By A. Hall.

T	1859, May $29^d{\cdot}0077$	Washington M.S.T.	
Log q	$9{\cdot}303310$		
ω	$281°\ 58'\ 10''{\cdot}7$	or $\pi =$	$75°\ 9'\ 46''{\cdot}1$
☊	$357\ \ 7\ 56{\cdot}8$		
i	$95\ 50\ 56{\cdot}8$	$\iota =$	$84\ \ 9\ \ 3{\cdot}2$

Motion Retrograde.

The comet will probably be visible after its perihelion passage.

On the Stability of the Motion of Saturn's Rings; an Essay which obtained the Adams' Prize for the Year 1856, *in the University of Cambridge.* By J. Clerk Maxwell, M.A. late Fellow of Trinity College, Cambridge: Professor of Natural Philosophy in the Marischal College and University of Aberdeen. Cambridge: Macmillan and Co., 1859.

The following abstract of an important paper has been kindly drawn up by the Astronomer Royal for the use of the readers of the *Monthly Notices:*—

The remarkable essay of which we have given the title was published in the beginning of the present year. The subject of it is so interesting, the difficulty of treating it in its utmost generality so considerable, and the results at which the author arrives so curious, that we think a brief abstract of it will be acceptable to the readers of the *Monthly Notices.* We shall commence with a very imperfect reference to preceding investigations on the same subject.

The first to which we shall allude is Laplace's, in the *Mécanique Céleste,* livre III. chapitre vi. Laplace considers a ring of *Saturn* as a solid, the form of which is investigated as if it were fluid (a mode of treatment whose result, in respect of the form of equilibrium, is evidently good for a solid), and finds, that if the breadth and thickness of the ring are very small in comparison with its distance from *Saturn,* its section may be an ellipse; and it appears that the formula for the proportion of the axes of the ellipse admits of its being considerably flattened. But Laplace rather inclines to the supposition that there are several rings, each existing by its own proper theory. Then remarking on the appearances noticed by some observers which seem to indicate irregularities in the rings, he adds, "J'ajoute que ces inégalités sont nécessaires pour maintenir l'anneau en équilibre autour de *Saturne,*" and gives an investigation which shows that, if the ring were rigid and uniform, the slightest disturbance would cause it to fall to the planet. Then he concludes thus, "Les divers anneaux qui entourent le globe de *Saturne* sont, par conséquent, des solides irreguliers d'une largeur inégale dans les différents points de leurs circonférences," &c. And in this state he leaves the theory.

Probably no competent mathematician has ever read this paper without remarking the want of ground for the last conclusion. Professor Maxwell has well given the following as the conclusion that ought to have been drawn: "If the rings were solid and uniform, their motion would be unstable, and they would be destroyed. But they are not destroyed, and their motion is stable; therefore they are either not uniform or not solid."

Omitting notices of some remarks by Plana and others which imply no important departure from Laplace's theory, we come to a paper by Mr. G. P. Bond, in the *Astronomical Journal*, Nos. 25 and 26. In No. 25 Mr. Bond gives arguments from observation tending to show that there are several rings. He then gives investigations, in a great measure similar to Laplace's, but leading more distinctly to a determination of the limits of density and dimensions of any ring: he finds that the number of rings must be considerable, and even makes calculations for eleven rings. He then alludes (without further remark or calculation) to Laplace's idea that a ring, if rigid, must be an irregular solid. And, finally, he gives his opinion that a fluid symmetrical ring is not necessarily unstable.

In No. 27 Professor Benjamin Peirce has commented upon Mr. Bond's paper. He "maintains, unconditionally, that there is no conceivable form of irregularity, and no combination of irregularities consistent with an actual ring, which would serve to retain it permanently about the primary if it were solid." He then refers to the theory of a fluid ring, and asserts that "there is nothing in the direct action of *Saturn* to prevent his ring from moving forward in its plane, in any direction and to any distance, until at last the rings would be brought into collision with the surface of the planet, and so be destroyed." Subsequently "the power which sustains the centre of gravity of *Saturn's* ring is not then to be sought in the planet itself, but in his satellites," and afterwards, "It follows, then, that no planet can have a ring, unless it is surrounded by a sufficient number of properly arranged satellites." This paper, it must be remarked, contains no symbolical investigations. Perhaps many mathematicians will think that a very complete mathematical treatment will be required to establish the two laws, that the attraction of *Saturn* will not support a fluid ring, and that the attraction of satellites will tend to support it.

In the same Journal, No. 86, is a paper of mathematical character on the same subject by Professor Pierce. But after some investigation, skilfully based on the method of potentials, as to the rotation of a solid ring, he refers merely to general reasoning in order to establish that "the solid ring must be excluded from any physical theory which rests upon a firm basis." He adds, however, "the conditions of permanence would lose the unstable element, if the two successive normals to the level surface of the rings [*quære*, to the surface of equal potentials] at the centre of *Saturn* intersected each other between the centre of *Saturn* and the centre of gravity of the ring." It seems possible that this may refer to the case of a very large protuberance at one point of the ring; but the interpretation of the expression as it stands is somewhat obscure. The paper then proceeds to treat of a fluid ring, and terminates with, "The attractions of the opposite elements of the ring upon *Saturn* balance each other, and the ring has no tendency to move *Saturn*, neither has *Saturn* any tendency to give a motion of translation to the ring, or to check a translation which the ring may have received from other causes. Hence the ring will yield to the slightest foreign action, and may pass through successive normal forms of possible equilibrium without being restrained by the action of *Saturn*. The fluid ring cannot then be regarded as one of real permanence without the aid of foreign support, although the action of the primary is not positively destructive to this, as it is to the solid ring." The inference that "the fluid ring cannot be regarded as one of real permanence without the aid of foreign support," is, we presume, to be limited strictly by the preceding sentence. Thus it permits one elliptic form to be changed, by foreign action, to another elliptic form, each being permanent so long as it is not disturbed by further foreign action. The equations and the conclusions as to attraction are exactly the same as they would be for a ring of satellites not perturbing each other. In any case, we do not perceive the advantage of "foreign support." The author, however, has spoken so briefly on the subject that it is possible that we have mistaken his meaning. An expectation was held out that this paper would be continued, but, so far as we are aware, no continuation has yet appeared.

Professor Maxwell commences his treatise with an abstract of Laplace's principal conclusions. He then states that he has confined himself to those parts of the subject which bear upon the question of the permanence of a given form of motion, and adds,

"There is a very general and very important problem in dynamics, the solution of which would contain all the results of this essay and a great deal more. It is this, 'Having found a particular solution of the equations of motion of any material system, to determine whether a slight disturbance of the motion indicated by the solution would cause a small periodic variation, or a total derangement of the motion.' The question may be made to depend upon the conditions of a maximum or a minimum of a function of many variables, but the theory of the tests for distinguishing maxima from minima by the calculus of variations becomes so intricate when applied to functions of several variables, that I think it doubtful whether the physical or the abstract problem will be first solved."

Part I. is headed "On the motion of a rigid body of any form about a sphere," attention being confined to the motion in the plane of reference. Equations being formed (founded on the method of potentials), they are then employed in the following propositions.

Prob. 1. "To find the conditions under which a uniform motion of the ring is possible."

Prob. 2. "To find the equations of the motion when slightly disturbed." Symbols for the disturbance of co-ordinates are attached to the symbols for co-ordinates found by the last problem, the new symbols being supposed liable to change with t, and a series of linear differential equations of the second order is formed.

Prob. 3. "To reduce the three simultaneous equations of motion to the form of a single equation."

Prob. 4. "To determine whether the motion of the ring is stable or unstable, by means of the relations of the co-efficients A, B, C, [introduced in prob. 3]."

Problems 3 and 4 give the key of the author's general method. In problem 3, the symbols of operation and of quantity are separated, n being then written for $\frac{d}{dt}$. (This amounts to the same as assuming that the variation of any element may be represented by $\varepsilon^{nt+\alpha}$.) Equations are thus formed which exhibit no differential co-efficients. Then it is remarked that, supposing the value of n found, it will consist generally of the sum of a real and an imaginary term. If the imaginary term $= 0$, the variations of elements will increase, with the time, indefinitely and without change of sign. If both terms have values, the variations of elements will be periodical, with constantly increasing co-efficients. If the real term $= 0$, the variations of elements will be periodical, with constant co-efficients. The last-mentioned case is evidently the only one which is compatible with the idea of stability. The relations of the co-efficients A, B, C, are therefore to be ascertained, which will insure that the values of n be purely imaginary. We commend these propositions to the study of the reader, as an interesting example of a beautiful method, applied with great skill to the solution of the difficult problems which follow.

Prob. 5 determines certain quantities required in the application of the theory to a rigid ring of constant or variable section: and Prob. 6 then takes up the question, "To determine the conditions of stability of the motion in terms of the coefficients f, g, h, [found in the last problem], which indicate the distribution of mass in the ring." On applying this to specific suppositions, it is found (1) that a ring of uniform section is unstable; (2) that a ring, thicker on one side than the other, and varying in section according to the simple law of sines, is unstable; (3) that a uniform ring, loaded with a heavy particle at a point of its circumference, may be stable, provided f be less than ·8279 and greater than ·8159 ($2f$ being the coefficient of cos θ in the expansion of the expression for the section of the ring at every point by Fourier's theorem). On calculating the case of $f = ·82$, it appears that the mass of the heavy particle is to the mass of the ring as 82 to 18. Such a law of structure evidently has no counterpart in *Saturn's* rings. It appears also that, in the case supposed, every variable co-ordinate may then be subject to two periodic variations, the period of one being 1·69 revolutions of the ring, and that of the other being 3·25 revolutions of the ring.

It appears then that, though a rigid ring is not impossible, yet it is impossible that *Saturn's* rings can be rigid rings.

Part II. treats "On the motion of a Ring, the parts of which are not rigidly connected."

The ring considered is small in section, and nearly circular and uniform, and revolving with nearly uniform velocity. The variations from circular form, uniform section, and uniform velocity, are expressed by an appropriate notation. The first step is "To express the position of an element of a variable ring at a given time in terms of the original position of the element in the ring." Fourier's theorem being applied to this, the problem becomes one of treating a number of separate variations, each of which is expressed by a circular function. Then the investigation is proposed "To find the magnitude and direction of the attraction between two elements of a disturbed ring." Having effected this, the author remarks, "We have found the expressions for the forces which act upon each member of a system of equal satellites, which originally formed a uniform ring, but are now affected with displacements depending on circular functions. If these displacements can be propagated round the ring in the form of waves with the velocity $\frac{m}{n}$, the quantities under the brackets will depend on t, and the complete expressions will be

$$\begin{aligned} \varrho &= \mathrm{A} \cos (ms + nt + \alpha) \\ \sigma &= \mathrm{B} \sin (ms + nt + \beta) \\ \zeta &= \mathrm{C} \cos (ms + nt + \gamma) \end{aligned}$$

[ϱ, σ, ζ, being perturbations of three elements of the particle's place]. Let us find in what cases expressions such as these will be true, and what will be the result when they are not true."

The conditions are shown to depend upon the possible or imaginary character of roots of an equation, nearly as in the first section; and the author soon arrives at this conclusion: "So that a ring of satellites can always be rendered stable by increasing the mass of the central body and the angular velocity of the ring." As an instance, supposing that there are 100 satellites, then the mass of the planet must not be less than 4352 times the entire mass of the ring.

Then the problem is considered "To determine the nature of the motion when the system of satellites is of small mass compared with the central body." After treating it generally, the author considers,—*First*, the motion of a single satellite with reference to its mean or undisturbed place; and shows that the motion is elliptic, the centre of the ellipse being at the mean place of the satellite, the major axis being in the tangential direction, the time of revolution being different from the time of revolution of the ring, and the direction of revolution being opposite to that of the ring. *Secondly*, the condition of the ring of satellites at a given instant: the author shows that the form of the ring at any instant is that of a string of beads, arranged as a circle affected by m regular waves of transversal displacement at equal intervals round the circle: besides these, there are waves of condensation and rarefaction, in which, according to the relation of coefficients, the points of greatest distance from the centre may be points of greatest or of least condensation. *Thirdly*, the condition of the ring of satellites through a duration of time: it is shown that the waves in question travel, with a motion which, in regard to the ring itself, is opposite to the ring's revolution.

There is then given a solution of the general problem, "From the state and motion of the satellites at one time, to find the state and motion at any future time."

The next problem is, "To consider the effect of an external disturbing force." A result soon reached is, "If the absolute angular velocity of the disturbing body is exactly or nearly equal to the absolute angular velocity of any of the free waves of the ring, that wave will increase till the ring be destroyed."

For the understanding of further results, it is almost necessary to refer to the theory of tides and waves. It is known in that theory that the action of the moon produces on the ocean waves whose periods and lengths are both dependent on the moon alone (called *forced* waves), and that these are or may be accompanied with *free* waves, whose periods depend on the moon but whose lengths do not depend on the moon or whose lengths depend on the moon but whose periods do not depend on the moon; and also with other free waves, entirely independent of the moon. And thus in the ring of satellites there will be forced waves and free waves.

The problem is then considered, "On the motion of a ring of satellites, when the conditions of stability are not fulfilled;" and the directions in which they tend to depart from their mean positions are found.

Satellites of unequal mass, it is considered, may be so grouped in our contemplation of their movements as to be treated nearly in the same manner as equal satellites.

The case of a fluid ring is then considered, and the conclusion arrived at is this:—"It appears, therefore, that a ring composed of a continuous liquid mass cannot revolve about a central body without being broken up, but that the parts of such a broken ring may, under certain conditions, form a permanent ring of satellites."

The author then treats "On the mutual perturbations of two rings." In this investigation arises a most curious complication of eight waves in each ring, with corresponding periods for the waves of the two rings, but with different relations of the coefficients in the two rings.

The author alludes shortly to "the effect of long-continued disturbances on a system of rings," and also to "the loss of energy due to internal friction in a broad fluid ring, the parts of which revolve about the planet, each with the velocity of a satellite at the same distance," which he concludes to be insensible.

The most important paragraph in the author's recapitulation is the following:—

"The final result, therefore, of the mechanical theory is, that the only system of rings which can exist is one composed of an indefinite number of unconnected particles, revolving round the planet with different velocities according to their respective distances. These particles may be arranged in series of narrow rings, or they may move through each other irregularly. In the first case, the destruction of the system will be very slow; in the second case, it will be more rapid; but there may be a tendency towards an arrangement in narrow rings, which may retard the process.

"We are not able to ascertain by observation the constitution of the two outer divisions of the system of rings; but the inner ring is certainly transparent, for the limb of *Saturn* has been observed through it. It is also certain that, though the space occupied by the ring is transparent, it is not through the material parts of it that *Saturn* was seen, for his limb was observed without distortion; which shows that there was no refraction, and therefore that the rays did not pass through a medium at all, but between the solid or liquid particles of which the ring is composed. Here, then, we have an optical argument in favour of the theory of independent particles as the material of the rings. The two outer rings may be of the same nature, but not so exceedingly rare that a ray of light can pass through their whole thickness without encountering one of the particles."

We have omitted allusion to a geometrical representation, by mechanism, of a disturbed ring of satellites, and also of a theorem by Professor W. Thomson, which, being expressed in a different notation, cannot without difficulty be compared with the investigations of Professor Maxwell.

The abstract which we have given will, we think, fully justify the opinion that the theory of *Saturn's* rings is now placed on a footing totally different from any that it has occupied before, and that the essay which we have abstracted is one of the most remarkable contributions to mechanical astronomy that has appeared for many years.

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