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# Signature of the Simplicial Supermetric

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# Abstract

We investigate the signature of the Lund-Regge metric on spaces of simplicial three-geometries which are important in some formulations of quantum gravity. Tetrahedra can be joined together to make a three-dimensional piecewise linear manifold. A metric on this manifold is specified by assigning a flat metric to the interior of the tetrahedra and values to their squared edge-lengths. The subset of the space of squared edge-lengths obeying triangle and analogous inequalities is simplicial configuration space. We derive the Lund-Regge metric on simplicial configuration space and show how it provides the shortest distance between simplicial three-geometries among all choices of gauge inside the simplices for defining this metric (Regge gauge freedom). We show analytically that there is always at least one physical timelike direction in simplicial configuration space and provide a lower bound on the number of spacelike directions. We show that in the neighborhood of points in this space corresponding to flat metrics there are spacelike directions corresponding to gauge

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freedom in assigning the edge-lengths. We evaluate the signature numerically for the simplicial configuration spaces based on some simple triangulations of the three-sphere  $(S^3)$  and three-torus  $(T^3)$ . For the surface of a four-simplex triangulation of  $S^3$  we find one timelike direction and all the rest spacelike over all of the simplicial configuration space. For the triangulation of  $T^3$  around flat space we find degeneracies in the simplicial supermetric as well as a few gauge modes corresponding to a positive eigenvalue. Moreover, we have determined that some of the negative eigenvalues are physical, i.e. the corresponding eigenvectors are not generators of diffeomorphisms. We compare our results with the known properties of continuum superspace.

### I. INTRODUCTION

The superspace of three-geometries on a fixed manifold [1] plays an important role in several formulations of quantum gravity. In Dirac quantization [2], wave functions on superspace represent states. In generalized quantum frameworks [3], sets of wave functions on superspace define initial and final conditions for quantum cosmology. The geometry of superspace is therefore of interest and has received considerable attention. The notion of distance that defines this geometry is induced from the DeWitt supermetric on the larger space of three-metrics [2,5]. While the properties of the supermetric on the space of metrics are explicit, the properties of the induced metric on the space of three-geometries are only partially understood [6,7]. For example, the signature of the metric on superspace, which is of special interest for defining spacelike surfaces in superspace, is known only in certain regions of this infinite dimensional space. In this paper we explore the signature of the metric on simplicial approximations to superspace generated by the methods of the Regge calculus [8].

Tetrahedra (three-simplices) can be joined together to make a three-dimensional, piecewise linear manifold. A metric on this manifold may be specified by assigning a flat metric to the interior of the simplices and values to their  $n_1$  squared edge-lengths. Not every value of the squared edge-lengths is consistent with a Riemannian metric (signature +++) on the simplicial manifold. Rather, the squared edge-lengths must be positive, satisfy the triangle inequalities, and the analogous inequalities for tetrahedra. The region of an  $\mathbb{R}^{n_1}$  whose axes are the squared edge-lengths  $t^i$ ,  $i = 1, \dots, n_1$ , where these inequalities are satisfied is a space of simplicial configurations we call simplicial configuration space.<sup>3</sup> The DeWitt supermetric induces a metric on simplicial configuration space. Lund and Regge [10] have given a simple expression for this metric and its properties have been explored by Piran and Williams [11] and Friedman and Jack [12]. In this paper we explore the signature of the Lund-Regge metric for several simplicial manifolds by a combination of analytical and numerical techniques. In contrast to the continuum problem, we are able to explore the signature over the whole of the finite dimensional simplicial configuration spaces.

In Section II we review the construction of the metric on the superspace of continuum three-geometries and summarize the known information on its signature. In Section III we show how the Lund-Regge simplicial metric is induced from the continuum metric and analytically derive a number of results limiting its signature. Section IV explores the signature numerically for a number of elementary, closed, simplicial manifolds. We study first the surface of a four-simplex. We find that throughout its 10-dimensional configuration space

<sup>&</sup>lt;sup>1</sup>For some representative earlier articles see [2,4,5]. For more recent articles that also review the current situation see [6,7].

<sup>&</sup>lt;sup>2</sup>The induced metric on the superspace of three metrics might also be called the DeWitt metric since it was first explored by DeWitt [5]. However, to avoid confusion we reserve the term "DeWitt metric" for the metric on the space of three-metrics.

<sup>&</sup>lt;sup>3</sup>This is the "truncated" superspace of [9]

that among a basis of orthogonal vectors there is one timelike direction and 9 spacelike ones. We next study the Lund-Regge metric of a three-torus at various lattice resolutions (ranging from a 189-dimensional to a 1764-dimensional simplicial configuration space) in the neighborhood of the single point representing a flat metric. We find that the Lund-Regge metric can be degenerate, change signature, and have more than one physical time-like directions. We conclude with a comparison with known continuum results.

## II. CONTINUUM SUPERSPACE

In this section we shall briefly review some of the known properties of the metric on the superspace of continuum geometries on a fixed three-manifold M. We do this to highlight the main features that we must address when analyzing the corresponding metric on the superspace of simplicial geometries. A more detailed account of the continuum situation can be readily found [1,2,4-7].

Geometries on M can be represented by three-metrics  $h_{ab}(x)$ , although, of course, different metrics describe the same geometry when related by a diffeomorphism. We denote the space of three-metrics on M by  $\mathcal{M}(M)$ . A point in  $\mathcal{M}$  is a particular metric  $h_{ab}(x)$  and we may consider the tangent space of vectors at a point. Infinitesimal displacements  $\delta h_{ab}(x)$  from one three-metric to another are particular examples of vectors. We denote such vectors generally by  $k_{ab}(x)$ ,  $k'_{ab}(x)$ , etc. A natural class of metrics on  $\mathcal{M}(M)$  emerges from the structure of the constraints of general relativity. Explicitly they are given by

$$(k',k) = \int_{M} d^{3}x N(x) \bar{G}^{abcd}(x) k'_{ab}(x) k_{cd}(x)$$
(2.1)

where  $\bar{G}^{abcd}(x)$ , called the inverse DeWitt supermetric, is given by

$$\bar{G}^{abcd}(x) = \frac{1}{2} h^{\frac{1}{2}}(x) \left[ h^{ac}(x) h^{bd}(x) + h^{ad}(x) h^{bc}(x) - 2h^{ab}(x) h^{cd}(x) \right]$$
 (2.2)

and N(x) is an essentially arbitrary but non-vanishing function called the lapse. Different metrics result from different choices of N(x). In the following we shall confine ourselves to the simplest choice, N(x) = 1.

The DeWitt supermetric (2.2) at a point x defines a metric on the six-dimensional space of three-metric components at x. This metric has signature (-,+,+,+,+,+) [2]. The signature of the metric (2.1) on  $\mathcal{M}$  therefore has an infinite number of negative signs and an infinite number of positive signs — roughly one negative sign and five positive signs for each point in M.

The space of interest, however, is not the space of three-metrics  $\mathcal{M}(M)$  but rather the superspace of three-geometries  $\mathrm{Riem}(M)$  whose "points" consist of classes of diffeomorphically equivalent metrics,  $h_{ab}(x)$ . A metric on  $\mathrm{Riem}(M)$  can be induced from the metric on  $\mathcal{M}(M)$ , (2.1), by choosing a particular perturbation in the metric  $\delta h_{ab}(x)$  to represent the infinitesimal displacement between two nearby three-geometries. However, a  $\delta h_{ab}$  is not fixed uniquely by the pair of nearby geometries. Rather, as is well known, there is an arbitrariness in  $\delta h_{ab}(x)$  corresponding to the arbitrariness in how the points in the two geometries are identified. That arbitrariness means that, for any vector  $\xi^a(x)$ , the "gauge-transformed" perturbation

$$\delta h'_{ab}(x) = \delta h_{ab}(x) + D_{(a}\xi_{b)}(x) , \qquad (2.3)$$

represents the same displacement in superspace as  $\delta h_{ab}(x)$  does, where  $D_a$  is the derivative in M.

The metric (2.1) on  $\mathcal{M}(M)$  is not invariant under gauge transformations of the form (2.3) even with N=1. Thus we may distinguish "vertical" directions in  $\mathcal{M}(M)$  which are pure gauge

$$k_{ab}^{\text{vertical}}(x) = D_{(a}\xi_{b)}(x) \tag{2.4}$$

and "horizontal" directions which are orthogonal to all of these in the metric (2.1).

Since the metric (2.1) is not invariant under gauge transformations, there are different notions of distance between points in superspace depending on what  $\delta h_{ab}(x)$  is used to represent displacements between them. The conventional choice [5] for defining a geometry on superspace has been to choose the *minimum* of such distances between points. That is the same as saying that distance is measured in "horizontal" directions in superspace. Equivalently, one could say that a gauge for representing displacements has been fixed. It is the gauge specified by the three conditions

$$D^b (k_{ab} - h_{ab}k_c^c) = 0. (2.5)$$

The signature of the metric defined by the above construction is an obvious first question concerning the geometry of superspace. The infinite dimensionality of superspace, however, makes this a non-trivial question to answer. The known results have been lucidly explained by Friedman and Higuchi [6] and Giulini [7] and we briefly summarize some of them here:

• At any point in superspace there is always at least one negative direction represented by constant conformal displacements of the form

$$k_{ab} = \delta \Omega^2 h_{ab}(x) . {2.6}$$

Evidently (2.6) satisfies (2.5) so that it is horizontal, and explicit computation from (2.1) shows  $(k, k) \leq 0$ ;

- If M is the sphere  $S^3$ , then for a neighborhood of the round metric on  $S^3$ , the signature has one negative sign corresponding to (2.6) and all other orthogonal directions are positive;
- Every *M* admits geometries with negative Ricci curvature (all eigenvalues strictly negative). In the open region of superspace defined by negative Ricci curvature geometries the signature has an *infinite* number of negative signs and an *infinite* number of positive signs. On the sphere, these results already show that there must be points in superspace where the metric is degenerate.

The above results are limited, covering only a small part of the totality of superspace. In the following we shall show that more complete results can be obtained in simplicial configuration space.

#### III. THE LUND-REGGE METRIC

#### A. Definition

In this Section we derive the form of the Lund-Regge metric on simplicial configuration space together with some analytic results on its signature. We consider a fixed closed simplicial three-manifold M consisting of  $n_3$  tetrahedra (three-simplices) joined together so that each neighborhood of a point in M is homeomorphic to a region of  $\mathbb{R}^3$ . The resulting collections of  $n_k$  k-simplices, (vertices, edges, triangles, and tetrahedra for k = 0, 1, 2, 3, respectively) we denote by  $\Sigma_k$ . A simplicial geometry is fixed by an assignment of values to the squared edge-lengths of M,  $t^m$ ,  $m = 1, \dots, n_1$  and a flat Riemannian geometry to the interior of each tetrahedron consistent with those values. The assignment of squared edge-lengths is not arbitrary. The squared edge-lengths are positive and constrained by the triangle inequalities and their analogs for the tetrahedra. Specifically if  $V_k^2(\sigma)$  is the squared measure (length, area, volume) of k-simplex  $\sigma$  expressed as a function of the  $t^m$ , we must have

$$V_k^2(\sigma) \ge 0 \ , \quad k = 1, 2, 3$$
 (3.1)

for all  $\sigma \in \Sigma_k$ . The space of three-geometries on M is therefore the subset of the space of  $n_1$  squared edge-lengths  $t^m$  in which (3.1) is satisfied. We call this *simplicial configuration space* and denote it by  $\mathcal{T}(M)$ . A point in  $\mathcal{T}(M)$  is a geometry on M; the  $\{t^m\}$  are coordinates locating points in  $\mathcal{T}(M)$ .

Distinct points in  $\mathcal{T}(M)$  correspond to different assignments of edge-lengths to the simplicial manifold M. In general distinct points correspond to distinct three-geometries and, in this respect,  $\mathcal{T}(M)$  is like a superspace of three-geometries. However, this is not always the case. Displacements of the vertices of a flat geometry in a flat embedding space result in a new assignment of the edge-lengths that corresponds to the same flat geometry. These variations in edge-lengths that preserve geometry are the simplicial analogs of diffeomorphisms [13,14]. Further, for large triangulations where the local geometry is near to flat we expect there to be approximate simplicial diffeomorphisms — small changes in the edge-lengths which approximately preserve the geometry [13,15,16]. Thus, the continuum limit of  $\mathcal{T}(M)$  is not the superspace of three-geometries but the space of three-metrics. It is for these reasons that we have used the term simplicial configuration space rather than simplicial superspace.

We now define a metric on  $\mathcal{T}(M)$  that gives the distance between points separated by infinitesimal displacements  $\delta t^m$  according to

$$\delta S^2 = G_{mn}(t^{\ell})\delta t^m \delta t^n \ . \tag{3.2}$$

Such a metric can be induced from the DeWitt metric on the space  $\mathcal{M}(M)$  of continuum three-metrics on M in the following way:

Every simplicial geometry can be represented in  $\mathcal{M}(M)$  by a metric which is piecewise flat in the tetrahedra and, indeed, there are many different metrics representing the same geometry. Every displacement  $\delta t^m$  between two nearly three-geometries can be represented by a perturbation  $\delta h_{ab}(x)$  of the metric in  $\mathcal{M}(M)$ . The DeWitt metric (2.1) which gives the notion of distance between nearby metrics in  $\mathcal{M}(M)$  can therefore be used to induce a notion of distance in  $\mathcal{T}(M)$  through the relation

$$G_{mn}(t^{\ell})\delta t^{m}\delta t^{n} = \int_{M} d^{3}x N(x) \,\bar{G}^{abcd}(x)\delta h_{ab}(x)\delta h_{cd}(x) . \qquad (3.3)$$

On the right hand side  $\bar{G}^{abcd}(x)$  is (2.2) evaluated at a piecewise flat metric representing the simplicial geometry and  $\delta h_{ab}(x)$  is a perturbation in the metric representing the change in that geometry corresponding to the displacement  $\delta t^m$ .

However, as the discussion of Section II should make clear, many different metrics on  $\mathcal{T}(M)$  can be induced by the identification (3.3). First, there is the choice of N(x). We choose N(x) = 1. Second, since the right hand side of (3.3) is not gauge invariant we must fix a gauge for the perturbations  $\delta h_{ab}$  to determine a metric  $G_{mn}(t^{\ell})$ . There are two parts to this. First, to evaluate the integral on the right hand side of (3.3) we must at least fix the gauge inside each tetrahedron. We shall refer to this as the Regge gauge freedom. It is important to emphasize, however, that any choice for the Regge gauge does not completely fix the total gauge freedom available. As discussed above, there still may be variations of the lengths of the edges which preserve the geometry – simplicial diffeomorphisms — and correspondingly  $\mathcal{T}(\mathcal{M})$  can still have both vertical and horizontal directions. Therefore, secondly, this gauge freedom must also be fixed.

A natural choice for the Regge gauge from the point of view of simplicial geometry is to require that the  $\delta h_{ab}$  are *constant* inside each tetrahedron,

$$D_c \delta h_{ab}(x) = 0$$
, inside each  $\tau \in \Sigma_3$ , (3.4)

but possibly varying from one tetrahedron to the next. The conditions (3.4) are, of course, more numerous than the three diffeomorphism conditions permitted at each point, but as we already mentioned, these two gauges are distinct. Thus, (3.4) is not the Regge calculus counterpart of (2.5). Nevertheless, this choice of Regge-gauge (3.4) has a beautiful property which we shall discuss below in Subsection B.

Assuming (3.4), the right hand side of (3.3) may be evaluated explicitly. Although not gauge invariant, the right-hand-side of (3.3) is coordinate invariant. We may therefore conveniently use coordinates in which the metric coefficients  $h_{ab}(x)$  satisfying (3.4) are constant in each tetrahedron. Then, using (2.2),

$$G_{mn}(t^{\ell})\delta t^{m}\delta t^{n} = \sum_{\tau \in \Sigma_{3}} V(\tau) \left\{ \delta h_{ab}(\tau) \delta h^{ab}(\tau) - \left[ \delta h_{a}^{a}(\tau) \right]^{2} \right\}$$
(3.5)

where  $V(\tau)$  is the volume of tetrahedron  $\tau$  and we have written  $h_{ab}(\tau)$ ,  $\delta h_{ab}(\tau)$ , etc for the constant values of these tensors inside  $\tau$ .

To proceed further we need explicit expressions for  $h_{ab}(\tau)$  in terms of  $t^{\ell}$ , and for  $\delta h_{ab}(\tau)$  in terms of  $t^{\ell}$  and  $\delta t^{\ell}$ . One way of making an explicit identification is to pick a particular vertex in each tetrahedron (0) and consider the vectors  $\mathbf{e}_a(\tau)$ , a=1,2,3 proceeding from this vertex to the other three vertices (1,2,3) along the edges of the tetrahedron. The metric  $h_{ab}(\tau)$  in the basis defined by these vectors is

$$h_{ab}(\tau) = \mathbf{e}_a(\tau) \cdot \mathbf{e}_b(\tau)$$
  
=  $\frac{1}{2} (t_{0a} + t_{0b} - t_{ab})$  (3.6)

where  $t_{AB}$  is the squared edge-length between vertices A and B. Eq (3.6) gives an explicit expression for the metric in each tetrahedron in terms of its squared edge-lengths in a basis adapted to its edges. The perturbation of (3.6),

$$\delta h_{ab}(\tau) = \frac{1}{2} \left( \delta t_{0a} + \delta t_{ab} - \delta t_{0b} \right) , \qquad (3.7)$$

gives an explicit expression for the perturbation in  $h_{ab}(\tau)$  induced by changes in the squared edge-lengths. (In general (3.7) changes discontinuously from tetrahedron to tetrahedron.) Eq (3.7) is an explicit realization of the gauge condition (3.4). Only trivial linear transformations of the form (2.3) inside the tetrahedra preserve (3.4), and there are none of these that preserve the simplicial structure in the sense that  $\xi^a(x)$  vanishes on the boundary of the tetrahedra. In this sense (3.7) fixes the Regge gauge for the perturbations.

An explicit expression for the  $G_{mn}(t^{\ell})$  defined by (3.3), (3.5), and (3.7) may be obtained by studying the expression<sup>4</sup> for the squared volume of tetrahedron  $\tau$ ,

$$V^{2}(\tau) = \frac{1}{(3!)^{2}} \det[h_{ab}(\tau)] . \tag{3.8}$$

Consider a perturbation  $\delta t^m$  in the squared edge-lengths. The left hand side of (3.8) may be expanded (dropping the label  $\tau$ ) as

$$V^{2}(t^{\ell} + \delta t^{\ell}) = V^{2}(t^{\ell}) + \frac{\partial V^{2}(t^{\ell})}{\partial t^{m}} \delta t^{m} + \frac{1}{2} \frac{\partial^{2} V^{2}(t^{\ell})}{\partial t^{m} \partial t^{n}} \delta t^{m} \delta t^{n} + \cdots$$
(3.9)

The right hand side may be expanded using the identity

$$\det A = \exp[Tr\log(A)] \tag{3.10}$$

as

$$\det(h_{ab} + \delta h_{ab}) = \det(h_{ab})\det(\delta_b^a + \delta h_b^a)$$

$$= \det(h_{ab}) \left\{ 1 + \delta h_a^a + \frac{1}{2} \left[ \delta h_{ab} \delta h^{ab} - (\delta h_a^a)^2 \right] + \cdots \right\} . \tag{3.11}$$

Equating (3.9) and (3.11) gives, at first order, the identity

$$\delta h_a^a(\tau) = \frac{1}{V^2} \frac{\partial V^2(\tau)}{\partial t^m} \, \delta t^m \,, \tag{3.12}$$

and at second order gives a relation which leads through (3.5) to the following elegant expression for the metric  $G_{mn}(t^{\ell})$ :

$$G_{mn}(t^{\ell}) = -\sum_{\tau \in \Sigma_2} \frac{1}{V(\tau)} \frac{\partial^2 V^2(\tau)}{\partial t^m \partial t^n} . \tag{3.13}$$

 $<sup>^{4}</sup>$ See, e.g. [13] for a derivation.

This is the Lund-Regge metric [10] on simplicial configuration space  $\mathcal{T}(M)$ . It is an explicit function of the squared edge-lengths  $t^m$  through (3.8) and (3.6). The metric may be reexpressed in a number of other ways of which a useful example is

$$G_{mn}(t^{\ell}) = -2 \left[ \frac{\partial^2 V_{\text{TOT}}}{\partial t^m \partial t^n} + \sum_{\tau \in \Sigma_3} \frac{1}{V(\tau)} \frac{\partial V(\tau)}{\partial t^m} \frac{\partial V(\tau)}{\partial t^n} \right]$$
(3.14)

where  $V_{\text{TOT}}$  is the total volume of M.

The metric (3.14) becomes singular at the boundary of  $\mathcal{T}(\mathcal{M})$  where  $V(\tau)$  vanishes for one or more tetrahedra. However, locally, since  $V^2$  is a third order polynomial in the t's,  $G_{mn} \sim (t^{\ell} - t_b^{\ell})^{-1/2}$  where  $t_b^{\ell}$  is a point on the boundary. A generic boundary point is therefore a finite distance from any other point in  $\mathcal{T}(\mathcal{M})$  as measured by the metric  $G_{mn}$ .

## B. Comparison with Nearby Regge Gauge Choices

The identification of points in a perturbed and unperturbed geometry is ambiguous up to a displacement  $\xi^a(x)$  in the point in the perturbed geometry identified with  $x^a$  in the unperturbed geometry. As a consequence any two perturbations  $\delta h_{ab}(x)$  which differ by a gauge transformation (2.3) represent the same displacement in the space  $\mathcal{T}(M)$  of three-geometries. In the continuum case of Section II, we followed DeWitt [5] and fixed this ambiguity by minimizing the right-hand-side of (3.3) over all possible gauge transformations  $\xi^a(x)$  so that distance between three-geometries was measured along "horizontal" directions in  $\mathcal{M}(M)$ . In the previous subsection we fixed the ambiguity in the comparison of the continuum space of piecewise metrics to simplicial lattices by requiring perturbations to be constant over tetrahedra [cf. (3.4)]. We can now show that the distance defined in this way is a local minimum with respect to other Regge gauge choices that preserve the simplicial structure, in a sense to be made precise below.

Consider the first variation of the right-hand-side of (3.3) with N(x) = 1 that is produced by an infinitesimal gauge transformation  $\xi^a(x)$ . This is

$$\int_{M} d^{3}x \bar{G}^{abcd}(x) \delta h_{ab}(x) D_{(c} \xi_{d)}(x) . \qquad (3.15)$$

Integrating by parts and making use of the symmetry of  $\bar{G}^{abcd}$ , this first variation can be written

$$-\sum_{\tau \in \Sigma_3} \int_{\tau} d^3x \bar{G}^{abcd}(x) D_c \delta h_{ab}(x) \xi_d(x) + \sum_{\sigma \in \Sigma_2} \int_{\sigma} d\Sigma \left[ \left| n_c \bar{G}^{abcd}(x) \delta h_{ab}(x) \right| \right] \xi_d(x) . \tag{3.16}$$

In this expression, the first term is a sum of volume integrals over the individual tetrahedra in M. The second term is an integral over triangles where, for a particular triangle,  $n^a$  is a unit outward pointing normal and  $\begin{bmatrix} & & \\ & & \end{bmatrix}$  denotes the discontinuity across the triangle. Such a term must be included since we do not necessarily assume that the non-gauge invariant argument of (3.15) is continuous from tetrahedron to tetrahedron. The conditions (3.4) make the first term vanish. The second vanishes when  $\xi^a(x)$  vanishes on the boundary of every tetrahedron. That means that the distance defined by the Lund-Regge metric is an extremum among all re-identifications of points in the *interiors* of the tetrahedra between the

perturbed and unperturbed geometries. It does not appear to necessarily be an extremum with respect to re-identifying points in the interior of triangles or edges. The Lund-Regge metric therefore provides the shortest distance between simplicial three-metrics among all choices of Regge gauge which vanish on the triangles. However it is not exactly "horizontal" in the sense of the continuum because of the possibility of simplicial diffeomorphisms. We shall see explicit consequences of this below.

## C. Analytic Results on the Signature

We are interested in the signature of  $G_{mn}$  on  $\mathcal{T}(M)$ . We shall calculate the signature numerically for some simple M in Section IV, but here we give a few analytic results which characterize it incompletely.

# 1. The Timelike Conformal Direction

The conformal perturbation defined by

$$\delta t^m = \delta \Omega^2 \ t^m \tag{3.17}$$

is always timelike. This can be seen directly from (3.5) by noting that (3.7) and (3.6) imply

$$\delta h_{ab}(\tau) = \delta \Omega^2 \ h_{ab}(\tau) \ . \tag{3.18}$$

However, it can also be verified directly from (3.13) using the fact that  $V^2$  is a homogeneous polynomial of degree three in the  $t^m$ . Then it follows easily from Euler's theorem that

$$G_{mn}(t^{\ell})t^{m}t^{n} = -6V_{\text{TOT}}(t^{\ell}) < 0$$
 (3.19)

The timelike conformal direction is not an eigenvector of  $G_{mn}$  because

$$G_{mn}t^n = -4\partial V_{TOT}/\partial t^m . (3.20)$$

We do not expect  $\partial V_{TOT}/\partial t^m$  to be proportional to  $t^m$  except for symmetric assignments of the edge lengths on highly symmetric triangulations.

The same relation shows that the conformal direction is orthogonal to any gauge direction  $\delta t^n$  because

$$G_{mn}t^m\delta t^n = -4(\partial V_{TOT}/\partial t^n)\delta t^n = -4\delta V_{TOT}. (3.21)$$

This vanishes for any change in edge lengths which does not change the geometry.

2. At Least  $n_1 - n_3$  Spacelike Directions

Eqs (3.5) and (3.12) can be combined to show that

$$\widetilde{G}_{mn}\delta t^{m}\delta t^{n} \equiv \left[G_{mn} + 4\sum_{\tau \in \Sigma_{3}} \frac{1}{V(\tau)} \frac{\partial V(\tau)}{\partial t^{m}} \frac{\partial V(\tau)}{\partial t^{n}}\right] \delta t^{m}\delta t^{n}$$

$$= \sum_{\tau \in \Sigma_{3}} V(\tau) \left[\delta h_{ab}(\tau)\delta h^{ab}(\tau)\right] \geq 0 \tag{3.22}$$

Thus

$$G_{mn} = \tilde{G}_{mn} - 4 \sum_{\tau \in \Sigma_3} \frac{1}{V(\tau)} \frac{\partial V(\tau)}{\partial t^m} \frac{\partial V(\tau)}{\partial t^n}$$
(3.23)

where  $\tilde{G}_{mn}$  is positive. Some displacements  $\delta t^m$  will leave the volumes of all the tetrahedra unchanged:

$$\frac{\partial V(\tau)}{\partial t^m} \, \delta t^m = 0 \quad , \quad \tau \in \Sigma_3. \tag{3.24}$$

These directions are clearly spacelike from (3.23). Since (3.24) is  $n_3$  conditions on  $n_1$  displacements  $\delta t^m$  we expect at least  $n_1 - n_3$  independent spacelike directions.

# 3. Signature of Diffeomorphism Modes

In general any change  $\delta t^m$  in the squared edge-lengths of M changes the three-geometry. A flat simplicial three-geometry is an exception. Locally a flat simplicial geometry may be embedded in Euclidean  $\mathbf{R}^3$  with the vertices at positions  $\mathbf{x}_A$ ,  $A=1,\cdots,n_0$ . Displacements of these locations result in new and different edge-lengths, but the flat geometry remains unchanged. Such changes in the edge-lengths  $\delta t^m$  are called gauge directions in simplicial configuration space. Each vertex may be displaced in three directions making a total of  $3n_0$  gauge directions. We shall now investigate whether these directions are timelike, spacelike, or null.

We evaluate  $\delta S^2$  defined by (3.2) and (3.14) for displacements  $\delta \mathbf{x}_A$  in the locations of the vertices. If an edge connects vertices A and B, its length is

$$t^{AB} = (\mathbf{x}_A - \mathbf{x}_B)^2 \equiv (\mathbf{x}_{AB})^2 \tag{3.25}$$

and the change in length  $\delta t^{AB}$  from a variation in position  $\delta \mathbf{x}_A$  follows immediately. The total volume is unchanged by any variation in position of the  $\mathbf{x}_A$ , which means that

$$\frac{\partial V_{TOT}}{\partial x_A^i} = 0 \qquad , \qquad \frac{\partial^2 V_{TOT}}{\partial x_A^i \partial x_B^j} = 0 \tag{3.26}$$

and so on. These derivatives are related to those with respect to the edge-lengths by the chain rule. Thus (3.26) does not imply that  $\partial^2 V_{TOT}/\partial t^{AC}\partial t^{BC}$  is zero, but only that

$$\frac{\partial^2 V_{TOT}}{\partial t^m \partial t^n} \delta t^m \delta t^n = -\frac{\partial V_{TOT}}{\partial t^{AB}} \frac{\partial^2 t^{AB}}{\partial x_A^i \partial x_B^j} \delta x_A^i \delta x_B^j$$
(3.27)

where  $\delta t^m = (\partial t^m/\partial x_A^i)\delta x_A^i$ , with summation over both A and i. Inserting (3.27) and the chain rule relations into (3.14), we obtain

$$\delta S^{2} \equiv G_{mn}(t^{\ell})\delta t^{m}\delta t^{n} =$$

$$-2\left[-\frac{\partial V_{TOT}}{\partial t^{n}} \frac{\partial^{2} t^{n}}{\partial x_{A}^{i} \partial x_{B}^{j}} + \sum_{\tau \in \Sigma_{3}} \frac{1}{V(\tau)} \frac{\partial V(\tau)}{\partial t^{m}} \frac{\partial V(\tau)}{\partial t^{n}} \frac{\partial t^{m}}{\partial x_{A}^{i}} \frac{\partial t^{n}}{\partial x_{B}^{j}}\right] \delta x_{A}^{i} \delta x_{B}^{j} .$$

$$(3.28)$$

To simplify this, we go from sums over edges to sums over the corresponding vertices, (C, D), with a factor of 1/2 for each sum. Then

$$\delta S^2 = \frac{\partial V_{TOT}}{\partial t^{CD}} \frac{\partial^2 t^{CD}}{\partial x_A^i \partial x_B^j} \delta x_A^i \delta x_B^j - \frac{1}{2} \sum_{\tau \in \Sigma_2} \frac{1}{V(\tau)} \left[ \frac{\partial V(\tau)}{\partial t^{CD}} \frac{\partial t^{CD}}{\partial x_A^i} \delta x_A^i \right]^2 . \tag{3.29}$$

Using the explicit relations (3.25), we find

$$\delta S^{2} = 2 \left\{ \frac{\partial V_{TOT}}{\partial t_{CD}} \left( \delta \mathbf{x}_{C} - \delta \mathbf{x}_{D} \right)^{2} - \sum_{\tau \in \Sigma_{3}} \frac{1}{V(\tau)} \left[ \frac{\partial V(\tau)}{\partial t_{CD}} \mathbf{x}_{CD} \cdot \left( \delta \mathbf{x}_{C} - \delta \mathbf{x}_{D} \right) \right]^{2} \right\}.$$
(3.30)

The second term of (3.30) is negative definite, but the first does not appear to have a definite sign, so that a general statement on the character of gauge modes does not emerge. However, more information is available in the specific cases to be considered below.

#### IV. NUMERICAL INVESTIGATION OF THE SIMPLICIAL SUPERMETRIC

The Lund-Regge metric can be evaluated numerically to give complete information about its signature over the whole of the simplicial configuration space which confirms the incomplete but more general analytic results obtained above. We consider specifically as manifolds the three-sphere  $(S^3)$ , and the three-torus  $(T^3)$ . For the simplest triangulation of  $S^3$  we investigate the signature over the whole of its simplicial configuration space. For several triangulations of  $T^3$  we investigate a limited region of their simplicial configuration space near flat geometries. Even though in both cases the triangulations are rather course, a number of basic and interesting features emerge.

The details of the triangulations in the two cases are described below but the general method of calculation is as follows. Initial edge-lengths are assigned (consistent with the triangle and tetrahedral equalities) and the supermetric calculated using (3.13). The eigenvalues of the metric  $G_{mn}$  are then calculated and the numbers of positive, negative and zero values counted. To explore other regions of the simplicial configuration space, the edgelengths are repeatedly updated (in such a manner so as to ensure that the squared-measures (see (3.1)) of all the triangles and tetrahedra are positive) and the eigenvalues and hence the signature of the supermetric are found. In the case of the  $T^3$  triangulations we also calculate the deficit angles which give information on the curvature of the simplicial geometry, and in addition, we explore the geometry of the neighboring points in simplicial configuration space along each eigenvector.

We now describe the details of the two numerical calculations for the two different manifolds.

# A. The 3-Sphere

The simplest triangulation of  $S^3$  is the surface of a four-simplex, which consists of five vertices, five tetrahedra, ten triangles, and ten edges. Thus the simplicial configuration space is 10-dimensional (Figure 1). Each point in this space represents a particular assignment of lengths to each of the ten edges of the 4-simplex. To numerically explore all of this 10-dimensional space would be foolish as the space contains redundant and ill-defined regions. To avoid the ill-defined parts we need only to restrict ourselves to those points in the configuration space that satisfy the two and three dimensional simplicial inequalities (Eq. 3.1). However, to avoid redundancy we must examine the invariance properties of the the eigenvalue spectrum of the Lund-Regge metric.

Let us begin by examining an invariance of the Lund-Regge eigenvalue spectrum under a global scale transformation. The Lund-Regge metric scales as  $G_{mn} \longrightarrow L^{-1}G_{mn}$  under an overall rescaling of the edges  $l_i \longrightarrow Ll_i$  where  $l_i = \sqrt{t^i}$ . The signature is scale invariant, so we may use this invariance to impose one condition on the  $l_i$ . We found it most convenient numerically to fix  $l_0 = 1$  as the longest edge. The ten-dimensional space has now collapsed to a nine-dimensional subspace. As this is the only invariance we have identified, we then further restrict our investigation to the points in this subspace which satisfy the various simplicial inequalities. Specifically, writing  $\overline{AB}$  as the edge between vertex A and vertex B:

- $l_0 \equiv \overline{34} = 1$  and  $l_0$  is the longest edge;
- $l_3 \equiv \overline{03} \in (0,1];$
- $l_4 \equiv \overline{04} \in [(1 l_3), 1];$
- $l_6 \equiv \overline{13} \in (0,1];$
- $l_7 \equiv \overline{14} \in [(1 l_6), 1];$
- $l_1 \equiv \overline{01} \in (\tau_-, \tau_+)$  where  $\tau_{\pm}$  is obtained from the tetrahedral inequality applied to edge  $\overline{01}$  of tetrahedron (0134),

$$\tau_{\pm}^{2} = \frac{1}{2} \left( -t_{0} + t_{3} + t_{4} + t_{6} - \frac{t_{3}t_{6}}{t_{0}} + \frac{t_{4}t_{6}}{t_{0}} + t_{7} + \frac{t_{3}t_{7}}{t_{0}} - \frac{t_{4}t_{7}}{t_{0}} + \frac{t_{4}t_{7}}{t_{0}} \right)$$

$$\pm \sqrt{(t_{0}^{2} - 2t_{0}t_{3} + t_{3}^{2} - 2t_{0}t_{4} - 2t_{3}t_{4} + t_{4}^{2})}$$

$$\sqrt{(t_{0}^{2} - 2t_{0}t_{6} + t_{6}^{2} - 2t_{0}t_{7} - 2t_{6}t_{7} + t_{7}^{2})/t_{0}}$$

$$(4.1)$$

- $l_8 \equiv \overline{23} \in (0,1];$
- $l_9 \equiv \overline{24} \in [(1 l_8), 1];$
- $l_2 \equiv \overline{02} \in (\tau_-, \tau_+)$ , where  $\tau_{\pm}$  is obtained from a tetrahedral inequality applied to edge  $\overline{02}$  of tetrahedron  $\{0234\}$ ;

•  $l_5 \equiv \overline{12} \in (s_-, s_+)$ , where  $s_{\pm}$  is obtained from the tetrahedral inequality as applied to the three tetrahedra ( (1234), (0123) and (0124) ) sharing edge  $\overline{12}$ , with

$$s_{-} = \max\{\tau_{-}^{(1234)}, \tau_{-}^{(0123)}, \tau_{-}^{(0124)}\}$$

$$s_{+} = \min\{\tau_{+}^{(1234)}, \tau_{+}^{(0123)}, \tau_{+}^{(0124)}\}.$$
(4.2)

Considering only this region, we were able to sample the whole of the configuration space. Subdividing the unit interval into ten points would ordinarily entail a calculation of the eigenvalues for  $10^{10}$  points. However, by using the scaling law for the Regge-Lund metric together with the various simplicial inequalities we only needed to calculate the eigenvalues for 102160 points. We observed exactly one timelike direction and nine spacelike directions for each point in the simplicial configuration space, even though the distortions of our geometry from sphericity were occasionally substantial — up to 10 to 1 deviations in the squared edge-lengths from their symmetric values. Recall that a conformal displacement is a timelike direction in simplicial configuration space [cf. (3.19)]. However this direction will coincide with the timelike eigenvector of  $G_{mn}$  only when all the edge lengths are equal as follows from (3.20).

There are clearly more than the  $n_1 - n_3 = 5$  spacelike directions required by the general result of III C 2. We conclude that in this case the signature of the Lund-Regge metric is (-, +, +, +, +, +, +, +, +, +) over the whole of simplicial configuration space.

More detailed information about the Lund-Regge metric beyond the signature is contained in the eigenvalues themselves. Predictions of their degeneracies arise from the symmetry group of the triangulation.

For the boundary of the 4-simplex the symmetry group is the permutation group on the five vertices,  $S_5$ . If all the edge lengths are assigned symmetrically — all equal edge-lengths in the present case — then the eigenvalues may be classified according to the irreducible representations of  $S_5$  and their degeneracies are given by the dimensions of those representations. This is because a permutation of the vertices can be viewed as a matrix in the 10-dimensional space of edges which interchanges the edges in accordance with the permutation of the vertices. The Regge-Lund metric  $G_{mn}$  may be viewed similarly and commutes with all the elements of  $S_5$  for a symmetric assignment of edges. The matrices representing the elements of  $S_5$  give a 10-dimensional reducible representation of it, which can be decomposed into irreducible representations by standard methods described in [17]. The result is that the reducible representation decomposes as 1+4+5=10, where the factors in this sum are dimensions of the irreducible representations, which we expect to be the multiplicities of the corresponding eigenvalues of  $G_{mn}$  at a symmetric assignment of edges.

The results of numerical calculations of the eigenvalues are illustrated in Figure 2 for a slice of simplicial configuration space. When all the edges are equal to 1 we found one eigenvalue of  $-1/2\sqrt{2}$ , four eigenvalues equal to  $1/3\sqrt{2}$ , and five of  $5/6\sqrt{2}$ . As expected, these degeneracies were broken when we departed from the spherical geometry (Figure 2). Nevertheless, even with aspect ratios on the order of 10 : 1 we always found a single timelike direction in this simplicial superspace.

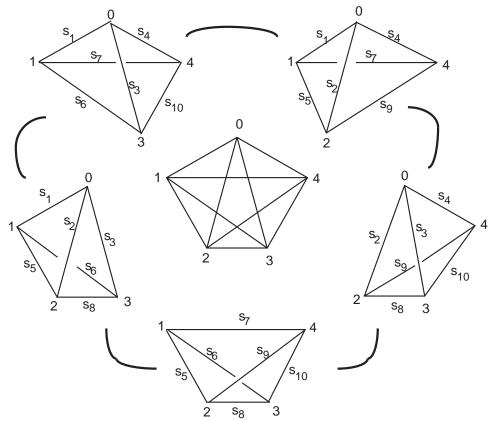


FIG. 1. The boundary of a 4-simplex as a 10-dimensional simplicial configuration space model for  $S^3$ . This figure shows the five tetrahedra corresponding to the boundary of the central 4-simplex (0,1,2,3,4) exploded off around its perimeter. The 4-simplex has 5 vertices, 10 edges, 10 triangles, and 5 tetrahedra. The topology of the boundary consisting of those tetrahedra is that of a 3-sphere. The specification of the 10 squared edge-lengths of the 4-simplex completely fixes its geometry, and represents a single point in the 10-dimensional simplicial configuration space. Here we analyze the geometry of this space using the Regge-Lund metric and show that there is one, and only one, timelike direction.

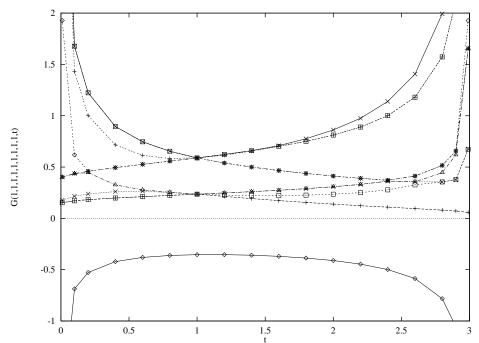


FIG. 2. The eigenvalue spectrum along a 1-dimensional cut through the 10-dimensional simplicial configuration space. We plot the 10 eigenvalues (not necessarily distinct) of the Lund-Regge metric  $G_{mn}$  with nine of the ten edges set to unity and the remaining squared edge-length (t)varies from 0 to 3 (a range obtained directly from the tetrahedral inequality). When t = 1 we have maximal symmetry and we have three distinct eigenvalues for the ten eigenvectors as expected with degeneracies 1, 4, and 5 as predicted. As we move away from this point one can see that the degeneracies are for the most part broken. However, there remains a twofold degeneracy in the 5 arising from the remaining symmetries in the assignments of the edge-lengths.

### V. THE 3-TORUS

In this section we analyze the Lund-Regge metric in the neighborhood of two different flat geometries on a common class of triangulations of the three-torus,  $T^3$ . We investigate the metric on triangulations of varying refinement in this class. We illustrate degeneracy of the metric and identify a few gauge modes (vertical directions) in flat space which correspond to positive eigenvalues.

The class of triangulations of  $T^3$  is constructed as follows: A lattice of cubes, with  $n_x$ ,  $n_y$  and  $n_z$  cubes in the x, y and z-directions, is given the topology of a 3-torus by identifying opposite faces in each of the three directions. Each cube is divided into six tetrahedra, by drawing in face diagonals and a body diagonal (for details, see Rŏcek and Williams [18]). The number of vertices is then  $n_0 = n_x n_y n_z$  and there are  $7n_0$  edges,  $12n_0$  triangles, and  $6n_0$  tetrahedra.

The geometry is flat when the squared edge lengths of the sides, face diagonals, and body diagonals take values of 1, 2, and 3 times the lattice scale respectively. We refer to this geometry as the *right-tetrahedron lattice*.

The flat 3-torus can also be tessellated by isosceles tetrahedra, each face of which is an isosceles triangle with a squared base edge of 1, the other two squared edge-lengths being 3/4. We refer to this as the *isosceles-tetrahedron lattice*. One can obtain this lattice from the right-tetrahedron lattice by the following construction on each cube. Compress the cube along its main diagonal in a symmetrical way, keeping all the "coordinate edges" at length 1, until the main diagonal is also of length 1. The face diagonals will then have squared edge-lengths of 4/3. An overall rescaling by a factor of  $\sqrt{3}/2$  then converts the lattice of these deformed cubes to the isosceles-tetrahedron lattice.

These two lattices correspond to distinct points in the  $7n_0$  dimensional simplicial configuration space of a triangulation of  $T^3$ . They are examples of inequivalent flat geometries on  $T^3$ . Even though they can be obtained from each other by the deformation procedure described, they are associated with distinct flat structures. For example, consider the geodesic structure. For the right-tetrahedron lattice, there are three orthogonal geodesics of extremal length at any point, parallel to the coordinate edges of the cubes of which the lattice is constructed, and corresponding to the different meridians of the torus. On the other hand, for the isosceles-tetrahedron lattice, the three extremal geodesics at any point will not be orthogonal to each other; they will actually be parallel to the "coordinate edges" which in the deformation process have moved into positions at angles of  $\arccos(-1/3)$  to each other. (It is perhaps easier to visualize the analogous situation in two dimensions where the geodesics will be at angles of  $\arccos(-1/2)$  to each other). If the metric structures on the two lattices we consider were diffeomorphically equivalent, the diffeomorphism would preserve geometric quantities, like the angles between the extremal-length geodesics, and this is clearly not the case. We shall see how this inequivalence between the flat tori manifests itself in the simplicial supermetric.

We first turn to a detailed examination of the the isosceles-tetrahedron lattice and, in particular, the eigenvalues of the Lund-Regge metric. Unlike the relatively simple  $S^3$  model described above where we used Mathematica to calculate the eigenvalue spectrum of the matrix  $G_{mn}$ , here we developed a C program utilizing a Householder method to determine the eigenvalue spectrum and corresponding eigenvectors. In addition, we calculated the

deficit angle (integrated curvature) associated to each edge. These deficit angles were used to identify diffeomorphism and conformal directions as well as to corroborate the analytic results for the continuum described in the Introduction. We performed various runs ranging from an isosceles-tetrahedron lattice with  $3 \times 3 \times 3$  vertices and 189 edges, up to a lattice with  $6 \times 6 \times 7$  vertices and 1764 edges. The points (simplicial 3-geometries) in such high dimensional simplicial configuration spaces cannot be systematically canvassed as we did in the 4-simplex model. For this reason we chose to search the neighborhood of flat space in two ways. First we explored the region around flat space by making random variations (up to 20 percent) in the squared edge-lengths of the isosceles-tetrahedron lattice, and secondly we perturbed the edge-lengths a short distance along selected flat-space eigenvectors.

Movement along the eigenvectors was performed in the following way. We start with a set of squared edge-lengths corresponding to zero curvature. All the  $7n_0$  deficit angles are zero. We calculate the  $7n_0$  eigenvectors,  $v = \{v_j, j = 1, 2, ..., 7n_0\}$  together with the corresponding eigenvalues  $\lambda_j$  and then adjust the squared edge-lengths along one of the  $v_j$  as specified in:

$$t_{new}^i = t_{flat}^i + \epsilon v^j, \quad \forall i \in \{1, 2, \dots, 7n_0\}.$$

$$(5.1)$$

Here we choose  $\epsilon \ll 1$ . We then calculate the deficit angles for this new point  $t_{new}$  and then repeat this procedure for each  $v_j$  in turn. In this way we can in principle identify which (if any) of the eigenvectors correspond to gauge directions. We observe the following:

- Eigenvectors corresponding to eigenvalues  $\lambda = 1/2$  appear to be diffeomorphisms to order  $\epsilon^3$  in the sense that the deficit angles are of order  $\epsilon^3$ .
- For an  $n_0 = n \times n \times n$  lattice there are 6n 4 eigenvectors corresponding to  $\lambda = 1/2$ .
- There are  $n_0 + 2$  eigenvectors corresponding to  $\lambda = 1$ .

A graphical representation of this movement along the eigenvectors away from the flat isosceles-tetrahedron lattice point is illustrated in Figure 3.

Although the eigenvalues corresponding to those gauge directions we have identified is positive, we have not been able to establish all gauge directions are spacelike. For the right-tetrahedron lattice we have simplified the expression (3.30) obtained for diffeomorphism modes as follows. The derivatives of  $V(\tau)$  with respect to the diagonal edges all vanish, so that only the edges of the cubic lattice need be included in the (C, D) sum. For lattice spacing of 1, these derivatives are all 1/12, giving  $\partial V_{TOT}/\partial t^{CD} \equiv 1/2$ , since each such edge is shared by 6 tetrahedra. Thus for this lattice we have

$$\delta S^2 = \sum_{(C,D)} (\delta \mathbf{x}_C - \delta \mathbf{x}_D)^2 - \frac{1}{12} \sum_{\tau \in \Sigma_3} \left[ \sum_{(C,D) \in \tau} \mathbf{x}_{CD} \cdot (\delta \mathbf{x}_C - \delta \mathbf{x}_D) \right]^2$$
 (5.2)

where (C, D) implies that C and D are connected by an edge. Alternatively, in terms of summation over edges, m, the expression is

$$\delta S^2 = 2 \left[ \sum_m \delta \mathbf{x}_m^2 - \frac{1}{6} \sum_{\tau \in \Sigma_3} \left( \sum_{m \in \tau} \mathbf{x}_m \cdot \delta \mathbf{x}_m \right)^2 \right]$$
 (5.3)

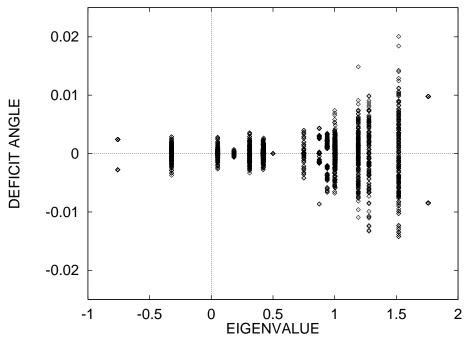


FIG. 3. The generation of curvature by motion in simplicial configuration space from a flat-space point to a neighboring point along one of the flat-space eigenvectors. Here we analyze the curvature of the 3-geometries of the  $3 \times 3 \times 3$   $T^3$  lattice in the neighborhood of the flat-space isosceles tetrahedral lattice point. The simplicial configuration space is 189 dimensional, and there are 189 deficit angles (integrated curvature) used as indicators of curvature change. This plot represents the deficit angle spectrum associated to motion along each of the eigenvectors associated to each of the eigenvalues  $\lambda_k$  Eq. (5.1). Here we chose  $\epsilon = 0.01$ . In the plot we notice that all the eigenvectors corresponding to eigenvalue  $\lambda = 1/2$  correspond to vertical or diffeomorphism directions.

Even though (5.3) is relatively simple, with definite signs for each term, we have not managed to prove a general result about the overall sign. We suspect that it is positive and in the following special cases have found this to be true:

1. If just one vertex moves through  $\delta \mathbf{r}$ 

$$\delta S^2 = 8\delta \mathbf{r}^2 \tag{5.4}$$

2. If only the edges in one coordinate direction of the lattice change,

$$\delta S^2 \ge \frac{1}{2} \sum \delta \mathbf{x}_{CD}^2 \tag{5.5}$$

3. If all the  $\delta \mathbf{x}_{CD}$ 's have the same magnitude  $\delta L$  (but not the same direction)

$$\delta S^2 > (\delta L)^2 \left( n_1 - \frac{3}{4} n_3 \right) > 0$$
 (5.6)

since  $n_1 > n_3$ 

If it could be proved that  $\delta S^2$  is always positive, it would still remain to calculate the number of independent directions in the space of edge-lengths, in order to see how many of the positive eigenvalues do indeed correspond to vertical gauge modes.

We next turn briefly to more general aspects of the right-tetrahedron lattice. As mentioned earlier, although the geometry of the right-tetrahedron is flat like the isoscelestetrahedron lattice, it is not diffeomorphic to it. It is therefore no surprise that when we calculate the eigenvalues of the Lund-Regge metric we find both their values and degeneracies to differ from the isosceles-tetrahedron case. Even if the lattices are rescaled so that their total volumes are equal, the eigenvalue spectra (which scale by the inverse length) are still not the same.

The right-tetrahedron case presents allows a particularly easy analysis of how the degeneracies of the eigenvalue spectrum are connected with the symmetry group of a lattice. To find the symmetry group of the right-tetrahedron lattice, consider the symmetries when one point, the origin say, is fixed. These are a "parity" transformation (when  $\mathbf{r}$  is mapped to  $-\mathbf{r}$ ), represented by  $Z_2$ , and a permutation of the three coordinate directions, represented by  $S_3$ . The full group is obtained by combining this  $Z_2 \times S_3$  subgroup with the subgroup of translations (mod 3) in the three coordinate directions. Thus the symmetry group of the  $T^3$  lattice with  $3 \times 3 \times 3$  vertices is the semi-direct product of the elementary Abelian normal subgroup of order  $3^3$  by the subgroup  $Z_2 \times S_3$ .

The action of the group on the vertices induces a permutation of the edges. This 189dimensional permutation representation of the edges decomposes as

$$3 \times 1_1 + 2 \times 2_1 + 3 \times 2_2 + 2 \times 4 + 5 \times (6_1 + 6_2 + 6_3 + 6_4) + 2 \times (6_5 + 6_6 + 6_7 + 6_8)$$
 (5.7)

where e.g.  $6_3$  is the third irreducible representation of dimension 6. When this is compared with the multiplicities found for the eigenvalues of the supermetric for the flat-space decomposition with right-angled tetrahedra, it can be seen that the numbers agree precisely

provided that the two multiplicities of 8 are interpreted as 6+2, and the two multiplicities of 3 are regarded as 2+1. We have no explanation for this unexpected degeneracy, although it has been observed before (for example for several triangulations of  $\mathbb{CP}^2$  [17,19]) and we suspect that there is a deep group theoretical reason for it. A detailed investigation of the eigenvalues found for the isosceles-tetrahedron lattice would almost certainly reveal similar accidental degeneracies, for example for the multiplicity of 29 found numerically for the eigenvalue  $\lambda = 1$ .

Finally we look at common properties of the isosceles-tetrahedron lattice and right tetrahedron lattice, in particular, the the signature of the Lund-Regge metric, which is the main point of these calculations. For both the isosceles- and right-tetrahedron lattices with  $3\times3\times3$  vertices, and therefore 189 edges, the supermetric (in a flat-space configuration, and in a neighborhood of flat space) has 176 positive eigenvalues and 13 negative ones (Fig. 4 illustrates these results for the isosceles-tetrahedron lattice, together with the other 32 runs we made). These are consistent with our analytical results and can be interpreted as follows.

- There are rather more than the required  $n_1 n_3 = 27$  spacelike directions.
- The negative eigenvalues include the conformal mode.
- We have shown that the eigenvalues corresponding to diffeomorphisms may sometimes be positive, and for the isosceles-tetrahedron lattice have identified each of the eigenvectors corresponding to the  $\lambda = 1/2$  eigenvalue as a generator of a diffeomorphism, or vertical direction.
- None of the eigenvectors corresponding to the 13 negative eigenvalues are generators of diffeomorphisms. This indicates that there are 13 horizontal directions corresponding to negative eigenvalues.

Furthermore, we consistently observe signature change in the supermetric as we depart from the flat space point. We also observed a few null eigenvalues for the right-tetrahedron lattice for flat space at various resolutions, including  $4 \times 4 \times 4$ . We are presumably seeing finite analogs of the infinite number of horizontal (i.e. non-gauge) directions predicted by Giulini [7] for regions of superspace in the neighborhood of a flat metric, where there is an open region with negative Ricci curvature. This illustrates the important fact that there are still timelike directions in the simplicial supermetric, beyond the conformal mode, which are always present.

# VI. GEOMETRIC STRUCTURE OF SUPERSPACE: RESULTS AND FUTURE DIRECTIONS

In this paper we used the simplicial supermetric of Lund and Regge as a tool for analyzing the geometry of simplicial configuration space, specifically the signature of the Lund-Regge metric. One way of summarizing our results is to compare them with the known results for the metric on continuum superspace described in the Introduction:

The conformal direction (3.17) is timelike as we showed analytically in (3.19). This coincides with the result for the continuum conformal displacements.

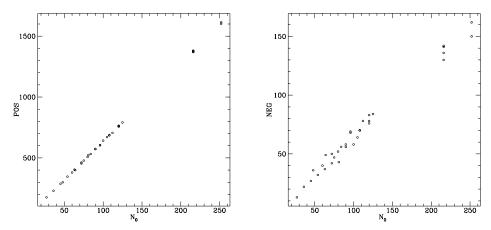


FIG. 4. An originally flat 3-torus isosceles-tetrahedron lattice with additional 10% random fluctuations induced on the squared edge-lengths. These two graphs show the number of positive (POS) and negative (NEG) eigenvalues as a function of the number of vertices (N<sub>0</sub>). Here we considered 33 different resolution  $T^3$  lattices ranging from a  $3 \times 3 \times 3$  lattice with  $N_0 = 27$  vertices to a 6x6x7 lattice with  $N_0 = 252$  vertices. The ratio of negative to positive eigenvalues ranges from  $\sim 0.074$  to  $\sim 0.123$ .

The simplest simplicial manifold with the topology of the three-sphere  $(S^3)$  is the surface of a four-simplex. Here, we showed that, in the 10-dimensional simplicial configuration space, among a set of orthogonal directions there is always a single timelike direction and nine spacelike ones, even for regions of simplicial configuration space corresponding to geometries distorted from spherical symmetry with aspect ratios exceeding 10:1 in squared edge-length. For the continuum such a result is known only in an arbitrarily small neighborhood of the round metric (analogous to all equal edges). However, we have no evidence that the situation of a single timelike direction in an orthogonal set extends to more refined triangulations of  $S^3$  such as the 600-cell. In particular, preliminary results indicate that there are 628 positive, 92 negative and no zero eigenvalues for the 600-cell model [20].

By investigating a neighborhood of the flat geometry in various triangulations of  $T^3$ , we exhibited exact simplicial diffeomorphisms for exactly flat geometries, and approximate simplicial diffeomorphisms for approximately flat assignments of the squared edge lengths. We showed that there was more than one orthogonal timelike non-gauge direction at the flat geometry. We showed that the Lund-Regge metric can become degenerate and change signature as one moves away from exactly flat geometries — a result that might have been expected at least on large triangulations of  $S^3$  from the combination of the continuum results on the signature near a round metric and the different signature on regions that correspond to negative curvature. Here, signature change is exhibited explicitly for  $T^3$ .

The principle advantage of casting the DeWitt supermetric into its simplicial form is to reduce the continuum infinite dimensional superspace to a finite dimensional simplicial configuration space. This simplicial configuration space is to be contrasted with "mini-" or "midi"-superspaces. Simplicial configuration space preserves elements of *both* the physical degrees of freedom and the diffeomorphisms. In the continuum limit of increasingly large triangulations we expect to recover the full content of both.

Our analysis provides motivation for further research. A potentially fruitful line of inves-

tigation is to define approximate notions of vertical and horizontal directions in simplicial configuration space in such a way that they coincide with the exact vertical and horizontal directions in the continuum limit. We already know that there are  $3n_0$  approximate diffeomorphism degrees of freedom for a simplicial 3-geometry – a fact that has been demonstrated analytically and illustrated numerically in Regge geometrodynamics via the freedom of choice of a shift vector per vertex [21,22].

Once armed with such a theory of approximate simplicial diffeomorphisms, it would be interesting to extend our  $S^3$  analysis to a simplicial model with arbitrarily large number of vertices. In this way we can be assured that the tessellation will have encoded in it all of the true dynamic degrees of freedom as well as the full diffeomorphism freedom.

#### VII. ACKNOWLEDGMENTS

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