Numerical Relativity for Inspiraling Binaries in Co-Rotating Coordinates: Test Bed for Lapse and Shift Equations

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Gravitational-wave data analysis requires a detailed understanding of the highly relativistic, late stages of inspiral of neutron-star and black-hole binaries. A promising method to compute the late inspiral and its emitted waves is numerical relativity in co-rotating coordinates. The coordinates must be kept co-rotating via an appropriate choice of numerical relativity's lapse and shift functions. This article proposes a model problem for testing the ability of various lapse and shift prescriptions to keep the coordinates co-rotating.

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I. INTRODUCTION

Patrick Brady, Jolien Creighton and I [1] have discussed the importance, for gravitational-wave data analysis, of understanding the evolution of inspiraling blackhole binaries. We showed that there is a crucial gap between the point, during inspiral, when a post-Newtonian analysis breaks down, and the point, just before merger, when conventional numerical relativity techniques can take over the analysis. We argued that this IBBH (Intermediate Binary Black Hole) gap might best be filled by numerical relativity computations carried out using co-rotating coordinates. In such coordinates, the quantities being computed (components of the 3-metric γ_{ij} and extrinsic curvature K_{ij}) change on the timescale of inspiral τ_* , which is much longer than the orbital timescale $1/\Omega = 1/(\text{orbital angular velocity})$. Therefore, in such coordinates one may be able to take long time steps, constrained only by τ_* , thereby speeding up the computer computations by a large factor relative to those carried out in the usual asymptotically inertial coordinates.

Numerical relativity in co-rotating coordinates may also be effective, and perhaps necessary, for analyses of the late stages of inspiral of neutron-star binaries and neutron-star / black-hole binaries.

A key foundation for such computations is a prescription for choosing the lapse and shift functions α and β^i , which generate the coordinate system as the computation proceeds. The shift function β^i must keep the coordinates co-rotating with the binary's orbital motion, and the lapse function α must appropriately retard the time slicing in regions of strengthening gravitational redshift; and this must be done quantitatively in such a way that the combined effects of α and β^i keep $\partial \gamma_{ij}/\partial t \sim \gamma_{ij}/\tau_*$ and $\partial K_{ij}/\partial t \sim K_{ij}/\tau_*$. For simplicity, I refer to such coordinates as co-rotating, even though they require more than just co-rotation.

Brady, Creighton and I [1] have proposed a class of equations for the lapse and shift, any one of which (we argued) may keep the coordinates co-rotating, if they have

been chosen co-rotating to begin with. The simplest of these are the "minimal-strain" lapse and shift equations:

$$D^{j}(D_{(i}\beta_{j)} - \alpha K_{ij}) = 0, \qquad (1a)$$

in which α is to be replaced by the following expression in terms of β_i

$$\alpha = \frac{K^{ij} D_i \beta_j}{K^{ab} K_{ab}} \,. \tag{1b}$$

Here D_i is the covariant derivative in the slice of constant time t, which has 3-metric γ_{ij} and extrinsic curvature K_{ij} .

Numerical relativists, in conversations with me, have suggested other choices of lapse and shift that might keep the coordinates co-rotating. A detailed tests of their proposals and of Brady, Creighton's and mine are now needed. The purpose of this Brief Report is to suggest a model problem that can serve as a test bed for the various proposed lapse and shift equations.

This test bed is a specific 4-dimensional spacetime that resembles that of a fully relativistic, inspiraling compact binary. More specifically, it is a spacetime with a 4-metric that possesses, semi-quantitatively correctly, all those features that are likely to significantly influence the behavior of the proposed lapse and shift equations—with one exception: the test-bed 4-metric does not possess black-hole horizons. Instead, it models a binary neutron-star system. This is because horizons may seriously complicate the task of implementing the proposed lapse and shift equations; so in initial tests it is best to omit them. It should not be difficult to generalize my test-bed 4-metric to include horizons, both nonrotating and rotating.

Any lapse and shift equations that stably generate and maintain a co-rotating coordinate system in my test-bed spacetime will very probably do so also in a real numerical-relativity calculation—i.e., in a calculation that is simultaneously evolving the binary's true spacetime geometry along with the coordinate system.

II. THE TEST-BED SPACETIME

I shall describe the test-bed spacetime in a co-rotating coordinate system $\{T,X,Y,Z\}$ that must not be confused with the coordinates $\{t,x,y,z\}$ generated by some chosen lapse and shift equations. Thinking of the spatial coordinates $\{X,Y,Z\}$ as Cartesian in some flat background metric, we can introduce the flat-space distance ϖ from the binary's rotation axis, spherical polar coordinates (R,θ,ϕ) , and unit vectors $\hat{\theta}$ and $\hat{\phi}$ along the θ and ϕ directions:

$$\overline{\omega} = \sqrt{X^2 + Y^2}, \quad R = \sqrt{X^2 + Y^2 + Z^2},
\theta = \arcsin(\overline{\omega}/R), \quad \phi = \arctan(Y/X);
\hat{\theta} = \frac{\partial_{\theta}}{R} = \frac{-\overline{\omega}\partial_Z + Z\partial_{\overline{\omega}}}{R} \quad \text{where } \partial_{\overline{\omega}} = \frac{X\partial_X + Y\partial_Y}{\overline{\omega}},
\hat{\phi} = \frac{\partial_{\phi}}{\overline{\omega}} = \frac{X\partial_Y - Y\partial_X}{\overline{\omega}}.$$
(2)

Here $\{\partial_X, \partial_Y, \partial_Z\}$ are the usual Cartesian coordinate basis vectors, $\{\partial_\theta, \partial_\phi\}$ are the usual spherical coordinate basis vectors, and ∂_{ϖ} is the unit vector pointing away from the rotation axis in the flat background metric.

The test-bed spacetime describes two identical stars in a circular orbit around each other. The spacetime's 4-metric depends on four parameters: the time-independent mass m of each star as measured gravitationally far from the binary, the time-independent stellar radius b and slowly evolving stellar separation 2a as measured in the flat background metric, and the slowly evolving orbital angular velocity Ω as measured far from the binary.

We require that

$$2m \lesssim b \lesssim a$$
, $\Omega a \lesssim 1$, (3)

so the stars are larger than their Schwarzschild radii, their separation is larger than the sum of their radii, and their orbital speeds are less than the speed of light. The "approximately less than" rather than strictly less than is because a and b are parameters measured in the flat background metric, not in the physical metric. We also require that

$$\frac{\partial a}{\partial T} \sim \frac{a}{\tau_*}$$
, $\frac{\partial \Omega}{\partial T} \sim \frac{\Omega}{\tau_*}$, where $\Omega \tau_* \gg 1$, (4)

so the inspiral is slow. To be somewhat realistic, one might want to set

$$\Omega = \sqrt{\frac{2m}{(2a)^3}} \tag{5}$$

(Kepler's law) and let the separation 2a evolve in the manner dictated by the quadrupolar description of radiation reaction [2]

$$a(T) = a_o \left(1 - \frac{4T}{\tau_{*o}}\right)^{1/4}$$
, where $\tau_{*o} = \frac{5}{8} \frac{a_o^4}{m^3}$; (6)

here $2a_o$ is the separation at time T=0, when the remaining time to merger is $\Delta T=\tau_{*o}/4$, and the inspiral timescale is $\tau_{*o}=a_o(da_o/dT)^{-1}$.

The test-bed 4-metric is not actually a solution of Einstein's field equations, but rather is cooked up to resemble the solutions that we expect numerical calculations to reveal.

We can write the 4-metric in 3+1 form in the corotating coordinates:

$$ds^{2} = -A^{2}dT^{2} + \Gamma_{ij}(dX^{i} + B^{i}dT)(dX^{j} + B^{j}dT).$$
 (7)

Here the co-rotating lapse A, shift B^i and 3-metric Γ_{ij} are not to be confused with the lapse α and shift β that are generated by some proposed prescription for keeping the coordinates (nearly) co-rotating, and the 3-metric γ_{ij} of that prescription.

The co-rotating A, B^i and Γ_{ij} will depend explicitly on the spatial coordinates $\{X,Y,Z\}$ and will depend on time T solely through the slowly changing separation 2aand angular velocity Ω .

To take account of the fact that information about the inspiral cannot propagate faster than light, in the 4-metric we shall regard a and Ω as functions of (approximately) retarded time T-R rather than time T:

$$a = a(T - R)$$
, $\Omega = \Omega(T - R)$. (8)

In numerical tests this might not be very important, since the the full coordinate grid is not likely to be larger than a few gravity-wave wavelengths, $\sim 10/\Omega$, and thus will likely be much smaller than τ_* (the timescale for changes of a and Ω), except possibly near the end of a test.

We choose the (truly) co-rotating lapse A to be 1 minus the (approximate) Newtonian potential of the binary, in accord with the Newtonian limit of general relativity:

$$A = 1 - mF_+ , \qquad (9)$$

where the function F_+ ("Newtonian potential per unit mass") and an associated function F_- are defined by

$$F_{\pm} = \frac{1}{\left[(X-a)^2 + Y^2 + Z^2 + b^2 \right]^{1/2}} \pm \frac{1}{\left[(X+a)^2 + Y^2 + Z^2 + b^2 \right]^{1/2}},$$
 (10)

and $a \equiv a(T - R)$. The b^2 rounds off the growth of the potential F_{\pm} as one moves inside each star.

If our coordinates were inertial rather than co-rotating, the shift $\bf B$ would consist solely of a piece $\bf B_{\rm drag}$ produced by the dragging of inertial frames due to the binary's orbital motion. The transformation $\bar{\phi} = \phi + \int \Omega dT$, from the inertial angular coordinate $\bar{\phi}$ to the co-rotating angular coordinate ϕ , augments this frame-dragging shift by $\Omega \partial_{\phi} = \Omega \varpi \hat{\phi}$, thereby giving rise to our following choice for the shift in co-rotating coordinates:

$$\mathbf{B} = \left(1 - \frac{4ma^2}{(R^2 + a^2)^{3/2}}\right)\Omega\varpi\hat{\phi} - \frac{16ma^5\Omega}{(R^2 + a^2)^2}F_-\partial_y.$$
(11)

Our choice here of $\mathbf{B}_{\mathrm{drag}}$ consists of two pieces. The first piece, $-4ma^2(R^2+a^2)^{-3/2}\Omega\varpi\hat{\phi}$, dominates at large radii $R\gg a$, where it has the standard form and value $-(2J/R^3)\partial_{\phi}$ for the dragging of inertial frames by the binary's total angular momentum $J=2ma^2\Omega$. The second piece, $-16ma^5\Omega(R^2+a^2)^{-2}F_-\partial_y$ dominates in the vicinity of each star, where it has the standard form and value $-4\mathbf{p}/(\mathrm{distance}$ to star) for the frame-dragging field produced by the star's linear momentum $\mathbf{p}=\pm ma\Omega\partial_y$ (+ for the star at x=+a; – for the star at -a). Because the vector fields ∂_y and $\varpi\hat{\phi}=\partial_\phi=X\partial_Y-Y\partial_X$ are both regular on the rotation axis and at the origin, our chosen shift \mathbf{B} is regular.

We split our co-rotating 3-metric into two parts:

$$\Gamma_{ij} = \Gamma_{ij}^{C} + \Gamma_{ij}^{TT} . \tag{12}$$

Here

$$\Gamma_{ij}^{\mathcal{C}} = \delta_{ij} (1 + mF_{+})^2 \tag{13}$$

is a conformally flat part with the form (flat metric) \times (1 - Newtonian potential)² expected from the Newtonian limit of general relativity;

$$\begin{split} \mathbf{\Gamma}^{\mathrm{TT}} &= \frac{-4ma^2\Omega^3(\Omega R)^4}{(1+\Omega^2R^2)^{5/2}} \\ &\times \left[(1+\cos^2\theta)\cos(2\phi+2\Omega R)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) \right. \\ &\left. + 2\cos\theta\sin(2\phi+2\Omega R)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}) \right] \end{split} \tag{14}$$

is a transverse-traceless (TT) part that represents outgoing gravitational waves in the radiation zone; and two vectors written side by side, e.g. $\hat{\theta}\hat{\phi}$, represent a tensor product.

The TT part of the 3-metric, $\mathbf{\Gamma}^{\mathrm{TT}}$, requires discussion: In the radiation zone, $R \gg 1/\Omega$, after transformation from the co-rotating angular coordinate ϕ to the inertial-frame angular coordinate $\bar{\phi} = \phi + \int \Omega dT \simeq \phi + \Omega T$, $\mathbf{\Gamma}^{\mathrm{TT}}$ takes the standard quadrupole-moment form for outgoing gravitational waves produced by a nearly Newtonian binary: $\mathbf{\Gamma}^{\mathrm{TT}} = h_{+}(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) + h_{\times}(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta})$. Here the waveforms h_{+} and h_{\times} are [4]

$$h_{+} = \frac{-4ma^{2}\Omega^{2}}{R} (1 + \cos^{2}\theta) \cos[2\bar{\phi} - 2\Omega(T - R)] ,$$

$$h_{\times} = \frac{-4ma^{2}\Omega^{2}}{R} 2\cos\theta \sin[2\bar{\phi} - 2\Omega(T - R)] .$$
 (15)

In co-rotating coordinates $\phi = \bar{\phi} - \Omega T$, the time oscillation of this radiation field is gone. Instead of oscillating in time, it consists of a simple spiral pattern in space that changes on the slow inspiral timescale τ_* . The fact that

it represents outgoing waves rather than ingoing shows up in the direction of the spiral and equivalently in the relation $(\partial_R - \Omega \partial_\phi)h_+ \sim h_+/\tau_* \ll \Omega h_+$ and similarly for h_\times . An ingoing-wave pattern would exhibit the opposite direction of spiral and would have $(\partial_R + \Omega \partial_\phi)h_+ \sim h_+/\tau_*$.

The specific functional form of $\mathbf{\Gamma}^{\mathrm{TT}}$, Eq. (14), is designed to have several properties: (i) It produces the desired radiation field (15) at $R \gg 1/\Omega$. (ii) It becomes very small in the near zone, $R \ll 1/\Omega$ —negligibly small compared to the conformally flat part of the 3-metric $\mathbf{\Gamma}^{\mathrm{C}}$. (iii) It is regular on the rotation axis, because as one approaches the axis $\cos(2\phi - 2\phi_o)(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \pm \sin(2\phi - 2\phi_o)(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta})$ becomes a simple tidal stretch along the ϕ_o direction. (iv) It is regular at the origin, because its angular dependence is that of the electric-type tensor spherical harmonic of order l = 2, $\mathbf{T}^{E2,lm}$, which is constructed via Clebsch-Gordon coupling from scalar harmonics of order l' = 0, 2 and 4 [5], and the R^4 dependence at the origin, multiplied by these scalar harmonics, produces regularity.

There is no attempt, in the chosen expression for $\Gamma^{\rm TT}$, to represent, even approximately correctly, the radiation-reaction field inside the binary. This is because the radiation-reaction field is so small inside the binary, $\sim \Omega^5 ma^2R^2 \ll m/R$ [6], that it is unlikely to have any significant influence on the behaviors of proposed lapses and shifts. Similarly, there is no attempt to approximate the details of the transition from near-zone field to radiation field. Except near the endpoint of inspiral, that transition field is small compared to m/R, the important part of $\Gamma^{\rm C}$.

In summary, our test-bed 4-metric (7) has the corotating lapse (9), shift (11), and 3-metric (12)—(14).

III. TESTING PROPOSED LAPSES AND SHIFTS

To test a proposed prescription for choosing the lapse α and shift β^i —for example, the Brady-Creighton-Thorne minimal-strain prescription (Eqs. (1) and associated boundary conditions discussed in Ref. [1])—one can proceed as follows:

Choose the 3-surface T=0 as an initial 3-slice in spacetime, and label it with the time coordinate t=0. On this initial slice, set the spatial coordinates $\{x,y,z\}$ equal to the co-rotating coordinates $\{X,Y,Z\}$. Correspondingly, the initial 3-metric and extrinsic curvature in the $\{x,y,z\}$ coordinates will be

$$\gamma_{ij}(t=0) = \Gamma_{ij}(T=0) , \qquad (16a)$$

$$K_{ij}(t=0) = \left[\frac{1}{2A} \left(-\frac{\partial \Gamma_{ij}}{\partial T} + 2D_{(i}B_{j)} \right) \right]_{T=0} , \quad (16b)$$

where the T-derivative acts solely through the dependence of Γ_{ij} on a(T-R) and $\Omega(T-R)$.

Choose the initial lapse $\alpha(t=0)$ and shift $\beta^i(t=0)$ to be as close as the proposed lapse and shift prescriptions allow to

$$\alpha(t=0) = A(T=0) , \quad \beta^i = B^i(T=0) .$$
 (17)

If the prescription does not allow these choices up to a fractional difference of order $1/(\Omega \tau_*) \ll 1$, then the prescription is suspect. The minimal-strain prescription (1) does have this property, as do all the other prescriptions discussed by Brady, Creighton and Thorne [1].

Beginning with these initial data, use the proposed prescription to evolve forward in time the lapse α and shift β^i , and the associated time slices t(T,X,Y,Z) = constant and the spatial coordinates $\{x,y,z\}$ on those time slices. At each step of the evolution, compute the 3-metric γ_{ij} and extrinsic curvature K_{ij} of the slice t = constant from the test-bed 4-metric $\{A, B^i, \Gamma_{ij}\}$. An acceptable lapse-shift prescription will keep $\alpha, \beta, \gamma_{ij}$, and K_{ij} all evolving on the slow timescale τ_* ; and most likely will achieve this by maintaining $t \simeq T$, $\alpha \simeq A$, $\beta^i \simeq B^i$, and $\gamma_{ij} \simeq \Gamma_{ij}$, aside from fractional differences of order $1/(\Omega \tau_*)$.

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P. R. Brady, J. D. E. Creighton and K. S. Thorne, Phys. Rev. D, in press (gr-qc/9804057).

^[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (W. H. Freeman, San Francisco, 1973), p. 988.

^[3] Ref. [2], Eq. (19.5).

^[4] T. A. Apostolatos, C. Cutler, G. J. Sussman, and K. S. Thorne, Phys. Rev. D, 49, 6274 (1994), Eqs. (2).

^[5] K. S. Thorne, Rev. Mod. Phys., 52, 299 (1980), Eqs. (2.30d) and (2.28a).

^[6] Ref. [2], Eq. (36.26).