Solutions of the Regge Equations on some Triangulations of CP^2

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Abstract

Simplicial geometries are collections of simplices making up a manifold together with an assignment of lengths to the edges that define a metric on that manifold. The simplicial analogs of the Einstein equations are the Regge equations. Solutions to these equations define the semiclassical approximation to simplicial approximations to a sum-over-geometries in quantum gravity. In this paper, we consider solutions to the Regge equations with cosmological constant that give Euclidean metrics of high symmetry on a family of triangulations of CP^2 presented by Banchoff and Kühnel. This family is characterized by a parameter p. The number of vertices grows larger with increasing p. We exhibit a solution of the Regge equations for $p = 2$ but find no solutions for $p = 3$. This example shows that merely increasing the number of vertices does not ensure a steady approach to a continuum geometry in the Regge calculus.

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I. INTRODUCTION

The sums over geometries that arise in quantum gravity may be approximated, and perhaps even defined, by the methods of the Regge Calculus.* Smooth, four-dimensional, geometries are approximated by simplicial geometries built from a finite number of four-simplices joined together so as to give a triangulation of a manifold. Different manifolds are represented by putting simplices together in different ways. Different metrics are obtained (in general) by different assignments of lengths to the edges of the simplices. A sum over topologies is approximated by a sum over different ways of assembling four-simplices. A sum over metrics on a given manifold is approximated by a multiple integral over the squared edge-lengths. In the Euclidean functional integral approach to quantum gravity typical integrals have the form

$$\int_C d\Sigma_1 A(s_i, M) \exp[-I(s_i, M)/\hbar] , \qquad (1.1)$$

where M is a closed, compact simplicial four-manifold, $I(s_i, M)$ is the Regge gravitational action, and $A(s_i, M)$ is some quantity of interest to be averaged in this way. Both I and A are functions of the squared edge-lengths $s_i, i = 1, ..., n_1$ which are integrated along a contour C with an appropriate measure $d\Sigma_1$. Such simplicial approximations to sums over geometries have been discussed in several places [3,4] and applied to a number of problems in quantum cosmology [5–7].

In some limits, the integral (1.1) may be evaluated semiclassically using the method of steepest descents. The value of the integral is then dominated by the contribution near one or more of stationary points of the action (through which the steepest descents distortion of the contour passes). At these,

$$\frac{\partial I(s_j, M)}{\partial s_i} = 0, \qquad i = 1, ..., n_1 . \qquad (1.2)$$

These algebraic equations are the simplicial analogs of the Einstein equations and are called the Regge equations. Solutions of the Regge equations on various triangulations of different manifolds are therefore of interest.

Large sets of algebraic equations like (1.2) are not always easy to solve, especially when the domain of the solution is constrained, as it is here because of the triangle inequalities and their higher dimensional analogs. However, when the triangulation has a symmetry, solutions consistent with that symmetry are more easily obtained. A number of solutions of this type on triangulations of the manifolds S^4 and CP^2 were exhibited in [8] (Paper II). Recently Banchoff and Kühnel [9] have exhibited a family of symmetric triangulations of the manifold CP^2. With such families, it is possible to observe the behavior of the Regge approximation to a given spacetime as the number of vertices is increasing. In this paper we investigate the solution of the Regge equations (1.2) on some of these triangulations.

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*The original paper is T. Regge [1]. For a review and extensive list of references see Williams and Tuckey [2].

For completeness, some requisite properties of CP^2 are summarized in Section II. The Banchoff-Kühnel triangulations are described briefly in Section III. They are characterized by an integer $p \geq 2$. With $n = p^2 + p + 1$, each triangulation CP^2_{n+3} has $p^2 + p + 4$ vertices, $3pn$ edges, $2(6p - 5)n$ triangles, $15(p - 1)n$ tetrahedra, $6(p - 1)n$ four-simplices, and a symmetry group of order $6n$. Maximally symmetric solutions with edges related by this symmetry group are specified by p independent edge-lengths. Thus, by varying p one has an increasingly large family of highly symmetric triangulations of CP^2.

We use a computer program to find analytic expressions for the action $I(s_i, CP^2_{n+3})$ as a function of the independent edges in a symmetric assignment of the edges. We exhibit a solution of the Regge equations (1.2) for $p = 2$ and compare it with that of one other triangulation [10] not in this family. For $p = 3$ we find no solution. This shows that, even with triangulations of high symmetry, one cannot always expect to find the discrete analogs of the continuum solutions of Einstein's equations. We discuss the reason this family of triangulations exhibits these phenomena.

II. THE FUBINI-STUDY METRIC OF CP^2

The complex projective plane CP^2 is defined by equivalence classes of points (Z^1, Z^2, Z^3) of the complex Euclidean 3-space C^3 with the origin excluded. The equivalence relation is $(Z^1, Z^2, Z^3) = (\lambda Z^1, \lambda Z^2, \lambda Z^3)$ for any non-zero complex λ. CP^2 possesses a continuum metric of high symmetry called the Fubini-Study metric. We follow Gibbons and Pope [11] in a brief review of its properties. Points in CP^2 may be labeled by coördinates

$$\zeta^i = Z^i/Z^3, \qquad i = 1, 2 \tag{2.1}$$

for $Z^3 \neq 0$.

As described in [11], the Fubini-Study metric on CP^2 is constructed from the Euclidean metric on the round 5-sphere in $\mathbf{C}^3$,

$$|Z^1|^2 + |Z^2|^2 + |Z^3|^2 = \frac{6}{\Lambda} , \tag{2.2}$$

where Λ is a constant. Defining Euler angles (ψ, θ, ϕ) and a radial coordinate r by

$$\zeta^1 = r \cos(\theta/2) \, e^{i(\psi+\phi)/2} , \tag{2.3}$$

$$\zeta^2 = r \sin(\theta/2) \, e^{i(\psi-\phi)/2} , \tag{2.4}$$

the Fubini-Study metric is

$$ds^2 = \frac{dr^2}{(1+\lambda r^2)^2} + \frac{r^2}{4(1+\lambda r^2)^2}(d\psi + \cos\theta d\phi)^2 + \frac{r^2}{4(1+\lambda r^2)}(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.5}$$

where $\lambda = \Lambda/6$. This coordinate system for $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, 0 \leq \psi \leq 4\pi$ and $0 \leq r \leq \infty$ covers $\mathbf{R}^4$ except at $r = 0, \theta = 0$ and $\theta = \pi$.

The Euclidean metric (2.5) satisfies the Einstein equation

$$R_{\alpha\beta} = 4\Lambda g_{\alpha\beta} \tag{2.6}$$

so that Λ is the cosmological constant. It is to this solution of the continuum Einstein equations that we shall compare our solutions of the Regge equations on triangulations of CP^2.

III. THE TRIANGULATION

A remarkable sequence of triangulations of the complex projective plane CP^2 has been presented by Banchoff and Kühnel [9]. It is based on Coxeter's regular map $\{3, 6\}_{1,p}$ of the flat torus T^2. The latter is defined by an array of

$$n = p^2 + p + 1 \qquad (3.1)$$

vertices (where $p = 2, 3, ...$) of a tessellation of the two-torus T^2 by equilateral triangles. Numbering these vertices by the non-negative integers $k \; (mod \; n)$, they may be located on a two dimensional "plane" in CP^2 at values of the coördinates (2.1)

$$\zeta^1 = e^{2\pi i k/n}, \qquad \zeta^2 = e^{-2\pi i p k/n} \; . \qquad (3.2)$$

The triangulation of CP^2_{n+3} is constructed as the union of three solid 4-balls B_1, B_2, B_3. The flat torus T^2 is the common intersection of the three balls. The triangulation of each of these balls consists of 4-simplices, four vertices of which lie on the boundary torus T^2 and one vertex at the point of CP^2 defined by the following points in C^3:

$$
\begin{aligned}
X &= (1, 0, 0) & &\text{for } B_1 \; , \\
Y &= (0, 1, 0) & &\text{for } B_2 \; , \\
Z &= (0, 0, 1) & &\text{for } B_3 \; . & &(3.3)
\end{aligned}
$$

Three kinds of chains of 4-simplices $C_m = C_m(X^i)$ are explicitly defined by one of the vertices $X^i = (X, Y, Z)$ and four of the vertices (3.2) as follows:

$$C_1 = \{j - 1, p + j - 1, (k+1)p + j - 1, (k+2)p + j - 1, X^i\} \; , \qquad (3.4\text{a})$$
$$C_2 = \{j - 1, p + j, (k+1)(p+1) + j - 1, (k+2)(p+1) + j - 1, X^i\} \; , \qquad (3.4\text{b})$$
$$C_3 = \{j - 1, j, k + j, j + k + 1, X^i\} \; (mod \; n) \; , \qquad j = 1, 2, ..., n \; , \; k = 1, 2, ..., p - 1. \quad (3.4\text{c})$$

For each p, the triangulation CP^2_{n+3} is given by the union of the three balls $B_i = C_j(X^i) \cup C_k(X^i)$, $\quad i = 1, 2, 3$, $\quad j, k \neq i$.

The number of 4-simplices in the triangulation CP^2_{n+3} is $n_4 = 6(p^3 - 1)$. One may employ the necessary relation for a combinatorial 4-manifold

$$5n_4 = 2n_3 \qquad (3.5)$$

together with the Euler number

$$n_0 - n_1 + n_2 - n_3 + n_4 = 3 \qquad (3.6)$$

and the Dehn-Sommerville relation

$$2n_1 - 3n_2 + 4n_3 - 5n_4 = 0 \qquad (3.7)$$

to compute the number n_k of k-simplices. The numbers are $n_0 = n + 3$, $n_1 = 3 p n$, $n_2 = 2(6p - 5) n$, $n_3 = 15(p - 1) n$, $n_4 = 6(p^3 - 1)$.

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Table I: The Orbits of Triangles for $p = 2$

Triangle Orbit	Are All Edges on T^2?	Number of Vertices on T^2	Length	Number of Incident 4-Simplices	Representative
α	yes	3	14	6	(0,1,3)
β	no	3	21	4	(0,1,4)
γ	no	2	21	5	(0,1,7)
δ	no	2	21	4	(0,1,8)
ϵ	no	2	21	3	(0,1,9)

The symmetry group of CP^2_{n+3} is generated by the cyclic permutations

$$
\begin{aligned}
T &= (k \to k + 1 \ (mod\, n)), \\
S &= (k \to pk \ (mod\, n)) \ (XYZ) \,, \\
R &= (k \to -k \ (mod\, n)) \,.
\end{aligned}
\tag{3.8}
$$

We denote the group by G_p.

When the symmetry group of the triangulation is an isometry of the simplicial geometry we say that we have a *maximally symmetric* geometry. We shall be interested only in maximally symmetric solutions of the Regge equations (1.1). As a consequence of this isometry, the edges of a maximally symmetric geometry lying on T^2 are all equal, and the length will be taken to be a. The cone edges from the vertices in T^2 to the ending vertices X, Y or Z are also all equal. The length of these will be taken to be b. There exist $p - 2$ trajectories of internal edges among the vertices of T^2 for $p > 2$. These will have the respective lengths c_k, $\quad k = 1, 2, ..., p - 2$.

The $2n(6p - 5)$ triangles are of three distinct types. Those with the edges lying on the boundary torus form a single orbit of length $2n$ under the isometries. Two orbits, from representative triangles $(0, 1, p+1)$ and $(0, p, p+1)$, are generated by the translations T and interchanged by the reflections R. For $p \geq 3$, there exist $p - 2$ orbits of interior triangles, each of length $3n$. The interior triangles have three vertices and two edges on the boundary torus and one edge inside a flat solid torus. The third type of triangle lies on a side of a cone, with one of the vertices being either X, Y or Z. The cone triangles form a single orbit of length $3n$ under the symmetries. There are $2p - 2$ orbits of the four-simplices defined by (3.4).

IV. SOLUTIONS AND NON-SOLUTIONS

In this Section we consider solutions to the Regge equations (1.2) on the triangulations CP^2_{n+3}. These equations define an extremum of the Regge action $I(s_i, M)$ that is the simplicial analog of the Euclidean continuum action

$$
\ell^2 I[g, M] = - \int_M d^4x \sqrt{g} \ (R - 2\Lambda)
\tag{4.1}
$$

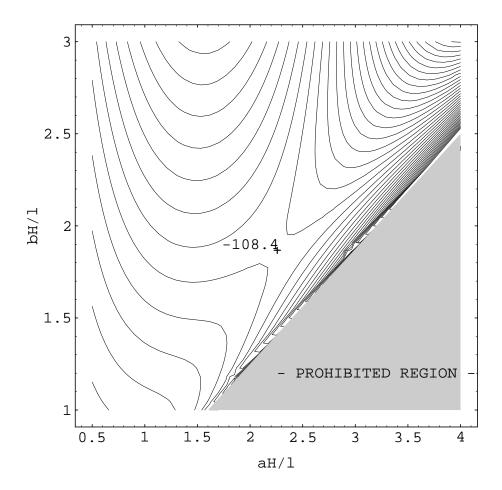*Figure 1. A contour map of the action for the the $p = 2$ triangulation CP_{10}^2. Contour lines of constant H^2I, where I is the action (4.2), are shown as a function of the two independent edge lengths a and b in a maximally symmetric assignment of edge lengths. The contour lines of H^2I are spaced by 2.35 dimensionless units. One or more of the triangle inequalities or their higher dimensional analogs are violated in the "prohibited region" at the lower right of the diagram. There is a saddle point and a solution of the Regge equations at $a = 2.26l/H$ and $b = 1.87l/H$.*

Here, ℓ is $\sqrt{16\pi}$ times the Planck length in units where $\hbar = c = 1$ and Λ is the cosmological constant. Specifically, the Regge action is, for a closed manifold,

$$\ell^2 I(s_i, M) = -2 \sum_{\sigma \in \Sigma_2} V_2(\sigma)\theta(\sigma) + \left(6H^2/\ell^2\right) \sum_{\tau \in \Sigma_4} V_4(\tau) \ . \tag{4.2}$$

The first sum is over all triangles and defines the curvature action $\ell^2 I_{\text{curv}}$. The area of triangle σ is $V_2(\sigma)$ and $\theta(\sigma)$ is its deficit angle which is 2π minus the sum of the dihedral angles between the tetrahedral faces of the four-simplices that meet σ. The second sum is over the four-simplices τ. $V_4(\tau)$ is the volume of the four-simplex τ, and we have introduced the dimensionless measure of the cosmological constant H^2 by writing $\Lambda = 3H^2/\ell^2$. For more details on the meaning of these quantities as well as an explanation of how to express them as functions of the squared edge-lengths s_i see, *e.g.* Paper I.

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Table II: Areas and Deficit Angles of the Triangles

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Triangle Orbit	Area	Deficit Angle
α	A_1	$2\pi - 6\theta_1$
β	A_1	$2\pi - 4\theta_1$
γ	A_2	$2\pi - 5\theta_2$
δ	A_2	$2\pi - 4\theta_2$
ϵ	A_2	$2\pi - 3\theta_2$

$$A_1 = \left(\sqrt{3}/4\right) a^2 \qquad A_2 = \tfrac{1}{4} a \left(4b^2 - a^2\right)^{\frac{1}{2}}$$
$$\theta_1 = \arccos\left\{ \sqrt{2a} / \left[4(3b^2 - a^2)^{\frac{1}{2}} \right] \right\}$$
$$\theta_2 = \arccos\left[(a^2 - 2b^2)/(2a^2 - 6b^2) \right]$$

We seek only maximally symmetric solutions of the Regge equations (1.1) as described in Section III. The simplest case is $p = 2$. The symmetry group G_2 is of order 42. Maximally symmetric geometries are characterized by the two independent edge lengths a and b. Thereare five orbits for the triangles under G_2 whose properties are spelled out in Table I. Their areas and deficit angles are given in Table II in terms of a and b. There are 2 orbits of length 21 for the four-simplices. In a maximally symmetric geometry all volumes have the same value:

$$V_4 = \frac{1}{4} a^3 (8b^2 - 3a^2)^{1/2} \ . \tag{4.3}$$

We developed a Mathematica program[†] to find various properties including the analytic form of the action (4.2) for an arbitrary triangulated spacetime in Regge calculus. The analytic expression for the action when $p = 2$ is

$$\ell^2 I = \frac{21\, a^3\, \sqrt{-3\, a^2 + 8\, b^2}\, H^2}{8\, l^2} - 35\, \sqrt{3}\, a^2\, \pi - 63\, \sqrt{-a^4 + 4\, a^2\, b^2}\, \pi$$
$$+ 84\, \sqrt{3}\, a^2\, \arccos\!\left(\frac{a}{2\, \sqrt{2}\, \sqrt{-a^2 + 3\, b^2}}\right)$$
$$+ 126\, \sqrt{-a^4 + 4\, a^2\, b^2}\, \arccos\!\left(\frac{-a^2 + 2\, b^2}{2\,(a^2 - 3\, b^2)}\right) \ . \tag{4.4}$$

For $p = 3$ the maximally symmetric geometries are specified by three independent edge-lengths a, b and c. The analytic expression for the Regge action is

$$\ell^2 I = \frac{39}{8} \frac{H^2}{l^2} \left(\sqrt{c^4(-2a^2 + c^2)(2a^2 - 8b^2 + c^2)} + \sqrt{a^2 c^2(-4a^4 + 12a^2 b^2 + a^2 c^2 - 4b^2 c^2)} \right)$$
$$- 13 \left(2\sqrt{3} a^2 \pi + 9\sqrt{-a^4 + 4a^2 b^2}\, \pi - 6\sqrt{3} a^2 \arccos\!\left(\frac{ac}{2\sqrt{(a^2 - 3b^2)(-3a^2 + c^2)}} \right) \right.$$
$$\left. - 3\sqrt{-a^4 + 4a^2 b^2}\, \arccos\!\left(\frac{-2a^4 + 6a^2 b^2 + a^2 c^2 - 4b^2 c^2}{2(a^4 - 3a^2 b^2)} \right) \right)$$

[†]This program is available from Z. Perjés

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$$- 12\sqrt{-a^4 + 4a^2b^2}\, \arccos\left(\frac{(-a^2+2b^2)c}{2\sqrt{(a^2-3b^2)A}}\right)$$

$$- 12\sqrt{-a^4 + 4a^2b^2}\, \arccos\left(\frac{(-a^2+2b^2)c^2}{2A}\right) + 9\pi\sqrt{4a^2c^2 - c^4}$$

$$- 12\arccos\left(\frac{c(2a^2-c^2)}{2\sqrt{2}\sqrt{(-2a^2+c^2)A}}\right)\sqrt{4a^2c^2 - c^4}$$

$$- 6\arccos\left(\frac{a^4 - a^2c^2/2}{\sqrt{a^2(-3a^2+c^2)A}}\right)\sqrt{4a^2c^2 - c^4} + 6\pi C$$

$$- 3\arccos\left(\frac{(a^2-2b^2)(-2a^2+c^2)}{2A}\right)C$$

$$- 6\arccos\left(\frac{-2a^4 + 8a^2b^2 - 6b^2c^2 + c^4}{2A}\right)C\bigg) \ , \tag{4.5}$$

where

$$A = \sqrt{a^4 - 4a^2b^2 + b^2c^2}, \qquad C = \sqrt{4b^2c^2 - c^4} \ . \tag{4.6}$$

And so on.

We now consider the solutions of the Regge equations (1.2) for $p = 2$ and $p = 3$. For maximally symmetric solutions it is enough to extremize the actions (4.4) and (4.5) as a function of the independent squared edge lengths. We searched for solutions by a combination of graphical and numerical techniques — using contour maps of the action to understand its behavior and then numerical evaluations of the Regge equations to find solutions. We studied the norm $N(s_i) \equiv \Sigma_i (\partial I/\partial s_i)^2$ to locate solutions accurately at its zeros. We compared results following from the analytic forms (4.4) and (4.5) with a general code which computes each of the Regge equations $\partial I/\partial s_i$, $i = 1, \cdots, n_1$ without assumption of symmetry. There was agreement.

A contour map of the action of the $p = 2$ triangulation CP^2_{10} is shown in *Figure 1*. Using the methods described above we found we found a maximally symmetric solution at

$$a = 2.26(\ell/H) \quad , \quad b = 1.87(\ell/H) \ . \tag{4.7}$$

The properties of this solution can be compared with the maximally symmetric continuum metric — the Fubini-Study metric described in Section II — and with the solution on the Kühnel-Lassmann triangulation (CP^2_9) of CP^2 discussed in Paper II. To do that we compare the values of two invariants among the different solutions. One is the total volume V_{tot} and the other is the curvature action defined by

$$V_{\mathrm{tot}} = \sum_{\tau \in \Sigma_4} V_4(\tau) \equiv \mathcal{B}\,(\ell/H)^4 \ , \tag{4.8}$$

$$I_{\mathrm{curv}} = -2\sum_{\sigma \in \Sigma_2} V_2(\sigma)\theta(\sigma) \equiv -\mathcal{A}\left(V_{\mathrm{tot}}/\ell^4\right)^{\frac{1}{2}} \ . \tag{4.9}$$

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Table III: Comparison of Solutions

	Curvature Action $\mathcal{A}$	Total Volume $\mathcal{B}$
CP_9^2	50.	18.
CP_{10}^2	51.	18.
Fubini-Study	53.29.	19.74.

Table III shows the dimensionless constants $\mathcal{A}$ and $\mathcal{B}$ for the various solutions.[‡] For the continuum Fubini-Study metric we have in particular

$$\mathcal{A} = 12\sqrt{2}\,\pi \quad , \quad \mathcal{B} = 2\pi^2 \, . \tag{4.10}$$

By these measures the progression from CP_9^2 to CP_{10}^2 is only slight and both give tolerable approximations to the continuum results.

For $p = 3$ we found no solution. A careful search of the norm $N(s_i) = \Sigma_i (\partial I / \partial s_i)^2$ showed it approaching a minimum value but that value was not zero. We do not find a saddle point for the triangulation CP_{16}^2. Moving from $p = 2$ to $p = 3$ therefore results in a worse approximation to the continuum Fubini-Study metric.

V. EIGENVALUES OF THE SECOND VARIATION

In addition to the extrema of the action, its second variation

$$I_{ij}^{(2)} = \frac{\partial^2 I}{\partial s_i \partial s_j} \tag{5.1}$$

is important for the evaluation of the semiclassical approximation to sums over simplicial geometries like (1.1). The prefactor of the semiclassical approximation is the inverse square root of the determinant of $I_{ij}^{(2)}$ evaluated at the contributing extrema. This determinant is the product of the eigenvalues of $I_{ij}^{(2)}$. The signs of these eigenvalues determine the steepest descents directions at the extrema. Small eigenvalues signal the approximate diffeomorphism invariance that occurs in Regge calculus [3]. For these and other reasons the eigenvalues of (5.1) at the extrema of the action are of interest. The eigenvalues of $I_{ij}^{(2)}$ for the $p = 2$ symmetric solution are shown in Table IV. The table shows the eigenvalues λ and their multiplicities ρ_λ. The eigenvalues of a maximally symmetric solution may be classified by the irreducible representations of that group and the multiplicities of these eigenvalues are given by the dimensions of these irreducible representations.

[‡]In Paper II, the constant $\mathcal{A}$ was denoted by a in eq.(3.4) and Table II. We use upper case here to avoid confusion with the edge-length a. The value of $\mathcal{A}$ for the Fubini-Study metric was quoted incorrectly in Table II of Paper II. Its correct value is in eq.(4.10) of this paper.

Table IV: Eigenvalues and Multiplicities of $\partial^2 I / \partial s_i \partial s_j$
Evaluated at the $p = 2$ Stationary Point

$\ell^4 H^2 \lambda$	ρ_λ	$\ell^4 H^2 \lambda$	ρ_λ
-1.74	1	-.12	6
-1.24	2	-.05	6
- .63	6	+.07	6
- .39	6	+.28	1
- .18	8		

There are seven classes of the group G_2 whose orders and characters are given in Table V. This shows the classes arranged horizontally with their orders on the top line and the value of the characters on those classes arranged vertically for the seven possible irreducible representations. An element of the symmetry group sends some edges into others thereby producing a 42 dimensional reducible representation of G_{42}. The irreducible representations it contains label the eigenvalues of I_{ij}. The number of times a given irreducible representation occurs in this reducible representation may be found by analyzing its characters into the characters of the irreducible representations. (For more explicit details see *e.g.* Paper II, Section IV.) The result is that the reducible representation decomposes as $2 \cdot 1_1 + 0 \cdot 1_2 + 0 \cdot 1_3 + 0 \cdot 1_4 + 2 \cdot 1_5 + 2 \cdot 1_6 + 6 \cdot 6_1$. Here, the individual terms in this sum correspond to the the irreducible representations in Table II and occur in the order listed there. The second factor in each summand is the dimension of the irreducible representation and multiplication indicates the number of times it occurs in the 42-dimensional reducible representation. The multiplicities of the eigenvalues calculated numerically and shown in Table II are consistent with this analysis although there is an unexplained degeneracy between a 6 and two 1's resulting in the multiplicity of 8 and between two 1's resulting in a multiplicity of 2. This kind of degeneracy also occurs with the eigenvalues of the symmetric solution of CP_9^2 (Paper II) in the flat, symmetric assignment of edge lengths to certain triangulations of T^3 [14]. It thus seems to be a feature of a wide range of triangulations.

Other features of the eigenvalues are immediately apparent from Table IV. Two of the eigenvalues are positive and the rest are negative. The largest positive eigenvalue corresponds to a uniform change in all edges. The number of positive eigenvalues available to represent true physical degrees of freedom is thus slightly larger than in the slightly smaller triangulation, CP_9^2, discussed in Paper II.

Some eigenvalues are small but none are exactly zero. There are thus directions in the space of edge lengths in which the action varies slowly as it would for an approximate diffeomorphism but none in which it is exactly constant.

Table V: Character Table for the Symmetry Group of the $p = 2$ Triangulation

1	6	7	7	7	7	7
1	1	1	1	1	1	1
1	1	-1	1	1	-1	-1
1	1	-1	ϵ	ϵ^2	$-\epsilon$	$-\epsilon^2$
1	1	-1	ϵ^2	ϵ	$-\epsilon^2$	$-\epsilon$
1	1	1	ϵ	ϵ^2	ϵ	ϵ^2
1	1	1	ϵ^2	ϵ	ϵ^2	ϵ
6	-1	0	0	0	0	0

$\epsilon = \exp(2\pi i/3)$

VI. DISCUSSION

One might have naively expected that the more refined the triangulation of a manifold M is, the better the approximation the solution to it's Regge equations will be to the solution in the continuum. The Banchoff-Kühnel sequence CP^2_{n+3} shows this is not generally the case. In the first two steps one moves from a solution which is a reasonable approximation by some measures to no solution at all! The reason for this is to be found in the nature of the refinement. As we take triangulations of higher p it is the triangulation of torus T^2 which is refined, while the number of "exterior" vertices (X, Y, Z) remain fixed. If the vertices are embedded in the Fubini-Study geometry as described in [9], the edges in T^2 better approximate the Fubini-Study distances between vertices with increasing p. However, the cone edges b is constant.

This example shows that sequences of triangulations of a given manifold will have to be chosen carefully if the solutions of the Regge equations are to converge to the continuum solution of the Einstein equations.

ACKNOWLEDGMENTS

REFERENCES

[1] T. Regge, *Nuovo Cimento*, **19**, 558 (1961).

[2] Williams and Tuckey, *Class. Quant. Grav.*, **9**, 1409 (1992).

[3] J.B. Hartle, *J. Math. Phys.*, **26**, 804 (1985). (Paper I)

[4] H. Hamber, *Nucl. Phys. B*, 400, 347 (1993); *ibid.* 435, 361 (1995).

[5] D. Eliezer, *Simplicial Lattice Methods in Cosmology*, unpublished Ph.D. dissertation, University of California, Santa Barbara, available from University Microfilms, Ann Arbor, MI.

[6] P. Morse, *Regge Calculus and Quantum Cosmology*, unpublished Ph.D. dissertation, University of California, Santa Barbara, available from University Microfilms, Ann Arbor, MI.

[7] Y. Furihata, *No Boundary Wave Function for Simplicial Anisotropic Universe* (preprint).

[8] J.B.Hartle, *J. Math. Phys.*, **27**, 287 (1986). (Paper II)

[9] T.F. Banchoff and W. Kühnel, *Geometriae Dedicata*, **44**, 313 (1992).

[10] W. Kühnel and G. Lassmann, *J. Combin. Theory Ser*, **A135**, 173 (1983).

[11] G.W. Gibbons and C.N. Pope, *Commun. Math. Phys.*, **61**, 239 (1978).

[12] H.S.M. Coxeter and W.G.J. Moser, *Generators and Relations for Discrete Groups*, Springer, Berlin (1980).

[13] J.B.Hartle, *J. Math. Phys.*, **30**, 452 (1989).

[14] J.B. Hartle, W. Miller, and R.M. Williams, *Signature of the Simplicial Supermetric* (to be published).