Large N and the Dine-Rajaraman problem

Jim McCarthy[a], Leonard Susskind[b] [1] and Andy Wilkins[a]

[a] *Department of Physics and Mathematical Physics*
University of Adelaide, Adelaide, SA 5005, Australia

[b] *Department of Physics*
Stanford University, Stanford, CA 94305-4060, USA

Abstract

We compute the effective action for scattering of three well-separated extremal brane solutions, in 11d supergravity, with zero p_- transfer and small transverse velocities. Using an interpretation of the conjecture of Maldacena, following Hyun, this can be viewed as the large N limit of the Matrix theory description of three supergraviton scattering at leading order. The result is consistent with the perturbative supergravity calculation.

1. Introduction

Matrix theory [1] proposes that M-theory is described by the maximally supersymmetric quantum mechanics of U(N) matrices in the large N limit. Moreover, a prescription for computations is given [1][2][3] in which it has been argued [4] that finite N corresponds to the so-called discrete light cone frame quantization of the M-theory (for a review see [5]). One test of these conjectures (see, e.g., [6][7]) is to compare low energy scattering of supergravitons in the Matrix theory with the corresponding results in eleven-dimensional supergravity – a subtle limit, see [8][9]. In particular, agreement is found for the scattering of two well-separated supergravitons with small transverse velocities [10][1][11]. However, supergravity seems to predict different behaviour to that of the matrix model (at any finite order in N) for processes such as the scattering of three supergravitons [12].

It has been suggested [13][14] that this discrepancy may vanish on taking the large N limit. However, only through the recent work of Maldacena [15] has it been possible to deal with this limit. In [15][16] brane configurations were studied in the limit where the field theory on the brane decouples from the bulk, and it was observed that when the number of branes N, becomes large, the curvature of spacetime around the brane becomes small (for earlier discussions in the conformal case see [17] and references therein). However, for small curvatures branes are well described by extremal black-hole type solutions of the associated supergravity. Moreover, as discussed in [18], this limit corresponds to the infinite boost limit in the DLCQ Matrix theory prescription mentioned above. Thus we are naturally led to the following conjecture — which we take to be the premise of this work — that in the large N limit of DLCQ Matrix theory, supergravitons are described by D0-brane solutions of IIA supergravity. Since these D0-branes are BPS states which can be identified with Kaluza-Klein supergraviton modes of 11d supergravity [19][20], this conjecture immediately implies a resolution of the problem in [12]. The leading order scattering amplitudes will just be proportional to those of point particles in 11d supergravity, as are those of supergravitons. In the rest of this paper we describe an explicit calculation of the three supergraviton amplitude as "extremal black hole" solutions, since the details may be of interest.

Thereto, we calculate the effective action for large separation and low transverse velocity scattering of these particles (neglecting spin effects as usual), following a "post-Newtonian" calculation similar to those in, e.g., [21] – with the slight twist that we work in the lightcone frame. The essence of the calculation is that we promote the centers in

the static solution to dynamical variables, and then determine corrections to the metric so that we have a solution

$$g_{MN} = \overline{g}_{MN} + g_{MN}^{(>)}, \qquad g_{MN}^{(>)} \equiv \sum_{n>0} g_{MN}^{(n)}, \tag{1.1}$$

order by order in an expansion in time (x^+) derivatives. We will say that $g^{(n)}$ "has $d_t = n$" in this expansion. This is a nontrivial solution since it corresponds to a nontrivial "tangent deformation" in the moduli space of static solutions. The corrections $g^{(n)}$ vanish on the spatial infinity.

The calculation is set up in Section 2, where we recall the uplifting of the static solution to 11d, and discuss the solution for moving sources to lowest order in transverse velocities. In section 3 we determine the leading large distance behaviour of the solution for up to three centers. In section 4 we give the result of the calculation for the leading order two and three particle scattering.

In the following V will stand for a typical (small) transverse speed and L a typical (large) transverse separation.

2. Uplifting to 11d

The static BPS solution of IIA supergravity with n_c D0-branes ([22], and references therein) is,

$$d\overline{s}^2 = \overline{g}_{\mu\nu}(x)dx^\mu dx^\nu = -f_0(\vec{y})^{-1/2}dt^2 + f_0(\vec{y})^{1/2}d\vec{y} \cdot d\vec{y}$$
$$e^{\overline{\Phi}(x)} = g_s f_0^{3/4}, \qquad \overline{A}_0(x) = f_0^{-1} - 1 \tag{2.1}$$
$$f_0(\vec{y}) = 1 + \sum_{i=1}^{n_c} f_0^{(i)}, \qquad f_0^{(i)} = \frac{\mu_i}{|\vec{y} - \vec{y}_i|^7}.$$

The statement that these are D0-branes means that the "charges" in this solution, μ_i, are determined in terms of string parameters [20] by $\mu_i \sim \ell_s^7 g_s$. Using the Kaluza-Klein relation we can lift this to a solution of 11d supergravity with the 11th direction compactified on a spacelike circle of radius R_c, with vanishing 3-form, and

$$ds^2 = \overline{g}_{MN}(x)dx^M dx^N = \overline{D}^{-1/8}\overline{g}_{\mu\nu}dx^\mu dx^\nu + \overline{D}(dx^{11} + \overline{A}_\mu dx^\mu)^2,$$
$$= (f_0(\vec{y}) - 2)dt^2 + f_0(\vec{y})(dx^{11})^2 - 2(f_0(\vec{y}) - 1)dtdx^{11} + d\vec{y} \cdot d\vec{y}, \tag{2.2}$$

3

where $\overline{D} = e^{4\overline{\Phi}/3} = f_0$. In lightcone coordinates, $x^{\pm} = x^{11} \pm t$, the result is suggestively simple [23],

$$ds^2 = dx^+ dx^- + f_0^B(\vec{y})dx^- dx^- + d\vec{y} \cdot d\vec{y}$$

$$f_0^B(\vec{y}) = f_0(\vec{y}) - 1 = \sum_{i=1}^{n_c} \frac{\mu_i}{|\vec{y} - \vec{y}_i|^7} \, . \tag{2.3}$$

The corresponding D0-brane source in 11d supergravity is (in the leading approximation where we neglect spin effects) a massless scalar with fixed nonzero momentum in the compact direction. The appropriate point particle action has been discussed in [7]. The point particle action in 11d for a *massive* "particle i" with mass m_i, is

$$S_m^{(i)} = -m \int d\tau \left[-g_{MN} \frac{dx^M}{d\tau} \frac{dx^N}{d\tau} \right]^{1/2} . \tag{2.4}$$

Choosing to parametrize the trajectory by x^+, this gives

$$p_-^i = m \frac{g_{-+} + g_{-a}v_i^a + g_{--}v_i^-}{[-g_{++} - g_{+a}v_i^a - g_{ab}v_i^a v_i^b - 2g_{+-}v_i^- - 2g_{-a}v_i^- v_i^a - g_{--}v_i^- v_i^-]^{1/2}} \, , \tag{2.5}$$

where $v_i^M = dy_i^M/dx^+$. Assuming no x^- dependence, then p_-^i is a cyclic variable, and the corresponding "Routhian" for the constant p_-^i physics of the *massless* particle is

$$S^{(i)} = \lim_{m \to 0} \left(S_m^{(i)} - \int dx^+ p_-^i v_i^- \right) .$$

To implement this limit is very easy, since the first term vanishes and v_i^- is determined at lowest order by the vanishing of the denominator in (2.5). Thus, with $p_-^i = Q_i$, the D0-brane source is

$$S^{(i)} = Q_i \int dx^+ \frac{1}{g_{--}} \left[g_{+-} + g_{-a}v_i^a - \sqrt{(g_{+-} + g_{-a}v_i^a)^2 - g_{--}(g_{++} + 2g_{+a}v_i^a + g_{ab}v_i^a v_i^b)} \right] . \tag{2.6}$$

In summary, then, the system of interest is described by

$$S = S_E + \sum_{i=1}^{n_c} S^{(i)} \, , \tag{2.7}$$

where $S_E = \kappa^{-2} \int d^{11}x \sqrt{-g}R(g)$ and $S^{(i)}$ is specified in (2.6). We work in the lightcone frame with $x^- \sim x^- + 2\pi R$. We may now proceed with the d_t-expansion of the x^--independent solution for centers moving with slow transverse velocity, and then substitute in to determine the effective action for the centers.

4

To zeroth order we simply have the static solution which fixes

$$Q_i = \frac{32\pi^4}{15}\mu_i \frac{2\pi R}{\kappa^2} \,. \tag{2.8}$$

To first order the solution is easily understood in terms of transverse boosts. A "Galilean" transversal boost with velocity $\vec{v}$ in lightcone time, as is appropriate for our discussion, gives

$$\begin{aligned} x'^+ &= x^+ \\ x'^- &= x^- + 2\vec{v}\cdot\vec{y} - v^2 x^+ \\ \vec{y}' &= \vec{y} - x^+\vec{v} \,. \end{aligned} \tag{2.9}$$

Taking x' as the coordinates of the "static frame", we obtain the metric for a single center moving transversally with constant velocity by using the boost as a coordinate transformation,

$$\begin{aligned} ds^2 =&(1 - 2v^2 f_0^B(r'))dx^+ dx^- + f_0^B(r')dx^- dx^- + v^4 f_0^B(r')dx^+ dx^+ + \\ &+ (\delta_{ab} + 4f_0^B(r')v_a v_b)dy^a dy^b + 4f_0^B(r')v_a dx^- dy^a - 4f_0^B(r')v^2 v_a dx^+ dy^a \,, \end{aligned} \tag{2.10}$$

where $r' = |\vec{y} - x^+\vec{v}|$. This is extended to n_c centers moving independently to give the ansatz

$$\begin{aligned} d\widehat{s}^2 =& \left(1 - 2\sum_{i=1}^{n_c} v_i^2 f_0^{(i)}(r_i)\right)dx^+ dx^- + \sum_{i=1}^{n_c} f_0^{(i)}(r_i)dx^- dx^- + \sum_{i=1}^{n_c} v_i^4 f_0^{(i)}(r_i)dx^+ dx^+ + \\ &+ \left(\delta_{ab} + 4\sum_{i=1}^{n_c} f_0^{(i)}(r_i)v_i^a v_i^b\right)dy^a dy^b + 4\sum_{i=1}^{n_c} f_0^{(i)}(r_i)v_i^a dx^- dy^a - \\ &- 4\sum_{i=1}^{n_c} f_0^{(i)}(r_i)v_i^2 v_i^a dx^+ dy^a \,, \end{aligned} \tag{2.11}$$

where $r_i = |\vec{y} - \vec{y}_i(x^+)|$. We will denote by $\widehat{g}_{MN}$ the metric specified by (2.11). In this way we find the possible nonzero first order corrections to the metric are just those which implement the "constraints" of the lightcone parametrization,

$$\begin{aligned} g_{+a}^{(1)} &= \widehat{g}_{+a}^{(1)} = 0 \\ g_{-a}^{(1)} &= \widehat{g}_{-a}^{(1)} = \sum_i 2\frac{\mu_i}{r_i^7}v_i^a \,. \end{aligned} \tag{2.12}$$

It is now straightforward to check that the independently boosted metric $\widehat{g}$ is a solution to first order in V of the 11d system (2.7).

5

3. Leading large-distance behaviour of the d_t expansion

In the d_t-expansion we must iteratively determine the expression for $\dot{\vec{v}}_i$, which clearly only receives corrections at even orders of d_t. It is well known that the $d_t = 2$ contribution vanishes (see, e.g., [24][25]) – this is "flatness of the moduli space". To see this in the present calculation we simply evaluate the effective action to second order, for which we only need the first order solution given previously. The resulting effective Lagrangian is a total derivative. Equivalently, note that from the equation of motion for the centers (the geodesic equation), using (2.12) we have (to second order)

$$\dot{v}_i^a = -\frac{1}{2}\partial_a g_{++}^{(2)} + O(V^4)\,. \tag{3.1}$$

But from the Einstein equations to second order, we find

$$-\frac{\pi R}{\kappa^2}\Delta_\perp g_{++}^{(2)} = T_{++}^{(2)} = 0\,, \tag{3.2}$$

and thus $\dot{\vec{v}}_i \sim O(V^4)$ as stated.

Further, a detailed calculation[2] [26] shows that

$$g_{ab}^{(2)} = \widehat{g}_{ab}\,, \qquad g_{+-}^{(2)} = \widehat{g}_{+-}\,, \tag{3.3}$$

and

$$g_{--}^{(2)} = \sum_{ij}\mu_i\mu_j\frac{|\vec{v}_i - \vec{v}_j|^2}{|r_i|^7|r_j|^7} + f^{(2)}\,, \tag{3.4}$$

where,

$$\Delta_\perp f^{(2)} = 2\partial_a\partial_b\sum_{ij}\mu_i\mu_j\frac{(v_i - v_j)^a(v_i - v_j)^b}{|r_i|^7|r_j|^7}\,.$$

The important result in the above is the observation that the solution at second order differs from the boosted metric (2.10), (2.11) by $O(\frac{V^2}{L^{14}})$.

At higher order the leading behaviour is equally simple. Let us separate out the "boosted" corrections $\overline{h}$ which are contained in (2.10) and (2.11); ie,

$$g_{MN}^{(>)} = \overline{h}_{MN} + h_{MN}\,, \tag{3.5}$$

where $\widehat{g}_{MN} = \overline{g}_{MN} + \overline{h}_{MN}$. Then

$$h_{MN}^{(n)} = O\left(\frac{V^n}{L^{14}}\right)\,. \tag{3.6}$$

---

[2] To regularize the point particle we replace $r_i \to (r_i^2 + \epsilon^2)^{1/2}$.

To see this one just calculates the leading terms in the Einstein equations at nth order in the expansion. For the Einstein tensor these are terms involving $g_{MN}^{(n)}$, since products of lower order (in d_t) contributions will be higher order in $1/L$. The $1/L$ expansion of the source is similarly straightforward, and the leading terms are just those required for $\overline{h}_{MN}^{(n)}$.

We now show that the leading term in the effective action is determined by $\widehat{g}_{MN}$. Expanding the action around $\widehat{g}_{MN}$, the above result implies that

$$S[g] = S[\widehat{g}] + \frac{\delta S}{\delta g}\big|_{\widehat{g}}[h] + \text{higher order} . \tag{3.7}$$

The second term can be further expanded around $\overline{g}_{MN}$, and only the first term in this expansion is required for the leading order result,

$$\frac{\delta S}{\delta g}\big|_{\overline{g}+\overline{h}}[h] = \frac{\delta^2 S}{\delta g \delta g}\big|_{\overline{g}}[\overline{h}, h] + \text{higher order} .$$

Now, the fact that the boosted single center metric is a solution of the Einstein equations for a constant transverse velocity source implies that, up to terms with derivatives on $\vec{v}$,

$$\frac{\delta S}{\delta g}\big|_{\widehat{g}_i}[h] = 0 ,$$

where $\widehat{g}_i$ denotes the boosted single center solution (for the ith center) – ie, the limit of $\widehat{g}_{MN}$ as $\mu_j \to 0$, $j \neq i$. Thus we must have, up to terms with derivatives on $\vec{v}$,

$$\frac{\delta^2 S}{\delta g \delta g}\big|_{\overline{g}}[\overline{h}, \cdot] = O(\frac{\mu_i \mu_j}{L^{14}}) ,$$

meaning that the RHS is at least quadratic in the μ_i since it vanishes if all but one of them is sent to zero. But then this is higher order, and can be ignored. Thus we only have to worry about the contribution of $\dot{\vec{v}}$ terms ($\ddot{\vec{v}}$ terms are irrelevant, as is easily seen by the following argument).

Using the previous results, we have so far shown that, in the second term of (3.7),

$$\frac{\delta S}{\delta g}\big|_{\overline{g}+\overline{h}} = O(\mu V^4) . \tag{3.8}$$

Thus, we only need the $\dot{\vec{v}}$ terms in the LHS of (3.8), and they need only be contracted with $h^{(2)}$ to this order. A calculation shows that in the Einstein tensor, $\dot{\vec{v}}$ terms only appear, at this order, in G_{+-} and G_{ab}. But, as summarized above, $h_{+-}^{(2)}$ and $h_{ab}^{(2)}$ vanish. Further, any contractions with off-diagonal metric components are higher order. Thus, finally, the result is proved – all terms but the first in (3.7) are higher order in $1/L$.

7

4. Computing the action

At this point we simply compute the leading contribution up to $O(V^6)$. The result for the leading $O(V^4)$ contribution to two particle scattering is (we don't write the "polarization" terms, with numerators containing $\vec{v} \cdot \vec{y}$)

$$S_{\text{eff}}^{(4)} = -\frac{32\pi^4}{15} \frac{2\pi R}{\kappa^2} \mu_1 \mu_2 \frac{|\vec{v}_1 - \vec{v}_2|^4}{|\vec{y}_1 - \vec{y}_2|^7} \,. \tag{4.1}$$

This is precisely the result reported in [11]. In the present calculation it results from a cancellation between Einstein and source contributions.

The result for the leading $O(V^6)$ contribution to three particle scattering, in the limit considered by Dine and Rajaraman ($|\vec{y}_3| >> |\vec{y}_1 - \vec{y}_2|$) is (for brevity we only write the "Dine-Rajaraman" term)

$$S_{\text{eff}}^{(6)} = -4\frac{32\pi^4}{15} \frac{2\pi R}{\kappa^2} \mu_1 \mu_2 \mu_3 \frac{|\vec{v}_1 - \vec{v}_2|^2 |\vec{v}_2 - \vec{v}_3|^2 |\vec{v}_1 - \vec{v}_3|^2}{|\vec{y}_3|^7 |\vec{y}_1 - \vec{y}_2|^7} \,. \tag{4.2}$$

It is interesting to note that there is clearly no contribution from the source action of this form.

Hence we see that the term required for agreement with the perturbative supergravity calculation does appear. The technical calculation in this paper is simply a check of the standard IIA/M-theory relation. The significance to Matrix theory rests on the conjectured relation to the large N limit, which it is clearly of interest to understand better.

After this paper was finished a number of papers have appeared which discuss the Dine-Rajaraman problem [27][28][29][30] from the finite N side. In [27] it was suggested that the supersymmetry cancellations proposed in the Matrix theory calculation of [12] would not occur, but this has been disputed in [28] and [29]. The technical calculation in the present paper has significant overlap with [30], where, further, the Matrix theory result is recalculated and shown to be in agreement at finite N. The present paper supports the supergravity side of this calculation.

References

[1] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *Phys. Rev.* **D55** (1997) 5112, `hep-th/9610043`.

[2] N. Seiberg, *Phys. Rev. Lett.* **79** (1997) 3577, `hep-th 9710009`.

[3] A. Sen, "D0 branes on T^n and matrix theory", `hep-th/9709220`.

[4] L. Susskind, "Another conjecture about M(atrix) theory", `hep-th/9704080`.

[5] A. Sen, "An introduction to nonperturbative string theory", `hep-th/9802051`.

[6] K. Becker, M. Becker, Nucl. Phys. **B506** (1997) 48, `hep-th/9705091`.

[7] K. Becker, M. Becker, J. Polchinski and A. Tseytlin, it Phys. Rev. **D56** (1997) 3174, `hep-th/9706072`.

[8] S. Hellerman and J. Polchinski, "Compactification in the lightlike limit," `hep-th/9711037`.

[9] A. Bilal, "DLCQ of M-theory as the light-like limit", `hep-th/9805070`.

[10] M. Douglas, D. Kabat, P. Pouliot and S. Shenker, Nucl. Phys. **B485** (1997) 85, `hep-th/9608024`.

[11] D. Berenstein and R. Corrado, *Phys. Lett. B* **406** (1997) 37, `hep-th/9702108`.

[12] M. Dine and A. Rajaraman, "Multigraviton scattering in the matrix model", `hep-th/9710174`.

[13] T. Banks, Nucl. Phys. Proc. Suppl. **67** (1998) 180, `hep-th/9710231`.

[14] D. Bigatti and L. Susskind, "Review of Matrix theory", `hep-th/9712072`.

[15] J. Maldacena, "The large N limit of superconformal field theories and supergravity", `hep-th/9711200`.

[16] N. Itzhaki, J. Maldacena, J. Sonnenschein and S. Yankielowicz, "Supergravity and the large N limit of theories with sixteen supercharges", `hep-th/9802042`.

[17] S. Gubser and I. Klebanov, Phys.Lett. **413B** (1997) 41, `hep-th/9708005`.

[18] S. Hyun, "The background geometry of DLCQ supergravity", `hep-th/9802026`.

[19] E. Witten, Nucl.Phys. B443 (1995) 85, `hep-th/9503124`.

[20] J. Polchinski, Phys. Rev. Lett. **75** (1995) 4724, `hep-th/9510017`.

[21] R. Ferrell and D. Eardley, Phys. Rev. Lett. **59** (1987) 1671.

[22] K.S. Stelle, "BPS branes in supergravity", `hep-th 9803116`.

[23] S. Hyun, Y. Kiem and H. Shin, *Phys. Rev.* **D57** (1998) 4856, `hep-th/9712021`.

[24] C. Bachas, Phys.Lett. **374B** (1996) 37, `hep-th/9511043`.

[25] R. Khuri and R. Myers, Nucl.Phys. **B466** (1996) 60, `hep-th/9512061`.

[26] J. McCarthy and A. Wilkins, to appear.

[27] M. Fabbrichesi, G. Ferretti and R. Iengo, "Supergravity and Matrix theory do not disagree on multi-graviton scattering", `hep-th/9806018`.

[28] W. Taylor IV and M. Raamsdonk, "Three-graviton scattering in Matrix theory revisited", `hep-th/9806066`.

[29] R. Echols and J. Gray, "Comment on multigraviton scattering in the Matrix model", `hep-th/9806109`.

[30] Y. Okawa and T. Yoneya, "Multi-body interactions of D-particles in supergravity and Matrix theory", `hep-th/9806108`.