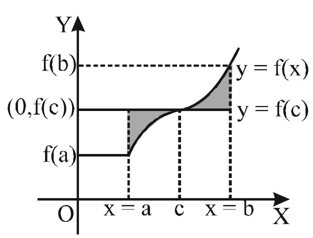
If $y = f(x)$ is a strictly monotonic function in (a, b), with $f'(x) \neq 0$, then the area bounded by the ordinates $x = a$, $x = b$, $y = f(x)$ and $y = f(c)$ (where $c \in (a, b)$) is minimum when $c = \frac{a+b}{2}$.

Proof Assume that f is strictly increasing in (a, b)

$$A = \int_a^c (f(c) - f(x))\,dx + \int_c^b (f(x) - f(c))\,dx$$

$$= f(c)(c-a) - \int_a^c (f(x))\,dx + \int_c^b (f(x))\,dx - f(c)(b-c)$$

$$\Rightarrow A = [2c - (a+b)]f(c) + \int_c^b (f(x))\,dx - \int_a^c (f(x))\,dx$$



Differentiating w.r.t c,

$$\frac{dA}{dc} = [2c - (a+b)]f'(c) + 2f(c) + 0 - f(c) - (f(c) - 0)$$

For maxima or minima $\frac{dA}{dc} = 0$

$\Rightarrow f'(c)[2c - (a+b)] = 0$ (as $f'(c) \neq 0$)

Hence, $c = \frac{a+b}{2}$.

Since f is strictly increasing $f'(c) > 0$.

Hence, for $c < \frac{a+b}{2}$, $\frac{dA}{dc} < 0$ and for $c > \frac{a+b}{2}$, $\frac{dA}{dc} > 0$

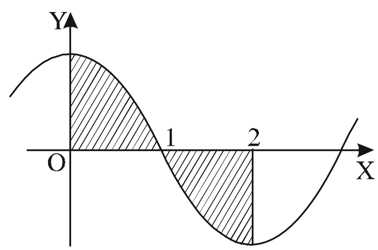
Hence, A is minimum when $c = \frac{a+b}{2}$.

We have a similar proof when f is strictly decreasing.

Example 10. If the area bounded by $f(x) = \frac{x^3}{3} - x^2 + a$ and the straight lines $x = 0$, $x = 2$ and the x-axis is minimum, then find the value of a.

Solution $f'(x) = x^2 - 2x = x(x-2)$. (note that $f(x)$ is monotonic in (0, 2)). Hence, for minimum area, $f(x)$ must cross the x-axis at

$$x = \frac{0+2}{2} = 1.$$



Hence, $f(1) = 1/3 - 1 + a = 0 \Rightarrow a = 2/3$.

Example 11. Find the value of the parameter 'a' for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$, and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, is the least.

Solution $f(x) = x^3 + 3x^2 + x + a$

$$f'(x) = 3x^2 + 6x + 1 = 0 \quad \Rightarrow \quad x = -1 \pm \frac{\sqrt{6}}{3}$$

Hence for minimum area, $f(x)$ must cross the x-axis at

$$x = \frac{1}{2}\left[\left(-1 + \frac{\sqrt{6}}{3}\right) + \left(-1 - \frac{\sqrt{6}}{3}\right)\right] = -1$$

Thus, $f(-1) = -1 + 3 - 1 + a = 0$

$\Rightarrow a = -1$.

Practice Problems G

1. Find the value of a for which the area of the figure bounded by the curve $y = \sin 2x$, the straight lines $x = \pi/6$, $x = a$, and the x-axis is equal to 1/2.
2. (i) Find the area of the region enclosed by the parabola $y = 2x - x^2$ and the x-axis.
   (ii) Find the value of m so that the line $y = mx$ divides the region in part (i) into two regions of equal area.
3. Find the values of c for which the area of the figure bounded by the curve $y = 8x^2 - x^5$, the straight line $x = 1$ and $x = c$ and the abscissa axis is equal to 16/3.
4. Find the values of c for which the area of the figure bounded by the curves $y = 4/x^2$, $x = 1$ and $y = c$ is equal to $2\frac{1}{4}$.
5. Find the value of k for which the area of the figure bounded by the curves $x = \pi/18$, $x = k$, $y = \sin 6x$ and the abscissa axis is equal to 1/6.
6. At what value of d is the area of the figure bounded by the curves $y = \cos 5x$, $y = 0$, $x = \pi/30$ and $x = d$ equal to 0.2 ?
7. For what value of a does the straight line $y = a$ bisects the area of the figure bounded by the lines $y = 0$, $y = 2 + x - x^2$?
8. Find all the values of the parameter b $(b > 0)$ for each of which the area of the figure bounded by the curves $y = 1 - x^2$ and $y = bx^2$ is equal to a. For what values of a does the problem have a solution?
9. For what value of a is the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight lines $y = 0$, $x = 0$, and $x = 1$ the least?
10. For what value of k is the area of the figure bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is the least. Determine the least area.
11. Find the least value of the area bounded by the line $y = mx + 1$ and the parabola $y = x^2 + 2x - 3$, where m is a parameter.
12. For what positive a does the area S of a curvilinear trapezoid bounded by the lines $y = \frac{x}{6} + \frac{1}{x^2}$, $y = 0$, $x = a$, $x = 2a$ assumes the least value?
13. For what values of a $(a \in [0, 1])$ does the area of the figure bounded by the graph of the function f(x) and the straight lines $x = 0$, $x = 1$, $y = f(a)$, is at a minimum, and for what values is it at a maximum, if $f(x) = \sqrt{1 - x^2}$?
14. For what value of a does the area of the figure, bounded by the straight lines $x = x_1$, $x = x_2$, the graph of the function $y = |\sin x + \cos x - a|$, and the abscissa axis, where x_1 and x_2 are two successive extrema of the function $f(x) = \sqrt{2}\sin(x + \pi/4)$, have the least value?

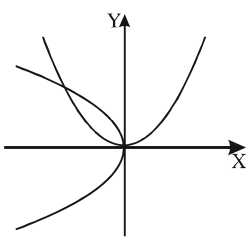
3.9 SHIFTING OF ORIGIN

Since area remains invariant even if the coordinates axes are shifted, hence shifting of origin in many cases proves to be convenient in computing the areas.

Example 1. Find the area enclosed between the parabolas $y^2 - 2y + 4x + 5 = 0$ and $x^2 + 2x - y + 2 = 0$.

Solution $(y - 1)^2 = -4(x + 1)$; $(x + 1)^2 = y - 1$

$Y^2 = -4X$ $\quad\quad$ $X^2 = Y$



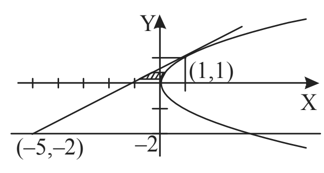
Solving the two equations we get the points of intersection as $(0, 0)$ and $(-4^{1/3}, 4^{2/3})$.

The required area $= \int_{-4^{1/3}}^{0} (\sqrt{-4X} - X^2)dX$

$$= -\frac{1}{3}\left(4(-X)^{3/2} + X^3\right)\Big|_{-4^{1/3}}^{0} = \frac{4}{3}.$$

Example 2. Find the area enclosed by the parabola $(y - 2)^2 = x - 1$ and the tangent to it at (2, 3) and x-axis.

Solution



Put $x-1=X$ and $y-2=Y$

Hence the parabola becomes $Y^2=X$

Also $x=2 \Rightarrow X=1$

and $y=2 \Rightarrow Y=1$

Also x-axis means $y=0 \Rightarrow Y=-2$

Equation of tangent : $YY_1 = 2\cdot\frac{1}{4}(X+X_1)$

$2YY_1 = X+X_1$

$2Y = X+1$

Hence, $A=\int_{-2}^{1}[Y^2-(2Y-1)]dY$

$$=\left(\frac{Y^3}{3}-Y^2+Y\right)\Bigg|_{-2}^{1}=9.$$

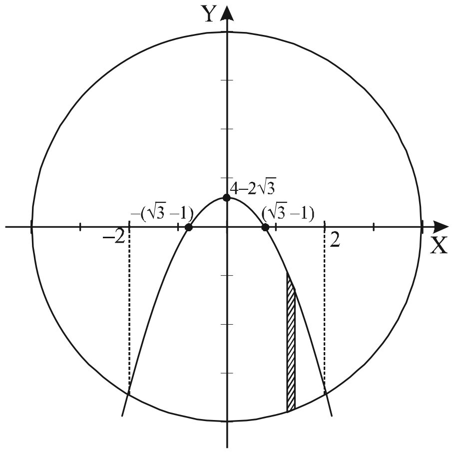
Example 3. Find the area enclosed between the smaller arc of the circle $x^2+y^2-2x+4y-11=0$ and the parabola $y=-x^2+2x+1-2\sqrt{3}$.

Solution Circle : $(x-1)^2+(y+2)^2=16$ ...(1)

Parabola : $y=-[x^2-2x-1+2\sqrt{3}]$

$=-[(x-1)^2-2+2\sqrt{3}]$

$y+2=(4-2\sqrt{3})-(x-1)^2$ ...(2)



Let $x-1=X$ and $y+2=Y$

Hence, Circle : $X^2+Y^2=16$;

Parabola : $Y=4-2\sqrt{3}-X^2$

Solving the circle and parabola

$X=2$ or -2

and $Y=-2\sqrt{3}$; $Y=1+2\sqrt{3}$ (rejected)

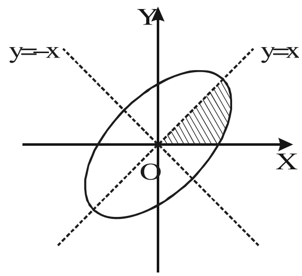
$$\therefore\ A=2\int_0^2\left[\left(4-2\sqrt{3}-X^2\right)-\left(-\sqrt{16-X^2}\right)\right]dX$$

$$=\frac{16-3\sqrt{3}+4\pi}{3}.$$

Example 4. Find the area of region enclosed by the curve $\frac{(x-y)^2}{a^2}+\frac{(x+y)^2}{b^2}=2$ $(a>b)$, the line $y=x$ and the positive x-axis.

Solution The given $\frac{(x-y)^2}{a^2}+\frac{(x+y)^2}{b^2}=2$ is an ellipse whose major and minor axes are $x-y=0$ and $x+y=0$ respectively.

The required area is shaded in the figure.



Instead of directly solving the problem we can solve an equivalent problem with an ellipse whose axes are along x-axis and y-axis.

The equivalent region is shown as $(OA'B'O')$ where

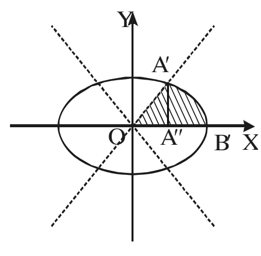
equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ ...(1)

$\therefore$ The required area = area $(\Delta OA'A''+A'A''B')$

where $A'=\left(\frac{ab}{\sqrt{a^2+b^2}},\frac{ab}{\sqrt{a^2+b^2}}\right)$. Note that A' is obtained by solving the equation (1) with the line $y=x$.

$$\text{Area } \Delta OA'A''=\frac{1}{2}\times\frac{ab}{\sqrt{a^2+b^2}}\times\frac{ab}{\sqrt{a^2+b^2}}$$

$$=\frac{1}{2}\left(\frac{a^2b^2}{a^2+b^2}\right) \quad ...(2)$$



$$\text{Area } A'B'A'' = \int_{\frac{ab}{\sqrt{a^2+b^2}}}^{a} b\sqrt{\left(1-\frac{x^2}{a^2}\right)}dx$$

$$= \int_{\frac{ab}{\sqrt{a^2+b^2}}}^{a} \sqrt{a^2-x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{ab/\sqrt{a^2+b^2}}^{a}$$

$$= \frac{b}{a}\left[0+\frac{a^2}{2}\cdot\frac{\pi}{2}-\frac{ab}{2\sqrt{a^2+b^2}}\cdot\sqrt{\left(a^2-\frac{a^2b^2}{a^2+b^2}\right)} - \frac{a^2}{2}\sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right]$$

$$= \frac{b}{a}\left[\frac{\pi a^2}{4}-\frac{a^2}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)-\frac{a^3b}{2(a^2+b^2)}\right]$$

$$= \frac{\pi ab}{4}-\frac{ab}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)-\frac{a^2b^2}{2(a^2+b^2)} \quad ...(3)$$

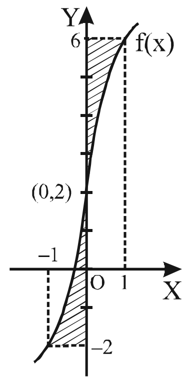
Hence the required area = sum of areas (2) and (3)

$$= \frac{\pi ab}{4}-\frac{ab}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right).$$

Area bounded by inverse function

Example 5. Let $f(x) = x^3 + 3x + 2$ and $g(x)$ be its inverse. Find the area bounded by $g(x)$, the x-axis and the ordinates $x = -2$ and $x = 6$.

Solution The required area will be equal to area enclosed by $y = f(x)$ and the y-axis, between the abscissa at $y = -2$ and $y = 6$.



$f(0) = 2$; $f(-1) = -2$ and $f(1) = 6$.

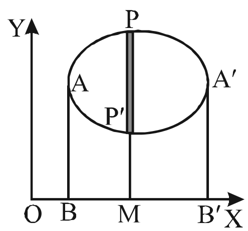
Note that f(x) is monotonic in $[-1, 1]$.

$$\text{Hence, } A = \int_0^1 (6-f(x))dx + \int_{-1}^{0}(f(x)-(-2))dx$$

$$= \int_0^1 (4-x^3-x)dx + \int_{-1}^{0}(x^3+3x+4)dx = \frac{5}{4}.$$

3.10 AREA BOUNDED BY A CLOSED CURVE

Let us now find the area of a closed curve, such as that represented in the figure,



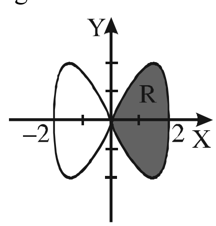
Let $PM = y_2$, $P'M = y_1$, the elemental area is represented by $(y_2 - y_1)\,dx$, and the entire area by

$$\int_{OB}^{OB'} (y_2 - y_1)dx,$$

in which OB, OB' are the limiting values of x.

Example 1. Find the area of the region bounded by one loop of the graph of the equation $y^2 = 4x^2 - x^4$.

Solution The graph is symmetric to both axes. Since $y = \pm\sqrt{4-x^2}$, it is clear that the total graph lies between $x = -2$ and $x = 2$, The point $(\sqrt{2}, 2)$ is a maximum point on the graph. We can plot a few points and sketch the rest of the curve by symmetry, as indicated in the figure.



Let us find the area of the shaded region R bounded by the loop between $x = 0$ and $x = 2$. The equation of the top half of this loop is $y = x\sqrt{4-x^2}$, whereas that of the lower half is $y = -x\sqrt{4-x^2}$.

$$\text{Thus, } A = \int_0^2 [x\sqrt{4-x^2}) - (-x\sqrt{4-x^2})]dx$$

$$= \int_0^2 2x\sqrt{4-x^2}\,dx$$

In order to evaluate this integral, we let

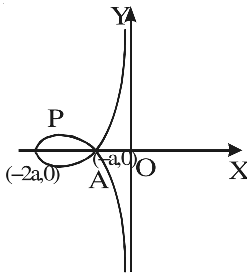
$u = 4 - x^2, \qquad du = -2x\,dx.$

Then $u = 4$ when $x = 0$, and $u = 0$ when $x = 2$. Hence,

$$A = -\int_4^0 u^{1/2}\,du = -\frac{2}{3}u^{3/2}\Big|_4^0 = \frac{16}{3}.$$

Example 2. Find the area of a loop of the curve $xy^2 + (x+a)^2(x+2a) = 0$.

Solution The curve is symmetrical about x-axis. Putting $y = 0$, we get $x = -a$ and $x = -2a$



The loop is formed between $x = -2a$ and $x = -a$.

To find the area of the loop, we first shift the origin to the point $(-a, 0)$. The equation of the curve then becomes

$$(x-a)y^2 + \{(x-a)+a\}^2(x-a+2a) = 0$$

or $\quad y^2(x-a) + x^2(x+a) = 0$

or $\quad y^2 = \dfrac{x^2(a+x)}{a-x} \qquad \ldots(1)$

Note: that the shifting of the origin only changes the equation of the curve and has no effect on its shape. now the origin being at the point A, the new limits for the loop are $x = -a$ to $x = 0$.

$\therefore$ The required area of the loop $= 2 \times$ area CPA

$$= 2\int_{-a}^0 y\,dx, \text{ [the value of y to be put from (1)}$$

$$= 2\int_{-a}^0 \left\{-x\sqrt{\left(\frac{a+x}{a-x}\right)}\right\}dx,$$

Note: that in the equation (1), for the portion CPA, $y = -x\sqrt{\left\{\dfrac{(a+x)}{(a-x)}\right\}}$]

$$= 2\int_{-a}^0 \frac{-x(a+x)}{\sqrt{(a^2-x^2)}}\,dx,$$

[multiplying the numerator and the denominator by $\sqrt{(a-x)}$]

$$= 2\int_{\pi/2}^0 \frac{-(-a\sin\theta)(a-a\sin\theta)}{a\cos\theta}\,.\,(-a\cos\theta)\,d\theta,$$

[putting $x = -a\sin\theta$ and $dx = -a\cos\theta\,d\theta$]

$$= -2a^2\int_{\pi/2}^0 (\sin\theta - \sin^2\theta)\,d\theta$$

$$= 2a^2\int_0^{\pi/2} (\sin\theta - \sin^2\theta)\,d\theta$$

$$= 2a^2\left[1 - \frac{1}{2}.\frac{1}{2}\pi\right] = 2a^2\left(1 - \frac{\pi}{4}\right).$$

Example 3. Trace the curve $y^2(2a-x) = x^3$ and find the entire area between the curve and its asymptotes.

Solution Tracing of the curve $y^2(2a-x) = x^3$.

(i) Since in the equation of the curve the powers of y that occur are all even, therefore the curve is symmetrical about the x-axis.

(ii) The curve passes through the origin. Equating to zero the lowest degree terms in the equation of the curve, we get the tangents at the origin as $2ay^2 = 0$ i.e., $y = 0$.

(iii) The curve cuts the coordinate axes only at the origin.

(iv) Solving the equation of the curve for y, we get

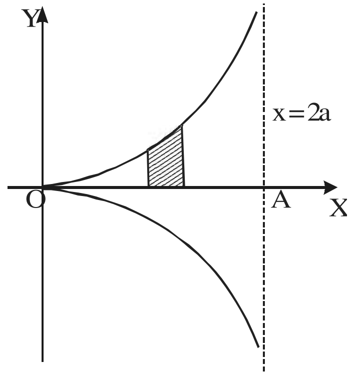
$$y^2 = \frac{x^3}{2a-x}.$$

When $x = 0$, $y = 0$.

When $x \to 2a$, $y^2 \to \infty$. Therefore $x = 2a$ is an asymptote of the curve.

When $0 \le x < 2a$, $y^2 \ge 0$ i.e., y is real. Therefore, the curve exists only in this region.

Combining all these facts, we see that the shape of the curve is as shown in the figure.



Now the required area

$= 2 \times$ area in the first quadrant

$$= 2\int_0^{2a} y\,dx$$

$$= 2\int_0^{2a} \frac{x^{3/2}}{\sqrt{(2a-x)}}\,dx, \left[\because\ y^2 = \frac{x^3}{2x-x}\right]$$

Now put $x = 2a\sin^2\theta$, so that $dx = 4a\sin\theta\cos\theta\,d\theta$.

$\therefore$ The required area

$$= 2\int_0^{\pi/2} \frac{(2a\sin^2\theta)^{3/2}}{\sqrt{(2a-2a\sin^2\theta)}}\,.\,4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2\int_0^{\pi/2} \frac{\sin^3\theta}{\cos\theta}\sin\theta\cos\theta\,d\theta = 16a^2\int_0^{\pi/2}\sin^4\theta\,d\theta$$

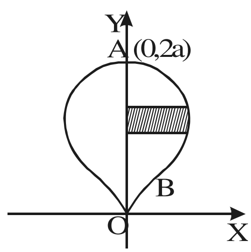
$$= 16a^2.\frac{3.1}{4.2}.\frac{\pi}{2}, \text{ by Wallis formula}$$

$$= 3\pi a^2.$$

Example 4. Find the area bounded by the curve $a^3x^2 = y^3(2a-y)$.

Solution The given curve is $a^2x^2 = y^3(2a-y)$ ...(1)

It is symmetrical about y-axis and it cuts the y-axis at the points (0, 0) and (0, 2a). The curve does not exist for $y > 2a$ and $y < 0$.



$\therefore$ The required area $= 2 \times$ area OBA

$$= 2\int_0^{2a} x\,dy$$

$$= 2\int_0^{2a} \frac{y^{3/2}\sqrt{(2a-y)}}{a}\,dy, \text{ from (1).}$$

Putting $y = 2a\sin^2\theta$, we get,

$$= \frac{2}{a}\int_0^{\pi/2} (2a)^{3/2}\sin^3\theta\,.\sqrt{(2a).\cos\theta.4a\sin\theta\cos\theta}\,d\theta$$

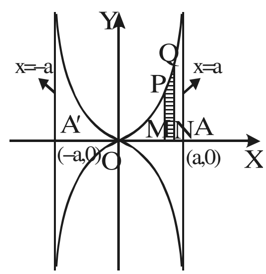
$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$$

$$= 32a^2.\frac{3.1.1}{6.4.2}.\frac{\pi}{2} \text{ by Wallis formula}$$

Thus, the required area $= \pi a^2$.

Example 5. Find the area between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

Solution The given curve is symmetrical about both the axes and passes through the origin. The tangent at (0, 0) are given by $y^2 - x^2 = 0$ i.e., $y = \pm x$ are the tangents at the origin. From the equation of the given curve, $y^2 = a^2x^2/(a^2 - x^2)$. Equating the denominator to zero, the asymptotes parallel to y-axis are given by $x^2 - a^2 = 0$ i.e., $x = \pm a$.



$\therefore$ The required area

$= 4 \times$ area lying in the first quadrant

$$= 4\int_0^a y\,dx = 4\int_0^a \sqrt{\left(\frac{a^2x^2}{a^2-x^2}\right)}\,dx,$$

$$= 4\int_0^a \frac{axdx}{\sqrt{(a^2-x^2)}} = -2a\int_0^a \frac{-2xdx}{\sqrt{(a^2-x^2)}}$$

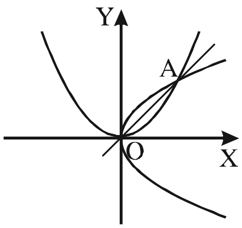
$$= 2a\left[\frac{(a^2-x^2)^{1/2}}{1/2}\right]_0^a$$

$$= -4a\,[0-a] = 4a^3.$$

Example 6. The curves $y = 4x^2$ and $y^2 = 2x$, meet at the origin O and the point A, forming a loop. Show that the straight line OA divides the loop into two parts of equal area.

Solution Solving the equations of the two given curves, we have $16x^2 = 2x$ or $16x^4 - 2x = 0$ i.e., $x = 0$ and $x = \frac{1}{2}$.

Thus, the points of intersection are (0, 0) and $\left(\frac{1}{2}, 1\right)$.



The equation of the line OA is $y - 0 = \dfrac{1-0}{\frac{1}{2}-0}(x - 0)$ i.e.

$y = 2x$. Now the area between the parabola $y^2 = 2x$ and the line $y = 2x$

$$= \int_0^{1/2} (y_1 - y_2)dx = \int_0^{1/2} (\sqrt{(2x)} - 3x)\,dx$$

$$= \left[\frac{2\sqrt{2}}{3} - x^{3/2} - x^2\right]_0^{1/2}$$

$$= \frac{2\sqrt{2}}{3}\cdot\frac{1}{2\sqrt{2}} - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \qquad ...(1)$$

Again the area between the parabola $y = 4x^2$ and the line $y = 2x$

$$= \int_0^{1/2} (2x - 4x^2)\,dx = [x^2]_0^{1/2} - \frac{4}{3}[x^3]_0^{1/2}$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12} \qquad ...(2)$$

From (1) and (2) we observe that the straight line OA divides the loop into two parts of equal area.

Example 7. Determine the area of the figure bounded by two branches of the curve $(y - x)^2 = x^3$ and the straight line $x = 1$.

Solution Two curves are given by $(y - x)^2 = x^3$

$\Rightarrow\ y = x \pm x^{3/2}$ i.e. $y - x = \pm x\sqrt{x}, x \geq 0$.

For the branch $y = x + x^{3/2}$, $\dfrac{dy}{dx} = 1 + \dfrac{3}{2}x^{1/2} > 0$.

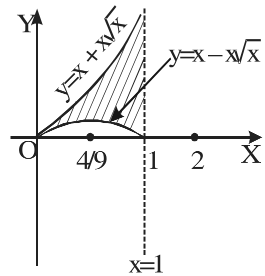
$\Rightarrow$ y is a strictly increasing function.

For the branch $y = x - x^{3/2}$, $\dfrac{dy}{dx} = 1 - \dfrac{3}{2}x^{1/2}$

$$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow x = \frac{4}{9}, \frac{d^2y}{dx^2} = -\frac{3}{4}x^{-\frac{1}{2}} < 0 \text{ at } x = \frac{4}{9}$$

$\therefore$ At $x = \frac{4}{9}$, $y = x - x^{3/2}$ has a maxima.

The graphs of $y = x + x\sqrt{x}$ and $y = x - x\sqrt{x}$ are shown in the figure.



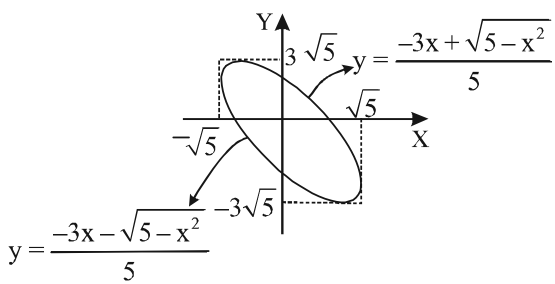
Hence, the required area

$$= \int_0^1 \{(x + x\sqrt{x}) - (x - x\sqrt{x})\}dx$$

$$= \int_0^1 (2x\sqrt{x})dx = 2\int_0^1 x^{3/2}dx = \frac{4}{5}.$$

Example 8. Find area contained by ellipse $2x^2 + 6xy + 5y^2 = 1$.

Solution The ellipse $5y^2 + 6xy + 2x^2 - 1 = 0$ is centred at origin, with slanted principal axes.



On solving the equation for y, we have

$$y = \frac{-6x \pm \sqrt{36x^2 - 20(2x^2 - 1)}}{10} = \frac{-3x \pm \sqrt{5 - x^2}}{5}$$

$\because$ y is real, $5 - x^2 \geq 0 \Rightarrow -\sqrt{5} \leq x \leq \sqrt{5}$

If $x = -\sqrt{5}$, $y = 3\sqrt{5}$

If $x = \sqrt{5}$, $y = -3\sqrt{5}$

The required area

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \left(\frac{-3x + \sqrt{5 - x^2}}{5} - \frac{-3x - \sqrt{5 - x^2}}{5} \right) dx$$

$$= \frac{2}{5} \int_{-\sqrt{5}}^{\sqrt{5}} \sqrt{5 - x^2}\, dx = \frac{4}{5} \int_{0}^{\sqrt{5}} \sqrt{5 - x^2}\, dx$$

Put $x = \sqrt{5} \sin\theta$, $dx = \sqrt{5} \cos\theta\, d\theta$

When $x = 0$, $\theta = 0$

When $x = \sqrt{5}$, $\theta = \frac{\pi}{2}$

Hence, the required area

$$= \frac{4}{5} \int_{\theta=0}^{\frac{\pi}{2}} \sqrt{5 - 5\sin^2\theta}\ \sqrt{5} \cos\theta\, d\theta$$

$$= 4 \int_{0}^{\frac{\pi}{2}} \cos^2\theta\, d\theta = 4\ \frac{1}{2}\ \frac{\pi}{2} = \pi.$$

Example 9. Find the area of the figure enclosed by the curve $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$

Solution The given equation denotes an ellipse, because $\Delta \neq 0$ and $h^2 < ab$. The equation of curve can be re-written as $2y^2 + 6(1 + x)y + 5x^2 + 7x + 6 = 0$

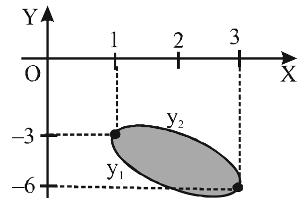
Solving for y,

$$y_1 = \frac{-3(1+x) - \sqrt{(3-x)(x-1)}}{2},$$

$$y_2 = \frac{-3(1+x) + \sqrt{(3-x)(x-1)}}{2}$$

Also, the curves y_1 and y_2 are defined for values of x for which $(3 - x)(x - 1) \geq 0$

i.e., $1 \leq x \leq 3$.



The required area is given by

$$A = \int_1^3 (y_1 - y_2)\,dx \Rightarrow A = \int_1^3 \sqrt{(3-x)(x-1)}\,dx$$

Put $x = 3\cos^2\theta + \sin^2\theta$ i.e., $dx = -2\sin 2\theta\, d\theta$

$$A = 2\int_0^{\pi/2} \sin^2 2\theta\, d\theta = \frac{\pi}{2}.$$

Practice Problems H

1. Find the area bounded by the curve g (x), the x-axis and the ordinate at x = – 1 and x = 4 where g (x) is the inverse of the function $f(x) = \frac{x^3}{24} + \frac{x^2}{8} + \frac{13x}{12} + 1$.
2. Find the smaller of the two areas enclosed between the ellipse $9x^2 + 4y^2 - 36x + 8y + 4 = 0$ and the line $3x + 2y - 10 = 0$.
3. Prove that the area included between the curve $x = y^2(1 - x)$ and the line $x = 1$ is π.
4. Find the area of loop $y^2 = x(x - 1)^2$.
5. Find the ratio in which the curve $x^{2/3} + y^{2/3} = a^{2/3}$ divides the area of the circle $x^2 + y^2 = a^2$.
6. Find the area bounded by the curve $x^4 + y^4 = x^2 + y^2$.
7. Compute the area of the figure bounded by two branches of the curve $(y - x)^2 = x^5$ and and the straight line $x = 4$.
8. Compute the area of the figure bounded by the curve $(y - x - 2)^2 = 9x$ and the coordinate axes.
9. Find the area of the figure enclosed by the curve $y^2 = (1 - x^2)^3$.
10. Find the area of the figure enclosed by the curve $x^4 - ax^3 + a^2y^2 = 0$.
11. Find the area of the finite portion of the figure bounded by the curve $x^2y^2 = 4(x - 1)$ and the straight line passing through its points of inflection.
12. Find the area of the figure enclosed by the curve $(y - \sin^{-1}x)^2 = x - x^2$.

13. Find the area of the figure contained between the curve $xy^2 = 8 - 4x$ and its asymptote.

14. Sketch the curve $|y| + (|x| - 1)^2 = 4$, and also find the area enclosed by this curve.

15. Find the area enclosed by curve $y^2 = x^2 - x^4$.

16. Show that the area of a loop of the curve $y^2 = x^2(4 - x^2)$ is 16/3.

17. Find the area enclosed by the curve $xy^2 = a^2(a - x)$ and y-axis.

18. Find the area between the curve $y^2(a + x) = (a - x)^3$ and its asymptote.

19. Show that the total area included between the two branches of the curve $y^2 = x^2/[(4 - x)(x - 2)]$ and the two asymptotes is 6π.

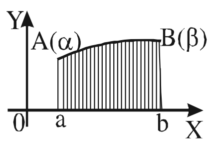
3.11 AREAS OF CURVES GIVEN BY PARAMETRIC EQUATIONS

Now let us compute the area of the curvilinear trapezoid bounded by a curve represented by parametric equations :

$x = \phi(t), y = \psi(t)$, t being parameter ...(1)

where $\alpha \le t \le \beta$ and $\phi(\alpha) = a$, $\psi(\beta) = b$. Let equation (1) define some non-negative function $y = f(x)$ on the interval [a, b] and, consequently, the area of the curvilinear trapezoid may be computed from the formula

$$A = \int_a^b f(x)\,dx$$



We change the variable in this integral :

$x = \phi(t)$, $dx = \phi'(t)\,dt$

From (1) we have $y = \psi(t)$

Consequently, $$A = \int_\alpha^\beta \psi(t)\phi'(t)\,dt$$

This is the formula for computing the area of a curvilinear trapezoid bounded by a curve represented parametrically.

Example 1. Compute the area of a region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose parametric equations are $x = a\cos t$, $y = b\sin t$.

Solution We compute the area of the upper half of the ellipse and double it. Here, x varies from $-a$ to a, and so t varies between π and 0.

$$A = 2\int_\alpha^\beta \psi(t)\phi'(t)\,dt = 2\int_\pi^0 (b\sin t)(-a\sin t\,dt)$$

$$= -2ab\int_\pi^0 \sin^2 t\,dt = 2ab\int_0^\pi \sin^2 t\,dt$$

$$= 2ab\int_0^\pi \frac{1-\cos 2t}{2}\,dt = 2ab\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^\pi = \pi ab.$$

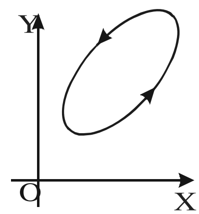
Now consider a closed curve represented by the parametric equations

$x = \phi(t)$, $y = \psi(t)$

We suppose that the curve does not intersect itself. Suppose that as the parameter 't' increases from a value t_1 to the vaue t_2, the point P(x, y) describes the curve completely in the anti-clockwise sense. The curve being closed, the point on it corresponding to the value t_2 of the parameter is the same as the point corresponding to the value t_1 of the parameter. The area of the region bounded by such a curve is given by the formula

$$A = \frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \frac{1}{2}\int_{t_1}^{t_2}[\phi(t)\psi'(t) - \psi(t)\phi'(t)]\,dt$$



The above formula gives the area enclosed by any closed curve whatsoever, provided only, that it does not intersect itself; there being no restriction as to the manner in which the curve is situated relative to the coordinate axes. For example, we again find the area enclosed be ellipse $x = a\cos t$, $y = b\sin t$. The ellipse is a closed curve and is completely described while t varies from 0 to 2π.

We have $x\dfrac{dy}{dt} - y\dfrac{dx}{dt} = ab(\cos^2 t + \sin^2 t) = ab$

Therefore, the required area

$$= \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt = \frac{1}{2}\int_0^{2\pi} ab\ dt = \pi ab.$$

Example 2. Compute the area bounded by the x-axis and an arch of the cycloid $x = a(t - \sin t)$ $y = a(1 - \cos t)$

Solution The variation of x from 0 to $2\pi a$ corresponds to the variation of t from 0 to 2π. We have

$$A = \int_\alpha^\beta \psi(t)\phi'(t)dt$$

$$= \int_0^{2\pi} a(1-\cos t)a(1-\cos t)dt = a^2\int_0^{2\pi}(1-\cos t)^2 dt$$

$$= a^2\left[\int_0^{2\pi} dt - 2\int_0^{2\pi}\cos t\,dt + \int_0^{2\pi}\cos^2 t\,dt\right]$$

Since, $\displaystyle\int_0^{2\pi} dt = 2\pi$, $\displaystyle\int_0^{2\pi}\cos t\,dt = 0$,

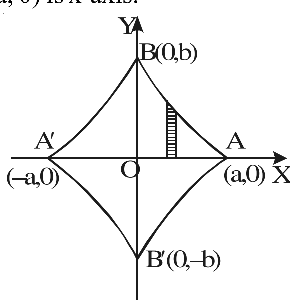
$\displaystyle\int_0^{2\pi}\cos^2 t\,dt = \int_0^{2\pi}\frac{1+\cos 2t}{2}dt = \pi$, we get

$A = a^2(2\pi + \pi) = 3\pi a^2.$

Example 3. Find the area bounded by the curve given by the equations $x = a\cos^3 t$, $y = b\sin^3 t$.

Solution Eliminating t from the equations the cartesian equation of the curve is obtained as $(x/a)^{2/3} + (y/b)^{2/3} = 1$

Since the powers of x and y are all even, the curve is symmetrical about both the axes. It does not pass through the origin. It cuts the axis of x at the points $(\pm a, 0)$ and the axis of y at the points $(0, \pm b)$. The tangent at the point $(a, 0)$ is x-axis.



At the point B, $x = 0$ and $t = \frac{1}{2}\pi$.

At the point A, $x = a$ and $t = 0$.

$\therefore$ The required area = 4 × area OAB

$$= 4\int_{x=0}^{a} y\,dx = 4\int_{t=\pi/2}^{0} y.\frac{dx}{dt}\,.\,dt$$

$$= 4\int_{\pi/2}^{0} b\sin^3 t\,.\,(-3a\cos^2 t\sin t)\,dt,$$

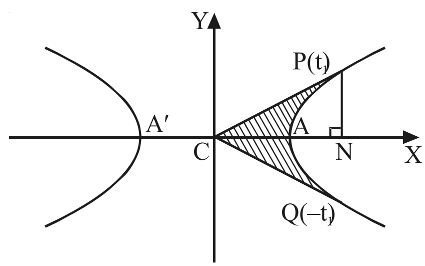
(putting for y and dx/dt)

$$= 12ab\int_0^{\pi/2}\sin^4 t\cos^2 t\,dt$$

$$= 12ab.\frac{3.1.1}{6.4.2}.\frac{\pi}{2} = \frac{3}{8}\pi ab.$$

Example 4. For any real t, $x = \dfrac{e^t + e^{-t}}{2}$, $y = \dfrac{e^t - e^{-t}}{2}$ is a point on the hyperbola $x^2 - y^2 = 1$. Find the area bounded by the hyperbola and the lines joining the centre to the points corresponding to t_1 and $-t_1$.

Solution We sketch the hyperbola $x^2 - y^2 = 1$, where $x = \dfrac{e^t + e^{-t}}{2}$, $y = \dfrac{e^t - e^{-t}}{2}$.



We have to find the area of the region bounded by the hyperbola and the lines joining the centre $x = 0$, $y = 0$ to the point (t_1) and $(-t_1)$.

The required area

= 2 [area of ΔPCN – area of PANP]

$$= 2\left[\text{area of } \Delta PCN - \int_1^{x_N} y\,dx\right]$$

$$= 2\left[\frac{1}{2}\left(\frac{e^{t_1}+e^{-t_1}}{2}\right)\left(\frac{e^{t_1}-e^{-t_1}}{2}\right) - \int_1^{t_1} y\frac{dx}{dt}dt\right]$$

$$= 2\left[\left(\frac{e^{2t_1}-e^{-2t_1}}{8}\right) - \int_0^{t_1}\left(\frac{e^t - e^{-t}}{2}\right)^2 dt\right]$$

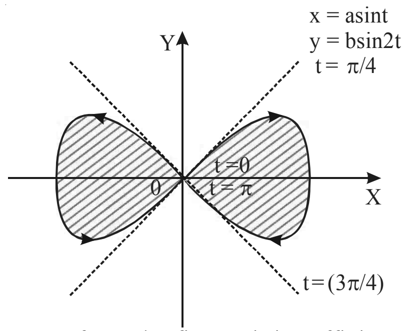
$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{1}{2}\int_0^{t_1}(e^{2t} + e^{-2t} - 2)dt$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{1}{2}\left[\frac{e^{2t}}{2} - \frac{e^{-2t}}{2} - 2t_1\right]_0^{t_1}$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{1}{2}\left[\frac{e^{2t_1}}{2} - \frac{e^{-2t_1}}{2} - 2t_1\right] = t_1.$$

Example 5. Compute the area of the region enclosed by the curve $x = a \sin t$, $y = b \sin 2t$.

Solution When constructing the curve one should bear in mind that it is symmetrical about the axes of coordinates. Indeed, if we substitute $\pi - t$ for t, the variable x remains unchanged, while y only changes its sign; consequently, the curve is symmetrical about the x-axis. When substituting $\pi + t$ for t the variable y remains unchanged, and x only changes its sign which means that the curve is symmetrical about the y-axis. Furthermore, since the functions $x = a \sin t$; $y = b \sin 2t$ have a common period 2π, it is sufficient to confine ourselves to the following interval of variation of the parameter : $0 \le t \le 2\pi$. From the equations of the curve it readily follows that the variables of the parameter : $0 \le t \le 2\pi$. From the equations of the curve it readily follows that the variables x and y simultaneously retain non-negative values only when the parameter t varies on the interval $\left[0, \frac{\pi}{2}\right]$, therefore at $0 \le t \le \frac{\pi}{2}$ we obtain the portion of the curve situated in the first quadrant. The curve is shown below.



As in seen from the figure, it is sufficient to evaluate the area enclosed by one loop of the curve corresponding to the variation of the parameter t from 0 to π and then to double the result

$$S = 2\int_0^{\pi} yx'dt = 2\int_0^{\pi} b\sin 2t \times a\cos t\, dt$$

$$= 4ab\int_0^{\pi} \cos^2 t \sin t dt$$

$$= -4ab\left(\frac{\cos^3 t}{3}\right)\Bigg|_0^{\pi} = \frac{8}{3}ab.$$

Example 6. Find the area enclosed by the curve $y = f(x)$ defined parametrically as $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$.

Solution Clearly t can be any real number.

Let $t = \tan\theta$

$\Rightarrow |x| = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$ and $y = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$

$\Rightarrow x^2 + y^2 = 1.$

Note that the variables x and y take both positive and negative values, so that the given equation represents the complete circle. Thus, the required area $= \pi.1^2 = \pi$.

3.12 AREAS OF CURVES GIVEN BY POLAR EQUATIONS

If $r = f(\theta)$ be the equation of a curve in polar coordinates where $f(\theta)$ is a single valued continuous function of θ, then the area of the sector enclosed by the curve and the two radii vectors $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2)$, is equal to $\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$.

Example 1. The equation of an ellipse in polar coordinates, the centre being pole, is $\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$. Find its area.

Solution The area is equal to $\frac{1}{2}\int_0^{2\pi} r^2 d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}d\theta$$

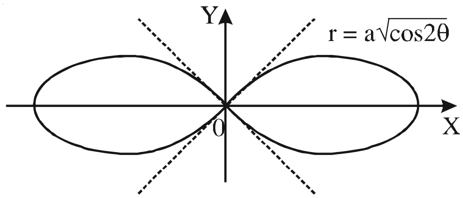
$$= 2a^2b^2\int_0^{\frac{1}{2}\pi} \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta}d\theta$$

$$= \pi ab$$

Hence the required area is πab.

Example 2. Compute the area bounded by the lemniscate $r = a\sqrt{\cos 2\theta}$.

Solution



The radius vector will describe one fourth of the required area if θ varies between 0 and $\pi/4$:

$$\frac{1}{4}A = \frac{1}{2}\int_0^{\pi/4} r^2 d\theta$$

$$= \frac{1}{2}a^2 \int_0^{\pi/4} \cos 2\theta d\theta = \frac{a^2}{2}\frac{\sin 2\theta}{2}\Bigg|_0^{\pi/4} = \frac{a^2}{4}$$

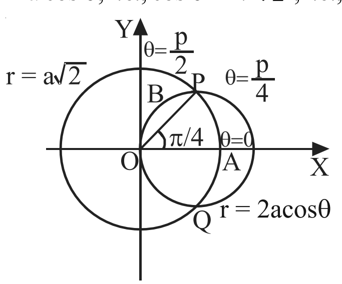
Hence, $A = a^2$.

Example 3. Find the area common to the circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$.

Solution The given equations of circles are $r = a\sqrt{2}$ and $r = 2a\cos\theta$. The first equation represents a circle with centre at pole and radius $a\sqrt{2}$.

The second equation represents a circle passing through the pole and the diameter through the pole as the initial line. Both these circles are symmetrical about the initial line. Eliminating r between the two equations, we have at the points of intersection

$$a\sqrt{2} = 2a\cos\theta, \text{ i.e., } \cos\theta = 1/\sqrt{2}, \text{ i.e., } \theta = \pm\pi/4.$$



At P, $\theta = \pi/4$. For the circle $r = 2a\cos\theta$, at O, $r = 0$ and so $\cos\theta = 0$ i.e., $\theta = \frac{1}{2}\pi$.

Now, the required area = Area OQAPBO

= 2(area OAPBO), (by symmetry)

= 2[Area OAP + Area OPBO]

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} r^2\, d\theta, \text{ for the circle } r = a\sqrt{2}\right.$$

$$\left. + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2 d\theta, \text{ for the circle } r = 2a\cos\theta\right]$$

$$= \int_0^{\pi/4} (a\sqrt{2})^2 d\theta + \int_{\pi/4}^{\pi/2} (2a\cos\theta)^2 d\theta$$

$$= 2a^2\,[\theta]_0^{\pi/4} + 2a^2\int_{\pi/4}^{\pi/2} (1+\cos 2\theta)\, d\theta$$

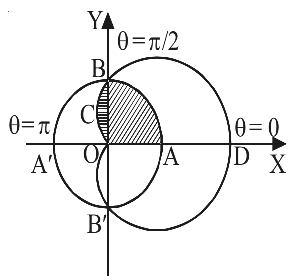
$$= 2a^2\left(\frac{\pi}{4}\right) + 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{\pi a^2}{2} + 2a^2\left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{1}{2}\pi a^2 + \frac{1}{2}\pi a^2 - a^2 = \pi a^2 - a^2 = a^2(\pi - 1).$$

Example 4. Find the area common to the circle $r = a$ and the cardioid $r = a(1 + \cos\theta)$.

Solution Eliminating r between the given equations, we get $a(1 + \cos\theta) = a$ or $\cos\theta = 0$ or $\theta = \pm\pi/2$. Thus the two curves cut each other at the points where $\theta = \pm\pi/2$.



Both the curves are symmetrical about the initial line.

Hence, the required area = 2 × area ABCOA

= 2 × (Area OABO + Area OBCO) ...(1)

Now area OABO

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} r^2\, d\theta, \text{ for } r = a$$

$$= \frac{1}{2}a^2\int_{\theta=0}^{\pi/2} d\theta = \frac{1}{2}a^2[\theta]_0^{\pi/2}$$

$$= \frac{1}{2}a^2 \frac{\pi}{2} = \frac{\pi a^2}{4}.$$

And area OBCO

$$= \frac{1}{2}\int_{\pi/2}^{\pi} r^2\, d\theta, \text{ for } r = a(1+\cos\theta)$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi} a^2 (1+\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}a^2\int_{\pi/2}^{\pi}\left\{1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right\} d\theta$$

$$= \frac{1}{2}a^2\int_{\pi/2}^{\pi}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{a^2}{2}\left[\frac{3\theta}{2}+\sin\theta+\frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$$

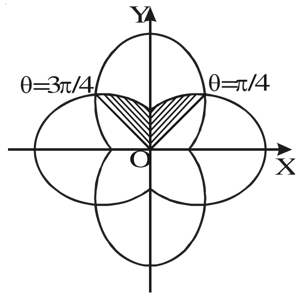
$$= \frac{a^2}{2}\left(\frac{3\pi}{2}-\frac{3\pi}{4}-2\right) = \frac{a^2}{8}(3\pi-8)$$

Hence, from (1), the required area

$$= 2\left[\frac{\pi a^2}{4}+\frac{a^2}{8}(3\pi-8)\right] = a^2\left(\frac{5\pi}{4}-2\right).$$

Example 5. The graph of $r = 2 + \cos 2\theta$ and its reflection over the line $y = x$ bound five regions in the plane. Find the area of the region containing the origin.

Solution



The original graph is closer to the origin than its reflection for $\theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$, and the region is symmetric about the origin. Therefore, the area we wish to find is the polar integral

$$4\int_{\pi/4}^{3\pi/4} \frac{1}{2}(2+\cos 2\theta)^2 d\theta$$

$$= 2\int_{\pi/4}^{3\pi/4} (4+4\cos 2\theta + \cos^2 2\theta) d\theta$$

$$= 2\int_{\pi/4}^{3\pi/4}\left(4+4\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right)d\theta$$

$$= \left[9\theta + 4\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_{\pi/4}^{3\pi/4}$$

$$= \left(\frac{27}{4}-4\right)-\left(\frac{9\pi}{4}+4\right) = \frac{9\pi}{2}-8.$$

Practice Problems I

1. Find the area of the loop formed by the curve given by $x = a(1-t^2)$, $y = at(1-t^2)$, $-1 \le t \le 1$.
2. If the curve given by parametric equation $x = t - t^3$, $y = 1 - t^4$ forms a loop for all values of $t \in [-1, 1]$, then find the area of the loop.
3. Find the area of the region bounded by the curve $r = a\cos 4\varphi$.
4. Find the area common to the cardiod $r = a(1+\cos\theta)$ and the circle $r = \frac{3}{2}a$, and also the area of the remainder of the cardiod.

3.13 AREAS OF REGIONS GIVEN BY INEQUALITIES

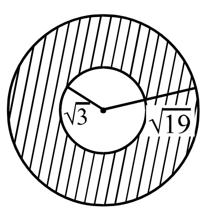
Example 1. Find the area of the region bounded by $|x^2 + y^2 - 2x + 4y - 6| < 8$.

Solution $|(x-1)^2 + (y+2)^2 - 11| < 8$

$$-8 < (x-1)^2 + (y+2)^2 - 11 < 8$$

$$3 < (x-1)^2 + (y+2)^2 < 19$$

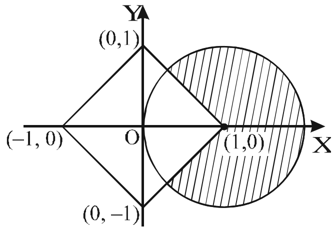
The bounded region is shown below :



It is an annular region having area $19\pi - 3\pi = 16\pi$.

Example 2. Find the area of the region defined by $1 \le |x| + |y|$ and $x^2 - 2x + 1 \le 1 - y^2$.

Solution $1 \le |x| + |y|$ represents the region outside the square and $x^2 - 2x + 1 \le 1 - y^2$ represents the region inside the circle, as shown in the figure.



The required area $= \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$.

Example 3. Find the area given by

$x + y \le 6$, $x^2 + y^2 \le 6y$ and $y^2 \le 8x$.

Solution Let us consider the curves :

$P \equiv y^2 - 8x = 0$ ...(1)

$C \equiv x^2 + y^2 = 6y$ i.e., $x^2 + (y - 3)^2 - 9 = 0$ ...(2)

and $L \equiv x + y - 6 = 0$ ...(3)

The intersection points of the curves (2) and (3) are given by $(6 - y)^2 + y^2 - 6y = 0$

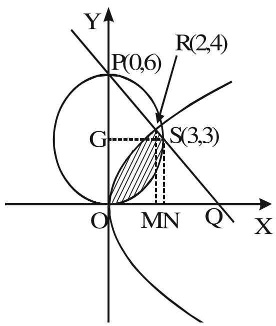
$\Rightarrow y = 3, 6$

Hence, the points are (0, 6) and (3, 3). The intersection points of the curve (1) and (3) are given by

$y^2 = 8(6 - y)$,

i.e. $y = 4, -12$

Hence, the point of intersection in 1st quadrant is (2, 4)



Now,

$C \le 0$ denotes the region, inside the circle $C = 0$.

$P \le 0$ denotes the region, inside the parabola $P = 0$.

$L \le 0$ denotes the half-plane containing origin.

$\therefore$ The required area = Area of OMRO + Area of MNSR – Area of ONSO.

$$= \int_0^2 \sqrt{8x}\,dx + \frac{1}{2}(MR + NS).MN$$

– [Area of square ONSGB – Area of quarter OSGO]

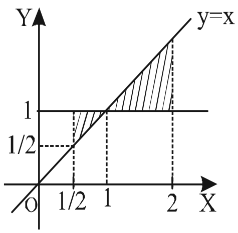
$$= \int_0^2 \sqrt{8x}\,dx + \frac{1}{2}(4+3).1 - \left(3^2 - \frac{\pi.3^2}{4}\right) = \frac{9\pi}{4} - \frac{1}{6}.$$

Example 4. Find the area of the region represented by the inequality $\log_2(\log_y x) > 0$ where $x \in \left(\frac{1}{2}, 2\right)$.

Solution $\log_2(\log_y x) > 0$

Case-I : $y \in (0, 1)$, $x < y$

Case-II : $y \in (1, \infty)$, $x > y$



The required area

$$= \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{2}(1)(1) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}.$$

Example 5. A point P moves in the xy plane in such a way that $[|x|] + [|y|] = 1$, where [.] denotes the greatest integer function. Find the area of the region representing all possible positions of the point P.

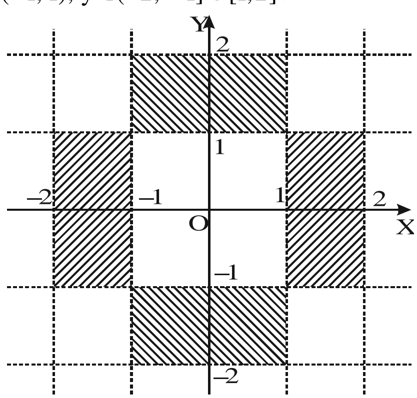
Solution If $[|x|] = 1$ and $[|y|] = 0$

then $1 \le |x| < 2$, $0 \le |y| < 1$

$\Rightarrow x \in (-2, -1] \cup [1, 2)$, $y \in (-1, 1)$

If $[|x|] = 0$, $[|y|] = 1$, then

$x \in (-1, 1)$, $y \in (-2, -1] \cup [1, 2]$.



Area of the required region $= 4(2 - 1)(1 - (-1)) = 8$.

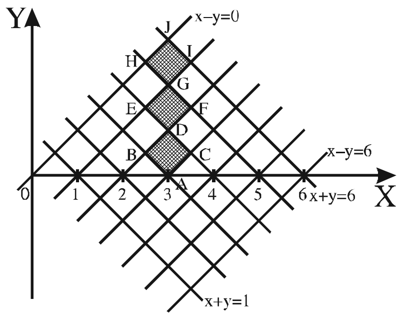
Example 6. Find the area of the region represented by $[x+y]+[x-y]=5$, for $x \geq y, x \geq 0, y \geq 0$.

Solution $[x+y]+[x-y]=5$

For $[x-y]=0, [x+y]=5$

$\Rightarrow 0 \leq x-y < 1, 5 \leq x+y < 6$

Similarly for $1 \leq x-y < 2, 4 \leq x+y < 5$ and so on.



The required area

= area of rectangle (ABCD + DEGF + GHJI)

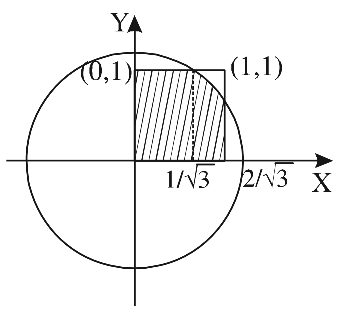
$= 3\left(\frac{1}{2}.1.1\right) = \frac{3}{2}$ square units.

Example 7. Consider the following regions in the plane: $R_1 = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$

$R_2 = \{(x, y) : x^2 + y^2 \leq 4/3\}$

The area of the region $R_1 \cap R_2$ can be expressed as $\frac{a\sqrt{3}+b\pi}{9}$, where a and b are integers. Then find the value of $(a+b)$.

Solution



The required area $A = \frac{1}{\sqrt{3}} \times 1 + A_1$ where

$$A_1 = \int_{1/\sqrt{3}}^{1} \left(\sqrt{(4/3)-x^2}\right)dx, \quad \text{put } x = \frac{2}{\sqrt{3}}\sin\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{2}{\sqrt{3}}\cos\theta \cdot \frac{2}{\sqrt{3}}\cos\theta\, d\theta$$

$$= \frac{2}{3}\int_{\pi/6}^{\pi/3} (\cos 2\theta + 1)\, d\theta = \frac{2}{3}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{2}{3}\left[\frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right] = \frac{2}{3}\cdot\frac{\pi}{6} = \frac{\pi}{9}$$

$$\therefore \quad A = \frac{1}{\sqrt{3}} + \frac{\pi}{9} = \frac{3\sqrt{3}+\pi}{9} = \frac{a\sqrt{3}+b\pi}{9}$$

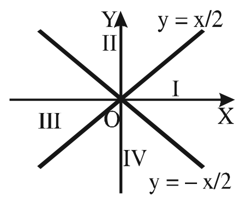
Hence, $a = 3, b = 1$

and $a + b = 4$.

Example 8. Find the area of the region given by $|x-2y|+|x+2y| \leq 8$ and $xy \geq 2$.

Solution We have $|x-2y|+|x+2y| \leq 8$

i.e. $|y-x/2|+|y+x/2| \leq 4$ ...(1)



Let us divide the reference frame into four regions (see figure) using the lines $y = x/2$ and $y = -x/2$.

Region I : $\left(y - \frac{x}{2} < 0 \text{ and } y + \frac{x}{2} > 0\right)$

The inequality (1) reduces to

$$-\left(y-\frac{x}{2}\right)+\left(y+\frac{x}{2}\right) \leq 4 \Rightarrow x \leq 4.$$

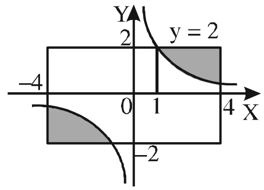
Region II : $\left(y - \frac{x}{2} > 0 \text{ and } y + \frac{x}{2} > 0\right)$

The inequality (1) reduces to

$$-\left(y-\frac{x}{2}\right)+\left(y+\frac{x}{2}\right) \leq 4 \Rightarrow y \leq 2.$$

Similarly, for regions III and IV, the inequality (1) respectively reduces to $x \geq -4$ and $y \geq -2$.

Thus, the region covered by the inequality (1) is a rectangle. The following figure shows the region satisfied by the inequality (1) and $xy \geq 2$.



The required area, is

$$A = 2\int_1^4 \left(2 - \frac{2}{x}\right)dx = 4[x - \ln x]_1^4 = 4(3 - \ln 4).$$

Example 9. Let O (0, 0), A(2, 0) and B$\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle. Let R be the region consisting of all those points P inside ΔOAB which satisfy d(P, OA) ≤ min {d(P, QB), d(P, AB)}, where 'd' denotes the distance from the point to the corresponding line. Sketch the region R and find its area.

Solution Let the coordinates of P be (x, y).

Equation of line OA : y = 0

Equation of line OB : $\sqrt{3}\,y = x$

Equation of line AB : $\sqrt{3}\,y = 2 - x$

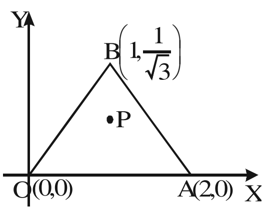
d(P, OA) = distance of P from line OA = y

d(P, OB) = distance of P from line OB

$$= \frac{|\sqrt{3}y - x|}{2}$$

d(P, AB) = distance of P from line AB

$$= \frac{|\sqrt{3}y + x - 2|}{2}$$



Given : d (P, OA) ≤ min. {d(P, OB), d(P, AB)}

$$y \le \min.\left\{\frac{|\sqrt{3}y - x|}{2}, \frac{|\sqrt{3}y + x - 2|}{2}\right\}$$

$$\Rightarrow \quad y \le \frac{|\sqrt{3}y - x|}{2} \qquad ...(1)$$

$$\text{and } y \le \frac{|\sqrt{3}y + x - 2|}{2} \qquad ...(2)$$

Case I : If $y \le \dfrac{|\sqrt{3}y - x|}{2}$

$\therefore \quad y \le \dfrac{x - \sqrt{3}y}{2}$,

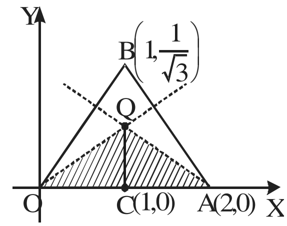
i.e., $x > \sqrt{3}\,y \qquad (\because \sqrt{3}\,y - x < 0)$

$\Rightarrow \quad (2 + \sqrt{3})\,y \le x$

$\Rightarrow \quad y \le (2 - \sqrt{3})x$

$\Rightarrow \quad y \le x.\tan 15^\circ \qquad ...(3)$

(y = x. tan 15° is an acute angle bisector of ∠AOB)



Case II : If $y \le \dfrac{|\sqrt{3}y + x - 2|}{2}$

$\Rightarrow \quad 2y \le 2 - x - \sqrt{3}\,y \qquad (\because \sqrt{3}\,y + x - 2 < 0)$

$\Rightarrow \quad (2 + \sqrt{3})y \le 2 - x$

$\Rightarrow \quad y \le -(2 - \sqrt{3})(x - 2)$

$\Rightarrow \quad y \le -(\tan 15^\circ)(x - 2) \qquad ...(4)$

(y = (x – 2) tan 15° is an acute angle bisector of CA)

From (3) and (4) we conclude that P moves inside the Δ QOA, (Q is the incentre of ΔABC), as shown in the figure.

As ∠QOA = ∠OAQ = 15°, ΔQOA is isoceles.

⇒ OC = AC = 1 unit

Area of shaded region

= area of ΔQOA

$= \frac{1}{2}(\text{base}) \times (\text{height})$

$= \frac{1}{2}(2)(1 \tan 15^\circ) = \tan 15^\circ$

$= (2 - \sqrt{3})$ sq. units.

Example 10. Consider a square with vertices at (1, 1), (–1, 1), (–1, –1) and (1, –1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area.

Solution Let the square be ABCD where the equations of the sides of the square are as follows :

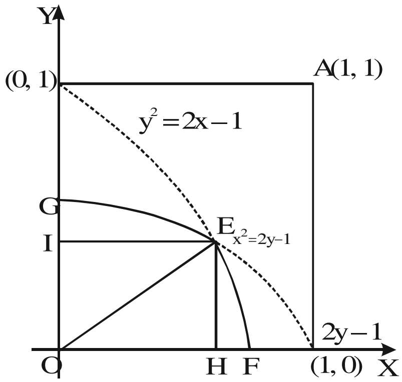
AB : $y = 1$, BC : $x = -1$,
CD : $y = -1$, DA : $x = 1$

Let the region be S and (x, y) is any point inside it. Then according to given conditions,

$$\sqrt{x^2 + y^2} < |1 - x|, |1 + x|, |1 - y|, |1 + y|$$

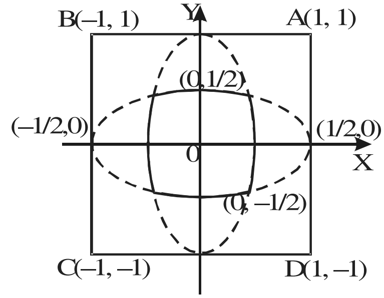
$\Rightarrow$ $x^2 + y^2 < x^2 - 2x + 1,\ x^2 + 2x + 1,\ y^2 - 2y + 1,\ y^2 + 2y + 1$

$\Rightarrow$ $y^2 < 1 - 2x,\ y^2 < 1 + 2x,\ x^2 < 1 - 2y$ and $x^2 < 2y + 1$



Now in $y^2 = 1 - 2x$ and $y^2 = 2x + 1$, the first equation represents a parabola with vertex (1/2, 0) and second equation represents a parabola with vertex (–1/2, 0) and in $x^2 = 1 - 2y$ and $x^2 = 1 + 2y$, the first equation represents parabola with vertex at (0, 1/2) and second equation represents a parabola with vertex at (0, –1/2). So, the region S is the region lying inside the four parabolas

$y^2 = 1 - 2x,\ y^2 = 1 + 2x,\ x^2 = 1 - 2y,\ x^2 = 1 + 2y$



Now, S is symmetrical in all four quadrants

$\therefore$ S = 4 × area lying in first quadrant

Now $y^2 = 1 - 2x$ and $x^2 = 1 - 2y$ intersect on $y = x$

The point of intersection is E $(\sqrt{2} - 1, \sqrt{2} - 1)$

$\therefore$ Area of region OEFO

= area of ΔOEH + area HEFH.

$$= \frac{1}{2}(\sqrt{2} - 1)^2 + \int_{\sqrt{2}}^{1/2} \sqrt{1 - 2x}\, dx$$

$$= \frac{1}{2}(2 + 1 - 2\sqrt{2}) + \frac{2}{3}\left[(1 - 2x)^{3/2}\right]_{\sqrt{2}-1}^{1/2}$$

$$= \frac{1}{2}(3 - 2\sqrt{2}) + \frac{1}{3}(3 - 2\sqrt{2})^{3/2}$$

$$= \frac{1}{2}(3 - 2\sqrt{2}) + \frac{1}{3}(\sqrt{2} - 1)^3$$

$$= \frac{1}{2}(3 - 2\sqrt{2}) + \frac{1}{3}(5\sqrt{2} - 7)$$

$$= \frac{1}{6}[4\sqrt{2} - 5]$$

Similarly, area OEGO = $\frac{1}{6}(4\sqrt{2} - 5)$

so, area of S lying in first quadrant = $\frac{2}{6}(4\sqrt{2} - 5)$

Hence, S = $\frac{4}{3}(4\sqrt{2} - 5)$.

Practice Problems J

1. Find the area of the region represented by

$$\begin{cases} x + y \le 2, \\ x + y \ge 1, \\ x \ge 0, \\ y \ge 0 \end{cases}$$

2. Find the area enclosed by $|x| + |y| \le 3$ and $xy \ge 2$.
3. Find are bounded by $x^2 + y^2 \le 2ax$ and $y^2 \ge ax$, $x \ge 0$.
4. Consider the closed figure C made by the line $|x| + |y| = \sqrt{2}$. Let S be the region inside the figure such that any point in it is nearer to the side $x + y = \sqrt{2}$ than the origin. Find the area of S.
5. Give a rough sketch of the region R consisting of points (x, y) satisfying $|x \pm y| \le 2$ and $x^2 + y^2 \ge 2$ and find the area of the region.

6. Calculate the area of a plane figure bounded by parts of the lines max (x, y) = 1 and $x^2 + y^2 = 1$ lying in the first quadrant :

$$\max(x,y) = \begin{cases} x, & \text{if } x \geq y, \\ y, & \text{if } x < y. \end{cases}$$

7. Find the area of the bounded region represented by $|x + y| = |y| - x$ and $y \geq x^2 - 1$.

8. Find the area between the curve $y = x^2 + x - 2$ and $y = 2x$ for which $|x^2 + x - 2| + |2x| = |x^2 + 3x - 2|$ is satisfied.

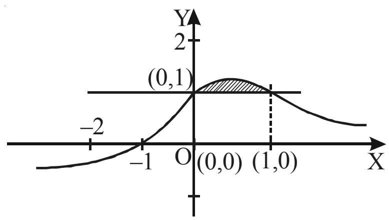
9. Let P (x, y) be a point satisfying these three inequalities, $|x + y| \leq 1$, $|y - x| \leq 1$ and $3x^2 + 3y^2 \geq 1$, sketch the region in which 'P' can move, also find the area of the region.

10. Find the area of the region which consists of all the points satisfying the conditions $|x - y| + |x + y| \leq 8$ and $xy \geq 2$.

Target Problems for JEE Advanced

Problem 1. Find the area of the region bounded by the curve $C: y = \dfrac{x+1}{x^2+1}$ and the line $L: y = 1$.

Solution



On solving $C: y = \dfrac{x+1}{x^2+1}$ and the line $L: y = 1$, we have

$$\frac{x+1}{x^2+1} = 1 \Rightarrow x + 1 = x^2 + 1 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$$

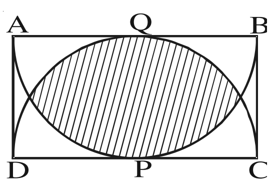
Thus, the points of intersection are at x = 0 and x = 1.

$$\text{The required area} = \int_0^1 \left(\frac{x+1}{x^2+1} - 1\right) dx$$

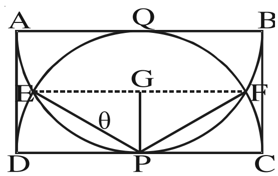
$$= \frac{1}{2}\int_0^1 \frac{2x}{x^2+1} dx + \int_0^1 \frac{1}{x^2+1} dx - \int_0^1 dx$$

$$= \left[\frac{1}{2}\ln(x^2+1) + (\tan^{-1} x) - x\right]_0^1 = \frac{1}{2}\ln 2 + \frac{\pi}{4} - 1.$$

Problem 2. ABCD is a rectangle with 2AD = AB = 2r. Find the area of the shaded region if $\overline{APB}$ and $\overline{DQC}$ are semicircles.



Solution



The required area is double of the area QEFQ.

In ΔEPG, $\cos\theta = \dfrac{r}{2r} = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$.

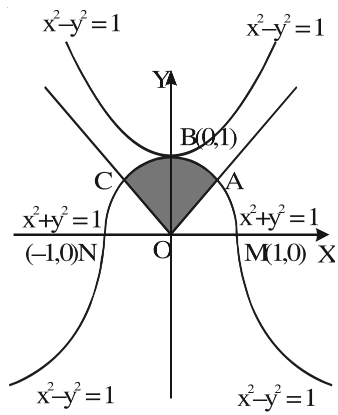
Since area QEFQ = area of segment PEQFP area of triangle PEF $= \dfrac{1}{2}r^2 \times \dfrac{2\pi}{3} - \dfrac{r}{2} \times \dfrac{r}{2}\tan 30°$

$\Rightarrow$ Area QEFQ $= \dfrac{1}{3}\pi r^2 - \dfrac{\sqrt{3}}{4}r^2$

The shaded area $= 2r^2\left[\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}\right] = \dfrac{r^2}{6}[4\pi - 3\sqrt{3}]$.

Problem 3. Find the area of the curve enclosed by the curves $y|y| + x|x| = 1$, $y|y| - x|x| = 1$ and $y = |x|$.

Solution In the first and second quadrants, we have $y^2 + x^2 = 1$ and $y^2 - x^2 = 1$. In the third and fourth quadrants, we have $x^2 - y^2 = 1$.



Area OABO $= \frac{1}{2}$ (area of OMABO) $= \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$

Area OBCO $= \frac{1}{2}$ (area of OBCNO) $= \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$

The required area OABCO = area OABO + area OBCO

$$= \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4}.$$

Problem 4. Sketch the graph of the cubic $y = x^3 - x^2 - x + 1$ and find the area of the region bounded by the curve and the line $y = x + 1$.

Solution $y = (x-1)^2(x+1)$ i.e. two concident roots are $x = 1, 1$ and other root is $x = -1$.

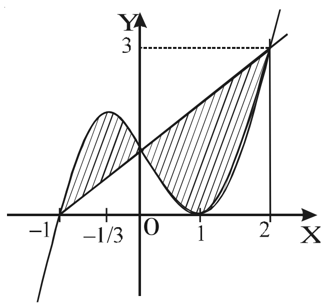
$$\frac{dy}{dx} = 3x^2 - 2x - 1 = (x-1)(3x+1) = 0$$

$\Rightarrow \; x = 1, \;\; x = -1/3$

Solving $y = x + 1$ and $x^3 - x^2 - x + 1 = x + 1$

$x(x^2 - x - 2) = 0$

$\Rightarrow \; x = 0, x = 2, -1$



$$A = \int_{-1}^{0} [(x^3 - x^2 - x + 1) - (x+1)]dx$$

$$+ \int_{0}^{2} [(x+1) - (x^3 - x^2 - x + 1)]dx$$

$$= \int_{-1}^{0} (x^3 - x^2 - 2x)dx + \int_{0}^{2} (2x + x^2 - x^3)dx$$

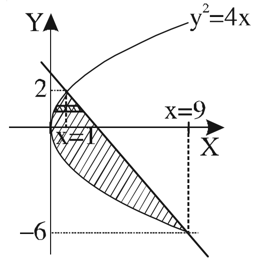
$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} + \left[x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_{0}^{2}$$

$$= \left[(0) - \left(\frac{1}{4} + \frac{1}{3} - 1\right)\right] + \left[\left(4 + \frac{8}{3} - 4\right) - 0\right]$$

$$= \left(1 - \frac{7}{12}\right) + \frac{8}{3} = \frac{11}{3} - \frac{7}{12} = \frac{44-7}{12} = \frac{37}{12}.$$

Problem 5. Find the area of the region bounded by the parabola $y^2 = 4x$ and a normal drawn to it with gradient -1.

Solution



Equation of normal to the parabola is $y = mx - 2am - am^3$

$y = -x + 2 + 1$ (since $a = 1$ and $m = -1$)

$y = 3 - x$. Solving it with $y^2 = 4x$,

$(3 - x)^2 = 4x \qquad \Rightarrow \; 9 + x^2 - 6x = 4x$

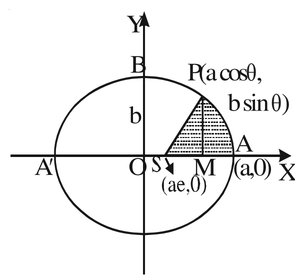
$\Rightarrow \; x^2 - 10x + 9 = 0 \qquad \Rightarrow \; x = 1 \;$ or $\; x = 9$

The values of y are 2 and –6.

$$\text{Now, } A = \int_{-6}^{2} x\,dy = \int_{-6}^{2} \left[(3-y) - \frac{y^2}{4}\right]dy = \frac{64}{3}.$$

Problem 6. Prove that the area of a sector of the ellipse of semi-axes a and b between the major axis and a radius vector from the focus is $\frac{1}{2}ab(\theta - e\sin\theta)$, where θ is the eccentric angle of the point to which the radius vector is drawn.

Solution Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$. Then, O is its centre and S(ae, 0) is the focus. Let θ be the eccentric angle of any point P(x, y) on the ellipse. Then $x = a\cos\theta$, $y = b\sin\theta$. Now SP is the radius vector of P drawn through the focus S and SA is the radius vector along the major axis. At the point A, $x = a$ and $\theta = 0$.



Draw PM perpendicular to the x-axis. The required area of the sector SAP

= area of the ΔSMP + area PMA

$$= \frac{1}{2} \text{SM . MP} + \int_{a\cos\theta}^{a} y\,dx, \text{ for the ellipse}$$

$$= \frac{1}{2}(\text{OM} - \text{OS}) . \text{MP} + \int_0^{\theta} y \frac{dx}{d\theta} d\theta$$

$$= \frac{1}{2}(a\cos\theta - ae) b\sin\theta + \int_0^{\theta} b\sin\theta . (-a\sin\theta)\, d\theta,$$

$$[\because x = a\cos\theta \text{ and } y = b\sin\theta]$$

$$= \frac{1}{2} ab(\cos\theta - e)\sin\theta + \int_0^{\theta} ab. \frac{1}{2}(1 - \cos 2\theta) d\theta$$

$$= \frac{1}{2} ab(\cos\theta - e)\sin\theta + \frac{1}{2} ab\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\theta}$$

$$= \frac{1}{2} ab(\cos\theta - e)\sin\theta + \frac{1}{2} ab\left(\theta - \frac{1}{2}.2\sin\theta\cos\theta\right)$$

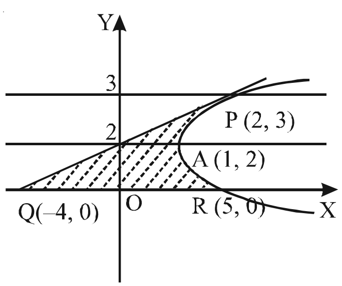
$$= \frac{1}{2} ab[\cos\theta\sin\theta - e\sin\theta + \theta - \sin\theta\cos\theta]$$

$$= \frac{1}{2} ab(\theta - e\sin\theta).$$

Problem 7. Find the area enclosed by the parabola $(y-2)^2 = x - 1$, the tangent to the parabola at (2, 3) and the x-axis.

Solution The given parabola is $(y-2)^2 = x - 1$ ...(1)

Its axis is $y = 2$ and vertex is (1, 2). Let P be the point (2, 3).



From (1), $2(y-2)\dfrac{dy}{dx} = 1$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2(y-2)}$$

At P(2, 3), $\dfrac{dy}{dx} = \dfrac{1}{2}$

$\therefore$ The equation of tangent at P(2, 3) is

$$y - 3 = \frac{1}{2}(x - 2)$$

or, $x - 2y + 4 = 0$ ...(2)

Line (2) cuts the x-axis at (–4, 0) and y-axis at (0, 2).

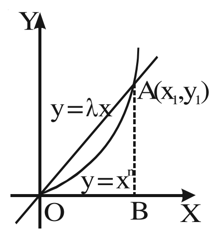
The required area, RQPAR $= \displaystyle\int_0^3 (x_1 - x_2)\,dy$

$$= \int_0^3 [(y-2)^2 + 1 - (2y - 4)]dy = \int_0^3 (y^2 - 6y + 9)dy$$

$$= \left[\frac{y^3}{3} - 3y^2 + 9y\right]_0^3 = (9 - 27 + 27) - 0 = 9.$$

Problem 8. Each line passing through the origin O (0, 0) and with slope $\lambda > 0$ cuts the curve $y = x^n$, $(x \geq 0, n \neq -1)$ at the point $A(x_1, y_1)$ which projects the point $B(x_1, 0)$ on the x-axis. ΔOAB is divided into two regions by the curve $y = x^n$. Demonstrate that ratio of the areas of these two regions is independent of λ. For which value of n are the areas of these two regions equal ?

Solution



Since $x_1^n = y_1 = \lambda x_1$, the area of the triangle OAB

$$= \frac{y_1 x_1}{2} = \frac{\lambda x_1^2}{2} \text{ and the area under the curve}$$

$$= \int_0^{x_1} x^n dx = \frac{x_1^{n+1}}{n+1} = \frac{\lambda x_1^2}{n+1}, \text{ and therefore the other}$$

portion of the triangle has area

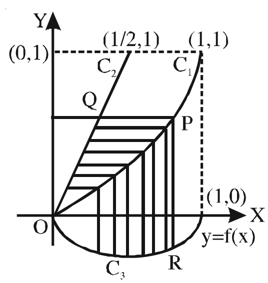
$$\frac{\lambda x_1^2}{2} - \frac{\lambda x_1^2}{n+1} = \frac{\lambda x_1^2 (n-1)}{2n+2}.$$

The ratio of both portions is therefore

$$\frac{\dfrac{\lambda x_1^2}{n+1}}{\dfrac{\lambda x_1^2(n-1)}{2n+2}} = \frac{2}{n-1}, \text{ independent of } \lambda.$$

If $\dfrac{2}{n-1} = 1$ then $n = 3$ and so the portions will have equal area for $n = 3$.

Problem 9. Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$; $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure). If for every position of P on (C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function f(x).



Solution On the curve C_1, i.e. $y = x^2$, let P be (α, α^2). Hence, ordinate of point Q on C_2 is also α^2. Now on C_2, i.e. $y = 2x$, the abscissae of Q is given by

$$x = \frac{y}{2} = \frac{\alpha^2}{2}.$$

$$\therefore \quad Q \text{ is } \left(\frac{\alpha^2}{2}, \alpha\right)$$

And R on C_3 is $(\alpha, f(\alpha))$. Now, Area OPQ

$$= \int_0^{\alpha^2} (x_1 - x_2)dy = \int_0^{\alpha^2} \left(\sqrt{y} - \frac{y}{2}\right)dy$$

$$= \frac{2}{3}\alpha^3 - \frac{\alpha^4}{4}$$

Area ORP $= \int_0^{\alpha} (y_1 - y_2)dx = \int_0^{\alpha} (x^2 - f(x))dx$

Thus, $\dfrac{2}{3}\alpha^3 - \dfrac{\alpha^4}{4} = \int_0^{\alpha} (x^2 - f(x))dx$

Differentiating both sides with respect to α we get

$$2\alpha^2 - \alpha^3 = \alpha^2 - f(\alpha)$$

$$\Rightarrow f(\alpha) = \alpha^3 - \alpha^2$$

$$\therefore f(x) = x^3 - x^2.$$

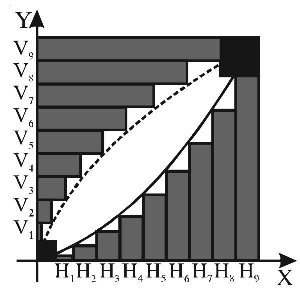
Problem 10. Prove, without using a calculator, that

$$\sum_{k=1}^{9}\left(\left(\frac{k}{10}\right)^2 + \sqrt{\frac{k}{10}}\right) < 9.5$$

Solution The inverse of the function

$f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$ is :

$f^{-1} : [0, \infty) \to [0, \infty)$, $f^{-1}(x) = \sqrt{x}$



In the diagram each rectangle V_k has its lower left corner $\left(0, \dfrac{k}{10}\right)$, base $\sqrt{\dfrac{k}{10}}$ and height $\dfrac{1}{10}$.

Each rectangle H_k has lower left corner at $\left(\dfrac{k}{10}, 0\right)$, base $\dfrac{1}{10}$ and height $\left(\dfrac{k}{10}\right)^2$.

The collective area of these rectangles is

$$\frac{1}{10}\left(\left(\frac{1}{10}\right)^2 + \sqrt{\frac{1}{10}} + \left(\frac{2}{10}\right)^2 + \sqrt{\frac{2}{10}} + \left(\frac{3}{10}\right)^2 + \sqrt{\frac{3}{10}} + \ldots + \left(\frac{9}{10}\right)^2 + \sqrt{\frac{9}{10}}\right)$$

Since these grey rectangles do not intersect with the black squares on the corners, their collective area is less than the area of the unit square minus these squares i.e.

$$1 - \left(\frac{1}{10}\right)^2 - \left(\frac{2}{10}\right)^2 = 1 - \frac{1}{100} - \frac{4}{100} = \frac{95}{100}.$$

We thus conclude that

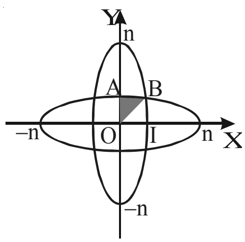
$$\frac{1}{10}\left(\left(\frac{1}{10}\right)^2+\sqrt{\frac{1}{10}}+\left(\frac{2}{10}\right)^2+\sqrt{\frac{2}{10}}+\left(\frac{3}{10}\right)^2+\sqrt{\frac{3}{10}}+\ldots+\left(\frac{9}{10}\right)^2+\sqrt{\frac{9}{10}}\right)$$

$$< \frac{95}{100}$$

$$\Rightarrow \sum_{k=1}^{9}\left(\left(\frac{k}{10}\right)^2+\sqrt{\frac{k}{10}}\right)<9.5.$$

Problem 11. For each positive integers $n > 1$, A_n represents the area of the region restricted to the following two inequalities : $\frac{x^2}{n^2}+y^2\le 1$ and $x^2+\frac{y^2}{n^2}\le 1$. Find $\lim_{n\to\infty} A_n$.

Solution



The intersecting points of the ellipses are :

$$(x.y)=\left(\frac{n}{\sqrt{n^2+1}}.\frac{n}{\sqrt{n^2+1}}\right),$$

$$\left(\frac{n}{\sqrt{n^2+1}},\frac{-n}{\sqrt{n^2+1}}\right),\left(\frac{-n}{\sqrt{n^2+1}},\frac{n}{\sqrt{n^2+1}}\right)$$

$$\left(\frac{-n}{\sqrt{n^2+1}},\frac{-n}{\sqrt{n^2+1}}\right)$$

Notice that the region is symmetric with respect to : x-axis, y-axis, $y=x$ and $y=-x$.

Hence, $A_n = 8 . B_n$,

where B_n = area between the y-axis, $y = x$, and $y=\frac{\sqrt{n^2-x^2}}{n}$.

As $n\to\infty$, $\lim_{n\to\infty}\frac{n}{n^2+1}=1$ and B_n approaches the area of ΔAOB where $O=(0, 0)$, $A=(1, 0)$ and $B=(1, 1)$.

$$\therefore \quad \lim_{n\to\infty} A_n = 8.\left(\frac{1}{2}.1.1\right)=4.$$

Problem 12. Find the area of an ellipse given by the general equation $ax^2+2hxy+by^2+2gx+2fy+c=0$

Solution Here, solving for y, we easily find

$$y_2-y_1=\frac{2}{b}\sqrt{(h^2-ab)x^2+2(hf-bg)x+f^2-bc}$$

Also, the limiting values of x are the roots of the quadratic expression under the radical sign. Accordingly, denoting these roots by α and β, and observing that $h^2 - ab$ is negative for an ellipse, the entire area is represented by

$$\frac{2\sqrt{ab-h^2}}{b}\int_\alpha^\beta\sqrt{(x-\alpha)(\beta-x)}\,dx.$$

To find this, assume $x - \alpha = (\beta - \alpha)\sin^2\theta$ then $\beta - x = (\beta - \alpha)\cos^2\theta$ and we get

$$\int_\alpha^\beta\sqrt{(x-\alpha)(\beta-x)}\,dx=2(\beta-\alpha)^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta$$

$$=\frac{\pi}{8}(\beta-\alpha)^2.$$

Again, $(\beta-\alpha)^2=4.\dfrac{(hf-bg)^2+(f^2-bc)(ab-h^2)}{(ab-h^2)^2}$

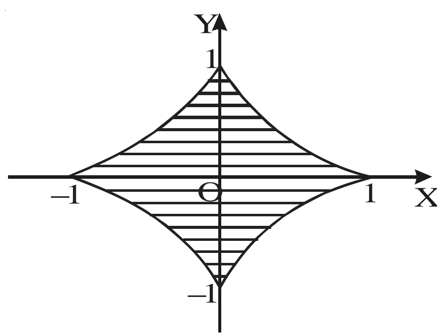
$$=\frac{4b(af^2+bg^2+ch^2-2fgh-abc)}{(ab-h^2)^2}$$

Hence, the area of the ellipse

$$=\frac{\pi(af^2+bg^2+ch^2-2fgh-abc)}{(ab-h^2)^{3/2}}.$$

Problem 13. Compute the area enclosed by the curve $y^2=(1-x^2)^3$.

Solution The curve is symmetric about x–axis as well as y–axis. When $x=-1,1 \Rightarrow y=0$, and when $x=0\Rightarrow y=\pm1$



$$I_1 = \int_0^1 (1-x^2)^{3/2}\,dx$$

$$I = 4I_1 = 4\int_0^1 (1-x^2)^{3/2}\,dx$$

Put $x = \sin\theta$

$$= 4\int_0^{\pi/2} \cos^3\theta \,.\, \cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2} \cos^4\theta\, d\theta$$

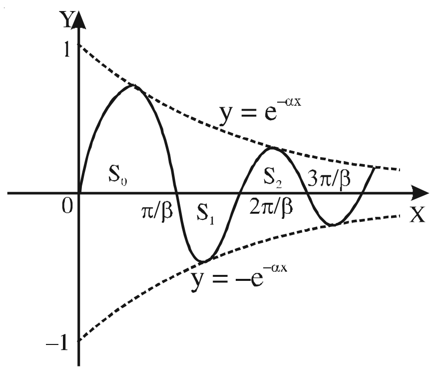
$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}.\frac{(n-3)}{n-2}........\frac{3}{4}.\frac{1}{2}\times\frac{\pi}{2}$$

if n is even

$$= 4\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{3}{4}\pi = 0.75\,\pi.$$

Problem 14. Prove that the areas $S_0, S_1, S_2 \ldots S_3$ .... bounded by the x-axis and half–waves of the curve $y = e^{-\alpha x}\sin\beta x$ $x \geq 0$ form a geometric progression with common ratio $q = e^{-\alpha\pi/\beta}$.

Solution The curve intersects the positive x-axis at the points where $\sin bx = 0$. Hence $x_n = \dfrac{n\pi}{\beta}$, $n \in W$.



The function $y = e^{-\alpha x}\sin\beta x$ is positive in the intervals $(x_{2k}\ x_{2k+1})$ the sign of the function in the interval $(x_n + x_{n+1})$ coincides with that of the number $(-1)^n$.

$$\therefore\quad S_n = \int_{\frac{n\pi}{\beta}}^{\frac{(n+1)\pi}{\beta}} |y|\,dx = (-1)^n \int_{\frac{n\pi}{\beta}}^{\frac{(n+1)\pi}{\beta}} e^{-\alpha x}.\sin\beta x\,dx$$

But the indefinite integral is equal to

$$\int e^{-\alpha x}.\sin\beta x\,dx = -\frac{e^{-\alpha x}}{\alpha^2+\beta^2}(\alpha\sin\beta x + \beta\cos\beta x) + c$$

Consequently

$$S_n = (-1)^{n+1}\left[\frac{e^{-\alpha x}}{\alpha^2+\beta^2}(\alpha\sin\beta x + \beta\cos\beta x)\right]_{n\pi/\beta}^{(n+1)\pi/\beta}$$

$$= \frac{(-1)^{n+1}}{\alpha^2+\beta^2}\left[e^{-\alpha(n+1)\pi/\beta}\,\beta\,(-1)^{n+1} - e^{\alpha n\,\pi/\beta}\,\beta(-1)^n\right]$$

$$= \frac{\beta}{\alpha^2+\beta^2}\, e^{-dn\,\pi/\beta}(1 + e^{\alpha\,\pi/\beta})$$

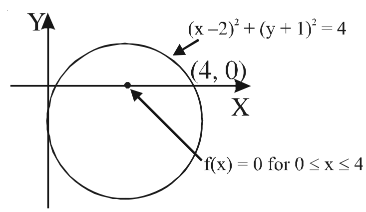
Hence, the ratio of area of two consecutive half waves

$$= \frac{S_{n+1}}{S_n} = \frac{e^{-\alpha(n+1)\pi\pi/\beta}}{e^{-\alpha n\,\pi/\beta}} = e^{-\alpha\pi/\beta},$$

which is independent of n. Hence, the areas form a geometric progression.

Problem 15. Find out the ratio of areas in which the function $f(x) = \left[\dfrac{x^3}{100} + \dfrac{x}{35}\right]$ divides the circle $x^2 + y^2 - 4x + 2y + 1 = 0$ ([.] denotes the greatest integer function).

Solution Circle : $x^2 + y^2 - 4x + 2y + 1 = 0$



or, $(x-2)^2 + (y+1)^2 = 4 = 2^2$ ...(1)

Now for $0 \leq x \leq 4$,

$$0 \leq \frac{x^3}{100} + \frac{x}{35} < 1 \Rightarrow \left[\frac{x^3}{100} + \frac{x}{35}\right] = 0\,.$$

So, we have to find out the ratio in which x axis divides the circle (1). Now, at x-axis, $y = 0$. So, $(x-2)^2 = 3$,

Hence, the circle cuts the x axis at the points $(2-\sqrt{3}, 0)$ and $(2+\sqrt{3}, 0)$

Let $A = \int_{2-\sqrt{3}}^{2+\sqrt{3}} \left(\sqrt{4-(x-2)^2} - 1 \right) dx = \dfrac{4\pi - 3\sqrt{3}}{3}$

The required ratio is $\dfrac{A}{4\pi - A} = \dfrac{4\pi - 3\sqrt{3}}{8\pi + 3\sqrt{3}}$.

Problem 16. For what value of 'a' does the area of the figure bounded by the straight lines, $x = x_1$, $x = x_2$, the graph of the function $y = |\sin x + \cos x - a|$ and the abscissa axis where x_1 and x_2 are two successive extrema of the function, $f(x) = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$, have the least value . Also find the least value .

Solution $f'(x) = \sqrt{2}\cos\left(x + \dfrac{\pi}{4}\right) = 0$

$$\Rightarrow \quad x + \frac{\pi}{4} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\Rightarrow x_1 = \frac{\pi}{4} \text{ and } x_2 = \frac{5\pi}{4}.$$

Hence, $A = \int_{\pi/4}^{5\pi/4} \left| \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - a \right| dx$

Let $a \geq \sqrt{2}$, then $A = \int_{\pi/4}^{5\pi/4} \left(a - \sqrt{2}\sin\left(\frac{\pi}{4} + x\right)\right) dx$

$A = a\pi \Rightarrow A_{min} = \sqrt{2}\,\pi$ where $a = \sqrt{2}$.

Let $a \leq -\sqrt{2}$,

$$A = \int_{\pi/4}^{5\pi/4} \left(\sqrt{2}\sin\left(\frac{\pi}{4} + x\right) - a \right) dx = -a\pi.$$

Hence $A_{min} = \sqrt{2}\,\pi$ when $a = -\sqrt{2}$

Let $-\sqrt{2} < a < \sqrt{2}$

$$A = \int_{\pi/4}^{5\pi/4} \left| \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - a \right| dx$$

Let $x + \dfrac{\pi}{4} = y = \int_{\pi/2}^{3\pi/2} \left| \sqrt{2}\sin y - a \right| dy$

$$= \int_{\pi/2}^{\pi - \sin^{-1}\frac{a}{\sqrt{2}}} \left(\sqrt{2}\sin y - a \right) dy + \int_{\pi - \sin^{-1}\frac{a}{\sqrt{2}}}^{3\pi/2} \left(a - \sqrt{2}\sin y \right) dy$$

$$= -\left[\sqrt{2}\cos y \right]_{\frac{\pi}{2}}^{\pi - \sin^{-1}\frac{a}{\sqrt{2}}} - a\left[\frac{\pi}{2} - \sin^{-1}\frac{a}{\sqrt{2}} \right]$$

$$+ \left[ay + \sqrt{2}\cos y \right]_{\pi - \sin^{-1}\frac{a}{\sqrt{2}}}^{\frac{3\pi}{2}}$$

This on simplification reduces to

$$A = 2\sqrt{2 - a^2} + 2a\sin^{-1}\frac{a}{\sqrt{2}}$$

$$\frac{dA}{da} = 2 \cdot \frac{1(-2a)}{2\sqrt{2 - a^2}} + 2a \frac{1}{\sqrt{1 - \frac{a^2}{2}}} \frac{1}{\sqrt{2}}$$

$$+ \sin^{-1}\frac{a}{\sqrt{2}} \cdot 2$$

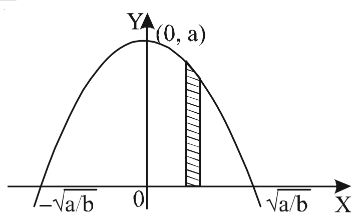
$$= -\frac{2a}{\sqrt{2 - a^2}} + \frac{2a}{\sqrt{2 - a^2}} + 2\sin^{-1}\frac{a}{\sqrt{2}} = 0$$

$$\Rightarrow \sin^{-1}\frac{a}{\sqrt{2}} = 0 \Rightarrow a = 0 \text{ and } A_{min} = 2\sqrt{2}.$$

Finally, the least value of area is $2\sqrt{2}$ and the value of a is 0.

Problem 17. Consider the collection of all curves of the form $y = a - bx^2$ that pass through the point (2, 1) where a and b are positive real numbers. If the minimum area of the region bounded by $y = a - bx^2$ and the x-axis is $\sqrt{m}$, find the value of M.

Solution



We have $y = a - bx^2$

Since the point (2, 1) lies on the curve,

$$1 = a - 4b \qquad \Rightarrow \quad a = 1 + 4b \qquad \text{...(1)}$$

$$A = 2\int_0^{\sqrt{a/b}} (a - bx^2)\,dx = 2\left[ax - \frac{bx^3}{3}\right]_0^{\sqrt{a/b}}$$

$$= 2\left[\frac{a\sqrt{a}}{\sqrt{b}} - \frac{\cancel{b}}{3}\cdot\frac{a\sqrt{a}}{\cancel{b}\sqrt{b}}\right]$$

$$= \frac{2}{3}\left[\frac{3a\sqrt{a}}{\sqrt{b}} - \frac{a\sqrt{a}}{\sqrt{b}}\right] = \frac{2}{3}\left[\frac{2a\sqrt{a}}{\sqrt{b}}\right]$$

$$= \frac{4}{3}\cdot\frac{(1+4b)\sqrt{1+4b}}{\sqrt{b}}, \qquad \text{...(2)}$$

$$\frac{dA}{db} = \frac{4}{3}\left[\frac{b^{1/2}\cdot\frac{3}{2}\sqrt{1+4b}\cdot 4 - (1+4b)^{3/2}\cdot\frac{1}{2}\frac{1}{\sqrt{b}}}{b\cdot 2\sqrt{b}}\right]$$

$$= \frac{4}{3}\left[\frac{12b\sqrt{1+4b} - (1+4b)^{3/2}}{2b^{3/2}}\right] = 0$$

$\Rightarrow$ $12b = 1 + 4b$

$\Rightarrow$ $8b = 1$ and hence

$\Rightarrow$ $b = 1/8$ and hence $a = 3/2$ using (1).

Putting these values in (2), we get

$$A_{\min} = \frac{4}{3}\cdot(3/2)\cdot\sqrt{3/2}\cdot 2\sqrt{2} = \frac{4\cdot\sqrt{3}\cdot\sqrt{2}}{\sqrt{2}} = \sqrt{48}.$$

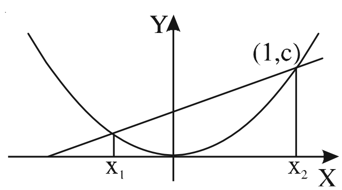
Hence, $\sqrt{m} = \sqrt{48}$ which m 48.

Problem 18. Let 'c' be the constant greater then 1. If the least area of the figure given by the line passing through the point (1, c) with gradient 'm' and the parabola $y = x^2$ is 36 sq. units find the value of $(c^2 + m^2)$.

Solution Equation of the line through (1, c) is

$$y - c = m(x - 1)$$

$$y = mx + (c - m) \qquad \text{...(1)}$$



$$A = \int_{x_1}^{x_2} (mx + (c - m) - x^2)\,dx,$$

where x_1 and x_2 are roots of $x^2 - mx + (m - c) = 0$.

$\Rightarrow$ $x_1 + x_2 = m$, $x_1 x_2 = m - c$

$$= (x_2 - x_1)\left[\frac{m(x_1 + x_2)}{2} + (c - m) - \frac{x_1^2 + x_2^2 + x_1x_2}{3}\right]$$

$$= \sqrt{m^2 + 4(c - m)}\left[\frac{m^2}{2} + (c - m) - \frac{m^2 - (m - c)}{3}\right]$$

$$= \sqrt{m^2 + 4(c - m)}\left[\frac{m^2 + 4(c - m)}{6}\right]$$

$$= \frac{[m^2 + 4(c - m)]^{3/2}}{6} = \frac{[m^2 - 4m + 4c]^{3/2}}{6}$$

$$= \frac{[(m - 2)^2 + 4(c - 1)]^{3/2}}{6}$$

$\therefore$ A is least if $m = 2$

$$A_{\text{least}} = \frac{[4(c - 1)]^{3/2}}{6} = \frac{2^3(c - 1)^{3/2}}{6} = \frac{4}{3}\cdot(c - 1)^{3/2}$$

$$\frac{4}{3}\cdot(c - 1)^{3/2} = 36$$

$\Rightarrow$ $(c - 1)^{3/2} = 27$

$\Rightarrow$ $(c - 1) = (3^3)^{2/3} = 9$

$\Rightarrow$ $c = 10$

Hence, $c = 10$, $m = 2$

$\therefore$ $c^2 + m^2 = 100 + 4 = 104$.

Problem 19. Let $0 \le a \le 4$. Prove that the area of the bounded region enclosed by the curves with equations $y = 1 - |x - 1|$ and $y = |2x - a|$ cannot exceed $\frac{1}{3}$.

Solution In the situation that $0 \le a \le 1$, the two curves intersect in the points (a/3, a/3) and (a, a), and the bounded region is the triangle with these two vertices and the vertex (a/2, 0). This triangle is contained in the triangle with vertices (0, 0), (1/2, 0) and (1, 1) with area 1/4. Hence, when $0 \le a \le 1$, the area of the bounded region cannot exceed 1/4.

Let $1 \le a \le 3$. In this case, the bounded region is a quadrilateral with the four vertices (a/3, a/3), (a/2, 0), ((a + 2)/3, (4 – a)/3) and (1, 1). Nothing that this quadrilateral is the result of removing two smaller triangle from a larger one, we find that its area is

$$1-\frac{1}{2}\cdot\frac{a}{3}\cdot\frac{a}{2}-\frac{1}{2}\cdot\frac{(4-a)}{3}\cdot\left(2-\frac{a}{2}\right)$$

$$=1-\frac{a^2}{12}-\frac{1}{12}(4-a)^2$$

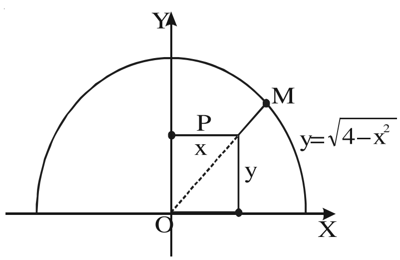
$$=-\frac{a^2-4a+2}{6}=\frac{1}{3}-\frac{(a-2)^2}{6}$$

whence we find the area does not exceed 1/3 and is equal to 1/3 exactly when a = 2.

The case $3 \le a \le 4$ is the symmetric image of the case $0 \le a \le 1$ and we find that the area of the bounded region cannot exceed 1/4.

Problem 20. Find the area between the curve $y=\sqrt{4-x^2}$ and the locus of the point P which moves such that the sum of its distances from the co-ordinate axes is equal to its distance from the curve $y=\sqrt{4-x^2}$.

Solution



Let the point P be (x, y)

$\Rightarrow\ |x|+|y| = PM = OM - OP$

$\Rightarrow\ |x|+|y| = 2-\sqrt{x^2+y^2}$

$\Rightarrow\ x^2+y^2+4-4|x|-4|y|+2|xy| = x^2+y^2$

$\Rightarrow\ 2|xy|-4|x|-4|y|+4=0$

$\Rightarrow\ |xy|-2|x|-2|y|+2=0$

Case I : If $x \ge 0$, $y \ge 0$, then locus of the point P is

$xy-2x-2y+2=0$

$\Rightarrow\ y(x-2)=2(x-1) \Rightarrow y=\dfrac{2(x-1)}{x-2}$

Case II : If $x \le 0$, $y \ge 0$, then locus of the point P is

$-xy+2x-2y+2=0$

$\Rightarrow\ y(-2-x)=-2(x+1)$

$\Rightarrow\ y=\dfrac{-2(x+1)}{-(2x-x)}=\dfrac{2(x+1)}{x+2}$.

The curve $y=\dfrac{2(x-1)}{x-2}$ is passing through the points (1, 0) and (0, 1) and the curve $y=\dfrac{2(x+1)}{x+2}$ is passing through (–1, 0) and (0,1).

Hence the required area = Area of circle – Area under the locus of the point P.

$$\Rightarrow\ \text{Area}=\frac{4\pi}{2}-\int_0^1\frac{2(x-1)}{x-2}dx-\int_{-1}^0\frac{2(x+1)}{x+2}dx$$

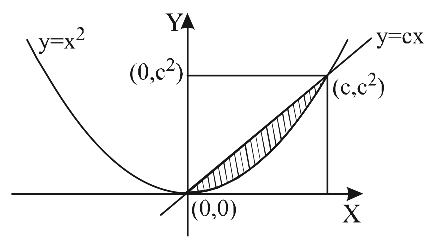
$$=2\pi-2(1+\ln 1-\ln 2)-2(1-\ln 2+\ln 1)$$

$$=2\pi-2+2\ln 2\ 2+2\ln 2=2\pi-4+4\ln 2$$

Problem 21. Let T be the triangle with vertices (0, 0), (0, c^2) and (c, c^2) and let R be the region between $y = cx$ and $y = x^2$ where $c > 0$ then show that

$$\text{Area (R)}=\frac{c^3}{6} \text{ and } \lim_{c\to 0^+}\frac{\text{Area (T)}}{\text{Area (R)}}=3.$$

Solution



$$\text{Area (T)}=\frac{c\cdot c^2}{2}=\frac{c^3}{2}$$

$$\text{Area (R)}=\frac{c^3}{2}-\int_0^c x^2\,dx=\frac{c^3}{2}-\frac{c^3}{3}=\frac{c^3}{6}$$

$$\therefore\quad \lim_{c\to 0^+}\frac{\text{Area (T)}}{\text{Area (R)}}=\lim_{c\to 0^+}\frac{c^3}{2}\cdot\frac{6}{c^3}=3.$$

Problem 22. Let $f(x)=\begin{cases}1-x & \text{if } 0\le x\le 1\\ 0 & \text{if } 1<x\le 2\\ (2-x)^2 & \text{if } 2<x\le 3\end{cases}$

and $F(x)=\int_0^x f(t)\,dt$ then find the area enclosed by the curve $y=F(x)$ and x-axis as x varies from 0 to 3.

Solution $F(x)=\int_0^x f(t)dt \Rightarrow F'(x)=f(x)$

For $0\le x\le 1, F(x)=\int_0^x (1-t)dt=\left[t-\frac{t^2}{2}\right]_0^x$

$$F(x)=x-\frac{x^2}{2} \text{ and } F(1)=\frac{1}{2}$$

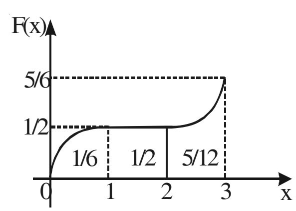
For $1<x\le 2, F(x)=\int_0^1 (1-t)dt+\int_1^x 0dt=\frac{1}{2}+0$

$\Rightarrow$ F(x) is constant

For $2<x\le 3, F(x)=\int_0^1 f(t)dt+\int_1^2 f(t)dt+\int_2^x f(t)dt$

$$=\frac{1}{2}+0+\int_0^x (t-2)^2dt=\frac{1}{2}+\frac{(x-2)^3}{3}$$

$$\therefore\ F(x)=\begin{cases}x-\frac{x^2}{2} & \text{if } 0\le x\le 1\\ \frac{1}{2} & \text{if } 1<x\le 2\\ \frac{1}{2}+\frac{(x-2)^3}{3} & \text{if } 2<x\le 3\end{cases}$$



$$\text{Area}=\int_0^1 F(x)dx+\frac{1}{2}+\int_2^3 F(x)dx$$

$$=\int_0^1\left(x-\frac{x^2}{2}\right)dx+\frac{1}{2}+\int_2^3\left(\frac{1}{2}\left(\frac{(x-2)^3}{3}\right)\right)dx$$

$$=\left[\frac{x^2}{2}-\frac{x^3}{6}\right]_0^1+\frac{1}{2}+\left[\frac{x}{2}+\frac{(x-2)^4}{12}\right]_2^3$$

$$=\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{6}\right)+\left[\left(\frac{3}{2}+\frac{1}{12}\right)-(1)\right]$$

$$=\frac{1}{2}+\frac{1}{3}+\frac{7}{12}=\frac{6+4+7}{12}=\frac{17}{12}.$$

Problem 23. Let f(x) be a differentiable function and satisfy $f(0)=2,\ f'(0)=3$ and $f''(x)=f(x)$. Find the area enclosed by $y=f(x)$ in the second quadrant.

Solution Given $f''(x)=f(x)$ or $2f'(x)f''(x)=2f(x)f'(x)$ [multiplying both sides by $2f'(x)$]

$$\left(\frac{d}{dx}\right)\left((f'(x))^2\right)=\left(\frac{d}{dx}\right)\left(f^2(x)\right)$$

Integrating, $(f'(x))^2=f^2(x)+C$ ...(1)

Put $x=0$, $(f'(0))^2=f^2(0)+C$

$9=4+C$ $\Rightarrow C=5.$

Hence equation (1) becomes

$$\left(\frac{dy}{dx}\right)^2=y^2+5 \text{ (where } f'(x)=\frac{dy}{dx} \text{ and } f(x)=y)$$

$$\frac{dy}{dx}=\sqrt{y^2+5} \text{ (cannot be} -\sqrt{y^2+5} \text{ as } f'(0)=3)$$

Integrating, $\int\frac{dy}{\sqrt{y^2+5}}=\int dx,$

$$\Rightarrow\ ln\left(y+\sqrt{y^2+5}\right)=x+C_1$$

Put $x=0,\ y(0)=2$

$ln(2+3)=0+C_1$

$\Rightarrow C_1=ln\,5$

$$\therefore \quad \ln\left(\frac{y+\sqrt{y^2+5}}{5}\right)=x$$

$$\Rightarrow \sqrt{y^2+5}+y=5e^x \qquad ...(2)$$

Rationalizing, $\dfrac{5}{\sqrt{y^2+5}-y}=5e^x$

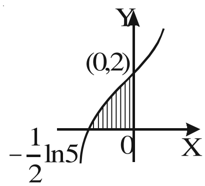
$$\Rightarrow \sqrt{y^2+5}-y=e^{-x} \qquad ...(3)$$

Now (2) – (3), gives $2y=5e^x-e^{-x}$

$$f(x)=\frac{5e^x-e^{-x}}{2},\ f'(x)=\frac{5e^x+e^{-x}}{2}>0\ \forall\ x\in R$$

$\Rightarrow$ f is increasing.

The graph of $y=f(x)$ is :



We have $f(0)=2$.

Now $f(x)=0$, When $5e^x-e^{-x}=0$

$$\Rightarrow 5e^{2x}=1 \Rightarrow e^{2x}=\frac{1}{5} \Rightarrow x=\frac{-1}{2}\ln 5.$$

Area in the second quadrant $=\displaystyle\int_{\frac{-1}{2}\ln 5}^{0}\left(\frac{5}{2}e^x-\frac{1}{2}e^{-x}\right)dx$

$$=\left.\frac{5}{2}e^x+\frac{1}{2}e^{-x}\right]_{\frac{-1}{2}\ln 5}^{0}=3-\left(\frac{5}{2}\frac{1}{\sqrt{5}}+\frac{\sqrt{5}}{2}\right)=3-\sqrt{5}.$$

Problem 24. Let f(x) be a function which satisfy the equation $f(xy)=f(x)+f(y)$ for all $x>0$, $y>0$ such that $f'(1)=2$. Find the area of the region bounded by the curves $y=f(x)$, $y=|x^3-6x^2+11x-6|$ and $x=0$.

Solution Take $x=y=1 \Rightarrow f(1)=0$

Now, put $y=\dfrac{1}{x} \Rightarrow 0=f\left(x\dfrac{1}{x}\right)=f(x)+f\left(\dfrac{1}{x}\right)$

$$\Rightarrow f\left(\frac{1}{x}\right)=-f(x)$$

$$\therefore \quad f\left(\frac{x}{y}\right)=f(x)+f\left(\frac{1}{y}\right)=f(x)-f(y)$$

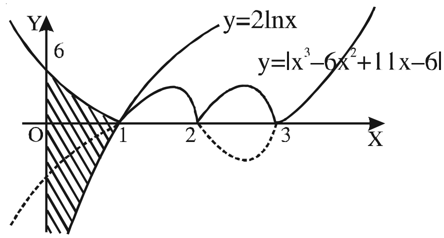
Now, $f'(x)=\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{1}{h}f\left(\frac{x+h}{x}\right)=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}x}$$

$$=\frac{f'(1)}{x}=\frac{2}{x}\Rightarrow f(x)=2\ln x+c \Rightarrow c=0$$

(since $f(1)=0$)

$\Rightarrow f(x)=2\ln x$



The required area

$$=\int_0^1(x^3-6x^2+11x-6)dx+\int_{-\infty}^{0}e^{y/2}dy=\frac{7}{4}.$$

(Using horizontal strip for the area below the x-axis)

Problem 25. Let $f(x+y)=f(x)+f(y)-xy\ \forall\ x,y\in R$ and $\displaystyle\lim_{h\to 0}\frac{f(h)}{h}=3$. Find the area bounded by the curves $y=f(x)$ and $y=x^2$.

Solution In $f(x+y)=f(x)+f(y)-xy$

Put $x=y=0$, so that $f(0)=0$.

Also, $f'(x)=\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(h)}{h}$

$$=\lim_{h\to 0}\frac{f(x)+f(h)-hx-f(x)}{h}=\lim_{h\to 0}\frac{f(h)}{h}-x$$

$\Rightarrow f'(x)=3-x$

$\Rightarrow f(x)=3x-\dfrac{x^2}{2}+c$ and $f(0)=0 \Rightarrow c=0$

$\Rightarrow f(x) = 3x - \frac{x^2}{2}$.

$f(x) \le x^2 \Rightarrow 3x - \frac{x^2}{2} \ge x^2 \Rightarrow 0 \le x \le 2$

The area is bounded before x = 0 and x = 2.

$$\text{Area} = \int_0^2 \left[\left(3x - \frac{x^2}{2}\right) - x^2\right] dx$$

$$= 3\int_0^2 \left[x - \frac{x^2}{2}\right] dx = 3\left(\frac{x^2}{2} - \frac{x^3}{6}\right)\Bigg|_0^2 = 2.$$

Problem 26. If $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ $\forall$ x, y $\in$ R and $f'(1)$ exists, and area under the curve f(x) bounded by x-axis x = 0 and x = 1 is $\frac{1}{3}$, then find $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n} e^{r/n} f\left(\frac{\sqrt{r}}{n}\right)$.

Solution If $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ $\forall$ x, y $\in$ R

$$\Rightarrow f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow f'(1) = \lim_{h\to 0}\frac{f(1+h) - 1}{h} \quad ...(1) \quad (\because f(1) = 1)$$

$$\text{Now } f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right) - 1}{h} \quad f(x) = f(x)\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right) - 1}{x\frac{h}{x}}$$

$$= \frac{f(x)}{x} f'(1) \qquad \text{[from equation (1)]}$$

$$\Rightarrow f'(x) = \frac{f(x)}{x}.k \text{ where } f'(1) = k$$

$\Rightarrow f(x) = x^k + c$

For x = 1, f(x) = 1 $\Rightarrow c = 0 \Rightarrow f(x) = x^k$.

$$\text{Now, area} = \int_0^1 f(x)dx = \frac{1}{3} = \left[\frac{x^{k+1}}{k+1}\right]_0^1 = \frac{1}{3} \Rightarrow k = 2.$$

So, $f(x) = x^2$.

$$\text{Now, } \lim_{n\to\infty}\sum_{r=1}^{n} e^{r/n} f\left(\frac{\sqrt{r}}{n}\right) = \lim_{n\to\infty}\sum_{r=1}^{n}\left(e^{r/n}.\frac{r}{n}.\frac{1}{n}\right)$$

$$= \int_0^1 e^x.x.dx = 1. \text{ (using integration by parts)}$$

Things to Remember

1. Curve Sketching
   (a) Extent
   (b) Intercepts
   (c) Sign Scheme of f(x)
   (d) Symmetry
      (i) The graph of F(x, y) = 0 is symmetric about the y-axis if on replacing x by – x, the equation of the curve does not change. i.e. F(x, y) = 0 implies F(–x, y) = 0.
      (ii) The graph of F(x, y) = 0 is symmetric about the x-axis if on replacing y by – y, the equation of the curve does not change. i.e. F(x, y) = 0 implies F(x,–y) = 0.
      (iii) The graph of F(x, y) = 0 is symmetric about the origin if on replacing x by – x and y by – y, the equation of the curve does not change. i.e. F(x, y) = 0 implies F(– x, –y) = 0.
      (iv) The graph of F(x, y) = 0 is symmetric about the line y = x if on interchanging x and y, the equation of the curve does not change. i.e. F(x, y) = 0 implies F(y, x) = 0.
      (v) The graph of F(x, y) = 0 is symmetric about the line y = –x if on replacing x by – y and y by – x, the equation of the curve does not change. i.e. F(x, y) = 0 implies F(– y, – x) = 0.
   (e) Periodicity
   (f) Monotonicity
   (g) Local Maximum and Minimum Values
   (h) Concavity and Points of Inflection
      (i) Asymptotes
      (i) If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to\infty} f(x) = L$, then the line y = L is a horizontal

asymptote of the curve $y = f(x)$.

(ii) The line $x = a$ is a vertical asymptote if at least one of the following statements is true :

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(iii) If there are limits

$$\lim_{x\to\infty} \frac{f(x)}{x} = m_1 \text{ and}$$

$$\lim_{x\to\infty} [f(x) - m_1x] = c_1,$$

then the straight line $y = m_1x + c_1$ will be an asymptote (a right inclined asymptote or, when $m_1 = 0$, a right horizontal asymptote).

If there are limits

$$\lim_{x\to-\infty} \frac{f(x)}{x} = m_2 \text{ and}$$

$$\lim_{x\to-\infty} [f(x) - m_2x] = c_2,$$

then the straight line $y = m_2x + c_2$ is an asymptote (a left inclined asymptote or, when $m_2 = 0$, a left horizontal asymptote).

2. If $f(x) \ge 0$ for $x \in [a, b]$, then the area bounded by curve $y = f(x)$, x-axis, $x = a$ and $x = b$ is

$$A = \int_a^b f(x)\,dx.$$

3. Let $f(x)$, $x \in [a, b]$, be a continuous function on $[a, b]$ whose graph intersects the interval $[a, b]$ of the x-axis at a finite number of points. Then, the area of the plane figure bounded by the graph of the function $f(x)$, the interval $[a,b]$ of the x-axis, and line segments $x = a$ and $x = b$ is computed by the formula

$$A = \int_a^b |f(x)|dx$$

4. If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \ge g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b (f(x) - g(x))dx$$

5. The area bounded by curves $y = f(x)$ and $y = g(x)$ between ordinates $x = a$ and $x = b$ is

$$\int_a^b |f(x) - g(x)|\,dx.$$

6. If $g(y) \ge 0$ for $y \in [c, d]$ then the area bounded by the curve $x = g(y)$, y–axis and the abscissa $y = c$ and $y = d$ is $\displaystyle\int_{y=c}^{d} g(y)dy$.

7. If $y = f(x)$ is a strictly monotonic function in (a, b), with $f'(x) \ne 0$, then the area bounded by the ordinates $x = a$, $x = b$, $y = f(x)$ and $y = f(c)$ (where $c \in (a, b)$) is minimum when $c = \dfrac{a+b}{2}$.

8. Since area remains invariant even if the coordinates axes are shifted, hence shifting of origin in many cases proves to be convenient in computing the areas.

9. The area of the curvilinear trapezoid bounded by a curve represented by $x = \phi(t)$, $y = \psi(t)$, where $\alpha \le t \le \beta$ and $\phi(\alpha) = a$, $\psi(\beta) = b$. is

$$A = \int_\alpha^\beta \psi(t)\phi'(t)dt.$$

The area of the region is also given by the formula

$$A = \frac{1}{2}\int_{t_1}^{t_2} \left[\phi(t)\psi'(t) - \psi(t)\phi'(t)\right]dt$$

10. If $r = f(\theta)$ be the equation of a curve in polar coordinates where $f(\theta)$ is a single valued continuous function of θ, then the area of the sector enclosed by the curve and the two radii vectors $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2)$, is equal to

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta.$$

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. If $f(x) = \begin{cases} \sqrt{\{x\}}, & x \notin I \\ 1, & x \in I \end{cases}$ and $g(x) = \{x\}^2$, (where $\{.\}$ denotes fractional part of x), then the area bounded by f(x) and g(x) for $x \in [0, 10]$ is
(A) 5/3 (B) 5
(C) 10/3 (D) none of these

2. The area enclosed by the curve $|y| = \sin 2x$, when $x \in [0, 2\pi]$ is
(A) 1 (B) 2
(C) 3 (D) 4

3. Let $f(x) = x^2$, $g(x) = \cos x$ and α, β $(\alpha < \beta)$ be the roots of the equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area bounded by the curves $y = fog(x)$, the ordinates $x = \alpha$, $x = \beta$ and the x-axis is
(A) $\frac{1}{2}(\pi - 3)$ (B) $\frac{\pi}{3}$
(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{12}$

4. The area common to the region determined by $y \geq \sqrt{x}$ and $x^2 + y^2 < 2$ has the value
(A) π (B) $(2\pi - 1)$
(C) $\frac{\pi}{4} - \frac{1}{6}$ (D) none of these

5. The graph of $y^2 + 2xy + 40|x| = 400$ divides the plane into regions. The area of the bounded region is
(A) 400 sq. units (B) 800 sq. units
(C) 600 sq. units (D) none of these

6. The area of the region defined by $||x| - |y|| \leq 1$ and $x^2 + y^2 \leq 1$ in the X-Y plane is
(A) π (B) 2π
(C) 3π (D) 1

7. The area defined by $1 \leq |x - 2| + |y + 1| \leq 2$ is
(A) 2 (B) 4
(C) 6 (D) none of these

8. The area enclosed by the curve $|y| = 5 - (1 - |x|)^2$ is
(A) $\frac{8}{3}(7 + 5\sqrt{5})$ sq. units
(B) $\frac{2}{3}(7 + 5\sqrt{5})$ sq. units
(C) $\frac{2}{3}(5\sqrt{5} - 7)$ sq. units
(D) none of these

9. A square ABCD is inscribed in a circle of radius 4. A point P moves inside the circle such that $d(P, AB) \leq \min(d(P, BC), d(P, CD), d(P, DA))$ where d(P, AB) is the distance of a point P from line AB. The area of region covered by tracing point P is
(A) 4π (B) 8π
(C) $8\pi - 16$ (D) None of these

10. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4$, $y = 4$ and the coordinate axes. If S_1, S_2, S_3 are respectively the areas of these parts numbered from top to bottom, then $S_1 : S_2 : S_3$ is
(A) 2 : 1 : 2 (B) 1 : 1 : 1
(C) 1 : 2 : 1 (D) 1 : 2 : 3

11. Consider the graph of continuous function $y = f(x)$ for $x \in [a - b, a + b]$ $a, b \in R^+$ and $b > a$. If the origin is shifted to $(a + b, 0)$ such that new axes are parallel to the old axes, then the area bounded by the given curve, the X–axis and the new ordinates $X = -a$, $X = -b$ can be written as
(A) $\int_{\frac{b-a}{2}}^{\frac{a+b}{2}} |f(x)|dx$ (B) $\int_{-a}^{-b} |f(x)|dx$
(C) $\int_{a}^{b} |f(a + b - x)|dx$ (D) $\frac{1}{2}\int_{a-b}^{a+b} |f(x)|dx$

12. The area of the region bounded by two branches of the curve $(y - x)^2 = x^3$ and the straight line $x = 1$ is
(A) $\frac{3}{5}$ (B) $\frac{2}{5}$
(C) $\frac{4}{5}$ (D) 1

13. If the tangent to the curve $y = 1 - x^2$ at $x = \alpha$, where $0 < \alpha < 1$, meets the axes at P and Q. Also α varies, the minimum value of the area of the triangle OPQ is k times the area bounded by the axes and the part of the curve for which $0 < x < 1$, then k is equal to

(A) $\frac{2}{\sqrt{3}}$ (B) $\frac{75}{16}$

(C) $\frac{25}{18}$ (D) $\frac{2}{3}$

14. Let f is a differentiable function such that $f(x + y) = f(x^2 + y^2)$, where $x, y \in R$. If 4 points A, B, C, D, are selected on curve $y = f(x)$, then area of $(\Delta ABC + \Delta BCD)$ can be equal to

(A) 2 (B) 1

(C) >2 (D) None

15. Suppose $y = f(x)$ and $y = g(x)$ are two functions whose graphs intersect at the three points (0, 4), (2, 2) and (4, 0) with $f(x) > g(x)$ for $0 < x < 2$ and $f(x) < g(x)$ for $2 < x < 4$. If $\int_0^4 [f(x) - g(x)]dx = 10$ and $\int_2^4 [g(x) - f(x)]dx = 5$, the area between two curves for $0 < x < 2$, is

(A) 5 (B) 10

(C) 15 (D) 20

16. 3 points O(0, 0), P(a, a^2), Q(–b, b^2) (a > 0, b > 0) are on the parabola $y = x^2$. Let S_1 be the area bounded by the line PQ and the parabola and let S_2 be the area of the triangle OPQ, the minimum value of S_1/S_2 is

(A) 4/3 (B) 5/3

(C) 2 (D) 7/3

17. The area bounded by the curve $f(x) = ||\tan x + \cot x| - |\tan x - \cot x||$ between the lines $x = 0$, $x = \frac{\pi}{2}$ and the x-axis, is

(A) ln 4 (B) $\ln\sqrt{2}$

(C) 2 ln 2 (D) $\sqrt{2}\ln 2$

18. The area bounded by the curves

$|y| = e^{-|x|} - \frac{1}{2}$ and $\frac{|x|+|y|}{2} + \left|\frac{|x|-|y|}{2}\right| \le 2$ is

(A) (7 + ln 4) sq. units

(B) (7 – ln 2) sq. units

(C) (14 + 2 ln 2) sq. units

(D) (14 – 2 ln 2) sq. units

19. A function $y = f(x)$ satisfies the condition $f'(x)\sin x + f(x)\cos x = 1$, f(x) being bounded when $x \to 0$. If $I = \int_0^{\pi/2} f(x)dx$ then

(A) $\frac{\pi}{2} < I < \frac{\pi^2}{4}$ (B) $\frac{\pi}{4} < I < \frac{\pi^2}{2}$

(C) $1 < I < \frac{\pi}{2}$ (D) $0 < I < 1$

20. Area enclosed by the graph of the function $y = ln^2x - 1$ lying in the 4^{th} quadrant is

(A) $\frac{2}{e}$ (B) $\frac{4}{e}$

(C) $2\left(e + \frac{1}{e}\right)$ (D) $4\left(e - \frac{1}{e}\right)$

21. The area bounded by the curve $y = f(x)$, the co-ordinate axes & the line $x = x_1$ is given by $x_1 \cdot e^{x_1}$. Therefore f(x) equals :

(A) e^x (B) xe^x

(C) $xe^x - e^x$ (D) $xe^x + e^x$

22. The slope of the tangent to a curve $y = f(x)$ at (x, f(x)) is 2x + 1. If the curve passes through the point (1, 2) then the area of the region bounded by the curve, the x-axis and the line x = 1 is

(A) $\frac{5}{6}$ (B) $\frac{6}{5}$

(C) $\frac{1}{6}$ (D) 1

23. Area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1-y^2}$ is

(A) 1 (B) 4/3

(C) 2/3 (D) 2

24. The area bounded by the curve $y = x e^{-x}$, $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point, is

(A) $1 - 3e^{-2}$ (B) $1 - 2e^{-2}$

(C) $1 - e^{-2}$ (D) 1

25. Area enclosed by the curves $y = ln x$, $y = ln|x|$, $y = |ln x|$ and $y = |ln|x||$ is equal to

(A) 2 (B) 4

(C) 8 (D) cannot be determined

26. If (a, 0); a > 0 is the point where the curve $y = \sin 2x - \sqrt{3}\sin x$ cuts the x-axis first, A is the area bounded by this part of the curve, the origin and the positive x-axis, then

(A) 4A + 8 cos a = 7 (B) 4A + 8 sin a = 7
(C) 4A – 8 sin a = 7 (D) 4A – 8 cos a = 7

27. A function y = f (x) satisfies the differential equation $\frac{dy}{dx} - y = \cos x - \sin x$, with initial condition that y is bounded when $x \to \infty$. The area enclosed by y = f(x), y = cos x and the y-axis in the 1st quadrant

(A) $\sqrt{2} - 1$ (B) $\sqrt{2}$
(C) 1 (D) $\frac{1}{\sqrt{2}}$

28. If the area bounded between x-axis and the graph of $y = 6x - 3x^2$ between the ordinates x = 1 and x = a is 19 square units then 'a' can take the value

(A) 4 or – 2
(B) two values are in (2, 3) and one in (–1, 0)
(C) two values one in (3,4) and one in (–2,–1)
(D) none of these

29. If area bounded by the line y = x, curve y = f(x) and lines x = 1, x = t, is $\left\{\sqrt{\sqrt{t^2+1}+t}+t\right\}^{3/2}$ $\forall\, t > 1$

then f(x) =

(A) $x + \frac{3}{4\sqrt{x^2+1}}\left\{\sqrt{\sqrt{x^2+1}+x}+x\right\}^{1/2}$

(B) $x + \frac{3}{4\sqrt{x^2+1}}\left\{\sqrt{\sqrt{x^2+1}+x}+2\sqrt{x^2+1}\right\}$

(C) $x + \frac{3}{4\sqrt{x^2+1}}\left\{\sqrt{\sqrt{x^2+1}+x}+x\right\}^{1/2}$
$\times\sqrt{\sqrt{x^2+1}+x}+2\sqrt{x^2+1}$

(D) None of these

30. The area bounded by the curve $y = \frac{3}{|x|}$ and $y + |2 - x| = 2$ is

(A) $\frac{4 - \ln 27}{3}$ (B) $2 - \ln 3$
(C) $2 + \ln 3$ (D) none of these

31. The area bounded by $y = x^2 + 2$ and $y = 2|x| - \cos \pi x$ is equal to

(A) 2/3 (B) 8/3
(C) 4/3 (D) 1/3

32. The graphs of $f(x) = x^2$ and $g(x) = cx^3$ (c > 0) intersect at the points (a, 0) & $\left(\frac{1}{c}, \frac{1}{c^2}\right)$. If the region which lies between these & over the interval [0, 1/c] has the area equal to 2/3 then the value of c is

(A) 1 (B) 1/3
(C) 1/2 (D) 2

33. The area between curve $y = 2x^4 - x^2$, x-axis and the ordinates of the two minima of the curve is

(A) $\frac{7}{120}$ sq. units (B) $\frac{11}{120}$ sq. units
(C) $\frac{13}{120}$ sq. units (D) None of these

34. If A(n) represents the area bounded by the curve $y = n\,ln x$, where $n \in N$ and n > 1, the x-axis and the lines x = 1 and x = e, then the value of A(n) + nA(n–1) is equal to:

(A) $\frac{n^2}{e+1}$ (B) $\frac{n^2}{e-1}$
(C) n^2 (D) $e.n^2$

35. If y = mx, equally divides the area bounded by $y = \cos^{-1}(\cos x)$ ($x \in [0, \pi]$) and y = 0, then m is equal to

(A) 1/2 (B) $1 - \sqrt{2}$
(C) 1/3 (D) None of these

36. Area enclosed by the curve $y = (x^2 + 2x)e^{-x}$ and the positive x-axis is

(A) 1 (B) 2
(C) 4 (D) 6

37. Let S(t) be the area of the ΔOAB with O (0, 0, 0), A (2, 2, 1) and B (t, 1, t + 1). The value of the definite integral $\int_1^e (S(t))^2\, ln\, t\, dt$, is equal to

(A) $\frac{2e^3+5}{2}$ (B) $\frac{e^3+5}{2}$

(C) $\frac{2e^3+15}{2}$ (D) $\frac{e^3+15}{2}$

38. The length of sub-normal at any point P(x, y) on the curve, which is passing through M(0, 1) is unity. The area bounded by the curves satisfying this condition is equal to

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$

(C) $\frac{4}{3}$ (D) $\frac{8}{3}$

39. A triangle has one vertex at (0, 0) and the other two on the graph of $y = -2x^2 + 54$ at (x, y) and (–x, y) where $0 < x < \sqrt{27}$. The value of x so that the corresponding triangle has maximum area is

(A) $\frac{\sqrt{27}}{2}$ (B) 3

(C) $2\sqrt{3}$ (D) None of these

40. The area of the region of the xy plane defined by the inequality $|x|+|y|+|x+y|\le 1$ is

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$

(C) 1 (D) None of these

41. The value of 'a' (a > 0) for which the area bounded by the curve $y=\frac{x}{6}+\frac{1}{x^2}$, y = 0, x = a and x = 2a has the least value, is

(A) 2 (B) $\sqrt{2}$

(C) $2^{1/3}$ (D) 1

42. The area included between the curve $xy^2 = a^2(a-x)$ and its asymptote is

(1) $\frac{\pi a^2}{2}$ (B) $2\pi a^2$

(C) πa^2 (D) none

43. The area bounded by the curves $y = x(1 - \ln x)$, $x = e^{-1}$ and a positive x-axis between $x = e^{-1}$ and x = e is

(A) $\left(\frac{e^2-4e^{-2}}{5}\right)$ (B) $\left(\frac{e^2-5e^{-2}}{4}\right)$

(C) $\left(\frac{4e^2-e^{-2}}{5}\right)$ (D) $\left(\frac{5e^2-e^{-2}}{4}\right)$

44. If the area enclosed between $f(x) = \min.\left(\cos^{-1}(\cos x), \cot^{-1}(\cot x)\right)$ and x-axis in $x \in (\pi, 2\pi)$ is $\frac{\pi^2}{k}$ where $k \in N$, then k is equal to

(A) 4 (B) 6

(C) 8 (D) 12

45. Area enclosed by the curve, $|x+y-1|+|2x+y+1| = 1$ is

(A) 2 sq. units (B) 1 sq. units

(C) 4 sq. units (D) none of these

46. The area of the region consisting of all points (x, y) so that $x^2+y^2 \le 1 \le |x|+|y|$, is

(A) π (B) $\pi - 1$

(C) $\pi - 2$ (D) $\pi - 3$

47. The area enclosed between the curves $y = \log_e(x+e)$, $x = \log_e\left(\frac{1}{y}\right)$ and the x–axis is

(A) 2 (B) 1

(C) 4 (D) None of these

48. The area of the region bounded in first quadrant by $y = x^{1/3}$, $y = -x^2 + 2x + 3$, $y = 2x - 1$ and the axis of ordinates is

(A) 12/55 (B) 55/12

(C) 32/55 (D) None of these

49. Let C be a curve passing through M (2, 2) such that the slope of the tangent at any point to the curve is reciprocal of the ordinate of the point. If the area bounded by curve C and line x = 2 is expressed as a rational $\frac{p}{q}$ (where p and q are in their lowest form), then (p + q) is equal to

(A) 19 (B) 18

(C) 9 (D) 6

50. Let C be the curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x–axis. If the area bounded by the curve C and x-axis in the first quadrant is $\frac{k\pi}{2}$ square units, then find the value of k.

(A) 2 (B) $2\sqrt{2}$

(C) 4 (D) 1

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

51. If area bounded by the curve $y = \cos[x]$, x-axis and the lines given by $x^2 - 3x + 2 = 0$ is A, then (where $[x]$ represents integer function)

(A) $A = \int_1^{\pi/2} \cos[x]dx + \left|\int_{\pi/2}^{2} \cos[x]dx\right|$

(B) $A = 2\cos 1$

(C) $A = \left|\int_1^{\pi/2} \cos[x]dx\right| + \int_{\pi/2}^{2} \cos[x]dx$

(D) $A < \sin 1$

52. Let $f(x) = x^2 - 5x + 6$ be a function $\forall\, x \in R$. C_1 and C_2 are two curves given by $|y| = |f(|x|)|$ and $x^2 + y^2 = \frac{25}{4}$ respectively. Then

(A) graph of C_1 and C_2 intersects each other at 8 distinct points

(B) area enclosed by C_1, C_2 and lying on left of the line $x - 2 = 0$ in 1st quadrant is less than 5 sq. units

(C) graph of C_1 and C_2 intersects each other at 12 distinct points

(D) both (B) and (C)

53. The area bounded by a curve, the axis of co-ordinates and the ordinate of some point of the curve is equal to the length of the corresponding arc of the curve. If the curve passes through the point P(0, 1) then the equation of this curve can be

(A) $y = \frac{1}{2}(e^x - e^{-x} + 2)$

(B) $y = \frac{1}{2}(e^x + e^{-x})$

(C) $y = 1$

(D) $y = \frac{2}{e^x + e^{-x}}$

54. The area enclosed by the currves $x = a\sin^3 t$ and $y = a\cos^3 t$ is equal to

(A) $12a\int_0^{\pi/2} \cos^4 t\, \sin^2 t\, dt$

(B) $12a\int_0^{\pi/2} \cos^2 t\, \sin^4 t\, dt$

(C) $2\int_{-a}^{a} (a^{2/3} - x^{2/3})^{3/2} dx$

(D) $4\int_0^{a} (a^{2/3} - x^{2/3})^{3/2} dx$

55. If A_i is the area bounded by $|x - a_i| + |y| = b_i$, $i \in N$, where $a_{i+1} = a_i + \frac{3}{2}b_i$ and $b_{i+1} = \frac{b_i}{2}$, $a_1 = 0$, $b_1 = 32$, then

(A) $A_3 = 128$

(B) $A_3 = 256$

(C) $\lim_{n\to\infty} \sum_{i=1}^{n} A_i = \frac{8}{3}(32)^2$

(D) $\lim_{n\to\infty} \sum_{i=1}^{n} A_i = \frac{4}{3}(16)^2$

56. Suppose f is defined from $R \to [-1, 1]$ as $f(x) = \frac{x^2 - 1}{x^2 + 1}$ where R is the set of real number. Then the statement which does not hold is

(A) f is many one onto

(B) f increases for $x > 0$ and decrease for $x < 0$

(C) minimum value is not attained even though f is bounded

(D) the area included by the curve $y = f(x)$ and the line $y = 1$ is π sq. units.

57. The curve $x^6 - x^2 + y^2 = 0$

(A) is symmetric about both axes

(B) meets the x-axis at three points.

(C) is symmetric about $y = -x$

(D) bounds an area $\frac{\pi}{2}$

58. Consider the functions f(x) and g(x), both defined from $R \to R$ and are defined as $f(x) = 2x - x^2$ and $g(x) = x^n$ where $n \in N$. If the area between f(x) and g(x) is 1/2 then *n* is a divisor of

(A) 12 (B) 15
(C) 20 (D) 30

59. Which of the following statement(s) is/are true for the function $f(x) = (x-1)^2(x-2) + 1$ defined on [0, 2]?

(A) Range of f is $\left[\frac{23}{27}, 1\right]$.

(B) The coordinates of the turning point of the graph of $y = f(x)$ occur at (1, 1) and $\left(\frac{5}{3}, \frac{23}{27}\right)$.

(C) The value of p for which the equation $f(x) = p$ has 3 distinct solutions lies in interval $\left(\frac{23}{27}, 1\right)$.

(D) The area enclosed by $y = f(x)$, the lines $x = 0$ and $y = 1$ as *x* varies from 0 to 1 is $\frac{7}{12}$.

60. Consider $f(x) = \begin{cases} \cos x & 0 \le x < \frac{\pi}{2} \\ \left(\frac{\pi}{2} - x\right)^2 & \frac{\pi}{2} \le x < \pi \end{cases}$ such that f is periodic with period π, then

(A) The range of f is $\left[0, \frac{\pi^2}{4}\right)$

(B) f is continuous for all real x, but not differentiable for some real x

(C) f is continuous for all real x

(D) The area bounded by $y = f(x)$ and the x-axis from $x = -n\pi$ to $x = n\pi$ is $2n\left(1 + \frac{\pi^3}{24}\right)$ for a given $n \in N$

61. Two circles have centres that are d units apart and each has diameter $\sqrt{d}$. Let A(D) be the area of the smallest circle that contains both these circles, then

(A) $A(D) = \pi\left(\frac{d + 2\sqrt{d}}{2}\right)^2$

(B) $A(D) = \pi\left(\frac{d + \sqrt{d}}{2}\right)^2$

(C) $\lim\limits_{d \to \infty} \frac{A(d)}{d^2} = \frac{\pi}{4}$

(D) $\lim\limits_{d \to \infty} \frac{A(d)}{d^2} = \frac{\pi}{2}$

62. Let $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ and $f(t) = \int_{-\infty}^{\infty} e^{-tx^2} dx, (t > 0)$

(A) f(x) is equal to $\frac{\sqrt{\pi}}{x}$

(B) f(x) is monotonic decreasing

(C) if area bounded by $y = f(x)$, $x = 2$, $x = 16$ and $y = c$ is minimum then $c = \frac{\sqrt{\pi}}{3}$

(D) equation of normal to curve $y = f(x)$ at $x = \pi$ is $y - 1 = 2\pi(x - \pi)$

63. Let $f(x) = x - x^2$ & $g(x) = ax$. If the region above the graph of g and below the graph of f has an area equal to 9/2 then 'a' is equal to

(A) 2 (B) 4
(C) −2 (D) 3

64. The equation $(y - x - 2)^2 = 9x$ represents a curve

(A) which is an ellipse

(B) which cuts the x-axis at two distinct points

(C) which touches the y-axis

(D) which encloses an area equal to unity with the co-ordinate axes.

65. The area of the figure bounded by the curve $y = 3x^3 + 2x$ and the straight lines $x = a$ and $y = 0$ is unity. Then value of 'a' is :

(A) 4 (B) $\sqrt{3}$

(C) $-\sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{2}{3}}$

66. Let f(x) be differentiable function on the interval $(-\infty, \infty)$ such that $f(1) = 5$

and $\lim\limits_{a\to x}\dfrac{af(x)-xf(a)}{a-x}=2, \forall x \in R$. Then which of the following alternative(s) is/are correct?
(A) f(x) has an inflection point.
(B) $f'(x)=3\ \forall x\in R$.
(C) $\int_0^2 f(x)dx=10$.
(D) Area bounded by f (x) with co-ordinate axes is $\frac{2}{3}$.

67. The parabola $x=y^2+ay+b$ intersect the parabola $x^2=y$ at (1, 1) at right angle.
Which of the following is/are correct?
(A) $a=4, b=-4$
(B) $a=2, b=-2$
(C) Equation of the director circle for the parabola $x=y^2+ay+b$ is $4x+1=0$.
(D) Area enclosed by the parabola $x=y^2+ay+b$ and its latus rectum is $\frac{1}{6}$.

68. If [λ] denotes the integral part of λ, and $f(x)=\sec^{-1}[-\sin^2(x)]$ then
(A) domain is $x \in R-\{n\pi\}$ where $n\in I$ and range is π
(B) thefunction has removable discontinuities at $x=n\pi, n\in I$
(C) area bounded by $y=x^2$ and $y=f(x)$ is $\frac{4\pi\sqrt{\pi}}{3}$
(D) none of these

69. Which of the statement(s) are true ?
(A) The area bounded by the curve $y = x\,|\,x\,|$, x – axis and the ordinates $x=1, x=-1$, is 2/3
(B) The area bounded by $y=x^2$, $y=[x+1]$, $x\le 1$ and the y-axis where [.] denotes the greatest integer function is 2/3
(C) The slope of the tangent to curve $y=f(x)$ at (x, f(x)) is $2x+1$. If the curve passes through the point (1, 2), then the area of the region bounded by the curve, the x-axis and the line x = 1, is 5/6.
(D) None of these

70. Which of the statement(s) are true ?
(A) The area bounded by the curve $y=\max\{2=2, 2, 1+x\}$ and the ordinates $x=-1$ and $x=1$, is 9/2
(B) The area bounded by the curve $y=\max\{x+|x|, x-[x]\}$, whee [.] denotes greatest integer function, and ordinates $x=-2$ and $x=2$, is 1
(C) The area bounded by the cureve $y=|x^3-3x^2+2x|$ and ordinates $x=0$ and $x=3$, is 11/4
(D) None of these

71. Which of the statement(s) are true ?
(A) The area bounded by $[x]=[y]$, $0\le x, y\le 10$ is 10
(B) The area bounded by $\max(|x|, |y|)\le 1$ is 4
(C) The area bounded by $|y+x|\le 1$, $|y-x|\le 1$, $2x^2+2y^2=1$ is $2-\frac{\pi}{2}$
(D) None of these

Comprehension - 1

Consider two curves $C_1\equiv[f(y)]^{2/3}+[f(x)]^{1/3}=0$ and $C_2\equiv[f(y)]^{2/3}+[f(x)]^{2/3}=12$, satisfying the relation $f(x-y)\,f(x+y)-(x+y)\,f(x-y)=4xy\,(x^2-y^2)$.

72. The area bounded by C_1 and C_2 is
(A) $2\pi-\sqrt{3}$ sq. units
(B) $2\pi+\sqrt{3}$ sq. units
(C) $\pi+\sqrt{6}$ sq. units
(D) $2\sqrt{3}-\pi$ sq. units

73. The area bounded by the curve C_2 and $|x|+|y|=\sqrt{12}$ is
(A) $12\pi-2\sqrt{12}$ sq. units
(B) $6-\sqrt{12}$ sq. units
(C) $2\sqrt{12}-6$ sq. units
(D) None of these

74. The area bounded by C_1 and $x+y+2=0$ is
(A) 5/2 sq. units
(B) 7/2 sq. units
(C) 9/2 sq. units
(D) None of these

Comprehension - 2

Consider the function defined implicitly by the equation $y^2-2ye^{\sin^{-1}x}+x^2-1+[x]+e^{2\sin^{-1}x}=0$ (where [x] denotes the greatest integer function).

75. The area of the region bounded by the curve and the lines $x=-1$ is
(A $\pi+1$ sq. units
(B) $\pi-1$ sq. units
(C) $\frac{\pi}{2}+1$ sq. units
(D) $\frac{\pi}{2}-1$ sq. units

76. Line x = 0 divides the region mentioned above in two parts. The ratio of area of left hand side of line to that of right hand side of line is

(A) $2+\pi:\pi$ (B) $2-\pi:\pi$

(C) $1:1$ (D) $\pi+2:\pi$

77. The area of the region bounded by the curve and lines x = 0 and x = 1/2 is

(A) $\frac{\sqrt{3}}{4}+\frac{\pi}{6}$ sq. units (B) $\frac{\sqrt{3}}{2}+\frac{\pi}{6}$ sq. units

(C) $\frac{\sqrt{3}}{4}-\frac{\pi}{6}$ sq. units (D) $\frac{\sqrt{3}}{2}-\frac{\pi}{6}$ sq. units

Comprehension - 3

Consider the polynomial $f(x) = x^6 - 2a^2x^4 + a^4x^2 + ax$. The graph of the function is tangent to a straight line at three point A_1, A_2, A_3 where A_2 lies between A_1 and A_3.

78. The equation of the straight line is

(A) $y = x + a$ (B) $2y = ax$

(C) $y = ax$ (D) None of these

79. The ratio of length of segments A_1A_2 and A_1A_3 is

(A) 1 : 2 (B) 1 : 9

(C) 2 : 3 (D) None of these

80. The ratio of areas of the figures bounded by line segments A_1A_2, A_2A_3 and the graph of the polynomial is

(A) 1 : 4 (B) 1 : 9

(C) 1 : 1 (D) None of these

Comprehension - 4

Consider f, g and h be three real valued functions defined on R. Let $f(x) = \begin{cases} -1, x<0 \\ 0, x=0 \\ 1, x>0 \end{cases}$,

$g(x) = x(1-x^2)$ and h(x) is such that $h''(x) = 6x - 4$. Also h(x) has local minimum value 5 at x = 1.

81. The equation of tangent at M(2, 7) to the curve y = h(x), is

(A) $5x + y = 17$ (B) $x + 5y = 37$

(C) $x - 5y + 33 = 0$ (D) $5x - y = 3$

82. The area bounded by $y = h(x)$, $y = g(f(x))$ between x = 0 and x = 2 equals

(A) $\frac{23}{2}$ (B) $\frac{20}{3}$

(C) $\frac{32}{3}$ (D) $\frac{40}{3}$

83. Range of the function $\sin^{-1}\sqrt{(fog(x))}$, is

(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left\{0, \frac{\pi}{2}\right\}$

(C) $\left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$ (D) $\left\{\frac{\pi}{2}\right\}$

Comprehension - 5

A variable line $y = l(x)$ intersects the parabola $y = x^2$ at points P and Q whose x-coordinates are α and β respectively with $\alpha < \beta$. The area of the figure enclosed by the segment PQ and the parabola is always equal to $\frac{4}{3}$. The variable segment PQ has its middle point as M.

84. The value of $(\beta - \alpha)$ is equal to

(A) 1 (B) 2

(C) 4 (D) 8

85. Equation of the locus of the mid-point of PQ, is

(A) $y = 1 + x^2$ (B) $y = 1 + 4x^2$

(C) $4y = 1 + x^2$ (D) $2y = 1 + x^2$

86. Area of the region enclosed between the locus of M and the pair of tangents on it from the origin, is

(A) $\frac{8}{3}$ (B) 2

(C) $\frac{4}{3}$ (D) $\frac{2}{3}$

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

87. f(x) is a polynomial of degree 3 passing through origin having local extrema at $x = \pm 2$.

Assertion (A) : Ratio of areas in which f(x) cuts the circle $x^2 + y^2 = 36$ is 1 : 1.

Reason (R) : Both y= f(x) and the circle are symmetric about origin.

88. Assertion (A) : The area enclosed between the parabolas $y^2 - 2y + 4x + 5 = 0$ and $x^2 + 2x - y + 2 = 0$ is same as that of bounded by curves $y^2 = -4x$ and $x^2 = y$.

Reason (R) : Shifting of origin to point (h, k) does not change the bounded area.

89. Assertion (A) : The area of the region bounded by the curve $2y = \log_e x$, $y = e^{2x}$ and the pair of lines $(x + y - 1) \times (x + y - 3) = 0$ is 2k sq. units.
Reason (R) : The area of the region bounded by the curves $y = e^{2x}$, $y = x$ and the pair of lines $x^2 + y^2 + 2xy - 4x - 4y + 3 = 0$ is k units.

90. Consider two regions
R_1 : Point P is nearer to (1, 0) than to x = – 1
R_2 : Point P is nearer to (0,0) than to (8,0).
Assertion (A) : Area of the region common to R_1 and R_2 is $\frac{128}{3}$ sq. units.
Reason (R) : Area bounded by $x = 4\sqrt{y}$ and $y = 4$ is $\frac{32}{3}$ sq. units.

91. Assertion (A) : (A) : Let A_n be the area outside a regular n-gon of side length 1 but inside its circumcircle and B_n be the area inside the n-gon but outside its inscribed circle. Then $\lim_{n\to\infty}\frac{A_n}{B_n} = 2$

Reason (R) : $A_n = \pi\left(\frac{1}{2}\operatorname{cosec}\frac{\pi}{n}\right)^2 - \frac{n}{4}\cot\frac{\pi}{n}$ and

$B_n = \frac{n}{4}\cot\frac{\pi}{n} - \pi\left(\frac{1}{2}\sec\frac{\pi}{n}\right)^2$ so $\lim_{n\to\infty}\frac{A_n}{B_n} = 2$

92. Assertion (A) **:** A convex quadrilateral is such that each of its vertices satisfies the equation $x^2 + y^2 = 73$ and $xy = 24$. The area of this quadrilateral is 110.
Reason (R) : The quadrilateral is a rectangle whose side lengths are $5\sqrt{2}$ and $11\sqrt{2}$.

93. Assertion (A) **:** Let $f(x) = x^3 + 2x^2 + 2x + 1$ and $g(x)$ be its inverse. The area bounded by g(x), x-axis, x = –3 and x = 6 is given by

$$\int_0^1 (5 - x^3 - 2x^2 - 2x)\,dx + \int_{-2}^{0} (x^3 + 2x^2 + 2x + 4)\,dx$$

Reason (R) : $\int_a^b f(x)\,dx = bf(b) - af(a)$

$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy$

94. Assertion (A) : The area of the region represented by the expression $\sqrt{2} \le |x + y| + |x - y| \le 2\sqrt{2}$ is equal to 6.
Reason (R) : This inequation gives a squared strip having inner side $\sqrt{2}$ unit and outer side $2\sqrt{2}$ unit.

95. Assertion (A) : Area enclosed by curve $(y - \sin^{-1}x)^2 = x - x^2$ is equal to

$$\int_0^1\left(\sin^{-1}x + \sqrt{x - x^2}\right)dx$$

Reason (R) : If $y = f(x)$ and $y = g(x)$ are two curves such that $f(x) \ge g(x)$ for $\forall\ x \in [a, b]$, where $f(x) = g(x) \Rightarrow x = a, b$ then area enclosed by these two curves is given by $\int_a^b \{f(x) - g(x)\}dx$

96. Assertion (A) : Area enclosed by the curve $\left(y - e^{-\sqrt{1-x^2}}\right)^2 = (1 - x^2)$ is equal to $2\int_{-1}^{1}\sqrt{1-x^2}\,dx$

Reason (R) : If $\alpha_1, \alpha_2, \ldots, \alpha_n \in$ domain of continuous functions f(x) and $\sqrt{g(x)}$, are the roots of equation $g(x) = 0$ and $\alpha_1 < \alpha_2 \ldots < \alpha_n$, then area enclosed by curve $(y - f(x))^2 = g(x)$ equals to

$$2\left[\left|\int_{\alpha_1}^{\alpha_2}\sqrt{g(x)}dx\right| + \left|\int_{\alpha_2}^{\alpha_3}\sqrt{g(x)}dx\right| + \ldots\ldots + \left|\int_{\alpha_{n-1}}^{\alpha_n}\sqrt{g(x)}dx\right|\right]$$

MATCH THE COLUMNS FOR JEE ADVANCED

97.

	Column-I	Column-II
(A)	Area enclosed by y = [x] and y = {x}, where [·] and {.} represent greatest integer and fractional part functions.	(P) 16/5 sq. units
(B)	The area bounded by the curves $y^2 = x^3$ and $\|y\| = 2x$.	(Q) 1 sq. units

(C) The smaller area included between the curves $\sqrt{x}+\sqrt{|y|}=1$ and $|x|+|y|=1$. — (R) 4 sq. units

(D) Area bounded by the curves $y=\left[\dfrac{x^2}{64}+2\right]$ (where [·] denotes the greatest integer function), $y=x-1$ and $x=0$ above the x-axis. — (S) 2/3 sq. units

98. **Column I** — **Column – II**

(A) A differentiable function satisfies $f'(x)=f(x)+2e^x$ with the condition $f(x)=0$, then the area bounded by $y=f(x)$ and the x-axis is — (P) 3

(B) $C_1: y=e^x$, $C_2: y=e^{a-x}$, where $a>0$. If A is the area bounded by y-axis, C_1 and C_2, then $\lim\limits_{a\to 0}\left(\dfrac{8A}{a^2}\right)$ is equal to — (Q) 2

(C) Let $y=f(x)$ and $y=g(x)$ are two continuous function intersecting at (0,4), (2,2) and (4,0) with $f(x)>g(x)$ $\forall x\in(2,4)$. If $\int_0^4(f(x)-g(x))dx=10$, $\int_2^4(g(x)-f(x))dx=5$ then the area between the two curves for $0<x<2$ is — (R) 15

(D) If the area bounded by the curves $y=x(1-\ln x)$, $x=e^{-1}$ and $x=e$ is $\left(\dfrac{e^a-be^{-a}}{4}\right)$, then roots of $x^2-(a+b)x+2b=0$ is /are — (S) 5

99. **Column I** — **Column – II**

(A) Area bounded by the curves $y=[\cos A+\cos B+\cos C]$, $y=\left[7\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}\right]$, (where [.] denotes the greatest integer function and A, B, C are the angles of a triangle) and curve $|x-4|+|y|=2$ is — (P) 0

(B) The area bounded by $|x|+|y|=1$ and $|x-1|+|y|=1$ is — (Q) $\dfrac{1}{3}$

(C) The area bounded by $y=f(x)$, y-axis and line $2y=\pi(x+1)$ where $f(x)=\dfrac{\pi}{6}+\sin^{-1}x+\tan^{-1}x+\cos^{-1}x+\tan^{-1}\dfrac{1}{x}$ is A, then $[A^2]$ is (where [.] is greatest integer function and $-1\le x\le 1$) — (R) $\dfrac{1}{2}$

(D) Area bounded by the curve $y=\sqrt{|x|}$ and $y=|x|$ is — (S) 3

100. Let consider the region max $\{|x|, |y|\}\le 2$ represented by f(x, y). Now, if the region f(x, y) undergoes the following transformations successively, then match the region obtained in column –II by transformations done in column–I after each step.

Column I	Column – II
(A) The region f(x, y) is a	(P) square of area 16 sq. unit
(B) The region given by $f_1(x,y) = f\left(\sqrt{x^2}, \sqrt{y^2}\right)$, is a	(Q) rectangle of area 12 sq. unit
(C) The region given by $f_2(x,y) = f_1(y, x)$, is a	(R) parallelogram
(D) The region given by $f_3(x, y) = f_2(x + 3y, y)$, is a	(S) rhombus of area 12 sq. unit
	(T) square of area 4 sq. unit

101.

Column - I	Column - II
(A) The positive value of a such that the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices (0, 0), (a, 0), $(a, a^2 + 1)$ and $(a, a^2 + 1)$ is	(P) $2 + \sqrt{2}$
(B) The graph of $x^2 - (y - 1)^2 = 1$ has one tangent at (a,b) with positive slope which pases through origin. Then a + b is	(Q) $3\sqrt{6}$
(C) The area of the largest rectangle that can be drawn inside a 3 – 4 – 5 right triangle with one of the rectangles sides along one of the legs of the triangle, is	(R) $\sqrt{3}$
(D) A ballon in the shape of a cube is blown up at a rate such that , at time t its surface area is 6t. The rate of pumping of air when the surface area is 144, equals	(S) 3

Review Exercises for JEE Advanced

1. Find the area of the closed figure bounded by the following curves.
   (i) $y = \dfrac{3(x-1)}{x-2}, x = \dfrac{2(y+1)}{y-1}, x = 3, x = 5$
   (ii) $y = 8\sin^4 x + 4\cos^2 x$, $x \in [0, \pi]$ and $y = 0$
   (iii) $y = \dfrac{|4 - x^2|}{4}, y = 7 - |x|$
   (iv) $y = 4 - \dfrac{6}{|1+x|}, y = |2 - x|$
2. Find the area bounded by the curve $y = \dfrac{x-1}{x+1}$ and its asymptote from x = 1 to x = 2.
3. Find the area bounded by the curve $y = 1 + 8/x^2$ and x–axis from x = 2 to x = 4. If the ordinate x = a divide the area into two equal parts, find ‘a’.
4. Compute the area of the figure bounded by straight lines x = 0, x = 2 and the curves $y = 2^x$ and $y = 2x - x^2$.
5. The line 3x + 2y = 13 divides the area enclosed by the curve, $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ into two parts. Find the ratio of the longer area to the smaller area.
6. Find the area of the closed figure bounded by the following curves.
   (i) $y = \dfrac{1}{2}x^2 - 2x + 2$, tangents to the parabola at $\left(1, \dfrac{1}{2}\right)$ and (4, 2)
   (ii) $y = 25^x + 16$, $y = b.5^x + 4$ whose tangent at x = 1 has slope $40\,\ell n\,5$.
   (iii) $y = x^4 - 2x^2 + 5, y = 1, x = 0, x = 1$
7. Find the area of the region bounded by the curves, $y = \log_e x$, $y = \sin^4 \pi x$ and x = 0.
8. Find the area bounded by the curves $y = \sqrt{1 - x^2}$ and $y = x^3 - x$. Also find the ratio in which the y–axis divided this area.
9. If the area enclosed by the parabolas $y = a - x^2$ and $y = x^2$ is $18\sqrt{2}$ sq. units. Find the value of ‘a’.

10. Through the point (x_0, y_0) of the graph of the function $y = \sqrt{1+\cos 2x}$ draw a normal to the graph, if it is known that the straight line $x = x_0$ divides the area bounded by the given curve, the x-axis, and the straight lines $x = 0$ and $x = 3/4\,\pi$ into equal parts.

11. Find all the values of the parameter a $(a > 0)$ for each of which the area of the figure bounded by the straight line $y = \frac{(a^2 - ax)}{1+a^4}$ and the parabola $y = \frac{(x^2 + 2ax + 3a^2)}{1+a^4}$ is the greatest.

12. Find the area of the smaller portion enclosed by the curves $x^2 + y^2 = 9$, $y^2 = 8x$.

13. Compute the area of the figure bounded by the curve $y = \ln x$ and $y = \ln^2 x$.

14. Find the area of the figure bounded by the pareabolas, $x = -2y^2$, $x = 1 - 3y^2$ and y–axis.

15. A normal to the curve, $x^2 + \alpha x - y + 2 = 0$ at the point whose abscissa is 1, is parallel to the line $y = x$. Find the area in the first quadrant bounded by the curve, this normal and the axis of 'x'.

16. Indicate the region bounded by the curves $y = x \ln x$ and $y = 2x - 2x^2$ and obtain the area enclosed by them.

17. Let us designate as S(k) the area contained between the parabola $y_1 = x^2 + 2x - 3$ and the straight line $y_2 = kx + 1$. Find S(–1) and calculate the least value of S(k).

18. Find the area enclosed by the curve $y^2 (x + 1) = x^2 (1 - x)$.

19. A rectangle of length $\frac{1}{4}\pi$ and height 4 is bisected by the x-axis and is in the first and fourth quadrants. The graph of $y = \sin x + C$ divides the area of the square in half. What is C ?

20. Calculate the area bounded by the curves $y = \cos^{-1}(\cos x)$ and $y = \pi^2 + x^2 - 2\pi x$.

21. The area from 0 to x under a certain graph is given to be $A = \sqrt{1+3x} - 1,\ x \geq 0$;

(i) Find the average rate of change of A w.r.t. x as x increases from 1 to 8.

(ii) Find the instantaneous rate of change of A w.r.t. x at $x = 5$.

(iii) Find the ordinate (height) y of the graph as a function of x.

(iv) Find the average value of the ordinate (height) y, w.r.t. x as x increases from 1 to 8

22. Find the area included between the curve $x^2 + y^2 = a^2$ and $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}\ (a > 0)$.

23. A figure is bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, $x = 1$. At what point of the curve $y = x^2 + 1$, must a tangent be drawn for it to cut off a trapezoid of the greatest area from the figure ?

24. Find the area of the region $\{(x, y): 0 \leq y \leq x^2 + 1, 0 \leq y \leq x + 1, 0 \leq x \leq 2\}$.

25. Find the value of c for which the area of the figure bounded by the curves $y = \sin 2x$, the straight lines $x = \pi/6$, $x = c$ and the abscissa axis is equal to 1/2.

26. A figure is bounded by the curves $y = \left|\sqrt{2}\sin\frac{\pi x}{4}\right|$, $y = 0$, $x = 2$ and $x = 4$. At what angles to the positive x–axis straight lines must be drawn through (4, 0) so that these lines partition the figure into three parts of the same size.

27. Find the value of p $(p < 0)$ for which the area of a figure bounded by the parabola $y = (1 + p^2)^2 x^2 + p$ and the line $y = 0$ attains its greatest value.

28. A figure is bounded by the curves $y = (x + 3)^2$, $y = 0$, $x = 0$. At what angles to the x-axis must straight lines be drawn through the point (0, 9) for them to partition the figure into three parts of the same size?

29. Let C be the curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x–axis. If the area bounded by the curve C and x-axis in the first quadrant is $\frac{k\pi}{2}$ square units, then find the value of k.

30. Let C be a curve passing through M (2, 2) such that the slope of the tangent at any point to the curve is reciprocal of the ordinate of the point. If the area bounded by curve C and line x = 2 is expressed as a rational $\frac{p}{q}$ (where p and q are in their lowest form), then find (p + q).

31. The functions $y = 1 + \cos x$ and $y = 1 + \cos(x - \alpha)$, where $0 < \alpha < \pi/2$, are given on the interval $[0, \pi]$. At what value of α is the figure bounded by the curves $y = 1 + \cos x$, $y = 1 + \cos(x - \alpha)$, $x = 0$, equivalent to the figure bounded by the curves $y = 1 + \cos(x - \alpha)$, $y = 1$, $x = \pi$?

32. Given two curves $y = |x - 1|$ and $4(y - b)^2 = x^2$, $b \le 1$. Find b so that the area included between the two curves in maximum.

33. What part of the area of a semi circle is cut off by the parabola passing through the end points of the diameter of the semicircle and touching the circumference at a point which is equidistant from the ends of the diameter?

34. Compute the area of the figure contained between the curve, $xy^2 = 8 - 4x$ and its asymptote

35. Find the area bounded by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote.

Target Exercises for JEE Advanced

1. The tangent to the graph of the function $y = \sqrt[3]{x^2}$ is such that the abscissa x_0 of the point of tangency belongs to the interval [1/2, 1]. For what value of x_0 is the area $S(x_0)$ of the triangle bounded by the tangent, the x-axis, and the straight line $x = 2$ the least and what is it equal to?

2. For what value of a ($a \in [0,1]$) does the area of the figure bounded by the graph of the function $y = f(x)$ and the straight line $x = 0$, $x = 1$, $y = f(a)$, have the greatest value and for which value does it have the least value if $f(x) = x^\alpha + 3x^\beta$, $\alpha, \beta \in R$ with $\alpha > 1$ $\beta > 1$?

3. Find the total area enclosed by the curve $a^2y^2 = x^2(a^2 - x^2)$.

4. Find the value of'a' (a > 2) for which the reciprocal of the area enclosed between $y = \frac{1}{x^2}$; $y = \frac{1}{4(x-1)}$; x = 2 and x = a is 'a' itself and for what values of $b \in (1, 2)$, the area of the figure bounded by the lines x = b and x = 2 is 1 –1/6.

5. Consider the curve $C : y = \sin 2x - \sqrt{3}\,|\sin x|$, C cuts the x–axis at (a, 0), $a \in (-\pi, \pi)$.

A_1 : The area bounded by the curve C and the positive x–axis between the origin & the ordinate at x = a.

A_2 : The area bounded by the curve C and the negative x–axis between the ordinate x = a and the origin. Prove that $A_1 + A_2 + 8A_1A_2 = 4$.

6. Find the area bounded by the curve $y = xe^{-x}$, $xy = 0$ and $x = c$ where c is the x–coordinate of the curve's inflection point.

7. Find the area bounded by the curve $y = xe^{-x^2}$, the x–axis, and the x = c where y(c) is maximum.

8. A polynomial function f(x) satisfies the condition $f(x + 1) = f(x) + 2x + 1$. Find f(x) if f(0) = 1. Find also the equations of the pair of tangents from the origin on the curve y = f(x) and compute the area enclosed by the curve and the pair of tangents.

9. Find the equation of the line passing through the origin and dividing the curvilinear triangle with vertex at the origin , bounded by the curves $y = 2x - x^2$, y = 0 and x = 1 into two parts of equal area .

10. Find the ratio in which the curve $x = \left[\frac{25}{12} - \sin y - \frac{1}{2}\cos 2y\right]$, $y \in [0, \pi]$ divides the area bounded by the curves $y = x^3 - x^2$ and $y = x^2$, where [.] denotes greatest integer function.

11. Consider the two curves $C_1 : y = 1 + \cos x$ and $C_2 : y = 1 + \cos(x - \alpha)$ for $\alpha \in \left(0, \frac{\pi}{2}\right)$, $x \in [0, \pi]$. Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 and x = 0 is same as that of the figure bounded by C_2, y = 1 and x = π. For this value of α , find the ratio in which the line y = 1 divides the area of the figure by the curves C_1, C_2 and x = π .

12. Find the area bounded by the curve $a^2y^2 = x^3(2a - x)$.

13. At what values of a is the area of the figure bounded by the curves $y = 1/x$, $y = 1/(2x - 1)$, $x = 2$ and $x = a$ equal to $\ell n \frac{4}{\sqrt{5}}$?

14. Consider the collection of all curve of the form $y = a - bx^2$ that pass through the point (2, 1), where a and b are positive constants. Determine the value of a and b that will minimise the area of the region bounded by $y = a - bx^2$ and x–axis. Also find the minimum area.

15. Show that the area bounded by the curve $y = \frac{\ln x - c}{x}$, the x–axis and the vertical line through the maximum point of the curve is independent of the constant c.

16. Compute the area of the loop of the curve $y^2 = x^2[(1+x)/(1-x)]$.

17. Find the area bounded by $y^2 = 4(x+1)$, $y^2 = -4(x-1)$ and $y = |x|$ above axis of x.

18. A differentiable function g(x) satisfies $g(x+y) = e^{y}g(x) + e^{x}g(y)$ for all x & y and $g'(0) = 2$. Find g(x) and determine the area bounded by the graph of the function, ordinate of its minima and the coordinate axes.

19. Let two curves $y^2 = 4a(x+2)$, $a > 0$ and $x^2 + y^2 = 4$ on each other at points A and B. Find the values of 'a' such that the area of the region bounded by the parabola and the chord is maximum.

20. Let S be the area included between the parabola $y = x^2 + 2x - 3$ and the line $y = \lambda x + 1$. Find the least value of S where λ is a parameter.

21. Let f be a function satisfying the condition $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$; $\forall\ x, y \in R, y \neq 0$ and $f'(1) = 2$. Find the area enclosed by $y \geq f(x)$, the x axis and the portion of $x^2 + y^2 \leq 2$ in the first quadrant.

22. Find the area of the region represented by :

(a) $|y| + 2|x| \leq x^2 + 1$, $|x| \leq 2$

(b) $\frac{y - 1 + |x-1|}{y - x^2 + 2x} \leq 0, -1 \leq x \leq 3$

23. Let f be a real valued function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)$ and $\lim_{x \to 0} \frac{f(1+x)}{x} = 3$. Find the area bounded by the curve $y = f(x)$, the y-axis and the line $y = 3$, where, $x, y \in R^+$.

24. Find the area of loop formed by $x = 3t^2$, $y = 3t - t^3$

25. Let $f(t) = |t-1| - |t| + |t+1|\ \forall\ t \in R$ and $g(x) = \max\{f(t) : x+1 \leq t \leq x+2\}\ \forall\ x \in R$. Find g(x) and the area bounded by the curve $y = g(x)$, the x-axis and the lines $x = -3/2$ and $x = 5$.

26. Find the area enclosed between the smaller arc of the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ and the parabola $y = -x^2 + 2x + 1 - 2\sqrt{3}$.

27. Draw a neat and clean graph of the function $f(x) = \cos^{-1}(4x^3 - 3x)$, $x \in [-1, 1]$ and find the area enclosed between the graph of the function and the x–axis as x varies from 0 to 1.

28. Let $f(x) = \min.\{e^x, 3/2, 1 + e^{-x}\}$, $0 \leq x \leq 1$. Find the area bounded by $y = f(x)$, x-axis, y-axis and the line $x = 1$.

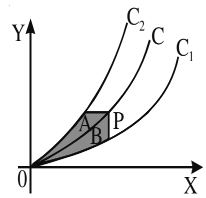
29. Find the area bounded by $y = f(x)$ and the curve $y = \frac{2}{1+x^2}$ where f is a continuous function satisfying the conditions $f(x) . f(y) = f(xy)$ $\forall\ x, y, \in R$ and $f'(1) = 2$, $f(1) = 1$.

30. Find the area of region enclosed by $|x+y| + |x-y| \leq 4$. $|x| \leq 1$, $y \geq \sqrt{x^2 - 2x + 1}$.

31. Find the area enclosed by the curve $[x] + [y] = 4$ in the first quadrant (where[.] denotes greatest integer function).

32. Sketch the region and find the area bounded by the curves $|y+x| \leq 1$, $|y-x| \leq 1$ and $2x^2 + 2y^2 = 1$.

33. Let C_1 and C_2 be two curves passing through the origin as indicated in the figure. A curve C is said to "bisect in area" the region between C_1 and C_2 if, for each point P of C, the two shaded regions A and B shown in the figure have equal areas. Determine the upper curve C_2 given that the bisecting curve C has the equation $y = x^2$ and that the lower curve C_1 has the equation $y = 1/2x^2$.



34. Find the area of the region bounded by the curves, $2^{|x|}|y| + 2^{|x|-1} \le 1$, within the square formed by the lines $|x| \le 1/2$, $|y| \le 1/2$.

35. Consider two curves $C_1 : y^2 = 4[\sqrt{y}]x$ and $C_2 : x^2 = 4[\sqrt{x}]y$, where [.] denotes the greatest integer function. Find the area of region enlosed by these two curves within the square formed by the lines ; $x = 1$, $y = 1$, $x = 4$, $y = 4$.

Previous Year's Questions (JEE Advanced)

A. Multiple Choice Question with ONE corect answer:

1. The area bounded by the curves $y = f(x)$, the x-axis and the ordintes $x = 1$ and $x = b$ is $(b – 1) \sin (3b + 4)$. Then f(x) is **[IIT - 1982]**
(A) $(x – 1) \cos (3x + 4)$
(B) $\sin (3x + 4)$
(C) $\sin (3x + 4) + (x – 1) \cos (3x + 4)$
(D) none of these

2. The area bounded by the curves $y = |x| – 1$ and $y = – |x| + 1$ is **[IIT - 2002]**
(A) 1 (B) 2 (C) $2\sqrt{2}$ (D) 4

3. The area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and x-axis in the 1st quadrant is **[IIT - 2003]**
(A) 9 (B) 27/4 (C) 36 (D) 18

4. The area bounded by the the parabolas $y = (x + 1)^2$ and $y = (x – 1)^2$ and the line $y = 1/4$ is **[IIT - 2005]**
(A) 4 sq. units (B) 1/6 sq. units
(C) 4/3 sq. units (D) 1/3 sq. units

5. The area enclosed between the curve $y = ax^2$ and $x = ay^2 (a > 0)$ is sq. units then the value of a is **[IIT - 2004]**
(A) $1/\sqrt{3}$ (B) 1/2 (C) 1 (D) 1/3

6. The area of the region between the curves $y = \sqrt{\frac{1+\sin x}{\cos x}}$ and $y = \sqrt{\frac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is **[IIT - 2008]**
(A) $\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$
(B) $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$
(C) $\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$
(D) $\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$

7. Let $f : [–1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1 – x)$ for all $x \in [–1, 2]$. Let $R_1 = \int_{-1}^{2} xf(x)\,dx$, and R_2 be the area of the region bounded by $y = f(x)$, $x = –1$, $x = 2$, and the x-axis. Then **[IIT - 2011]**
(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$
(C) $2R_1 = R_2$ (D) $3R_1 = R_2$

8. Let the straight line $x = b$ divide the area enclosed by $y = (1 – x)^2$, $y = 0$, and $x = 0$ into two parts $R_1 (0 \le x \le b)$ and $R_2 (b \le x \le 1)$ such that $R_1 – R_2 = \frac{1}{4}$. Then b equals : **[IIT - 2011]**
(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

B. Multiple Choice Question with ONE or MORE THAN ONE correct answer:

9. For which of the following values of m, is the area of the region bounded by the curve $y = x – x^2$ and the line $y = mx$ equals 9/2? **[IIT - 1999]**
(A) –4 (B) –2 (C) 2 (D) 4

10. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then **[IIT - 2012]**
(A) $S \ge \frac{1}{e}$ (B) $S \ge 1 - \frac{1}{e}$
(C) $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (D) $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

C. Subjective Problems:

11. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$. **[IIT - 1981]**

12. For any real t, $x = \frac{e^t + e^{-t}}{2}, y = \frac{e^t - e^{-t}}{2}$ is a point on the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by this hyperbola and the lines joining its centre to the points correspoinding to t_1 and $-t_1$ is t_1. **[IIT - 1982]**

13. Find the area bounded by the x-axis, part of the curve $y = \left(1 + \frac{8}{x^2}\right)$ and the ordinates at $x = 2$ and $x = 4$. If the ordinate at $x = a$ divideds the area into two equal parts, find the a. **[IIT - 1983]**

14. Find the area of the region bounded by the x-axis and the curves defined by

$$y = \tan x, -\frac{\pi}{3} \le x \le \frac{\pi}{3};\ y = \cot x, \frac{\pi}{6} \le x \le \frac{3\pi}{2}$$

[IIT - 1984]

15. Sketch the region bounded by the curves $y = \sqrt{5 - x^2}$ any $y = |x - 1|$ and find its area. **[IIT - 1985]**

16. Find the area bounded by the curves $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}\, y$ and $x = y$. **[IIT - 1986]**

17. Find the area bounded by the curves, $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$ above the x-axis **[IIT - 1988]**

18. Find the area of the region bounded by the curve $C : y = \tan x$, tangent drawn to C at $x = \frac{\pi}{4}$ and the x-axis. **[IIT - 1988]**

19. Compute the area of the region bounded by the curves $y = ex \ln x$ and $y = \frac{\ln x}{ex}$ **[IIT - 1990]**

20. Sketch the curves and identify the region bounded by $x = 1/2$, $x = 2$, $y = \ln x$ and $y = 2^x$. Find the area of this region **[IIT - 1991]**

21. Sketch the region bounded by the curves $y = x^2$ and $y = \frac{2}{1 + x^2}$. Find the area. **[IIT - 1992]**

22. In what ratio does the x-axis divide the are a of the region bounded by the parabola $y = 4x - x^2$ and $y = x^2 - x$. **[IIT - 1994]**

23. Consider a square with vertices at (1, 1), (–1, 1), (–1, –1) and (1, –1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area. **[IIT - 1995]**

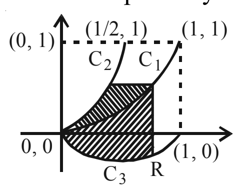
24. Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \pi/4$. Prove the for $n > 2$, $A_n + A_{n+2} = \frac{1}{n-1}$ and deduce $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$ **[IIT - 1996]**

25. Find all possible values of $b > 0$, so that the area of the bounded region enclosed between the parabola $y = (x - bx^2)/b$ and $y = x^2/b$ is maximum. **[IIT - 1997]**

26. Let O (0, 0), A(2, 0) and B $(1, \frac{1}{\sqrt{3}})$ be the vertices of a triangle. Let R be the region consisting of all those points P inside Δ OAB which satisfy d(P, OA) $\ge$ min {d (P, OB), d(P, AB)}, where d denotes the distance from the point to the corresponding line. Sketch the region R and find its area. **[IIT - 1997]**

27. Let $f(x) = \text{maximum}\ \{x^2, (1 - x)^2, 2x(1 - x)\}$ where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$ and $x = 1$. **[IIT - 1997]**

28. Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respestively.



Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes meet C_2 and C_3 at Q and R respectivley (see in figure). If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function f(x). **[IIT - 1998]**

29. Let f(x) be a continuous function given by $f(x)=\begin{cases} 2x & , |x|\le 1 \\ x^2+ax+b & , |x|>1 \end{cases}$. Find the area of the region in the third quadrant bounded by the curves $x=-2y^2$ and $y=f(x)$ lying on the left of the line $8x+1=0$. **[IIT - 1999]**

30. Let $b\neq 0$ and for $j=0, 1, 2 .... n$, let S_j be the area of the region bounded by the y-axis and the curve $xe^{ay}=\sin by$, $\frac{j\pi}{b}\le y\le \frac{(j+1)\pi}{b}$. Show that $S_0, S_1, S_2, ... S_n$ are in geometric progression. Also, find their sum for $a=-1$ and $b=\pi$. **[IIT - 2001]**

31. Find the curve passing through (2, 0) and having slope of tangent at any point P(x, y) as $\frac{(x+1)^2+y-3}{x+1}$. Find also the area enclosed by the curve and x-axis in the IV quadrant. **[IIT - 2004]**

32. Find the are a bounded by the curves $x^2=y$, $x^2=-y$ and $y^2=4x-3$. **[IIT - 2005]**

33. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix}=\begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$ f(x) is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of y = f(x) with x-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by f(x) and chord AB. **[IIT - 2005]**

Comperehension (Q. No. 33–35) **[IIT - 2008]**

Consider the functions defined implicitly by the equation $y^3-3y+x=0$ on various intervals in the real line.

If $x\in(-\infty,-2)\cup(2,\infty)$, the equation implicitly defines a unique real valued differentiable function $y=f(x)$.

If $x\in(-2, 2)$, the equation implicity defines a unique real valued differentiable function $y=g(x)$ satisfying $g(0)=0$.

34. If $f(-10\sqrt{2})=2\sqrt{2}$, then $f''(-10\sqrt{2})=$

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ (B) $-\frac{4\sqrt{2}}{7^3 3^2}$

(C) $\frac{4\sqrt{2}}{7^3 3}$ (D) $-\frac{4\sqrt{2}}{7^3 3}$

35. The area of the region bounded by the curve y = f(x), the x–axis, and the lines x= a and x = b, where $-\infty<a<b<-2$, is

(A) $\int_a^b \frac{x}{3(f(x))^2-1}dx+b\,f(b)-a\,f(a)$

(B) $-\int_a^b \frac{x}{3(f(x))^2-1}dx+b\,f(b)-a\,f(a)$

(C) $\int_a^b \frac{x}{3(f(x))^2-1}dx-b\,f(b)+a\,f(a)$

(D) $-\int_a^b \frac{x}{3(f(x))^2-1}dx-b\,f(b)+a\,f(a)$

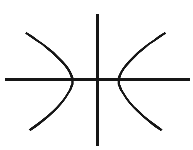
36. $\int_{-1}^{1} g'(x)dx=$

(A) 2g(–1) (B) 0
(C) –2g(1) (D) 2g(1)

ANSWERS

PRACTICE PROBLEMS—A

1. (i)



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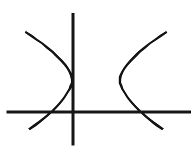

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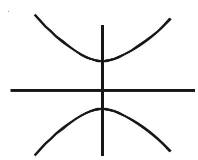

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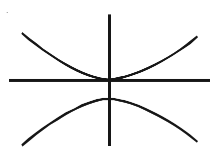

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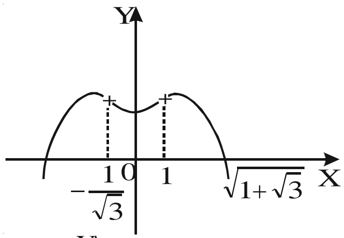

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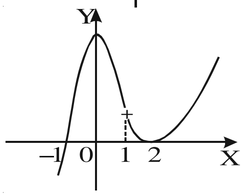

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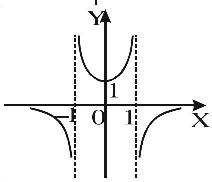

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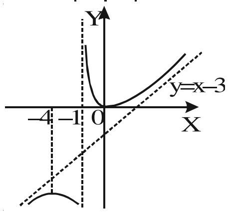

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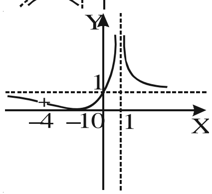

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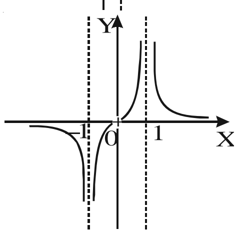
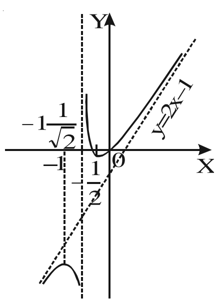


(ii)



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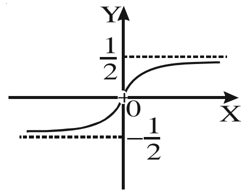

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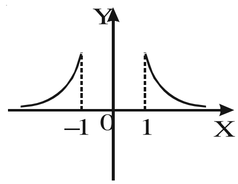

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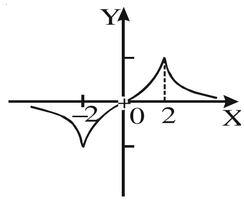

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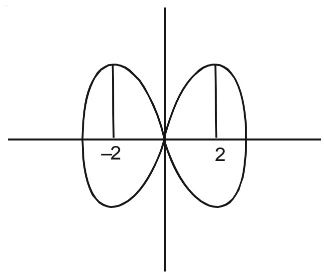

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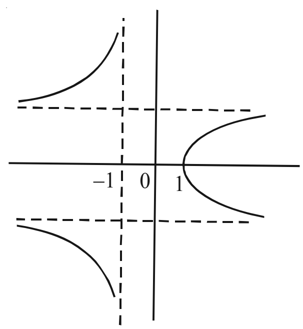

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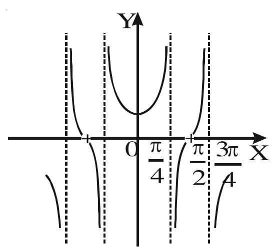

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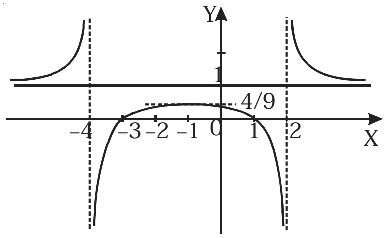
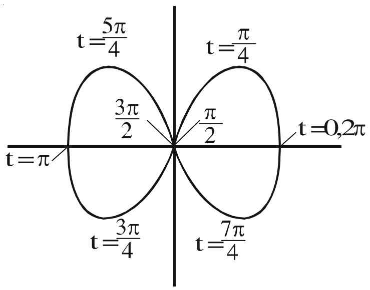


8. (i)



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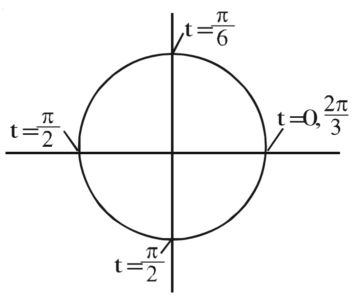
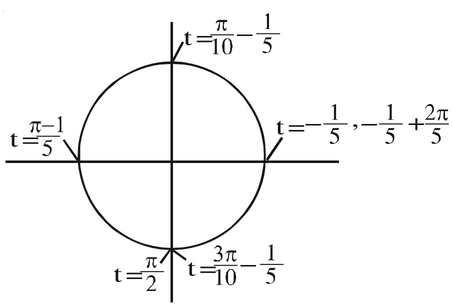
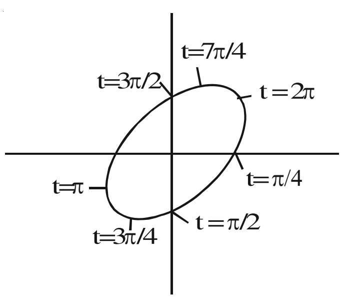


(iii)



(iv)



CONCEPT PROBLEMS—A

1. 3
2. $f(x) \geq 0$ for all $x \in [a, b]$, if $a < b$; $f(x) \leq 0$ for all $x \in [b, a]$ if $a > b$.
3. (i), (iii), (iv), (vi)
4. (i) $\int_0^3 (x^2 + 1)dx$ (ii) $\int_a^b (e^x + 2)dx$
6. $10\frac{2}{3}$
7. (i) $\frac{1}{6}$ (ii) 1
8. $\frac{1}{2}$
9. $20\frac{5}{6}$ cm

PRACTICE PROBLEMS—B

10. $\frac{3}{14}$
11. $\frac{5}{3}\sqrt{2}$.
12. 4
14. $\frac{3}{e}(e^3 - 4)$

PRACTICE PROBLEMS—C

1. $\frac{32}{3}$
2. 1
3. $9 - 8 \ln 2$
4. $[\ell og_2 3, \infty)$
5. 2/3
6. 125/2
7. $3.5 - 12\, \ell n\frac{4}{3}$
8. $\frac{9}{2} - \sqrt{2}\, \ln(3 + \sqrt{8})$
10. 1.18
11. (a) 1800 ft (b) $\frac{3T^2}{2} - \frac{T^3}{60}$ ft
13. 9
14. $2\left(\frac{\pi}{2} - \frac{1}{3}\right)$

CONCEPT PROBLEMS—B

1. (a) the integral gives the net signed area (a) the integral gives the area bounded between the curves.
2. $\frac{25}{12}$

3. (i) 121.5 (ii) 9

(iii) 7/4 (iv) $\frac{16\sqrt{3}}{15}$

4. 11/3 **5.** $\sqrt{2}-1$

6. $\sqrt{2}+4$

PRACTICE PROBLEMS—D

7. (i) $\frac{6-\pi}{3\pi}$ (ii) $\frac{33}{2}-8\ell n2$

(iii) $1.5-\ell n2$ (iv) $\frac{8}{9}$

8. $\frac{4-3\ln 3}{2}$ **9.** $\frac{3-2\ln 2-2\ln^2 2}{16}$

10. 2 **11.** $2\pi-\frac{8}{5}$

12. $2(1-\ell n2)$ **13.** $0<m<1; m-\ell n m-1$

14. 32/27 **15.** $2-\sin^{-1}(2/\sqrt{5})$

16. $\frac{1}{3}+\frac{2}{\pi}$ **17.** $\frac{e^{\pi}+1}{2(e^{\pi}-1)}$

PRACTICE PROBLEMS—E

1. $\frac{4}{3}$ **2.** 1/5

3. $e-(1/e)+\frac{10}{3}$ **4.** 9

5. 10/3 **6.** $\ln(3+2\sqrt{2})-\frac{\pi}{2}$.

7. $\frac{1}{2}$

8. (i) $\frac{331}{4}$ (ii) $2\ln 1.6-\sin^{-1}0.6$.

9. (a) $\frac{13}{6}$ (b) $\frac{10}{3}$

(c) $\frac{16}{3}$

10. $\frac{\pi}{3}-\frac{\sqrt{3}}{2}-\frac{2}{3}$ **11.** $\frac{2(3\pi-2)}{3}$

PRACTICE PROBLEMS—F

1. $2\sqrt{2}a^2\sin^{-1}\frac{1}{\sqrt{3}}$

2. $a^2\left[\frac{\pi}{6}-\frac{\sqrt{2}}{8}\ln(\sqrt{3}+\sqrt{2})\right]$;

$a^2\left[\frac{\pi}{6}-\frac{\sqrt{2}}{8}\ln(\sqrt{3}+\sqrt{2})\right]$;

$a^2\left[\frac{2\pi}{3}+\frac{\sqrt{2}}{4}\ln(\sqrt{3}+\sqrt{2})\right]$

3. $\frac{28}{3}$ **4.** $\left(\frac{1}{2},\frac{5}{4}\right)$

5. $\tan^{-1}\frac{27}{2}$, $\tan^{-1}\frac{27}{4}$ **6.** 3/4

8. The parabola partitions the square into two parts whose areas are related as 1 : 2.

9. $S=b$ for a $\sqrt{8/3b-1}$; the problem has a solution for $b\in(0, 8/3)$.

14. $S_1=S_3=\pi-\frac{\sqrt{2}}{2}\ln 3-2\sin^{-1}\sqrt{\frac{2}{3}}\approx 0.46$; $S_2=2(\pi-S_1)$.

15. $c=\frac{1}{4}$ or $\frac{49}{4}$

PRACTICE PROBLEMS—G

1. $-\pi/6, \pi/3$

2. (i) 4/3 (ii) $m=2\sqrt[3]{4}$

3. $\{-1, \sqrt[3]{8-\sqrt{17}}\}$ **4.** $\{1/4, 49/4\}$.

5. $\{-\pi/18, \pi/9\}$ **6.** $\left\{-\frac{\pi}{30},\frac{\pi}{6}\right\}$

7. $\frac{9}{4}\left(1-\frac{1}{\sqrt[3]{4}}\right)$

8. $a=\in\left(0,\frac{4}{3}\right)$; $b=\frac{16}{9a^2}-1$

9. 3/4 **10.** $k=0, A=\frac{20\sqrt{5}}{3}$

11. $\frac{32}{3}$ **12.** $a = 1$

13. For $a = 1/2$ the area has the least value and for $a = 0$ it has its greatest value.

14. $a = 0$

PRACTICE PROBLEMS—H

1. $\frac{16}{3}$ **2.** $3\pi/2 - 3$

4. $\frac{8}{15}$ **5.** $3:5$

6. $\pi\sqrt{2}$ **7.** $73\frac{1}{7}$

8. 1 (the figure consists of two equal portions).

9. $\frac{3}{4}\pi$ **10.** $\frac{\pi a^2}{8}$

11. $8\left(\sqrt{1+\frac{2}{3}\sqrt{3}} - \tan^{-1}\sqrt{1+\frac{2}{3}\sqrt{3}}\right)$

12. $\frac{\pi}{4}$. **13.** 4π

14. 36 **15.** 4/3

17. πa^2 **18.** $3a^2\pi$.

PRACTICE PROBLEMS—I

1. $\frac{8a^2}{15}$ **2.** $\frac{16}{35}$

3. $\frac{\pi a^2}{4}$ **4.** $a^2\left\{\frac{9\sqrt{3}}{8} - \frac{1}{4}\pi\right\}$

PRACTICE PROBLEMS—J

1. 3/2. **2.** $3 - 4\ell n2$

3. $\left(\frac{3\pi-8}{6}\right)a^2$ **4.** $\frac{4}{3}$

5. $2(4-\pi)$ **6.** $1 - \pi/4$

7. $\frac{5(3+\sqrt{5})}{12}$ **8.** $\frac{7}{3}$

9. $2 - \pi/3$ **10.** $4(7 - 3\, ln2)$

OBJECTIVE EXERCISES

1. C	**2.** D	**3.** D
4. C	**5.** B	**6.** A
7. C	**8.** A	**9.** A
10. B	**11.** C	**12.** C
13. A	**14.** D	**15.** C
16. A	**17.** A	**18.** C
19. A	**20.** B	**21.** D
22. A	**23.** D	**24.** A
25. B	**26.** A	**27.** A
28. C	**29.** C	**30.** A
31. B	**32.** C	**33.** A
34. C	**35.** C	**36.** C
37. B	**38.** C	**39.** B
40. B	**41.** D	**42.** C
43. B	**44.** A	**45.** A
46. C	**47.** A	**48.** B
49. A	**50.** D	**51.** A,C,D
52. A,B	**53.** B,C	**54.** A,C,D
55. A	**56.** A,C,D	**57.** A,B,D
58. B,C,D	**59.** B,C,D	**60.** A,D
61. B,C	**62.** B,C,D	**63.** B,C
64. B,C,D	**65.** C,D	**66.** B,C,D
67. C,D	**68.** A,B,C	**69.** A,B,C
70. A,B,C	**71.** A,B,C	**72.** B
73. A	**74.** C	**75.** A
76. D	**77.** A	**78.** C
79. A	**80.** C	**81.** D
82. C	**83.** B	**84.** B
85. A	**86.** D	**87.** A
88. A	**89.** A	**90.** D
91. C	**92.** A	**93.** B
94. A	**95.** D	**96.** A

97. $A \to Q$. $B \to P$. $C \to S$. $D \to R$.

98. A–(Q); B –(Q); C – (R); D–(Q), (S)

99. A–(S); B –(R); C – (P); D–(Q)

100. A–P; B –(T); C – (T); D–(R)

101. $A \to R$, $B \to P$, $C \to S$, $D \to Q$

REVIEW EXERCISES for JEE ADVANCED

1. (i) $4-\ell n3$ (ii) 3π
(iii) 16 (iv) $11-6\,\ell n3$

2. $ln\ 9/4$

3. 4 sq. units, $a=2\sqrt{2}$

4. $\left(\dfrac{3}{\log_e 2}-\dfrac{4}{3}\right)$

5. $\dfrac{3\pi+2}{\pi-2}$

6. (i) $\dfrac{9}{8}$ (ii) $4\log_5\dfrac{e^4}{27}$
(iii) $\dfrac{53}{15}$

7. $\dfrac{11}{8}$ sq. units

8. $\dfrac{\pi}{2};\dfrac{\pi-1}{\pi+1}$

9. $a=9$

10. $y=\left[x-\sin^{-1}\left(1-\dfrac{\sqrt{2}}{4}\right)\right]\dfrac{4}{\sqrt{2}(4-\sqrt{2})}+\dfrac{\sqrt{2}}{4}\sqrt{8\sqrt{2}-2}$.

11. $a=\sqrt[4]{3}$

12. $2\left(\dfrac{\sqrt{2}}{3}+\dfrac{9\pi}{4}-\dfrac{9}{2}\sin^{-1}\dfrac{1}{3}\right)$

13. $(3-e)$

14. $\dfrac{4}{3\sqrt{3}}(\sqrt{3}-1)$

15. $\dfrac{7}{6}$

16. 7/12

17. $S(-1)=125/6$, min. $S(k)=S(2)=32/3$.

18. $\dfrac{1}{2}(4-\pi)$

19. $\dfrac{2\sqrt{2}-4}{\pi}$

20. $\dfrac{\pi^2-\alpha^2}{2}+\alpha\pi(\pi-\alpha)-\dfrac{\pi^3-\alpha^3}{3}$, $\alpha=\dfrac{2\pi+1+\sqrt{4\pi+1}}{2}$

21. (i) $\dfrac{3}{7}$ (ii) $\dfrac{3}{8}$
(iii) $y=\dfrac{3}{2\sqrt{1+3x}}$ (iv) $\dfrac{3}{7}$

22. $\left(\pi-\dfrac{2}{3}\right)a^2$

23. $\left(\dfrac{1}{2},\dfrac{5}{4}\right)$

24. 23/6

25. $c=-\pi/6$ or $\pi/3$

26. $\pi-\tan^{-1}\dfrac{2\sqrt{2}}{3\pi}$; $\pi-\tan^{-1}\dfrac{4\sqrt{2}}{3\pi}$

27. $p=-1$

28. $\tan^{-1}\dfrac{27}{2}$, $\tan^{-1}\dfrac{27}{4}$

29. 1

30. 19

31. $\{\pi/3\}$

32. $b=1$

33. $\dfrac{S_n}{S}=\dfrac{3\pi-8}{3\pi}$, where S is the area cut off by the parabola from the semicircle.

34. 4π

35. $4\pi a^2$

TARGET EXERCISES for JEE ADVANCED

1. $\min\limits_{x_0\in[1/2,1]} S(x_0)=S\left(\dfrac{4}{5}\right)=\sqrt[3]{\dfrac{5}{4}\cdot\dfrac{48}{25}}$

2. For $a=1$ the area assumes the greatest value and for $a=1/2$ it assumes its least value.

3. $4a^2/3$

4. $1+e^{-2}$

6. $1-3e^{-2}$

7. $\dfrac{1}{2}(1-e^{1/2})$

8. $f(x)=x^2+1;\ y=\pm 2x;\ A=\dfrac{2}{3}$

9. $y=2x/3$

10. 5 : 11

11. $\alpha=\pi/3$

12. πa^2

13. $\{(12-2\sqrt{21})/5,\ 8\}$

14. $b=1/8$ $A_{min}=4\sqrt{3}$

15. 1/2

16. $2-(\pi/2)$

17. $\dfrac{8}{3}-\dfrac{8}{3}(3-2\sqrt{2})^{3/2}-(2\sqrt{2}-2)^2$

18. $y=2xe^x$, $2-\dfrac{4}{e}$

19. $\dfrac{64a}{9}(3-4a)^3$, max at $a=\dfrac{3}{16}$

20. $\frac{32}{3}$

21. $\frac{3\pi}{4} - \frac{1}{6}$

22. (a) $\frac{8}{3}$ (b) 6

23. 3e sq. units

24. $\frac{72\sqrt{3}}{5}$

25. $g(x) = \begin{cases} -x-1, & x \le -5/2 \\ 4+x, & -5/2 < x \le -2 \\ 2, & -2 < x \le -1 \\ 1-x, & -1 < x \le -1/2 \\ 1+x, & x > -1/2 \end{cases}$ and area $= \frac{101}{4}$ sq. units

26. $\frac{32}{3} - 4\sqrt{3} + \frac{8\pi}{3}$

27. $3(\sqrt{3}-1)$ sq. units

28. $\left[2 + \ln\left(\frac{4}{3\sqrt{3}}\right) - \frac{1}{e}\right]$ sq. units

29. $\left(\pi - \frac{2}{3}\right)$ sq. units

30. 2 sq. units.

31. 5. sq. units

32. $\left(2 - \frac{\pi}{2}\right)$ sq. units

33. $y = 16x^2/9$

34. $\left[\frac{4}{\ln 2}(1-2^{-1/2}) - 1\right]$ sq. units

35. $\frac{11}{3}$ sq. units.

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. C
2. B
3. D
4. D
5. A
6. B
7. C
8. B
9. BD
10. ABD
11. 9/8 sq. units
13. $a = 2\sqrt{2}$
14. log 2 units
15. $\frac{5\pi - 2}{4}$ sq. units
16. $\pi + 1/3$ sq. units.
17. $4 + 25\sin^{-1} 4/5$
18. 1/2 [log 2 – 1/2] sq. units
19. $\frac{e^2-5}{4e}$
20. $\frac{4-\sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}$
21. $\left(\pi - \frac{2}{3}\right)$ sq. units
22. 4 : 121
23. $\frac{16\sqrt{2}-20}{3}$
25. $b = 1$
26. $2 - \sqrt{2}$
27. 17/27 sq. units
28. $f(x) = -x^2 + x^3$
29. $\frac{257}{192}$ sq. units
30. $\frac{(e+1)\pi(e^{n+1}-1)}{(\pi^2+1)(e-1)}$ sq. units
31. 4/3 sq. units
32. 1/3 sq. units
33. 125/3 sq. units.
34. B
35. A
36. D

For example, the differential equation :

$$\frac{dy}{dx} = \sin 2x + \cos x \text{ is of order 1,}$$

$$\frac{d^3y}{dx^3} + 2\frac{dy}{dx} + y = e^x \text{ is of order 3.}$$

CHAPTER

4

DIFFERENTIAL EQUATIONS

4.1 INTRODUCTION

An equation involving independent and dependent variables and the derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation. The following relations are some of the examples of differential equations:

(i) $\frac{dy}{dx} = \sin 2x + \cos x$

(ii) $k\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

(iii) $\frac{dy}{dx} + \frac{dz}{dx} = y + z$

(iv) $\frac{\partial^3 v}{\partial t^3} = k\left(\frac{\partial^2 v}{\partial x^2}\right)^2$

There are two main classes of differential equations

(i) Ordinary differential equations

A differential equation which involves derivatives with respect to a single independent variable is known as an ordinary differential equation.

For example, $\frac{dy}{dx} + xy = \sin x$,

$$\frac{d^3y}{dx^3} + 2\frac{dy}{dx} + y = e^x.$$

(ii) Partial differential equations

A differential equation which contains two or more independent variables and partial derivatives with respect to them is called a partial differential equation.

For example, $y^2\frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial y^2} = x$,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

Classification of differential equations

Differential equations are classified into order and degree.

Order of a differential equation

The order of the highest order derivative involved in a differential equation is called the order of a differential equation.
For example, the differential equation :

$$\frac{dy}{dx} = \sin 2x + \cos x \text{ is of order 1,}$$

$$\frac{d^3y}{dx^3} + 2\frac{dy}{dx} + y = e^x \text{ is of order 3.}$$

Degree of a Differential Equation

The degree of differential equation is the degree of the highest order derivative present in the equation, after the differential equation has been made free from the radicals and fractions as far as the derivatives are concerned.
To get the degree of the differential equation, we

first try to convert it into the following form :

$$f_1(x, y)\left[\frac{d^m y}{dx^m}\right]^{n_1} + f_2(x, y)\left[\frac{d^{m-1}y}{dx^{m-1}}\right]^{n_2} + \ldots.$$

$$\ldots. + f_k(x, y)\left[\frac{dy}{dx}\right]^{n_k} = 0,$$

where m, $n_1, n_2, \ldots, n_k$ are positive integers.

The degree of the differential equation is n_1. Futher, the equation is of order m.

For example, the differential equation

$$\left(y - x\frac{dy}{dx}\right)^2 = k^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] \text{ is of degree 2.}$$

Note that in the differential equation

$$e^{(d^3y/dx^3)^2} + x\frac{d^2y}{dx^2} + y = 0$$

order is three but degree does not apply.

Example 1. Find the order and degree of the following differential equations

(i) $(y'')^5 + 2(y')^6 + 3x^2 - y = 0$

(ii) $(y''')^3 + \ln(y'' - xy') = 0$

(iii) $\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right)$

(iv) $\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx} + 3}$

Solution

(i) Clearly, the order and degree of the given differential equation are 2 and 5 respectively.

(ii) The order of the given differential equation is 3 and degree is not defined as it cannot be written as a polynomial in derivatives.

(iii) The order of the given differential equation is 2 and degree is not defined as it cannot be written as a polynomial equation in derivatives.

(iv) The order of the differential equation is 2 and for degree we rewrite the given differential equation as

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx} + 3\right)^2$$

Hence the degree of the given differential equation is 3.

Example 2. Find the order and degree of following differential equations.

(i) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$ (ii) $y = e^{\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\right)}$

(iii) $\sin\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\right) = y$ (iv) $\cos\frac{d^2y}{dx^2} - xy = \frac{dy}{dx}$.

Solution

(i) $\left(\frac{d^2y}{dx^2}\right)^4 = y + \left(\frac{dy}{dx}\right)^6$

order = 2, degree = 4

(ii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \ell ny$

order = 2, degree = 1

(iii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin^{-1} y$

order = 2, degree = 1

(iv) $\cos\frac{d^2y}{dx^2} - xy = \frac{dy}{dx}$

This equation cannot be expressed as a polynomial in differential coefficients, so degree is not applicable but order is 3.

Example 3. Find the degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^{2/3} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$.

Solution $\left(\frac{d^3y}{dx^3}\right)^{2/3} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$

$$\Rightarrow \left(\frac{d^3y}{dx^3}\right)^2 = \left[3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4\right]^3$$

It is a differential equation of degree 2.

Linear and Non linear Differential Equations

A differential equation in which the dependent variables and all its derivatives present occur in the first degree only and no products of dependent variables and/or derivatives occur, is known as a linear differential equation. A differential equation which is not linear is called a non linear differential equation.

For example, the differential equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \text{ is linear, while}$$

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + x = 0 \text{ is non linear.}$$

Concept Problems A

1. Find the order and degree of the following differential equations :

(i) $\frac{dy}{dx} = 1 + x + y$

(ii) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$

(iii) $\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$

(iv) $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}}$

2. Find the order and degree of the following differential equations :

(i) $\frac{dy}{dx} + xy = \cot x.$

(ii) $e^{\left(\frac{dy}{dx} - \frac{d^3y}{dx^3}\right)} = \ln\left(\frac{d^5y}{dx^5} + 1\right)$

3. Find the order and degree of the following differential equations :

(i) $\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^4\right\}^{\frac{5}{3}}$

(ii) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x\,\ell n\frac{d^2y}{dx^2}$

(iii) $\frac{d^2y}{dx^2} = x\,\ell n\left(\frac{dy}{dx}\right)$

(iv) $\left(\frac{d^4y}{dx^4}\right)^3 + 4\left(\frac{dy}{dx}\right)^7 + 6y = 5\cos 3x.$

4.2 FORMATION OF A DIFFERENTIAL EQUATION

Differential equations are used to represent a family of curves. Suppose that a family of curves is represented using a set of n arbitrary constants, then the differential equation corresponding to that family is a relation between x, y and derivatives of y upto order n, not containing any of the n arbitrary constants.
Given a relation between the variables x, y and n arbitrary constants $C_1, C_2, \ldots C_n$, if we differentiate n times in succession with respect to x, we have altogether n + 1 equations between which the n arbitrary constants can be eliminated. The result is a differential equation of the nth order. From this point of view the original equation is called the primitive.
For example, from the primitive $y^2 = 4ax + C$, where C is an arbitrary constant, and a is a given constant, on differentiation, we deduce that $y\frac{dy}{dx} = 2a$.
The equations obtained by varying the arbitrary constants in the primitive represent a certain system or family of curves ; the differential equation (in which these constants do not appear) expresses some property common to all these curves.

In the above example, the primitive represents a system of equal parabolas having their axes coincident with the axis of x, but their vertices at different points of it. The differential equation expresses a property common to all these curves, i.e. the subnormal has a given constant value 2a.

Note: If arbitrary constants appear in addition, subtraction, multiplication or division, then we can club them to reduce into one new arbitrary constant. Hence, the differential equation corresponding to a family of curves will have order exactly same as number of essential arbitrary constants in the equation of curve.
On the other hand, it is evident that a differential equation of the nth order cannot have more than n arbitrary constants in its solution; for, if it had, say n + 1, on eliminating them there would appear, not an equation of the nth order, but one of the (n + 1)th order.
Finally, the differential equation corresponding to a family of curves is obtained by using the following steps:
(a) Identify the number of essential arbitrary constants (say n) in the equation of the curve.
(b) Differentiate the equation n times.
(c) Eliminate the arbitrary constants from the equation of curve and n additional equations obtained in step (b) above.

We now work out in detail some examples to illustrate the method of forming differential equations.

Example 1. If the primitive be $x\cos\alpha + y\sin\alpha = a$, where α is arbitrary, find the differential equation.

Solution $x\cos\alpha + y\sin\alpha = a$

$$\cos\alpha + \frac{dy}{dx}\sin\alpha = 0.$$

These give

$$\left(y - x\frac{dy}{dx}\right)\sin\alpha = a, \qquad ...(1)$$

$$\left(y - x\frac{dy}{dx}\right)\cos\alpha = -a\frac{dy}{dx} \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$\left(y - x\frac{dy}{dx}\right)^2 = a^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}.$$

Example 2. Find the differential equation of the family of curves $y = c_1e^{2x} + c_2e^{-2x}$, where c_1 and c_2 are arbitrary constants.

Solution Given $y = c_1e^{2x} + c_2e^{-2x}$ ...(1)

Differentiating (1) twice with respect to x, we get

$$\frac{dy}{dx} = 2c_1e^{2x} - 2c_2e^{-2x} \qquad ...(2)$$

$$\frac{d^2y}{dx^2} = 4c_1e^{2x} + 4c_2e^{-2x} = 4(c_1e^{2x} + c_2e^{-2x}) \qquad ...(3)$$

From (1) and (3), we obtain

$$\frac{d^2y}{dx^2} - 4y = 0 \qquad ...(4)$$

Thus, the two arbitrary constants c_1 and c_2 have been eliminated from (1) and (2). Hence, (4) is the required equation of the family of curves given by (1).

Example 3. Obtain the differential equation of the family of curves represented by $y = Ae^x + Be^{-x} + x^2$, where A and B are parameters.

Solution $y = Ae^x + Be^{-x} + x^2$

Differentiating with respect to x,

$$\frac{dy}{dx} = Ae^x - Be^{-x} + 2x$$

Again differentiating with respect to x,

$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x} + 2$$

$$\therefore \quad \frac{d^2y}{dx^2} = -x^2 + y + 2$$

Thus, $\frac{d^2y}{dx^2} - y + x^2 - 2 = 0$, is the required differential equation of the family of curves.

Example 4. Find the differential equation corresponding to the family of curves $y = c(x - c)^2$, where c is an arbitrary constant.

Solution Given $y = c(x - c)^2$ ...(1)

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 2c(x - c) \qquad ...(2)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4c^2(-c)^2 \qquad ...(3)$$

Dividing (3) by (1), we obtain

$$\frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 4c \Rightarrow c = \frac{1}{4y}\left(\frac{dy}{dx}\right)^2$$

Substituting this value of c in (2), we get

$$\frac{dy}{dx} = 2.\frac{1}{4y}\left(\frac{dy}{dx}\right)^2\left[x - \frac{1}{4y}\left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow 2y = \frac{dy}{dx}\left[x - \frac{1}{4y}\left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow 8y^2 = 4xy\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3$$

which is the required differential equation for the family of curves (1).

Example 5. By the elimination of the parameters h and k, find the differential equation of which $(x - h)^2 + (y - k)^2 = a^2$, is a solution.

Solution Three relations are necessary to eliminate two constants . Thus, besides the given relation, we require two more and they will be obtained by differentiating the given relation twice successively.

Thus, we have $(x - h) + (y - k)\frac{dy}{dx} = 0$ ...(1)

$$1 + (y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad ...(2)$$

From (1) and (2), we obtain

$$y-k=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

$$\text{and } x-h=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}}.$$

Substituting these values in the given relation, we obtain

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2.$$

which is the required differential equation.

Example 6. Form the differential equation that represent all parabolas each of which has a latus rectum 4a and whose axes are parallel to x-axis.

Solution Equation of the family of such parabolas is

$(y-k)^2=4a(x-h)$, ...(1)

where h and k are arbitrary constant.

Differentiating, $(y-k)\frac{dy}{dx}=2a.$ ...(2)

Differentiating again, $(y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)=0$ ...(3)

Putting value of $y-k$ from (2) in (3), we get

$$2a\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0,$$

which is the required differential equation.

Example 7. Find the differential equation of the system of hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{1}=1.$

Solution Differentiating this equation with respect to x we get $\frac{2x}{a^2}-2yy'=0$ or $\frac{x}{a^2}=yy'.$

We multiply both sides by x, then $x^2/a^2 = xyy'$. Substituting into the equation of the family we find that $xyy' - y^2 = 1$.

Forming differential equation in geometrical problems

When forming differential equations in geometrical problems, we can frequently make use of the geometrical meaning of the derivative as the tangent of an angle formed by the tangent line to the curve in the positive x-direction. In many cases this makes it possible straightway to establish a relationship between the ordinate y of the desired curve, its abscissa x, and the tangent of the angle of the tangent line y′. In other instances, use is made of the geometrical significance of the definite integral as the area of a curvilinear trapezoid. In this case, by hypothesis we have a simple integral equation (since the desired function is under the sign of the integral); however, we can readily pass to a differential equation by differentiating both sides.

Example 8. Set up the differential equation of the family of straight lines lying at a distance equal to unity away from the origin of coordinates.

Solution We start from the normal equation of the straight line $x\cos\alpha+y\sin\alpha-1=0$, ...(1)

where α is a parameter.

Differentiating (1) with respect to x we find that

$\cos\alpha+y'\sin\alpha=0$,

$\Rightarrow$ $y'=-\cot\alpha$, consequently,

$$\sin\alpha=\frac{1}{\sqrt{1+y'^2}},\ \cos\alpha=-\frac{y'}{\sqrt{1+y'^2}}.$$

On substituting sin α and cos α into (1) we get

$$\frac{-xy'}{\sqrt{1+y'^2}}+\frac{y}{\sqrt{1+y'^2}}-1=0$$

$$\Rightarrow\ y=xy'+\sqrt{1+y'^2}.$$

Example 9. Form a differential equation of family of circles touching x-axis at the origin ?

Solution Equation of family of circles touching x-axis at the origin is $x^2+y^2+\lambda y=0$ ...(1),

where λ is parameter

$$2x+2y\frac{dy}{dx}+\lambda\frac{dy}{dx}=0 \quad ...(2)$$

Eliminating ‘λ’ from (1) and (2)

$$\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$$

which is required differential equation.

Example 10. Form the differential equation representing all tangents to the parabola $y^2 = 2x$. State the order and degree of this differential equation. Also indicate whether the equation is linear or not .

Solution The equation of a tangent is ,

$$ty = x + \frac{1}{2}t^2 \text{ where } \frac{1}{t} = \frac{dy}{dx}$$

$$\Rightarrow \frac{y}{t} = \frac{x}{t^2} + \frac{1}{2}$$

$$\Rightarrow y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + \frac{1}{2}$$

$$\Rightarrow 2x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} + 1 = 0$$

The order is one and degree is two. The differential equation is nonlinear.

Physical origins of differential equations

In the above discussions we have seen how differential equations arise through geometrical considerations and, now with the help of some examples, we shall look for the physical origins of differential equations. For example :

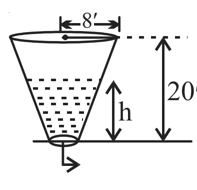
(i) We know that freely falling objects, close to the surface of the earth, accelerate at a constant rate g. Thus, if we assume that the upward direction is positive, the equation $\frac{d^2y}{dt^2} = -g$ is the differential equation governing the vertical distance that the falling body travels. The negative sign is used since the weight of the body is a force directed opposite to the positive direction.

(ii) The rate at which a population P expands is proportional to the population which is present at any time t. Roughly speaking, the more people there are, the greater will be the increase in population. Thus, one model for population growth is given by the differential equation $\frac{dP}{dt} = kP$ where k is the constant of proportionality. Since we also expect the population to expand, we must have $dP/dt > 0$, and thus $k > 0$.

(iii) Newton's law of cooling states that the time rate at which a body cools in proportional to the difference between the temperature of the body and the temperature of the surrounding medium. If T(t) denotes the temperature of the body at any time t, and T_0 is the constant temperature of the outside medium, then $\frac{dT}{dt} = k(T - T_0)$ where k is the usual constant of proportionality. However, in this case T(t) is decreasing and so we require $k < 0$.

Concept Problems B

1. Find the differential equations of the family of curves
   (i) $cy^2 + 4y = 2x^2$
   (ii) $xy = c_1e^x - c_2e^{-x} + x^2$
   (iii) $r = c(1 + \cos\theta)$
2. Find the differential equations of the following families of curves :
   (i) $y = e^x(ax + b)$.
   (ii) $ax^2 + by^2 = 1$
   (iii) $y = c_1x + c_2x^2$
   (iv) $xy = ae^x + be^{-x} + x^2$
3. Obtain a differential equation of the family of curves y = a sin (bx + c) where a and c being arbitrary constant.
4. Show that the primitive $y = mx + \frac{a}{m}$, where m is arbitrary, leads to $x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + a = 0$.
5. Show the differential equation of the system of parabolas $y^2 = 4a(x - b)$ is given by
$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$
6. A conical tank loses water out of an orifice at its bottom. If the cross-sectional area of the orifice is 1/4 ft^2, obtain the differential equation representing the height h of water to any time.



Practice Problems A

7. Find the differential equations of the following families of curves :

(i) $y = C_1 x + \frac{C_2}{x} + C_3$

(ii) $y = C_1 e^{3x} + C_2 e^{2x} + C_3 e^{x}$

(iii) $y = (\sin^{-1} x)^2 + A \sin^{-1} x + B$

8. Obtain a differential equation of all straight lines which are at a fixed distance 'p' from the origin.

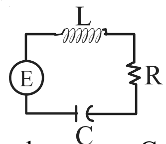
9. Find a differential equation of circles passing through the points of intersection of unit circle with centre at origin and the line bisecting the first quadrant.

10. Prove that the differential equation of all hyperbolas which pass through the origin, and have their asymptotes parallel to the coordinate axes, is

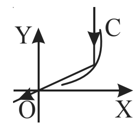
$$xy\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} = 0.$$

11. Form a differential equation of family of parabolas with focus origin and axis of symmetry along the x-axis.

12. A series circuit contains an inductor, resistor and capacitor. Determine the differential equation for the charge q(t).



13. Light strikes a plane curve C such that all beams parallel to the y axis are reflected to a single point O, obtain the differential equation for the function y = f(x) describing the shape of the curve.



14. Obtain the differential equation for the velocity ν of a body of mass m falling vertically downward through a medium offering a resistance proportional to the square of the instantaneous velocity.

4.3 SOLUTION OF A DIFFERENTIAL EQUATION

The process of passing from a given differential equation to the general relation between the variables which it implies, is called solving, or integrating the equation. The solution or the integral of a differential equation is, therefore, a relation between dependent and independent variables (free from derivatives) such that it satisfies the given differential equation.

For example, $y = ce^{2x}$ is a solution of the differential equation $dy/dx - 2y = 0$, because $dy/dx = 2ce^{2x}$ and $y = ce^{2x}$ satisfy the given differential equation.

Note: The solution of the differential equation is also called its primitive, because the differential equation can be regarded as a relation derived from it.

There can be three types of solution of a differential equation:

(i) **General solution (or complete integral or complete primitive) :** A relation in x and y satisfying a given differential equation and involving exactly the same number of arbitrary constants as the order of the differential equation.

(ii) **Particular Solution :** A solution obtained by assigning values to one or more of the arbitrary constants found in the general solution.
For example, $y = c_1 e^{2x} + c_2 e^{-2x}$ is a general solution and if we take $c_1 = 4$ and $c_2 = 0$, we get $y = 4e^{2x}$ as the particular solution.

(iii) **Singular Solution :** A solution which cannot be obtained from the general solution but still is a solution of the given differential equation is called a singular solution. Geometrically, singular solution acts as an envelope to the general solution.
For example, $x^2 + y^2 = a^2$ is the singular solution of

$$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

and is such that it cannot be obtained from the general solution $y = cx + a\sqrt{1+c^2}$ by assigning values to the arbitrary constant.

Example 1. Show that the function $2x^2y + e^{-x^2} = 0$ is a solution of the differential equation $xy' + 2(1+x^2)y = 0$.

Solution We have $2x^2y + e^{-x^2} = 0$ ...(1)

Differentiating equation (1) w.r.t. x, we have

$$4xy + 2x^2y' - 2xe^{-x^2} = 0$$

$\Rightarrow 2y + xy' - e^{-x^2} = 0$

$\Rightarrow 2y + xy' + 2x^2y = 0$ [from (1)]

$\Rightarrow xy' + 2(1+x^2)y = 0$

which is the desired result.

Example 2. Show that $y = ae^x + be^{2x} + ce^{-3x}$ is a solution of the equation $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$.

Solution We have $y = ae^x + be^{2x} + ce^{-3x}$. ...(1)

$\Rightarrow y_1 = ae^x + 2be^{2x} - 3ce^{-3x}$ ...(2)

$(2) - (1) \Rightarrow y_1 - y = be^{2x} - 4ce^{-3x}$ ...(3)

$\Rightarrow y_2 - y_1 = 2be^{2x} + 12ce^{-3x}$ ...(4)

$(4) - 2\times(3) \Rightarrow y_2 - y_1 - 2(y_1 - y) = 20\,ce^{-3x}$

$\Rightarrow y_2 - 3y_1 + 2y = 20ce^{-3x}$ ...(5)

$\Rightarrow y_3 - 3y_2 + 2y_1 = -60ce^{-3x}$ ...(6)

$(6) + 3\times(5) \Rightarrow y_3 - 3y_2 + 2y_1 + 3(y_2 - 3y_1 + 2y) = 0$

$\Rightarrow y_3 - 7y_1 + 6y = 0$

$$\Rightarrow \frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0.$$

Thus, $y = ae^x + be^{2x} + ce^{-3x}$ is a solution of the given differential equation.

General solution

Let us consider a differential equation concerning the reproduction of bacteria. The experiments on bacteria have ascertained that the rate of their reproduction is proportional to their quantity.

If we denote by x(t) the mass of all bacteria at time t, then $\frac{dx}{dt}$ will be the rate of reproduction of these bacteria. Since the rate of reproduction of bacteria is proportional to their quantity, there exists a constant k such that

$$\frac{dx}{dt} = kx \quad \text{...(1)}$$

By hypothesis, x(t) and x'(t) are nonnegative, therefore the coefficient k is also nonnegative. It is obvious that only the case $k > 0$ is of interest, since for $k = 0$ there takes place no reproduction.

Equation (1) is a simplest example of differential equation. The sought for unknown of (1) is the function $x = x(t)$ which enters into the equation together with its derivative x'(t).

It is easy to verify that any function of the form

$x = Ce^{kt}$,

where C is a certain constant, is a solution of (1). Indeed

$$\frac{dx}{dt} = \frac{d}{dt}(Ce^{kt}) = C\frac{d}{dt}e^{kt} = Cke^{kt} = k(Ce^{kt}) = kx.$$

All the solutions of equation (1) are given by the above formula. The above function, where C is an arbitrary constant, is called the general solution of equation (1)

The general type of a **differential equation of the first order** may be written in the form :

$y' = f(x, y)$ ...(1)

Such equations are said to be resolved with respect to the derivative. The function F(x), $x \in (a, b)$, is called a solution of differential equation (1) if it has a derivative F'(x) on (a, b) and if for any $x \in (a, b)$ the following equality holds true:

$F'(x) = f(x, F(x))$

In other words, the function F(x), $x \in (a, b)$ is called a solution of differential equation (1) if the latter, when substituted instead of y, turns into an identity with respect to x on the interval (a, b).

The definition of an equation of form (1) is equivalent to that of a function f(x, y) of the variables x, y. Geometrically, the function f of the variables x, y is a function defined on a certain set D of points of the plane with the coordinates x, y.

Any curve specified by the equation $y = F(x)$, $x \in (a, b)$, where F(x) is a certain solution of equation (1), is called the **integral curve** of differential equation (1).

It follows from this definition that the integral curve of equation (1) lies entirely in the domain D in which the function f is defined, and that the integral curve, at each of its points P(x, y), has a tangent whose slope is equal to the value of the function f at this point P.

The problem of determining the particular solution of equation (1) satisfying the condition

$y(x_0) = y_0$ ...(2)

where x_0, y_0 are given numbers, is referred to as **Cauchy's initial value problem.** Condition (2) is called the initial condition. The solution of equation (1), satisfying the initial condition (2), is called the solution of Cauchy's problem (1)-(2).

The solution of Cauchy's problem has a simple geometrical meaning. Indeed, according to the given definitions, to solve Cauchy's problem means to find the integral curve of equation (1) which passes through the given point P_0 (x_0, y_0).

Now consider a second order differential equation

$$\frac{d^2y}{dx^2} + y = 0 \qquad ...(3)$$

and the function defined by

$$y = A \sin x + B \cos x \qquad ...(4)$$

where A and B are constants. It can be easily verified that (4) is a solution of equation (3) regardless of the values assigned to A and B.

As before these constants are referred to as arbitrary constants. Thus the second order differential equation (3) has a solution which involves two arbitrary constants and these constants cannot be replaced by a smaller number of constants. Such constants are called essential arbitrary constants. Whenever arbitrary constants will be referred, it will mean essential arbitrary constants.

In order to be more clear about the concept of essential arbitrary constants consider the functions

$$y = C_1e^x + C_2e^{-x} \qquad ...(5)$$
$$y = (C_1 + C_2)e^{2x} \qquad ...(6)$$
$$y = C_1e^{x+c2} \qquad ...(7)$$

(5) contains two arbitrary constants C_1 and C_2 which are essential since C_1, C_2 cannot be replaced by a single arbitrary constant.

In (6), $C_1 + C_2$ can be replaced by another constant C_3 (say) and giving certain values to C_1, C_2 amounts to the same thing as giving one certain value to C_3. Thus the two constants in (4) are not essential.

In (7), $y = C_1 e^x . e^{C_2} = (C_1 e^{C_2}) e^x = C_3e^x$

So again the constants C_1, C_2 are not essential.

With the concept of essential arbitrary constants in mind, the solution (4) is called the general solution of equation (3).

Note:

1. The general solution of a differential equation of the n^{th} order contains 'n' and only 'n' essential arbitrary constants.
2. In counting the arbitrary constants in the general solution, it must be seen that they are essential and are not equivalent to a lesser number of constants. The arbitrary constants in the solution of a differential equation are said to be essential, when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants.

Thus, the solution $y = c_1 \cos x + c_2 \sin (x + c_3)$ appears to have three constants but they are actually equivalent to two, because

$c_1 \cos x + c_2 \sin (x + c_3)$

$= c_1 \cos x + c_2 \sin x \cos c_3 + c_2 \cos x \sin c_3$

$= (c_1 + c_2 \sin c_3) \cos x + c_2 \cos c_3 \sin x$

$= A \cos x + B \sin x$

Hence, three constants c_1, c_2 and c_3 are equivalent to two constants A and B. Thus, constants c_1, c_2 and c_3 are not independent.

3. The general solution of a differential equation can have more than one form, but arbitrary constants in one form will be related to arbitrary constants in another form.
   Thus, $y = c_1 \cos (x + c_2)$ and $y = c_3 \sin x + c_4 \cos x$ are both solutions of the differential equation $d^2y/dx^2 + y = 0$. Each is a general solution containing two arbitrary constants. Expanding the first equation and comparing it with the second, we get,
   $c_1 \cos c_2 = c_4$, $\quad - c_1 \sin c_2 = c_3$
   $c_1 = \sqrt{c_3^2 + c_4^2}$, $\quad c_2 = - \tan^{-1} \dfrac{c_3}{c_4}$

which shows the relationship between the constants appearing in two solutions.

Example 3. Find the order of the differential equation whose general solution is given by

$$y = (c_1 + c_2) \cos (x + c_3) - c_4 e^{x+c_5}$$

where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants.

Solution The given equation can be rewritten as

$$y = A \cos (x + B) - Ce^x,$$

where $A = c_1 + c_2$, $B = c_3$, $C = c_4e^{c_5}$

As the minimum number of parameters is 3, order of the differential equation = 3.

Example 4. Find the coinciding solutions of the two equations :

(a) $y' = y^2 + 2x - x^4$; (b) $y' = - y^2 - y + 2x + x^2 + x^4$.

Solution If y(x) is a solution of a differential equation, then it turns the equation into an identity. Therefore, if equations (a) and (b) have coinciding solutions, then their left and right hand sides are identically equal:

$$y^2 + 2x - x^4 = - y^2 - y + 2x + x^2 + x^4.$$

Hence we find that $y = x^2$ or $y = - x^2 - 1/2$.

The second function does not satisfy equation (a) and so must be discarded. We get $y = x^2$.

Singular Solution

In order to see the existence of the singular solution, consider the relation,

$$y = (x + c)^2, \text{ (c is constant)} \qquad ...(1)$$

which is the general solution of the equation

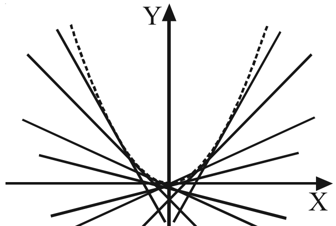
$$\left(\frac{dy}{dx}\right)^2 - 4y = 0 \qquad ...(2)$$

Evidently, $y = 0$ ...(3)

is also a solution of equation (2) and cannot be obtained from (1) by any choice of c. Thus, (3) is a singular solution of equation (2).

Consider another example. A family of nonparallel straight lines is shown in the figure. These are integral curves of the differential equation

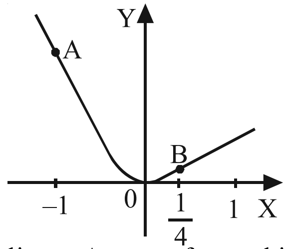
$$y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^2 \qquad ...(4)$$



and a one parameter family of solutions is given by

$$y = Cx - \frac{1}{4}C^2 \qquad ...(5)$$

This family is one which possesses an **envelope**, that is, a curve having the property that at each of its points it is tangent to one of the members of the family. The envelope here is $y = x^2$ and its graph is indicated by the dotted curve in the figure. The envelope of a family of integral curves is itself an integral curve because the slope and coordinates at a point of the envelope are the same as those of one of the integral curves of the family. In this example, it is each to verify directly that $y = x^2$ is a solution of (4). Note that this particular solution is not a member of the family in (5). Further solutions, not members of the family, may be obtained by piecing together members of the family with portions of the envelope. An example is shown in the figure below, which provides a solution of the equation (4) that is not a member of the family in equation (5).



The tangent line at A comes from taking C = –2 in (5) and the tangent at B comes from $C = \frac{1}{2}$. The resulting solution, y = f(x), is given as follows :

$$f(x) = \begin{cases} -2x-1 & \text{if} \quad x \le -1, \\ x^2 & \text{if} \quad -1 \le x \le \frac{1}{4} \\ \frac{1}{2}x - \frac{1}{16} & \text{if} \quad x \ge \frac{1}{4} \end{cases}$$

This function has a derivative and satisfies the differential equation (4) for every real x. It is clear that an infinite number of similar examples could be constructed in the same way. This example shows that it may not be easy to exhibit all possible solutions of a differential equation.

We should note that there are differential equations which donot have solutions at all and others which donot have a general solution.

For example,

the equation $\left(\frac{dy}{dx}\right)^2 + 1 = 0$ has no solution, and

the equation $\left|\frac{dy}{dx}\right| + |y| = 0$ does not have a general solution; the only solution is y = 0.

Concept Problems C

1. In the following problems show that the given functions are solution of the indicated differential equations :

   (i) $y = \frac{\sin x}{x}$, $xy' + y = \cos x$.

   (ii) $y = Ce^{-2x} + \frac{1}{3}e^x$, $y' + 2y = e^x$

   (iii) $y = 2 + C\sqrt{1-x^2}$, $(1-x^2)y' + xy = 2x$.

2. Are the following functions solutions of the equation $y' + y\cos x = \frac{1}{2}\sin 2x$?

   (a) $y = \sin x - 1$ (b) $y = e^{-\sin x}$

   (c) $y = \sin x$

3. Show that V = (A/r) + B is a solution of the differential equation $\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0$.

4. Show that $y = \cos x$, $y = \sin x$, $y = c_1 \cos x$, $y = c_2 \sin x$ are all solutions of the differential equation $y_2 + y = 0$.

5. For what value of the exponent a is the function $y = x^a$ a solution to the differential equation $\frac{dy}{dx} = -y^2$?

Practice Problems B

6. Find the values of α for which the given function is a solution of the indicated equation :

   (a) $y = e^{\alpha x} + \frac{1}{3}e^x$; $y' + 2y = e^x$

   (b) $y = (x^2 - x)^{\alpha}$; $y' = \frac{x^2 + y^2}{2xy}$

   (c) $y = x^{\alpha}$; $x^2y'' + 2xy' - 6y = 0$

7. Show that $y = x - x^{-1}$ is a solution of the differential equation $xy' + y = 2x$.

8. A function of x is a solution of a differential equation if it and its derivatives make the equation true. For what value (or values) of m is $y = e^{mx}$ a solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$?

9. Verify that $y = \sin x \cos x - \cos x$ is a solution of the initial value problem $y' + (\tan x)y = \cos^2 x$, $y(0) = -1$, on the interval $-\pi/2 < x < \pi/2$

10. (a) For what nonzero values of k does the function $y = \sin kt$ satisfy the differential equation $y'' + 9y = 0$?

    (b) For those values of k, verify that every member of the family of functions $y = A \sin kt + b \cos kt$ is also a solution.

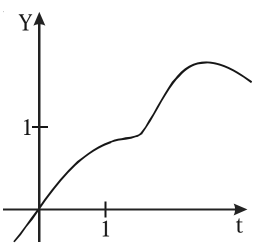
11. A function y(t) satisfies the differential equation $\frac{dy}{dt} = y^4 - 6y^3 + 5y^2$.

    (a) What are the constant solutions of the equation?

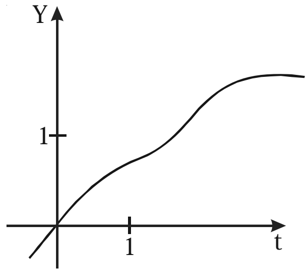
    (b) For what values of y is y decreasing ?

12. Explain why the functions with the given graphs cannot be solutions of the differential equation $\frac{dy}{dt} = e^t(y-1)^2$

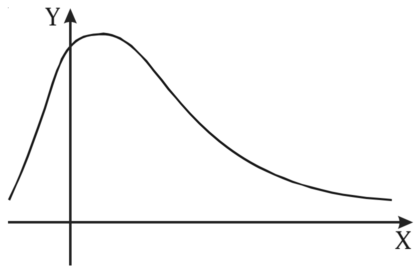
(a)



(b)



13. The function with the given graph is a solution of one of the following differential equations. Decide which is the correct equation and justify your answer.



(a) $y' = 1 + xy$ (b) $y' = -2xy$

(c) $y' = 1 - 2xy$

14. Find functions continuous on the whole real axis, which satisfy the given conditions :

    (a) $g(x) = 2 + \int_1^x f(t)\,dt$.

    (b) $g^2(x) + [g'(x)]^2 = 1$,
    note that : $g(x) = -1$ is one solution.

15. If y_h is a solution of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ and y_p is a solution of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = h(x)$, show that $y_h + y_p$ is also a solution of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = h(x)$.

4.4 FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

An ordinary differential equation of the first order and first degree is of the form

$$\frac{dy}{dx} = f(x, y) \qquad ...(1)$$

which is sometimes conveniently written as

$$Mdx + Ndy = 0 \qquad ...(2)$$

where M and N are functions of x and y, or constants. If f(x, y) be real and single-valued for all values of x and y, then corresponding to any point in the plane xy we have a definite direction, assigned by the equation (1). If we imagine a point, starting from any position in the plane, to move always in the direction thus indicated, it will trace out a curve, which constitutes a particular solution, or primitive, of the proposed equation. It appears moreover, that in the present case no two curves of the system will intersect.

Note that equation (1) cannot be solved in every case. However, the solution exists if these equations belong to any of the standard form discussed here.

One variable absent

The form $\frac{dy}{dx} = f(x)$, where y does not appear explicitly, requires merely an ordinary integration. Thus,

$$y = \int f(x)\, dx + C$$

where C is an arbitrary constant.

(ii) The equation $\frac{dy}{dx} = f(y)$ in which x does not appear explicitly, may be written

$$\frac{dy}{f(y)} = dx,$$

$$\Rightarrow \int \frac{dy}{f(y)} = x + C.$$

Example 1. To find the curves whose subtangent has a given constant value a. We have

$$y\frac{dx}{dy} = a \text{ or, } \frac{dy}{y} = \frac{dx}{a}$$

Hence, ln $y = \frac{x}{a} + C$ or, $y = be^{x/a}$, where $b = e^C$, is arbitrary.

Variables separable differential equations

If the coefficient of dx is only a function of x and that of dy is only a function of y in the given differential equation, then the equation can be solved using variable separable method. Such a differential equation can be expressed as

$$g(y)dy = f(x)dx.$$

To find the general solution of a differential equation with variables separable :

(i) separate the variables, i.e. transform the given equation to the form
$$g(y)dy = f(x)\, dx \qquad ...(1)$$

(ii) integrate both sides of the obtained equation with respect to x and y, respectively, i.e. find a certain primitive G(y) of the function g(y) and a certain primitive F(x) of the function f(x),
$$\int g(y)dy = \int f(x)dx + C.$$
Write the equation as
$$G(y) = F(x) + C, \qquad ...(2)$$
where C is an arbitrary constant.

(iii) Solving equation (2) with respect to y, we obtain the general solution of differential equation (1)
$$y = \phi(x, C).$$

Example 2. Solve $\frac{dy}{dx} = (e^x + 1)(1 + y^2)$

Solution The equation can be written as

$$\frac{dy}{1+y^2} = (e^x + 1)dx$$

Integrating both sides, we get $\tan^{-1} y = e^x + x + c$.

$\Rightarrow \; y = \tan(e^x + x + c)$.

Example 3. Solve the differential equation

$$(1 + x)\, y\, dx = (y - 1)\, x\, dy$$

Solution The equation can be written as

$$\Rightarrow \left(\frac{1+x}{x}\right)dx = \left(\frac{y-1}{y}\right)dy$$

$$\Rightarrow \int\left(\frac{1}{x}+1\right)dx = \int\left(1-\frac{1}{y}\right)dy$$

$$\Rightarrow \ell n\,|x| + x = y - \ell n\,|y| + c$$

$$\Rightarrow \ell n\,|y| + \ell n\,|x| = y - x + c$$

$$\Rightarrow |xy| = e^{y-x+c}.$$

$$\Rightarrow xy = a\, e^{y-x}.$$

Example 4. Solve

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0.$$

Solution $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$

$$\Rightarrow -\frac{\sqrt{1+x^2}}{x}dx = \frac{ydy}{\sqrt{1+y^2}}$$

Integrating, we get

$$\Rightarrow -\int \frac{\sqrt{1+x^2}}{x}dx = \int \frac{y\,dy}{\sqrt{1+y^2}}$$

$$\Rightarrow -\left[\sqrt{1+x^2}+\frac{1}{2}\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1+x^2}+1}\right|\right] = \sqrt{1+y^2}+C.$$

This is the solution to the given differential equation.

Example 5. Solve $y - x\frac{dy}{dx} = a\left(y^2+\frac{dy}{dx}\right)$

Solution The equation can be written as

$$y - ay^2 = (x+a)\frac{dy}{dx}$$

$$\frac{dx}{x+a} = \frac{dy}{y-ay^2} \Rightarrow \frac{dx}{x+a} = \frac{1}{y(1-ay)}\,dy$$

$$\frac{dx}{x+a} = \left(\frac{1}{y}+\frac{a}{1-ay}\right)dy$$

Integrating both sides,

$$\ell n\,|x+a| = \ell n\,|y| - \ell n\,|1-ay| + \ell n\,|c|$$

$$\ell n\,|x+a| = \ell n\left|\frac{cy}{1-ay}\right|$$

$$|cy| = |(x+a)(1-ay)|$$

$$\Rightarrow\ ky = (x+a)(1-ay)$$

where 'k' is an arbitrary constant.

Example 6. Solve $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$.

Solution The given equation may be written as

$(\sin y + y\cos y)\,dy = x(2\log x+1)\,dx$

Integrating, we have

$$\int \sin y\,dy + \int y\cos y\,dy = 2\int x\log x\,dx + \int x\,dx \text{ or,}$$

$$-\cos y + y\sin y + \cos y = 2\left(\frac{1}{2}x^2\log x - \frac{1}{4}x^2\right) + \frac{x^2}{2}$$

$+ c$ or, $y\sin y = x^2\log x + c$, which is the required solution.

CAUTION

Consider a variable separable differentiable equation of the form $\varphi_1(x)\,\psi_1(y)\,dx = \varphi_2(x)\,\psi_2(y)\,dy$ in which the coefficients of the differentials are factors depending on x alone and on y alone.

Dividing by the product $\psi_1(y)\,\varphi_2(x)$ reduces it to an equation with separated variables :

$$\frac{\varphi_1(x)}{\varphi_2(x)}dx = \frac{\psi_2(y)}{\psi_1(y)}dy.$$

The general integral of this equation is of the form

$$\int \frac{\varphi_1(x)}{\varphi_2(x)}dx - \int \frac{\psi_2(y)}{\psi_1(y)}dy = C.$$

Note that, dividing by $\psi_1(y)\,\varphi_2(x)$ may lead to the loss of particular solutions making the product $\psi_1(y)\,\varphi_2(x)$ zero. For instance, let $\psi_1(y_0)$ vanish for $y = y_0$ i.e. $\psi_1(y_0) = 0$. Then, the constant function $y = y_0$ is a solution of the equation. Indeed, we have $dy = 0$, and hence the substitution of $y = y_0$ into equation leads to an identity. If x and y are regarded as playing equivalent roles in the equation, analogous argument shows that if $\varphi_2(x_0)$, then $x = x_0$ is also a particular solution of the equation.

Example 7. Solve the equation $y' = -\frac{y}{x}$. In particular, find the solution that satisfies the initial conditions $y(1) = 2$.

Solution $y' = -\frac{y}{x}$ ...(1)

Equation (1) may be written in the form $\frac{dy}{dx} = -\frac{y}{x}$.

Separating the variables, we have $\frac{dy}{dx} = -\frac{dx}{x}$

and, consequently, $\ln|y| = -\ln|x| + \ln C_1$,

where the arbitrary constant $\ln C_1$ is taken in logarithmic form. After taking antilogarithms we get

the general solution $y = \frac{C}{x}$, where $C = \pm C_1$ ...(2)

When dividing by y we could lose the solution $y = 0$, but the latter is contained in the formula (2) for $C = 0$. Utilizing the given initial conditions, we get $C = 2$;

and, hence, the desired particular solution is $y = \frac{2}{x}$.

Example 8. Solve the equation

$$3e^x\tan y\,dx + (2-e^x)\sec^2 y\,dy = 0$$

Solution We divide both sides of the equation

by the product $\tan y \times (2-e^x)$: $\frac{3e^x dx}{2-e^x} + \frac{\sec^2 y\,dy}{\tan y} = 0.$

This is an equation with separated variables. Integrating it, we find that

$$-3\ln|2-e^x| + \ln|\tan y| = C_1.$$

On taking the exponent of both sides we have

$$\frac{|\tan y|}{|2-e^x|^3} = e^{C_1} \Rightarrow \left|\frac{\tan y}{(2-e^x)^3}\right| = e^{C_1}$$

$$\Rightarrow \frac{\tan y}{(2-e^x)^3} = \pm e^{C_1}.$$

Denoting $\pm e^{C_1} = C$ we get,

$$\frac{\tan y}{(2-e^x)^3} = C \Rightarrow \tan y - C(2-e^x)^3 = 0.$$

This is the general solution of the given equation.
It was assumed in dividing by the product $\tan y \times (2 - e^x)$ that neither of the factors vanishes. Equating each factor to zero gives respectively $y = k\pi$, $k \in I$, and $x = \ln 2$.

Direct substitution in the original equation shows that $y = k\pi$ and $x = \ln 2$ are solutions of this equation. They can be obtained formally from the general solution when $C = 0$ and $C = \infty$. This means that the constant C is replaced by $1/C_2$, which makes the general solution assume the form

$$\tan y - \frac{1}{C_2}(2-e^x)^3 = 0$$

$\Rightarrow \; C_2 \tan y - (2-e^x)^3 = 0.$

Putting in the last equation $C_2 = 0$, which corresponds to $C = \infty$, we have $(2-e^x)^3 = 0$.

From here, we obtain the solution $x = \ln 2$ of the original equation. So the functions $y = k\pi$, $k \in I$ and $x = \ln 2$ are particular solutions of the given equation.

Therefore, the final answer is $\tan y - C(2-e^x)^3 = 0$.

Example 9. Find all solutions of the equation

$$y' = 2\sqrt{y}$$

Solution This is an equation with variables separable. Obviously, the function $y = 0$ is its solution. Let now $y > 0$. The differential equation has the form

$$dy = 2\sqrt{y}\,dx$$

Separating the variables in this equation: $\dfrac{dy}{2\sqrt{y}} = dx$

and integrating, we get $\sqrt{y} = x + C$, where C is an arbitrary constant. Hence it follows that

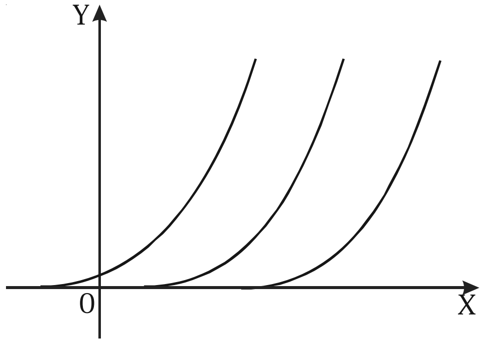
$$y = (x+C)^2, \text{ where } x + C \geq 0.$$

Thus, for each fixed value of the constant C the function

$$y = (x+C)^2,\; x \geq -C$$

is a solution. The equation has no other solutions in the half-plane $y > 0$.

The arrangement of the integral curves of the equation is shown in the figure.



In the half plane $y > 0$ each integral curve is obtained from the parabola branch $y = x^2$, $x > 0$, by translating it either to the left or to the right along the x-axis. The straight line $y = 0$ is also an integral curve.

Example 10. Solve the equation $y' = xy$ ...(1)

Solution This equation is an equation with variables separable. Separating the variables.

$$\frac{dy}{y} = x\,dx$$

and integrating, we get $\ell n|y| = \frac{1}{2}x^2 + C_1$

where C_1 is an arbitrary constant. Hence it follows

that $|y| = e^{C_1}.e^{\frac{1}{2}x^2}$, or, $y = Ce^{\frac{1}{2}x^2}$ ...(2)

where $C = \pm e^{C_1}$

The right member of equation (1) vanishes for $y = 0$, therefore it has a solution $y = 0$. This solution is obtained from (2) for $C = 0$. Thus, formula (2), where C is an arbitrary constant, gives all the solutions of equation (1).

Example 11. Solve the equation $y' = \dfrac{xy\cos x}{1+y}$

Solution The constant function $y = 0$ is, obviously, one of its solutions.

Let now $y \neq 0$. First of all we separate the variables:

$$\left(1+\frac{1}{y}\right)dy = x\cos x\,dx.$$

Integrating the left member of this equation with respect to y, and the right member with respect to x, we get the equation $y + \ln|y| = x\sin x + \cos x + C$, where C is an arbitrary constant.

To find y is forms of x we have to solve the above equation for y. Unfortunately, it is impossible. But the problem of finding the general solution of a differential equation has been reduced to solving an equation which does not contain derivatives. In this case we say that the general solution is defined by this formula.

Example 12. Solve the equation $\frac{dy}{y} = 3x^2\,dx$ and find its solution satisfying the condition $y|_{x=0} = 2$.

Solution On integrating and writing the arbitrary constant in the form $\ell n|C|$ $(C \neq 0)$, which is convenient for taking antilogarithms, we obtain

$$\ell n\,|y| = x^3 + \ell n\,|C|,\ y = C\,e^{x^3}$$

If we took the arbitrary constant in the form $\ell n\,|C|$ the general solution would be written as $y = \pm\, C\,e^{x^3}$ because in this case we must have $C > 0$.

Substituting the given initial values into the general solution we determine C :

$$2 = C$$

Thus, the function $y = 2\,e^{x^3}$ is the sought for particular solution of the given equation. We could also have used definite integrals, which would lead to

$$\int_2^y \frac{dy}{y} = \int_0^x 3x^2 dx$$

that is, $\ell n\,y\Big|_2^y = x^3\Big|_0^x$, or $\ell n\,y - \ell n\,2 = x^3$.

Now, taking antilogarithms we arrive at the same particular solution : $y = 2\,e^{x^3}$.

Example 13. Solve the initial value problem $y' = y^{1/3}$, $y(0) = 0$ for $x \geq 0$. ...(1)

Solution This problem is easily solved since the differential equation is separable. Thus, we have

$y^{-1/3}\,dy = dx$, so $\frac{3}{2}y^{2/3} = x + c$ and $y = \left[\frac{2}{3}(x+c)\right]^{3/2}$.

The initial condition is satisfied if $c = 0$ so

$$y = \left(\frac{2}{3}x\right)^{3/2},\ x \geq 0$$

This satisfies both of equations (1). On the other hand, the function

$$y = -\left(\frac{2}{3}x\right)^{3/2},\ x \geq 0$$

is also a solution of the initial value problem. Moreover, the function $y = 0,\ x \geq 0$

is yet another solution. Indeed it is not hard to show that, for an arbitrary positive x_0, the functions

$$y = \begin{cases} 0, & \text{if } 0 \leq x < x_0 \\ \pm[2/3(x - x_0)]^{3/2}, & \text{if } x \geq x_0 \end{cases}$$

are continuous, differentiable (in particular at $x = x_0$), and are solutions of the initial value problem (1). Hence this problem has an infinite family of solutions.

Example 14. Solve the initial value problem $y' = y^2$, $y(0) = 1$, and determine the interval in which the solution exists.

Solution $y^{-2}\,dy = dx$

Then $-y^{-1} = x + c$ and $y = -\frac{1}{x+c}$

To satisfy the initial condition we must choose $c = -1$, so $y = \frac{1}{1-x}$ is the solution of the initial value problem.

Clearly, the solution becomes unbounded as $x \to 1$. Therefore, the solution exists only in the interval $-\infty < x < 1$.

Example 15. Find a particular solution of the equation $(1 + e^x)\,yy' = e^x$ satisfying the initial condition $y(0) = 1$.

Solution We have $(1 + e^x)\,y\frac{dy}{dx} = e^x$. Separating the variables we get $y\,dy = \frac{e^x dx}{1+e^x}$. Integrating we find the general integral $\frac{y^2}{2} = \ln(1 + e^x) + C$. ...(1)

Putting in (1) $x = 0$ and $y = 1$ we have

$$1/2 = \ln 2 + C \Rightarrow C = 1/2 - \ln 2.$$

Substituting the obtained value of C in (1) we get the particular solution

$$y^2 = 1 + \ln\left(\frac{1+e^x}{2}\right)^2 \Rightarrow y = \pm\sqrt{1 + \ln\left(\frac{1+e^x}{2}\right)^2}.$$

It follows from the initial condition that $y > 0$, therefore we use the positive sign before the radical. So, the sought-for particular solution is

$$y = \sqrt{1 + \ln\left(\frac{1+e^x}{2}\right)^2}.$$

Example 16. Find particular solutions of the equation $y'\sin x = y\ln y$ satisfying the initial conditions:

(a) $y(\pi/2) = e$ (b) $y(\pi/2) = 1$

Solution We have $\frac{dy}{dx}\sin x = y\ln y$.

We separate the variables $\frac{dy}{y\ln y} = \frac{dx}{\sin x}$.

Integrating, we find the general integral
$\ln|\ln y| = \ln|\tan x/2| + \ln C.$
On taking the exponent of both sides

we get $\ln y = C\tan x/2 \Rightarrow y = e^{C\tan\frac{x}{2}}$
which is the general solution of the original equation.

(a) Put $x = \pi/2, y = e$, then $e = e^{C\tan\frac{x}{4}} \Rightarrow C = 1.$

The required particular solution is $y = e^{\tan\frac{x}{2}}$.

(b) Putting $x = \pi/2, y = 1$, in the general solution

we have $1 = e^{C\tan\frac{x}{4}} \Rightarrow C = 0.$

The required particular solution is $y = 1$.

Notice that in the process of obtaining the general solution the constant C was under the logarithm sign and hence C = 0 should be regarded as the limiting value. This particular solution y = 1 is contained among the zeros of the product y ln y sin x by which we divided both sides of the given equation.

Example 17. Find the solution of the equation $x^3 \sin y \cdot y' = 2$, satisfying the condition $y \to \pi/2$ as $x \to \infty$.

Solution Separating the variables and integrating we find the general integral of the equation as

$$\cos y = \frac{1}{x^2} + C.$$

The condition gives $\cos\frac{\pi}{2} = C$, i.e. $C = 0$, so that the particular integral will be of the form $\cos y = 1/x^2$. There are an infinite number of particular solutions of the form $y = \pm\cos^{-1}\frac{1}{x^2} + 2\pi n,\ n \in I$ ...(1)

corresponding to it. Among these solutions there is only one satisfying the condition. This solution will be found proceeding to the limit as $x \to \infty$ in equation (1):

$$\frac{\pi}{2} = \pm\cos^{-1}0 + 2\pi n$$

$$\Rightarrow \frac{\pi}{2} = \pm\frac{\pi}{2} + 2\pi n$$

$$\Rightarrow \frac{1}{2} = \pm\frac{1}{2} + 2n. \quad ...(2)$$

It can easily be seen that equation (2) has two solutions, namely, n = 0 and n = 1/2, the solution n = 1/2, corresponding to the negative sign before $\cos^{-1}\frac{1}{x^2}$, being unsuitable (n must be an integer or zero). Thus, the required particular solution is

$$y = \cos^{-1}\frac{1}{x^2}.$$

Example 18. Given $y(0) = 2000$ and $\frac{dy}{dx} = 32000 - 20y^2$, then find the value of $\lim_{x\to\infty} y(x)$.

Solution We have $\frac{dy}{dx} = 20(1600 - y^2)$

$$\Rightarrow \int\frac{dy}{(40)^2 - y^2} = 20\int dx$$

$$\Rightarrow \frac{1}{80}\ln\frac{40+y}{40-y} = 20x + C'$$

$$\Rightarrow \ell n\frac{40+y}{40-y} = 1600x + C$$

$$\Rightarrow \frac{40+y}{40-y} = \frac{ke^{1600x}}{1}, \text{ where } k = e^C \text{ (say)}$$

$$\Rightarrow \frac{2y}{80} = \frac{ke^{1600x} - 1}{ke^{1600x} + 1}$$ (using componendo & dividendo)

$$\therefore \lim_{x\to\infty} y = 40\lim_{x\to\infty}\left[\frac{k - e^{-1600x}}{k - e^{-1600x}}\right] = 40.$$

Practice Problems C

Solve the following differential equations :

1. $ydx - xdy = xydx$
2. $(1 - x^2)(1 - y)dx = xy(1 + y)dy$
3. $xy\,dx + (1 + x^2)dy = 0$
4. $(e^y + 1)\cos x\,dx + e^y\sin x\,dy = 0$
5. $(x^2y + x^2)dx + (y^2x - y^2)\,dy = 0$
6. $\sin x.\frac{dy}{dx} = y.\ell ny$ if $y = e$, when $x = \frac{\pi}{2}$
7. $e^{(dy/dx)} = x + 1$ given that when $x = 0, y = 3$
8. $\left(y - \frac{xdy}{dx}\right) = 3\left(1 - x^2\frac{dy}{dx}\right)$
9. $\left(y - \frac{xdy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$
10. $x^2y\frac{dy}{dx} = (x+1)(y+1)$

11. $\frac{dy}{dx} = (e^{x+y} + y^2 e^x)^{-1}$

12. $\frac{\ell n(\sec x + \tan x)}{\cos x} dx = \frac{\ell n(\sec y + \tan y)}{\cos y} dy$

13. $xy^2 \frac{dy}{dx} = 1 - x^2 + y^2 - x^2 y^2$

14. $x\sqrt{1-y^2}\, dx + y\sqrt{1-x^2}\, dy = 0,\ y(0) = 1.$

15. $e^x \sin^3 y + (1 + e^{2x}) \cos y \cdot y' = 0$

16. $(a^2 + y^2)\, dx + 2x\sqrt{ax - x^2}\, dy = 0,\ y(a) = 0.$

17. $x^2 y' \cos y + 1 = 0,\ y \to \frac{16}{3}\pi$, as $x \to \infty$.

18. $(1 + x^2)\, y' - 1/2 \cos^2 2y = 0,\ y \to \frac{7}{2}\pi,\ x \to -\infty.$

4.5 REDUCIBLE TO VARIABLE SEPARABLE

Type 1 : A differential equation of the form

$$\frac{dy}{dx} = f(ax + by + c), \qquad ...(1)$$

where a, b and c are constants, is converted into an equation with variables separable by the replacement of the variables $v = ax + by + c$. It is called reducible to variable separable differential equation.

$v = ax + by + c.$

$$\therefore \quad \frac{dv}{dx} = a + b\frac{dy}{dx} \quad \text{or,} \quad \frac{dy}{dx} = \frac{\frac{dv}{dx} - a}{b}$$

$$\Rightarrow \quad \frac{\frac{dv}{dx} - a}{b} = f(v) \quad \Rightarrow \quad \frac{dv}{dx} = bf(v) + a$$

$$\Rightarrow \quad \frac{dv}{bf(v) + a} = dx \qquad ...(2)$$

In the differential equation (2), the variables x and v are separated. Integrating (2), we get

$$\int \frac{dv}{bf(v) + a} = \int dx + C$$

$$\int \frac{dv}{bf(v) + a} = x + C, \text{ where } v = ax + by + c$$

This represents the general solution of the differential equation (1).

Example 1. Solve $(x + y)^2 \frac{dy}{dx} = a^2.$

Solution Put $x + y = v$, i.e. $y = v - x$

$$\therefore \quad \frac{dy}{dx} = \frac{dv}{dx} - 1.$$

The equation reduces to

$$v^2\left(\frac{dv}{dx} - 1\right) = a^2 \quad \text{or} \quad \frac{dv}{dx} = 1 + \frac{a^2}{v^2} = \frac{a^2 + v^2}{v^2}$$

$$\Rightarrow \quad dx = \frac{v^2}{a^2 + v^2} dv \Rightarrow dx = \left(1 - \frac{a^2}{a^2 + v^2}\right) dv$$

Integrating both sides,

$$\int dx = \int dv - a^2 \int \frac{dv}{a^2 + v^2},$$

$$\Rightarrow \quad x + C = v - a^2 . \frac{1}{a} \tan^{-1} \frac{v}{a}$$

$$\Rightarrow \quad x + C = x + y - a \tan^{-1} \frac{x + y}{a}$$

$\therefore \quad y = a \tan^{-1} \frac{x + y}{a} + C$ is the required solution.

Example 2. Solve the differential equation

$$\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$$

Solution We have $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$ ...(1)

Let $z = x + y$

$$\Rightarrow \quad \frac{dz}{dx} = 1 + \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dz}{dx} - 1$$

$\therefore$ From (1) $\frac{dz}{dx} - 1 = \sin z + \cos z$

$$\Rightarrow \quad \frac{dz}{\sin z + \cos z + 1} = dx \quad \text{(variables are separated)}$$

Integrating both sides, we get

$$\int \frac{dz}{\sin z + \cos z + 1} = \int 1.dx + C$$

$$\Rightarrow \quad \int \frac{dz}{\frac{2\tan z/2}{1 + \tan^2 z/2} + \frac{1 - \tan^2 z/2}{1 + \tan^2 z/2} + 1} = x + C$$

$$\Rightarrow \quad \int \frac{\sec^2 z/2 dz}{2\tan z/2 + 2} = x + C$$

$\Rightarrow \int \frac{dt}{t+1} = x + C$, where $t = \tan\frac{z}{2}$.

$\Rightarrow \ln|t+1| = x + C$

$\Rightarrow \ln\left|\tan\frac{x+y}{2} + 1\right| = x + C.$

This is the required general solution.

Example 3. Solve the differential equation

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \text{ where } \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}.$$

Solution Consider the differential equation

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \text{ where } \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \quad ...(1)$$

Let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$ (say)

$\Rightarrow a_1 = \lambda a_2,\ b_1 = \lambda b_2$

$\therefore$ (1) becomes $\frac{dy}{dx} = \frac{\lambda a_2x + \lambda b_2y + c_1}{a_2x + b_2y + c_2}$

$$\Rightarrow \frac{dy}{dx} = \frac{\lambda(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2} \quad ...(2)$$

Let $z = a_2x + b_2y$

$\therefore$ (2) becomes $\frac{\frac{dz}{dx} - a_2}{b_2} = \frac{\lambda z + c_1}{z + c_2}$

$\Rightarrow \frac{dz}{dx} = \frac{b_2(\lambda z + c_1)}{z + c_2} + a_2$

$\Rightarrow \frac{dz}{dx} = \frac{\lambda b_2 z + b_2c_1 + a_2z + a_2c_2}{z + c_2}$

$$\Rightarrow \frac{z + c_2}{(\lambda b_2 + a_2)z + b_2c_1 + a_2c_2}dz = dx \quad ...(3)$$

In the differential equation (3), the variables x and z are separated.

Integrating (3), we get

$$\int \frac{z + c_2}{(\lambda b_2 + a_2)z + b_2c_1 + a_2c_2}dz = \int 1.dx + C$$

$$\Rightarrow \int \frac{z + c_2}{(\lambda b_2 + a_2)z + b_2c_1 + a_2c_2}dz = x + C,$$

where $z = a_2x + b_2y$.

This represents the general solution of the differential equation (1)

Example 4. Solve the differential equation

$$\frac{dy}{dx} = \frac{2x - y + 2}{2y - 4x + 1}$$

Solution We have $\frac{dy}{dx} = \frac{2x - y + 2}{2y - 4x + 1}$

$$\Rightarrow \frac{dy}{dx} = \frac{2x - y + 2}{-4x + 2y + 1} \quad ...(1)$$

Here $\frac{a_1}{a_2} = \frac{2}{-4} = -\frac{1}{2}$ and $\frac{b_1}{b_2} = \frac{-1}{2}$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2}$

$$(1) \Rightarrow \frac{dy}{dx} = \frac{(2x - y) + 2}{-2(2x - y) + 1} \quad ...(2)$$

Let $z = 2x - y$

$\therefore \frac{dz}{dx} = 2 - \frac{dy}{dx}$ or $\frac{dy}{dx} = 2 - \frac{dz}{dx}$

$\therefore (2) \Rightarrow 2 - \frac{dz}{dx} = \frac{z + 2}{-2z + 1}$

$\Rightarrow \frac{2z - 1}{5z}dz = dx$ (variables are separate)

Integrating both sides, we get $\int \frac{2z - 1}{5z}dz = \int dx$

$\Rightarrow \int \left(\frac{2}{5} - \frac{1}{5z}\right)dz = x + C$

$\Rightarrow \frac{2}{5}z - \frac{1}{5}\ln|z| = x + C$

$\Rightarrow 2z - \ln|z| = 5x + 5C$

$\Rightarrow 2(2x - y) - \ln|2x - y| = 5x + C_1$ where $C_1 = 5C$

$\Rightarrow x + 2y + \ln|2x - y| + C_1 = 0$

Type 2 : Sometimes transformation to the polar co–ordinates facilitates separation of variables. In this connection, it is convenient to remember the following differentials :

If $x = r\cos\theta$; $y = r\sin\theta$ then ,

(i) $x\,dx + y\,dy = r\,dr$

(ii) $dx^2 + dy^2 = dr^2 + r^2\,d\theta^2$

(iii) $x\,dy - y\,dx = r^2\,d\theta$

If $x = r\sec\theta$ and $y = r\tan\theta$ then,

(i) $x\,dx - y\,dy = r\,dr$

(ii) $x\,dy - y\,dx = r^2\sec\theta\,d\theta$.

Example 5. Solve the differential equation

$$xdx + ydy = x\,(xdy - ydx)$$

Solution Taking

$$x = r\cos\theta,\ y = r\sin\theta$$
$$x^2 + y^2 = r^2$$
$$2x\,dx + 2ydy = 2rdr$$
$$xdx + ydy = rdr \qquad ...(1)$$

$$\frac{y}{x} = \tan\theta$$

$$\frac{d\frac{dy}{dx} - y}{x^2} = \sec^2\theta \,.\, \frac{d\theta}{dx}$$

$$xdy - ydx = x^2\sec^2\theta \,.\, d\theta$$
$$xdy - ydx = r^2\,d\theta \qquad ...(2)$$

Using (1) and (2) in the given differential equation then it becomes $r\,dr = r\cos\theta.\ r^2\,d\theta$

$$\frac{dr}{r^2} = \cos\theta\,d\theta \Rightarrow -\frac{1}{r} = \sin\theta + \lambda$$

$$-\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + \lambda \Rightarrow \frac{y+1}{\sqrt{x^2+y^2}} = c$$

where $-\lambda' = c$

$$(y+1)^2 = c(x^2+y^2)$$

Example 6. Solve

$$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}\,.$$

Solution Here we change to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$,

$$x^2 + y^2 = r^2 \Rightarrow x\,dx + y\,dy = r\,dr.$$

$$\frac{y}{x} = \tan\theta \Rightarrow \frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\,d\theta$$

$\Rightarrow$ $x\,dy - y\,dx = r^2\,d\theta$.

$\therefore$ The equation becomes $\dfrac{1}{r}\dfrac{dr}{d\theta} = \sqrt{\left(\dfrac{a^2 - r^2}{r^2}\right)}$.

Separating the variables, $\dfrac{dr}{\sqrt{(a^2 - r^2)}} = d\theta$.

Integrating, $\sin^{-1}(r/a) = \theta + C$

$\Rightarrow$ $r = a\sin(\theta + C)$,

$\Rightarrow$ $\sqrt{(x^2 + y^2)} = a\sin\,[\tan^{-1}(y/x) + C$.

Example 7. Solve $x\dfrac{dy}{dx} - y = x\sqrt{(x^2 + y^2)}$.

Solution The equation can be put as

$$x\,dy - y\,dx = x\,\sqrt{x^2 + y^2}\,dx$$

Changing to polar coordinates, the equation becomes

$$x^2\sec^2\theta\,d\theta = xr\,dx$$

or, $x\sec^2\theta\,d\theta = r\,dx$ or $\cos\theta\sec^2\theta\,d\theta = r\,dx$

or, $\sec\theta\,d\theta = dx$, variables separated.

Integrating, $\ln(\sec\theta + \tan\theta) = x + \ln c$.

$\therefore$ $\sec\theta + \tan\theta = ce^x$

or, $\sqrt{(1 + y^2/x^2)} + y/x = ce^x$.

Type 3 : In an equation of the form

$yf_1(xy)\,dx + xf_2(xy)\,dy = 0$, the variables can be separated by the substitution $xy = v$.

Proof $xy = v \Rightarrow y = \dfrac{v}{x}$

$$\Rightarrow \quad dy = \frac{x\,dv - v\,dx}{x^2},$$

$$\Rightarrow \quad x\,dy = dv - \frac{v}{x}\,dx.$$

$$\therefore \quad \frac{v}{x}f_1(v)\,dx + f_2(v)\left\{dv - \frac{v}{x}dx\right\} = 0$$

$$\therefore \quad \frac{f_2(v)\,dv}{v\{f_1(v) - f_2(v)\}} + \frac{dx}{x} = 0.$$

Thus, the variables are separated.

Example 8. Solve

$$(x^3y^3 + x^2y^2 + xy + 1)y\,dx + (x^3y^3 - x^2y^2 - xy + 1)x\,dy = 0.$$

Solution Here we put $xy = v$

$$\Rightarrow \quad y = \frac{v}{x}$$

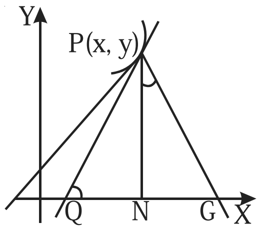
$$\Rightarrow \quad dy = \frac{x\,dv - v\,dx}{x^2}$$

$$\Rightarrow \quad x\,dy = dv - \frac{v}{x}\,dx.$$

The equation reduces to $\displaystyle\int \frac{v^2 - 2v + 1}{v^2}dv = -2\int\frac{dx}{x}$

$$\Rightarrow \quad xy - 2\ln y - \frac{1}{xy}dx = C.$$

Geometrical details of a curve



(i) Slope of tangent at any point P(x, y) = $\frac{dy}{dx}$.

(ii) Equation of tangent PQ at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

(iii) Equation of normal PG at (x, y) is

$$Y - y = -\frac{dx}{dy}(X - x)$$

(iv) Length of tangent PQ at

$$(x, y) = |y|\sqrt{1 + \left(\frac{dx}{dy}\right)^2}.$$

(v) Length of normal PG at

$$(x, y) = |y|\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(vi) Subtangent QN at $(x, y) = \left|y \cdot \frac{dx}{dy}\right|$.

(vii) Subnormal NG at $(x, y) = \left|y \cdot \frac{dy}{dx}\right|$

Example 9. Find the equation of the curve, slope of whose tangent at any point (x, y) is $\frac{2y}{x}$ $\forall$ x, y > 0 and which passes through the point (1, 1).

Solution We have, $\frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$

Integrating both sides,

$\ln y = 2\ln x + \ln c \Rightarrow \ln y = \ln cx^2$,

$\Rightarrow y = cx^2$

The curve passes through the point (1, 1), hence c = 1.

So, the curve is $y = x^2$.

Example 10. A curve 'C' has the property that if the tangent at any point 'P' on 'C' meets the coordinate axis at A and B, then P is the mid point of AB. If the curve passes through the point (1, 1), then find the equation of curve.

Solution Equation of tangent at P (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

It meets the coordinate axis in A and B

$$\therefore \quad A = \left(x - y\frac{dx}{dy},\ 0\right) \text{ and } B = \left(0,\ y - x\frac{dy}{dx}\right)$$

Since P is the mid point of AB,

$$2x = x - y\frac{dx}{dy} \text{ and } 2y = y - x\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0$$

Integrating both sides,

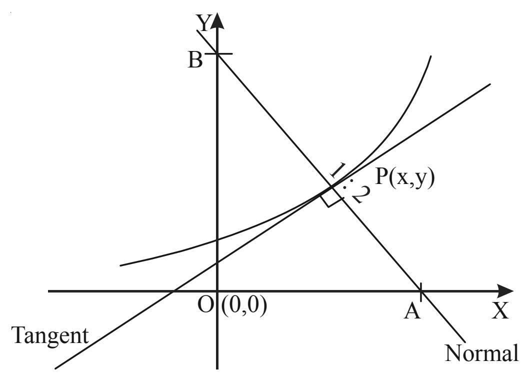
$\ln y + \ln x = \ln c \Rightarrow \ln(xy) = \ln c \Rightarrow xy = c$

As curve passes through (1, 1), c = 1

So equation of curve is xy = 1

Example 11. Let C be a curve such that the normal at any point P on it meets x-axis and y-axis at A and B respectively. If BP : PA = 1 : 2 (internally) and the curve passes through the point (0, 4), then show that the curve is a hyperbola and it passes through $\left(\sqrt{10}, -6\right)$.

Solution



The equation of normal at P(x, y) is

$$(Y - y) = \frac{-1}{\frac{dy}{dx}}(X - x)$$

$$\therefore \quad A\left(x + y\frac{dy}{dx}, 0\right) \text{ and } B\left(0, y + \frac{x}{\frac{dy}{dx}}\right)$$

$$\text{Now } \frac{1\left(x + y\frac{dy}{dx}\right) + 2(0)}{1 + 2} = x \quad \Rightarrow x + y\frac{dy}{dx} = 3x$$

$\therefore \quad y\dfrac{dy}{dx} = 2x$

$\Rightarrow \int y\,dy = \int 2x\,dx$

$\Rightarrow \dfrac{y^2}{2} = x^2 + C$

Also (0, 4) satisfy it, so C = 8.

$\therefore$ $y^2 = 2x^2 + 16$ is the equation of the curve which represents a hyperbola.

Example 12. A conic C passes through the point (2, 4) and is such that the segment of any of its tangents at any point contained between the co-ordinate axes is bisected at the point of tangency . Find the equation of a circle described on the latus rectum of the conic C as diameter .

Solution $Y - y = m(X - x)$

if, $Y = 0$ then $X = x - \dfrac{y}{m}$ and

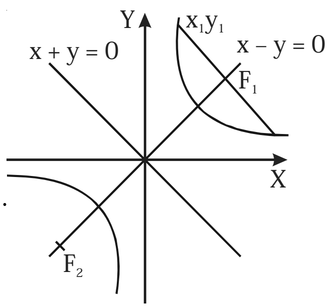
if, $X = 0$ then $Y = y - mx$

Hence, $x - \dfrac{y}{m} = 2x \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$

$$\int \frac{dy}{y} + \int \frac{dx}{x} = c \Rightarrow xy = c$$

Since it passes through (2, 4) the equation of conic is

$xy = 8$. It is a rectangular hyperbola with $e = \sqrt{2}$



Hence, the two focii are (4, 4) and (– 4, 4) .

Equation to the latus rectum is $x + y = 8$.

Solving it with $xy = 8$ we get $x^2 - 8x + 8 = 0$.

Similarly $y^2 - 8y + 8 = 0$.

Hence the equation of circle is,

$x^2 + y^2 - 8(x + y) + 16 = 0$.

Example 13. A normal is drawn at a point P(x, y) of a curve, it meets the x-axis and the y-axis in point A and B, respectively, such that $\dfrac{1}{OA} + \dfrac{1}{OB} = 1$, where O is the origin, find the equation of such a curve passing through (5, 4).

Solution The equation of the normal at(x, y) is

$$(X - x) + (Y - y)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{X}{x + y\dfrac{dy}{dx}} + \frac{Y}{\dfrac{(x + y\,dy/dx)}{dy/dx}} = 1$$

$$\Rightarrow OA = x + y\frac{dy}{dx},\ OB = \frac{\left(x + y\dfrac{dy}{dx}\right)}{\dfrac{dy}{dx}}$$

Also, $\dfrac{1}{OA} + \dfrac{1}{OB} = 1$

$\Rightarrow 1 + \dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$

$\Rightarrow (y - 1)\dfrac{dy}{dx} + (x - 1) = 0$

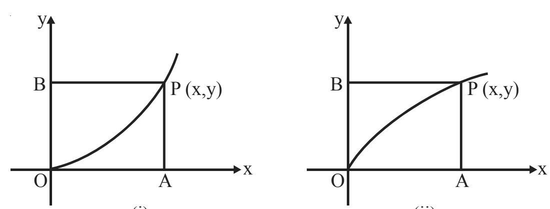
Integrating, we get $(y - 1)^2 + (x - 1)^2 = c$

Since the curve passes through (5, 4), c = 25.

Hence, the curve is $(x - 1)^2 + (y - 1)^2 = 25$.

Example 14. Through any point (x, y) of a curve which passes through the origin, lines are drawn parallel to the coordinate axes. Find the curve given that it divides the rectangle formed by the two lines and the axes into two areas, one of which is three times the other.

Solution There are two cases illustrated in the figure :



(i) (ii)

(i) Area OPB = 3 (area OAP)

$\therefore \quad xy - \int_0^x y\,dx = 3\int_0^x y\,dx$

or $\quad xy = 4\int_0^x y\,dx$

Differentiating w.r.t. x

$y + x\dfrac{dy}{dx} = 4y$

or $\quad \dfrac{dy}{y} = 3\dfrac{dx}{x}$

This on integrating yields $y = cx^3$.

Similarly (ii) yields $x = cy^3$.

Concept Problems D

1. Solve the following differential equations :

(i) $\cos(x+y)\,dy = dx$ (ii) $\frac{dy}{dx}+1=e^{x+y}$

(iii) $\frac{dy}{dx} = e^{x+y} + x^2$

(iv) $xy\frac{dy}{dx} = 1+x+y+xy$

2. Solve the following differential equations :

(i) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

(ii) $\frac{dy}{dx} = \frac{xy+y}{xy+x}$ (iii) $\frac{dy}{dx} = (x+y)^2$

(iv) $\frac{dy}{dx}\tan y = \sin(x+y) + \sin(x-y)$

3. Solve the following differential equations :

(i) $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$

(ii) $\left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx} = \frac{x+y+1}{x+y+2}$

(iii) $(2x+y+1)dy+(4x+2y-1)dx=0$

(iv) $(x+y)(dx-dy) = dx+dy$

4. Show that the equation $\frac{dy}{dx} = \frac{y}{x}$ subject to the initial condition $y(0) = 0$ has an infinite number of solutions of the form $y = Cx$. The same equation subject to the initial condition $y(0) = a \neq 0$ has no solution.

5. Show that the problem $\frac{dy}{dx} = y^{\alpha}$, $y(0) = 0$ has at least two solutions for $0 < \alpha < 1$ and one solution for $\alpha = 1$.

6. Find the solution of the equation $\frac{dy}{dx} = y\,|\ln y|^{\alpha}$, $\alpha > 0$ satisfying the initial condition $y(0) = 0$. For what values of α has the problem a unique solution?

7. Prove that a curve possessing the property that all its normals pass through a fixed point is a circle.

8. Find the equation of the curve which passes through the point $(a, 1)$ and has a subtangent with a constant length c.

Practice Problems D

9. Solve the following differential equations :

(i) $\frac{dy}{dx} = (4x+y+1)^2, y(0) = 1$

(ii) $\left(\frac{x+y-a}{x+y-b}\right)\frac{dy}{dx} = \left(\frac{x+y+a}{x+y+b}\right)$

(iii) $\frac{dy}{dx} + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$

(iv) $\frac{dy}{dx} - x\tan(y-x) = 1$

10. Solve the differential equation

$$\frac{xdx+ydy}{\sqrt{x^2+y^2}} = \frac{ydx-xdy}{x^2}$$

11. Solve $y'\sqrt{1+x+y} = x+y-1$.

12. A certain amount of substance containing 3 kg of moisture was placed into a room with a capacity of 100 cu. m and an initial air humidity of 25%. At the same temperature the saturated air contains 0.12 kg of moisture per 1 cu. m. If in twenty four hours the substance lost half of its moisture, how much moisture would remain in it at the end of another twenty four hours ?

Note: Moisture contained in a porous substance evaporates into the environment at a rate proportional to the amount of moisture in the substance as well as to the difference between the humidity of the ambient air and that of the saturated air.

13. A brick wall is 30 cm thick. Find the dependence of temperature on the distance of a point from the outer edge of the wall if the temperature is 20°C at the inner surface of the wall and 0°C at its outer surface. Find also the amount of heat the wall gives of (per 1 sq. cm) in twenty-four hours.

Note: By virtue of Newton's law the velocity Q with which heat spreads over an area element A perpendicular to the Ox axis is

$Q = -kS\frac{dT}{dt}$, where k is the thermal conductivity coefficient of the substance (k = 0.0015), T is the temperature, t is the time, and S is the area of A.

14. Show that the tangents to all integral curves of the differential equation $y' + y\tan x = x\tan x + 1$ at the points of intersection with the y-axis are parallel. Determine the angle at which the integral curves cut the y-axis.

15. A curve $y = f(x)$ passes through the origin. Lines drawn parallel to the coordinate axes through an arbitrary point of the curve form a rectangle with two sides on the axes. The curve divides every such rectangle into two region A and B, one of which has an area equal to n times the other. Find the function f.

16. A normal at P(x, y) on a curve meets the x-axis at Q and N is the foot of the ordinate at P. If

$NQ = \dfrac{x(1+y^2)}{(1+x^2)}$ find the equation of the curve,

given that it passes through the point (3, 1).

17. A particle moves on the parabola $y = x^2$, and its horizontal component of velocity is given by

$$x'(t) = \frac{1}{(t+1)^2},\ t \le 0.$$

At time t = 0 the particle is at the origin.

(a) What are the x and y coordinates of the particle when t = 1 ? When t = 3 ?

(b) As t increases without bound what happens to the particle ?

18. Find the equation of a curve for which the area contained between the x-axis, the curve and two ordinates, one of which is a constant and the other a variable, is equal to the ratio of the cube of the variable ordinate to the appropriate abscissa.

19. Discontinuous Coefficients : Linear differential equations sometimes occur in which one or both of the functions P(x) and Q(x) have jump discontinuities. If x_0 is such a point of discontinuity, then it is necessary to solve the equation separately for $x < x_0$ and for $x > x_0$. Afterwards the two solutions are matched so that y is continuous at x_0; this is accomplished by a proper choice of the arbitrary constants. The following two problems illustrate this situation.

Note: In each case that it is impossible also to make y′ continuous at x_0.

(a) Solve the initial value problem

$y' + 2y = Q(x),\ y(0) = 0$

$$\text{where } Q(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$$

(b) Solve the initial value problem

$y' + P(x)\,y = 0,\ y(0) = 1$

$$P(x) = \begin{cases} 2, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

4.6 HOMOGENEOUS DIFFERENTIAL EQUATIONS

A function f(x, y) is said to be a homogeneous function of its variables of degree n if the identity $f(tx, ty) \equiv t^n f(x,y)$ is valid.

For instance, the function $f(x, y) = x^2 + y^2 - xy$ is a homogeneous function of the second degree, since

$$f(tx, ty) = (tx)^2 + (ty)^2 - (tx)(ty)$$
$$= t^2(x^2 + y^2 - xy) = t^2 f(x,y).$$

Also, $f(x, y) = ax^{2/3} + hx^{1/3}y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3 .

For n = 0 we have the function of zero degree. For instance, $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ is a function of zero degree

$$\text{since } f(tx, ty) = \frac{(tx)^2 - (ty)^2}{(tx)^2 + (ty)^2} = \frac{t^2(x^2 - y^2)}{t^2(x^2 + y^2)}$$
$$= \frac{x^2 - y^2}{x^2 + y^2} = f(x,y).$$

A differential equation of the form $\dfrac{dy}{dx} = f(x, y)$ is homogeneous if f(x , y) is a homogeneous function of degree zero i.e. $f(tx, ty) = t^{\circ} f(x, y) = f(x, y)$. The function f does not depend on x and y separately, but only on their ratio y/x or x/y.

In other words, an equation of the form $dy/dx = \varphi(y/x)$ is said to be homogeneous.

Examples of homogeneous differential equations are the following :

$$\frac{dy}{dx} = \left(\frac{x^2 + y^2}{xy}\right)^3,\quad \frac{dy}{dx} = \frac{x}{y}\sin\frac{x^2 + y^2}{x^2 - y^2}.$$

Consider the following examples :

(a) $\dfrac{dy}{dx} = \dfrac{y^2 + 2xy}{x^2} = \left(\dfrac{y}{x}\right)^2 + 2\dfrac{y}{x};$

(b) $\dfrac{dy}{dx} = \ln x - \ln y + \dfrac{x+y}{x-y} = \ln\dfrac{1}{y/x} + \dfrac{1 + (y/x)}{1 - (y/x)};$

(c) $\dfrac{dy}{dx} = \dfrac{y^3 + 2xy}{x^2} = y\left(\dfrac{y}{x}\right)^2 + 2\dfrac{y}{x}.$

Equation (a) and (b) are homogeneous, since the right side of each can be expressed as a function of y/x; since (c) cannot be so written, it is not homogeneous. Such an equation can be solved by the substitution y = vx. We first represent the homogeneous equation

as $\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$ ...(1)

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \varphi(v)$$

or $\frac{dv}{\varphi(v) - v} = \frac{dx}{x}$

The variables have now been separated and the solution is

$$\int \frac{dv}{\varphi(v) - v} = \ln x + c \quad ...(2)$$

After the integration v should be replaced by y/x to get the required solution.

Note:

1. When solving homogeneous equations it is not obligatory to reduce them to the form (1). It is possible to make a direct substitution y = vx.
2. In most cases it is impossible to find an explicity expression for v. Then, after the integration, we should replace v in the left-hand side by $\frac{y}{x}$, which leads to the solution of the equation in implicit form.
3. It is of course supposed that f(v) – v ≠ 0. If f(v) ≡ v then $f\left(\frac{y}{x}\right) \equiv \frac{y}{x}$, and no substitution is needed since the given equation $\frac{dy}{dx} = \frac{y}{x}$ is one with the variables separable.
   As has been already pointed out, if the denominator f(v) – v turns into zero only for an isolated value v_0, the function $v = v_0$ is a solution of the transformed equation and the function $y = v_0x$ (a straight line through the origin of coordinates), satisfies the original equation.
4. It is unnecessary to learn by heart the formula (2) since it is simpler to carry out all the calculations in every case.

Example 1. Solve the equation

$$xy' = \sqrt{x^2 - y^2} + y.$$

Solution Write the equation in the form

$$y' = \sqrt{1 - \left(\frac{y}{x}\right)^2} + \frac{y}{x},$$

Put v = y/x or y = vx. Then y′ = xv′ = v. Substituting the expressions for y and y′ in the equation, we obtain

$$x\frac{dv}{dx} = \sqrt{1 - v^2}.$$

We separate the variables : $\frac{dv}{\sqrt{1 - v^2}} = \frac{dx}{x}$.

Hence, we find by integration

$\sin^{-1} v = \ln|x| + \ln C_1$ $(C_1 > 0)$ or

$\sin^{-1} v = \ln C_1|x|$.

Since $C_1|x| = \pm C_1x$,

then denoting $\pm C_1 = C$, we have $\sin^{-1} v = \ln Cx$,

where $|\ln Cx| \le \pi/2$ or, $e^{-\pi/2} \le Cx \le e^{\pi/2}$.

Replacing v by y/x we obtain the general solution

$$\sin^{-1}\frac{y}{x} = \ln Cx.$$

Hence, the general solution is y = x sin ln Cx.

When separating the variables we divided both sides of the equation by the product $x\sqrt{1 - v^2}$, we might therefore lose the solutions making this product zero.

Now put x = 0 and $\sqrt{1 - v^2} = 0$.

From the given differention equation x ≠ 0. and from $\sqrt{1 - v^2}$, we find that $1 - \frac{y^2}{x^2} = 0$,

$\Rightarrow$ y = ± x.

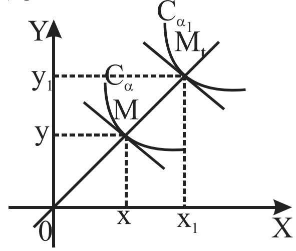
A direct check shows that the functions y = – x and y = x are also solution of the given equation.

Example 2. Consider a family of integral curves C_α of the homogeneous equation

$$y' = \varphi(y/x) \quad ....(1)$$

Show that the tangents to the curves determined by (1) are parallel at **corresponding points** (those points on curves C_α which lie on one ray issuing from the origin of coordinates).

Solution By the definition of corresponding points we have $\frac{y}{x} = \frac{y_1}{x_1}$, so that by virtue of equation (1) itself $y' = y_1'$, where y′ and y_1' are the slopes of the tangents to the integral curves C_α and C_{α_1} at the points M and M_1 respectively.



In the geometrical interpretation, the general solution of a homogeneous differential equation must represent a system of similar and similarly situated curves, the origin being a centre of similitude. For the equation $dy/dx = \varphi(y/x)$ shows that where the curves cross any arbitrary straight line $(y/x = m)$ through the origin, dy/dx has the same value for each, that is, the tangents are parallel.

Example 3. Solve $(x^2 - y^2)\,dx + 2xydy = 0$.

Solution The given equation can be written as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \qquad ...(1)$$

which is a homogeneous differential equation.
Putting $y = vx$ in (1), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2xy} = \frac{v^2 - 1}{2v}$$

Separating the variables, we have

$$\frac{2v}{v^2+1}dv = -\frac{dx}{x}$$

Integrating, we obtain $\ln(v^2 + 1) = -\ln x + \ln c$

$\Rightarrow$ $\ln(v^2 + 1) + \ln x = \ln c$

$\Rightarrow$ $\ln[(v^2 + 1)x] = \ln c$

$\Rightarrow$ $(v^2 + 1)\,x = c$

$\Rightarrow$ $(y^2 + x^2) = cx$

which is the required solution.

Example 4. Solve

$$\left(x\frac{dy}{dx} - y\right)\tan^{-1}\frac{y}{x} = x, \text{ given } y(1) = 0.$$

Solution $x\left(\frac{dy}{dx} - \frac{y}{x}\right).\tan^{-1}\frac{y}{x} = x$

$$\Rightarrow \left(\frac{dy}{dx} - \frac{y}{x}\right)\tan^{-1}\frac{y}{x} = 1$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \left(v + x\frac{dv}{dx} - v\right)\tan^{-1} v = 1$$

$$\Rightarrow x\frac{dv}{dx}\tan^{-1} v = 1$$

$$\Rightarrow \int \tan^{-1} v\, dv = \int \frac{dx}{x}$$

$$\Rightarrow v\tan^{-1}v - \int\frac{v}{1+v^2}dv = \ln x + C$$

$$\Rightarrow v.\tan^{-1}v - \frac{1}{2}\ln(1+v^2) = \ln x + C$$

where $x = 1$, $y = 0 \Rightarrow v = 0 \Rightarrow C = 0$

$$\Rightarrow \frac{y}{x}\tan^{-1}\frac{y}{x} = \ln\sqrt{1+\frac{y^2}{x^2}} + \ln x$$

$$= \ln x\sqrt{1+\frac{y^2}{x^2}} = \ln\sqrt{x^2+y^2}$$

$$\therefore \sqrt{x^2+y^2} = e^{\frac{y}{x}\tan^{-1}\frac{y}{x}}.$$

Example 5. Solve $xdy - ydx = \sqrt{x^2+y^2}\,dx$.

Solution The given equation can be written as

$$x\frac{dy}{dx} - y = \sqrt{x^2+y^2}$$

Substituting $y = vx$ in this equation and separating the variables, we get $\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

Integrating, we obtain

$$\ln[v + \sqrt{v^2+1}] = \ln x + \ln c$$

$$y + \sqrt{y^2+x^2} = cx^2 \qquad \left(\text{since } v = \frac{y}{x}\right)$$

This is the required solution.

Example 6. A line is drawn from a point P(x, y) on curve y = f(x), making an angle with the x-axis which is supplementary to the one made by the tangent to the curve at P(x, y). The line meets the x-axis at A. Another lien perpendicular to the first, is drawn from P(x, y) meeting the y-axis at B. If OA = OB, where O is the origin, find the curve which passes through (1, 1).

Solution The equation of the line through P(x, y) making an angle with the x-axis which is suplementary to the angle made by the tangent at P(x, y) is

$$Y - y = -\frac{dy}{dx}(X - x) \qquad ...(1)$$

At the point where it meets the x-axis

$$Y = 0, X = x + \frac{y}{\frac{dy}{dx}} \Rightarrow OA = x + \frac{y}{\frac{dy}{dx}}. \qquad ....(2)$$

The line through P(x, y) and perpendicular to (1) is

$$Y - y = \frac{dy}{dx}(X - x).$$

At the point where it meets the y-axis

$$X = 0, Y = y - \frac{x}{\frac{dy}{dx}} \Rightarrow OB = y\frac{x}{\frac{dy}{dx}}.$$

Since OA = OB,

$$x + \frac{y}{\frac{dy}{dx}} = y - \frac{x}{\frac{dy}{dx}} \Rightarrow (y - x) = \frac{y + x}{\frac{dy}{dx}} \Rightarrow \frac{dy}{dx} = \frac{y + x}{y - x}$$

Writing $y = vx$ this equation becomes

$$v + x\frac{dv}{dx} = \frac{1+v}{v-1} \qquad \Rightarrow x\frac{dv}{dx} = \frac{1 + 2v - v^2}{v - 1}$$

$$\Rightarrow \frac{(1-v)dv}{1 + 2v - v^2} + \frac{dx}{x} = 0$$

$\Rightarrow$ $\ell n\,(12v - v^2) + \ell n\; x^2 = \text{constant}$

$\Rightarrow$ $x^2 + 2xy - y^2 = c.$

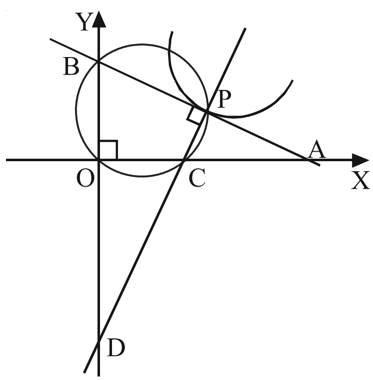
Since the curve passes through (1, 1), c = 2.

Hence, the required curve is $x^2 - y^2 + 2xy = 2$.

Example 7. The tangent and a normal to a curve at any point P meet the x and y axis at A, B, C and D respectively. Find the equation of the curve passing through (1, 0) if the centre of circle through O, C, P and B lies on the line $y = x$ (where O is origin).

Solution Let P(x, y) be a point on the curve.

$$C \equiv \left(x + y\frac{dy}{dx}, 0\right) \qquad B \equiv \left(0, y - x\frac{dy}{dx}\right)$$



Circle passing through O, C, P and B has its centre at mid-point of BC.

Let the centre of the circle be (α, β)

$$\Rightarrow 2\alpha = x + y\frac{dy}{dx} \text{ and } 2\beta = y - x\frac{dy}{dx}$$

and since $\beta = \alpha$, $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{y - x}{y + x}$

Let $y = vx$

$$\Rightarrow x\frac{dv}{dx} = \frac{(1+v^2)}{1+v}$$

$$\Rightarrow \frac{1+v}{v^2+1}dv = -\frac{dx}{x}$$

Integrating both sides we get,

$$\int \frac{1+v}{v^2+1}dv = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int \frac{2v}{v^2+1}dv + \int \frac{dv}{v^2+1} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\ln|v^2 + 1| + \tan^{-1} v = -\ln x + c$$

$$\Rightarrow \ln\{(\sqrt{v^2+1})\,x\} + \tan^{-1} v = c$$

$$\Rightarrow \ln\sqrt{x^2+y^2} + \tan^{-1}\frac{y}{x} = c$$

Given that if $x = 1, y = 0$, we have

$\Rightarrow$ $\ln 1 + \tan^{-1} 0 = c \Rightarrow c = 0$

$\therefore$ The required curve is

$$\ln\sqrt{x^2+y^2} + \tan^{-1}\left(\frac{y}{x}\right) = 0.$$

Example 8. Find a curve for which the product of the abscissa of any point P and the intercept made by a normal at P on the y-axis plus twice the product of the square of the distance of the point P from the origin and gradient of the normal is equal to zero.

Solution $Y - y = -\dfrac{1}{m}(X - x)$

$$x = 0 \quad \Rightarrow \quad Y = y + \frac{x}{m}$$

$$\text{Hence, } x\left(y + \frac{x}{m}\right) - \frac{2\,(x^2 + y^2)}{m} = 0$$

$$\Rightarrow m = \frac{x^2 + 2y^2}{xy} = \frac{dy}{dx}$$

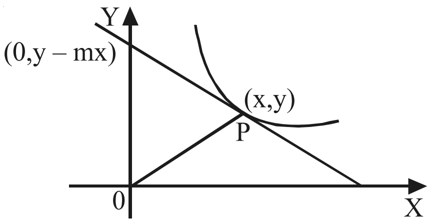
We solve using $y = vx$, to get $x^2 + y^2 = cx^4$.

Example 9. Show that the curve for which the ratio between the length of a segment intercepted by a tangent on the y–axis and the length of the radius vector is constant k, is given by

$$y = \frac{c}{2}x^{1\pm k} - \frac{1}{2c}x^{1\mp k}$$

where c is an arbitrary constant.

Solution



$$|y - mx| = k\sqrt{x^2 + y^2}$$

or $\dfrac{dy}{dx} = \dfrac{y \pm k\sqrt{x^2+y^2}}{x}$ Put $y = vx$

$$\Rightarrow \int \frac{dv}{\sqrt{1+v^2}} = \pm k \int \frac{dx}{x}$$

$$\Rightarrow \ln\left[v + \sqrt{1+v^2}\right] = \pm k \,.\, \ln x + c$$

$$\Rightarrow \ln \frac{y + \sqrt{x^2+y^2}}{x} = \pm k \ln x + c = \ln c \,.\, x^{\pm k}$$

$$\Rightarrow y + \sqrt{x^2+y^2} = c \,.\, x^{1\pm k}$$

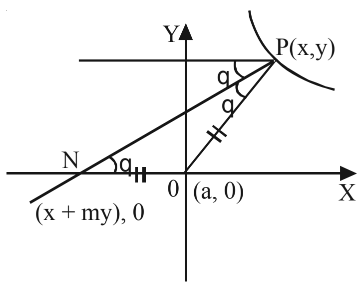
$$\Rightarrow x^2 + y^2 = (c \,.\, x^{1\pm k} - y)^2$$

$$\Rightarrow x^2 + y^2 = c^2 x^{2(1\pm k)} + y^2 - 2cy\,x^{1\pm k}$$

$$\Rightarrow y = \frac{c}{2} x^{1\pm k} - \frac{1}{2c} x^{1\mp k}$$

Example 10. The light rays emanating from a point source situated at origin when reflected from the mirror of a search light are reflected as beam parallel to the x – axis . Show that the surface is parabolic, by first forming the differential equation and then solving it .

Solution



Equation of normal at P

$$Y - y = -\frac{1}{m}(X - x)$$

Put $Y = 0$

$\Rightarrow X = x + my$

Now, $(OP)^2 = (ON)^2$

$x^2 + y^2 = (x + my)^2$

$$\Rightarrow m = -\frac{x + \sqrt{x^2 - y^2}}{y} = \frac{dy}{dx}$$

Put $y = vx$ to get

$$\int \frac{v\,dv}{\sqrt{1+v^2}\left[\pm 1 - \sqrt{1+v^2}\right]} = \int \frac{dx}{x}$$

Put $\pm 1 - \sqrt{1+v^2} = t$

$$-\int \frac{dt}{t} = \ln x + c$$

Now we proceed to get $y^2 = c^2 \pm 2cx$.

Example 11. Solve $(1 + 2e^{x/y})dx + 2e^{x/y}(1 - x/y)dy = 0$.

Solution The appearance of x/y in the equation suggests the substitution $x = vy$ or $dx = v\,dy + y\,dv$.

$\therefore$ The given equation is

$(1 + 2e^v)(v\,dy + y\,dv) + 2e^v(1 - v)\,dy = 0$

i.e. $y(1 + 2e^v)dv + (v + 2e^v)dy = 0$

i.e. $\dfrac{1+2e^v}{v+2e^v}dv + \dfrac{dy}{y} = 0$

Integrating, $\displaystyle\int \frac{1+2e^v}{v+2e^v}dv + \int \frac{dy}{y} = 0$

$\ln|v + 2e^v| + \ln|y| = \ln|c|$

$$\Rightarrow (v + 2e^v)\,y = c \left\{v = \frac{x}{y}\right\}$$

$$\Rightarrow \left(\frac{x}{y} + 2e^{x/y}\right) y = c$$

$\Rightarrow (x + 2ye^{x/y}) = c.$

Example 12. Solve the differential equation

$$\frac{dy}{dx} = \frac{(x+y)^2}{(x+2)(y-2)}.$$

Solution On putting $X = x + 2$ and $Y = y - 2$, the given differential eqution reduces to

$$\frac{dY}{dX} = \frac{(X - 2 + Y + 2)}{XY} = \frac{(X+Y)^2}{XY}$$

Put $Y = VX \Rightarrow \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

$$V + X\frac{dV}{dX} = \frac{(1+V)^2}{V} \Rightarrow \frac{V}{2V+1}dV = \frac{dX}{X},$$

$$\left(1 - \frac{1}{1+2V}\right)dV = \frac{2dX}{X}$$

Integrating, $\int\left(1-\frac{1}{1+2V}\right)dV = \int\frac{2dX}{X}$

$\Rightarrow V - \frac{1}{2}\ln(1+2V) = 2\ln X + C$

$\Rightarrow X^4\left(1+\frac{2Y}{X}\right) = Ke^{2Y/X}$

where $X = x+2$, and $Y = y-2$.

Differential Equations Reducible to Homogeneous Form

If the differential equation is of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad ...(1)$$

it can be reduced to a homogeneous differential equation as follows :

Put $x = X+h,\ y = Y+k$ ...(2)

where X and Y are new variables and h and k are constants yet to be chosen. From (2)

$dx = dX, dy = dY$

Equation (1), thus reduces to

$$\frac{dY}{dX} = \frac{aX+bY+(ah+bc+c)}{a'X+b'Y+(a'h+b'k+c')} \quad ...(3)$$

In order to have equation (3) as a homogeneous differential equation, choose h and k such that the following equations are satisfied :

$$\left.\begin{aligned} ah+bk+c=0 \\ a'h+b'k+c'=0 \end{aligned}\right\} \quad ...(4)$$

Now, (3) becomes

$$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y} \quad ...(5)$$

which is a homogeneous differential equation and can be solved by putting $Y = vX$.

Note: Solving equations (4), we obtain

$$\frac{h}{bc'-b'c} = \frac{k}{a'c-ac'} = \frac{1}{ab'-a'b}$$

If $ab'-a'b = 0$, the above method does not apply. In such cases, we have $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{t}$

Equation (1) now becomes

$$\frac{dy}{dx} = \frac{ax+by+c}{t(ax+by)+c'} \quad ...(6)$$

which can be solved by putting $ax+by = v$ so that

$$a + b\frac{dy}{dx} = \frac{dv}{dx}.$$

Hence, Eq. (6) takes the form

$$\frac{dv}{dx} = a + b\frac{v+c}{tv+c'}$$

Now separate the variables and integrate to get the required solution.

Example 13. Solve $(2x+y-3)\,dy = (x+2y-3)\,dx$.

Solution The given differential equation is

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

Putting $x = X+h$, $y = Y+k$, we get

$$\frac{dY}{dX} = \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \quad ...(1)$$

Choose h and k such that

$h+2k-3=0, \qquad 2h+k-3=0$

Solving these equations, we get $h = k = 1$. Equation (1) can now be written as $\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$ ...(2)

where $X = x-1$ and $Y = y-1$.

Putting $Y = vX$ in (2) and simplifying, we get

$$\frac{2+v}{1-v^2}dv = \frac{dX}{X}$$

Resolving into partial fraction and integrating, we get

$$\int\left(\frac{1}{2}\frac{1}{1+v}+\frac{3}{2}\frac{1}{1-v}\right)dv = \int\frac{dX}{X}$$

$\Rightarrow \frac{1}{2}\ln(1+v) - \frac{3}{2}\ln(1-v) = \ln X + \ln c$

$\Rightarrow \ln\left[\frac{1+v}{(1-v)^3}\right] = 2\ln Xc = \ln(cX)^2$

$\Rightarrow \frac{1+v}{(1-v)^3} = (cX)^2$

Now substituting $v = Y/X = (y-1)/(x-1)$, we get

$(x+y-2) = c^2(x-y)^3$

which is the required solution.

Example 14. Solve $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$.

Solution Here, the coefficients of x and y, in the numerator and denominator of the right-hand side are proportional. Therefore, we put

$x-y+3 = v$ ...(1)

Differentiating (1), we get $\frac{dy}{dx} = 1 - \frac{dv}{dx}$

Substituting these values in the given differential equation, we get $\frac{2v-1}{v-1}dv = dx$

$\Rightarrow \left(2+\frac{1}{v-1}\right)dv = dx$

Integration yields $2v + \ln(v-1) = x + c$

$\Rightarrow 2(x-y+3) + \ln(x-y+3-1) = x + c$

$\Rightarrow (x-2y) + \ln(x-y+2) = c_1,\ (c_1 = c-6)$

Substitution

Sometimes an equation can be reduced to a homogeneous one by substituting z^α for y. This is the case when all the terms in the equation are of the same degree once the variable x is assigned degree 1, the variable y degree α, and the derivative

$\frac{dy}{dx}$ degree $\alpha - 1$.

Example 15. Solve the equation

$(x^2y^2 - 1)\,dy + 2xy^3\,dx = 0$ ...(1)

Solution We replace y by z^α, dy by $\alpha z^{\alpha-1}dz$, where α is as yet an arbitrary number to be chosen later on. Substituting in the equation the expressions for y and dy we get

$(x^2z^{2\alpha} - 1)\,\alpha z^{\alpha-1}dz + 2xz^{3\alpha}dx = 0$

or $(x^2z^{3\alpha-1} - z^{\alpha-1})\alpha\,dz + 2xz^{3\alpha}dx = 0.$

Notice that $x^2z^{3\alpha-1}$ is of degree $2 + 3\alpha - 1 = 3\alpha + 1$, $z^{\alpha-1}$ is of degree $\alpha - 1$, and $xz^{3\alpha}$ is of degree $1 + 3\alpha$. The equation obtained will be homogeneous if all the terms are of the same degree, i.e. if the condition

$3\alpha + 1 = \alpha - 1$

or, $\alpha = -1$ is fulfilled.

Putting $y = 1/z$, the original equation will take the form

$$\left(\frac{1}{z^2} - \frac{x^2}{z^2}\right)dz + 2\frac{x}{z^3}dx = 0$$

or, $(z^2 - x^2)\,dz + 2zx\,dx = 0$

Now put $z = ux$, $dz = u\,dx + x\,du$.

Then this equation will assume the form

$(u^2 - 1)(u\,dx + x\,du) + 2u\,dx = 0,$

$u(u^2+1)\,dx + x(u^2-1)\,du = 0.$

We separate the variables in the equation :

$$\frac{dx}{x} + \frac{u^2-1}{u^3+u}du = 0.$$

Integrating we find,

$\ln|x| + \ln(u^2+1) - \ln|u| = \ln C$

or, $\frac{x(u^2+1)}{u} = C.$

Replacing u by $1/xy$ we obtain the general integral of the given equation : $1 + x^2y^2 = Cy$.

Equation (1) has another obvious solution $y = 0$ which is obtained from the general integral when $C \to \infty$, if we write the integral as $y = (1 + x^2y^2)/C$ and then proceed to the limit when $C \to \infty$. Thus, the function $y = 0$ is a particular solution of the original equation.

Practice Problems E

1. Show that the equation $y' = f(x,y)$ is homogeneous if $f(x,y)$ is such that $f(x, tx) = f(1, t)$, where t is a real parameter. Use this fact to determine whether each of the following equation is homogeneous.

   (i) $y' = \frac{x^3 + xy + y^3}{x^2y + xy^2}$

   (ii) $y' = \ln x - \ln y + \frac{x+y}{x-y}$

   (iii) $y' = \frac{(x^2 + 3xy + 4y^2)^{1/2}}{x + 2y}$

   (iv) $y' = \frac{\sin(xy)}{x^2 + y^2}$

2. Solve the following differential equations :

   (i) $(3xy + y^2)dx + (x^2 + xy)dy = 0$

   (ii) $x\frac{dy}{dx} = y - x\tan\frac{y}{x}$

   (iii) $\frac{dy}{dx} = \frac{6x^2 + 2y^2}{x^2 + 4xy}$

   (iv) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

3. Solve the following differential equations :

   (i) $\frac{dy}{dx} = \frac{6x - 2y - 7}{3y + 2x - 6}$

   (ii) $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$

   (iii) $(x + y)dx + (3x + 3y - 4)dy = 0$

   (iv) $(x - y)dy = (x + y + 1)dx.$

4. Solve $(x^2 - y^2)dx + 2\,xy\,dy = 0$, $y(1) = 2$

5. Solve $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

6. Find a curve for which the product of the abscissa of some point and the length of the segment

intercepted by a normal on the y-axis is twice the square of the distance from that point to the origin.

7. Show that the differential equation $y^3dy + (x + y^2)\,dx = 0$ can be reduced to a homogeneous equation. Hence, solve it.
8. Find the curve such that the initial ordinate of any tangent is equal to the corresponding subnormal.
9. Solve the following differential equations :
   (i) $2xy'(x - y^2) + y^3 = 0$
   (ii) $4y^6 + x^3 = 6xy^5y'$
   (iii) $y\left(1+\sqrt{x^2y^4+1}\right)dx + 2x\,dy = 0$
   (iv) $(x + y^3)\,dx + 3(y^3 - x)\,y^2\,dy = 0$
10. Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hourse for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely ?
11. Show that the half-life H of a radioactive substance can be obtained from two measurements $y_1 = y(t_1)$ and $y_2 = y(t_2)$ of the amount present at times t_1 and t_2 by the formula $H = [(t_2 - t_1)\ln 2]/\ln(y_1/y_2)$.
12. Suppose that a moth ball loses volume by evaporation at a rate proportional to its instantaneous area. If the diameter of the ball decreases from 2 to 1 cm in 3 months, how long will it take until the ball has practically gone (say until its diameter is 1 mm)?
13. A body in a room at 60° cools from 200° to 120° in half an hour.
    (a) Show that its tmperature after t minutes is $60 + 140e^{-kt}$, where $k = (\ln 7 - \ln 3)/30$.
    (b) Show that the time t required to reach a temperature of T degrees is given by the formula $t = [\ln 140 - \ln(T - 60)]/k$, where $60 < T \le 200$.
    (c) Find the time at which the temperature is 90°.
    (d) Find a formula for the temperature of the body at time t if the room temperature is not kept constant but falls at a rate of 1° each ten minutes. Assume the room temperature is 60° when the body temperature is 200°.
14. In a tank are 100 litres of brine containing 50 kg of dissolved salt. Water runs into the tank at the rate of 3 litres per minute, and the concentration is kept uniform by stirring. How much salt is in the tank at the end of one hour if the mixture runs out at a rate of 2 litres per minute ?
15. A motorboat moves in still water with a speed v = 10 km/h. At full speed its engine was cut off and in 20 seconds the speed was reduced to v_1 = 6 km/h. Assuming that the force of water resistance to the moving boat is proportional to its speed, find the speed of the boat in two minutes after the engine was shut off; find also the distance travelled by the boat during one minute with the engine dead.

4.7 LINEAR DIFFERENTIAL EQUATIONS

The linear differential equations are those in which the dependent variable and its derivative appear only in their first degree and are not multiplied together. The coefficients may be constants or functions of x alone. The general linear differential equation of order n is of the form $y^{(n)} + P_1(x)y^{(n-1)} + P_2(x)y^{(n-2)} + \ldots + P_n(x)y = Q(x)$ ...(1)

where $P_1, P_2, \ldots, P_n$, Q are either constants or functions of x alone. Here $P_1, P_2, \ldots, P_n$ are called the coefficients of the differential equation.

Note: That a linear differential equation is always of the first degree but every differental equation of the first degree need not be linear. For example, the differential equation

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$$ is not linear, though its degree is 1. A differential equation that is not of the above form is a **non linear** differential equation. Consider another example of a nonlinear equation :

$$\frac{d^2y}{dx^2} + \frac{y}{x}\sin y = x^2.$$

The linear differential equation of the first order is of the form $\frac{dy}{dx} + P(x)y = Q(x)$ ...(1)

where P and Q are constants or functions of x alone.

Method 1

To solve the differential equation (1), we multiply both sides of (1) by $e^{\int P(x)dx}$. Then we get

$$e^{\int P(x)dx}\left(\frac{dy}{dx} + P(x)y\right) = e^{\int P(x)dx}\cdot Q(x)$$

or $\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = e^{\int P(x)dx}.Q(x)$

Integrating both sides, we get

$$ye^{\int P(x)dx} = \int Q(x)\, e^{\int P(x)dx}\, dx + C$$

which is the general solution of (1).

Note: The factor $e^{\int P(x)dx}$, which on multiplying to the left hand side of (1), makes the left hand side the exact derivative of some function of x and y, is called the **integrating factor** of the given differential equation.

Example 1. Solve $(1+x)\frac{dy}{dx} - xy = 1 - x$.

Solution The given equation can be written as

$$\frac{dy}{dx} - \frac{x}{1+x}y = \frac{1-x}{1+x}$$

Here $P = -\frac{x}{1+x} = \frac{1}{1+x} - 1,\quad Q = \frac{1-x}{1+x}$

The integrating factor (I. F.) is thus, given by

$$\text{I.F.} = e^{\int Pdx} = \exp\left[\int\left(\frac{1}{1+x} - 1\right)dx\right] = e^{\ln(1+x) - x}$$

Therefore, the solution of the given differential is

$$ye^{\int Pdx} = \int Qe^{\int Pdx}\, dx + C$$

or, $y(1+x)\,e^{-x} = \int \frac{1-x}{1+x}(1+x)\,e^{-x}\,dx + C$

$$= \int (e^{-x} - xe^{-x})\,dx + C = xe^{-x} + C$$

or, $y(1+x) = x + Ce^{x}$.

Proof Let us now estabilish the method given abovee. The following artificial technique reduces equation (1) to two equations with variables separable. Let us represent the function y as a product of two functions : $y = u(x)v(x)$. One of these functions can be taken quite arbitrarily and the other should then be determined, depending on the former, in such a way that the whole product satisfies the given linear equation. The arbitrariness in the choice of one of the functions u and v is used for simplifying the resultant equation appearing after the substitution.

From the equality $y = uv$ we find the derivative y':

$$y' = u'v + uv'$$

Substituting this expression into equation (1) we obtain

$$u'v + uv' + p(x)uv = q(x),$$

i.e. $u'v + u(v' + p(x)v) = q(x)$

Now let us take as v an arbitrary particular solution of the equation

$$v' + p(x)v = 0 \qquad ...(2)$$

Then we obtain the equation

$$u'v = q(x) \qquad ...(3)$$

for determining u.

We shall begin with finding v from equation(2). On separating the variables we obtain

$$\frac{dv}{v} = -p(x)dx$$

$\Rightarrow \quad \ell nv = -\int p(x)dx$, that is $v = e^{-\int p(x)dx}$,

As was agreed, the sign of the indefinite integral designates here a particular antiderivative of p(x), and therefore v is a completely determined function of x. Now knowing v, we readily find u from equation(3):

$$\frac{du}{dx} = \frac{q(x)}{v} = q(x)e^{\int p(x)dx},\ du = q(x)e^{\int p(x)dx}dx$$

and hence , $u = \int q(x)e^{\int p(x)dx}dx + C$

In the latter formula expressing u we take the family of all the antiderivatives. Finally, knowing u and v we determine the sought for function y :

$$y = uv = e^{-\int p(x)dx}\left[\int q(x)e^{\int p(x)dx}dx + C\right]$$

This formula expresses the general solution of linear equation (1).

The final result remains unchanged if we add an arbitrary constant C_1 to the integrals standing as exponents. Indeed, this second arbitrary constant is cancelled out in the ultimate formula since one factor has e^{C1} in the denominator while the other has it in the numerator.

Let us solve the equation $y' + \frac{1}{x}y = \frac{\sin x}{x}$.

We put $y = uv$; then $y' = u'v + uv'$.

Furthermore, $u'v + uv' + \frac{1}{x}uv = \frac{\sin x}{x}$, and thus

$$u'v + u\left(v' + \frac{v}{x}\right) = \frac{\sin x}{x}.$$

Let $v' + \frac{v}{x} = 0$. Then $\frac{dv}{v} = -\frac{dx}{x}$, and, hence,

$$\ell n\,|\,v\,| = -\ell n|x|,\ \text{i.e. } v = \frac{1}{x}.$$

Consequently. $u'.\frac{1}{x} = \frac{\sin x}{x}$, which gives

$$u' = \sin x \text{ and } u = -\cos x + c.$$

Finally, we obtain $y = uv = \frac{1}{x}(-\cos x + C)$.

Example 2. Solve the differential equation

$$x^2\frac{dy}{dx} - 3xy - 2x^2 = 4x^4$$

Solution To put this in the standard form, we add $2x^2$ to both sides and then divide by x^2.

The result is $\dfrac{dy}{dx} - \dfrac{3}{x}y = 4x^2 + 2$,

where $P(x) = -\dfrac{3}{x}$ and $Q(x) = 4x^2 + 2$.

An antiderivative of P(x) is given by

$$\int P(x)dx = \int -\frac{3}{x}dx = -3\ln|x| = \ln|x^{-3}|,$$

and it follows that the function

$$\varphi(x) = e^{\int P(x)dx} = e^{\ln|x^{-3}|} = |x^{-3}|$$

is an integrating factor, It is seen that if $\varphi(x)$ is an integrating factor, then so also is $-\varphi(x)$. Hence we may drop the absolute values and write simply

$\varphi(x) = x^{-3}$.

Multiplying both sides by x^{-3}, we obtain

$$x^{-3}\frac{dy}{dx} - 3x^{-4}y = 4x^{-1} + 2x^{-3}.$$

It is easy to see that the left side of this equation is equal to $\dfrac{d}{dx}(x^{-3}y)$. Hence $\dfrac{d}{dx}(x^{-3}y) = 4x^{-1} + 2x^{-3}$, and

$$\text{so, } x^{-3}y = \int(4x^{-1} + 2x^{-3})dx + c = 4\ln|x| + 2\frac{x^{-2}}{-2} + c$$

$$= 4\ln|x| - \frac{1}{x^2} + c,$$

where c is an arbitrary constant. It follows that

$y = 4x^3\ln|x| - x + cx^3$ is the general solution.

Example 3. Solve $ydx - xdy + \ln xdx = 0$.

Solution The given equation can be written as

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x}\ln x$$

Here $\quad P = \dfrac{1}{x}, Q = \dfrac{1}{x}\ln x$

Therefore, I.F. $= e^{\int Pdx} = e^{-\int Pdx} = e^{-\ln x} = \dfrac{1}{x}$

Hence, the solution is $y\,e^{\int Pdx} = \displaystyle\int Q\,e^{\int Pdx}dx + C$

or $\quad y\dfrac{1}{x} = \displaystyle\int\left(\frac{1}{x}\ln x\right)\frac{1}{x}dx + c$

Putting, $\ln x = t$ and integrating by parts the integral on right hand side, we get

$y = cx - (1 + \ln x)$.

Example 4. Solve $(1-x^2)\dfrac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$.

Solution The given equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x(1-x^2)^{1/2}}{1-x^2}$$

where

$$\text{I.F.} = e^{\int Pdx} = \exp\left(\int\frac{2x}{1-x^2}dx\right) = e^{-\ln(1-x^2)} = \frac{1}{1-x^2}$$

Therefore, the solution of the given differential equation is $ye^{\int Pdx} = \displaystyle\int Q\,e^{\int Pdx}dx + C$

or, $\quad y\dfrac{1}{1-x^2} = \displaystyle\int\frac{x(1-x^2)^{1/2}}{1-x^2}\frac{1}{1-x^2}dx + C$

Putting, $t = 1 - x^2$ and integrating the integral on the right hand side, we get

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + C \text{ or } y = \sqrt{1-x^2} + C(1-x^2).$$

Note: Sometimes a given differential equation becomes linear if we take x as the dependent variable and y as the independent variable, i.e. it can be written in the form $\dfrac{dx}{dy} + P_1x = Q_1$

where P_1 and Q_1 are constants or functions of y alone.

In this case, the solution is $xe^{\int P_1dy} = \displaystyle\int Q_1e^{\int P_1dy}dy + c$

Example 5. Solve $(1 + y^2)\,dx = (\tan^{-1}y - x)\,dy$.

Solution The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}.$$

Here I.F. $= e^{\int Pdy} = \exp\left(\displaystyle\int\frac{1}{1+y^2}dy\right) = e^{\tan^{-1}y}$

The solution of the given differential equation, therefore, is

$$xe^{\int Pdy} = \int Qe^{\int Pdy}dy + C$$

$$xe^{\tan^{-1}y} = \int\frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}}dy + c$$

Put $t = \tan^{-1}y$ in the integral on the right-hand side, and integrate, we get

$$xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + C.$$

Method 2

A linear differential equations of the first order can be written in the form

$$y' = f(x)y + g(x) \quad ...(1)$$

If $g(x) \equiv 0$, then the linear differential equation (1) is called homogeneous. It has the form

$$y' = f(x)y \quad ...(2)$$

Equation (2) is an equation with variables separable. It can be shown that all solutions of this equation are given by the formula

$$y = Ce^{F(x)}, \quad ...(3)$$

where F(x) is a certain primitive of the function f(x), and C is an arbitrary constant. In particular, if the function f(x) is constant, say, f(x) = k for any x, then the equation y' = ky has the general solution $y = Ce^{kx}$

If $f(x) \equiv 0$, then equation (1) takes the form y' = g(x)

As is known, the general solution of this equation will be $y = G(x) + C$

where G(x) is a certain primitive of the function g(x) and C is an arbitrary constant (the constant of integration)

Theorem If $y = \phi(x)$ is a certain solution of equation (1), then all solutions of this equation are given by the formula

$$y = Ce^{F(x)} + \phi(x) \quad ...(4)$$

where $Ce^{F(x)}$ is the general solution of homogeneous equation (2).

Proof First of all we check whether for any value of the constant C, function (4) is a solution of equation (1). Indeed,

$$y' = Ce^{F(x)}F'(x) + \phi'(x) = Ce^{F(x)}f(x) + f(x)\phi(x) + g(x)$$
$$= f(x)(Ce^{F(x)} + \phi(x)) + g(x) = f(x)y + g(x).$$

Let now $\psi(x)$ be a certain solution of equation (1). We are going to show that the function $y = \psi(x) - \phi(x)$ is a solution of homogeneous equation (2):

$$y' = \psi'(x) - \phi'(x) = f(x)\,\psi(x) + g(x) - f(x)\phi(x) - g(x)$$
$$= f(x)(\psi(x) - \phi(x)) = f(x)y.$$

Therefore, there exists a constant C such that

$$\psi(x) - \phi(x) = Ce^{F(x)}$$

and, consequently,

$$\psi(x) = Ce^{F(x)} + \phi(x).$$

Thus, any solution of equation (1) is obtained by formula (4) for a certain value of the constant C.

The theorem has been proved.

It follows from this theorem that, in order to find the general solution of equation (1), it is sufficient to find at least one of its particular solutions.

Example 6. Find the general solution of the differential equation $y' + xy = 4x$

Solution By trial and error, we find that the function y = 4 is the solution of the given nonhomogeneous equation. Let us now find the general solution of the corresponding homogeneous equation $y' + xy = 0$

By formula (4), the general solution of this equation has the form

$$y = Ce^{-\frac{1}{2}x^2}$$

According to the above proved theorem, the general solution of the given nonhomogeneous linear equation is given by the formula

$$y = Ce^{-\frac{1}{2}x^2} + 4$$

where C is an arbitrary constant.

Now we are going to indicate the method for finding the particular solution of the non-homogeneous equation

$$y' = f(x)y + g(x) \quad ...(1)$$

Let $y = Ce^{F(x)}$ ...(2)

be the general solution of the homogeneous linear equation

$$y' = f(x)y, \quad ...(3)$$

then we shall find the particular solution of equation (1) in the following form :

$$y = u(x)e^{F(x)}, \quad ...(4)$$

where u(x) is an unknown function. Substituting function (4) into equation (1), we get

$$u'e^F + ue^F f = fue^F + g$$

and finally, $u' = g(x)e^{-F(x)}$

Consequently, the function u(x) is certain primitive of the function $g(x)\,e^{-F(x)}$.

Thus, in order to find the particular solution of nonhomogeneous equation (1), we have to replace the constant C by a certain primitive for the function $g(x)\,e^{-F(x)}$ in the general solution (2) of the corresponding homogeneous equation (3).

This way of finding the particular solution is called the method of variation of the constant.

Example 7. Find a solution of the equation $y' - y = \cos x - \sin x$ satisfying the condition that y should be bounded when $x \to \infty$.

Solution The general solution of the equation is

$$y = Ce^x + \sin x.$$

Any solution of the equation obtained from the general solution when $C \neq 0$ will be unbounded since when $x \to \infty$ the function sin x is bounded and $e^x \to \infty$. It follows that the given equation has a unique solution $y = \sin x$, bounded when $x \to \infty$, which is obtained from the general solution when C = 0.

Example 8. If (y_1, y_2) are two solutions of the differential equation $\frac{dy}{dx} + P(x)\,.\,y = Q(x)$. Then prove that $y = y_1 + c(y_1 - y_2)$ is the general solution of the equation where c is any constant. For what relation between the constant α, β will the linear combination $\alpha y_1 + \beta y_2$ also be a solution.

Solution As y_1, y_2 are the solutions of the differential equation

$$\frac{dy}{dx}+P(x).y=Q(x) \quad ...(1)$$

$$\frac{dy_1}{dx}+P(x).y_1=Q(x) \quad ...(2)$$

and $$\frac{dy_2}{dx}+P(x).y_2=Q(x) \quad ...(3)$$

From (1) and (2)

$$\left(\frac{dy}{dx}-\frac{dy_1}{dx}\right)+P(x)(y-y_1)=0$$

$$\therefore \frac{d}{dx}(y-y_1)+P(x).(y-y_1)=0 \quad ...(4)$$

From (2) and (3),

$$\frac{d}{dx}(y_1-y_2)+P(x)(y_1-y_2)=0 \quad ...(5)$$

From (4) and (5), $$\frac{\frac{d}{dx}(y-y_1)}{\frac{d}{dx}(y_1-y_2)}=\frac{y-y_1}{y_1-y_2}$$

$$\Rightarrow \frac{d(y-y_1)}{y-y_2}=\frac{d(y_1-y_2)}{y_1-y_2}$$

Integrating both sides we get,

$\ln(y-y_1)=\ln(y_1-y_2)$

$\therefore\ y=y_1+c(y_1-y_2)$

Now, $y=\alpha y_1+\beta y_2$ will be a solution if

$$\frac{d}{dx}(\alpha y_1+\beta y_2)+P(x)(\alpha y_1+\beta y_2)=Q(x)$$

or $$\alpha\left(\frac{dy_1}{dx}+P(x)y_1\right)+\beta\left(\frac{dy_2}{dx}+P(x).y_2\right)=Q(x)$$

or $\alpha Q(x)+\beta Q(x)=Q(x)$ using (2) and (3)

$(\alpha+\beta)Q(x)=Q(x)$

Hence, $\alpha+\beta=1$.

Example 9. Show that if y_1 and y_2 be solutions of the equation $\frac{dy}{dx}+Py=Q$.where P and Q are functions of x alone, and $y_2=y_1z$, then $z=1+ae^{\int -Q/y_1\,dx}$, where a is an arbitrary constant.

Solution $y_2=y_1z$.

$$\frac{dy_2}{dx}=z\frac{dy_1}{dx}+y_1\frac{dz}{dx}$$

As y_2 is a solution of the given equation,

$$\frac{dy_2}{dx}+Py_2=Q$$

Substituting in this value of $\frac{dy_2}{dx}$ and y_2, we get

$$z\frac{dy_1}{dx}+y_1\frac{dz}{dx}+P\,y_1z=Q$$

or $$z\left(\frac{dy_1}{dx}+Py_1\right)+y_1\frac{dz}{dx}=Q$$

or $$zQ+y_1\frac{dz}{dx}=Q \text{ as } \frac{dy_1}{dx}+Py_1=Q$$

or $$\frac{dz}{z-1}=-\frac{Q}{y_1}dx$$

Integrating, $\ln(z-1)=C+\int -\frac{Q}{y_1}dx$

or $z=1+e^{\int -Q/y_1}dx$.

This proves the result.

Example 10. A family C_α of integral curves of the linear equation $y'+p(x)y=q(x)$ is given. Show that the tangents at corresponding points to the curves C_α determined by the linear equation intersect at one point. Corresponding points of curves C_α are such points that lie on the same straight line parallel to the y-axis.

Solution Consider the tangent to some curve C_α at a point M (x,y). The equation of the tangent at a point M(x, y) is of the form

$Y-q(x)(X-x)=y[1-p(x)(X-x)]$,

Where X and Y are moving coordinates of a tangency point. By the definition, at corresponding points x is constant and y is variable. Taking any two tangents to curves C_α at corresponding points we get for the coordinates of the point S of their intersection.

$$X=x+\frac{1}{p(x)},\quad Y=\frac{q(x)}{p(x)}. \quad ...(1)$$

Hence it is seen that all the tangents to the curves C_α at corresponding points (x being fixed) intersect at the same point.

$$S\left(x+\frac{1}{p(x)},\frac{q(x)}{p(x)}\right).$$

Eliminating the variable x we obtain the equation of the locus of points S : f(X, Y) = 0.

Differential Equations Reducible to Linear Form

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \quad ...(1)$$

where P and Q are constants or functions of x alone and n is constant except 0 and 1, is called a **Bernoulli's equation**.

The solution of (1) is obtained as follows :

Divide (1) by y^n and put $y^{-n+1} = v$,

so that $(-n + 1)y^{-n}\, dy/dx = dv/dx$ and (1) thus, reduces

to $\frac{1}{-n+1}\frac{dv}{dx} + Pv = Q$

or $\frac{dv}{dx} + (1-n)Pv = (1-n)Q$

which is a linear differential equation in v and can be solved by the method described earlier.

Example 11. Solve $(1-x^2)\frac{dy}{dx} + xy = xy^2$.

Solution The given equation can be written as

$$y^{-2}\frac{dy}{dx} + \frac{x}{1-x^2}y^{-1} = \frac{x}{1-x^2}$$

Put $y^{-1} = v$, so that $-y^{-2}\, dy/dx = dv/dx$ and the given equation reduces to

$$\frac{dv}{dx} - \frac{x}{1-x^2}v = -\frac{x}{1-x^2}$$

which is a linear equation in v. Its integrating factor is

$$\text{I.F.} = xe^{\int P dx} = \exp\left(-\int\frac{x}{1-x^2}dx\right) = e^{(1/2)\ln(1-x^2)}$$

$$= (1-x^2)^{1/2}$$

The required solution of the given equation is

$$ve^{\int P dx} = \int Qe^{\int P dx}\, dx + c$$

$$\Rightarrow \quad v(1-x^2)^{1/2} = \int -\frac{x}{1-x^2}(1-x^2)^{1/2}\, dx + c$$

In the integral on the right hand side, put $t = 1 - x^2$ and integrate. We get

$$v(1-x^2)^{1/2} = (1-x^2)^{1/2} + c$$

$$\Rightarrow \quad \frac{1}{y}(1-x^2)^{1/2} = (1-x^2)^{1/2} + c$$

$$\Rightarrow \quad (1-y)\sqrt{1-x^2} = cy.$$

Example 12. Solve $\{xy^3(1+\cos x) - y\}\, dx + x dy = 0$.

Solution The given equation can be written as

$$\frac{dy}{dx} + y^3(1+\cos x) - \frac{y}{x} = 0$$

$$\Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} - \frac{1}{y^2 x} = -(1+\cos x)$$

Using the substitution

$$\frac{-1}{y^2} = u, \text{ we get } \frac{2}{y^3}dy = du$$

the above equation reduces to

$$\frac{1}{2}\frac{du}{dx} + \frac{u}{x} = -(1+\cos x)$$

whose I.F. $= e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

Hence, the solution of the given differential equation is given by

$$ux^2 = -2\int x^2(1+\cos x)dx$$

$$\Rightarrow \quad \frac{x^2}{2y^2} = \int x^2 dx + \int x^2\cos x dx$$

$$= \frac{x^3}{3} + x^2\sin x - \int 2x\sin x dx$$

$$= \frac{x^3}{3} + x^2\sin x + 2x\cos x - 2\int \cos x + C$$

$$= \frac{x^3}{3} + x^2\sin x + 2x\cos x - 2\sin x + C$$

Example 13. Solve $\frac{dy}{dx}(x^2y^2 + xy) = 1$

Solution The equation can be written as

$$\frac{dx}{dy} - xy = x^2y^2.$$

Dividing by x^2, $x^{-2}\frac{dx}{dy} - \frac{1}{x}y = y^3$. Put $-\frac{1}{x} = v$

$$\therefore \quad x^{-2}\frac{dx}{dy} = \frac{dv}{dy}.$$

$\therefore$ The equation becomes $\frac{dv}{dy} + vy = y^3$

It is linear in v and y. I.F. $= e^{\int y dy} = e^{\frac{1}{2}y^2}$.

$$\therefore \quad ve^{\frac{1}{2}y^2} = \int y^3 e^{\frac{1}{2}y^2} dy + C, \text{ put } \frac{1}{2}y^2 = t, \; y\, dy = dt$$

$$= 2\int te^t dt + C = 2e^t(t-1) + C$$

$$-\frac{1}{x}e^{\frac{1}{2}y^2} = 2e^{\frac{1}{2}y^2}\left(\frac{1}{2}y^2 - 1\right) + C$$

or, $\frac{1}{x} = (2 - y^2) - Ce^{-\frac{1}{2}y^2}$ is the solution.

Substitution

A well found substitution of variables may help to reduce some nonlinear equations of the first order to linear equations or to Bernoulli equations.

Equation $f'(y)\frac{dy}{dx} + Pf(y) = Q$

where P and Q are functions of x or constants.

Put $f(y) = v$ so that $f'(y)\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore$ The equation becomes $\dfrac{dv}{dx} + Pv = Q$,

which is a linear equation in v and x.

Note: In each of these equations, single out Q (function on the right) and then make suitable substitution to reduce the equation in linear form.

Example 14. Solve $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

Solution The given equation can be written as

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3 \quad ...(1)$$

Put $\tan y = v$ so that $\sec^2 y(dy/dx) = dv/dx$, and (1) reduces to

$$\frac{dv}{dx} + 2xv = x^3 \quad ...(2)$$

which is a linear differential equation. Now

$$\text{I.F.} = e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$$

Multiplying (2) by I.F. and then integrating, we get

$$ve^{x^2} = \int x^3 e^{x^2}dx + c = \int \frac{1}{2}te^t dt + c$$

$(x^2 = t)$

$$= \frac{1}{2}e^t(t-1) + c = \frac{1}{2}e^{x^2}(x^2-1) + c$$

or $(\tan y)e^{x^2} = \dfrac{1}{2}e^{x^2}(x^2-1) + c$

or $\tan y = \dfrac{1}{2}(x^2-1) + ce^{-x^2}$

which is the required solution.

Example 15. Solve

$\sec^2\theta\, d\theta + \tan\theta(1 - r\tan\theta)dr = 0$.

Solution The given equation can be written as

$$\frac{d\theta}{dr} + \frac{\tan\theta}{\sec^2\theta} = \frac{r\tan^2\theta}{\sec^2\theta}$$

$$\Rightarrow \left(\frac{\sec^2\theta}{\tan^2\theta}\right)\frac{d\theta}{dr} + \frac{1}{\tan\theta} = r$$

$$\Rightarrow \operatorname{cosec}^2\theta\frac{d\theta}{dr} + \cot\theta = r$$

Using the substitution

$\cot\theta = u$, we get $-\operatorname{cosec}^2\theta\, d\theta = du$

the equation reduces to $\dfrac{du}{dr} - u = -r$

whose I.F. $= e^{\int -1 dr} = e^{-r}$.

Hence, the solution of the given differential equation is given by

$$ue^{-r} = -\int re^{-r}dr = re^{-r} + \int e^{-r}dr = re^{-r} - e^{-r} + C$$

$\Rightarrow u = r - 1 + Ce^r$

$\Rightarrow \cot\theta = r - 1 + Ce^r$.

Example 16. Solve

$\sin y\, dy/dx - 2\cos y\cos x = -\cos x\sin^2 x$

Solution The given equation can be written as

$\sin y\, dy/dx - 2\cos y\cos x = -\cos x\sin^2 x$

Using the substitution $-\cos y = u$

we have $\sin y\dfrac{dy}{dx} = \dfrac{du}{dx}$

Then, the equation reduces to

$$\frac{du}{dx} + u(2\cos x) = -\cos x\sin^2 x$$

whose I.F. $= e^{\int 2\cos x\, dx} = e^{2\sin x}$

Introducing it, the equation becomes exact, and its primitive is $ue^{2\sin x} = -\int e^{2\sin x}\cos x\sin^2 x\, dx = -\int t^2 e^{2t} dt$

[putting $\sin x = t$, $\cos x\, dx = dt$] $= -t^2\dfrac{e^{2t}}{2} + \int te^{2t}dt$

$$= -\frac{t^2e^{2t}}{2} + \frac{te^{2t}}{2} - \frac{e^{2t}}{4}$$

$$\Rightarrow -\cos y\, e^{2\sin x} = -e^{2\sin x}\left(\frac{\sin^2 x}{2} - \frac{\sin x}{2} + \frac{1}{4}\right) + C$$

$\Rightarrow 4\cos y = 2\sin^2 x - 2\sin x + 1 + Ce^{2\sin x}$.

Example 17. Solve $\dfrac{dy}{dx} = 1 - x(y-x) - x^3(y-x)^3$.

Solution Put $y - x = v$,

$$\frac{dy}{dx} - 1 = \frac{dv}{dx}$$

$\therefore$ The equation is $\dfrac{dv}{dx} + 1 = 1 - xv - x^3v^3$

or $\dfrac{dv}{dx} + xv = -x^3v^2$

or $v^{-3}\dfrac{dv}{dx} + xv^{-2} = -x^3$.

Put $v^{-2} = u,\ -2v^{-3}\dfrac{dv}{dx} = \dfrac{du}{dx}$.

$\therefore$ The equation is $-\dfrac{1}{2}\dfrac{du}{dx} + xu = -x^3$.

or $\dfrac{du}{dx} - 2xu = 2x^3$.

Linear in u and x. I.F. $= e^{\int -2x\,dx} = e^{-x^2}$

$\therefore\ ue^{-x^2} = \int 2x^3e^{-x^2} + C$, Put $-x^2 = t, -2x\,dx = dt$

$= \int te^t\,dt + C = e^t(t-1) + C$

or, $(y-x)^{-2} = Ce^{x^2} - (1+x^2)$ is the solution.

Example 18. Make the substitution $x = r\cos\theta$ and $y = r\sin\theta$ to convert the differential equation $(x^2 + y^2 + y)\,dx = x\,dy$ into a linear differential equation with θ as independent and r as a dependent variable and solve it. Verify the solution by the substitution $y = tx$.

Solution $(x^2 + y^2)\,dx = x\,dy - y\,dx$

Put $x = r\cos\theta,\ y = r\sin\theta \Rightarrow \tan\theta = \dfrac{y}{x}$

$r^2[-r\sin\theta\,d\theta + \cos\theta\,dr] = r^2\,d\theta$

$\dfrac{dr}{d\theta} - r\tan\theta = \sec\theta$ which is linear.

$IF = e^{-\int\tan\theta\,d\theta} = \cos\theta$

$\Rightarrow r\cos\theta = \int d\theta = \theta + c$

$\Rightarrow \dfrac{y}{x} = \tan(x+c)$

$\Rightarrow y = x\tan(x+c)$ ...(1)

With $y = tx,\ \dfrac{dy}{dx} = t + x\dfrac{dt}{dx}$

$(x^2 + t^2x^2 + tx) = x\left[t + x\dfrac{dt}{dx}\right] = tx + x^2\dfrac{dt}{dx}$

$\Rightarrow \dfrac{dt}{dx} = 1 + t^2 \qquad \Rightarrow \tan^{-1}t = x + c$

$y = x\tan(x+c)$ which is same as (1)

Concept Problems E

1. Solve the following differential equations :

   (i) $\dfrac{dy}{dx} = y\tan x - 2\sin x$

   (ii) $(1-x^2)\dfrac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$

   (iii) $(x+a)\dfrac{dy}{dx} - 3y = (x+a)^5$

   (iv) $(x+1)\dfrac{dy}{dx} - ny = e^x(x+1)^{n+1}$.

2. Solve the differential equation

   $\dfrac{dy}{dx} = \dfrac{1}{x\cos y + \sin 2y}$

3. Find the curves possessing the property that the intercept the tangent at any point of a curve cuts off on the y-axis is equal to the square of the abscissa of the tangency point.

4. Find the general solution of the linear equation of the first order $y' + p(x)\,y = q(x)$ if one particular solution, $y_1(x)$, is known.

5. Find the general solution of the first order non-homogeneous linear equation $y' + p(x)\,y = q(x)$ if two particular solutions of it, $y_1(x)$ and $y_2(x)$, are known.

6. (a) Find the general solution y_h of the homogeneous differential equation

   $\dfrac{dy}{dx} + 2xy = 0$.

   (b) Show that the general solution of the nonhomogeneous equation

   $\dfrac{dy}{dx} + 2xy = 3e^{-x^2}$ is equal to the solution y_h in part (a) plus a particular solution to the nonhomogeneous equation.

7. Show that a linear equation remains linear whatever replacements of the independent variable $x = \varphi(t)$, where $\varphi(t)$ is a differentiable function, are made.

8. Show that a linear equation remains linear whatever linear transformations of the sought-for function $y = \alpha(x)\,z + \beta(x)$, where $\alpha(x)$ and $\beta(x)$

are arbitrary differentiable functions, with $\alpha(x) \neq 0$ in the interval under consideration, take place.

9. Given three particular solutions y, y_1, y_2 of a linear equation. Prove that the expression $\frac{y_2 - y}{y - y_1}$ remains unchanged for any x. What is the geometrical significance of this result ?

Practice Problems F

10. Solve the following differential equations :

(i) $y' - y \tan x = \frac{1}{\cos^3 x}$, $y(0) = 0$.

(ii) $t(1 + t^2)dx = (x + xt^2 - t^2)\,dt$; $x(1) = \frac{\pi}{4}$.

(iii) $y' - \frac{y}{1 - x^2} = 1 + x$, $y(0) = 1$

(iv) $2x y' = y + 6x^{5/2} - 2\sqrt{x}$, $y(1) = 3/2$.

11. Solve the following differential equations :

(i) $y' - y \ln 2 = 2^{\sin x}(\cos x - 1)\ln 2$, y being bounded when $x \to \infty$.

(ii) $y' \sin x - y \cos x = -\frac{\sin^2 x}{x^2}$, $y \to 0$ as $x \to \infty$

(iii) $x^2 y' \cos\frac{1}{x} - y \sin\frac{1}{x} = -1$, $y \to 1$ as $x \to \infty$.

(iv) $x^2 y' + y = (x^2 + 1)e^x$, $y \to 1$ as $x \to \infty$

12. Solve the following differential equations :

(i) $(1 + xy + x^2y^2)\,dx = x^2\,dy$.

(ii) $y' + \frac{2y}{x} = \frac{2\sqrt{y}}{\cos^2 x}$

(iii) $(x^2y^2 - 1)\,y' + 2xy^3 = 0$.

(iv) $y' = \frac{y^3}{2(xy^2 - x^2)}$

13. Solve the following differential equations :

(i) $x\,dx = \left(\frac{x^2}{y} - y^3\right)dy$

(ii) $\frac{y}{x}\,dx + (y^3 - \ln x)dy = 0$

(iii) $\frac{2x\,dx}{y^3} + \frac{y^2 - 3x^2}{y^4}dy = 0$

(iv) $y - y' \cos x = y^2 \cos x\,(1 - \sin x)$.

14. Solve the following differential equations :

(i) $yy' + 1 = (x - 1)\,e^{-y^2/2}$

(ii) $y' + x \sin 2y = 2xe^{-x^2}\cos^2 y$.

(iii) $yy' \sin x = \cos x\,(\sin x - y^2)$

(iv) $y' = \frac{y^2 - x}{2y(x + 1)}$

15. Find all solutions of $y' \sin x + y \cos x = 1$ on the interval $(0, \pi)$. Prove that exactly one of these solutions has a finite limit as $x \to 0$, and another has a finite limit as $x \to \pi$.

16. Find all solutions of $y' + y \cot x = 2 \cos x$ on the interval $(0, \pi)$. Prove that exactly one of these is also a solution on $(-\infty, \infty)$

17. Find all solutions of $(x - 2)(x - 3)y' + 2y = (x - 1)(x - 2)$ on each of the following intervals : (a) $(-\infty, 2)$ (b) $(2, 3)$ (c) $(3, \infty)$. Prove that all solutions tend to a finite limit as $x \to 2$, but that none has a finitie limit as $x \to 3$.

18. Find the curve such that the area of the rectangle constructed on the abscissa of any point and the initial ordinate of the tangent at this point is a constant (a^2).

19. A curve is such that the intercept a tangent cuts off on the ordinate axis is half the sum of the co-ordinates of the tangency point . Form the differential equation and obtain the equation of the curve if it passes through (1, 2) .

4.8 SOLUTION BY INSPECTION

Exact differential equation

An equation $M dx + N dy = 0$ ...(1)

is said to be exact if it is equivalent to

$du = 0$ ...(2)

and its integral is

$u = C$. ...(3)

It may be shown that every equation of the type (1) is either exact, or can be rendered exact by a suitable 'integrating factor'. The number of such factors is unlimited ; for if we suppose the equation (1) to have been brought to the form (3), it will still be exact when multiplied by $f'(u)$, where $f(u)$ may be any function of

u. The integral of

$f'(u)\,du = 0$ ...(4)

is $f(u) = C.$ ...(5)

which is obviously equivalent to (4).

Example 1. Solve $(2x\ln y)dx + \left(\frac{x^2}{y} + 3y^2\right)dy = 0$

Solution The given differential equation is

$$(\ln y)\,2x\,dx + \frac{x^2}{y}\,dy + 3y^2\,dy = 0$$

$\Rightarrow$ $(\ln y)\,d(x^2) + x^2\,d(\ln y) + d\,(y^3) = 0$

$\Rightarrow$ $d(x^2\ln y) + d\,(y^3) = 0 \Rightarrow x^2\ln y + y^3 = c$

Example 2. Solve the differential equation

$(x^3 + xy^2)\,dx + (x^2y + y^3)\,dy = 0.$ ...(1)

Solution The equation is easy to reduce to the form $du = 0$ by immediate grouping of its terms. For this purpose we rewrite it as :

$x^3\,dx + xy\,(y\,dx + x\,dy) + y^3\,dy = 0.$

Obviously, $x^3dx = d\left(\frac{x^4}{4}\right)$,

$$xy(y\,dx + x\,dy) = xy\,d\,(xy) = d\left(\frac{(xy)^2}{2}\right),$$

$$y^3dy = d\left(\frac{y^2}{4}\right).$$

Therefore equation (1) may be written in the form

$$d\left(\frac{x^4}{4}\right) + d\left(\frac{(xy)^2}{2}\right) + d\left(\frac{y^2}{4}\right) = 0$$

or, $d\left[\frac{x^4}{4} + \frac{(xy)^2}{2} + \frac{y^4}{4}\right] = 0.$

Consequently, $x^4 + 2(xy)^2 + y^4 = C$ is the general solution of the equation.

Example 3. Solve $xdx + ydy = k(xdy - ydx)$

Solution This may be written as

$$d(x^2 + y^2) = 2kx^2 d\left(\frac{y}{x}\right),$$

and so the equation becomes exact on division by $x^2 + y^2$, thus

$$\frac{d(x^2 + y^2)}{x^2 + y^2} = \frac{2kd\left(\frac{y}{x}\right)}{1 + \frac{y^2}{x^2}}$$

Hence, integrating both sides,

$$\ln(x^2 + y^2) = 2k\tan^{-1}\frac{y}{x} + C$$

The given equation may also be solved as follows. The form suggests the substitutions

$x = r\cos\theta,\ y = r\sin\theta,$

which give $xdx + ydy = rdr,\ xdy - ydx = r^2d\theta$

The equation therefore reduces to

$$\frac{dr}{r} = k\,d\theta$$

$\Rightarrow$ $\ln r = k\theta + C$

This is obviously equivalent to the previous solution.

Integrating Factors

A non-exact differential equation can always be made exact by multiplying it by some function of x and y. Such a function is called an integrating factor. Although a differential equation of the type $Mdx + Ndy = 0$ always has an integrating factor, there is no general method of finding them.

In some cases the integrating factor is found by inspection. Using the following exact differentials, it is easy to find the integrating factors :

(a) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(b) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(c) $d(xy) = xdy + ydx$

(d) $d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$

(e) $d\left(\frac{x^2}{y}\right) = \frac{2yxdx - x^2dy}{y^2}$

(f) $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2ydy - 2xy^2dx}{x^4}$

(g) $d\left(\frac{x^2}{y^2}\right) = \frac{2xy^2dx - 2x^2ydy}{y^4}$

(h) $d\left(\frac{1}{xy}\right) = -\frac{xdy + ydx}{x^2y^2}$

(i) $d\left(\ln\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$

(j) $d\left(\ln\frac{x}{y}\right) = \frac{ydx - xdy}{xy}$

(k) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

(l) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2+y^2}$

(m) $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$

(n) $d\left(\frac{e^y}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$

(o) $d\left[\frac{1}{2}\ln(x^2+y^2)\right] = \frac{xdx + ydy}{x^2+y^2}$

Example 4. Solve $(1+xy)y\,dx + (1-xy)x\,dy = 0$.

Solution The given equation can be written as

$(ydx + xdy) + (xy^2 dx - x^2 ydy) = 0$

$\Rightarrow d(yx) + xy^2 dx - x^2 ydy = 0$

Dividing by x^2y^2, we get

$$-d\left(\frac{1}{xy}\right) + \frac{1}{x}dx - \frac{1}{y}dy = 0$$

Integrating, we get

$$-\frac{1}{xy} + \ln x - \ln y = c$$

$$\Rightarrow \frac{1}{xy} + \ln\frac{x}{y} = c \Rightarrow \ln\frac{x}{y} = c + \frac{1}{xy}$$

which is the required solution.

Example 5. Solve $x\,dy - y\,dx = a(x^2+y^2)\,dy$.

Solution The given equation can be written as

$$\frac{x\,dy - y\,dx}{x^2+y^2} = a\,dy$$

$$\Rightarrow d\left(\tan^{-1}\frac{y}{x}\right) = a\,dy$$

Integrating, we get the required solution as

$$\tan^{-1}\frac{y}{x} = ay + c.$$

Example 6. Solve

$x\,dy + y\,dx + y^2(x\,dy - y\,dx) = 0.$

Solution $d(xy) + x^2y^2\left(\frac{x\,dy - y\,dx}{x^2}\right) = 0$

$$d(xy) + x^2y^2\,d\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \frac{d(xy)}{x^2y^2} + d\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow (y^2 - 1) = cxy \Rightarrow -\frac{1}{xy} + \frac{y}{x} = c$$

Example 7. Solve the differential equation,

$(2y + xy^3)\,dx + (x + x^2y^2)\,dy = 0.$

Solution $(2y\,dx + x\,dy) + xy^3\,dx + x^2y^2\,dy = 0$

Multiplying by x,

$2xy\,dx + x^2\,dy + x^2y^2(y\,dx + x\,dy) = 0$

$d(x^2y) + x^2y^2\,d(xy) = c$

$d(x^2y) + t^2\,dt = c$

where $xy = t$

$$x^2y + \frac{t^3}{3} = c \Rightarrow x^2y + \frac{x^3y^3}{3} = c$$

Alternative:

Put $xy = t \Rightarrow x\frac{dy}{dx} + y = \frac{dt}{dx}$

$$\left(2\frac{t}{x} + t\,.\frac{t^2}{x^2}\right) + \frac{1}{x}(x + t^2) = 0$$

$$\frac{2t}{x} + \frac{t^3}{x^2} + \frac{dt}{dx} - \frac{t}{x} + \frac{t^2}{x}\frac{dt}{dx} - \frac{t^3}{x^2} = 0$$

$$\frac{t}{x} + \frac{dt}{dx}\left(\frac{t^2}{x} + 1\right) = 0$$

$$(t^2 + x)\frac{dt}{dx} + t = 0$$

$$\Rightarrow \frac{dx}{dt} + \frac{x}{t} = -t,$$

which is a linear differential equation. which can be found easily.

Example 8. Solve

$$\frac{y + \sin x\cos^2(xy)}{\cos^2(xy)}dx + \left(\frac{x}{\cos^2(xy)} + \sin y\right)dy = 0.$$

Solution The given differential equation can be written as

$$\frac{ydx + xdy}{\cos^2(xy)} + \sin x\,dx + \sin y\,dy = 0.$$

$\Rightarrow \sec^2(xy)\,d(xy) + \sin x\,dx + \sin y\,dy = 0$

$d(\tan(xy)) + d(-\cos x) + d(-\cos y) = 0$

$\Rightarrow \tan(xy) - \cos x - \cos y = c.$

Example 9. Solve $\dfrac{x + y\frac{dy}{dx}}{x - y\frac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$.

Solution The given equation can be written as

$$\frac{x\,dx + y\,dy}{(x^2+y^2)^2} = \frac{y\,dx - x\,dy}{y^2}.\frac{y^2}{x^2}$$

$$\Rightarrow \int \frac{d(x^2+y^2)}{(x^2+y^2)^2} = 2\int \frac{1}{x^2/y^2} d\left(\frac{x}{y}\right)$$

Integrating both sides, we get

$$-\frac{1}{(x^2+y^2)} = -\frac{1}{(x/y)} + c \Rightarrow \frac{y}{x} - \frac{1}{x^2+y^2} = c$$

Example 10. Solve $(ax+hy+g)dx+(hx+by+f)dy=0$.

Solution This is equivalent to

$d(ax^2+2hxy+by^2+2gx+2fy)=0$.

Integrating both sides, we get

$ax^2+2hxy+by^2+2gx+2fy=C$.

Example 11. Solve $(x^3e^x-my^2)\,dx+m\,xy\,dy=0$.

Solution The given equation can be written as

$x^3e^x\,dx+m(xy\,dy-y^2\,dx)=0$

Dividing by x^3, we get

$$\Rightarrow e^x\,dx + m\frac{xy\,dy - y^2dx}{x^3} = 0$$

$$\Rightarrow e^x\,dx = \frac{m}{2}\frac{x^2 2y\,dy - y^2 2x\,dx}{x^4} = 0$$

$$\Rightarrow e^x\,dx + \frac{1}{2}\,m\,d\left(\frac{y^2}{x^2}\right) = 0$$

Integrating, we get

$$e^x + \frac{1}{2}\,m\frac{y^2}{x^2} = c \Rightarrow 2x^2e^x + my^2 = 2cx^2$$

which is the required solution.

Example 12. Solve

$y(2xy+1)\,dx+x(1+2xy+x^2y^2)\,dy=0$.

Solution The given differential equation is

$\Rightarrow y(2xy+1)\,dx+x(1+2xy+x^2y^2)\,dy=0$

$\Rightarrow y(2xy+1)\,dx+x(1+2xy)\,dy+x^3y^2\,dy=0$

$\Rightarrow (2xy+1)(ydx+xdy)+x^3y^2\,dy=0$

$$\Rightarrow (2xy+1)\,d(xy) = -x^3y^3\frac{1}{y}\,dy$$

$$\Rightarrow \frac{2xy+1}{(xy)^3}d(xy) + \frac{dy}{y} = 0$$

Integrating, we get $-\dfrac{2}{xy} - \dfrac{1}{2(xy)^2} + \ell n\, y = c$

$$\Rightarrow \ell n\, y = c + \frac{2}{xy} + \frac{1}{2(xy)^2}.$$

Example 13. Find the acute angle between the two lines passing through the origin and satisfying the differential equation,

$(4x-3y)\,dx + (2y-3x)\,dy = 0$.

Solution $4x\,dx+2y\,dy-3(y\,dx+x\,dy)=0$

$4x\,dx + 2y\,dy - 3\,d(xy) = 0$

$2x^2+y^2-3xy=c$,

since the lines pass through the origin $c = 0$.

Hence $(x-y)(2x-y)=0$

Hence, the acute angle between the lines is

$$\theta = \tan^{-1}\frac{1}{3}.$$

Example 14. Solve $(y + x\sqrt{xy}\ (x+y))\,dx + (y\sqrt{xy}\ (x+y)-x)\,dy=0$.

Solution The given equation is written as

$$\Rightarrow ydx - xdy + x\sqrt{xy}\ (x+y)dx + y\sqrt{xy}\ (x+y)dy = 0$$

$$\Rightarrow y\,dx - x\,dy + (x+y)\sqrt{xy}\ (x\,dx + y\,dy) = 0$$

$$\Rightarrow \frac{ydx - xdy}{y^2} + \left(\frac{x}{y}+1\right)\sqrt{\frac{x}{y}}\left(d\left(\frac{x^2+y^2}{2}\right)\right) = 0$$

$$\Rightarrow d\left(\frac{x^2+y^2}{2}\right) + \frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y}+1\right)\sqrt{\frac{x}{y}}} = 0$$

$$\Rightarrow \frac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\frac{x}{y}} = c.$$

Example 15. Solve $\left[\dfrac{1}{x} - \dfrac{y^2}{(x-y)^2}\right]dx + \left[\dfrac{x^2}{(x-y)^2} - \dfrac{1}{y}\right]dy = 0$.

Solution $\dfrac{dx}{x} - \dfrac{dy}{y} + \dfrac{x^2\,dy - y^2\,dx}{(x-y)^2} = 0$

$$\frac{dx}{x} - \frac{dy}{y} + \frac{x^2\,dy - y^2\,dx}{x^2y^2\left(\frac{1}{y}-\frac{1}{x}\right)^2} = 0$$

or $$\frac{dx}{x} - \frac{dy}{y} + \frac{\frac{dy}{y^2} - \frac{dx}{x^2}}{\left(\frac{1}{y}-\frac{1}{x}\right)^2} = 0$$

$$\frac{dx}{x} - \frac{dy}{y} + d\left(\frac{xy}{x-y}\right) = 0$$

Integrating, we get

$$\ln|x| - \ln|y| + \frac{xy}{x-y} = c.$$

Practice Problems G

1. Solve the following differential equations :
   (i) $y(xy+1)dx + x(1+xy+x^2y^2)dy = 0$
   (ii) $x(2x^2+y^2) + y(x^2+2y^2)y' = 0$

2. Solve the following differential equations :
   (i) $yx^{y-1}\,dx + x^y \ln x\,dy = 0$
   (ii) $ye^{-x/y}\,dx - (xe^{-x/y} + y^3)\,dy = 0$

3. Solve the following differential equations :
   (i) $(2x^3 - xy^2)\,dx + (2y^3 - x^2y)\,dy = 0$
   (ii) $(3x^2 - 2x - y)\,dx + (2y - x + 3y^2)\,dy = 0$

4. Solve the following differential equations :
   (i) $(2\,x\cos y + y^2\cos x)\,dx$
   $+ (2y\sin x - x^2\sin y)\,dy = 0$
   (ii) $\dfrac{x^3dx + yx^2dy}{\sqrt{x^2+y^2}} = ydx - xdy$

5. Solve the following differential equations :
   (i) $(x^2\sin^3 y - y^2\cos x)dx + (x^3\cos y\sin^2 y - 2y\sin x)dy = 0$
   (ii) $(\sin y + y\sin x + 1/x)\,dx + (x\cos y - \cos x + 1/y)\,dy = 0$

6. Solve the following differential equations :
   (i) $\dfrac{x\,dy}{x^2+y^2} = \left(\dfrac{y}{x^2+y^2} - 1\right)dx$
   (ii) $\dfrac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \dfrac{y\,dx - x\,dy}{x^2}$

7. Solve $\left(\dfrac{x}{\sqrt{x^2+y^2}} + \dfrac{1}{x} + \dfrac{1}{y}\right)dx$
   $+\left(\dfrac{y}{\sqrt{x^2+y^2}} + \dfrac{1}{y} - \dfrac{x}{y^2}\right)dy = 0.$

8. Solve $\left(3x^2\tan y - \dfrac{2y^3}{x^3}\right)dx$
   $+\left(x^3\sec^2 y + 4y^3 + \dfrac{3y^2}{x^2}\right)dy = 0.$

9. Solve $\left(\dfrac{\sin 2x}{y} + x\right)dx + \left(y - \dfrac{\sin^2 x}{y^2}\right)dy = 0.$

4.9 FIRST ORDER HIGHER DEGREE DIFFERENTIAL EQUATION

The most general form of a differential equation of the first order and higher degree (say nth degree) is

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2}$$

$$..... + P_{n-1}\left(\frac{dy}{dx}\right) + P_n = 0 \qquad ...(1)$$

or, $p^n + P_1p^{n-1} + P_2p^{n-2} + ... + P_{n-1}\,p + P_n = 0$
where $p = dy/dx$ and $P_1, P_2, ...,P_n$ are functions of x and y. This equation can also be written as

$$F(x, y, p) = 0 \qquad ...(2)$$

For example, the equation $y\left(\dfrac{dy}{dx}\right)^2 + 2x\dfrac{dy}{dx} - y = 0$ is a differential equation of second degree.

Geometrical meaning of a differential equation of higher degree

The equation being of the n^{th} degree in p, indicates that n branches of the primitive curves go through any assigned point in the plane xy. Some of these branches may of course be imaginary, and for some ranges of x and y all may be imaginary.

If $F\left(x, y, \dfrac{dy}{dx}\right) = 0$ is of the second degree in $\dfrac{dy}{dx}$, there will be two values of $\dfrac{dy}{dx}$ belonging to each particular point (x_1, y_1). Therefore the moving point can pass through each point of the plane in either of two directions; and hence, two curves of the system which is, the locus of the general solution pass through each point. The general solution,

$$\phi(x, y, c) = 0,$$

must therefore have two different values of c for each point ; and hence, c must appear in that solution in the second degree.

In general, it may be said that a differential equation,

$$F\left(x, y, \frac{dy}{dx}\right) = 0,$$

which is of the nth degree in $\frac{dy}{dx}$ and which has

$$\phi(x, y, c) = 0$$

for its general solution, has for its locus infinite number of curves, there being but one arbitrary constant in ϕ, n of these curves pass through each point of the plane, since $\frac{dy}{dx}$ has n values at any point; and hence the constant c must appear in the nth degree in the general solution.

The equation (2) cannot be solved in the general form. We will discuss different situations where a solution of this equation exists. We have two cases :

Case I : In this case, the left hand side of (1) can be resolved into rational factors of the first degree.

Case II : Here the member cannot be thus factored.

Case I Equation solvable for p

Suppose a differential equation can be solved for p and is of the form

$$[p - f_1(x, y)]\,[p - f_2(x, y)] \;....\; [p - f_n(x, y)] = 0$$

Equating each factor to zero we get equations of the first order and the first degree.

Let their solutions be

$$\phi_1(x, y, c_1) = 0, \qquad \phi_2(x, y, c_2) = 0, \ldots.,$$
$$\phi_1(x, y, c_n) = 0$$

Without any loss of generality, we can write

$$c_1 = c_2 = \ldots = c_n = c$$

as they are arbitrary constants. Therefore, the solution of (1) can be put in the form

$$\phi_1(x, y, c)\,\phi_2(x, y, c) \ldots\ldots \phi_n(x, y, c) = 0.$$

Example 1. Solve $(p+y+x)(xp+y+x)(p+2x)=0.$

Solution Here, we have

$$p + y + x = 0, \quad xp + y + x = 0, \quad p + 2x = 0$$

If $p + y + x = 0$, then $dy/dx + y + x = 0$.

Put $x + y = v$, then this equation becomes

$$\frac{dv}{1 - v} = dx$$

$\Rightarrow$ $-\ln(1 - v) = x + c_1$

$\Rightarrow$ $(1 - v) = e^{-x-c1} = ce^{-x}$

$\Rightarrow$ $1 - x - y - ce^{-x} = 0$ ...(1)

If $xp + y + x = 0$, then $dy/dx + (1/x)y = 1$, whose solution is $yx = \frac{1}{2}c_2 - \frac{1}{2}x^2$

$\Rightarrow$ $2xy + x^2 - c_2 = 0$ ...(2)

Finally, if $p + 2x = 0$, then the solution is

$y + x^2 - c_3 = 0$ ...(3)

From (1), (2), (3), the solution of the given equation is

$$(1 - x - y - ce^{-x})\,(2xy + x^2 - c)(y + x^2 - c) = 0.$$

Example 2. Solve $p^2 + 2py \cot x = y^2$.

Solution Solving the given equation for p, we get

$$p = \frac{1}{2}\left[-2y\cot x \pm \sqrt{(4y^2\cot^2 x + 4y^2)}\right]$$

$$= -y\cot\frac{x}{2},\; y\tan\frac{x}{2}$$

If $p = -y\cot(x/2)$, then by integrating, we have

$$\ln y = -2\ln\sin\frac{x}{2} + \ln A$$

Solving, we get $y = \dfrac{A}{\sin^2(x/2)}$

$\Rightarrow$ $y(1 - \cos x) = 2A = c_1$

If $p = y\tan(x/2)$, then by integrating, we get

$$\ln y = 2\ln\sec\frac{x}{2} + \ln B$$

$\Rightarrow$ $y(1 + \cos x) = 2B = c_2$

Therefore, the required solution is

$$[y(1 - \cos x) - c]\,[y(1 + \cos x) - c] = 0.$$

Case II

The differential equation $f(x, y, p) = 0$ may have one or more of the following properties :

(a) It may be solvable for y.
(b) It may be solvable for x.
(c) It may be of the first degree in x and y.

(a) Equations solvable for y

If the differential equation $F(x, y, p) = 0$ is solvable for y, then $y = f(x, p)$ ...(1)

Differentiating with respect to x, gives

$$p = \frac{dy}{dx} = \varphi\left(x, p, \frac{dp}{dx}\right) \qquad ...(2)$$

which is an equation in two variables x and p, and it will give rise to a solution of the form

$$\psi(x, p, c) = 0 \qquad ...(3)$$

The elimination of p between equations (1) and (3) gives a relation between x, y and c, which is the required solution. When the elimination of p between these equations is not easily done, the values of x and y in terms of p can be found, and these together will constitute the required solution.

Example 3. Solve $y = yp^2 + 2px$.

Solution The given equation can be written as

$$y = \frac{2px}{1 - p^2} \qquad ...(1)$$

Differentiating (1) with respect to x, we get

$$\frac{2dp}{p(p^2-1)} = \frac{dx}{x} \Rightarrow \left(\frac{1}{p-1} + \frac{1}{p+1} - \frac{2}{p}\right) dp = \frac{dx}{x}$$

which on integration gives

$$\ln (p-1) + \ln (p+1) - 2 \ln p = \ln x + \ln c$$

$$\Rightarrow \frac{p^2-1}{p^2} = cx \Rightarrow p = \frac{1}{\sqrt{1-cx}}$$

Substituting this value of p in (1), we get

$$2x\sqrt{1-cx} + c\,xy = 0$$

which is the required solution.

Example 4. Solve the equation

$$2y'^2 - 2xy' - 2y + x^2 = 0.$$

Solution We solve the equation for y :

$$y = y'^2 - xy' + \frac{x^2}{2}$$

Let us put $y' = p$, then we get

$$y = p'^2 - xp + \frac{x^2}{2} \qquad ...(1)$$

Differentiating (1) we find that

$$dy = 2p\,dp - p\,dx - x\,dp + x\,dx.$$

But since $dy = p\,dx$, we have

$$p\,dx = 2p\,dp - p\,dx - x\,dp + x\,dx$$

or $2p\,dp - 2p\,dx - x\,dp + x\,dx = 0,$

$$2p\,(dp - dx) - x\,(dp - dx) = 0,$$

$$(2p - x)\,(dp - dx) = 0.$$

Consider two cases :

(i) $dp - dx = 0$, where $p = x + C$, where C is an arbitrary constant. Substituting the value of p in (1) we obtain the general solution of the given equation :

$$y = Cx + C^2 + \frac{x^2}{2}.$$

CAUTION

In the equation $p = x + C$ one cannot replace p by y' and integrate the resulting equation $y' = x + C$ (there appearing another arbitrary constant, which is inadmissible since the differential equation considered is a first order equation).

(ii) $2p - x = 0$, which gives $p = x/2$. Substituting into (1) we obtain one more solution $y = x^2/4$.

(b) Equations solvable for x

When the differential equation $F(x, y, p) = 0$ is solvable for x, then we have

$$x = f(y, p)$$

Differentiating with respect to y, gives

$$\frac{1}{p} = \varphi\left(y, p, \frac{dp}{dy}\right)$$

from which a relation between p and y may be obtained

$$\psi(y, p, c) = 0 \text{ (say)}$$

Between this and the given equation, p may be eliminated, or x and y expressed in terms of p.

Example 5. Solve $pxy = y^2 \ln y - p^2$

Solution The given equation can be written as

$$x = \frac{1}{p}\, y \ln y - \frac{p}{y}$$

Differentiating with respect to y, we get, after simplification,

$$\frac{dp}{p} = \frac{dy}{y}$$

which on integration gives

$$\ln p = \ln y + \ln c = \ln cy$$

or $p = cy$

Eliminating p from this and the given equation, we obtain

$$\ln y = cx + c^2$$

which is the required solution.

(C) Equations of the first degree in x and y

The Lagrange's equation

When a differential equation is of the first degree in x and y, then

$$y = xf_1(p) + f_2(p) \qquad ...(1)$$

where $p = \dfrac{dy}{dx}$

Equation (1) is known as Lagrange's equation. To solve it, we differentiate with respect to x to obtain

$$p = \frac{dy}{dx} = f_1(p) + xf_1'(p) + f'_2(p)\frac{dp}{dx}$$

or $$\frac{dp}{dx} - \frac{f'_1(p)}{p - f_1(p)}x = \frac{f'_2(p)}{p - f_1(p)} \qquad ...(2)$$

which is a linear equation in x and p, and hence can be solved in the form $x = \phi(p, C)$ ...(3)

Eliminating p from the equations (1) and (2), we get the required solution. If it is not possible to eliminate p, then the values of x and y in terms of p can be found from the equations (1) and (3), and these will constitute the required solution.

In addition the Lagrange equation may have some singular solutions of the form $y = f_1(C)x + f_2(C)$, where c is the root of the equation $C = f_1(C)$.

Example 6. Solve the equation $y = 2xy' + \ln y'$.

Solution We set $y' = p$ then $y = 2xp + \ln p$. Differentiating we find that

$$p\,dx = 2p\,dx + 2x\,dp + \frac{dp}{p},$$

$$\Rightarrow\quad p\frac{dx}{dp} = -2x - \frac{1}{p}$$

or $$\frac{dx}{dp} = -\frac{2}{p}x - \frac{1}{p^2}.$$

We have obtained a first order equation linear in x; solving it we find that $x = \frac{C}{p^2} - \frac{1}{p}$.

Substituting the obtained value of x in the expression for y we finally get

$$x = \frac{C}{p^2} - \frac{1}{p},\; y = \ln p + \frac{2C}{p} - 2.$$

The Clairaut's equation

If $f_1(p) = p$ and $f_2(p) = f(p)$, then (1) reduces to

$$y = px + f(p) \qquad ...(4)$$

Equation (4) is known as Clairaut's equation. To solve it, we differentiate with respect to x to obtain

$$p = \frac{dy}{dx} = [x + f'(p)]p' + p$$

or $$[x + f'(p)]\frac{dp}{dx} = 0 \qquad ...(5)$$

If $dp/dx = 0$, then $p = C$ = constant. Eliminating p between this and (4), we get

$$y = Cx + f(C) \qquad ...(6)$$

which is the required solution of Clairaut's equation. Clairaut's equation is one where a curve is defined by some property of the tangent.

Hence any equation of the form (4) expresses a relation between intercept and the direction of the tangent.

Now it is evident that this relation is satisfied by any straight line whose intercepts have the given relation. Along any such straight line we have p = C and we thus get the solution y = Cx + f(C), involving an arbitrary constant C.

But the equation will also be satisfied by the curve which has the family (6) of straight lines as its tangents, in other words, by the envelope of this family. This envelope is found when we eliminate p between

$x + f'(p) = 0$ and (4). We get a solution which does not contain any arbitrary constant, and hence, is not a particular case of solution (5), but a singular solution.

Example 7. Solve the equation $y = xy' + \frac{a}{2y'}$ where a is a constant.

Solution Setting $y' = p$ we get $y = xp + \frac{a}{2p}$.

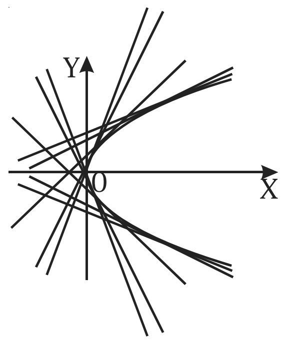
Differentiating this latter equation and replacing dy by p dx we find that

$$p\,dx = p\,dx + x\,dp - \frac{a}{2p^2}dp, \Rightarrow dp\left(x - \frac{a}{2p^2}\right) = 0.$$

Equation the first factor to zero we get dp = 0, which gives p = C and the general solution of the original equation is $y = Cx + \frac{a}{2c}$, a one parameter family of straight lines.

Equating the second factor to zero we have $x = a/2p^2$. Eliminating p from this equation and from the equation

$y = xp + \frac{a}{2p}$ we get $y^2 = 2ax$, which is also a solution of our equation (a singular one).



From a geometrical point of view the curve $y^2 = 2ax$ is the envelope of the family of straight lines given by the general solution.

Example 8. Solve $y = px + p - p^2$.

Solution Differentiating both sides with respect to x,

$$p = p + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}.$$

$$\Rightarrow\quad \frac{dp}{dx}(x + 1 - 2p) = 0.$$

$$\Rightarrow\quad \text{either } \frac{dp}{dx} = 0$$

$$\Rightarrow\quad p = C \qquad ...(1)$$

or, $$x + 1 - 2p = 0$$

$$\Rightarrow\quad p = \frac{1}{2}(x + 1). \qquad ...(2)$$

Eliminating p between (1) and the given equation, we get $y = Cx + C - C^2$ as the general solution and eliminating p between (2) and the given equation, we get

$$y = \frac{1}{2}(x+1)x + \frac{1}{2}(x+1) - \frac{1}{4}(x+1)^2 = \frac{1}{4}(x+1)^2,$$

$\Rightarrow$ $4y = (x+1)^2$ as the singular solution.

Note: It can easily be verified that the family of straight lines represented by the general solution touches the parabola represented by the singular solution.

Example 9. Reduce $xyp^2 - (x^2 + y^2 + 1)p + xy = 0$ to Clairaut's form and find its singular solution.

Solution Let $x^2 = u$ and $y^2 = v$, the the given equation becomes

$$u\left(\frac{dv}{du}\right)^2 - (u + v - 1))\frac{dv}{du} + v = 0$$

or $up^2 - (u + v + 1)p + v = 0$, where $p = \dfrac{dv}{du}$

or $v = up + \dfrac{p}{p-1}$ ...(1)

which is in Clairaut's form. Differentiating it with respect to u, we get

$$\left[u - \frac{1}{(p-1)^2}\right]\frac{dp}{du} = 0 \quad ...(2)$$

To get the singular solution, we consider

$$u - \frac{1}{(p-1)^2} = 0$$

which gives $p = 1 + (1/\sqrt{u})$. Putting this in (1), we get the required solution as $y^2 = (x+1)^2$

Practice Problems H

Solve the following differential equations :

1. $y'^2 - 2xy' - 8x^2 = 0$
2. $y'^3 - yy'^2 - x^2y' + x^2y = 0$
3. $y'^2 - 4xy' + 2y + 2x^2 = 0.$
4. $y = xy' + \dfrac{a}{y'^2}$
5. $y = xy' + y'^2.$
6. $xy'^2 - yy' - y' + 1 = 0$
7. $y = xy' + a\sqrt{1 + y'^2}.$
8. $x = \dfrac{y}{y'} + \dfrac{1}{y'^2}$
9. $y\left(\dfrac{dy}{dx}\right)^2 + 2x\dfrac{dy}{dx} - y = 0;\ y(0) = \sqrt{5}.$
10. $y = xy' + \sqrt{1 + y'^2}.$
11. $y = xy' + \sin y'$
12. $2yy' = x(y'^2 + 4).$
13. Find the curve such that the square of the intercept cut by any tangent off the y-axis is equal to the product of the coordinates of the point of tangency.
14. Find a curve each tangent of which forms with the coordinate axes a triangle of constant area $S = 2a^2$.
15. Find a curve for which the segment of a tangent to it contained between the coordinate axes is of constant length a.

4.10 HIGHER ORDER DIFFERENTIAL EQUATION

Form $\dfrac{d^2y}{dx^2} = f(x)$

This requires merely two ordinary integrations with respect to x, thus

$$\frac{dy}{dx} = \int f(x)\,dx + A,$$

$$y = \int\left\{\int f(x)\,dx\right\}dx + Ax + B$$

where the constants A, B are arbitrary.

Example 1. $\dfrac{d^2y}{dx^2} = xe^x + \cos x.$

Solution Here $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = xe^x + \cos x.$

Hence, $\dfrac{dy}{dx} = \int (xe^x + \cos x)\,dx$

$= xe^x - e^x + \sin x + C_1,$

and another integration yields

$y = xe^x - 2e^x - \cos x + C_1x + C_2.$

Example 2. Find all twice differentiable functions f(x) such that $f''(x) = 0$, $f(0) = 19$, and $f(1) = 99$.

Solution Since $f''(x) = 0$ we must have

$f(x) = ax + b$ for some real numbers a, b. Thus $f(0) = b = 19$ and $f(1) = a + 19 = 99$, so $a = 80$. Therefore $f(x) = 80x + 19$.

Example 3. Find the general solution of the equation $y''' = \frac{\ln x}{x^2}$ and separate out the solution satisfying the initial conditions $y(1) = 0, y'(1) = 1, y''(1) = 2$.

Solution We integrate this equation three times in succession :

$$y'' = \int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x^2} - \frac{1}{x} + C_1, \quad ...(1)$$

$$y' = -\frac{1}{2}\ln^2 x - \ln x + C_1 x + C_2, \quad ...(2)$$

$$y = -\frac{x}{2}\ln^2 x + C_1\frac{x_2}{2} + C_2 x + C_3. \quad ...(3)$$

We find the solution satisfying the given initial conditions. Substituting the initial data $y(1) = 0$, $y'(1) = 1, y''(1) = 2$ in (1), (2) and (3) we have

$$-1 + C_1 = 2, C_1 + C_2 = 1, \frac{C_1}{2} + C_2 + C_3 = 0$$

Hence $C_1 = 3, C_2 = -2, C_3 = 1/2$.

The desired solution is

$$y = -\frac{x}{2}\ln^2 x + \frac{3}{2}x^2 - 2x + \frac{1}{2}.$$

Form $\frac{d^2y}{dx^2} = f(y)$

If the equation be of the type

$$\frac{d^2y}{dx^2} = f(y) \quad ...(1)$$

a first integral may be obtained in two ways.

In one of these we multiply both sides by dy/dx, and then integrate with respect to x ; thus

$$\frac{dy}{dx}\frac{d^2y}{dx^2} = f(y)\frac{dy}{dx},$$

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 = \int f(y)\frac{dy}{dx}dx + A = \int f(y)dy + A \quad ...(2)$$

The second method is to introduce a special symbol p for dy/dx. Since this makes

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy} \quad ...(3)$$

we have, in place of (1),

$$p\left(\frac{dp}{dy}\right) = f(y), \quad ...(4)$$

which may be regarded as an equation of the first order, with p as dependent, and y as independent, variable. Integrating (3) with respect to y, we have

$$\frac{1}{2}p^2 = \int f(y)dy + A \quad ...(5)$$

which is equivalent to (2).

To complete the solution, we write (2) in the form

$$\frac{dy}{\sqrt{2\int f(y)dy + 2A}} = \pm dx \quad ...(6)$$

The variable are here separated.

Example 4. Solve $\frac{d^2y}{dx^2} + a^2y = 0$.

Solution Here $\frac{d^2y}{dx^2} = -a^2y$

The equation may be written as

$y'\,dy' = -a^2y\,dy$

$1/2\,y'^2 = -1/2\,a^2y^2 + C.$

$$y' = \sqrt{2C - a^2y^2},$$

$$\frac{dy}{dx} = \sqrt{C_1 - a^2y^2},$$

setting $2C = C_1$ and taking the positive sign of the radical. Separating the variables, we get

$$\frac{dy}{\sqrt{C_1 - a^2y^2}} = dx.$$

Integrating, $\frac{1}{a}\sin^{-1}\frac{ay}{\sqrt{C_1}} = x + C_2,$

or $\sin^{-1}\frac{ay}{\sqrt{C_1}} = ax + aC_2.$

This is the same as $\frac{ay}{\sqrt{C_1}} = \sin(ax + aC_2)$

$= \sin ax \cos aC_2 + \cos ax \sin aC_2,$

or, $y = \frac{\sqrt{C_1}}{a}\cos aC_2 \cdot \sin ax + \frac{\sqrt{C_1}}{a}\sin aC_2 \cdot \cos ax.$

$y = c_1 \sin ax + c_2 \cos ax.$

Example 5. Solve $\frac{d^2y}{dx^2} - 2y = 0.$

Solution Since $\frac{d}{dx}[(y')^2] = 2y'y'',$

we can multiply the given equation by $2y'$ to obtain

$2y'y'' = 4yy'$,

and integrate to obtain

$$(y')^2 = 4\int yy'\,dx = 4\int y\,dy = 2y^2 + C_1.$$

Then $\frac{dy}{dx} = \sqrt{2y^2 + C_1}$, so that $\frac{dy}{\sqrt{2y^2+C_1}} = dx$

and

$$\ln\left|\sqrt{2}\,y + \sqrt{2y^2 + C_1}\right| = \sqrt{2} + \ln C_2.$$

Then last equation yields

$$\sqrt{2}\,y + \sqrt{2y^2 + C_1} = C_2 e^{\sqrt{2}x}.$$

Example 6. Solve $y'' = -1/y^3$.

Solution Multiply by $2y'$ to obtain $2y'y'' = -\frac{2y'}{y^3}$.

Then integration yields

$$(y')^2 = \frac{1}{y^2} + C_1 \text{ so that } \frac{dy}{dx} = \frac{\sqrt{1+C_1y^2}}{y}$$

or $$\frac{y\,dy}{\sqrt{1+C_1y^2}} = dx$$

Another integration gives $\sqrt{1+C_1y^2} = C_1x + C_2$

or $(C_1x + C_2)^2 - C_1y^2 = 1$.

Example 7. Solve the initial value problem

$y'' = 2y^3$; $y(0) = 1$, $y'(0) = 1$.

Solution Setting $y' = p$ we get

$$p\frac{dp}{dy} = 2y^3 \Rightarrow p^2 = y + C_1 \text{ or } \frac{dy}{dx} = \sqrt{y^4 + C_1}.$$

Separating the variables we find that

$x + C_2 = \int (y^4 + C_1)^{-1/2}\,dy$.

The right hand side of the last equation contains an integral of binomial differential. Hence $m = 0$, $n = 4$, $p = -1/2$, i.e. we have the nonintegrable case. Consequently, this integral cannot be expressed as a finite combination of elementary functions.

However, if we use the initial conditions, then we get

$$C_1 = 0. \text{ So } \frac{dy}{dx} = y^2.$$

Thus, taking into account the initial conditions, we finally find that $y = \frac{1}{1-x}$.

Equations involving only the first and second derivatives

If the equation be of the type

$$\varphi\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}\right) = 0, \quad \text{...(1)}$$

i.e., the variables x, y do not appear (explicitly), then, writing p for $\frac{dy}{dx}$, we have

$$\varphi\left(\frac{dp}{dx}, p\right) = 0 \quad \text{...(2)}$$

which is an equation of the first order with p as dependent variable.

The equation (1) may also be reduced to an equation of the first order, with y as independent variable writing

$$p\frac{dp}{dy} \text{ for } \frac{d^2y}{dx^3}$$

thus $$\phi\left(p\frac{dp}{dy}, p\right) = 0 \quad \text{...(3)}$$

To determine the rectilinear motion of a particle subject to a force which is a given function of the velocity. The equation of motion is of the form

$$\frac{d^2x}{dt^2} = f\left(\frac{dx}{dt}\right)$$

Writing v for $\frac{dx}{dt}$, we have

$$\frac{dv}{dt} = f(v), \quad \frac{dv}{f(v)} = dt, \quad \int \frac{dv}{f(v)} = t + C$$

For example, if the particle be subject solely to a resistance varying as the velocity, we have

$$\frac{dv}{dt} = -kv \Rightarrow \frac{dx}{dt} = v = Ae^{-kt},$$

$$x = -\frac{1}{k}Ae^{-kt} + B.$$

If we follow the alternative method,

the equation $\frac{d^2x}{dt^2} = f\left(\frac{dx}{dt}\right)$ is replaced by

$$v\frac{dv}{dx} = f(v)$$

Thus, in the case of resistance varying as the velocity, we get

$$\frac{dv}{dx} = -k, \quad v = -kx + C$$

Hence, $\frac{dx}{dt} + kx = C$ and therefore, $x = \frac{C}{x} + De^{-kx}$ where C, D are arbitrary constants.

Example 8. Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = a$.

Solution Let $p = \frac{dy}{dx}$, then $\frac{d^2y}{dx^2} = \frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{dp}{dx}$ and the given equation becomes

$$x^2\frac{dp}{dx} + xp = a \text{ or } x\,dp + p\,dx = \frac{a}{x}dx.$$

Then integration yields $xp = a \ln |x| + C_1$,

or $x\frac{dy}{dx} = a \ln |x| + C_1$.

When this is written as $dy = a \ln |x| \frac{dx}{x} + C_1\frac{dx}{x}$,

integration gives $y = \frac{1}{2} a \ln^2 |x| + C_1 \ln |x| + C_2$.

Example 9. Solve $xy'' + y' + x = 0$.

Solution Let $p = \frac{dy}{dx}$. Then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$ and the given equation becomes $x\frac{dp}{dx} + p + x = 0$

or $x\,dp + p\,dx = -x\,dx$.

Integration gives $xp = -\frac{1}{2}x^2 + C_1$,

substitution for p gives $\frac{dy}{dx} = -\frac{1}{2}x + \frac{C_1}{x}$,

and another integration yields

$$y = \frac{1}{4}x^2 + C_2 \ln |x| + C_2.$$

Example 10. Find the particular solution of the equation $xy'' + y' + x = 0$, that satisfies the conditions $y = 0$, $y' = 0$ when $x = 0$.

Solution Putting $y' = p$, we have $y'' = p'$, thus
$xp' + p + x = 0$.
Solving the latter equation as a linear equation in the function p, we get

$$px = C_1 - \frac{x^2}{2}.$$

From the fact that $y' = p = 0$ when $x = 0$, we have $0 = C_1 - 0$, i.e., $C_1 = 0$.

Hence $p = -\frac{x}{2}$ or $\frac{dy}{dx} = -\frac{x}{2}$, Integrating once again,

we obtain $y = -\frac{x^2}{4} + C_2$.

Putting $y = 0$ when $x = 0$, we find $C_2 = 0$.

Hence, the desired particular solution is $y = -\frac{1}{4}x^2$.

Example 11. Find the particular solution of the equation $yy'' - y'^2 = y^4$ provided that $y = 1, y' = 0$ when $x = 0$.

Solution Put $y' = p$, then $y'' = p\frac{dp}{dy}$ and our equation becomes $yp\frac{dp}{dy} - p^2 = y^4$.

We have obtained an equation of the Bernoulli type in p(y is considered the argument). Solving it, we find

$$p = \pm y\sqrt{C_1 + y^2}.$$

From the fact that $y' = p = 0$ when $y = 1$, we have $C_1 = -1$. Hence,

$$p = \pm y\sqrt{y^2 - 1} \text{ or } \frac{dy}{dx} = \pm y\sqrt{y^2 - 1}.$$

Integrating, we have

$$\cos^{-1}\frac{1}{y} \pm x = C_2$$

Putting $y = 1$ and $x = 0$, we obtain $C_2 = 0$,

Hence, $\frac{1}{y} = \cos x$ or $y = \sec x$.

Practice Problems I

1. Solve the following differential equations :
   (i) $y'' = x + \sin x$ (ii) $y'' = 1 + y'^2$
   (iii) $2(y')^2 = y''(y - 1)$ (iv) $y''' + y''^2 = 0$
2. Solve the following differential equations :
   (i) $a\frac{d^2y}{dx^2} = \frac{dy}{dx}$ (ii) $2a\frac{dy}{dx}\frac{d^2y}{dx^2} = 1$
   (iii) $\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0$ (iv) $x\frac{d^2y}{dx^2} = \frac{dy}{dx}$
3. Solve the following differential equations :
   (i) $(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

(ii) $(1-y)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$

(iii) $y\frac{d^2y}{dx^2}=2\left(\frac{dy}{dx}\right)^2$

(iv) $y\frac{d^2y}{dx^2}=1-\left(\frac{dy}{dx}\right)^2$

4. Solve the following differential equations :
   (i) $y''=y'+x$
   (ii) $xy''=y'\ln\frac{y'}{x}$
   (iii) $2xy'y''=(y')^2+1$
   (iv) $xy''+x(y')^2-y'=0$
5. Solve $y''=e^{2y}$, $y(0)=0$, $y'(0)=1$.
6. Solve $y'''=(y'')^3$.

4.11 INTEGRAL EQUATION

In some equations the sought for function y(x) may be under the integral sign. In such cases it is sometimes possible to reduce a given equation to a differential one by differentiation.

Example 1. Let $f : R^+ \to R$ be a differentiable function where $f(x)=e-(x-1)(\ln x-1)+\int_1^x f(x)\,dx$. Find f(x).

Solution We have

$$f(x)=e-(x-1)(\ln x-1)+\int_1^x f(x)dx \quad ...(1)$$

and $f'(x)=\frac{1-x}{x}-(\ln x-1)+f(x)$

$\Rightarrow \frac{df}{dx}-f=\frac{1}{x}-\ln x$

whose I.F. $=e^{\int -1dx}=e^{-x}$.

$\therefore\ e^{-x}f(x)=\int e^{-x}\left(\frac{1}{x}-\ln x\right)dx=e^{-x}\ln x+C$

This gives $f(x)=\ln x+Ce^x$ ...(2)

Putting x = 1 in equation (1) and (2), we have

$f(1)=e=Ce$

gives C = 1 and hence

$f(x)=\ln x+e^x$.

Example 2. Given

$\int_0^1 f(tx)\,dt=nf(x)$ then find f(x) where x > 0.

Solution $\int_0^1 f(tx)\,dt=n\cdot f(x)$

Put $tx=y \Rightarrow dt=\frac{1}{x}dy$

$\therefore\ \frac{1}{x}\int_0^x f(y)\,dy=n\,f(x)\ \therefore\ \int_0^x f(y)\,dy=x\cdot n\cdot f(x)$

Differentiating,

$f(x)=n[f(x)+x f'(x)]=f(x)(1-n)=nx f'(x)$

$\therefore\ \frac{f'(x)}{f(x)}=\frac{1-n}{nx}$

Integrating $\ln f(x)=\left(\frac{1-n}{n}\right)\ln cx=\ln (cx)^{\frac{1-n}{n}}$

$\therefore\ f(x)=c.x^{\frac{1-n}{n}}$.

Example 3. Let $f:(0,\infty)\to(0,\infty)$ be a differentiable function satisfying,

$$x\int_0^x (1-t)f(t)\,dt=\int_0^x tf(t)\,dt\ \forall\ x\in R^+ \text{ and } f(1)=1.$$

Determine f(x).

Solution We have $x\int_0^x (1-t)f(t)\,dt=\int_0^x tf(t)\,dt$

Differentiating both sides w.r.t. x, we get

$$x(1-x)f(x)+\int_0^x (1-t)f(t)\,dt=xf(x)$$

$\Rightarrow x^2f(x)=\int_0^x (1-t)f(t)\,dt$

Differentiate both sides w.r.t x again, we get

$x^2f'(x)+2xf(x)=(1-x)f(x)$

$\Rightarrow \frac{f'(x)}{f(x)}=\frac{1-3x}{x^2} \Rightarrow \int\frac{f'(x)}{f(x)}dx=\int\frac{1-3x}{x^2}dx$

$\Rightarrow \ln f(x)=-\frac{1}{x}-3\ln x+\ln c$

$\Rightarrow \ln\left[\frac{f(x)}{c}\right]=-\frac{1}{x}-3\ln x$

Given $f(1)=1\Rightarrow \ln\left(\frac{1}{c}\right)=-1\Rightarrow c=e$

$\Rightarrow \ln\left(\frac{f(x)x^3}{e}\right)=-\frac{1}{x}\Rightarrow f(x)=\frac{1}{x^3}e^{\left(1-\frac{1}{x}\right)}$.

Example 4. Solve the equation

$$x\int_0^x y(t)\,dt = (x+1)\int_0^x t y(t)\,dt \quad x>0.$$

Solution Differentiating both sides of this equation with respect to x we get.

$$\int_0^x y(t)\,dt + xy(x) = \int_0^x ty(t)\,dt + (x+1)\,xy(x)$$

or $\int_0^x y(t)\,dt = \int_0^x ty(t)\,dt + x^2y(x).$

Differentiating once again with respect to x we obtain a differential equation in y(x) :

$y(x) = xy(x) + x^2y'(x) + 2xy(x)$

or $x^2y'(x) + (3x-1)\,y(x) = 0.$

Separating the variables and integrating we find that

$$y = C\,\frac{1}{x^3},\ e^{-1/x}.$$

It is easy to verify that this solution satisfies the original equation.

Example 5. Let f (x) is a continuous function which takes positive values for $x \ge 0$ and satisfy

$$\int_0^x f(t)\,dt = x\sqrt{f(x)} \text{ with } f(1) = \frac{1}{2}.$$

Then find the value of $f(\sqrt{2}+1)$.

Solution We have $\int_0^x f(t)\,dt = x\sqrt{f(x)}$ ...(1)

Differentiating both the sides w.r.t. x, we get

$$f(x) = \frac{x f'(x)}{2\sqrt{f(x)}} + \sqrt{f(x)}$$

Let $f(x) = y^2 \Rightarrow f'(x) = 2y\dfrac{dy}{dx}$

$$y^2 = x\cdot 2y\cdot\frac{dy}{dx}\cdot\frac{1}{2y} + y$$

$$\Rightarrow y^2 = x\cdot\frac{dy}{dx} + y \Rightarrow y^2 - y = x\cdot\frac{dy}{dx}$$

$$\int\frac{dy}{y(y-1)} = \int\frac{dx}{x}$$

$$\Rightarrow \int\frac{y-(y-1)}{y(y-1)}dy = \int\frac{dx}{x}$$

$$\Rightarrow \ln\frac{(y-1)}{y} = \ln cx \Rightarrow \frac{(y-1)}{y} = cx$$

$$\Rightarrow 1-\frac{1}{y} = cx \Rightarrow \frac{1}{y} = 1-cx \Rightarrow y = \frac{1}{1-cx}$$

$$\therefore \sqrt{f(x)} = \frac{1}{1-cx} \quad ...(2)$$

Since, $f(1) = \frac{1}{2}$ (given)

$1-c = \sqrt{2} \Rightarrow c = 1-\sqrt{2}$

$$\sqrt{f(x)} = \frac{1}{1+(\sqrt{2}-1)x}$$

$$\Rightarrow f(x) = \frac{1}{[1+(\sqrt{2}-1)x]^2}$$

$$\Rightarrow f(\sqrt{2}+1) = \frac{1}{4}.$$

Example 6. Suppose f and q are differentiable functions such that for all real x,

$xg(f(x))\,f'(g(x))g'(x) = f(g(x))g'(f(x))\,f'(x).$

Moreover, f is nonnegative and g is positive. Furthermore, $\int_0^a f(g(x))dx = 1-\dfrac{e^{-2a}}{2}$ for all reals a. Given that $g(f(0)) = 1$, compute the value of $g(f(4))$.

Solution Differentiating the given integral with respect to a gives $f(g(a)) = e^{-2a}$.

Now $x\dfrac{d[\ln(f(g(x)))]}{dx} = x\dfrac{f'(g(x))g'(x)}{f(g(x))}$

$$= \frac{g'(f(x))f'(x)}{g(f(x))} = \frac{d[\ln(g(f(x)))]}{dx}$$

where the second equals sign follows from the given. Since $\ln(f(g))) = -2x$, we have $-x^2 + C = \ln(g(f(x)))$, so $g(f(x)) = Ke^{-x^2}$. It follows that K = 1 and $g(f(4)) = e^{-16}$.

Example 7. If a continuous function f(x) satisfies the relation, $\int_0^x t f(x-t)\,dt = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1$, for all real numbers x, then find f(x).

Solution $\int_0^x t f(x-t)\,dt = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1$

$$\int_0^x (x-t)f(t)\,dt = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1$$

[using property P-5 of definite integral]

$$x\int_0^x f(t)\,dt - \int_0^x t f(t)\,dt =$$

$$\int_0^x f(t)\,dt + \sin x + \cos x - x - 1$$

Differentiating both sides w.r.t. x

$$x\cdot f(x) + \int_0^x f(t)\,dt - x\cdot f(x)$$

$$= f(x) + \cos x - \sin x - 1 \quad ...(1)$$

Again differentiating both sides w.r.t. x

$$f(x) = f'(x) - \sin x - \cos x \quad ...(2)$$

which is a linear differential equation

Let $f(x) = y$

$\therefore \dfrac{dy}{dx} - y = \sin x + \cos x \qquad$ I.F. $= e^{-x}$

$$y\cdot e^{-x} = \int e^{-x}(\sin x + \cos x)\,dx$$

Put $x = -t = \int e^{t}(\sin t - \cos t)\,dt$

$y\cdot e^{-x} = -e^{t}\cos t + C$

$y\cdot e^{-x} = -e^{-x}\cos x + C$

$f(x) = y = C e^{x} - \cos x$

If $x = 0, f(0) = 0$ from (1)

$\therefore C = 1$

$f(x) = e^{x} - \cos x.$

Example 8. Let f(x) is periodic function such that $\int_0^x (f(t))^3\,dt = \dfrac{1}{x^2}\left(\int_0^x (f(t)dt)\right)^3 \ \forall\ x \in R - \{0\}$. Find the function f(x) if (1) = 1.

Solution Let $\int_0^x f(t)dt = F(x)$

$\Rightarrow f(x) = F'(x) \quad ...(1)$

$\therefore \int_0^x (f(t))^3\,dt \int_0^x \{F'(t)\}^3\,dt \quad ...(2)$

and $\dfrac{1}{x^2}\left(\int_0^x f(t)dt\right)^3 = \dfrac{(F(x))^3}{x^2} \quad ...(3)$

From (2) and (3), $\int_0^x (F'(t))^3\,dt = \dfrac{1}{x^2}(F(x))^3$

Differentiating both sides w.r.t. x, we get

$$(F'(x))^3 = \frac{x^2.3(F(x))^2F'(x) - (F(x))^3.2x}{x^4}$$

$$= \frac{3(F(x))^2F'(x) - (F(x))^3}{x^3}$$

or $(x F'(x))^3 = 2x(F(x))^2 F'(x) - 2(F(x))^3$

or $\left\{\dfrac{xF'(x)}{F(x)}\right\}^3 = 3\left\{\dfrac{xF'(x)}{F(x)}\right\} - 2$

$\Rightarrow \lambda^3 - 3\lambda + 2 = 0$ where $\lambda = \dfrac{xF'(x)}{F(x)}$

$\Rightarrow (\lambda - 1)^2(\lambda + 2) = 0 \Rightarrow \lambda = 1, -2$

For $\lambda = 1$, $\dfrac{xF'(x)}{F(x)} = 1$

$\Rightarrow \dfrac{F'(x)}{F(x)} = \dfrac{1}{x} \Rightarrow ln\,F(x) = ln\,x + ln\,c$

$\Rightarrow F(x) = cx \Rightarrow F'(x) = c \Rightarrow f(x) = c$ {from (1)}

$f(1) = 1 = c\ (\because f(1) = 1)$

$\therefore f(x) = 1 \quad ...(4)$

For $\lambda = -2$, $\dfrac{x F'(x)}{F(x)} = -2$

$\Rightarrow \dfrac{F'(x)}{F(x)} = -\dfrac{2}{x} \Rightarrow ln F(x) = -2\,ln\,x + ln\,c_1$

$\Rightarrow F(x) = \dfrac{c_1}{x^2} \Rightarrow F'(x) = -\dfrac{2c_1}{x^3}$

$\Rightarrow f(x) = -\dfrac{2c_1}{x^3} \Rightarrow f(1) = 1 = -2c_1$

then $f(x) = \dfrac{1}{x^3} \quad ...(5)$

But the given f(x) is a periodic function

Hence f(x) = 1.

Practice Problems J

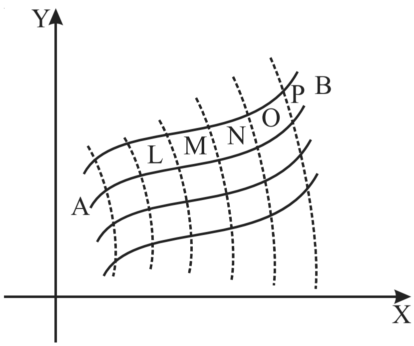
1. Solve $\int_0^x t y(t)\,dt = x^2 y(x)$.
2. Solve $y(x) = \int_0^x y(t)\,dt + e^x$.
3. Solve $\int_0^x t y(t)\,dt = x^2 + y(x)$.
4. If f(x) is a function such that $x\int_0^x (1 - t) f(t)\,dt = \int_0^x t\cdot f(t)\,dt$ and f(1) = 1 then find f(x).
5. Find an initial-value problem whose solution is $y = \cos x + \int_0^x e^{-t^2}dt$

4.12 PROBLEMS IN TRAJECTORIES

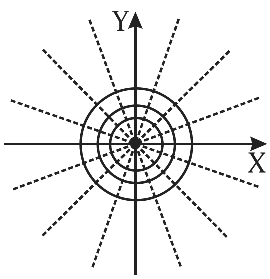
Suppose that a family of curves $\phi(x, y, a) = 0$ is given. A curve making at each of its points a fixed angle α with the curve of the family passing through that point, is called an **isogonal trajectory** of that family.
In particular if $\alpha = \pi/2$, it is called an orthogonal trajectory.

Orthogonal trajectories

If we are given a family of curves (heavy lines in figure), we may think of another family of curves (dashed lines) such that each member of this family cuts each member of the another family at right angles. For example, AB meets several members of the dashed family at right angles at the points L, M, N, O and P. We say that the families are mutually orthogonal, or that either family forms a set of orthogonal trajectories of the other family.



As an example, consider the family of all circles having their centre at origin (a few such circles appear in figure below)



The orthogonal trajectories for this family of circles would be members of the family of straight lines (shown by dashed lines) Similarly, the orthogonal trajectories of the family of straight lines passing through the origin are circles having centre at the origin.

Example 1. Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

Solution The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis. The first step is to find the differential equation satisfied by the members of the family.
If we differentiate $x = ky^2$, we get

$$1 = 2ky\frac{dy}{dx} \text{ or } \frac{dy}{dx} = \frac{1}{2ky}$$

To eliminate k we note that, from the equation of the given general parabola $x = ky^2$, we have $k = x/y^2$ and so the differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\frac{x}{y^2}y} \text{ or } \frac{dy}{dx} = \frac{y}{2x}$$

This means that the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = y/(2x)$. On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope. Therefore, the orthogonal trajectories must satisfy the differential

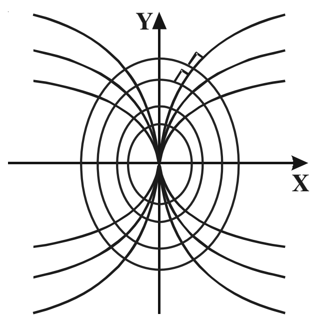
equation $\dfrac{dy}{dx} = -\dfrac{2x}{y}$

This differential equation is separable, and we solve it as follows :

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{2} = -x^2 + C,\; x^2 + \frac{y^2}{2} = C$$

where C is an arbitrary positive constant. Thus, the orthogonal trajectories are the family of ellipses given above. See the sketch below.



Steps to find the orthogonal trajectories

(i) Let the family of given curves be $\phi(x, y, a) = 0$.
(ii) We form the differential equation of this family by eliminating a, say $F(x, y, y') = 0$.

(iii) We replace y' by $-\frac{1}{y'}$ in F(x, y, y') = 0, to obtain the differential equation of the orthogonal family of curves, i.e. $F(x, y, -\frac{1}{y'}) = 0$.

(iv) We solve the above equation to obtain the orthogonal trajectories.

Example 2. Find the orthogonal trajectories of the family $y = x + ce^{-x}$ and determine that particular member of each family that passes through (0, 3).

Solution Differentiation of the given equation yields

$$y' = 1 - ce^{-x}$$

Elimination of c gives

$$y' = 1 + x - y$$

Thus, the differential equation for the family of orthogonal trajectories is $-\frac{1}{y'} = 1 + x - y$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{1+x-y} \qquad \Rightarrow \frac{dx}{dy} + x = y - 1$$

which is a linear differential equation having the solution as $xe^y - e^y(y-2) = c_1$

Therefore, the required curves passing through (0, 3) are found to be

$$y = x + 3e^{-x},\ x - y + 2 + e^{3-y} = 0.$$

Example 3. Find the curves orthogonal to the circles $x^2 + y^2 + 2\mu y - k^2 = 0$, where μ is the variable parameter.

Solution Differentiating, we have

$$x\,dx + (y + \mu)\,dy = 0,$$

and therefore, for the trajectory,

$$x\,dy - (y + \mu)\,dx = 0.$$

Eliminating μ between this and the given equation, we find $2xy\frac{dy}{dx} + (x^2 - y^2 - k^2) = 0$

$$\Rightarrow x\frac{d(y^2)}{dx} - y^2 = -x^2 + k^2$$

This is linear, with y^2 as the independent variable. The integrating factor, by inspection, is $1/x^2$. Introducing this we have $\frac{d}{dx}\left(\frac{y^2}{x}\right) = -1 + \frac{k^2}{x^2}$,

$$\Rightarrow \frac{y^2}{x} = -x - \frac{k^2}{x} + 2\lambda,$$

or $x^2 + y^2 - 2\lambda x + k^2 = 0$, λ being arbitrary.

The original equation represents a system of coaxial circles, cutting the axis of x in the points (± k, 0). The orthogonal trajectories consists of a second system of coaxial circles having these as 'limiting points, ; viz. if we put λ = ± k we get the point-circles

$$(x \mp k)^2 + y^2 = 0.$$

Note: To find the isogonal trajectory at an angle α to the family F(x, y, y') = 0, we replace y' by $\frac{y' - \tan\alpha}{1 + y'\tan\alpha}$ to obtain the required differential equation, $F\left(x, y, \frac{y' - \tan\alpha}{1 + y'\tan\alpha}\right) = 0$.

Example 4. Determine the 45° trajectories of the family of concentric circles $x^2 + y^2 = c$.

Solution We differentiate the equation of the family w.r.t. x $2x + 2y\frac{dy}{dx} = 0$. ...(1)

Replacing $\frac{dy}{dx}$ by $\frac{\frac{dy}{dx} - \tan 45°}{1 + \frac{dy}{dx}\tan 45°} = \frac{\frac{dy}{dx} - 1}{1 + \frac{dy}{dx}}$

we get $x + y\left\{\frac{\frac{dy}{dx} - 1}{1 + \frac{dy}{dx}}\right\} = 0$ or $(x - y)\,dx + (y + x)\,dy = 0$

or $\frac{dy}{dx} = \frac{y - x}{y + x}$ ...(2)

which is the differential equation of the desired trajectories. Putting y = vx, equation (2) reduces to

$$\therefore \frac{dx}{x} + \frac{v+1}{v^2+1}dv = 0.$$

which on integration yields

$$\ln x + (1/2)\ln(v^2 + 1) + \tan^{-1} v = \text{constant}$$

or $x^2 + y^2 = c.\, e^{-2\tan^{-1} y/x}$

Practice Problems K

1. Find the orthogonal trajectories of the family of curves :
   (i) $x^2 - y^2 = c^2$ (ii) $y^2 = 4cx$ (iii) $y = \frac{C}{x^2}$ (iv) $y = C\sqrt{x}$

2. Find the orthogonal trajectory of family of circles touching x-axis at the origin.

3. Find the orthogonal trajectories of the given family of curves : all circles through the points (1, 1) and (–1, –1).

4. Find the curves for which $\frac{dy}{dx} = \frac{y^2 + 3x^2y}{x^2 + 3xy^2}$, and determine their orthogonal trajectories.

5. Prove that the differential equation of the confocal parabolas $y^3 = 4a(x + a)$, is $yp^2 + 2xp - y = 0$, where $p = dy/dx$.
Show that this coincides with the differential equation of the orthogonal curves and interpret the result.

6. Prove that the differential equation of the confocal conics $\frac{x^3}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$, is $xyp^2 + (x^2 - y^2 - a^2 + b^2)p - xy = 0$.
Show that this coincides with the differential equation of the orthogonal curves, and interpret the result.

7. A system of rectangular hyperbolas pass through the fixed points ($\pm$ a, 0) and have the origin as centre; prove that their orthogonal trajectories are the curve $(x^2 + y^2)^2 = 2a^2(x^2 - y^2) + C$.

8. Find the family of trajectories intersecting the curves $x^2 = 2a\,(y - x\sqrt{3})$ at an angle $\alpha = 60°$.

4.13 APPLICATIONS OF DIFFERENTIAL EQUATION

Growth and Decay Problems

Let N(t) denote the amount of substance (or population) that is either growing or decaying. If we assume that dN/dt, the rate of change of this amount of substance, is proportional to the amount of substance present, then $dN/dt = kN$, or $\frac{dN}{dt} - kN = 0$ where k is the constant of proportionality. We are assuming that N(t) is a differentiable, hence continuous, function of time.

Example 1. A culture initially has N_0 number of bacteria. At t = 1 hr. the number of bacteria is measured to be $(3/2)N_0$. If the rate of growth is proportional to the number of bacteria present, determine the time necessary for the number of bacteria to triple.

Solution The present problem is governed by the differential equation

$$\frac{dN}{dt} = kN \qquad ...(1)$$

subject to $N(0) = N_0$.
Separating the variables in (1) and solving, we have

$N = N(t) = ce^{kt}$

At $t = 0$, we have $N_0 = ce^0 = c$ and so $N(t) = N_0e^{kt}$.
At $t = 1$, we have $(3/2)\,N_0 = N_0e^k$
or $e^k = 3/2$, which gives $k = \ln(3/2) = 0.4055$. Thus $N(t) = N_0e^{0.4055t}$

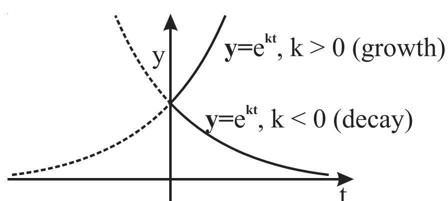
To find the time at which the bacteria have tripled, we solve $3N_0 = N_0e^{0.4055t}$

for t, to get $0.4055t = \ln 3$ or $t = \frac{\ln 3}{0.4055} = 2.71$ hr (approx.)

Note: The function N(t), using the laws of exponents, can also be written as

$$N(t) = N_0(e^k)^t = N_0\left(\frac{3}{2}\right)^t$$

since, $e^k = 3/2$. This latter solution provides a convenient method for computing N(t) for small positive integral values of t; it also shows the influence of the subsequent experimental observation at t = 1 on the solution for all time.
Also, it may be noticed that the actual number of bacteria present at time t = 0 is quite irrelevant in finding the time required to triple the number in the culture. The necessary time to triple, say, 100 or 1000 bacteria is still approximately 2.71 hours.
As shown in the figure, exponential function e^{kt} increases as t increases for k > 0, and decreases as t decreases. Thus, problem describing growth, such as population, bacteria, or even capital, are characterized by a positive value of k, whereas problems involving decay, as in radioactive disintegration, will yield a negative value.



Exponential growth and decay

Example 2. In a culture of yeast, the amount A of active yeast grows at a rate proportional to the amount present. If the original amount A_0 doubles in 2 hours, how long does it take for the original amount to triple ?

Solution The amount A_0 grows exponentially according to $A = A_0e^{kt}$. Since, $A = 2A_0$ when $t = 2$, $2A_0 = A_0e^{2k}$ and $e^{2k} = 2$.
Thus, at time t

$$A = A_0(e^{2k})^{t/2} = A_0(2)^{t/2}$$

Setting $A = 3A_0$ and solving for t, we get

$$3A_0 = A_0(2)^{t/2} \text{ or } 3A_0 = A_0(2)^{t/2}$$

$$\ln 3 = \frac{t}{2}\ln 2,\ t = \frac{2\ln 3}{\ln 2} = 3.17 \text{ hr.}$$

Example 3. A breeder reactor converts the relatively stable uranium 238 into the isotope plutonium 239. After 15 years it is found that 0.043 per cent of the intital amount A_0 of the plutonium has disintegrated. Find the half-life of this isotope, if the rate of disintegration is proportional to the remaining amount.

Solution Let A(t) denote the amount of the plutonium remaining at any time. Then, the solution of the initial value problem

$$\frac{dA}{dt} = kA,\quad A(0) = A_0 \text{ is } A(t) = A_0e^{kt}.$$

If 0.043 per cent of the atoms of A_0 have disintegrated the 99.957 per cent of the substance remains.
To find k, we solve $0.99957\,A_0 = A_0k^{15k}$ to get $e^{15k} = 0.99957$ or, $k = \ln(0.99957)/15 = -0.00002867$.
Hence, $A(t) = A_0e^{-0.00002867t}$.
Now, the half-life is the corresponding value of time for which $A(t) = A_0/2$.

Solving for t, we get $\dfrac{A_0}{2} = A_0e^{-0.00002867t}$

or $t = \dfrac{\log 2}{0.00002867} = 24176.74156$ years.

Example 4. The concentration of the potassium in a kidney is 0.0025 mg/cm³. The kidney is placed in a vessel in which the potassium concentration is 0.0040 mg/cm³. In 2 hours, the potassium concentration in the kidney is found to be 0.0030 mg/cm³. What would be the concentration of potassium in the kidney 4 hours after it was placed in the vessel? How long does it take for the concentration to reach 0.0035 mg/cm³? Assume that the vessel is sufficiently large and that the vessel concentration a = 0.0040 mg/cm³ remains constant.

Solution This problem can be solved using equation

$$y = a - ce^{-kt} \qquad \text{...(1)}$$

Here, $t = 0$, $y = 0.0025$ and $a = 0.0040$.
Put these in (1) to get $c = 0.0015$.
Use this value of c with $t = 2$ and $y = 0.0030$ in (1) to obtain $k = 0.088$.
Now the concentration after 4 hours is

$$y = a - ce^{-kt} = 0.004 - (0.0015)\,e^{(-0.088)(4)}$$

or $y = 0.0033$ mg/cm³
Also, the time required to reach the concentration level

$y = 0.0035$ is $0.0035 = 0.0040 - (0.0015)\,e^{(-0.088)t}$

or $t = 5.42$ hr. (approx.)

Example 5. A student carrying a flue virus returns to an isolated college hostel of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number N_i of infected students but also to the students not infected. Find the number of infected students after 6 days when it is further observed that after 4 days $N_i(4) = 50$.

Solution Assuming that no one leaves the hostel throughout the duration of the disease, we must then solve the initial value problem.

$$\frac{dN_i}{dt} = kN_i(N_0 - N_i),$$
$$= kN_i(1000 - N_i)$$

On solving, we have,

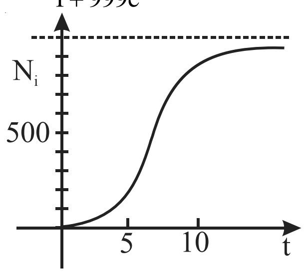
$$N_i = N(t) = \frac{1000}{1 + 999e^{-1000kt}} \qquad \text{...(1)}$$

Now, using $N_i = N(4) = 50$, we can determine k.

$$50 = \frac{1000}{1 + 999e^{-1000k\times 4}} \text{ or } k = 0.0009906$$

Thus, (1) becomes $N_i = N(t) = \dfrac{1000}{1 + 999e^{-0.9906t}}$

or $N_i = N(6) = \dfrac{1000}{1 + 999e^{-5.9436}} = 276$ students



Example 6. A wet porous substance in the open air loses its moisture at a rate proportional to the moisture content . If a sheet hung in the wind loses half its moisture during the first hour, then find the time when it would have lost 99.9 % of its moisture (weather conditions remaining same).

Solution $\dfrac{dM}{d\ell} = -KM$

$M = c\,e^{-kt}$

When $t = 0$; $M = M_0$ $\Rightarrow C = M_0$

$M = M_0\,e^{-kt}$

When $t = 1$, $M = \dfrac{M_0}{2}$ $\Rightarrow k = \ln 2$

$M = M_0\,e^{-t\ln 2}$

When $M = \dfrac{M_0}{1000}$, then $t = \log_2 1000$.

Solution problems

Let Q denote the number of undissolved grams of a solute in a solution at time t. In many important cases, the rate at which Q decreases with respect to t is proportional to the amount Q at t ant to the difference between the saturation concentration and the concentration at t.

Example 7. 100 g of a certain solvent is capable of dissolving 50 g of a particular solute. Given that 25 g of the undissolved solute is contained in the solvent at time t = 0 and that 10 g dissolves in 2 hour, find the amount Q of the undissolved solute at any time t and at t = 6.

Solution The saturation concentration is 50/100 and the concentration at time t is (25 – Q)/100. Thus, the differential equation of the present problem is

$$\frac{dQ}{dt} = kQ\left(\frac{50}{100} - \frac{25-Q}{100}\right) = kQ\frac{25+Q}{100}$$

where the constant of proportionality k is negative.

Separating the variables, we get $\dfrac{100dQ}{Q(25+Q)} = k\,dt$

Resolving the left-hand side into a partial fraction,

we get $4\left(\dfrac{1}{Q} - \dfrac{1}{25+Q}\right) dQ = k\,dt$

Integrating it, we obtain

$$\ln Q - \ln(25+Q) = \frac{kt}{4} - \ln c$$

or $\ln\dfrac{25+Q}{cQ} = -\dfrac{kt}{4}$

Put t = 0 and Q = 25 to obtain c = 2. Again put t = 2 and Q = 25 – 10 = 15 in (25 + Q)/2Q = $e^{-kt/4}$ to get $e^{-k/4} = (4.3)^{1/2}$. Hence $\dfrac{25+Q}{cQ} = e^{-kt/4}$

becomes $\dfrac{25+Q}{cQ} = 2\left(\dfrac{4}{3}\right)^{t/2}$

Therefore, $Q = Q(t) = \dfrac{25}{2(4/3)^{t/2} - 1}$

$$Q(6) = \frac{25}{2(4/3)^{6/2} - 1} = 6.68 \text{ (approx.)}$$

Example 8. 2 g of substance Y combines with 1g substance X to form 3 g of substance Z. When 100 g Y is thoroughly mixed with 50 g X, it is found that in 10 min. 50 g Z has been formed. How many grams of Z can be formed in 20 min.? How long does it take to form 60 g Z?

Solution We have

$$\frac{dQ}{dt} = k\left(50 - \frac{Q}{3}\right)\left(100 - \frac{2Q}{3}\right) = R(150-Q)^2$$

where R is a constant. Separating the variables and integrating, yields $(150 - Q)^{-1} = Rt + c$

Putting t = 0 and Q = 0 to obtain c = 1/150, and t = 10, Q = 50 yields R = 1/3000.

Now, putting t = 20 in

$$\frac{1}{150-Q} = \frac{t}{3000} + \frac{1}{150} \quad ...(1)$$

and solving for Q, we get Q(20) = 75 g.

Putting Q = 60 in (1) and solve for t, we get t = 40/3 min.

Newton's law of cooling

The law states that the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium.

Let T denote the temperature of the body and let T_s denote the temperature of the surrounding medium. Then the time rate of change temperature of the body is $\dfrac{dT}{dt}$.

Newton's law of cooling can be formulated as

$$\frac{dT}{dt} = -k(T - T_s),$$

where k is a positive constant of proportionality. Once k is chosen positive, the minus sign is required in Newton's law to make $\dfrac{dT}{dt}$ negative in a cooling process, when T is greater than T_s, and positive in a heating process, when T is less than T_s.

Example 9. A body whose temperature T is initially 200° C is immersed in a liquid when temperature T_0 is constantly 100°C. If the temperature of the body is 150°C at t = 1 minute, what is its temperature at t = 2 minutes ?

Solution We have $\dfrac{dT}{dt} = -k(T - 100)$

Separating the variables, we get

$$\frac{dT}{T-100} = -\,kdt$$

and the solution is

$$\ln(T-100) = -kt + C \qquad ...(1)$$

When t = 0, T = 200, we find that C = ln 100.
Also, at t = 1, T = 150 and (1) gives

$$\ln 50 = -k(1) + \ln 100$$

or $k = \ln 2$

Now, substituting C = ln 100 and k = ln 2 in (1), we obtain $\ln(T-100) = -t\ln 2 + \ln 100$

or $T = 100(1 + 2^{-t})$

Thus, at t = 2 min, T = 125°C.

Example 10. A body at an unknown temperature is placed in a room which is held at a constant temperature of 30^0 F. If after 10 minutes the temperature of the body is 0^0 F and after 20 minutes the temperature of the body is 15^0 F, find the unknown initial temperature.

Solution We have $\frac{dT}{dt} + kT = 30k$

Solving, we obtain $T = ce^{-kt} + 30$ ...(1)

At t = 10, we are given that T = 0. Hence, form (1)

$$0 = ce^{-10k} + 30 \text{ or } ce^{-100} = -30 \qquad ...(2)$$

At t = 20, we are given that T = 15. Hence, from (1) again, $15 = ce^{-20k} + 30$ or $ce^{-20} = -15$ ...(3)

Solving (2) and (3) for k and c, we find

$$k = \frac{1}{10}\ell n\ 2 \text{ and } c = -30\,e^{10k} = -30(2) = -60$$

Substituting these values into (1), we have for the temperature of the body at any time t

$$T = -60e^{-1/2\ln 2t} + 30 \qquad ...(4)$$

Since we require T at the initial time t = 0, it follows from (4) that

$$T_0 = -60e^{(-1/2\ln 2\ (0))} + 30 = -60 + 30 = -30^0\text{F}.$$

Dilution problems

Consider a tank which initially holds V_0 litre of brine that contains m_0 kg of salt. Another brine solution, containing p kg of salt per litre, is poured into the tank at the rate of a litre/min while, simultaneously, the well–stirred solution leaves the tank at the rate of b litre/min. The problem is to find the amount of salt in the tank at any time t.

Let m denote the amount of salt in the tank at any time. The time rate of change of m, dm/dt, equals the rate at which salt enters the tank minus the rate at which salt leaves the tank. Salt enters the tank at the rate of pa kg/min. To determine the rate at which salt leaves the tank, we first calculate the volume of brine in the tank at any time t, which is the initial volume V_0 plus the volume of brine added at minus the volume of brine removed bt. Thus, the volume of brine at any time is $V_0 + at - bt$ ...(1)

The concentration of salt in the tank at any time is $m/(V_0 + at - bt)$, from which it follows that salt leaves the tank at the rate of $b\left(\frac{m}{V_0 + at - bt}\right)$ kg / min

Thus, $\frac{dm}{dt} = pa - b\left(\frac{m}{V_0 + at - bt}\right)$

or $\frac{dm}{dt} + \frac{b}{V_0 + (a-b)t}m = pa$ ...(2)

Example 11. A tank contains 100 litres brine in which 10 kg of salt are dissolved. Brine containing 2 kg salt per litre flows into the tank at 5 litre/min. If the well-stirred mixture is drawn off at 4 litre/min., find : (i) the amount of the salt in the tank at time t, and (ii) the amount of the salt in the tank at t = 10 min.

Solution Let m(t) denote the number of kgs of salt in the tank and V(t), the number of litres of brine at time t. Then V(t) = 100 + t. Also m(0) = 10 and V(0) = 100. Since 5(2) = 10 kg salt is added to the tank per minute and m(100 + t)(4) kg salt per minute is extracted from the tank, m the satisfies the differential equation

$$\frac{dm}{dt} = 10 - \frac{4m}{100+t} \qquad ...(1)$$

Equation (1) can be written as

$$\frac{dm}{dt} + \frac{4m}{100+t} = 10$$

which is a linear differential equation having solution

$$m(100+t)^4 = \int 10(100+t)^4\,dt = 2(100+t)^5 + c$$

Putting t = 0 and m = 10, we get $c = -190(100)^4$. Thus

(a) $m(t) = 2(100+t) - 190(100)^4(100+t)^{-4}$

(b) $m(10) = 2(100+10) - 190(100)^4 + (100+10)^{-4}$
$= 90.2$ kg.

Miscellaneous problems

Example 12. A hollow spherical brass (thermal conductivity k = 0.26) shell has an inner radius 4 cm and an outer radius 10 cm. If the inner surface temperature is kept at 100°C and the outer surface temperature at 20°C, what is the temperature T in terms of r, the radial distance from the centre of the shell? What is the temperature on the sphere, where r = 7 cm and for what value of r is T = 60°C?

Solution The same amount of heat per second flows across every spherical surface having its centre at the centre of the shell, radius r, and the surface area $A = 4\pi r^2$. The flow is in the radial direction and hence,

$$Q = -kA\frac{dT}{dx}$$

$$Q = -(0.26)(4\pi r^2)\frac{dT}{dr}$$

Separation of the variables yields

$$dT = Br^{-2}\,dr,$$

$$B = -\frac{Q}{(0.26)(4\pi r^2)}$$

Integration yields $T = -(B/r) + C$.

Setting $r = 4$, $T = 100$ and $r = 10$, $T = 20$, we obtain $100 = -0.25\,B + C$ and $20 = -0.1\,B + C$ which gives

$B = -1600/3$ and $C = -100/3$.

Thus, $T = T(r) = \dfrac{1600}{3r} - \dfrac{100}{3}$

When $r = 7$, $T \cong 42.9°C$;

and $T = 60$, where $r \cong 5.7$ cm

Example 13. Suppose that a sky diver falls from rest towards the earth and the parachute opens at an instant, call it t = 0, when the sky diver's speed is $v(0) = v_0 = 10.0$ m/s. Find v(t) of the sky diver at any later time t. Does v(t) increase indefinitely?

Suppose that the weight of the man plus the equipment is W = 712 N, the air resistance R is proportional to v^2, say $R = bv^2$ N, where b is the constant of proportionality and depends mainly upon the parachute. We also assume that $b = 30.0\ Ns^2/m^2 = 30.0$ kg/m.

Solution We now set up the differential equation of the problem as follows.

Newton's second law states that

mass × acceleration = force

where 'force' means the resultant of the forces acting on the sky diver at any instant. These forces are the weight W and the resistance R. The weight W = mg, $g = 9.8\ m/s^2$. Hence, the mass of the man plus equipment is m = W/g = 72.7 kg. The air resistance R acts upward (against the direction of motion), so that the resultant is $W - R = mg - bv^2$

The acceleration a = dv/dt. Hence, by Newton's second law $m\dfrac{dv}{dt} = mg - bv^2$

This is the differential equation of the problem. From the given condition $v(0) = v_0 = 10$, we will now solve this differential equation. This equation can be written as $\dfrac{dv}{dt} = -\dfrac{b}{m}(v^2 - k^2)$, where $k^2 = \dfrac{mg}{b}$

Separating the variables, we get $\dfrac{dv}{v^2 - k^2} = -\dfrac{b}{m}dt$

or $\dfrac{1}{2k}\left(\dfrac{1}{v-k} - \dfrac{1}{v+k}\right)dv = -\dfrac{b}{m}dt$

Integrating, we get

$$\frac{1}{2k}[\ln(v-k) - \ln(v+k)] = -\frac{b}{m}t + c_1$$

or $$\frac{v-k}{v+k} = ce^{-pt},\quad p = \frac{2kb}{m},\quad c = e^{2kc_1} \qquad ...(1)$$

Solving this for v, we have $v(t) = k\dfrac{1+ce^{-pt}}{1-ce^{-pt}}$ ...(2)

Note that as $t \to \infty$, $v(t) \to k$;

i.e. v(t) does not increase indefinitely but approaches a limit k. This limit is independent of the initial condition $v(0) = v_0$.

We now find c in (2) such that we obtain the particular solution satisfying the initial condition.

From (1) with t = 0, we have

$$c = \frac{v_0 - k}{v_0 + k}$$

With this c, (2) represents the solution we are looking for. From the given numerical data, we obtain

$$k^2 = \frac{mg}{b} = \frac{W}{b} = \frac{712}{30} = 23.7\ m^2/s^2$$

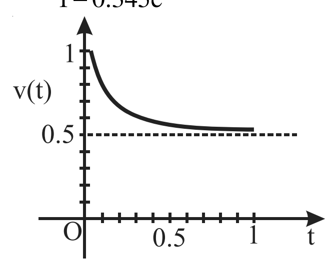
Hence, k = 4.87 m/s. This is the limiting speed. Practically speaking, this is the speed of the sky diver after a sufficiently long time, called as terminal velocity. To $v(0) = v_0 = 10$ m/s, there corresponds

$$c = \frac{v_0 - k}{v_0 + k} = 0.345$$

Finally $p = \dfrac{2kb}{m} = \dfrac{2 \times 4.87 \times 30.0}{72.7} = 4.02\ s^{-1}$

This altogether yields the result

$$v(t) = 4.87\,\frac{1+0.345e^{-4.02t}}{1-0.345e^{-4.02t}} \quad \text{(see figure)}$$



The speed v of a sky diver at any time t.

Example 14. A boat rowed with a velocity u directly across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the path of the boat and the distance down stream to the point where it lands.

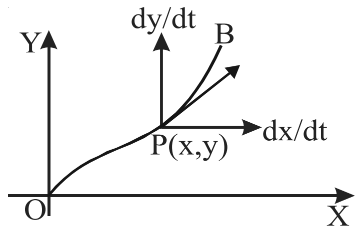
Solution Let us take the origin at the point from where the boat starts.

At any time t after its start from O, let the boat be at P(x, y), so that $\frac{dx}{dt}$ = velocity of the current = $ky(a-y)$

and $\frac{dy}{dt}$ = velocity with which the boat is being rowed = u

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y}{ky(a-y)}$

This gives the direction of the resultant velocity of the boat which is also the direction of the tangent to the path of the boat.



Now from above, $y(a-y)\,dy = \frac{u}{k}\,dx$

$\Rightarrow a\frac{y^2}{2} - \frac{y^3}{3} = \frac{u}{k}x + c$

Since, $y = 0$, when $x = 0 \Rightarrow c = 0$

Hence the equation to the path of the boat is

$$x = \frac{k}{6u}\,y^2(3a-2y)$$

Putting y = a, we get the distance AB, down steam where the boat lands = $\frac{ka^2}{6u}$.

Practice Problems L

1. A person places Rs. 20000 in a saving account which pays 5 percent interest per annum, compounded continuously. Find (a) the amount in the account after three years, and (b) the time required for the account to double in value, presuming no withdrawals and no additional deposits. ($\ln 2 = 0.693$)

2. The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled, and after three years the population is 20,000, estimate the number of people initially living in the country.

3. A radioactive substance decays with time such that at any moment the rate of decay of volume is proportional to the volume at that time. Calculate the half-life of the substance, if 20% of it disappears in 15 years.

4. A yeast grows at a rate proportional to its present size. If the original amount doubles in two hours, in how many hours will it triple ?

5. A depositor places Rs. 10,000 in a certificate of deposit which pay 6 percent interest per annum, compounded continuously. How much will be in the account at the end of seven years assuming no additional deposits or withdrawal ?

6. How long will it take a bank deposit to triple in value if interest is compounded continuously at a constant rate of $5\frac{1}{4}$ percent per annum?

7. A body at a temperature of 50^0 F is placed outdoors where the temperature is 100^0 F. If after 5 minutes the temperature of the body is 60^0 F, find the temperature of the body after 20 minutes.

8. A cup of tea is prepared in a preheated cup with hot water so that the temperature of both the cup and the brewing tea is initially 190^o F. The cup is then left to cool in a room kept at a constant 72^oF. Two minutes later, the temperature of the tea is 150^oF. Determine (a) the temperature of the tea after 5 minutes. (b) the time required for the tea to reach 100^oF.

9. A tank initially contains 50 litres of fresh water. Brine contains 2 kg per litre of salt, flows into the tank at the rate of 2 litre per minutes and the mixture kept uniform by stirring runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 kg.

10. A 50 litre tank initially contains 10 litre of fresh water. At $t = 0$, a brine solution containing 1 kg of salt per litre is poured into the tank at the rate of 4

litre/min, while the well–stirred mixture leaves the tank at the rate of 2 litre/min. Find (a) the amount of time required for overflow to occur and (b) the amount of salt in the tank at the moment of overflow.

11. A motorboat moves in still water with a speed $v = 10$ km/h. At full speed its engine was cut off and in 20 seconds the speed was reduced to $v_1 = 6$ km/h. Assuming that the force of water resistance to the moving boat is proportional to its speed, find the speed of the boat in two minutes after the engine was shut off; find also the distance travelled by the boat during one minute with the engine dead.

Target Problems for JEE Advanced

Problem 1. Find the differential equation which represents the family $xy = ae^x + be^{-x}$.

Solution $xy = ae^x + be^{-x}$ ...(1)

Differentiating (1) w.r.t. x, we get

$$x\frac{dy}{dx} + y = ae^x - be^{-x} \quad ...(2)$$

Differentiating (2) w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}.1 + \frac{dy}{dx} = ae^x + be^{-x} \quad ...(3)$$

Using (1) and (3), we get $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = xy$,

which is the required differential equation.

Problem 2. Find the differential equation whose solution represents the family $c(y + c)^2 = x^3$

Solution $c\,(y + c)^2 = x^3$ ...(1)

Differentiating we get, $c[2(y + c)]\dfrac{dy}{dx} = 3x^2$

using (1), we get $\dfrac{2x^3}{(y+c)^2}(y+c)\dfrac{dy}{dx} = 3x^2$

$$\Rightarrow \frac{2x^3}{y+c}\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{2x}{y+c}\frac{dy}{dx} = 3 \Rightarrow \frac{2x}{3}\left[\frac{dy}{dx}\right] = y + c$$

Hence $c = \dfrac{2x}{3}\left[\dfrac{dy}{dx}\right] - y$ Substituting back into equation (1), we get $\left[\dfrac{2x}{3}\left(\dfrac{dy}{dx}\right) - y\right]\left[\dfrac{2x}{3}\dfrac{dy}{dx}\right]^2 = x^3$

which is the required differential equation.

Problem 3. Find the differential equation corresponding to the family of curves $x^2 + y^2 + 2c_1 + 2c_2y + c_3 = 0$, where c_1, c_2 and c_3 are arbitrary constants.

Solution The equation of the curve is

$$x^2 + y^2 + 2c_1 + 2c_2y + c_3 = 0 \quad ...(1)$$

Differentiating (1) with respect to x, we get

$$2x + 2y\frac{dy}{dx} + 2c_1 + 2c_2\frac{dy}{dx} = 0 \quad ...(2)$$

Differentiating again, we obtain

$$2 + 2\left(y\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{dy}{dx}\right) + 2c_2\frac{d^2y}{dx^2} = 0$$

$$\text{or, } 1 + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + c_2\frac{d^2y}{dx^2} = 0 \quad ...(3)$$

Differentiating (3) with respect to x yields

$$(y + c_2)\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\frac{d^2y}{dx^2} = 0 \quad ...(4)$$

From (3) $c_2\dfrac{d^2y}{dx^2} = -\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{d^2y}{dx^2} - 1$

and (4) gives $c_2\dfrac{d^3y}{dx^3} = -y\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}$

Dividing these two equations, we get

$$\frac{d^3y}{dx^3}\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

Problem 4. Solve $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$

Solution Let $x = r\cos\theta$, and $y = r\sin\theta$.

Then $r^2 = x^2 + y^2$ and $y/x = \tan\theta$.

Differentiating these relations and substituting the respective value in the given equation, we get after simplification, the relation $2rdr = 2a^2\,d\theta$

which on integration yields $r^2 = 2a^2\theta + C$

$$\Rightarrow x^2 + y^2 = 2a^2\tan^{-1}\frac{y}{x} + C$$

which is the required solution.

Problem 5. Let $\frac{d}{dx}(x^2y) = x - 1$ where $x \neq 0$ and $y = 0$ when $x = 1$. Find the set of values of x for which $\frac{dy}{dx}$ is positive.

Solution $\frac{d}{dx}(x^2y) = x - 1$

$$x^2 y = \int (x-1)\,dx = \frac{x^2}{2} - x + C$$

If $x = 1, y = 0$

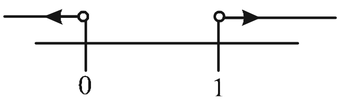
$$\therefore\ 0 = \frac{1}{2} - 1 + C \Rightarrow C = \frac{1}{2}$$

$$x^2 y = \frac{x^2}{2} - x - \frac{1}{2}$$

$$y = \frac{1}{2} - \frac{1}{x} + \frac{1}{2x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^3} = \frac{x-1}{x^3}$$

$$\therefore\ \frac{x-1}{x^3} > 0$$



$$\therefore\ x \in (-\infty, 0) \cup (1, \infty)]$$

Problem 6. Solve

$$2y\frac{dy}{dx} = e^{\frac{x^2+y^2}{x}} + \frac{x^2+y^2}{x} - 2x.$$

Solution Put $x^2 + y^2 = z$

$$\Rightarrow \frac{dz}{dx} = 2x + 2y\frac{dy}{dx}$$

$$\frac{dz}{dx} = e^{z/x} + \frac{z}{x} \qquad \text{Put } z = tx$$

$$\Rightarrow \frac{dz}{dx} = t + x\frac{dt}{dx}$$

$$t + x\frac{dt}{dx} = e^t + t \quad \Rightarrow \frac{dt}{e^t} = \frac{dx}{x}$$

$$-e^{-t} = \ln x + c$$

$$\Rightarrow -e^{-\frac{x^2+y^2}{x}} = \ln x + c$$

$$\Rightarrow \ln|cx| = -e^{-\frac{x^2+y^2}{x}}.$$

Problem 7. Solve the differential equation,

$$\left(2xy + x^2y + \frac{y^3}{3}\right)dx + (x^2 + y^2)\,dy = 0.$$

Solution Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$2vx^2 + vx^3 + \frac{v^3x^3}{3} + (x^2 + v^2x^2)\left(v + x\frac{dv}{dx}\right) = 0$$

Cancelling x^2 and rearranging

$$\left(v + \frac{v^3}{3}\right)x + 3\left(v + \frac{v^3}{3}\right) + x\frac{dv}{dx}(1 + v^2) = 0$$

Put $v + \frac{v^3}{3} = t$

$$(1 + v^2)\frac{dv}{dx} = \frac{dt}{dx}$$

$$\Rightarrow tx + 3t + x\frac{dt}{dx} = 0$$

$$\Rightarrow \frac{t(x+3)}{x} + \frac{dt}{dx} = 0$$

$$\Rightarrow \left(1 + \frac{3}{x}\right)dx + \frac{dt}{t} = 0$$

$$\Rightarrow x + 3\ln x + \ln t = c_1 \qquad \Rightarrow \ln x^3 = c_1 - x$$

$$\Rightarrow \left(\frac{y}{x} + \frac{y^3}{3x^3}\right)x^3 = c\,e^{-x}$$

$$\Rightarrow yx^2 + \frac{y^3}{3} = c.e^{-x}.$$

Problem 8. Solve the equation
$y' + \sin y + x\cos y + x = 0.$

Solution We write the equation in the form

$$y' + 2\sin\frac{y}{2}\cos\frac{y}{2} + x.2\cos^2\frac{y}{2} = 0.$$

By dividing both sides of the latter equation by

$$2\cos^2\frac{y}{2}, \text{ we get } \frac{y'}{2\cos^2\frac{y}{2}} + \tan\frac{y}{2} + x = 0.$$

Put $\tan\frac{y}{2} = z \Rightarrow \frac{dz}{dx} = \frac{y'}{2\cos^2\frac{y}{2}}$

$$\Rightarrow \frac{dz}{dx} + z = -x.$$

This is a linear equation whose general solution is

$$z = 1 - x + Ce^{-x}.$$

Replacing z by its expression in terms of y we obtain the general solution of the given equation.

$$\tan\frac{y}{2} = 1 - x + Ce^{-x}.$$

Problem 9. Solve

$(x\cos y - y\sin y)\,dy + (x\sin y + y\cos y)\,dx = 0.$

Solution Put $x\sin y + y\cos y = t$

$$(x\cos y - y\sin y)\frac{dy}{dx} + \sin y + \cos y\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dt}{dx} - \left(\sin y + \cos y\frac{dy}{dx}\right) + t = 0$$

Put $\sin y = z \Rightarrow \cos y\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$$\Rightarrow \frac{dt}{dx} - \frac{dz}{dx} + t - z = 0 \Rightarrow d(t-z) \neq -(t-z)\,dx$$

$$\Rightarrow \int\frac{d(t-z)}{t-z} = -\int dx$$

$$\Rightarrow ln(t-z) = -x + c$$

$$ln(x\sin y + y\cos y - \sin y) = -x + c$$

$$\Rightarrow e^x(x\sin y + y\cos y - \sin y) = c.$$

Problem 10. Solve $x^2\dfrac{dy}{dx} + y^2 e^{\frac{x(y-x)}{y}} = 2y(x-y)$

Solution $x^2\dfrac{dy}{dx} - 2y(x-y) + y^2 e^{\frac{x(y-x)}{y}} = 0$

$$\Rightarrow \frac{x^2dy}{y^2dx} - 2\frac{(x-y)}{y} + e^{\frac{x(y-x)}{y}} = 0$$

$$\Rightarrow \frac{x^2dy}{y^2dx} - \frac{2x}{y} + 2 + e^{-x^2/y}e^x = 0 \qquad ...(1)$$

Let $e^{x^2/y} = t$

$$\Rightarrow e^{x^2/y}\frac{\left(2xy - \frac{x^2dy}{dx}\right)}{y^2} = \frac{dt}{dx}$$

$$\Rightarrow \frac{x^2}{y^2}\frac{dy}{dx} - \frac{2x}{y} = \frac{-1}{t}\frac{dt}{dx}$$

So from equation (1)

$$-\frac{1}{t}\frac{dt}{dx} + 2\frac{1}{t}e^x = 0 \qquad \Rightarrow \frac{dt}{dx} - 2t = e^x \qquad ...(2)$$

$$I.F = e^{-\int 2dx} = e^{-2x}$$

The solution to equation (2) is given by

$$t.e^{-2x} = \int e^{-2x}e^x dx \Rightarrow t.e^{-2x} = -e^{-x} + c$$

$$\Rightarrow t = -e^x + c\,.\,e^{2x}$$

$$\Rightarrow e^{x^2/y} = -e^x + c\,.\,e^{2x}$$

$$\Rightarrow x(x-y) = y\,\ell n(ce^x - 1).$$

Problem 11. Solve

$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0,\quad y\big|_{x=0} = \sqrt{5}$$

Solution $\dfrac{dy}{dx} = \dfrac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y}$

$$\Rightarrow \frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + y^2}}{y} \quad \text{which is homogeneous}$$

Put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{-x \pm x\sqrt{1+v^2}}{vx} = \frac{-1 \pm \sqrt{1+v^2}}{v}$$

$$\therefore x\frac{dv}{dx} = \frac{-1 \pm \sqrt{1+v^2}}{v} - v = \frac{-1 \pm \sqrt{1+v^2} - v^2}{v}$$

Taking positive sign,

$$\int\frac{v\,dv}{\sqrt{v^2+1} - (1+v^2)} = \int\frac{dx}{x}$$

$$\Rightarrow \int\frac{v\,dv}{\sqrt{v^2+1}\left(\sqrt{1+v^2} - 1\right)} = -\int\frac{dx}{x}$$

Put $\sqrt{1+v^2} - 1 = t \Rightarrow \dfrac{v}{\sqrt{1+v^2}}dv = dt$

$$\Rightarrow \int\frac{dt}{t} = -\int\frac{dx}{x}$$

$$ln\,tx = C$$

$$\ln\left[\left(\sqrt{1 + \frac{y^2}{x^2}} - 1\right)x\right] = C$$

$$\Rightarrow \ln\left[\sqrt{x^2 + y^2} - x\right] = C$$

For $x = 0, y = \sqrt{5}$

$\ln\ [\sqrt{5}-0] \Rightarrow C = \ln\sqrt{5}$

$\therefore\ \sqrt{x^2+y^2} = \sqrt{5} + x$

$\Rightarrow\ x^2+y^2 = 5 + x^2 + 2\sqrt{5}\,x$

$\Rightarrow\ y^2 = 5 + 2\sqrt{5}\,x$

Similarly, with negative sign ,

$y^2 = 5 - 2\sqrt{5}\,x.$

Problem 12. Solve $xy^2(p^2+2) = 2py^3 + x^3$

Solution The given equation can be written as

$(xy^2p^2 - x^3) + 2(xy^2 - py^3) = 0$

$\Rightarrow\ x(y^2p^2 - x^2) + 2y^2(x - py) = 0$

$\Rightarrow\ (py - x)\,\{x(py + x) - 2y^2\} = 0$

If $py - x = 0$ then $y\,dy - x\,dx = 0$

$\Rightarrow\ y^2 - x^2 = c$

If $xyp + x^2 - 2y^2 = 0$ then $2y\dfrac{dy}{dx} - \dfrac{4y^2}{x} = -2x$

$\Rightarrow\ \dfrac{dt}{dx} - \dfrac{4}{x}t = -2x$, where $t = y^2$

$$\text{I.F.} = e^{-\int \frac{4}{x}dx} = e^{-4\ln x} = \frac{1}{x^4}$$

Its solution is $t\left(\dfrac{1}{x^4}\right) = \displaystyle\int -2x.\frac{1}{x^4}dx$

$\Rightarrow\ \dfrac{t}{x^4} = \dfrac{1}{x^2} + c \Rightarrow y^2 = x^2 + c\,x^4$

Hence the required solution is

$(y^2 - x^2 - c)\,(y^2 - x^2 - cx^4) = 0.$

Problem 13. The function $y(x)$ satisfies the equation $y(x) + 2x\displaystyle\int_0^x \frac{y(u)}{1+u^2}du = 3x^2 + 2x + 1$. Show that the substitution $z(x) = \displaystyle\int_0^x \frac{y(u)}{1+u^2}du$ converts the equation into a first order linear differential equation for $z(x)$ solve for $z(x)$. Hence solve the original equation for $y(x)$.

Solution $y(x) + 2x\displaystyle\int_0^x \frac{y(u)}{1+u^2}du = 3x^2 + 2x + 1$ ...(1)

Let $z(x) = \displaystyle\int_0^x \frac{y(u)}{1+u^2}du$

$\therefore\ z'(x) = \dfrac{y(x)}{1+x^2}$

$\Rightarrow\ y(x) = (1+x^2)\,z'(x)$ ...(2)

Substituting $y(x)$ in equation (1)

$(1+x^2)\,z'(x) + 2x\cdot z(x) = 3x^2 + 2x + 1$

Let $z'(x) = \dfrac{dz}{dx}$

$$\frac{dz}{dx} + \frac{2x}{1+x^2}\,z(x) = \frac{3x^2+2x+1}{1+x^2}$$

which is a linear differential equation.

$$\text{I.F.} = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1 + x^2$$

Hence $z(1+x^2) = \displaystyle\int (3x^2 + 2x + 1)\,dx$

$z(1+x^2) = x^3 + x^2 + x + C$

but $z = 0$ when $x = 0 \Rightarrow C = 0$

Hence $z(x) = \dfrac{x^3+x^2+x}{1+x^2}$ ...(3)

Now $y(x) = (1+x^2)\cdot z'(x)$ ...(4)

Also $z(x) = \dfrac{x(1+x^2)+x^2}{1+x^2} = x + \dfrac{x^2+1-1}{1+x^2}$

$= x + 1 - \dfrac{1}{1+x^2}$

$\therefore\ z'(x) = 1 + \dfrac{2x}{(1+x^2)^2} = \dfrac{(1+x^2)+2x}{(1+x^2)^2}$

$\therefore\ y(x) = \dfrac{(1+x^2)^2+2x}{(1+x^2)}$ [from (4)]

Problem 14. $\left(\dfrac{1}{y}\sin\dfrac{x}{y} - \dfrac{y}{x^2}\cos\dfrac{y}{x} + 1\right)dx$

$+\left(\dfrac{1}{x}\cos\dfrac{y}{x} - \dfrac{x}{y^2}\sin\dfrac{x}{y} + \dfrac{1}{y^2}\right)dy = 0$

Solution Isolating $\sin\dfrac{x}{y}$ and $\cos\dfrac{y}{x}$

$$\left(\frac{dx}{y} - \frac{x\,dy}{y^2}\right)\sin\frac{x}{y} + \left(\frac{dy}{x} - \frac{y\,dx}{x^2}\right)\cos\frac{y}{x} + dx + \frac{dy}{y^2} = 0$$

$$\frac{ydx - xdy}{y^2}.\sin\frac{x}{y}\ \ \frac{xdy - ydx}{x^2}.\cos\frac{y}{x} + dx + \frac{dy}{y^2} = 0$$

$$\sin\frac{x}{y}.d\left(\frac{x}{y}\right) + \cos\frac{y}{x}.d\left(\frac{y}{x}\right) + dx + \frac{dy}{y^2} = 0$$

$\Rightarrow\ -\cos\dfrac{x}{y} + \sin\dfrac{y}{x} + x - \dfrac{1}{y} = C.$

Problem 15. Given a function g which has a derivative $g'(x)$ for every real x and which satisfy $g'(0) = 2$ and $g(x+y) = e^y . g(x) + e^x . g(y)$ for all x and y, find $g(x)$ and determine the area bounded by the graph of the function, ordinate of its minima and the co-ordinate axes .

Solution Putting $x = 0$ and $y = 0 \Rightarrow g(0) = 0$

and $g'(0) = \lim\limits_{h \to 0} \dfrac{g(h)}{h} = 2$

Now $\quad g'(x) = \lim\limits_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$

$$= \lim_{h \to 0} \frac{e^h . g(x) + e^x . g(h) - g(x)}{h}$$

$$= g(x) \lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) + e^x \lim_{h \to 0} \frac{g(h)}{h}$$

Hence, $g'(x) = g(x) + 2e^x$

If $\quad g(x) = y$ then $\dfrac{dy}{dx} = y + 2e^x$

$$\Rightarrow e^{-x}\frac{dy}{dx} - y\,e^{-x} = 2$$

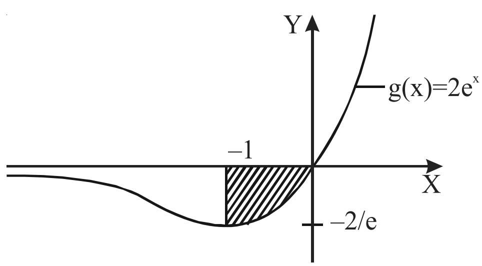
$$\Rightarrow \frac{d}{dx}(y\,e^{-x}) = 2$$

$$\Rightarrow y\,e^{-x} = 2x + c$$

Now, $\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right] \Rightarrow c = 0$

$$\frac{dy}{dx} = 2[e^x + x\,e^x] = 0 \Rightarrow x = -1$$

Hence, $y = 2x\,e^x = g(x)$



Area $\quad A = \left| 2\displaystyle\int_{-1}^{0} x\,e^x \, dx \right|$

Using integration by parts, we get

$$A = \left(2 - \frac{4}{e}\right) \text{ sq. units.}$$

Problem 16. A differentiable function f satisfies $(x-y)\,f(x+y) - (x+y)\,f(x-y) = 4xy(x^2 - y^2)\ \forall\ x,\ y \in R$, where $f(1) = 1$. Find $f(x)$.

Solution We have

$$(x-h)\,f(x+h) - (x+h)f(x-h) = 4xh(x^2 - h^2)$$

$$\Rightarrow xf(x+h) - x(f(x-h) - hf(x+h) - hf(x-h) = 4xh(x^2-h^2)$$

$$\Rightarrow \lim_{h \to 0} x\left(\frac{f(x+h) - f(x)}{h}\right) + x\left(\frac{f(x-h) - f(x)}{-h}\right)$$

$$= \lim_{h \to 0} f(x+h) + f(x+h)$$

$$\Rightarrow x.2f'(x) = 2f(x) + 4x^3$$

Assuming $f(x)$ as y,

$\dfrac{dy}{dx} - \dfrac{y}{x} = 2x^2$ which is a linear diff. equation

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\ell nx} = \frac{1}{x}$$

$$\Rightarrow y . \frac{1}{x} = \int 2x^2 . \frac{1}{x} dx$$

$$\Rightarrow \frac{y}{x} = x^2 + C$$

$$\Rightarrow y = x^3 + Cx$$

$$f(1) = 1 \Rightarrow C = 0$$

Hence, $f(x) = x^3$.

Problem 17. A differentiable function f satisfies the relation $f(x+y) + f(x) . f(y) = f(xy + 1)$. Given $f(0) = -1$, $f'(0) = f'(1) = 1$, find f.

Solution We have $f(0) = -1$.

Putting $x = 0$ and $y = 1$, we get $f(1) = 0$

Now $\quad f(x+h) + f(x) . f(h) = f(xh) + 1)$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + f(x)\left\{\frac{f(h) + 1}{h}\right\}$$

$$= \lim_{h \to 0} \frac{f(xh+1) - f(1)}{xh} . x$$

$$\Rightarrow f'(x) + f(x) . f'(0) = xf'(1)$$

$$\Rightarrow \frac{dy}{dx} + py = qx$$

$$\Rightarrow y.\, e^{px} = \int qx.e^{-x}dx = 2\left[x.\frac{e^{px}}{p} - \frac{e^{px}}{p^2}\right] + C$$

$$\Rightarrow ye^x = xe^x - e^x + C$$

$$f(0) = -1$$

$$\Rightarrow -1 = -1 + C \Rightarrow C = 0$$

$$\Rightarrow y = x - 1.$$

Problem 18. A differentiable function f satisfies the relation $f\left(\frac{x+y}{1+xy}\right) = f(x) \,.\, f(y) \; \forall\, x, y \in R - \{-1\}$, where $f(0) \neq 0$ and $f'(0) = 1$. Find f(x).

Solution Put $x = y = 0$, $f(0) = f(0)^2$

$\therefore$ $f(0) = 0, 1 \Rightarrow f(0) = 1$

We have $f\left(\frac{x+h}{1+xh}\right) - f(x)\,.\,f(h) = 0$

$$\Rightarrow \frac{\left[f\left(x+\frac{h-x^2h}{1+xh}\right)-f(x)\right]}{\frac{h-x^2h}{1+xh}.h}\;\frac{h-x^2h}{(1+xh)}$$

$$= \frac{f(x).f(h)-f(x)}{h}$$

$$\Rightarrow f'(x)\,.\,(1-x^2) = f(x)\,.\,f'(0) \Rightarrow \frac{dy}{y} = \frac{dx}{1-x^2}$$

Integrating both sides,

$$\ell ny = \ell n\left(\frac{1-x}{1+x}\right) \Rightarrow y = C\left(\frac{1-x}{1+x}\right)$$

Since, $f(0) = 1$ we get $1 = C \Rightarrow y = \frac{1-x}{1+x}$.

Problem 19. A differentiable function f satisfies the relation $f(xy) = xf(y) + yf(x) \; \forall\, x, y \in R^+$. Find f, if $f'(1) = 1$.

Solution Put $x = y = 1 \Rightarrow f(1) = 2f(1) \Rightarrow f(1) = 0$.

Now, $f(x(1+h)) = x\,f(1+h) + (1+h)\,f(x)$

$$x\,.\frac{f(x+xh)-f(x)}{xh} = \frac{x[f(1+h)-0]}{h} + \frac{hf(x)}{h}$$

Applying limits $h \to 0$ on both sides, $xf'(x) = xf'(1) + f(x)$. Put $f'(1) = 1$

$$\frac{dy}{dx} = 1 + \frac{y}{x} \quad \text{Homogeneous diff. equation}$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + \frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = 1 + v$

$$\Rightarrow dv = \frac{dx}{x}$$

$$\ell n\,|x| = v + C$$

$$x = \pm\, e^{\frac{v}{x}+C}$$

$$f(1) = 0 \Rightarrow C = 0$$

$$\therefore \quad x = e^{y/x} \Rightarrow \ell nx = \frac{y}{x}$$

$$\Rightarrow y = x\ell nx.$$

Problem 20. Let $f: R \to R$ be a smooth function such that $f'(x)^2 = f(x)\,f''(x)$ for all x. Suppose $f(0) = 1$ and $f^{(4)}(0) = 9$. Find all possible values of $f'(0)$.

Solution Let $f'(0) = a$. Then the equation gives $f''(0) = a^2$. Differentiating the given equation gives $2f'(x)\,f''(x) = f(x)\,f'''(x) + f'(x)\,f''(x)$, or $f'(x)\,f''(x) = f(x)\,f'''(x)$. Differentiating once more gives

$$f'(x)\,f'''(x) + f''(x)^2 = f(x0\,f^{(4)}(x) + f'(x)\,f'''(x)$$

or $\quad f''(x)^2 = f(x)\,f^{(4)}(x)$, giving $9 = f^{(4)}(0) = a^4$.

Thus $a = \pm\sqrt{3}$. These are indeed both attainable by

$$f(x) = e^{\pm x\sqrt{3}}.$$

Alternative: Rewrite the given equation as $\frac{f''(x)}{f'(x)} = \frac{f'(x)}{f(x)}$. Integrating both sides gives $\ln f'(x) = \ln f(x) + C_1$, and exponentiating gives $f'(x) = Cf(x)$. This has solution $f(x) = Ae^{Cx}$ for constants A and C.

Since $f(0) = 1$, $A = 1$, and differentiating we find that

$$C^4 = f^{(4)}(0) = 9, \text{ yielding } f'(0) = C = \pm\sqrt{3}.$$

Problem 21. The function $f: R \to R$ is twice differentiable. Also, $f\lambda^2 - 2f'\lambda + f'' = 0$ gives two equal values of λ for every x. Find f(x), if $f(0) = 1$, $f'(0) = 2$.

Solution The equation $f\lambda^2 - 2f'\lambda + f'' = 0$ gives two equal values of λ for every x. Therefore, we have discriminant = 0.

$$\Rightarrow (f')^2 = ff'' = f\left(f'\frac{df'}{df}\right)$$

$$\Rightarrow f' = f\frac{df'}{df}$$

$$\Rightarrow \frac{df'}{f'} - \frac{df}{f} = 0$$

$\Rightarrow \ln f' - \ln f = C$ [integrating]

Using the condition $f(0) = 1$, $f'(0) = 2$, gives $C = \ln 2$. Thus, we have

$$\ln\frac{f'}{f} = \ln 2$$

$$\Rightarrow \frac{f'}{f} = 2$$

$$\Rightarrow \frac{df}{f} = 2\,dx$$

$\Rightarrow \ln f = 2x + C'$ [integrating]

Using the condition $f(0) = 1$, gives $C' = 0$. Hence, we have $f(x) = e^{2x}$

Problem 22. Suppose $\begin{vmatrix} f'(x) & f(x) \\ f''(x) & f'(x) \end{vmatrix} = 0$ where f (x) is continuously differentiable function with $f'(x) \neq 0$ and satisfies $f(0) = 1$ and $f'(0) = 2$ then find f(x).

Solution $f'(x) \cdot f'(x) - f(x) \cdot f''(x) = 0$

or $\dfrac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = 0$

$$\frac{d}{dx}\left[\frac{f(x)}{f'(x)}\right] = 0$$

Integrating, $\dfrac{f(x)}{f'(x)} = C$ ...(1)

Put $x = 0$, $\dfrac{f(0)}{f'(0)} = C \Rightarrow C = \dfrac{1}{2}$,

Hence $\dfrac{f(x)}{f'(x)} = \dfrac{1}{2}$.

From (1), $2f(x) = f'(x)$

$\therefore \dfrac{f'(x)}{f(x)} = 2$

Again integrating $\ln[f(x)] = 2x + k$

Put $x = 0$ to get $k = 0$

$f(x) = e^{2x}$.

Problem 23. A function $y = f(x)$ satisfies $f'(x) + \dfrac{2f(x)}{x} = \dfrac{2\sqrt{f(x)}}{\cos^2 x}$ with $f(0) = 1$ then find the value of $f(1)$.

Solution We have $\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{2\sqrt{y}}{\cos^2 x}$ where $y = f(x)$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{2\sqrt{y}}{x} = 2\sec^2 x$$

Put $2\sqrt{y} = t \Rightarrow \dfrac{1}{\sqrt{y}}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

Hence, $\dfrac{dt}{dx} + \dfrac{1}{x} \cdot t = 2\sec^2 x$, which is a linear differential equation

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\therefore\ x \cdot t = 2\int x\sec^2 x\,dx = 2\left[x\tan x - \int \tan x\,dx\right]$$

$$2 \cdot \sqrt{y} \cdot x = 2[x\tan x + \ln(\cos x)] + C$$

$$\Rightarrow yx^2 = [x\tan x + \ln(\cos x)]^2.$$

Problem 24. Prove the identity $\int_0^x e^{zx-z^2}dz = e^{x^2/4}\int_0^x e^{-z^2/4}dz$ deriving for the function $I(x) = \int_0^x e^{zx-z^2}dz$ a differential equation and solving it.

Solution $I(x) = \int_0^x e^{zx-z^2}dz$

We have $\left[zx - z^2 = \dfrac{x^2}{4} - \left(z - \dfrac{x}{2}\right)^2\right]$

$$\therefore\ I(x) = e^{x^2/4}\int_0^x e^{-\left(z-\frac{x}{2}\right)^2}dz$$

Put $z - \dfrac{x}{2} = t$, $dz = dt$ $= e^{x^2/4}\displaystyle\int_{-\frac{x}{2}}^{\frac{x}{2}} e^{-t^2}dt$

$$I(x) = 2.e^{x^2/4}\int_0^{\frac{x}{2}} e^{-t^2}dt \qquad ...(1)$$

(f(t) is an even function)

Differentiating (1) with respect to x.

$$\frac{dI}{dx} = 2\left[e^{\frac{x^2}{4}}.e^{-\frac{x^2}{4}}.\frac{1}{2} + \left(\int_0^{\frac{x}{2}} e^{-t^2}dt\right).e^{\frac{x^2}{4}}.\frac{x}{2}\right]$$

$$= 1 + x\,e^{\frac{x^2}{4}}\int_0^{\frac{x}{2}} e^{-t^2}dt = 1 + x.\frac{1}{2}$$

$$\frac{dI}{dx} - \frac{x}{2}I = 1,\ \text{IF} = e^{-\frac{x^2}{4}}$$

$$\therefore\ I.\,e^{-\frac{x^2}{4}} = \int e^{-\frac{x^2}{4}}dx + C = F(x) + C$$

[say, where $\int e^{-\frac{x^2}{4}}dx = F(x)$]

when $x = 0$ then $I = 0 \Rightarrow C = -F(0)$

$$I.\,e^{-\frac{x^2}{4}} = F(x) - F(0) = \int_0^x e^{-\frac{z^2}{4}}dz$$

$$\therefore\ I = e^{\frac{x^2}{4}}.\int_0^x e^{-\frac{z^2}{4}}dz.$$

Problem 25. Let the function $ln(f(x))$ is defined where $f(x)$ exists for $x \geq 2$ and k is fixed positive real number, prove that if $\frac{d}{dx}(x\,f(x)) \leq -k\,f(x)$ then $f(x) \leq A x^{-1+k}$ where A is independent of x.

Solution $x f'(x) + f(x) \leq -k f(x)$

$\Rightarrow \; x f'(x) + (k+1) f(x) \leq 0$

I.F. $= e^{\int P dx} = e^{\int \frac{K+1}{x} dx} = e^{(K+1)\ln x} = e^{\ln x^{K+1}} = x^{K+1}$

$\Rightarrow \; x^{k+1} f'(x) + (k+1) x^k f(x) \leq 0$

$$\Rightarrow \frac{d[x^{k+1} \cdot f(x)]}{dx} \leq 0$$

Let $F(x) = x^{k+1} \cdot f(x)$

$F(x)$ is decreasing $x \geq 2$

$F(x) \leq F(2)$ for all $x \geq 2$

$\Rightarrow \; F(x) \leq A$

$x^{k+1} \cdot f(x) \leq A \Rightarrow f(x) \leq A x^{-1+k}$.

Problem 26. Let

$$I = \int \frac{1-y}{\ln x^x + xy^{-1}} dx \text{ and } J = \int \frac{\ln x^x + xy^{-1}}{1-y} dy,$$

where $\frac{x}{y} = x^y$. Show that $I \cdot J = (x+d)(y+c)$ where $c, d \in R$. Hence show that $\frac{d}{dx}(IJ) = I + J\frac{dy}{dx}$.

Solution $\frac{x}{y} = x^y \Rightarrow ln\,x - ln\,y = y\,ln\,x$

Differentiating w.r.t. x,

$$\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} = \frac{y}{x} + ln\,x\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(ln\,x + y^{-1}) = \frac{1-y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-y}{x(\ln x + y^{-1})} = \frac{1-y}{\ln x^x + xy^{-1}} \quad ...(1)$$

Integrating w.r.t. x

$$\int \frac{1-y}{\ln x^x + xy^{-1}} dx = \int dy \Rightarrow I = y + c$$

$$\text{Again } \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{\ln x^x + xy^{-1}}{1-y}$$

$$\text{Integrating w.r.t.y, } \int \frac{\ln x^x + xy^{-1}}{1-y} dy = \int dx$$

$J = x + d$

$\therefore \; IJ = (x+d)(y+c)$

where $c, d \in R$. Hence proved

$$\text{Now } \frac{d}{dx}(IJ) = (y+c) + (x+d)\frac{dy}{dx}$$

$$\frac{d}{dx}(IJ) = I + J\frac{dy}{dx}.$$

Problem 27. Let $u(x)$ and $\upsilon(x)$ satisfy the differential equations $\frac{du}{dx} + p(x)u = f(x)$ and $\frac{d\upsilon}{dx} + p(x)\upsilon = g(x)$ respectively where $p(x)$, $f(x)$ and $g(x)$ are continuous functions. If $u(x_1) > \upsilon(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$, prove that any point (x,y), where $x > x_1$, does not satisfy the equations $y = u(x)$ and $y = \upsilon(x)$.

Solution Here, $\frac{du(x)}{dx} + p(x)u(x) = f(x)$

and $\frac{d\upsilon(x)}{dx} + p(x)\upsilon(x) = g(x)$

$$\therefore \frac{d\{u(x) - \upsilon(x)\}}{dx} + p(x)\{u(x) - \upsilon(x)\} = f(x) - g(x)$$

or $\frac{d}{dx}[\{u(x) - \upsilon(x)\} \cdot e^{\int p(x)dx}]$

$= \{f(x) - g(x)\}\, e^{\int p(x)dx}$ ($\because$ it is in linear form)

$\therefore \; [\{u(x) - \upsilon(x)\} \cdot e^{\int p(x)dx}]_{x_1}^{x}$

$$= \int_{x_1}^{x} \{f(x) - g(x)\}\, e^{\int p(x)dx} \quad ...(1)$$

Let $\left(e^{\int p(x)dx}\right)_x = \lambda$,

$\left(e^{\int p(x)dx}\right)_{x_1} = \mu$

Clearly λ, μ are positive.

$\therefore \; \{u(x) - \upsilon(x)\}\lambda - \{u(x_1) - \upsilon(x_1)\}\mu > 0$

$[\because f(x) > g(x)$ for $x > x_1$ and $e^{\int p(x)dx} > 0]$

$\therefore \; \{u(x) - \upsilon(x)\}\lambda > \{u(x_1) - \upsilon(x)\}\mu > 0$

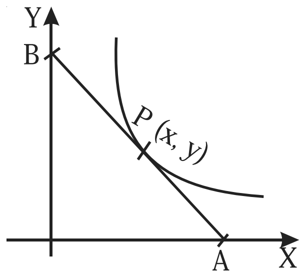
because $u(x_1) > \upsilon(x_1)$

$\therefore \; u(x) \neq \upsilon(x)$ when $x > x_1$.

Hence proved.

Problem 28. Find the curves for which the portion of the tangent included between the co-ordinate axes is bisected at the point of contact.

Solution Let P (x, y) be any point on the curve.



Equation of tangent at P (x, y) is

$$Y - y = m(X - x)$$

where $m = \frac{dy}{dx}$ is slope of the tangent at P (x, y).

Co-ordinates : $A\left(\frac{mx-y}{m}, 0\right)$ and $B(0, y-mx)$

P is the middle point of A & B

$\therefore \quad \frac{mx-y}{m} = 2x \Rightarrow mx - y = 2mx \Rightarrow mx = -y$

$\Rightarrow \quad \frac{dy}{dx} x = -y \Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0 \Rightarrow \ell n x + \ell n y = \ell n c$

$\therefore \quad xy = c$

Problem 29. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1, 1).

Solution Let P (x, y) be any point on the curve

Equation of tangent at 'P' is

$$Y - y = m(X - x)$$

$$mX - Y + y - mx = 0$$

Now, $\left(\frac{y - mx}{\sqrt{1+m^2}}\right) = x$

$$y^2 + m^2x^2 - 2mxy = x^2(1 + m^2)$$

$\frac{y^2 - x^2}{2xy} = \frac{dy}{dx}$ which is a homogeneous equation

Putting $y = vx \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = \frac{v^2-1}{2v} \Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$

$\Rightarrow \quad \int \frac{2v}{v^2+1} dv = -\int \frac{dx}{x}$

$\Rightarrow \quad \ell n (v^2 + 1) = -\ell n x + \ell n c$

$\Rightarrow \quad x\left(\frac{y^2}{x^2} + 1\right) = c$

The curve passes through (1, 1)

$\therefore \quad c = 2$

$$x^2 + y^2 - 2x = 0.$$

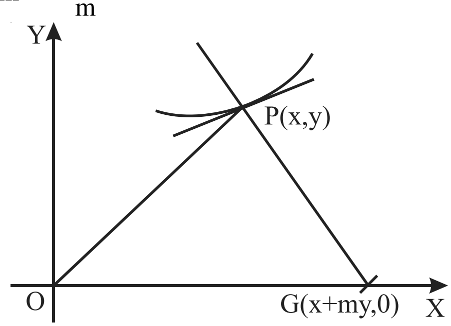
Problem 30. Find the nature of the curve for which the length of the normal at a point 'P' is equal to the radius vector of the point 'P'.

Solution Let the equation of the curve be y = f(x). P(x, y) be any point on the curve.

Slope of the tangent at P(x, y) is $\frac{dy}{dx} = m$

$\therefore$ Slope of the normal at P is

$$m' = -\frac{1}{m}$$



Equation of the normal at 'P' : $Y - y = -\frac{1}{m}(X - x)$

Co-ordinates of G are (x + my, 0)

Now, $\quad OP^2 = PG^2 \Rightarrow x^2 + y^2 = m^2y^2 + y^2$

$$m = \pm\frac{x}{y} \Rightarrow \frac{dy}{dx} = \pm\frac{x}{y}$$

Taking positive sign, $\frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow \quad y \,.\, dy = x \,.\, dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \lambda$

$\Rightarrow \quad x^2 - y^2 = -2\lambda$. It is a rectangular hyperbola.

Now, taking negative sign, $\frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow \quad y\, dy = -x\, dx$

$\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + \lambda'$

$\Rightarrow \quad x^2 + y^2 = 2\lambda'$

$\Rightarrow \quad x^2 + y^2 = c'$. It is a circle.

Problem 31. Find the equation of the curve passing through (2, 2) such that the slope of the tangent at any point to the curve is reciprocal of the ordinate of the point.

Solution Let any point on the curve be P(x, y)

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{y}$

$\Rightarrow y\,dy = dx$

$\Rightarrow \dfrac{y^2}{2} = x + c$

Since the curve passes through (2, 2)

$4 = 4 + c$

$\Rightarrow c = 0$

$\Rightarrow y^2 = 2x$ is the required curve.

Problem 32. A conic passing through the point A(1, 4) is such that the segment joining a point P(x, y) on the conic and the point of intersection of the normal at P with the abscissa axis is bisected by the y-axis. Find the equation of the conic and the equation of a circle touching the conic at A(1, 4) and passing through its focus.

Solution $Y - y = -\dfrac{1}{m}(X - x)$

$Y = 0$ gives $X = x + my$ and

$X = 0$ gives $Y = \dfrac{x + my}{m}$

Hence $\dfrac{x + x + my}{2} = 0$

$\Rightarrow 2x + y\dfrac{dy}{dx} = 0$

$\Rightarrow x^2 + \dfrac{y^2}{2} = c$

It passes through (1, 4)

$\Rightarrow$ The conic is $\dfrac{x^2}{9} + \dfrac{y^2}{18} = 1$ with $e = \dfrac{1}{\sqrt{2}}$ and focii (0, 3) & (0, –3).

Equation of the circles are ;

$(x-1)^2 + (y-4)^2 + \lambda(x + 2y - 9) = 0$

where $x + 2y - 9 = 0$ is the tangent to the ellipse at (1, 4).

Problem 33. Find the orthogonal trajectories of the curves $y^2 = 4a(x + a)$.

Solution We set up the differential equation of the given family of curves.

$2yy_1 = 4a$ ...(1)

$y^2 = 2yy_1\left(x + \dfrac{yy_1}{2}\right)$

$y^2 = 2xyy_1 + y^2 \cdot y_1^2$

$y^2\left(\dfrac{dy}{dx}\right)^2 + 2xy\left(\dfrac{dy}{dx}\right) - y^2 = 0$

The differential equation of the orthogonal trajectories is

$y\left(\dfrac{dx}{dy}\right)^2 - 2x\left(\dfrac{dx}{dy}\right) - y = 0$

$\dfrac{dx}{dy} = \dfrac{2x \pm \sqrt{4x^2 + 4y^2}}{2y} = \dfrac{x \pm \sqrt{x^2 + y^2}}{y}$

Let $x = vy$

$\Rightarrow \dfrac{dx}{dy} = v + y\dfrac{dv}{dx}$

$v + y\dfrac{dv}{dy} = v \pm \sqrt{1 + v^2}$

$\int \dfrac{dv}{\sqrt{1+v^2}} = \pm \int \dfrac{dy}{y}$

$ln\,(v + \sqrt{1+v^2}) = \pm\, ln\,(cy)$

$\dfrac{x + \sqrt{x^2 + y^2}}{y} = cy$

$x + \sqrt{x^2 + y^2} = cy^2.$

Problem 34. A tank contains 200 litres of brine in which 20 gms of salt dissolved. Brine containing $\dfrac{1}{4}$ gm of salt/litre runs into the tank at the rate of 2 litre/min. The mixture is kept stirring runs out at the same rate. Express the concentration of solution in tank in grams as a function of time t. What is the limiting value approached by the amount of salt as t → ∞. When was the amount of salt in solution equal to 20 gm ?

Solution $t = 0 \;:\; y = 20$

$\dfrac{dy}{dx} = \dfrac{1}{2} - \dfrac{y}{200}\cdot 2$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2} - \dfrac{y}{100}$

Hence $\dfrac{dy}{50 - y} = \dfrac{dt}{100}$

$\Rightarrow -\ln(50 - y) = \dfrac{t}{100} + c$

$\Rightarrow \ln(50-y) = -\frac{t}{100} + c$

We have $t = 0$; $y = 20$

$\Rightarrow c = \ln 30$

Hence $\ln \frac{50-y}{30} = \frac{t}{100}$

$$\frac{50-y}{30} = e^{-\frac{t}{100}}$$

$$y = 50 - 30\cdot e^{-\frac{t}{100}}.$$

The limiting value approached by the amount of salt y, when $t \to \infty$ is 50 gms.

Since $y > 20$ for all $t > 0$, it is not possible.

Problem 35. A and B are two separate reservoirs of water. The capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in the reservoir A is $1\frac{1}{2}$ times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water.

Solution Let v represent the volume of water in the reservoir. Given that $\frac{dv}{dt} \propto v$ for each reservoir

For reservoir A

$$\frac{dv_A}{dt} \propto v_A \Rightarrow \frac{dv_A}{dt} = -kv_A$$

(k_1 is a positive proportionality constant)

$$\Rightarrow \int_{v_A}^{v'} \frac{dv_A}{v_A} = -k_1\int_0^t dt$$

$$\Rightarrow \ln\left(\frac{v'_A}{v_A}\right) = -k_1 t \Rightarrow v'_A = v_A . e^{-k_1 t} \qquad ...(1)$$

Similarly for B, $v'_B = v_B . e^{-k_2 t}$ ...(2)

Dividing (1) by (2) we get

$$\frac{v'_A}{v'_B} = \frac{v_A}{v_B} . e^{(k_1-k_2)t} \qquad ...(3)$$

It is given that at $t = 0$, $v_A = 2v_B$ and at

$$t = 3/2,\ v'_A = \frac{3}{2}v'_B$$

Thus, $\frac{3}{2} = 2 . e^{-(k_1-k_2)t}$

$$\Rightarrow e^{-(k_1-k_2)} = \frac{3}{4}$$

Now let at $t = t_0$ both the reservoirs have same quantity of water. Then $v'_A = v'_B$

Hence from (3) $2e^{-(k_1-k_2)t_0} = 1$

$$\Rightarrow \frac{1}{2} = \left\{e^{-(k_1-k_2)}\right\}^{t_0} = \left(\frac{3}{4}\right)^{t_0}$$

$$\Rightarrow t_0 = \log_{3/4}\left(\frac{1}{2}\right).$$

Problem 36. The force of resistance encountered by water on a motor boat of mass 'm' going in still water with velocity 'v' is proportional to the velocity v. At $t=0$ when its velocity is v_0, the engine is shut off. Find an expression for the position of motor boat at time t and also the distance travelled by the boat before it comes to rest. Take the proportionality constant as $k > 0$.

Solution The resistance force opposing the motion $= m \times$ acceleration $= m\frac{dv}{dt}$.

Hence, the differential equation is,

$$m\frac{dv}{dt} = -kv$$

$$\Rightarrow \frac{dv}{v} = -\frac{k}{m}dt$$

Integrating $\ln v = -\frac{k}{m}\cdot t + c$

At $t = 0$, $v = v_0$, hence $c = \ln v_0$

$$\therefore \ln\frac{v}{v_0} = -\frac{k}{m}\cdot t$$

$$v = v_0\, e^{-\frac{k}{m}\cdot t} \qquad ...(1)$$

where v is the velocity at time t

Now, $\frac{ds}{dt} = v_0 e^{-\frac{k}{m}\cdot t}$

$\Rightarrow ds = v_0 e^{-\frac{k}{m}\cdot t}\, dt$

The body's position at time t is

$$s(t) = -\frac{v_0 m}{k}e^{-\frac{kt}{m}} + c$$

If $t = 0,\ s = 0$

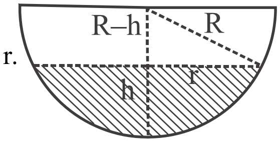
$$\Rightarrow \quad c = \frac{v_0 m}{k}$$

$$\therefore \quad s(t) = \frac{v_0 m}{k}\left[1 - e^{-\frac{kt}{m}}\right] \qquad \text{....(2)}$$

To find how far the boat go, we have to find $\lim\limits_{t \to \infty} s(t)$ which equals $\dfrac{mv_0}{k}$.

Problem 37. A hemispherical tank of radius 2 metres is initially full of water and has an outlet of 12 cm² cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t) = 0.6\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$ are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to empty the tank.

Solution Let the depth of water be h at time t ; the radius of water surface be r.



Then $\quad r^2 = R^2 - (R - h)^2$

$\Rightarrow \ r^2 = 2Rh - h^2$

Now, if in time dt the decrease in water level is dh then

$$-\pi r^2\, dh = 0.6\sqrt{2gh}\ .\ a\ .\ dt$$

(a is cross-sectional area of the outlet)

$$\Rightarrow \quad \frac{-\pi}{(0.6)a\sqrt{2g}}(2Rh - h^2)\frac{dh}{\sqrt{h}} = dt$$

$$\Rightarrow \quad \frac{\pi}{(0.6)a\sqrt{2g}}\int_R^0 (h^{3/2} - 2Rh^{1/2})d = \int_0^1 dt$$

$$\Rightarrow \quad \frac{\pi}{(0.6)a\sqrt{2g}}\left[\frac{2}{5}h^{5/2} - \frac{4}{3}Rh^{3/2}\right]_R^0 = t$$

$$\Rightarrow \quad t = \frac{\pi}{(0.6)a\sqrt{2g}}\left[0 - R^{5/2}\left(\frac{2}{5} - \frac{4}{3}\right)\right]$$

$$= \frac{7\pi \times 10^5}{135\sqrt{g}}$$

Things to Remember

1. The order of the highest order derivative involved in a differential equation is called the order of a differential equation.
2. The degree of differential equation is the degree of the highest order derivative present in the equation, after the differential equation has been made free from the radicals and fractions as far as the derivatives are concerned.
3. The differential equation corresponding to a family of curves is obtained by using the following steps:
   (A) Identify the number of essential arbitrary constants (say n) in the equation of the curve.
   (B) Differentiate the equation n times.
   (C) Eliminate the arbitrary constants from the equation of curve and n additional equations obtained in step (B) above.
4. A differential equation of the form $\dfrac{dy}{dx} = f(ax + by + c)$ is converted into an equation with variables separable by the substitution $v = ax + by + c$.
5. Sometimes transformation to the polar co–ordinates facilitates separation of variables. In this connection, it is convenient to remember the following differentials :
   If $x = r\cos\theta$; $y = r\sin\theta$ then,
   (i) $x\,dx + y\,dy = r\,dr$
   (ii) $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$
   (iii) $x\,dy - y\,dx = r^2 d\theta$
   If $x = r\sec\theta$ and $y = r\tan\theta$ then,
   (i) $x\,dx + y\,dy = r\,dr$
   (i) $x\,dx - y\,dy = r\,dr$
   (ii) $x\,dy - y\,dx = r^2 \sec\theta\, d\theta$.
6. In an equation of the form $yf_1(xy)\,dx + xf_2(xy)\,dy = 0$, the variables can be separated by the substitution $xy = v$.
7. A homogeneous differential equation $\dfrac{dy}{dx} = \varphi\left(\dfrac{y}{x}\right)$ is solved using the substitution $y = vx$.
8. If the differential equation is of the form $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$ it can be reduced to a homogeneous differential equation as follows :
   Put $x = X + h,\ y = Y + k$
   In order to make the equation homogeneous, choose h and k such that the following equations are satisfied :
   $$\left.\begin{aligned} ah + bk + c &= 0 \\ a'h + b'k + c' &= 0 \end{aligned}\right\}$$

Now, the equation becomes $\frac{dY}{dX}=\frac{aX+bY}{a'X+b'Y}$ which is a homogeneous differential equation and can be solved by putting Y = vX.

9. Sometimes an equation can be reduced to a homogeneous one by substituting z^α for y. This is the case when all the terms in the equation are of the same degree once the variable x is assigned degree 1, the variable y degree α, and the derivative $\frac{dy}{dx}$ degree α – 1.

10. The linear differential equation of the first order is of the form $\frac{dy}{dx}+P(x)y = Q(x)$, where P and Q are constants or functions of x alone. Its solution is given by $y e^{\int P(x)dx} = \int Q(x)\, e^{\int P(x)dx}\, dx + C$

11. Bernoulli's equation $\frac{dy}{dx}+Py=Qy^n$ where P and Q are constants or functions of x alone is solved using the substitution $v=\frac{1}{y^{n-1}}$.

12. The equation $f'(y)\frac{dy}{dx}+Pf(y)=Q$, where P and Q are functions of x or constants is solved by putting f(y) = v.

13. A non-exact differential equation can be made exact by multiplying it by some function of x and y. Such a function is called an integrating factor.

14. Suppose a differential equation can be solved for p = y' and is of the form
$[p-f_1(x,y)]\,[p-f_2(x,y)] \ldots [p-f_n(x,y)]=0$.
Equating each factor to zero we get equations of the first order and the first degree. The solution can be put in the form $\phi_1(x, y, c)\ \phi_2(x, y, c) \ldots \phi_n(x, y, c)=0$

15. The differential equation f(x, y, p) = 0 may have one or more of the following properties :
(A) It may be solvable for y.
(B) It may be solvable for x.
(C) It may be of the first degree in x and y.

16. The differential equation y = px + f(p) is known as Clairaut's equation. To solve it, we differentiate with respect to x. The general solution is y = Cx + f(C).

17. Steps to find the orthogonal trajectories :
(i) Let the family of given curves be f(x, y, a) = 0.
(ii) We form the differential equation of this family by eliminating a, say F(x, y, y') = 0.
(iii) We replace y' by $-\frac{1}{y'}$ in F(x, y, y') = 0, to obtain the differential equation of the orthogonal family of curves, i.e. $F\left(x, y, -\frac{1}{y'}\right)=0$.
(iv) We solve the above equation to obtain the orthogonal trajectories.

Objective Exercises

SINGLE CORRECT ANSWER TYPE

1. The differential equation of all circles in the first quadrant which touch the coordinates axes is
(A) $(x-y)^2(1+(y')^2)=(x+yy')^2$
(B) $(x-y)^2(1+(y')^2)=(x+y')^2$
(C) $(x-y)^2(1+y')=(x+yy')^2$
(D) None of these

2. The differential equation satisfying the curve $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, is
(A) $(x+yy_1)(xy_1-y)=(a^2-b^2)y_1$
(B) $(x+yy_1)(x-yy_1)=y_1$
(C) $(x-yy_1)(xy_1+y)=(a^2-b^2)y_1$
(D) None of these

3. The differential equation having $y=(\sin^{-1}x)^2 + A(\cos^{-1}x)^2 + B$, where A and B are arbitrary constants, is
(A) $(1-x^2)y_2-xy_1=2$ (B) $(1-x^2)y_2+yy_1=0$
(C) $(1-x)y_2+xy_1=0$ (D) None of these

4. The degree of the differential equation
$$\left(\frac{d^3y}{dx^3}\right)^{2/3}+4-3\frac{d^2y}{dx^2}+5\frac{dy}{dx}=0,\text{ is}$$
(A) 1 (B) 2
(C) 3 (D) none

5. The differential equation of the curve $\frac{x}{c-1}+\frac{y}{c+1}=1$ is given by
(A) $\left(\frac{dy}{dx}-1\right)\left(y+x\frac{dy}{dx}\right)=2\frac{dy}{dx}$
(B) $\left(\frac{dy}{dx}+1\right)\left(y-x\frac{dy}{dx}\right)=\frac{dy}{dx}$

(C) $\left(\frac{dy}{dx}+1\right)\left(y-x\frac{dy}{dx}\right)=2\frac{dy}{dx}$

(D) None of these

6. The solution of $x^2\,dy - y^2\,dx + xy^2(x-y)dy = 0$, is

(A) $\ln\left|\frac{x-y}{xy}\right| = \frac{y^2}{2}+c$ (B) $\ln\left|\frac{xy}{x-y}\right| = \frac{x^2}{2}+c$

(C) $\ln\left|\frac{x-y}{xy}\right| = \frac{x^2}{2}+c$ (D) $\ln\left|\frac{x-y}{xy}\right| = x+c$

7. The solution of the differential equation $ydx - xdy + xy^2\,dx = 0$, is

(A) $\frac{x}{y}+x^2=\lambda$ (B) $\frac{x}{y}+\frac{x^2}{2}=\lambda$

(C) $\frac{x}{2y^2}+\frac{x^2}{4}=\lambda$ (D) None of these

8. The solution of differential equation $xdy\,(y^2\,e^{xy} + e^{x/y}) = ydx(e^{x/y} - y^2e^{xy})$, is

(A) $xy = \ln(e^x+\lambda)$ (B) $x^2/y = \ln(e^{x/y}+\lambda)$

(C) $xy = \ln(e^{x/y}+\lambda)$ (D) $xy^2 = \ln(e^{x/y}+\lambda)$

9. The solution of the differential equation $(y+x\sqrt{xy}\,(x+y))dx + (y\sqrt{xy}\,(x+y)-x)dy = 0$, is

(A) $\frac{x^2+y^2}{2}+2\tan^{-1}\sqrt{\frac{x}{2y}}=c$

(B) $\frac{x^2+y^2}{2}+2\tan^{-1}\sqrt{\frac{x}{y}}=c$

(C) $\frac{x^2+y^2}{\sqrt{2}}+2\tan^{-1}\sqrt{\frac{x}{y}}$

(D) None of these

10. The solution of, $ydx - xdy + (1+x^2)\,dx + x^2\sin y\,dy = 0$, is given by

(A) $x+1-y^2+\cos y+c=0$

(B) $y+1-x^2+x\cos y+c=0$

(C) $\frac{x}{y}+\frac{1}{y}-y+\cos y+c=0$

(D) $\frac{y}{x}+\frac{1}{x}-x+\cos y+c=0$

11. The solution of $(1+x\sqrt{x^2+y^2})\,dx + (-1+\sqrt{x^2+y^2})ydy = 0$, is

(A) $2x-y^2+\frac{2}{3}(x^2+y^2)^{3/2}=c$

(B) $2x-y+\frac{2}{3}(x^2+y^2)^{3/2}=c$

(C) $2y-x^2+\frac{2}{3}(x^2+y^2)^{3/2}=c$

(D) None of these

12. The solution of, $\frac{xdy}{x^2+y^2}=\left(\frac{y}{x^2+y^2}-1\right)dx$, is given by

(A) $\tan^{-1}\left(\frac{x}{y}\right)+x=c$

(B) $\tan^{-1}\left(\frac{y}{x}\right)+x=c$

(C) $\tan^{-1}\left(\frac{y}{x}\right)+xy=c$

(D) $\tan^{-1}\left(\frac{y}{x}\right)+x^2=c$

13. If f(x) be a positive, continuous and differentiable on the interval (a, b). If $\lim_{x\to a^+} f(x) = 1$ and $\lim_{x\to b^-} f(x) = 3^{1/4}$. Also $f'(x) \geq f^3(x) + \frac{1}{f(x)}$, then

(A) $b-a \geq \pi/4$ (B) $b-a \leq \pi/4$

(C) $b-a \leq \pi/24$ (D) None of these

14. The solution of the differential equation $\frac{dy}{dx}=\frac{\sin y + x}{\sin 2y - x\cos y}$ is

(A) $\sin^2 y = x\sin y + \frac{x^2}{2}+c$

(B) $\sin^2 y = x\sin y - \frac{x^2}{2}+c$

(C) $\sin^2 y = x+\sin y + \frac{x^2}{2}+c$

(D) $\sin^2 y = x-\sin y + \frac{x^2}{2}+c$

15. The equation of curve passing through (1, 0) and satisfying $\left(y\frac{dy}{dx}+2x\right)^2 = (y^2+2x^2)\left(1+\left(\frac{dy}{dx}\right)^2\right)$, is given by

(A) $\sqrt{2}x^{\pm\frac{1}{\sqrt{2}}}=\frac{y+\sqrt{y^2+2x^2}}{x}$

(B) $\sqrt{2}x^{\pm\frac{1}{\sqrt{2}}}=\frac{y+\sqrt{y^2+\sqrt{2}x^2}}{x}$

(C) $\sqrt{2}y^{\pm\frac{1}{\sqrt{2}}} = \dfrac{y+\sqrt{x^2+2y^2}}{x}$

(D) None of these

16. The solution of $\dfrac{dy}{dx} = (x+y-1) + \dfrac{x+y}{\ln(x+y)}$, is given by

(A) $\{1+\ln(x+y)\} - \ln\{1+\ln(x+y)\} = x+c$
(B) $\{1-\ln(x+y)\} - \ln\{1-\ln(x+y)\} = x+c$
(C) $\{1+\ln(x+y)\}^2 - \ln\{1+\ln(x+y)\} = x+c$
(D) None of the these

17. The solution of $(2x^2+3y^2-7)\,x\,dx - (3x^2+2y^2-8)\,y\,dy = 0$, is given

(A) $(x^2+y^2-1) = (x^2+y^2-3)^5 c$
(B) $(x^2+y^2-1)^2 = (x^2+y^2-3)^5 c$
(C) $(x^2+y^2-3) = (x^2+y^2-1)^5 c$
(D) None of these

18. The solution of $x\sin\left(\dfrac{y}{x}\right)dy = \left\{y\sin\left(\dfrac{y}{x}\right) - x\right\}dx$, is given by

(A) $\ln x - \cos\left(\dfrac{y}{x}\right) = \ln c$

(B) $\ln x - \sin\left(\dfrac{y}{x}\right) = c$

(C) $\ln\left(\dfrac{x}{y}\right) - \cos\left(\dfrac{y}{x}\right) = \ln c$

(D) None of these

19. The solution of $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{(1+\ln x+\ln y)^2}$ is given by

(A) $xy(1+\ln(xy)) = c$
(B) $xy^2(1+\ln(xy)) = c$
(C) $xy(1+\ln(xy))^2 = c$
(D) $xy(1+(\ln xy))^2 = c$

20. The solution of differential equation $yy' = x\left(\dfrac{y^2}{x^2} + \dfrac{f(y^2/x^2)}{f'(y^2/x^2)}\right)$ is

(A) $f(y^2/x^2) = cx^2$ (B) $x^2 f(y^2/x^2) = c^2y^2$
(C) $x^2 f(y^2/x^2) = c$ (D) $f(y^2/x^2) = cy/x$

21. A function $y = f(x)$ satisfies

$(x+1)\,f'(x) - 2(x^2+x)\,f(x) = \dfrac{e^{x^2}}{(x+1)}$, " $x > -1$.

If $f(0) = 5$, then $f(x)$ is

(A) $\left(\dfrac{3x+5}{x+1}\right)e^{x^2}$ (B) $\left(\dfrac{6x+5}{x+1}\right)e^{x^2}$

(C) $\left(\dfrac{6x+5}{(x+1)^2}\right)e^{x^2}$ (D) $\left(\dfrac{5-6x}{x+1}\right)e^{x^2}$

22. The family of curves represented by

$\dfrac{dy}{dx} = \dfrac{x^2+x+1}{y^2+y+1}$ and $\dfrac{dy}{dx} = \dfrac{y^2+y+1}{x^2+x+1} = 0$

(A) Touch each other
(B) Are orthogonal
(C) Are one and the same
(D) None of these

23. A normal at P(x,y) on a curve meets the x-axis at Q,S and N is the foot of the ordinate at P. If $NQ = \dfrac{x(1+y^2)}{1+x^2}$, then the equation of curve given that it passes through the point (3, 1) is

(A) $x^2 - y^2 = 8$ (B) $x^2 + 2y^2 = 11$
(C) $x^2 - 5y^2 = 4$ (D) None of these

24. An object falling from rest in air is subject not only to the gravitational force but also to air resistance. Assume that the air resistance is is proportional to the velocity with constant of proportionality as $k > 0$, and acts in a direction opposite to motion ($g = 9.8$ m/s^2). Then velocity cannot exceed.

(A) 9.8 k m/s
(B) 98/k m/s
(C) $\dfrac{k}{9.8}$ m/s
(D) None of these

25. A curve passing through (2, 3) and satisfying the differential equation $\int_0^x t\,y(t)\,dt = x^2y(x)$, $(x > 0)$ is

(A) $x^2 + y^2 = 13$ (B) $y^2 = \dfrac{9}{2}x$

(C) $\dfrac{x^2}{8} + \dfrac{y^2}{18} = 1$ (D) $xy = 6$

26. The solution of the differential equation $\dfrac{d^2y}{dx^2} = \sin 3x + e^x + x^2$ when $y_1(0) = 1$ and $y(0) = 0$ is

(A) $\dfrac{-\sin 3x}{9} + e^x + \dfrac{x^4}{12} + \dfrac{1}{3}x - 1$

(B) $\dfrac{-\sin 3x}{9} + e^x + \dfrac{x^4}{12} + \dfrac{1}{3}x$

(C) $\frac{-\cos 3x}{3}+e^x+\frac{x^4}{12}+\frac{1}{3}x+1$

(D) None of these

27. Solution of the differential equation

$\left(e^{x^2}+e^{y^2}\right)y\frac{dy}{dx}+e^{x^2}(xy^2-x)=0$ is

(A) $e^{x^2}(y^2-1)+e^{y^2}=C$

(B) $e^{y^2}(x^2-1)+e^{x^2}=C$

(C) $e^{y^2}(y^2-1)+e^{x^2}=C$

(D) $e^{x^2}(y-1)+e^{y^2}=C$

28. If $y_1(x)$ and $y_2(x)$ are two solutions of $\frac{dy}{dx}+f(x)y=r(x)$, then $y_1(x)+y_2(x)$ is solution of

(A) $\frac{dy}{dx}+f(x)y=0$

(B) $\frac{dy}{dx}+2f(x)y=r(x)$

(C) $\frac{dy}{dx}+f(x)y=2r(x)$

(D) $\frac{dy}{dx}+2f(x)y=2r(x)$

29. The solution of the differential equation,

$x^2\frac{dy}{dx}\cos\frac{1}{x}-y\sin\frac{1}{x}=-1$, where $y\to -1$ as $x\to\infty$ is

(A) $y=\sin\frac{1}{x}-\cos\frac{1}{x}$ (B) $y=\frac{x+1}{x\sin\frac{1}{x}}$

(C) $y=\cos\frac{1}{x}+\sin\frac{1}{x}$ (D) $y=\frac{x+1}{x\cos\frac{1}{x}}$

30. The general solution of the differential equation $(2x^2y-2y^4)dx+(2x^3+3xy^3)\,dy=0$, is

(A) $y^2.\ \ell n(x^2y^2)-y^3=cx^2$

(B) $x^2.\ \ell n(x^2y^2)-y^3=cx^2$

(C) $y^2.\ \ell n(x^2y^2)+y^3=cx^2$

(D) $x^2.\ \ell n(x^2y^2)+y^3=cx^2$

31. The general solution of the differential equation $2xy^3dx+y^7dx+(xy^6-3x^2y^2)dy=0$ is

(A) $x^2+xy^4=c$ (B) $\frac{x^2}{y^3}+xy=c$

(C) $\frac{x^2}{y^3}-\frac{x}{y}=c$ (D) None of these

32. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality $k>0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k=\frac{1}{15}$ then the time to drain the tank if the water is 4 meter deep to start with is

(A) 1

(B) 2

(C) A rational number

(D) An irrational number

33. The solution of $3x(1-x^2)y^2\,dy/dx+(2x^2-1)y^3=ax^3$ is

(A) $y^3=ax+c\sqrt{1-x^2}$

(B) $y^3=ax+cx\sqrt{1-x^2}$

(C) $y^2=ax+c\sqrt{1-x^2}$

(D) None of these

34. The solution of $y=x\frac{dy}{dx}+\frac{dy}{dx}-\left(\frac{dy}{dx}\right)^2$, is

(A) $y=(x-1)^2$ (B) $4y=(x+1)^2$

(C) $(y-1)^2=4x$ (D) none of these

35. A curve $y=f(x)$ passes through the origin. Through any point (x, y) on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in the ratio m : n, then the equation of curve is

(A) $y=cx^{m/n}$ (B) $my^2=cx^{m/n}$

(C) $y^3=cx^{m/n}$ (D) None of these

36. The equation of the curve passing through the points (3a, a) $(a>0)$ in the form $x=f(y)$ which satisfy the differential equation

$\frac{a^2}{xy}.\frac{dx}{dy}=\frac{x}{y}+\frac{y}{x}-2$, is

(A) $x=y+a\left(\frac{1+e^{y-k}}{1-2e^{y-k}}\right)$

(B) $x = y + a\left(\frac{1+e^{y-k}}{1-e^{y-k}}\right)$

(C) $y = x + a\left(\frac{1+e^{y-k}}{1-e^{y-k}}\right)$

(D) None of these

37. The family of curves, the subtangent at any point of which is the arithmetic mean of the coordinates of the point of tangency, is given by

(A) $(x-y)^2 = cy$ (B) $(y+x)^2 = cx$
(C) $(x-y)^2 = cxy$ (D) None of these

38. The solution of the equation $\frac{dy}{dx} = \frac{x(2\ln x + 1)}{\sin y + y\cos y}$ is

(A) $y\sin y = x^2 \ln x + \frac{x^2}{2} + c$

(B) $y\cos y = x^2(\ln x + 1) + c$

(C) $y\cos y = x^2\ln x + \frac{x^2}{2} + c$

(D) $y\sin y = x^2\ln x + c$

39. If f(x) and g(x) are two solutions of the differential equation $\left(a\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = e^x\right)$, then f(x) – g(x) is the solution of

(A) $a^2\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^x$

(B) $a^2\frac{d^2y}{dx^2} + y = e^x$

(C) $a\frac{d^2y}{dx^2} + y = e^x$

(D) $a\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0$

40. The integrating factor of $x(1-x)^2dy + (2x^2y - y - ax^3)dx = 0$ is $e^{\int P dx}$, then P is

(A) $\frac{2x^2 - a^3}{x(1-x^2)}$ (B) $2x^3 - 1$

(C) $\frac{2x^2-1}{ax^3}$ (D) $\frac{2x^2-1}{x(1-x^2)}$

41. The orthogonal trajectory of the curve $a^{n-1}y = x^n$, is

(A) $nx^2 + y^2 = 0$ (B) $ny^2 + x^2 = \text{constant}$
(C) $y^2 - x^2 = \text{constant}$ (D) None of these

42. The solution of $y(xy + e^{xy})dx + x(xy - e^{xy})dy = 0$ is

(A) $\frac{1}{2}(\ln|xy| + xy) = \ln|y| + c$

(B) $\frac{1}{2}(\ln|xy| - xy) = \ln|y| + c$

(C) $(\ln|xy| + xy) = \ln|y| + c$

(D) None of these

43. If the solution of the differential equation $\frac{dy}{dx} = \frac{1-3y-3x}{1+x+y}$, is $lx + y + m\ln|1-x-y| = C$ (C is arbitrary constant), then l and m respectively are

(A) 2, 3 (B) 3, 2
(C) 1, 1 (D) 2, 2

44. The solution of

$\sin\left(\frac{x}{y}\right)(y\,dx - x\,dy) = xy^3(x\,dy + y\,dx)$, is

(A) $\frac{xy}{2} + \sin\left(\frac{x}{y}\right) + C = 0$

(B) $\frac{(xy)^2}{2} + \sin\left(\frac{x}{y}\right) + C = 0$

(C) $\frac{(xy)^2}{2} + \cos\left(\frac{x}{y}\right) + C = 0$

(D) $(xy) + \tan\left(\frac{x}{y}\right) + C = 0$

45. The solution of differential equation

$\frac{x + y\,.\frac{dy}{dx}}{y - x\,.\frac{dy}{dx}} = x^2 + 2y^2 + \frac{y^2}{x^2}$, is

(A) $\frac{y}{x} - \frac{1}{2(x^2+y^2)} = c$ (B) $\frac{y}{x} + \frac{1}{(x^2+y^2)} = c$

(C) $\frac{y}{x} + \left(\frac{1}{x^2-y^2}\right) = c$ (D) None of these

46. The curve satisfying the equation

$\frac{dy}{dx} = \frac{y(x+y^3)}{x(y^3-x)}$ and passing through the point (4, –2) is

(A) $y^2 = -2x$ (B) $y = -2x$
(C) $y^3 = -2x$ (D) None of these

47. The equation of the curve in which the perpendicular from the origin upon the tangent is equal to the abscissa of the point of contact is
(A) $x^2+y^2=cx$ (B) $x^2+y^2=cy$
(C) $x^2+y^2=c$ (D) None of these

48. The equation $f(t)=\frac{d}{dt}\int_0^t \frac{dx}{1-\cos t\cos x}$ satisfies the differential equation
(A) $\frac{df}{dt}+2f(t)\cot t=0$ (B) $\frac{df}{dt}+2f(t)=0$
(C) $\frac{df}{dt}-2f(t)\cot t=0$ (D) $\frac{df}{dt}-2f(t)=0$

49. The solution of
$(y(1+x^{-1})+\sin y)\,dx+(x+\ln x+x\cos y)\,dy=0$ is
(A) $(1+y^{-1}\sin y)+x^{-1}\ln x=C$
(B) $(y+\sin y)+xy\ln x=C$
(C) $xy+y\ln x+x\sin y=C$
(D) None of these

50. The solution of
$(2x-10y^3)\frac{dy}{dx}+y=0$ is xy^2+ly^5+C. Then l is
(A) 3 (B) 4
(C) 5 (D) None of these

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

51. In which of the following differential equation degree is not defined
(A) $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2=x\log\frac{d^2y}{dx^2}$
(B) $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^2=x\sin\left(\frac{d^2y}{dx^2}\right)$
(C) $x=\sin\left(\frac{dy}{dx}-2y\right), |x|<1$
(D) $x-2y=\log\left(\frac{dy}{dx}\right)$

52. If m and n are order and degree of the equation
$$\left(\frac{d^2y}{dx^2}\right)^5+4.\frac{\left(\frac{d^2y}{dx^2}\right)^3}{\frac{d^3y}{dx^3}}+\frac{d^3y}{dx^3}=x^2-1, \text{ then}$$
(A) $m=3$ (B) $n=5$
(C) $n=2$ (D) $n=1$

53. The general solution of the differential equation,
$x\left(\frac{dy}{dx}\right)=y.\ln\left(\frac{y}{x}\right)$ is :
(A) $y=xe^{1-cx}$ (B) $y=xe^{1+cx}$
(C) $y=ex.e^{cx}$ (D) $y=xe^{cx}$
where c is an arbitrary constant.

54. Which of the following equation(s) is/are linear.
(A) $\frac{dy}{dx}+\frac{y}{x}=\ln x$ (B) $y\left(\frac{dy}{dx}\right)+4x=0$
(C) $dx+dy=0$ (D) $\frac{d^2y}{dx^2}=\cos x$

55. Identify the statement(s) which is/are true.
(A) $f(x, y)=e^{y/x}+\tan\frac{y}{x}$ is homogeneous of degree zero
(B) $x.\ln\frac{y}{x}\,dx+\frac{y^2}{x}\sin^{-1}\frac{y}{x}\,dy=0$ is homogeneous differential equation
(C) $f(x, y)=x^2+\sin x.\cos y$ is not homogeneous
(D) $(x^2+y^2)\,dx-(xy^2-y^3)\,dy=0$ is a homogeneous differential equation.

56. Taking y as the dependent and x as the independent variable, which of the following ordinary differential equations are not linear?
(A) $y'y''+y^2=x^2$ (B) $x^2y''-xy'+6y=\ln x$
(C) $[1+(y')^2]^{1/2}=5y$ (D) $y'=\sqrt{x}+\sqrt{y}$

57. The differential equation of the curve for which the initial ordinate of any tangent is equal to the corresponding subnormal
(A) is linear
(B) is homogeneous of first degree
(C) has separable variables
(D) is second order

58. The graph of the function $y=f(x)$ passing through the point (0, 1) and satisfying the differential equation $\frac{dy}{dx}+y\cos x=\cos x$ is such that
(A) It is a constant function
(B) It is periodic
(C) It is neither an even nor an odd function
(D) It is continuous and differentiable for all x

59. If f(x), g(x) be twice differential functions on [0, 2] satisfying $f''(x)=g''(x)$, $f'(1)=2g'(1)=4$ and $f(2)=3g(2)=9$, then
(A) $f(4)-g(4)=10$
(B) $|f(x)-g(x)|<2\Rightarrow-2<x<0$
(C) $f(2)=g(2)\Rightarrow x=-1$
(D) $f(x)-g(x)=2x$ has real root

60. The solution of $\left(\frac{dy}{dx}\right)(x^2y^3 + xy) = 1$ is

(A) $\frac{1}{x} = 2 - y^2 + Ce^{-\frac{y^2}{2}}$

(B) The solution of an equation which is reducible to linear equation

(C) $\frac{2}{x} = 1 - y^2 + \frac{e^{-y}}{2}$

(D) $\frac{1-2x}{x} = y^2 + Ce^{-\frac{y^2}{2}}$

61. The solution of $\frac{dy}{dx} = \sqrt{y-x}$ is given by

(A) $x + C = 2\sqrt{y-x} + 2\log(\sqrt{y-x} - 1)$

(B) $x^2 + C = \sqrt{y-x} + \log(\sqrt{y-x} - 1)$

(C) $x + C = (y-x)^2 + \log(y-x-1)$

(D) $\sqrt{y-x} - 1 = C'e^{x/2 - \sqrt{y-x}}$

62. The function f(x) is defined for $x \geq 0$ and has its inverse g (x) which is differentiable. If f(x) satisfies

$\int_0^{g(x)} f(t)\,dt = x^2$ and $g(0) = 0$ then

(A) f(x) is an odd linear polynomial

(B) f(x) is some quadratic polynomial

(C) $f(2) = 1$

(D) $g(2) = 4$

63. Family of curves whose tangent at a point with its intersection with the curve $xy = c^2$ form an angle of $\frac{\pi}{4}$ is

(A) $y^2 - 2xy - x^2 = k$ (B) $y^2 + 2xy - x^2 = k$

(C) $y = x - 2c\tan^{-1}\left(\frac{x}{c}\right) + k$

(D) $y = c\ln\left|\frac{c+x}{c-x}\right| - x + k$

where k is an arbitrary constant .

64. The function f(x) satisfying the equation, $f^2(x) + 4f'(x) \cdot f(x) + [f'(x)]^2 = 0$

(A) $f(x) = c \cdot e^{(2-\sqrt{3})x}$ (B) $f(x) = c \cdot e^{(2+\sqrt{3})x}$

(C) $f(x) = c \cdot e^{(\sqrt{3}-2)x}$ (D) $f(x) = c \cdot e^{-(2+\sqrt{3})x}$

where c is an arbitrary constant .

65. The equation of the curve passing through (3 , 4) and satisfying the differential equation,

$y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$ can be

(A) $x - y + 1 = 0$ (B) $x^2 + y^2 = 25$

(C) $x^2 + y^2 - 5x - 10 = 0$ (D) $x + y - 7 = 0$

66. The solution of $\frac{dy}{dx} + x = xe^{(n-1)y}$ is

(A) $\frac{1}{n-1}\log\left(\frac{e^{(n-1)y} - 1}{e^{(n-1)y}}\right) = \frac{x^2}{2} + C$

(B) $e^{(n-1)y} = Ce^{(n-1)y + (n-1)x^2/2} + 1$

(C) $\log\left(\frac{e^{(n-1)y} - 1}{(n-1)e^{(n-1)y}}\right) = x^2 + C$

(D) $e^{(n-1)y} = Ce^{(n-1)x^2/2} + 1$

67. The solution of $\left(\frac{dy}{dx}\right)^2 + 2y\cot x\frac{dy}{dx} = y^2$ is

(A) $y - \frac{c}{1+\cos x} = 0$ (B) $y = \frac{c}{1-\cos x}$

(C) $x = 2\sin^{-1}\sqrt{\frac{c}{2y}}$ (D) $x = 2\cos^{-1}\sqrt{\frac{c}{2y}}$

68. The tangent at any point P on a curve $f(x, y) = 0$ cuts the y-axis at T. If the distance of the point T from P equals the distance of T from the origin then the curve with this property represents a family of circles. Which of the following is/are correct?

(A) Any arbitrary line $y = mx$ cuts every member of this family at the points where the slopes of these members are equal.

(B) $f(x, y) = 0$ is orthogonal to the family of circles $x^2 + y^2 - ky = 0 \ \forall \ k \in R$

(C) If $f(x, y) = 0$ passes through (2, 2) then the intercept made by its director circle on the y-axis is equal to 8.

(D) If $f(x, y) = 0$ passes through (–1, 1) then image of its centre in the line $y = x$, is (1, 0)

69. The graph of the function $y = f(x)$ passing through the point (0 , 1) and satisfying the differential equation $\frac{dy}{dx} + y\cos x = \cos x$ is such that

(A) It is a constant function

(B) It is periodic

(C) It is neither an even nor an odd function

(D) It is continuous & differentiable for all x.

70. A function $y = f(x)$ satisfying the differential equation $\frac{dy}{dx}\cdot\sin x - y\cos x + \frac{\sin^2 x}{x^2} = 0$ is such that, $y \to 0$ as $x \to \infty$ then the statement which is correct is

(A) $\lim_{x\to 0} f(x) = 1$

(B) $\int_0^{\pi/2} f(x)\,dx$ is less than $\frac{\pi}{2}$

(C) $\int_0^{\pi/2} f(x)\,dx$ is greater than unity

(D) f(x) is an odd function

Comprehension - 1

Let f(x) be non-positive continuous function and

$F(x) = \int_0^x f(t)\,dt\ \forall x \ge 0$ and $f(x) \ge cF(x)$ where $c > 0$ and

let $g : [0, \infty) \to R$ be a function such that $\frac{dg(x)}{dx} < g(x)$ $\forall x > 0$ and $g(0) = 0$.

71. The total number of roots of the equation $f(x) = g(x)$ is

(A) ∞ (B) 1
(C) 2 (D) 0

72. The number of solutions of the equation $|x^2 + x - 6| = f(x) + g(x)$ is

(A) 2 (B) 1
(C) 0 (D) 3

73. The solution set of inequation $g(x)(\cos^{-1}x - \sin^{-1}x) \le 0$

(A) $\left[-1, \frac{1}{\sqrt{2}}\right]$ (B) $\left[\frac{1}{\sqrt{2}}, 1\right]$

(C) $\left[0, \frac{1}{\sqrt{2}}\right]$ (D) $\left(0, \frac{1}{\sqrt{2}}\right]$

Comprehension - 2

Consider a differential equation

$f'(x) = 2\sqrt{(f(x))^2 + x}$, $f(1) = 1$.

74. The order and degree of the differential equation respectively are

(A) 2, 1 (B) 1, 2
(C) 1, 1 (D) None of these

75. The solution of the differential equation is

(A) $f^2(x) = \frac{25}{36}4^x - \frac{4x}{3} - \frac{4}{9}$

(B) $f^2(x) = \frac{25}{36}4^x - \frac{16}{9}$

(C) $f^2(x) = \frac{25}{27}4^x - \frac{4x}{3} + \frac{4}{9}$

(D) None of these

76. The value of $\sum_{n=1}^{9} f^2(n)$ is

(A) $\frac{25(4^9 - 9)}{36} - 54$ (B) $\frac{25(4^9 - 9)}{27} - 64$

(C) $\frac{25(4^9 - 9)}{27} + 64$ (D) None of these

Comprehension - 3

Consider a continuously differentiable function f(x) satisfying the equation

$f^2(x) = \int_0^x (f^2(t) - f^4(t) + (f'(t))^2)\,dt + 100$ where $f(x) \le f'(x)$ and $f(0) = 10$.

77. The function satisfies the differential equation

(A) $f - f' = f^2$ (B) $f + f' = f^2$
(C) $f - f' = 2f^2$ (D) None of these

78. The function f is

(A) $\frac{10}{10 - 9e^{-x}}$ (B) $\frac{-10}{10 - 9e^{-x}}$

(C) $\frac{10}{10 - 11e^{-x}}$ (D) $\frac{-10}{10 - 11e^{-x}}$

79. The range of the function f is

(A) $(-\infty, 0) \cup (1, \infty)$ (B) $(-\infty, -1) \cup (0, \infty)$
(C) $(-\infty, -1) \cup (1, \infty)$ (D) None of these

Comprehension - 4

Suppose f and g are differentiable function such that $x\,g(f(x))\,f'(g(x))\,g'(x) = f(g(x))\,g'(f(x))\,f'(x)$ for all real x.

Moreover $f(x) \ge 0$, $g(x) > 0$ and $g(f(0)) = 1$. Also

$\int_0^a f(g(x))\,dx = 1 - \frac{e^{-2a}}{2}$ for all real a.

80. The function f(g(x)) is

(A) e^{2x} (B) $x\ln x$
(C) e^{-2x} (D) x^2

81. $x \dfrac{d}{dx}(\ell n\, f(g(x)))$ is equal to

(A) $\dfrac{g'(f(x))}{g(f(x))}$ (B) $\dfrac{g'(f(x))}{f(g(x))}$

(C) $\dfrac{g'(f(x))\,f'(x)}{g(f(x))}$ (D) None of these

82. g(f(4)) is equal to

(A) $\dfrac{1}{2}e^{-16}$ (B) e^{-16}

(C) e^{-4} (D) None

Comprehension - 5

The family of curves $y = f(x)$ has a property that any chord joining $A(x_1, f(x_1))$ and $B(x_2, f(x_2))$ intersects the y-axis at the point $(0, -x_1x_2)$. Determine the family of curves and answer the following questions

83. The family of curves $y = f(x)$ represents
(A) Circles centered at origin
(B) Circles passing through origin
(C) Parabolas with focus at origin
(D) Parabolas passing through origin

84. If $f(1) = -1$ then the area bounded by $y = f(x)$ and x-axis is given by

(A) $\dfrac{2}{3}$ (B) $\dfrac{4}{3}$

(C) $\dfrac{8}{3}$ (D) $\dfrac{16}{3}$

85. The differential equation of the family of curves $y = f(x)$ is of
(A) Order 1 and degree 2
(B) Order 2 and degree 1
(C) Order 1 and degree 1
(D) Order 2 and degree 2

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

86. Assertion (A) : The solution of the differential equation $\dfrac{dy}{dx} + \dfrac{y}{x} = x^{-2}$ is $x = e^{xy}$ if $y(1) = 0$.
Reason (R) : There is no arbitrary constant in the particular solution of a first order differential equation.

87. Assertion (A) : Differential equation whose solution is $y = cx + c - c^3$ is $y = x\dfrac{dy}{dx} + \dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^3$
Reason (R) : Integrating factor of $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3 - 3$ is x.

88. Assertion (A) : The differential equation of all parabolas having latus rectum 4a, whose axes are parallel to the x-axis, opens in positive x direction and vertex lying on y-axis is of order 2 and degree 2.
Reason (R) : The general equation of such parabolas is $(y - k)^2 = 4ax$ which has two arbitrary constants. Hence the order must be two.

89. Assertion (A) : All the solutions of the differential equation $\phi_1(x)\,\psi_1(y)\,dx = \phi_2(x)\,\psi_2(y)\,dy$ are given by $\displaystyle\int \frac{\varphi_1(x)}{\varphi_2(x)}\,dx - \int \frac{\psi_1(y)}{\psi_2(y)}\,dy = c.$
Reason (R) : Dividing by $\psi_1(y)\,\phi_2(x)$ may lead to loss of particular solutions making the product $\psi_1(y)\,\phi_2(x)$ zero

90. Assertion (A) : The tangents to the curves represented by the homogeneous differential equation $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$ at the corresponding points are parallel.
Note that corresponding points are points on the curve which lie on one ray issuing from the origin.
Reason (R) : Consider two corresponding points (x_1, y_1) and (x_2, y_2) on two curves c_1 & c_2. Since $\dfrac{y_1}{x_1} = \dfrac{y_2}{x_2}$ we have $\left(\dfrac{dy}{dx}\right)_{c_1} = \left(\dfrac{dy}{dx}\right)_{c_2}$

91. Consider a differentiable function $y = f(x)$ which satisfies $f(x) = \displaystyle\int_0^x \left(f(t)\sin t - \sin(t - x)\right)dt$
Assertion (A) : The differential equation corresponding to $y = f(x)$ is a first order linear differential equation.
Reason (R) : The differential equation corresponding to $y = f(x)$ is of degree one.

92. Assertion (A) : If f is twice differentiable such that $f''(x) = -f(x)$, $f'(x) = g(x)$ and $h'(x) = (f(x))^2 + (g(x))^2$ and $h(0) = 2$; $h(1) = 4$, then $y = h(x)$ is a line with slope '2'.

Reason (R) : Solution for differential equation $\frac{dy}{dx} = c$ is a family of lines.

93. Assertion (A) : If f(x) is twice differentiable function such that $f''(x) = g(x)$ and $h'(x) = (f(x))^2 + (g(x))^2$ and $h(0) = 2$; $h(1) = 4$, then $y = h(x)$ is a line with slope '4'.

Reason (R) : Solution for differential equation $\frac{dy}{dx} = c$ is a family of lines.

94. Assertion (A) : The function $f : R \to R$ satisfies $f(x^2)\, f''(x) = f'(x)\, f'(x^2)$ for all real x. If $f(1) = 1$ and $f'''(1) = 8$ then $f'(1) + f''(1) = 6$.

Reason (R) : From the given equation, we get $a^2 = b$ and $ab = 8$, when $f'(1) = a$ and $f''(1) = b$.

95. Assertion (A) : Let f be a differentiable function such that $f(x) + f'(x) \le 1$ for all x, and $f(0) = 0$. The largest possible value of f(1) is $1 - \frac{1}{e}$.

Reason (R) : Consider $g(x) = e^x f(x)$. Then $g'(x) \le e^x$. Integrating we get $g(1) - g(0) \le e - 1$.

MATCH THE COLUMNS FOR JEE ADVANCED

96.

	Column - I	Column - II
(A)	The order of the differential equation of all conics whose centre lie at the origin is equal to	(P) 1
(B)	The order of the differential equation of all circles of radius a is equal to	(Q) 2
(C)	The order of the differential equation of all parabolas whose axis of symmetry is parallel to x-axis is equal to	(R) 3
(D)	The order of the differential equation of all conics whose axes coincide with the axes of coordinates is equal to	(S) 4
		(T) Can't be determined

97.

	Column–I	Column–II
(A)	The differential equation of all parabolas having heir axis of t symmetry coinciding with the axis of x has its degree	(P) 1
(B)	The differential equation of all parabolas each of which has a latus rectum '4a' & whose axes are parallel to x-axis is of degree	(Q) 2
(C)	Degree of the differential equation $y = a(1 - e^{-x/a})$, a being the parameter is	(R) 3
(D)	The polynomial f(x) satisfies the condition $f(x + 1) = x^2 + 4x$. The area enclosed by $y = f(x - 1)$ and the curve $x^2 + y = 0$, is $\frac{16\sqrt{2}}{k}$ where k is	(S) Does not exist

98.

	Column-I	Column-II
(A)	If the function $y = e^{4x} + 2e^{-x}$ is a solution of the differential equation $\dfrac{\frac{d^3y}{dx^3} - 13\frac{dy}{dx}}{y} = K$ then the value of K/3 is	(P) 3
(B)	Number of straight lines which satisfy the differential equation $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$ is	(Q) 4

(C) If the substitution, $y = u^m$ will transforms the differential equation, $2x^4y\frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation then the value of 2m is — (R) 2

(D) If the solution of differential equation $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 12y$ is $y = Ax^m + Bx^{-n}$ then $|m + n|$ is — (S) 1

99.

	Column-I	Column-II
(A)	Number of integers which do not lie in the range of the function $f(x) = \sec\left(2\sin^{-1}\frac{1}{x}\right)$	(P) 0
(B)	Let $f : (0, \infty)$ onto $(0, \infty)$ be a derivable function for which there exists its primitive F such that $2\left(F(x) - f(x)\right) = f^2(x)$ for any real positive x. Then $\lim\limits_{x\to\infty}\frac{f(x)}{x}$ equals	(Q) 1
(C)	How many of the following derivatives are correct (on their domains)? **I** $\frac{d}{dx}\ln\lvert\sec x\rvert = \tan x$ **II** $\frac{d}{dx}\ln(x + e^x) = 1 + \frac{1}{x}$ **III** $\frac{d}{dx}x^{\ln x} = (\ln x)x^{\ln(x)-1}$	(R) 2
(D)	A differentiable function satisfies $f'(x) = f(x) + 2e^x$ with initial conditions $f(0) = 0$. The area enclosed f(x) and the x-axis is	(S) 3

100.

	Column-I	Column-II
(A)	Let f(x) is a derivable function satisfying $f(x) = \int_0^x e^t \sin(x - t)dt$ and $g(x) = f''(x) - f(x)$ then the possible integers in the range of g(x) is	(P) –1
(B)	If the substitution $x = \tan^{-1}(t)$ transforms the differential equation $\frac{d^2y}{dx^2} + xy\frac{dy}{dx} + \sec^2 x = 0$ into a differential equation $(1 + t^2)\frac{d^2y}{dt^2}$ $(2t + y\tan^{-1}(t))$ $\frac{dy}{dt} = k$ then k is equal to	(Q) 0

(C) If $a^2 + b^2 = 1$ then $(a^3b - ab^3)$ can be equal to (R) 1

(D) If the system of equations $\left.\begin{array}{l} x - \lambda y - z = 0 \\ \lambda x - y - z = 0 \\ x + y - z = 0 \end{array}\right\}$ has a unique solution, then the value of λ can be (S) 2

Review Exercises for JEE Advanced

1. Solve the following differential equations :
   (i) $x\dfrac{dy}{dx} + y\ln y = xye^x$
   (ii) $\dfrac{dy}{dx} = e^{x-y}(e^x - e^y)$
   (iii) $2(x - y\sin 2x)\,dx + (3y^2 + \cos 2x)\,dy = 0$
2. Solve the following differential equations :
   (i) $x\,dx + y\,dy = a^2\dfrac{x\,dy - y\,dx}{x^2 + y^2}$
   (ii) $\left(x\cos\dfrac{y}{x} + y\sin\dfrac{y}{x}\right)ydx + \left(x\cos\dfrac{y}{x} - y\sin\dfrac{y}{x}\right)xdy = 0$
   (iii) $(1 + \tan y)\,[xdx - e^{-y}.\,dy] + x^2\,dy = 0.$
3. Solve the following differential equations :
   (i) $(x^2 - 1)\sin x\dfrac{dy}{dx} + [\,2x\sin x + (x^2 - 1)\cos x\,]\,y - (x^2 - 1)\cos x = 0.$
   (ii) $y' + \left(\dfrac{\sin x - \cos x}{e^{-x} - \cos x}\right)y = \dfrac{1}{e^{-x} - \cos x}.$
   (iii) $(x\sqrt{x^2 - y^2} - y)\,dx + (y\sqrt{x^2 - y^2} - x)\,dy = 0$
4. Solve the following differential equations :
   (i) $(x^2 - 1)\dfrac{dy}{dx}\sin y - 2x\cos y = 2x - 2x^3$
   (ii) $(x^2 + y^2 + 1)\,dy + xy\,dx = 0.$
5. Solve the following differential equations :
   (i) $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = \dfrac{x\sin^2(x^2 + y^2)}{y^3}$
   (ii) $(x^6y^4 + x^2)\,dy = (1 - x^5y^5 - xy)\,dx$
6. Solve the differential equation $(x + 1)\,y' = y - 1$, y being bounded when $x \to \infty$.
7. Solve the differential equation $x^2y' + \sin 2y = 1$, $y \to \dfrac{11}{4}\pi,\ x \to \infty.$
8. Solve the following differential equations :
   (i) $y'' - 2\cot x \,.\, y' = \sin^3 x$
   (ii) $yy'' - yy'\ln y = (y')^2$
9. Solve
   $$\frac{y + \sin x\cos^2(xy)}{\cos^2(xy)}\,dx + \frac{x\,dy}{\cos^2 xy} + \sin y\,dy = 0$$
10. A differentiable function f satisfies the relation $f(x + y) + f(xy) = f(x)\,.\,f(y) + 1$. If $f'(0) = 1$, find f.
11. Let $f : R^+ \to R$ satisfies the functional equation $f(xy) = e^{xy - x - y}\,(e^y f(x) + e^x f(y))\ \forall\ x, y \in R^+$. If $f'(1) = e$, determine f(x).
12. (A) Find a polynomial P(x) such that $P'(x) - 3P(x) = 4 - 5x + 3x^2$. Prove that there is only one solution.
    (B) If Q(x) is a given polynomial, prove that there is one and only one polynomial P(x) such that $P'(x) - 3P(x) = Q(x)$.
13. Let $y = f(x)$ be that solution of the differential equation $y' = \dfrac{2y^2 + x}{3y^2 + 5}$ which satisfies the initial condition $f(0) = 0$.
    (A) The differential equation shows that $f'(0) = 0$. Discuss whether f has a local maximum or minimum or neither at 0.
    (B) Notice that $f'(x) \geq 0$ and that $f'(x) \geq \dfrac{2}{3}$ for

each $x \geq \frac{10}{3}$. Exhibit two positive numbers a and b such that $f(x) > ax - b$ for each $x \geq \frac{10}{3}$.

(C) Show that $x/y^2 \to 0$ as $x \to \infty$. Give full details of your reasoning.

(D) Show that y/x tends to a finite limit as $x \to \infty$ and determine this limit.

14. A periodic function with period a satisfies $f(x + a) = f(x)$ for all x in its domain. What can you conclude about a function which has a derivative everywhere and satisfies an equation of the form $f(x + a) = bf(x)$ for all x, where a and b are positive constants ?

15. Solve the differential equation $(1 + y^3e^{2x})y' + y = 0$ by introducing a change of variable of the form $y = ue^{mx}$, where m is constant and u is a new unknown function.

16. Find the curve such that the y-intercept cut off by the tangent on any arbitrary point is

(A) proportional to the square of the ordinate of the point of tangency.

(B) proportional to the cube of the ordinate of the point of tangency.

(C) Show that the solution of the differential equation $(1 - x^2)\frac{dy}{dx} + xy = ax$ are ellipses or hyperbolas with the centres at the point (0, a) and the axes parallel to the co-ordinate axes, each curve having one constant axis whose length is equal to 2.

17. Find the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point.

18. Find the curve $y = f(x)$, where $f(x) \geq 0$, $f(0) = 0$ and $f(1) = 1$, bounding a curvilinear trapezoid with the base [0, x], if the area bounded by curve, the coordinate axes and the variable ordinate is proportional to $(f(x))^{n+1}$

19. A normal is drawn at a point P(x, y) of a curve. If meets the x-axis and the y-axis in point A and B, respectively, such that $\frac{1}{OA} + \frac{1}{OB} = 1$, where O is the origin. Find the equation of such a curve passing through (5, 4).

20. The tangent at a point P of a curve meets the y-axis at A, and the line parallel to the y axis through P meet the x-axis at B. If area of triangle OAB is constant (O being the origin) then from the differential equation of the curve and show that the curve is a hyperbola.

21. Find the equation of the curve intersecting with the x-axis at the point x = 1 and possesses the property that the subnormal at any point of the curve is equal to the A.M. of the co-ordinates of this point.

22. Find the curve such that the sum of the intercepts made by the tangent on the coordinate axes is equal to a.

23. Find the curve such that the product of the perpendiculars from the points ($\pm$c, 0) on any tangent is equal b^2.

24. Find out area bounded by the curves,

$$y = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt, \quad 0 \leq x \leq \frac{\pi}{2}$$

and the curve satisfying the differential equation, $y(x + y^3)\,dx = x(y^3 - x)\,dy$ passing through (4, –2).

25. Find the orthogonal trajectories of the following families :

(i) $x^2 - \frac{1}{3}y^2 = a^2$

(ii) $y = a e^{\sigma x}$, where σ is a constant and a is a parameter.

(iii) $\cos y = ae^{-x}$

(iv) $y^2 = 4(x - a)$

26. A storage tank contains 2000 litres of gasoline which initially has 100 gms of an additive dissolved in it. Gasoline containing 2 gms of additive per litre is pumped into the tank at a rate of 40 litre/min. The well mixed solution is pumped out at a rate of 45 litre/min. Form the differential equation and express the amount of additive in gasoline as a function of t.

27. Show that the half-life H of a radioactive substance can be obtained from two measurements $y_1 = y(t_1)$ and $y_2 = y(t_2)$ of the amount present at times t_1 and t_2 by the formula $H = [(t_2 - t_1)\ln 2]/\ln(y_1/y_2)$.

28. Suppose that a moth ball loses volume by evaporation at a rate proportional to its instantaneous area. If the diameter of the ball decreases from 2 to 1 cm in 3 months, how long will it take until the ball has practically gone (say until its diameter is 1 mm)?

29. A body in a room at 60° cools from 200° to 120° in half an hour.
(A) Show that its temperature after t minutes is $60 + 140e^{-kt}$, where $k = (\ln 7 - \ln 3)/30$.
(B) Show that the time t required to reach a temperature of T degrees is given by the formula $t = [\ln 140 - \ln (T - 60)]/k$, where $60 < T \le 200$.
(C) Find the time at which the temperature is 90°.
(D) Find a formula for the temperature of the body at time t if the room temperature is not kept constant but falls at a rate of 1° each ten minutes. Assume the room temperature is 60° when the body temperature is 200°.

30. In a tank are 100 litres of brine containing 50 kg of dissolved salt. Water runs into the tank at the rate of 3 litres per minute, and the concentration is kept uniform by stirring. How much salt is in the tank at the end of one hour if the mixture runs out at a rate of 2 litres per minute ?

31. Water is heated to the boiling point temperature 100ºC. It is then removed from heat and kept in a room which is at a constant temperature of 60ºC. After 3 minutes, the temperature of the water is 90ºC.
(A) Find the temperature of water after 6 minutes.
(B) When will the temperature of water be 75ºC.

32. A certain solvent weighs 120 g and dissolves 40 g of particular solute. Given the 20 g of solute is contained in the solvent at time t = 0, find the amount Q of undissolved solute at t = 4 hr. if 5 g solute dissolves in one hour. Find also the time required for 10 g solute to dissolve.

33. Three grams of substance Y combine with 2 g of substance X to form a 5 g of substance Z. When 90 g of Y are thoroughly mixed with 60 g of X, it is found that in 20 min, 50 g of Z have been formed. How many grams of Z can be formed in 30 min ? How long does it take to form 100 g of Z.

34. A sheet of aluminium (k = 0.49) is 10 cm thick. One face is kept at 20ºC and the other face at 80ºC. Assuming that the sheet is sufficiently large for heat flow to be perpendicular to those two faces, find the temperature T in terms of the distance x from the cooler faces. What is the amount of heat transmitted per second across a sq. cm of a section parallel to these faces ?

35. A liquid carries a drug into an organ of volume 500 cm^3 at a rate of 10 cm^3/s and leaves at the same rate. The concentration of the drug in the entering liquid is 0.08 g/cm^3. Assuming that the drug is not present in the organ initially, find
(A) the concentration of the drug after 30 s and 120 s,
(B) the steady state concentration,
(C) how long would it take for the concentration of the drug in the organ to reach 0.04 g/cm^3 and 0.06 g/cm^3 ?

Target Exercises for JEE Advanced

1. Solve the following differential equations :

(i) $y' = \dfrac{1 + y^2}{xy(1 + x^2)}$

(ii) $\dfrac{dx}{x^2 - xy + x^2} = \dfrac{dy}{2y^2 - xy}$

(iii) $yy' + x = \dfrac{1}{2}\left(\dfrac{x^2 + y^2}{x}\right)^2$

2. Solve the following differential equations :

(i) $\left(2xy + x^2y + \dfrac{y^3}{3}\right) dx + (x^2 + y^2)\, dy = 0$

(ii) $2yy' = e^{\frac{x^2+y^2}{x}} + \dfrac{x^2 + y^2}{x} - 2x$

(iii) $y' = \dfrac{(1 + y)^2}{x(y + 1) - x^2}$

3. Solve the following differential equations :

(i) $\dfrac{dy}{dx} = \dfrac{e^x}{2}\dfrac{e^{y^2/x}}{y} - \dfrac{1}{2}\left(\dfrac{x}{y} + \dfrac{y}{x}\right)$

(ii) $\left[\dfrac{1}{x} - \dfrac{y^2}{(x - y)^2}\right] dx + \left[\dfrac{x^2}{(x - y)^2} - \dfrac{1}{y}\right] dy = 0$

4. Solve the following differential equations

(i) $\dfrac{y}{x} + \dfrac{dy}{dx} = \dfrac{1}{\sin^4(xy) + \cos^2(xy)}$

(ii) $$\frac{dy}{dx}=\frac{(x-1)^2+(y-2)^2\tan^{-1}\left(\frac{y-2}{x-1}\right)}{(xy-2x-y+2)\tan^{-1}\left(\frac{y-2}{y-1}\right)}$$

5. Solve the following differential equations :

(i) $$\left(\frac{1}{y}\sin\frac{x}{y}-\frac{y}{x^2}\cos\frac{y}{x}+1\right)dx+\left(\frac{1}{x}\cos\frac{y}{x}-\frac{x}{y^2}\sin\frac{x}{y}+\frac{1}{y^2}\right)dy=0.$$

(ii) $$\frac{dy}{dx}=\frac{2y^2\cos x+y\sin 2x+2\cos x.\sin^2 x}{\sin^2 x}$$

6 Solve

$$\frac{dy}{dx}=\sqrt{\frac{1}{2}+\left\{\int_{\cos^4\theta}^{-\sin^4\theta}\frac{\sqrt{f(\varphi)}.d\varphi}{\sqrt{f(\cos(2\theta-\varphi)}+\sqrt{f(\varphi)}}\right\}}$$

7. Solve $2xyy'+(1+x)y^2=e^x$ on $(0,\infty)$, with

(A) $y=\sqrt{e}$ when $x=1$;

(B) $y=-\sqrt{e}$ when $x=1$;

(C) A finite limit as $x\to 0$.

8. Find all solutions of $x(x+1)y'+y=x(x+1)^2\,e^{-x^2}$ on the interval $(-1,0)$. Prove that all solutions approach 0 as $x\to -1$, but that only one of them has a finite limit as $x\to 0$.

9. Solve the following differential equations :

(i) $x^2y'''=(y'')^2$

(ii) $y'''[1+(y')^2]=3y'(y'')^2$

10. If $f:R\to[0,\infty)$ be a function satisfying , $f(x+y)-f(x-y)=f(x)\,[f(y)-f(-y)]$; $f'(0)=\log a$, $f(0)=1$ (for all values of a except 1), then solve the differential $$\frac{dy}{dx}=\frac{\{\log_a(f(x)f(y))\}^2}{(\log_a f(x)+2)(\log_a f(y)-2)}$$

11. A differentiable function f satisfies the relation $[1+f(x)+f(y)]\,f(x+y)=f(x)+f(y)\ \forall\ x, y\in R$. If $f(0)=0$ and $f'(0)=1$, find f(x).

12. Find all functions f such that f' is continuous and

$$[f(x)]^2=100+\int_0^x\{[f(t)]^2+[f'(t)]^2\}dt \text{ for all real x}$$

13. If $f:R-\{-1\}\to R$ be a differentiable function which satisfies : $f((x)+f(y)+xf(y))=y+f(x)+yf(x)\ \forall\ x, y\in R-[-1]$. Find f(x) ?

14. Let $s(x)=(\sin x)/x$ if $x\neq 0$, and let $s(0)=1$.

Define $T(x)=\int_0^x s(t)dt$. Prove that the function $f(x)=xT(x)$ satisfies the differential equation $xy'-y=x\sin x$ on the interval $(-\infty,\infty)$ and find all solutions on this interval. Prove that the differential equation has no solution satisfying the initial condition $f(0)=1$.

15. Show that the complete solution of $\frac{d^2s}{dt^2}=\frac{1}{\sqrt{s}}$ can be put in the form

$3t=2(\sqrt{s}-2C_1)\sqrt{\sqrt{s}+C_1}+C_2$.

16. Prove that there is exactly one function f, continuous on the positive real axis, such that

$f(x)=1+\frac{1}{x}\int_1^x f(t)dt$ for all $x>0$ and find this function.

17. Given a function g which has a derivative $g'(x)$ for every real x and which satisfies the following equations : $g'(0)=2$ and $g(x+y)=e^yg(x)+e^xg(y)$ for all x and y.

(A) Show that $g(2x)=2e^xg(x)$ and find a similar formula for $g(3x)$.

(B) Generalize (A) by finding a formula relating $g(nx)$ to $g(x)$, valid for every positive integer n.

(C) Show that $g(0)=0$ and find the limit of $g(h)/h$ as $h\to 0$.

(D) There is a constant C such that $g'(x)=g(x)+Ce^x$ for all x. Prove this statement and find the value of C.

18. The Riccati equation $y'+y+y^2=2$ has two constant solutions. Start with each of these and find further solutions as follows:

(A) If $-2\le b<1$, find a solution on $(-\infty,\infty)$ for which $y=b$ when $x=0$.

(B) if $b\ge 1$ or $b<-2$, find a solution on the interval $(-\infty,\infty)$ for which $y=b$ when $x=0$.

19. Given a function f which satisfies the differential equation $xf''(x)+3x[f'(x)]^2=1-e^{-x}$ for all real x.

(A) If f has an extrem um at a point $c \neq 0$, show that this extremum is a minimum.
(B) If f has an extremum at 0, is it a maximum or a minimum ? Justify your conclusion.
(C) If $f(0) = f'(0) = 0$, find the smallest constant A such that $f(x) \leq Ax^2$ for all $x \geq 0$.

20. The function f defined by the equation

$$f(x) = xe^{(1-x^2)/2} - xe^{-x^2/2}\int_1^x t^{-2}e^{t^2/2}dt \text{ for } x > 0 \text{ has}$$

the properties that (i) it is continuous on the positive real axis, and (ii) it satisfies the equation

$$f(x) = 1 - x\int_1^x f(t)\,dt \text{ for all } x > 0.$$ Find all functions with these two properties.

21. Solve the system of differential equations

$$\begin{cases} 2\dfrac{dx}{dt} + 5\dfrac{dy}{dt} = t \\ \dfrac{dx}{dt} + 3\dfrac{dy}{dt} = 7\cos t \end{cases}$$

22. Find the curve which passes through the origin of coordinates and is such that the area of the triangle formed by the tangent to the curve at some point, the ordinate of that point and the x-axis is proportional to the area of the curvilinear trapezoid formed by the curve, the x-axis and the ordinate of the point.

23. Find the curve such that the ordinate of any of its points is the proportional mean between the abscissa and the sum of the abscissa and subnormal at the point.

24. Find the curve such that the angle, formed with the x-axis by the tangent to the curve at any of its points, is twice the angle formed by the polar radius of the point of tangency with the x-axis. Interpret the curve.

25. Find the curve such. that the ratio of the subnormal at any point to the sum of its abscissa and ordinates is equal to the ratio of the ordinate of this point to its abscissa. If the curve passes through (1, 0) find all possible equations in the form $y = f(x)$.

26. Find the curve such that the initial ordinate of any tangent is less than the abscissa of the point of tangency by two units.

27. Find the curve such that the area of the rectangle constructed on the abscissa of any point and the initial ordinate of the tangent at this point is a^2.

28. The area of the figure bounded by a curve, the x-axis and two ordinates, one of which is constant, the other variable is equal to the ratio of the cube of the variable ordinate to the variable abscissa. Find the curve if it passes through (1, 0).

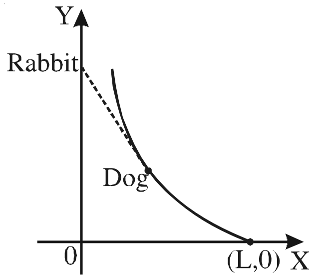
29. Show that the curve passing through (1, 2) for which the segment of the tangent between P (point of contact) and T (point of intersection of the tangent with the x-axis) is bisected at its point of intersection with the y-axis is a parabola. Further show that if, P, Q, R are three points on this parabola such that three normals at them are concurrent on the line $y = d$ then the sides of the ΔPQR touch another parabola $x^2 = 2ky$.

30. Find the isogonal trajectories of the family of parabolas $y^2 = 4ax$; the angle of intersection $\alpha = 45°$.

31. Show that any differential equation of the type

$f\left(x, y, p - \dfrac{1}{p}\right) = 0$ represents two systems of orthogonal curves.

32. A dog sees a rabbit running in a straight line across an open field and gives chase. In a rectangular coordinate system (as shown in the figure), assume:
(i) The rabbit is at the origin and the dog is at the point (L, 0) at the instant the dog first sees the rabbit.
(ii) The rabbit runs up the y-axis and the dog always runs straight for the rabbit.
(iii) The dog runs at the same speed as the rabbit.
(A) Show that the dog's path is the graph of the function $y = f(x)$, where y satisfies the differential equation $x\dfrac{d^2y}{dx^2} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$
(B) Determine the solution of the equation in part
(C) That satisfies the initial conditions $y = y' = 0$ when $x = L$.
Hint: Let $z = dy/dx$ in the differential equation and solve the resulting first-order equation to find z; then integrate z to find y]
(D) Does the dog ever catch the rabbit ?



33. A reservoir contains 50 litres of pure water initially. Salted water flows into the reservoir at the rate of 2 litres per minute. It contains 2 grams of salt per litre. The mixture is kept uniform by stirring and it flows out of the bottom of the reservoir at the same rate. Find the time taken for the quantity of salt in the reservoir to increase from 40 grams to 80 grams.

34. Denote by y = f(t) the amount of a substance present at time t. Assume it disintegrates at a rate proportional to the amount present. If n is a positive integer, the number T for which f(T) = f(0)/n is called the 1/nth life of the substance.

(A) Prove that the 1/nth life of the same for every sample of a given material, and compute T in terms of n and the decay constant k.

(B) If a and b are given, prove that f can be expressed in the form $f(t) = f(A)^{w(t)} f(B)^{1-w(t)}$ and determine w(t). This shows that the amount present at time t is a weighted geometric mean of the amounts present at two instants t = a and t = b.

35. Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hourse for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely ?

Previous Year's Questions (JEE Advanced)

A. Fill in the blanks :

1. A spherical rain drop evaporates at a rate proportional to its surface area at any instant t. The differential equation giving the rate of change of the radius of the rain drop is....... **[IIT - 1997]**

B. Multiple Choice Question with ONE correct answer :

2. The order of the differential equation whose general solution is given by

$y = (c_1 + c_2)\cos(x + c_3) - c_4\, e^{x+c_5}$ where c_1, c_2, c_3, c_4, c_5, are arbitrary constants is **[IIT - 1998]**

(A) 5 (B) 4
(C) 3 (D) 2

3. If $x^2 + y^2 = 1$, then **[IIT - 2000]**

(A) $yy'' - 2(y')^2 + 1 = 0$
(B) $yy'' + (y')^2 + 1 = 0$
(C) $yy'' + (y')^2 - 1 = 0$
(D) $yy'' + 2(y')^2 + 1 = 0$

4. If y(t) is a solution of (1 + t) dy/dt – ty = 1 and y(0) = – 1, then y(1) is equal to - **[IIT - 2003]**

(A) –1/2 (B) e + 1/2
(C) e – 1/2 (D) 1/2

5. If y = y(x) and $\dfrac{2+\sin x}{y+1}\left(\dfrac{dy}{dx}\right) = -\cos x$, y(0) = 1, then $y\left(\dfrac{\pi}{2}\right)$ equals - **[IIT - 2004]**

(A) 1/3 (B) 2/3
(C) –1/3 (D) 1

6. If y = y(x) and it follows the relation x cos y + y cos x = π then y″ (0) **[IIT - 2005]**

(A) 1 (B) –1
(C) π (D) –π

7. The solution of primitive integral equation $(x^2 + y^2)\,dy = xy\,dx$, is y = y(x). If y(1) = 1 and $y(x_0) = e$, then x_0 is **[IIT - 2005]**

(A) $\sqrt{2(e^2-1)}$ (B) $\sqrt{2(e^2+1)}$

(C) $\sqrt{3}e$ (D) $\sqrt{\dfrac{e^2+1}{2}}$

8. For the primitive integral equation $ydx + y^2 dy = x\,dy$; $x \in R$, $y > 0$, y = y(x), y(1) = 1, then y(–3) is **[IIT - 2005]**

(A) 3 (B) 2
(C) 1 (D) 5

9. The differential equation $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y}$ determines a family of circles with **[IIT - 2005]**
(A) Variable radii and a fixed centre at (0, 1)
(B) Variable radii and a fixed centre at (0, –1)
(C) Fixed radius 1 and variable centres along the x-axis.
(D) Fixed radius 1 and variable centres along the y-axis.

10. Let f(x) be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$, and $\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1$ for each $x > 0$. Then f(x) is **[IIT - 2007]**
(A) $\frac{1}{3x}+\frac{2x^2}{3}$ (B) $-\frac{1}{3x}+\frac{4x^2}{3}$
(C) $-\frac{1}{x}+\frac{2}{x^2}$ (D) $\frac{1}{x}$

11. The differential equation $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y}$ determines a family of circles with **[IIT - 2007]**
(A) Variable radii and a fixed centre (0, 1)
(B) Variable radii and a fixed centre (0, – 1)
(C) Fixed radius 1 and a variable centres along the x-axis
(D) Fixed radius 1 and variable centres along the y-axis.

C. Multiple choice Questions with ONE or MORE THAN ONE correct answer :

12. A solution of the differential equation $\left(\frac{dy}{dx}\right)^2-x\frac{dy}{dx}+y=0$ is- **[IIT - 1999]**
(A) $y = 2$ (B) $y = 2x$
(C) $y = 2x - 4$ (D) $y = 2x^2 - 4$

13. The differential equation representing the family or curves $y^2 = 2c\,(x + \sqrt{c}\,)$ where c is a positive parameter is of **[IIT - 1999]**
(A) Order 1 (B) Order 2
(C) Degree 3 (D) Degrees 4

14. A curve y = f(x) passes through (1, 1) and tangent at P(x, y) cuts the x-axis and y-axis at A and B respectively such that BP : AP = 3 : 1, then **[IIT-2006]**
(A) Equation of curve is $xy' - 3y = 0$
(B) Normal (1, 1) is $x + 3y = 4$
(C) Curve passes through (2, 1/8)
(D) Equation of curve is $xy' + 3y = 0$

15. If y(x) satisfies the differential equation $y' - y\tan x = 2x\sec x$ and $y(0) = 0$ then **[IIT - 2012]**
(A) $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{8\sqrt{2}}$ (B) $y'\left(\frac{\pi}{4}\right)=\frac{\pi^2}{18}$
(C) $y\left(\frac{\pi}{3}\right)=\frac{\pi^2}{9}$ (D) $y'\left(\frac{\pi}{3}\right)=\frac{4\pi}{3}+\frac{2\pi^2}{3\sqrt{3}}$

D. Subjective Problems :

16. If $(a + bx)e^{y/x} = x$, then prove that
$$x^2\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2$$ **[IIT - 1983]**

17. A normal is drawn at a point P (x, y) of a curve, It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is, $y\frac{dy}{dx}=\pm\sqrt{k^2-y^2}$. Find the equation of such a curve passing through (0, k). **[IIT - 1994]**

18. Let y = f(x) be a curve passing through (1, 1) such that the triangle formed by the coordinate axes & the tangent at any point of the curve lies in the first quadrant & has area 2. Form the diff. equation & determine all such possible curves. **[IIT - 1995]**

19. Determine the equation of the curve passing through the origin in the form y = f(x), which satisfies the differential equation, $dy/dx = \sin(10x + 6y)$. **[IIT - 1996]**

20. A curve y = f (x) passes through the point P(1,1). The normal to the curve at P is ; $a(y - 1) + (x - 1) = 0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve & the normal to the curve at P. **[IIT - 1996]**

21. A and B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filed completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at the time. One hour after the water is released, the quantity of water in reservoir A is 1½ times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water ? **[IIT - 1997]**

22. Let u (x) & v (x) satisfy the differential equations $\frac{du}{dx} + p(x)\,u = f(x)$ & $\frac{dv}{dx} + p(x),\, v = g(x)$ where p(x), f(x) & g(x) are continuous functions. If $u(x_1) > v(x_1)$ for some x_1 and f(x) > g(x) for all $x > x_1$, prove that any point (x, y) where $x > x_1$ does not satisfy the equations y = u (x) & y = v (x). **[IIT - 1997]**

23. A curve C has the property that if the tangent drawn at any point P on C meets the co-ordinates axes at A and B, then P is the mid-point of AB. The curve passes through the point (1, 1). Determine the equation of the curve. **[IIT - 1998]**

24. A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x-axis. Determine the equation of the curve. **[IIT - 1999]**

25. A country has a food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after 'n' is years, where n is the smallest integer bigger than or equal to, $\frac{\ell n10 - \ell n9}{\ell n(1.04) - 0.03}$ **[IIT - 2000]**

26. A hemispherical tank of radius 2 meters is initially full of water and has an outlet of 12 cm² cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t) = 0.6\sqrt{2gh(t)}$, where v(t) and h(t) are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to empty the tank. (Hint : Form a differential equation by relating the decreasing of water level to the outflow.). **[IIT - 2001]**

27. A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = k > 0). Find the time after which the cone is empty. **[IIT - 2003]**

28. A curve 'C' passes through (2, 0) and the slope of at (x, y) as $\frac{(x+1)^2 + y - 3}{(x+1)}$. Find the equation of the curve and area enclosed by the curve and the x-axis in the fourth quadrant. **[IIT - 2004]**

29. If length of tangent at any point on the curve y = f(x) intercepted between the point and the x-axis is of length 1. Find the equation of the curve. **[IIT - 2005]**

E. Assertion A and Reason R

30. Let a solution y = y(x) of the differential equation $x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$ satisfy $y(2) = \frac{2}{\sqrt{3}}$.

Assertion (A) : $y(x) = \sec\left(\sec^{-1}x - \frac{\pi}{6}\right)$

Reason (R) : y(x) is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$

[IIT - 2008]

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

F. Integer Answer Type

31. Let f be a real valued differentiable function on R (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x,y) on the curve y = f(x) is equal to the cube of the abscissa of P, then the value of f(–3) is equal to **[IIT - 2010]**

32. Let $y'(x) + y(x)\,g'(x) = g(x).g'(x)$, y(0) = 0, $x \in R$, where f′(x) denotes $\frac{df(x)}{dx}$ and g(x) is a given non-constant differentiable function on R with g(0) = g(2) = 0. Then the value of y(2) is **[IIT - 2011]**

33. Let $f : [1, \infty) \to [2, \infty)$ be a differentiable function such that f(1) = 2. If $6\int_1^x f(t)\,dt = 3xf(x) - x^3 - 5$ for all $x \ge 1$, then the value of f(2) is. **[IIT - 2011]**

G. Matrix Match Type

34.

Column - I	**Column - II**
	[IIT - 2006]
(A) $\int_0^{\pi/2} (\sin x)^{\cos x}\,[\cos x \cot x - \log(\sin x)^{\sin x}]\,dx$	(P) 1
(B) Area bounded by $-4y^2 = x$ and $x - 1 = 5y^2$	(Q) 0
(C) Tangent of the angle of intersection of curves $y = 3^{x-1}\log x$ and $y = x^{x-1}$ equals	(R) 16/e
(D) If y satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{2}{x+y}$, $y(1) = 1$ then $(x + y + 2)^2 e^{-y}$ equals	(S) 4/3

35.

Column - I	**Column - II**
	[IIT - 2009]
(A) The number of solutions of the equation $xe^{\sin x} - \cos x = 0$ in the interval $\left(0, \dfrac{\pi}{2}\right)$	(P) 1
(Q)	2
(B) Values(s) of k for which the planes $kx + 4y + z = 0$, $4x + ky + 2z = 0$ and $2x + 2y + z = 0$ intersect in a straight line	(R) 3
(C) Values(s) of k for which $\|x - 1\| + \|x - 2\| + \|x + 1 + \|x + 2\|$ $= 4k$ has integer solution(s)	(S) 4
(D) If $y' = y + 1$ and $y(0) = 1$, then value(s) of $y(\ln 2)$	(T) 5

ANSWERS

CONCEPT PROBLEMS—A

1. (i) order = 1, degree = 1
(ii) order = 2, degree = 2
(iii) order = 1, degree = 2
(iv) order = 1, degree = 2

2. (i) order = 1, degree = 1 ,
(ii) order = 5, degree is not applicable,
(iii) order = 2, degree = 2,
(iv) order = 2, degree = 3.

(iii) $$\left[\left(\frac{dy}{dx}\right)^{1/2} + y\right]^2 = \frac{d^2y}{dx^2}$$

(iv) $$\left(\frac{d^2y}{dx^2}\right)^3 - 3\left(\frac{dy}{dx}\right)^5 + 2y = x \quad \sin x.$$

3. (i) order = 2, degree = 3
(ii) order = 2, degree is not applicable
(iii) order = 2, degree is not applicable
(iv) order = 4, degree = 3

CONCEPT PROBLEMS—B

1. (i) $(x^2 - y)y' = xy$
(ii) $xy'' + 2y' = xy - x^2 + 2$
(iii) $(1 + \cos\theta)\dfrac{dr}{d\theta} + r\sin\theta = 0$

2. (i) $y'' - 2y' + y = 0$
(ii) $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$
(iii) $y = xy_1 - \dfrac{1}{2}x^2y_2$

(iv) $xy'' + 2y' - xy + x^2 - 2 = 0$

3. $\frac{d^2y}{dx^2} + b^2y = 0$

6. $\frac{dh}{dt} = -\frac{25}{16\pi}\sqrt{\frac{2g}{h^3}}$

PRACTICE PROBLEMS—A

7. (i) $y''' + \frac{3}{x}y'' = 0$

(ii) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

(iii) $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$

8. $(y - xy')^2 = a^2(1 + y'^2)$

9. $(x^2 + y^2 - 1)(y' - 1) - 2(x + yy')(y - x) = 0$

11. $y^2 = y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$

12. $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$

13. $x\left(\frac{dx}{dy}\right)^2 + 2y\frac{dx}{dy} = x$

14. $\frac{dv}{dt} = g - \frac{k}{m}v^2$

CONCEPT PROBLEMS—C

2. (A) yes ; (B) no;
(C) no

5. -1

PRACTICE PROBLEMS—B

6. (A) $\alpha = -2$; (B) $\alpha = \frac{1}{2}$
(C) $\alpha = 2, \alpha = -3$

8. $m = 1, 2$

10. (A) ± 3

11. (A) $y = 0, 1, 5$ (B) $1 < y < 5$

13. (C)

14. (A) $f(x) = 2e^{x-1}$
(B) $g(x) = \pm 1$; $g(x) = \sin(x + C)$; also, those continuous functions whose graphs may be obtained by piecing together portions of the curves $y = \sin(x + C)$ with portions of the lines $y = \pm 1$. One such example is $g(x) = -1$ for $x \le 0$, $g(x) = \sin(x - \frac{1}{2}\pi)$ for $0 \le x \le 3\pi$, $g(x) = 1$ for $x \ge 3\pi$

PRACTICE PROBLEMS—C

1. $y = cxe^{-x}$

2. $\ell n x(1 - y)^2 = c - \frac{1}{2}y^2 - 2y + \frac{1}{2}x^2$

3. $y^2(1 + x^2) = c$

4. $(e^y + 1)\sin x = c$

5. $(x + 1)^2 + (y - 1)^2 + 2\log(x - 1)(y + 1) = c$

6. $y = e^{\tan(x/2)}$

7. $y = (x + 1).\ \ell n(x + 1) - x + 3$

8. $(y - 3)(1 - 3x) = cx$

9. $y = c(a + x)(1 - ay)$

10. $y\,\ell n\,(y + 1) = \ell n x - \frac{1}{x} + c$

11. $e^{-x} + e^y + y^3/3 = c$

12. $\ell n^2(\sec x + \tan x) - \ell n^2(\sec y + \tan y) = c$

13. $y - \tan^{-1} y = \log x - \frac{x^2}{2} + c$

14. $\sqrt{1 - x^2} + \sqrt{1 - y^2} = 1, y = 1$

15. $\tan^{-1} e^x = \frac{1}{2\sin^2 y} + C.$

16. $y = a\tan\sqrt{\frac{a}{x} - 1}$

17. $y = \sin^{-1}\left(\frac{\sqrt{3}}{2} - \frac{1}{x}\right) + 5\pi.$

18. $y = \frac{1}{2}\tan^{-1}(\pi/2 + \tan^{-1} x) + \frac{7\pi}{2}$

CONCEPT PROBLEMS—D

1. (i) $-\operatorname{cosec}(x + y) + \cot(x + y) + y = c$
(ii) $(x + c)e^{x+y} + 1$
(iii) $e^y - \frac{1}{e^y} = e^x + \frac{x^3}{3} + c$
(iv) $y = x + \ln|x(1 + y)| + c$

2. (i) $y = k(1 - ay)(a + x)$
(ii) $y + \ln y = x + \ln x + c$
(iii) $\tan^{-1}(x + y) = x + c$
(iv) $\sec y = -2\cos x + c$

3. (i) $y - x + \ln|x + y| = c$

(ii) $y + \frac{1}{2}\log|(x+y)^2 - 2| = x + c$

(iii) $(2x + y + 1)^2 = 6x + c$

(iv) $(x - y + c) = \ln(x + y)$

6. $y = 0$, for $\alpha \le 1$ the solution is unique.

8. $y = e^{(x-a)/c}$

PRACTICE PROBLEMS—D

9. (i) $\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$

(ii) $y + \left(\frac{b-a}{2}\right)\log\left|(x+y)^2 - ab\right| = x + c$

(iii) $\ell n\left|\tan\frac{y}{4}\right| = c - 2\sin\frac{x}{2}$

(iv) $\ln|\sin(y-x)| = \frac{x^2}{2} + c$

10. $\sqrt{x^2+y^2} + \frac{y}{x} = c$

11. $x + C = 2u + \frac{2}{3}\ln|u-1| - \frac{8}{3}\ln(u+2)$, where

$u = \sqrt{1+x+y}$

12. 0.82 kg; $\frac{ds}{dt} = ks(s+6)$.

13. $T = 2/3\,x$; $864{,}000 \times 4.2$ J; $\frac{dT}{dx} = -\frac{Q}{ks}$, where

Q = const.

15. $f(x) = Cx^n$, or $f(x) = Cx^{1/n}$

16. $5(1 + y^2) = (1 + x^2)$

18. $(2y^2 - x^2)^2 = Cx^2$.

19. (A) $y = \begin{cases} \frac{1}{2}(1-e^{-2x}), & 0 \le x \le 1 \\ \frac{1}{2}(e^2-1)e^{-2x}, & x > 1 \end{cases}$

(B) $y = \begin{cases} e^{-2x}, & 0 \le x \le 1 \\ e^{-(x+1)}, & x > 1 \end{cases}$

PRACTICE PROBLEMS—E

1. (B) and (C) are homogeneous

2. (i) $x^2(y^2 + 2xy) = c$ (ii) $x\sin\frac{y}{x} = C$

(iii) $(y + 2x)(2y - 3x) = cx$

(iv) $y = ce^{y/x}$

3. (i) $6(y-1)^2 + 4(2x-3)(y-1) - 3(2x-3)^2 = c$

(ii) $3(2y - x) + \ln(3x + 7y + 4) = C$

(iii) $x + 3y + 2\log(2 - x - y) = c$

(iv) $\tan^{-1}\frac{2y+1}{2x+1} = \ell nc\sqrt{x^2+y^2+x+y+\frac{1}{2}}$

4. $x^2 + y^2 = 5x$

5. $x + y\,.\,e^{\frac{x}{y}} = C$

6. $x^2 + y^2 = Cx^4$

7. $\frac{1}{2}\ln(y^4 + 2y^2x + 2x^2) - \tan^{-1}\frac{y^2+x}{x} = c.$

8. $x = y\ln|Cy|$

9. (i) $y^2 = x\ln Cy^2$. (ii) $Cx^4 = y^6 + x^3$.

(iii) $\sqrt{x^2y^4+1} = Cx^2y^2 - 1$

(iv) $2\tan^{-1}\frac{y^3}{x} = \ln(x^2 + y^6)$

10. $(3/\sqrt[3]{2} - 1) \approx 11\frac{1}{2}$ h

12. $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt} = kA = 4k\pi r^2$, 5.7 months.

13. (C) 54.5 min

(D) $T = \frac{1}{10k}[1 + (600 - t)k + (1400k - 1)e^{-kt}]$

14. 19.5 kg

15. 0.467 km/h ; 85.2 m.

CONCEPT PROBLEMS—E

1. (i) $y = \sec x(-\sin^2 x + c)$

(ii) $y = c(1 - x^2) + \sqrt{1-x^2}$

(iii) $y = c(x + a)^3 + \frac{1}{2}(x + a)^5$

(iv) $y = (C + e^x)(1 + x)^n$

2. $x = Ce^{\sin y} - 2(1 + \sin y)$.

3. $y = Cx - x^2$; $y - xy' = x^2$.

4. $y(x) = y_1(x) + Ce^{-\int p(x)dx}$

5. $y = y_1(x) + C[y_2(x) - y_1(x)]$

6. (A) $y_h = ce^{-x^2}$

PRACTICE PROBLEMS—F

10. (i) $y = \dfrac{\sin x}{\cos^2 x}$ (ii) $x = -t\tan^{-1} t$.

(iii) $y = \dfrac{1}{2}\sqrt{\dfrac{1+x}{1-x}}[2+\sqrt{1-x^2}+\sin^{-1} x]$

(iv) $y = \dfrac{3}{2}x^{5/2} - \sqrt{x}\ln x$

11. (i) $y = 2^{\sin x}$ (ii) $y = \dfrac{\sin x}{x}$

(iii) $y = \dfrac{x+1}{x\cos\frac{1}{x}}$ (iv) $y = e^x + e^{1/x}$

12. (i) $y = \dfrac{1}{x}\tan(\ln|Cx|)$.

(ii) $y = \left(\dfrac{C + \ln|\cos x|}{x} + \tan x\right)^2$

(iii) $x^2y^2 + 1 = Cy$

(iv) $y^2 e^{-\frac{y^2}{x}} = C$

13. (i) $x^2 = y^2(C - y^2)$ (ii) $\dfrac{y^2}{2} + \dfrac{\ln x}{y} = C$

(iii) $x^2 - y^2 = Cy^3$ (iv) $y = \dfrac{\tan x + \sec x}{C + \sin x}$

14. (i) $x - 2 + Ce^{-x} = e^{y^2/2}, z = 2e^{-y^2/2}$

(ii) $\tan y = (C + x^2)e^{-x^2}; z = \tan y$

(iii) $y^2 = \dfrac{2}{3}\sin x + \dfrac{C}{\sin^2 x}$

(iv) $y^2 = x + (x+1)\ln\dfrac{C}{x+1}$

15. $y = (x + C)/\sin x$ **16.** $y = \sin x + C/\sin x$

17. $y = \left(\dfrac{x-2}{x-3}\right)^2\left(x + \dfrac{1}{x-2} + C\right)$

18. $y = Cx \pm \dfrac{a^2}{2x}$.

19. $y = 3\sqrt{x} - x$

PRACTICE PROBLEMS—G

1. (i) $\ln y = \dfrac{1}{xy} + \dfrac{1}{2x^2y^2} + c$ (ii) $x^4 + x^2y^2 + y^4 = c$

2. (i) $x^y = C$ (ii) $2e^{-x/y} + y^2 = c$

3. (i) $x^4 - x^2y^2 + y^4 = C$
(ii) $x^3 + y^3 + -x^2 + xy + y^2 = C$

4. (i) $x^2\cos y + y^2\sin x = c$ (ii) $\sqrt{x^2+y^2} = Cx$

5. (i) $\dfrac{x^3\sin^3 y}{3} = y^2\sin + c$

(ii) $x\sin y - y\cos x + \ln|xy| = C$

6. (i) $x + \tan^{-1}\dfrac{y}{x} = C$

(ii) $\sqrt{x^2+y^2} + \dfrac{y}{x} = C$

7. $\sqrt{x^2+y^2} + \ln|xy| + \dfrac{x}{y} = C$.

8. $x^3\tan y + y^4 + \dfrac{y^3}{x^2} = C$.

9. $\dfrac{\sin^2 x}{y} + \dfrac{x^3+y^2}{2} = C$

PRACTICE PROBLEMS—H

1. $y = 2x^2 + C, y = -x^2 + C$. **2.** $y = \dfrac{x^2}{2} + C$.

3. $y = Cx + \dfrac{1}{2}(x^2 - C^2), y = x^2$

4. $y = Cx + \dfrac{a}{C_2}; 4y^3 = 27ax^2$.

5. $y = Cx + C^2, y = -\dfrac{x^2}{4}$

6. $y = Cx - \dfrac{C-1}{C}; (y+1)^2 = 4x$.

7. $y = Cx + a\sqrt{1+C_2}; x^2 + y^2 = a^2$.

8. $x = Cy + C^2; 4x = -y^2$.

9. $y^2 = 5 \pm 2\sqrt{5}\,x.$ **10.** $y = Cx + \sqrt{1 + C^2}$

11. $y = Cx + \sin C$; $y = x(\pi - \cos^{-1}x) + \sqrt{1 - x^2}$.

12. $y = Cx^2 + \dfrac{1}{C}$, $y^2 - 4x^2 = 0.$

13. $x = Ce^{\pm 2\sqrt{\frac{y}{x}}}$ **14.** $xy = \pm a^2.$

15. $x^{2/3} + y^{2/3} = a^{2/3}.$

PRACTICE PROBLEMS—I

1. (i) $y = \dfrac{x^3}{6} - \sin x + C_1x + C_2$

(ii) $y = C_2 - \ln|\cos(C_1 + x)|$

(iii) $y = \dfrac{x+1}{x}$

(iv) $y = (x + C_1)\ln|x + C_1| + C_2x + C_3$

2. (i) $y = A + Be^{x/a}$ (ii) $a(y - \beta)^2 = \dfrac{4}{9}(x - a)^2$

(iii) $V = A\ln r + B$ (iv) $y = Ax^2 + B$

3. (i) $y = A\tan^{-1}x + B$ (ii) $y = \dfrac{x + A}{x + B}$

(iii) $y = \dfrac{A}{x + B}$ (iv) $y^2 = x^2 + Ax + B$

4. (i) $y = C_1e^x + C_2 - x - \dfrac{x^2}{2}$

(ii) $y = (C_1x - C_1^2)e^{\frac{x}{C_1}+1} + C_2$

(iii) $y = \dfrac{2}{3C_1}\sqrt{(C_1x - 1)^3} + C_2)$

(iv) $y = 2 + \ln\dfrac{x^2}{4}$

5. $y = -\ln|x - 1|$

6. $y = \dfrac{1}{3}(C_1 - 2x)^{3/2} + C_2x + C_3.$

PRACTICE PROBLEMS—J

1. $xy = C$ **2.** $y = (x + 1)e^x$

3. $y = 2 - (2 + a^2)\,e^{\frac{x^2 - a^2}{2}}$ **4.** $\dfrac{1}{x^3}\cdot e^{1-\frac{1}{x}}$

5. $\dfrac{dy}{dx} = -\sin x + e^{-x^2}$, $y(0) = 1$

PRACTICE PROBLEMS—K

1. (i) $xy = k$

(ii) $x^2 + \frac{1}{2}y^2 = k(k > 0)$

(iii) $x^2 - 2y^2 = c$ (iv) $2x^2 + y = 2c$

2. $x^2 + y^2 = cx$, where c is an arbitrary constant.

3. $x^2 + y^2 - C(x + y) + 2 = 0$

4. $(x^2 - y^2)^2 = Cxy$; $x^4 + 6x^2y^2 + y^4 = C.$

8. Measuring the angle a in one of the two possible directions, we get the equation of the family

$$xy - \frac{\sqrt{3}}{2}(x^2 + y^2) = C.$$

PRACTICE PROBLEMS—L

1. (A) 23,23668 (B) 13.86 years

2. ≈ 7071

3. 46.6 years

4. 3.17 hr.

5. 15219.62

6. 20.93 yrs.

7. $100 - 50e^{1/5 \ln 4/5\,(20)}$

8. (A) 113.9°F (B) 6.95 min.

9. 25 ln 3 min.

10. (A) 20 min (B) 48 kg

11. 0.467 km/h ; 85.2 m.

OBJECTIVE EXERCISES

1. A	**2.** A	**3.** A
4. B	**5.** C	**6.** A
7. B	**8.** C	**9.** B
10. B	**11.** A	**12.** A
13. C	**14.** A	**15.** A
16. A	**17.** C	**18.** A
19. D	**20.** A	**21.** B
22. B	**23.** C	**24.** A
25. D	**26.** A	**27.** A
28. C	**29.** A	**30.** D
31. B	**32.** C	**33.** B
34. B	**35.** A	**36.** B
37. A	**38.** D	**39.** D
40. D	**41.** B	**42.** A
43. A	**44.** C	**45.** A
46. C	**47.** A	**48.** A
49. C	**50.** D	**51.** A,B

52. A,C **53.** A,B,C **54.** A,C,D
55. A,B,C **56.** A,C,D **57.** A,B
58. A,B,D **59.** A,B,C **60.** A,B,D
61. A,D **62.** A,C,D **63.** A,B,C,D
64. C,D **65.** A,B **66.** A,B
67. A,B,C,D **68.** A,B **69.** A,B,D
70. A,B,C **71.** B **72.** C
73. A **74.** B **75.** A
76. B **77.** D **78.** D
79. B **80.** C **81.** C
82. B **83.** D **84.** B
85. C **86.** B **87.** B
88. C **89.** D **90.** A
91. B **92.** B **93.** D
94. A **95.** A
96. (A) $\to$ (R), (B) $\to$ (Q), (C) $\to$ (R), (D) $\to$ (Q)
97. (A) $\to$ P; (B) $\to$ P; (C) $\to$ S; (D) $\to$ R
98. (A) $\to$ Q. (B) $\to$ R, (C) $\to$ P. (D) $\to$ S
99. (A) $\to$ R; (B) $\to$ Q; (C) $\to$ Q; (D) $\to$ R
100. (A) $\to$ PQR, (B) $\to$ P, (C) $\to$ Q, (D) $\to$ QS

REVIEW EXERCISES for JEE ADVANCED

1. (i) $x\,ln\,y = e^x(x-1) + c$
(ii) $e^y = c \cdot \exp(-e^x) + e^x - 1$
(iii) $x^2 + y^3 + y\cos 2x = c$

2. (i) $x^2 + y^2 = 2a^2 \tan^{-1}\dfrac{y}{x} + C$

(ii) $xy \cos\dfrac{y}{x} = C$

(iii) $x^2 e^y [\cos y + \sin y] + 2[\cos y - \sin y] = C$

3. (i) $y(x^2-1)\sin x = (x^2-3)\sin x + 2x\cos x + c$

(ii) $y = \dfrac{e^x + C}{1 - e^x \cos x}$ (iii) $(x^2+y^2)^{3/2} = 3xy + c$

4. (i) $(x^2-1)\cos y = \dfrac{x^4}{2} - x^2 + c$

(ii) $y^4 + 2x^2y^2 + 2y^2 = C$

5. (i) $(x-y)^2 = cxy^2$

(ii) $xy + \dfrac{(xy)^5}{5} = \ln x = c$

6. $y = 1$ **7.** $y = \cot^{-1}\dfrac{1}{2x} + \dfrac{9}{4}\pi$

8. (i) $y = -\dfrac{1}{3}\sin^3 x + C_1\left(\dfrac{x}{2} - \dfrac{\sin 2x}{4}\right) + C_2$

(ii) $\dfrac{x + C_3}{2} = C_1 \tan^{-1}(C_1 \ln y),\ C_1 > 0$

9. $\tan(xy) = \cos x + \cos y + C$

10. $f(x) = x + 1$ **11.** $f(x) = e^x \ln x$

12. (A) $p(x) = -x^2 + x - 1$

13. (A) local minimum at 0

(B) $a = \dfrac{2}{3}, b = \dfrac{20}{9}$ (D) $\dfrac{2}{3}$

14. $f(x) = b^{x/a} g(x)$, where g is periodic with period a

15. $m = -1$; $y^2 \ln|y| = \dfrac{1}{2}e^{-2x} + Cy^2$

16. (A) $\dfrac{a}{x} + \dfrac{b}{y} = 1$); (B) $\dfrac{a}{x^2} + \dfrac{b}{y^2} = 1$

17. $\sqrt{x^2 + y^2} = Ce^{\pm\tan^{-1}\frac{y}{x}}$ **18.** $(x) = x^{1/n}$

19. $(x-1)^2 + (y-1)^2 = 25)$ **21.** $(y-x)^2(x+2y) = 1$

22. The parabola $(x-y)^2 - 2a(x+y) + a^2 = 0$.

23. The conics $\dfrac{x^2}{b^2 + c^2} + \dfrac{y^2}{b^2} = 1$, $\dfrac{x^2}{c^2 - b^2} - \dfrac{y^2}{b^2} = 1$

24. $\dfrac{1}{8}\left(\dfrac{3\pi}{16}\right)^4$

25. (i) $xy^3 = c$ (ii) $\sigma y^2 + 2x = c$
(iii) $\sin y = c\,e^{-x}$ (iv) $y = c\,e^{-x/2}$

26. $\dfrac{dy}{dt} = 80 - \dfrac{y}{(2000-5t)} \cdot 45$;

$y = 10(400 - t) - 3900\left(1 - \dfrac{t}{400}\right)^9$.

28. $\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt} = 4\pi r^2 \dfrac{dr}{dt} = kA = 4k\pi r^2$, 5.7 months.

29. (C) 54.5 min

(D) $T = \dfrac{1}{10k}[1 + (600 - t)k + (1400k - 1)e^{-kt}]$

30. 19.5 kg

31. (A) 82.5ºC (B) 10.2 min.

32. $Q(4) \approx 7.39$ g, $t \approx 2.63$ hr. **33.** 63.3 g, 80 min.

34. -2.94 cal/s

35. (A) $0.08\,(1-e^{-0.6}),\ 0.08(1-e^{-2.4})$,
(B) 0.08 (C) $50\log 2,\ 100\log 2$.

TARGET EXERCISES for JEE ADVANCED

1. (i) $(1+x^2)(1+y^2)=Cx^2$ (ii) $y(y-2x)^3=C(y-x)^2$

(iii) $Cx = 1-\dfrac{x}{x^2+y^2}$

2. (i) $y\left(x^2+\dfrac{1}{3}y^2\right)=C^{e^{-x}}$

(ii) $\ln|Cx| = -e^{-\frac{x^2+y^2}{x}}$

(iii) $\ln|1+y|-\dfrac{1+y}{x}=C$

3. (i) $cx = e^{-e^{-\left(\frac{x^2+y^2}{x}\right)}}$

(ii) $\ell nx-\ell ny-\left(\dfrac{1}{x}-\dfrac{1}{y}\right)=C$

4. (i) $\dfrac{7}{8}xy+\dfrac{\sin 4xy}{32}+\dfrac{x^2}{2}+C$

(ii) $[(x-1)^2+(y-2)^2]\tan^{-1}\left(\dfrac{y-2}{x-1}\right)$
$=(x-1)(y-2)+2(x-1)^2\ln C(x-1)$

5. (i) $\sin\dfrac{y}{x}-\cos\dfrac{x}{y}+x-\dfrac{1}{y}=C$

(ii) $\dfrac{2}{\sqrt5}\tan^{-1}\left(\dfrac{y}{\sin x}+\dfrac{1}{4}\right)=\ln(\sin x)+c$

6. $\dfrac{2}{\sqrt3}\tan^{-1}\left(\dfrac{2\tan\theta-1}{\sqrt3}\right)=x+c$

7. (A) $\left(\dfrac{e^x+e^{2-x}}{2x}\right)^{1/2}$ (B) $-\left(\dfrac{e^x+e^{2-x}}{2x}\right)^{1/2}$

(C) $y^2=\dfrac{\sinh x}{x}$

8. $y=\dfrac{1}{2}\left(1+\dfrac{1}{x}\right)(C-e^{-x^2})$

9. (i) $y=C_1\dfrac{x^2}{2}+C_2x+C_3-C_1^{\,2}(x+C_1)\ln|x+C_1|$

(ii) $(x+C_2)^2+(y+C_3)^2=C_1^{\,2}$

10. $2\ln|x+2|+C)$ **11.** $f(x)=\dfrac{e^{2x}-1}{e^{2x}+1}$.

12. $f(x)=\pm 10e^x$. **13.** $f(x)=-\left(\dfrac{x}{1+x}\right)$

14. $y=xT(x)+Cx$ **16.** $f(x)=1+\ln x$

17. (A) $g(3x)=3e^{2x}g(x)$ (B) $g(nx)=ne^{(n-1)x}g(x)$
(C) 2 (D) $C=2$

18. (A) $y=\dfrac{Ce^{3x}+2}{Ce^{3x}-1}$ with $C=\dfrac{b+2}{b-1}$

(B) $\dfrac{e^{3x}+2C}{e^{3x}-C}$ with $C=\dfrac{b-1}{b+2}$

19. (B) Minimum (C) $\dfrac{1}{2}$

20. Only the given function

21. Solve for dx/dt and dy/dt algebraically, and then integrate.

22. $Cx=y^{2k-1}$ $(k>1/2)$.

23. $y^2=\dfrac{x^4+c}{2x^2}$ or $y^2+2x^2\,ln x=cx^2$

24. $x^2+y^2-2cy=0$. The curve is a circle with centre on y-axis and touching the x-axis at the origin.

25. $y=x\,ln|x|;\ x^2+2xy=1$.

26. $y=cx-x\,ln|x|-2$ **27.** $y=cx\pm\dfrac{a^2}{2x}$

28. $(2y-x^2)^3=cx^2$ **29.** $y^2=4x$

30. Measuring the angle α in one of the two possible directions, we get the equation of the family

$$\ln\ (2x^2+xy+y^2)+\frac{6}{\sqrt7}\tan^{-1}\frac{x+2y}{x\sqrt7}=C.$$

32. (B) $f(x)=(x^2-L^2)/(4L)-(L/2)\,\ell n(x/L)$
(C) No

33. $25\ln 3$

34. (A) $T = (\ln n)/k$ (B) $w(t) = (b - t)/(b - a)$

35. $(3/\sqrt[3]{2} - 1) \approx 11\frac{1}{2}$ h

PREVIOUS YEAR'S QUESTIONS (JEE ADVANCED)

1. $\frac{dr}{dt} = -k$	**2.** C	**3.** B
4. A	**5.** A	**6.** C
7. C	**8.** A	**9.** C
10. A	**11.** C	**12.** C
13. A, C	**14.** C, D	**15.** A, D

17. $x^2 + y^2 = k^2$

18. $xy = 1; \sqrt{1 - xy} + 1 = x; \sqrt{1 - xy} + x = 1$

19. $y = \frac{1}{3}\tan^{-1}\left(\frac{5\tan 4x}{4 - 3\tan 4x}\right) - \frac{5x}{3}$

20. $e^{a(x-1)}$, $\frac{1}{a}\left[a - \frac{1}{2} + e^{-a}\right]$ sq. unit

21. $T = \left(\frac{\log 2}{\log\frac{4}{3}}\right)$

23. $xy = 1$

24. $x^2 + y^2 - 2x = 0$

26. $\frac{14\pi \times 10^5}{27\sqrt{g}}$ unit.

27. H/K

28. 4/3 sq. units

29. $\log\left|\frac{1 - \sqrt{1 - y^2}}{y}\right| + \sqrt{1 - y^2} = \pm x + c$

30. C

31. 9

32. 0

33. 6

34. (A) $\to$ (P), (B) $\to$ (S), (C) $\to$ (Q), (D) $\to$ (R)

35. (A) $\to$ (P), (B) $\to$ (QS), (C) $\to$ (QRST), (D) $\to$ (R)