IV.—*On the Equilibrium of Elastic Solids.* By JAMES CLERK MAXWELL, Esq.

(Read 18th February, 1850).

There are few parts of mechanics in which theory has differed more from experiment than in the theory of elastic solids.

Mathematicians, setting out from very plausible assumptions with respect to the constitution of bodies, and the laws of molecular action, came to conclusions which were shewn to be erroneous by the observations of experimental philosophers. The experiments of ŒRSTED proved to be at variance with the mathematical theories of NAVIER, POISSON, and LAMÉ and CLAPEYRON, and apparently deprived this practically important branch of mechanics of all assistance from mathematics.

The assumption on which these theories were founded may be stated thus:—

Solid bodies are composed of distinct molecules, which are kept at a certain distance from each other by the opposing principles of attraction and heat. When the distance between two molecules is changed, they act on each other with a force whose direction is in the line joining the centres of the molecules, and whose magnitude is equal to the change of distance multiplied into a function of the distance which vanishes when that distance becomes sensible.

The equations of elasticity deduced from this assumption contain only *one* coefficient, which varies with the nature of the substance.

The insufficiency of one coefficient may be proved from the existence of bodies of different degrees of solidity.

No effort is required to retain a liquid in any form, if its volume remain unchanged; but when the form of a solid is changed, a force is called into action which tends to restore its former figure; and this constitutes the difference between elastic solids and fluids. Both tend to recover their *volume*, but fluids do not tend to recover their *shape.*

Now, since there are in nature bodies which are in every intermediate state from perfect solidity to perfect liquidity, these two elastic powers cannot exist in every body in the same proportion, and therefore all theories which assign to them an invariable ratio must be erroneous.

I have therefore substituted for the assumption of NAVIER the following axioms as the results of experiments.

If three pressures in three rectangular axes be applied at a point in an elastic solid,—

1. *The sum of the three pressures is proportional to the sum of the compressions which they produce.*

2. *The difference between two of the pressures is proportional to the difference of the compressions which they produce.*

The equations deduced from these axioms contain two coefficients, and differ from those of NAVIER only in not assuming any invariable ratio between the cubical and linear elasticity. They are the same as those obtained by Professor STOKES from his equations of fluid motion, and they agree with all the laws of elasticity which have been deduced from experiments.

In this paper *pressures* are expressed by the number of units of weight to the unit of surface; if in English measure, in pounds to the square inch, or in atmospheres of 15 pounds to the square inch.

Compression is the proportional change of any dimension of the solid caused by pressure, and is expressed by the quotient of the change of dimension divided by the dimension compressed.*

Pressure will be understood to include tension, and compression dilatation; pressure and compression being reckoned positive.

Elasticity is the force which opposes pressure, and the *equations of elasticity* are those which express the relation of pressure to compression.†

Of those who have treated of elastic solids, some have confined themselves to the investigation of the laws of the bending and twisting of rods, without considering the relation of the coefficients which occur in these two cases; while others have treated of the general problem of a solid body exposed to any forces.

The investigations of LEIBNITZ, BERNOULLI, EULER, VARIGNON, YOUNG, LA HIRE, and LAGRANGE, are confined to the equilibrium of bent rods; but those of NAVIER, POISSON, LAMÉ and CLAPEYRON, CAUCHY, STOKES, and WERTHEIM, are principally directed to the formation and application of the general equations.

The investigations of NAVIER are contained in the seventh volume of the *Memoirs of the Institute*, page 373; and in the *Annales de Chimie et de Physique*, 2e *Série*, xv., 264, and xxxviii., 435; *L'Application de la Mécanique*, tom. i.

Those of POISSON in *Mém. de l'Institut*, viii., 429; *Annales de Chimie*, 2e *Série*, xxxvi., 334; xxxvii., 337; xxxviii., 338; xlii. *Journal de l'École Polytechnique*, cahier xx., with an abstract in *Annales de Chimie* for 1829.

The memoir of MM. LAMÉ and CLAPEYRON is contained in CRELLE's *Mathematical Journal*, vol. vii.; and some observations on elasticity are to be found in LAMÉ's *Cours de Physique.*

M. CAUCHY's investigations are contained in his *Exercises de Analyse*, vol. iii., p. 180, published in 1828.

Instead of supposing each pressure proportional to the linear compression which it produces, he supposes it to consist of two parts, one of which is propor-

* The laws of pressure and compression may be found in the Memoir of Lamé and Clapeyron. See note A.

† See note B.

tional to the linear compression in the direction of the pressure, while the other is proportional to the diminution of volume. As this hypothesis admits two coefficients, it differs from that of this paper only in the values of the coefficients selected. They are denoted by K and k, and $K=\mu-\frac{1}{3}m$, $k=m$.

The theory of Professor STOKES is contained in Vol. viii., Part 3, of the *Cambridge Philosophical Transactions*, and was read April 14, 1845.

He states his general principles thus:—"The capability which solids possess of being put into a state of isochronous vibration, shews that the pressures called into action by small displacements depend on homogeneous functions of those displacements of one dimension. I shall suppose, moreover, according to the general principle of the superposition of small quantities, that the pressures due to different displacements are superimposed, and, consequently, that the pressures are linear functions of the displacements."

Having assumed the proportionality of pressure to compression, he proceeds to define his coefficients.—"Let $-A\delta$ be the pressures corresponding to a uniform linear dilatation δ when the solid is in equilibrium, and suppose that it becomes $m A \delta$, in consequence of the heat developed when the solid is in a state of rapid vibration. Suppose, also, that a displacement of shifting parallel to the plane xy, for which $\delta x=kx$, $\delta y=-ky$, and $\delta z=0$, calls into action a pressure $-Bk$ on a plane perpendicular to the axis of x, and a pressure Bk on a plane perpendicular to the axis of y; the pressure on these planes being equal and of contrary signs; that on a plane perpendicular to z being zero, and the tangential forces on those planes being zero." The coefficients A and B, thus defined, when expressed as in this paper, are $A=3\mu$, $B=\frac{m}{2}$.

Professor STOKES does not enter into the solution of his equations, but gives their results in some particular cases.

1. A body exposed to a uniform pressure on its whole surface.
2. A rod extended in the direction of its length.
3. A cylinder twisted by a statical couple.

He then points out the method of finding A and B from the two last cases.

While explaining why the equations of motion of the luminiferous ether are the same as those of incompressible elastic solids, he has mentioned the property of *plasticity* or the tendency which a constrained body has to relieve itself from a state of constraint, by its molecules assuming new positions of equilibrium. This property is opposed to linear elasticity; and these two properties exist in all bodies, but in variable ratio.

M. WERTHEIM, in *Annales de Chimie*, 3e *Série*, xxiii., has given the results of some experiments on caoutchouc, from which he finds that $K=k$, or $\mu=\frac{4}{3}m$; and concludes that $k=K$ in all substances. In his equations, μ is therefore made equal to $\frac{4}{3}m$.

The accounts of experimental researches on the values of the coefficients are so numerous that I can mention only a few.

CANTON, PERKINS, ŒRSTED, AIMÉ, COLLADON and STURM, and REGNAULT, have determined the cubical compressibilities of substances; COULOMB, DULEAU, and GIULIO, have calculated the linear elasticity from the torsion of wires; and a great many observations have been made on the elongation and bending of beams.

I have found no account of any experiments on the relation between the doubly refracting power communicated to glass and other elastic solids by compression, and the pressure which produces it;* but the phenomena of bent glass seem to prove, that, in homogeneous singly-refracting substances exposed to pressures, the principal axes of pressure coincide with the principal axes of double refraction; and that the difference of pressures in any two axes is proportional to the difference of the velocities of the oppositely polarised rays whose directions are parallel to the third axis. On this principle I have calculated the phenomena seen by polarised light in the cases where the solid is bounded by parallel planes.

In the following pages I have endeavoured to apply a theory identical with that of STOKES to the solution of problems which have been selected on account of the possibility of fulfilling the conditions. I have not attempted to extend the theory to the case of imperfectly elastic bodies, or to the laws of permanent bending and breaking. The solids here considered are supposed not to be compressed beyond the limits of perfect elasticity.

The equations employed in the transformation of co-ordinates may be found in GREGORY'S *Solid Geometry*.

I have denoted the displacements by δx, δy, δz. They are generally denoted by α, β, γ; but as I had employed these letters to denote the principal axes at any point, and as this had been done throughout the paper, I did not alter a notation which to me appears natural and intelligible.

The laws of elasticity express the relation between the changes of the dimensions of a body and the forces which produce them.

These forces are called Pressures, and their effects Compressions. Pressures are estimated in pounds on the square inch, and compressions in fractions of the dimensions compressed.

Let the position of material points in space be expressed by their co-ordinates x, y, and z, then any change in a system of such points is expressed by giving to these co-ordinates the variations δx, δy, δz, these variations being functions of x, y, z.

The quantities δx, δy, δz, represent the absolute motion of each point in the directions of the three co-ordinates; but as compression depends not on absolute, but on relative displacement, we have to consider only the nine quantities—

* See note C.

$$\frac{d\,\delta x}{dx} \quad \frac{d\,\delta x}{dy} \quad \frac{d\,\delta x}{dz}$$
$$\frac{d\,\delta y}{dx} \quad \frac{d\,\delta y}{dy} \quad \frac{d\,\delta y}{dz}$$
$$\frac{d\,\delta z}{dx} \quad \frac{d\,\delta z}{dy} \quad \frac{d\,\delta z}{dz}$$

Since the number of these quantities is nine, if nine other independent quantities of the same kind can be found, the one set may be found in terms of the other. The quantities which we shall assume for this purpose are—

1. Three compressions, $\frac{\delta\alpha}{\alpha}, \frac{\delta\beta}{\beta}, \frac{\delta\gamma}{\gamma}$, in the directions of three principal axes α, β, γ.

2. The nine *direction-cosines* of these axes, with the *six connecting equations*, leaving three independent quantities. (See GREGORY'S *Solid Geometry*).

3. The small angles of rotation of this system of axes about the axes of x, y, z,

The cosines of the angles which the axes of x, y, z make with those of α, β, γ are—

$$\cos(\alpha\,0\,x)=a_1,\ \cos(\beta\,0\,x)=b_1,\ \cos(\gamma\,0\,x)=c_1,$$
$$\cos(\alpha\,0\,y)=a_2,\ \cos(\beta\,0\,y)=b_2,\ \cos(\gamma\,0\,y)=c_2,$$
$$\cos(\alpha\,0\,z)=a_3,\ \cos(\beta\,0\,z)=b_3,\ \cos(\gamma\,0\,z)=c_3,$$

These *direction-cosines* are connected by the six equations,

$$a_1^2+b_1^2+c_1^2=1 \qquad a_1a_2+b_1b_2+c_1c_2=0$$
$$a_2^2+b_2^2+c_2^2=1 \qquad a_2a_3+b_2b_3+c_2c_3=0$$
$$a_3^2+b_3^2+c_3^2=1 \qquad a_3a_1+b_3b_1+c_3c_1=0$$

The rotation of the system of axes α, β, γ, round the axis of

$$x,\ \text{from } y \text{ to } z,\ =\delta\theta_1,$$
$$y,\ \text{from } z \text{ to } x,\ =\delta\theta_2,$$
$$z,\ \text{from } x \text{ to } y,\ =\delta\theta_3;$$

By resolving the displacements $\delta\alpha, \delta\beta, \delta\gamma, \theta_1, \theta_2, \theta_3$, in the directions of the axes x, y, z, the displacements in these axes are found to be

$$\delta x = a_1\,\delta\alpha + b_1\,\delta\beta + c_1\,\delta\gamma - \theta_2 z + \theta_3 y$$
$$\delta y = a_2\,\delta\alpha + b_2\,\delta\beta + c_2\,\delta\gamma - \theta_3 x + \theta_1 z$$
$$\delta z = a_3\,\delta\alpha + b_3\,\delta\beta + c_2\,\delta\gamma - \theta_1 y + \theta_2 x$$

But $$\delta\alpha = \alpha\frac{\delta\alpha}{\alpha},\ \delta\beta = \beta\frac{\delta\beta}{\beta},\ \text{and } \delta\gamma = \gamma\frac{\delta\gamma}{\gamma},$$

and $$\alpha=a_1x+a_2y+a_3z,\ \beta=b_1x+b_2y+b_3z,\ \text{and } \gamma=c_1x+c_2y+c_3z.$$

Substituting these values of $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ in the expressions for δx, δy,

δz, and differentiating with respect to x, y, and z, in each equation, we obtain the equations—

$$\left.\begin{aligned}
\frac{d\,\delta x}{d x} &= \frac{\delta \alpha}{\alpha} a_1^2 + \frac{\delta \beta}{\beta} b_1^2 + \frac{\delta \gamma}{\gamma} c_1^2 \\
\frac{d\,\delta y}{d y} &= \frac{\delta \alpha}{\alpha} a_2^2 + \frac{\delta \beta}{\beta} b_2^2 + \frac{\delta \gamma}{\gamma} c_2^2 \\
\frac{d\,\delta z}{d z} &= \frac{\delta \alpha}{\alpha} a_3^2 + \frac{\delta \beta}{\beta} b_3^2 + \frac{\delta \gamma}{\gamma} c_3^2
\end{aligned}\right\} \quad (1).$$

Equations of compression.

$$\left.\begin{aligned}
\frac{d\,\delta x}{d y} &= \frac{\delta \alpha}{\alpha} a_1 a_2 + \frac{\delta \beta}{\beta} b_1 b_2 + \frac{\delta \gamma}{\gamma} c_1 c_2 + \delta \theta_3 \\
\frac{d\,\delta x}{d z} &= \frac{\delta \alpha}{\alpha} a_1 a_3 + \frac{\delta \beta}{\beta} b_1 b_3 + \frac{\delta \gamma}{\gamma} c_1 c_3 - \delta \theta_2 \\
\frac{d\,\delta y}{d z} &= \frac{\delta \alpha}{\alpha} a_2 a_3 + \frac{\delta \beta}{\beta} b_2 b_3 + \frac{\delta \gamma}{\gamma} c_2 c_3 + \delta \theta_1 \\
\frac{d\,\delta y}{d x} &= \frac{\delta \alpha}{\alpha} a_2 a_1 + \frac{\delta \beta}{\beta} b_2 b_1 + \frac{\delta \gamma}{\gamma} c_2 c_1 - \delta \theta_3 \\
\frac{d\,\delta z}{d x} &= \frac{\delta \alpha}{\alpha} a_3 a_1 + \frac{\delta \beta}{\beta} b_3 b_1 + \frac{\delta \gamma}{\gamma} c_3 c_1 + \delta \theta_2 \\
\frac{d\,\delta z}{d y} &= \frac{\delta \alpha}{\alpha} a_3 a_2 + \frac{\delta \beta}{\beta} b_3 b_2 + \frac{\delta \gamma}{\gamma} c_3 c_2 - \delta \theta_1
\end{aligned}\right\} \quad (2).$$

Equations of the equilibrium of an element of the solid.

The forces which may act on a particle of the solid are:—

1. Three attractions in the direction of the axes, represented by X, Y, Z.
2. Six pressures on the six faces.
3. Two tangential actions on each face.

Let the six faces of the small parallelopiped be denoted by x_1, y_1, z_1, x_2, y_2, and z_2, then the forces acting on x_1 are:—

1. A normal pressure p_1 acting in the direction of x on the area $d y\, d z$.
2. A tangential force q_3 acting in the direction of y on the same area.
3. A tangential force $q_2{}^1$ acting in the direction of z on the same area, and so on for the other five faces, thus:—

Forces which act in the direction of the axes of

	x	y	z
On the face x_1	$-p_1\, d y\, d z$	$-q_3\, d y\, d z$	$-q_2{}^1\, d y\, d z$
... ... x_2	$(p_1 + \frac{d p_1}{d x} d x)\, d y\, d z$	$(q_3 + \frac{d q_3}{d x} d x)\, d y\, d x$	$(q_2{}^1 + \frac{d q_2{}^1}{d x} d x)\, d y\, d z$
... ... y_1	$-q_3{}^1\, d z\, d x$	$-p_2\, d z\, d x$	$-q_1\, d z\, d x$
... ... y_2	$(q_3 + \frac{d q_3{}^1}{d y} d y)\, d z\, d x$	$(p_2 + \frac{d p_2}{d y} d y)\, d z\, d x$	$(q_1 + \frac{d q_1}{d y} d y)\, d z\, d x$

On the face z_1	$-q_2\, dx\, dy$	$-q^1\, dx\, dy$	$-p_3\, dx\, dy$
... ... z_2	$(q_2 + \frac{dq_2}{dz} dz)\, dx\, dy$	$(q^1 + \frac{dp^1}{dz} dz)\, dx\, dy$	$(p_3 + \frac{dp_3}{dz} dz)\, dx\, dy$
Attractions,	$\rho\, X\, dx\, dy\, dz$	$\rho\, Y\, dx\, dy\, dz$	$\rho\, Z\, dx\, dy\, dz$

Taking the moments of these forces round the axes of the particle, we find

$$q_1{}^1 = q_1 \qquad q_1{}^2 = q_2 \qquad q_3{}^1 = q_3\,;$$

and then equating the forces in the directions of the three axes, and dividing by dx, dy, dz, we find the equations of pressures.

$$\left.\begin{aligned} \frac{dp_1}{dx} + \frac{dq_3}{dy} + \frac{dq_2}{dz} + \rho\, X = 0 \\ \frac{dp_2}{dy} + \frac{dq_1}{dz} + \frac{dq_3}{dx} + \rho\, Y = 0 \\ \frac{dp_3}{dz} + \frac{dq_2}{dx} + \frac{dq_1}{dy} + \rho\, Z = 0 \end{aligned}\right\} \text{Equations of Pressures.} \quad . \ . \ . \ (3.)$$

The resistance which the solid opposes to these pressures is called Elasticity, and is of two kinds, for it opposes either change of *volume* or change of *figure*. These two kinds of elasticity have no necessary connection, for they are possessed in very different ratios by different substances. Thus *jelly* has a cubical elasticity little different from that of water, and a linear elasticity as small as we please; while *cork*, whose cubical elasticity is very small, has a much greater linear elasticity than jelly.

Hooke discovered that the elastic forces are proportional to the changes that excite them, or, as he expressed it, "Ut tensio sic vis."

To fix our ideas, let us suppose the compressed body to be a parallelopiped, and let pressures P_1, P_2, P_3 act on its faces in the direction of the axes α, β, γ, which will become the principal axes of compression, and the compressions will be $\frac{\delta\alpha}{\alpha}$, $\frac{\delta\beta}{\beta}$, $\frac{\delta\gamma}{\gamma}$.

The fundamental assumption from which the following equations are deduced is an extension of Hooke's law, and consists of two parts.

I. The sum of the compressions is proportional to the sum of the pressures.

II. The difference of the compressions is proportional to the difference of the pressures.

These laws are expressed by the following equations:—

$$\text{I. } (P_1 + P_2 + P_3) = 3\,\mu\left(\frac{\delta\alpha}{\alpha} + \frac{\delta\beta}{\beta} + \frac{\delta\gamma}{\gamma}\right) \quad . \ (4.)$$

Equations of Elasticity.

$$\text{II.} \left\{\begin{aligned} (P_1 - P_2) &= m\left(\frac{\delta\alpha}{\alpha} - \frac{\delta\beta}{\beta}\right) \\ (P_2 - P_3) &= m\left(\frac{\delta\beta}{\beta} - \frac{\delta\gamma}{\gamma}\right) \\ (P_3 - P_1) &= m\left(\frac{\delta\gamma}{\gamma} - \frac{\delta\alpha}{\alpha}\right) \end{aligned}\right\} \quad . \ (5.)$$

The quantity μ is the coefficient of cubical elasticity, and m that of linear elasticity.

By solving these equations, the values of the pressures P_1, P_2, P_3, and the compressions $\frac{\delta\alpha}{\alpha}$, $\frac{\delta\beta}{\beta}$, $\frac{\delta\gamma}{\gamma}$ may be found.

$$\left.\begin{aligned}
P_1&=(\mu-\tfrac{1}{3}m)\left(\frac{\delta\alpha}{\alpha}+\frac{\delta\beta}{\beta}+\frac{\delta\gamma}{\gamma}\right)+m\frac{\delta\alpha}{\alpha}\\
P_2&=(\mu-\tfrac{1}{3}m)\left(\frac{\delta\alpha}{\alpha}+\frac{\delta\beta}{\beta}+\frac{\delta\gamma}{\gamma}\right)+m\frac{\delta\beta}{\beta}\\
P_3&=(\mu-\tfrac{1}{3}m)\left(\frac{\delta\alpha}{\alpha}+\frac{\delta\beta}{\beta}+\frac{\delta\gamma}{\gamma}\right)+m\frac{\delta\gamma}{\gamma}
\end{aligned}\right\}\quad .\quad (6.)$$

$$\left.\begin{aligned}
\frac{\delta\alpha}{\alpha}&=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)(P_1+P_2+P_3)+\frac{1}{m}P_1\\
\frac{\delta\beta}{\beta}&=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)(P_1+P_2+P_3)+\frac{1}{m}P_2\\
\frac{\delta\gamma}{\gamma}&=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)P_1+P_2+P_3)+\frac{1}{m}P_3
\end{aligned}\right\}\quad .\quad (7.)$$

From these values of the pressures in the axes α, β, γ, may be obtained the equations for the axes x, y, z, by resolution of pressures and compressions.*

For $\qquad p=a^2 P_1 + b^2 P_2 + c^2 P_3$

and $\qquad q=a\,a P_1 + b\,b P_2 + c\,c P_3$

$$\left.\begin{aligned}
p_1&=(\mu-\tfrac{1}{3}m)\left(\frac{d\,\delta x}{dx}+\frac{d\,\delta y}{dy}+\frac{d\,\delta z}{dz}\right)+m\frac{d\,\delta x}{dx}\\
p_2&=(\mu-\tfrac{1}{3}m)\left(\frac{d\,\delta x}{dx}+\frac{d\,\delta y}{dy}+\frac{d\,\delta z}{dz}\right)+m\frac{d\,\delta y}{dy}\\
p_2&=(\mu-\tfrac{1}{3}m)\left(\frac{d\,\delta x}{dx}+\frac{d\,\delta y}{dy}+\frac{d\,\delta z}{dx}\right)+m\frac{d\,\delta z}{dz}
\end{aligned}\right\}\quad .\quad (8.)$$

$$\left.\begin{aligned}
q_1&=\frac{m}{2}\left(\frac{d\,\delta y}{dz}+\frac{d\,\delta z}{dy}\right)\\
q_2&=\frac{m}{2}\left(\frac{d\,\delta z}{dx}+\frac{d\,\delta x}{dz}\right)\\
q_3&=\frac{m}{2}\left(\frac{d\,\delta x}{dy}+\frac{d\,\delta y}{dx}\right)
\end{aligned}\right\}\quad .\quad (9.)$$

$$\left.\begin{aligned}
\frac{d\,\delta x}{dx}&=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)(p_1+p_2+p_3)+\frac{1}{m}p_1\\
\frac{d\,\delta y}{dy}&=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)(p_1+p_2+p_3)+\frac{1}{m}p_2\\
\frac{d\,\delta z}{dz}&=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)(p_1+p_2+p_3)+\frac{1}{m}p_3
\end{aligned}\right\}\quad .\quad (10.)$$

* See the Memoir of Lamé and Clapeyron, and note A.

$$\left.\begin{aligned} \frac{d\,\delta x}{dy} - \delta\theta_3 &= \frac{d\,\delta y}{dx} + \delta\theta_3 = \frac{1}{m} q_3 \\ \frac{d\,\delta y}{dz} - \delta\theta_1 &= \frac{d\,\delta z}{dy} + \delta\theta_1 = \frac{1}{m} q_1 \\ \frac{d\,\delta z}{dx} - \delta\theta_2 &= \frac{d\,\delta x}{dz} + \delta\theta_2 = \frac{1}{m} q_2 \end{aligned}\right\} \quad . \quad (11.)$$

By substituting in Equations (3.) the values of the forces given in Equations (8.) and (9.), they become

$$\left.\begin{aligned} (\mu + \tfrac{1}{6} m) \left(\frac{d}{dx}\left(\frac{d\,\delta x}{dx} + \frac{d\,\delta y}{dy} + \frac{d\,\delta z}{dz}\right)\right) + \frac{m}{2}\left(\frac{d^2}{dx^2}\delta x + \frac{d^2}{dy^2}\delta y + \frac{d^2}{dz^2}\delta z\right) + \rho\,\mathrm{X} = 0 \\ (\mu + \tfrac{1}{6} m) \left(\frac{d}{dy}\left(\frac{d\,\delta x}{dx} + \frac{d\,\delta y}{dy} + \frac{d\,\delta z}{dz}\right)\right) + \frac{m}{2}\left(\frac{d^2}{dx^2}\delta x + \frac{d^2}{dy^2}\delta y + \frac{d^2}{dz^2}\delta z\right) + \rho\,\mathrm{Y} = 0 \\ (\mu + \tfrac{1}{6} m) \left(\frac{d}{dz}\left(\frac{d\,\delta x}{dx} + \frac{d\,\delta y}{dy} + \frac{d\,\delta z}{dz}\right)\right) + \frac{m}{2}\left(\frac{d^2}{dx^2}\delta x + \frac{d^2}{dy^2}\delta y + \frac{d^2}{dz^2}\delta z\right) + \rho\,\mathrm{Z} = 0 \end{aligned}\right\} \quad (12.)$$

These are the general equations of elasticity, and are identical with those of M. Cauchy, in his *Exercises d'Analyse*, vol. iii., p. 180, published in 1828, when k stands for m, and K for $\mu - \frac{m}{2}$, and those of Mr Stokes, given in the *Cambridge Philosophical Transactions*, vol. viii., part 3, and numbered (30.); in his equations $\mathrm{A} = 3\mu$, $\mathrm{B} = \frac{m}{2}$.

If the temperature is variable from one part to another of the elastic solid, the compressions $\frac{d\,\delta x}{dx}$, $\frac{d\,\delta y}{dy}$, $\frac{d\,\delta z}{dz}$, at any point will be diminished by a quantity proportional to the temperature at that point. This principle is applied in Cases X. and XI. Equations (10.) then become

$$\left.\begin{aligned} \frac{d\,\delta x}{dx} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(p_1 + p_2 + p_3) + c_3 v + \frac{1}{m} p_1 \\ \frac{d\,\delta y}{dy} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(p_1 + p_2 + p_3) + c_3 v + \frac{1}{m} p_2 \\ \frac{d\,\delta x}{dz} &= \left(\frac{1}{9\mu} - \frac{1}{3m}\right)(p_1 + p_2 + p_3) + c_3 v + \frac{1}{m} p_3 \end{aligned}\right\} \quad . \quad . \quad . \quad (13.)$$

$c_3 v$ being the linear expansion for the temperature v.

Having found the general equations of the equilibrium of elastic solids, I proceed to work some examples of their application, which afford the means of determining the coefficients μ, m, and ω, and of calculating the stiffness of solid figures. I begin with those cases in which the elastic solid is a hollow cylinder exposed to given forces on the two concentric cylindric surfaces, and the two parallel terminating planes.

In these cases the co-ordinates x, y, z are replaced by the co-ordinates

$x=\mathrm{X}$, measured along the axis of the cylinder.

$y=r$, the radius of any point, or the distance from the axis.

$z=r\,\theta$, the arc of a circle measured from a fixed plane passing through the axis.

$\frac{d\,\delta\,x}{d\,x}=\frac{d\,\delta\,x}{d\,x}$, $p_1=0$, are the compression and pressure in the direction of the axis at any point.

$\frac{d\,\delta\,y}{d\,y}=\frac{d\,\delta\,r}{d\,r}$, $p_2=p$, are the compression and pressure in the direction of the radius.

$\frac{d\,\delta\,z}{d\,z}=\frac{d\,\delta\,r\,\theta}{d\,r\,\theta}=\frac{\delta\,r}{r}$, $p_3=q$, are the compression and pressure in the direction of the tangent.

Equations (9.) become, when expressed in terms of these co-ordinates—

$$\left.\begin{aligned} q_1 &= \frac{m}{2}\,r\,\frac{d\,\delta\,\theta}{d\,r} \\ q_2 &= \frac{m}{2}\,r\,\frac{d\,\delta\,\theta}{d\,x} \\ q_3 &= \frac{m}{2}\cdot\frac{d\,\delta\,x}{d\,r} \end{aligned}\right\} \quad . \quad (14.)$$

The length of the cylinder is b, and the two radii a_1 and a_2 in every case.

Case I.

The first equation is applicable to the case of a hollow cylinder, of which the outer surface is fixed, while the inner surface is made to turn through a small angle $\delta\,\theta$, by a couple whose moment is M.

The twisting force M is resisted only by the elasticity of the solid, and therefore the whole resistance, in every concentric cylindric surface, must be equal to M.

The resistance at any point, multiplied into the radius at which it acts, is expressed by $r\,q_1=\frac{m}{2}\,r^2\frac{d\,\delta\,\theta}{d\,r}$.

Therefore for the whole cylindric surface

$$\frac{d\,\delta\,\theta}{d\,r}\,m\,\pi\,r^3\,b=\mathrm{M}.$$

Whence $$\delta\,\theta=\frac{\mathrm{M}}{2\,\pi\,m\,b}\left(\frac{1}{a_1^2}-\frac{1}{a_2^2}\right)$$

and $$m=\frac{\mathrm{M}}{2\,\pi\,b\,\delta\,\theta}\left(\frac{1}{a_1^2}-\frac{1}{a_2^2}\right) \quad . \quad . \quad (15.)$$

The optical effect of the pressure of any point is expressed by

$$\mathrm{I}=\omega\,q_1\,b=\omega\frac{\mathrm{M}\,b}{\pi\,r^2} \quad . \quad . \quad (16.)$$

Therefore, if the solid be viewed by polarized light (transmitted parallel to the axis), the difference of retardation of the oppositely polarized rays at any point in the solid will be inversely proportional to the square of the distance from the axis of the cylinder, and the planes of polarization of these rays will be inclined 45° to the radius at that point.

The general appearance is therefore a system of coloured rings arranged oppositely to the rings in uniaxal crystals, the tints ascending in the scale as they approach the centre, and the distance between the rings decreasing towards the centre. The whole system is crossed by two dark bands inclined 45° to the plane of primitive polarization, when the plane of the analysing plate is perpendicular to that of the first polarizing plate.

A jelly of isinglass poured when hot between two concentric cylinders forms, when cold, a convenient solid for this experiment; and the diameters of the rings may be varied at pleasure by changing the force of torsion applied to the interior cylinder.

By continuing the force of torsion while the jelly is allowed to dry, a hard plate of isinglass is obtained, which still acts in the same way on polarized light, even when the force of torsion is removed.

It seems that this action cannot be accounted for by supposing the interior parts kept in a state of constraint by the exterior parts, as in unannealed and heated glass; for the optical properties of the plate of isinglass are such as would indicate a strain preserving in every part of the plate the direction of the original strain, so that the strain on one part of the plate cannot be maintained by an opposite strain on another part.

Two other uncrystallised substances have the power of retaining the polarizing structure developed by compression. The first is a mixture of wax and resin pressed into a thin plate between two plates of glass, as described by Sir David Brewster, in the *Philosophical Transactions* for 1815 and 1830.

When a compressed plate of this substance is examined with polarized light, it is observed to have no action on light at a perpendicular incidence; but when inclined, it shews the segments of coloured rings. This property does not belong to the plate as a whole, but is possessed by every part of it. It is therefore similar to a plate cut from a uniaxal crystal perpendicular to the axis.

I find that its action on light is like that of a *positive* crystal, while that of a plate of isinglass similarly treated would be *negative*.

The other substance which possesses similar properties is gutta percha. This substance in its ordinary state, when cold, is not transparent even in thin films; but if a thin film be drawn out gradually, it may be extended to more than double its length. It then possesses a powerful double refraction, which it retains so strongly that it has been used for polarizing light.* As one of its refractive in-

* By Dr Wright, I believe.

dices is nearly the same as that of Canada balsam, while the other is very different, the common surface of the gutta percha and Canada balsam will transmit one set of rays much more readily than the other, so that a film of extended gutta percha placed between two layers of Canada balsam acts like a plate of nitre treated in the same way. That these films are in a state of constraint may be proved by heating them slightly, when they recover their original dimensions.

As all these permanently compressed substances have passed their limit of perfect elasticity, they do not belong to the class of elastic solids treated of in this paper; and as I cannot explain the method by which an uncrystallised body maintains itself in a state of constraint, I go on to the next case of twisting, which has more practical importance than any other. This is the case of a cylinder fixed at one end, and twisted at the other by a couple whose moment is M.

CASE II.

In this case let $\delta\theta$ be the angle of torsion at any point, then the resistance to torsion in any circular section of the cylinder is equal to the twisting force M.

The resistance at any point in the circular section is given by the second Equation of (14.)

$$q_2 = \frac{m}{2} r \frac{d\delta\theta}{dx}.$$

This force acts at the distance r from the axis; therefore its resistance to torsion will be $q_2 r$, and the resistance in a circular annulus will be

$$q_2 r\, 2\pi r\, dr = m\pi r^3 \frac{d\delta\theta}{dx} dr$$

and the whole resistance for the hollow cylinder will be expressed by

$$M = \frac{m\pi}{4} \frac{d\delta\theta}{dx} (a_1{}^4 - a_2{}^4) \quad . \quad . \quad (16.)$$

$$m = 4\,M \frac{1}{\pi \frac{\delta\theta}{b} (a_1{}^4 - a_2{}^4)}$$

$$m = \frac{720}{\pi^2} \frac{M}{n} \left(\frac{b}{a_1{}^4 - a_2{}^4}\right) \quad . \quad . \quad (17.)$$

In this equation, m is the coefficient of linear elasticity; a_1 and a_2 are the radii of the exterior and interior surfaces of the hollow cylinder in inches; M is the moment of torsion produced by a weight acting on a lever, and is expressed by the product of the number of pounds in the weight into the number of inches in the lever; b is the distance of two points on the cylinder whose angular motion is measured by means of indices, or more accurately by small mirrors attached to the cylinder; n is the difference of the angle of rotation of the two indices in degrees.

This is the most accurate method for the determination of m independently of μ, and it seems to answer best with thick cylinders which cannot be used with the balance of torsion, as the oscillations are too short, and produce a vibration of the whole apparatus.

CASE III.

A hollow cylinder exposed to normal pressures only. When the pressures parallel to the axis, radius, and tangent are substituted for p_1, p_2, and p_3, Equations (10) become

$$\frac{d\,\delta x}{dx}=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)(o+p+q)+\frac{1}{m}\,o \quad . \quad . \quad (18.)$$

$$\frac{d\,\delta r}{dr}=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)(o+p+q)+\frac{1}{m}\,p \quad . \quad . \quad (19.)$$

$$\frac{d\,\delta(r\,\theta)}{d\,(r\,\theta)}=\frac{\delta r}{r}=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)(o+p+q)+\frac{1}{m}\,q \quad . \quad . \quad (20.)$$

By multiplying Equation (20) by r, differentiating with respect to r, and comparing this value of $\frac{d\,\delta r}{dr}$ with that of Equation (19.)

$$\frac{p-q}{rm}=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)\left(\frac{do}{dr}+\frac{dp}{dr}+\frac{dq}{dr}\right)-\frac{1}{m}\,\frac{dq}{dr}$$

The equation of the equilibrium of an element of the solid is obtained by considering the forces which act on it in the direction of the radius. By equating the forces which press it outwards with those pressing it inwards, we find the equation of the equilibrium of the element,

$$\frac{q-p}{r}=\frac{dp}{dr} \quad . \quad . \quad . \quad . \quad (21.)$$

By comparing this equation with the last, we find

$$\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)\frac{do}{dr}+\left(\frac{1}{9\,\mu}+\frac{2}{3\,m}\right)\left(\frac{dp}{dr}+\frac{dq}{dr}\right)=0$$

Integrating,

$$\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)o+\left(\frac{1}{9\,\mu}+\frac{2}{3\,m}\right)(p+q)=\mathrm{C}_1$$

Since o, the longitudinal pressure, is supposed constant, we may assume

$$c_2=\frac{c_1-\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)0}{\frac{1}{9\,\mu}+\frac{2}{3\,m}}=(p+q)$$

$$\therefore \quad q - p = c_2 - 2p, \quad \text{therefore by (21.)}$$

$$\frac{dp}{dr} + \frac{2p}{r} = \frac{c_2}{r}$$

a linear equation, which gives

$$p = c_3 \frac{1}{r_2} + \frac{c_2}{2}$$

The coefficients c_2 and c_3 must be found from the conditions of the surface of the solid. If the pressure on the exterior cylindric surface whose radius is a_1 be denoted by h_1, and that on the interior surface whose radius is a_2 by h_2,

$$\text{then } p = h_1 \text{ when } r = a_1$$
$$\text{and } p = h_2 \text{ when } r = a_2$$

and the general value of p is

$$p = \frac{a_1^2 h_1 - a_2^2 h_2}{a_1^2 - a_2^2} - \frac{a_1^2 a_2^2}{r^2} \frac{h_1 - h_2}{a_1^2 - a_2^2} \quad . \quad . \quad . \quad (22.)$$

$$r \frac{dp}{dr} = q - p = 2 \frac{a_1^2 a_2^2}{r^2} \frac{h_1 - h_2}{a_1^2 - a_1^2} \quad \text{by (21.)}$$

$$q = \frac{a_1^2 h_1 - a_2^2 h}{a_1^2 - a_2^2} + \frac{a_1^2 a_2^2}{r^2} \frac{h_1 - h_2}{a_1^2 - a_2^2} \quad . \quad . \quad . \quad (23.)$$

$$\mathrm{I} = b\,\omega\,(p - q) = b\,\omega \frac{a_1^2 a_2^2}{r^2} \frac{h_1 - h_2}{a_1^2 - a_2^2} \quad . \quad . \quad . \quad (24.)$$

This last equation gives the optical effect of the pressure at any point. The law of the magnitude of this quantity is the inverse square of the radius, as in Case I.; but the direction of the principal axes is different, as in this case they are parallel and perpendicular to the radius. The dark bands seen by polarized light will therefore be parallel and perpendicular to the plane of polarization, instead of being inclined at an angle of 45°, as in Case I.

By substituting in Equations (18.) and (20.), the values of p and q given in (22.) and (23.), we find that when $r = a_1$,

$$\left.\begin{aligned} \frac{\delta x}{x} &= \left(\frac{1}{9\mu}\right)\left(o + 2\frac{a_1^2 h_1 - a_2^2 h^2}{a_1^2 - a_2^2}\right) + \frac{2}{3m}\left(o - \frac{a_1^2 h_1 - a_2^2 h_2}{a_1^2 - a_2^2}\right) \\ &= o\left(\frac{1}{9\mu} + \frac{2}{3m}\right) + 2\left(h_1 a_1^2 - h_1 a_2^2\right)\frac{1}{a_1^2 - a_2^2}\left(\frac{1}{9\mu} - \frac{1}{3m}\right) \end{aligned}\right\} \quad (25.)$$

When
$$\left.\begin{aligned} r = a_1, \frac{\delta r}{r} &= \frac{1}{9\mu}\left(o + 2\frac{a_1^2 h_1 - a_2^2 h_2}{a_1^2 - a_2^2}\right) + \frac{1}{3m}\left(\frac{a_1^2 h_1 + 3a_2^2 h_1 - 4a_2^2 h_2}{a_1^2 - a_2^2} - o\right) \\ &= o\left(\frac{1}{9\mu} - \frac{1}{3m}\right) + h_1 \frac{1}{a_1^2 - a_2^2}\left(\frac{2a_1}{9\mu} + \frac{a_1^2 + 3a_2^1}{3m}\right) + h_2 \frac{a_2}{2a_1 - a_2}\left(\frac{2}{9\mu} - \frac{4}{3m}\right) \end{aligned}\right\} \quad (26.)$$

From these equations it appears that the longitudinal compression of cylindric tubes is proportional to the longitudinal pressure referred to unit of surface when the lateral pressures are constant, so that for a given pressure the compression is inversely as the sectional area of the tube.

These equations may be simplified in the following cases:—

1. When the external and internal pressures are equal, or $h_1=h_2$.

2. When the external pressure is to the internal pressure as the square of the interior diameter is to that of the exterior diameter, or when $a_1^2 h_1=a_2^2 h_2$.

3. When the cylinder is solid, or when $a_2=0$.

4. When the solid becomes an indefinitely extended plate with a cylindric hole in it, or when a_2 becomes infinite.

5. When pressure is applied only at the plane surfaces of the solid cylinder, and the cylindric surface is prevented from expanding by being inclosed in a strong case, or when $\frac{\delta r}{r}=0$.

6. When pressure is applied to the cylindric surface, and the ends are retained at an invariable distance, or when $\frac{\delta x}{x}=0$.

1. When $h_1=h_2$, the equations of compression become

$$\left.\begin{aligned}
\frac{\delta x}{x}&=\frac{1}{9\mu}(o+2h_1)+\frac{2}{3m}(o-h_1)\\
&o=\left(\frac{1}{9\mu}+\frac{2}{3m}\right)+2h_1\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\\
\frac{\delta r}{r}&=\frac{1}{9\mu}(o+2h_1)+\frac{1}{3m}(h_1-o)\\
&o=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)+h_1\left(\frac{2}{9\mu}+\frac{1}{3m}\right)
\end{aligned}\right\}\quad . \quad . \quad (27.)$$

When $h_1=h_2=o$, then

$$\frac{\delta x}{x}=\frac{\delta r}{r}=\frac{h_1}{3\mu}.$$

The compression of a cylindrical vessel exposed on all sides to the same hydrostatic pressure is therefore independent of m, and it may be shewn that the same is true for a vessel of any shape.

2. When $a_1^2 h_1=a_2^2 h_2$,

$$\left.\begin{aligned}
\frac{\delta x}{x}&=o\left(\frac{1}{9\mu}+\frac{2}{3m}\right)\\
\frac{\delta r}{r}&=\frac{1}{9\mu}(o)+\frac{1}{3m}(3h_1-o)\\
&=o\left(\frac{1}{9\mu}-\frac{1}{3m}\right)+\frac{h_1}{m}
\end{aligned}\right\}\quad . \quad . \quad (28.)$$

In this case, when $o=0$, the compressions are independent of μ.

3. In a solid cylinder, $a_2=0$,

$$p=q=h_1.$$

The expressions for $\frac{\delta x}{x}$ and $\frac{\delta r}{r}$ are the same as those in the first case, when $h_1=h_2$.

When the longitudinal pressure o vanishes,

$$\frac{\delta x}{x} = 2h_1\left(\frac{1}{9\mu}-\frac{1}{3m}\right)$$

$$\frac{\delta r}{r} = h_1\left(\frac{2}{9\mu}+\frac{1}{3m}\right)$$

When the cylinder is pressed on the plane sides only,

$$\frac{\delta x}{x} = o\left(\frac{1}{9\mu}+\frac{2}{3m}\right)$$

$$\frac{\delta r}{r} = o\left(\frac{1}{9\mu}-\frac{1}{3m}\right)$$

4. When the solid is infinite, or when a_2 is infinite,

$$\left.\begin{aligned}
p &= h_2 + \frac{1}{r^2}a_1^2(h_1-h_2)\\
q &= h_2 - \frac{1}{r^2}a_1^2(h_1-h_2)\\
\mathrm{I} &= \omega(q-p) = -\frac{2\omega}{r^2}a_1^2(h_1-h_2)\\
\frac{dx}{x} &= \frac{1}{9\mu}(o+2h_2)+\frac{2}{3m}(o-h_2)\\
&= o\left(\frac{1}{9\mu}+\frac{2}{3m}\right)+2h_2\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\\
\frac{dr}{r} &= \frac{1}{9\mu}(o+2h_2)+\frac{1}{3m}(4h_2-3h_2-o)\\
&= o\left(\frac{1}{9\mu}-\frac{1}{3m}\right)+2h_2\left(\frac{1}{9\mu}+\frac{2}{3m}\right)-\frac{h_1}{m}
\end{aligned}\right\} \quad . \; . \; (29.)$$

5. When $\delta r = 0$ in a solid cylinder,

$$\frac{\delta x}{x} = \frac{3o}{2m+3\mu}$$

6. When $\quad \frac{\delta x}{x} = o \quad \frac{\delta r}{r} = \frac{3h}{m+6\mu}$ $\quad . \; . \; (30.)$

Since the expression for the effect of a longitudinal strain is

$$\frac{\delta x}{x} = o\left(\frac{1}{9\mu}+\frac{2}{3m}\right)$$

if we make $\quad \mathrm{E} = \frac{9m\mu}{m+6\mu}$, then $\frac{\delta x}{x} = o\frac{1}{\mathrm{E}}$ $\quad . \; . \; . \; (31.)$

The quantity E may be deduced from experiment on the extension of wires or rods of the substance, and μ is given in terms of m and E by the equation,

$$\mu = \frac{\mathrm{E}m}{9m-6\mathrm{E}} \quad . \; . \; . \; (32.)$$

and

$$\mathrm{E} = \frac{\mathrm{P}b}{s\,\delta x} \quad . \; . \; . \; . \; (33.)$$

P being the extending force, b the length of the rod, s the sectional area, and δx the elongation, which may be determined by the deflection of a wire, as in the apparatus of S' GRAVESANDE, or by direct measurement.

CASE IV.

The only known direct method of finding the compressibility of liquids is that employed by CANTON, ŒRSTED, PERKINS, AIMÉ, &c.

The liquid is confined in a vessel with a narrow neck, then pressure is applied, and the descent of the liquid in the tube is observed, so that the difference between the change of volume of the liquid and the change of internal capacity of the vessel may be determined.

Now, since the substance of which the vessel is formed is compressible, a change of the internal capacity is possible. If the pressure be applied only to the contained liquid, it is evident that the vessel will be distended, and the compressibility of the liquid will appear too great. The pressure, therefore, is commonly applied externally and internally at the same time, by means of a hydrostatic pressure produced by water compressed either in a strong vessel or in the depths of the sea.

As it does not necessarily follow, from the equality of the external and internal pressures, that the capacity does not change, the equilibrium of the vessel must be determined theoretically. ŒRSTED, therefore, obtained from POISSON his solution of the problem, and applied it to the case of a vessel of lead. To find the cubical elasticity of lead, he applied the theory of POISSON to the numerical results of TREDGOLD. As the compressibility of lead thus found was greater than that of water, ŒRSTED expected that the apparent compressibility of water in a lead vessel would be *negative*. On making the experiment the apparent compressibility was *greater* in lead than in glass. The quantity found by TREDGOLD from the extension of rods was that denoted by E, and the value of μ deduced from E alone by the formulæ of POISSON cannot be true, unless $\frac{\mu}{m}=\frac{5}{6}$; and as $\frac{\mu}{m}$ for lead is probably more than 3, the calculated compressibility is much too great.

A similar experiment was made by Professor FORBES, who used a vessel of caoutchouc. As in this case the apparent compressibility vanishes, it appears that the cubical compressibility of caoutchouc is equal to that of water.

Some who reject the mathematical theories as unsatisfactory, have conjectured that if the sides of the vessel be sufficiently thin, the pressure on both sides being equal, the compressibility of the vessel will not affect the result. The following calculations shew that the apparent compressibility of the liquid depends on the compressibility of the vessel, and is independent of the thickness when the pressures are equal.

A hollow sphere, whose external and internal radii are a_1 and a_2, is acted on by external and internal normal pressures h_1 and h_2, it is required to determine the equilibrium of the elastic solid.

The pressures at any point in the solid are:—

1. A pressure p in the direction of the radius;
2. A pressure q in the perpendicular plane.

These pressures depend on the distance from the centre, which is denoted by r.

The compressions at any point are $\frac{d\,\delta r}{d\,r}$ in the radial direction, and $\frac{\delta r}{r}$ in the tangent plane, the values of these compressions are:—

$$\frac{d\,\delta r}{d\,r} = \left(\frac{1}{9\,\mu} - \frac{1}{3\,m}\right)(p+2\,q) + \frac{1}{m}p \quad . \quad . \quad . \quad (34.)$$

$$\frac{\delta r}{r} = \left(\frac{1}{9\,\mu} - \frac{1}{3\,m}\right)(p+2\,q) + \frac{1}{m}q \quad . \quad . \quad . \quad (35.)$$

Multiplying the last equation by r, differentiating with respect to r, and equating the result with that of the first equation, we find

$$r\left(\frac{1}{9\,\mu} - \frac{1}{3\,m}\right)\left(\frac{d\,p}{d\,r} + 2\frac{d\,q}{d\,r}\right) + \frac{1}{m}\left(r\frac{dq}{d\,r} + q-p\right) = 0$$

Since the forces which act on the particle in the direction of the radius must balance one another, or $2\,q\,d\,r\,d\,\theta + p\,(r\,d\,\theta)^2 = (p + \frac{dp}{d\,r}d\,r)(r+d\,r)^2\,\theta$

$$\therefore \qquad q-p = \frac{r}{2}\frac{dp}{d\,r} \quad . \quad . \quad . \quad (36.)$$

Substituting this value of $q-p$ in the preceding equation, and reducing,

$$\therefore \qquad \frac{dp}{d\,r} + 2\frac{d\,q}{d\,r} = 0$$

Integrating, $\qquad p+2\,q = c_1$

But $\qquad q = \frac{r}{2}\frac{dp}{d\,r} + p$, and the equation becomes

$$\frac{dp}{d\,r} + 3\frac{p}{r} + \frac{c_1}{r} = 0$$

$$\therefore \qquad p = c_2\frac{1}{r^3} + \frac{c_1}{3}$$

Since $p=h_1$ when $r=a_1$, and $p=h_2$ when $r=a_2$, the value of p at any time is found to be

$$p = \frac{a_1^3\,h_1 - a_2^3\,h_2}{a_1^3 - a_2^3} - \frac{a_1^3\,a_2^3}{r^3}\,\frac{h_1-h_2}{a^3-b^3} \quad . \quad . \quad . \quad . \quad (37.)$$

$$q = \frac{a_1^3\,h_1 - a_2^3 h_2}{a_1^3 - a_2^3} - \frac{a_1^3\,a_2^3}{r^3}\,\frac{h_1-h_2}{a_1^3-a_2^3} \quad . \quad . \quad (38.)$$

$$\frac{\delta\,\mathrm{V}}{\mathrm{V}} = 3\frac{\delta\,r}{r} = \frac{a_1^3\,h_1 - a_2^3\,h_2}{a_1^3 - a_2^3}\,\frac{1}{\mu} + \frac{3}{2}\,\frac{a_1^3\,a_2^3}{r^3}\,\frac{h_1-h_2}{a_1^3-a_2^3}\,\frac{1}{m}$$

$$\left.\begin{aligned} \text{When } r=a_1\frac{\delta V}{V} &= \frac{a_2^3 h_1 - a_2^3 h_2}{a_1^3-a_2^3}\frac{1}{\mu}+\frac{3}{2}a_2^3\frac{h_1-h_2}{a_1^3-a_2^3}\frac{1}{m} \\ &= \frac{h_1}{a_1^3-a_2^3}\left(\frac{a_1^3}{\mu}+\frac{3a_2^3}{2m}\right)-\frac{h_2 a_2^3}{a_1^3-a_2^3}\left(\frac{1}{\mu}+\frac{3}{2m}\right) \end{aligned}\right\} \quad . \quad . \quad (39.)$$

When the external and internal pressures are equal

$$h_1=h_2=p=q, \text{ and } \frac{\delta V}{V}=\frac{h_1}{u}=\frac{h_1}{u} \quad . \quad . \quad (40.)$$

the change of internal capacity depends entirely on the cubical elasticity of the vessel, and not on its thickness or its linear elasticity.

When the external and internal pressures are inversely as the cubes of the radii of the surfaces on which they act,

$$\left.\begin{aligned} a_1^3 h_1 = a_2^3 h_2,\ p=\frac{a^3}{r^3}h_1,\ q=-\frac{1}{2}\frac{a_1^3}{r^3}h_1 \\ \frac{\delta V}{V}=-\frac{3}{2}\frac{a_1^3}{r^3}\frac{h_1}{m} \\ \text{when} \qquad r=a_1\ \frac{\delta V}{V}=-\frac{3}{2}\frac{h_1}{m} \end{aligned}\right\} \quad . \quad . \quad (41.)$$

In this case the change of capacity depends on the linear elasticity alone.

M. REGNAULT, in his researches on the theory of the steam engine, has given an account of the experiments which he made in order to determine with accuracy the compressibility of mercury.

He considers the mathematical formulæ very uncertain, because the theories of molecular forces from which they are deduced are probably far from the truth; and even were the equations free from error, there would be much uncertainty in the ordinary method by measuring the elongation of a rod of the substance, for it is difficult to ensure that the material of the rod is the same as that of the hollow sphere.

He has, therefore, availed himself of the results of M. LAMÉ for a hollow sphere in three different cases, in the first of which the pressure acts on the interior and exterior surface at the same time, while in the other two cases the pressure is applied to the exterior or interior surface alone. Equation (39.) becomes in these cases,—

1. When $h_1=h_2\ \frac{\delta V}{V}=\frac{h_1}{\mu}$ and the compressibility of the enclosed liquid being μ_2, and the apparent diminution of volume $\delta' V$, $\frac{\delta' V}{V}=h_1\left(\frac{1}{\mu_2}-\frac{1}{\mu}\right)$ . . (42.)

2. When $\quad h_1=o,\ \frac{\delta V}{V}=\frac{\delta' V}{V}=-h_2\frac{a_2^3}{a_1^3-a_2^3}\left(\frac{1}{\mu}+\frac{3}{2m}\right)$ . . . . (43.)

3. When $\quad h_2=o,\ \frac{\delta V}{V}=\frac{h_1}{a_1^3-a_2^3}\left(\frac{a_1^3}{\mu}+\frac{3}{2}\frac{a_2^3}{m}\right)$

$$\frac{\delta' V}{V}=\frac{h_1}{a_1^3-a_2^3}\left(\frac{a_1^3}{\mu}+\frac{3}{2}\frac{a_2^3}{m}+(a_2^3-a_1^3)\frac{1}{\mu_2}\right)$$

M. LAMÉ's equations differ from these only in assuming that $\mu = \frac{5}{6} m$. If this assumption be correct, then the coefficients μ, m, and μ_2, may be found from two of these equations; but since one of these equations may be derived from the other two, the *three* coefficients cannot be found when μ is supposed independent of m. In Equations (39.), the quantities which may be varied at pleasure are h_1 and h_2, and the quantities which may be deduced from the apparent compressions are,

$$c_1 = \left(\frac{1}{\mu} + \frac{3}{2\,m}\right) \text{ and } \left(\frac{1}{\mu} - \frac{1}{\mu_2}\right) = c_2$$

therefore some independent equation between these quantities must be found, and this cannot be done by means of the sphere alone; some other experiment must be made on the liquid, or on another portion of the substance of which the vessel is made.

The value of μ_2, the elasticity of the liquid, may be previously known.

The linear elasticity m of the vessel may be found by twisting a rod of the material of which it is made;

Or, the value of E may be found by the elongation or bending of the rod, and $\frac{1}{E} = \frac{1}{9\,\mu} + \frac{2}{3\,m}$.

We have here five quantities, which may be determined by experiment.

$$\left.\begin{array}{lll} (43.)\ 1. & c_1 = \left(\frac{1}{\mu} + \frac{3}{2\,m}\right) & \text{by external pressure} \\ (42.)\ 2. & c_2 = \left(\frac{1}{\mu} - \frac{1}{\mu_2}\right) & \text{equal pressures} \end{array}\right\} \text{on sphere.}$$

$$\begin{array}{lll} (31.)\ 3. & \frac{1}{E} = \left(\frac{1}{9\mu} + \frac{2}{3\,m}\right) & \text{by elongation of a rod.} \\ (17.)\ 4. & m & \text{by twisting the rod.} \\ 5. & \mu_2 & \text{the elasticity of the liquid.} \end{array}$$

When the elastic sphere is solid, the internal radius a_1 vanishes, and $h_2 = p = q$, and $\frac{\delta V}{V} = \frac{h_2}{\mu}$.

When the case becomes that of a spherical cavity in an infinite solid, the external radius a_2 becomes infinite, and

$$\left.\begin{aligned} p &= h_2 + \frac{a_1^{\,3}}{r^3}(h_1 - h_2) \\ q &= h_2 - \frac{1}{2}\frac{a_1^{\,3}}{r^3}(h_1 - h_2) \\ \frac{\delta r}{r} &= h_2 \frac{1}{3\,\mu} - \frac{1}{2}\frac{a_1^{\,3}}{r^3}(h_1 - h_2)\frac{1}{m} \\ \frac{\delta V}{V} &= \frac{h_2}{\mu} - \frac{1}{2}\frac{h_1 - h_2}{m} \end{aligned}\right\} \quad . \quad . \quad . \quad (46.)$$

The effect of pressure on the surface of a spherical cavity on any other part of an elastic solid is therefore inversely proportional to the cube of its distance from the centre of the cavity.

When one of the surfaces of an elastic hollow sphere has its radius rendered invariable by the support of an incompressible sphere, whose radius is a_1, then

$$\frac{\delta r}{r}=0, \quad \text{when } r=a_1$$

$$\left.\begin{aligned}
\therefore \quad p &= h_2 \frac{3 a_2^3 \mu}{2 a_1^3 m+3 a_2^3 \mu}+h_2 \frac{a_1^3 a_2^3}{r^3} \frac{2 m}{2 a_1^3 m+3 a_2^3 \mu} \\
q &= h_2 \frac{3 a_2^3 \mu}{2 a_1^3 m+3 a_2^3 \mu}-h_2 \frac{a_1^3 a_2^3}{r^3} \frac{m}{2 a_1^3 m+3 a_2^3 \mu} \\
\frac{\delta r}{r} &= h_2 \frac{a_2^3}{2 a_1^3 m+3 a_2^3 \mu}-h_2 \frac{a_1^3 a_2^3}{r^3} \frac{1}{2 a_1^3 m+3 a_2^3 \mu} \\
\text{When } r=a_2 \quad \frac{\delta \mathrm{V}}{\mathrm{V}} &= h_2 \frac{3 a_2^3-3 a_1^3}{2 a_1^3 m+3 a_2^3 \mu}
\end{aligned}\right\} \quad . \; . \; . \quad (45.)$$

Case V.

On the equilibrium of an elastic beam of rectangular section uniformly bent.

By supposing the bent beam to be produced till it returns into itself, we may treat it as a hollow cylinder.

Let a rectangular elastic beam, whose length is $2 \pi c$, be bent into a circular form, so as to be a section of a hollow cylinder, those parts of the beam which lie towards the centre of the circle will be longitudinally compressed, while the opposite parts will be extended.

The expression for the tangential compression is therefore

$$\frac{\delta r}{r} = \frac{c-r}{c}.$$

Comparing this value of $\frac{\delta r}{r}$ with that of Equation (20.)

$$\frac{c-r}{r}=\left(\frac{1}{9 \mu}-\frac{1}{3 m}\right)(o+p+q)+\frac{q}{m}$$

and by (21.) $\qquad q=p+r\frac{d p}{d r}.$

By substituting for q its value, and dividing by $r\left(\frac{1}{9 \mu}+\frac{1}{3 m}\right)$, the equation becomes

$$\frac{d p}{d r}+\frac{2 m+3 \mu}{m+6 \mu} \frac{p}{r}=\frac{9 m \mu-(m-3 \mu) o}{(m+6 \mu) r}-\frac{9 m \mu}{(m+6 \mu) c}$$

a linear differential equation, which gives

$$p=\mathrm{C}_1 r^{-\frac{2 m+3 \mu}{m+6 \mu}}-\frac{3 m \mu}{m+3 \mu} \frac{r}{c}+\frac{9 \mu m-(m-3 \mu) o}{2 m+3 \mu} \quad . \; . \quad (46.)$$

C_1 may be found by assuming that when $r=a_1$ $p=h_1$, and q may be found from p by Equation (21.)

As the expressions thus found are long and cumbrous, it is better to use the following approximations:—

$$q=-\left(\frac{9\,m\,\mu}{m+6\,\mu}\right)\frac{y}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47.)$$

$$p=\left(\frac{9\,m\,\mu}{m+6\,\mu}\right)\frac{1}{2\,c}\left(\frac{c^2-a^2}{y-c}+c+y\right) \quad . \quad (48.)$$

In these expressions a is half the depth of the beam, and y is the distance of any part of the beam from the neutral surface, which in this case is a cylindric surface, whose radius is c.

These expressions suppose c to be large compared with a, since most substances break when $\frac{a}{c}$ exceeds a certain small quantity.

Let b be the breadth of the beam, then the force with which the beam resists flexure $=$M.

$$\mathrm{M}=\int b\,y\,q=\frac{q\,m\,\mu}{m+6\,\mu}\frac{b}{c}\frac{a^3}{3}=\mathrm{E}\frac{a^3\,b}{3\,c} \quad . \quad . \quad . \quad (49.)$$

which is the ordinary expression for the stiffness of a rectangular beam.

The stiffness of a beam of any section, the form of which is expressed by an equation between x and y, the axis of x being perpendicular to the plane of flexure, or the osculating plane of the axis of the beam at any point, is expressed by

$$\mathrm{M}\,c=\mathrm{E}\int y^2\,d\,x, \quad . \quad . \quad . \quad (50.)$$

M being the moment of the force which bends the beam, and c the radius of the circle into which it is bent.

Case VI.

At the meeting of the British Association in 1839, Mr James Nasmyth described his method of making concave specula of silvered glass by bending.

A circular piece of silvered plate-glass was cemented to the opening of an iron vessel, from which the air was afterwards exhausted. The mirror then became concave, and the focal distance depended on the pressure of the air.

Buffon proposed to make burning-mirrors in this way, and to produce the partial vacuum by the combustion of the air in the vessel, which was to be effected by igniting sulphur in the interior of the vessel by means of a burning-glass. Although sulphur evidently would not answer for this purpose, phosphorus might; but the simplest way of removing the air is by means of the air-pump. The mirrors which were actually made by Buffon, were bent by means of a screw acting on the centre of the glass.

To find an expression for the curvature produced in a flat, circular, elastic plate, by the difference of the hydrostatic pressures which act on each side of it,—

Let t be the thickness of the plate, which must be small compared with its diameter.

Let the form of the middle surface of the plate, after the curvature is produced, be expressed by an equation between r, the distance of any point from the axis, or normal to the centre of the plate, and x the distance of the point from the plane in which the middle of the plate originally was, and let $ds=\sqrt{(dx)^2+(dr^2)}$.

Let h_1 be the pressure on one side of the plate, and h_2 that on the other.

Let p and q be the pressures in the plane of the plate at any point, p acting in the direction of a tangent to the section of the plate by a plane passing through the axis, and q acting in the direction perpendicular to that plane.

By equating the forces which act on any particle in a direction parallel to the axis, we find

$$tp\frac{dr}{ds}\frac{dx}{ds}+tr\frac{dp}{ds}\frac{dx}{ds}+trp\frac{d^2x}{ds^2}+r(h_1-h_2)\frac{dr}{ds}=0$$

By making $p=0$ when $r=0$ in this equation,

$$p=-\frac{r}{2t}\frac{ds}{dx}(h_1-h_2) \quad . \quad . \quad . \quad (51.)$$

The forces perpendicular to the axis are

$$tp\left(\frac{dr}{ds}\right)^2+tr\frac{dp}{ds}\frac{dr}{ds}+trp\frac{d^2r}{ds^2}-(h_1-h_2)r\frac{dx}{ds}-qt=0$$

Substituting for p its value, the equation gives

$$q=-\frac{(h_1-h_2)}{t}r\left(\frac{dr}{ds}\frac{dr}{dx}+\frac{dx}{ds}\right)+\frac{(h_1-h_2)}{2t}r^2\left(\frac{dr}{dx}\frac{ds}{dx}\frac{d^2x}{ds^2}-\frac{ds}{dx}\frac{d^2r}{ds^2}\right) \quad . \quad . \quad . \quad (52.)$$

The equations of elasticity become

$$\frac{d\delta s}{ds}=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(p+q+\frac{h_1+h_2}{2}\right)+\frac{p}{m}$$

$$\frac{\delta r}{r}=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(p+q+\frac{h_1+h_2}{2}\right)+\frac{q}{m}$$

Differentiating $\frac{d\delta r}{dr}=\frac{d}{dr}\left(\frac{\delta r}{r}r\right)$, and in this case

$$\frac{d\delta r}{dr}=1-\frac{dr}{ds}+\frac{dr}{ds}\frac{d\delta s}{ds}$$

By a comparison of these values of $\frac{d\delta r}{ds}$

$$\left(1-\frac{dr}{ds}\right)\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(p+q+\frac{h_1+h_2}{2}\right)+\frac{q}{m}+\frac{dr}{ds}\frac{p}{m}+r\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(\frac{dp}{dr}+\frac{dq}{dr}\right)$$

$$+\frac{r}{m}\frac{dq}{dr}+\frac{dr}{ds}-1=0.$$

To obtain an expression for the curvature of the plate at the vertex, let a be the radius of curvature, then, as an approximation to the equation of the plate, let

$$x = \frac{r^2}{2a}.$$

By substituting the value of x in the values of p and q, and in the equation of elasticity, the approximate value of a is found to be

$$a = \frac{t}{h_1 - h_2} \frac{(h_1 + h_2)\left(\frac{1}{9\mu} - \frac{1}{3m}\right) - 2}{10\left(\frac{1}{9\mu} - \frac{1}{3m}\right) - \frac{9}{m}}$$

$$a = \frac{t}{h_1 - h_2} \frac{-18m\mu}{10m + 51\mu} + t\frac{h_1 + h_2}{h_1 - h_2} \frac{m - 3\mu}{10m + 51\mu} \quad . \quad . \quad . \quad (53.)$$

Since the focal distance of the mirror, or $\frac{a}{2}$, depends on the difference of pressures, a telescope on Mr NASMYTH's principle would act as an aneroid barometer, the focal distance varying inversely as the pressure of the atmosphere.

CASE VII.

To find the conditions of torsion of a cylinder composed of a great number of parallel wires bound together without adhering to one another.

Let x be the length of the cylinder, a its radius, r the radius at any point, $\delta\theta$ the angle of torsion, M the force producing torsion, δx the change of length, and P the longitudinal force. Each of the wires becomes a helix whose radius is its angular rotation $\delta\theta$, and its length along the axis $x - \delta\theta$.

Its length is therefore $\sqrt{(r\delta\theta)^2 + x\left(1 - \frac{\delta x}{x}\right)^2}$

and the tension is $= \mathrm{E}\left(1 - \sqrt{\left(1 - \frac{\delta x}{x}\right)^2 + r^2\left(\frac{\delta\theta}{x}\right)^2}\right)$

This force, resolved parallel to the axis, is

$$\frac{d}{d\theta}\frac{d}{dr}\mathrm{P} = \mathrm{E}\left(\frac{1}{\sqrt{\left(1 - \frac{\delta x}{x}\right)^2 + r^2\left(\frac{\delta\theta}{x}\right)^2}} - 1\right)$$

and since $\frac{\delta x}{x}$ and $r\frac{\delta\theta}{x}$ are small, we may assume

$$\frac{d}{d\theta}\frac{d}{dr}\mathrm{P} = \mathrm{E}\left(\frac{\delta x}{x} - \frac{r^2}{2}\left(\frac{\delta\theta}{x}\right)^2\right)$$

$$\mathrm{P} = \pi\mathrm{E}\left(r^2\frac{\delta x}{x} - \frac{r^4}{4}\left(\frac{\delta\theta}{x}\right)^2\right) \quad . \quad . \quad . \quad (54.)$$

The force, when resolved in the tangential direction, is approximately

$$\frac{1}{r^2}\frac{d}{d\theta}\frac{d}{dx}\mathrm{M} = \mathrm{E}\left(r\frac{\delta\theta}{x}\frac{\delta x}{x} - \frac{r^3}{2}\left(\frac{\delta\theta}{x}\right)^3\right)$$

$$\mathrm{M} = \pi\mathrm{E}\left(\frac{r^4}{2}\frac{\delta\theta}{x}\frac{\delta x}{x} - \frac{r^6}{6}\left(\frac{\delta\theta}{x}\right)^3\right) \quad . \quad . \quad . \quad (55.)$$

By eliminating $\frac{\delta x}{x}$ between (54.) and (55.) we have

$$M=\frac{r^2}{2}\frac{\delta\theta}{x}P-E\pi\frac{r^6}{24}\left(\frac{\delta\theta}{x}\right)^3 \quad . \quad . \quad . \quad (56.)$$

When $P=0$, M depends on the sixth power of the radius and the cube of the angle of torsion, when the cylinder is composed of separate filaments.

Since the force of torsion for a homogeneous cylinder depends on the fourth power of the radius and the first power of the angle of torsion, the torsion of a wire having a fibrous texture will depend on both these laws.

The parts of the force of torsion which depend on these two laws may be found by experiment, and thus the difference of the elasticities in the direction of the axis and in the perpendicular directions may be determined.

A calculation of the force of torsion, on this supposition, may be found in YOUNG'S *Mathematical Principles of Natural Philosophy;* and it is introduced here to account for the variations from the law of Case II., which may be observed in a twisted rod.

CASE VIII.

It is well known that grindstones and fly-wheels are often broken by the centrifugal force produced by their rapid rotation. I have therefore calculated the strains and pressure acting on an elastic cylinder revolving round its axis, and acted on by the centrifugal force alone.

The equation of the equilibrium of a particle (see Equation (21.)), becomes

$$q-p=r\frac{dp}{dr}-\frac{4\pi^2 k}{g t^2}r^2;$$

where q and p are the tangential and radial pressures, k is the weight in pounds of a cubic inch of the substance, g is twice the height in inches that a body falls in a second, t is the time of revolution of the cylinder in seconds.

By substituting the value of g and $\frac{dq}{dr}$ in Equations (19.), (20.), and neglecting o,

$$0=\left(\frac{1}{9\mu}-\frac{1}{3m}\right)\left(3\frac{dp}{dr}-2\frac{4\pi^2 k}{g t^2}r+r\frac{d^2p}{dr^2}\right)+\frac{1}{m}\left(3\frac{dp}{dr}-3\frac{4\pi^2 k}{g t^2}r+r\frac{d^2p}{dr^2}\right)$$

which gives

$$\left.\begin{aligned} p&=c_1\frac{1}{r^2}+\frac{\pi^2 k}{2 g t^2}\left(2+\frac{E}{m}\right)r^2+c_2 \\ \therefore \quad q-p&=-c_1\frac{1}{r^2}+\frac{\pi^2 k}{2 g t}\left(-4+\frac{2E}{m}\right)r^2 \\ q&=-c_1\frac{1}{r^2}+\frac{\pi^2 k}{2 g t^2}\left(-2+\frac{3E}{m_3}\right)r^2+c_2 \quad . \end{aligned}\right\} \quad . \quad . \quad . \quad (57.)$$

If the radii of the surfaces of the hollow cylinder be a_1 and a_2, and the pressures acting on them h_1 and h_2, then the values of c_1 and c_2 are

$$\left.\begin{aligned} c_1 &= a_1^2\, a_2^2 \frac{\pi^2 k}{2 g t^2}\left(2+\frac{\mathrm{E}}{m}\right) - a_1^2\, a_2^2 \frac{h_1 - h_2}{a_1^2 - a_2^2} \\ c_2 &= \frac{a_1^2 h_1 - a_2^2 h_2}{a_1^2 - a_2^2} - (a_1^2 + a_2^2)\frac{\pi^2 k}{2 g t^2}\left(2+\frac{\mathrm{E}}{m}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (58.)$$

When $a_2=0$, as in the case of a solid cylinder, $c_1=0$, and

$$c_2 = h_1 - a_1^2 \frac{\pi^2 k}{2 g t^2}\left(2+\frac{\mathrm{E}}{m}\right)$$

$$q = h_1 + \frac{\pi^2 k}{2 g t^2}\left\{2\,(r^2+a_1^2) + \frac{\mathrm{E}}{m}\,(3\,r^2 - a_1^2)\right\} \quad . \quad . \quad . \quad . \quad (59.)$$

When $h_1=0$, and $r=a_1$,

$$q = \frac{\pi^2 k\, a^2}{g\, t^2}\left(\frac{\mathrm{E}}{m} - 2\right) \quad . \quad . \quad . \quad . \quad (60.)$$

When q exceeds the tenacity of the substance in pounds per square inch, the cylinder will give way; and by making q equal to the number of pounds which a square inch of the substance will support, the velocity may be found at which the bursting of the cylinder will take place.

Since $\mathrm{I} = b\,\omega\,(q-p) = \frac{\pi^2 k\,\omega}{g\,t}\left(\frac{\mathrm{E}}{m} - 2\right) b\, r^2$, a transparent revolving cylinder, when polarized light is transmitted parallel to the axis, will exhibit rings whose diameters are as the square roots of an arithmetical progression, and brushes parallel and perpendicular to the plane of polarization.

CASE IX.

A hollow cylinder or tube is surrounded by a medium of a constant temperature while a liquid of a different temperature is made to flow through it. The exterior and interior surfaces are thus kept each at a constant temperature till the transference of heat through the cylinder becomes uniform.

Let v be the temperature at any point, then when this quantity has reached its limit,

$$\frac{r\, d\, v}{d\, r} = c_1$$

$$x = c_1 \log r + c_2 \quad . \quad . \quad . \quad . \quad (61.)$$

Let the temperatures at the surfaces be θ_1 and θ_2, and the radii of the surfaces a_1 and a_2, then

$$c_1 = \frac{\theta_1 - \theta_2}{\log a_1 - \log a_2} \qquad c_2 = \frac{\log a_1\, \theta_2 - \log a_2\, \theta_1}{\log a_1 - \log a_2}$$

Let the coefficient of linear dilatation of the substance be c_3, then the proportional dilatation at any point will be expressed by $c_3\, v$, and the equations of elasticity (18.), (19.), (20.), become

$$\frac{d\,\delta\, x}{d\, x} = \left(\frac{1}{9\mu} - \frac{1}{3\, m}\right)(o+p+q) + \frac{o}{m} - c_3\, v$$

$$\frac{d\,\delta\,r}{d\,r}=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)(o+p+q)+\frac{p}{m}-c_3\,v$$

$$\frac{\delta\,r}{r}=\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)(o+p+q)+\frac{q}{m}-c_3\,v$$

The equation of equilibrium is

$$q=p+r\frac{d\,p}{d\,r} \quad . \quad . \quad (21.)$$

and since the tube is supposed to be of a considerable length

$$\frac{d\,\delta\,x}{d\,x}=c_4 \text{ a constant quantity.}$$

From these equations we find that

$$o=\frac{c_4+c_3\,v-\left(\frac{1}{9\,\mu}-\frac{1}{3\,m}\right)\left(2\,p+r\frac{d\,p}{d\,r}\right)}{\frac{1}{9\,\mu}+\frac{2}{3\,m}}$$

and hence $v=c_1\log r+c_2$, p may be found in terms of r.

$$p=\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)^{-1}c_1\,c_3\log r+c_5\frac{1}{r^2}+c_6$$

Hence $$q=\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)^{-1}c_1\,c_3\log r-c_5\frac{1}{r^2}+c_6+\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)c_1\,c_3$$

Since $$\mathrm{I}=b\,\omega\,(q+p)=b\,\omega\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)^{-1}c_1\,c_3-2\,b\,\omega\,c_5\frac{1}{r^2}$$

the rings seen in this case will differ from those described in Case III. only by the addition of a constant quantity.

When no pressures act on the exterior and interior surfaces of the tube $h_1=h_2=0$, and

62. $$\begin{cases} p=\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)^{-1}c_1\,c_3\left\{\log r+\frac{a_1^2\,a_2^2}{r^2}\frac{\log a_1-\log a_2}{a_1^2-a_2^2}+\frac{a_1^2\log a_1-a_2^2\log a_2}{a_1^2-a_2^2}\right\} \\ q=\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)^{-1}c_1\,c_3\left\{\log r-\frac{a_1^2\,a_2^2}{r^2}\frac{\log a_1-\log a_2}{a_1-a_2}+\frac{a_1^2\log a_1-a_2^2\log a_2}{a_1^2-a_2^2}+1\right\} \\ \mathrm{I}=b\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)^{-1}c_1\,c_3\,\omega\left\{1-2\,\frac{a_1^2\,a_2^2}{r^2}\frac{\log a_1-\log a_2}{a_1^2-a_2^2}\right\} \end{cases}$$

There will, therefore, be no action on polarized light for the ring whose radius is r when

$$r^2=2\frac{a_1^2\,a_2^2}{a_1^2-a_2^2}\log\frac{a_1}{a_2}$$

CASE X.

Sir DAVID BREWSTER has observed (*Edinburgh Transactions*, vol. viii.), that when a solid cylinder of glass is suddenly heated at the cylindric surface a polarizing force is developed, which is at any point proportional to the square of the distance from the axis of the cylinder; that is to say, that the difference of retardation of the oppositely polarized rays of light is proportional to the square of the radius r, or

$$\mathrm{I}=b\,c_1\,\omega\,r^2=b\,\omega\,(q-p)=b\,\omega\,r\frac{dp}{dr}$$

$$\therefore \frac{dp}{dr}=c_1\,r \therefore p=\frac{c_1}{2}\,r^2+c_2$$

Since if a be the radius of the cylinder, $p=o$ when $r=a$,

$$p=\frac{c_1}{2}\,(r^2-a^2)$$

Hence
$$q=\frac{c_1}{2}\,(3\,r^2-a^2)$$

By substituting these values of p and q in equations (19) and (20), and making $\frac{d}{dr}\frac{\delta r}{r}r=\frac{d\,\delta r}{dr}$, I find,

$$v=\frac{2\,c_1}{c_3}\left(\frac{1}{9\,\mu}+\frac{2}{3\,m}\right)r^2+c_4 \tag{63.}$$

c_4 being the temperature of the axis of the cylinder, and c_3 the coefficient of linear expansion for glass.

CASE XI.

Heat is passing uniformly through the sides of a spherical vessel, such as the ball of a thermometer, it is required to determine the mechanical state of the sphere. As the methods are nearly the same as in Case IX., it will be sufficient to give the results, using the same notation.

$$r^2\frac{dv}{dv}=c_1 \therefore v=c_2-\frac{c_1}{r}$$

$$c_1=a_1\,a_2\,\frac{\theta_1-\theta_2}{a_1-a_2} \qquad c_2=\frac{\theta_1\,a_1-\theta_2\,a_2}{a_1-a_2}$$

$$p=c_4\,\frac{1}{r^3}+\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)^{-1}c_1\,c_3=\frac{1}{r}-c_5$$

When $h_1=h_2=0$ the expression for p becomes

$$p=\left(\frac{2}{9\,\mu}+\frac{1}{3\,m}\right)^{-1}c_3\,(\theta_1-\theta_2)\left\{\frac{a_1^{\,3}\,a_2^{\,3}}{a_1^{\,3}-a_2^{\,3}}\,\frac{1}{r^3}+\frac{a_1\,a_2}{a_1-a_2}\,\frac{1}{r}-a_1\,a^2\,\frac{a_1^{\,2}-a_2^{\,2}}{(a_1-a_2)\,(a_1^{\,3}-a_2^{\,3})}\right\} \tag{64.}$$

From this value of p the other quantities may be found, as in Case IX., from the equations of Case IV.

CASE XII.

When a long beam is bent into the form of a closed circular ring (as in Case V.), all the pressures act either parallel or perpendicular to the direction of the length of the beam, so that if the beam were divided into planks, there would be no tendency of the planks to slide on one another.

But when the beam does not form a closed circle, the planks into which it may be supposed to be divided will have a tendency to slide on one another, and the amount of sliding is determined by the linear elasticity of the substance. The deflection of the beam thus arises partly from the bending of the whole beam, and partly from the sliding of the planks; and since each of these deflections is small compared with the length of the beam, the total deflection will be the sum of the deflections due to bending and sliding.

Let $$A = M\,c = E\int x\,y^2\,dy \quad . \quad . \quad (65.)$$

A is the stiffness of the beam as found in Case V., the equation of the transverse section being expressed in terms of x and y, y being measured from the neutral surface.

Let a horizontal beam, whose length is $2\,l$, and whose weight is $2\,w$, be supported at the extremities and loaded at the middle with a weight W.

Let the deflection at any point be expressed by $\delta_1 y$, and let this quantity be small compared with the length of the beam.

At the middle of the beam, $\delta_1 y$ is found by the usual methods to be

$$\delta_1 y = \frac{1}{A}\left(\frac{5}{24}\,l^3\,w + \frac{1}{6}\,l^3 W\right) \quad . \quad . \quad . \quad . \quad (66.)$$

Let $$B = \frac{m}{2}\int x\,dy = \frac{m}{2}\,\text{(sectional area)}. \quad . \quad . \quad (67.)$$

B is the resistance of the beam to the sliding of the planks. The deflection of the beam arising from this cause is

$$\delta_2 y = \frac{l}{2\,B}(w + W) \quad . \quad . \quad . \quad . \quad . \quad . \quad (68.)$$

The quantity is small compared with $\delta_1 y$, when the depth of the beam is small compared with its length.

The whole deflection $\Delta y = \delta_1 y + \delta_2 y$

$$\Delta y = \frac{l^3}{6\,A}\left(\frac{5}{4}\,w + W\right) + \frac{l}{2\,B}(w + W)$$

$$\Delta y = w\left(\frac{5}{24}\frac{l^3}{A} + \frac{1}{2}\frac{l}{B}\right) + W\left(\frac{l^3}{6\,A} + \frac{1}{2}\frac{l}{B}\right) \quad . \quad (69.)$$

Case XIII.

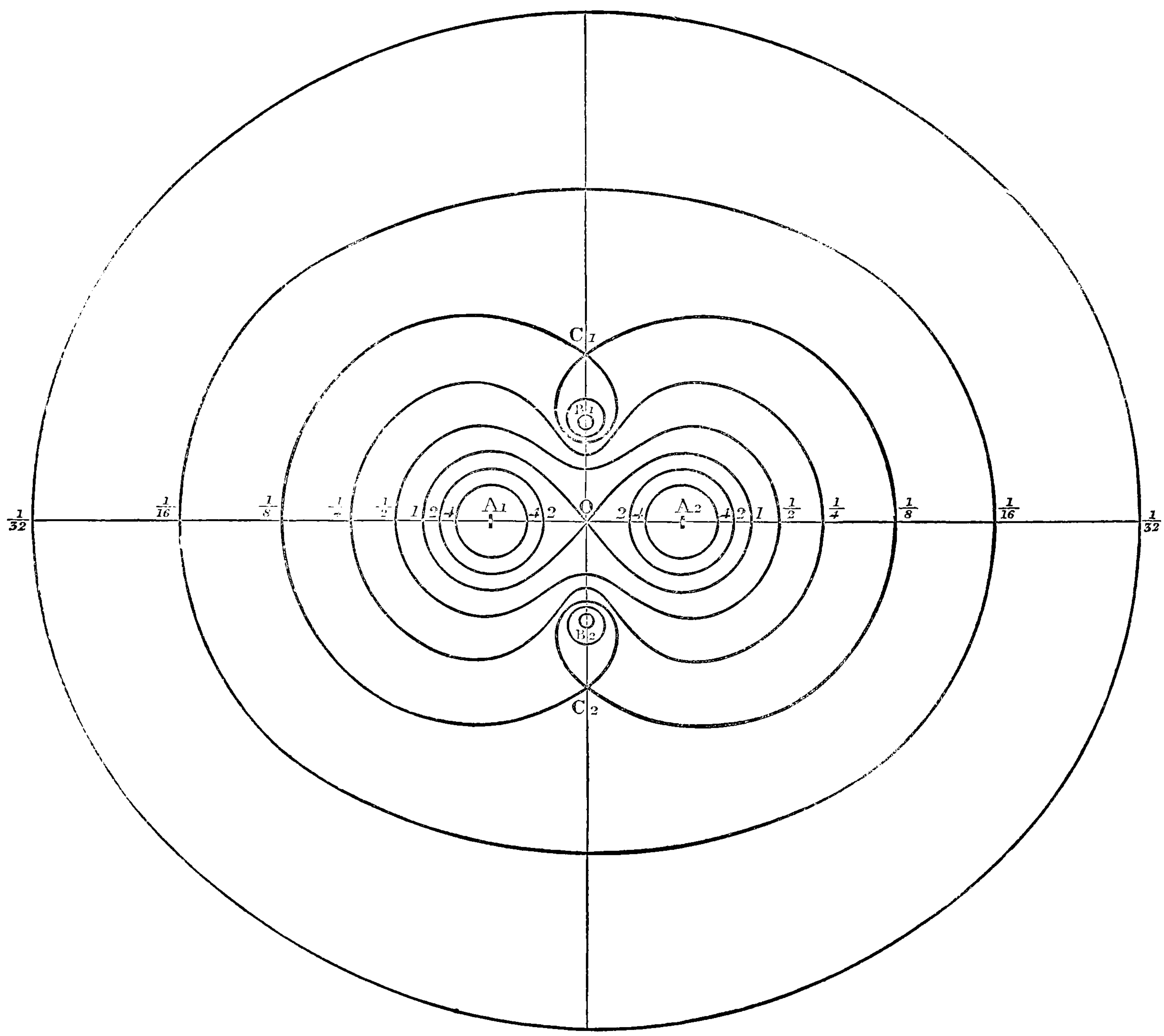
When the values of the compressions at any point have been found, when two different sets of forces act on a solid separately, the compressions, when the forces act at the same time, may be found by the composition of compressions, because the small compressions are independent of one another.

It appears from Case I., that if a cylinder be twisted as there described, the compressions will be inversely proportional to the square of the distance from the centre.

If *two* cylindric surfaces, whose axes are perpendicular to the plane of an indefinite elastic plate, be equally twisted in the same direction, the resultant compression in any direction may be found by adding the compression due to each resolved in that direction.

The result of this operation may be thus stated geometrically. Let A_1 and A_2 (fig. 1.) be the centres of the twisted cylinders. Join A_1 A_2, and bisect A_1 A_2 in O. Draw OBC at right angles, and cut off OB_1 and OB_2 each equal to OA_1.

Fig. 1.



Then the difference of the retardation of oppositely polarized rays of light passing perpendicularly through any point of the plane varies directly as the product of its distances from B_1 and B_2, and inversely as the square of the product of its distances from A_1 and A_2.

The isochromatic lines are represented in the figure.

The retardation is infinite at the points A_1 and A_2; it vanishes at B_1 and B_2;

and if the retardation at o be taken for unity, the isochromatic curves 2, 4, surround A_1 and A_2; that in which the retardation is unity has two loops, and passes through O; the curves $\frac{1}{2}$, $\frac{1}{4}$ are continuous, and have points of contrary flexure; the curve $\frac{1}{8}$ has multiple points at C_1 and C_2, where $A_1 C_1 = A_1 A_2$, and two loops surrounding B_1 and B_2; the other curves, for which $I=\frac{1}{16}, \frac{1}{32}$, &c., consists each of two ovals surrounding B_1 and B_2, and an exterior portion surrounding all the former curves.

I have produced these curves in the jelly of isinglass described in Case I. They are best seen by using circularly polarized light, as the curves are then seen without interruption, and their resemblance to the calculated curves is more apparent. To avoid crowding the curves toward the centre of the figure, I have taken the values of I for the different curves, not in an arithmetical, but in a geometrical progression, ascending by powers of 2.

Case XIV.

On the determination of the pressures which act in the interior of transparent solids, from observations of the action of the solid on polarized light.

Sir David Brewster has pointed out the method by which polarized light might be made to indicate the strains in elastic solids; and his experiments on bent glass confirm the theories of the bending of beams.

The phenomena of heated and unannealed glass are of a much more complex nature, and they cannot be predicted and explained without a knowledge of the laws of cooling and solidification, combined with those of elastic equilibrium.

In Case X. I have given an example of the inverse problem, in the case of a cylinder in which the action on light followed a simple law; and I now go on to describe the method of determining the pressures in a general case, applying it to the case of a triangle of unannealed plate-glass.

The lines of equal intensity of the action on light are seen without interruption, by using circularly polarized light. They are represented in fig. 2, where A, BBB, DDD are the neutral points, or points of no action on light, and CCC, EEE are the points where that action is greatest; and the intensity of the action at any other point is determined by its position with respect to the isochromatic curves.

The direction of the principal axes of pressure at any point is found by transmitting plane polarized light, and analysing it in the plane perpendicular to that of polarization. The light is then restored in every part of the triangle, except in those points at which one of the principal axes is parallel to the plane of polarization. A dark band formed of all these points is seen, which shifts its position as the triangle is turned round in its own plane. Fig. 3 represents these

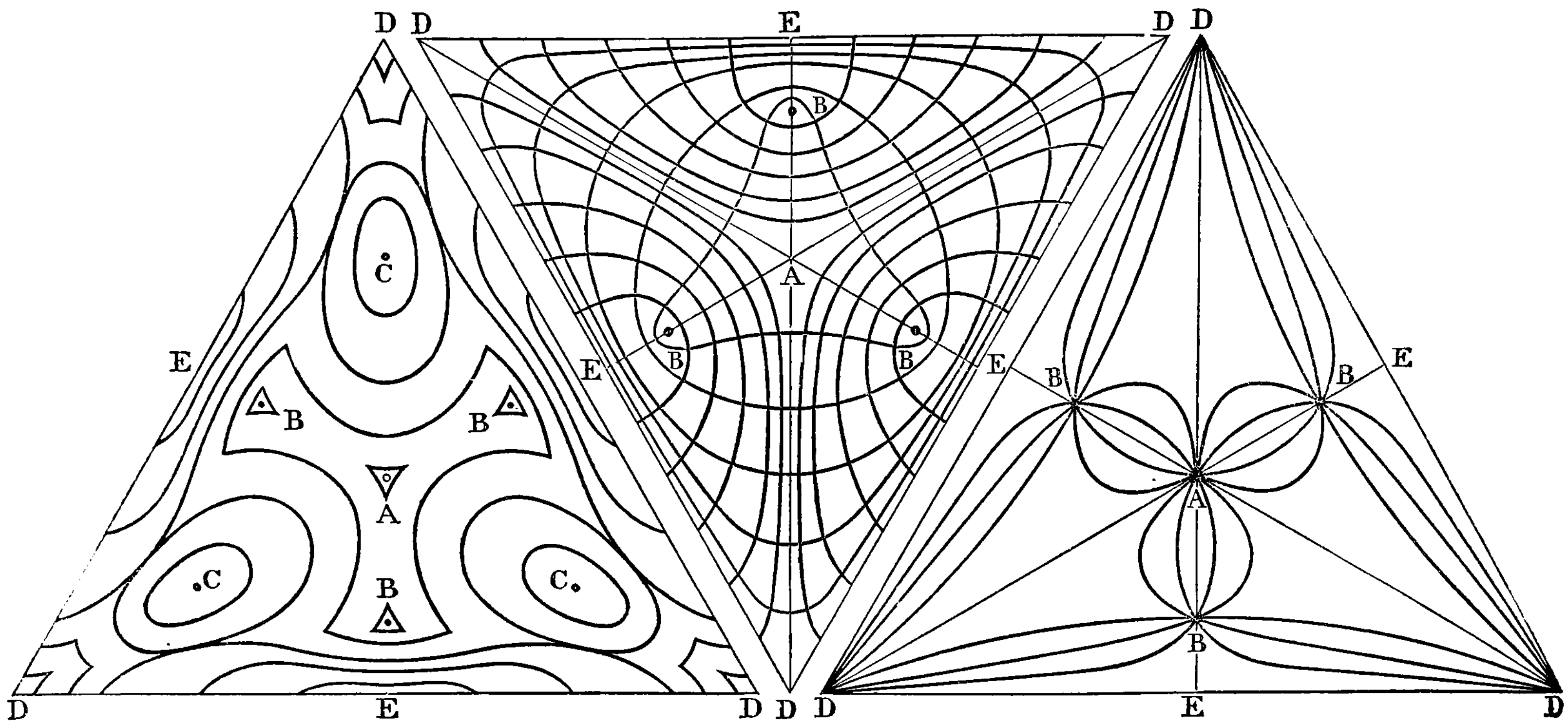


Fig. 2. Fig. 4. Fig. 3.

curves for every fifteenth degree of inclination. They correspond to the lines of equal variation of the needle in a magnetic chart.

From these curves others may be found which shall indicate, by their own direction, the direction of the principal axes at any point. These curves of direction of compression and dilatation are represented in fig. 4; the curves whose direction corresponds to that of *compression* are concave toward the centre of the triangle, and intersect at right angles the curves of dilatation.

Let the isochromatic lines in fig. 2 be determined by the equation

$$\phi_1 (x_1 y) = \mathrm{I}\frac{1}{z} = \omega (q - p) \frac{1}{z}$$

where I is the difference of retardation of the oppositely polarized rays, and q and p the pressure in the principal axes at any point, z being the thickness of the plate.

Let the lines of equal inclination be determined by the equation

$$\phi_2 (x_1 y) = \tan \theta$$

θ being the angle of inclination of the principal axes; then the differential equation of the curves of direction of compression and dilatation (fig. 4) is

$$\phi_2 (x_1 y) = \frac{dy}{dx}$$

By considering any particle of the plate as a portion of a cylinder whose axis passes through the centre of curvature of the curve of compression, we find

$$q - p = r \frac{dp}{dr} \quad . \quad . \quad (21.)$$

Let R denote the radius of curvature of the curve of compression at any point, and let S denote the length of the curve of dilatation at the same point,

$$\phi_3 (x_1 y) = \mathrm{R} \qquad \phi_4 (x_1 y) = \mathrm{S}$$

$$q - p = \mathrm{R} \frac{dp}{ds}$$

and since $(q-p)$, R and S are known, and since at the surface, where $\phi_5(x_1 y)=0$, $p=0$, all the data are given for determining the absolute value of p by integration.

Though this is the best method of finding p and q by graphic construction, it is much better, when the equations of the curves have been found, that is, when ϕ_1 and ϕ_2 are known, to resolve the pressures in the direction of the axes.

The new quantities are p_1, p_2, and q_3; and the equations are

$$\tan\theta=\frac{q_3}{p_1-p_2}, \qquad (p-q)^2=q_3{}^2+(p_1-p_2)^2, \qquad p_1+p_2=p+q$$

It is therefore possible to find the pressures from the curves of equal tint and equal inclination, in any case in which it may be required. In the meantime the curves of figs. 2, 3, 4 shew the correctness of Sir JOHN HERSCHELL'S ingenious explanation of the phenomena of heated and unannealed glass.

NOTE A.

As the mathematical laws of compressions and pressures have been very thoroughly investigated, and as they are demonstrated with great elegance in the very complete and elaborate memoir of MM. LAMÉ and CLAPEYRON, I shall state as briefly as possible their results.

Let a solid be subjected to compressions or pressures of any kind, then, if through any point in the solid lines be drawn whose lengths, measured from the given point, are proportional to the compression or pressure at the point resolved in the directions in which the lines are drawn, the extremities of such lines will be in the surface of an ellipsoid, whose centre is the given point.

The properties of the system of compressions or pressures may be deduced from those of the ellipsoid.

There are three diameters having perpendicular ordinates, which are called the *principal axes* of the ellipsoid.

Similarly, there are always three directions in the compressed particle in which there is no tangential action, or tendency of the parts to slide on one another. These directions are called the *principal axes* of *compression* or of *pressure*, and in homogeneous solids they always coincide with each other.

The compression or pressure in any other direction is equal to the sum of the products of the compressions or pressures in the principal axes multiplied into the squares of the cosines of the angles which they respectively make with that direction.

NOTE B.

The fundamental equations of this paper differ from those of NAVIER, POISSON, &c., only in not assuming an invariable ratio between the linear and the cubical elasticity; but since I have not attempted to deduce them from the laws of molecular action, some other reasons must be given for adopting them.

The experiments from which the laws are deduced are—

1st, Elastic solids put into motion vibrate isochronously, so that the sound does not vary with the amplitude of the vibrations.

2d, REGNAULT'S experiments on hollow spheres shew that both linear and cubic compressions are proportional to the pressures.

3d, Experiments on the elongation of rods and tubes immersed in water, prove that the elongation, the decrease of diameter, and the increase of volume, are proportional to the tension.

4th, In COULOMB'S balance of torsion, the angles of torsion are proportional to the twisting forces.

It would appear from these experiments, that compressions are always proportional to pressures.

Professor STOKES has expressed this by making one of his coefficients depend on the cubical elasticity, while the other is deduced from the displacement of shifting produced by a given tangential force.

M. CAUCHY makes one coefficient depend on the linear compression produced by a force acting in one direction, and the other on the change of volume produced by the same force.

Both of these methods lead to a correct result; but the coefficients of STOKES seem to have more of a real signification than those of CAUCHY; I have therefore adopted those of STOKES, using the symbols m and μ, and the fundamental equations (4.) and (5.), which define them.

NOTE C.

As the coefficient ω, which determines the optical effect of pressure on a substance, varies from one substance to another, and is probably a function of the linear elasticity, a determination of its value in different substances might lead to some explanation of the action of media on light.