Øyvind Grøn

LECTURE NOTES IN PHYSICS 772

Lecture Notes on the General Theory of Relativity

From Newton's Attractive Gravity to the Repulsive Gravity of Vacuum Energy

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From Newton's Attractive Gravity to the Repulsive Gravity of Vacuum Energy

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Preface

These notes are a transcript of lectures delivered by Øyvind Grøn during the spring of 1997 at the University of Oslo.

The present version of this document is an extended and corrected version of a set of Lecture Notes which were typesetted by S. Bard, Andreas O. Jaunsen, Frode Hansen and Ragnvald J. Irgens using LaTeX2ϵ. Svend E. Hjelmeland has made many useful suggestions which have improved the text. I would also like to thank Jon Magne Leinaas and Sigbjørn Hervik for contributing with problems, and Gorm Krogh Johnsen for help with finishing the manuscript. I also want to thank prof. Finn Ravndal for inspiring lectures on general relativity.

While we hope that these typeset notes are of benefit particularly to students of general relativity and look forward to their comments, we welcome all interested readers and accept all feedback with thanks.

All comment may be sent to the author by e-mail.

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Contents

List of Figures

List of Definitions

List of Examples

Chapter 1
Newton's Law of Universal Gravitation

1.1 The Force Law of Gravitation

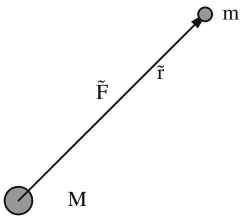

Fig. 1.1 Newton's law of universal gravitation states that the force between two masses is attractive, acts along the line joining them and is inversely proportional to the distance separating the masses

$$\vec{F} = -mG\frac{M}{r^3}\vec{r} = -mG\frac{M}{r^2}\vec{e}_r \,. \tag{1.1}$$

Let V be the potential energy of m (see Fig. 1.1). Then

$$\vec{F} = -\nabla V(\vec{r}), \quad F_i = -\frac{\partial V}{\partial x_i} \,. \tag{1.2}$$

For a spherical mass distribution, $V(\vec{r}) = -mG\frac{M}{r}$, with zero potential infinitely far from the centre of M. Newton's law of gravitation is valid for "small" velocities, i.e. velocities much smaller than the velocity of light and "weak" fields. Weak fields are fields in which the gravitational potential energy of a test particle is very small compared to its rest mass energy. (Note that here one is interested only in the absolute values of the above quantities and not their sign.)

$$mG\frac{M}{r} \ll mc^2 \quad \Rightarrow \quad r \gg \frac{GM}{c^2} \,. \tag{1.3}$$

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The **Schwarzschild radius** for an object of mass M is $R_s = \frac{2GM}{c^2}$. Far outside the Schwarzschild radius we have a weak field. To get a feeling for magnitudes consider that $R_s \approxeq 1\,\mathrm{cm}$ for the Earth which is to be compared with $R_E \approxeq 6400\,\mathrm{km}$. That is, the gravitational field at the Earth's surface can be said to be weak! This explains, in part, the success of the Newtonian theory.

1.2 Newton's Law of Gravitation in Its Local Form

Let P be a point in the field (see Fig. 1.2) with position vector $\vec{r} = x^i \vec{e}_i$ and let the gravitating point source be at $\vec{r'} = x^{i'} \vec{e}_{i'}$. Newton's law of gravitation for a continuous distribution of mass is

$$\begin{aligned}\vec{F} &= -mG \int_r^\infty \rho(\vec{r'}) \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} d^3 r' \\ &= -\nabla V(\vec{r}) \,.\end{aligned} \tag{1.4}$$

See Fig. (1.2) for symbol definitions.

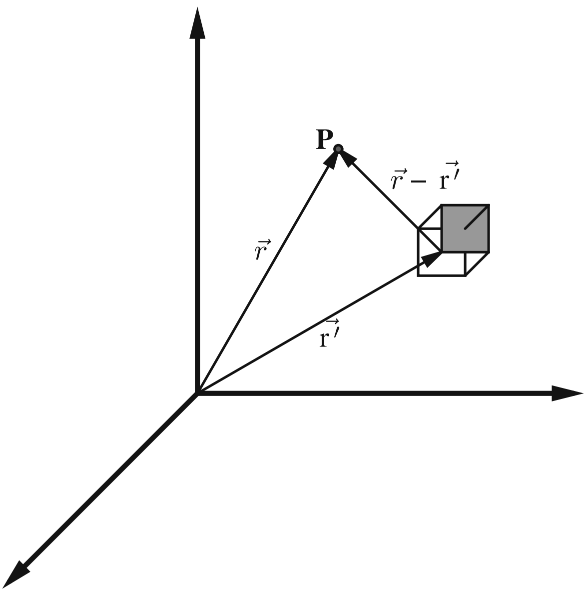

Fig. 1.2 Newton's law of gravitation in its local form

Let's consider Eq. (1.4) term by term:

$$\begin{aligned}
\nabla \frac{1}{|\vec{r}-\vec{r}'|} &= \vec{e}_i \frac{\partial}{\partial x_i} \frac{1}{\left[(x^j - x^{j'})(x_j - x_{j'})\right]^{1/2}} \\
&= \vec{e}_i \frac{\partial}{\partial x_i} \left[(x^j - x^{j'})(x_j - x_{j'})\right]^{-1/2} \\
&= \vec{e}_i \frac{-1}{2} 2(x_j - x_{j'}) \frac{\partial x^j}{\partial x_i} \left[(x^k - x^{k'})(x_k - x_{k'})\right]^{-3/2} \\
&= -\vec{e}_i \frac{(x^j - x^{j'})\delta^i_j}{\left[(x^k - x^{k'})(x_k - x_{k'})\right]^{3/2}} \\
&= -\vec{e}_i \frac{(x^i - x^{i'})}{\left[(x^j - x^{j'})(x_j - x_{j'})\right]^{3/2}} \\
&= -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \,.
\end{aligned} \tag{1.5}$$

Now Eqs. (1.4) and (1.5) together $\Rightarrow$

$$V(\vec{r}) = -mG \int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r' \,. \tag{1.6}$$

Gravitational potential at point P:

$$\begin{aligned}
\phi(\vec{r}) \equiv \frac{V(\vec{r})}{m} &= -G \int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r' \\
\Rightarrow \quad \nabla\phi(\vec{r}) &= G \int \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} d^3r' \\
\Rightarrow \quad \nabla^2\phi(\vec{r}) &= G \int \rho(\vec{r}') \nabla \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} d^3r' \,.
\end{aligned} \tag{1.7}$$

The above equation simplifies considerably if we calculate the divergence in the integrand:

$$\begin{aligned}
\nabla \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} &= \frac{\nabla \cdot \vec{r}}{|\vec{r}-\vec{r}'|^3} + (\vec{r}-\vec{r}') \cdot \nabla \frac{1}{|\vec{r}-\vec{r}'|^3} \\
&= \frac{3}{|\vec{r}-\vec{r}'|^3} - (\vec{r}-\vec{r}') \cdot \frac{3(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^5} \\
&= \frac{3}{|\vec{r}-\vec{r}'|^3} - \frac{3}{|\vec{r}-\vec{r}'|^3} \\
&= 0 \quad \forall \quad \vec{r} \neq \vec{r}' \,.
\end{aligned} \tag{1.8}$$

Note that "∇" operates on $\vec{r}$ only!

We conclude that the Newtonian gravitational potential at a point in a gravitational field outside a mass distribution satisfies Laplace's equation

$$\boxed{\nabla^2 \phi = 0}\,. \tag{1.9}$$

Digression 1.2.1 (Dirac's delta function) The Dirac delta function has the following properties:

1. $\delta(\vec{r}-\vec{r'}) = 0 \quad \forall \quad \vec{r} \neq \vec{r'}$.
2. $\int \delta(\vec{r}-\vec{r'})d^3r' = 1$ when $\vec{r} = \vec{r'}$ is contained in the integration domain. The integral is identically zero otherwise.
3. $\int f(\vec{r'})\delta(\vec{r}-\vec{r'})d^3r' = f(\vec{r})$.

A calculation of the integral $\int \nabla \cdot \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} d^3r'$ which is valid also in the case where the field point is inside the mass distribution is obtained through the use of Gauss' integral theorem:

$$\int\limits_{v} \nabla \cdot \vec{A} d^3r' = \oint\limits_{s} \vec{A} \cdot d\vec{s}\,, \tag{1.10}$$

where s is the boundary of v ($s = \partial v$ is an area).

Definition 1.2.1 (Solid angle)

$$d\Omega \equiv \frac{ds'_\perp}{|\vec{r}-\vec{r'}|^2}\,, \tag{1.11}$$

where $ds'_\perp$ is the projection of the area ds' normal to the line of sight. $\vec{ds'}_\perp$ is the component vector of $\vec{ds'}$ along the line of sight which is equal to the normal vector of $ds'_\perp$ (see Fig. 1.3).

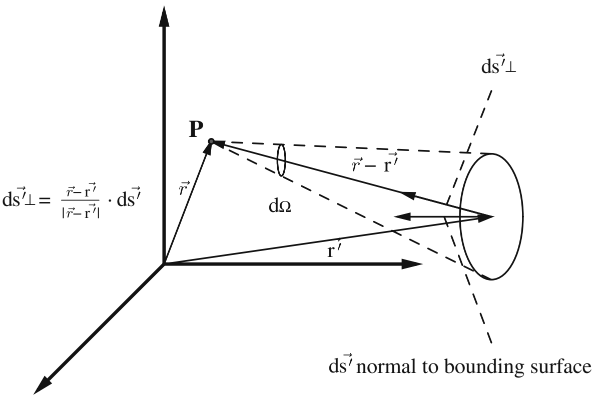

Fig. 1.3 The solid angle $d\Omega$ is defined such that the surface of a sphere subtends 4π at the centre

Now, let's apply Gauss' integral theorem:

$$\int_{\mathrm{V}} \nabla \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} d^3 r' = \oint_{\mathrm{S}} \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \cdot d\vec{s}' = \oint_{\mathrm{S}} \frac{ds'_{\perp}}{|\vec{r}-\vec{r}'|^2} = \oint_{\mathrm{S}} d\Omega \, . \tag{1.12}$$

So that,

$$\int_{\mathrm{V}} \nabla \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} d^3 r' = \begin{cases} 4\pi & \text{if P is inside the mass distribution,} \\ 0 & \text{if P is outside the mass distribution.} \end{cases} \tag{1.13}$$

The above relation is written concisely in terms of the Dirac delta function:

$$\nabla \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = 4\pi\delta(\vec{r}-\vec{r}') \, . \tag{1.14}$$

We now have

$$\begin{aligned} \nabla^2 \phi(\vec{r}) &= G \int \rho(\vec{r}') \nabla \cdot \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} d^3 r' \\ &= G \int \rho(\vec{r}') 4\pi\delta(\vec{r}-\vec{r}') d^3 r' \\ &= 4\pi G \rho(\vec{r}) \, . \end{aligned} \tag{1.15}$$

Newton's theory of gravitation can now be expressed very succinctly indeed!

1. Mass generates gravitational potential according to

$$\nabla^2 \phi = 4\pi G \rho \, . \tag{1.16}$$

2. Gravitational potential generates motion according to

$$\vec{g} = -\nabla \phi \, , \tag{1.17}$$

where $\vec{g}$ is the field strength of the gravitational field.

1.3 Tidal Forces

Tidal force is the difference of gravitational force on two neighbouring particles in a gravitational field. The tidal force is due to the inhomogeneity of a gravitational field.

In Fig. 1.4 two points have a separation vector $\vec{\zeta}$. The position vectors of 1 and 2 are $\vec{r}$ and $\vec{r}+\vec{\zeta}$, respectively, where $|\vec{\zeta}| \ll |\vec{r}|$. The gravitational forces on a mass m at 1 and 2 are $\vec{F}(\vec{r})$ and $\vec{F}(\vec{r}+\vec{\zeta})$. By means of a Taylor expansion to lowest order in $|\vec{\zeta}|/|\vec{r}|$ we get for the i-component of the tidal force

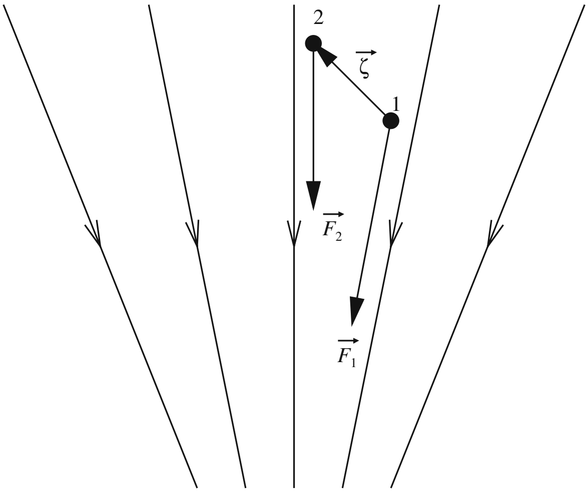

Fig. 1.4 Tidal forces

$$f_i = F_i(\vec{r}+\vec{\zeta}) - F_i(\vec{r}) = \zeta_j \left(\frac{\partial F_i}{\partial x^j}\right)_{\vec{r}} . \tag{1.18}$$

The corresponding vector equation is

$$\vec{f} = (\vec{\zeta}\cdot\nabla)_{\vec{r}}\vec{F} . \tag{1.19}$$

Using that

$$\vec{F} = -m\nabla\phi , \tag{1.20}$$

the tidal force may be expressed in terms of the gravitational potential according to

$$\vec{f} = -m(\vec{\zeta}\cdot\nabla)\nabla\phi . \tag{1.21}$$

It follows that in a local Cartesian coordinate system, the *i*-coordinate of the relative acceleration of the particles is

$$\frac{d^2\zeta_i}{dt^2} = -\left(\frac{\partial^2\phi}{\partial x^i\partial x^j}\right)_{\vec{r}}\zeta_j . \tag{1.22}$$

Let us look at a few simple examples. In the first one $\vec{\zeta}$ has the same direction as $\vec{g}$. Consider a small Cartesian coordinate system at a distance R from a mass M (see Fig. 1.5). If we place a particle of mass m at a point $(0,0,+z)$, it will, according to Eq. (1.1), be acted upon by a force

$$F_z(+z) = -m\frac{GM}{(R+z)^2} \tag{1.23}$$

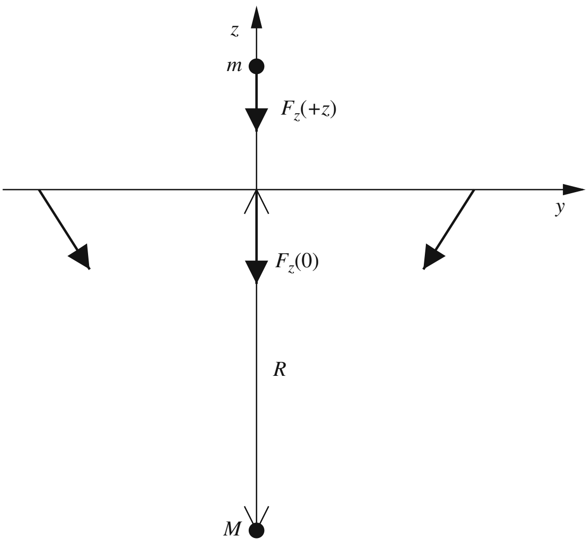

Fig. 1.5 A small Cartesian coordinate system at a distance R from a mass M

while an identical particle at the origin will be acted upon by the force

$$F_z(0) = -m\frac{GM}{R^2} . \tag{1.24}$$

If this little coordinate system is falling freely towards M, an observer at the origin will say that the particle at $(0,0,+z)$ is acted upon by a force

$$f_z = F_z(z) - F_z(0) \approx 2mz\frac{GM}{R^3} \tag{1.25}$$

directed away from the origin, along the positive z-axis. We have assumed $z \ll R$. This is the tidal force.

In the same way particles at the points $(+x,0,0)$ and $(0,+y,0)$ are attracted towards the origin by tidal forces

$$f_x = -mx\frac{GM}{R^3}, \tag{1.26}$$

$$f_y = -my\frac{GM}{R^3}. \tag{1.27}$$

Equations (1.25)–(1.27) have among others the following consequence: If an elastic, circular ring is falling freely in the Earth's gravitational field, as shown in Fig. 1.6, it will be stretched in the vertical direction and compressed in the horizontal direction.

In general, tidal forces cause changes of shape.

The tidal forces from the Sun and the Moon cause flood and ebb on the Earth. Let us consider the effect due to the Moon. We then let M be the mass of the Moon,

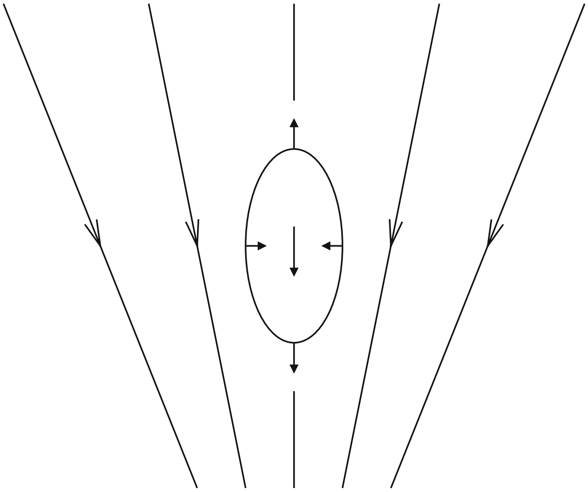

Fig. 1.6 An elastic, circular ring falling freely in the Earth's gravitational field

and choose a coordinate system with origin at the Earth's centre. The tidal force per unit mass at a point is the negative gradient of the tidal potential

$$\phi(\vec{r}) = -\frac{GM}{R^3}\left(z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) = -\frac{GM}{2R^3}r^2(3\cos^2\theta - 1)\,, \tag{1.28}$$

where we have introduced spherical coordinates, $z = r\cos\theta$, $x^2 + y^2 = r^2\sin^2\theta$, R is the distance between the Earth and the Moon, and the radius r of the spherical coordinate is equal to the radius of the Earth.

The potential at a height h above the surface of the Earth has one term, mgh, due to the attraction of the Earth and one given by Eq. (1.28), due to the attraction of the Moon. Thus,

$$\Theta(r) = gh - \frac{GM}{2R^3}r^2(3\cos^2\theta - 1)\,. \tag{1.29}$$

At equilibrium, the surface of the Earth will be an equipotential surface, given by $\Theta =$ constant. The height of the water at flood, $\theta = 0$ or $\theta = \pi$, is therefore

$$h_{\text{flood}} = h_0 + \frac{GM}{gR}\left(\frac{r}{R}\right)^2\,, \tag{1.30}$$

where h_0 is an unknown constant. The height of the water at ebb ($\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$) is

$$h_{\text{ebb}} = h_0 - \frac{1}{2}\frac{GM}{gR}\left(\frac{r}{R}\right)^2\,. \tag{1.31}$$

The height difference between flood and ebb is therefore

$$\Delta h = \frac{3}{2}\frac{GM}{gR}\left(\frac{r}{R}\right)^2\,. \tag{1.32}$$

For a numerical result we need the following values:

$$M_{\text{Moon}} = 7.35 \cdot 10^{25}\,\text{g}, \qquad g = 9.81\,\text{m/s}^2, \tag{1.33}$$

$$R = 3.85 \cdot 10^{5}\,\text{km}, \qquad r_{\text{Earth}} = 6378\,\text{km}\,. \tag{1.34}$$

With these values we find $\Delta h = 53$ cm, which is typical of tidal height differences.

1.4 The Principle of Equivalence

Galilei investigated experimentally the motion of freely falling bodies. He found that they moved in the same way, regardless what sort of material they consisted of and what mass they had.

In Newton's theory of gravitation, mass appears in two different ways: as gravitational mass, m_G, in the law of gravitation, analogous to charge in Coulomb's law, and as inertial mass, m_I in Newton's second law.

The equation of motion of a freely falling particle in the field of gravity from a spherical body with mass M then takes the form

$$\frac{d^2\vec{r}}{dt^2} = -G\frac{m_G}{m_I}\frac{M}{r^3}\vec{r}\,. \tag{1.35}$$

The results of Galilei's measurements imply that the quotient between gravitational and inertial mass must be the same for all bodies. With a suitable choice of units, we then obtain

$$m_G = m_I\,. \tag{1.36}$$

Measurements performed by the Hungarian baron Eötvös around the turn of the century indicated that this equality holds with an accuracy better than 10^{-8}. More recent experiments have given the result $|\frac{m_I}{m_G} - 1| < 9 \cdot 10^{-13}$.

Einstein assumed the exact validity of Eq. (1.36). He did not consider this as an accidental coincidence, but rather as an expression of a fundamental principle, called *the principle of equivalence*.

A consequence of this principle is the possibility of removing the effect of a gravitational force by being in free fall. In order to clarify this, Einstein considered a homogeneous gravitational field in which the acceleration of gravity, g, is independent of the position. In a freely falling, non-rotating reference frame in this field, all free particles move according to

$$m_I\frac{d^2\vec{r}}{dt^2} = (m_G - m_I)\vec{g} = 0\,, \tag{1.37}$$

where Eq. (1.36) has been used.

This means that an observer in such a freely falling reference frame will say that the particles around him are not acted upon by forces. They move with constant velocities along straight paths. In other words, such a reference frame is inertial.

Einstein's heuristic reasoning suggests equivalence between inertial frames in regions far from mass distributions, where there are no gravitational fields, and inertial frames falling freely in a gravitational field. This equivalence between all types of inertial frames is so intimately connected with the equivalence between gravitational and inertial mass that the term "principle of equivalence" is used whether one talks about masses or inertial frames. The equivalence of different types of inertial frames encompasses all types of physical phenomena, not only particles in free fall.

The principle of equivalence has also been formulated in an "opposite" way. An observer at rest in a homogeneous gravitational field and an observer in an accelerated reference frame in a region far from any mass distributions will obtain identical results when they perform similar experiments. An inertial field caused by the acceleration of the reference frame is equivalent to a field of gravity caused by a mass distribution, as far as tidal effects can be ignored.

1.5 The General Principle of Relativity

The principle of equivalence led Einstein to a generalization of the special principle of relativity. In his general theory of relativity Einstein formulated a general principle of relativity, which says that not only velocities but accelerations too are relative.

Consider two formulations of the special principle of relativity:

S1 All laws of Nature are the same (may be formulated in the same way) in all inertial frames.
S2 Every inertial observer can consider himself to be at rest.

These two formulations may be interpreted as different formulations of a single principle. But the generalization of S1 and S2 to the general case, which encompasses accelerated motion and non-inertial frames, leads to two different principles G1 and G2:

G1 The laws of Nature are the same in all reference frames.
G2 An observer with arbitrary motion may consider himself to be at rest and the environment as moving.

In the literature both G1 and G2 are mentioned as *the general principle of relativity*. But G2 is a stronger principle (i.e. stronger restriction on natural phenomena) than G1. Generally the course of events of a physical process in a certain reference frame depends upon the laws of physics, the boundary conditions, the motion of the reference frame and the geometry of spacetime. The *two* latter properties are described by means of a metrical tensor. By formulating the physical laws in a metric-independent way, one obtains that G1 is valid for all types of physical phenomena.

Even if the laws of Nature are the same in all reference frames, the course of events of a physical process will, as mentioned above, depend upon the motion of the reference frame. As to the spreading of light, for example, the law is that light follows null-geodesic curves (see Chap. 5). This law implies that the path of a light particle is curved in non-inertial reference frames and straight in inertial frames.

The question whether G2 is true in the general theory of relativity has been thoroughly discussed recently, and the answer is not clear yet [1]. It should be noted, however, that G1 is one of the fundamental principles of the general theory of relativity. Hence, for the remainder of this book G1 will be the starting point for all further considerations.

1.6 The Covariance Principle

The principle of relativity is a physical principle. It is concerned with physical phenomena. This principle motivates the introduction of a formal principle, called the *covariance principle*: The equations of a physical theory shall have the same form in every coordinate system.

This principle is not concerned directly with physical phenomena. The principle may be fulfilled for every theory by writing the equations in a form-invariant, i.e. covariant, way. This may be done by using tensor (vector) quantities, only, in the mathematical formulation of the theory.

The covariance principle and the equivalence principle may be used to obtain a description of what happens in the presence of gravitation. We then start with the physical laws as formulated in the special theory of relativity. Then the laws are written in a covariant form, by writing them as tensor equations. They are then valid in an arbitrary, accelerated system. But the inertial field ("fictive force") in the accelerated frame is equivalent to a gravitational field. So, starting with a description of an inertial frame, we have obtained a description valid in the presence of a gravitational field.

The tensor equations have in general a coordinate-independent form. Yet, such form-invariant, or covariant, equations need not fulfil the principle of relativity.

This is due to the following circumstances. A physical principle, for example the principle of relativity, is concerned with observable relationships. Therefore, when one is going to deduce the observable consequences of an equation, one has to establish relations between the tensor components of the equation and observable physical quantities. Such relations have to be defined; they are not determined by the covariance principle.

From the tensor equations, that are covariant, and the defined relations between the tensor components and the observable physical quantities, one can deduce equations between physical quantities. The special principle of relativity, for example, demands that the laws which these equations express must be the same with reference to every inertial frame.

The relationships between physical quantities and tensors (vectors) are theory dependent. The relative velocity between two bodies, for example, is a vector within Newtonian kinematics. However, in the relativistic kinematics of four-dimensional spacetime, an ordinary velocity, which has only three components, is not a vector. Vectors in spacetime, so-called 4-vectors, have four components. Equations between physical quantities are not covariant in general.

For example, Maxwell's equations in three-vector-form are not invariant under a Galilei transformation. However, if these equations are rewritten in tensor form, then neither a Galilei transformation nor any other transformation will change the form of the equations.

If all equations of a theory are tensor equations, the theory is said to be given a *manifestly covariant form*. A theory that is written in a manifestly covariant form will automatically fulfil the covariance principle, but it need not fulfil the principle of relativity.

1.7 Mach's Principle

Einstein gave up Newton's idea of an absolute space. According to Einstein all motion is relative. This may sound simple, but it leads to some highly non-trivial and fundamental questions.

Imagine that there are only two particles connected by a spring, in the universe. What will happen if the two particles rotate about each other? Will the spring be stretched due to centrifugal forces? Newton would have confirmed that this is indeed what will happen. However, when there is no longer any absolute space that the particles can rotate relatively to, the answer is not so obvious. If we, as observers, rotate around the particles, and they are at rest, we would not observe any stretching of the spring. But this situation is now kinematically equivalent to the one with rotating particles and observers at rest, which leads to stretching.

Such problems led Mach to the view that all motion is relative. The motion of a particle in an empty universe is not defined. All motion is motion relatively to something else, i.e. relatively to other masses. According to Mach this implies that inertial forces must be due to a particle's acceleration relative to the great masses of the universe. If there were no such cosmic masses, there would not exist inertial forces, like the centrifugal force. In our example with two particles connected by a string, there would not be any stretching of the spring, if there were no cosmic masses that the particles could rotate relatively to.

Another example may be illustrated by means of a turnabout. If we stay on this, while it rotates, we feel that the centrifugal forces lead us outwards. At the same time we observe that the heavenly bodies rotate. According to Mach identical centrifugal forces should appear if the turnabout is static and the heavenly bodies rotate.

Einstein was strongly influenced by Mach's arguments, which probably had some influence, at least with regards to motivation, on Einstein's construction of his general theory of relativity. Yet, it is clear that general relativity does not fulfil all requirements set by Mach's principle. For example there exist general relativistic, rotating cosmological models, where free particles will tend to rotate relative to the cosmic masses of the model.

However, some Machian effects have been shown to follow from the equations of the general theory of relativity. For example, inside a rotating, massive shell the inertial frames, i.e. the free particles, are dragged on and tend to rotate in the same direction as the shell. This was discovered by Lense and Thirring in 1918 and is therefore called the Lense–Thirring effect. More recent investigations of this effect have, among others, led to the following result [2]: "A massive shell with radius equal to its Schwarzschild radius has often been used as an idealized model of our universe. Our result shows that in such models local inertial frames near the center cannot rotate relatively to the mass of the universe. In this way our result gives an explanation in accordance with Mach's principle, of the fact that the 'fixed stars' is at rest on heaven as observed from an inertial reference frame".

Problems

1.1. *A tidal force pendulum*

(a) Find the expression of the period of a mathematical pendulum with a length ℓ in an area where the acceleration of gravity equals g. How long is the period of such a mathematical pendulum at the Earth's surface when the length of the pendulum is 0.25 m?

(b) A tidal force pendulum consists of two equally massive points that are connected through a stiff, massless rod. The length of the rod is 2ℓ. The pendulum oscillates with respect to the centre of the rod which again is fixed at a constant distance from the centre of the Earth. The oscillations takes place in a firm (i.e. non-rotating) vertical plane. The Earth is considered as a spherically symmetric mass distribution. The respective forces on the two massive points are shown in Fig. 1.7.

Find the two equilibrium positions of the pendulum and perform a stability analysis with respect to these two points. Then use the definition of torque to find an expression of the period of the harmonic motion of the rod. Find the period at the surface of the Earth. Is it possible to have a similar harmonic motion in a homogeneous gravitational field?

1.2. *Newtonian potentials for spherically symmetric bodies*

(a) Calculate the Newtonian potential $\phi(r)$ for a spherical shell of matter. Assume that the thickness of the shell is negligible, and the mass per unit area, σ, is constant on the spherical shell. Find the potential both inside and outside the shell.

(b) Let R and M be the radius and the mass of the Earth. Find the potential $\phi(r)$ for $r < R$ and $r > R$. The mass–density is assumed to be constant for $r < R$. Calculate the gravitational acceleration on the surface of the Earth. Compare with the actual value of $g = 9.81\ \mathrm{m/s^2}$ ($M = 6.0 \cdot 10^{24}$ kg and $R = 6.4 \cdot 10^6$ m).

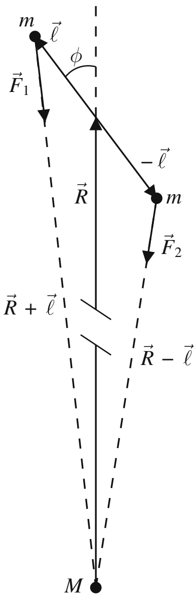

Fig. 1.7 A tidal force pendulum

(c) Assume that a hollow tube has been drilled right through the centre of the Earth. A small solid ball is then dropped into the tube from the surface of the Earth. Find the position of the ball as a function of time. What is the period of the oscillations of the ball?

(d) We now assume that the tube is not passing through the centre of the Earth, but at a closest distance s from the centre. Find how the period of the oscillations vary as a function of s. Assume for simplicity that the ball is sliding without friction (i.e. no rotation) in the tube.

1.3. *The Earth–Moon system*

(a) Assume that the Earth and the Moon are point objects and isolated from the rest of the Solar system. Put down the equations of motion for the Earth–Moon system. Show that there is a solution where the Earth and the Moon are moving in perfect circular orbits around their common centre of mass. What is the radii of the orbits when we know the mass of the Earth and the Moon, and the orbital period of the Moon?

(b) Find the Newtonian potential along the line connecting the two bodies. Draw the result in a plot, and find the point on the line where the gravitational interactions from the bodies exactly cancel each other.

(c) The Moon acts with a different force on a 1 kg measure on the surface of the Earth, depending on whether it is closest to or farthest from the Moon. Find the difference in these forces.

1.4. *The Roche limit*

(a) A spherical moon with a mass m and radius R is orbiting a planet with mass M. Show that if the moon is closer to its parent planet's centre than

$$r = \left(\frac{2M}{m}\right)^{1/3} R\,,$$

then loose rocks on the surface of the moon will be elevated due to tidal effects.

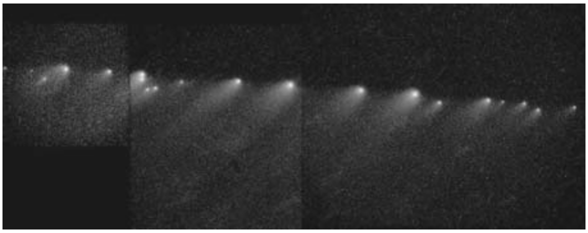

Fig. 1.8 Photograph of the comet Shoemaker-Levy 9 taken by the Hubble telescope, March 1994

(b) The comet Shoemaker-Levy 9, that in July 1994 collided with Jupiter, was ripped apart already in 1992 after having passed near Jupiter (Fig. 1.8). Assuming the comet had a mass of $m = 2.0 \cdot 10^{12}$ kg and that the closest passing in 1992 was at a distance of 96000 km from the centre of Jupiter, it is possible to estimate the size of the comet. Use that the mass of Jupiter is $M = 1.9 \cdot 10^{27}$ kg.

1.5. *The strength of gravity compared to the Coulomb force*

(a) Determine the difference in strength between the Newtonian gravitational attraction and the Coulomb force of the interaction of the proton and the electron in a hydrogen atom.
(b) What is the gravitational force of attraction of two objects of 1 kg at a separation of 1 m. Compare with the corresponding electrostatic force of two charges of 1 C at the same distance.
(c) Compute the gravitational force between the Earth and the Sun. If the attractive force was not gravitational but caused by opposite electric charges, then what would the charges be?

1.6. *Falling objects in the gravitational field of the Earth*

(a) Two test particles are in free fall towards the centre of the Earth. They both start from rest at a height of 3 Earth radii and with a horizontal separation of 1 m. How far have the particles fallen when the distance between them is reduced to 0.5 m?

(b) Two new test particles are dropped from the same height with a time separation of 1 s. The first particle is dropped from rest. The second particle is given an initial velocity equal to the instantaneous velocity of the first particle, and it follows after the first one in the same trajectory. How far and how long have the particles fallen when the distance between them is doubled?

1.7. *A Newtonian black hole*

In 1783 the English physicist John Michell used Newtonian dynamics and laws of gravity to show that for massive bodies which were small enough, the escape velocity of the bodies are larger than the speed of light. (The same was emphasized by the French mathematician and astronomer Pierre Laplace in 1796).

(a) Assume that the body is spherical with mass M. Find the largest radius, R, that the body can have in order for it to be a "black hole", i.e. so that light cannot escape. Assume naively that photons have kinetic energy $\frac{1}{2}mc^2$.
(b) Find the tidal force on two bodies m at the surface of a spherical body, when their internal distance is ζ. What would the tidal force be on the head and the feet of a 2 m tall human, standing upright, in the following cases (consider the head and feet as point particles, each weighing 5 kg):
   1. The human is standing on the surface of a black hole with 10 times the Solar mass.
   2. On the Sun's surface.
   3. On the Earth's surface.

References

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2. Brill, D. R. and Cohen, J. M. 1966. Rotating masses and their effect on inertial frames, *Phys. Rev.* **143**, 1011–1015. 13

Chapter 2
The Special Theory of Relativity

In this chapter we shall give a short introduction to the fundamental principles of the special theory of relativity and deduce some of the consequences of the theory.

The special theory of relativity was presented by Albert Einstein in 1905. It was founded on two postulates:

1. The laws of physics are the same in all Galilean frames.
2. The velocity of light in empty space is the same in all Galilean frames and independent of the motion of the light source.

Einstein pointed out that these postulates are in conflict with Galilean kinematics, in particular with the Galilean law for addition of velocities. According to Galilean kinematics two observers moving relative to each other cannot measure the same velocity for a certain light signal. Einstein solved this problem by a thorough discussion of how two distant clocks should be synchronized.

2.1 Coordinate Systems and Minkowski Diagrams

The most simple physical phenomenon that we can describe is called an event. This is an incident that takes place at a certain point in space and at a certain point in time. A typical example is the flash from a flashbulb.

A complete description of an event is obtained by giving the position of the event in space and time. Assume that our observations are made with reference to a reference frame. We introduce a coordinate system into our reference frame. Usually it is advantageous to employ a Cartesian coordinate system. This may be thought of as a cubic lattice constructed by measuring rods. If one lattice point is chosen as origin, with all coordinates equal to zero, then any other lattice point has three spatial coordinates equal to the distances of that point along the coordinate axes that pass through the origin. The spatial coordinates of an event are the three coordinates of the lattice point at which the event happens.

It is somewhat more difficult to determine the point of time of an event. If an observer is sitting at the origin with a clock, then the point of time when he catches sight of an event is not the point of time when the event happened. This is because

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the light takes time to pass from the position of the event to the observer at the origin. Since observers at different positions have to make different such corrections, it would be simpler to have (imaginary) observers at each point of the reference frame such that the point of time of an arbitrary event can be measured locally.

But then a new problem appears. One has to synchronize the clocks, so that they show the same time and go at the same rate. This may be performed by letting the observer at the origin send out light signals so that all the other clocks can be adjusted (with correction for light travel time) to show the same time as the clock at the origin. These clocks show the *coordinate time* of the coordinate system, and they are called *coordinate clocks*.

By means of the lattice of measuring rods and coordinate clocks, it is now easy to determine four coordinates ($x^0 = ct, x, y, z$) for every event. (We have multiplied the time coordinate t by the velocity of light c in order that all four coordinates shall have the same dimension.)

This coordinatization makes it possible to describe an event as a point P in a so-called *Minkowski diagram*. In this diagram we plot ct along the vertical axis and one of the spatial coordinates along the horizontal axis.

In order to observe particles in motion, we may imagine that each particle is equipped with a flash-light and that they flash at a constant frequency. The flashes from a particle represent a succession of events. If they are plotted into a Minkowski diagram, we get a series of points that describe a curve in the continuous limit. Such a curve is called a *world line* of the particle. The world line of a free particle is a straight line, as shown to left of the time axis in Fig. 2.1.

A particle acted upon by a net force has a curved world line as the velocity of the particle changes with time. Since the velocity of every material particle is less than the velocity of light, the tangent of a world line in a Minkowski diagram will always make an angle less than 45° with the time axis.

A flash of light gives rise to a light front moving onwards with the velocity of light. If this is plotted in a Minkowski diagram, the result is a light cone. In Fig. 2.1 we have drawn a light cone for a flash at the origin. It is obvious that we could have

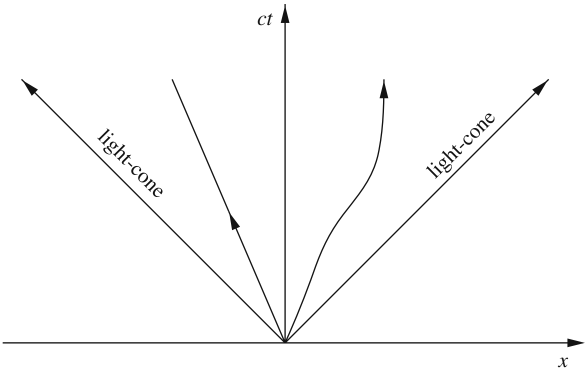

Fig. 2.1 World lines

drawn light cones at all points in the diagram. An important result is that *the world line of any particle at a point is inside the light cone of a flash from that point*. This is an immediate consequence of the special principle of relativity, and is also valid locally in the presence of a gravitational field.

2.2 Synchronization of Clocks

There are several equivalent methods that can be used to synchronize clocks. We shall here consider the radar method.

We place a mirror on the x-axis and emit a light signal from the origin at time t_A. This signal is reflected by the mirror at t_B, and received again by the observer at the origin at time t_C. According to the second postulate of the special theory of relativity, the light moves with the same velocity in both directions, giving

$$t_B = \frac{1}{2}(t_A + t_C) \,. \tag{2.1}$$

When this relationship holds we say that the clocks at the origin and at the mirror are *Einstein synchronized*. Such synchronization is presupposed in the special theory of relativity. The situation corresponding to synchronization by the radar method is illustrated in Fig. 2.2.

The radar method can also be used to measure distances. The distance L from the origin to the mirror is given by

$$L = \frac{c}{2}(t_C - t_A) \,. \tag{2.2}$$

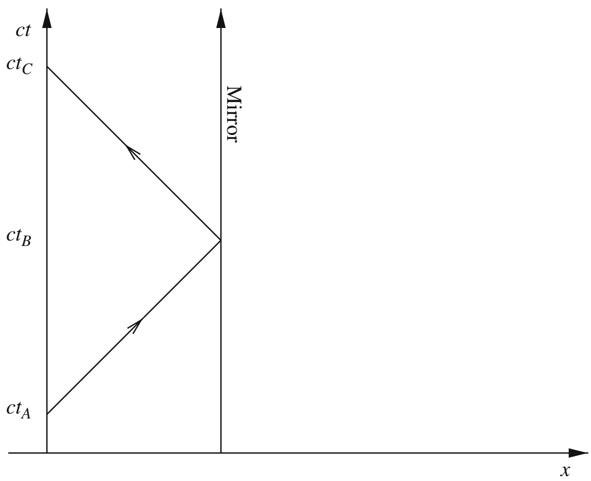

Fig. 2.2 Clock synchronization by the radar method

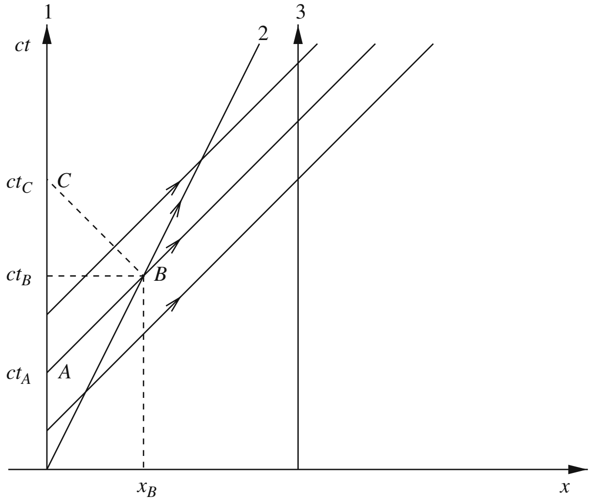

Fig. 2.3 The Doppler effect

2.3 The Doppler Effect

Consider three observers (1, 2 and 3) in an inertial frame. Observers 1 and 3 are at rest, while 2 moves with constant velocity along the x-axis. The situation is illustrated in Fig. 2.3.

Each observer is equipped with a clock. If observer 1 emits light pulses with a constant period τ_1, then observer 2 receives them with a longer period τ_2 according to his or her[1] clock. The fact that these two periods are different is a well-known phenomenon, called the *Doppler effect*. The same effect is observed with sound; the tone of a receding vehicle is lower than that of an approaching one.

We are now going to deduce a relativistic expression for the Doppler effect. First, we see from Fig. 2.3 that the two periods τ_1 and τ_2 are proportional to each other,

$$\tau_2 = K\tau_1 \,. \tag{2.3}$$

The constant $K(v)$ is called Bondi's K-factor. Since observer 3 is at rest, the period τ_3 is equal to τ_1 so that

$$\tau_3 = \frac{1}{K}\tau_2 \,. \tag{2.4}$$

These two equations imply that if 2 moves away from 1, so that $\tau_2 > \tau_1$, then $\tau_3 < \tau_2$. This is because 2 moves towards 3.

[1] For simplicity we shall – without any sexist implications – follow the grammatical convention of using masculine pronouns, instead of the more cumbersome "his or her".

The K-factor is most simply determined by placing observer 1 at the origin, while letting the clocks show $t_1 = t_2 = 0$ at the moment when 2 passes the origin. This is done in Fig. 2.3. The light pulse emitted at the point of time t_A, is received by 2 when his clock shows $\tau_2 = Kt_A$. If 2 is equipped with a mirror, the reflected light pulse is received by 1 at a point of time $t_C = K\tau_2 = K^2 t_A$. According to Eq. (2.1) the reflection event then happens at a point of time

$$t_B = \frac{1}{2}(t_C + t_A) = \frac{1}{2}(K^2 + 1)t_A \, . \tag{2.5}$$

The mirror has then arrived at a distance x_B from the origin, given by Eq. (2.2),

$$x_B = \frac{c}{2}(t_C - t_A) = \frac{c}{2}(K^2 - 1)t_A \, . \tag{2.6}$$

Thus, the velocity of observer 2 is

$$v = \frac{x_B}{t_B} = c\frac{K^2 - 1}{K^2 + 1} \, . \tag{2.7}$$

Solving this equation with respect to the K-factor we get

$$K = \left(\frac{c + v}{c - v}\right)^{1/2} \, . \tag{2.8}$$

This result is relativistically correct. The special theory of relativity was included through the tacit assumption that the velocity of the reflected light is c. This is a consequence of the second postulate of special relativity; the velocity of light is isotropic and independent of the velocity of the light source.

Since the wavelength λ of the light is proportional to the period τ, Eq. (2.3) gives the observed wavelength λ' for the case when the observer moves away from the source,

$$\boxed{\lambda' = K\lambda = \left(\frac{c + v}{c - v}\right)^{1/2} \lambda.} \tag{2.9}$$

This Doppler effect represents a redshift of the light. If the light source moves towards the observer, there is a corresponding blueshift given by K^{-1}.

It is common to express this effect in terms of the relative change of wavelength,

$$z = \frac{\lambda' - \lambda}{\lambda} = K - 1 \tag{2.10}$$

which is positive for redshift. If $v \ll c$, Eq. (2.9) gives

$$\frac{\lambda'}{\lambda} = K \approx 1 + \frac{v}{c} \tag{2.11}$$

to lowest order in v/c. The redshift is then

$$z = \frac{v}{c} \,. \tag{2.12}$$

This result is well known from non-relativistic physics.

2.4 Relativistic Time Dilation

Every periodic motion can be used as a clock. A particularly simple clock is called the light clock. This is illustrated in Fig. 2.4.

The clock consists of two parallel mirrors that reflect a light pulse back and forth. If the period of the clock is defined as the time interval between each time the light pulse hits the lower mirror, then $\Delta t' = 2L_0/c$.

Assume that the clock is at rest in an inertial reference frame Σ' where it is placed along the y-axis, as shown in Fig. 2.4. If this system moves along the ct-axis with a velocity v relative to another inertial reference frame Σ, the light pulse of the clock will follow a zigzag path as shown in Fig. 2.5.

The light signal follows a different path in Σ than in Σ'. The period Δt of the clock as observed in Σ is different from the period $\Delta t'$ which is observed in the rest frame. The period Δt is easily found from Fig. 2.5. Since the pulse takes the time $(1/2)\Delta t$ from the lower to the upper mirror and since the light velocity is always the same, we find

$$\left(\frac{1}{2}c\Delta t\right)^2 = L_0^2 + \left(\frac{1}{2}v\Delta t\right)^2 \,, \tag{2.13}$$

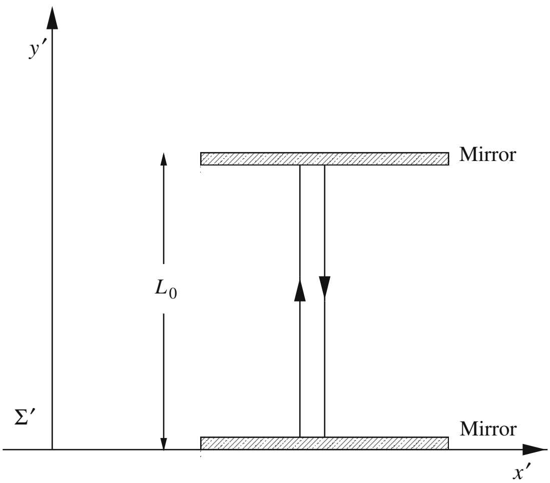

Fig. 2.4 Light clock

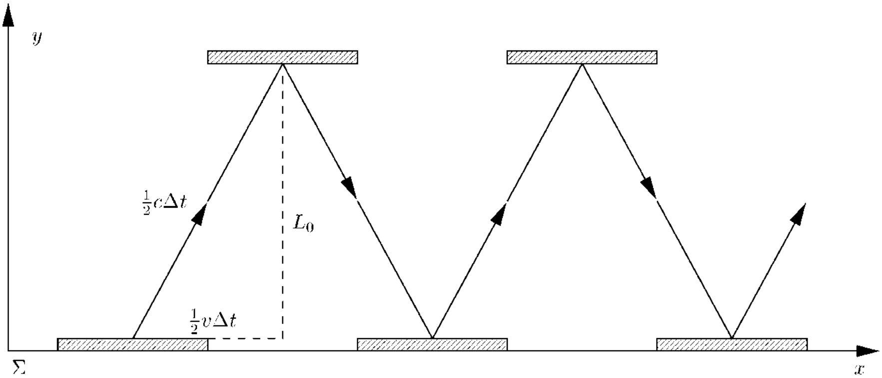

Fig. 2.5 Moving light clock

i.e.

$$\Delta t = \gamma \frac{2L_0}{c}, \quad \gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{2.14}$$

The γ factor is a useful short-hand notation for a term which is often used in relativity theory. It is commonly known as the Lorentz factor.

Since the period of the clock in its rest frame is $\Delta t' = 2L_0/c$, we get

$$\boxed{\Delta t = \gamma \Delta t'.} \tag{2.15}$$

Thus, we have to conclude that the period of the clock when it is observed to move (Δt) is greater that its rest period ($\Delta t'$). In other words: *a moving clock goes slower than a clock at rest*. This is called *the relativistic time dilation*. The period $\Delta t'$ of the clock as observed in its rest frame is called the proper period of the clock. The corresponding time t' is called the proper time of the clock.

One might be tempted to believe that this surprising consequence of the special theory of relativity has something to do with the special type of clock that we have employed. This is not the case. If there had existed a mechanical clock in Σ that did not show the time dilation, then an observer at rest in Σ might measure his velocity by observing the different rates of his light clock and this mechanical clock. In this way he could measure the absolute velocity of Σ. This would be in conflict with the special principle of relativity.

2.5 The Relativity of Simultaneity

Events that happen at the same point of time are said to be *simultaneous events*. We shall now show that according to the special theory of relativity, events that are simultaneous in one reference frame are not simultaneous in another reference

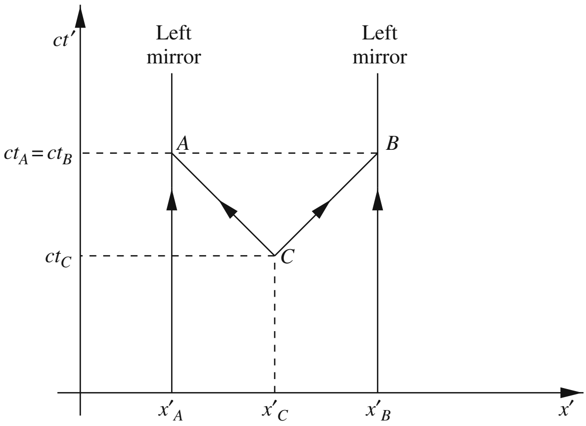

Fig. 2.6 Simultaneous events A and B

frame moving with respect to the first. This is what is meant by the expression "the relativity of simultaneity".

Consider again two mirrors connected by a line along the x'-axis, as shown in Fig. 2.6. Halfway between the mirrors there is a flash-lamp emitting a spherical wave front at a point of time t_C.

The points at which the light front reaches the left-hand and the right-hand mirrors are denoted by A and B, respectively. In the reference frame Σ' of Fig. 2.6 the events A and B are simultaneous.

If we describe the same course of events from another reference frame (Σ), where the mirror moves with constant velocity v in the positive x-direction, we find the Minkowski diagram shown in Fig. 2.7. Note that the light follows world lines making an angle of 45° with the axes. This is the case in every inertial frame.

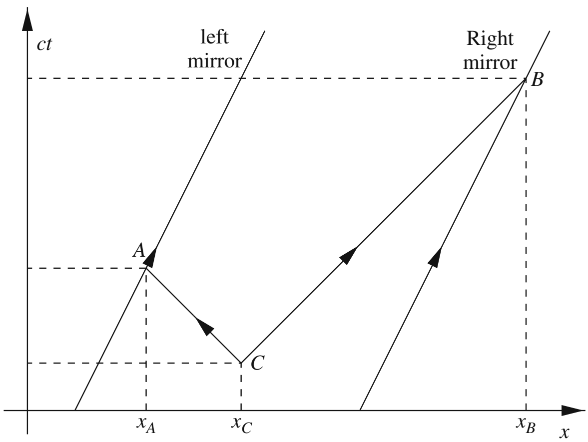

Fig. 2.7 The simultaneous events of Fig. 2.6 in another frame

In Σ the light pulse reaches the left mirror, which moves towards the light, before it reaches the right mirror, which moves in the same direction as the light. In this reference frame the events when the light pulses hit the mirrors are not simultaneous.

As an example illustrating the relativity of simultaneity, Einstein imagined that the events A, B and C happen in a train which moves past the platform with a velocity v. The event C represents the flash of a lamp at the mid-point of a wagon. A and B are the events when the light is received at the back end and at the front end of the wagon, respectively. This situation is illustrated in Fig. 2.8.

As observed in the wagon, A and B happen simultaneously. As observed from the platform the rear end of the wagon moves towards the light which moves backwards, while the light moving forwards has to catch up with the front end. Thus, as observed from the platform A will happen before B.

The time difference between A and B as observed from the platform will now be calculated. The length of the wagon, as observed from the platform, will be denoted by L. The time coordinate is chosen such that $t_C = 0$. The light moving backwards hits the rear wall at a point of time t_A. During the time t_A the wall has moved a distance vt_A forwards, and the light has moved a distance ct_A backwards. Since the distance between the wall and the emitter is $L/2$, we get

$$\frac{L}{2} = vt_A + ct_A \ . \tag{2.16}$$

Thus

$$t_A = \frac{L}{2(c+v)} \ . \tag{2.17}$$

In the same manner one finds

$$t_B = \frac{L}{2(c-v)} \ . \tag{2.18}$$

It follows that the time difference between A and B as observed from the platform is

$$\Delta t = t_B - t_A = \frac{\gamma^2 vL}{c^2} \ . \tag{2.19}$$

As observed from the wagon A and B are simultaneous. As observed from the platform the rear event A happens at a time interval Δt before the event B. This is the relativity of simultaneity.

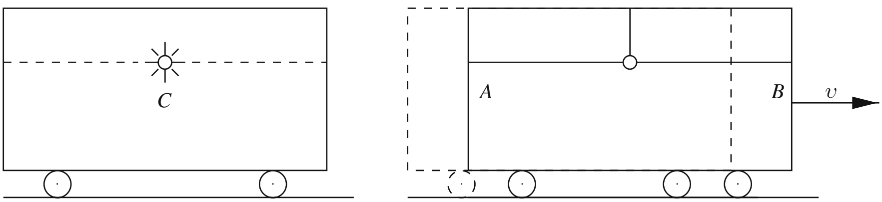

Fig. 2.8 Light flash in a moving train

2.6 The Lorentz Contraction

During the first part of the nineteenth century the so-called luminiferous ether was introduced into physics to account for the propagation and properties of light. After J.C. Maxwell showed that light is electromagnetic waves the ether was still needed as a medium in which electromagnetic waves propagated.

It was shown that Maxwell's equations do not obey the principle of relativity, when coordinates are changed using the Galilean transformations. If it is assumed that the Galilean transformations are correct, then Maxwell's equations can only be valid in one coordinate system. This coordinate system was the one in which the ether was at rest. Hence, Maxwell's equations in combination with the Galilean transformations implied the concept of "absolute rest". This made the measurement of the velocity of the Earth relative to the ether of great importance.

An experiment sufficiently accurate to measure this velocity to order v^2/c^2 was carried out by Michelson and Morley in 1887. A simple illustration of the experiment is shown in Fig. 2.9.

Our earlier photon clock is supplied by a mirror at a distance L along the x-axis from the emitter. The apparatus moves in the x-direction with a velocity v. In the rest frame (Σ') of the apparatus, the distance between A and B is equal to the distance between A and C. This distance is denoted by L_0 and is called the *rest length* between A and B.

Light is emitted from A. Since the velocity of light is isotropic and the distances to B and C are equal in Σ', the light reflected from B and that reflected from C have the same travelling time. This was the result of the Michelson–Morley experiment, and it seems that we need no special effects such as the Lorentz contraction to explain the experiment.

However, before 1905 people believed in the physical reality of absolute velocity. The Earth was considered to move through an "ether" with a velocity that changed with the seasons. The experiment should therefore be described under the assumption that the apparatus is moving.

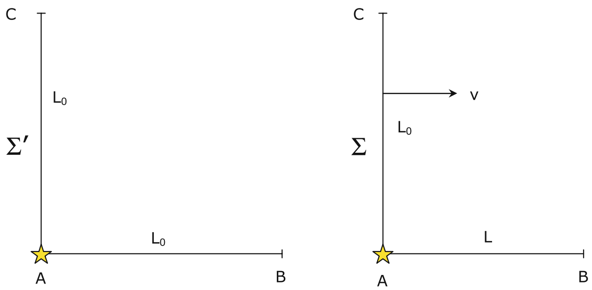

Fig. 2.9 Length contraction

Let us therefore describe an experiment from our reference frame Σ, which may be thought of as at rest in the "ether". Then according to Eq. (2.14) the travel time of the light being reflected at C is

$$\Delta t_C = \gamma \frac{2L_0}{c} \,. \tag{2.20}$$

For the light moving from A to B we may use Eq. (2.18), and for the light from B to A Eq. (2.17). This gives

$$\Delta t_B = \frac{L}{c-v} + \frac{L}{c+v} = \gamma^2 \frac{2L}{c} \,. \tag{2.21}$$

If length is independent of velocity, then $L = L_0$. In this case the travelling times of the light signals will be different. The travelling time difference is

$$\Delta t_B - \Delta t_C = \gamma(\gamma - 1)\frac{2L_0}{c} \,. \tag{2.22}$$

To the lowest order in v/c, $\gamma \approx 1 + \frac{1}{2}(v/c)^2$, so that

$$\Delta t_B - \Delta t_C \approx \frac{1}{2}\left(\frac{v}{c}\right)^2 \,, \tag{2.23}$$

which depends upon the velocity of the apparatus.

According to the ideas involving an absolute velocity of the Earth through the ether, if one lets the light reflected at B interfere with the light reflected at C (at the position A) then the interference pattern should vary with the season. This was not observed. On the contrary, observations showed that $\Delta t_B = \Delta t_C$.

Assuming that length varies with velocity, Eqs. (2.20) and (2.21), together with this observation, gives

$$\boxed{L = \gamma^{-1} L_0.} \tag{2.24}$$

The result that $L < L_0$ (i.e. the length of a rod is less when it moves than when it is at rest) is called the *Lorentz contraction*.

2.7 The Lorentz Transformation

An event P has coordinates $(t', x', 0, 0)$ in a Cartesian coordinate system associated with a reference frame Σ'. Thus the distance from the origin of Σ' to P measured with a measuring rod at rest in Σ' is x'. If the distance between the origin of Σ' and the position at the x-axis where P took place is measured with measuring rods at rest in a reference frame moving with velocity v in the x-direction relative to Σ', one finds the length $\gamma^{-1}x$ due to the Lorentz contraction. Assuming that the origin of Σ and Σ' coincided at the point of time $t = 0$, the origin of Σ' has an x-coordinate vt at a point of time t. The event P thus has an x-coordinate

$$x = vt + \gamma^{-1} x' \tag{2.25}$$

or

$$x' = \gamma(x - vt) \; . \tag{2.26}$$

The x-coordinate may be expressed in terms of t' and x' by letting $v \to -v$,

$$\boxed{x = \gamma(x' + vt').} \tag{2.27}$$

The y and z coordinates are associated with axes directed perpendicular to the direction of motion. Therefore, they are the same in the two-coordinate systems

$$y = y' \;\text{ and }\; z = z' \; . \tag{2.28}$$

Substituting x' from Eq. (2.26) in to Eq. (2.27) reveals the connection between the time coordinates of the two-coordinate systems,

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{2.29}$$

and

$$\boxed{t = \gamma\left(t' + \frac{vx'}{c^2}\right).} \tag{2.30}$$

The latter term in this equation is nothing but the deviation from simultaneity in Σ for two events that are simultaneous in Σ'.

The relations (2.27)–(2.30) between the coordinates of Σ and Σ' represent a special case of the *Lorentz transformations*. The above relations are special since the two-coordinate systems have the same spatial orientation, and the x and x'-axes are aligned along the relative velocity vector of the associated frames. Such transformations are called *boosts*.

For non-relativistic velocities $v \ll c$, the Lorentz transformations (2.27)–(2.30) pass over into the corresponding Galilei transformations.

The Lorentz transformation gives a connection between the relativity of simultaneity and the Lorentz contraction. The *length* of a body is defined as the difference between the coordinates of its end points, *as measured by simultaneity in the rest frame of the observer*.

Consider the wagon of Sect. 2.5. Its rest length is $L_0 = x'_B - x'_A$. The difference between the coordinates of the wagon's end points, $x_A - x_B$ as measured in Σ, is given implicitly by the Lorentz transformation

$$x'_B - x'_A = \gamma[x_B - x_A - v(t_B - t_A)] \; . \tag{2.31}$$

According to the above definition the length (L) of the moving wagon is given by $L = x_B - x_A$ with $t_B = t_A$.

From Eq. (2.31) we then get

$$L_0 = \gamma L \tag{2.32}$$

which is equivalent to Eq. (2.24).

The Lorentz transformation will now be used to deduce the relativistic formulae for velocity addition. Consider a particle moving with velocity u along the x'-axis of Σ'. If the particle was at the origin at $t' = 0$, its position at t' is $x' = u't'$. Using this relation together with Eqs. (2.27) and (2.28) we find the velocity of the particle as observed in Σ

$$\boxed{u = \frac{x}{t} = \frac{u' + v}{1 + \frac{u'v}{c^2}}.} \tag{2.33}$$

A remarkable property of this expression is that by adding velocities less than c one cannot obtain a velocity greater than c. For example, if a particle moves with a velocity c in Σ' then its velocity in Σ is also c regardless of Σ's velocity relative to Σ'.

Equation (2.33) may be written in a geometrical form by introducing the so-called *rapidity* η defined by

$$\boxed{\tanh \eta = \frac{u}{c}} \tag{2.34}$$

for a particle with velocity u. Similarly the rapidity of Σ' relative to Σ is

$$\tanh \theta = \frac{v}{c} \,. \tag{2.35}$$

Since

$$\tanh(\eta' + \theta) = \frac{\tanh \eta' + \tanh \theta}{1 + \tanh \eta' \tanh \theta} \,, \tag{2.36}$$

the relativistic velocity addition formula, Eq. (2.33), may be written as

$$\eta = \eta' + \theta \,. \tag{2.37}$$

Since rapidities are additive, their introduction simplifies some calculations and they have often been used as variables in elementary particle physics.

With these new hyperbolic variables we can write the Lorentz transformation in a particularly simple way. Using Eq. (2.35) in Eqs. (2.27) and (2.30) we find

$$\boxed{x = x' \cosh \theta + ct' \sinh \theta, \quad ct = x' \sinh \theta + ct' \cosh \theta.} \tag{2.38}$$

2.8 Lorentz-Invariant Interval

Let two events be given. The coordinates of the events, as referred to two different reference frames Σ and Σ', are connected by a Lorentz transformation. The coordinate differences are therefore connected by

$$\begin{aligned}\Delta t &= \gamma(\Delta t' + \tfrac{v}{c^2}\Delta x'), & \Delta x &= \gamma(\Delta x' + v\Delta t'),\\ \Delta y &= \Delta y', & \Delta z &= \Delta z'.\end{aligned} \tag{2.39}$$

Just like $(\Delta y)^2 + (\Delta z)^2$ is invariant under a rotation about the x-axis, $-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ is invariant under a Lorentz transformation, i.e.

$$\begin{aligned}(\Delta s)^2 &= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\\ &= -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \,.\end{aligned} \tag{2.40}$$

This combination of squared coordinate intervals is called the spacetime interval, or the *interval*. It is invariant under both rotations and Lorentz transformations.

Due to the minus sign in Eq. (2.40), the interval between two events may be positive, zero or negative. These three types of intervals are called

$$\begin{aligned}&(\Delta s)^2 > 0 \text{ space-like}\\ &(\Delta s)^2 = 0 \text{ light-like}\\ &(\Delta s)^2 < 0 \text{ time-like.}\end{aligned} \tag{2.41}$$

The reasons for these names are the following. Given two events with a space-like interval (A and B in Fig. 2.10), there exists a Lorentz transformation to a new reference frame where A and B happen simultaneously. In this frame the distance between the events is purely spatial. Two events with a light-like interval (C and D in Fig. 2.10) can be connected by a light signal, i.e. one can send a photon from C to D. The events E and F have a time-like interval between them, and can be observed from a reference frame in which they have the same spatial position, but occur at different points of time.

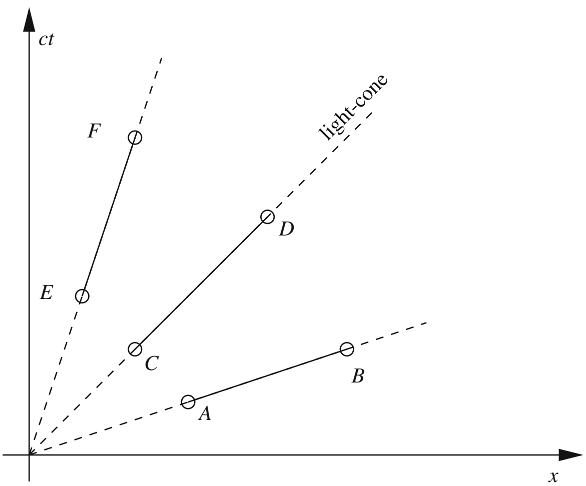

Fig. 2.10 The interval between A and B is space-like, between C and D light-like and between E and F time-like

Since all material particles move with a velocity less than that of light, the points on the world line of a particle are separated by time-like intervals. The curve is then said to be time-like. All time-like curves through a point pass inside the light cone from that point.

If the velocity of a particle is $u = \Delta x/\Delta t$ along the x-axis, Eq. (2.40) gives

$$(\Delta s)^2 = -\left(1 - \frac{u^2}{c^2}\right)(c\Delta t)^2 \; . \tag{2.42}$$

In the rest frame Σ' of the particle, $\Delta x' = 0$, giving

$$(\Delta s)^2 = -(c\Delta t')^2 \; . \tag{2.43}$$

The time t' in the rest frame of the particle is the same as the time measured on a clock carried by the particle. It is called the *proper time* of the particle, and denoted by τ. From Eqs. (2.42) and (2.43) it follows that

$$\Delta\tau = \sqrt{1 - \frac{u^2}{c^2}}\Delta t = \gamma^{-1}\Delta t \tag{2.44}$$

which is an expression of the relativistic time dilation.

Equation (2.43) is important. It gives the physical interpretation of a time-like interval between two events. The interval is a measure of the proper time interval between the events. This time is measured on a clock that moves such that it is present at both events. In the limit $u \to c$ (the limit of a light signal), $\Delta\tau = 0$. This shows that $(\Delta s)^2 = 0$ for a light-like interval.

Consider a particle with a variable velocity, $u(t)$, as indicated in Fig. 2.11. In this situation we can specify the velocity at an arbitrary point of the world line. Equation (2.44) can be used with this velocity, in an infinitesimal interval around this point,

$$d\tau = \sqrt{1 - \frac{u^2(t)}{c^2}}dt \; . \tag{2.45}$$

This equation means that the acceleration has no local effect upon the proper time of the clock. Here the word “local” means as measured by an observer at the position of the clock. Such clocks are called *standard clocks*.

If a particle moves from A to B in Fig. 2.11, the proper time as measured on a standard clock following the particle is found by integrating Eq. (2.45)

$$\tau_B - \tau_A = \int_A^B \sqrt{1 - \frac{u^2(t)}{c^2}}dt \; . \tag{2.46}$$

The relativistic time dilation has been verified with great accuracy by observations of unstable elementary particles with short lifetimes [1].

An infinitesimal spacetime interval

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{2.47}$$

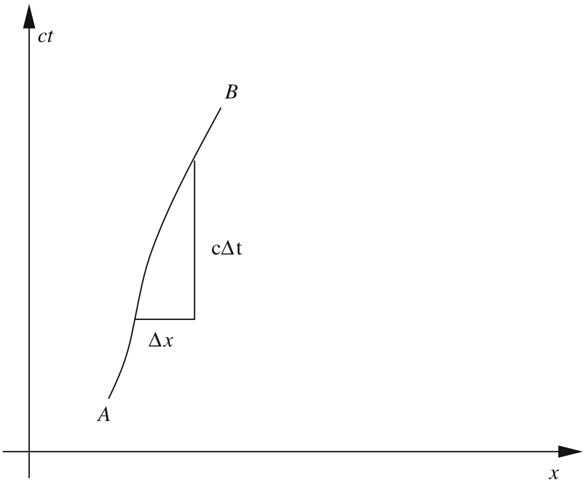

Fig. 2.11 World line of an accelerating particle

is called a *line element*. The physical interpretation of the line element between two infinitesimally close events on a time-like curve is

$$\boxed{ds^2 = -c^2 d\tau^2,} \tag{2.48}$$

where $d\tau$ is the proper time interval between the events, measured with a clock following the curve. The spacetime interval between two events is given by the integral (2.46). It follows that *the proper time interval between two events is path dependent*. This leads to the following surprising result: A time-like interval between two events is *greatest* along the straightest possible curve between them.

2.9 The Twin Paradox

Rather than discussing the life time of elementary particles, we may as well apply Eq. (2.46) to a person. Let her name be Eva. Assume that Eva is rapidly accelerating from rest at the point of time $t = 0$ at origin to a velocity v along the x-axis of a (ct, x) coordinate system in an inertial reference frame Σ. (See Fig. 2.12.)

At a point of time t_P she has come to a position x_P. She then rapidly decelerates until reaching a velocity v in the negative x-direction. At a point of time t_Q, as measured on clocks at rest in Σ, she has returned to her starting location. If we neglect the brief periods of acceleration, Eva's travelling time as measured on a clock which she carries with her is

$$t_{\mathrm{Eva}} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} t_Q\,. \tag{2.49}$$

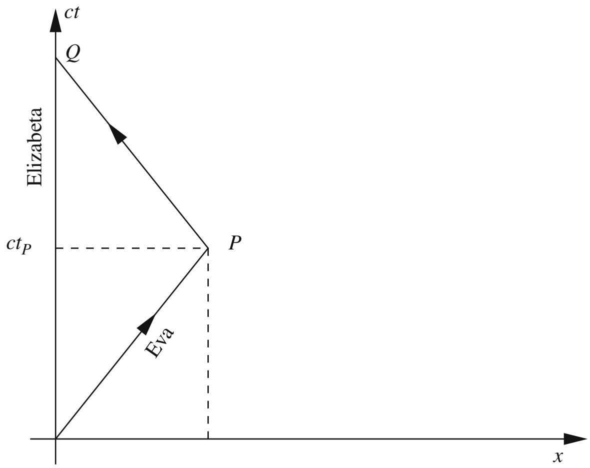

Fig. 2.12 World lines of the twin sisters Eva and Elizabeth

Now assume that Eva has a twin sister named Elizabeth who remains at rest at the origin of Σ.

Elizabeth has become older by $\tau_{\text{Elizabeth}} = t_Q$ during Eva's travel, so that

$$\tau_{\text{Eva}} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \tau_{\text{Elizabeth}} \,. \tag{2.50}$$

For example, if Eva travelled to Alpha Centauri (the Sun's nearest neighbour at four light years) with a velocity $v = 0.8c$, she would be gone for 10 years as measured by Elizabeth. Therefore Elizabeth has aged 10 years during Eva's travel. According to Eq. (2.50), Eva has only aged 6 years. According to Elizabeth, Eva has aged less than herself during her travels.

The principle of relativity, however, tells that Eva can consider herself as at rest and Elizabeth as the traveller. According to Eva it is Elizabeth who has only aged by 6 years, while Eva has aged by 10 years during the time they are apart.

What happens? How can the twin sisters arrive at the same prediction as to how much each of them age during the travel? In order to arrive at a clear answer to these questions, we shall have to use a result from the general theory of relativity. The twin paradox will be taken up again in Chap. 5.

2.10 Hyperbolic Motion

With reference to an inertial reference frame it is easy to describe relativistic accelerated motion. The special theory of relativity is in no way limited to describe motion with constant velocity.

Let a particle move with a variable velocity $u(t) = dx/dt$ along the x-axis in Σ. The frame Σ' moves with velocity v in the same direction relative to Σ. In this frame the particle velocity is $u'(t') = dx'/dt'$. At every moment the velocities u and u' are connected by the relativistic formula for velocity addition, Eq. (2.33). Thus, a velocity change du' in Σ' and the corresponding velocity change du in Σ are related – using Eq. (2.30) by

$$dt = \frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1+\frac{u'v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}dt' \, . \tag{2.51}$$

Combining these expressions we obtain the relationship between the acceleration of the particle as measured in Σ and in Σ'

$$a = \frac{du}{dt} = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{u'v}{c^2}\right)^3}a' \, . \tag{2.52}$$

Until now the reference frame Σ' has had an arbitrary velocity. Now we choose $v = u(t)$ so that Σ' is the instantaneous rest frame of the particle at a point of time t. At this moment $u' = 0$. Then Eq. (2.52) reduces to

$$a = \left(1-\frac{u^2}{c^2}\right)^{3/2} a' \, . \tag{2.53}$$

Here a' is the acceleration of the particle as measured in its instantaneous rest frame. It is called *the rest acceleration* of the particle. Equation (2.53) can be integrated if we know how the rest acceleration of the particle varies with time.

We shall now focus on the case where the particle has uniformly accelerated motion and moves along a straight path in space. The rest acceleration of the particle is constant, say $a' = g$. Integration of Eq. (2.53) with $u(0) = 0$ then gives

$$u = \left[1+\frac{g^2}{c^2}t^2\right]^{-1/2} gt \, . \tag{2.54}$$

Integrating once more gives

$$x = \frac{c^2}{g}\left[1+\frac{g^2}{c^2}t^2\right]^{1/2} + x_0 - \frac{c^2}{g} \, , \tag{2.55}$$

where x_0 is a constant of integration corresponding to the position at $t = 0$.

Equation (2.55) can be given the form

$$\boxed{\left(x - x_0 + \frac{c^2}{g}\right)^2 - c^2t^2 = \frac{c^4}{g^2}.} \tag{2.56}$$

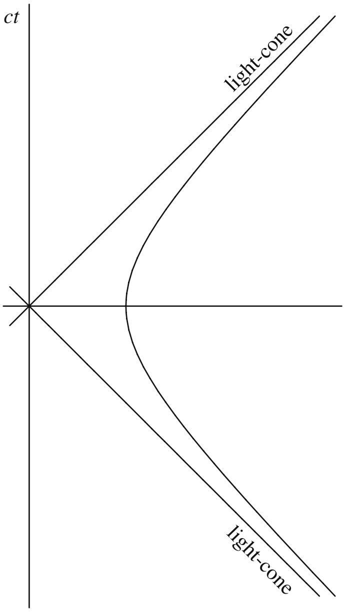

Fig. 2.13 World line of particle with constant rest acceleration

As shown in Fig. 2.13, this is the equation of a hyperbola in the Minkowski-diagram.

Since the world line of a particle with uniformly accelerated, rectilinear motion has the shape of a hyperbola, this type of motion is called *hyperbolic motion*.

Using the proper time τ of the particle as a parameter, we may obtain a simple parametric representation of its world line. Substituting Eq. (2.54) into Eq. (2.45) we get

$$d\tau = \frac{dt}{\sqrt{1+\frac{g^2}{c^2}t^2}}\,. \tag{2.57}$$

Integration with $\tau(0) = 0$ gives

$$\tau = \frac{c}{g}\operatorname{arcsinh}\left(\frac{gt}{c}\right) \tag{2.58}$$

or

$$\boxed{t = \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right).} \tag{2.59}$$

Inserting this expression into Eq. (2.55), we get

$$\boxed{x = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right) + x_0 - \frac{c^2}{g}.} \tag{2.60}$$

These expressions shall be used later when describing uniformly accelerated reference frames.

Note that *hyperbolic motion* results when the particle moves with *constant rest acceleration*. Such motion is usually called *uniformly accelerated motion*. Motion with constant acceleration as measured in the "laboratory frame" Σ gives rise to the usual parabolic motion.

2.11 Energy and Mass

The existence of an electromagnetic radiation pressure was well known before Einstein formulated the special theory of relativity. In black body radiation with energy density ρ there is an isotropic pressure $p = (1/3)\rho c^2$. If the radiation moves in a certain direction (laser), then the pressure in this direction is $p = \rho c^2$.

Einstein gave several deductions of the famous equation connecting the inertial mass of a body with its energy content. A deduction he presented in 1906 is as follows.

Consider a box with a light source at one end. A light pulse with radiation energy E is emitted to the other end where it is absorbed. (See Fig. 2.14.)

The box has a mass M and a length L. Due to the radiation pressure of the shooting light pulse the box receives a recoil. The pulse is emitted during a time interval Δt. During this time the radiation pressure is

$$p = \rho c^2 = \frac{E}{V} = \frac{E}{Ac\Delta t}\,, \tag{2.61}$$

where V is the volume of the radiation pulse and A the area of a cross-section of the box. The recoil velocity of the box is

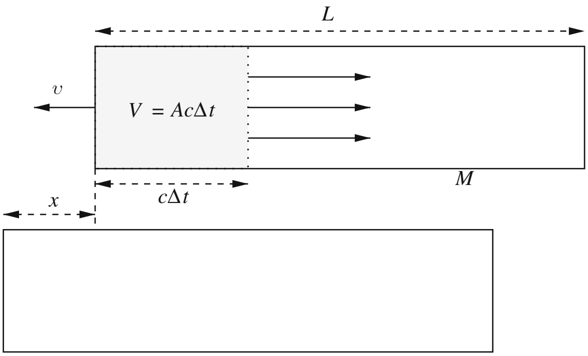

Fig. 2.14 Light pulse in a box

$$\begin{aligned}\Delta v &= -a\Delta t = -\frac{F}{M}\Delta t = -\frac{pA}{M}\Delta t \\ &= -\left(\frac{E}{Ac\Delta t}\right)\left(\frac{A\Delta t}{M}\right) = -\frac{E}{Mc}\,.\end{aligned} \tag{2.62}$$

The pulse takes the time L/c to move to the other side of the box. During this time the box moves a distance

$$\Delta x = \Delta v \frac{L}{c} = -\frac{EL}{Mc^2}\,. \tag{2.63}$$

Then the box is stopped by the radiation pressure caused by the light pulse hitting the wall at the other end of the box.

Let m be the mass of the radiation. Before Einstein one would put $m = 0$. Einstein, however, reasoned as follows. Since the box and its contents represents an isolated system, the mass centre has not moved. The mass centre of the box with mass M has moved a distance Δx to the left, the radiation with mass m has moved a distance L to the right. Thus

$$mL + M\Delta x = 0 \tag{2.64}$$

which gives

$$m = -\frac{M}{L}\Delta x = -\left(\frac{M}{L}\right)\left(-\frac{EL}{Mc^2}\right) = \frac{E}{c^2} \tag{2.65}$$

or

$$\boxed{E = mc^2.} \tag{2.66}$$

Here we have shown that radiation energy has an innate mass given by Eq. (2.65). Einstein derived Eq. (2.66) using several different methods showing that it is valid in general for all types of systems.

The energy content of even small bodies is enormous. For example, by transforming 1 g of matter to heat, one may heat 300,000 metric tons of water from room temperature to the boiling point. (The energy corresponding to a mass m is enough to change the temperature by ΔT of an object of mass M and specific heat capacity c_V: $mc^2 = Mc_V\Delta T$.)

2.12 Relativistic Increase of Mass

In the special theory of relativity, force is defined as rate of change of momentum. We consider a body that gets a change of energy dE due to the work performed on it by a force F. According to Eq. (2.66) and the definition of work (force times distance) the body gets a change of mass dm, given by

$$c^2 dm = dE = F ds = F v dt = v d(mv) = mv dv + v^2 dm\,, \tag{2.67}$$

which gives

$$\int_{m_0}^{m} \frac{dm}{m} = \int_0^v \frac{v dv}{c^2 - v^2}\,, \tag{2.68}$$

where m_0 is the rest mass of the body – i.e. its mass as measured by an observer comoving with the body – and m its mass when its velocity is equal to v. Integration gives

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_0 \ . \tag{2.69}$$

In the case of small velocities compared to the velocity of light we may use the approximation

$$\sqrt{1-\frac{v^2}{c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \ . \tag{2.70}$$

With this approximation Eqs. (2.66) and (2.69) give

$$E \approx m_0 c^2 + \frac{1}{2} m_0 v^2 \ . \tag{2.71}$$

This equation shows that the total energy of a body encompasses its rest energy m_0 and its kinetic energy. In the non-relativistic limit the kinetic energy is $m_0 v^2/2$. The relativistic expression for the kinetic energy is

$$E_K = E - m_0 c^2 = (\gamma - 1) m_0 c^2 \ . \tag{2.72}$$

Note that $E_K \rightarrow \infty$ when $v \rightarrow c$.

According to Eq. (2.33), it is not possible to obtain a velocity greater than that of light by adding velocities. Equation (2.72) gives a dynamical reason that material particles cannot be accelerated up to and above the velocity of light.

2.13 Tachyons

Particles cannot pass the velocity-barrier represented by the velocity of light. However, the special theory of relativity permits the existence of particles that have *always* moved with a velocity $v > c$. Such particles are called *tachyons*.

Tachyons have special properties that have been used in the experimental searches for them. There is currently no observational evidence for the physical existence of tachyons.

There are also certain theoretical difficulties with the existence of tachyons. The special theory of relativity applied to tachyons leads to the following paradox. Using a tachyon telephone a person, A, emits a tachyon to B at a point of time t_1. B moves away from A. The tachyon is reflected by B and reach A before it was emitted, see Fig. 2.15. If the tachyon could carry information it might bring an order to destroy the tachyon emitter when it arrives back at A.

To avoid similar problems in regards to the energy-exchange between tachyons and ordinary matter, a reinterpretation principle is introduced for tachyons. For certain observers a tachyon will move backwards in time, i.e. the observer finds that

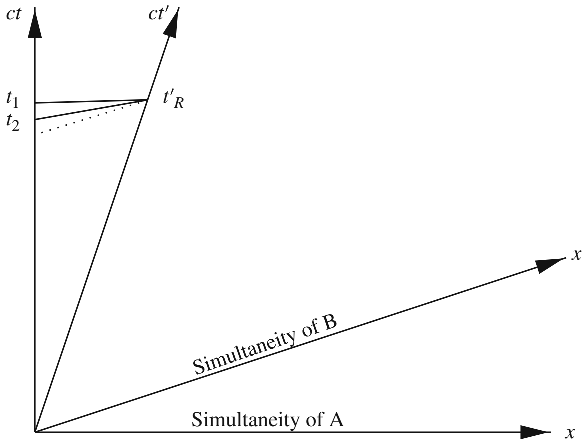

Fig. 2.15 *A* emits a tachyon at the point of time t_1. It is reflected by *B* and arrives at *A* at a point of time t_2 before t_1. Note that the arrival event at *A* is later than the reflection event as measured by *B*

the tachyon is received before it was emitted. Special relativity tells us that such a tachyon is always observed to have negative energy.

According to the reinterpretation principle, the observer will interpret his observations to mean that a tachyon with positive energy moves forward in time. In this way, one finds that the energy-exchange between tachyons and ordinary matter proceeds in accordance with the principle of causality.

However, the reinterpretation principle cannot be used to remove the problems associated with exchange of information between tachyons and ordinary matter. The tachyon telephone paradox cannot be resolved by means of the reinterpretation principle. The conclusion is that if tachyons exist, they cannot be carriers of information in our slowly moving world.

2.14 Magnetism as a Relativistic Second-Order Effect

Electricity and magnetism are described completely by Maxwell's equations of the electromagnetic field,

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho_q \tag{2.73}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.74}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.75}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{2.76}$$

together with Lorentz's force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ . \tag{2.77}$$

However, the relation between the magnetic and the electric force was not fully understood until Einstein had constructed the special theory of relativity. Only then could one clearly see the relationship between the magnetic force on a charge moving near a current carrying wire and the electric force between charges.

We shall consider a simple model of a current carrying wire in which we assume that the positive ions are at rest while the conducting electrons move with the velocity v. The charge per unit length for each type of charged particle is $\hat{\lambda} = Sne$ where S is the cross-sectional area of the wire, n the number of particles of one type per unit length and e the charge of one particle. The current in the wire is

$$J = Snev = \hat{\lambda} v \ . \tag{2.78}$$

The wire is at rest in an inertial frame $\hat{\Sigma}$. As observed in $\hat{\Sigma}$ it is electrically neutral. Let a charge q move with a velocity u along the wire in the opposite direction of the electrons. The rest frame of q is Σ. The wire will now be described from Σ (see Figs. 2.16 and 2.17).

Note that the charge per unit length of the particles as measured in their own rest frames, Σ_0, is

$$\lambda_{0-} = \hat{\lambda}\left(1 - \frac{v^2}{c^2}\right)^{1/2}, \quad \lambda_{0+} = \hat{\lambda} \tag{2.79}$$

since the distance between the electrons is Lorentz contracted in $\hat{\Sigma}$ compared to their distances in Σ_0.

The velocities of the particles as measured in Σ are

$$v_- = -\frac{v+u}{1+\frac{uv}{c^2}} \quad \text{and} \quad v_+ = -u \ . \tag{2.80}$$

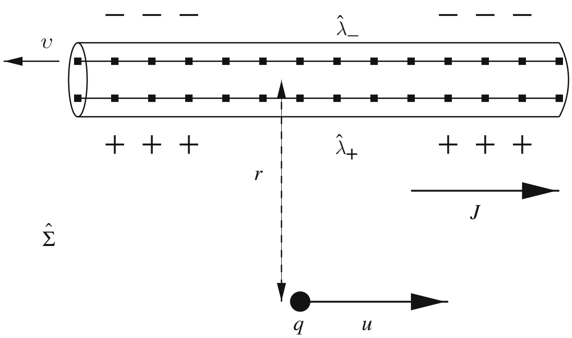

Fig. 2.16 Wire seen from its own rest frame

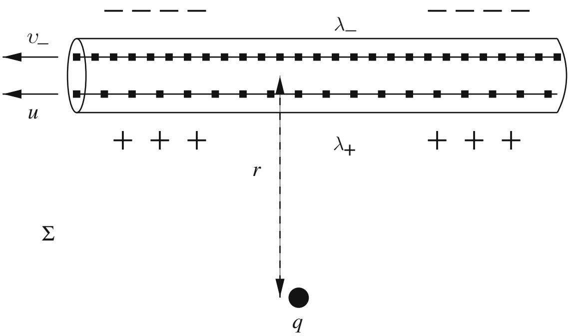

Fig. 2.17 Wire seen from rest frame of moving charge

The charge per unit length of the negative particles as measured in Σ is

$$\lambda_- = \left(1 - \frac{v_-^2}{c^2}\right)^{-1/2} \lambda_0 \,. \tag{2.81}$$

Substitution from Eqs. (2.79) and (2.80) gives

$$\lambda_- = \gamma\left(1 + \frac{uv}{c^2}\right)\hat{\lambda} \,, \tag{2.82}$$

where $\gamma = \left(1 - u^2/c^2\right)^{-1/2}$. In a similar manner, the charge per unit length of the positive particles as measured in Σ is found to be

$$\lambda_+ = \gamma\hat{\lambda} \,. \tag{2.83}$$

Thus, as observed in the rest frame of q the wire has a net charge per unit length

$$\lambda = \lambda_- - \lambda_+ = \frac{\gamma uv}{c^2}\hat{\lambda} \,. \tag{2.84}$$

As a result of the different Lorentz contractions of the positive and negative ions when we transform from their respective rest frames to Σ, a current carrying wire which is electrically neutral in the laboratory frame is observed to be electrically charged in the rest frame of the charge q.

As observed in this frame there is a radial electrical field with field strength

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \,. \tag{2.85}$$

Then a force F acts on q, this is given by

$$F = qE = \frac{q\lambda}{2\pi\epsilon_0 r} = \frac{\hat{\lambda} v}{2\pi\epsilon_0 c^2 r}\gamma q u \,. \tag{2.86}$$

If a force acts upon q as observed in $\hat{\Sigma}$ then a force also acts on q as observed in Σ. According to the relativistic transformation of a force component in the same direction as the relative velocity between $\hat{\Sigma}$ and Σ, this force is

$$\hat{F} = \gamma^{-1}F = \frac{\hat{\lambda}v}{2\pi\epsilon_0 c^2 r}qu \,. \tag{2.87}$$

Inserting $J = \hat{\lambda}v$ from Eq. (2.78) and using $c^2 = (\epsilon_0\mu_0)^{-1}$ (where μ_0 is the permeability of a vacuum) we obtain

$$\hat{F} = \frac{\mu_0 J}{2\pi r}qu \,. \tag{2.88}$$

This is exactly the expression obtained if we calculate the magnetic flux-density $\hat{B}$ around the current carrying wire using Ampere's circuit law

$$\hat{B} = \mu_0 \frac{J}{2\pi r} \tag{2.89}$$

and use the force law (Eq. (2.77)) for a charge moving in a magnetic field

$$\hat{F} = qu\hat{B} \,. \tag{2.90}$$

We have seen here how a magnetic force appears as a result of an electrostatic force and the special theory of relativity. The considerations above have also demonstrated that a force which is identified as electrostatic in one frame of reference is observed as a magnetic force in another frame. In other words, the electric and the magnetic force are really the same. What an observer names it depends upon his state of motion.

Problems

2.1. *The twin paradox*
On New Years day 2004, an astronaut (A) leaves Earth on an interstellar journey. He is travelling in a spacecraft at the speed of $v = 4/5c$ heading towards Alpha Centauri. This star is at a distance of 4 ly(ly =light years) measured from the reference frame of the Earth. As A reaches the star, he immediately turns around and heads home. He reaches the Earth New Years day 2016 (in Earth's time frame).
The astronaut has a brother (B), who remains on Earth during the entire journey. The brothers have agreed to send each other a greeting every new years day with the aid of radio-telescope.

(a) Show that A only sends 6 greetings (including the last day of travel), while B sends 10.
(b) Draw a Minkowski diagram where A's journey is depicted with respect to the Earth's reference frame. Include all the greetings that B is sending. Show with

the aid of the diagram that while A is outbound, he only receives one greeting, while on his way home he receives nine.

(c) Draw a new diagram, still with respect to Earth's reference frame, where A's journey is depicted. Include the greetings that A is sending to B. Show that B is receiving one greeting every third year the first 9 years after A has left, while the last year before his return he receives three.

(d) Show how the results from (b) and (c) can be deduced from the Doppler-effect.

2.2. *Faster than the speed of light?*

The quasar 3C273 emits a jet of matter that moves with the speed v_0 towards Earth making an angle ϕ to the line of sight (see Fig. 2.18).

(a) Assume that two signals are sent towards the Earth simultaneously, one from A and one from B. How much earlier will the signal from B reach the Earth compared to that from A?

(b) Find an expression of the transverse distance that the emitted part has moved when it reaches B. How much time (relative to the Earth) has this part been travelling?

(c) Let v be the transverse velocity and relate it to v_0. The observed (the transverse) speed of the light source is $v = 10c$. Find v_0 when we assume that $\phi = 10°$. What is the largest possible ϕ?

2.3. *Two successive boosts in different directions*

Let us consider Lorentz transformations without rotation ("boosts"). A boost in the x-direction is given by

$$x = g(x' + bct'), \quad y = y', \quad z = z' \quad t = g(t' + bx'/c)$$
$$\gamma = \frac{1}{\sqrt{1-b^2}} \quad b = \frac{v}{c}\,. \tag{2.91}$$

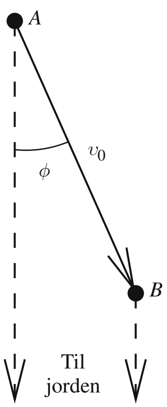

Fig. 2.18 A Quasar emitting a jet of matter

This can be written as

$$x^{\mu} = \Lambda^{\mu}{}_{\mu'} x^{\mu'} , \tag{2.92}$$

where $\Lambda^{\mu}{}_{\mu'}$ is the matrix

$$\Lambda^{\mu}{}_{\mu'} = \begin{bmatrix} g & gb & 0 & 0 \\ gb & g & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{2.93}$$

(a) Show that Eqs. (2.92) and (2.93) yield Eq. (2.91). Find the transformation matrix, $\bar{\Lambda}^{\mu}{}_{\mu'}$, for a boost in the negative y-direction.

(b) Two successive Lorentz transformations are given by the matrix product of each matrix. Find $\Lambda^{\mu}_{a}\Lambda^{a}{}_{\mu'}$ and $\Lambda^{\mu}_{a}\bar{\Lambda}^{a}{}_{\mu'}$. Are the product of two boosts a boost? The matrix for a general boost in arbitrary direction is given by

$$\begin{aligned} \Lambda^{0}{}_{0} &= g, \\ \Lambda^{0}{}_{m} &= \Lambda^{m}{}_{0} = gb_{m}, \\ \Lambda^{m}{}_{m'} &= \delta^{m}_{m'} + \frac{b_m b_{m'}}{b^2}(g-1), \\ g &= \frac{1}{\sqrt{1-b^2}}, \quad b^2 = b^m b_m, \quad m, m' = 1,2,3 . \end{aligned} \tag{2.94}$$

Does the set of all possible boosts form a group?

2.4. *Length contraction and time dilation*

(a) A rod with length ℓ is moving with constant velocity $\mathbf{v}$ with respect to the inertial frame Σ. The length of the rod is parallel to $\mathbf{v}$, which we will for the sake of simplicity assume is parallel to the x-axis. At time $t = 0$, the rear end of the rod is in the origin of Σ. What do we mean by the length of such a moving rod? Describe how an observer can find this length. Draw the rod in a Minkowski diagram and explain how the length of the rod can be read from the diagram. Using the Lorentz transformations, calculate the position of the end points of the rod as a function of time t. Show that the length of the rod, as measured in Σ, is shorter than its rest length ℓ.

(b) The rod has the same velocity as before, but now the rod makes an angle with $\mathbf{v}$. In an inertial frame which follows the movement of the rod (Σ'), this angle is $a' = 45°$ (with the x-axis in Σ). What is the angle between the velocity $\mathbf{v}$ and the rod when measured in Σ? What is the length of the rod as a function of a', as measured from Σ?

(c) We again assume that $a' = 0$. At the centre of the rod there is a flash that sends light signals with a time interval τ_0 between every flash. In the frame Σ', the light signals will reach the two ends simultaneously. Show that these two events are not simultaneous in Σ. Find the time difference between these two events. Show that the time-interval τ measured from Σ between each flash is larger than the interval τ_0 measured in Σ'.

An observer in Σ is located at the origin. He measures the time-interval Δt between every time he receives a light signal. Find Δt in terms of the speed v, and check whether Δt is greater or less than τ.

(d) The length of the rod is now considered to be $ell = 1\,\mathrm{m}$ and its speed, as measured in Σ, is $v = \frac{3}{5}c$. As before, we assume that the rod is moving parallel to the x-axis, but this time at a distance of $y = 10\,\mathrm{m}$ from the axis. A measuring ribbon is stretched out along the trajectory of the rod. This ribbon is at rest in Σ. An observer at the origin sees the rod move along the background ribbon. The ribbon has tick-marks along it which correspond to the x coordinates. The rod length can be measured by taking a photograph of the rod and the ribbon. Is the length that is directly measured from the photograph identical to the length of the rod in Σ?
In one of the photographs the rod is symmetrically centred with respect to $x = 0$. What is the length of the rod as measured using this photograph? Another photograph shows the rod with its trailing edge at $x = 10$ m. At what point will the leading edge of the rod be on this photograph? Compare with the length of the rod in the Σ frame.

(e) At one point along the trajectory the rod passes through a box which is open at both ends and stationary in Σ. This box is shorter than the rest length of the rod, but longer than the length of the rod as measured in Σ. At a certain time in Σ, the entire rod is therefore inside the box. At this time the box is closed at both ends, trapping the rod inside. The rod is also brought to rest. It is assumed that the box is strong enough to withstand the impact with the rod.
What happens to the rod? Describe what happens as observed from Σ and Σ'. Draw a Minkowski diagram. This is an example of why the theory of relativity has difficulty with the concept of absolute rigid bodies. What is the reason for this difficulty?

2.5. *Reflection angles off moving mirrors*

(a) The reflection angle of light equals the incidence angle of the light. Show that this is also the case for mirrors that are moving parallel to the reflection surface.

(b) A mirror is moving with a speed v in a direction orthogonal to the reflection surface. Light is sent towards the mirror with an angle ϕ. Find the angle of the reflected light as a function of v and ϕ. What is the frequency of the reflected light expressed in terms of its original frequency f?

2.6. *Minkowski diagram*

The reference frame Σ' is moving relative to the frame Σ at a speed of $v = 0.6c$. The movement is parallel to the x-axes of the two frames.
Draw the x' and the ct'-axis in the Minkowski diagram of Σ. Points separated by 1 m are marked along both axes. Draw these points in the Minkowski diagram as for both frames.
Show where the lines of simultaneity for Σ' are in the diagram. Also show where the $x' =$ constant line is.

Assume that the frames are equipped with measuring rods and clocks that are at rest in their respective frames. How can we use the Minkowski diagram to measure the length contraction of the rod that is in rest at Σ'? Similarly, how can we measure the length contraction of the rod in Σ when measured from Σ? Show how the time dilation of the clocks can be measured from the diagram.

2.7. *Robb's Lorentz-invariant spacetime interval formula* (A.A. Robb, 1936)
Show that the spacetime interval between the origin event and the reflection event in Fig. 2.2 is $s = c\sqrt{t_A t_B}$.

2.8. *The Doppler effect*
A radar antenna emits radio pulses with a wavelength of $\lambda = 1.0$ cm, at a time-interval $\tau = 1.0$ s. An approaching spacecraft is being registered by the radar.
Draw a Minkowski diagram for the reference frame Σ. The antenna is at rest in this frame. In this diagram, indicate the position of

1. the antenna,
2. the spacecraft and
3. the outgoing and reflected radar pulses.

Calculate the time difference Δt_1 between two subsequent pulses as measured in the spacecraft. What is the wavelength of these signals?
Calculate the time difference Δt_2 between two reflected signals, as it is measured from the antenna's receiver? At what wavelength will these signals be?

2.9. *Aberration and Doppler effect*
We shall describe light emitted from a spherical surface that expands with ultra-relativistic velocity. Consider a surface element dA with velocity $v = \beta c$ in the laboratory frame F (i.e. the rest frame of the observer), as shown in Fig. 2.19 .

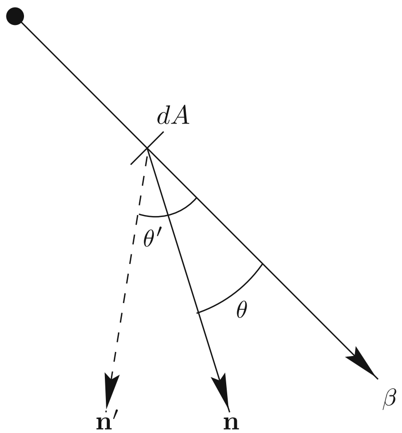

Fig. 2.19 Light is emitted in the direction $\mathbf{n}'$ as measured in the rest frame F' of the emitting surface element. The light is measured to propagate in the $\mathbf{n}$-direction in the rest frame F of the observer

(a) Show by means of the relativistic formula for velocity addition that the relationship between the directions of propagation measured in F and F' is

$$\cos\theta = \frac{\cos\theta' + b}{1 + b\cos\theta'} \ . \tag{2.95}$$

This is the aberration formula.

(b) Show that an observer far away from the surface will only observe light from a spherical cap with opening angle (see Fig. 2.20)

$$\theta_0 = \arccos\beta = \arcsin\frac{1}{\gamma} \approx \frac{1}{\gamma} \text{ for } \gamma \gg 1 \ . \tag{2.96}$$

(c) Assume that the expanding shell emits mononromatic light with frequency ν' in F'. Show that the observer in F will measure an angle-dependent frequency

$$\nu = \frac{\nu'}{\gamma(1 - b\cos\theta)} = \gamma(1 + b\cos\theta')\nu' \ . \tag{2.97}$$

(d) Let the measured frequency of light from M and Q be ν_M and ν_Q, respectively. This is the maximal and minimal frequency. Show that the expansion velocity can be found from these measurements, as

$$v = \frac{\nu_M - \nu_Q}{\nu_Q} c \ . \tag{2.98}$$

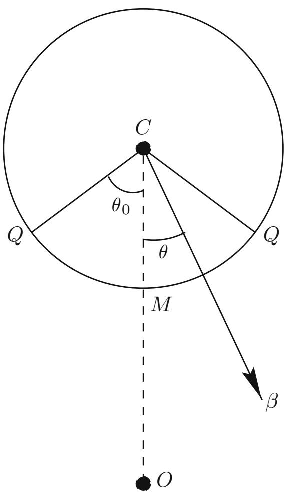

Fig. 2.20 The faraway observer, O, can only see light from the spherical cap with opening angle θ_0

2.10. *A traffic problem*
A driver is in court for driving through a red light. In his defence, the driver claims that the traffic signal appeared green as he was approaching the junction. The judge says that this does not strengthen his case stronger as he would have been travelling at the speed of ...
At what speed would the driver have to travel for the red traffic signal ($\lambda = 6000$ Å) to Doppler shift to a green signal ($\lambda = 5000$ Å)?

2.11. *Work and rotation*
A circular ring is initially at rest. It has radius r, rest mass m and a constant of elasticity k. Find the work that has to be done to give the ring an angular velocity ω. We assume that the ring is accelerated in such a way that its radius is constant. Compare with the non-relativistic case. How can we understand that in the relativistic case we also have to do elastic work?

2.12. *Muon experiment*
How many of the 10 million muons created 10 km above sea level will reach the Earth? If there are initially n_0 muons, $n = n_0 2^{-t/T}$ will survive for a time t (T is the half-lifetime).

(a) Compute the non-relativistic result.
(b) What is the result of a relativistic calculation by an Earth observer?
(c) Make a corresponding calculation from the point of view of an observer co-moving with the muon. The muon has a rest half-lifetime $T = 1.56 \cdot 10^{-6} s$ and moves with a velocity $v = 0.98c$.

2.13. *Cerenkov radiation*
When a particle moves through a medium with a velocity greater than the velocity of light in the medium, it emits a cone of radiation with a half-angle θ given by $\cos\theta = c/nv$ (see Fig. 2.21).

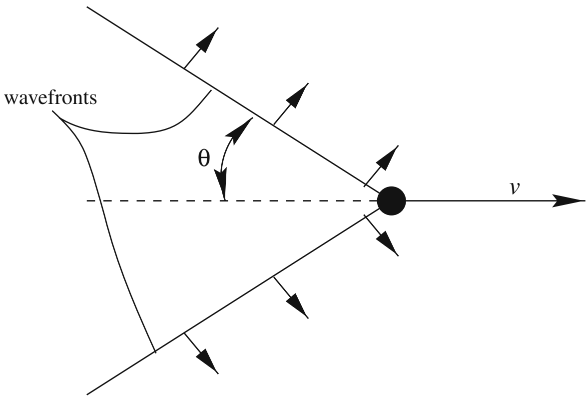

Fig. 2.21 Cerenkov radiation from a particle

(a) What is the threshold kinetic energy (in MeV) of an electron moving through water in order that it shall emit Cerenkov radiation? The index of refraction of water is $n = 1.3$. The rest energy of an electron is $m_e = 0.511$ MeV.
(b) What is the limiting half-angle of the cone for high-speed particles moving through water?

Reference

1. Frisch, D. H. and Smith, J. H. 1963. Measurement of relativistic time-dilation using mu-mesons, *Am. J. Phys.* **31**, 342. 31

Chapter 3
Vectors, Tensors and Forms

3.1 Vectors

An expression of the form $a^\mu \vec{e}_\mu$, where a^μ, $\mu = 1,2,\ldots,n$ are real numbers, is known as a **linear combination** of the vectors $\vec{e}_\mu$.

The vectors $\vec{e}_1,\ldots,\vec{e}_n$ are said to be linearly independent if there does **not** exist real numbers $a^\mu \neq 0$ such that $a^\mu \vec{e}_\mu = 0$ (Fig. 3.1).

Geometrical interpretation: A set of vectors are **linearly independent** if it is **not** possible to construct a closed polygon of the vectors (even by adjusting their lengths).

A set of vectors $\vec{e}_1,\ldots,\vec{e}_n$ are said to be **maximally linearly independent** if $\vec{e}_1,\ldots,\vec{e}_n,\vec{v}$ are linearly dependent for all vectors $\vec{v} \neq \vec{e}_\mu$. We define the **dimension**

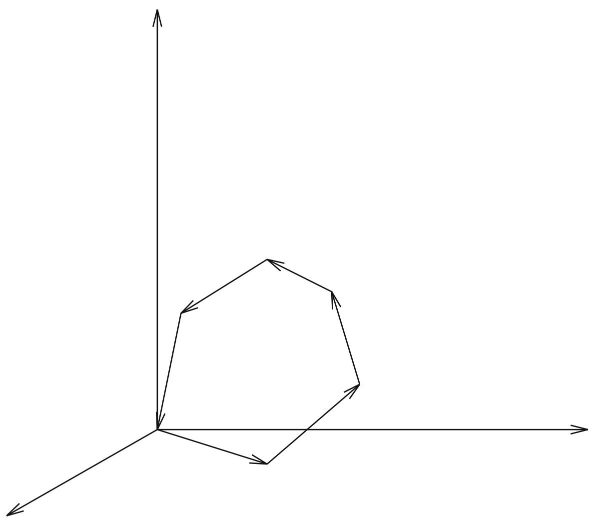

Fig. 3.1 Closed polygon (linearly dependent)

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of a vector-space as the number of vectors in a maximally linearly independent set of vectors of the space. The vectors $\vec{e}_\mu$ in such a set are known as the **basis vectors** of the space:

$$\begin{aligned} \vec{v} + a^\mu \vec{e}_\mu &= 0 \\ &\Downarrow \\ \vec{v} &= -a^\mu \vec{e}_\mu \ . \end{aligned} \tag{3.1}$$

The components of $\vec{v}$ are the numbers v^μ defined by $v^\mu = -a^\mu \Rightarrow \vec{v} = v^\mu \vec{e}_\mu$.

3.1.1 4-Vectors

4-Vectors are vectors which exist in (4-dimensional) spacetime. A 4-vector equation represents four independent component equations.

Example 3.1.1 (Photon clock) Carriage at rest:

$$\Delta t_0 = \frac{2L}{c} \ .$$

Carriage with velocity $\vec{v}$:

$$\begin{aligned} \Delta t &= \frac{2\sqrt{\left(v\frac{\Delta t}{2}\right)^2 + L^2}}{c} \\ &\Downarrow \\ c^2 \Delta t^2 &= v^2 \Delta t^2 + 4L^2 \\ &\Downarrow \\ \Delta t &= \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L/c}{\sqrt{1 - v^2/c^2}} = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \ . \end{aligned} \tag{3.2}$$

The proper time-interval is denoted by $d\tau$ (above it was denoted Δt_0). The proper time-interval for a particle is measured with a standard clock which follows the particle (Fig. 3.2).

Definition 3.1.1 (4-Velocity)

$$\vec{U} = c\frac{dt}{d\tau}\vec{e}_t + \frac{dx}{d\tau}\vec{e}_x + \frac{dy}{d\tau}\vec{e}_y + \frac{dz}{d\tau}\vec{e}_z \ , \tag{3.3}$$

where t is the coordinate time, measured with clocks at rest in the reference frame.

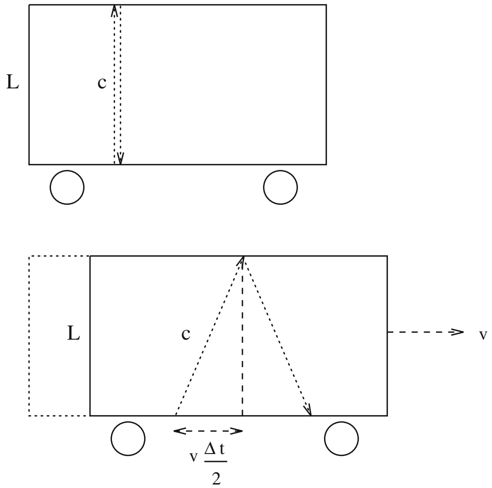

Fig. 3.2 Carriage at rest (*top*) and with velocity $\vec{v}$ (*bottom*)

$$\vec{U} = U^{\mu}\vec{e}_{\mu} = \frac{dx^{\mu}}{d\tau}\vec{e}_{\mu}, \quad x^{\mu} = (ct, x, y, z), \quad x^0 \equiv ct,$$
$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \gamma\,. \tag{3.4}$$

$\vec{U} = \gamma(c, \vec{v})$, where $\vec{v}$ is the common 3-velocity of the particle.

Definition 3.1.2 (4-Momentum)

$$\vec{P} = m_0\vec{U}\ , \tag{3.5}$$

where m_0 is the rest mass of the particle.

$\vec{P} = (\frac{E}{c}, \vec{p})$, where $\vec{p} = \gamma m_0 \vec{v} = m\vec{v}$ and E is the relativistic energy.

The 4-force or Minkowski force $\vec{F} \equiv \frac{d\vec{P}}{d\tau}$ and the "common force" $\vec{f} = \frac{d\vec{p}}{dt}$. Then

$$\vec{F} = \gamma\left(\frac{1}{c}\vec{f}\cdot\vec{v}, \vec{f}\right)\ . \tag{3.6}$$

Definition 3.1.3 (4-Acceleration)

$$\vec{A} = \frac{d\vec{U}}{d\tau}\ . \tag{3.7}$$

The 4-velocity has the scalar value c so that

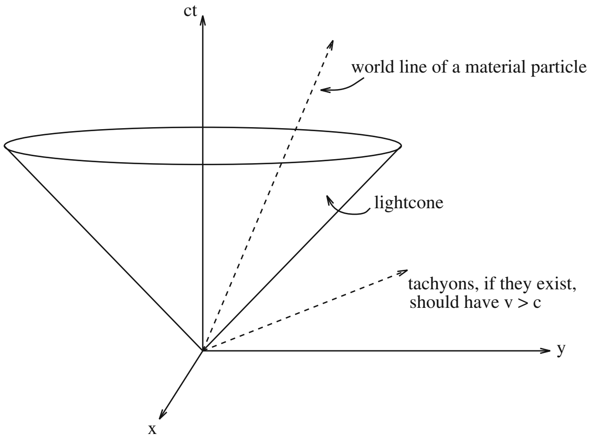

Fig. 3.3 World lines in a Minkowski diagram

$$\vec{U} \cdot \vec{U} = -c^2. \tag{3.8}$$

The *4-velocity identity* Eq. (3.8) gives $\vec{U} \cdot \vec{A} = 0$, in other words $\vec{A} \perp \vec{U}$ and $\vec{A}$ is space-like.

The line element for Minkowski spacetime (flat spacetime) with Cartesian coordinates is (Fig. 3.3)

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \,. \tag{3.9}$$

In general relativity theory, gravitation is not considered a force. Gravitation is instead described as motion in a curved spacetime.

A particle in free fall is in Newtonian gravitational theory, said to be only influenced by the gravitational force. According to general relativity theory the particle is not influenced by any force.

Such a particle has no 4-acceleration. $\vec{A} \neq 0$ implies that the particle is not in free fall. It is then influenced by non-gravitational forces.

One has to distinguish between observed acceleration, i.e. common 3-acceleration, and the absolute 4-acceleration.

3.1.2 Tangent Vector Fields and Coordinate Vectors

In a curved space position vectors with finite length do not exist. (See Fig. 3.4.)

Different points in a curved space have different tangent planes. Finite vectors do only exist in these tangent planes (see Fig. 3.5). However, infinitesimal position vectors $d\vec{r}$ do exist.

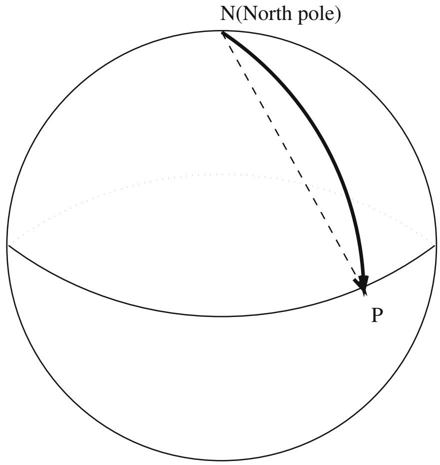

Fig. 3.4 In curved space, vectors can only exist in tangent planes. The vectors in the tangent plane of N, do not contain the vector $\overrightarrow{NP}$ (*dashed line*)

Definition 3.1.4 (Reference frame) A **reference frame** is defined as a continuum of non-intersecting time-like world lines in spacetime.

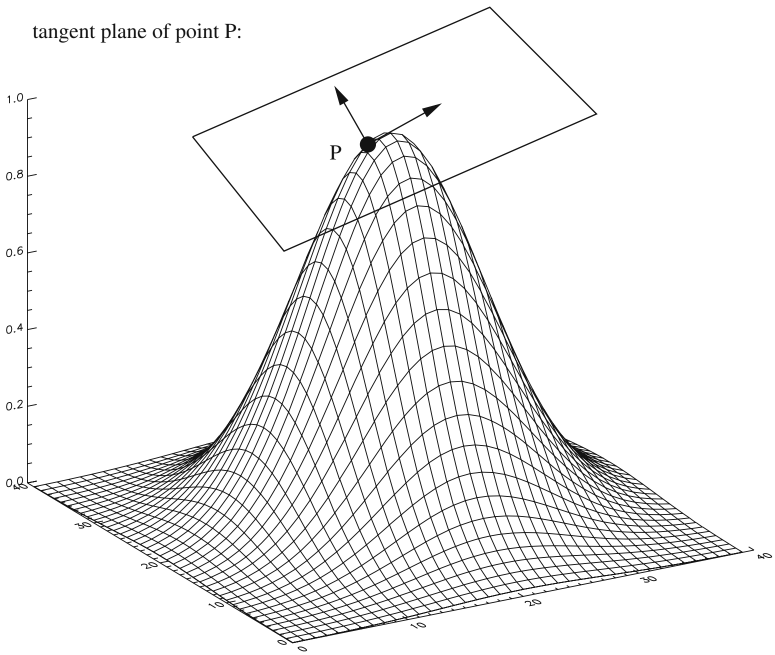

Fig. 3.5 In curved space, vectors can only exist in tangent planes

We can view a reference frame as a set of reference particles with a specified motion. An *inertial reference frame* is a non-rotating set of free particles.

Definition 3.1.5 (Coordinate system) A **coordinate system** is a continuum of 4-tuples giving a unique set of coordinates for events in spacetime .

Definition 3.1.6 (Comoving coordinate system) A **comoving coordinate system** in a frame is a coordinate system where the particles in the reference frame have constant spatial coordinates .

Definition 3.1.7 (Orthonormal basis) An **orthonormal basis** $\{\vec{e}_{\hat{\mu}}\}$ in spacetime is defined by

$$\begin{aligned} \vec{e}_{\hat{t}} \cdot \vec{e}_{\hat{t}} &= -1 (c=1), \\ \vec{e}_{\hat{i}} \cdot \vec{e}_{\hat{j}} &= \delta_{\hat{i}\hat{j}}, \end{aligned} \tag{3.10}$$

where $\hat{i}$ and $\hat{j}$ are space indices .

Definition 3.1.8 (Coordinate basis vectors) Temporary definition of coordinate basis vector:

Assume any coordinate system $\{x^\mu\}$:

$$\vec{e}_\mu \equiv \frac{\partial \vec{r}}{\partial x^\mu}. \tag{3.11}$$

A *vector field* is a continuum of vectors in a space, where the components are continuous and differentiable functions of the coordinates. Let $\vec{v}$ be a tangent vector to the curve $\vec{r}(\lambda)$:

$$\vec{v} = \frac{d\vec{r}}{d\lambda}, \qquad \text{where} \quad \vec{r} = \vec{r}[x^\mu(\lambda)]. \tag{3.12}$$

The chain rule for differentiation yields

$$\vec{v} = \frac{d\vec{r}}{d\lambda} = \frac{\partial \vec{r}}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\lambda}\vec{e}_\mu = v^\mu \vec{e}_\mu. \tag{3.13}$$

Thus, the components of the tangent vector field along a curve, parameterized by λ, is given by

$$v^\mu = \frac{dx^\mu}{d\lambda}. \tag{3.14}$$

In the theory of relativity, the invariant parameter is often chosen to be the proper time. Tangent vector to the world line of a material particle:

$$u^\mu = \frac{dx^\mu}{d\tau}. \tag{3.15}$$

These are the components of the 4-velocity of the particle!

Digression 3.1.1 (Proper time of the photon) Minkowski space:

$$\begin{aligned} ds^2 &= -c^2dt^2 + dx^2 \\ &= -c^2dt^2\left(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right) \\ &= -\left(1 - \frac{v^2}{c^2}\right)c^2dt^2. \end{aligned} \tag{3.16}$$

For a photon, $v = c$ so

$$\lim_{v\to c} ds^2 = 0. \tag{3.17}$$

Thus, the spacetime interval between two points on the world line of a photon is zero! This also means that the proper time for the photon is zero!! (see *Example 3.1.2*) .

Digression 3.1.2 (Relationships between spacetime intervals, time and proper time)
Physical interpretation of the spacetime interval for a time-like interval is

$$ds^2 = -c^2d\tau^2, \tag{3.18}$$

where $d\tau$ is the proper time-interval between two events measured on a clock moving in a way such that it is present at both events (Fig. 3.6):

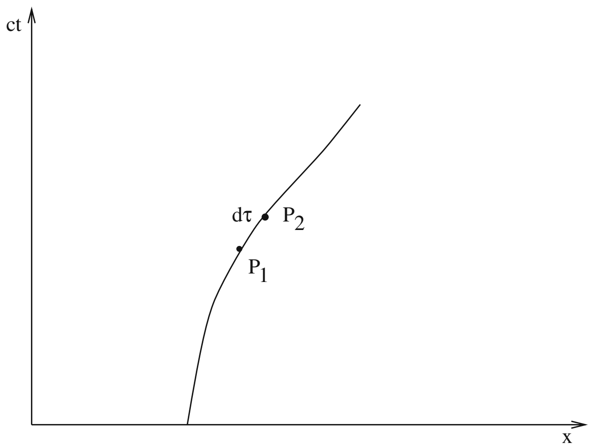

Fig. 3.6 P_1 and P_2 are two events in spacetime, separated by a proper time-interval $d\tau$

$$-c^2 d\tau^2 = -c^2\left(1-\frac{v^2}{c^2}\right)dt^2$$
$$\Rightarrow d\tau = \sqrt{1-\frac{v^2}{c^2}}dt\,. \tag{3.19}$$

The time-interval between two events in the laboratory is smaller measured on a moving clock than measured on a stationary one, because the moving clock is ticking slower!

3.1.3 Coordinate Transformations

Given two coordinate systems $\{x^\mu\}$ and $\{x^{\mu'}\}$:

$$\vec{e}_{\mu'} = \frac{\partial \vec{r}}{\partial x^{\mu'}}. \tag{3.20}$$

Suppose there exists a coordinate transformation, such that the primed coordinates are functions of the unprimed, and vice versa. Then we can apply the chain rule

$$\vec{e}_{\mu'} = \frac{\partial \vec{r}}{\partial x^{\mu'}} = \frac{\partial \vec{r}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial x^{\mu'}} = \vec{e}_{\mu}\frac{\partial x^{\mu}}{\partial x^{\mu'}}. \tag{3.21}$$

This is the transformation equation for the basis vectors. $\frac{\partial x^\mu}{\partial x^{\mu'}}$ are elements of the transformation matrix. Indices that are *not* sum-indices are called "free indices".

Rule: In *all terms* on each side in an equation, the free indices should behave identically (high or low), **and** there should be exactly the *same* indices in all terms!

Applying this rule, we can now find the inverse transformation:

$$\vec{e}_{\mu} = \vec{e}_{\mu'}\frac{\partial x^{\mu'}}{\partial x^{\mu}},$$
$$\vec{v} = v^{\mu'}\vec{e}_{\mu'} = v^{\mu}\vec{e}_{\mu} = v^{\mu'}\vec{e}_{\mu}\frac{\partial x^{\mu}}{\partial x^{\mu'}}.$$

So, the transformation rules for the *components* of a vector becomes

$$\boxed{v^{\mu} = v^{\mu'}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\,; \qquad v^{\mu'} = v^{\mu}\frac{\partial x^{\mu'}}{\partial x^{\mu}}}\,. \tag{3.22}$$

The directional derivative along a curve, parameterized by λ, is

$$\frac{d}{d\lambda} = \frac{\partial}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = v^\mu \frac{\partial}{\partial x^\mu}, \tag{3.23}$$

where $v^\mu = \frac{dx^\mu}{d\lambda}$ are the components of the tangent vector of the curve. Directional derivative along a coordinate curve:

$$\lambda = x^\nu, \qquad \frac{\partial}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\nu}\,. \tag{3.24}$$

In the primed system,

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu}. \tag{3.25}$$

Definition 3.1.9 (Coordinate basis vectors) We define the *coordinate basis vectors* as

$$\boxed{\vec{e}_\mu = \frac{\partial}{\partial x^\mu}}\,. \tag{3.26}$$

This definition is not based upon the existence of finite position vectors. It applies in curved spaces as well as in flat spaces .

Example 3.1.2 (Coordinate transformation) From Fig. 3.7 we see that

$$\boxed{x = r\cos\theta,\ y = r\sin\theta}\,. \tag{3.27}$$

Coordinate basis vectors were defined by

$$\vec{e_\mu} \equiv \frac{\partial}{\partial x^\mu}\,. \tag{3.28}$$

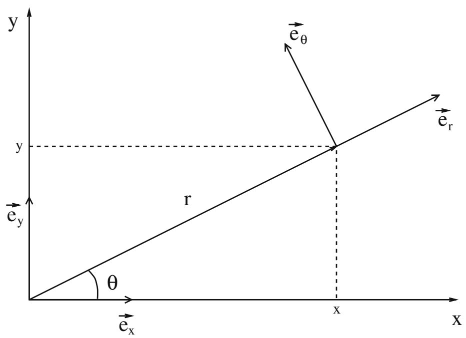

Fig. 3.7 Coordinate transformation, flat space

This means that we have

$$\begin{aligned} \vec{e}_x &= \frac{\partial}{\partial x}\,, \quad \vec{e}_y = \frac{\partial}{\partial y}\,, \quad \vec{e}_r = \frac{\partial}{\partial r}\,, \quad \vec{e}_\theta = \frac{\partial}{\partial \theta}, \\ \vec{e}_r &= \frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}. \end{aligned} \tag{3.29}$$

Using the chain rule and Eqs. (3.27) and (3.29) we get

$$\begin{aligned} \vec{e}_r &= \cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y, \\ \vec{e}_\theta &= \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} \\ &= -r\sin\theta\,\vec{e}_x + r\cos\theta\,\vec{e}_y. \end{aligned} \tag{3.30}$$

But are the vectors in Eq. (3.30) also unit vectors?

$$\vec{e}_r \cdot \vec{e}_r = \cos^2\theta + \sin^2\theta = 1\ . \tag{3.31}$$

So $\vec{e}_r$ is a unit vector, $|\vec{e}_r| = 1$:

$$\vec{e}_\theta \cdot \vec{e}_\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2 \tag{3.32}$$

and we see that $\vec{e}_\theta$ is **not** a unit vector, $|\vec{e}_\theta| = r$. But we have that $\vec{e}_r \cdot \vec{e}_\theta = 0 \Rightarrow \vec{e}_r \perp \vec{e}_\theta$. **Coordinate basis vectors are not generally unit vectors.**

Definition 3.1.10 (Orthonormal basis) An orthonormal basis is a vector basis consisting of unit vectors that are normal to each other. To show that we are using an orthonormal basis we will use "hats" over the indices, $\{\vec{e}_{\hat{\mu}}\}$.

Orthonormal basis associated with planar polar coordinates:

$$\vec{e}_{\hat{r}} = \vec{e}_r\,, \quad \vec{e}_{\hat{\theta}} = \frac{1}{r}\vec{e}_\theta. \tag{3.33}$$

Example 3.1.3 (Relativistic Doppler effect) The Lorentz transformation is known from special relativity and relates the reference frames of two systems where one is moving with a constant velocity v with regard to the other:

$$\begin{aligned} x' &= \gamma(x - vt), \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right). \end{aligned}$$

According to the vector component transformation (3.22), the 4-momentum for a particle moving in the x-direction, $P^\mu = (\frac{E}{c}, p, 0, 0)$, transforms as

$$\begin{aligned} P^{\mu'} &= \frac{\partial x^{\mu'}}{\partial x^\mu}P^\mu, \\ E' &= \gamma(E - vp). \end{aligned}$$

Using the fact that a photon has energy $E = h\nu$ and momentum $p = \frac{h\nu}{c}$, where h is Planck's constant and ν is the photon's frequency, we get the equation for the frequency shift known as the relativistic Doppler effect:

$$\nu' = \gamma(\nu - \frac{v}{c}\nu) = \frac{\left(1-\frac{v}{c}\right)\nu}{\sqrt{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}},$$

$$\boxed{\frac{\nu'}{\nu} = \sqrt{\frac{c-v}{c+v}}}\,. \tag{3.34}$$

3.1.4 Structure Coefficients

Definition 3.1.11 (Commutators between vectors) The commutator between two vectors, $\vec{u}$ and $\vec{v}$, is defined as

$$[\vec{u}\,,\,\vec{v}] \equiv \vec{u}\vec{v} - \vec{v}\vec{u}\,, \tag{3.35}$$

where $\vec{u}\vec{v}$ is defined as

$$\vec{u}\vec{v} \equiv u^{\mu}\vec{e}_{\mu}(v^{\nu}\vec{e}_{\nu}) = u^{\mu}\frac{\partial}{\partial x^{\mu}}\left(v^{\nu}\frac{\partial}{\partial x^{\nu}}\right)\,. \tag{3.36}$$

We can think of a vector as a linear combination of partial derivatives. We get

$$\begin{aligned}\vec{u}\vec{v} &= u^{\mu}\frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}} + u^{\mu}v^{\nu}\frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}\\ &= u^{\mu}\frac{\partial v^{\nu}}{\partial x^{\mu}}\vec{e}_{\nu} + u^{\mu}v^{\nu}\frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}\,.\end{aligned} \tag{3.37}$$

Due to the last term, $\vec{u}\vec{v}$ is **not** a vector:

$$\begin{aligned}\vec{v}\vec{u} &= v^{\nu}\frac{\partial}{\partial x^{\nu}}\left(u^{\mu}\frac{\partial}{\partial x^{\mu}}\right)\\ &= v^{\nu}\frac{\partial u^{\mu}}{\partial x^{\nu}}\vec{e}_{\mu} + v^{\nu}u^{\mu}\frac{\partial^2}{\partial x^{\nu}\partial x^{\mu}},\\[2ex] \vec{u}\vec{v} - \vec{v}\vec{u} &= u^{\mu}\frac{\partial v^{\nu}}{\partial x^{\mu}}\vec{e}_{\nu} - \underbrace{v^{\nu}\frac{\partial u^{\mu}}{\partial x^{\nu}}\vec{e}_{\mu}}_{v^{\mu}\frac{\partial u^{\nu}}{\partial x^{\mu}}\vec{e}_{\nu}}\\ &= \left(u^{\mu}\frac{\partial v^{\nu}}{\partial x^{\mu}} - v^{\mu}\frac{\partial u^{\nu}}{\partial x^{\mu}}\right)\vec{e}_{\nu}\,.\end{aligned} \tag{3.38}$$

Here we have used that

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\mu} \,. \tag{3.39}$$

The Einstein comma notation $\Rightarrow$

$$\vec{u}\vec{v} - \vec{v}\vec{u} = (u^\mu v^\nu{}_{,\mu} - v^\mu u^\nu{}_{,\mu})\vec{e_\nu} \,. \tag{3.40}$$

As we can see, the commutator between two vectors is itself a vector.

Definition 3.1.12 (Structure coefficients $c^\rho_{\ \mu\nu}$) The structure coefficients $c^\rho_{\ \mu\nu}$ in an arbitrary basis $\{\vec{e_\mu}\}$ are defined by

$$[\vec{e_\mu}\,,\,\vec{e_\nu}] \equiv c^\rho_{\ \mu\nu}\vec{e_\rho} \,. \tag{3.41}$$

Structure coefficients in a coordinate basis:

$$\begin{aligned}[\vec{e_\mu}\,,\,\vec{e_\nu}] &= \left[\frac{\partial}{\partial x^\mu}\,,\,\frac{\partial}{\partial x^\nu}\right] \\ &= \frac{\partial}{\partial x^\mu}\left(\frac{\partial}{\partial x^\nu}\right) - \frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x^\mu}\right) \\ &= \frac{\partial^2}{\partial x^\mu \partial x^\nu} - \frac{\partial^2}{\partial x^\nu \partial x^\mu} = 0 \,.\end{aligned} \tag{3.42}$$

The commutator between two coordinate basis vectors is zero, so the structure coefficients are zero in coordinate basis.

Example 3.1.4 (Structure coefficients in planar polar coordinates) We will find the structure coefficients of an orthonormal basis in planar polar coordinates. In Eq. (3.33) we found that

$$\vec{e}_{\hat{r}} = \vec{e}_r \,, \quad \vec{e}_{\hat{\theta}} = \frac{1}{r}\vec{e}_\theta. \tag{3.43}$$

We will now use this to find the structure coefficients:

$$\begin{aligned}[\vec{e}_{\hat{r}}\,,\,\vec{e}_{\hat{\theta}}] &= \left[\frac{\partial}{\partial r}\,,\,\frac{1}{r}\frac{\partial}{\partial \theta}\right] \\ &= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial r}\right) \\ &= -\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial \theta \partial r} \\ &= -\frac{1}{r^2}\vec{e}_\theta = -\frac{1}{r}\vec{e}_{\hat{\theta}}.\end{aligned} \tag{3.44}$$

To find the structure coefficients in an orthonormal basis we must use $[\vec{e}_{\hat{r}}\,,\,\vec{e}_{\hat{\theta}}] = -\frac{1}{r}\vec{e}_{\hat{\theta}}$:

$$[\vec{e}_{\hat{\mu}}\,,\,\vec{e}_{\hat{\nu}}] = c^{\hat{\rho}}_{\ \hat{\mu}\hat{\nu}}\vec{e}_{\hat{\rho}}. \tag{3.45}$$

Using Eqs. (3.44) and (3.45) we get

$$c^{\hat{\theta}}_{\ \hat{r}\hat{\theta}} = -\frac{1}{r}. \tag{3.46}$$

From the definition of $c^{\rho}_{\ \mu\nu}$ ($[\vec{u}\,,\,\vec{v}] = -[\vec{v}\,,\,\vec{u}]$) we see that the structure coefficients are antisymmetric in their lower indices:

$$\boxed{c^{\rho}_{\ \mu\nu} = -c^{\rho}_{\ \nu\mu}}, \tag{3.47}$$

$$c^{\hat{\theta}}_{\ \hat{\theta}\hat{r}} = \frac{1}{r} = -c^{\hat{\theta}}_{\ \hat{r}\hat{\theta}}. \tag{3.48}$$

3.2 Tensors

A 1-form basis $\underline{\omega}^1,\ldots,\underline{\omega}^n$ is defined by

$$\underline{\omega}^{\mu}(\vec{e}_{\nu}) = \delta^{\mu}_{\ \nu}\,. \tag{3.49}$$

An **arbitrary** 1-form can be expressed, in terms of its components, as a linear combination of the basis forms:

$$\underline{\alpha} = \alpha_{\mu}\underline{\omega}^{\mu}\,, \tag{3.50}$$

where α_{μ} are the components of $\underline{\alpha}$ in the given basis.
Using Eqs. (3.49) and (3.50), we find

$$\begin{aligned}\underline{\alpha}(\vec{e}_{\nu}) &= \alpha_{\mu}\underline{\omega}^{\mu}(\vec{e}_{\nu}) = \alpha_{\mu}\delta^{\mu}_{\nu} = \alpha_{\nu},\\ \underline{\alpha}(\vec{v}) &= \underline{\alpha}(v^{\mu}\vec{e}_{\mu}) = v^{\mu}\underline{\alpha}(\vec{e}_{\mu}) = v^{\mu}\alpha_{\mu} = v^1\alpha_1 + v^2\alpha_2 + \ldots\,.\end{aligned} \tag{3.51}$$

We will now look at functions of multiple variables.

Definition 3.2.1 (Multilinear function, tensors) A multilinear function is a function that is linear in all its arguments and maps 1-forms and vectors into real numbers. A tensor is a multilinear function that maps 1-forms and vectors into real numbers .

- A **covariant tensor** only maps vectors.
- A **contravariant tensor** only maps forms.
- A **mixed tensor** maps both vectors and forms into R.

A tensor of **rank** $\binom{N}{N'}$ maps N 1-forms and N' vectors into R. It is usual to say that a tensor is of rank $(N+N')$. A 1-form, for example, is a covariant tensor of rank 1:

$$\underline{\alpha}(\vec{v}) = v^{\mu}\alpha_{\mu} \ . \tag{3.52}$$

Definition 3.2.2 (Tensor product) The basis of a tensor R of rank q contains a **tensor product**, $\otimes$. If T and S are two tensors of rank m and n, the tensor product is defined by

$$T \otimes S(\vec{u_1},\ldots,\vec{u_m},\vec{v_1},\ldots,\vec{v_n}) \equiv T(\vec{u_1},\ldots,\vec{u_m})S(\vec{v_1},\ldots,\vec{v_n}) \ , \tag{3.53}$$

where T and S are tensors of rank m and n, respectively. $T \otimes S$ is a tensor of rank $(m+n)$.

Let $R = T \otimes S$. We then have

$$R = R_{\mu_1,\ldots,\mu_q}\underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} \otimes \cdots \otimes \underline{\omega}^{\mu_q} \ . \tag{3.54}$$

Notice that $S \otimes T \neq T \otimes S$. We get the components of a tensor (R) by using the tensor on the basis vectors:

$$R_{\mu_1,\ldots,\mu_q} = R(\vec{e_{\mu_1}},\ldots,\vec{e_{\mu_q}}) \ . \tag{3.55}$$

The indices of the components of a contravariant tensor are written as upper indices, and the indices of a covariant tensor as lower indices.

Example 3.2.1 (Example of a tensor) Let $\vec{u}$ and $\vec{v}$ be two vectors and $\underline{\alpha}$ and $\underline{\beta}$ two 1-forms:

$$\vec{u} = u^{\mu}\vec{e}_{\mu}; \quad \vec{v} = v^{\mu}\vec{e}_{\mu}; \quad \underline{\alpha} = \alpha_{\mu}\underline{\omega}^{\mu}; \quad \underline{\beta} = \beta_{\mu}\underline{\omega}^{\mu} \ . \tag{3.56}$$

From these we can construct tensors of rank 2 through the relation $R = \vec{u} \otimes \vec{v}$ as follows: The components of R are

$$\begin{aligned}
R^{\mu_1\mu_2} &= R(\underline{\omega}^{\mu_1},\underline{\omega}^{\mu_2}) \\
&= \vec{u} \otimes \vec{v}(\underline{\omega}^{\mu_1},\underline{\omega}^{\mu_2}) \\
&= \vec{u}(\underline{\omega}^{\mu_1})\vec{v}(\underline{\omega}^{\mu_2}) \\
&= u^{\mu}\vec{e}_{\mu}(\underline{\omega}^{\mu_1})v^{\nu}\vec{e}_{\nu}(\underline{\omega}^{\mu_2}) \\
&= u^{\mu}\delta^{\mu_1}_{\mu}v^{\nu}\delta^{\mu_2}_{\nu} \\
&= u^{\mu_1}v^{\mu_2}.
\end{aligned} \tag{3.57}$$

3.2.1 Transformation of Tensor Components

We shall not limit our discussion to coordinate transformations. Instead, we will consider arbitrary transformations between bases, $\{\vec{e}_\mu\} \longrightarrow \{\vec{e}_{\mu'}\}$. The elements of transformation matrices are denoted by $M^\mu_{\ \mu'}$ such that

$$\vec{e}_{\mu'} = \vec{e}_\mu M^\mu_{\ \mu'} \quad \text{and} \quad \vec{e}_\mu = \vec{e}_{\mu'} M^{\mu'}_{\ \mu} , \tag{3.58}$$

where $M^{\mu'}_{\ \mu}$ are elements of the inverse transformation matrix. Thus, it follows that

$$M^\mu_{\ \mu'} M^{\mu'}_{\ \nu} = \delta^\mu_\nu . \tag{3.59}$$

If the transformation is a coordinate transformation, the elements of the matrix become

$$\boxed{M^{\mu'}_{\ \mu} = \frac{\partial x^{\mu'}}{\partial x^\mu}} . \tag{3.60}$$

3.2.2 Transformation of Basis 1-Forms

$$\begin{aligned} \underline{\omega}^{\mu'} &= M^{\mu'}_{\ \mu} \underline{\omega}^\mu , \\ \underline{\omega}^\mu &= M^\mu_{\ \mu'} \underline{\omega}^{\mu'} . \end{aligned} \tag{3.61}$$

The components of a tensor of higher rank transform such that every contravariant index (upper) transforms as a basis 1-form and every covariant index (lower) as a basis vector. Also, all elements of the transformation matrix are multiplied with one another.

Example 3.2.2 (A mixed tensor of rank 3)

$$T^{\alpha'}_{\ \mu'\nu'} = M^{\alpha'}_{\ \alpha} M^\mu_{\ \mu'} M^\nu_{\ \nu'} T^\alpha_{\ \mu\nu} . \tag{3.62}$$

The components in the **primed basis** are linear combinations of the components in the **unprimed basis** .

Tensor transformation of components means that tensors have a basis-**independent** existence. That is, if a tensor has non-vanishing components in a **given basis** then it has non-vanishing components in **all bases**. This means that tensor equations have a basis-independent form. **Tensor equations are invariant**. A basis transformation might result in the vanishing of one or more tensor components. Equations in **component** form may differ from one basis to another. But an equation expressed in tensor components can be transformed from one basis to another using the tensor component transformation rules. An equation that is expressed only in terms of tensor components is said to be **covariant**.

3.2.3 The Metric Tensor

Definition 3.2.3 (The metric tensor) The scalar product of two vectors $\vec{u}$ and $\vec{v}$ is denoted by g($\vec{u}$,$\vec{v}$) and is defined as a symmetric linear mapping which for each pair of vectors gives a scalar g($\vec{v}$,$\vec{u}$) = g($\vec{u}$,$\vec{v}$) .

The value of the scalar product g($\vec{u}$,$\vec{v}$) is given by specifying the scalar products of each pair of basis vectors in a basis.

g is a symmetric **covariant** tensor of rank 2. This tensor is known as the **metric tensor**. The components of this tensor are

$$\boxed{g(\vec{e}_\mu,\vec{e}_\nu) = g_{\mu\nu}}\,, \tag{3.63}$$

$$\vec{u}\cdot\vec{v} = g(\vec{u},\vec{v}) = g(u^\mu\vec{e}_\mu, v^\nu\vec{e}_\nu) = u^\mu v^\nu g(\vec{e}_\mu,\vec{e}_\nu) = u^\mu v^\nu g_{\mu\nu}\,. \tag{3.64}$$

Usual notation:

$$\boxed{\vec{u}\cdot\vec{v} = g_{\mu\nu}u^\mu v^\nu}\,. \tag{3.65}$$

The absolute value of a vector:

$$|\vec{v}| = \sqrt{g(\vec{v},\vec{v})} = \sqrt{|g_{\mu\nu}v^\mu v^\nu|}\,. \tag{3.66}$$

Example 3.2.3 (Cartesian coordinates in a plane)

$$\begin{gathered}\vec{e}_x\cdot\vec{e}_x = 1, \qquad \vec{e}_y\cdot\vec{e}_y = 1, \qquad \vec{e}_x\cdot\vec{e}_y = \vec{e}_y\cdot\vec{e}_x = 0,\\ g_{xx} = g_{yy} = 1, \qquad g_{xy} = g_{yx} = 0,\\ g_{\mu\nu} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{gathered} \tag{3.67}$$

Example 3.2.4 (Basis vectors in plane polar coordinates) (Fig. 3.8)

$$\vec{e}_r\cdot\vec{e}_r = 1, \quad \vec{e}_\theta\cdot\vec{e}_\theta = r^2, \quad \vec{e}_r\cdot\vec{e}_\theta = 0\,. \tag{3.68}$$

The metric tensor in plane polar coordinates:

$$g_{\mu\nu} = \begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix}\,. \tag{3.69}$$

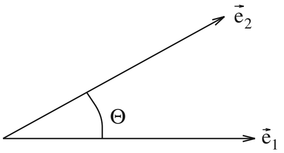

Fig. 3.8 Basis vectors $\vec{e}_1$ and $\vec{e}_2$

Example 3.2.5 (Non-diagonal basis vectors)

$$\vec{e}_1 \cdot \vec{e}_1 = 1, \qquad \vec{e}_2 \cdot \vec{e}_2 = 1, \qquad \vec{e}_1 \cdot \vec{e}_2 = \cos\theta = \vec{e}_2 \cdot \vec{e}_1,$$
$$g_{\mu\nu} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} . \tag{3.70}$$

Definition 3.2.4 (Contravariant components) The **contravariant components** $g^{\mu\alpha}$ of the metric tensor are defined as

$$g^{\mu\alpha} g_{\alpha\nu} \equiv \delta^\mu{}_\nu \quad g^{\mu\nu} = \vec{w}^\mu \cdot \vec{w}^\nu , \tag{3.71}$$

where $\vec{w}^\mu$ is defined by

$$\vec{w}^\mu \cdot \vec{w}_\nu \equiv \delta^\mu{}_\nu . \tag{3.72}$$

$g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$.

It is possible to define a **mapping** between tensors of different type (e.g. covariant on contravariant) using the metric tensor (Fig. 3.9).

We can for instance map a vector on a 1-form:

$$v_\mu = g(\vec{v}, \vec{e}_\mu) = g(v^\alpha \vec{e}_\alpha, \vec{e}_\mu) = v^\alpha g(\vec{e}_\alpha, \vec{e}_\mu) = v^\alpha g_{\alpha\mu} . \tag{3.73}$$

This is known as lowering of an index. Raising of an index becomes

$$v^\mu = g^{\mu\alpha} v_\alpha . \tag{3.74}$$

The mixed components of the metric tensor become

$$g^\mu{}_\nu = g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu . \tag{3.75}$$

We now define distance along a curve. Let the curve be parameterized by λ (proper time τ for time-like curves). Let $\vec{v}$ be the tangent vector field of the curve.

The squared distance ds^2 between the points along the curve is defined as

$$ds^2 \equiv g(\vec{v}, \vec{v}) d\lambda^2 \tag{3.76}$$

Equations (3.64) and (3.76) give

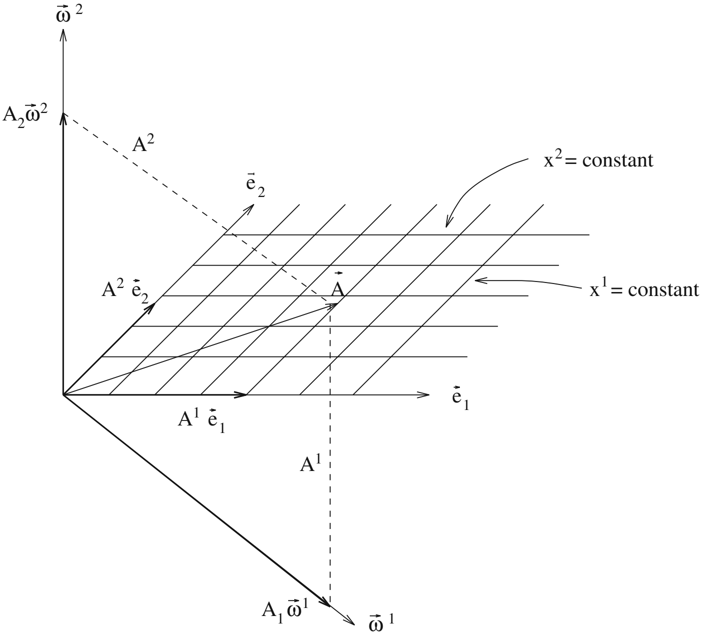

Fig. 3.9 The covariant and contravariant components of a vector

$$ds^2 = g_{\mu\nu} v^\mu v^\nu d\lambda^2 \,. \tag{3.77}$$

The tangent vector has components $v^\mu = \frac{dx^\mu}{d\lambda}$, which give

$$\boxed{ds^2 = g_{\mu\nu} dx^\mu dx^\nu} \,. \tag{3.78}$$

The expression ds^2 is known as the **line element**.

Example 3.2.6 (Cartesian coordinates in a plane)

$$\begin{aligned} g_{xx} = g_{yy} = 1, \qquad g^x_{\ y} = g^y_{\ x} = 0, \\ ds^2 = dx^2 + dy^2 \,. \end{aligned} \tag{3.79}$$

Example 3.2.7 (Plane polar coordinates)

$$\begin{aligned} g_{rr} = 1, \quad g_{\theta\theta} = r^2, \\ ds^2 = dr^2 + r^2 d\theta^2 \,. \end{aligned} \tag{3.80}$$

Cartesian coordinates in the (flat) Minkowski spacetime:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{3.81}$$

In an arbitrary curved space, an orthonormal basis can be adopted in any point. If $\vec{e}_{\hat{t}}$ is tangent vector to the world line of an observer, then $\vec{e}_{\hat{t}} = \vec{u}$, where $\vec{u}$ is the 4-velocity of the observer. In this case, we are using what we call the *comoving orthonormal basis* of the observer. In a such basis, we have the Minkowski metric:

$$ds^2 = \eta_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} \ . \tag{3.82}$$

3.3 Forms

An **antisymmetric tensor** is a tensor whose sign changes under an arbitrary exchange of two arguments:

$$A(\cdots, \vec{u}, \cdots, \vec{v}, \cdots) = -A(\cdots, \vec{v}, \cdots, \vec{u}, \cdots) \ . \tag{3.83}$$

The components of an antisymmetric tensor change sign under exchange of two indices:

$$A_{\ldots\mu\ldots\nu\ldots} = -A_{\ldots\nu\ldots\mu\ldots} \ . \tag{3.84}$$

Definition 3.3.1 (p-form) A **p-form** is defined to be an antisymmetric, covariant tensor of rank p.

An antisymmetric tensor product $\wedge$ is defined by

$$\underline{\omega}^{[\mu_1} \otimes \cdots \otimes \underline{\omega}^{\mu_p]} \wedge \underline{\omega}^{[\nu_1} \otimes \cdots \otimes \underline{\omega}^{\nu_q]} \equiv \frac{(p+q)!}{p!q!} \underline{\omega}^{[\mu_1} \otimes \cdots \otimes \underline{\omega}^{\nu_q]} \ , \tag{3.85}$$

where [ ] denotes antisymmetric combinations defined by

$$\underline{\omega}^{[\mu_1} \otimes \cdots \otimes \underline{\omega}^{\mu_p]} \equiv \frac{1}{p!} \cdot (\text{the sum of terms with all possible permutations of indices with “+” for even and “-” for odd permutations}). \tag{3.86}$$

Example 3.3.1 (Antisymmetric combinations)

$$\underline{\omega}^{[\mu_1} \otimes \underline{\omega}^{\mu_2]} = \frac{1}{2}(\underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} - \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_1}) \ . \tag{3.87}$$

Example 3.3.2 (Antisymmetric combinations)

$$\underline{\omega}^{[\mu_1} \otimes \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_3]} =$$
$$\frac{1}{3!}(\underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_3} + \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_2} + \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_1}$$
$$- \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_3} - \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_2} \otimes \underline{\omega}^{\mu_1} - \underline{\omega}^{\mu_1} \otimes \underline{\omega}^{\mu_3} \otimes \underline{\omega}^{\mu_2})$$
$$= \frac{1}{3!}\epsilon_{ijk}(\underline{\omega}^{\mu_i} \otimes \underline{\omega}^{\mu_j} \otimes \underline{\omega}^{\mu_k}) \,. \quad (3.88)$$

Example 3.3.3 (A 2-form in 3-space)

$$\begin{aligned} \underline{\alpha} = {} & \alpha_{12}\underline{\omega}^1 \otimes \underline{\omega}^2 + \alpha_{21}\underline{\omega}^2 \otimes \underline{\omega}^1 + \alpha_{13}\underline{\omega}^1 \otimes \underline{\omega}^3 + \alpha_{31}\underline{\omega}^3 \otimes \underline{\omega}^1 \\ & + \alpha_{23}\underline{\omega}^2 \otimes \underline{\omega}^3 + \alpha_{32}\underline{\omega}^3 \otimes \underline{\omega}^2 \,. \end{aligned} \quad (3.89)$$

Now the antisymmetry of $\underline{\alpha}$ means that

$$+\underline{\alpha}_{21} = -\underline{\alpha}_{12}; \quad +\underline{\alpha}_{31} = -\underline{\alpha}_{13}; \quad +\underline{\alpha}_{32} = -\underline{\alpha}_{23} \,, \quad (3.90)$$

$$\begin{aligned} \underline{\alpha} = {} & \underline{\alpha}_{12}(\underline{\omega}^1 \otimes \underline{\omega}^2 - \underline{\omega}^2 \otimes \underline{\omega}^1) \\ & + \underline{\alpha}_{13}(\underline{\omega}^1 \otimes \underline{\omega}^3 - \underline{\omega}^3 \otimes \underline{\omega}^1) \\ & + \underline{\alpha}_{23}(\underline{\omega}^2 \otimes \underline{\omega}^3 - \underline{\omega}^3 \otimes \underline{\omega}^2) \\ & = \alpha_{|\mu\nu|} 2\underline{\omega}^{[\mu} \otimes \underline{\omega}^{\nu]}, \end{aligned} \quad (3.91)$$

where $|\mu\nu|$ means summation only for $\mu < \nu$ (see [1]). We now use the definition of $\wedge$ with $p = q = 1$. This gives

$$\boxed{\underline{\alpha} = \alpha_{|\mu\nu|}\underline{\omega}^\mu \wedge \underline{\omega}^\nu} \,.$$

$\underline{\omega}^\mu \wedge \underline{\omega}^\nu$ is the form basis.

We can also write

$$\boxed{\underline{\alpha} = \frac{1}{2}\alpha_{\mu\nu}\underline{\omega}^\mu \wedge \underline{\omega}^\nu} \,.$$

A tensor of rank 2 can always be split up into a symmetric and an antisymmetric part. (Note that tensors of higher rank cannot be split up in this way.)

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) + \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \\ &= A_{\mu\nu} + S_{\mu\nu}. \end{aligned} \quad (3.92)$$

We thus have

$$
\begin{aligned}
S_{\mu\nu}A^{\mu\nu} &= \frac{1}{4}(T_{\mu\nu}+T_{\nu\mu})(T^{\mu\nu}-T^{\nu\mu}) \\
&= \frac{1}{4}(T_{\mu\nu}T^{\mu\nu}-T_{\mu\nu}T^{\nu\mu}+T_{\nu\mu}T^{\mu\nu}-T_{\nu\mu}T^{\nu\mu}) \\
&= 0.
\end{aligned}
\tag{3.93}
$$

In general, summation over indices of a symmetric and an antisymmetric quantity vanishes. In a summation $T_{\mu\nu}A^{\mu\nu}$ where $A^{\mu\nu}$ is antisymmetric and $T_{\mu\nu}$ has no symmetry, only the antisymmetric part of $T_{\mu\nu}$ contributes. So that, in

$$
\underline{\alpha} = \frac{1}{2}\alpha_{\mu\nu}\underline{\omega}^{\mu}\wedge\underline{\omega}^{\nu} \tag{3.94}
$$

only the antisymmetric elements $\alpha_{\nu\mu} = -\alpha_{\mu\nu}$ contribute to the summation. These antisymmetric elements are the **form components**.

Forms are antisymmetric covariant tensors. Because of this antisymmetry a form with two identical components must be a **null form** (= zero), e.g. $\alpha_{131} = -\alpha_{131} \Rightarrow \alpha_{131} = 0$.

In an n-dimensional space all p-forms with $p > n$ are null forms.

Problems

3.1. *1-Forms*

(a) Show that the following holds:

$$
\underline{p}(A^{\alpha}\vec{e}_{\alpha}) = A^{\alpha}\underline{p}(\vec{e}_{\alpha})\ . \tag{3.95}
$$

(b) Let the components of $\underline{p}$, $\vec{A}$ and $\vec{B}$ be $(-1,1,2,0)$, $(2,1,0,-1)$ and $(0,2,0,0)$, respectively. Find (i) $\underline{p}(\vec{A})$; (ii) $\underline{p}(\vec{B})$; (iii) $\underline{p}(\vec{A}-3\vec{B})$; (iv) $\underline{p}(\vec{A})-3\underline{p}(\vec{B})$.

3.2. *The tensor product*

(a) Given 1-forms α and β. Assume that the components of α and β are $(1,1,0,0)$ and $(-1,0,1,0)$, respectively. Show – by using two vectors as arguments – that $\alpha\otimes\beta \neq \beta\otimes\alpha$. Find also the components of $\alpha\otimes\beta$.
(b) Find also the components of the symmetric and antisymmetric part of $\alpha\otimes\beta$, defined above.

3.3. *The wedge product*
Given the following forms in a 4-dimensional spacetime

$$
\begin{aligned}
\underline{\alpha} &= \alpha_{\mu}\underline{\omega}^{\mu}, \\
\underline{\beta} &= \beta_{\mu\nu}\underline{\omega}^{\mu}\wedge\underline{\omega}^{\nu}, \\
\underline{\gamma} &= \gamma_{\mu}\underline{\omega}^{\mu}, \\
\underline{\delta} &= \delta_{\mu\nu}\underline{\omega}^{\mu}\wedge\underline{\omega}^{\nu}\ ,
\end{aligned}
\tag{3.96}
$$

what is the correspondence between

$$\begin{aligned} \underline{\alpha}\wedge\underline{\beta} \quad &\text{and} \quad \underline{\beta}\wedge\underline{\alpha} \quad , \\ \underline{\alpha}\wedge\underline{\gamma} \quad &\text{and} \quad \underline{\gamma}\wedge\underline{\alpha} \quad , \\ \underline{\beta}\wedge\underline{\delta} \quad &\text{and} \quad \underline{\delta}\wedge\underline{\beta} \quad ? \end{aligned} \tag{3.97}$$

3.4. *Contractions of tensors*
Assume that **A** is an antisymmetric tensor of rank $\binom{2}{0}$, **B** a symmetric tensor of rank $\binom{0}{2}$, **C** an arbitrary tensor of rank $\binom{0}{2}$ and **D** an arbitrary tensor of rank $\binom{2}{0}$. Show that

$$\begin{aligned} A^{ab}B_{ab} &= 0, \\ A^{ab}C_{ab} &= A^{ab}C_{[ab]}, \end{aligned}$$

and

$$B_{ab}D^{ab} = B_{ab}D^{(ab)}.$$

3.5. *Symmetric and antisymmetric tensors*

(a) Let the components of the tensor $M^{\alpha\beta}$ be given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} . \tag{3.98}$$

Find (let the metric tensor have components $\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$):

1. the components of the symmetric tensor $M^{(\alpha\beta)}$ and the antisymmetric tensor $M^{[\alpha\beta]}$;
2. the components of $M^{\alpha}{}_{\beta}$;
3. the components of $M_{\alpha}{}^{\beta}$;
4. the components of $M_{\alpha\beta}$.

(b) Consider now the tensor with components $M^{\alpha}{}_{\beta}$. Does it make sense to talk about the symmetric and antisymmetric parts of this tensor? If yes, define them. If no, explain why.

3.6. *4-Vectors*

(a) Given three 4-vectors

$$\begin{aligned} \mathbf{A} &= 4\mathbf{e}_t + 3\mathbf{e}_x + 2\mathbf{e}_y + \mathbf{e}_z, \\ \mathbf{B} &= 5\mathbf{e}_t + 4\mathbf{e}_x + 3\mathbf{e}_y, \\ \mathbf{C} &= \mathbf{e}_t + 2\mathbf{e}_x + 3\mathbf{e}_y + 4\mathbf{e}_z, \end{aligned}$$

where

$$\mathbf{e}_x \cdot \mathbf{e}_x = \mathbf{e}_y \cdot \mathbf{e}_y = \mathbf{e}_z \cdot \mathbf{e}_z = 1,$$

while

$$\mathbf{e}_t \cdot \mathbf{e}_t = -1,$$

show that **A** is time-like, **B** is light-like and **C** is space-like.

(b) Assume that **A** and **B** are two non-zero orthogonal 4-vectors, $\mathbf{A} \cdot \mathbf{B} = 0$. Show the following:
- If **A** is time-like, then **B** is space-like.
- If **A** is light-like, then **B** is space-like or light-like.
- If **A** and **B** are light-like, then they are proportional.
- If **A** is space-like, then **B** is time-like, light-like or space-like.

Illustrate this in a 3-dimensional Minkowski diagram.

(c) A change of basis is given by

$$\begin{aligned}
\mathbf{e}_{t'} &= \cosh a \mathbf{e}_t + \sinh a \mathbf{e}_x, \\
\mathbf{e}_{x'} &= \sinh a \mathbf{e}_t + \cosh a \mathbf{e}_x, \\
\mathbf{e}_{y'} &= \mathbf{e}_y, \quad \mathbf{e}_{z'} = \mathbf{e}_z.
\end{aligned}$$

Show that this describes a Lorentz transformation along the x-axis, where the relative velocity v between the reference frames are given by $v = \tanh a$. Draw the vectors in a 2-dimensional Minkowski diagram and find what type of curves the $\mathbf{e}_{t'}$ and $\mathbf{e}_{x'}$ describe as a varies.

(d) The 3-vector **v** describing the velocity of a particle is defined *with respect to an observer*. Explain why the 4-velocity **u** is defined *independent* of any observer. The 4-momentum of a particle, with rest mass m, is defined by $\mathbf{p} = m\mathbf{u} = md\mathbf{r}/d\tau$, where τ is the comoving time of the particle. Show that **p** is time-like and that $\mathbf{p} \cdot \mathbf{p} = -m^2$. Draw, in a Minkowski diagram, the curve to which **p** must be tangent to and explain how this is altered as $m \longrightarrow 0$.

Assume that the energy of the particle is being observed by an observer with 4-velocity **u**. Show that the energy he measures is given by

$$E = -\mathbf{p} \cdot \mathbf{u}\,. \tag{3.99}$$

This is an expression which is very useful when one wants to calculate the energy of a particle in an arbitrary reference frame.

3.7. *Wedge products of forms*
Given the 1-forms

$$a = x^2\omega^1 - y\omega^2, \quad b = y\omega^1 - xz\omega^2 + y^2\omega^3, \quad \sigma = y^2 z\omega^2\ ,$$

the 2-form

$$\eta = xy\omega^1 \wedge \omega^3 + x\omega^2 \wedge \omega^3$$

and the 3-form

$$\theta = xyz\omega^1 \wedge \omega^2 \wedge \omega^3\ ,$$

calculate the wedge products

$$a \wedge b, \quad a \wedge b \wedge \sigma, \quad a \wedge \eta, \quad a \wedge \theta \ .$$

3.8. *Coordinate transformations in a 2-dimensional Euclidean plane*
In this problem we will investigate vectors $\mathbf{x}$ in the 2-dimensional Euclidean plane E^2. The set $\{\mathbf{e}_m | m = x, y\}$ is an orthonormal basis in E^2, i.e.

$$\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{mn} \ .$$

The components of a vector $\mathbf{x}$ in this basis is given by x and y, or x^m:

$$\mathbf{x} = x^m \mathbf{e}_m = x\mathbf{e}_x + y\mathbf{e}_y \ .$$

A skew basis set, $\{\mathbf{e}_\mu | \mu = 1, 2\}$, is also given. In this basis

$$\mathbf{x} = x^\mu \mathbf{e}_\mu = x^1 \mathbf{e}_1 + x^2 \mathbf{e}_2 \ .$$

The transformation between these to coordinates are

$$\begin{aligned} x^1 &= 2x - y, \\ x^2 &= x + y. \end{aligned}$$

(a) Find $\mathbf{e}_1$ and $\mathbf{e_2}$ expressed in terms of $\mathbf{e}_x$ and $\mathbf{e}_y$. Determine the transformation matrix M, defined by

$$x^m = M^m_{\ \mu} x^\mu \ .$$

What is M^{-1}?

(b) The metric tensor $\mathbf{g}$ is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{mn} dx^m dx^n \ ,$$

where ds is the distance between $\mathbf{x}$ and $\mathbf{x} + \mathbf{dx}$. Show that we have

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = g_{\mu\nu} \ .$$

What is the relation between the matrices $(g_{\mu\nu})$ and (g_{mn}) and the transformation matrix M?

The scalar product between two vectors can therefore be expressed as

$$\mathbf{v} \cdot \mathbf{u} = g_{\mu\nu} v^\mu u^\nu = g_{mn} v^m u^n \ .$$

Verify this equation for the case $\mathbf{u} = 2\mathbf{e}_1$ and $\mathbf{v} = 3\mathbf{e}_2$.

(c) Using the basis vectors $\mathbf{e}_\mu$, we can define a new set ω^μ by

$$\omega^\mu \cdot \mathbf{e}_\nu = \delta^\mu_{\ \nu} \ .$$

Find ω^1 and ω^2 expressed in terms of $\mathbf{e}_x$ and $\mathbf{e}_y$. Why is $\omega^m = \mathbf{e}_m$, while $\omega^\mu \neq \mathbf{e}_\mu$?
A vector $\mathbf{x}$ can now be expressed as

$$\mathbf{x} = x^\mu \mathbf{e}_\mu = x_\mu \omega^\mu \ .$$

What is the relation between the contravariant components x^μ and the covariant components x_μ? Determine both sets of components for the vector $\mathbf{A} = 3\mathbf{e}_x + \mathbf{e}_y$. In a (x,y)-diagram, draw the three set of basis vectors $\{\mathbf{e}_\mu\}$, $\{\mathbf{e}_m\}$ and $\{\omega_\mu\}$. What is the geometrical interpretation of the relation between the two sets $\{\mathbf{e}_\mu\}$ and $\{\omega_\mu\}$? Depict also the vector $\mathbf{A}$ and explain how the components of $\mathbf{A}$ in the three basis sets can be seen from the diagram.

(d) Find the matrix $(g^{\mu\nu})$ defined by

$$\omega^\mu \cdot \omega^\nu = g^{\mu\nu} \ .$$

Verify that this matrix is the inverse to $\left(g_{\mu\nu}\right)$.
The metric tensor is a symmetric tensor of rank 2 and can therefore be expressed with the basis vectors $\mathbf{e}_m \otimes \mathbf{e}_n$ in the tensor product space $E^2 \otimes E^2$,

$$\mathbf{g} = g_{mn} \mathbf{e}_m \otimes \mathbf{e}_n \ .$$

Show that we can also express it as

$$\mathbf{g} = g_{\mu\nu} \omega^\mu \otimes \omega^\nu$$

and

$$\mathbf{g} = g^{\mu\nu} \mathbf{e}_\mu \otimes \mathbf{e}_\nu \ .$$

What is the dimension of the space spanned by the vectors $\mathbf{e}_m \otimes \mathbf{e}_n$?
The antisymmetric tensors span a 1-dimensional subspace. Show this by proving that an antisymmetric tensor A_{mn} is a linear combination of the basis vector

$$\mathbf{e}_x \wedge \mathbf{e}_y = \mathbf{e}_x \otimes \mathbf{e}_y - \mathbf{e}_y \otimes \mathbf{e}_x \ .$$

Find $\mathbf{u} \wedge \mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are the vectors from (b), expressed in terms of the basis vector $\mathbf{e}_x \wedge \mathbf{e}_y$. What is the relation between this and the area that is spanned by $\mathbf{u}$ and $\mathbf{v}$? Calculate also $\omega^1 \wedge \omega^2$.

Reference

1. Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1973. *Gravitation*, first edn, W. H. Freeman and company, San Francisco. 70

Chapter 4
Accelerated Reference Frames

4.1 Rotating Reference Frames

4.1.1 The Spatial Metric Tensor

Let $\vec{e}_{\hat{0}}$ be the 4-velocity field ($x^0 = ct, c = 1, x^0 = t$) of the reference particles in a reference frame R. We are going to find the metric tensor γ_{ij} in a tangent space orthogonal to $\vec{e}_{\hat{0}}$, expressed by the metric tensor $g_{\mu\nu}$ of spacetime.

In an arbitrary coordinate basis $\{\vec{e}_\mu\}$, $\{\vec{e}_i\}$ is not necessarily orthogonal to $\vec{e}_0$. We choose $\vec{e}_0 \| \vec{e}_{\hat{0}}$. Let $\vec{e}_{\perp i}$ be the component of $\vec{e}_i$ orthogonal to $\vec{e}_0$, that is, $\vec{e}_{\perp i} \cdot \vec{e}_0 = 0$. The metric tensor of space is defined by

$$\begin{aligned}
\gamma_{ij} &= \vec{e}_{\perp i} \cdot \vec{e}_{\perp j}, \gamma_{i0} = 0, \gamma_{00} = 0, \\
\vec{e}_{\perp i} &= \vec{e}_i - \vec{e}_{\| i}, \\
\vec{e}_{\| i} &= \frac{\vec{e}_i \cdot \vec{e}_0}{\vec{e}_0 \cdot \vec{e}_0} \vec{e}_0 = \frac{g_{i0}}{g_{00}} \vec{e}_0,
\end{aligned}$$

$$\begin{aligned}
\gamma_{ij} &= (\vec{e}_i - \vec{e}_{\| i}) \cdot (\vec{e}_j - \vec{e}_{\| j}) \\
&= \left(\vec{e}_i - \frac{g_{i0}}{g_{00}} \vec{e}_0 \right) \cdot \left(\vec{e}_j - \frac{g_{j0}}{g_{00}} \vec{e}_0 \right) \\
&= \vec{e}_i \cdot \vec{e}_j - \frac{g_{j0}}{g_{00}} \vec{e}_0 \cdot \vec{e}_i - \frac{g_{i0}}{g_{00}} \vec{e}_0 \cdot \vec{e}_j + \frac{g_{i0} g_{j0}}{g_{00}^2} \vec{e}_0 \cdot \vec{e}_0 \\
&= g_{ij} - \frac{g_{i0} g_{j0}}{g_{00}} - \frac{g_{i0} g_{j0}}{g_{00}} + \frac{g_{i0} g_{j0}}{g_{00}}.
\end{aligned}$$

$$\Rightarrow \gamma_{ij} = g_{ij} - \frac{g_{i0} g_{j0}}{g_{00}} . \tag{4.1}$$

(Note: $g_{ij} = g_{ji} \Rightarrow \gamma_{ij} = \gamma_{ji}$).

The line element in space

$$dl^2 = \gamma_{ij} dx^i dx^j = \left(g_{ij} - \frac{g_{i0} g_{j0}}{g_{00}} \right) dx^i dx^j \tag{4.2}$$

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gives the distance between simultaneous events in a reference frame where the metric tensor of spacetime in a comoving coordinate system is $g_{\mu\nu}$.

The line element for spacetime can be expressed as

$$ds^2 = -c^2 d\hat{t}^2 + dl^2. \tag{4.3}$$

It follows that $d\hat{t} = 0$ represents the simultaneity defining the spatial line element. The temporal part of the spacetime line element may be expressed as

$$\begin{aligned}
d\hat{t}^2 &= dl^2 - ds^2 = (\gamma_{\mu\nu} - g_{\mu\nu})dx^\mu dx^\nu \\
&= (\gamma_{ij} - g_{ij})dx^i dx^j + 2(\gamma_{i0} - g_{i0})dx^i dx^0 + (\gamma_{00} - g_{00})dx^0 dx^0 \\
&= \left(g_{ij} - \frac{g_{i0}g_{j0}}{g_{00}} - g_{ij}\right) dx^i dx^j - 2g_{i0}dx^i dx^0 - g_{00}(dx^0)^2 \\
&= -g_{00}\left[(dx^0)^2 + 2\frac{g_{i0}}{g_{00}}dx^0 dx^i + \frac{g_{i0}g_{j0}}{g_{00}^2}dx^i dx^j\right] \\
&= \left[(-g_{00})^{1/2}\left(dx^0 + \frac{g_{i0}}{g_{00}}dx^i\right)\right]^2 .
\end{aligned}$$

So finally we get

$$d\hat{t} = (-g_{00})^{1/2}\left(dx^0 + \frac{g_{i0}}{g_{00}}dx^i\right). \tag{4.4}$$

The 3-space orthogonal to the world lines of the reference particles in R, $d\hat{t} = 0$, corresponds to a coordinate time-interval $dt = -\frac{g_{i0}}{g_{00}}dx^i$. This is not an exact differential, that is, dt is not integrable, which means that one cannot in general define a 3-space orthogonal to the world lines of the reference particles, i.e. a “simultaneity space”, in an arbitrary reference frame. We must also conclude that unless g_{i0}/g_{00} is constant, it is not possible to Einstein-synchronize clocks around closed curves.

In particular, it is not possible to Einstein synchronize clocks around a closed curve in a rotating reference frame. If this is attempted, contradictory boundary conditions in the non-rotating lab frame will arise, due to the relativity of simultaneity. (See Fig. 4.1.)

The distance in the laboratory frame between two points is (Fig. 4.2)

$$\Delta x = \frac{2\pi r}{n} . \tag{4.5}$$

Lorentz transformation from the instantaneous rest frame (x', t') to the laboratory system (x, t):

$$\begin{aligned}
\Delta t &= \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right), \quad \gamma = \frac{1}{\sqrt{1 - \frac{r^2\omega^2}{c^2}}}, \\
\Delta x &= \gamma(\Delta x' + v\Delta t') .
\end{aligned} \tag{4.6}$$

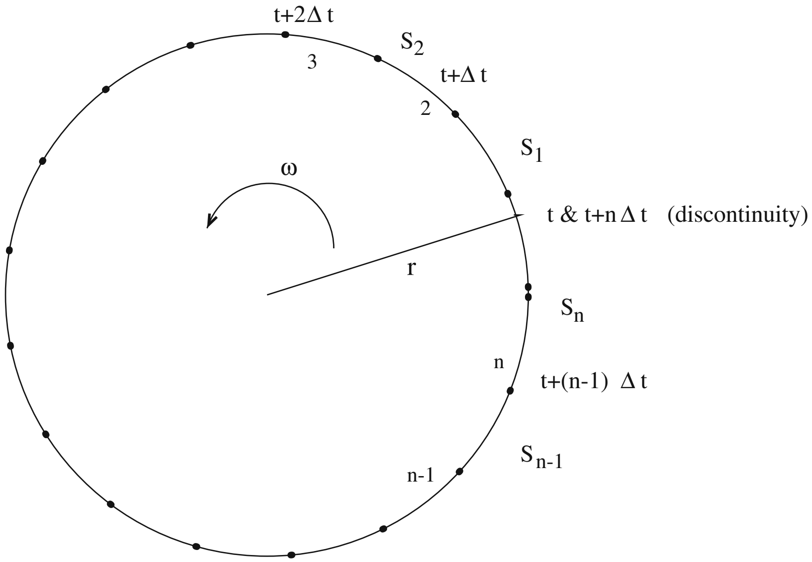

Fig. 4.1 Events simultaneous in the rotating reference frame. 1 comes before 2, before 3, etc. Note the discontinuity at *t*

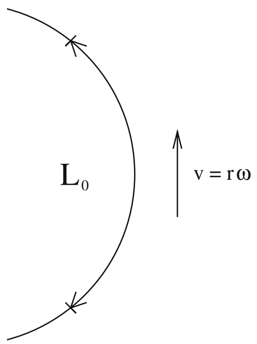

Fig. 4.2 The distance between two points on the circumference is L_0

Since we for simultaneous events in the rotating reference frame have $\Delta t' = 0$, and proper distance $\Delta x' = \gamma \Delta x$, we get in the laboratory frame

$$\Delta t = \gamma^2 \frac{r\omega}{c^2} \Delta x = \gamma^2 \frac{r\omega}{c^2} \frac{2\pi r}{n} . \tag{4.7}$$

The fact that $\Delta t' = 0$ and $\Delta t \neq 0$ is an expression of the relativity of simultaneity. Around the circumference this is accumulated to

$$n\Delta t = \gamma^2 \frac{2\pi r^2 \omega}{c^2} \tag{4.8}$$

and we get a discontinuity in simultaneity, as shown in Fig. 4.3. Let IF be an inertial frame with cylinder coordinates (T, R, Θ, Z). The line element is then given by

$$ds^2 = -c^2dT^2 + dR^2 + R^2d\Theta^2 + dZ^2 \quad (c = 1)\,. \tag{4.9}$$

In a rotating reference frame, RF, we have cylinder coordinates (t, r, θ, z). We then have the following coordinate transformation:

$$t = T, \quad r = R, \quad \theta = \Theta - \omega T, \quad z = Z\,. \tag{4.10}$$

The line element in the comoving coordinate system in RF is then

$$\begin{aligned} ds^2 &= -c^2dt^2 + dr^2 + r^2(d\theta + \omega dt)^2 + dz^2 \\ &= -(1 - r^2\omega^2/c^2)dt^2 + dr^2 + r^2d\theta^2 + dz^2 + 2r^2\omega d\theta dt \quad (c = 1)\,. \end{aligned} \tag{4.11}$$

The metric tensor has the following components:

$$\begin{aligned} &g_{tt} = -(1 - r^2\omega^2/c^2),\ g_{rr} = 1,\ g_{\theta\theta} = r^2,\ g_{zz} = 1, \\ &g_{\theta t} = g_{t\theta} = r^2\omega\,. \end{aligned} \tag{4.12}$$

$dt = 0$ gives

$$ds^2 = dr^2 + r^2d\theta^2 + dz^2. \tag{4.13}$$

This represents the Euclidean geometry of the 3-space (simultaneity space, $t = T$) in IF.

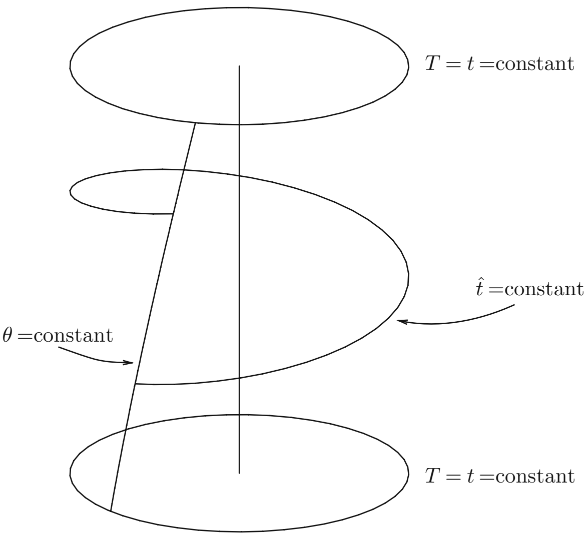

Fig. 4.3 Discontinuity in simultaneity

As applied to the rotating system the spatial line element takes the form

$$
\begin{aligned}
dl^2 &= \left(g_{ij} - \frac{g_{i0}g_{j0}}{g_{00}} \right) dx^i dx^j , \\
\gamma_{rr} &= g_{rr} = 1, \ \gamma_{zz} = g_{zz} = 1, \\
\gamma_{\theta\theta} &= g_{\theta\theta} - \frac{g_{\theta 0}^2}{g_{00}} \\
&= r^2 - \frac{(r^2\omega)^2/c^2}{-(1 - r^2\omega^2)/c^2} = \frac{r^2}{1 - r^2\omega^2/c^2}
\end{aligned}
$$

$$
\Rightarrow \quad dl^2 = dr^2 + \frac{r^2 d\theta^2}{1 - r^2\omega^2/c^2} + dz^2 . \tag{4.14}
$$

It describes the geometry of a local 3-space orthogonal to the world line of a reference particle in RF. This 3-space cannot be extended to a finite 3-dimensional space in RF since Einstein synchronization is not integrable in RF. From the line element (4.14) it is seen that the geometry of this local simultaneity space in RF is non-Euclidean. The circumference of a circle with radius r is

$$
l_\theta = \frac{2\pi r}{\sqrt{1 - r^2\omega^2/c^2}} > 2\pi r . \tag{4.15}
$$

We see that the quotient between circumference and radius is $> 2\pi$ which means that the spatial geometry is *hyperbolic*. (For spherical geometry we have $l_\theta < 2\pi r$.)

4.1.2 Angular Acceleration in the Rotating Frame

We will now investigate what happens when we give RF an angular acceleration. Then we consider a rotating circle made of standard measuring rods, as shown in Fig. 4.4. All points on a circle are accelerated simultaneously in IF (the laboratory system). We let the angular velocity increase from ω to $\omega + d\omega$, measured in IF. Lorentz transformation to an instantaneous rest frame for a point on the circumference then gives an increase in velocity in this system:

$$
r d\omega' = \frac{r d\omega}{1 - r^2\omega^2/c^2} , \tag{4.16}
$$

where we have used that the initial velocity in this frame is zero.

The time difference for the accelerations of the front and back ends of the points on the periphery of the rotating disc (the front end is accelerated first) in the instantaneous rest frame is

$$
\Delta t' = \frac{r\omega L_0}{\sqrt{1 - r^2\omega^2/c^2}} , \tag{4.17}
$$

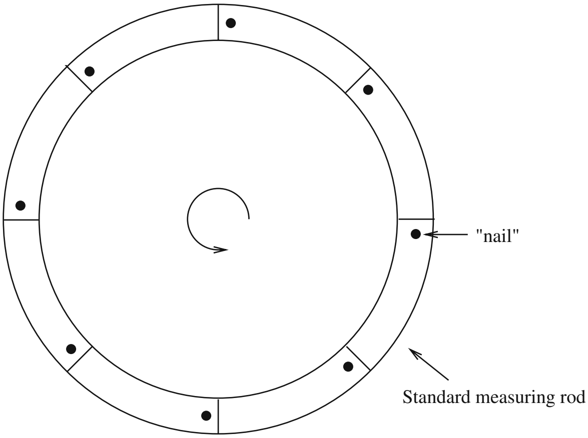

Fig. 4.4 A non-rotating disc with measuring rods. The standard measuring rods are fastened with nails at one end. We will see what then happens when we have an angular acceleration

where L_0 is the distance between points on the circumference when at rest (= the length of the rods when at rest), $L_0 = \frac{2\pi r}{n}$. In IF all points on the circumference are accelerated simultaneously. In RF, however, this is not the case. Here the distance between points on the circumference will increase, see Fig. 4.5. The rest distance increases by

$$dL' = r d\omega' \Delta t' = \frac{r^2 \omega L_0 d\omega / c^2}{(1 - r^2\omega^2/c^2)^{3/2}} \,. \tag{4.18}$$

(It may be noted that each point on an arbitrary measuring rod is accelerated simultaneously in the rest frame of the rod to preserve its rest length. In the laboratory frame the rear point of the rod is accelerated first, giving the rod a Lorentz contraction.)

The increase of the distance during the acceleration (in an instantaneous rest frame) is

$$L' = \frac{r^2 L_0}{c^2} \int_0^\omega \frac{\omega d\omega}{(1 - r^2\omega^2/c^2)^{3/2}} = \left(\frac{1}{\sqrt{1 - r^2\omega^2/c^2}} - 1 \right) L_0 \,. \tag{4.19}$$

Hence, after the acceleration there is a proper distance L' between the rods. In the laboratory system (IF) the distance between the rods is

$$L = \sqrt{1 - r^2\omega^2/c^2} L' = \sqrt{1 - r^2\omega^2/c^2} \left(\frac{1}{\sqrt{1 - r^2\omega^2/c^2}} - 1 \right) L_0 = L_0 - L_0\sqrt{1 - r^2\omega^2/c^2}, \tag{4.20}$$

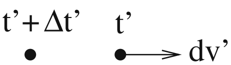

Fig. 4.5 In RF two points on the circumference are accelerated at different times. Thus the distance between them is increased

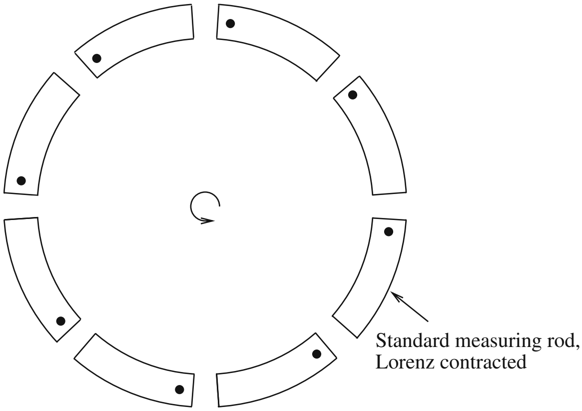

Fig. 4.6 The standard measuring rods have been Lorentz contracted

where L_0 is the rest length of the rods and $L_0\sqrt{1-r^2\omega^2/c^2}$ is their Lorentz contracted length. We now have the situation shown in Fig. 4.6.

Thus, there is room for more standard rods around the periphery the faster the disk rotates. This means that as measured with measuring rods at rest in the rotating frame the measured length of the periphery (number of standard rods) gets larger with increasing angular velocity. This is how an inertial observer would explain the measuring result of the rotating observer. The rotating observer, however, previous that the disc material has been stretched in the tangential direction. Note that as measured by the inertial observer the length of the periphery is $2\pi r$ independent of the angular velocity of the disc, since the inertial observer uses measuring rods at rest in the non-rotating reference frame. The Lorentz contraction of tangential lengths on the disc just compensates for the stretching of the disc (increase of the length), making the length of the periphery independent of the rotating velocity.

4.1.3 Gravitational Time Dilation

$$ds^2 = -\left(1-\frac{r^2\omega^2}{c^2}\right)c^2dt^2 + dr^2 + r^2d\theta^2 + dz^2 + 2r^2\omega d\theta dt\ . \tag{4.21}$$

We now look at *standard clocks* with constant r and z:

$$ds^2 = c^2dt^2\left[-\left(1-\frac{r^2\omega^2}{c^2}\right) + \frac{r^2}{c^2}\left(\frac{d\theta}{dt}\right)^2 + 2\frac{r^2\omega}{c^2}\frac{d\theta}{dt}\right]\ . \tag{4.22}$$

Let $\frac{d\theta}{dt} \equiv \dot{\theta}$ be the angular velocity of the clock in RF. The proper time-interval measured by the clock is then

$$ds^2 = -c^2 d\tau^2. \tag{4.23}$$

From this we see that

$$\boxed{d\tau = dt\sqrt{1 - \frac{r^2\omega^2}{c^2} - \frac{r^2\dot{\theta}^2}{c^2} - 2\frac{r^2\omega\dot{\theta}}{c^2}}}. \tag{4.24}$$

A non-moving standard clock in RF: $\dot{\theta} = 0 \Rightarrow$

$$d\tau = dt\sqrt{1 - \frac{r^2\omega^2}{c^2}}. \tag{4.25}$$

Seen from IF, the non-rotating laboratory system, Eq. (4.25) represents the **velocity-dependent time dilation** from the special theory of relativity.

But how is Eq. (4.25) interpreted in RF? The clock does not move relative to an observer in this system, hence what happens cannot be interpreted as a velocity-dependent phenomenon. According to Einstein, the fact that standard clocks slow down the farther away from the axis of rotation they are is due to a **gravitational effect**.

We will now find the gravitational potential at a distance r from the axis. The centripetal acceleration is v^2/r, $v = r\omega$ so

$$\Phi = -\int_0^r g(r)dr = -\int_0^r r\omega^2 dr = -\frac{1}{2}r^2\omega^2.$$

We then get

$$d\tau = dt\sqrt{1 - \frac{r^2\omega^2}{c^2}} = dt\sqrt{1 + \frac{2\Phi}{c^2}}\,. \tag{4.26}$$

In RF the position-dependent time dilation is interpreted as a **gravitational time dilation***: Time flows slower further down in a gravitational field.*

4.1.4 Path of Photons Emitted from Axes in the Rotating Reference Frame (RF)

We start with description in the inertial frame (IF). In IF photon paths are radial. Consider a photon path with $\Theta = 0$, $R = T$ with light source at $R = 0$. Transforming to RF,

$$\begin{aligned} &t = T, \quad r = R, \quad \theta = \Theta - \omega T \\ \Rightarrow \quad &r = t, \quad \theta = -\omega t\,. \end{aligned} \tag{4.27}$$

The orbit equation is thus $\theta = -\omega r$ which is the equation for an Archimedean spiral. The time used by a photon out to distance r from axis is $t = \frac{r}{c}$.

4.1.5 The Sagnac Effect

IF Description:

Here the velocity of light is isotropic, but the emitter/receiver moves due to the disc's rotation as shown in Fig. 4.7. Photons are emitted/received in/from opposite directions. Let t_1 be the travel time of photons which move *with* the rotation.

Then

$$\begin{aligned} 2\pi r + r\omega t_1 &= ct_1 \\ \Rightarrow \quad t_1 &= \frac{2\pi r}{c - r\omega}\,. \end{aligned} \tag{4.28}$$

Let t_2 be the travel time for photons moving against the rotation of the disc. The difference in travel time is

$$\begin{aligned} \Delta t = t_1 - t_2 &= 2\pi r\left(\frac{1}{c - r\omega} - \frac{1}{c + r\omega}\right) \\ &= \frac{2\pi r 2r\omega}{c^2 - r^2\omega^2} \\ &= \gamma^2\frac{4A\omega}{c^2}\,. \end{aligned} \tag{4.29}$$

A is the area enclosed by the photon path or orbit.

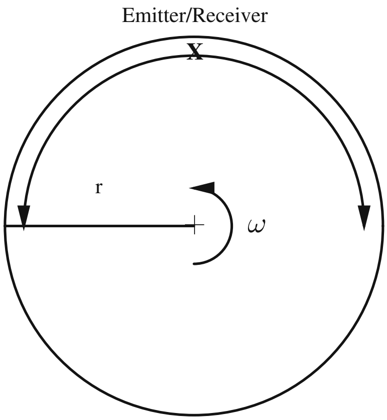

Fig. 4.7 The Sagnac effect demonstrates the **anisotropy** of the speed of light when measured in a rotating reference frame

RF Description:

$$ds^2 = -\left(1 - \frac{r^2\omega^2}{c^2}\right)c^2dt^2 + r^2d\theta^2 + 2r^2\omega d\theta dt.$$

Let $\dot{\theta} = \dfrac{d\theta}{dt}$. Since $ds^2 = 0$ along the world line of a photon,

$$r^2\dot{\theta}^2 + 2r^2\omega\dot{\theta} - (c^2 - r^2\omega^2) = 0,$$

$$\dot{\theta} = \frac{-r^2\omega \pm \sqrt{(r^4\omega^2 + r^2c^2 - r^4\omega^2)}}{r^2},$$

$$\begin{aligned} \dot{\theta} &= -\omega \pm \frac{rc}{r^2} \\ &= -\omega \pm \frac{c}{r}\,. \end{aligned} \tag{4.30}$$

The speed of light is, $v_{\pm} = r\dot{\theta} = -r\omega \pm c$. We see that in the rotating frame RF, the measured (coordinate) velocity of light is NOT isotropic. The difference in the travel time of the two beams is

$$\begin{aligned} \Delta t &= \frac{2\pi r}{c - r\omega} - \frac{2\pi r}{c + r\omega} \\ &= \gamma^2 \frac{4A\omega}{c^2}\,. \end{aligned} \tag{4.31}$$

The coordinate clocks are not Einstein synchronized in RF, but they represent a globally well-defined time. As measured with locally Einstein synchronized clocks the velocity of light is isotropic. But as shown, it is not possible to Einstein synchronize clocks around a closed curve in RF. (See Phil. Mag. series 6, vol. 8 (1904) for Michelson's article.)

4.2 Hyperbolically Accelerated Reference Frames

Consider a particle moving along a straight line with velocity u and acceleration $a = \frac{du}{dT}$. Rest acceleration is $\hat{a}$:

$$\Rightarrow \quad a = \left(1 - u^2/c^2\right)^{3/2}\hat{a}\,. \tag{4.32}$$

Assume that the particle has constant rest acceleration $\hat{a} = g$, that is

$$\frac{du}{dT} = \left(1 - u^2/c^2\right)^{3/2} g, \tag{4.33}$$

which on integration with $u(0) = 0$ gives

$$u = \frac{gT}{\left(1 + \frac{g^2T^2}{c^2}\right)^{1/2}} = \frac{dX}{dT}$$

$$\Rightarrow \quad X = \frac{c^2}{g}\left(1 + \frac{g^2T^2}{c^2}\right)^{1/2} + k$$

$$\Rightarrow \quad \frac{c^4}{g^2} = (X-k)^2 - c^2T^2 \ . \tag{4.34}$$

In its final form the above equation describes a hyperbola in the Minkowski diagram as shown in Fig. 4.8.

The proper time-interval as measured by a clock which follows the particle is

$$d\tau = \left(1 - \frac{u^2}{c^2}\right)^{1/2} dT \ . \tag{4.35}$$

Substitution for $u(T)$ and integration with $\tau(0) = 0$ gives

$$\begin{aligned} \tau &= \frac{c}{g}\operatorname{arcsinh}\left(\frac{gT}{c}\right) \\ \text{or} \quad T &= \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right) \\ \text{and} \quad X &= \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right) + k \ . \end{aligned} \tag{4.36}$$

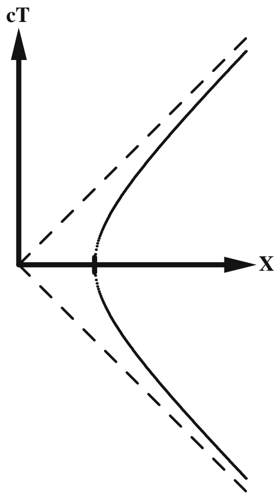

Fig. 4.8 Hyperbolically accelerated reference frames are so called because the loci of particle trajectories in spacetime are hyperbolae

We now use this particle as the origin of space in an hyperbolically accelerated reference frame.

Definition 4.2.1 (Born-stiff motion) Born-stiff motion of a system is a motion such that every element of the system has constant rest length. We demand that our accelerated reference frame is Born-stiff.

Let the inertial frame have coordinates (T,X,Y,Z) and the accelerated frame have coordinates (t,x,y,z). We now denote the X-coordinate of the "origin particle" by X_0:

$$1+\frac{gX_0}{c^2}=\cosh\frac{g\tau_0}{c}, \tag{4.37}$$

where τ_0 is the proper time for this particle and k is set to $\frac{-c^2}{g}$. (These are Møller coordinates. Setting $k=0$ gives Rindler coordinates.)

Let us denote the accelerated frame by Σ. The coordinate time at an arbitrary point in Σ is defined by $t=\tau_0$. That is coordinate clocks in Σ run identically with the standard clock at the "origin particle". Let $\vec{X}_0$ be the position 4-vector of the "origin particle". Decomposed in the laboratory frame, this becomes

$$\vec{X}_0=\left\{\frac{c^2}{g}\sinh\frac{gt}{c},\frac{c^2}{g}\left(\cosh\frac{gt}{c}-1\right),0,0\right\}\ . \tag{4.38}$$

P is chosen such that P and P_0 are simultaneous in the accelerated frame Σ. The distance (see Fig. 4.9) vector from P_0 to P, decomposed into an orthonormal comoving

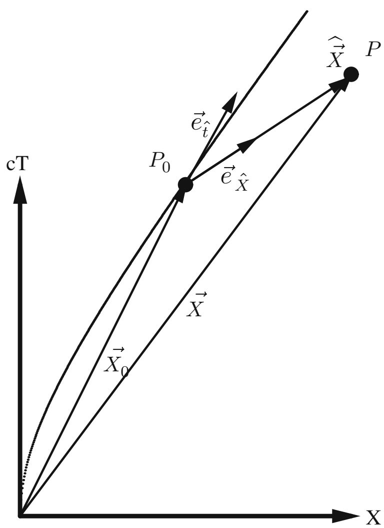

Fig. 4.9 Simultaneity in hyperbolically accelerated reference frames. The vector $\widehat{\vec{X}}$ lies along the "simultaneity line" which makes the same angle with the X-axis as does $\vec{e}_{\hat{t}}$ with the cT-axis.

basis of the "origin particle", is $\hat{X} = (0, \hat{x}, \hat{y}, \hat{z})$ where $\hat{x}, \hat{y}$ and $\hat{z}$ are physical distances measured simultaneously in Σ. The space coordinates in Σ are defined by

$$x \equiv \hat{x}, \quad y \equiv \hat{y}, \quad z \equiv \hat{z}\,. \tag{4.39}$$

The position vector of P is $\vec{X} = \vec{X}_0 + \hat{\vec{X}}$. The relationship between basis vectors in IF and the comoving orthonormal basis is given by a Lorentz transformation in the x-direction:

$$\begin{aligned}\vec{e}_{\hat{\mu}} &= \vec{e}_{\mu}\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \\ &= (\vec{e}_T, \vec{e}_X, \vec{e}_Y, \vec{e}_Z,)\begin{pmatrix}\cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix},\end{aligned} \tag{4.40}$$

where θ is the **rapidity** defined by

$$\tanh\theta \equiv \frac{U_0}{c}. \tag{4.41}$$

U_0 is the velocity of the "origin particle":

$$\begin{aligned} U_0 &= \frac{dX_0}{dT_0} = c\tanh\frac{gt}{c}. \\ \therefore \theta &= \frac{gt}{c}\,. \end{aligned} \tag{4.42}$$

So the basis vectors can be written as follows:

$$\begin{aligned}\vec{e}_{\hat{t}} &= \vec{e}_T\cosh\frac{gt}{c} + \vec{e}_X\sinh\frac{gt}{c}, \\ \vec{e}_{\hat{x}} &= \vec{e}_T\sinh\frac{gt}{c} + \vec{e}_X\cosh\frac{gt}{c}, \\ \vec{e}_{\hat{y}} &= \vec{e}_Y, \\ \vec{e}_{\hat{z}} &= \vec{e}_Z\,.\end{aligned} \tag{4.43}$$

The equation $\vec{X} = \vec{X}_0 + \hat{\vec{X}}$ can now be decomposed in IF:

$$\begin{aligned}&cT\vec{e}_T + X\vec{e}_X + Y\vec{e}_Y + Z\vec{e}_Z = \\ &\quad \frac{c^2}{g}\sinh\frac{gt}{c}\vec{e}_T + \frac{c^2}{g}\left(\cosh\frac{gt}{c} - 1\right)\vec{e}_X + x\sinh\frac{gt}{c}\vec{e}_T + x\cosh\frac{gt}{c}\vec{e}_X + y\vec{e}_Y + z\vec{e}_Z.\end{aligned} \tag{4.44}$$

This then gives the coordinate transformations:

$$
\begin{aligned}
T &= \frac{c}{g}\sinh\frac{gt}{c} + \frac{x}{c}\sinh\frac{gt}{c},\\
X &= \frac{c^2}{g}\left(\cosh\frac{gt}{c} - 1\right) + x\cosh\frac{gt}{c},\\
Y &= y,\\
Z &= z\\
\Rightarrow \quad \frac{gT}{c} &= \left(1+\frac{gx}{c^2}\right)\sinh\frac{gt}{c}\\
1+\frac{gX}{c^2} &= \left(1+\frac{gx}{c^2}\right)\cosh\frac{gt}{c}\ .
\end{aligned}
$$

Now dividing the last two of the above equations we get

$$
\frac{gT}{c} = \left(1+\frac{gX}{c^2}\right)\tanh\frac{gt}{c}\ , \tag{4.45}
$$

showing that the coordinate curves t = constant are straight lines in the T,X-frame passing through the point T = 0, X = $-\frac{c^2}{g}$. Using the identity $\cosh^2\theta - \sinh^2\theta = 1$ we get

$$
\left(1+\frac{gX}{c^2}\right)^2 - \left(\frac{gT}{c}\right)^2 = \left(1+\frac{gx}{c^2}\right)^2, \tag{4.46}
$$

showing that the coordinate curves x = constant are hypebolae in the T,X-diagram (Fig. 4.10).

The line element (the metric) gives

$$
\begin{aligned}
ds^2 &= -c^2dT^2 + dX^2 + dY^2 + dZ^2\\
&= -\left(1+\frac{gx}{c^2}\right)^2 c^2dt^2 + dx^2 + dy^2 + dz^2\ .
\end{aligned} \tag{4.47}
$$

ds^2 is an invariant quantity. Note: When the metric is diagonal the unit vectors are orthogonal.

Clocks at rest in the accelerated system:

$$
\begin{gathered}
dx = dy = dz = 0, \quad ds^2 = -c^2d\tau^2\\
\Downarrow\\
-c^2d\tau^2 = -\left(1+\frac{gx}{c^2}\right)^2 c^2dt^2\\
\Downarrow
\end{gathered}
$$

$$
\boxed{d\tau = \left(1+\frac{gx}{c^2}\right)dt}\ . \tag{4.48}
$$

Here $d\tau$ is the **proper time** and dt the **coordinate time**.

An observer in the accelerated system Σ experiences a gravitational field in the negative x-direction. When $x < 0$ then $d\tau < dt$. The coordinate clocks tick equally

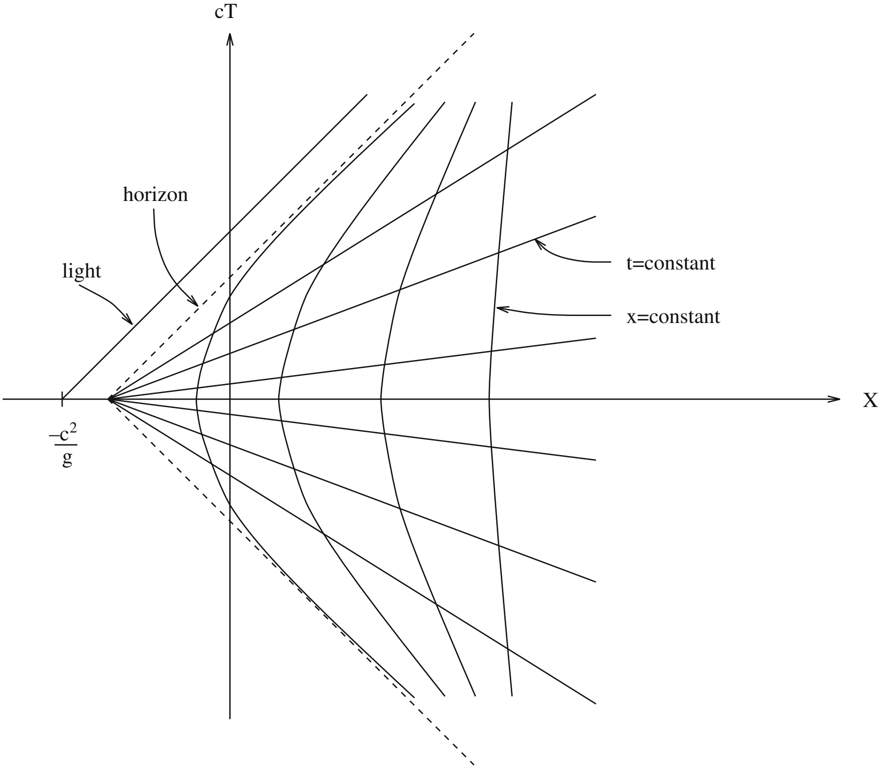

Fig. 4.10 The hyperbolically accelerated reference system

fast independent of their position. This implies that time passes slower further down in a gravitational field.

Consider a standard clock moving in the x-direction with velocity $v = dx/dt$. Then

$$\begin{aligned} -c^2 d\tau^2 &= -\left(1+\frac{gx}{c^2}\right)^2 c^2 dt^2 + dx^2 \\ &= -\left[\left(1+\frac{gx}{c^2}\right)^2 - \frac{v^2}{c^2}\right] c^2 dt^2 \, . \end{aligned} \tag{4.49}$$

Hence

$$d\tau = \sqrt{\left(1+\frac{gx}{c^2}\right)^2 - \frac{v^2}{c^2}}\, dt \, . \tag{4.50}$$

This expresses the combined effect of the gravitational and the kinematic time dilation.

Problems

4.1. *Relativistic rotating disc*

A disc rotates with constant angular velocity in ω in its own plane and around a fixed axis A. The axis is chosen to be the origin in a non-rotating Cartesian coordinate system (x, y). (The z-coordinate will be neglected from now on.) The motion of a given point on the disc can be expressed as

$$x = r\cos(\omega t + \phi), \tag{4.51}$$

$$y = r\sin(\omega t + \phi), \tag{4.52}$$

where (r, ϕ) are coordinates specifying the point at the disc.

(a) An observer is able to move on the disc and performs measurements of distance between neighbouring points at different locations on the disc. The measurements are performed when the observer is stationary with respect to the disc. The result is assumed to be the same as that measured in an inertial frame with the same velocity as the observer at the time of the measurement. The lengths measured by the observer are given by

$$d\ell^2 = f_1(r, \phi)dr^2 + f_2(r, \phi)d\phi^2. \tag{4.53}$$

Find $f_1(r, \phi)$ and $f_2(r, \phi)$.

We now assume that the observer measures the distance from the axis A to a point $(R, 0)$ along the line $\phi = 0$, by adding the result of measurements between neighbouring points. What is the result the observer finds?

Furthermore the observer measures the distance around the circle $r = R$. What is then found? In what way, based on this result, is it possible to deduce that the metric considered by the observer is non-Euclidean? Will the observer find a negative or positive curvature of the disc?

(b) We introduce the coordinates $(\tilde{x}, \tilde{y})$ that follow the rotating disc. They are given by

$$\tilde{x} = r\cos\phi, \tag{4.54}$$

$$\tilde{y} = r\sin\phi. \tag{4.55}$$

Find the invariant interval $ds^2 = dx^2 + dy^2 - c^2dt^2$ in terms of the coordinates $(\tilde{x}, \tilde{y}, t)$. Write down the relativistic Lagrangian of a free particle and find the equation of motion (EOM) expressed in the same coordinates. Show that the equations enter a non-relativistic form, with a Coriolis term and a centrifugal term, when we assume that $\omega r/c \ll 1$ and that the velocity of the particle satisfies $v/c \ll 1$.

(c) Light signals are sent from the axis A. How will the paths of the light be as seen from the $(\tilde{x}, \tilde{y})$-system? Draw a figure that illustrates this. A light signal with the frequency ν_0 is received by the observer in $r = R$, $\phi = 0$. Which frequency ν will be measured by the observer?

(d) We now assume that clocks are tightly packed around the disc $r = R$. The clocks are non-moving with respect to the disc, and are standard clocks, measuring proper time of the clock. We now want to synchronize the clocks and start out with a clock at the point $(R, 0)$. The clocks are then synchronized in the direction of increasing ϕ in the following way: When the clock is tuned at the point ϕ, the clock at the neighbouring point $\phi + d\phi$ is also tuned so that they show the same time at simultaneity in the instantaneous rest frame of the two clocks.
Show that there is a problem with synchronization when this process is performed around the entire circle, by the fact that the clock we started out with is no longer synchronous with the neighbouring clock which is tuned according to the synchronization process. Find the time difference between these two clocks.

(e) Locally around a point (r, ϕ, t) we can define an inertial system being an instantaneous rest frame of the point (r, ϕ) on the disc. We introduce an orthonormal set of basis vectors $\vec{e}_{\hat{\lambda}}$, $\vec{e}_{\hat{\eta}}$ and $\vec{e}_{\hat{\xi}}$ in this frame. The vector $\vec{e}_{\hat{\lambda}}$ points along the time axis of the system, $\vec{e}_{\hat{\xi}}$ points radially and $e_{\hat{\eta}}$ tangentially. Find the vectors expressed by $\vec{e}_t$, $\vec{e}_x$ and $\vec{e}_y$.
The path of a light signal from A is studied at the local inertial frames along the path. Find how the spatial direction of the light signal changes relative to the basis vectors $\vec{e}_{\hat{\xi}}$ and $\vec{e}_{\hat{\eta}}$ (outwards) along the paths. Is this result in accordance with what was earlier found on the path of the light in the $(\tilde{x}, \tilde{y})$-system?

4.2. *Free particle in a hyperbolic reference frame*
The metric for a 2-dimensional space is given by

$$ds^2 = -V^2 dU^2 + dV^2. \tag{4.56}$$

(a) Find the Euler–Lagrange equations for the motion of a free particle using this metric. Show that they admit the following solution:

$$\frac{1}{V} = \frac{1}{V_0} \cosh(U - U_0).$$

What is the physical interpretation of the constants V_0 and U_0?

(b) Show that these are straight lines in the coordinate system (t, x) given by

$$\begin{aligned} x &= V \cosh U, \\ t &= V \sinh U. \end{aligned} \tag{4.57}$$

Express the speed of the particle in terms of U_0, and its x-component at $t = 0$ in terms of V_0 and U_0. Find the interval ds^2 expressed in terms of x and t and show that the space in which the particle is moving is a Minkowski space with one time and one spatial dimension.

(c) Express the covariant component p_U of the momentum using $p_t = -E$ and $p_x = p$, and show that it is a constant of motion. How can this fact be directly extracted from the metric? Show further that the contravariant component p^U is not a constant of motion. Are p_V or p^V constants of motion?

4.3. *Uniformly accelerated system of reference*
We will now study a curved coordinate system (U,V) in a 2-dimensional Minkowski space. The connection with the Cartesian system (t,x) is given by

$$t = V \sinh(aU), \tag{4.58}$$
$$x = V \cosh(aU), \tag{4.59}$$

where a is a constant. (See Problem 4.2.)

(a) Draw the coordinate lines U and V in a (t,x)-diagram. Calculate the basis vectors $\vec{e}_U$ and $\vec{e}_V$ and draw them at some chosen points in the diagram. Find the metric $ds^2 = dx^2 - c^2dt^2$ expressed by U and V.
(b) We now assume that a particle has a path in spacetime so that it follows one of the curves $V =$ constant. Such a motion is called hyperbolic motion. Why?
Show that the particle has constant acceleration g along the path when the acceleration is measured in the instantaneous rest frame of the particle. Find the acceleration. Find also the velocity and accelereation of the particle in the stationary system (t,x).
(c) Show that at any point on the particle trajectory, the direction of the (U,V)-coordinate axis will overlap with the time and spatial axis of the instantaneous rest frame of the particle. Explain why it is possible to see from the line element that the V-coordinate measures length along the spatial axis, whereas the U-coordinate, which is the coordinate time, is in general not the proper time of the particle? For what value of V is the coordinate equal to the proper time?
The (U,V)-coordinate system can be considered as an attempt to construct, from the instantaneous rest frames along the path, a coordinate system covering the entire spacetime. Explain why this is not possible for the entire space. (Hint: There is a coordinate singularity at a certain distance from the trajectory of the particle.)
(d) A rod is moving in the direction of its own length. At the time $t = 0$ the rod is at rest, but still accelerated. The length of the rod measured in the stationary system is L at this time. The rod moves so that the forwards point of the rod has constant rest acceleration measured in the instantaneous rest frame.
We assume that the acceleration of the rod finds place so that the infinitesimal distance $d\ell$ between neighbouring points on the rod measured in the instantaneous rest frame are constant. Find the motion of the rear point of the rod in the stationary reference system. Why is there a maximal length of the rod, $L_{\max}$?
If the rear point of the rod has constant acceleration and the rod is accelerated as previously in this exercise, then is there a maximal value of L?
(e) A spaceship leaves the Earth at the time $t = 0$ and moves with a constant acceleration g, equal to the gravitational constant at the Earth, into space. Find how far the ship has travelled during 10 years of proper time of the ship.

Radiosignals are sent from the Earth towards the spaceship. Show that signals that are sent after a given time T will never reach the ship (even if the signals travel with the speed of light). Find T. At what time are the signals sent from the Earth if they reach the ship after 10 years (proper time of the ship)?
Calculate the frequency of the radiosignals received by the ship, given by the frequency ν_0 (emitter frequency) and the time t_0 (emitter time). Investigate the behaviour of the frequency when $t_0 \to T$.

4.4. *The projection tensor*
Let the metric tensor of the spacetime in a coordinate system K have components $g_{\mu\nu}$. An observer has a 4-velocity given by $\vec{u}$.
An arbitrary vector $\vec{a}$ can be decomposed into a component $\vec{a}_{\parallel}$ parallel to $\vec{u}$ and a component $\vec{a}_{\perp}$ orthogonal to $\vec{u}$, so that $\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$.

(a) Show that

$$\vec{a}_{\parallel} = (\vec{a}\cdot\vec{u})\vec{u}/u^2 = -(\vec{a}\cdot\vec{u})\vec{u}, \tag{4.60}$$
$$\vec{a}_{\perp} = \vec{a} + (\vec{a}\cdot\vec{u})\vec{u}. \tag{4.61}$$

Equation (4.61) can be rewritten by the *projection tensor*

$$P = \underline{I} + \vec{u}\otimes\underline{u}, \tag{4.62}$$

where $\underline{I}$ is the vectorial 1-form that can be written as

$$\underline{I} = \delta^{\mu}{}_{\nu}\vec{e}_{\mu}\otimes\underline{\omega}^{\nu} \tag{4.63}$$

and $\underline{u}$ has components $u_{\mu} = g_{\mu\nu}u^{\nu}$.
Show that Eq. (4.61) can be written as

$$\vec{a}_{\perp} = P(\ \ ,\vec{a}) \tag{4.64}$$

so that the components of $\vec{a}_{\perp}$ and $\vec{a}$ are related via

$$a^{\mu}_{\perp} = P^{\mu}{}_{\nu}a^{\nu}. \tag{4.65}$$

Since $\vec{u}$ is tangent vector of the world line of the observer, then $P^{\mu}{}_{\nu}a^{\nu}$ is the projection of $\vec{a}$ in the spatial plane of simultaneity orthogonal to the time vector in the local orthonormal basis of the observer.

(b) Assume that the observer is non-moving in K. Find the mixed and covariant components of P.

(c) Let $\vec{a}$ be the 4-acceleration of a particle. What kind of motion does the covariant equation $P^{\mu}{}_{\nu}\frac{da^{\nu}}{d\tau} = 0$ describe. Explain! (Hint: Find the time and space components of this equation. In an instantaneous rest frame of the particle, $d\vec{a}/d\tau = (g^2, d\vec{g}/d\tau)$ where the 3-vector $\vec{g}$ is the rest acceleration.)

(d) Consider an interval dx^μ in spacetime. It has a component orthogonal to a 4-velocity $\vec{u}$ given by $d\sigma^\mu = P^\mu{}_\alpha dx^\alpha$. The invariant spatial line element $d\ell$ is given by $d\ell^2 = d\sigma_\mu d\sigma^\mu$, and the components of the spatial metric tensor, i.e. the metric tensor in the spatial plane orthogonal to $\vec{u}$, is defined by $d\ell^2 = \gamma_{\mu\nu} dx^\mu dx^\nu$. Find $\gamma_{\mu\nu}$ given by $g_{\mu\nu}$ and u^α.
Find in particular γ_{ij} when $\vec{u}$ is the 4-velocity of an observer at rest in K.

Chapter 5
Covariant Differentiation

5.1 Differentiation of Forms

We must have a method of differentiation that maintains the anti-symmetry, thus making sure that what we end up with after differentiation is still a form.

5.1.1 *Exterior Differentiation*

The exterior derivative of a 0-form, i.e. a scalar function, f, is given by

$$\underline{d}f = \frac{\partial f}{\partial x^\mu}\,\underline{\omega}^\mu = f_{,\mu}\underline{\omega}^\mu \;, \tag{5.1}$$

where $\underline{\omega}^\mu$ are coordinate basis forms

$$\underline{\omega}^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu \;. \tag{5.2}$$

We then (in general) get

$$\underline{\omega}^\mu = \delta^\mu_\nu\underline{\omega}^\nu = \frac{\partial x^\mu}{\partial x^\nu}\underline{\omega}^\nu = \underline{d}x^\mu \;. \tag{5.3}$$

In coordinate basis we can always write the basis forms as exterior derivatives of the coordinates. The differential dx^μ is given by

$$\underline{d}x^\mu(d\vec{r}) = dx^\mu \;, \tag{5.4}$$

where $d\vec{r}$ is an infinitesimal position vector. $\underline{d}x^\mu$ are *not* infinitesimal quantities. In coordinate basis the exterior derivative of a p-form

$$\underline{\alpha} = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}\underline{d}x^{\mu_1}\wedge\cdots\wedge\underline{d}x^{\mu_p} \tag{5.5}$$

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will have the following component form:

$$\underline{d}\,\underline{\alpha} = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p,\mu_0}\underline{d}x^{\mu_0}\wedge\underline{d}x^{\mu_1}\wedge\cdots\wedge\underline{d}x^{\mu_p}\ , \tag{5.6}$$

where $\mu_0 \equiv \frac{\partial}{\partial x^{\mu_0}}$. **The exterior derivative of a p-form is a $(p+1)$-form.**

Consider the exterior derivative of a p-form $\underline{\alpha}$:

$$\underline{d\alpha} = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p,\mu_0}\underline{d}x^{\mu_0}\wedge\cdots\wedge\underline{d}x^{\mu_p}\ . \tag{5.7}$$

Let $(\underline{d\alpha})_{\mu_0\cdots\mu_p}$ be the form components of $\underline{d\alpha}$. They must, by definition, be antisymmetric under an arbitrary interchange of indices:

$$\begin{aligned} \underline{d\alpha} &= \frac{1}{(p+1)!}(\underline{d\alpha})_{\mu_0\cdots\mu_p}\underline{d}x^{\mu_0}\wedge\cdots\wedge\underline{d}x^{\mu_p} \\ \text{which by Eq. (5.7)}\qquad &= \frac{1}{p!}\alpha_{[\alpha_{\mu_1}\cdots\mu_p,\mu_0]}\underline{d}x^{\mu_0}\wedge\cdots\wedge\underline{d}x^{\mu_p}\ . \end{aligned}$$

$$\boxed{\therefore (\underline{d\alpha})_{\mu_0\cdots\mu_p} = (p+1)\alpha_{[\mu_1\cdots\mu_p,\mu_0]}}\ . \tag{5.8}$$

The form equation $\underline{d\alpha} = 0$ in component form is

$$\alpha_{[\mu_1\cdots\mu_p,\mu_0]} = 0\ . \tag{5.9}$$

Example 5.1.1 (Outer product of 1-forms in 3-space)

$$\begin{aligned} \underline{\alpha} &= \alpha_i\underline{d}x^i, \qquad x^i = (x,y,z), \\ \underline{d\alpha} &= \alpha_{i,j}\underline{d}x^j\wedge\underline{d}x^i\ . \end{aligned} \tag{5.10}$$

Also, assume that $\underline{d\alpha} = 0$. The corresponding component equation is

$$\begin{aligned} &\alpha_{[i,j]} = 0 \qquad \Rightarrow \qquad \alpha_{i,j} - \alpha_{j,i} = 0 \\ \Rightarrow \frac{\partial\alpha_x}{\partial y} - \frac{\partial\alpha_y}{\partial x} = 0,\quad &\frac{\partial\alpha_x}{\partial z} - \frac{\partial\alpha_z}{\partial x} = 0,\quad \frac{\partial\alpha_y}{\partial z} - \frac{\partial\alpha_z}{\partial y} = 0, \end{aligned} \tag{5.11}$$

which corresponds to

$$\nabla\times\vec{\alpha} = 0\ . \tag{5.12}$$

The outer product of an outer product:

$$\begin{aligned} d^2\underline{\alpha} &\equiv \underline{d}(\underline{d\alpha}), \\ d^2\underline{\alpha} &= \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p,\nu_1\nu_2}\underline{d}x^{\nu_2}\wedge\underline{d}x^{\nu_1}\wedge\cdots\wedge\underline{d}x^{\mu_p}, \end{aligned} \tag{5.13}$$

$$\boxed{{}_{,\nu_1\nu_2} \equiv \frac{\partial^2}{\partial x^{\nu_1}\partial x^{\nu_2}}}. \tag{5.14}$$

Since

$$_{,\nu_1\nu_2} \equiv \frac{\partial^2}{\partial x^{\nu_1}\partial x^{\nu_2}} =_{,\nu_2\nu_1} \equiv \frac{\partial^2}{\partial x^{\nu_2}\partial x^{\nu_1}} \tag{5.15}$$

summation over ν_1 and ν_2 which are symmetric in $\alpha_{\mu_1\cdots\mu_p,\nu_1\nu_2}$ and antisymmetric in the basis gives **Poincaré's lemma** (valid only for **scalar fields**):

$$\boxed{\underline{d}^2\underline{\alpha} = 0}. \tag{5.16}$$

This corresponds to the vector equation

$$\nabla\cdot(\nabla\times\vec{A}) = 0\,. \tag{5.17}$$

Let $\underline{\alpha}$ be a p-form and $\underline{\beta}$ be a q-form. Then

$$\underline{d}(\underline{\alpha}\wedge\underline{\beta}) = \underline{d\alpha}\wedge\underline{\beta} + (-1)^p\underline{\alpha}\wedge\underline{d\beta}\,. \tag{5.18}$$

5.1.2 Covariant Derivative

The general theory of relativity contains a **covariance principle** which states that all equations expressing laws of nature must have the same form irrespective of the coordinate system in which they are derived. This is achieved by writing all equations in terms of tensors. Let us see if the partial derivative of vector components transform as tensor components. Given a vector $\vec{A} = A^\mu\vec{e}_\mu = A^{\mu'}\vec{e}_{\mu'}$ with the transformation of basis given by

$$\frac{\partial}{\partial x^{\nu\prime}} = \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial}{\partial x^\nu}\,. \tag{5.19}$$

So that

$$\begin{aligned}
A^{\mu'}_{\ ,\nu'} &\equiv \frac{\partial}{\partial x^{\nu\prime}}\left(A^{\mu'}\right)\\
&= \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial}{\partial x^\nu}\left(A^{\mu'}\right)\\
&= \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial}{\partial x^\nu}\left(\frac{\partial x^{\mu\prime}}{\partial x^\mu}A^\mu\right)\\
&= \frac{\partial x^\nu}{\partial x^{\nu\prime}}\frac{\partial x^{\mu\prime}}{\partial x^\mu}A^\mu_{\ ,\nu} + \frac{\partial x^\nu}{\partial x^{\nu\prime}}A^\mu\frac{\partial^2 x^{\mu'}}{\partial x^\nu\partial x^\mu}\,.
\end{aligned} \tag{5.20}$$

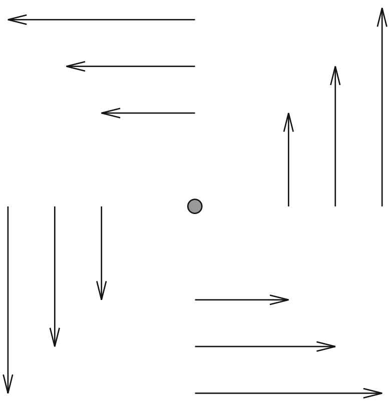
The first term corresponds to a tensorial transformation. The existence of the last term shows that $A^{\mu}_{,\nu}$ does not, in general, transform as the components of a tensor. Note that $A^{\mu}_{,\nu}$ will transform as a tensor under linear transformations such as the Lorentz transformations.

The partial derivative must be generalized so as to ensure that when it is applied to tensor components it produces tensor components.

Example 5.1.2 (The derivative of a vector field with rotation) We have a vector field

$$\vec{A} = kr\vec{e}_{\theta}.$$

The chain rule for derivation gives



$$\frac{d}{d\tau} = \frac{\partial}{\partial x^{\nu}} \cdot \frac{dx^{\nu}}{d\tau} = u^{\nu} \frac{\partial}{\partial x^{\nu}},$$

$$\begin{aligned} \frac{d\vec{A}}{d\tau} &= u^{\nu}\left(A^{\mu}\vec{e}_{\mu}\right)_{,\nu} \\ &= u^{\nu}\left(A^{\mu}_{,\nu}\vec{e}_{\mu} + A^{\mu}\vec{e}_{\mu,\nu}\right) . \end{aligned}$$

The change of the vector field with a displacement along a coordinate curve is expressed by

$$\frac{\partial \vec{A}}{\partial x^{\nu}} = \vec{A}_{,\nu} = A^{\mu}_{,\nu}\vec{e}_{\mu} + A^{\mu}\vec{e}_{\mu,\nu}.$$

The change in $\vec{A}$ with the displacement in the θ-direction is

$$\frac{\partial \vec{A}}{\partial \theta} = A^{\mu}_{,\theta}\vec{e}_{\mu} + A^{\mu}\vec{e}_{\mu,\theta}.$$

For our vector field, with $A^{r} = 0$, we get

$$\frac{\partial \vec{A}}{\partial \theta} = \underbrace{A^{\theta}_{,\theta}}_{=0} \vec{e}_{\theta} + A^{\theta} \vec{e}_{\theta,\theta}$$

and since $A^{\theta}_{,\theta} = 0$ because $A^{\theta} = kr$ we end up with

$$\frac{\partial \vec{A}}{\partial \theta} = A^{\theta} \vec{e}_{\theta,\theta} = kr\vec{e}_{\theta,\theta}.$$

We now need to calculate the derivative of $\vec{e}_{\theta}$. We have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Using $\vec{e}_{\mu} = \frac{\partial}{\partial x^{\mu}}$ we can write

$$\begin{aligned}
\vec{e}_{\theta} &= \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} \\
&= -r\sin\theta \vec{e}_x + r\cos\theta \vec{e}_y,
\end{aligned}$$

$$\vec{e}_r = \frac{\partial}{\partial r} = \cos\theta \vec{e}_x + \sin\theta \vec{e}_y,$$

which gives

$$\begin{aligned}
\vec{e}_{\theta,\theta} &= -r\cos\theta \vec{e}_x - r\sin\theta \vec{e}_y \\
&= -r(\cos\theta \vec{e}_x + \sin\theta \vec{e}_y) = -r\vec{e}_r\ .
\end{aligned}$$

This gives us finally

$$\frac{\partial \vec{A}}{\partial \theta} = -kr^2 \vec{e}_r.$$

Thus $\frac{\partial \vec{A}}{\partial \theta} \neq 0$ even if $\vec{A} = A^{\theta}\vec{e}_{\theta}$ and $A^{\theta}_{,\theta} = 0$.

5.2 The Christoffel Symbols

The covariant derivative was introduced by Christoffel to be able to differentiate tensor fields. It is defined in coordinate basis by generalizing the partial derivative $A^{\mu}_{,\nu}$ to a derivative written as $A^{\mu}_{;\nu}$ and which transforms tensorially,

$$A^{\mu'}_{;\nu'} \equiv \frac{\partial x^{\mu'}}{\partial x^{\mu}} \cdot \frac{\partial x^{\nu}}{\partial x^{\nu'}} A^{\mu}_{;\nu}\ . \tag{5.21}$$

The covariant derivative of the contravariant vector components is written as

$$A^{\mu}_{;\nu} \equiv A^{\mu}_{,\nu} + A^{\alpha}\Gamma^{\mu}_{\alpha\nu}\ . \tag{5.22}$$

This equation defines the Christoffel symbols $\Gamma^{\mu}_{\alpha\nu}$, which are also called the "connection coefficients in coordinate basis". From the transformation formulae for the two first terms it follows that the Christoffel symbols transform as

$$\Gamma^{\alpha'}_{\mu'\nu'} = \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \Gamma^{\alpha}_{\mu\nu} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x^{\mu'} \partial x^{\nu'}} \,. \tag{5.23}$$

The Christoffel symbols do not transform as tensor components. It is possible to cancel *all* Christoffel symbols by transforming into a locally Cartesian coordinate system which is comoving in a locally non-rotating reference frame in free fall. Such coordinates are known as **Gaussian coordinates**.

In general relativity theory an inertial frame is defined as a non-rotating frame in free fall. The Christoffel symbols are 0 (zero) in a locally Cartesian coordinate system which is comoving in a local inertial frame. Local Gaussian coordinates are indicated with a bar over the indices, giving

$$\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = 0 \,. \tag{5.24}$$

A transformation from local Gaussian coordinates to any coordinates leads to

$$\Gamma^{\alpha'}_{\mu'\nu'} = \frac{\partial x^{\alpha'}}{\partial x^{\bar{\alpha}}} \frac{\partial^2 x^{\bar{\alpha}}}{\partial x^{\mu'} \partial x^{\nu'}} \,. \tag{5.25}$$

This equation shows that the Christoffel symbols are symmetric in the two lower indices, i.e.

$$\Gamma^{\alpha'}_{\mu'\nu'} = \Gamma^{\alpha'}_{\nu'\mu'} \,. \tag{5.26}$$

Example 5.2.1 (The Christoffel symbols in plane polar coordinates)

$$\begin{aligned} x &= r\cos\theta, & y &= r\sin\theta, \\ r &= \sqrt{x^2+y^2}, & \theta &= \arctan\frac{y}{x}, \end{aligned}$$

$$\begin{aligned} &\frac{\partial x}{\partial r} = \cos\theta, \quad &&\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad &&\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad &&\frac{\partial r}{\partial y} = \sin\theta, \\ &\frac{\partial y}{\partial r} = \sin\theta, \quad &&\frac{\partial y}{\partial \theta} = r\cos\theta, \quad &&\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \quad &&\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}, \end{aligned}$$

$$\begin{aligned} \Gamma^{r}_{\theta\theta} &= \frac{\partial r}{\partial x}\frac{\partial^2 x}{\partial \theta^2} + \frac{\partial r}{\partial y}\frac{\partial^2 y}{\partial \theta^2} \\ &= \cos\theta(-r\cos\theta) + \sin\theta(-r\sin\theta) \\ &= -r(\cos\theta^2 + \sin\theta^2) = -r, \end{aligned}$$

$$\begin{aligned}\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} &= \frac{\partial\theta}{\partial x}\frac{\partial^2 x}{\partial\theta\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial^2 y}{\partial\theta\partial r} \\ &= -\frac{\sin\theta}{r}(-\sin\theta) + \frac{\cos\theta}{r}(\cos\theta) \\ &= \frac{1}{r}\ .\end{aligned}$$

The geometrical interpretation of the covariant derivative was given by Levi-Civita.

Consider a curve S in any (e.g. curved) space. It is parameterized by λ, i.e. $x^{\mu} = x^{\mu}(\lambda)$. λ is invariant and chosen to be the curve length.

The tangent vector field of the curve is $\vec{u} = (dx^{\mu}/d\lambda)\vec{e}_{\mu}$. The curve passes through a vector field $\vec{A}$. The covariant directional derivative of the vector field along the curve is defined as

$$\nabla_{\vec{u}}\vec{A} = \frac{d\vec{A}}{d\lambda} \equiv A^{\mu}_{;\nu}\frac{dx^{\nu}}{d\lambda}\vec{e}_{\mu} = A^{\mu}_{;\nu}u^{\nu}\vec{e}_{\mu}\ . \tag{5.27}$$

The vectors in the vector field are said to be connected by parallel transport along the curve if

$$A^{\mu}_{;\nu}u^{\nu} = 0.$$

$$\vec{u} = \frac{dx^{\mu}}{d\lambda}\vec{e}_{\mu}\ . \tag{5.28}$$

According to the geometrical interpretation of Levi-Civita, the covariant directional derivative is

$$\nabla_{\vec{u}}\vec{A} = A^{\mu}_{;\nu}u^{\nu}\vec{e}_{\mu} = \lim_{\Delta\lambda\to 0}\frac{\vec{A}_{\|}(\lambda+\Delta\lambda) - \vec{A}(\lambda)}{\Delta\lambda}\ , \tag{5.29}$$

where $\vec{A}_{\|}(\lambda+\Delta\lambda)$ means the vector $\vec{A}$ parallel transported from Q to P (Fig. 5.1).

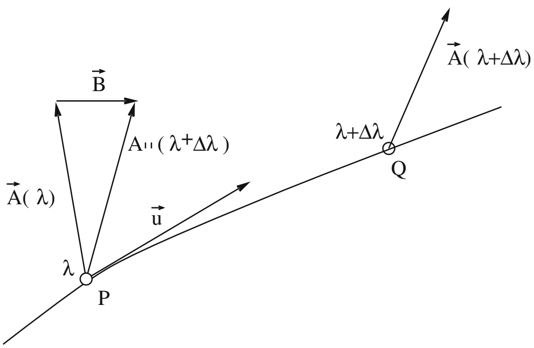


Fig. 5.1 Parallel transport from P to Q. The vector $\vec{B} = A^{\mu}_{;\nu}u^{\nu}\Delta\lambda\vec{e}_{\mu}$

5.3 Geodesic Curves

Definition 5.3.1 (Geodesic curves) A geodesic curve is defined in such a way that the vectors of the tangent vector field of the curve are connected by parallel transport.

This definition says that geodesic curves are "as straight as possible".
If vectors in a vector field $\vec{A}(\lambda)$ are connected by parallel transport by a displacement along a vector $\vec{u}$, we have $A^{\mu}_{;\nu}u^{\nu} = 0$. For geodesic curves, we then have

$$\boxed{u^{\mu}_{;\nu}u^{\nu} = 0}\,, \tag{5.30}$$

which is the *geodesic equation*.

$$(u^{\mu}_{,\nu} + \Gamma^{\mu}_{\alpha\nu}u^{\alpha})u^{\nu} = 0\,. \tag{5.31}$$

Then we are using that $\frac{d}{d\lambda} \equiv \frac{dx^{\nu}}{d\lambda}\frac{\partial}{\partial x^{\nu}} = u^{\nu}\frac{\partial}{\partial x^{\nu}}$:

$$\frac{du^{\mu}}{d\lambda} = u^{\nu}\frac{\partial u^{\mu}}{\partial x^{\nu}} = u^{\nu}u^{\mu}_{,\nu}\,. \tag{5.32}$$

The geodesic equation can also be written as

$$\frac{du^{\mu}}{d\lambda} + \Gamma^{\mu}_{\alpha\nu}u^{\alpha}u^{\nu} = 0\,. \tag{5.33}$$

Usual notation: $\dot{} = \frac{d}{d\lambda}$

$$u^{\mu} = \frac{dx^{\mu}}{d\lambda} = \dot{x}^{\mu}\,, \tag{5.34}$$

$$\boxed{\ddot{x}^{\mu} + \Gamma^{\mu}_{\alpha\nu}\dot{x}^{\alpha}\dot{x}^{\nu} = 0}\,. \tag{5.35}$$

Geodesic curves on a flat surface and on a spherical surface are shown in Figs. 5.2 and 5.3, respectively.

Example 5.3.1 Inserting the Christoffel symbols $\Gamma^{x}_{tt} = (1 + \frac{gx}{c^2})g$ from Example (5.5.3) into the geodesic equation for a vertical geodesic curve in a hyperbolically accelerated reference frame, we get

$$\ddot{x} + \left(1 + \frac{gx}{c^2}\right)g\dot{t}^2 = 0.$$

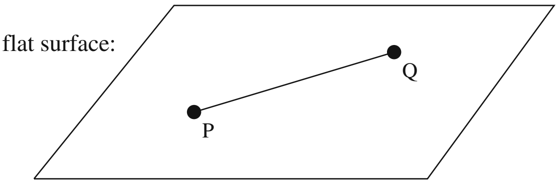


Fig. 5.2 On a flat surface, the geodesic curve is the minimal distance between P and Q

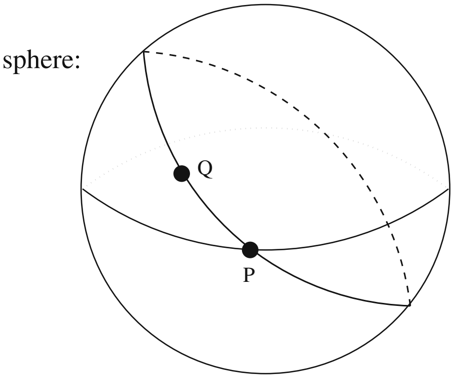


Fig. 5.3 On a sphere, the geodesic curves are great circles

5.4 The Covariant Euler–Lagrange Equations

Let a particle have a world line (in spacetime) between two points (events) P_1 and P_2. Let the curves be described by an invariant parameter λ (proper time τ is used for particles with a rest mass).

The Lagrange function is a function of coordinates and their derivatives,

$$L = L(x^\mu, \dot{x}^\mu), \qquad \dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \,. \tag{5.36}$$

(Note: if $\lambda = \tau$ then $\dot{x}^\mu$ are the 4-velocity components.)

The action integral is $S = \int L(x^\mu, \dot{x}^\mu)d\lambda$. The principle of extremal action (Hamiltons principle): The world line of a particle is determined by the condition that S shall be extremal for all infinitesimal1 variations of curves which keep P_1 and P_2 rigid, i.e.

$$\delta \int_{\lambda_1}^{\lambda_2} L(x^\mu, \dot{x}^\mu)d\lambda = 0 \,, \tag{5.37}$$

where λ_1 and λ_2 are the parameter values at P_1 and P_2 (Fig. 5.4). For all the variations the following condition applies:

$$\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0 \,. \tag{5.38}$$

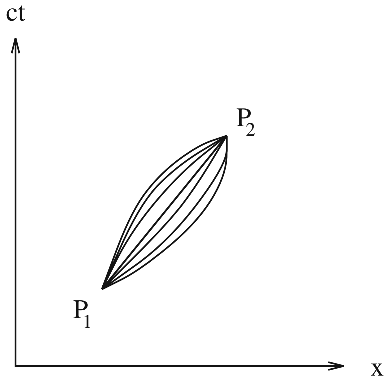


Fig. 5.4 Different world lines connecting P_1 and P_2 in a Minkowski diagram

We write Eq. (5.37) as

$$\delta \int_{\lambda_1}^{\lambda_2} L d\lambda = \int_{\lambda_1}^{\lambda_2} \left[\frac{\partial L}{\partial x^\mu} \delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu} \delta \dot{x}^\mu \right] d\lambda \ . \tag{5.39}$$

Partial integration of the last term:

$$\int_{\lambda_1}^{\lambda_2} \frac{\partial L}{\partial \dot{x}^\mu} \delta \dot{x}^\mu d\lambda = \left[\frac{\partial L}{\partial \dot{x}^\mu} \delta x^\mu \right]_{\lambda_1}^{\lambda_2} - \int_{\lambda_1}^{\lambda_2} \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) \delta x^\mu d\lambda \ . \tag{5.40}$$

Due to the conditions $\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0$ the first term becomes zero. Then we have

$$\delta S = \int_{\lambda_1}^{\lambda_2} \left[\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) \right] \delta x^\mu d\lambda \ . \tag{5.41}$$

The world line the particle follows is determined by the condition $\delta S = 0$ for any variation δx^μ. Hence, the world line of the particle must be given by

$$\boxed{\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) = 0} \ . \tag{5.42}$$

These are the covariant **Euler–Lagrange** equations.

The canonical momentum p_μ conjugated to a coordinate x^μ is defined as

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} \ . \tag{5.43}$$

The Lagrange equations can now be written as

$$\boxed{\frac{dp_\mu}{d\lambda} = \frac{\partial L}{\partial x^\mu} \quad \text{or} \quad \dot{p}_\mu = \frac{\partial L}{\partial x^\mu}} \ . \tag{5.44}$$

A coordinate which the Lagrange function does not depend on is known as a **cyclic coordinate**. Hence, $\frac{\partial L}{\partial x^\mu} = 0$ for a cyclic coordinate. From this follows:

> The canonical momentum conjugated to a cyclic coordinate is a **constant of motion**.

That is, $p_\mu = C$ (constant) if x^μ is cyclic.

A free particle in spacetime (curved spacetime includes gravitation) has the Lagrange function

$$L = \frac{1}{2}\vec{u}\cdot\vec{u} = \frac{1}{2}\dot{x}_\mu \dot{x}^\mu = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \; . \tag{5.45}$$

An integral of the Lagrange equations is obtained readily from the 4-velocity identity:

$$\begin{cases} \dot{x}_\mu \dot{x}^\mu = -c^2 & \text{for a particle with rest mass} \\ \dot{x}_\mu \dot{x}^\mu = 0 & \text{for light} \end{cases} \; . \tag{5.46}$$

The line element is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\lambda^2 = 2L d\lambda^2 \; . \tag{5.47}$$

Thus the Lagrange function of a free particle is obtained from the line element.

5.5 Application of the Lagrangian Formalism to Free Particles

To describe the motion of a free particle, we start by setting up the line element of the spacetime in the chosen coordinate system. There are coordinates on which the metric does not depend. For example, given axial symmetry we may choose the angle θ which is a cyclic coordinate here and the conjugate (covariant) impulse P_θ is a constant of the motion (the orbital spin of the particle). If, in addition, the metric is time independent (**stationary metric**) then t is also cyclic and p_t is a constant of the motion (the mechanical energy of the particle).

A **static metric** is time independent and unchanged under time reversal (i.e. $t \to -t$). A stationary metric changed under time reversal. Examples of static metrics are Minkowski and hyperbolically accelerated frames. The rotating cylindrical coordinate system is stationary.

5.5.1 Equation of Motion from Lagrange's Equations

The Lagrange function for a free particle is

$$L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \; , \tag{5.48}$$

where $g_{\mu\nu} = g_{\mu\nu}(x^\lambda)$. And the Lagrange equations are

$$\begin{aligned}\frac{\partial L}{\partial x^\beta} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\beta}\right) &= 0, \\ \frac{\partial L}{\partial \dot{x}^\beta} &= g_{\beta\nu}\dot{x}^\nu, \\ \frac{\partial L}{\partial x^\beta} &= \frac{1}{2} g_{\mu\nu,\beta}\dot{x}^\mu \dot{x}^\nu .\end{aligned} \tag{5.49}$$

$$\begin{aligned}\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\beta}\right) \equiv \left(\frac{\partial L}{\partial \dot{x}^\beta}\right)^{\bullet} &= \dot{g}_{\beta\nu}\dot{x}^\nu + g_{\beta\nu}\ddot{x}^\nu \\ &= g_{\beta\nu,\mu}\dot{x}^\mu \dot{x}^\nu + g_{\beta\nu}\ddot{x}^\nu \quad .\end{aligned} \tag{5.50}$$

Now, Eqs. (5.50) and (5.49) together give

$$\frac{1}{2} g_{\mu\nu,\beta}\dot{x}^\mu \dot{x}^\nu - g_{\beta\nu,\mu}\dot{x}^\mu \dot{x}^\nu - g_{\beta\nu}\ddot{x}^\nu = 0 . \tag{5.51}$$

The second term on the left-hand side of (5.51) may be rewritten making use of the fact that $\dot{x}^\mu \dot{x}^\nu$ is symmetric in $\mu\nu$, as follows:

$$\begin{aligned} g_{\beta\nu,\mu}\dot{x}^\mu \dot{x}^\nu &= \frac{1}{2}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu})\dot{x}^\mu \dot{x}^\nu \\ \Rightarrow \; g_{\beta\nu}\ddot{x}^\nu + \frac{1}{2}(g_{\beta\mu,\nu} &+ g_{\beta\nu,\mu} - g_{\mu\nu,\beta})\dot{x}^\mu \dot{x}^\nu = 0 .\end{aligned} \tag{5.52}$$

Finally, since we are free to multiply (5.52) through by $g^{\alpha\beta}$, we can isolate $\ddot{x}^\alpha$ to get the equation of motion in a particularly elegant and simple form:

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 , \tag{5.53}$$

where the **Christoffel symbols** $\Gamma^\alpha_{\mu\nu}$ in (5.53) are defined by

$$\Gamma^\alpha_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) . \tag{5.54}$$

Comparison with Eq. (5.35) shows that Eq. (5.53) describes a **geodesic** curve. Hence, free particles follow geodesic **curves** in spacetime.

5.5.2 Geodesic World Lines in Spacetime

Consider two time-like curves between two events in spacetime. In Fig. 5.5 they are drawn in a Minkowski diagram which refers to an inertial reference frame.

The general interpretation of the line element for a time-like interval is: The spacetime distance between O and P (see Fig. 5.5) equals the proper time-interval

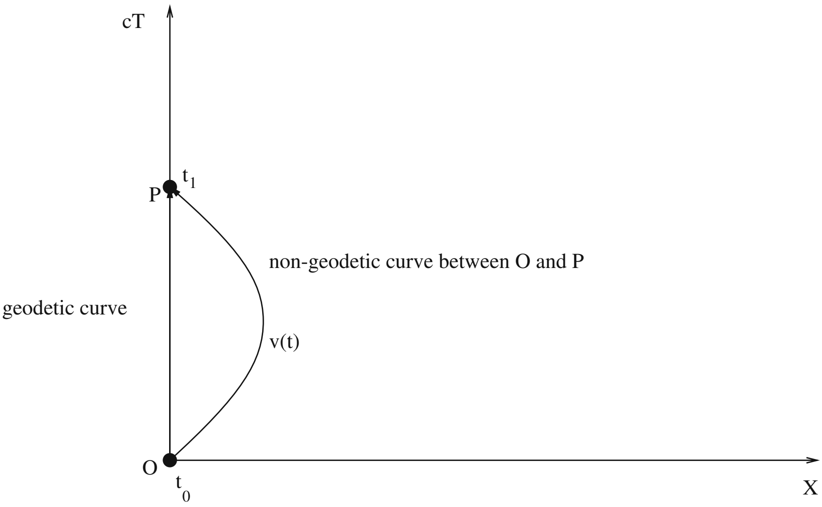


Fig. 5.5 Time-like curves in spacetime

between two events O and P measured on a clock moving in a such way that it is present both at O and P.

$$\boxed{ds^2 = -c^2 d\tau^2}\,, \tag{5.55}$$

The proper time interval between two events at coordinate times T_0 and T_1 are

$$\tau_{0-1} = \int_{T_0}^{T_1} \sqrt{1 - \frac{v^2(T)}{c^2}} dT\ . \tag{5.56}$$

We can see that τ_{0-1} is maximal along the geodesic curve with $v(T) = 0$. Time-like geodesic curves in spacetime have maximal distance between two points.

Example 5.5.1 (How geodesics in spacetime can give parabolas in space) A geodesic curve between two events O and P has maximal proper time. Consider the last expression in Sect. 4.2 of the proper time-interval of a particle with position x and velocity v in a gravitational field with acceleration of gravity g:

$$d\tau = dt\sqrt{\left(1 + \frac{gx}{c^2}\right)^2 - \frac{v^2}{c^2}}.$$

This expression shows that the proper time of the particle proceeds faster the higher up in the field the particle is, and it proceeds slower the faster the particle moves. Consider Fig. 5.6. The path a free particle follows between the events O and P is a compromise between moving as slowly as possible in space, in order to keep the velocity-dependent time dilation small, and moving through regions high up in the gravitational field, in order to prevent the slow proceeding of proper time far down. However, if the particle moves too high up, its velocity becomes so large that it

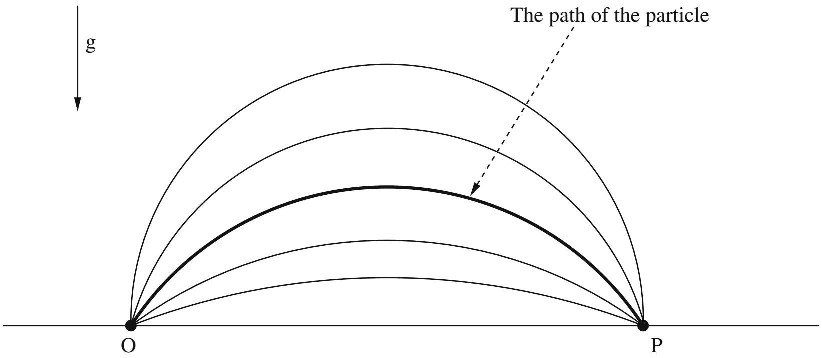


Fig. 5.6 The particle moves between two events O and P at fixed points in time. The path chosen by the particle between O and P is such that the proper time taken by the particle between these two events is as large as possible. Thus the goal of the particle is to follow a path such that its comoving standard clocks goes as fast as possible. If the particle follows the horizontal line between O and P it goes as slowly as possible and the kinematical time dilation is as small as possible. Then the slowing down of its comoving standard clocks due to the kinematical time dilation is as small as possible, but the particle is far down in the gravitational field and its proper time goes slowly for that reason. Paths further up lead to a greater rate of proper time. But above the curve drawn as a thick line, the kinematical time dilation will dominate, and the proper time proceeds more slowly

proceeds slower again. The compromise between kinematic and gravitational time dilation which gives maximal proper time between O and P is obtained for the thick curve in Fig. 5.6. This is the curve followed by a free particle between the events O and P.

We shall now deduce the mathematical expression of what has been said above. Time-like geodesic curves are curves with maximal proper time, i.e.

$$\tau = \int_0^{\tau_1} \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\tau$$

is maximal for a geodesic curve. However, the action

$$J = -2\int_0^{\tau_1} L d\tau = -\int_0^{\tau_1} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\tau$$

is maximal for the same curves and this gives an easier calculation.

In the case of a vertical curve in a hyperbolically accelerated reference frame the Lagrangian is

$$L = \frac{1}{2}\left(-\left(1+\frac{gx}{c^2}\right)^2 \dot{t}^2 + \frac{\dot{x}^2}{c^2}\right) . \tag{5.57}$$

Using the Euler–Lagrange equations now gives

$$\ddot{x} + \left(1+\frac{gx}{c^2}\right) g\dot{t}^2 = 0,$$

which is the equation of the geodesic curve in Example 5.3.1.

Since spacetime is flat, the equation represents straight lines in spacetime. The projection of such curves into the three space of arbitrary inertial frames gives straight paths in 3-space, in accordance with Newton's 1st law. However, projecting it into an accelerated frame where the particle also has a horizontal motion, and taking the Newtonian limit, one finds the parabolic path of projectile motion.

Example 5.5.2 (Spatial geodesics described in the reference frame of a rotating disc)

In Fig. 5.7, we see a rotating disc. We can see two geodesic curves between P_1 and P_2. The dashed line is the geodesic for the non-rotating disc. The other curve is a geodesic for the 3-space of a rotating reference frame. We can see that the geodesic is curved inward when the disc is rotating. The curve has to curve **inward** since the measuring rods are longer there (because of Lorentz contraction). Thus, the minimum distance between P_1 and P_2 will be achieved by an inwardly bent curve.

We will show this mathematically, using the Lagrangian equations. The line element for the space $d\hat{t} = dz = 0$ of the rotating reference frame is

$$dl^2 = dr^2 + \frac{r^2 d\theta^2}{1 - \frac{r^2\omega^2}{c^2}} .$$

Lagrangian function:

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{r^2\dot{\theta}^2}{1 - \frac{r^2\omega^2}{c^2}} .$$

We will also use the identity

$$\dot{r}^2 + \frac{r^2\dot{\theta}^2}{1 - \frac{r^2\omega^2}{c^2}} = 1 . \tag{5.58}$$

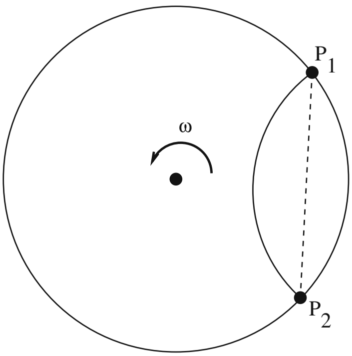


Fig. 5.7 Geodesic curves on a non-rotating (*dashed line*) and rotating (*solid line*) disc

(We got this from using $\vec{u}\cdot\vec{u}=1$.) We see that θ is cyclic ($\frac{\partial L}{\partial \theta}=0$), implying

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{r^2\dot{\theta}}{1-\frac{r^2\omega^2}{c^2}} = \text{constant.}$$

This gives

$$\dot{\theta} = \left(1-\frac{r^2\omega^2}{c^2}\right)\frac{p_\theta}{r^2} = \frac{p_\theta}{r^2} - \frac{\omega^2 p_\theta}{c^2}\,. \tag{5.59}$$

Inserting Eq. (5.59) into Eq. (5.58) we get

$$\dot{r}^2 = 1 + \frac{\omega^2 p_\theta^2}{c^2} - \frac{p_\theta^2}{r^2}\,. \tag{5.60}$$

This gives us the equation of the geodesic curve between P_1 and P_2 (Fig. 5.8):

$$\frac{\dot{r}}{\dot{\theta}} = \pm\frac{dr}{d\theta} = \frac{r^2\sqrt{1+\frac{\omega^2 p_\theta^2}{c^2}-\frac{p_\theta^2}{r^2}}}{p_\theta\left(1-\frac{r^2\omega^2}{c^2}\right)}\,. \tag{5.61}$$

Boundary conditions:

$$\dot{r}=0, r=r_0,\ \text{for } \theta=0.$$

Inserting this into Eq. (5.60) gives

$$\frac{p_\theta}{r_0} = \sqrt{1+\frac{p_\theta^2\omega^2}{c^2}}\,. \tag{5.62}$$

Rearranging Eq. (5.61), using Eq. (5.62), gives

$$\frac{dr}{r\sqrt{r^2-r_0^2}} - \frac{\omega^2}{c^2}\frac{rdr}{\sqrt{r^2-r_0^2}} = \frac{d\theta}{r_0}.$$

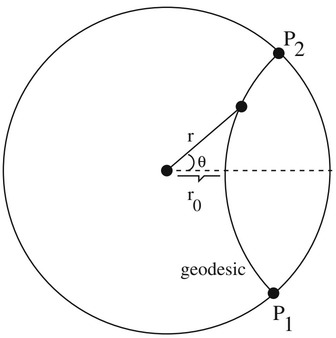


Fig. 5.8 Geodesic curves on a rotating disc, coordinates

Integrating this yields

$$\theta = \pm \frac{r_0 \omega^2}{c^2} \sqrt{r^2 - r_0^2} \mp \arccos \frac{r_0}{r}.$$

Example 5.5.3 (Christoffel symbols in a hyperbolically accelerated reference frame)
The Christoffel symbols were defined in Eq. (5.54):

$$\Gamma^{\alpha}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}).$$

In this example

$$g_{tt} = -\left(1 + \frac{gx}{c^2}\right)^2 c^2, \qquad g_{xx} = g_{yy} = g_{zz} = 1$$

and only the term $\frac{\partial g_{tt}}{\partial x}$ contributes to $\Gamma^{\alpha}_{\mu\nu}$ (Fig. 5.9). Thus the only non-vanishing Christoffel symbols are

$$\begin{aligned}
\Gamma^t_{xt} = \Gamma^t_{tx} &= \frac{1}{2} g^{tt} \left(\frac{\partial g_{tt}}{\partial x} \right) \\
&= \frac{1}{2g_{tt}} \frac{\partial g_{tt}}{\partial x} \\
&= \frac{2\left(1 + \frac{gx}{c^2}\right) g}{2\left(1 + \frac{gx}{c^2}\right)^2 c^2} \\
&= \frac{1}{\left(1 + \frac{gx}{c^2}\right)} \frac{g}{c^2}, \\
\Gamma^x_{tt} &= -\frac{1}{2} g^{xx} \left(\frac{\partial g_{tt}}{\partial x} \right) \\
&= -\frac{1}{2} \left\{ -2 \left(1 + \frac{gx}{c^2}\right) \frac{g}{c^2} c^2 \right\} \\
&= \left(1 + \frac{gx}{c^2}\right) g \, .
\end{aligned}$$

Example 5.5.4 (Vertical projectile motion in a hyperbolically accelerated reference frame)

$$ds^2 = -\left(1 + \frac{gx}{c^2}\right)^2 c^2 dt^2 + dx^2 + dy^2 + dz^2 \, . \tag{5.63}$$

Vertical motion implies that $dy = dz = 0$ and the Lagrange function becomes

$$\begin{aligned}
L &= \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \\
&= -\frac{1}{2} \left(1 + \frac{gx}{c^2}\right)^2 c^2 \dot{t}^2 + \frac{1}{2} \dot{x}^2,
\end{aligned}$$

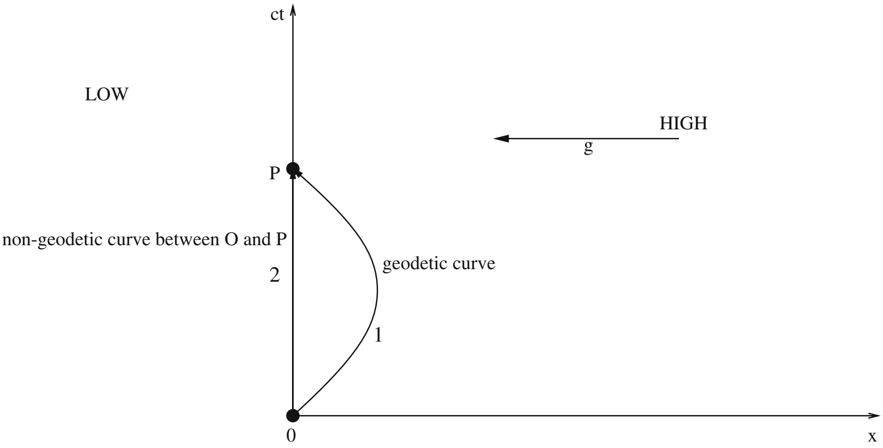


Fig. 5.9 Vertical throw in the accelerated reference frame

where the dots imply differentiation w.r.t the particle's proper time, τ. And the initial conditions are

$$x(0) = 0, \quad \dot{x}(0) = (u^0, u^x, 0, 0)$$
$$= \gamma(c, v, 0, 0),$$
$$\text{where} \quad \gamma = \left(1 - v^2/c^2\right)^{-1/2}.$$

What is the maximum height, h, reached by the particle?
Newtonian description: $\frac{1}{2}mv^2 = mgh \Rightarrow h = \frac{v^2}{2g}$.
Relativistic description: t is a cyclic coordinate $\Rightarrow x^0 = ct$ is cyclic and $p_0 =$ constant.

$$p_0 = \frac{\partial L}{\partial \dot{x}^0} = \frac{1}{c}\frac{\partial L}{\partial \dot{t}} = -c\left(1 + \frac{gx}{c^2}\right)^2 \dot{t}\ . \tag{5.64}$$

Now the 4-velocity identity is

$$\vec{u} \cdot \vec{u} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2\ , \tag{5.65}$$

so

$$-\frac{1}{2}\left(1 + \frac{gx}{c^2}\right)^2 c^2 \dot{t}^2 + \frac{1}{2}\dot{x}^2 = -\frac{1}{2}c^2 \tag{5.66}$$

and given that the maximum height h is reached when $\dot{x} = 0$ we get

$$\left(1 + \frac{gh}{c^2}\right)^2 \dot{t}^2_{x=h} = 1\ . \tag{5.67}$$

Now, since p_0 is a constant of the motion, it preserves its initial value throughout the flight (i.e. $p_0 = -c\dot{t}(0) = -\gamma c$) and particularly at $x = h$:

$$(5.64) \Rightarrow p_0 = -\gamma c = -c\left(1 + \frac{gh}{c^2}\right)^2 \dot{t}_{x=h} . \tag{5.68}$$

Finally, dividing Eq. (5.67) by Eq. (5.68) and substituting back in Eq. (5.67) gives

$$h = \frac{c^2}{g}(\gamma - 1) . \tag{5.69}$$

In the Newtonian limit (5.69) becomes

$$h = \frac{c^2}{g}\left(\frac{1}{(1 - v^2/c^2)^{1/2}} - 1\right) \approxeq \frac{c^2}{g}\left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) \Rightarrow h \approxeq \frac{v^2}{2g}.$$

Example 5.5.5 (The twin "paradox") Eva travels to Alpha Centauri, 4 light years from the Earth, with a velocity $v = 0.8c$ ($\gamma = 1/0.6$). The trip takes 5 years out and 5 years back. This means that Eli, who remains at Earth, is 10 years older when she meets Eva at the end of her journey. Eva, on the other hand, is $10(1 - v^2/c^2)^{1/2} = 10(0.6) = 6$ years older (Fig. 5.10).

According to the general principle of relativity (see G2 in Sect. 1.5), Eva can consider herself as being stationary and Eli as the one whom undertakes the long journey. In this picture it seems that Eva and Eli must be 10 and 6 years older, respectively, upon their return.

Let us accept the principle of general relativity as applied to accelerated reference frames and review the twin "paradox" in this light.

Eva's description of the trip when she sees herself as stationary is as follows.

Eva perceives a Lorentz-contracted distance between the Earth and Alpha Centauri, namely, 4 light years $\times 1/\gamma = 2.4$ light years. The Earth and Eli travel with $v = 0.8c$. Her travel time in one direction is then $\frac{2.4 \text{ light years}}{0.8c} = 3$ years. So the

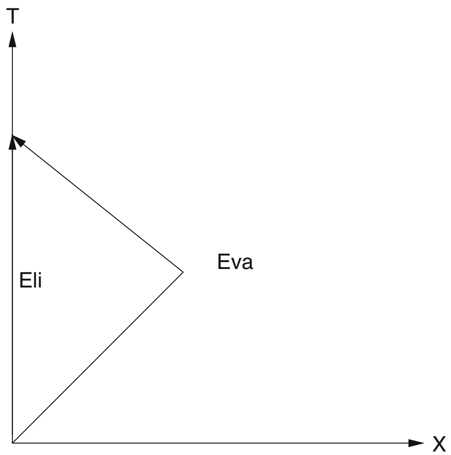


Fig. 5.10 The twins Eli and Eva each travel between two fixed events in space–time

round trip takes 6 years according to Eva. That is Eva is 6 years older when they meet again. This is in accordance with the result arrived at by Eli. According to Eva, Eli ages by only 6 years $\times 1/\gamma = 3.6$ years during the round trip, not 10 years as Eli found.

On turning about Eva experiences a force which reduces her velocity and accelerates her towards the Earth and Eli. This means that she experiences a gravitational force directed **away** from the Earth. Eli is higher up in this gravitational field and ages **faster** than Eva, because of the gravitational time dilation. We assume that Eva has constant proper acceleration and is stationary in a hyperbolically accelerated frame as she turns about.

The canonical momentum p_t for Eli is then (see Eq. (5.64))

$$p_t = -\left(1+\frac{gx}{c^2}\right)^2 c\dot{t}.$$

Inserting this into the 4-velocity identity gives

$$p_t^2 - c^2\left(1+\frac{gx}{c^2}\right)^2 = \left(1+\frac{gx}{c^2}\right)^2 \dot{x}^2 \tag{5.70}$$

or

$$d\tau = \frac{1+\frac{gx}{c^2}}{\sqrt{p_t^2 - c^2\left(1+\frac{gx}{c^2}\right)^2}}dx.$$

Now, since $\dot{x} = 0$ for $x = x_2$ (x_2 is Eli's turning point according to Eva), we have that

$$p_t = c\left(1+\frac{gx_2}{c^2}\right).$$

Let x_1 be Eli's position according to Eva just as Eva begins to notice the gravitational field, that is when Eli begins to slow down in Eva's frame.

Integration from x_1 to x_2 and inserting the value of p_t gives

$$\tau_{1-2} = \frac{c}{g}\sqrt{\left(1+\frac{gx_2}{c^2}\right)^2 - \left(1+\frac{gx_1}{c^2}\right)^2}$$
$$\Rightarrow \lim_{g\to\infty} \tau_{1-2} = \frac{1}{c}\sqrt{x_2^2 - x_1^2}\,.$$

Now setting $x_2 = 4$ and $x_1 = 2.4$ light years, respectively, we get

$$\lim_{g\to\infty} \tau_{1-2} = 3.2 \quad \text{years.}$$

Eli's aging as she turns about is, according to Eva,

$$\Delta\tau_{Eli} = 2\lim_{g\to\infty} \tau_{1-2} = 6.4 \quad \text{years.}$$

So Eli's has aged by a total of $\tau_{Eli} = 3.6 + 6.4 = 10$ years, according to Eva, which is just what Eli herself found.

5.5.3 Gravitational Doppler Effect

This concerns the frequency shift of light traversing up or down in a gravitational field. The 4-momentum of a particle with relativistic energy E and spatial velocity $\vec{w}$ (3-velocity) is given by

$$\vec{P} = E(1, \vec{w}) \qquad (c = 1)\ . \tag{5.71}$$

Let $\vec{U}$ be the 4-velocity of an observer. In a comoving orthonormal basis of the observer we have $\vec{U} = (1, 0, 0, 0)$. This gives

$$\vec{U} \cdot \vec{P} = -\hat{E}\ . \tag{5.72}$$

The energy of a particle with 4-momentum $\vec{P}$ measured by an observer with 4-velocity $\vec{U}$ is

$$\hat{E} = -\vec{U} \cdot \vec{P}\ . \tag{5.73}$$

Let $E_S = -(\vec{U} \cdot \vec{P})_S$ and $E_a = -(\vec{U} \cdot \vec{P})_a$ be the energy of a photon, measured locally by observers in rest in the transmitter and receiver positions, respectively. This gives[1]

$$\frac{E_S}{(\vec{U} \cdot \vec{P})_S} = \frac{E_a}{(\vec{U} \cdot \vec{P})_a}\ . \tag{5.74}$$

Let the angular frequency of the light, measured by the transmitter and receiver, be w_s and w_a, respectively. We then have

$$w_s = \frac{E_S}{\hbar}, \qquad w_a = \frac{E_a}{\hbar}\ , \tag{5.75}$$

which gives

$$w_a = \frac{(\vec{U} \cdot \vec{P})_a}{(\vec{U} \cdot \vec{P})_s} w_s\ . \tag{5.76}$$

This formula may be applied to observers with arbitrary motion [1]. We shall here restrict ourselves to an observer at rest in a time-independent orthogonal metric. Then we have

$$\vec{U} \cdot \vec{P} = U^t P_t = \frac{dt}{d\tau} P_t\ , \tag{5.77}$$

where P_t is a constant of motion (since t is a cyclic coordinate) for photons and

[1] $\vec{A} \cdot \vec{B} = A_0 B^0 + A_1 B^1 + \cdots = g_{00} A^0 B^0 + g_{11} A^1 B^1 + \cdots$, an orthonormal basis gives $\vec{A} \cdot \vec{B} = -A^0 B^0 + A^1 B^1 + \cdots$.

hence has the same value in transmitter and receiver positions. The line element is

$$ds^2 = g_{tt}dt^2 + g_{ii}(dx^i)^2 \ . \tag{5.78}$$

Using the physical interpretation (5.55) of the line element for a timelike interval, we obtain for the proper time of an observer at rest

$$d\tau^2 = -g_{tt}dt^2 \Rightarrow d\tau = \sqrt{-g_{tt}}dt \ . \tag{5.79}$$

Hence

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{-g_{tt}}} \ , \tag{5.80}$$

which gives

$$\vec{U}\cdot\vec{P} = \frac{1}{\sqrt{-g_{tt}}}P_t \ . \tag{5.81}$$

Inserting this into the expression for angular frequency (5.76) gives

$$w_a = \sqrt{\frac{(g_{tt})_s}{(g_{tt})_a}}w_s \ . \tag{5.82}$$

Note: we have assumed an orthogonal and time-independent metric, i.e. $P_{t_1} = P_{t_2}$. Inserting the metric of a hyperbolically accelerated reference system with

$$g_{tt} = -\left(1+\frac{gx}{c^2}\right)^2 \tag{5.83}$$

gives

$$w_a = \frac{1+\frac{gx_s}{c^2}}{1+\frac{gx_a}{c^2}}w_s \tag{5.84}$$

or

$$\frac{w_a - w_s}{w_s} = \frac{1+\frac{gx_s}{c^2}}{1+\frac{gx_a}{c^2}} - 1 = \frac{\frac{g}{c^2}(x_s - x_a)}{1+\frac{gx_a}{c^2}} \approx \frac{g}{c^2}H \ , \tag{5.85}$$

where $H = x_s - x_a$ is the difference in height between transmitter and receiver.

Example 5.5.6 (Measurements of gravitational Doppler effects [2])

$$H \approx 20m, \qquad g = 10m/s^2$$

gives

$$\frac{\Delta w}{w} = \frac{200}{9\times 10^{16}} = 2.2\times 10^{-15}.$$

This effect was measured by Pound and Rebka in 1960.

5.6 The Koszul Connection

The covariant directional derivative of a scalar field f in the direction of a vector $\vec{u}$ is defined as

$$\nabla_{\vec{u}} f \equiv \vec{u}(f) \, . \tag{5.86}$$

Here the vector $\vec{u}$ should be taken as a differential operator. (In coordinate basis, $\vec{u} = u^\mu \frac{\partial}{\partial x^\mu}$). The directional derivative along a basis vector $\vec{e}_\nu$ is written as

$$\nabla_\nu \equiv \nabla_{\vec{e}_\nu} \, . \tag{5.87}$$

Hence $\nabla_\mu (\quad) = \nabla_{\vec{e}_\mu} (\quad) = \vec{e}_\mu (\quad)$.

Definition 5.6.1 (Koszul's connection coefficients in an arbitrary basis) In an arbitrary basis the Koszul connection coefficients are defined by

$$\boxed{\nabla_\nu \vec{e}_\mu \equiv \Gamma^\alpha_{\mu\nu} \vec{e}_\alpha} \, , \tag{5.88}$$

which may also be written $\vec{e}_\nu(\vec{e}_\mu) = \Gamma^\alpha_{\mu\nu}\vec{e}_\alpha$. In coordinate basis, $\Gamma^\alpha_{\mu\nu}$ is reduced to Christoffel symbols and one often writes $\vec{e}_{\mu,\nu} = \Gamma^\alpha_{\mu\nu}\vec{e}_\alpha$. In an arbitrary basis, $\Gamma^\alpha_{\mu\nu}$ has no symmetry.

Example 5.6.1 (The connection coefficients in a rotating reference frame) Coordinate transformation: (T, R, Θ are coordinates in the non-rotating reference frame, t, r, θ in the rotating) Corresponding Cartesian coordinates: X, Y and x, y (Fig. 5.11).

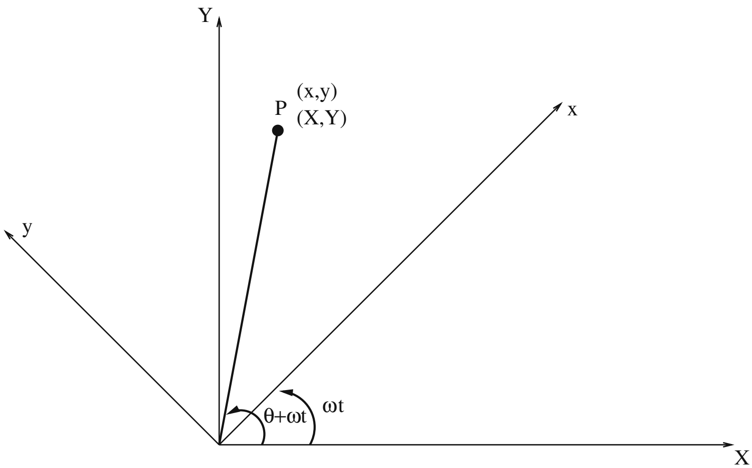


Fig. 5.11 The non-rotating coordinate system (X,Y) and the rotating system (x,y), rotating with angular velocity ω

$$t = T, r = R, \theta = \Theta - \omega T,$$
$$X = R\cos\Theta, Y = R\sin\Theta,$$
$$X = r\cos(\theta + \omega t), Y = r\sin(\theta + \omega t).$$

$$\vec{e}_t = \frac{\partial}{\partial t} = \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial t}\frac{\partial}{\partial Y} + \frac{\partial T}{\partial t}\frac{\partial}{\partial T}$$

gives

$$\begin{aligned}\vec{e}_t &= -r\omega\sin(\theta+\omega t)\vec{e}_X + r\omega\cos(\theta+\omega t)\vec{e}_Y + \vec{e}_T,\\ \vec{e}_r &= \frac{\partial X}{\partial r}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial r}\frac{\partial}{\partial Y}\\ &= \cos(\theta+\omega t)\vec{e}_X + \sin(\theta+\omega t)\vec{e}_Y,\\ \vec{e}_\theta &= \frac{\partial X}{\partial \theta}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial \theta}\frac{\partial}{\partial Y}\\ &= -r\sin(\theta+\omega t)\vec{e}_X + r\cos(\theta+\omega t)\vec{e}_Y.\end{aligned}$$

We are going to find the Christoffel symbols, which involves differentiation of basis vectors. This coordinate transformation makes this easy, since $\vec{e}_X, \vec{e}_Y, \vec{e}_T$ are constant. Differentiation gives

$$\nabla_t \vec{e}_t = -r\omega^2\cos(\theta+\omega t)\vec{e}_X - r\omega^2\sin(\theta+\omega t)\vec{e}_Y\ . \tag{5.89}$$

The connection coefficients are (see *Eq. 5.88*)

$$\nabla_\nu \vec{e}_\mu \equiv \Gamma^\alpha_{\mu\nu}\vec{e}_\alpha\ . \tag{5.90}$$

So, to calculate $\Gamma^\alpha_{\mu\nu}$, the right-hand side of Eq. (5.89) has to be expressed by the basis that we are differentiating.
By inspection, the right-hand side is $-r\omega^2\vec{e}_r$.
That is $\nabla_t\vec{e}_t = -r\omega^2\vec{e}_r$ giving $\Gamma^r{}_{tt} = -r\omega^2$.
The other non-zero Christoffel symbols are

$$\Gamma^\theta_{rt} = \Gamma^\theta_{tr} = \frac{\omega}{r}, \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r},$$
$$\Gamma^r{}_{\theta t} = \Gamma^r{}_{t\theta} = -r\omega, \Gamma^r{}_{\theta\theta} = -r.$$

Example 5.6.2 (Acceleration in a non-rotating reference frame (Newton))

$$\ddot{\vec{r}} = \dot{\vec{v}} = (\dot{v}^i + \Gamma^i{}_{\alpha\beta}v^\alpha v^\beta)\vec{e}_i,$$

where $\dot{} \equiv \frac{d}{dt}$. i, j and k are space indices. Inserting the Christoffel symbols for plane polar coordinates (see Example 5.2.1) gives

$$\vec{a}_{inert} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + \left(\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}\right)\vec{e}_\theta.$$

Example 5.6.3 (The acceleration of a particle, relative to the rotating reference frame) Inserting the Christoffel symbols from Example 5.6.1,

$$\begin{aligned}\vec{a}_{rot} &= (\ddot{r} - r\dot{\theta}^2 - \Gamma^r_{tt}\dot{t}^2 + \Gamma^r_{\theta t}\dot{\theta}\dot{t} + \Gamma^r_{t\theta}\dot{t}\dot{\theta})\vec{e}_r + \left(\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} + \Gamma^{\theta}_{rt}\dot{r}\dot{t} + \Gamma^{\theta}_{tr}\dot{t}\dot{r}\right)\vec{e}_\theta \\ &= (\ddot{r} - r\dot{\theta}^2 - r\omega^2 - 2r\omega\dot{\theta})\vec{e}_{\hat{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} + 2\dot{r}\omega)\vec{e}_{\hat{\theta}} \\ &= \vec{a}_{inert} - (r\omega^2 + 2r\omega\dot{\theta})\vec{e}_{\hat{r}} + 2\dot{r}\omega\vec{e}_{\hat{\theta}}.\end{aligned}$$

The angular velocity of the reference frame, is $\vec{\omega} = \omega\vec{e}_z$. We also introduce $\vec{r} = r\vec{e}_r$. The velocity relative to the rotating reference frame is then

$$\dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r.$$

Furthermore

$$\dot{\vec{e}}_r = \frac{d\vec{e}_r}{dt} = \frac{\partial\vec{e}_r}{\partial x^i}\frac{dx^i}{dt} = v^i\vec{e}_{r,i}.$$

Using Definition 5.6.1 in a coordinate basis, this may be written as

$$\dot{\vec{e}}_r = v^i\Gamma^j_{ri}\vec{e}_j.$$

Using the expressions of the Christoffel symbols in Example 5.6.1, we get

$$\dot{\vec{e}}_r = v^{\theta}\Gamma^{\theta}_{r\theta}\vec{e}_\theta = \dot{\theta}\frac{1}{r}\vec{e}_\theta = \dot{\theta}\vec{e}_{\hat{\theta}}.$$

Hence

$$\vec{v} = \dot{\vec{r}} = \dot{r}\vec{e}_{\hat{r}} + r\dot{\theta}\vec{e}_{\hat{\theta}}.$$

Inserting this into the expression for the acceleration gives

$$\boxed{\ddot{\vec{r}}_{rot} = \ddot{\vec{r}}_{inert} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2\vec{\omega}\times\vec{v}}.$$

We can see that the centrifugal acceleration (the term in the middle) and the coriolis acceleration (last term) is contained in the expression for the covariant derivative.

5.7 Connection Coefficients $\Gamma^{\alpha}_{\mu\nu}$ and Structure Coefficients $c^{\alpha}_{\mu\nu}$ in a Riemannian (Torsion-Free) Space

The commutator of two vectors, $\vec{u}$ and $\vec{v}$, expressed by covariant directional derivatives is given by

$$[\vec{u},\vec{v}] = \nabla_{\vec{u}}\vec{v} - \nabla_{\vec{v}}\vec{u}\,. \tag{5.91}$$

Let $\vec{u} = \vec{e}_\mu$ and $\vec{v} = \vec{e}_\nu$. We then have

$$[\vec{e}_\mu,\vec{e}_\nu] = \nabla_\mu\vec{e}_\nu - \nabla_\nu\vec{e}_\mu\,. \tag{5.92}$$

Using the definitions of the connection and structure coefficients we get

$$c^{\alpha}{}_{\mu\nu}\vec{e}_{\alpha} = (\Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu})\vec{e}_{\alpha} \ . \tag{5.93}$$

Thus in a torsion-free space

$$c^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu}. \tag{5.94}$$

In **coordinate basis** we have

$$\vec{e}_{\mu} = \frac{\partial}{\partial x^{\mu}} \ , \quad \vec{e}_{\nu} = \frac{\partial}{\partial x^{\nu}} \ . \tag{5.95}$$

And therefore

$$\begin{aligned} [\vec{e}_{\mu}, \vec{e}_{\nu}] &= \left[\frac{\partial}{\partial x^{\mu}}, \frac{\partial}{\partial x^{\nu}}\right] \\ &= \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial}{\partial x^{\nu}}\right) - \frac{\partial}{\partial x^{\nu}}\left(\frac{\partial}{\partial x^{\mu}}\right) \\ &= \frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}} - \frac{\partial^2}{\partial x^{\nu}\partial x^{\mu}} = 0 \ . \end{aligned} \tag{5.96}$$

Equation (5.96) shows that $c^{\alpha}{}_{\mu\nu} = 0$ and that the connection coefficients in Eq. (5.94) therefore are symmetrical in a coordinate basis:

$$\Gamma^{\alpha}{}_{\nu\mu} = \Gamma^{\alpha}{}_{\mu\nu}. \tag{5.97}$$

5.8 Covariant Differentiation of Vectors, Forms and Tensors

5.8.1 Covariant Differentiation of a Vector in an Arbitrary Basis

$$\begin{aligned} \nabla_{\nu}\vec{A} &= \nabla_{\nu}(A^{\mu}\vec{e}_{\mu}) \\ &= \nabla_{\nu}A^{\mu}\vec{e}_{\mu} + A^{\alpha}\nabla_{\nu}\vec{e}_{\alpha}, \end{aligned} \tag{5.98}$$

$$\nabla_{\nu}A^{\mu} = \vec{e}_{\nu}(A^{\mu}), \quad \vec{e}_{\nu} = M^{\mu}{}_{\nu}\frac{\partial}{\partial x^{\mu}} \ , \tag{5.99}$$

where $M^{\mu}{}_{\nu}$ are the elements of a transformation matrix between a coordinate basis $\{\frac{\partial}{\partial x^{\mu}}\}$ and an arbitrary basis $\{\vec{e}_{\nu}\}$. (If $\vec{e}_{\nu}$ had been a coordinate basis vector, we would have gotten $\vec{e}_{\nu}(A^{\mu}) = \frac{\partial}{\partial x^{\nu}}(A^{\mu}) = A^{\mu}{}_{,\nu}$.)

$$\nabla_{\nu}\vec{A} = [\vec{e}_{\nu}(A^{\mu}) + A^{\alpha}\Gamma^{\mu}{}_{\alpha\nu}]\vec{e}_{\mu} \ . \tag{5.100}$$

Definition 5.8.1 (Covariant derivative of a vector) The covariant derivative of a vector in an arbitrary basis is defined by

$$\nabla_\nu \vec{A} \equiv A^\mu{}_{;\nu} \vec{e}_\mu \; . \tag{5.101}$$

So

$$\begin{aligned} A^\mu_{;\nu} &= \vec{e}_\nu (A^\mu) + A^\alpha \Gamma^\mu_{\alpha\nu}, \\ \text{where } \nabla_\nu \vec{e}_\alpha &\equiv \Gamma^\mu_{\alpha\nu} \vec{e}_\mu \; . \end{aligned} \tag{5.102}$$

5.8.2 *Covariant Differentiation of Forms*

Definition 5.8.2 (Covariant directional derivative of a 1-form field) Given a vector field $\vec{A}$ and a 1-form field $\underline{\alpha}$, the covariant directional derivative of $\underline{\alpha}$ in the direction of the vector $\vec{u}$ is defined by

$$(\nabla_{\vec{u}} \underline{\alpha})(\vec{A}\,) \equiv \nabla_{\vec{u}} [\underbrace{\underline{\alpha}(\vec{A}\,)}_{\alpha_\mu A^\mu}] - \underline{\alpha}(\nabla_{\vec{u}} \vec{A}) \; . \tag{5.103}$$

Let $\underline{\alpha} = \underline{\omega}^\mu$ (basis form), $\underline{\omega}^\mu(\vec{e}_\nu) \equiv \delta^\mu{}_\nu$ and let $\vec{A} = \vec{e}_\nu$ and $\vec{u} = \vec{e}_\lambda$. We then have

$$(\nabla_\lambda \underline{\omega}^\mu)(\vec{e}_\nu) = \nabla_\lambda [\underbrace{\underline{\omega}^\mu(\vec{e}_\nu)}_{\delta^\mu{}_\nu}] - \underline{\omega}^\mu(\nabla_\lambda \vec{e}_\nu) \; . \tag{5.104}$$

The covariant directional derivative ∇_λ of a constant scalar field is zero, $\nabla_\lambda \delta^\mu{}_\nu = 0$. We therefore get

$$\begin{aligned} (\nabla_\lambda \underline{\omega}^\mu)(\vec{e}_\nu) &= -\underline{\omega}^\mu(\nabla_\lambda \vec{e}_\nu) \\ &= -\underline{\omega}^\mu(\Gamma^\alpha_{\nu\lambda} \vec{e}_\alpha) \\ &= -\Gamma^\alpha_{\nu\lambda} \underline{\omega}^\mu(\vec{e}_\alpha) \\ &= -\Gamma^\alpha_{\nu\lambda} \delta^\mu{}_\alpha \\ &= -\Gamma^\mu_{\nu\lambda} \; . \end{aligned} \tag{5.105}$$

The contraction between a 1-form and a basis vector gives the components of the 1-form, $\underline{\alpha}(\vec{e}_\nu) = \alpha_\nu$. Equation (5.105) tells us that the ν-component of $\nabla_\lambda \underline{\omega}^\mu$ is equal to $-\Gamma^\mu_{\nu\lambda}$, and that we therefore have

$$\boxed{\nabla_\lambda \underline{\omega}^\mu = -\Gamma^\mu_{\nu\lambda} \underline{\omega}^\nu} \; . \tag{5.106}$$

Equation (5.106) gives the directional derivatives of the basis forms. Using the product of differentiation gives

$$\begin{aligned}\nabla_\lambda \underline{\alpha} &= \nabla_\lambda (\alpha_\mu \underline{\omega}^\mu) \\ &= \nabla_\lambda (\alpha_\mu) \underline{\omega}^\mu + \alpha_\mu \nabla_\lambda \underline{\omega}^\mu \\ &= \vec{e}_\lambda (\alpha_\mu) \underline{\omega}^\mu - \alpha_\mu \Gamma^\mu_{\ \nu\lambda} \underline{\omega}^\nu \, .\end{aligned} \tag{5.107}$$

Definition 5.8.3 (Covariant derivative of a 1-form) The covariant derivative of a 1-form $\underline{\alpha} = \alpha_\mu \underline{\omega}^\mu$ is therefore given by Eq. (5.108) below, when we let $\mu \to \nu$ in the first term on the right-hand side in (5.107):

$$\boxed{\nabla_\lambda \underline{\alpha} = [\vec{e}_\lambda (\alpha_\nu) - \alpha_\mu \Gamma^\mu_{\ \nu\lambda}] \underline{\omega}^\nu} \, . \tag{5.108}$$

The covariant derivative of the 1-form components α_μ is denoted by $\alpha_{\nu;\lambda}$ and defined by

$$\nabla_\lambda \underline{\alpha} \equiv \alpha_{\nu;\lambda} \underline{\omega}^\nu \, . \tag{5.109}$$

It then follows that

$$\boxed{\alpha_{\nu;\lambda} = \vec{e}_\lambda (\alpha_\nu) - \alpha_\mu \Gamma^\mu_{\ \nu\lambda}} \, . \tag{5.110}$$

It is worth to note that $\Gamma^\mu_{\ \nu\lambda}$ in Eq. (5.110) are not Christoffel symbols. In coordinate basis we get

$$\alpha_{\nu;\lambda} = \alpha_{\nu,\lambda} - \alpha_\mu \Gamma^\mu_{\ \lambda\nu} \, , \tag{5.111}$$

where $\Gamma^\mu_{\ \lambda\nu} = \Gamma^\mu_{\ \nu\lambda}$ are Christoffel symbols.

5.8.3 Generalization for Tensors of Higher Rank

Definition 5.8.4 (Covariant derivative of a tensor) Let A and B be two tensors of arbitrary rank. The covariant directional derivative along a basis vector $\vec{e}_\lambda$ of a tensor $A \otimes B$ of arbitrary rank is defined by

$$\nabla_\lambda (A \otimes B) \equiv (\nabla_\lambda A) \otimes B + A \otimes (\nabla_\lambda B) \, . \tag{5.112}$$

We will use Eq. (5.112) to find the formula for the covariant derivative of the components of a tensor of rank 2:

$$\begin{aligned}\nabla_\alpha S &= \nabla_\alpha (S_{\mu\nu} \underline{\omega}^\mu \otimes \underline{\omega}^\nu) \\ &= (\nabla_\alpha S_{\mu\nu}) \underline{\omega}^\mu \otimes \underline{\omega}^\nu + S_{\mu\nu} (\nabla_\alpha \underline{\omega}^\mu) \otimes \underline{\omega}^\nu + S_{\mu\nu} \underline{\omega}^\mu \otimes (\nabla_\alpha \underline{\omega}^\nu) \\ &= (S_{\mu\nu,\alpha} - S_{\beta\nu} \Gamma^\beta_{\ \mu\alpha} - S_{\mu\beta} \Gamma^\beta_{\ \nu\alpha}) \underline{\omega}^\mu \otimes \underline{\omega}^\nu ,\end{aligned} \tag{5.113}$$

where $S_{\mu\nu,\alpha} = \vec{e}_\alpha (S_{\mu\nu})$. Defining the covariant derivative $S_{\mu\nu;\alpha}$ by

$$\nabla_\alpha S = S_{\mu\nu;\alpha} \underline{\omega}^\mu \otimes \underline{\omega}^\nu \tag{5.114}$$

we get

$$S_{\mu\nu;\alpha} = S_{\mu\nu,\alpha} - S_{\beta\nu} \Gamma^\beta_{\ \mu\alpha} - S_{\mu\beta} \Gamma^\beta_{\ \nu\alpha} \, . \tag{5.115}$$

For the metric tensor we get

$$g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - g_{\beta\nu}\Gamma^{\beta}_{\ \mu\alpha} - g_{\mu\beta}\Gamma^{\beta}_{\ \nu\alpha} \,. \tag{5.116}$$

From

$$g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu \tag{5.117}$$

we get

$$\begin{aligned} g_{\mu\nu,\alpha} &= (\nabla_\alpha \vec{e}_\mu)\cdot\vec{e}_\nu + \vec{e}_\mu(\nabla_\alpha \vec{e}_\nu) \\ &= \Gamma^{\beta}_{\ \mu\alpha}\vec{e}_\beta \cdot \vec{e}_\nu + \vec{e}_\mu \cdot \Gamma^{\beta}_{\ \nu\alpha}\vec{e}_\beta \\ &= g_{\beta\nu}\Gamma^{\beta}_{\ \mu\alpha} + g_{\mu\beta}\Gamma^{\beta}_{\ \nu\alpha} \,. \end{aligned} \tag{5.118}$$

This means that

$$g_{\mu\nu;\alpha} = 0 \,. \tag{5.119}$$

So the metric tensor is a (covariant) constant tensor.

5.9 The Cartan Connection

Definition 5.9.1 (Exterior derivative of a basis vector)

$$\underline{d}\vec{e}_\mu \equiv \Gamma^{\nu}_{\ \mu\alpha}\vec{e}_\nu \otimes \underline{\omega}^\alpha \,. \tag{5.120}$$

Exterior derivative of a vector field:

$$\underline{d}\vec{A} = \underline{d}(\vec{e}_\mu A^\mu) = \vec{e}_\nu \otimes \underline{d}A^\nu + A^\mu \underline{d}\vec{e}_\mu \,. \tag{5.121}$$

In arbitrary basis,

$$\underline{d}A^\nu = \vec{e}_\lambda(A^\nu)\underline{\omega}^\lambda \tag{5.122}$$

(in coordinate basis, $\vec{e}_\lambda(A^\nu) = \frac{\partial}{\partial x^\lambda}(A^\nu) = A^\nu_{\ ,\lambda}$)
giving

$$\begin{aligned} \underline{d}\vec{A} &= \vec{e}_\nu \otimes [\vec{e}_\lambda(A^\nu)\underline{\omega}^\lambda] + A^\mu \Gamma^{\nu}_{\ \mu\lambda}\vec{e}_\nu \otimes \underline{\omega}^\lambda \\ &= (\vec{e}_\lambda(A^\nu) + A^\mu\Gamma^{\nu}_{\ \mu\lambda})\vec{e}_\nu \otimes \underline{\omega}^\lambda , \end{aligned} \tag{5.123}$$

$$\boxed{\underline{d}\vec{A} = A^\nu_{\ ;\lambda}\vec{e}_\nu \otimes \underline{\omega}^\lambda} \,. \tag{5.124}$$

Definition 5.9.2 (Connection forms $\underline{\Omega}^\nu_{\ \mu}$) The connection forms $\underline{\Omega}^\nu_{\ \mu}$ are 1-forms, defined by

$$\begin{aligned} &\underline{d}\vec{e}_\mu \equiv \vec{e}_\nu \otimes \underline{\Omega}^\nu_{\ \mu} , \\ &\Gamma^{\nu}_{\ \mu\alpha}\vec{e}_\nu \otimes \underline{\omega}^\alpha = \vec{e}_\nu \otimes \Gamma^{\nu}_{\ \mu\alpha}\underline{\omega}^\alpha = \vec{e}_\nu \otimes \underline{\Omega}^\nu_{\ \mu} , \end{aligned} \tag{5.125}$$

$$\boxed{\underline{\Omega}^{\nu}{}_{\mu} = \Gamma^{\nu}{}_{\mu\alpha}\underline{\omega}^{\alpha}} . \tag{5.126}$$

The exterior derivatives of the components of the metric tensor:

$$\underline{d}g_{\mu\nu} = \underline{d}(\vec{e}_\mu \cdot \vec{e}_\nu) = \vec{e}_\mu \cdot \underline{d}\vec{e}_\nu + \vec{e}_\nu \cdot \underline{d}\vec{e}_\mu \ , \tag{5.127}$$

where the meaning of the dot is defined as follows.

Definition 5.9.3 (Scalar product between vector and 1-form) The scalar product between a vector $\vec{u}$ and a (vectorial) 1-form $\underline{A} = A^{\mu}{}_{\nu}\vec{e}_\mu \otimes \underline{\omega}^\nu$ is defined by

$$\vec{u} \cdot \underline{A} \equiv u^\alpha A^{\mu}{}_{\nu}(\vec{e}_\alpha \cdot \vec{e}_\mu)\underline{\omega}^\nu \ . \tag{5.128}$$

Using this definition, we get

$$\begin{aligned} \underline{d}g_{\mu\nu} &= (\vec{e}_\mu \cdot \vec{e}_\lambda)\underline{\Omega}^{\lambda}{}_{\nu} + (\vec{e}_\nu \cdot \vec{e}_\gamma)\underline{\Omega}^{\gamma}{}_{\mu} \\ &= g_{\mu\lambda}\underline{\Omega}^{\lambda}{}_{\nu} + g_{\nu\gamma}\underline{\Omega}^{\gamma}{}_{\mu}. \end{aligned} \tag{5.129}$$

Lowering an index gives

$$\underline{d}g_{\mu\nu} = \underline{\Omega}_{\mu\nu} + \underline{\Omega}_{\nu\mu} \ . \tag{5.130}$$

In an orthonormal basis field there is Minkowski metric:

$$g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \ , \tag{5.131}$$

which is constant. Then we have

$$\underline{d}g_{\hat{\mu}\hat{\nu}} = 0 \Rightarrow \boxed{\underline{\Omega}_{\hat{\nu}\hat{\mu}} = -\underline{\Omega}_{\hat{\mu}\hat{\nu}}} , \tag{5.132}$$

where we write $\underline{\Omega}_{\hat{\nu}\hat{\mu}} = \Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}}\underline{\omega}^{\hat{\alpha}}$. It follows that $\Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}} = -\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}}$.
It also follows that

$$\begin{aligned} &\Gamma^{\hat{t}}{}_{\hat{i}\hat{j}} = -\Gamma_{\hat{t}\hat{i}\hat{j}} = \Gamma_{\hat{i}\hat{t}\hat{j}} = \Gamma^{\hat{i}}{}_{\hat{t}\hat{j}}, \\ &\Gamma^{\hat{i}}{}_{\hat{j}\hat{k}} = -\Gamma^{\hat{j}}{}_{\hat{i}\hat{k}}. \end{aligned} \tag{5.133}$$

Cartans 1st structure equation (without proof):

$$\begin{aligned} \underline{d\omega}^\rho &= -\frac{1}{2}c^{\rho}{}_{\mu\nu}\underline{\omega}^\mu \wedge \underline{\omega}^\nu \\ &= -\frac{1}{2}(\Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu})\underline{\omega}^\mu \wedge \underline{\omega}^\nu \\ &= -\Gamma^{\rho}{}_{\nu\mu}\underline{\omega}^\mu \wedge \underline{\omega}^\nu \\ &= -\underline{\Omega}^{\rho}{}_{\nu} \wedge \underline{\omega}^\nu. \end{aligned} \tag{5.134}$$

$$\boxed{\underline{d\omega}^\rho = -\underline{\Omega}^{\rho}{}_{\nu} \wedge \underline{\omega}^\nu} \quad \text{and} \quad \boxed{\underline{d\omega}^\rho = \Gamma^{\rho}{}_{\mu\nu}\underline{\omega}^\mu \wedge \underline{\omega}^\nu} . \tag{5.135}$$

In coordinate basis, we have $\underline{\omega}^\rho = \underline{d}x^\rho$.
Thus, $\underline{d\omega}^\rho = \underline{d}^2 x^\rho = 0$.
We also have $c^\rho{}_{\mu\nu} = 0$, and C1 is reduced to an identity. This formalism cannot be used in coordinate basis!

Example 5.9.1 (Cartan connection in an orthonormal basis field in plane polar coord.)

$$ds^2 = dr^2 + r^2 d\theta^2.$$

Introducing basis forms in an orthonormal basis field (where the metric is $g_{\hat{r}\hat{r}} = g_{\hat{\theta}\hat{\theta}} = 1$),

$$ds^2 = g_{\hat{r}\hat{r}}\underline{\omega}^{\hat{r}} \otimes \underline{\omega}^{\hat{r}} + g_{\hat{\theta}\hat{\theta}}\underline{\omega}^{\hat{\theta}} \otimes \underline{\omega}^{\hat{\theta}} = \underline{\omega}^{\hat{r}} \otimes \underline{\omega}^{\hat{r}} + \underline{\omega}^{\hat{\theta}} \otimes \underline{\omega}^{\hat{\theta}}$$
$$\Rightarrow \underline{\omega}^{\hat{r}} = \underline{d}r, \underline{\omega}^{\hat{\theta}} = r\underline{d}\theta.$$

Exterior differentiation gives

$$\underline{d\omega}^{\hat{r}} = \underline{d}^2 r = 0, \underline{d\omega}^{\hat{\theta}} = \underline{d}r \wedge \underline{d}\theta = \frac{1}{r}\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\theta}}.$$

C1:

$$\begin{aligned} \underline{d\omega}^{\hat{\mu}} &= -\underline{\Omega}^{\hat{\mu}}_{\hat{\nu}} \wedge \underline{\omega}^{\hat{\nu}} \\ &= -\underline{\Omega}^{\hat{\mu}}_{\hat{r}} \wedge \underline{\omega}^{\hat{r}} - \underline{\Omega}^{\hat{\mu}}_{\hat{\theta}} \wedge \underline{\omega}^{\hat{\theta}}. \end{aligned}$$

We have that $\underline{d\omega}^{\hat{r}} = 0$, which gives

$$\underline{\Omega}^{\hat{r}}{}_{\hat{\theta}} = \Gamma^{\hat{r}}{}_{\hat{\theta}\hat{\theta}}\underline{\omega}^{\hat{\theta}} \tag{5.136}$$

since $\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\theta}} = 0$. ($\underline{\Omega}^{\hat{r}}_{\hat{r}}$=0 because of the antisymmetry $\underline{\Omega}_{\hat{\nu}\hat{\mu}} = -\underline{\Omega}_{\hat{\mu}\hat{\nu}}$.)
We also have $\underline{d\omega}^{\hat{\theta}} = -\frac{1}{r}\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{r}}$. C1:

$$\underline{d\omega}^{\hat{\theta}} = -\underline{\Omega}^{\hat{\theta}}_{\hat{r}} \wedge \underline{\omega}^{\hat{r}} - \underbrace{\underline{\Omega}^{\hat{\theta}}_{\hat{\theta}}}_{=0} \wedge \underline{\omega}^{\hat{\theta}},$$

$$\underline{\Omega}^{\hat{\theta}}_{\hat{r}} = \Gamma^{\hat{\theta}}{}_{\hat{r}\hat{\theta}}\underline{\omega}^{\hat{\theta}} + \Gamma^{\hat{\theta}}{}_{\hat{r}\hat{r}}\underline{\omega}^{\hat{r}}, \tag{5.137}$$

giving $\Gamma^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} = \frac{1}{r}$.
We have $\underline{\Omega}^{\hat{r}}{}_{\hat{\theta}} = -\underline{\Omega}^{\hat{\theta}}_{\hat{r}}$. Using Eqs. (5.136) and (5.137) we get

$$\Gamma^{\hat{\theta}}_{\hat{r}\hat{r}} = 0$$
$$\Rightarrow \Gamma^{\hat{r}}{}_{\hat{\theta}\hat{\theta}} = -\frac{1}{r},$$

giving $\underline{\Omega}^{\hat{r}}{}_{\hat{\theta}} = -\underline{\Omega}^{\hat{\theta}}_{\hat{r}} = -\frac{1}{r}\underline{\omega}^{\hat{\theta}}$.

Problems

5.1. *Spatial geodesics in a rotating RF*
We studied a rotating reference frame in the beginning of this chapter and will now consider spatial geodesics in this reference frame. Consider the spatial metric

$$d\ell^2 = dr^2 + \frac{r^2}{1-\frac{\omega^2 r^2}{c^2}}d\theta^2 + dz^2\,. \tag{5.138}$$

Using the Lagrangian $L = \frac{1}{2}\left(\frac{d\ell}{d\lambda}\right)^2$ the shortest distance curves between points will be calculated. We will for the sake of simplicity assume that $\frac{dz}{d\lambda} = 0$, i.e. the curve is *planar*.

(a) Assume that the parameter λ is the arc length of the curve. What is the "3-velocity" identity in this case?
(b) The system possesses a cyclic coordinate. Which coordinate is that? Set down the expression for the corresponding constant of motion.
(c) Find the expressions for $\frac{dr}{d\lambda}$ and $\frac{d\theta}{d\lambda}$ as a function of r. Deduce the differential equation for the curve.
(d) Use the initial condition $\frac{dr}{d\lambda} = 0$ for $r = r_0$ and show that

$$\frac{p_\theta}{r_0} = \sqrt{1+\frac{\omega^2 p_\theta^2}{c^2}}.$$

(e) Show that the differential equation can be written as

$$\frac{dr}{r\sqrt{r^2-r_0^2}} - \frac{\omega^2}{c^2}\frac{rdr}{\sqrt{r^2-r_0^2}} = \frac{d\theta}{r_0}\,. \tag{5.139}$$

Integrate this equation and find the equation for the curve. Finally, draw the curve.

5.2. *Dual forms*
Let $\{\mathbf{e}_i\}$ be a Cartesian basis in the 3-dimensional Euclidean space. Using a vector $\mathbf{a} = a^i\mathbf{e}_i$ there are two ways of constructing a form:

i) By constructing a 1-form from its covariant components $a_j = g_{ji}a^i$:

$$\mathbf{A} = a_i\mathbf{dx}^i.$$

ii) By constructing a 2-form from its dual components, defined by $a_{ij} = \varepsilon_{ijk}a^k$:

$$\mathbf{a} = \frac{1}{2}a_{ij}\mathbf{dx}^i \wedge \mathbf{dx}^j.$$

We write this form as $\mathbf{a} = \star\mathbf{A}$ where $\star$ means to take the dual form.

(a) Given the vectors $\mathbf{a} = \mathbf{e}_x + 2\mathbf{e}_y - \mathbf{e}_z$ and $\mathbf{b} = 2\mathbf{e}_x - 3\mathbf{e}_y + \mathbf{e}_z$. Find the corresponding 1-forms $\mathbf{A}$ and $\mathbf{B}$, and the dual 2-forms $\mathbf{a} = \star\mathbf{A}$ and $\mathbf{b} = \star\mathbf{B}$. Find also the dual form θ to the 1-form $\sigma = \mathbf{dx} - 2\mathbf{dy}$.

(b) Take the exterior product $\mathbf{A} \wedge \mathbf{B}$ and show that

$$\theta_{ij} = \varepsilon_{ijk} C^k,$$

where $\theta = \mathbf{A} \wedge \mathbf{B}$ and $\mathbf{C} = \mathbf{a} \times \mathbf{b}$. Show also that the exterior product $\mathbf{A} \wedge \star\mathbf{B}$ is given by the 3-form

$$\mathbf{A} \wedge \star\mathbf{B} = (\mathbf{a} \cdot \mathbf{b})\, \mathbf{dx} \wedge \mathbf{dy} \wedge \mathbf{dz}.$$

5.3. *Poincaré's lemma*

(a) Use Poincaré's lemma $\underline{d}^2 \underline{\alpha} = 0$ for the specific case where $\underline{\alpha}$ is a 0-form, that is a scalar function $f(x,y,z)$, and show that it corresponds to the vector identity $\nabla \times \nabla f = 0$.

(b) If $\underline{A}$ is a 1-form with corresponding vector $\vec{A}$, show that Poincaré's lemma $\underline{d}^2 \underline{A} = 0$ corresponds to the vector identity $\nabla \cdot (\nabla \times \vec{A}) = 0$.

5.4. *Parabolic coordinates*

The connection between the Cartesian coordinates (x,y) (in a plane) and the *parabolic coordinates* (ξ,η) is given by the equations

$$x = \xi\eta, \tag{5.140}$$

$$y = \frac{1}{2}(\xi^2 - \eta^2). \tag{5.141}$$

(a) Find the basis vectors $\vec{e}_\xi$ and $\vec{e}_\eta$, and also the metric tensor $g_{\xi\eta}$ in the parabolic coordinate system. Draw the coordinates for ξ and η in a (x,y)-plane. Then draw the vectors $\vec{e}_\xi$ and $\vec{e}_\eta$ (with their correct lengths) at some points.

(b) Find the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}$.
Find a transformation to an orthonormal basis $\{\vec{e}_{\hat{\xi}}, \vec{e}_{\hat{\eta}}\}$ so that $\vec{e}_{\hat{\xi}}$ is parallel to $\vec{e}_\xi$ and that $\vec{e}_{\hat{\eta}}$ is parallel to $\vec{e}_\eta$.
What is the correspondence between the forms $\underline{\omega}^{\hat{\xi}}, \underline{\omega}^{\hat{\eta}}$ and $\underline{\omega}^{\xi}, \underline{\omega}^{\eta}$?
Find $\underline{\omega}^{\xi}$ and $\underline{\omega}^{\eta}$ expressed by $\underline{\omega}^x$ and $\underline{\omega}^y$.

(c) Find the components of a vector $\vec{A}$ in the orthonormal basis $(A^{\hat{\xi}}, A^{\hat{\eta}})$ expressed by the Cartesian coordinates A^x and A^y.
Let $\underline{A}$ be the corresponding 1-form, $\underline{A} = A_x \underline{\omega}^x + A_y \underline{\omega}^y = A_{\hat{\xi}} \underline{\omega}^{\hat{\xi}} + A_{\hat{\eta}} \underline{\omega}^{\hat{\eta}}$. What is the connection between $(A^{\hat{\xi}}, A^{\hat{\eta}})$ and $(A_{\hat{\xi}}, A_{\hat{\eta}})$?

What is the correspondence between the exterior derivative of $\underline{A}$ and $\nabla \times \vec{A}$ (see Problem 5.2)? Use this correspondence in order to find $\nabla \times \vec{A}$ in parabolic coordinates expressed by $A^{\hat{\xi}}$ and $A^{\hat{\eta}}$.

5.5. *Covariant derivative*

(a) Assume that $A^{\mu\nu}{}_{\lambda}$ transforms as a tensor. Show that $A^{\mu\nu}{}_{\nu}$ then transforms as a vector, but $A^{\mu\mu}{}_{\nu}$ does not.
(b) Show, by using the expression

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2}\left(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}\right) , \tag{5.142}$$

that $\Gamma^{\mu}{}_{\nu\lambda}$ is not a tensor.
(c) Assume that $A^{\mu}(x)$ is a vector field. Show that $A^{\mu}{}_{,\nu} = \frac{\partial A^{\mu}}{\partial x^{\nu}}$ does not transform as a tensor, but that the covariant derivative,

$$A^{\mu}{}_{;\nu} = A^{\mu}{}_{,\nu} + \Gamma^{\mu}{}_{\alpha\nu}A^{\alpha}, \tag{5.143}$$

does.
(d) Show that

$$g_{\mu\nu;\lambda} = 0 . \tag{5.144}$$

(e) Show that covariant derivative satisfies the following:

$$(A^{\mu}B_{\nu})_{;\lambda} = A^{\mu}{}_{;\lambda}B_{\nu} + A^{\mu}B_{\nu;\lambda} . \tag{5.145}$$

(f) Show that the covariant derivative can be expressed as

$$\nabla \cdot \vec{A} \equiv A^{\mu}{}_{;\mu} = \frac{1}{\sqrt{|g|}}\partial_{\mu}\left(\sqrt{|g|}A^{\mu}\right) , \tag{5.146}$$

where now g is the determinant of the metric tensor, $g = \det(g_{\mu\nu})$. (Hint: Use that $\partial_{\alpha} g \equiv g_{,\alpha} = g\, g^{\mu\nu} g_{\mu\nu,\alpha}$.)

5.6. *Geodetic curves in space*

(a) In the 2-dimensional Euclidean plane the line element is given by $ds^2 = dx^2 + dy^2$. A curve $y = y(x)$ connects two points A and B in the plane. The distance between A and B along the curve is therefore

$$S = \int_A^B ds = \int_A^B \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx . \tag{5.147}$$

If we vary the shape of this curve slightly, but keeping the end points A and B fixed, it would lead to a change δS of the length of the curve. Whenever $\delta S = 0$ for all small arbitrary variations with respect to a given curve, then the curve is a geodetic curve. Find the Euler–Lagrange equation which corresponds to $\delta S = 0$ and show that geodetic curves in the plane are straight lines.

(b) A particle with mass m is moving without friction on a 2-dimensional surface embedded in the 3-dimensional space. Write down the expressions for the Lagrangian, L, and the corresponding Euler–Lagrange equations for the particle. Show that L is a constant of motion and explain this by referring to the forces acting on the particle.
The geodetic curves are found by variation of $S = \int_A^B ds$. Show, using the Euler–Lagrange equations, that the particle is moving along a geodetic curve with constant speed.

(c) A particle is moving without friction on a sphere. Express the Lagrange function in terms of the polar angles θ and ϕ and find the corresponding Euler–Lagrange equations.
The coordinate axes can be chosen so that at the time $t = 0$, $\theta = \pi/2$ and $\dot{\theta} = 0$. Show, using the Euler–Lagrange equations, that this implies that θ is constant and equal to $\pi/2$ for all t. Hence, the particle is moving on a great circle, i.e. on a geodetic curve on the sphere.
Assume further that at $t = 0$, $\theta = \pi/2$ and $\phi = 0$, and at $t = t_1 > 0$, $\phi = \theta = \pi/2$. Along what type of different curves can the particle have travelled for $0 < t < t_1$ so that $\delta \int_0^{t_1} L dt = 0$ for the different curves? Find the action integral $S = \int_0^{t_1} L dt$ for the different curves. Do all the curves correspond to local minima for the total length S?

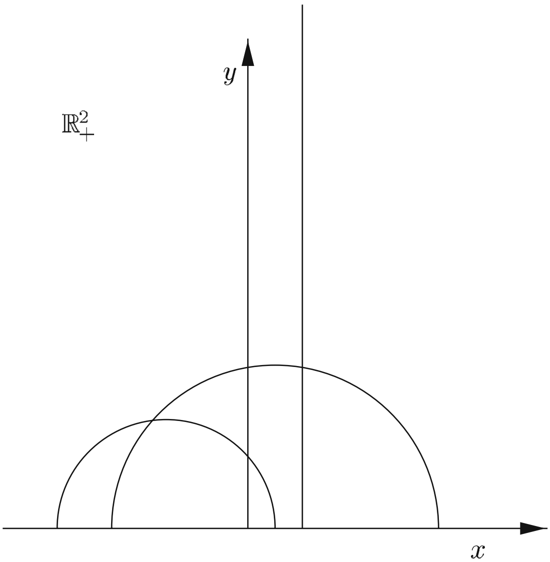


Fig. 5.12 Geodesics in the Poincaré half-plane

5.7. *The Poincaré half-plane*
The Poincaré half-plane is the upper half of R^2 given by $R^2_+ = \{(x,y) \in R^2 | y > 0\}$ equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \,. \tag{5.148}$$

(a) Use the orthonormal frame formalism and calculate the rotation forms.
(b) Using for instance the variational principle, show that the geodesics are semi-circles centred at $y = 0$ or lines of constant x.

Reference

1. Grøn, Ø. and Hervik, S. 2007. *Einstein's General Theory of Relativity*, first edn, Springer, New York. 117
2. Pound, R. V. and Rebka, G. A. 1960. Apparent weight of photons, *Phys. Rev. Lett.* **4**, 337–341. xvii, 118

Chapter 6
Curvature

6.1 The Riemann Curvature Tensor

The covariant directional derivative of a vector field $\vec{A}$ along a vector $\vec{u}$ was defined and interpreted geometrically in Sect. 5.2, as follows:

$$\begin{aligned} \nabla_{\vec{v}}\vec{A} &= \frac{d\vec{A}}{d\lambda} = A^{\mu}{}_{;\nu} v^{\nu} \vec{e}_{\mu} \\ &= \lim_{\Delta\lambda \to 0} \frac{\vec{A}_{QP}(\lambda + \Delta\lambda) - \vec{A}(\lambda)}{\Delta\lambda}\,. \end{aligned} \tag{6.1}$$

Let $\vec{A}_{QP}$ be the parallel transported of $\vec{A}$ from Q to P (Figs. 6.1 and 6.2). Then to first order in $\Delta\lambda$ we have $\vec{A}_{QP} = \vec{A}_P + (\nabla_{\vec{v}}\vec{A})_P \Delta\lambda$ and

$$\vec{A}_{PQ} = \vec{A}_Q - (\nabla_{\vec{v}}\vec{A})_Q \Delta\lambda\,. \tag{6.2}$$

To second order in $\Delta\lambda$ we have

$$\vec{A}_{PQ} = \left(1 - \nabla_{\vec{v}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2\right)\vec{A}_Q. \tag{6.3}$$

If $\vec{A}_{PQ}$ is parallel transported further onto R we get

$$\begin{aligned} \vec{A}_{PQR} = &\left(1 - \nabla_{\vec{u}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{u}}\nabla_{\vec{u}}(\Delta\lambda)^2\right) \\ &\cdot \left(1 - \nabla_{\vec{v}}\Delta\lambda + \frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2\right)\vec{A}_R, \end{aligned} \tag{6.4}$$

where $\vec{A}_Q$ is replaced by $\vec{A}_R$ because the differential operator always shall be applied to the vector in the first position. If we parallel transport $\vec{A}$ around the whole polygon in Fig. 6.3 we get

Grøn, Ø.: *Curvature*. Lect. Notes Phys. **772**, 133–147 (2009)
DOI 10.1007/978-0-387-88134-8_6

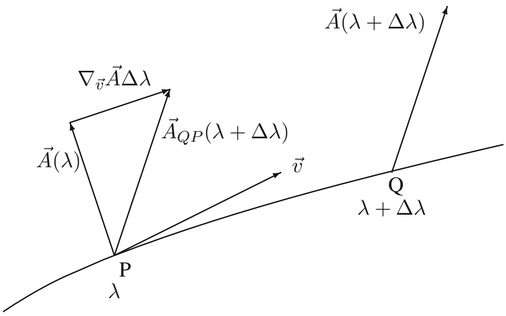


Fig. 6.1 Parallel transport of $\vec{A}$

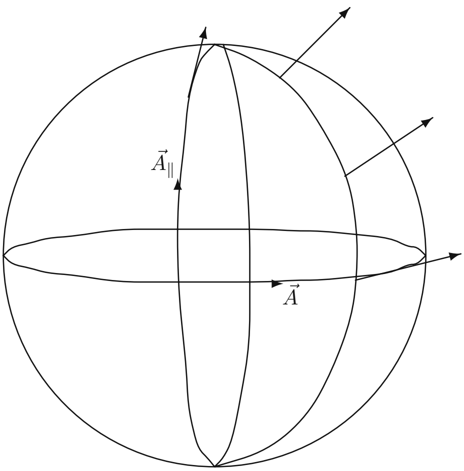


Fig. 6.2 A vector which is parallel transported around a triangle of angles 90° is rotated 90°

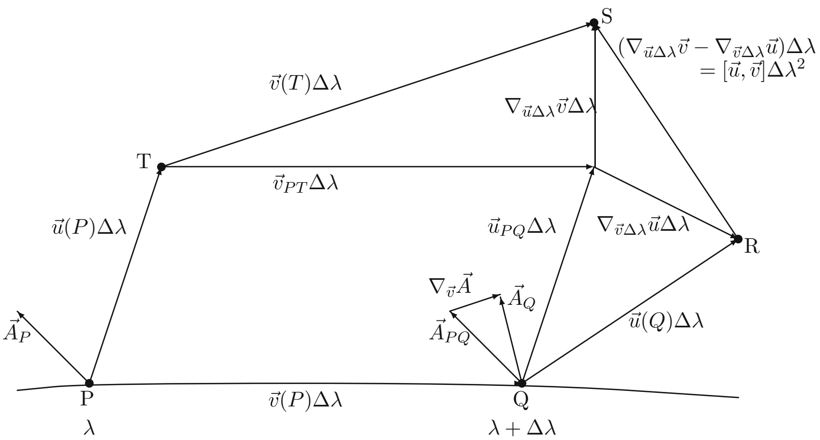


Fig. 6.3 Geometrically implied curvature from non-zero differences between vectors along a curve (parameterized by λ) and their parallel-transported equivalents

$$
\begin{aligned}
\vec{A}_{PQRSTP} = {} & \left(1+\nabla_{\vec{u}}\Delta\lambda+\frac{1}{2}\nabla_{\vec{u}}\nabla_{\vec{u}}(\Delta\lambda)^2\right) \\
& \cdot\left(1+\nabla_{\vec{v}}\Delta\lambda+\frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2\right) \\
& \cdot\left(1-\nabla_{[\vec{u},\vec{v}]}(\Delta\lambda)^2\right)\cdot\left(1-\nabla_{\vec{u}}\Delta\lambda+\frac{1}{2}\nabla_{\vec{u}}\nabla_{\vec{u}}(\Delta\lambda)^2\right) \\
& \cdot\left(1-\nabla_{\vec{v}}\Delta\lambda+\frac{1}{2}\nabla_{\vec{v}}\nabla_{\vec{v}}(\Delta\lambda)^2\right)\vec{A}_P\,.
\end{aligned}
\tag{6.5}
$$

Calculating to second order in $\Delta\lambda$ gives

$$\vec{A}_{PQRSTP} = \vec{A}_P + \left([\nabla_{\vec{u}},\nabla_{\vec{v}}] - \nabla_{[\vec{u},\vec{v}]}\right)(\Delta\lambda)^2\vec{A}_P. \tag{6.6}$$

There is a variation of the vector under parallel transport around the closed polygon:

$$\delta\vec{A} = \vec{A}_{PQRSTP} - \vec{A}_P = \left([\nabla_{\vec{u}},\nabla_{\vec{v}}] - \nabla_{[\vec{u},\vec{v}]}\right)\vec{A}_P(\Delta\lambda)^2. \tag{6.7}$$

We now introduce the Riemann's curvature tensor as

$$R(\quad,\vec{A},\vec{u},\vec{v}) \equiv \left([\nabla_{\vec{u}},\nabla_{\vec{v}}] - \nabla_{[\vec{u},\vec{v}]}\right)(\vec{A}). \tag{6.8}$$

The components of the Riemann curvature tensor is defined by applying the tensor on basis vectors

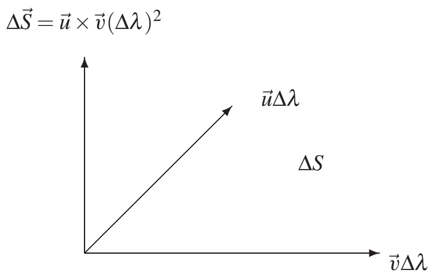
$$R^{\mu}_{\ \nu\alpha\beta}\vec{e}_\mu \equiv \left([\nabla_\alpha,\nabla_\beta] - \nabla_{[\vec{e}_\alpha,\vec{e}_\beta]}\right)(\vec{e}_\nu). \tag{6.9}$$

Antisymmetry follows from the definition

$$R^{\mu}_{\ \nu\beta\alpha} = -R^{\mu}_{\ \nu\alpha\beta}\,. \tag{6.10}$$

The expression for the change of $\vec{A}$ under parallel transport around the polygon, Eq. (6.7), can now be written as

$$
\begin{aligned}
\Delta\vec{A} &= R(\quad,\vec{A},\vec{u},\vec{v})(\Delta\lambda)^2 \\
&= R(\quad,A^\nu\vec{e}_\nu,u^\alpha\vec{e}_\alpha,v^\beta\vec{e}_\beta)(\Delta\lambda)^2 \\
&= \vec{e}_\mu R^{\mu}_{\ \nu\alpha\beta}A^\nu u^\alpha v^\beta\cdot(\Delta\lambda)^2 \\
&= \frac{1}{2}\vec{e}_\mu R^{\mu}_{\ \nu\alpha\beta}A^\nu\left(u^\alpha v^\beta - u^\beta v^\alpha\right)(\Delta\lambda)^2.
\end{aligned}
\tag{6.11}
$$



The area of the parallelogram defined by the vectors $\vec{u}\Delta\lambda$ and $\vec{v}\Delta\lambda$ is

$$\Delta\vec{S} = \vec{u}\times\vec{v}(\Delta\lambda)^2.$$

Using that

$$(\vec{u}\times\vec{v})^{\alpha\beta} = u^\alpha v^\beta - u^\beta v^\alpha \ ,$$

we can write Eq. (6.11) as

$$\boxed{\Delta\vec{A} = \frac{1}{2}A^\nu R^\mu{}_{\nu\alpha\beta}\Delta S^{\alpha\beta}\vec{e}_\mu} \ . \tag{6.12}$$

The components of the Riemann tensor expressed by the connection and structure coefficients are given below:

$$\begin{aligned}
\vec{e}_\mu R^\mu{}_{\nu\alpha\beta} &= [\nabla_\alpha, \nabla_\beta]\vec{e}_\nu - \nabla_{[\vec{e}_\alpha,\vec{e}_\beta]}\vec{e}_\nu \\
&= (\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha - c^\rho{}_{\alpha\beta}\nabla_\rho)\vec{e}_\nu \\
&= \nabla_\alpha\nabla_\beta\vec{e}_\nu - \nabla_\beta\nabla_\alpha\vec{e}_\nu - c^\rho{}_{\alpha\beta}\nabla_\rho\vec{e}_\nu \\
\text{(Kozul connection)} \quad &= \nabla_\alpha\Gamma^\mu{}_{\nu\beta}\vec{e}_\mu - \nabla_\beta\Gamma^\mu{}_{\nu\alpha}\vec{e}_\mu - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho}\vec{e}_\mu \\
&= (\nabla_\alpha\Gamma^\mu{}_{\nu\beta})\vec{e}_\mu + \Gamma^\mu{}_{\nu\beta}\nabla_\alpha\vec{e}_\mu \\
&\quad - (\nabla_\beta\Gamma^\mu{}_{\nu\alpha})\vec{e}_\mu - \Gamma^\mu{}_{\nu\alpha}\nabla_\beta\vec{e}_\mu - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho}\vec{e}_\mu \\
&= \vec{e}_\alpha(\Gamma^\mu{}_{\nu\beta})\vec{e}_\mu + \Gamma^\rho{}_{\nu\beta}\Gamma^\mu{}_{\rho\alpha}\vec{e}_\mu \\
&\quad - \vec{e}_\beta(\Gamma^\mu{}_{\nu\alpha})\vec{e}_\mu - \Gamma^\rho{}_{\nu\alpha}\Gamma^\mu{}_{\rho\beta}\vec{e}_\mu - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho}\vec{e}_\mu \ .
\end{aligned} \tag{6.13}$$

This gives (in arbitrary basis)

$$\begin{aligned}
R^\mu{}_{\nu\alpha\beta} &= \vec{e}_\alpha(\Gamma^\mu{}_{\nu\beta}) - \vec{e}_\beta(\Gamma^\mu{}_{\nu\alpha}) \\
&\quad + \Gamma^\rho{}_{\nu\beta}\Gamma^\mu{}_{\rho\alpha} - \Gamma^\rho{}_{\nu\alpha}\Gamma^\mu{}_{\rho\beta} - c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho} \ .
\end{aligned} \tag{6.14}$$

In coordinate basis Eq. (6.14) is reduced to

$$R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu{}_{\nu\beta,\alpha} - \Gamma^\mu{}_{\nu\alpha,\beta} + \Gamma^\rho{}_{\nu\beta}\Gamma^\mu{}_{\rho\alpha} - \Gamma^\rho{}_{\nu\alpha}\Gamma^\mu{}_{\rho\beta} \ , \tag{6.15}$$

where $\Gamma^\mu{}_{\nu\beta} = \Gamma^\mu{}_{\beta\nu}$ are the Christoffel symbols.

Due to the antisymmetry (6.10) we can define a matrix of *curvature forms*:

$$\underline{R}^\mu{}_\nu = \frac{1}{2}R^\mu{}_{\nu\alpha\beta}\underline{\omega}^\alpha \wedge \underline{\omega}^\beta. \tag{6.16}$$

Inserting the components of the Riemann tensor from Eq. (6.14) gives

$$\underline{R}^\mu{}_\nu = \left(\vec{e}_\alpha(\Gamma^\mu{}_{\nu\beta}) + \Gamma^\rho{}_{\nu\beta}\Gamma^\mu{}_{\rho\alpha} - \frac{1}{2}c^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\nu\rho}\right)\underline{\omega}^\alpha \wedge \underline{\omega}^\beta. \tag{6.17}$$

The connection forms:

$$\underline{\Omega}^{\mu}_{\ \nu} = \Gamma^{\mu}_{\ \nu\alpha}\underline{\omega}^{\alpha}. \tag{6.18}$$

Exterior derivatives of basis forms:

$$\underline{d\omega}^{\rho} = -\frac{1}{2}c^{\rho}_{\ \alpha\beta}\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta}. \tag{6.19}$$

Exterior derivatives of connection forms (C1: $\underline{d\omega}^{\rho} = -\underline{\Omega}^{\rho}_{\ \alpha} \wedge \underline{\omega}^{\alpha}$):

$$\begin{aligned} \underline{d\Omega}^{\mu}_{\ \nu} &= \underline{d}\Gamma^{\mu}_{\ \nu\beta} \wedge \underline{\omega}^{\beta} + \Gamma^{\mu}_{\ \nu\rho}\underline{d\omega}^{\rho} \\ &= \vec{e}_{\alpha}(\Gamma^{\mu}_{\ \nu\beta})\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} - \frac{1}{2}c^{\rho}_{\ \alpha\beta}\Gamma^{\mu}_{\ \nu\rho}\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta}. \end{aligned} \tag{6.20}$$

The curvature forms can now be written as

$$\boxed{\underline{R}^{\mu}_{\ \nu} = \underline{d\Omega}^{\mu}_{\ \nu} + \underline{\Omega}^{\mu}_{\ \lambda} \wedge \underline{\Omega}^{\lambda}_{\ \nu}}\,. \tag{6.21}$$

This is Cartans 2nd structure equation.

6.2 Differential Geometry of Surfaces

Imagine an arbitrary surface embedded in an Euclidean 3-dimensional space. (See Fig. 6.4.) Coordinate vectors on the surface:

$$\vec{e}_u = \frac{\partial}{\partial u}, \vec{e}_v = \frac{\partial}{\partial v}, \tag{6.22}$$

where u and v are coordinates on the surface.

Line element on the surface:

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{6.23}$$

with $x^1 = u$ and $x^2 = v$.
(1st fundamental form)

The directional derivatives of the basis vectors are written as

$$\vec{e}_{\mu,\nu} = \Gamma^{\alpha}_{\mu\nu}\vec{e}_{\alpha} + K_{\mu\nu}\vec{N}, \alpha = 1, 2. \tag{6.24}$$

Greek indices run through the surface coordinates, $\vec{N}$ is a unit vector orthogonal to the surface.

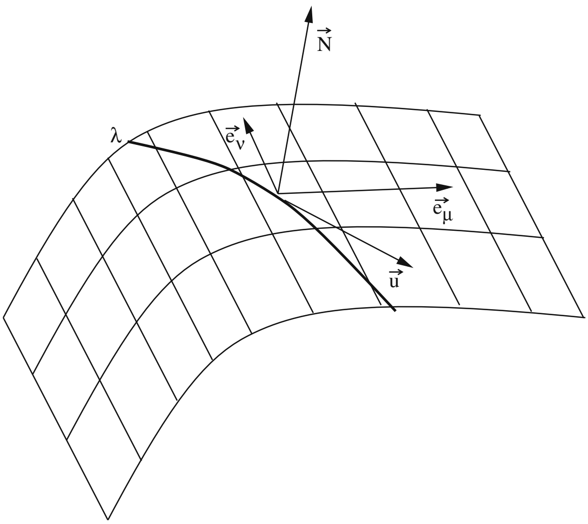


Fig. 6.4 The geometry of a surface. We see the normal vector and the unit vectors of the tangent plane of a point on the surface

The equation above is called Gauss' equation. We have $K_{\mu\nu} = \vec{e}_{\mu,\nu} \cdot \vec{N}$. In coordinate basis, we have $\vec{e}_{\mu,\nu} = \frac{\partial^2}{\partial x^\mu \partial x^\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\mu} = \vec{e}_{\nu,\mu}$. It follows that

$$K_{\mu\nu} = K_{\nu\mu}. \tag{6.25}$$

Let $\vec{u}$ be the unit tangent vector to a curve on the surface, parameterized by λ. Differentiating $\vec{u}$ along the curve,

$$\frac{d\vec{u}}{d\lambda} = u^\mu_{;\nu} u^\nu \vec{e}_\mu + \underbrace{K_{\mu\nu} u^\mu u^\nu}_{\text{2nd fundamental form}} \vec{N}. \tag{6.26}$$

We define κ_g and κ_N by

$$\frac{d\vec{u}}{d\lambda} = \kappa_g \vec{e} + \kappa_N \vec{N}. \tag{6.27}$$

κ_g is called *geodesic curvature*. κ_N is called *normal curvature* (external curvature). $\kappa_g = 0$ for geodesic curves on the surface:

$$\begin{aligned} \kappa_g \vec{e} &= u^\mu_{;\nu} u^\nu \vec{e}_\mu = \nabla_{\vec{u}} \vec{u}, \\ \kappa_N &= K_{\mu\nu} u^\mu u^\nu \\ \text{and } \kappa_N &= \frac{d\vec{u}}{d\lambda} \cdot \vec{N}. \end{aligned} \tag{6.28}$$

We also have that $\vec{u} \cdot \vec{N} = 0$ along the whole curve. Differentiation gives

$$\frac{d\vec{u}}{d\lambda} \cdot \vec{N} + \vec{u} \cdot \frac{d\vec{N}}{d\lambda} = 0, \tag{6.29}$$

which gives

$$\kappa_N = -\vec{u} \cdot \frac{d\vec{N}}{d\lambda}, \tag{6.30}$$

which is called Weingarten's equation.

κ_g and κ_N together give a complete description of the geometry of a surface in a flat 3-dimensional space. We are now going to consider geodesic curves through a point on the surface. Tangent vector $\vec{u} = u^\mu \vec{e}_\mu$ with $\vec{u} \cdot \vec{u} = g_{\mu\nu} u^\mu u^\nu = 1$. Directions with maximum and minimum values for the normal curvatures are found by extremalizing κ_N under the condition $g_{\mu\nu} u^\mu u^\nu = 1$. We then solve the variation problem $\delta F = 0$ for arbitrary u^μ, where $F = K_{\mu\nu} u^\mu u^\nu - k(g_{\mu\nu} u^\mu u^\nu - 1)$. Here k is the Lagrange multiplicator. Variation with respect to u^μ gives

$$\delta F = 2(K_{\mu\nu} - k g_{\mu\nu}) u^\nu \delta u^\mu .$$

$\delta F = 0$ for arbitrary δu^μ demands

$$(K_{\mu\nu} - k g_{\mu\nu}) u^\nu = 0. \tag{6.31}$$

For this system of equations to have non-zero solutions, we must have

$$\det(K_{\mu\nu} - k g_{\mu\nu}) = 0, \tag{6.32}$$

$$\begin{vmatrix} K_{11} - k g_{11} & K_{12} - k g_{12} \\ K_{21} - k g_{21} & K_{22} - k g_{22} \end{vmatrix} = 0. \tag{6.33}$$

This gives the following quadratic equation for k:

$$k^2 \det(g_{\mu\nu}) - (g_{11} K_{22} - 2 g_{12} K_{12} + g_{22} K_{11}) k + \det(K_{\mu\nu}) = 0 \tag{6.34}$$
$$(K \text{ symmetric } K_{12} = K_{21}).$$

The equation has two solutions, k_1 and k_2. These are the extremal values of k.

To find the meaning of k, we multiply Eq. (6.31) by u^μ:

$$\begin{aligned} 0 &= (K_{\mu\nu} - k g_{\mu\nu}) u^\mu u^\nu \\ &= K_{\mu\nu} u^\mu u^\nu - k g_{\mu\nu} u^\mu u^\nu \\ &= \kappa_N - k \Rightarrow k = \kappa_N . \end{aligned} \tag{6.35}$$

The extremal values of κ_N are called the principal curvatures of the surface. Let the directions of the geodesics with extreme normal curvature be given by the tangent vectors $\vec{u}$ and $\vec{v}$. Equation (6.31) gives

$$K_{\mu\nu} u^\nu = k g_{\mu\nu} u^\nu . \tag{6.36}$$

We then get

$$\begin{aligned} K_{\mu\nu}u^\nu v^\mu &= k_1 g_{\mu\nu}u^\nu v^\mu \\ &= k_1 u_\mu v^\mu = k_1(\vec{u}\cdot\vec{v}), \\ K_{\mu\nu}v^\nu u^\mu &= k_2 g_{\mu\nu}v^\nu u^\mu = k_2(\vec{u}\cdot\vec{v}), \end{aligned}$$

which give

$$\begin{aligned} (k_1-k_2)(\vec{u}\cdot\vec{v}) &= K_{\mu\nu}(u^\nu v^\mu - v^\nu u^\mu) \\ &= 2K_{\mu\nu}u^{[\nu}v^{\mu]}. \end{aligned} \tag{6.37}$$

$K_{\mu\nu}$ is symmetric in μ and ν. So we get $(k_1-k_2)(\vec{u}\cdot\vec{v}) = 0$. For $k_1 \neq k_2$ we have to demand $\vec{u}\cdot\vec{v} = 0$. So the geodesics with extremal normal curvature are orthogonal to each other.

The *Gaussian curvature* (at a point) is defined as

$$\boxed{K = \kappa_{N1}\cdot\kappa_{N2}}\ . \tag{6.38}$$

Since κ_{N1} and κ_{N2} are solutions of the quadratic equation above, we get

$$K = \frac{\det(K_{\mu\nu})}{\det(g_{\mu\nu})}. \tag{6.39}$$

6.2.1 Surface Curvature Using the Cartan Formalism

In each point on the surface we have an orthonormal set of basis vectors. Greek indices run through the surface coordinates (2-dimensional) and Latin indices through the space coordinates (3-dimensional):

$$\vec{e}_a = (\vec{e}_1, \vec{e}_2, \vec{N}), \quad \vec{e}_\mu = \{\vec{e}_1, \vec{e}_2\}. \tag{6.40}$$

Using the exterior derivative and form formalism, we find how the unit vectors on the surface change:

$$\begin{aligned} \underline{d}\vec{e}_\nu &= \vec{e}_a \otimes \underline{\Omega}^a{}_\nu \\ &= \vec{e}_\alpha \otimes \underline{\Omega}^\alpha{}_\nu + \vec{N}\otimes\underline{\Omega}^3{}_\nu\,, \end{aligned} \tag{6.41}$$

where $\underline{\Omega}^\mu{}_\nu = \Gamma^\mu{}_{\nu\alpha}\underline{\omega}^\alpha$ are the connection forms on the surface, i.e. the intrinsic connection forms. The extrinsic connection forms are

$$\underline{\Omega}^3{}_\nu = K_{\nu\alpha}\underline{\omega}^\alpha\,, \underline{\Omega}^\mu{}_3 = K^\mu_\alpha\underline{\omega}^\alpha. \tag{6.42}$$

We let the surface be embedded in an Euclidean (flat) 3-dimensional space. This means that the curvature forms of the 3-dimensional space are zero:

$$\underline{R}^{a}{}_{3b} = 0 = \underline{d}\,\underline{\Omega}^{a}{}_{b} + \underline{\Omega}^{a}{}_{k} \wedge \underline{\Omega}^{k}{}_{b}, \tag{6.43}$$

which gives

$$\begin{aligned} R^{\mu}{}_{3\nu} = 0 &= \underline{d}\,\underline{\Omega}^{\mu}{}_{\nu} + \underline{\Omega}^{\mu}{}_{\alpha} \wedge \underline{\Omega}^{\alpha}{}_{\nu} + \underline{\Omega}^{\mu}{}_{3} \wedge \underline{\Omega}^{3}{}_{\nu} \\ &= \underline{R}^{\mu}{}_{\nu} + \underline{\Omega}^{\mu}{}_{3} \wedge \underline{\Omega}^{3}{}_{\nu}\,, \end{aligned} \tag{6.44}$$

where $\underline{R}^{\mu}{}_{\nu}$ are the **curvature forms of the surface**. We then have

$$\frac{1}{2} R^{\mu}{}_{\nu\alpha\beta}\,\underline{\omega}^{\alpha} \wedge \underline{\omega}^{\beta} = -\underline{\Omega}^{\mu}{}_{3} \wedge \underline{\Omega}^{3}{}_{\nu}. \tag{6.45}$$

Inserting the components of the extrinsic connection forms, we get (when we remember the antisymmetry of α and β in $\underline{R}^{\mu}{}_{\nu\alpha\beta}$)

$$R^{\mu}{}_{\nu\alpha\beta} = K^{\mu}{}_{\alpha} K_{\nu\beta} - K^{\mu}{}_{\beta} K_{\nu\alpha}. \tag{6.46}$$

We now lower the first index:

$$R_{\mu\nu\alpha\beta} = K_{\mu\alpha} K_{\nu\beta} - K_{\mu\beta} K_{\nu\alpha}. \tag{6.47}$$

$R_{\mu\nu\alpha\beta}$ are the components of a curvature tensor which **only** refer to the dimensions of the surface. In particular we have

$$R_{1212} = K_{11} K_{22} - K_{12} K_{21} = \det K. \tag{6.48}$$

We then have the following connection between this component of the Riemann curvature tensor of the surface and the Gaussian curvature of the surface:

$$K = \kappa_{N1} \cdot \kappa_{N2} = \frac{\det K_{\mu\nu}}{\det g_{\mu\nu}} = \frac{R_{1212}}{\det g_{\mu\nu}}. \tag{6.49}$$

Since the right-hand side refers to the intrinsic curvature and the metric on the surface, we have proved that the Gaussian curvature of a surface is an intrinsic quantity. It can be measured by observers on the surface without embedding the surface in a 3-dimensional space. This is the contents of Gauss' **theorema egregium**.

6.3 The Ricci Identity

$$\vec{e}_{\mu} R^{\mu}{}_{\nu\alpha\beta} A^{\nu} = (\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha} - \nabla_{[\vec{e}_{\alpha},\vec{e}_{\beta}]})(\vec{A}). \tag{6.50}$$

In coordinate basis this is reduced to

$$\vec{e}_\mu R^\mu{}_{\nu\alpha\beta} A^\nu = (A^\mu{}_{;\beta\alpha} - A^\mu{}_{;\alpha\beta})\vec{e}_\mu \,, \tag{6.51}$$

where

$$A^\mu{}_{;\alpha\beta} \equiv (A^\mu{}_{;\beta})_{;\alpha}. \tag{6.52}$$

The **Ricci identity** on component form is

$$A^\nu R^\mu{}_{\nu\alpha\beta} = A^\mu{}_{;\beta\alpha} - A^\mu{}_{;\alpha\beta}. \tag{6.53}$$

We can write this as

$$\underline{d}^2\vec{A} = \frac{1}{2} R^\mu{}_{\nu\alpha\beta} A^\nu \vec{e}_\mu \otimes \underline{\omega}^\alpha \wedge \underline{\omega}^\beta. \tag{6.54}$$

This shows us that the 2nd exterior derivative of a vector is equal to zero only in a *flat* space. Equations (6.53) and (6.54) *both* represent the Ricci identity.

6.4 Bianchi's 1st Identity

Cartan's 1st structure equation:

$$\underline{d}\,\underline{\omega}^\mu = -\underline{\Omega}^\mu_\nu \wedge \underline{\omega}^\nu. \tag{6.55}$$

Cartan's 2nd structure equation:

$$\underline{R}^\mu_\nu = \underline{d}\,\underline{\Omega}^\mu_\nu + \underline{\Omega}^\mu_\lambda \wedge \underline{\Omega}^\lambda_\nu. \tag{6.56}$$

Exterior differentiation of Eq. (6.55) and use of *Poincaré's* lemma (5.16) gives ($\underline{d}^2\,\underline{\omega}^\mu = 0$)

$$0 = \underline{d}\,\underline{\Omega}^\mu_\nu \wedge \underline{\omega}^\nu - \underline{\Omega}^\mu_\lambda \wedge \underline{d}\,\underline{\omega}^\lambda. \tag{6.57}$$

Use of (6.55) gives

$$\underline{d}\,\underline{\Omega}^\mu_\nu \wedge \underline{\omega}^\nu + \underline{\Omega}^\mu_\lambda \wedge \underline{\Omega}^\lambda_\nu \wedge \underline{\omega}^\nu = 0. \tag{6.58}$$

From this we see that

$$(\underline{d}\,\underline{\Omega}^\mu_\nu + \underline{\Omega}^\mu_\lambda \wedge \underline{\Omega}^\lambda_\nu) \wedge \underline{\omega}^\nu = 0. \tag{6.59}$$

We now get **Bianchi's 1st identity**:

$$\boxed{\underline{R}^\mu_\nu \wedge \underline{\omega}^\nu = 0}\,. \tag{6.60}$$

On component form Bianchi's 1st identity is

$$\underbrace{\frac{1}{2} R^\mu{}_{\nu\alpha\beta}\,\underline{\omega}^\alpha \wedge \underline{\omega}^\beta}_{R^\mu_\nu} \wedge \underline{\omega}^\nu = 0. \tag{6.61}$$

The component equation is (remember the antisymmetry in α and β)

$$R^{\mu}_{\ [\nu\alpha\beta]} = 0 \tag{6.62}$$

or

$$R^{\mu}_{\ \nu\alpha\beta} + R^{\mu}_{\ \alpha\beta\nu} + R^{\mu}_{\ \beta\nu\alpha} = 0, \tag{6.63}$$

where the antisymmetry $R^{\mu}_{\ \nu\alpha\beta} = -R^{\mu}_{\ \nu\beta\alpha}$ has been used. Without this antisymmetry we would have gotten six, and not three, terms in this equation.

6.5 Bianchi's 2nd Identity

Exterior differentiation of (6.56) $\Rightarrow$

$$\begin{aligned} \underline{d}\,\underline{R}^{\mu}_{\ \nu} &= \underline{R}^{\mu}_{\ \lambda} \wedge \underline{\Omega}^{\lambda}_{\ \nu} - \underline{\Omega}^{\mu}_{\ \rho} \wedge \underline{\Omega}^{\rho}_{\ \lambda} \wedge \underline{\Omega}^{\lambda}_{\ \nu} - \underline{\Omega}^{\mu}_{\ \lambda} \wedge \underline{R}^{\lambda}_{\ \nu} + \underline{\Omega}^{\mu}_{\ \lambda} \wedge \underline{\Omega}^{\lambda}_{\ \rho} \wedge \underline{\Omega}^{\rho}_{\ \nu} \\ &= \underline{R}^{\mu}_{\ \lambda} \wedge \underline{\Omega}^{\lambda}_{\ \nu} - \underline{\Omega}^{\mu}_{\ \lambda} \wedge \underline{R}^{\lambda}_{\ \nu}. \end{aligned} \tag{6.64}$$

We now have **Bianchi's 2nd identity** as a form equation:

$$\boxed{\underline{d}\,\underline{R}^{\mu}_{\ \nu} + \underline{\Omega}^{\mu}_{\ \lambda} \wedge \underline{R}^{\lambda}_{\ \nu} - \underline{R}^{\mu}_{\ \lambda} \wedge \underline{\Omega}^{\lambda}_{\ \nu} = 0}\,. \tag{6.65}$$

As a component equation Bianchi's 2nd identity is given by

$$R^{\mu}_{\ \nu[\alpha\beta;\gamma]} = 0\,. \tag{6.66}$$

Definition 6.5.1 (Contraction) "Contraction" is a tensor operation defined by

$$R_{\nu\beta} \equiv R^{\mu}_{\ \nu\mu\beta}. \tag{6.67}$$

We must here have summation over μ. What we do, then, is constructing a new tensor from another given tensor, with a rank 2 lower than the given one.

The tensor with components $R_{\nu\beta}$ is called **the Ricci curvature tensor**. Another contraction gives **the Ricci curvature scalar**, $R = R^{\mu}_{\ \mu}$.

Riemann curvature tensor has four symmetries. The definition of the Riemann tensor implies that $R^{\mu}_{\ \nu\alpha\beta} = -R^{\mu}_{\ \nu\beta\alpha}$.
Bianchi's 1st identity: $R^{\mu}_{\ [\nu\alpha\beta]} = 0$.

From Cartan's 2nd structure equation follows:

$$\begin{aligned} \underline{R}_{\,\mu\nu} &= \underline{d\Omega}_{\,\mu\nu} + \underline{\Omega}_{\mu\lambda} \wedge \underline{\Omega}^{\lambda}_{\ \nu} \\ \Rightarrow R_{\,\mu\nu\alpha\beta} &= -R_{\,\nu\mu\alpha\beta}\,. \end{aligned} \tag{6.68}$$

By choosing a locally Cartesian coordinate system in an inertial frame we get the following expression for the components of the Riemann curvature tensor:

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}(g_{\mu\beta,\nu\alpha} - g_{\mu\alpha,\nu\beta} + g_{\nu\alpha,\mu\beta} - g_{\nu\beta,\mu\alpha}) , \tag{6.69}$$

from which it follows that $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$. Contraction of μ and α leads to

$$\begin{aligned} R^{\alpha}_{\ \nu\alpha\beta} &= R^{\alpha}_{\ \beta\alpha\nu} \\ \Rightarrow R_{\nu\beta} &= R_{\beta\nu}, \end{aligned} \tag{6.70}$$

that is the Ricci tensor is symmetric. In 4-D the Ricci tensor has 10 independent components.

Problems

6.1. *Parallel transport*

(a) A curve $P(\lambda)$ runs through a point $P = P(0)$, and a vector $\vec{A}$ is defined at this point. The vector is parallel transported along the curve so that in each point $P(\lambda)$ there is a well-defined vector $\vec{A}(\lambda)$. Express the condition of parallelity of the vectors along the curve as an equation of the components of the vector $A^{\mu}(\lambda)$. Show that the change in the components of the vector by an infinitesimal displacement dx^{μ} is given by

$$dA^{\mu} = -\Gamma^{\mu}_{\ \lambda\nu}(x)A^{\lambda}dx^{\nu}. \tag{6.71}$$

(b) A closed curve has the shape of a parallelogram with the sides $d\vec{a}$ and $d\vec{b}$. The corners of the parallelogram are denoted by A, B, C and D, respectively. A vector $\vec{A}$ is parallel transported from A and C along the two curves ABC and ADC. Show that the result in these two cases is in general not the same. Then use this fact to show that the change of $\vec{A}$, by parallel transporting it along the closed curve $ABCDA$, is

$$\delta A^{\alpha} = -R^{\alpha}_{\ \beta\gamma\delta}A^{\beta}da^{\gamma}db^{\delta}, \tag{6.72}$$

where $R^{\alpha}_{\ \beta\gamma\delta}$ is the Riemann curvature tensor.

6.2. *A non-Cartesian coordinate system in two dimensions*
Consider the following metric on a 2-dimensional surface:

$$ds^2 = v^2du^2 + u^2dv^2. \tag{6.73}$$

You are going to show, in two different ways, that this is only the flat Euclidean plane in disguise.

(a) Use the orthonormal frame approach and find the connection 1-forms $\Omega^{\hat{a}}{}_{\hat{b}}$. Find also the curvature 2-forms $\mathbf{R}^{\hat{a}}{}_{\hat{b}}$ and show that they are identically zero.
(b) Show that the metric can be put onto the form

$$ds^2 = dx^2 + dy^2$$

by finding a transformation matrix $\mathsf{M} = \left(M^i{}_a\right)$ connecting the basis vectors $\mathbf{e}_u$ and $\mathbf{e}_v$, and $\mathbf{e}_x$ and $\mathbf{e}_y$. This can be done using the following relations:

$$\begin{aligned} g_{ab} &= g_{ij} M^i{}_a M^j{}_b, \\ \partial d M^i{}_a x^b &= \partial d M^i{}_b x^a. \end{aligned} \tag{6.74}$$

Where do these relations come from?

6.3. *Rotation matrices*
Show that

$$e^{t\mathsf{J}_{(ab)}} = 1 + t\mathsf{J}_{(ab)} + \frac{1}{2}t^2\mathsf{J}^2_{(ab)} + \ldots = \mathsf{R}_{(ab)}(t). \tag{6.75}$$

6.4. *The curvature of a curve*
Consider a curve $y = y(x)$ in a 2-dimensional plane. Utilize that the differential of the arc length is $ds = \left(1 + y'^2\right)^{1/2} dx$ and show that the tangent vector, the curvature vector and the curvature of the curve are given by

$$\begin{aligned} \mathbf{t} &= \left(1 + y'^2\right)^{-1/2} \left(\mathbf{e}_x + y'\mathbf{e}_y\right), \\ \mathbf{k} &= \left(1 + y'^2\right)^{-2} y'' \left(y'\mathbf{e}_x + \mathbf{e}_y\right), \\ \kappa &= \left(1 + y'^2\right)^{-3/2} y'', \end{aligned} \tag{6.76}$$

respectively.

6.5. *The Poincaré half-space*
Consider half of $\mathbb{R}^3$, $z > 0$ with metric

$$ds^2 = \frac{1}{z^2}\left(dx^2 + dy^2 + dz^2\right). \tag{6.77}$$

(a) Calculate the connection forms and the curvature forms using the structural equations of Cartan.
(b) Calculate the Riemann tensor, the Ricci tensor and the Ricci scalar.
(c) Show that

$$R_{abcd} = -(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{6.78}$$

Compare this with the 3-dimensional hyperbolic space. Are there any way we can differentiate between these two cases? Are they different manifestations of the same space?

6.6. *The tidal force pendulum and the curvature of space*
We will again consider the tidal force pendulum. Here we shall use the equation for geodesic deviation to find the period of the pendulum.

(a) Why can the equation for geodesic equation be used to find the period of the pendulum in spite of the fact that the particles do not move along geodesics? Explain also why the equation can be used even though the centre of the pendulum does not follow a geodesic.
(b) Assume that the centre of the pendulum is fixed at a distance R from the centre of mass of the Earth. Introduce an orthonormal basis $\{\mathbf{e}_{\hat{a}}\}$ with the origin at the centre of the pendulum (see Fig. 6.5).
Show that, to first order in v/c and ϕ/c^2, where v is the 3-velocity of the masses and ϕ the gravitational potential at the position of the pendulum, the equation of geodesic equation takes the form

$$\frac{d^2\ell_{\hat{i}}}{dt^2} + R_{\hat{i}\hat{0}\hat{j}\hat{0}}\ell^{\hat{j}} = 0. \tag{6.79}$$

(c) Find the period of the pendulum expressed in terms of the components of Riemann's curvature tensor.

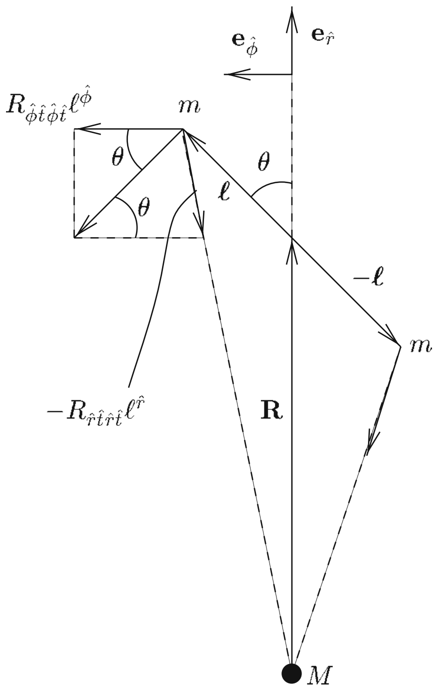


Fig. 6.5 The tidal force pendulum

6.7. *The Weyl tensor vanishes for spaces of constant curvature*
Use the definition of the Weyl curvature in four dimensions to show that the constant curvature spaces S^4, $\mathbb{E}^4$ and $\mathbb{H}^4$ all have zero Weyl tensor.

6.8. *Frobenius' theorem*
In this problem we will consider integrability conditions for vector fields. In particular, we shall obtain a necessary and sufficient criterion for when a vector field is hypersurface orthogonal. Assume that we have an m-dimensional manifold M and a smooth collection of r-dimensional subspaces $\mathscr{D}_p \in T_pM$, one for each $p \in M$. A submanifold N of M is called an *integral submanifold* if $\mathscr{D}_p = T_pN$ for each $p \in N$. Frobenius' theorem can now be stated:

Frobenius' Theorem:

Given a smooth collection of spaces $\mathscr{D}_p$ as above. Then there exists an integral submanifold N at every p if and only if $\{\mathscr{D}_p\}$ is involute; i.e. for all $p \in M$,

$$[\mathbf{X}_i, \mathbf{X}_j] \in \mathscr{D}_p, \quad \forall X_k \in \mathscr{D}_p.$$

(a) Assume that ω^a is a set of 1-forms that span the orthogonal complement of $\mathscr{D}_p$; i.e. $\omega^a(\mathbf{X}_i) = 0$ for all $\mathbf{X}_i \in \mathscr{D}_p$. Show that $\mathbf{d}\omega^a$ must be of the form

$$\mathbf{d}\omega^a = \sum_b \omega^b \wedge \rho^b,$$

where each ρ^b is in T_p^*M. (Hint: Show first that $X^\mu Y^\nu \nabla_{[\mu} \omega^a_{\nu]} = 0$ for all $X^\mu, Y^\nu \in T_pN$.)

(b) In particular, this means if M is an m-dimensional manifold and ξ_μ is a covariant vector field, then ξ_μ is hypersurface orthogonal to some $(m-1)$-dimensional submanifold if and only if $\nabla_{[\mu}\xi_{\nu]} = \xi_{[\mu}\rho_{\nu]}$. Show that a sufficient condition for ξ_μ to be hypersurface orthogonal is

$$\xi_{[\mu}\nabla_\nu \xi_{\sigma]} = 0. \tag{6.80}$$

Chapter 7
Einstein's Field Equations

7.1 Energy–Momentum Conservation

7.1.1 Newtonian Fluid

Energy–momentum conservation for a Newtonian fluid in terms of the divergence of the energy momentum tensor can be shown as follows. The total derivative of a velocity field is

$$\frac{D\vec{v}}{Dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \,. \tag{7.1}$$

$\frac{\partial \vec{v}}{\partial t}$ is the local derivative which gives the change in $\vec{v}$ as a function of time at a given point in space. $(\vec{v} \cdot \vec{\nabla})\vec{v}$ is called the **convective** derivative of $\vec{v}$. It represents the change of $\vec{v}$ for a moving fluid particle due to the inhomogeneity of the fluid velocity field. In component notation the above becomes

$$\frac{Dv^i}{Dt} \equiv \frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \,. \tag{7.2}$$

The continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v^i)}{\partial x^i} = 0 \,. \tag{7.3}$$

Euler's equation of motion (ignoring gravity):

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p \quad \text{or} \quad \rho \left(\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \right) = -\frac{\partial p}{\partial x^i} \,. \tag{7.4}$$

The **energy–momentum tensor** is a symmetric tensor of rank 2 that describes material characteristics:

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} , \tag{7.5}$$

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T^{00} represents energy density.
T^{i0} represents momentum density.
T^{ii} represents pressure ($T^{ii} > 0$).
T^{ii} represents stress ($T^{ii} < 0$).
T^{ij} represents shear forces ($i \neq j$).

Example 7.1.1 (Energy–momentum tensor for a Newtonian fluid)

$$\begin{aligned} T^{00} &= \rho, \qquad T^{i0} = \rho v^i, \\ T^{ij} &= \rho v^i v^j + p\delta^{ij}, \end{aligned} \tag{7.6}$$

where p is pressure, assumed isotropic here. We choose a locally Cartesian coordinate system in an inertial frame such that the covariant derivatives are reduced to partial derivatives. The divergence of the momentum–energy tensor, $T^{\mu\nu}_{;\nu}$, has four components, one for each value of μ.

The zeroth component is

$$\begin{aligned} T^{0\nu}_{;\nu} &= T^{0\nu}_{,\nu} = T^{00}_{,0} + T^{0i}_{,i} \\ &= \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i}, \end{aligned} \tag{7.7}$$

which by comparison to Newtonian hydrodynamics implies that $T^{0\nu}_{;\nu} = 0$ is the continuity equation. This equation represents the conservation of energy.

The ith component of the divergence is

$$\begin{aligned} T^{i\nu}_{,\nu} &= T^{i0}_{,0} + T^{ij}_{,j} \\ &= \frac{\partial(\rho v^i)}{\partial t} + \frac{\partial(\rho v^i v^j + p\delta^{ij})}{\partial x^j} \\ &= \rho\frac{\partial v^i}{\partial t} + v^i\frac{\partial \rho}{\partial t} + v^i\frac{\partial \rho v^j}{\partial x^j} + \rho v^j\frac{\partial v^i}{\partial x^j} + \frac{\partial p}{\partial x^i}; \end{aligned} \tag{7.8}$$

now, according to the continuity equation

$$\begin{aligned} &\frac{\partial(\rho v^i)}{\partial x^i} = -\frac{\partial \rho}{\partial t} \\ \Rightarrow \quad & T^{i\nu}_{,\nu} = \rho\frac{\partial v^i}{\partial t} + v^i\frac{\partial \rho}{\partial t} - v^i\frac{\partial \rho}{\partial t} + \rho v^j\frac{\partial v^i}{\partial x^j} + \frac{\partial p}{\partial x^i} \\ & \qquad = \rho\frac{Dv^i}{Dt} + \frac{\partial p}{\partial x^i}. \\ \therefore \quad & T^{i\nu}_{;\nu} = 0 \Rightarrow \rho\frac{Dv^i}{Dt} = -\frac{\partial p}{\partial x^i}, \end{aligned} \tag{7.9}$$

which is Euler's equation of motion. It expresses the conservation of momentum.

The equations $T^{\mu\nu}_{;\nu} = 0$ are general expressions for energy and momentum conservation.

7.1.2 Perfect Fluids

A perfect fluid is a fluid with no viscosity and is given by the energy–momentum tensor

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + p g_{\mu\nu} \ , \tag{7.10}$$

where ρ and p are the mass–density and the stress, respectively, measured in the fluids rest frame and u_μ are the components of the 4-velocity of the fluid.

In a comoving orthonormal basis the components of the 4-velocity are $u^{\hat{\mu}} = (c,0,0,0)$. Then the energy–momentum tensor is given by

$$T_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} , \tag{7.11}$$

where $p > 0$ is pressure and $p < 0$ is tension.

There are three different types of perfect fluids that are useful:

1. **Dust** or non-relativistic gas is given by $p = 0$ and the energy–momentum tensor $T_{\mu\nu} = \rho u_\mu u_\nu$.
2. **Radiation** or ultra-relativistic gas is given by a traceless energy–momentum tensor, i.e. $T^\mu{}_\mu = 0$. It follows that $p = \frac{1}{3}\rho c^2$.
3. **Vacuum energy**: If we assume that no velocity can be measured relatively to vacuum, then all the components of the energy–momentum tensor must be Lorentz invariant. It follows that $T_{\mu\nu} \propto g_{\mu\nu}$. If vacuum is defined as a perfect fluid we get $p = -\rho c^2$ so that $T_{\mu\nu} = p g_{\mu\nu} = -\rho c^2 g_{\mu\nu}$.

7.2 Einstein's Curvature Tensor

The field equations are assumed to have the form

spacetime curvature $\propto$ momentum–energy tensor.

Also, it is demanded that energy and momentum conservation should follow as a consequence of the field equation. This puts the following constraints on the curvature tensor: It must be a symmetric, divergence-free tensor of rank 2.

Bianchi's 2nd identity:

$$R^\mu{}_{\nu\alpha\beta;\sigma} + R^\mu{}_{\nu\sigma\alpha;\beta} + R^\mu{}_{\nu\beta\sigma;\alpha} = 0 \ . \tag{7.12}$$

Contraction of μ and $\alpha \Rightarrow$

$$\begin{aligned} R^\mu{}_{\nu\mu\beta;\sigma} - R^\mu{}_{\nu\mu\sigma;\beta} + R^\mu{}_{\nu\beta\sigma;\mu} &= 0, \\ R_{\nu\beta;\sigma} - R_{\nu\sigma;\beta} + R^\mu{}_{\nu\beta\sigma;\mu} &= 0; \end{aligned} \tag{7.13}$$

further contraction of ν and σ gives

$$\begin{aligned} R^{\sigma}_{\beta;\sigma} - R^{\sigma}_{\sigma;\beta} + R^{\sigma\mu}_{\sigma\beta;\mu} &= 0, \\ R^{\sigma}_{\beta;\sigma} - R_{;\beta} + R^{\sigma}_{\beta;\sigma} &= 0. \\ \therefore \quad 2R^{\sigma}_{\beta;\sigma} &= R_{;\beta}\,. \end{aligned} \tag{7.14}$$

Thus, we have calculated the divergence of the Ricci tensor,

$$R^{\sigma}_{\beta;\sigma} = \frac{1}{2} R_{;\beta}\,. \tag{7.15}$$

Now we use this expression together with the fact that the metric tensor is covariant and divergence free to construct a new divergence-free curvature tensor:

$$R^{\sigma}_{\beta;\sigma} - \frac{1}{2} R_{;\beta} = 0\,. \tag{7.16}$$

Keeping in mind that $(g^{\sigma}_{\beta} R)_{;\sigma} = g^{\sigma}_{\beta} R_{;\sigma}$ we multiply Eq. (7.16) by g^{β}_{α} to get

$$\begin{aligned} g^{\beta}_{\alpha} R^{\sigma}_{\beta;\sigma} - g^{\beta}_{\alpha} \frac{1}{2} R_{;\beta} &= 0, \\ \left(g^{\beta}_{\alpha} R^{\sigma}_{\beta}\right)_{;\sigma} - \frac{1}{2}\left(g^{\beta}_{\alpha} R\right)_{;\beta} &= 0; \end{aligned} \tag{7.17}$$

interchanging σ and β in the first term of the last equation we get

$$\begin{aligned} \left(g^{\sigma}_{\alpha} R^{\beta}_{\sigma}\right)_{;\beta} - \frac{1}{2}\left(g^{\beta}_{\alpha} R\right)_{;\beta} &= 0 \\ \Rightarrow \quad \left(R^{\beta}_{\alpha} - \frac{1}{2}\delta^{\beta}_{\alpha} R\right)_{;\beta} &= 0, \end{aligned} \tag{7.18}$$

since $g^{\sigma}_{\alpha} R^{\beta}_{\sigma} = \delta^{\sigma}_{\alpha} R^{\beta}_{\sigma} = R^{\beta}_{\alpha}$. So $R^{\beta}_{\alpha} - \frac{1}{2}\delta^{\beta}_{\alpha} R$ is the divergence-free curvature tensor desired.

This tensor is called the Einstein tensor and its covariant components are denoted by $E_{\alpha\beta}$, that is

$$\boxed{E_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R}\,. \tag{7.19}$$

Note that: $E^{\mu\nu}_{;\nu} = 0 \to 4$ equations, giving only 6 equations from $E_{\mu\nu}$, which secures a free choice of coordinate system.

7.3 Einstein's Field Equations

Einstein's field equations:

$$E_{\mu\nu} = \kappa T_{\mu\nu} \tag{7.20}$$

or

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \ . \tag{7.21}$$

Contraction gives

$$\begin{aligned} R - \frac{1}{2} 4R = \kappa T \ , \qquad & \text{where } T \equiv T^{\mu}_{\mu}, \\ R = -\kappa T, & \end{aligned} \tag{7.22}$$

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} (-\kappa T) + \kappa T_{\mu\nu} \ \ , \tag{7.23}$$

thus the field equations may be written in the form

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \ . \tag{7.24}$$

In the Newtonian limit the metric may be written as

$$ds^2 = -\left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 + (1 + h_{ii})(dx^2 + dy^2 + dz^2) \ , \tag{7.25}$$

where the Newtonian potential $|\phi| \ll c^2$, and h_{ii} is a perturbation of the metric satisfying $|h_{ii}| << 1$. We also have $T_{00} \gg T_{kk}$ and $T \approx -T_{00}$. Then the 00-component of the field equations becomes

$$R_{00} \approx \frac{\kappa}{2} T_{00} \ . \tag{7.26}$$

Furthermore we have

$$\begin{aligned} R_{00} = R^{\mu}{}_{0\mu 0} &= R^{i}{}_{0i0} \\ &= \Gamma^{i}{}_{00,i} - \Gamma^{i}{}_{0i,0} \\ &= \frac{\partial \Gamma^{k}{}_{00}}{\partial x^k} = \frac{1}{c^2} \nabla^2 \phi \ . \end{aligned} \tag{7.27}$$

Since $T_{00} \approx \rho c^2$, Eq. (7.26) can be written as $\nabla^2 \phi = \frac{1}{2} \kappa c^4 \rho$. Comparing this equation with the Newtonian law of gravitation on local form, $\nabla^2 \phi = 4\pi G \rho$, we see that $\kappa = \frac{8\pi G}{c^4}$.

In classical vacuum we have $T_{\mu\nu} = 0$, which gives

$$\boxed{E_{\mu\nu} = 0 \quad \text{or} \quad R_{\mu\nu} = 0 \ .} \tag{7.28}$$

These are the "vacuum field equations". Note that $R_{\mu\nu} = 0$ does *not* imply $R_{\mu\nu\alpha\beta} = 0$.

Digression 7.3.1 (Lagrange variation principle) It was shown by Hilbert that the field equations may be deduced from a variation principle with action

$$\int R\sqrt{-g}d^4x \ , \tag{7.29}$$

where $R\sqrt{-g}$ is the Lagrange density. One may also include a so-called cosmological constant Λ:

$$\int (R+2\Lambda)\sqrt{-g}d^4x \, . \tag{7.30}$$

The field equations with cosmological constant are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \ . \tag{7.31}$$

7.4 The "Geodesic Postulate" as a Consequence of the Field Equations

The principle that free particles follow geodesic curves has been called the "geodesic postulate". We shall now show that the "geodesic postulate" follows as a consequence of the field equations.

Consider a system of free particles in curved spacetime. This system can be regarded as a pressure-free gas. Such a gas is called *dust*. It is described by an energy–momentum tensor

$$T^{\mu\nu} = \rho u^\mu u^\nu \ , \tag{7.32}$$

where ρ is the rest density of the dust as measured by an observer at rest in the dust and u^μ are the components of the 4-velocity of the dust particles.

Einstein's field equations as applied to spacetime filled with dust take the form

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa\rho u^\mu u^\nu \ . \tag{7.33}$$

Because the divergence of the left-hand side is zero, the divergence of the right-hand side must be zero, too:

$$(\rho u^\mu u^\nu)_{;\nu} = 0 \tag{7.34}$$

or

$$(\rho u^\nu u^\mu)_{;\nu} = 0 \, . \tag{7.35}$$

We now regard the quantity in the parenthesis as a product of ρu^ν and u^μ. By the rule for differentiating a product we get

$$(\rho u^\nu)_{;\nu} u^\mu + \rho u^\nu u^\mu_{;\nu} = 0 \, . \tag{7.36}$$

Since the 4-velocity of any object has a magnitude equal to the velocity of light we have

$$u_\mu u^\mu = -c^2 \ . \tag{7.37}$$

Differentiation gives

$$(u_\mu u^\mu)_{;\nu} = 0 \,. \tag{7.38}$$

Using, again, the rule for differentiating a product, we get

$$u_{\mu;\nu} u^\mu + u_\mu u^\mu_{;\nu} = 0 \,. \tag{7.39}$$

From the rule for raising an index and the freedom of changing a summation index from α to μ, say, we get

$$u_{\mu;\nu} u^\mu = u^\mu u_{\mu;\nu} = g^{\mu\alpha} u_\alpha u_{\mu;\nu} = u_\alpha g^{\mu\alpha} u_{\mu;\nu} = u_\alpha u^\alpha_{;\nu} = u_\mu u^\mu_{;\nu} \,. \tag{7.40}$$

Thus the two terms of Eq. (7.39) are equal. It follows that each of them are equal to zero. So we have

$$u_\mu u^\mu_{;\nu} = 0 \,. \tag{7.41}$$

Multiplying Eq. (7.36) by $u_{\mu,}$ we get

$$(\rho u^\nu)_{;\nu} u_\mu u^\mu + \rho u^\nu u_\mu u^\mu_{;\nu} = 0 \,. \tag{7.42}$$

Using Eq. (7.37) in the first term and Eq. (7.41) in the last term, which then vanishes, we get

$$(\rho u^\nu)_{;\nu} = 0 \,. \tag{7.43}$$

Thus the first term in Eq. (7.36) vanishes and we get

$$\rho u^\nu u^\mu_{;\nu} = 0 \,. \tag{7.44}$$

Since $\rho \neq 0$ we must have

$$u^\nu u^\mu_{;\nu} = 0 \,. \tag{7.45}$$

This is just the geodesic equation.

Conclusion: It follows from Einstein's field equations that free particles move along paths corresponding to geodesic curves of spacetime.

Problems

7.1. *Lorentz transformation of a perfect fluid*
Consider a homogeneous perfect fluid. In the rest frame of the fluid the equation of state is $p = w\rho$ (with $c = 1$), and the energy–momentum tensor has the form

$$T_{\mu\nu} = \rho\, diag(1, w, w, w) \,. \tag{7.46}$$

(a) Make a Lorentz transformation in the 1-direction with velocity v and show that the transformed energy–momentum tensor has the form

$$T_{\mu'\nu'} = \rho \begin{bmatrix} \gamma^2(1+v^2w) & \gamma^2 v(1+w) & 0 & 0 \\ \gamma^2 v(1+w) & \gamma^2(v^2+w) & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{bmatrix}, \gamma = \frac{1}{\sqrt{1-v^2}} . \tag{7.47}$$

(b) The weak energy condition requires that the energy density is positive. What restriction does this put on w?

(c) Which value of w makes the components of the energy–momentum tensor Lorentz invariant?

7.2. *Geodesic equation and constants of motion*
Show that the covariant components of the geodesic equation have the form

$$\dot{u}_\mu = \frac{1}{2} g_{\alpha\beta,\mu} u^\alpha u^\beta .$$

What does this equation tell about constants of motion for free particles?
7.3. *The electromagnetic energy–momentum tensor*
Consider a general energy–momentum tensor $T_{\mu\nu}$ and a time-like vector u^μ. We can always decompose $T_{\mu\nu}$ as follows:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + 2u_{(\mu} q_{\nu)} + \pi_{\mu\nu} , \tag{7.48}$$

where ρ is the energy density, p is the isotropic pressure, q_μ is the energy flux, $\pi_{\mu\nu}$ is the anisotropic stress tensor and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the 3-metric tensor on the hypersurfaces orthogonal to u_μ. These fulfil the following relations:

$$u^\mu q_\mu = u^\mu \pi_{\mu\nu} = u^\mu h_{\mu\nu} = \pi^\mu{}_\mu = 0, \quad \pi_{\nu\mu} = \pi_{\mu\nu} .$$

(a) Show that

$$\begin{aligned} \rho &= T_{\mu\nu} u^\mu u^\nu, & p &= \frac{1}{3} T_{\mu\nu} h^{\mu\nu}, \\ q_\nu &= -u^\mu h^\rho{}_\nu T_{\mu\rho}, & \pi_{\mu\nu} &= T_{\rho\sigma}\left(h^\rho{}_\mu h^\sigma{}_\nu - \frac{1}{3} h^{\rho\sigma} h_{\mu\nu} \right) . \end{aligned} \tag{7.49}$$

(b) Consider the electromagnetic field tensor in an orthonormal basis $\{\mathbf{e}_t, \mathbf{e}_i\}$. Assume that u^μ is aligned with the time-like basis vector $\mathbf{e}_t$. Use the electromagnetic energy–momentum tensor to show that

$$\begin{aligned} \rho = 3p &= \frac{1}{2}(E^k E_k + B^k B_k), \\ q_i &= -\varepsilon_{ijk} E^j B^k, \\ \pi_{ij} &= -E_i E_j - B_i B_j + \frac{1}{3}\delta_{ij}(E^k E_k + B^k B_k) . \end{aligned} \tag{7.50}$$

What is the physical interpretation of q_i?

7.4. *Lorentz-invariant radiation*
Consider a region filled with photons of all frequencies and moving in all directions. Let $n(\nu, \mathbf{e})$ be the number of photons per unit volume, per frequency interval and per unit solid angle moving in the direction $\mathbf{e}$, as referred to an orthonormal basis Σ. Let primed quantities be measured in a basis Σ' moving with a velocity $\mathbf{v}$ relative to Σ. A comoving volume element dV has a velocity $\mathbf{u}$ in Σ. The corresponding rest volume is dV_0.
The quantity $n(\nu, \mathbf{e}) dV d\nu d\Omega$ represents the number of photons occupying a volume dV, with frequencies between ν and $\nu + d\nu$, and moving with directions within a solid angle $d\Omega = \sin\theta d\theta d\phi$. It is an invariant quantity. Hence,

$$n(\nu, \mathbf{e}) dV d\nu d\Omega = n'(\nu', \mathbf{e}') dV' d\nu' d\Omega'.$$

(a) Use that dV_0 is invariant to show that $dV' = \gamma_{\mathbf{u}'}^{-1} \gamma_{\mathbf{u}} dV$ where $\gamma_{\mathbf{u}'} = (1 - |\mathbf{u}'|^2/c^2)^{-1/2}$ and $\gamma_{\mathbf{u}} = (1 - |\mathbf{u}|^2/c^2)^{-1/2}$.
(b) Choose the x'- and x-axes to be directed along $\mathbf{v}$ (so that $\mathbf{v} = v\mathbf{e}_x$) and use the transformation formulae of the velocity components,

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}, \quad u'_y = \frac{u_y}{\gamma_{\mathbf{v}} \left(1 - \frac{u_x v}{c^2}\right)}, \quad \gamma_{\mathbf{v}} = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}},$$

to show that

$$\gamma_{\mathbf{u}'} = \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} \left(1 - \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}\right)^{-1};$$

and hence that

$$dV' = \gamma_{\mathbf{v}}^{-1} \left(1 - \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}\right)^{-1} dV.$$

(c) Let θ be the angle between the transformation velocity $\mathbf{v}$ and the velocity $\mathbf{u}$ of the volume. Since the volume is comoving with photons moving in the $\mathbf{u}$-direction, we now set $|\mathbf{u}| = c$. Show that this leads to $dV' = \kappa^{-1} dV$, where $\kappa = \gamma_{\mathbf{v}}[1 - (v/c)\cos\theta]$.
(d) Use the relativistic equations for the Doppler effect and aberration (see Problem 2.9), and show that the transformation equations for the differentials of frequency and solid angle are

$$d\nu' = \kappa d\nu, \quad d\Omega' = \kappa^{-2} d\Omega.$$

(e) Deduce that $n'(\nu', \mathbf{e}') = \kappa^2 n(\nu, \mathbf{e})$ and use the transformation equation for the frequency to show that $n(\nu, \mathbf{e})/\nu^2$ is a Lorentz-invariant quantity.
(f) Since ν is not Lorentz invariant it follows that $n(\nu, \mathbf{e})/\nu^2$ must be independent of ν. Use this, together with the fact that the energy of a photon is given by $h\nu$, to find how the energy density per frequency interval and solid angle of a Lorentz-invariant radiation depend upon the frequency.

Chapter 8
The Schwarzschild Spacetime

8.1 Schwarzschild's Exterior Solution

This is a solution of the vacuum field equations $E_{\mu\nu} = 0$ for a static spherically symmetric spacetime. One can then *choose* the following form of the line element (employing units so that c=1):

$$\begin{aligned} ds^2 &= -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2, \\ d\Omega^2 &= d\theta^2 + \sin^2\theta d\phi^2 \, . \end{aligned} \tag{8.1}$$

These coordinates are chosen so that the area of a sphere with radius r is $4\pi r^2$.

Physical distance in radial direction, corresponding to a coordinate distance dr, is $dl_r = \sqrt{g_{rr}}dr = e^{\beta(r)}dr$.

Here follows a stepwise algorithm to determine the components of the Einstein tensor by using the Cartan formalism:

1. Using orthonormal basis we find

$$\underline{\omega}^{\hat{t}} = e^{\alpha(r)}\underline{dt} \, , \quad \underline{\omega}^{\hat{r}} = e^{\beta(r)}\underline{dr} \, , \quad \underline{\omega}^{\hat{\theta}} = r\underline{d}\theta \, , \quad \underline{\omega}^{\hat{\phi}} = r\sin\theta\underline{d}\phi. \tag{8.2}$$

2. Computing the connection forms by applying Cartan's 1st structure equations

$$\underline{d\omega}^{\hat{\mu}} = -\underline{\Omega}^{\hat{\mu}}_{\ \hat{\nu}} \wedge \underline{\omega}^{\hat{\nu}}. \tag{8.3}$$

$$\begin{aligned} \underline{d\omega}^{\hat{t}} &= e^{\alpha}\alpha'\underline{dr} \wedge \underline{dt} \\ &= e^{\alpha}\alpha' e^{-\beta}\underline{\omega}^{\hat{r}} \wedge e^{-\alpha}\underline{\omega}^{\hat{t}} \\ &= -e^{-\beta}\alpha'\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}} \\ &= -\underline{\Omega}^{\hat{t}}_{\ \hat{r}} \wedge \underline{\omega}^{\hat{r}}. \end{aligned} \tag{8.4}$$

$$\therefore \underline{\Omega}^{\hat{t}}_{\ \hat{r}} = e^{-\beta}\alpha'\underline{\omega}^{\hat{t}} + f_1\underline{\omega}^{\hat{r}}. \tag{8.5}$$

3. To determine the f-functions we apply the antisymmetry

$$\underline{\Omega}_{\hat{\mu}\hat{\nu}} = -\underline{\Omega}_{\hat{\nu}\hat{\mu}} \, . \tag{8.6}$$

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This gives (the non-zero connection forms)

$$\begin{aligned}
\underline{\Omega}^{\hat{r}}{}_{\hat{\phi}} &= -\underline{\Omega}^{\hat{\phi}}{}_{\hat{r}} = -\frac{1}{r}e^{-\beta}\underline{\omega}^{\hat{\phi}}, \\
\underline{\Omega}^{\hat{\theta}}{}_{\hat{\phi}} &= -\underline{\Omega}^{\hat{\phi}}{}_{\hat{\theta}} = -\frac{1}{r}\cot\theta\,\underline{\omega}^{\hat{\phi}}, \\
\underline{\Omega}^{\hat{t}}{}_{\hat{r}} &= +\underline{\Omega}^{\hat{r}}{}_{\hat{t}} = e^{-\beta}\alpha'\underline{\omega}^{\hat{t}}, \\
\underline{\Omega}^{\hat{r}}{}_{\hat{\theta}} &= -\underline{\Omega}^{\hat{\theta}}{}_{\hat{r}} = -\frac{1}{r}e^{-\beta}\underline{\omega}^{\hat{\theta}}.
\end{aligned} \tag{8.7}$$

4. We then proceed to determine the curvature forms by applying Cartan's 2nd structure equations
$$\underline{R}^{\hat{\mu}}{}_{\hat{\nu}} = \underline{d\Omega}^{\hat{\mu}}{}_{\hat{\nu}} + \underline{\Omega}^{\hat{\mu}}{}_{\hat{\alpha}} \wedge \underline{\Omega}^{\hat{\alpha}}{}_{\hat{\nu}}, \tag{8.8}$$
which gives
$$\begin{aligned}
\underline{R}^{\hat{t}}{}_{\hat{r}} &= -e^{-2\beta}(\alpha'' + \alpha'^2 - \alpha'\beta')\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}}, \\
\underline{R}^{\hat{t}}{}_{\hat{\theta}} &= -\frac{1}{r}e^{-2\beta}\alpha'\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta}}, \\
\underline{R}^{\hat{t}}{}_{\hat{\phi}} &= -\frac{1}{r}e^{-2\beta}\alpha'\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\phi}}, \\
\underline{R}^{\hat{r}}{}_{\hat{\theta}} &= \frac{1}{r}e^{-2\beta}\beta'\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\theta}}, \\
\underline{R}^{\hat{r}}{}_{\hat{\phi}} &= \frac{1}{r}e^{-2\beta}\beta'\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\phi}}, \\
\underline{R}^{\hat{\theta}}{}_{\hat{\phi}} &= \frac{1}{r^2}(1 - e^{-2\beta})\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\phi}}.
\end{aligned} \tag{8.9}$$
5. By applying the following relation
$$\underline{R}^{\hat{\mu}}{}_{\hat{\nu}} = \frac{1}{2}R^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}\hat{\beta}}\underline{\omega}^{\hat{\alpha}} \wedge \underline{\omega}^{\hat{\beta}}, \tag{8.10}$$
we find the components of Riemann's curvature tensor.
6. Contraction gives the components of Ricci's curvature tensor:
$$R_{\hat{\mu}\hat{\nu}} \equiv R^{\hat{\alpha}}{}_{\hat{\mu}\hat{\alpha}\hat{\nu}}\,. \tag{8.11}$$
7. A new contraction gives Ricci's curvature scalar:
$$R \equiv R^{\hat{\mu}}{}_{\hat{\mu}}. \tag{8.12}$$
8. The components of the Einstein tensor can then be found:
$$E_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2}\eta_{\hat{\mu}\hat{\nu}}R\,, \tag{8.13}$$
where $\eta_{\hat{\mu}\hat{\nu}} = diag(-1,1,1,1)$. We then have

$$
\begin{aligned}
E_{\hat{t}\hat{t}} &= \frac{2}{r}e^{-2\beta}\beta' + \frac{1}{r^2}(1-e^{-2\beta}),\\
E_{\hat{r}\hat{r}} &= \frac{2}{r}e^{-2\beta}\alpha' - \frac{1}{r^2}(1-e^{-2\beta}),\\
E_{\hat{\theta}\hat{\theta}} = E_{\hat{\phi}\hat{\phi}} &= e^{-2\beta}\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{\alpha'}{r} - \frac{\beta'}{r}\right).
\end{aligned}
\tag{8.14}
$$

We want to solve the equations $E_{\hat{\mu}\hat{\nu}} = 0$. We get only two independent equations, and choose to solve those:

$$E_{\hat{t}\hat{t}} = 0 \qquad \text{and} \qquad E_{\hat{r}\hat{r}} = 0. \tag{8.15}$$

By adding the two equations we get

$$
\begin{aligned}
&E_{\hat{t}\hat{t}} + E_{\hat{r}\hat{r}} = 0\\
&\Rightarrow \frac{2}{r}e^{-2\beta}(\beta' + \alpha') = 0\\
&\Rightarrow (\alpha+\beta)' = 0 \Rightarrow \alpha + \beta = K_1 \quad \text{(const)}.
\end{aligned}
\tag{8.16}
$$

We now have

$$ds^2 = -e^{2\alpha}dt^2 + e^{2(k_1-\alpha)}dr^2 + r^2 d\Omega^2\ . \tag{8.17}$$

By choosing a suitable coordinate time, we can achieve

$$K_1 = 0 \Rightarrow \alpha = -\beta.$$

Since we now have $ds^2 = -e^{2\alpha}dt^2 + e^{-2\alpha}dr^2 + r^2 d\Omega^2$, this means that $g_{rr} = -\frac{1}{g_{tt}}$. We still have to solve one more equation to get the complete solution, and choose the equation $E_{\hat{t}\hat{t}} = 0$, which gives

$$\frac{2}{r}e^{-2\beta}\beta' + \frac{1}{r^2}(1-e^{-2\beta}) = 0.$$

This equation can be written as

$$
\begin{aligned}
&\frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\beta})] = 0.\\
&\therefore r(1-e^{-2\beta}) = K_2 \qquad \text{(const)}\ .
\end{aligned}
\tag{8.18}
$$

If we choose $K_2 = 0$ we get $\beta = 0$ giving $\alpha = 0$ and

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2\ , \tag{8.19}$$

which is the Minkowski spacetime described in spherical coordinates. In general, $K_2 \neq 0$ and $1 - e^{-2\beta} = \frac{K_2}{r} \equiv \frac{K}{r}$, giving

$$e^{2\alpha} = e^{-2\beta} = 1 - \frac{K}{r}$$

and

$$ds^2 = -\left(1-\frac{K}{r}\right)dt^2 + \frac{dr^2}{1-\frac{K}{r}} + r^2 d\Omega^2. \tag{8.20}$$

We can find K by going to the Newtonian limit. We calculate the gravitational acceleration (that is the acceleration of a free particle instantaneously at rest) in the limit of a weak field of a particle at a distance r from a spherical mass M. Newtonian:

$$g = \frac{d^2r}{dt^2} = -\frac{GM}{r^2}. \tag{8.21}$$

We anticipate that $r >> K$. Then the proper time τ of a particle will be approximately equal to the coordinate time, since $d\tau = \sqrt{1-\frac{K}{r}}dt$.
The acceleration of a particle in 3-space is given by the geodesic equation:

$$\begin{aligned} \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0, \\ u^\alpha = \frac{dx^\alpha}{d\tau}. \end{aligned} \tag{8.22}$$

For a particle instantaneously at rest in a weak field, we have $d\tau \approx dt$. Using $u^\mu = (1,0,0,0)$, we get

$$g = \frac{d^2r}{dt^2} = -\Gamma^r_{tt}. \tag{8.23}$$

This equation gives a physical interpretation of Γ^r_{tt} as the gravitational acceleration. This is a mathematical way to express the principle of equivalence: The gravitational acceleration can be transformed to 0 since the Christoffel symbols always can be transformed to 0 locally, in a freely falling non-rotating frame, i.e. a local inertial frame:

$$\begin{aligned} \Gamma^r_{tt} &= \frac{1}{2}\underbrace{g^{r\alpha}}_{\frac{1}{g_{r\alpha}}}(\underbrace{\frac{\partial g_{\alpha t}}{\partial t}}_{=0} + \underbrace{\frac{\partial g_{\alpha t}}{\partial t}}_{=0} - \frac{\partial g_{tt}}{\partial x^\alpha}) \\ &= -\frac{1}{2g_{rr}}\frac{\partial g_{tt}}{\partial r}, \\ g_{tt} &= -(1-\frac{K}{r}), \quad \frac{\partial g_{tt}}{\partial r} = -\frac{K}{r^2}, \\ g &= -\Gamma^r_{tt} = -\frac{K}{2r^2} = -\frac{GM}{r^2}, \\ &\text{hence } K = 2GM \\ &\text{or with } c: \ K = \frac{2GM}{c^2}. \end{aligned} \tag{8.24}$$

Then we have the line element of the exterior Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2 d\Omega^2 \,. \tag{8.25}$$

$R_S \equiv \frac{2GM}{c^2}$ is the Schwarzschild radius of a mass M.
Weak field: $r >> R_S$.
For the Earth: $R_S \sim 0.9\,\text{cm}$
For the Sun: $R_S \sim 3\,\text{km}$
A standard clock at rest in the Schwarzschild spacetime shows a proper time τ:

$$d\tau = \sqrt{1 - \frac{R_S}{r}} dt. \tag{8.26}$$

So the coordinate clocks showing t are ticking with the same rate as the standard clocks far from M. Coordinate clocks are running equally fast no matter where they are. If they hadn't, the distance between simultaneous events with given spatial coordinates would depend on the time of the measuring of the distance. Then the metric would be time dependent. Hence Eq. (8.26) shows that the rate of proper time is slower for decreasing value of r, i. e. farther down in the gravitational field. Time is not running at the Schwarzschild radius.

Definition 8.1.1 (Physical singularity) A physical singularity is a point where the curvature is infinitely large.

Definition 8.1.2 (Coordinate singularity) A coordinate singularity is a point (or a surface) where at least one of the components of the metric tensor is infinitely large, but where the curvature of spacetime is finite.

Kretschmann's curvature scalar is $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. From the Schwarzschild metric, we get

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{48G^2M^2}{r^8} \,, \tag{8.27}$$

which diverges only at the origin. Since there is no physical singularity at $r = R_S$, the singularity here is just a coordinate singularity and can be removed by a transformation to a coordinate system falling inward (Eddington–Finkelstein coordinates, Kruskal–Szekers analytical extension of the description of Schwarzschild spacetime to include the area inside R_S).

8.2 Radial Free Fall in Schwarzschild Spacetime

The Lagrangian function of a particle moving radially in Schwarzschild space–time is

$$L = -\frac{1}{2}\left(1 - \frac{R_S}{r}\right) c^2 \dot{t}^2 + \frac{1}{2}\frac{\dot{r}^2}{(1 - \frac{R_S}{r})} \quad , \quad \dot{} \equiv \frac{d}{d\tau}, \tag{8.28}$$

where τ is the time measured on a standard clock which the particle is carrying.
The momentum p_t conjugate to the cyclic coordinate t is a constant of motion:

$$p_t = \frac{\partial L}{\partial \dot{t}} = -\left(1 - \frac{R_S}{r}\right) c^2 \dot{t}. \tag{8.29}$$

4-Velocity identity: $u_\mu u^\mu = -c^2$:

$$-\left(1 - \frac{R_S}{r}\right) c^2 \dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{R_S}{r}} = -c^2. \tag{8.30}$$

Inserting the expression for $\dot{t}$ gives

$$\dot{r}^2 - \frac{p_t^2}{c^2} = -\left(1 - \frac{R_S}{r}\right) c^2 \,. \tag{8.31}$$

Boundary conditions: the particle is falling from rest at $r = r_0$:

$$p_t = -\left(1 - \frac{R_S}{r_0}\right) \underbrace{\frac{c^2}{\sqrt{1 - \frac{R_S}{r_0}}}}_{\dot{t}(r=r_0)} = -\sqrt{1 - \frac{R_S}{r_0}} c^2 \tag{8.32}$$

giving

$$\dot{r} = \frac{dr}{d\tau} = -c\sqrt{\frac{R_S}{r_0}}\sqrt{\frac{r_0 - r}{r}} \tag{8.33}$$

$$\int \frac{dr}{\sqrt{\frac{r_0 - r}{r}}} = -c\sqrt{\frac{R_S}{r_0}}\tau \,. \tag{8.34}$$

Integration with $\tau = 0$ for $r = 0$ gives

$$\tau = -\frac{r_0}{c}\sqrt{\frac{r_0}{R_S}}\left(\arcsin\sqrt{\frac{r}{r_0}} - \sqrt{\frac{r}{r_0}}\sqrt{1 - \frac{r}{r_0}}\right) \,. \tag{8.35}$$

τ is the proper time that the particle spends on the part of the fall which is from r to $r = 0$. The proper travelling time from the initial point $r = r_0$ to $r = 0$ is

$$|\tau(r_0)| = -\frac{\pi}{2}\sqrt{\frac{r_0}{R_S}}\frac{r_0}{c} \,. \tag{8.36}$$

If the particle falls from $r_0 = R_S$ the travelling time is

$$|\tau| = \frac{\pi}{2}\frac{R_S}{c} = \frac{\pi G m}{c^3} \,. \tag{8.37}$$

8.3 Light Cones in Schwarzschild Spacetime

The Schwarzschild line element (with $c = 1$) is

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) dt^2 + \frac{dr^2}{(1 - \frac{R_S}{r})} + r^2 d\Omega^2 \,. \tag{8.38}$$

We will look at **radially moving photons** ($ds^2 = d\Omega^2 = 0$). We then get

$$\begin{gathered} \frac{dr}{\sqrt{1 - \frac{R_S}{r}}} = \pm\sqrt{1 - \frac{R_S}{r}} dt \Leftrightarrow \frac{r^{\frac{1}{2}} dr}{\sqrt{r - R_S}} = \pm \frac{\sqrt{r - R_S}}{r^{\frac{1}{2}}} dt, \\ \frac{r dr}{r - R_S} = \pm dt, \end{gathered} \tag{8.39}$$

with $+$ for outward motion and $-$ for inward motion. For inwardly moving photons, integration yields

$$r + t + R_S \ln\left|\frac{r}{R_S} - 1\right| = k = \text{constant} \,. \tag{8.40}$$

We now introduce a new time coordinate t' such that the equation of motion for photons moving **inward** takes the following form:

$$\begin{gathered} r + t' = k \Rightarrow \frac{dr}{dt'} = -1. \\ \therefore \quad t' = t + R_S \ln\left|\frac{r}{R_S} - 1\right|. \end{gathered} \tag{8.41}$$

The coordinate t' is called an ingoing Eddington–Finkelstein coordinate. The photons here always move with the *local* velocity of light, c. For photons moving **outwards** we have

$$r + R_S \ln\left|\frac{r}{R_S} - 1\right| = t + k \,. \tag{8.42}$$

Making use of $t = t' - R_S \ln\left|\frac{r}{R_S} - 1\right|$ we get

$$\begin{aligned} & r + 2R_S \ln\left|\frac{r}{R_S} - 1\right| = t' + k \\ \Rightarrow \quad & \frac{dr}{dt'} + \frac{2R_S}{r - R_S}\frac{dr}{dt'} = 1 \Leftrightarrow \frac{r + R_S}{r - R_S}\frac{dr}{dt'} = 1 \\ \Leftrightarrow \quad & \frac{dr}{dt'} = \frac{r - R_S}{r + R_S}. \end{aligned} \tag{8.43}$$

Making use of ordinary Schwarzschild coordinates we would have gotten the following coordinate velocities for inwardly and outwardly moving photons:

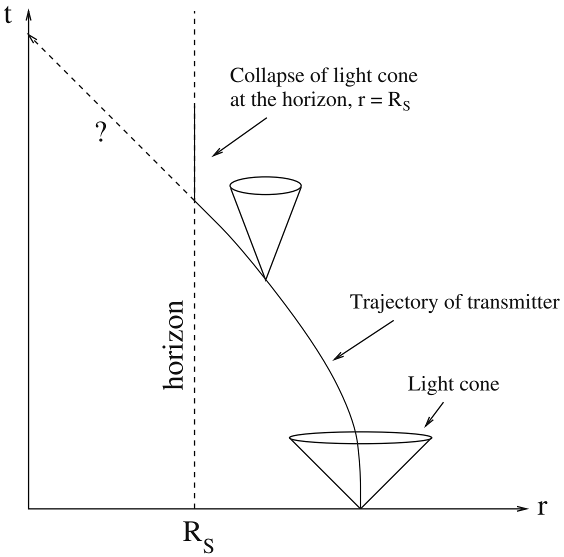

Fig. 8.1 At a radius $r = R_S$ the light cones collapse, and nothing can any longer escape, when we use the Schwarzschild coordinate time

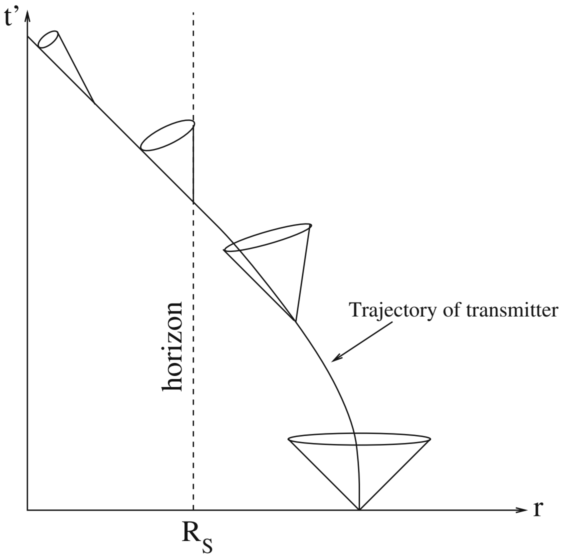

Fig. 8.2 Using the ingoing Eddington–Finkelstein time coordinate there is no collapse of the light cone at $r = R_S$. Instead we get a collapse at the singularity at $r = 0$. The angle between the left part of the light cone and the t'-axis is always 45°. We also see that once the transmitter gets inside the horizon at $r = R_S$, no particles can escape

$$\frac{dr}{dt} = \pm \left(1 - \frac{R_S}{r}\right), \tag{8.44}$$

which shows us how light is decelerated in a gravitational field. Figure 8.1 shows how this is viewed by a non-moving observer located far away from the mass. In Fig. 8.2 we have instead used the alternative time coordinate t'. The special theory of relativity is valid locally, and all material particles thus have to remain inside the light cone.

8.4 Analytical Extension of the Schwarzschild Coordinates

The Schwarzschild coordinates are comoving with a static reference frame outside a spherical mass distribution. If the mass has collapsed to a black hole there exist a horizon at the Schwarzschild radius. As we have seen in Sect. 8.3 there do not exist static observers at finite radii inside the horizon. Hence, the Schwarzschild coordinates are well defined only outside the horizon.

Also the rr-component of the metric tensor has a coordinate singularity at the Schwarzschild radius. The curvature of spacetime is finite here.

Kruskal and Szekeres have introduced new coordinates that are well defined inside as well as outside the Schwarzschild radius, and with the property that the metric tensor is non-singular for all $r > 0$.

In order to arrive at these coordinates we start by considering a photon moving radially inwards. From Eq. (8.40) we then have

$$t = -r - R_S \ln\left|\frac{r}{R_S} - 1\right| + v, \tag{8.45}$$

where v is a constant along the world line of the photon. We introduce a new radial coordinate

$$r^* \equiv r + R_S \ln\left|\frac{r}{R_S} - 1\right|. \tag{8.46}$$

Then the equation of the world line of the photon takes the form

$$t + r^* = v. \tag{8.47}$$

The value of the constant v does only depend upon the point of time when the photon was emitted. We may therefore use v as a new time coordinate.

For an outgoing photon we get in the same way

$$t - r^* = u, \tag{8.48}$$

where u is a constant of integration, which may be used as a new time coordinate for outgoing photons. The coordinates u and v are the generalization of the **light cone coordinates** of Minkowski spacetime to the Schwarzchild spacetime.

From Eqs. (8.47) and (8.48) we get

$$dt = \frac{1}{2}(dv + du), \tag{8.49}$$

$$dr^* = \frac{1}{2}(dv - du) \tag{8.50}$$

and from Eq. (8.46)

$$dr = \left(1 - \frac{R_s}{r}\right) dr^*. \tag{8.51}$$

Inserting these differentials into Eq. (8.38) we arrive at a new form of the Schwarzschild line element,

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) du\, dv + r^2 d\Omega^2 \ . \tag{8.52}$$

The metric is still not well behaved at the horizon. Introducing the coordinates

$$U = -e^{-\frac{u}{2R_S}}, \tag{8.53}$$

$$V = e^{\frac{v}{2R_S}}, \tag{8.54}$$

gives

$$UV = -e^{\frac{v-u}{2R_S}} = -e^{\frac{r^*}{R_S}} = -\left|\frac{R_s}{r} - 1\right| e^{\frac{r}{R_S}} \tag{8.55}$$

and

$$du\, dv = -4R_s^2 \frac{dU dV}{UV}. \tag{8.56}$$

The line element (8.52) then takes the form

$$ds^2 = -\frac{4R_s^3}{r} e^{-\frac{r}{R_S}} dU dV + r^2 d\Omega^2 \ . \tag{8.57}$$

This is the first form of the Kruskal–Szekeres line element. Here there is no coordinate singularity, only a physical singularity at $r = 0$.

We may furthermore introduce two new coordinates:

$$T = \frac{1}{2}(V + U) = \left|\frac{r}{R_s} - 1\right|^{\frac{1}{2}} e^{\frac{r}{2R_S}} \sinh \frac{t}{2R_s}, \tag{8.58}$$

$$Z = \frac{1}{2}(V - U) = \left|\frac{r}{R_s} - 1\right|^{\frac{1}{2}} e^{\frac{r}{2R_S}} \cosh \frac{t}{2R_s} \ . \tag{8.59}$$

Hence

$$V = T + Z, \tag{8.60}$$

$$U = T - Z, \tag{8.61}$$

giving

$$dUdV = dT^2 - dZ^2. \tag{8.62}$$

Inserting this into Eq. (8.57) we arrive at the second form of the Kruskal–Szekeres line element

$$ds^2 = -\frac{4R_s^3}{r}e^{-\frac{r}{R_s}}\left(dT^2 - dZ^2\right) + r^2 d\Omega^2. \tag{8.63}$$

The inverse transformations of Eqs. (8.58) and (8.59) are

$$\left|\frac{r}{R_s} - 1\right| e^{\frac{r}{R_s}} = Z^2 - T^2, \tag{8.64}$$

$$\tanh\frac{t}{2R_s} = \frac{T}{Z}. \tag{8.65}$$

Note from Eq. (8.63) that with the Kruskal–Szekeres coordinates T and Z the equation of the radial null geodesics has the same form as in flat spacetime:

$$Z = \pm T + \text{constant}. \tag{8.66}$$

8.5 Embedding of the Schwarzschild Metric

We will now look at a static, spherically symmetric space. A curved simultaneity plane ($dt = 0$) through the equatorial plane ($d\theta = 0$) has the line element

$$ds^2 = g_{rr}dr^2 + r^2 d\phi^2 \tag{8.67}$$

with a radial coordinate such that a circle with radius r has a circumference of length $2\pi r$.

We now embed this surface in a flat 3-dimensional space with cylinder coordinates (z, r, ϕ) and line element

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2. \tag{8.68}$$

The surface described by the line element in Eq. (8.67) has the equation $z = z(r)$. The line element in (8.68) is therefore written as

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2\,. \tag{8.69}$$

Demanding that (8.69) be in agreement with (8.67) we get

$$g_{rr} = 1 + \left(\frac{dz}{dr}\right)^2 \Leftrightarrow \frac{dz}{dr} = \pm\sqrt{g_{rr} - 1}\,. \tag{8.70}$$

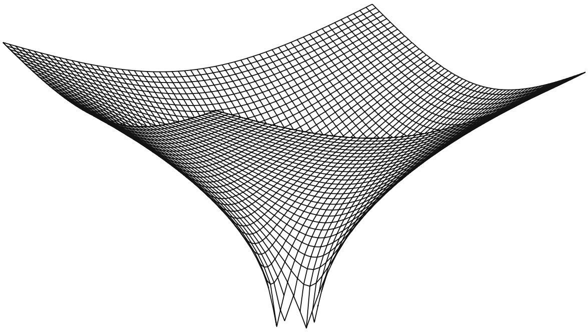

Fig. 8.3 Embedding of the Schwarzschild metric

Choosing the positive solution gives

$$\boxed{dz = \sqrt{g_{rr} - 1}dr} \, . \tag{8.71}$$

In the Schwarzschild spacetime we have

$$g_{rr} = \frac{1}{1 - \frac{R_S}{r}} \, . \tag{8.72}$$

Making use of this we find z:

$$z = \int_{R_S}^{r} \frac{dr}{\sqrt{\frac{r}{R_S} - 1}} = \sqrt{4R_S(r - R_S)} \, . \tag{8.73}$$

This is shown in Fig. 8.3.

8.6 Deceleration of Light

The radial speed of light in Schwarzschild coordinates found by putting $ds^2 = d\Omega^2 = 0$ in Eq. (8.38) is

$$\bar{c} = 1 - \frac{R_S}{r}. \tag{8.74}$$

To measure this effect one can look at how long it takes for light to get from Mercury to the Earth [1]. This is illustrated in Fig. 8.4. The travel time from z_1 to z_2 is

$$\begin{aligned}\Delta t &= \int_{z_1}^{z_2} \frac{dz}{1 - \frac{R_S}{r}} \approx \int_{z_1}^{z_2} \left(1 + \frac{R_S}{r}\right) dz = \int_{z_1}^{z_2} \left(1 + \frac{R_S}{\sqrt{b^2 + z^2}}\right) dz \\ &= z_2 + |z_1| + R_S \ln \frac{\sqrt{{z_2}^2 + b^2} + z_2}{\sqrt{{z_1}^2 + b^2} - |z_1|},\end{aligned} \tag{8.75}$$

where R_S is the Schwarzschild radius of the Sun.

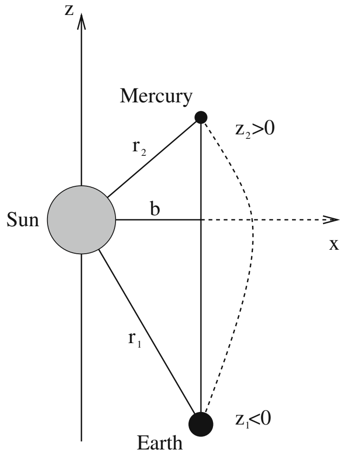

Fig. 8.4 General relativity predicts that light travelling from Mercury to the Earth will be delayed due to the effect of the Sun's gravity field on the speed of light. This effect has been measured

The deceleration is greatest when Earth and Mercury (where the light is reflected) are on nearly opposite sides of the Sun. The impact parameter b is then small. A series expansion to the lowest order of b/z gives

$$\Delta t = z_2 + |z_1| + R_S \ln \frac{4|z_1|z_2}{b^2} . \tag{8.76}$$

The last term represents the extra travelling time due to the effect of the Sun's gravity field on the speed of light. The journey takes longer time:

$R_S =$	the Schwarzschild radius of the Sun	$\sim 3\,\text{km}$	
$\lvert z_1 \rvert =$	the radius of Earth's orbit	$= 15 \times 10^{10}\,\text{m}$	
$z_2 =$	the radius of Mercury's orbit	$= 5.8 \times 10^{10}\,\text{m}$	
$b =$	$R_\odot = 7 \times 10^8\,\text{m}$		

give a delay of 1.1×10^{-4}s. In addition to this one must also, of course, take into account among other things the effects of the curvature of spacetime near the Sun and atmospheric effects on Earth.

8.7 Particle Trajectories in Schwarzschild 3-Space

$$\begin{aligned} L &= \frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \\ &= -\frac{1}{2}\left(1 - \frac{R_s}{r}\right)\dot{t}^2 + \frac{\frac{1}{2}\dot{r}^2}{1 - \frac{R_s}{r}} + \frac{1}{2} r^2 \dot{\theta}^2 + \frac{1}{2} r^2 \sin^2\theta \dot{\phi}^2 . \end{aligned} \tag{8.77}$$

Since t is a cyclic coordinate

$$-p_t = -\frac{\partial L}{\partial \dot{t}} = \left(1 - \frac{R_s}{r}\right)\dot{t} = \text{constant} = E\ , \tag{8.78}$$

where E is the particle's energy as measured by an observer "far away" ($r \gg R_s$). Also ϕ is a cyclic coordinate so that

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = r^2 \sin^2\theta\dot{\phi} = \text{constant}, \tag{8.79}$$

where p_ϕ is the particle's orbital angular momentum.

Making use of the 4-velocity identity $\vec{U}^2 = g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = -1$ we transform the above to get

$$-\left(1 - \frac{R_s}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{R_s}{r}} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 = -1, \tag{8.80}$$

which on substitution for $\dot{t} = \frac{E}{1-\frac{R_s}{r}}$ and $\dot{\phi} = \frac{p_\phi}{r^2\sin^2\theta}$ becomes

$$-\frac{E^2}{1-\frac{R_s}{r}} + \frac{\dot{r}^2}{1-\frac{R_s}{r}} + r^2\dot{\theta}^2 + \frac{p_\phi^2}{r^2\sin^2\theta} = -1\ . \tag{8.81}$$

Now, referring back to the Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{X}^\mu}\right) - \frac{\partial L}{\partial X^\mu} = 0 \tag{8.82}$$

we get, for θ

$$\begin{aligned}(r^2\dot{\theta})^\bullet &= r^2\sin\theta\cos\theta\dot{\phi}^2 \\ &= \frac{p_\phi^2\cos\theta}{r^2\sin^3\theta}\ .\end{aligned} \tag{8.83}$$

Multiplying this by $r^2\dot{\theta}$ we get

$$(r^2\dot{\theta})(r^2\dot{\theta})^\bullet = \frac{\cos\theta\dot{\theta}}{\sin^3\theta}p_\phi^2\ , \tag{8.84}$$

which, on integration, gives

$$(r^2\dot{\theta})^2 = k - \left(\frac{p_\phi}{\sin\theta}\right)^2\ , \tag{8.85}$$

where k is the constant of integration.

Because of the spherical geometry we are free to choose a coordinate system such that the particle moves in the equatorial plane and along the equator at a given time $t = 0$. That is $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$ at time $t = 0$. This determines the constant of integration and $k = p_\phi^2$ such that

$$(r^2\dot{\theta})^2 = p_\phi^2 \left(1 - \frac{1}{\sin^2\theta}\right) . \tag{8.86}$$

The right-hand side is negative for all $\theta \neq \frac{\pi}{2}$. It follows that the particle cannot deviate from its original (equatorial) trajectory. Also, since this particular choice of trajectory was arbitrary we can conclude, quite generally, that any motion of free particles in a spherically symmetric gravitational field is planar motion.

8.7.1 Motion in the Equatorial Plane

$$-\frac{E^2}{1-\frac{R_s}{r}} + \frac{\dot{r}^2}{1-\frac{R_s}{r}} + \frac{p_\phi^2}{r^2} = -1 , \tag{8.87}$$

that is

$$\dot{r}^2 = E^2 - \left(1 - \frac{R_s}{r}\right)\left(1 + \frac{p_\phi^2}{r^2}\right) . \tag{8.88}$$

This corresponds to an energy equation with an effective potential $V(r)$ given by

$$\begin{aligned} V^2(r) &= \left(1 - \frac{R_s}{r}\right)\left(1 + \frac{p_\phi^2}{r^2}\right), \\ \dot{r}^2 + V^2(r) &= E^2 \\ \Rightarrow V &= \sqrt{1 - \frac{r_s}{r} + \frac{p_\phi^2}{r^2} - \frac{R_s p_\phi^2}{r^3}} \\ &\approx 1 - \frac{1}{2}\frac{R_s}{r} + \frac{1}{2}\frac{p_\phi^2}{r^2}. \end{aligned} \tag{8.89}$$

Newtonian potential V_N is defined by using the last expression in

$$V_N = V - 1 \Rightarrow V_N = -\frac{GM}{r} + \frac{p_\phi^2}{2r^2} . \tag{8.90}$$

The possible trajectories of particles in the Schwarzschild 3-space are shown schematically in Fig. 8.5 as functions of position and energy of the particle in the Newtonian limit.

To take into account the relativistic effects the above picture must be modified. We introduce dimensionless variables (Fig. 8.6)

$$X = \frac{r}{GM} \quad \text{and} \quad k = \frac{p_\phi}{GMm} . \tag{8.91}$$

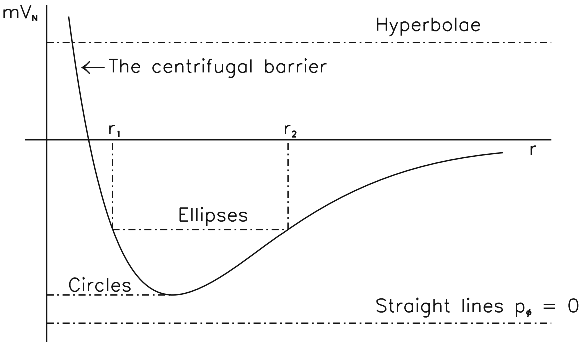

Fig. 8.5 Newtonian particle trajectories are functions of the position and energy of the particle. Note the **centrifugal barrier**. Due to this particles with $p_\phi \neq 0$ cannot arrive at $r = 0$

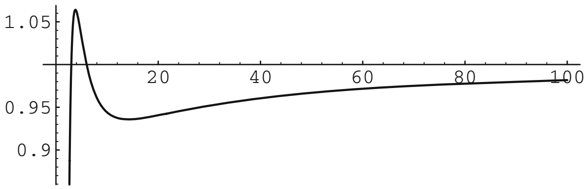

Fig. 8.6 When relativistic effects are included there is no longer a limit to the values that r can take and collapse to a singularity is "possible". Note that V^2 is plotted here

The potential $V^2(r)$ now takes the form

$$V = \left(1 - \frac{2}{X} + \frac{k^2}{X^2} - \frac{2k^2}{X^3}\right)^{1/2} . \tag{8.92}$$

For r equal to the Schwarzschild radius ($X = 2$) we have

$$V(2) = \sqrt{1 - 1 + \frac{k^2}{4} - \frac{2k^2}{8}} = 0 . \tag{8.93}$$

For $k^2 < 12$ particles will fall in towards $r = 0$.

An **orbit equation** is one which connects r and ϕ. So for motion in the equatorial plane for weak fields we have

$$\frac{d\phi}{dt} = \frac{p_\phi}{mr^2}, \qquad \bullet \equiv \frac{d}{dt} = \frac{p_\phi}{mr^2}\frac{d}{d\phi} . \tag{8.94}$$

Introducing the new radial coordinate $u \equiv \frac{1}{r}$ our equations transform to

$$\begin{aligned} \frac{du}{d\phi} &= -\frac{1}{r^2}\frac{dr}{d\phi} = -\frac{1}{r^2}\frac{mr^2}{p_\phi}\frac{dr}{dt} = -\frac{m}{p_\phi}\dot{r} \\ \Rightarrow \dot{r} &= -\frac{p_\phi}{m}\frac{du}{d\phi} . \end{aligned} \tag{8.95}$$

Substitution from above for $\dot{r}$ in the energy equation yields the orbit equation

$$\left(\frac{du}{d\phi}\right)^2 + (1-2GMu)\left(u^2 + \frac{m^2}{p_\phi^2}\right) = \frac{E^2}{p_\phi^2} . \tag{8.96}$$

Differentiating this, we find

$$\frac{d^2u}{d\phi^2} + u = \frac{GMm^2}{p_\phi^2} + 3GMu^2 . \tag{8.97}$$

The last term on the right-hand side is a relativistic correction term.

8.8 Classical Tests of Einstein's General Theory of Relativity

8.8.1 The Hafele–Keating Experiment

Hafele and Keating measured the difference in time shown on moving and stationary atomic clocks [2]. This was done by flying around the Earth in the East–West direction comparing the time on the clock in the plane with the time on a clock on the ground.

The proper time-interval measured on a clock moving with a velocity $v^i = \frac{dx^i}{dt}$ in an arbitrary coordinate system with metric tensor $g_{\mu\nu}$ is given by

$$\begin{aligned} d\tau &= (-\frac{g_{\mu\nu}}{c^2}dx^\mu dx^\nu)^{\frac{1}{2}} , \quad dx^0 = cdt \\ &= (-g_{00} - 2g_{i0}\frac{v^i}{c} - \frac{v^2}{c^2})^{\frac{1}{2}} dt, \\ v^2 &\equiv g_{ij}v^i v^j . \end{aligned} \tag{8.98}$$

For a diagonal metric tensor ($g_{i0} = 0$) we get

$$d\tau = \left(-g_{00} - \frac{v^2}{c^2}\right)^{\frac{1}{2}} dt , \quad v^2 = g_{ii}(v^i)^2 . \tag{8.99}$$

We now look at an idealized situation where a plane flies at constant altitude and with constant speed along the equator:

$$d\tau = \left(1 - \frac{R_S}{r} - \frac{v^2}{c^2}\right)^{\frac{1}{2}} dt , \quad r = R + h . \tag{8.100}$$

To the lowest order in $\frac{R_S}{r}$ and $\frac{v^2}{c^2}$ we get

$$d\tau = \left(1 - \frac{R_S}{2r} - \frac{1}{2}\frac{v^2}{c^2}\right) dt . \tag{8.101}$$

The speed of the moving clock is

$$v = (R+h)\Omega + u\,, \tag{8.102}$$

where Ω is the angular velocity of the Earth and u is the speed of the plane. A series expansion and use of this value for v gives

$$\Delta\tau = \left(1 - \frac{GM}{Rc^2} - \frac{1}{2}\frac{R^2\Omega^2}{c^2} + \frac{gh}{c^2} - \frac{2R\Omega u + u^2}{2c^2}\right)\Delta t\,, \quad g = \frac{GM}{R^2} - R\Omega^2\,. \tag{8.103}$$

$u > 0$ when flying in the direction of the Earth's rotation, i.e. eastwards. For a clock that is left on the airport (stationary, $h = u = 0$) we get

$$\Delta\tau_0 = \left(1 - \frac{GM}{Rc^2} - \frac{1}{2}\frac{R^2\Omega^2}{c^2}\right)\Delta t\,. \tag{8.104}$$

To the lowest order the relative difference in travel time is

$$\boxed{k = \frac{\Delta\tau - \Delta\tau_0}{\Delta\tau_0} \cong \frac{gh}{c^2} - \frac{2R\Omega u + u^2}{2c^2}}\,. \tag{8.105}$$

Measurements:		
Travel time:	$\Delta\tau_0$	$= 1.2 \times 10^5$ s (a little over 24h)
Travelling eastwards:	k_e	$= -1.0 \times 10^{-12}$
Travelling westwards:	k_w	$= 2.1 \times 10^{-12}$
$(\Delta\tau - \Delta\tau_0)_e$		$= -1.2 \times 10^{-7}$ s ≈ -120 ns
$(\Delta\tau - \Delta\tau_0)_w$		$= 2.5 \times 10^{-7}$ s ≈ 250 ns

8.8.2 Mercury's Perihelion Precession

The orbit equation for a planet orbiting a star of mass M is given by Eq. (8.97),

$$\frac{d^2u}{d\phi^2} + u = \frac{GMm^2}{{p_\phi}^2} + ku^2\,, \tag{8.106}$$

where $k = 3GM$. We will be slightly more general and allow k to be a theory- or situation-dependent term. This equation has a circular solution, such that

$$u_0 = \frac{GMm^2}{{p_\phi}^2} + k{u_0}^2\,. \tag{8.107}$$

With a small perturbation from the circular motion u is changed by u_1, where $u_1 \ll u_0$. To lowest order in u_1 we have

$$\frac{d^2u_1}{d\phi^2} + u_0 + u_1 = \frac{GMm^2}{p_\phi{}^2} + ku_0{}^2 + 2ku_0u_1 \tag{8.108}$$

or

$$\frac{d^2u_1}{d\phi^2} + u_1 = 2ku_0u_1 \quad \Leftrightarrow \quad \frac{d^2u_1}{d\phi^2} + (1-2ku_0)u_1 = 0\,. \tag{8.109}$$

For $ku_0 \ll 1$ the equilibrium orbit is stable and we get a periodic solution:

$$u_1 = \epsilon u_0 \cos[\sqrt{1-2ku_0}(\phi - \phi_0)]\,, \tag{8.110}$$

where ϵ and ϕ_0 are integration constants. ϵ is the *eccentricity* of the orbit. We can choose $\phi_0 = 0$ and then have

$$\frac{1}{r} = u = u_0 + u_1 = u_0[1 + \epsilon\cos(\sqrt{1-2ku_0}\phi)]\,. \tag{8.111}$$

Let $f \equiv \sqrt{1-2ku_0} \;\Rightarrow$

$$\frac{1}{r} = \frac{1}{r_0}(1 + \epsilon\cos f\phi)\,. \tag{8.112}$$

For $f = 1$ ($k = 0$, no relativistic term) this expression describes a non-precessing elliptic orbit (a Kepler orbit).

For $f < 1$ ($k > 0$) the ellipse is not closed. To give the same value for r as on a given starting point, ϕ has to increase by $\frac{2\pi}{f} > 2\pi$. The extra angle per rotation is $2\pi(\frac{1}{f} - 1) = \Delta\phi_1$:

$$\Delta\phi_1 = 2\pi\left(\frac{1}{\sqrt{1-2ku_0}} - 1\right) \approx 2\pi k u_0\,. \tag{8.113}$$

Using general relativity we get for Mercury

$$k = 3GM \quad \Rightarrow \quad \Delta\phi = 6\pi GMu_0 \approx 6\pi GM\frac{GMm^2}{p_\phi{}^2}\,, \tag{8.114}$$

$$\boxed{\Delta\phi = 6\pi\left(\frac{GMm}{p_\phi}\right)^2 \text{per orbit}}\,, \tag{8.115}$$

which in Mercury's case amounts to $(\Delta\phi)_{\text{century}} = 43''$.

8.8.3 Deflection of Light

The orbit equation for a free particle with mass $m = 0$ is

$$\frac{d^2u}{d\phi^2} + u = ku^2\,. \tag{8.116}$$

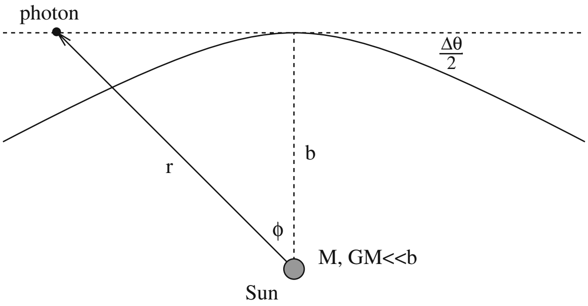

Fig. 8.7 Light travelling close to a massive object is deflected

If light is not deflected it will follow the straight line

$$\cos\phi = \frac{b}{r} = b\,u_0, \tag{8.117}$$

where b is the impact parameter of the path. This is the horizontal dashed line in Fig. 8.7. The zero th-order solution (8.117) fullfils

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0\,. \tag{8.118}$$

Hence it is a solution of (8.116) with $k = 0$.

The perturbed solution is

$$u = u_0 + u_1\,, \quad |u_1| \ll u_0\,. \tag{8.119}$$

Inserting this into the orbit equation gives

$$\frac{d^2u_0}{d\phi^2} + \frac{d^2u_1}{d\phi^2} + u_0 + u_1 = ku_0^2 + 2ku_0u_1 + ku_1^2. \tag{8.120}$$

The first and third terms at the left-hand side cancel each other due to Eq. (8.118) and the last term at the right-hand side is small to second order in u_1 and will be neglected. Hence we get

$$\frac{d^2u_1}{d\phi^2} + u_1 = ku_0^2 + 2ku_0u_1\,. \tag{8.121}$$

The last term at the right-hand side is much smaller than the first, and will also be neglected. Inserting for u_0 from (8.117) we then get

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{k}{b^2}\cos^2\phi\,. \tag{8.122}$$

This equation has a particular solution of the form

$$u_{1p} = A + B\cos^2\phi. \tag{8.123}$$

Inserting this into (8.122) we find

$$A = \frac{2k}{3b^2}\,, \quad B = -\frac{k}{3b^2}\,. \tag{8.124}$$

Hence

$$u_{1p} = \frac{k}{3b^2}\left(2 - \cos^2\phi\right) \tag{8.125}$$

giving

$$\frac{1}{r} = u = u_0 + u_1 = \frac{\cos\phi}{b} + \frac{k}{3b^2}\left(2 - \cos^2\phi\right)\,. \tag{8.126}$$

The deflection of the light $\Delta\theta$ is assumed to be small. We therefore put $\phi = \frac{\pi}{2} + \frac{\Delta\theta}{2}$ where $\Delta\theta \ll \pi$ (see Fig. 8.7). Hence

$$\cos\phi = \cos\left(\frac{\pi}{2} + \frac{\Delta\theta}{2}\right) = -\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}\,. \tag{8.127}$$

Thus, the term $\cos^2\phi$ in (8.126) can be neglected. Furthermore, the deflection of the light is found by letting $r \to \infty$, i.e. $u \to 0$. Then we get

$$\Delta\theta = \frac{4k}{3b}. \tag{8.128}$$

For motion in the Schwarzschild spacetime outside the Sun, $k = \frac{3}{2}R_S$ where R_S is the Schwarzschild radius of the Sun. And for light passing the surface of the Sun $b = R_\odot$ where $R_\odot$ is the actual radius of the Sun. The deflection is then

$$\Delta\theta = 2\frac{R_S}{R_\odot} = 1.75''\,. \tag{8.129}$$

Problems

8.1. *The curvature tensor of a sphere*
Introduce an orthonormal basis on the sphere, S^2, and use Cartan's structural equations to find the physical components of the Riemann curvature tensor.

8.2. *The curvature scalar of a surface of simultaneity*
The spatial line element of a rotating disc is

$$d\ell^2 = dr^2 + \frac{r^2}{1 - \frac{\omega^2 r^2}{c^2}}d\phi^2\,. \tag{8.130}$$

Introduce an orthonormal basis on this surface and use Cartan's structural equations to find the Ricci scalar.

8.3. *Non-relativistic Kepler motion*

(a) In the first part of this exercise we will consider the gravitational potential in a distance of r from the Sun to be given by $V(r) = -\frac{GM}{r}$, where M is the mass of the Sun. Write down the classical Lagrangian in spherical coordinates (r, θ, ϕ) for a planet with mass m moving in this field. The Sun is assumed to be stationary.
What is the physical interpretation of the canonical momenta $p_\phi = \ell$? How is it possible, by just looking at the Lagrangian, to state that p_ϕ is a constant of motion? Find the Euler equation for θ and show that it can be written into the following form:

$$\frac{d}{dt}\left(mr^4\dot{\theta}^2 + \frac{\ell^2}{m\sin^2\theta}\right) = 0\,. \tag{8.131}$$

Based on the above equation, show that the planet moves in a plane by choosing a direction of the z-axis so that at a given time, $t = 0$, we have that $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$.

(b) Write down the Euler equation for r and use this equation to find $u = \frac{1}{r}$ as a function of ϕ. Show that the orbits that describe bound states are elliptic. Find the period T_0 for a circular orbit given by the radius R in the circle.

(c) If the Sun is not entirely spherical, but rather a bit deformed (i.e. more flat near the poles), the gravitational field in the plane where the Sun has its greatest extension will be modified into

$$V(r) = -\frac{GM}{r} - \frac{Q}{r^3}\,. \tag{8.132}$$

Q is a small constant. We now assume that the motion of the planet takes place in the plane where the expression of $V(r)$ is valid. Show that a circular motion is still possible. What is the period T now, expressed by the radius R?

(d) We now assume that the motion deviates slightly from a pure circular orbit, that is $u = \frac{1}{R} + u_1$, where $u_1 \ll \frac{1}{R}$. Show that u_1 varies periodically around the orbit,

$$u_1 = k\sin(f\phi)\,. \tag{8.133}$$

Find f and show that the path rotates in space for each orbit. What is the size of the angle $\Delta\phi$ that the planetary motion rotates for each orbit?
The constant Q can be written as $Q = \frac{1}{2}J_2 GMR_s^2$ where J_2 is a parameter describing the quadrupole moment and R_s is the radius of the Sun. Observational data indicate that $J_2 \lesssim 3\cdot 10^{-5}$. Calculate how large the rotation $\Delta\phi$ of the orbit of Mercury this can cause maximally. Is this sufficient to explain the observed perihelion motion of Mercury?

8.4. *The linear field approximation*
We now assume that the gravitational field is weak, and introduce a near-Cartesian coordinate system. The metric then describes small deviations from Minkowski spacetime and the metric tensor is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric, $h_{\mu\nu} \ll 1$ and $h_{\mu\nu,\lambda}$ is small.

Einstein's field equations are $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$, where the units are chosen so that $c = 1$. $R_{\mu\nu}$ is the Ricci tensor and $R \equiv R^{\beta}_{\ \beta}$ is the scalar curvature tensor. $R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu}$ where

$$R^{\alpha}_{\ \mu\beta\nu} = \Gamma^{\alpha}_{\ \mu\nu,\beta} - \Gamma^{\alpha}_{\ \mu\beta,\nu} + \Gamma^{\alpha}_{\ \lambda\beta}\Gamma^{\lambda}_{\ \mu\nu} - \Gamma^{\alpha}_{\ \lambda\nu}\Gamma^{\lambda}_{\ \mu\beta} \tag{8.134}$$

is the Riemann curvature tensor.
In the linear approximation we will only calculate to first order in the metric perturbation h. Show that the Ricci tensor can now be written as

$$R_{\mu\nu} = \frac{1}{2}\left(h_{\mu\alpha,\nu}^{\ \ \ \ \alpha} + h_{\nu\alpha,\mu}^{\ \ \ \ \alpha} - h_{\mu\nu,\alpha}^{\ \ \ \ \alpha} - h_{,\mu\nu}\right), \tag{8.135}$$

where $h \equiv h_{\alpha}^{\alpha} = \eta^{\alpha\beta}h_{\alpha\beta}$.
Show that Einstein's field equations in this linear approximation can be written as

$$h_{\mu\alpha,\nu}^{\ \ \ \ \alpha} + h_{\nu\alpha,\mu}^{\ \ \ \ \alpha} - h_{\mu\nu,\alpha}^{\ \ \ \ \alpha} - h_{,\mu\nu} - \eta_{\mu\nu}(h_{\alpha\beta,}^{\ \ \ \ \alpha\beta} - h_{,\beta}^{\ \ \beta}) = 16\pi G T_{\mu\nu}. \tag{8.136}$$

This equation can be simplified by introducing $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. Assume that $\bar{h}$ satisfies the condition $\bar{h}_{\mu\alpha,}^{\ \ \ \alpha} = 0$ (Lorenz gauge). Show that the field equations can be written as

$$\bar{h}_{\mu\nu,\alpha}^{\ \ \ \ \alpha} = -16\pi G T_{\mu\nu}\ . \tag{8.137}$$

8.5. *The gravitational potential outside a static mass distribution*
A point mass with the mass m is situated in the origin of the nearly Cartesian coordinate system seen in Problem 8.4. Its energy–momenta tensor is $T_{00} = m\delta(\vec{r})$, $T_{\mu j} = 0$ ($\mu = \{0, i\}$), where $\delta(\vec{r})$ is the 3-dimensional δ-function.

(a) Show that when writing the field equations in the linear field approximation of the form

$$\bar{h}_{\mu\nu,\alpha}^{\ \ \ \ \alpha} = -16\pi G T_{\mu\nu}\ , \tag{8.138}$$

the solution in this case is

$$\bar{h}_{00} = \frac{4Gm}{r}, \qquad \text{where} \quad r = (x^2 + y^2 + z^2)^{1/2}, \tag{8.139}$$

$$\bar{h}_{\mu i} = 0. \tag{8.140}$$

(b) Show that $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$ and find the expression of the infinitesimal line element outside the point mass.
(c) Introduce spherical coordinates and compare with the line elements in 8.25 when these metrics are written to the first. order in M/r

8.6. *The Schwarzschild solution expressed in isotropic coordinates*

(a) We introduce a new and radial coordinate ρ so that the Schwarzschild metric gets the following form (with units $G = c = 1$):

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{8.141}$$

$$= -\left(1-\frac{2M}{r(\rho)}\right)dt^2 + f^2(\rho)(d\rho^2 + \rho^2 d\Omega^2), \tag{8.142}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Find the functions $r(\rho)$ and $f(\rho)$, and write down the explicit expression of the line element with ρ as the radial coordinate.

(b) What is the value of ρ at the Schwarzschild horizon $r = 2M$ and at the origin, $r = 0$? The Schwarzschild coordinates t and r interchange their roles as $r < 2M$. What is the behaviour of ρ inside the horizon?

8.7. *The perihelion precession of Mercury and the cosmological constant*

(a) When the cosmological constant Λ is non-zero, the Einstein equation takes the form $E_{\hat{\mu}\hat{\nu}} = -\Lambda g_{\hat{\mu}\hat{\nu}}$. Generalize the Schwarzschild solution in this case by showing that the metric is

$$ds^2 = -\left(1-\frac{2M}{r}-\frac{1}{3}\Lambda r^2\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}-\frac{1}{3}\Lambda r^2} + r^2 d\Omega^2\,, \tag{8.143}$$

where units are chosen so that $c = G = 1$.

(b) Show that the orbit equation for free particles in the metric given in (a) has the form (when $u = \frac{1}{r}$)

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{L^2} + 3Mu^2 - \frac{\Lambda}{3L^2u^3}\,. \tag{8.144}$$

L is the angular momentum per mass unit for the particle.

(c) Find the perihelion precession per rotation given by the new orbit equation. (Hint: Assume that the orbit can be described as a perturbation of a circle.)

(d) Estimations without the Λ-term lead to a correlation with observations to a precision of 1 arc second per century. Which limitations on the value of Λ does this lead to?

8.8. *Gravitational waves*

We will here consider gravitational waves in the weak field approximation of Einstein's equations using the Maxwell equations for the gravitoelectromagnetic fields.

(a) Use these equations for vacuum, and the Lorentz gauge condition,

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\cdot\mathbf{A}$$

to show that ϕ and $\mathbf{A}$ satisfy the wave equations

$$\Box\phi = 0,$$
$$\Box\mathbf{A} = 0, \tag{8.145}$$

where $\Box$ is the d'Alembert operator in Minkowski space. Hence, not surprisingly, gravitational waves travel with the speed of light. We can assume that $\mathbf{A}$ has in general complex components. The physical vector potential is the real part of $\mathbf{A}$.

(b) Consider waves far from any sources, so that $\phi = 0$. Find particular solutions where the wave describes a plane wave with wave-vector $\mathbf{k}$. What does the Lorentz gauge condition tell us about the nature of these gravitational waves?

(c) A test particle is initially at rest as one of the plane waves with wave-vector $\mathbf{k} = k\mathbf{e}_x$ passes by. The wave is plane-polarized so that $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{A}_0 e^{ik(x-ct)}, \tag{8.146}$$

where $\mathbf{A}_0 = A_0\mathbf{e}_y$, and A_0 is real. Assume that the test particle is placed at the origin and that the deviation from the origin as the wave passes by is very small compared to the wavelength of the wave. Hence, we can assume that $e^{ik(x-ct)} \approx e^{-ikct}$. Assume also that the speed of the particle v is non-relativistic: $v/c \ll 1$ (thus A_0 has to be sufficiently small). Use the Lorentz force law for GEM fields and derive the position of the particle to lowest order in A_0/c^2 as the wave passes by.

(d) Explain why gravitational waves cannot have only a Newtonian part, and thus that there are no gravitational waves in the Newtonian theory (or that they move with infinite speed).

8.9. *Plane-wave spacetimes*

We will in this problem consider the *plane-wave* metric

$$ds^2 = 2du(dv + Hdu) + dx^2 + dy^2, \tag{8.147}$$

where the function $H = H(u,x,y)$ does not depend on v (but is otherwise arbitrary).

(a) From the above we have the metric components given by $g_{uv} = 1$, $g_{uu} = 2H$ and $g_{xx} = g_{yy} = 0$. What is $g^{\mu\nu}$? Calculate also the Christoffel symbols and show that the vector $\mathbf{k} = \partial dv$ is covariantly constant; i.e. $k_{\mu;\nu} = 0$. Is $\mathbf{k}$ time-like, null or space-like?

(b) Use the Christoffel symbols to calculate the Einstein tensor and show that in vacuum ($\Lambda = 0$) the Einstein's field equations reduce to

$$\left(\partial d^2x^2 + \partial d^2y^2\right) H = 0\,. \tag{8.148}$$

What are the linearized field equations in this case?

(c) Show that

$$H = C_1(u)(x^2 - y^2) + 2C_2(u)xy,$$

where $C_1(u)$ and $C_2(u)$ are arbitrary functions of u, is a solution to Einstein's field equations.

Consider the complex coordinate $z = x + iy$ and an arbitrary analytic complex function $f(u,z)$. Show that

$$H = f(u,z) + \bar{f}(u,\bar{z}),$$

where a bar means complex conjugate, is a solution to Einstein's field equations. Is it also a solution to the linear field equations?

8.10. *Embedding of the interior Schwarzschild metric*
Make an embedding of the 3-dimensional spatial section $t = 0$ of the internal Schwarzschild solution. Join the resulting surface to the corresponding embedding of the external Schwarzschild solution.

8.11. *The Schwarzschild–de Sitter metric*
In this problem we will solve the Einstein equations with a cosmological constant. The Einstein equations with a cosmological constant Λ can be written as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa G T_{\mu\nu}. \tag{8.149}$$

(a) Use Schwarzschild coordinates and solve the Einstein vacuum equations with a cosmological constant.
(b) Show that in this case there are two horizons. Set the mass parameter equal to zero, and show that the spatial sections $dt = 0$ can be considered as a 3-sphere, S^3.

8.12. *Proper radial distance in the external Schwarzschild space*
Show that the proper radial distance from a coordinate position r to the horizon R_S in the external Schwarzschild space is

$$\ell_r = \sqrt{r}\sqrt{r - R_S} + R_S \ln\left(\sqrt{\frac{r}{R_S}} - \sqrt{\frac{r}{R_S} - 1}\right).$$

Find the limit of this expression for $R_S \ll r$.

8.13. *Gravitational redshift in the Schwarzschild spacetime*
Define z, describing the redshift of light, by

$$z = \frac{\Delta\lambda}{\lambda_e}, \tag{8.150}$$

where $\Delta\lambda$ is the change in the photons wavelength and λ_e the wavelength of the photon when emitted.
Show that the gravitational redshift of light emitted at r_E and received at r_R in the Schwarzschild spacetime outside a star of mass M is

$$z = \left(\frac{r_R - R_S}{r_E - R_S}\right)^{\frac{1}{2}} - 1,$$

where $R_S = 2M$ is the Schwarzschild radius of the star. What is the gravitational redshift of light emitted from the surface of a neutron star as observed by a faraway observer? A neutron star has typically a mass of 1.2 solar masses and a radius of about 20 km.

Reference

1. Shapiro, I. I., Ingalls, R. P., Smith, W. B., Campbell, D. B., Dyce, R. E., Jurgens, R. B., and Pettengill, G. H. 1971. Fourth test of general relativity – new radar result, *Phys. Rev. Lett.* **26**, 1132–1135. 170
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Chapter 9
Black Holes

9.1 "Surface Gravity": Gravitational Acceleration on the Horizon of a Black Hole

Surface gravity is denoted by κ_1 and is defined by

$$\kappa = \lim_{r \to r_+} \frac{a}{u^t} \qquad a = \sqrt{a_\mu a^\mu}\ , \tag{9.1}$$

where r_+ is the horizon radius, $r_+ = R_S$ for the Schwarzschild spacetime, u^t is the time component of the 4-velocity.

The 4-velocity of a free particle instantaneously at rest in the Schwarzschild spacetime:

$$\vec{u} = u^t \vec{e}_t = \frac{dt}{d\tau}\vec{e}_t = \frac{1}{\sqrt{-g_{tt}}}\vec{e}_t = \frac{\vec{e}_t}{\sqrt{1-\frac{R_S}{r}}}\ . \tag{9.2}$$

The only component of the 4-acceleration different from zero is a_r. The 4-acceleration: $\vec{a} = \nabla_{\vec{u}}\vec{u} = u^\mu{}_{;\nu} u^\nu \vec{e}_\mu = (u^\mu{}_{,\nu} + \Gamma^\mu_{\alpha\nu} u^\alpha) u^\nu \vec{e}_\mu$.

$$\begin{aligned} a_r &= (u_{r,\nu} + \Gamma_{r\alpha\nu} u^\alpha) u^\nu \\ &= \underbrace{u_{r,\nu} u^\nu}_{=0} + \Gamma_{rtt} (u^t)^2 \\ &= \frac{\Gamma_{rtt}}{1-\frac{R_S}{r}}, \\ \Gamma_{rtt} &= -\frac{1}{2}\frac{\partial g_{tt}}{\partial r} = -\frac{R_S}{2r^2}, \\ a_r &= \frac{\frac{R_S}{2r^2}}{1-\frac{R_S}{r}}, \\ a^r &= g^{rr} a_r = \frac{a_r}{g_{rr}} = (1-\frac{R_S}{r}) a_r = \frac{R_S}{2r^2}. \end{aligned} \tag{9.3}$$

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The acceleration scalar: $a = \sqrt{a_r a^r} = \frac{\frac{R_S}{2r^2}}{\sqrt{1-\frac{R_S}{r}}}$ (measured with standard instruments: at the horizon, time is not running).

$$\frac{a}{u^t} = \frac{R_S}{2r^2} . \tag{9.4}$$

With c:

$$\frac{a}{u^t} = \frac{c^2 R_S}{2r^2} = \frac{GM}{r^2} , \tag{9.5}$$

$$\kappa = \lim_{r \to R_S} \frac{a}{u^t} = \frac{1}{2R_S} = \frac{1}{4GM} . \tag{9.6}$$

Including c the expression is $\kappa = \frac{c^2}{4GM}$. On the horizon of a black hole with one solar mass, we get $\kappa_\odot = 2 \times 10^{13} \frac{m}{s^2}$.

9.2 Hawking Radiation: Radiation from a Black Hole

The radiation from a black hole has a thermal spectrum. We are going to "find" the temperature of a Schwarzschild black hole of mass M. The Planck spectrum has an intensity maximum at a wavelength given by Wien's displacement law:

$$\Lambda = \frac{N\hbar c}{kT} \text{ where } k \text{ is the Boltzmann constant and } N = 0.2014.$$

For radiation emitted from a black hole, Hawking derived the following expression for the wavelength at a maximum intensity:

$$\Lambda = 4\pi N R_S = \frac{8\pi NGM}{c^2} . \tag{9.7}$$

Inserting Λ from Wien's displacement law gives

$$T = \frac{\hbar c^3}{8\pi GkM} = \frac{\hbar c}{2\pi k} \kappa . \tag{9.8}$$

Inserting values for $\hbar$, c and k gives

$$T \approx \frac{2 \times 10^{-4} m}{R_S} K . \tag{9.9}$$

For a black hole with one solar mass, we have $T_\odot \approx 10^{-7}$. When the mass is decreasing because of the radiation, the temperature is *increasing*. So a black hole has a negative heat capacity. The energy loss of a black hole because of radiation is given by the Stefan–Boltzmann law:

$$-\frac{dM}{dt} = \sigma T^4 \frac{A}{c^2} \, , \tag{9.10}$$

where A is the surface of the horizon

$$A = 4\pi R_S^2 = \frac{16\pi G^2 M^2}{c^4} \, , \tag{9.11}$$

which gives

$$\begin{gathered} -\frac{dM}{dt} = \frac{1}{15360\pi}\frac{\hbar c^6}{G^2 M^2} \equiv \frac{Q}{M^2}, \\ M(t) = (M_0^3 - 3Qt)^{1/3}, \;\; M_0 = M(0). \end{gathered} \tag{9.12}$$

A black hole with mass M_0 early in the history of the universe which is about to explode now had to have a starting mass

$$M_0 = (3Qt_0)^{1/3} \approx 10^{12}\,\mathrm{kg} \tag{9.13}$$

about the mass of a mountain. They are called “mini black holes”.

9.3 Rotating Black Holes: The Kerr Metric

This solution was found by Roy Kerr in 1963.

A time-independent, time-orthogonal metric is known as a **static** metric. A time-independent metric is known as a **stationary** metric. A stationary metric allows rotation.

Consider a stationary metric which describes an axial-symmetric space

$$ds^2 = -e^{2\nu}dt^2 + e^{2\mu}dr^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\lambda}d\theta^2 \;\; , \tag{9.14}$$

where ν, μ, ψ, λ and ω are functions of r and θ.

By solving the vacuum field equations for this line element, Kerr found the solution:

$$\begin{aligned} e^{2\nu} = \frac{\rho^2\Delta}{\Sigma^2} \, , \quad e^{2\mu} = \frac{\rho^2}{\Delta} \, , \quad e^{2\psi} = \frac{\Sigma^2}{\rho^2}\sin^2\theta \, , \quad e^{2\lambda} = \rho^2 \, , \\ \omega = \frac{2Mar}{\Sigma^2} \, , \quad \text{where} \quad \rho^2 = r^2 + a^2\cos^2\theta, \\ \Delta = r^2 + a^2 - 2Mr, \\ \Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta. \end{aligned} \tag{9.15}$$

This is the Kerr solution expressed in Boyer–Lindquist coordinates. The function ω is the angular velocity. The Kerr solution is the metric for spacetime outside a rotating mass distribution. The constant a is spin per mass unit for the mass distribution and M is its mass.

Line element:

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{2Mr}{\rho^2}\right)dt^2+\frac{\rho^2}{\Delta}dr^2-\frac{4Mar}{\rho^2}\sin^2\theta dt d\phi+\rho^2 d\theta^2 \\
&+\left(r^2+a^2+\frac{2Ma^2r}{\rho^2}\sin^2\theta\right)\sin^2\theta d\phi^2.
\end{aligned}
\tag{9.16}
$$

(Here M is a measure of the mass so that $M = G\cdot\text{mass}$, i.e. $G = 1$.)

Light is emitted from the surface, $r = r_0$, where $g_{tt} = 0$ is infinitely redshifted further out. Observed from the outside time stands still:

$$
\begin{aligned}
\rho^2 = 2Mr_0 \quad \Rightarrow \quad & r_0^2 + a^2\cos^2\theta = 2Mr_0, \\
& r_0 = M \pm \sqrt{M^2 - a^2\cos^2\theta}\ .
\end{aligned}
\tag{9.17}
$$

This is the equation for the surface which represents infinite redshift.

9.3.1 Zero-Angular-Momentum-Observers

The Lagrange function of a free particle in the equator plane, $\theta = \frac{\pi}{2}$, is

$$
L = -\frac{1}{2}(e^{2\nu}-\omega^2 e^{2\Psi})\dot{t}^2+\frac{1}{2}e^{2\mu}\dot{r}^2+\frac{1}{2}e^{2\Psi}\dot{\phi}^2+\frac{1}{2}e^{2\lambda}\dot{\theta}^2-\omega e^{2\Psi}\dot{t}\dot{\phi}\ . \tag{9.18}
$$

Here $\dot{\theta} = 0$. The momentum p_ϕ of the cyclic coordinates ϕ is

$$
p_\phi \equiv \frac{\partial L}{\partial\dot{\phi}} = e^{2\Psi}(\dot{\phi}-\omega\dot{t})\ , \quad \dot{t} = \frac{dt}{d\tau}\ , \quad \dot{\phi} = \frac{d\phi}{d\tau}. \tag{9.19}
$$

The angular speed of the particle relative to the coordinate system is

$$
\begin{aligned}
\Omega = \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}}\ , \quad \dot{\phi} = \Omega\dot{t} \\
\Rightarrow p_\phi = e^{2\Psi}\dot{t}(\Omega-\omega);
\end{aligned}
\tag{9.20}
$$

p_ϕ is conserved during the motion.

$$
\begin{aligned}
\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{(r^2+a^2)^2-a^2(r^2+a^2-2Mr)}\ , \\
\omega \to 0 \quad \text{when} \quad r\to\infty\ .
\end{aligned}
\tag{9.21}
$$

When studying the Kerr metric one finds that Kerr $\rightarrow$ Minkowski for large r. The coordinate clocks in the Kerr spacetime show the same time as the standard clocks at rest in the asymptotic Minkowski spacetime.

A ZAMO is per definition a particle or observer with $p_\phi = 0$. Consider a far-away observer who let a stone fall with vanishing initial velocity. p_ϕ is a constant of motion, so the stone remains a ZAMO during the movement. A local reference frame which coincides with the stone is a member of the class of inertial frames that are at rest in the asymptotic Minkowski region. These ZAMO inertial frames may be used to define "the state of motion of the space". They have

$$p_\phi = 0 \Rightarrow \Omega = \frac{d\phi}{dt} = \omega \,. \tag{9.22}$$

That is the local inertial frame obtains an angular speed relative to the BL-system (Boyer–Lindquist system).

Since the Kerr metric is time independent, the BL-system is stiff. The distant observer has no motion relative to the BL-system. To this observer the BL-system will appear stiff and non-rotating. The observer will observe that the local inertial system of the stone obtains an angular speed

$$\frac{d\phi}{dt} = \omega = \frac{2Mar}{(r^2+a^2)^2 - a^2(r^2+a^2-2Mr)}; \tag{9.23}$$

In other words, inertial systems at finite distances from the rotating mass M are dragged with it in the same direction. This is known as **inertial dragging** or the Lense–Thirring effect (about 1920).

9.3.2 Does the Kerr Space Have a Horizon?

Definition 9.3.1 (Horizon) A surface one can enter, but not exit.

Consider a particle in an orbit with constant r and θ. Its 4-velocity is

$$\begin{aligned}\vec{u} &= \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{x}}{dt} \\ &= (-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2)^{-\frac{1}{2}}(1,\Omega)\,, \qquad \text{where } \Omega = \frac{d\phi}{dt}.\end{aligned} \tag{9.24}$$

To have stationary orbits the following must be true:

$$g_{\phi\phi}\Omega^2 + 2g_{t\phi}\Omega + g_{tt} < 0\,. \tag{9.25}$$

This implies that Ω must be in the interval

$$\Omega_{min} < \Omega < \Omega_{max}\ , \tag{9.26}$$

where $\Omega_{min} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}$ and $\Omega_{max} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}$ since $g_{t\phi} = -\omega g_{\phi\phi}$.

Outside the surface with infinite redshift $g_{tt} < 0$. That is Ω can be negative, zero and positive. Inside the surface $r = r_0$ with infinite redshift $g_{tt} > 0$. Here $\Omega_{min} > 0$

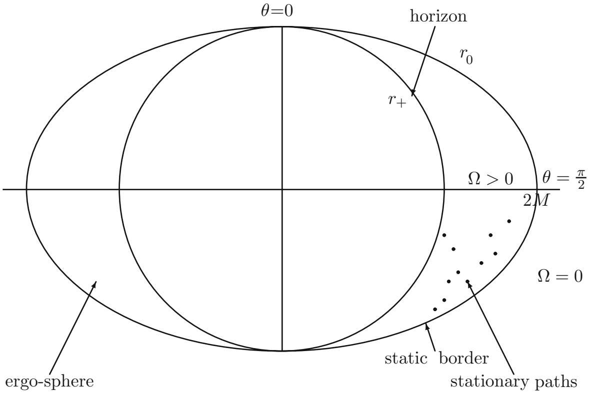

Fig. 9.1 Static border and horizon of a Kerr black hole

and static particles, $\Omega = 0$, cannot exist. This is due to the inertial dragging effect. The surface $r = r_0$ is therefore known as "the static border".

The interval of Ω, where stationary orbits are allowed, is reduced to zero when $\Omega_{min} = \Omega_{max}$, that is $\omega^2 = \frac{g_{tt}}{g_{\phi\phi}} \Rightarrow g_{tt} = \omega^2 g_{\phi\phi}$ (equation for the horizon).

For the Kerr metric we have

$$g_{tt} = \omega^2 g_{\phi\phi} - e^{2\nu} . \tag{9.27}$$

Therefore the horizon equation becomes

$$e^{2\nu} = 0 \quad \Rightarrow \Delta = 0 \qquad \therefore r^2 - 2Mr + a^2 = 0 . \tag{9.28}$$

The largest solution is $r_+ = M + \sqrt{M^2 - a^2}$ and this is the equation for a spherical surface. The static border is $r_0 = M + \sqrt{M^2 - a^2 \cos\theta}$ (Fig. 9.1).

Problems

9.1. *A spaceship falling into a black hole*

(a) In this problem we will consider a spaceship (A) falling radially into a Schwarzschild black hole with mass $M = 5M_{\text{Sun}}$ (set $c = 1$). What is the Schwarzschild radius of the black hole? Find the equations of motion of the spaceship in Schwarzschild coordinates r and t, using the proper time τ as time parameter. At the time $t = \tau = 0$ the spaceship is located at $r = 10^{10}M$. The total energy is equal to its rest energy. Solve the equations of motion with these initial conditions. When (in terms of τ) does the spaceship reach the Schwarzschild radius? And the singularity?

(b) Show that the spaceship, from the Schwarzschild radius to the singularity, uses maximally $\Delta\tau = \pi GM$ no matter how it is manoeuvred. How should the spaceship be manoeuvred to maximalize this time?

(c) The spaceship (A) has radio contact with a stationary space station (B) at $r_B = 1$ light years. The radio-signals are sent with intervals ΔT and with frequency ω from both A and B. The receivers at A and B receive signals with frequency ω_A and ω_B, respectively. Find ω_A and ω_B as a function of the position of the spaceship. (Hint: Perform the calculation in two steps. At first find the change in frequency between two stationary inertial systems in the points r_A (the pos. of the spaceship) and r_B. Then calculate the change in frequency due to a transfer into an inertial system with the velocity of the spaceship.) Investigate whether something particular is happening as the spaceship passes the Schwarzschild radius. Discuss what these results tell us about how the events in the spaceship is described in the space station, and vice versa.

9.2. *The spacetime inside and outside a rotating spherical shell*
A spherical shell with mass M and radius R is rotating with a constant angular velocity ω. In this problem the metric inside and outside the shell shall be found using the linearized Einstein's field equations

$$\Box \bar{h}_{\alpha\beta} = -2\kappa T_{\mu\nu} \, , \tag{9.29}$$

where $\bar{h}_{\alpha\beta}$ is the metric perturbation with respect to the Minkowski metric and $\Box$ represents the d'Alembert operator. The rotation is assumed to be non-relativistic, thus the calculations should be made to first order in $R\omega$.
Assume that the shell is composed of dust, so that the energy–momentum tensor can be expressed as

$$\begin{aligned} T_{\alpha\beta} &= \rho u_\alpha u_\beta, \quad \rho = \frac{M}{4\pi R^2}\delta(r-R), \\ u_\alpha &\approx (-1, -R\omega\sin\theta\sin\phi, R\omega\sin\theta\cos\phi, 0), \end{aligned} \tag{9.30}$$

where (r,θ,ϕ) are spherical coordinates:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad r = \cos\theta \, . \tag{9.31}$$

Find the metric inside and outside the rotating shell and show that

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{R}\right)dt^2 + \left(1 + \frac{2M}{R}\right)(dx^2 + dy^2 + dz^2) \\ &\quad - \frac{8M\omega}{3R} r^2 \sin^2\theta d\phi dt, \qquad r < R, \\ ds^2 &= -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)(dx^2 + dy^2 + dz^2) \\ &\quad - \frac{4J}{r}\sin^2\theta d\phi dt, \qquad r > R, \end{aligned} \tag{9.32}$$

where $J = (2/3)MR^2\omega$ is the angular momentum of the shell. (Hint: Estimate $g_{0\phi}$ by first finding $\bar{h}_{0y}$. Due to the axial symmetry we have that $g_{0\phi} = g_{\phi\phi}^{1/2}[g_{0y}]_{\phi=0}$. Assume that $\bar{h}_{0y}$ has the form $\bar{h}_{0y} = f(r)\sin\theta\cos\phi$.)

9.3. *Physical interpretation of the Kerr metric*
In this problem we shall use the linearized solution of the spacetime outside a rotating shell derived in Problem 9.2.

(a) Show that the Kerr metric is reduced to the metric (9.32) in the limit $r > R$ and $r \gg M$ and identify thereby the constant a with the angular momentum per unit mass of the rotating shell. (Hint: Expand the Kerr metric to first order in J/Mr, introduce isotropic coordinates ($r \to \rho$ see Problem 8.6) and expand the result to first order in M/ρ).
(b) Find the angular velocity

$$\omega_L = -\frac{g_{0\phi}}{g_{\phi\phi}} \tag{9.33}$$

that local reference frames are rotating with, with respect to reference frames at infinity.

9.4. *Kinematics in 3-space*
A Kerr black hole is a electrically neutral, rotating black hole. When spacetime outside a Kerr black hole is described in Boyer–Lindquist (BL) coordinates, the line element is the following:

$$ds^2 = -e^{2\nu}dt^2 + e^{2\mu}dr^2 + e^{2\lambda}d\theta^2 + e^{2\psi}(d\phi - \omega dt)^2\,, \tag{9.34}$$

where

$$\begin{aligned} e^{2\nu} &= \frac{\rho^2\Delta}{\Sigma^2}, \quad e^{2\mu} = \frac{\rho^2}{\Delta}, \quad e^{2\lambda} = \rho^2, \\ e^{2\psi} &= \left(\frac{\Sigma^2}{\rho^2}\right)\sin^2\theta, \quad \omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{\Sigma^2}, \\ \rho^2 &= r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - 2Mr, \qquad \Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta. \end{aligned}$$

Here M is the mass of the hole and a its spin per unit mass.

(a) Consider light moving in negative and positive direction of ϕ. What is the coordinate velocity

$$c_\phi = \frac{d\phi}{dt} \tag{9.35}$$

of light?
We now want to investigate the Sagnac effect in the Kerr space. An emitter-receiver is attached to a point in the BL-coordinate system. Light signals with the frequency ν are sent by means of mirrors in both directions along the circle $r = r_0$, $\theta = \pi/2$. Find the phase difference of light travelling in opposite directions, when the signals reach the receiver.

The result shows that the BL-system looks like a rotating coordinate system when observed from a finite distance r from the black hole. Does the black hole appear to be rotating when observed from infinity?

(b) Consider a free particle in the Kerr space. Assume that it is moving in the direction of ϕ, with an angular velocity

$$\Omega = d\phi/dt \tag{9.36}$$

given in the BL-coordinates. Write down the Lagrangian L of the particle and find its canonical momentum,

$$p_\phi = \partial L/\partial\dot{\phi} \ . \tag{9.37}$$

What is the angular velocity of a particle with vanishing momentum?

(c) A *ZAMO* (*Zero Angular Momentum Observer*) is an observer with vanishing angular momentum. Consider in particular a *ZAMO* with fixed r- and θ-coordinates. Is this *ZAMO* freely falling?

In the following *ZAMO* will be describing the particular observers with fixed r- and θ-coordinates. Introduce an orthonormal basis $(\vec{e}_{\hat{t}'}, \vec{e}_{\hat{r}'}, \vec{e}_{\hat{\theta}'}, \vec{e}_{\hat{\phi}'})$, where $\vec{e}_{\hat{t}'}$ is the 4-velocity of a *ZAMO*. The dual basis 1-forms are

$$\begin{aligned} \underline{\omega}^{\hat{t}'} &= e^{\nu}\,\underline{\omega}^{t}, \qquad & \underline{\omega}^{\hat{r}'} &= e^{\mu}\,\underline{\omega}^{r}, \\ \underline{\omega}^{\hat{\theta}'} &= e^{\lambda}\,\underline{\omega}^{\theta}, \qquad & \underline{\omega}^{\hat{\phi}'} &= e^{\psi}(\underline{\omega}^{\phi} - \omega\underline{\omega}^{t}). \end{aligned} \tag{9.38}$$

Show that

$$\begin{aligned} \vec{e}_{\hat{t}'} &= e^{-\nu}(\vec{e}_t + \omega\vec{e}_\phi), \qquad & \vec{e}_{\hat{r}'} &= e^{-\mu}\vec{e}_r, \\ \vec{e}_{\hat{\theta}'} &= e^{-\lambda}\vec{e}_\theta, \qquad & \vec{e}_{\hat{\phi}'} &= e^{-\psi}\vec{e}_\phi, \end{aligned} \tag{9.39}$$

where $(\vec{e}_t, \vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ are the coordinate basis vectors in the BL-coordinate system.

Show that the physical velocity of the particle from part (b), measured by a *ZAMO*, is

$$v^{\hat{\phi}'} = e^{\psi-\nu}(\Omega - \omega) \ . \tag{9.40}$$

What is the velocity $v_0^{\hat{\phi}'}$ of a fixed coordinate point measured by a *ZAMO*?

(d) Introduce an orthonormal basis field given by the expressions

$$\vec{e}_{\hat{0}} = (-g_{00})^{-1/2}\vec{e}_0, \qquad \vec{e}_{\hat{i}} = (\gamma_{ii})^{-1/2}[\vec{e}_i - (g_{i0}/g_{00})\vec{e}_0] \ , \tag{9.41}$$

where

$$\gamma_{ii} = g_{ii} - g_{i0}^2/g_{00}.$$

Show that

$$\vec{e}_{\hat{t}} = \hat{\gamma} e^{-\nu} \vec{e}_t, \qquad \vec{e}_{\hat{r}} = e^{-\mu} \vec{e}_r,$$
$$\vec{e}_{\hat{\theta}} = e^{-\lambda} \vec{e}_r, \qquad \vec{e}_{\hat{\phi}} = \hat{\gamma}^{-1} e^{-\psi} \vec{e}_\phi + \hat{\gamma} e^{-\nu} v_0^{\hat{\phi}'} \vec{e}_t, \tag{9.42}$$

where $\hat{\gamma} = (1 - (v_0^{\hat{\phi}'})^2)^{-1/2}$. Find the dual basis 1-forms. The vector $\vec{e}_{\hat{t}}$ is the 4-velocity of a particle at rest in the BL-coordinate system, that is of a *static* particle.

(e) Find the physical velocity $v^{\hat{\phi}}$ of the particle from point (b), measured by a static observer. What is the correspondence between $v^{\hat{\phi}'}$, $v^{\hat{\phi}}$ and $v_0^{\hat{\phi}'}$? Show that the orthonormal basis field associated with a static observer and that associated with a *ZAMO* are related through a Lorentz transformation.

9.5. *The lifetime of a black hole*
Consider a black hole in a zero-temperature heat-bath. Including c, $\hbar$, k_B and G we have $M = Gmc^{-2}$, the black hole temperature is

$$T = \frac{\hbar}{2\pi k_B c} \kappa \tag{9.43}$$

and the Stefan–Boltzmann constant is

$$\sigma = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} \,. \tag{9.44}$$

(a) Assume that a black hole has a mass m_0 at $t = 0$. Find $m(t)$.
(b) If we consider a time span of approximately $\tau = 10^{10}$ years, what mass would the black hole have at $t = 0$ to have $m(\tau) = 0$?
(c) Show that a black hole of mass m cannot disintegrate into two smaller black holes of mass m_1 and m_2 where $m = m_1 + m_2$.

9.6. *A gravitomagnetic clock effect*
This problem is concerned with the difference of proper time shown by two clocks moving freely in opposite directions in the equatorial plane of the Kerr spacetime outside a rotating body. The clocks move along a path with $r =$ constant and $\theta = \pi/2$.

(a) Show that in this case the radial geodesic equation reduces to

$$\Gamma^r{}_{tt} dt^2 + 2\Gamma^r{}_{\phi t} d\phi dt + \Gamma^r_{\phi\phi} d\phi^2 = 0.$$

(b) Calculate the Christoffel symbols and show that the equation takes the form

$$\left(\frac{dt}{d\phi}\right)^2 - 2a\frac{dt}{d\phi} + a^2 - \frac{r^3}{M} = 0,$$

where M is the mass of the rotating body and a its angular momentum per unit mass, $a = J/M$.

(c) Use the solution of the geodesic equation and the 4-velocity identity to show that the proper time-interval $d\tau$ shown on a clock moving an angle $d\phi$ is

$$d\tau = \pm\sqrt{1 - \frac{3M}{r} \pm 2a\omega_0} d\phi,$$

where $\omega_0 = \left(M/r^3\right)^{1/2}$ is the angular velocity of a clock moving in the Schwarzschild spacetime in accordance with Kepler's 3rd law. The plus and minus sign apply to direct and retrograde motion, respectively.

(d) Show that to first order in a the proper time difference for one closed orbit ($\phi \to \phi + 2\pi$) in the direct and the retrograde direction is $\tau_+ - \tau_- \approx 4\pi a = 4\pi J/M$, or in S.I. units, $\tau_+ - \tau_- \approx 4\pi a = 4\pi J/mc^2$.
Estimate this time difference for clocks in satellites moving in the equatorial plane of the Earth. (The mass of the Earth is $m = 6 \cdot 10^{26}$ kg and its angular momentum $J = 10^{34}\,\mathrm{kg\,m^2/s}$.)

9.7. *The photon sphere radius of a Reissner–Nordström black hole*
Show that there exists a sphere of radius

$$r_{PS} = \frac{3M}{2}\left(1 + \sqrt{1 - \frac{8Q^2}{9M^2}}\right) \tag{9.45}$$

in the Reissner–Nordström black hole spacetime where photons will have circular orbits around the black hole.

9.8. *The Reissner–Nordström repulsion*
Consider a radially infalling neutral particle in the Reissner–Nordström space–time with $M > |Q|$. Show that when the particle comes inside the radius $r = Q^2/M$ it will feel a repulsion away from $r = 0$ (i.e. $d^2r/d\tau^2 < 0$ for τ the proper time of the particle). Is this inside or outside the outer horizon r_+? Show further that the particle can never reach the singularity at $r = 0$.

9.9. *Gravitational mass*

(a) Use the general Schwarzschild line element and show that the surface gravity of a Schwarzschild black hole can be written as

$$\kappa = -e^{\alpha-\beta}\alpha' \,. \tag{9.46}$$

(b) Show, using Einstein's field equations, that

$$4\pi r^2 e^{\alpha+\beta}\left(T^0_0 - T^1_1 - T^2_2 - T^3_3\right) = \left(r^2 e^{\alpha-\beta}\alpha'\right)' \,. \tag{9.47}$$

Hence, deduce that the surface gravity can be written as

$$\kappa = -\frac{4\pi}{r^2}\int_0^r \left(T^0_{\ 0} - T^1_{\ 1} - T^2_{\ 2} - T^3_{\ 3}\right) e^{\alpha+\beta} r^2 dr\,. \tag{9.48}$$

(c) Define the gravitational mass M_G inside a radius r of a spherical mass distribution by

$$\kappa = -\frac{M_G}{r^2}\,, \tag{9.49}$$

and deduce that

$$M_G = 4\pi\int_0^r \left(T^0_{\ 0} - T^1_{\ 1} - T^2_{\ 2} - T^3_{\ 3}\right) e^{\alpha+\beta} r^2 dr\,. \tag{9.50}$$

This is the *Tolman–Whittaker* expression for the gravitational mass of a system. What is the condition for repulsive gravitation?

9.10. *The river model for black holes*
In this problem you are going to picture space as flowing like a river into a Schwarzschild black hole. "Space" is then represented by a continuum of local inertial frames falling freely from zero velocity at infinity.

(a) Show that the Schwarzschild metric may be written in the Gullstrand–Painlevé form

$$ds^2 = -d\tau^2 + (dr + \beta d\tau)^2 + r^2(d\theta^2 + \sin^2 d\phi^2), \quad \beta = \sqrt{\frac{r_S}{r}},$$

by introducing a new coordinate time

$$\tau = t + 2r_S\left(\frac{1}{\beta} - \operatorname{arctanh}\beta\right).$$

(b) Show that $\beta(r) = -dr/d\tau$ is the velocity of an inertial frame falling freely from rest at infinity, i.e. the river velocity. What happens at the horizon of the black hole? Show that $d\tau$ is the proper time-interval as measured by a clock comoving with the inertial frames that define the river model of the space.

Chapter 10
Schwarzschild's Interior Solution

10.1 Newtonian Incompressible Star

$$\begin{aligned}\nabla^2\phi &= 4\pi G\rho, \quad \phi=\phi(r),\\ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) &= 4\pi G\rho.\end{aligned} \tag{10.1}$$

Assuming $\rho=$ constant,

$$\begin{aligned} d\left(r^2\frac{d\phi}{dr}\right) &= 4\pi G\rho r^2 dr,\\ r^2\frac{d\phi}{dr} &= \frac{4\pi}{3}G\rho r^3+K\\ &= M(r)+K.\end{aligned} \tag{10.2}$$

Gravitational acceleration: $\vec{g}=-\nabla\phi=-\frac{d\phi}{dr}\vec{e}_r$

$$g=\frac{M(r)}{r^2}+\frac{K_1}{r^2}=\frac{4\pi}{3}G\rho r+\frac{K_1}{r^2}\,. \tag{10.3}$$

Finite g in $r=0$ demands $K_1=0$.

$$g=\frac{4\pi}{3}G\rho r, \quad \frac{d\phi}{dr}=\frac{4\pi}{3}G\rho r\,. \tag{10.4}$$

Assume that the mass distribution has a radius R.

$$\phi=\frac{2\pi}{3}G\rho r^2+K_2\,. \tag{10.5}$$

Demand continuous potential at $r=R$.

$$\begin{aligned}\frac{2\pi}{3}G\rho R^2+K_2=\frac{M(R)}{R} &= -\frac{4\pi}{3}G\rho R^2\\ &\Rightarrow K_2=-2\pi G\rho R^2\end{aligned} \tag{10.6}$$

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(with zero level at infinite distance). Given the potential inside the mass distribution:

$$\phi = \frac{2\pi}{3} G\rho (r^2 - 3R^2) \, . \tag{10.7}$$

The star is in hydrostatic equilibrium, that is the pressure forces are in equilibrium with the gravitational forces.

Consider a mass element, $dm = \rho dV = \rho dAdr$, in the shell depicted in Fig. 10.1. The pressure force on the mass element is $dF = dAdp$, and the gravitational force is

$$dG = gdm = \frac{Gm(r)}{r^2} dm \, , \tag{10.8}$$

where $m(r)$ is the mass inside the shell. With constant density $m(r) = (4\pi/3)\rho r^3$. Hence

$$dG = gdm = \frac{4\pi}{3} G\rho^2 rdAdr \, . \tag{10.9}$$

Equilibrium, $dF = -dG$, demands that

$$dp = -\frac{4\pi}{3} G\rho^2 rdr \, . \tag{10.10}$$

Integrating this gives

$$p = K_3 - \frac{2\pi G}{3} \rho^2 r^2 \, . \tag{10.11}$$

$p(R) = 0$ gives the value of the constant of integration K_3

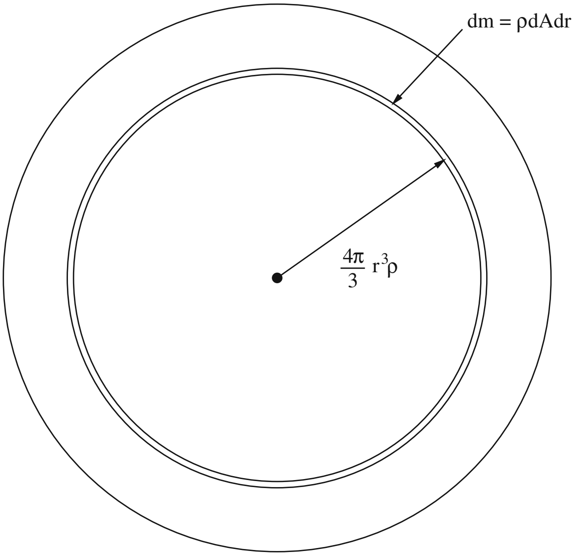

Fig. 10.1 The shell with thickness *dr*, is affected by both gravitational and pressure forces

$$K_3 = \frac{2\pi G}{3}\rho^2 R^2 \tag{10.12}$$

and we find

$$p(r) = \frac{2\pi G}{3}\rho^2 (R^2 - r^2) \ . \tag{10.13}$$

No matter how massive the star is, it is possible for the pressure forces to keep the equilibrium with gravity. In Newtonian theory, gravitational collapse is not a necessity.

10.2 The Pressure Contribution to the Gravitational Mass of a Static, Spherically Symmetric System

We now give a new definition of the gravitational acceleration (not equivalent to (8.23)):

$$g = -\frac{a}{u^t} \ , \qquad a = \sqrt{a_\mu a^\mu} \ . \tag{10.14}$$

We have the line element:

$$\begin{aligned} ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2, \\ g_{tt} = -e^{2\alpha} \ , \quad g_{rr} = e^{2\beta}, \end{aligned} \tag{10.15}$$

which gives (because of the gravitational acceleration)

$$g = -e^{\alpha-\beta}\alpha' \ . \tag{10.16}$$

From the expressions for $E_{\hat{t}\hat{t}}$, $E_{\hat{r}\hat{r}}$, $E_{\hat{\theta}\hat{\theta}}$, $E_{\hat{\phi}\hat{\phi}}$ it follows (see Sect. 8.1)

$$E^{\hat{t}}_{\ \hat{t}} - E^{\hat{r}}_{\ \hat{r}} - E^{\hat{\theta}}_{\ \hat{\theta}} - E^{\hat{\phi}}_{\ \hat{\phi}} = -2e^{-2\beta}\left(\frac{2\alpha'}{r} + \alpha'' + \alpha'^2 - \alpha'\beta'\right) \ . \tag{10.17}$$

We also have

$$(r^2 e^{\alpha-\beta}\alpha')' = r^2 e^{\alpha-\beta}\left(\frac{2\alpha'}{r} + \alpha'' + \alpha'^2 - \alpha'\beta'\right) \ , \tag{10.18}$$

which gives

$$g = \frac{1}{2r^2}\int (E^{\hat{t}}_{\ \hat{t}} - E^{\hat{r}}_{\ \hat{r}} - E^{\hat{\theta}}_{\ \hat{\theta}} - E^{\hat{\phi}}_{\ \hat{\phi}}) r^2 e^{\alpha+\beta} dr \ . \tag{10.19}$$

By applying Einstein's field equations

$$E^{\hat{\mu}}_{\ \hat{\nu}} = 8\pi G T^{\hat{\mu}}_{\ \hat{\nu}} \tag{10.20}$$

we get

$$g = \frac{4\pi G}{r^2}\int (T^{\hat{t}}_{\ \hat{t}} - T^{\hat{r}}_{\ \hat{r}} - T^{\hat{\theta}}_{\ \hat{\theta}} - T^{\hat{\phi}}_{\ \hat{\phi}}) r^2 e^{\alpha+\beta} dr \ . \tag{10.21}$$

This is the Tolman–Whittaker expression for gravitational acceleration.

The corresponding Newtonian expression is

$$g_N = -\frac{4\pi G}{r^2}\int \rho r^2 dr \,. \tag{10.22}$$

The relativistic gravitational mass–density is therefore defined as

$$\rho_G = -T^{\hat{t}}_{\hat{t}} + T^{\hat{r}}_{\hat{r}} + T^{\hat{\theta}}_{\hat{\theta}} + T^{\hat{\phi}}_{\hat{\phi}}. \tag{10.23}$$

For an isotropic fluid with

$$T^{\hat{t}}_{\hat{t}} = -\rho \,, \quad T^{\hat{r}}_{\hat{r}} = T^{\hat{\theta}}_{\hat{\theta}} = T^{\hat{\phi}}_{\hat{\phi}} = p \tag{10.24}$$

we get $\rho_G = \rho + 3p$ (with $c = 1$), which becomes

$$\boxed{\rho_G = \rho + \frac{3p}{c^2}} \,. \tag{10.25}$$

It follows that in relativity, pressure has a gravitational effect. Greater pressure gives increasing gravitational attraction. Strain ($p < 0$) decreases the gravitational attraction.

In the Newtonian limit, $c \to \infty$, pressure has no gravitational effect.

10.3 The Tolman–Oppenheimer–Volkoff Equation

With spherical symmetry the spacetime line element may be written as

$$\begin{aligned} ds^2 &= -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2, \\ E_{\hat{t}\hat{t}} &= 8\pi G T_{\hat{t}\hat{t}} \,, \quad T^{\hat{\mu}}_{\hat{\nu}} = diag(-\rho, p, p, p) \,. \end{aligned} \tag{10.26}$$

From $E_{\hat{t}\hat{t}}$ we get

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\beta})] &= 8\pi G\rho, \\ r(1-e^{-2\beta}) &= 2G\int_0^r 4\pi\rho r^2 dr \,, \end{aligned} \tag{10.27}$$

where $m(r) = \int_0^r 4\pi\rho r^2 dr$ giving

$$e^{-2\beta} = 1 - \frac{2Gm(r)}{r} = \frac{1}{g_{rr}} \,. \tag{10.28}$$

From $E_{\hat{r}\hat{r}}$ we have

$$
\begin{aligned}
E_{\hat{r}\hat{r}} &= 8\pi G T_{\hat{r}\hat{r}}, \\
\frac{2}{r}\frac{d\alpha}{dr}e^{-2\beta} - \frac{1}{r^2}(1-e^{-2\beta}) &= 8\pi G p\,.
\end{aligned} \tag{10.29}
$$

We get

$$
\begin{aligned}
\frac{2}{r}\frac{d\alpha}{dr}\left(1-\frac{2Gm(r)}{r}\right) - \frac{2Gm(r)}{r^3} &= 8\pi G p, \\
\frac{d\alpha}{dr} &= G\frac{m(r)+4\pi r^3 p(r)}{r(r-2Gm(r))}\,.
\end{aligned} \tag{10.30}
$$

The relativistic generalized equation for hydrostatic equilibrium is $T^{\hat{r}\hat{\nu}}_{;\hat{\nu}} = 0$, giving

$$
\begin{aligned}
T^{\hat{r}\hat{\nu}}_{,\hat{\nu}} + \Gamma^{\hat{\nu}}_{\hat{\alpha}\hat{\nu}}T^{\hat{r}\hat{\alpha}} + \Gamma^{\hat{r}}_{\hat{\alpha}\hat{\nu}}T^{\hat{\alpha}\hat{\nu}} &= 0, \\
T^{\hat{r}\hat{\nu}}_{,\hat{\nu}} = T^{\hat{r}\hat{r}}_{,\hat{r}} = p_{,\hat{r}} &= \frac{1}{\sqrt{g_{rr}}}\frac{\partial p}{\partial r}, \\
T^{\hat{r}\hat{\nu}}_{,\hat{\nu}} &= e^{-\beta}\frac{dp}{dr}, \\
\Gamma^{\hat{\nu}}_{\hat{\alpha}\hat{\nu}}T^{\hat{r}\hat{\alpha}} = \Gamma^{\hat{\nu}}_{\hat{r}\hat{\nu}}p &= \Gamma^{\hat{t}}_{\hat{r}\hat{t}}p + \Gamma^{\hat{\alpha}}_{\hat{r}\hat{\alpha}}p, \\
\Gamma^{\hat{r}}_{\hat{\alpha}\hat{\nu}}T^{\hat{\alpha}\hat{\nu}} = \Gamma^{\hat{r}}_{\hat{\nu}\hat{\nu}}T^{\hat{\nu}\hat{\nu}} &= \Gamma^{\hat{r}}_{\hat{t}\hat{t}}\rho + \Gamma^{\hat{r}}_{\hat{\alpha}\hat{\alpha}}p\,.
\end{aligned} \tag{10.31}
$$

In orthonormal basis we have

$$
\begin{aligned}
\underline{\Omega}_{\hat{\nu}\hat{\mu}} = -\underline{\Omega}_{\hat{\mu}\hat{\nu}} &\Rightarrow \Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} = -\Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}}, \\
\Gamma^{\hat{\alpha}}_{\hat{r}\hat{\alpha}} = \Gamma_{\hat{\alpha}\hat{r}\hat{\alpha}} &= -\Gamma_{\hat{r}\hat{\alpha}\hat{\alpha}} = -\Gamma^{\hat{r}}_{\hat{\alpha}\hat{\alpha}}\,.
\end{aligned} \tag{10.32}
$$

$T^{\hat{r}\hat{\nu}}_{;\hat{\nu}} = 0$ now takes the form

$$
e^{-\beta}\frac{dp}{dr} + \Gamma^{\hat{t}}_{\hat{r}\hat{t}}p + \Gamma^{\hat{r}}_{\hat{t}\hat{t}}\rho = 0\,. \tag{10.33}
$$

We have

$$
\Gamma^{\hat{t}}_{\hat{r}\hat{t}} = -\Gamma_{\hat{t}\hat{r}\hat{t}} = \Gamma_{\hat{r}\hat{t}\hat{t}} = \Gamma^{\hat{r}}_{\hat{t}\hat{t}} \tag{10.34}
$$

and we also have $\Gamma^{\hat{r}}_{\hat{t}\hat{t}} = e^{-\beta}\frac{d\alpha}{dr}$, giving

$$
\frac{dp}{dr} + (p+\rho)\frac{d\alpha}{dr} = 0\,. \tag{10.35}
$$

Inserting Eq. (10.30) into Eq. (10.35) gives

$$
\boxed{\frac{dp}{dr} = -G(\rho+p)\frac{m(r)+4\pi r^3 p(r)}{r(r-2Gm(r))}}\,. \tag{10.36}
$$

This is the Tolman–Oppenheimer–Volkoff (TOV) equation. The component $g_{tt} = -e^{2\alpha(r)}$ may now be calculated as follows:

$$\begin{aligned}\frac{dp}{\rho+p} &= -d\alpha\ , \qquad \rho = \text{constant}, \\ ln(\rho+p) &= K-\alpha, \\ \rho+p &= K_1 e^{-\alpha}\ , \quad p = K_1 e^{-\alpha} - \rho\ .\end{aligned} \tag{10.37}$$

Hence

$$e^{\alpha} = e^{\alpha(R)}\left(1+\frac{p}{\rho}\right)^{-1}\ , \tag{10.38}$$

where R is the radius of the mass distribution.

10.4 An Exact Solution for Incompressible Stars – Schwarzschild's Interior Solution

The mass inside a radius r for an incompressible star is

$$m(r) = \frac{4}{3}\pi\rho r^3\ , \tag{10.39}$$

$$e^{-2\beta} = 1 - \frac{2Gm(r)}{r} \equiv 1 - \frac{r^2}{a^2}\ , \tag{10.40}$$

where

$$a^2 = \frac{3}{8\pi G\rho},\ m(r) = \frac{r^3}{2Ga^2},\ r_s = 2Gm = \frac{r^2}{a^2}r\ . \tag{10.41}$$

TOV equation:

$$\begin{aligned}\frac{dp}{dr} &= -G\frac{\frac{4}{3}\pi\rho r^3 + 4\pi r^3 p(r)}{r(r-2G\frac{4}{3}\pi\rho r^3)}(\rho+p(r)) \\ &= -G\frac{4}{3}\pi\frac{\rho+3p(r)}{1-G\frac{8}{3}\pi\rho r^2}r(\rho+p(r)) \\ &= -\frac{1}{2a^2\rho}\frac{\rho+3p(r)}{1-\frac{r^2}{a^2}}r(\rho+p(r)) \\ &\Rightarrow \int_0^p \frac{dp}{(\rho+3p)(\rho+p)} = -\frac{1}{2a^2\rho}\int_R^r \frac{r}{1-\frac{r^2}{a^2}}dr, \\ \frac{p+\rho}{3p+\rho} &= \sqrt{\frac{a^2-R^2}{a^2-r^2}}.\end{aligned} \tag{10.42}$$

So the relativistic pressure distribution is

$$p(r) = \frac{\sqrt{a^2-r^2}-\sqrt{a^2-R^2}}{3\sqrt{a^2-R^2}-\sqrt{a^2-r^2}}\rho,\ \forall r \le R\ , \tag{10.43}$$

also

$$a^2 = \frac{3}{8\pi G\rho}, \quad \frac{a^2}{r^2} = \frac{r}{r_s} > 1 \Rightarrow a > r\,. \qquad (10.44)$$

To satisfy the condition for hydrostatic equilibrium we must have $p > 0$ or $p(0) > 0$ which gives

$$p(0) \equiv p_c = \frac{a - \sqrt{a^2 - R^2}}{3\sqrt{a^2 - R^2} - a} > 0 \qquad (10.45)$$

in which the numerator is positive so that

$$\begin{aligned} 3\sqrt{a^2 - R^2} &> a, \\ 9a^2 - 9R^2 &> a^2, \\ R &< \sqrt{\frac{8}{9}}a, \\ R^2 &< \frac{8}{9}a = \frac{8}{9}\frac{3}{8\pi G\rho} = \frac{1}{3\pi G\rho}\,. \end{aligned} \qquad (10.46)$$

Stellar mass:

$$\begin{aligned} M = \frac{4}{3}\pi\rho R^3 &< \frac{4}{3}\pi\rho R\frac{1}{3\pi G\rho} = \frac{4R}{9G}, \\ M &< \frac{4}{9G}\frac{1}{\sqrt{3\pi G\rho}}\,. \end{aligned} \qquad (10.47)$$

For a neutron star we can use $\rho \approx 10^{17}\,\mathrm{g/cm^3}$. An upper limit on the mass is then $M < 2.5\,M_\odot$.

Substitution for p in the expression for e^α gives

$$e^\alpha = \frac{3}{2}\sqrt{1 - \frac{R_s}{R}} - \frac{1}{2}\sqrt{1 - \frac{R_s}{R^3}r^2}\,. \qquad (10.48)$$

The line element for the interior Schwarzschild solution is

$$ds^2 = -\left(\frac{3}{2}\sqrt{1 - \frac{R_s}{R}} - \frac{1}{2}\sqrt{1 - \frac{R_s}{R^3}r^2}\right)^2 dt^2 + \frac{dr^2}{1 - \frac{R_s}{R^3}r^2} + r^2 d\Omega, \; r \leq R\,. \qquad (10.49)$$

Problems

10.1. *Curvature of 3-space and 2-surfaces of the internal and the external Schwarzschild spacetimes*

(a) The 3-space of the internal Schwarzschild solution has a geometry given by the line element

$$d\ell_I^2 = \frac{dr^2}{1 - \frac{R_S}{R^2}r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where $R_S = 2M$ is the Schwarzschild radius of the mass distribution and R its radius. The corresponding line element for the external Schwarzschild solution is

$$d\ell_E^2 = \frac{dr^2}{1 - \frac{R_S}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Find the spatial curvature $k = k(r) = \frac{1}{6}R$ of the 3-spaces, where R is the Ricci scalar.

(b) We shall now consider the equatorial surfaces $\theta = \pi/2$. The line elements of these surfaces are for the internal solution

$$d\sigma_I^2 = \frac{dr^2}{1 - \frac{R_S}{R^2}r^2} + r^2 d\phi^2$$

and for the external solution

$$d\sigma_E^2 = \frac{dr^2}{1 - \frac{R_S}{r}} + r^2 d\phi^2.$$

For these line elements the Gaussian curvatures of the surfaces they describe are given by

$$K = -\frac{1}{2g}g'_{\phi\phi} + \frac{g_{\phi\phi}}{4g^2}g'_{rr}g'_{\phi\phi} + \frac{g_{rr}}{4g^2}\left(g'_{\phi\phi}\right)^2,$$

where $g = g_{rr}g_{\phi\phi}$ and differentiation is with respect to r. Show that the Gaussian curvature of the equatorial surfaces is for
The internal solution: $K = R_S/R^3$. What sort of surface is this?
The external solution: $K = -(1/2)(R_S/r^3)$.

(c) The equatorial surfaces shall now be compared to the embedding surfaces. The Gaussian curvature of a surface of revolution given by $z = z(r)$ is

$$K = \frac{z'z''}{r(1 + z'^2)^2}.$$

Calculate the Gaussian curvatures of the embedding surfaces of the internal Schwarzschild solution, as given in Problem 8.10, and of the external solution. Compare the results with those of the previous point.

Chapter 11
Cosmology

11.1 Comoving Coordinate System

We will consider expanding homogeneous and isotropic models of the universe. We introduce an expanding frame of reference with the galactic clusters as reference particles. Then we introduce a "comoving coordinate system" in this frame of reference with spatial coordinates χ, θ, ϕ. We use time measured on standard clocks carried by the galactic clusters as coordinate time (cosmic time). The line element can then be written in the form

$$ds^2 = -dt^2 + a(t)^2[d\chi^2 + r(\chi)^2 d\Omega^2] \ . \tag{11.1}$$

(For standard clocks at rest in the expanding system, $d\chi = d\Omega = 0$ and $ds^2 = -d\tau^2 = -dt^2$.) The function $a(t)$ is called the expansion factor, and t is called cosmic time.

The physical distance to a galaxy with coordinate distance $d\chi$ from an observer at the origin is

$$dl_x = \sqrt{g_{\chi\chi}} d\chi = a(t) d\chi \ . \tag{11.2}$$

Even if the galactic clusters have no coordinate velocity, they do have a radial velocity expressed by the expansion factor.

The value of χ determines which cluster we are observing and $a(t)$ how it is moving. 4-velocity of a reference particle (galactic cluster) is

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} = (1,0,0,0) \ . \tag{11.3}$$

This applies at an arbitrary time, that is $\frac{du^\mu}{dt} = 0$. Geodesic equation $\frac{du^\mu}{dt} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0$ which reduces to $\Gamma^\mu_{tt} = 0$:

$$\Gamma^\mu_{tt} = \frac{1}{2} g^{\mu\nu} (\overbrace{g_{\nu t,t}}^{0} + \overbrace{g_{t\nu,t}}^{0} + \overbrace{g_{tt,\nu}}^{0}) = 0 \ . \tag{11.4}$$

This shows that the reference particles are freely falling.

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11.2 Curvature Isotropy – The Robertson–Walker Metric

Introduce orthonormal form basis:

$$\underline{\omega}^{\hat{t}} = \underline{d}t, \quad \underline{\omega}^{\hat{\chi}} = a(t)\underline{d}\chi, \quad \underline{\omega}^{\hat{\theta}} = a(t)r(\chi)\underline{d}\theta,$$
$$\underline{\omega}^{\hat{\phi}} = a(t)r(\chi)\sin\theta\underline{d}\phi. \tag{11.5}$$

Use Cartan's 1st equation,

$$\underline{d\omega}^{\hat{\mu}} = -\underline{\Omega}^{\hat{\mu}}_{\ \hat{\nu}} \wedge \underline{\omega}^{\hat{\nu}} \tag{11.6}$$

to find the connection forms. Then use Cartan's 2nd structure equation to calculate the curvature forms,

$$\underline{R}^{\hat{\mu}}_{\ \hat{\nu}} = \underline{d\Omega}^{\hat{\mu}}_{\ \hat{\nu}} + \underline{\Omega}^{\hat{\mu}}_{\ \hat{\lambda}} \wedge \underline{\Omega}^{\hat{\lambda}}_{\ \hat{\nu}}\ . \tag{11.7}$$

Calculations give (notation: $\dot{} = \frac{d}{dt}$, $' = \frac{d}{d\chi}$)

$$\begin{aligned}
\underline{R}^{\hat{t}}_{\ \hat{i}} &= \frac{\ddot{a}}{a}\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{i}}, \quad \underline{\omega}^{\hat{i}} = \underline{\omega}^{\hat{\chi}}, \underline{\omega}^{\theta}, \underline{\omega}^{\phi}, \\
\underline{R}^{\hat{\chi}}_{\ \hat{j}} &= \left(\frac{\dot{a}^2}{a^2} - \frac{r''}{ra^2}\right)\underline{\omega}^{\hat{\chi}} \wedge \underline{\omega}^{\hat{j}}, \quad \underline{\omega}^{\hat{j}} = \underline{\omega}^{\hat{\theta}}, \underline{\omega}^{\hat{\phi}}, \\
\underline{R}^{\hat{\theta}}_{\ \hat{\phi}} &= \left(\frac{\dot{a}^2}{a^2} + \frac{1}{r^2a^2} - \frac{r'^2}{r^2a^2}\right)\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\phi}}.
\end{aligned} \tag{11.8}$$

The curvature of 3-space ($dt = 0$) can be found by putting $a = 1$. That is

$$\begin{aligned}
{}_3\underline{R}^{\hat{\chi}}_{\ \hat{j}} &= -\frac{r''}{r}\underline{\omega}^{\hat{\chi}} \wedge \underline{\omega}^{\hat{j}}, \\
{}_3\underline{R}^{\hat{\theta}}_{\ \hat{\phi}} &= \left(\frac{1}{r^2} - \frac{r'^2}{r^2}\right)\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\phi}}\ .
\end{aligned} \tag{11.9}$$

The 3-space is assumed to be isotropic and homogeneous. This demands

$$-\frac{r''}{r} = \frac{1 - r'^2}{r^2} = k\ \ , \tag{11.10}$$

where k represents the constant curvature of the 3-space,

$$\therefore\ r'' + kr = 0 \quad \text{and} \quad r' = \sqrt{1 - kr^2}\ . \tag{11.11}$$

Solutions with $r(0) = 0$, $r'(0) = 1$:

$$\begin{aligned}
\sqrt{-k}r &= \sinh(\sqrt{-k}\chi) \quad (k < 0), \\
r &= \chi \quad (k = 0), \\
\sqrt{k}r &= \sin(\sqrt{k}\chi) \quad (k > 0).
\end{aligned} \tag{11.12}$$

The solutions can be characterized by the following three cases:

$$\begin{aligned} &r = \sinh\chi, \quad dr = \sqrt{1+r^2}d\chi \quad (k=-1), \\ &r = \chi, \quad dr = d\chi \quad (k=0), \\ &r = \sin\chi, \quad dr = \sqrt{1-r^2}d\chi \quad (k=1)\,. \end{aligned} \tag{11.13}$$

In all three cases one may write $dr = \sqrt{1-kr^2}d\chi$, which is just the last equation above.

We now set $d\chi^2 = \frac{dr^2}{1-kr^2}$ into the line element :

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\left(d\chi^2 + r^2(\chi)d\Omega^2\right) \\ &= -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)\,. \end{aligned} \tag{11.14}$$

The first expression is known as the standard form of the line element, the second is called the Robertson–Walker line element.

The 3-space has constant curvature. 3-Space is spherical for $k=1$, Euclidean for $k=0$ and hyperbolic for $k=-1$.

Universe models with $k=1$ are known as "closed" and models with $k=-1$ are known as "open". Models with $k=0$ are called "flat" even though also these models have curved spacetime.

11.3 Cosmic Dynamics

11.3.1 Hubble's Law

The observer is placed in origo of the coordinate system; $\chi_0 = 0$. The proper distance to a galaxy with radial coordinate χ_e is $D = a(t)\chi_e$. The galaxy has a radial velocity

$$v = \frac{dD}{dt} = \dot{a}\chi_e = \frac{\dot{a}}{a}D = HD, \quad \text{where} H = \frac{\dot{a}}{a}\,. \tag{11.15}$$

The expansion velocity v is proportional to the distance D. This is Hubble's law.

11.3.2 Cosmological Redshift of Light

Δt_e : the time-interval in transmitter position at transmission time
Δt_0 : the time-interval in receiver position at receiving time

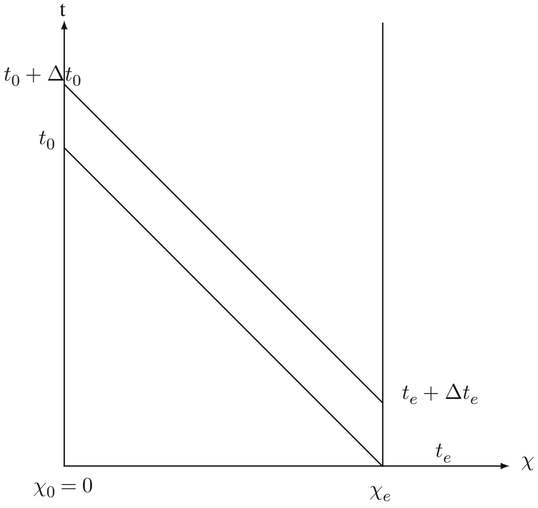

Fig. 11.1 Schematic representation of cosmological redshift

Light follows curves with $ds^2 = 0$, with $d\theta = d\phi = 0$, and we have (Fig. 11.1)

$$dt = -a(t)d\chi \; . \tag{11.16}$$

Integration from emitter event to receiver event:

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = -\int_{\chi_e}^{\chi_0} d\chi = \chi_e,$$
$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} = -\int_{\chi_e}^{\chi_0} d\chi = \chi_e \; ,$$

which give

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a} - \int_{t_e}^{t_0} \frac{dt}{a} = 0 \tag{11.17}$$

or

$$\int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a} - \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a} = 0. \tag{11.18}$$

Under the integration from t_e to $t_e + \Delta t_e$ the expansion factor $a(t)$ can be considered a constant with value $a(t_e)$ and under the integration from t_0 to $t_0 + \Delta t_0$ with value $a(t_0)$, giving

$$\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_0}{a(t_0)} \; . \tag{11.19}$$

Δt_0 and Δt_e are intervals of the light at the receiving and emitting time. Since the wavelength of the light is $\lambda = c\Delta t$ we have

$$\frac{\lambda_0}{a(t_0)} = \frac{\lambda_e}{a(t_e)} \; . \tag{11.20}$$

This can be interpreted as a "stretching" of the electromagnetic waves due to the expansion of space [1]. The cosmological redshift is denoted by z and is given by

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{a(t_0)}{a(t_e)} - 1 \; . \tag{11.21}$$

Using $a_0 \equiv a(t_0)$ we can write this as

$$1 + z(t) = \frac{a_0}{a} \; . \tag{11.22}$$

11.3.3 Cosmic Fluids

The energy–momentum tensor for a perfect fluid (no viscosity and no thermal conductivity) is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \; . \tag{11.23}$$

In an orthonormal basis

$$T_{\hat{\mu}\hat{\nu}} = (\rho + p)u_{\hat{\mu}} u_{\hat{\nu}} + p\eta_{\hat{\mu}\hat{\nu}} \; , \tag{11.24}$$

where $\eta_{\hat{\mu}\hat{\nu}}$ is the Minkowski metric. We consider three types of cosmic fluid:

1. Dust: $p = 0$,
$$T_{\hat{\mu}\hat{\nu}} = \rho u_{\hat{\mu}} u_{\hat{\nu}} \; . \tag{11.25}$$

2. Radiation: $p = \frac{1}{3}\rho$,
$$\begin{aligned} T_{\hat{\mu}\hat{\nu}} &= \frac{4}{3}\rho u_{\hat{\mu}} u_{\hat{\nu}} + p\eta_{\hat{\mu}\hat{\nu}} \\ &= \frac{\rho}{3}(4u_{\hat{\mu}} u_{\hat{\nu}} + \eta_{\hat{\mu}\hat{\nu}}) \; . \end{aligned} \tag{11.26}$$

   The trace
$$T = T^{\hat{\mu}}_{\hat{\mu}} = \frac{\rho}{3}(4u^{\hat{\mu}} u_{\hat{\mu}} + \delta^{\mu}_{\mu}) = 0 \; . \tag{11.27}$$

3. Vacuum: $p = -\rho$,
$$T_{\hat{\mu}\hat{\nu}} = -\rho\eta_{\hat{\mu}\hat{\nu}} \; . \tag{11.28}$$

If vacuum can be described as a perfect fluid we have $p_v = -\rho_v$, where ρ is the energy density. It can be related to Einstein's cosmological constant $\Lambda = 8\pi G\rho_v$.

One has also introduced a more general type of vacuum energy given by the equation of state $p_\phi = w\rho_\phi$, where ϕ denotes that the vacuum energy is connected to a scalar field ϕ. In a homogeneous universe the pressure and the density are given by

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \; , \tag{11.29}$$

where $V(\phi)$ is the potential for the scalar field. Then we have

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \,. \tag{11.30}$$

The special case $\dot{\phi} = 0$ gives the Lorentz-invariant vacuum with $w = -1$. The more general vacuum is called "quintessence".

11.3.4 Isotropic and Homogeneous Universe Models

We will discuss isotropic and homogenous universe models with perfect fluid and a non-vanishing cosmological constant Λ. Calculating the components of the Einstein tensor from the line-element (11.14) we find in an orthonormal basis

$$E_{\hat{t}\hat{t}} = \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2}, \tag{11.31}$$

$$E_{\hat{m}\hat{m}} = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} \,. \tag{11.32}$$

The components of the energy–momentum tensor of a perfect fluid in a comoving orthonormal basis are

$$T_{\hat{t}\hat{t}} = \rho \,, \quad T_{\hat{m}\hat{m}} = p \,. \tag{11.33}$$

Hence the $\hat{t}\hat{t}$ component of Einstein's field equations is

$$3\frac{\dot{a}^2 + k}{a^2} = 8\pi G\rho + \Lambda \tag{11.34}$$

and $\hat{m}\hat{m}$ component is

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi G p - \Lambda, \tag{11.35}$$

where ρ is the energy density and p is the pressure. The equations with vanishing cosmological constant are called the Friedmann equations. Inserting Eq. (11.34) into Eq. (11.35) gives

$$\ddot{a} = -\frac{4\pi G}{3} a(\rho + 3p) \,. \tag{11.36}$$

If we interpret ρ as the mass–density and use the speed of light c, we get

$$\ddot{a} = -\frac{4\pi G}{3} a(\rho + 3p/c^2) \,. \tag{11.37}$$

Inserting the gravitational mass–density ρ_G from Eq. (10.25) this equation takes the form

$$\ddot{a} = -\frac{4\pi G}{3} a \rho_G \; . \tag{11.38}$$

Inserting $p = w\rho c^2$ into Eq. (10.25) gives

$$\rho_G = (1 + 3w)\rho \; , \tag{11.39}$$

which is negative for $w < -1/3$, i.e. for $\dot{\phi}^2 < V(\phi)$. Special cases:

- dust: $w = 0$, $\rho_G = \rho$
- radiation: $w = \frac{1}{3}$, $\rho_G = 2\rho$
- Lorentz-invariant vacuum: $w = -1$, $\rho_G = -2\rho$

In a universe dominated by a Lorentz-invariant vacuum the acceleration of the cosmic expansion is

$$\ddot{a}_v = \frac{8\pi G}{3} a \rho_v > 0 \; , \tag{11.40}$$

that is *accelerated expansion*. This means that vacuum acts upon itself with repulsive gravitation.

The field equations can be combined into

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} - \frac{k}{a^2}, \tag{11.41}$$

where ρ_m is the density of matter, $\Lambda = 8\pi G \rho_\Lambda$ where ρ_Λ is the vacuum energy with constant density. $\rho = \rho_m + \rho_\Lambda$ is the total mass–density. Then we may write

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \; . \tag{11.42}$$

The critical density ρ_{cr} is the density in a universe with Euclidean space-like geometry, $k = 0$, which gives

$$\rho_{cr} = \frac{3H^2}{8\pi G} \; . \tag{11.43}$$

We introduce the relative densities

$$\Omega_m = \frac{\rho_m}{\rho_{cr}}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}} \; . \tag{11.44}$$

Furthermore we introduce a dimensionless parameter that describes the curvature of 3-space

$$\Omega_k = -\frac{k}{a^2 H^2}. \tag{11.45}$$

Equation (11.42) can now be written as

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1 \; . \tag{11.46}$$

From the Bianchi identity and Einstein's field equations, it follows that the energy–momentum density tensor is covariant divergence free. The time component expresses the equation of continuity and takes the form

$$[(\rho+p)u^{\hat{t}}u^{\hat{\nu}}]_{;\hat{\nu}}+(p\eta^{\hat{t}\hat{\nu}})_{;\hat{\nu}}=0\,. \tag{11.47}$$

Since $u^{\hat{t}}=1$, $u^{\hat{m}}=0$ and $\eta^{\hat{t}\hat{t}}=-1$, $\eta^{\hat{t}\hat{m}}=0$, we get

$$(\rho+p)^{\cdot}+(\rho+p)u^{\hat{\nu}}{}_{;\hat{\nu}}-\dot{p}=0 \tag{11.48}$$

or

$$\dot{\rho}+(\rho+p)(u^{\hat{\nu}}{}_{,\hat{\nu}}+\Gamma^{\hat{\nu}}{}_{\hat{t}\hat{\nu}})=0\,. \tag{11.49}$$

Here $u^{\nu}{}_{,\nu}=0$ and $\Gamma^{\hat{t}}{}_{\hat{t}\hat{t}}=0$. Calculating $\Gamma^{\hat{m}}{}_{\hat{t}\hat{m}}$ for $\underline{d\omega}^{\hat{\mu}}=\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\underline{\omega}^{\hat{\alpha}}\wedge\underline{\omega}^{\hat{\beta}}$ we get

$$\Gamma^{\hat{m}}{}_{\hat{t}\hat{m}}=\Gamma^{\hat{r}}{}_{\hat{t}\hat{r}}+\Gamma^{\hat{\theta}}{}_{\hat{t}\hat{\theta}}+\Gamma^{\hat{\phi}}{}_{\hat{t}\hat{\phi}}=3\frac{\dot{a}}{a}\,. \tag{11.50}$$

Hence

$$\dot{\rho}+3(\rho+p)\frac{\dot{a}}{a}=0\,, \tag{11.51}$$

which may be written as

$$(\rho a^3)^{\cdot}+p(a^3)^{\cdot}=0\,. \tag{11.52}$$

Let $V=a^3$ be a comoving volume in the universe and $U=\rho V$ be the energy in the comoving volume. Then we may write

$$dU+pdV=0\,. \tag{11.53}$$

This is the first law of thermodynamics for an adiabatic expansion. It follows that the universe expands adiabatically. The adiabatic equation can be written as

$$\frac{\dot{\rho}}{\rho+p}=-3\frac{\dot{a}}{a}\,. \tag{11.54}$$

Assuming $p=w\rho$ we get

$$\begin{aligned}\frac{d\rho}{\rho}&=-3(1+w)\frac{da}{a},\\ \ln\frac{\rho}{\rho_0}&=\ln\left(\frac{a}{a_0}\right)^{-3(1+w)}.\end{aligned}$$

It follows that

$$\rho=\rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)}. \tag{11.55}$$

This equation tells how the density of different types of matter depends on the expansion factor

$$\rho a^{3(1+w)}=\text{constant}\,. \tag{11.56}$$

Special cases:

- Dust: $w = 0$ gives $\rho_d a^3 = \text{constant}$.
  Thus, the mass in a comoving volume is constant.
- Radiation: $w = \frac{1}{3}$ gives $\rho_r a^4 = \text{constant}$.
  Thus, the radiation energy density decreases faster than the case with dust when the universe is expanding. The energy in a comoving volume is decreasing because of the thermodynamic work on the surface. In a remote past, the density of radiation must have exceeded the density of dust:
- Lorentz-invariant vacuum: $w = -1$ gives $\rho_\Lambda = \text{constant}$.
  The vacuum energy in a comoving volume is increasing by $\propto a^3$.

11.4 Some Cosmological Models

11.4.1 Radiation-Dominated Model

The energy–momentum tensor for radiation is trace free. According to the Einstein's field equations the Einstein tensor must then be trace free:

$$\begin{aligned} a\ddot{a} + \dot{a}^2 + k &= 0, \\ (a\dot{a} + kt)^{\cdot} &= 0 \,. \end{aligned} \tag{11.57}$$

Integration gives

$$a\dot{a} + kt = B \,. \tag{11.58}$$

Another integration gives

$$\frac{1}{2}a^2 + \frac{1}{2}kt^2 = Bt + C \,. \tag{11.59}$$

The initial condition $a(0) = 0$ gives $C = 0$. Hence

$$a = \sqrt{2Bt - kt^2} \,. \tag{11.60}$$

For $k = 0$ we have

$$a = \sqrt{2Bt} \,, \quad \dot{a} = \sqrt{\frac{B}{2t}} \,. \tag{11.61}$$

The expansion velocity reaches infinity at $t = 0$ ($\lim_{t\to 0} \dot{a} = \infty$) (Fig. 11.2):

$$\begin{aligned} \rho_R a^4 = K \,, \quad a = \sqrt{2Bt}, \\ 4\rho_R B^2 t^2 = K \,. \end{aligned} \tag{11.62}$$

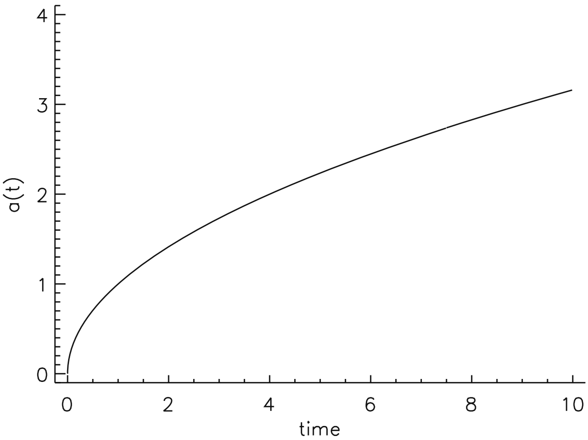

Fig. 11.2 In a radiation-dominated universe the expansion velocity reaches infinity at $t = 0$

According to the Stefan–Boltzmann law we then have

$$\rho_R = \sigma T^4 \to 4B^2 \sigma T^4 t^2 = K \Rightarrow$$
$$t = \frac{K_1}{T^2} \Leftrightarrow T = \sqrt{\frac{K_1}{t}}, \tag{11.63}$$

where T is the temperature of the background radiation.

11.4.2 Dust-Dominated Model

From the first of the Friedmann equations we have

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 . \tag{11.64}$$

We now introduce a time parameter η given by

$$\frac{dt}{d\eta} = a(\eta) \Rightarrow \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta}.$$
$$\text{So,} \quad \dot{a} = \frac{da}{dt} = \frac{1}{a}\frac{da}{d\eta} . \tag{11.65}$$

We also introduce $A \equiv \frac{8\pi G}{3}\rho_0 {a_0}^3$. The first Friedmann equation then gives

$$a\dot{a}^2 + ka = \frac{8\pi G}{3}\rho a^3 = \frac{8\pi G}{3}\rho_0 {a_0}^3 = A . \tag{11.66}$$

Using η we get

$$\begin{aligned}
\frac{1}{a}\left(\frac{da}{d\eta}\right)^2 &= A - ka, \\
\frac{1}{a^2}\left(\frac{da}{d\eta}\right)^2 &= \frac{A}{a} - k, \\
\frac{1}{a}\frac{da}{d\eta} &= \sqrt{\frac{A}{a} - k} = \sqrt{\frac{A}{a}}\sqrt{1 - \frac{a}{A}k},
\end{aligned} \tag{11.67}$$

where we choose the positive root. We now introduce u, given by $a = Au^2$, $u = \sqrt{\frac{a}{A}}$. We then get

$$\frac{da}{d\eta} = 2Au\frac{du}{d\eta}\ , \tag{11.68}$$

which together with the equation above gives

$$\begin{gathered}
\frac{1}{Au^2}2Au\frac{du}{d\eta} = \frac{1}{u}\sqrt{1 - ku^2} \\
\Downarrow \\
\frac{du}{\sqrt{1 - ku^2}} = \frac{1}{2}d\eta\ .
\end{gathered} \tag{11.69}$$

This equation will first be integrated for $k < 0$. Then $k = -|k|$, so that

$$\int \frac{du}{\sqrt{1 + |k|u^2}} = \frac{\eta}{2} + K \tag{11.70}$$

or $\operatorname{arcsinh}(\sqrt{-k}u) = \frac{\eta}{2} + K$. The condition $u(0) = 0$ gives $K = 0$. Hence

$$-\frac{k}{A}a = \sinh^2\frac{\eta}{2} = \frac{1}{2}(\cosh\eta - 1) \tag{11.71}$$

or

$$a = -\frac{A}{2k}(\cosh\eta - 1)\ . \tag{11.72}$$

From Eqs. (11.43), (11.44) and (11.66) we have

$$A = \frac{8\pi G}{3}\rho_{m0} = H_0^2\frac{\rho_{m0}}{\rho_{cr0}} = H_0^2\Omega_{m0}\ . \tag{11.73}$$

From Eqs. (11.45) and (11.46) we get

$$k = H_0^2(\Omega_{m0} - 1). \tag{11.74}$$

Hence, the scale factor of the negatively curved, dust-dominated universe model is

$$a(\eta) = \frac{1}{2}\frac{\Omega_{m0}}{1 - \Omega_{m0}}(\cosh\eta - 1)\ . \tag{11.75}$$

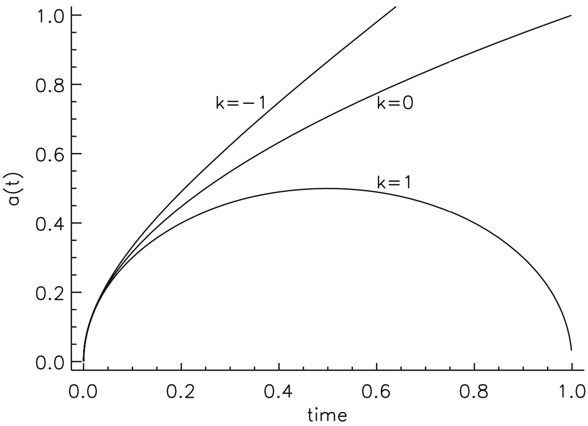

Fig. 11.3 For $k = 1$ the density is larger than the critical density, and the universe is closed. For $k = 0$ we have $\rho = \rho_{cr}$ and the expansion velocity of the universe will approach zero as $t \to \infty$. For $k = -1$ we have $\rho < \rho_{cr}$. The universe is then open and will continue expanding forever

Inserting this into Eq. (11.65) and integrating with $t(0) = \eta(0)$ leads to

$$t(\eta) = \frac{\Omega_{m0}}{2H_0(1-\Omega_{m0})^{3/2}}(\sinh\eta - \eta)\ . \tag{11.76}$$

Integrating Eq. (11.69) for $k = 0$ leads to an Einstein–de Sitter universe

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}. \tag{11.77}$$

Finally integrating Eq. (11.69) for $k > 0$ gives, in a similar way as for $k < 0$,

$$a(\eta) = \frac{1}{2}\frac{\Omega_{m0}}{\Omega_{m0}-1}(1-\cos\eta), \tag{11.78}$$

$$t(\eta) = \frac{\Omega_{m0}}{2H_0(\Omega_{m0}-1)^{3/2}}(\eta - \sin\eta). \tag{11.79}$$

We see that this is a parametric representation of a cycloid.

In the Einstein–de Sitter model the Hubble factor is

$$\boxed{H = \frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t}\ , \quad t = \frac{2}{3}\frac{1}{H} = \frac{2}{3}t_H.} \tag{11.80}$$

The critical density in the Einstein–de Sitter model is given by the first Friedmann equation

$$\begin{gathered}H^2 = \frac{8\pi G}{3}\rho_{\text{cr}}\,, \quad k = 0 \\ \Downarrow \\ \rho_{\text{cr}} = \frac{3H^2}{8\pi G}\,, \quad \Omega = \frac{\rho}{\rho_{\text{cr}}}\,.\end{gathered} \tag{11.81}$$

Example 11.4.1 (Age–redshift relation for dust-dominated universe with $k = 0$)

$$\begin{aligned}1 + z = \frac{a_0}{a} \Rightarrow a = \frac{a_0}{1+z}, \\ da = -\frac{a_0}{(1+z)^2}dz = -\frac{a}{1+z}dz.\end{aligned} \tag{11.82}$$

Equation (11.34) gives (Fig. 11.3)

$$\begin{aligned}\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho &= \frac{8\pi G}{3}\frac{\rho_0 a_0^3}{a^3} \\ &= \frac{8\pi G}{3}\rho_0(1+z)^3.\end{aligned} \tag{11.83}$$

Using $H_0^2 = \frac{8\pi G}{3}\rho_0$ gives $\frac{\dot a}{a} = H_0(1+z)^{\frac{3}{2}}$. From $\dot a = \frac{da}{dt}$ we get

$$dt = \frac{da}{\dot a} = \frac{da}{a\frac{\dot a}{a}} = -\frac{dz}{H_0(1+z)^{\frac{5}{2}}}\,. \tag{11.84}$$

Integration gives the age of the universe:

$$t_0 = -\frac{1}{H_0}\int_\infty^0 \frac{dz}{(1+z)^{\frac{5}{2}}} = \frac{2}{3}\frac{1}{H_0}\left[\frac{1}{(1+z)^{\frac{3}{2}}}\right]_\infty^0 = \frac{2}{3}t_H\,, \tag{11.85}$$

where the Hubble time $t_H \equiv \frac{1}{H_0}$ is the age of the universe if the expansion rate had been constant (Fig. 11.4). The "lookback time" to a source with redshift z is

$$\Delta t = t_H \int_0^z \frac{dz}{(1+z)^{\frac{5}{2}}} = \frac{2}{3}t_H\left[1 - \frac{1}{(1+z)^{\frac{3}{2}}}\right]\,, \tag{11.86}$$

$$\Delta t = t_0\left[1 - \frac{1}{(1+z)^{3/2}}\right]\,. \tag{11.87}$$

Hence, the redshift of an object with lookback time Δt is

$$z = \frac{1}{(1-\frac{\Delta t}{t_0})^{2/3}} - 1\,. \tag{11.88}$$

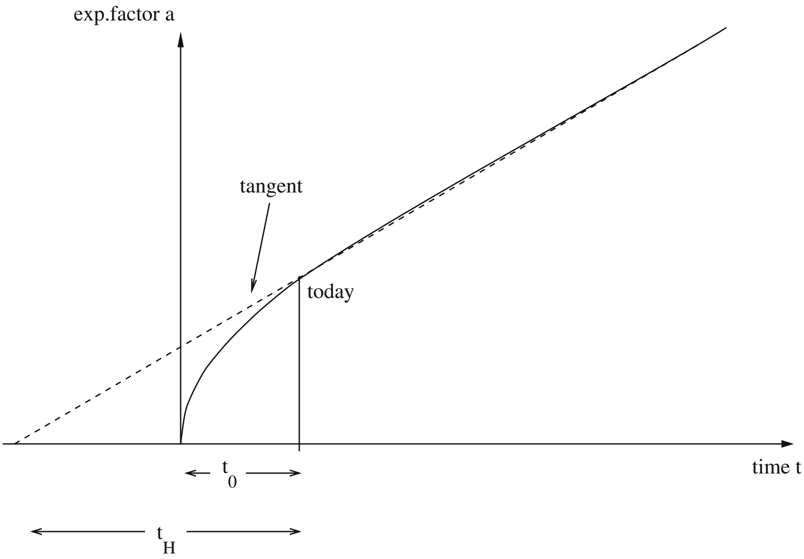

Fig. 11.4 t_H is the age of the universe if the expansion had been constant. BUT the exp.rate was faster, closer to the Big Bang, so the age is lower

11.4.3 Transition from Radiation – To Matter-Dominated Universe

We consider the early universe filled with radiation and matter, but where vacuum energy can be neglected. The universe is assumed to be flat. Then Friedmann's 1st equation takes the form

$$\dot{a}^2 = \frac{8\pi G}{3}(\rho_M + \rho_R)a^2 . \tag{11.89}$$

For matter,

$$\rho_M a^3 = \rho_{M0} . \tag{11.90}$$

For radiation,

$$\rho_R a^4 = \rho_{R0} . \tag{11.91}$$

Hence

$$a^2\dot{a}^2 = \frac{8\pi G}{3}(\rho_{M0}\, a + \rho_{R0}) . \tag{11.92}$$

The present values of the critical density and the density parameters are

$$\rho_{cr0} = \frac{3H_0^2}{8\pi G}, \tag{11.93}$$

$$\Omega_{M0} = \frac{\rho_{M0}}{\rho_{cr0}}, \tag{11.94}$$

$$\Omega_{R0} = \frac{\rho_{R0}}{\rho_{cr0}}, \tag{11.95}$$

giving

$$a\dot{a} = H_0 \left(\Omega_{M0}\, a + \Omega_{R0}\right)^{1/2} \; . \tag{11.96}$$

Integration with $a(0) = 0$ leads to

$$H_0 t = \frac{4}{3}\frac{\Omega_{R0}^{3/2}}{\Omega_{M0}^2} + \frac{2}{3}\frac{\left(\Omega_{M0}\, a - 2\Omega_{R0}\right)\left(\Omega_{M0}\, a + \Omega_{R0}\right)^{1/2}}{\Omega_{M0}^2} \; . \tag{11.97}$$

From Eqs. (11.90) and (11.91) it follows that at the transition time t_{eq} when $\rho_M = \rho_R$, the scale factor has the value

$$a_{eq} = \frac{\rho_{R0}}{\rho_{M0}} = \frac{\Omega_{R0}}{\Omega_{M0}} \; . \tag{11.98}$$

Inserting this into Eq. (11.97) gives

$$t_{eq} = \frac{2}{3}\left(2 - \sqrt{2}\right)\frac{\Omega_{R0}^{3/2}}{\Omega_{M0}^2} t_H \; . \tag{11.99}$$

The microwave background radiation has a temperature 2.73 K corresponding to a density parameter $\Omega_{R0} = 8.4 \cdot 10^{-5}$. In a flat universe without vacuum energy $\Omega_{M0} = 1 - \Omega_{R0}$. From the value of H_0 as determined by measurements we have $t_H \approx 14 \cdot 10^9$ years. This leads to $t_{eq} = 47 \cdot 10^3$ years.

11.4.4 Friedmann–Lemaître Model

The dynamics of galaxies and clusters of galaxies has made it clear that far stronger gravitational fields are needed to explain the observed motions than those produced by visible matter [2]. At the same time it has become clear that the density of this dark matter is only about 30% of the critical density, although it is a prediction by the usual versions of the inflationary universe models that the density ought to be equal to the critical density [3]. Also the recent observations of the temperature fluctuations of the cosmic microwave radiation have shown that space is either flat or very close to flat [4, 5, 6]. The energy that fills up to the critical density must be evenly distributed in order not to affect the dynamics of the galaxies and the clusters.

Furthermore, in 1998 observations of supernovae of type Ia with high cosmic redshifts indicated that the expansion of the universe is accelerating [7, 8]. This was explained as a result of repulsive gravitation due to some sort of vacuum energy. Thereby the missing energy needed to make space flat was identified as vacuum energy. Hence, it seems that we live in a flat universe with vacuum energy having a density around 70% of the critical density and with matter having a density around 30% of the critical density.

Until the discovery of the accelerated expansion of the universe the standard model of the universe was assumed to be the Einstein–de Sitter model, which is a flat universe model dominated by cold matter. This universe model is thoroughly presented in nearly every textbook on general relativity and cosmology. Now it seems that we must replace this model with a new "standard model" containing both dark matter and vacuum energy [9].

Recently several types of vacuum energy or so-called quintessence energy have been discussed [10, 11]. However, the most simple type of vacuum energy is the Lorentz-invariant vacuum energy (LIVE), which has constant energy density during the expansion of the universe [12, 13]. This type of energy can be mathematically represented by including a cosmological constant in Einstein's gravitational field equations. The flat universe model with cold dark matter and this type of vacuum energy is the Friedmann–Lemaître model.

The field equations for the flat Friedmann–Lemaître is found by putting $k = p = 0$ in Eq. (11.35). This gives

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \Lambda \,. \tag{11.100}$$

Integration leads to

$$a\dot{a}^2 = \frac{\Lambda}{3}a^3 + K, \tag{11.101}$$

where K is a constant of integration. Since the amount of matter in a volume comoving with the cosmic expansion is constant, $\rho_M a^3 = \rho_{M0} a_0^3$, where the index 0 refers to measured values at the present time. Normalizing the expansion factor so that $a_0 = 1$ and comparing Eqs. (11.42) and (11.101) then give $K = (8\pi G/3)\rho_{M0}$. Introducing a new variable x by $a^3 = x^2$ and integrating once more with the initial condition $a(0) = 0$ we obtain

$$a^3 = \frac{3K}{\Lambda}\sinh^2\left(\frac{t}{t_\Lambda}\right), \quad t_\Lambda = \frac{2}{\sqrt{3\Lambda}} \,. \tag{11.102}$$

The vacuum energy has a constant density ρ_Λ given by

$$\Lambda = 8\pi G \rho_\Lambda. \tag{11.103}$$

The critical density, which is the density making the 3-space of the universe flat, is

$$\rho_{cr} = \frac{3H^2}{8\pi G} \,. \tag{11.104}$$

The relative density, i.e. the density measured in units of the critical density, of the matter and the vacuum energy are, respectively,

$$\Omega_M = \frac{\rho}{\rho_{cr}} = \frac{8\pi G \rho_M}{3H^2}, \tag{11.105}$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}} = \frac{\Lambda}{3H^2} \,. \tag{11.106}$$

Since the present universe model has flat space, the total density is equal to the critical density, i.e. $\Omega_M + \Omega_\Lambda = 1$. Equation (11.101) with the normalization $a(t_0) = 1$, where t_0 is the present age of the universe, gives $3H_0^2 = 3K + \Lambda$. Equation (11.34) with $k = 0$ gives $8\pi G\rho_0 = 3H_0^2 - \Lambda$. Hence $K = 8\pi G\rho_0/3$ and $\frac{3K}{\Lambda} = \frac{8\pi G\rho_0}{\Lambda} = \frac{\rho_0}{\rho_\Lambda} = \frac{\Omega_{M0}}{\Omega_{\Lambda 0}}$. In terms of the values of the relative densities at the present time the expression for the expansion factor then takes the form

$$a = A^{1/3} \sinh^{2/3}\left(\frac{t}{t_\Lambda}\right), \quad A = \frac{\Omega_{M0}}{\Omega_{\Lambda 0}} = \frac{1 - \Omega_{\Lambda 0}}{\Omega_{\Lambda 0}} . \tag{11.107}$$

Using the identity $\sinh(x/2) = \sqrt{(\cosh x - 1)/2}$ this expression may be written as

$$a^3 = \frac{A}{2}\left[\cosh\left(\frac{2t}{t_\Lambda}\right) - 1\right] . \tag{11.108}$$

The age t_0 of the universe is found from $a(t_0) = 1$, which, by use of the formula $arc\tanh x = arc\sinh(x/\sqrt{1-x^2})$, leads to the expression

$$t_0 = t_\Lambda arc\tanh\sqrt{\Omega_{\Lambda 0}} . \tag{11.109}$$

Inserting typical values of $t_0 = 15 \cdot 10^9$years and $\Omega_{\Lambda 0} = 0.7$ we get $A = 0.43$ and $t_\Lambda = 12 \cdot 10^9$years. With these values the expansion factor is $a = 0.75\sinh^{2/3}(1.2t/t_0)$. This function is plotted in Fig. 11.5. The Hubble parameter as a function of time is

$$H = (2/3t_\Lambda)\coth(t/t_\Lambda). \tag{11.110}$$

Inserting $t_0 = 1.2t_\Lambda$ we get $Ht_0 = 0.8\coth(1.2t/t_0)$, which is plotted in Fig. 11.6. The Hubble parameter decreases all the time and approaches a constant value $H_\infty = 2/3t_\Lambda$ in the infinite future. The present value of the Hubble parameter is

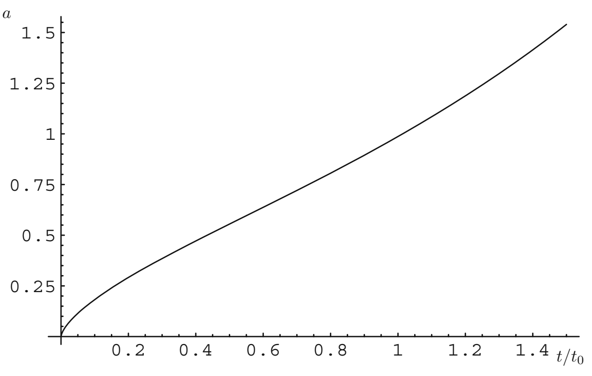

Fig. 11.5 The expansion factor as function of cosmic time in units of the age of the universe

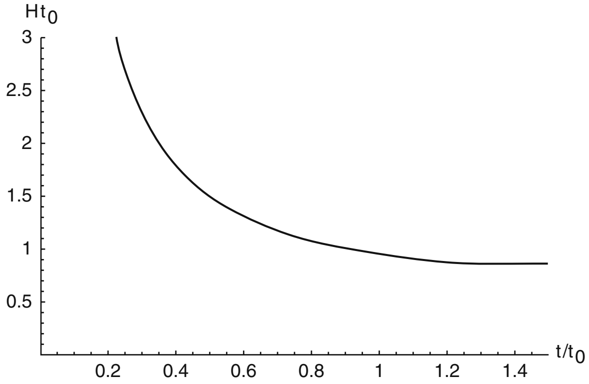

Fig. 11.6 The Hubble parameter as function of cosmic time

$$H_0 = \frac{2}{3t_\Lambda\sqrt{\Omega_{\Lambda 0}}} . \tag{11.111}$$

The corresponding Hubble age is $t_{H0} = (3/2)t_\Lambda\sqrt{\Omega_{\Lambda 0}}$. Inserting our numerical values gives $H_0 = 64\,\mathrm{km/s/Mpc}$ and $t_{H0} = 15.7 \cdot 10^9$years. In this universe model the age of the universe is nearly as large as the Hubble age, while in the Einstein–de Sitter model the corresponding age is $t_{0ED} = (2/3)t_{H0} = 10.5 \cdot 10^9$years. The reason for this difference is that in the Einstein–de Sitter model the expansion is decelerated all the time, while in the Friedmann–Lemaître model the repulsive gravitation due to the vacuum energy has made the expansion accelerate lately (see below). Hence, for a given value of the Hubble parameter the previous velocity was larger in the Einstein–de Sitter model than in the Friedmann–Lemaître model.

The ratio of the age of the universe and its Hubble age depends on the present relative density of the vacuum energy as follows:

$$\frac{t_0}{t_{H0}} = H_0 t_0 = \frac{2}{3}\frac{arc\tanh\sqrt{\Omega_{\Lambda 0}}}{\sqrt{\Omega_{\Lambda 0}}} . \tag{11.112}$$

This function is depicted graphically in Fig. 11.7. The age of the universe increases with increasing density of vacuum energy. In the limit that the density of the vacuum approaches the critical density, there is no dark matter, and the universe model approaches the de Sitter model with exponential expansion and no Big Bang. This model behaves in the same way as the steady-state cosmological model and is infinitely old.

A dimensionless quantity representing the rate of change of the cosmic expansion velocity is the deceleration parameter, which is defined as $q = -\ddot{a}/aH^2$. For the present universe model the deceleration parameter as a function of time is

$$q = \frac{1}{2}[1 - 3\tanh^2(t/t_\Lambda)], \tag{11.113}$$

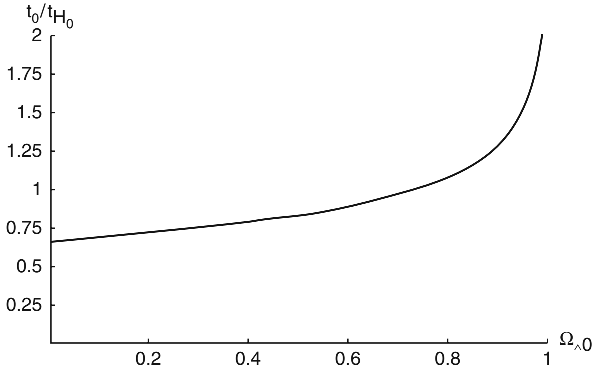

Fig. 11.7 The ratio of the age of the universe and the Hubble age as function of the present relative density of the vacuum energy

which is shown graphically in Fig. 11.8. The inflection point of time t_1 when deceleration turned into acceleration is given by $q = 0$. This leads to

$$t_1 = t_\Lambda arc\tanh(1/\sqrt{3}) \tag{11.114}$$

or is expressed in terms of the age of the universe

$$t_1 = \frac{arc\tanh(1/\sqrt{3})}{arc\tanh\sqrt{\Omega_{\Lambda 0}}} t_0. \tag{11.115}$$

The corresponding cosmic redshift is

$$z(t_1) = \frac{a_0}{a(t_1)} - 1 = \left(\frac{2\Omega_{\Lambda 0}}{1 - \Omega_{\Lambda 0}}\right)^{1/3} - 1. \tag{11.116}$$

Inserting $\Omega_{\Lambda 0} = 0.7$ gives $t_1 = 0.54 t_0$ and $z(t_1) = 0.67$.

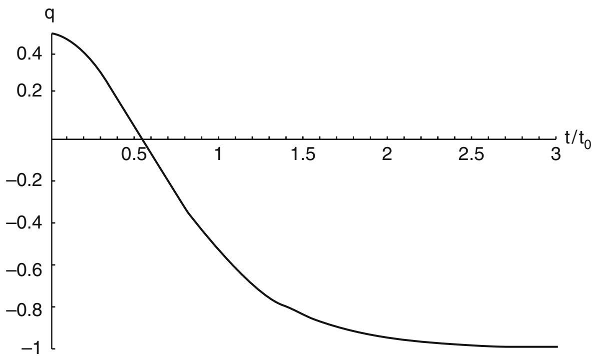

Fig. 11.8 The deceleration parameter as function of cosmic time

The results of analysing the observations of supernova SN 1997 at $z = 1.7$, corresponding to an emission time $t_e = 0.30t_0 = 4.5 \cdot 10^9$ years, have provided evidence that the universe was decelerated at that time [14]. M. Turner and A.G. Riess [15] have recently argued that the other supernova data favour a transition from deceleration to acceleration for a redshift around $z = 0.5$.

Note that the expansion velocity given by Hubble's law, $v = Hd$, always decreases as seen from Fig. 11.6. This is the velocity away from the Earth of the cosmic fluid at a fixed physical distance d from the Earth. The quantity $\dot{a}$, on the other hand, is the velocity of a fixed fluid particle comoving with the expansion of the universe. If such a particle accelerates, the expansion of the universe is said to accelerate. While $\dot{H}$ tells how fast the expansion velocity changes at a fixed distance from the Earth, the quantity $\ddot{a}$ represents the acceleration of a free particle comoving with the expanding universe. The connection between these two quantities is $\ddot{a} = a(\dot{H} + H^2)$.

The ratio of the inflection point of time and the age of the universe, as given in Eq. (11.115), is depicted graphically as a function of the present relative density of vacuum energy in Fig. 11.9. The turnover point of time happens earlier the greater the vacuum density is. The change from deceleration to acceleration would happen at the present time if $\Omega_{\Lambda 0} = 1/3$.

The redshift of the inflection point given in Eq. (11.116) as a function of vacuum energy density is plotted in Fig. 11.10. Note that the redshift of future points of time is negative, since then $a > a_0$. If $\Omega_{\Lambda 0} < 1/3$ the transition to acceleration will happen in the future.

The critical density is

$$\rho_{cr} = \rho_\Lambda \tanh^{-2}(t/t_\Lambda) \ . \qquad (11.117)$$

This is plotted in Fig. 11.11. The critical density decreases with time.

Equation (11.117) shows that the relative density of the vacuum energy is

$$\Omega_\Lambda = \tanh^2(t/t_\Lambda) \ , \qquad (11.118)$$

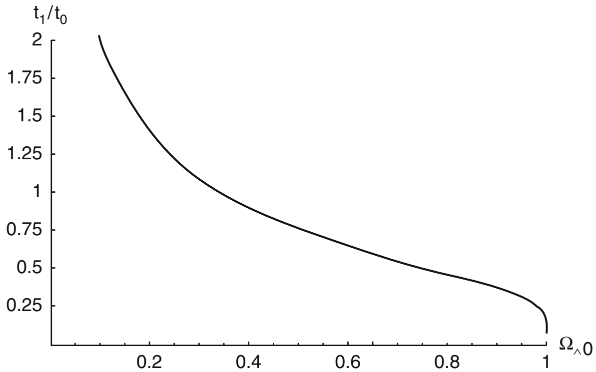

Fig. 11.9 The ratio of the point of time when cosmic decelerations turn over to acceleration to the age of the universe

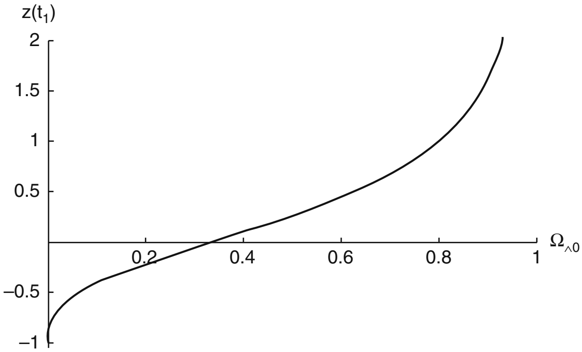

Fig. 11.10 The cosmic redshift of light emitted at the turnover time from deceleration to acceleration as function of the present relative density of vacuum energy

which is plotted in Fig. 11.12. The density of the vacuum energy approaches the critical density. Since the density of the vacuum energy is constant, this is better expressed by saying that the critical density approaches the density of the vacuum energy. Furthermore, since the total energy density is equal to the critical density all the time, this also means that the density of matter decreases faster than the critical density. The density of matter as function of time is

$$\rho_M = \rho_\Lambda \sinh^{-2}(t/t_\Lambda) \,, \tag{11.119}$$

which is shown graphically in Fig. 11.13. The relative density of matter as function of time is

$$\Omega_M = \cosh^{-2}(t/t_\Lambda), \tag{11.120}$$

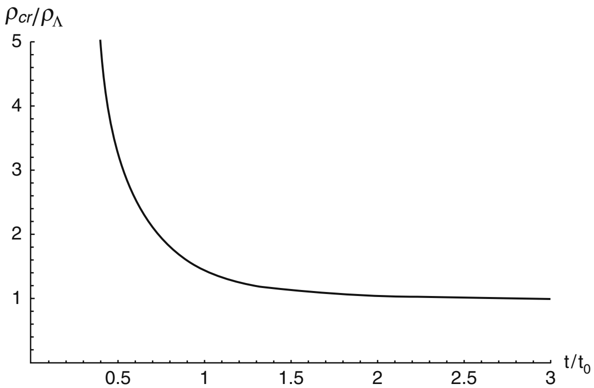

Fig. 11.11 The critical density in units of the constant density of the vacuum energy as function of time

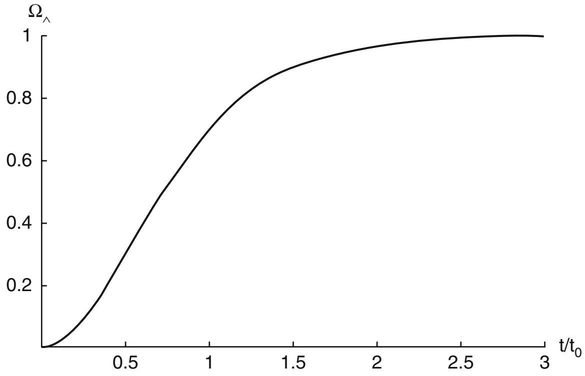

Fig. 11.12 The relative density of the vacuum energy density as function of time

which is depicted in Fig. 11.14. Adding the relative densities of Figs. 11.13 and 11.14 or the expressions (11.118) and (11.120) we get the total relative density $\Omega_{TOT} = \Omega_M + \Omega_\Lambda = 1$.

The universe became vacuum dominated at a point of time t_2 when $\rho_\Lambda(t_2) = \rho_M(t_2)$. From Eq. (11.119) it follows that this point of time is given by $\sinh(t_2/t_\Lambda) = 1$. According to Eq. (11.109) we get

$$t_2 = \frac{arc\sinh(1)}{arc\tanh(\sqrt{\Omega_{\Lambda 0}})} t_0. \tag{11.121}$$

From Eq. (11.107) it follows that the corresponding redshift is

$$z(t_2) = A^{-1/3} - 1\,. \tag{11.122}$$

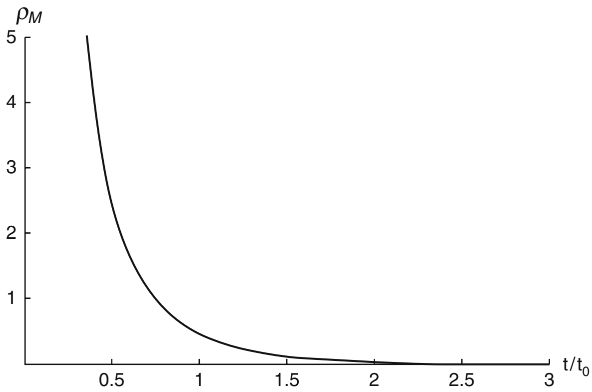

Fig. 11.13 The density of matter in units of the density of vacuum energy as function of time

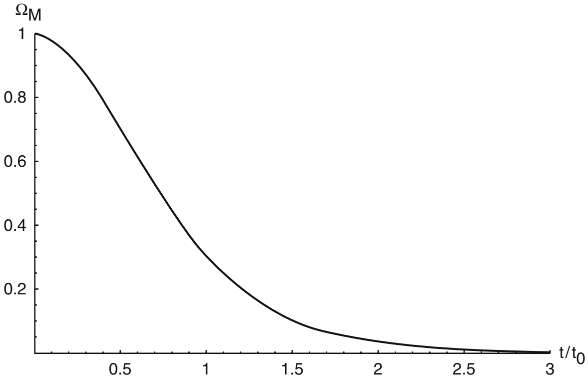

Fig. 11.14 The relative density of matter as function of time

Inserting $\Omega_{\Lambda 0} = 0.7$ gives $t_2 = 0.73t_0$ and $z(t_2) = 0.32$. The transition to accelerated expansion happens before the universe becomes vacuum dominated.

Note from Eqs. (11.113) and (11.118) that in the case of the flat Friedmann–Lemaître universe model, the deceleration parameter may be expressed in terms of the relative density of vacuum only, $q = (1/2)(1 - 3\Omega_\Lambda)$. The supernova Ia observations have shown that the expansion is now accelerating. Hence if the universe is flat, this alone means that $\Omega_{\Lambda 0} > 1/3$.

As mentioned above, many different observations indicate that we live in a universe with critical density, where cold matter contributes with about 30% of the density and vacuum energy with about 70%. Such a universe is well described by the Friedmann–Lemaître universe model that has been presented above.

However, this model is not quite without problems in explaining the observed properties of the universe. In particular there is now much research directed at solving the so-called *coincidence problem*. As we have seen, the density of the vacuum energy is constant during the expansion, while the density of the matter decreases inversely proportional to a volume comoving with the expanding matter. Yet, one observes that the density of matter and the density of the vacuum energy are of the same order of magnitude at the present time. This seems to be a strange and unexplained coincidence in the model. Also just at the present time the critical density is approaching the density of the vacuum energy. At earlier times the relative density was close to zero, and now it changes approaching the constant value 1 in the future. S.M. Carroll [16] has illustrated this aspect of the coincidence problem by plotting $\dot{\Omega}_\Lambda$ as a function of $\ln(t/t_0)$. Differentiating the expression (11.118) we get

$$\frac{t_\Lambda}{2}\frac{d\Omega_\Lambda}{dt} = \frac{\sinh(t/t_\Lambda)}{\cosh^3(t/t_\Lambda)}\,, \tag{11.123}$$

which is plotted in Fig. 11.15.

Putting $\ddot{\Omega}_\Lambda = 0$ we find that the rate of change of Ω_Λ was maximal at the point of time t_1 when the deceleration of the cosmic expansion turned into acceleration.

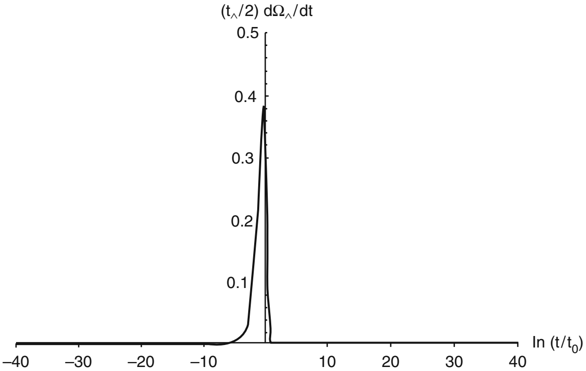

Fig. 11.15 Rate of change of Ω_Λ as function of $\ln(\frac{t}{t_0})$. The value $\ln(\frac{t}{t_0}) = -40$ corresponds to the cosmic point of time $t_0 \sim 1\,\mathrm{s}$

There is now a great activity in order to try to explain these coincidences by introducing more general forms of vacuum energy called quintessence and with a density determined dynamically by the evolution of a scalar field [17].

However, the simplest type of vacuum energy is the LIVE. One may hope that a future theory of quantum gravity may settle the matter and let us understand the vacuum energy. In the meantime we can learn much about the dynamics of a vacuum-dominated universe by studying simple and beautiful universe models such as the Friedmann–Lemaître model.

11.5 Inflationary Cosmology

11.5.1 Problems with the Big Bang Models

The Horizon Problem

The cosmic microwave background (CMB) radiation from two points A and B in opposite directions has the same temperature. This means that it has been radiated by sources of the same temperature in these points. Thus, the universe must have been in thermic equilibrium at the decoupling time, $t_d = 3 \cdot 10^5$years. This implies that points A and B, "at opposite sides of the universe", had been in causal contact already at that time. That is, a light signal must have had time to move from A to B during the time from $t = 0$ to $t = 3 \cdot 10^5$ years. The points A and B must have been within each other's horizons at the decoupling.

Consider a photon moving radially in space described by the Robertson–Walker metric (11.14) with $k = 0$. Light follows a null-geodesic curve, i.e. the curve is defined by $ds^2 = 0$. We get

$$dr = \frac{dt}{a(t)} \,. \tag{11.124}$$

The coordinate distance the photon has moved during the time t is

$$\Delta r = \int_0^t \frac{dt}{a(t)}. \tag{11.125}$$

The physical distance the light has moved at the time t is called the *horizon distance* and is

$$l_h = a(t)\Delta r = a(t) \int_0^t \frac{dt}{a(t)}. \tag{11.126}$$

To find a quantitative expression for the "horizon problem", we may consider a model with critical mass–density (Euclidean space-like geometry). Using $p = w\rho$ and $\Omega = 1$, integration of Eq. (11.36) gives

$$a \propto t^{\frac{2}{3+3w}}. \tag{11.127}$$

Inserting this into the expression for l_h and integrating gives

$$l_h = \frac{3w+3}{3w+1} t. \tag{11.128}$$

Let us call the volume inside the horizon the "horizon volume" and denote it by V_H. From Eq. (11.128) it follows that $V_H \propto t^3$. At the decoupling time, the horizon volume may therefore be written as

$$(V_H)_d = \left(\frac{t_d}{t_0}\right)^3 V_0, \tag{11.129}$$

where V_0 is the size of the present horizon volume. Events within this volume are causally connected, and a volume of this size may be in thermal equilibrium at the decoupling time.

Let $(V_0)_d$ be the size, at the decoupling, of the part of the universe that corresponds to the present horizon volume, i.e. the observable universe. For our Euclidean universe, Eq. (11.127) holds, giving

$$(V_0)_d = \frac{a^3(t_d)}{a^3(t_0)} V_0 = \left(\frac{t_d}{t_0}\right)^{\frac{2}{w+1}} V_0. \tag{11.130}$$

From Eqs. (11.129) and (11.130), we get

$$\frac{(V_0)_d}{(V_H)_d} = \left(\frac{t_d}{t_0}\right)^{-\frac{3w+1}{w+1}}. \tag{11.131}$$

Using that $t_d = 10^{-4} t_0$ and inserting $w = 0$ for dust, we find $\frac{V_0)_d}{(V_H)_d} = 10^4$. Thus, there was room for 10^4 causally connected areas at the decoupling time within what presently represents our observable universe. Points at opposite sides of our observable universe were therefore not causally connected at the decoupling, according to the Friedmann models of the universe. These models therefore cannot explain that the temperature of the radiation from such points is the same.

The Flatness Problem

According to Eq. (11.42), the total mass parameter $\Omega = \frac{\rho}{\rho_{cr}}$ is given by

$$\Omega - 1 = \frac{k}{\dot{a}^2}. \tag{11.132}$$

By using the expansion factor (11.127) for a universe near critical mass–density, we get

$$\frac{\Omega - 1}{\Omega_0 - 1} = \left(\frac{t}{t_0}\right)^{2\left(\frac{3w+1}{3w+3}\right)} . \tag{11.133}$$

For a radiation-dominated universe, we get

$$\frac{\Omega - 1}{\Omega_0 - 1} = \frac{t}{t_0} . \tag{11.134}$$

Measurements indicate that $\Omega_0 - 1$ is of order of magnitude 1. The age of the universe is about $t_0 = 10^{17}$s. When we stipulate initial conditions for the universe, it is natural to consider the Planck time, $t_P = 10^{-43}$ s, since this is the limit to the validity of general relativity. At earlier time, quantum effects will be important, and one cannot give a reliable description without using quantum gravitation. The stipulated initial condition for the mass parameter then becomes that $\Omega - 1$ is of order 10^{-60} at the Planck time. Such an extreme fine tuning of the initial value of the universe's mass–density cannot be explained within standard Big Bang cosmology.

Other Problems

The Friedmann models can neither explain questions about why the universe is nearly homogeneous and has an isotropic expansion nor say anything about why the universe is expanding.

11.5.2 Cosmic Inflation

Spontaneous Symmetry Breaking and the Higgs Mechanism

The particles responsible for the electroweak force, $W^{\pm}$ and Z^0, are massive (causing the weak force to only have short-distance effects). This was originally a problem for the quantum field theory describing this force, since it made it difficult to create a renormalizable theory.[1] This was solved by Higgs and Kibble in 1964 by introducing the so-called Higgs mechanism.

The main idea is that the massive bosons $W^{\pm}$ and Z^0 are given a mass by interacting with a *Higgs field* ϕ. The effect causes the mass of the particles to be proportional to the value of the Higgs field in vacuum. It is therefore necessary that the Higgs field has a value different from zero in the vacuum (the *vacuum expectation value* must be non-zero).

Let us see how the Higgs field can get a non-zero vacuum expectation value. The important thing for our purpose is that the potential for the Higgs field may be temperature dependent. Let us assume that the potential for the Higgs field is described by the function

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 , \tag{11.135}$$

where the sign of μ^2 depends on whether the temperature is above or below a critical temperature T_c. This sign has an important consequence on the shape of the potential V. The potential is shown in Fig. 11.16 for two different temperatures. For $T > T_c$, $\mu^2 > 0$, and the shape is like in Fig. 11.16(a), and there is a stable minimum for $\phi = 0$. However, for $T < T_c$, $\mu^2 < 0$, and the shape is like in Fig. 11.16(b). In this case the potential has stable minima for $\phi = \pm\phi_0 = \pm\frac{|\mu|}{\sqrt{\lambda}}$ and an unstable maximum at $\phi = 0$. For both cases, the potential $V(\phi)$ is invariant under the symmetry transformation $\phi \mapsto -\phi$ (i.e. $V(\phi) = V(-\phi)$).

The "real" vacuum state of the system is at a stable minimum of the potential. For $T > T_c$, the minimum is in the "symmetric" state $\phi = 0$. On the other hand, for $T < T_c$ this state is unstable. It is therefore called a "false vacuum". The system will

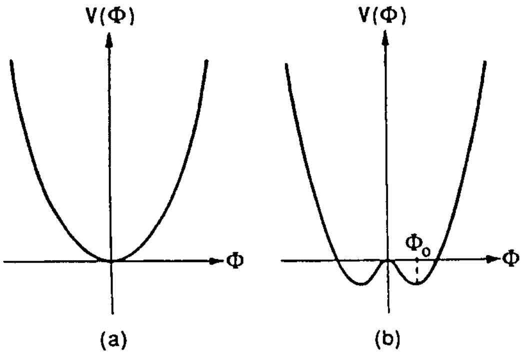

Fig. 11.16 The shape of the potential depends on the sign of μ^2. (**a**) Higher temperature than the critical, with $\mu^2 > 0$. (**b**) Lower temperature than the critical, with $\mu^2 < 0$

[1] The problem is that the Lagrangian for the gauge bosons cannot include terms like $m^2W_{\mu}^2$, which are not gauge invariant.

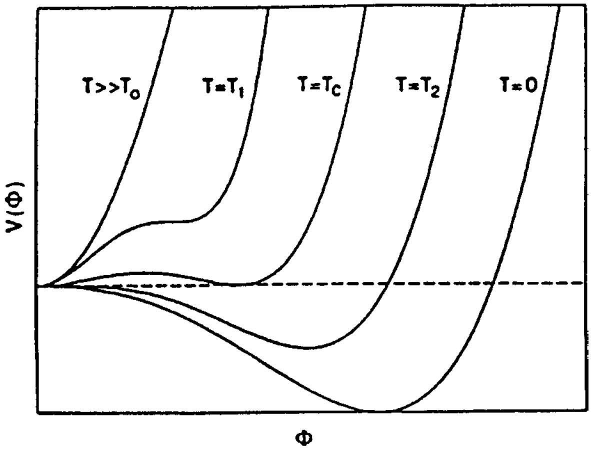

Fig. 11.17 The temperature dependence of a Higgs potential with a first-order phase transition

move into one of the stable minimas at $\phi = \pm\phi_0$. When the system is in one of these states, it is no longer symmetric under the change of sign of ϕ. Such a symmetry, which is not reflected in the vacuum state, is called *spontaneously broken*. Note that from Fig. 11.16(b) we see that the energy of the false vacuum is larger than for the real vacuum.

The central idea, from which originated the "inflationary cosmology", was to take into consideration the consequences of the unified quantum field theories, the gauge theories, at the construction of relativistic models for the early universe. According to the Friedmann models, the temperature was extremely high in the early history of the universe. If one considers Higgs fields associated with GUT models (grand unified theories), one finds a critical temperature T_c corresponding to the energy $kT_c = 10^{14}$ GeV, where k is the Boltzmann's constant. Before the universe was about $t_1 = 10^{-35}$ s old, the temperature was larger than this. Thus, the Higgs field was in the symmetric ground state. According to most of the inflation models, the universe was dominated by radiation at this time.

When the temperature decreases, the Higgs potential changes. This could happen as shown in Fig. 11.17. Here, there is a potential barrier at the critical temperature, which means that there cannot be a classical phase transition. The transition to the stable minimum must happen by quantum tunnelling. This is called a first-order phase transition.

Guth's Inflation Model

Alan Guth's original inflation model [18] was based on a first-order phase transition.

According to most of the inflationary models, the universe was dominated by radiation during the time before 10^{-35} s. The universe was then expanding so fast that there was no causal contact between the different parts of the universe that became our observable universe. Probably, the universe was rather homogeneous, with considerable space-like variations in temperature. There was also areas of false vacuum, with energy densities characteristic of the GUT energy scale, which also controls its critical temperature. While the energy density of the radiation decreased quickly, as a^{-4}, the energy density of vacuum was constant. At the time $t = 10^{-35}$ s, the energy density of the radiation became less than that of the vacuum.

At the same time, the potential started to change, such that the vacuum went from being stable to being an unstable false vacuum. Thus, there was a first-order phase transition to the real vacuum. Because of the inhomogeneity of the universe's initial condition, this happened with different speeds at differing places. The potential barrier slowed down the process, which happened by tunnelling, and the universe was at several places considerably undercooled, and there appeared "bubbles" dominated by the energy of the false vacuum. These areas acted on themselves with repulsive gravity.

By integrating the equation of motion for the expansion factor in such a vacuum-dominated bubble, one gets

$$a = e^{Ht}, \qquad H = \sqrt{\frac{8\pi G\rho_c}{3}} \,. \tag{11.136}$$

By inserting the GUT value above, we get $H = 6.6 \cdot 10^{34}\,\mathrm{s}^{-1}$, i.e. $H^{-1} = 1.5 \cdot 10^{-35}$ s. With reference to field theoretical works by Sidney Coleman and others, Guth reasoned that a realistic duration of the nucleation process happening during the phase transition is 10^{-33} s. During this time, the expansion factor increases by a factor of 10^{28}. This vacuum-dominated epoch is called the *inflation era*.

Let us look closer at what happens with the energy of the universe in the course of its development, according to the inflationary models. To understand this we first have to consider what happens at the end of the inflationary era. When the Higgs field reaches the minimum corresponding to the real vacuum, it starts to oscillate. According to the quantum description of the oscillating field, the energy of the false vacuum is converted into radiation and particles. In this way the equation of state for the energy dominating the development of the expansion factor changes from $p = -\rho$, characteristic for vacuum, to $p = \frac{1}{3}\rho$, characteristic of radiation.

The energy density and the temperature of the radiation is then increased enormously. Before and after this short period around the time $t = 10^{-33}$ s the radiation energy increases adiabatically, such that $\rho a^4 =$ constant. According to Stefan–Boltzmann law of radiation, $\rho \propto T^4$. Therefore, $aT =$ constant during adiabatic expansion. This means that during the inflationary era, while the expansion factor increases exponentially, the energy density and temperature of radiation decrease exponentially. At the end of the inflationary era, the radiation is reheated so that it returns to the energy it had when the inflationary era started.

It may be interesting to note that the Newtonian theory of gravitation does not allow an inflationary era, since stress has no gravitational effect according to it.

The Inflation Models' Answers to the Problems of the Friedmann Models

The horizon problem will be investigated here in the light of this model. The problem was that there was room for about 10000 causally connected areas inside the area spanned by our presently observable universe at the time. Let us calculate the horizon radius l_h and the radius a of the region presently within the horizon, $l_h = 15 \cdot 10^9 ly = 1.5 \cdot 10^{26}$ cm, at the time $t_1 = 10^{-35}$ s when the inflation started. From Eq. (11.128) for the radiation-dominated period before the inflationary era, one gets

$$l_h = 2t_1 = 6 \cdot 10^{-25}\text{cm} \,. \tag{11.137}$$

The radius, at time t_1, of the region corresponding to our observable universe is found by using $a \propto e^{Ht}$ during the inflation era from $t_1 = 10^{-35}$ s to $t_2 = 10^{-33}$ s, $a \propto t^{\frac{1}{2}}$ in the radiation-dominated period from t_2 to $t_3 = 10^{11}$ s and $a \propto t^{\frac{2}{3}}$ in the matter-dominated period from t_3 until now, $t_0 = 10^{17}$ s. This gives

$$a_1 = \frac{e^{Ht_1}}{e^{Ht_2}} \left(\frac{t_2}{t_3}\right)^{\frac{1}{2}} \left(\frac{t_3}{t_0}\right)^{\frac{2}{3}} l_h(t_0) = 1.5 \cdot 10^{-28}\,\text{cm} \,. \tag{11.138}$$

We see that at the beginning of the inflationary era the horizon radius, l_h, was larger than the radius a of the region corresponding to our observable universe. The whole of this region was then causally connected, and thermic equilibrium was established. This equilibrium has been kept since then and explains the observed isotropy of the cosmic background radiation.

We will now consider the flatness problem. This problem was the necessity, in the Friedmann models, of fine tuning the initial density in order to obtain the closeness of the observed mass–density to the critical density. Again, the inflationary models give another result. Inserting the expansion factor (11.136) into Eq. (11.132), we get

$$\Omega - 1 = \frac{k}{H^2} e^{-2Ht}, \tag{11.139}$$

where H is constant and given in Eq. (11.136). The ratio of $\Omega - 1$ at the end of the inflationary era to the beginning of the inflationary era becomes

$$\frac{\Omega_2 - 1}{\Omega_1 - 1} = e^{-2H(t_2 - t_1)} = 10^{-56} \,. \tag{11.140}$$

Contrary to the Friedmann models, where the mass–density moves *away* from the critical density as time increases, the density approaches the critical density exponentially during the inflationary era. Within a large range of initial conditions, this means that according to the inflation models the universe should still have almost critical mass–density.

Problems

11.1. *Cosmic redshift*
We shall in this problem study the cosmic redshift in an expanding FRW universe and show that this redshift, for small distances between emitter and receiver, can be split into a gravitational and a kinematic part.

(a) Show that the assumption that the distance between emitter and receiver is small can be expressed as

$$H_0(t_0 - t_e) \ll 1.$$

Here, the lower indices of 0 and e mean evaluated at the receiver and emitter, respectively.
In the following, include only terms to 2nd order in $H_0(t_0 - t_e)$.

(b) Light is emitted at wavelength λ_e and received at λ_0. Show that the redshift, z, can be written as

$$z = H_0(t_0 - t_e) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t_e)^2 , \tag{11.141}$$

where q is the deceleration parameter. We introduce z_K and z_G, the kinematic and the gravitational redshift, defined as follows. z_K is the redshift of light emitted due to the velocity with respect to the observer, of the emitter. z_G is the redshift of light for an emitter who has a fixed distance to the receiver. Show that $z \approx z_K + z_G$ for $z_K \ll 1$ and $z_G \ll 1$. Use the Doppler shift formula from the special theory of relativity

$$z_K = [(1 + V_e)/(1 - V_e)]^{1/2} - 1 \tag{11.142}$$

to find z_K. Show further that

$$z_G = -\frac{1}{2} q_0 H_0^2 (t_0 - t_e)^2 . \tag{11.143}$$

Why has z_G the sign it has?
The universe is matter dominated at t_0, and thus $p \ll \rho$, where p is the pressure and ρ is the energy density of the cosmic fluid. Show that, using the Friedmann equations,

$$q_0 H_0^2 = \frac{4\pi}{3} G \rho_0, \quad \rho_0 = \rho(t_0) . \tag{11.144}$$

Define the cosmic gravitational potential, ϕ_e, in the Newtonian approximation such that $\phi_e = 0$ at the position of the receiver. Show lastly that

$$z_G = -\phi_e . \tag{11.145}$$

11.2. *Gravitational collapse*
In this problem we shall find a solution to Einstein's field equations describing a spherical symmetric gravitational collapse. The solution shall describe the space-

time both exterior and interior to the star. To connect the exterior and interior solutions, the metrics must be expressed in the same coordinate system. We will assume that the interior solution has the same form as a Friedmann solution. The Friedmann solutions are expressed in comoving coordinates, thus freely falling particles have constant spatial coordinates.
Let (ρ, τ) be the infalling coordinates. τ is the proper time to a freely falling particle starting at infinity with zero velocity. These coordinates are connected to the Schwarzschild coordinates via the requirements

$$\begin{aligned} \rho &= r, \quad \text{for} \quad \tau = 0, \\ \tau &= t, \quad \text{for} \quad r = 0. \end{aligned} \tag{11.146}$$

(a) Show that the transformation between the infalling coordinates and the Schwarzschild coordinates is given by

$$\begin{aligned} \tau &= \frac{2}{3}(2M)^{-\frac{1}{2}}\left(\rho^{\frac{3}{2}} - r^{\frac{3}{2}}\right), \\ t &= \tau - 4M\left(\frac{r}{2M}\right)^{\frac{1}{2}} + 2M\ln\left[\frac{\left(\frac{r}{2M}\right)^{\frac{1}{2}}+1}{\left(\frac{r}{2M}\right)^{\frac{1}{2}}-1}\right], \end{aligned}$$

where M is the Schwarzschild mass of the star. Show that the Schwarzschild metric in these coordinates takes the form

$$\begin{aligned} ds^2 = -d\tau^2 &+ \left[1 - \frac{3}{2}(2M)^{\frac{1}{2}}\tau\rho^{-\frac{3}{2}}\right]^{-\frac{2}{3}} d\rho^2 \\ &+ \left[1 - \frac{3}{2}(2M)^{\frac{1}{2}}\tau\rho^{-\frac{3}{2}}\right]^{\frac{4}{3}} \rho^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \end{aligned} \tag{11.147}$$

Show that the metric is not singular at the Schwarzschild radius. Where is it singular?

(b) Assume the star has a position-dependent energy density $\rho(\tau)$, and that the pressure is zero. Assume further that the interior spacetime can be described with a Friedmann solution with Euclidean geometry ($k = 0$). Find the solution when the radius of the star is R_0 at $\tau = 0$.

11.3. *Physical significance of the Robertson–Walker coordinate system*
Show that the reference particles with fixed spatial coordinates move along geodesic world lines, and hence are free particles.

11.4. *The volume of a closed Robertson–Walker universe*
Show that the volume of the region contained inside a radius $r = a\chi = a\arcsin r$ is

$$V = 2\pi a^3\left(\chi - \frac{1}{2}\sin 2\chi\right).$$

Find the maximal volume. Find also an approximate expression for V when $\chi \ll R$.

11.5. *The past light cone in expanding universe models*

(a) Show that the radial standard coordinate of the past light cone is

$$\chi_{\mathrm{lc}}(t_e) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \,. \tag{11.148}$$

(b) The proper distance at a point of time t to a particle at a radial coordinate χ is $d = a(t)\chi$. Differentiation gives $\dot{d} = \dot{a}\chi + a\dot{\chi}$ which may be written as

$$v_{\mathrm{tot}} = v_{\mathrm{rec}} + v_{\mathrm{pec}} \,, \tag{11.149}$$

where v_{tot} is the total radial velocity of the particle; v_{rec} is its *recession velocity*; and v_{pec} is its *peculiar velocity*.
Show that the recession velocity of a light source with redshift z is given by

$$v_{\mathrm{rec}}(z) = c\frac{E(z)}{1+z} \int_0^z \frac{dy}{E(y)} \,. \tag{11.150}$$

Can this velocity be greater than the speed of light? What is the total velocity of a photon emitted towards $\chi = 0$? Is it possible to observe a galaxy with recession velocity greater than the speed of light?

(c) Make a plot of the past light cone, i.e. of t_e as a function of the proper distance

$$d_{\mathrm{lc}} = a(t_e)\chi_{\mathrm{lc}}(t_e),$$

for a flat, matter-dominated universe model. Explain the shape of the light cone using its slope to be equal to the total velocity of a photon emitted towards an observer at the origin.

(d) Introduce conformal time and calculate the coordinate distance of the past light cone as a function of conformal time for the flat, matter-dominated universe model. Make a plot of the past light cone in these variables.

11.6. *Lookback time*

The lookback time of an object is the time required for light to travel from an emitting object to the receiver. Hence, it is $t_L \equiv t_0 - t_e$, where t_0 is the point of time the object was observed and t_e is the point of time the light was emitted.

(a) Show that the lookback time is given by

$$t_L = \frac{1}{H_0} \int_0^z \frac{dy}{(1+y)E(y)} \,, \tag{11.151}$$

where z is the redshift of the object.

(b) Show that $t_L = t_0\left[1-(1+z)^{-3/2}\right]$, where $t_0 = 2/(3H_0)$, in a flat, matter-dominated universe.

(c) Show that the lookback time in the Milne universe model with $a(t) = (t/t_0)$, $k < 0$, is

$$t_L = \frac{1}{H_0}\frac{z}{1+z}.$$

(d) Make a plot with t_L as a function of z for the last two universe models.

11.7. *The FRW models with a w-law perfect fluid*
In this problem we will investigate FRW models with a perfect fluid. We will assume that the perfect fluid obeys the equation of state

$$p = w\rho \ , \tag{11.152}$$

where $-1 \leq w \leq 1$.

(a) Write down the Friedmann equations for a FRW model with a w-law perfect fluid. Express them in terms of the scale factor a only.
(b) Assume that $a(0) = 0$. Show that when $-1/3 < w \leq 1$, the closed model will recollapse. Explain why this does not happen in the flat and open models.
(c) Solve the Friedmann equation for a general $w \neq -1$ in the flat case. What is the Hubble parameter and the deceleration parameter? Also write down the time evolution for the matter density.
(d) Find the particle horizon distance in terms of H_0, w and z.
(e) Specialize the above to the dust- and radiation-dominated universe models.

11.8. *Age–density relations*

(a) Show that the age of a radiation-dominated universe model is given by

$$t_0 = \frac{1}{H_0}\cdot\frac{1}{1+\sqrt{\Omega_{\gamma 0}}} \ , \tag{11.153}$$

for all values of k.
(b) Show that the age of a matter-dominated universe model with $k > 0$ may be expressed by

$$t_0 = \frac{\Omega_{m0}}{2H_0(\Omega_{m0}-1)^{\frac{3}{2}}}\left[\arccos\left(\frac{2}{\Omega_{m0}}-1\right) - \frac{2}{\Omega_{m0}}(\Omega_{m0}-1)^{\frac{1}{2}}\right] \tag{11.154}$$

and of a matter-dominated universe model with $k < 0$

$$t_0 = \frac{\Omega_{m0}}{2H_0(1-\Omega_{m0})^{\frac{3}{2}}}\left[\frac{2}{\Omega_{m0}}(1-\Omega_{m0})^{\frac{1}{2}} - \operatorname{arccosh}\left(\frac{2}{\Omega_{m0}}-1\right)\right] . \tag{11.155}$$

(c) Show that the lifetime of the closed universe is

$$T = \frac{\pi}{H_0}\frac{\Omega_{m0}}{(\Omega_{m0}-1)^{\frac{3}{2}}} , \tag{11.156}$$

and that the scale factor at maximum expansion is

$$a_{\max} = \frac{\Omega_{m0}}{\Omega_{m0} - 1} \,. \tag{11.157}$$

11.9. *Redshift–luminosity relation for matter-dominated universe*
Show that the luminosity distance of an object with redshift z in a matter-dominated universe with relative density Ω_0 and Hubble parameter H_0 is

$$d_L = \frac{2c}{H_0 \Omega_0^2} \left[\Omega_0 z + (\Omega_0 - 2)(\sqrt{1 + \Omega_0 z} - 1) \right] \,. \tag{11.158}$$

For the Einstein–de Sitter universe, with $\Omega_0 = 1$, this relation reduces to

$$d_L = \frac{2c}{H_0} (1 + z - \sqrt{1+z}) \,. \tag{11.159}$$

Plot this distance in light years as a function of z for a universe with Hubble parameter $H_0 = 20$ km/s per light years.

11.10. *Newtonian approximation with vacuum energy*
Show that Einstein's linearized field equation for a static spacetime containing dust with density ρ and vacuum energy with density ρ_Λ takes the form of a modified Poisson equation

$$\nabla^2 \phi = 4\pi G(\rho - 2\rho_\Lambda). \tag{11.160}$$

11.11. *Universe models with constant deceleration parameter*

(a) Show that the universe with constant deceleration parameter q has expansion factor

$$a = \left(\frac{t}{t_0} \right)^{\frac{1}{1+q}} , \quad q \neq -1, \text{ and } a \propto e^{Ht}, \quad q = -1.$$

(b) Find the Hubble length $\ell_H = H^{-1}$ and the radius of the particle horizon as functions of time for these models.

11.12. *Relative densities as functions of the expansion factor*
Show that the relative densities of matter and vacuum energy as functions of a are

$$\begin{aligned} \Omega_v &= \frac{\Omega_{v0} a^3}{\Omega_{v0} a^3 + (1 - \Omega_{v0} - \Omega_{m0})\, a + \Omega_{m0}}, \\ \Omega_m &= \frac{\Omega_{m0}}{\Omega_{v0} a^3 + (1 - \Omega_{v0} - \Omega_{m0})\, a + \Omega_{m0}} \,. \end{aligned} \tag{11.161}$$

What can you conclude from these expressions concerning the universe at early and late times?

11.13. *FRW universe with radiation and matter*
Show that the expansion factor and the cosmic time as functions of conformal time of a universe with radiation and matter are

$$k>0: \begin{cases} a = a_0\left[\alpha(1-\cos\eta)+\beta\sin\eta\right] \\ t = a_0\left[\alpha(\eta-\sin\eta)+\beta(1-\cos\eta)\right], \end{cases} \tag{11.162}$$

$$k=0: \begin{cases} a = a_0\left[\frac{1}{2}\alpha\eta^2+\beta\eta\right] \\ t = a_0\left[\frac{1}{6}\alpha\eta^3+\frac{1}{2}\beta\eta^2\right], \end{cases} \tag{11.163}$$

$$k<0: \begin{cases} a = a_0\left[\alpha(\cosh\eta-1)+\beta\sinh\eta\right] \\ t = a_0\left[\alpha(\sinh\eta-\eta)+\beta(\cosh\eta-1)\right], \end{cases} \tag{11.164}$$

where $\alpha = a_0^2 H_0^2 \Omega_{m0}/2$ and $\beta = (a_0^2 H_0^2 \Omega_{\gamma 0})^{1/2}$, and $\Omega_{\gamma 0}$ and Ω_{m0} are the present relative densities of radiation and matter, H_0 is the present value of the Hubble parameter.

11.14. *Matter–vacuum transition in the Friedmann–Lemaître model*
Find the point of time of transition from matter domination to vacuum domination of the flat Friedmann–Lemaître universe model and the corresponding redshift.

11.15. *Event horizons in de Sitter universe models*
Show that the coordinate distances to the event horizons of the de Sitter universe models with $k>0$, $k=0$ and $k<0$ are (assuming $\Lambda = 3$)

$$\begin{aligned} r_{EH} &= \tfrac{1}{\cosh t},\ k>0,\ t\geq 0, \\ r_{EH} &= e^{-t},\ \ k=0, \\ r_{EH} &= \tfrac{1}{\sinh t},\ k<0,\ t\geq 0, \end{aligned}$$

respectively.

11.16. *Light travel time*
In this problem you are going to calculate the light travel time of light from an object with redshift z in a flat Friedmann–Lemaître model with age t_0 and a present relative density of LIVE, $\Omega_{\Lambda 0}$.
Show that the point of time of the emission event is

$$t_e = t_0 \frac{arcsinh\sqrt{\frac{\Omega_{\Lambda 0}}{(1-\Omega_{\Lambda 0})(1+z)^3}}}{arcsinh\sqrt{\frac{\Omega_{\Lambda 0}}{1-\Omega_{\Lambda 0}}}}, \tag{11.165}$$

and calculate the light travel time $t_0 - t_e$. Make a plot of $(t_0 - t_e)/t_0$ as a function of z.

11.17. *Superluminal expansion*
Show from Hubble's law that all objects in a flat Friedmann–Lemaître model with redshifts $z > z_c$ are presently receding faster than the speed of light, where z_c is given by

$$\int_0^{1+z_c} \frac{dy}{\sqrt{\Omega_{\Lambda 0} + \Omega_{m0} y^3}} = 1. \tag{11.166}$$

Find z_c for a universe model with $\Omega_{\Lambda 0} = 0.7$ and $\Omega_{m0} = 0.3$.

11.18. *Flat universe model with radiation and vacuum energy*

(a) Find the expansion factor as a function of time for a flat universe with radiation and Lorentz-invariant vacuum energy represented by a cosmological constant Λ, and with present relative density of vacuum energy Ω_{v0}.
(b) Calculate the Hubble parameter, H, as a function of time, and show that the model approaches a de Sitter model in the far future. Find also the deceleration parameter, $q(t)$.
(c) When is the inflection point, t_1, for which the universe went from deceleration to acceleration? What is the corresponding redshift observed at the time t_0?

11.19. *Creation of radiation and ultra-relativistic gas at the end of the inflationary era*
Assume that the vacuum energy can be described by a decaying cosmological parameter $\Lambda(t)$. Show from energy conservation that if the density of radiation and gas is negligible at the final period of the inflationary era compared to after it, then the density immediately after the inflationary era is

$$\rho = \frac{1}{8\pi G a(t)^4} \int_{t_1}^{t_2} \dot{\Lambda} a(t)^4 dt \, , \tag{11.167}$$

where $t_2 - t_1$ is the duration of the period with $\dot{\Lambda} \neq 0$.

11.20. *Phantom energy*
Consider a flat universe model dominated by quintessence energy with equation of state, $p = w\rho$, $w =$ constant. We shall consider "phantom energy" with $w < -1$:

(a) Use the normalization $a(t_0) = 1$ and find the scale factor a as a function of cosmic time.
(b) Find the energy density as a function of time.
(c) Show that the scale factor and the density blow up to infinity at a time

$$t_r = t_0 - \frac{2}{3(1+w)H_0\sqrt{\Omega_{P0}}} \, , \tag{11.168}$$

where Ω_{P0} is the relative density of the phantom energy at the present time. The cosmic catastrophe at the time t_r is called "the Big Rip". What is $t_r - t_0$ for $H_0 = 20\,\mathrm{km/s}$ per $10^6 ly$, $\Omega_{P0} = 0.64$ and $w = -3/2$?
(d) A planet in an orbit of radius R around a star of mass M will become unbound roughly when $-(4\pi/3)(\rho + 3p)R^3 \approx M$, where ρ and p are the density and the pressure of the phantom energy.
Show that a gravitationally bound system of mass M and radius R will be stripped at a time t_s before the big rip, given by

$$t_s \approx -\frac{\sqrt{-2(1+3w)}}{6\pi(1+w)} T,$$

where T is the period of a circular orbit with radius R around the system. Find t_s for the Milky Way galaxy with $w = -3/2$.

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Index