### Particle Growth and BPS Saturated States

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Consistency of the Bekenstein bound on entropy requires the physical dimensions of particles to grow with momentum as the particle is boosted to transplanckian energies. In this paper the problem of particle growth in heterotic string theory is mapped into a problem involving the properties of BPS saturated black holes as the charge is increased. Explicit calculation based on the black hole solutions of Sen are shown to lead to a growth pattern consistent with the holographic speculation described in earlier work.

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### 1. Introduction

According to the holographic principle of 't Hooft and the author, the physical dimensions of a transplanckian particle must depend on the relative motion of the particle and observer in an unconventional manner. In particular, the transverse cross sectional area must grow at least as fast as the longitudinal momentum. In free string theory, logarithmic growth has been known since the earliest days of string theory. Arguments were given in [1] that nonperturbative effects would cause the growth to accelerate to the more rapid behavior. The purpose of this paper is to present additional evidence that this is so.

The speculations of [1] can be summarized as follows:

- 1) Strings can be described by Hamiltonian quantum mechanics in the light cone frame.
- 2) Strings are composed of "string-bits". Each string-bit is a segment of string of length  $\sqrt{\alpha'}$ . The information content of a string never exceeds one bit per string-bit.
- 3) The transverse density of string bits in the light cone frame never exceeds one stringbit per Planck area. In what follows we will set G=1 so that the planck length is unity.
- 4) A system of longitudinal momentum  $P_{-}$  may be described as a collection of string-bits of total length  $\alpha'P_{-}$  and therefore contains  $\sqrt{\alpha'}P_{-}$  string-bits.

Evidently, the transverse area occupied by such a system must be at least

$$A_{min} \approx P_{-} \sqrt{\alpha'} \tag{1.1}$$

The precise framework for this discussion is the light cone frame. In this gauge, relativistic mechanics has a galilean structure, [2], which allows concepts of particle size, wave functions of composites and other similar nonrelativistic concepts to be formulated. Furthermore, the light cone frame is is the only known hamiltonian formulation of string theory. Light cone coordinates are defined by the line element

$$ds^{2} = g_{+-}dx^{+}dx^{-} + g_{+i}dx^{+}dx^{i} + g_{ij}dx^{i}dx^{j}$$
(1.2)

We assume that spacetime is flat at infinity. In this situation we may further fix the gauge so that  $g_{-+} = 1$ ,  $g_{+i} = 0$  and  $g_{ij} = -\delta_{ij}$  as  $X_{-} \to \infty$ . It is easy to see that the trajectories  $X^{i} = const$ ,  $X^{+} = const$  are light like geodesics.

In these coordinates the transverse spread of a particle  $R_{\perp}^{2}$  should grow like

$$R_{\perp}^{2} \to P_{-} \tag{1.3}$$

The symmetries of the light cone formulation include various kinematical or nondynamical symmetries such as transverse rotation and translation which are realized in a trivial manner. Among these are the transverse galilean boosts under which each transverse momentum  $P_{\perp}$  is shifted by an amount proportional to the corresponding longitudinal momentum  $P_{\perp}$ 

$$P_{\perp} \to P_{\perp} + vP_{-} \tag{1.4}$$

Under these symmetries, transverse dimensions are unchanged.

In the light cone frame the role of time is played by  $X^+$ . The initial data surfaces  $X^+ = const$  are composed of light rays. The transverse coordinate of an event p may be obtained as follows:

Choose a 2+1 dimensional surface at spatial infinity defined by

$$X^{+} + X^{-} = L (1.5)$$

where L is a large constant which eventually tends to infinity. In Ref.1 this surface was called "the screen". Assuming flat conditions at infinity, the screen may be equipped with cartesian coordinates with  $g_{ij} = -\delta_{ij}$ . The transverse light cone coordinates of p are found by passing a light ray through p which intersects the screen at right angles. The transverse coordinates of this "image" point on the screen are also the transverse coordinates  $X_{\perp}(p)$ . It will not be important in what follows to define the longitudinal coordinate of p. The statement that a particle grows with  $P_{-}$  should be taken to mean that its image on the distant screen grows.

## 2. Particle Size and BPS Black Holes

We will consider heterotic string theory compactified on a 6 torus. The four noncompact dimensions are  $X^{\mu} = (X^+, X^-, X^i)$  where i = 1, 2. We also use the notation  $X^0 = (X^+ + X^-)$ ,  $X^3 = (X^+ - X^-)$ , and  $X^m = (X^1, X^2, X^3)$ . One of the six compact coordinates will be singled out and called Y. The others play no role and will be ignored. The compactification radius for Y is called  $R_c$  and is assumed to be larger than the fundamental string length  $l_s = \sqrt{\alpha'}$ . In particular  $R_c$  is assumed large enough to easily contain a fundamental particle such as a graviton.

Let us consider a state with such a particle with vanishing transverse momentum and longitudinal momentum  $P_-$ . We wish to know if  $R_{\perp}^2(P_-)$  increases with  $P_-$  and if so how. To answer this, let us make a transverse galilean boost along the Y direction until  $P_y = P_-$ . This should have no effect on the size. The result of such a transverse boost is to bring the particle to rest in the  $X_3$  direction. To see this note that  $P_+ = (P_{\perp}^2 + P_y^2)/P_-$ . Therefore in the present case  $P_+ = P_-$  and therefore,  $P_3 = 0$ .

The object we have constructed can be described in another way. It is a Kaluza Klein charged particle with charge  $Q = P_y = P_-$  and mass equal to its charge. In heterotic string theory it is a BPS saturated particle with  $Q_L = Q_R$  [4].

Let us first consider the transverse size of these objects in the free string approximation. By T-duality we can map the system to one which is compactified on a circle of radius  $R'_c = \frac{\alpha'}{R_c^2}$  with winding number  $P_-R_c = N$ . The total length of the wound string is  $NR'_c = P_-\alpha'$ . It is well known that such a string will fluctuate in the orthogonal directions giving a growth proportional to  $\log(P_-\alpha')$ . Thus we recover the usual logarithmic growth of weakly coupled strings. However this calculation is inadequate when the winding number becomes large. In this case, no matter how small the string coupling is the number of overlapping strings passing through the same place will be so large that interactions can not be ignored. Fortunately, in this limit, a semiclassical description of these objects is available. For large Q these objects are considered to be extreme black holes whose large distance behavior are described by classical solutions of low energy field theory. The classical solutions have been given by Sen [4].

## 3. Properties of Sen's Black holes

In the limit of large mass and charge, these extreme BPS objects are described by a metric with the form

$$ds^{2} = g_{00}dt^{2} - d\rho^{2} - \rho^{2}d\omega^{2} = g_{00}dt^{2} - dX^{m}dX^{m}$$
(3.1)

where  $\rho$  is the proper distance from the horizon and

$$g_{00} = \frac{\rho^4}{\rho^4 + M\rho^3} \tag{3.2}$$

The horizon at  $\rho = 0$  is singular. According to Sen [4], the classical solution should only be believed for  $\rho > \sqrt{\alpha'}$ . At  $\rho = \sqrt{\alpha'}$  a stringy stretched horizon, Ref 6, 5 is present. The stretched horizon contains whatever distinctions exist between different states. We will identify the region of the stretched horizon as the object whose growth we are interested in.

At first sight the situation seems disappointing. According to eq.(3.1) the area of the stretched horizon is of order  $\alpha'$  for all M which if taken at face value would say that the dimensions are independent of  $P_-$ . However we are not yet in the light cone frame. To obtain the transverse spread in the light cone frame we need to project the black hole to the screen. This requires solving for the light like geodesics.

# 4. Projection to the Screen

To find the light like geodesics in the background (3.1) we use the usual action principle.

$$\delta \int \left[ g_{\mu\nu} \frac{dX^{\mu}}{d\tau} \frac{dX^{\nu}}{d\tau} \right] d\tau = 0 \tag{4.1}$$

which gives

$$\delta \int \left[ g_{00} \left( \frac{dt}{d\tau} \right)^2 - \frac{dX^m}{d\tau} \frac{dX^m}{d\tau} \right] d\tau = 0 \tag{4.2}$$

The equation of motion for t gives

$$g_{00}\frac{dt}{d\tau} = 1\tag{4.3}$$

In eq.(4.3) we have arbitrarily normalized  $\tau$  so that the right hand side is unity.

The equations of motion for  $X^m$  are

$$2\frac{d^2X^m}{d\tau^2} = -\frac{\partial g_{00}}{\partial X^m} \left(\frac{dt}{d\tau}\right)^2$$

. Using eqn.(4.3), this becomes

$$\frac{d^2X^m}{d\tau^2} = \frac{1}{2} \frac{\partial g_{00}^{-1}}{\partial X^m} \tag{4.4}$$

This is the equation of motion for a Newtonian particle in a potential

$$V = -\frac{1}{2q_{00}} \tag{4.5}$$

The condition that the geodesic be light like reduces to the condition that the total energy vanish.

$$\frac{1}{2} \left[ \frac{dX}{d\tau} \right]^2 + V = 0 \tag{4.6}$$

Using eq (3.2)we find that the potential V is given by the familiar coulomb potential plus an additive constant.

$$2V = -1 - \frac{M}{\rho} \tag{4.7}$$

Thus we see that light like geodesics correspond to nonrelativistic particles in a coulomb potential which approach from infinity with unit velocity. The problem of the size of the image on the screen can now be solved. Assume that a light ray leaves the screen with impact parameter b relative to the center of the coulomb potential. It will hit the stretched horizon if b is less than some maximum value  $R_{\perp}$ . For a given b the distance of closest approach of the trajectory can easily be obtained. It is given by the equation

$$V(\rho) + \frac{b^2}{\rho^2} = 0 (4.8)$$

If the distance of closest approach is less than  $\sqrt{\alpha'}$  than the trajectory hits the stretched horizon and the original point is in image of the black hole. This gives the size of the image as

$$R_{\perp}^2 = \sqrt{\alpha'} M = P_{-} \alpha' \tag{4.9}$$

for large M. Thus we see that the light cone frame transverse size of a particle grows with  $P_{-}$  in the required way.

#### 5. Two Particles

In Ref.1 a speculative picture was given for the behavior of a system of well separated particles moving with small relative motion when the combined system is boosted to extreme transplanckian momentum. The boost has no effect on the relative center of mass energies of the particles and should also have no effect on the invariant scattering amplitudes. Consider, for example, a pair of particles moving slowly past one another it the rest frame. For simplicity we assume that in this frame the particles move purely in the transverse directions. Also assume they pass one another at a large macroscopic transverse distance Z so that for all practical purposes they never interact.

Now boost the entire system along the longitudinal direction (orthogonal to their motion). Each particle grows so large that the image discs overlap as they pass. Nevertheless they must eventually emerge from the region of overlap with no scattering. This seems quite remarkable. We shall see that it is closely related to the very special properties of BPS saturated states.

To analyze this system, again boost along the compact Y direction until the components  $P_Y$  are equal to the longitudinal momenta. The system then consists of two BPS black holes. Ignoring the slow relative motion, we can write the metric for this system in the form of eq.(3.1). The only difference is that  $g_{00}$  is now given by

$$g_{00}^{-1} = 1 + \frac{M}{|z - z_1|} + \frac{M}{|z - z_2|}$$

$$(5.1)$$

For simplicity the two longitudinal momenta and therefore the BPS masses have been chosen equal. In eq.(5.1) z represents cartesian coordinates and  $z_i$  are chosen so that the images of the black holes on the screen are centered at the transverse locations of the two particles in the light cone frame. Once again the motion of light rays is mapped into a nonrelativistic coulomb problem, this time with two force centers. The motion can be analyzed by elementary methods.

If the separation between the transverse light cone locations of the centers of the particles is large enough and  $P_{-}$  is not too large, the images will not overlap. We will also find that  $|z_1 - z_2| > \sqrt{\alpha'}$ . As the mass increases the individual images will grow and for fixed distance between them the value of  $|z_1 - z_2|$  will decrease. Eventually, when M becomes large enough the images will begin to overlap. At this point  $|z_1 - z_2|$  will be about  $\sqrt{\alpha'}$ . In other words the stretched horizons begin to touch. Beyond this the images presumably merge into a single blob.

We are considering a system of particles which are slowly moving and well separated in the rest frame. How, from the boosted viewpoint, can we understand the fact that the overlapping objects do not interact? The answer lies in the very special properties of BPS states. Since the particles were boosted in the same direction along the Y axis they are mapped into same sign BPS particles. It is a well known fact that the forces between such objects exactly cancel if they are at rest no matter how close the are. The cancellation of the long range massless exchanges is easily seen from the low energy equations of motion. However, much more is involved. In particular cases such as D-branes [7] it can be seen that the entire tower of masses exchanges cancels exactly. In any case, for particles at rest, the cancellation is thought to be an exact consequence of supersymmetry.

On the other hand if the original particles are not at rest the forces will not cancel. In particular, if the relative velocity between them is large we expect that they will undergo a violent collision, perhaps leading to genuine nonextremal black hole formation. This is exactly what we expect of BPS particles if they collide with appreciable relative velocity.

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