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Monday, 7th February 1870.

DR CHRISTISON, President, in the Chair.

The following Communications were read:—

1. On Reciprocal Figures, Frames, and Diagrams of Forces. By J. Clerk Maxwell, Esq., F.R.SS. L. & E.

The reciprocal figures treated of in this paper are plane rectilinear figures, such that every line in one figure is perpendicular to the corresponding line in the other, and lines which meet in a point in one figure correspond to lines which form a closed polygon in the other.

By turning one of the figures round 90°, the corresponding lines become parallel, and are more easily recognised. The practical use of these figures depends on the proposition known as the "Polygon of Forces." If we suppose one of the reciprocal figures to represent a system of points acted on by tensions or pressures along the lines of the figure, then, if the forces which act along these lines are represented in magnitude, as they are in direction, by the corresponding lines of the other reciprocal figure, every point of the first figure will be in equilibrium. For the forces which act at that point are parallel and proportional to the sides of a polygon formed by the corresponding lines in the other figure.

In all cases, therefore, in which one of the figures represents a frame, or the skeleton of a structure which is in equilibrium under the action of pressures and tensions in its several pieces, the other figure represents a system of forces which would keep the frame in equilibrium; and, if the known data are sufficient to determine these forces, the reciprocal figure may be drawn so as to represent, on a selected scale, the actual values of all these forces.

In this way a practical method of determining the tensions and pressures in structures has been developed. The "polygon of forces" has been long known. The application to polygonal frames, with a system of forces acting on the angles, and to several other cases, was made by Professor Rankine in his Applied Mechanics. Mr W. P. Taylor, a practical draughtsman, has independently worked out more extensive applications of the method. Starting from Professor Rankine's examples, I taught the method to the class of Applied Mechanics in King's College, London, and published a short account of it in the "Philosophical Magazine" for April 1864. Professor Fleeming Jenkin, in a paper recently presented to the Society, has fully explained the application of the method to the most important cases occurring in practice, and I believe that it has been found to have three important practical advantages. It is easily taught to any person who can use a ruler and scale. It is quite sufficiently accurate for all ordinary calculations, and is much more rapid than the trigonometrical method. When the figure is drawn the whole process remains visible, so that the accuracy of the drawing of any single line can be afterwards tested; and if any mistake has been made, the figure cannot be completed. Hence the verification of the process is much easier than that of a long series of arithmetical operations, including the use of trigonometric tables.

In the present paper I have endeavoured to develope the idea of reciprocal figures, to show its connection with the idea of reciprocal polars as given in pure mathematics, and to extend it to figures in three dimensions, and to cases in which the stresses, instead of being along certain lines only, are distributed continuously throughout the interior of a solid body. In making this extension of the theory of reciprocal figures, I have been led to see the connection of this theory with that of the very important function introduced into the theory of stress in two dimensions by Mr Airy, in his paper "On the Strains in the Interior of Beams" (Phil. Trans. 1863).

If a plane sheet is in equilibrium under the action of internal stress of any kind, then a quantity, which we shall call Airy's Function of Stress, can always be found, which has the following properties.

At each point of the sheet let a perpendicular be erected proportional to the function of stress at that point, so that the extremities of such perpendiculars lie in a certain surface, which we may call the surface of stress. In the case of a plane frame the surface of stress is a plane-faced polyhedron, of which the frame is the projection. On another plane, parallel to the sheet, let a perpendicular be erected of height unity, and from the extremity of this perpendicular let a line be drawn normal to the tangent plane at a point of the surface of stress, and meeting the plane at a certain point.

Thus, if points be taken in the plane sheet, corresponding points may be found by this process in the other plane, and if both points are supposed to move, two corresponding lines will be drawn, which have the following property:—The resultant of the whole stress exerted by the part of the sheet on the right hand side of the line on the left hand side, is represented in direction and magnitude by the line joining the extremities of the corresponding line in the other figure. In the case of a plane frame, the corresponding figure is the reciprocal diagram described above.

From this property the whole theory of the distribution of stress in equilibrium in two dimensions may be deduced.

In the most general case of three dimensions, we must use three such functions, and the method becomes cumbrous. I have, however, used these functions in forming equations of equilibrium of elastic solids, in which the stresses are considered as the quantities to be determined, instead of the displacements, as in the ordinary form.

These equations are especially useful in the cases in which we wish to determine the stresses in uniform beams. The distribution of stress in such cases is determined, as in all other cases, by the elastic yielding of the material; but if this yielding is small and the beam uniform, the stress at any point will be the same, whatever be the actual value of the elasticity of the substance.

Hence the coefficients of elasticity disappear from the ultimate values of the stresses.

In this way I have obtained values for the stresses in a beam supported in a specified way, which differ only by small quantities from the values obtained by Mr Airy, by a method involving certain assumptions, which were introduced in order to avoid the consideration of elastic yielding.

2. On the Extension of Brouncker's Method. By Edward Sang, Esq.

The operation in use by the ancient geometers for finding the numerical expression for the ratio of two quantities, was to repeat each of them until some multiple of the one agreed with a multiple of the other; the numbers of the repetitions being inversely proportional to the magnitudes.

The modern process, introduced by Lord Brouncker, under the name of continued fractions, is to seek for that submultiple of the one which may be contained exactly in the other; the numbers being then directly proportional to the quantities compared.

On applying this method to the roots of quadratic equations, the integer parts of the denominators were found to recur in periods; and Lagrange showed that, while all irrational roots of quadratics give recurring chain-fractions, every recurring chain-fraction expresses the root of a quadratic; and hence it was argued that this phenomenon of recurrence is exhibited by quadratic equations alone.

The author of this paper had supplemented Lagrange's proposition, by showing that when the progression of fractions converging to one root of a quadratic is continued backwards, the convergence is toward the other root. The singularity of this exclusive property of quadratic equations led him to consider whether some analogous property may not be possessed by equations of higher degrees. Putting aside the idea of the chain-fraction as being merely accidental to the subject, and attending to the series of converging fractions, he came upon a kind of recurrence which extends to equations of all orders; and which proceeds by operating on two, three, or more contiguous terms according to the rank of the equation. In this way a ready means of approximating to the greatest and to the least root of any equation was obtained.

The following cases were cited:—

If we begin with the terms $\frac{1}{0}$, $\frac{1}{1}$, and form a progression by