Background independence in a nutshell:
the dynamics of a tetrahedron

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Abstract

We study how physical information can be extracted from a background independent quantum system. We use an extremely simple 'minimalist' system that models a finite region of 3d euclidean quantum spacetime with a single equilateral tetrahedron. We show that the physical information can be expressed as a boundary amplitude. We illustrate how the notions of "evolution" in a boundary proper-time and "vacuum" can be extracted from the background independent dynamics.

1   Introduction

To understand quantum gravity, we have to learn how to do quantum physics in a background independent context. On a background, distance and time separation are described by the independent variables x and t that coordinatize the background. In a background independent field theory, on the contrary, distance and time separation must be extracted from the dynamical variables. In a classical theory we know how to do so: we compare observations with coordinate-independent quantities. In a quantum theory, we don't. As a consequence, we still lack a general technique for extracting physical information and computing, say, particle scattering amplitudes, even when the basic formalism of a background-independent quantum field theory is defined, as in loop quantum gravity and in the spinfoam formalism [1, 2, 3].

An idea for solving this problem is to study the quantum propagator of a finite spacetime region, as a function of the boundary data [1, 4]. The key observation [5] is that in gravity the boundary data include the gravitational field, hence the geometry of the boundary, hence all relevant relative distances and time separations. In other words, the boundary formulation realizes very elegantly in the quantum context the complete identification between spacetime geometry and dynamical fields, which is Einstein's great discovery.

Formally, the idea consists in extracting the physical information from a background independent quantum field theory in terms of the quantity

$$W[\varphi] = \int_{\phi|_\Sigma = \varphi} D\phi \ \ e^{-\frac{i}{\hbar} S[\phi]} \ . \tag{1}$$

Here ϕ represents the ensemble of the dynamical fields, $S[\phi]$ their diffeomorphism invariant and background independent action, the integral is over the fields in a finite spacetime region, bounded by a

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compact surface Σ, and φ is the value of ϕ on Σ. $W[\varphi]$ does not depend on (local deformations of) Σ because of diffeomorphism invariance. The boundary field φ can be viewed as expressing initial, final as well as boundary values of ϕ and $W[\varphi]$ expresses the corresponding amplitude. This is in the same sense in which the nonrelativistic Feynman propagator $W(x, x', t)$ expresses the amplitude for given initial and final positions [6]. The difference is that while $W(x, x', t)$ depends on the background variable t, here there is no distinction between background variables, initial, final or boundary data. In the context of a finite number of degrees of freedom, a covariant generalization of the Feynman propagator, viable when there is no distinction between independent (t) and dependend (x) variable, was illustrated in [7]. In field theory, we can assume that $W[\varphi]$, formally defined in (1), expresses the amplitude of having a certain set of initial and final fields, as well as boundary fields, measured by apparatus that are located in spacetime in the manner described by (the geometry of) the surface Σ, this geometry being determined by φ itself. That is, the independent variable is simply hidden in the boundary values of the field. This picture closely describes what happens in a laboratory experiment, where, say, scattering events are confined in a finite-size spacetime region, around which we measure incoming and outgoing particles (that is, matter-field variables) as well as distances between instruments and elapsed time (that is, gravitational-field variables).

We expect that particle scattering amplitudes can be effectively computed from $W[\varphi]$ in quantum gravity; details will be given elsewhere [8]. The relation between particle states defined in such a finite context and the usual particle states of quantum field theory, defined on an infinite spacelike region, will be discussed in [9]. For a theory which is not diffeomorphism invariant, the amplitude (1) depends also on Σ and, appropriately defined, can be proven [10, 11] to satisfy a generalized Tomonaga–Schwinger equation [12]. This equation becomes a generalized Wheeler-DeWitt equation in the background independent context [1].

This boundary picture is pithy and appealing, but its implementation in the full 4d quantum gravity theory is difficult because of the technical complexity of the theory. It is useful to test and illustrate it in a simple context. This is what we do in this paper. We consider riemannian general relativity in three dimensions. Since the theory is topological, the integral (1) is trivial. To further simplify the context, we triangulate spacetime, reducing the field variables to a finite number [13, 14, 15]. Furthermore, we take a 'minimalist' triangulation: a single tetrahedron with four equal edges. In this way the number of variables we deal with is reduced to a bare minimum. The result is an extremely simple system, which, nevertheless, is sufficient to realize the conceptual complexity of a background independent theory of spacetime geometry.

We show that this simple system has in fact a background independent classical and quantum dynamics. The classical dynamics is governed by the relativistic Hamilton function [1], the quantum dynamics is governed by the relativistic propagator (1). We compute both these functions explicitly. The classical dynamics, which is equivalent to the Einstein equations, fixes relations between quantities that can be measured on the boundary of the tetrahedron. The quantum dynamics gives probability amplitudes for ensembles of boundary measurements.

The model and its interpretation are well-defined with no need of picking a particular variable as a time variable. However, we can also identify an elapsed proper "time" T among the boundary variables, and reinterpret the background independent theory as a theory describing evolution in the observable time T (observables in the sense of [16], see [1].) We describe the two (equivalent) interpretations of the model, in the classical as well in the quantum theory. Furthermore, we concretely illustrate the distinction between the nonperturbative vacuum state and the "Minkowski" vacuum that minimizes the energy associated with the evolution in T, and we show that the technique suggested in [5] for computing the Minkowski vacuum state from the nonperturbative vacuum state works in this context.

Thus, the system captures the essence of background independent physics in a nutshell.

The classical theory is discussed in Section 2; classical time evolution in Section 3; the quantum theory in Section 4; quantum time evolution in Section 5. As a preliminary step, we describe below the geometry of an equilateral tetrahedron.
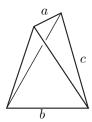
Figure 1: The equilateral tetrahedron.

1.1 Elementary geometry of an equilateral tetrahedron

Consider a tetrahedron immersed in euclidean three-dimensional space. Let a be the length of one of the edges (we call it the "top" edge) and b the length of the opposite ("bottom") edge, namely the edge disjoint from the top edge. Assume that the other four ("side") edges have equal length c. See Figure 1. We call such a tetrahedron "equilateral". We call $\theta_a, \theta_b, \theta_c$, the (respectively "bottom", "top" and "side") dihedral angles at the edges (with length) a, b, c. Elementary geometry gives

$$\sin\frac{\theta_a}{2} = \frac{b}{\sqrt{4c^2 - a^2}}, \qquad \sin\frac{\theta_b}{2} = \frac{a}{\sqrt{4c^2 - b^2}}, \qquad \cos\theta_c = \frac{ab}{\sqrt{(4c^2 - a^2)(4c^2 - b^2)}}. \tag{2}$$

(The last equation can be easily obtained from the scalar product of the normals to two adjacent triangles, by working in the orthonormal basis determined by the top and bottom edges, and the tetrahedron axis.) It follows from (2) that

$$\cos\theta_c = \sin\frac{\theta_a}{2} \sin\frac{\theta_b}{2}. \tag{3}$$

For later purpose, we consider also the case in which $c \gg a, b$. In this case, we have, to the first relevant order,

$$\theta_a = \frac{b}{c}, \qquad \theta_b = \frac{a}{c}, \qquad \theta_c = \frac{\pi}{2} - \frac{ab}{4c^2} \tag{4}$$

and

$$\theta_c = \frac{\pi}{2} - \frac{\theta_a\theta_b}{4}. \tag{5}$$

We consider also the three external angles at the edges

$$k_a(a, b, c) = \pi - \theta_a(a, b, c), \qquad k_b(a, b, c) = \pi - \theta_b(a, b, c), \qquad k_c(a, b, c) = \pi - \theta_c(a, b, c). \tag{6}$$

Notice that they express the discretized extrinsic curvature of the surface of the tetrahedron. This is why we have denoted them with the letter k, often used for the extrinsic curvature. Using (2) and (6), the relation between the edge lengths a, b, c and the external angles k_a, k_b, k_c can be written in the form

$$\begin{aligned}
a &= \sqrt{4c^2 - b^2} \, \cos\frac{k_b}{2}, \\
b &= \sqrt{4c^2 - a^2} \, \cos\frac{k_a}{2}, \\
ab &= -\sqrt{(4c^2 - a^2)(4c^2 - b^2)} \, \cos k_c \, ;
\end{aligned} \tag{7}$$

while (3) reads

$$\cos k_c = -\cos\frac{k_a}{2} \, \cos\frac{k_b}{2}. \tag{8}$$

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2 Classical theory

2.1 Regge action

Consider the action of general relativity, in the case of a simply connected finite spacetime region $\mathcal{R}$. Recall that in the presence of a boundary $\Sigma = \partial\mathcal{R}$ we have to add a boundary term to the Einstein–Hilbert action, in order to have well defined equations of motion. The full action reads

$$S_{\mathrm{GR}}[g] = \int_{\mathcal{R}} d^n x \sqrt{\det g}\ R + \int_{\Sigma} d^{n-1} x \sqrt{\det q}\ k. \tag{9}$$

Here g is the metric field, R is the Ricci scalar, n is the number of spacetime dimensions, while q is the metric, and k the trace of the extrinsic curvature, induced by g on Σ. (For a discussion on different choices of boundary terms in three-dimensional gravity, see [18]; here we are interested in variations of the action at fixed value of the boundary metric.)

In general, the *Hamilton function* of a finite dimensional dynamical system is the value of the action of a solution of the equations of motion, viewed as a function of the initial and final coordinates; the general solution of the equations of motion can be obtained from the Hamilton function by simple derivations [19]. In field theory, the Hamilton function can be defined as the value of the action of a solution of the equations of motion, integrated on a finite region $\mathcal{R}$, viewed as a function of value of the field on the boundary Σ (see [1]). In general relativity, the Hamilton function $S[q]$ is the value of the action (9), computed on the solution g_q of the equations of motion determined by the boundary value q

$$S[q] = S_{\mathrm{GR}}[g_q]. \tag{10}$$

If g_q is not unique, $S[q]$ is multivalued. Notice that $S[q]$ is independent from (local deformations of) Σ, because of diffeomorphism invariance.

Since the bulk action vanishes on a vacuum solution of the equations of motion, the Hamilton function of general relativity reads

$$S[q] = \int_{\Sigma} d^{n-1} x\ \sqrt{\det q}\ k[q]. \tag{11}$$

where the extrinsic curvature $k[q]$ is a nonlocal function, determined by the Ricci-flat metric g_q bounded by q.

In the following we consider only the three-dimensional riemannian case, where $n = 3$ and the signature of g is $[+++]$. In this case, we must add an overall minus sign in (9) and (11), see for instance the Appendix C of reference [17]. Furthermore, we consider the discretization of the theory provided by a Regge triangulation [13]. Let i be the index labelling the links of the triangulation and call l_i the length of the link i. In three dimensions, the bulk Regge action is

$$S_{\mathrm{Regge}}(l_i) = -\sum_i l_i \left(2\pi - \sum_t \theta_{i,t}(l) \right), \tag{12}$$

where $\theta_{i,t}(l)$ is the dihedral angle of the tetrahedron t at the link i, and the angle in the parenthesis is therefore the deficit angle at i. The boundary term is

$$S_{\mathrm{boundary}}(l_i) = - \sum_{\mathrm{boundary}\ i} l_i \left(\pi - \sum_t \theta_{i,t}(l) \right). \tag{13}$$

Notice that the angle in the parenthesis is the angle formed by the boundary, which can be seen as a discretization of the extrinsic curvature.

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We choose the minimalist triangulation formed by a single tetrahedron, and, furthermore, consider only the case in which the tetrahedron is equilateral. Then there are no internal links, the Regge action is the same as the Regge Hamilton function, and is given by

$$S(a, b, c) = -a\, k_a(a, b, c) - b\, k_b(a, b, c) - 4c\, k_c(a, b, c). \tag{14}$$

The expression for the dihedral angles as functions of the edges length, for a flat interior geometry, is given in (2) and (6). Inserting these equations into (14) gives the Hamilton function

$$
\begin{aligned}
S(a, b, c) &= a\left(2\arcsin\frac{b}{\sqrt{4c^2 - a^2}} - \pi\right) + b\left(2\arcsin\frac{a}{\sqrt{4c^2 - b^2}} - \pi\right) \\
&\quad + 4c\left(\arccos\frac{ab}{\sqrt{4c^2 - a^2}\,\sqrt{4c^2 - b^2}} - \pi\right).
\end{aligned}
\tag{15}
$$

2.2 The dynamical model and its physical meaning

The Hamilton function (15) defines a simple relativistic dynamical model. The model has three variables, a, b and c. These are partial observables in the sense of [16]. That is, they include both the independent ("time") and the dependent (dynamical) variables, all treated on equal footing. (This paper is self contained, but the general formalism and the interpretation of these general relativistic systems is discussed in detail in [1].)

The equations of motion are obtained following the general algorithm of the relativistic Hamilton–Jacobi theory [1]: define the momenta

$$p_a(a, b, c) = \frac{\partial S(a, b, c)}{\partial a}, \qquad p_b(a, b, c) = \frac{\partial S(a, b, c)}{\partial b}, \qquad p_c(a, b, c) = \frac{\partial S(a, b, c)}{\partial c}, \tag{16}$$

and equate them to constants

$$p_a(a, b, c) = p_a, \qquad p_b(a, b, c) = p_b, \qquad p_c(a, b, c) = p_c. \tag{17}$$

These equations give the dynamics, namely the solution of the equations of motion. Explicitly, the calculation of the momenta is simplified by the observation that the action is a homogeneous function of degree one, hence

$$S(a, b, c) = a\,\frac{\partial S(a, b, c)}{\partial a} + b\,\frac{\partial S(a, b, c)}{\partial b} + c\,\frac{\partial S(a, b, c)}{\partial c}; \tag{18}$$

this allows us to identify immediately

$$p_a(a, b, c) = -k_a(a, b, c), \qquad p_b(a, b, c) = -k_b(a, b, c), \qquad p_c(a, b, c) = -4\,k_c(a, b, c). \tag{19}$$

Inserting the explicit form (2) of the angles, we obtain the evolution equations

$$
\begin{aligned}
a &= \sqrt{4c^2 - b^2}\,\cos\frac{p_b}{2}, \\
b &= \sqrt{4c^2 - a^2}\,\cos\frac{p_a}{2}, \\
ab &= -\sqrt{(4c^2 - a^2)(4c^2 - b^2)}\,\cos\frac{p_c}{4},
\end{aligned}
\tag{20}
$$

which reproduce (7). This result deserves various comments.

(i) We begin with a technical comment. Notice that the variation of the action with respect to the lengths is completely determined by the variation of the first length factor in (13): the variation of the length in the argument of the angles has no effect on the action. The fact that this variation vanishes was already pointed out by Regge [13]. It is the discrete analog of the well-known fact that in deriving the Einstein equations from the Einstein–Hilbert action we can ignore the change of the Levi–Civita connection under a variation of the metric.

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(ii) Notice that boundary lengths a, b, c determine the intrinsic geometry of the boundary surface. Their conjugate momenta p_a, p_b, p_c, are determined by the dihedral angles and are given by the external angles at the links. That is, they measure the extrinsic curvature of the boundary surface. This is precisely as in the ADM hamiltonian framework [20], where the momentum variable conjugate to the metric is the extrinsic curvature. Equation (19) is the discrete analog of the ADM relation between momenta and extrinsic curvature.

(iii) The evolution equations (20) are not independent, as is always the case in relativistic systems (for instance, out of the four equations of motion of a relativistic particle, only three are independent). We can take the first two equations as the independent ones. They express relations between the lengths and dihedral angles of the tetrahedron.

(iv) How are the evolution equations (20) related to the Einstein equations? They are essentially equivalent. In three dimensions, the vacuum Einstein equations $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor, imply that the Riemann tensor vanishes, namely that spacetime is flat. This implies that the tetrahedron is immersed in a *flat* 3d spacetime. But if spacetime is flat, the extrinsic curvature of the boundary at the edge is exactly equal to π minus the dihedral angle. Hence these equations express the flatness of spacetime, namely they have the same content as the Einstein equations $R_{\mu\nu} = 0$. In other words, we have derived the relation (2) between length and angles assuming a *flat* 3d space: viceversa, the fact that these relations are satisfied implies that, in the approximation captured by the triangulation, 3d space is flat, namely the Einstein equations hold.

(v) The physical interpretation of the model is as follows. We assume that we can measure the three lengths a, b and c and the three external angles k_a, k_b and k_c (these are six partial observables in the sense of [16]). These are all local observations that can be made on the boundary surface. They refer to the intrinsic as well as the extrinsic geometry of the surface itself. The classical theory establishes relations between these measurable quantities. These relations are the physical content of the theory and are given by the equations (20). They are equivalent to the statement that spacetime is flat (to the given approximation).

(vi) The fact that the equations of motion are not independent is reflected in a relation between the momenta. The relation is of course the one given by equation (8), that is

$$H(p_a, p_b, p_c) = \cos\frac{p_c}{4} + \cos\frac{p_a}{2} \ \cos\frac{p_b}{2} = 0. \tag{21}$$

From this we can directly read out the Hamilton–Jacobi equation satisfied by $S(a, b, c)$.

$$\cos\frac{1}{4}\frac{\partial S}{\partial c} + \cos\frac{1}{2}\frac{\partial S}{\partial a} \ \cos\frac{1}{2}\frac{\partial S}{\partial b} \ = \ 0. \tag{22}$$

The function $H(p_a, p_b, p_c)$ given in (21) is the *relativistic hamiltonian* [1], or hamiltonian constraint, of the system.

(vii) Finally, in the limit in which $c \gg a, b$, the action is given simply by

$$S(a, b, c) = \frac{ab}{c} - (a + b + 2c)\,\pi, \tag{23}$$

and the evolution equations (20) become

$$a = c\,(p_b + \pi), \qquad b = c\,(p_a + \pi), \qquad ab = -c^2\,(p_c + 2\pi). \tag{24}$$

3   Time evolution

In the description given so far, no reference to evolution in a preferred time variable was considered. We now introduce it here. We decide to regard the direction of the axis of the equilateral tetrahedron as a temporal direction. In particular, we decide to interpret b as an initial variable and a as a final variable (b for *before* and a for *after*). The length c of the side links can then be regarded as a (proper!) length measured in the temporal direction, namely as the physical time elapsed from the measurement of a to the measurement of b. Indeed, had we considered a spacetime with signature $[+ + -]$, and assuming we had oriented the tetrahedron axis in a timelike direction, c would *precisely* be the physical time measured by a real clock on the boundary of the spatial region considered, the worldline of the clock running along one of the side edges.

To emphasize this interpretation of the variable c, in this section we change its name, renaming c as T. The Hamilton function reads then $S(a, b, T)$ and can now be interpreted as the Hamilton function that determines the evolution in T of a variable a. The variable b is interpreted as measured at time $T = 0$ and the variable a at time T; therefore b can be viewed as an integration constant for the evolution of a in T. (Notice that b is not necessarily the same variable as a, namely $T = 0$ does not imply $a = b$).

For comparison, recall that the Hamilton function of a free particle moving from a position b to a position a in a time T is

$$S_{\text{free particle}}(a, b, T) = \frac{m(a - b)^2}{2T} \tag{25}$$

which completely describes the free particle dynamics: equations (17) give in fact

$$p_a(a, b, T) \quad = \quad \frac{\partial S(a, b, T)}{\partial a} = m\frac{a - b}{T} = p_a, \tag{26}$$

$$p_b(a, b, T) \quad = \quad \frac{\partial S(a, b, b)}{\partial a} = m\frac{b - a}{T} = p_b, \tag{27}$$

$$p_c(a, b, T) \quad = \quad \frac{\partial S(a, b, T)}{\partial T} = -\frac{m(a - b)^2}{2T^2} = p_T, \tag{28}$$

which can be readily recognized as the evolution equation for coordinate and momentum

$$a(T) = a_0 + VT, \qquad p_a(T) = mV, \tag{29}$$

where $a_0 = b$ and $V = -p_b/m$, and the relation between energy ($E \equiv -p_T$) and momentum

$$E = H(p_b) = \frac{p_b^2}{2m}, \tag{30}$$

which defines the hamiltonian function $H(p_b)$.

Returning to our system, the hamiltonian that evolves the system in the time T, which we can call "proper-time hamiltonian", can obtained from the energy

$$E = -p_T = -\frac{\partial S(a, b, T)}{\partial T} = 4\,\pi - 4\,\arccos\frac{ab}{\sqrt{(4T^2 - a^2)(4T^2 - b^2)}} \tag{31}$$

by using the equations of motion to express the initial position b as a function of the position a and momentum p_a. This gives

$$H(a, p_a, T) \quad = \quad 4\,\pi - 4\arccos\left(\frac{a\cos(p_a/2)}{\sqrt{4T^2 - (4T^2 - a^2)\cos^2(p_a/2)}}\right) \tag{32}$$

Notice that the angle θ_c can vary between 0 and $\pi/2$, and therefore so does the arccos. Therefore the energy can vary between 2π and 4π. The fact that the domain of the energy is bounded has important consequences. For instance, we should expect time to become discrete in the quantum theory.

Figure 2: For large T, at constant b and θ_b, we have $\theta_a \to 0$ and $a \sim T$.

In this way, the relativistic background independent system can be reinterpreted as an evolution system, where the "proper time" on the boundary of the region of interest is taken as the independent time variable. The Hamilton equation generated by the hamiltonian for $a(T)$ and $p_a(T)$ are:

$$
\begin{aligned}
\frac{da(T)}{dT} = \frac{\partial H}{\partial p_a} &= \frac{4aT}{4T^2 \sin^2(p_a/2) + a^2 \cos^2(p_a/2)}, \\
\frac{dp_a(T)}{dT} = -\frac{\partial H}{\partial a} &= -\frac{4T \sin(p_a)}{4T^2 \sin^2(p_a/2) + a^2 \cos(p_a/2)}.
\end{aligned}
\tag{33}
$$

The solution of these equations is

$$
\begin{aligned}
a(T) &= \sqrt{4T^2 - b^2} \, \cos\frac{p_b}{2}, \\
p_a(T) &= -2\arccos\frac{b}{\sqrt{4T^2 \sin^2(p_b/2) + b^2 \cos^2(p_b/2)}},
\end{aligned}
\tag{34}
$$

where b and p_b are integration constants. These solutions are immediately recognized as the equations (7). Therefore the dynamics generated by the hamiltonian is the same as the general relativistic dynamics defined in a-temporal terms in the previous section.

It is interesting to consider the long time evolution of the system. In the large T limit we have the behavior

$$
a(T) \to \text{const } T, \qquad p_a(T) \to \frac{\text{const}}{T} - \pi,
\tag{35}
$$

which is precisely (4), identifying the two integration constants with the initial data θ_a and a. Therefore $p_a(T)$ tends to $-\pi$ as T increases. It is easy to understand this behavior geometrically. See Figure 2: at fixed values of the bottom length b and bottom angle $\theta_b = \pi + p_b$, as the side length T grows, we have that the top angle $\theta_a = \pi + p_a \to 0$ and a grows proportionally to T.

The energy is not constant (there is no reason for the energy to be constant) and tends to

$$
E(T) \to 2\pi
\tag{36}
$$

which is its minimal value. This result can also be obtained by considering the hamiltonian for large T. Starting from (23), we obtain

$$
H = -\frac{\partial S(a, b, T)}{\partial T} = \frac{ab}{T^2} + 2\pi = \frac{a(\pi - p_a)}{T} + 2\pi.
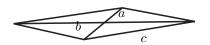\tag{37}
$$

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Figure 3: The flat tetrahedron: the bottom and top edges touch.

The equations of motion

$$\frac{da(T)}{dT} = \frac{\partial H}{\partial p_a} = \frac{a}{T}, \qquad\qquad \frac{dp_a(T)}{dT} = -\frac{\partial H}{\partial a} = -\frac{\pi + p_a}{T}, \qquad (38)$$

are solved by (35) and yield (36).

Notice that the convergence of the "velocity" to the attraction point $p_a \to -\pi$ and the energy to its minimal value, resembles a dissipative system, such as a point particle under a constant force in a fluid.

3.1 Phase space and extremal configurations

Viewed as a dynamical system evolving in T, our system has a phase space coordinatized by $a \in [0, \infty[$ and $p_a \in [0, -\pi]$. The maximum value of the energy (32) on this phase space is $E_{min} = 4\pi$, which is attained along the boundary $p_a = 0$ of Γ. These are states with vanishing external angle at the top edge. They are configuration in which the tetrahedron is "flattened": its volume is zero, and the upper and bottom edges touch. The value of a is arbitrary. See Figure 3. Notice that these configurations evolve into one another. In fact, if $p_a = 0$, (34) gives

$$\begin{aligned} a(T) &= \sqrt{4T^2 - b^2}, \\ p_a(T) &= 0. \end{aligned} \qquad (39)$$

Therefore these states grow in T remaining flattened and with the the energy remaining constant in T at the value $E = 4\pi$.

In all the other states, the energy changes with time. As T grows a generic state evolves towards a state of the form

$$\begin{aligned} a &= 2T \, \cos\frac{p_b}{2}, \\ p_a &= -\pi, \end{aligned} \qquad (40)$$

with the energy converging to the value $E = 2\pi$. These states minimize the energy and form the boundary $p_a = -\pi$ of Γ. We call these states "Minkowski vacuum states", since they minimize the energy. Notice that their definition depends on the choice of the time variables made.

Therefore the 2d phase space has two notable subsets: the line $p_a = 0$ forms an independent sector evolving into itself, given by the energy-maximizing states; while the line $p_a = -\pi$ is an attractor for the rest of the phase space, and is formed by the energy-minimizing states that we have called "Minkowski" states. See Figure 4

Notice that the variable T is bounded by $|T| > b/2$ from (34), therefore we cannot continue the solution for arbitrarily small T. It is natural to introduce the time variable

$$t = \sqrt{T^2 - b^2/4}, \text{ for } T > b/2 \qquad (41)$$

which geometrically represents the height of the triangular face of the tetrahedron with base b, and
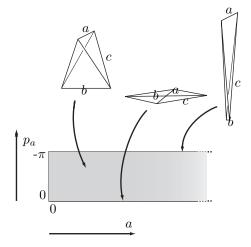
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Figure 4: The phase space of the system with some typical configurations. The "Minkowski" states are the ones along the $p_a = -\pi$ boundary.

which arrives at zero. The evolution equations read then

$$
\begin{aligned}
a(T) &= 2t \cos \frac{p_b}{2}, \\
p_a(T) &= -2 \arccos \frac{b}{\sqrt{4t^2 \sin^2(p_b/2) + b^2}}.
\end{aligned}
\tag{42}
$$

Notice that the equations of motions can be extended also for *negative t* and *negative a* and p_a. It is natural to interpret this as an evolution in which the tetrahedron crosses the point $a = 0, p_a = 0$ in which it has zero volume, and grows "on the other side", overturned as a glove. See Figure 5. If we consider this extension, we can take the phase space to be given by $a \in R$ and $p_a \in [-\pi, \pi]$. In the following, we will not consider this extension.
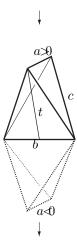

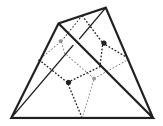Figure 5: The extension to negative a and negative t.

Figure 6: The tetrahedrons T (continuous lines) and T^* (pointed lines).

4 Quantum theory

4.1 Kinematics

We begin by constructing the boundary Hilbert space $\mathcal{K}$, on which the operators representing the boundary (partial) observables are defined. Consider the triad formalism for three-dimensional euclidean general relativity. The variables are a the triad $e^i_\mu(x)$, $\mu = 1, 2, 3$, $i = 1, 2, 3$ and its $SO(3)$ spin connection $A^i_\mu(x)$. The canonical boundary variables can be taken to be $SO(3)$ connection $A^i_a(x)$ $a = 1, 2$ and the inverse densitized triad $E^a_i(x)$ induced on the boundary surface. Let us then discretize spacetime in terms of a single tetrahedron T.

Call f^p, $p = 1, 2, 3, 4$, the faces of the tetrahedron, e^{pq} the oriented edge separating the face p from the face q (say oriented rightward in going from p to q). To define the discrete dynamical variables, consider the dual tetrahedron T^* defined by vertices v_p in the face f^p of T. The edges e_{pq} of T^* connect the vertex p to the vertex q; they are dual, and cut the corresponding edges e^{pq} of T. We can discretize the boundary field $A^i_a(x)$ by replacing it with six group elements U_{pq} associated to the six edges e_{pq}, interpreted as the parallel transport matrix of the connection along e_{pq}. As usual in quantum gravity, we take $U_{pq} \in SU(2)$ (the classical theory is determined by the algebra, not the group). We write $U_{pq} = U_{qp}^{-1}$. Gauge transformations act on the vertices v_p; they are determined by four group elements V_p and the group elements U_{pq} transform as

$$U_{pq} \;\rightarrow\; V_p U_{pq} V_q^{-1}. \tag{43}$$

The quantum theory can be defined starting from the Hilbert space $\mathcal{K}$ of the Haar-square-integrable functions $\psi(U_{pq})$ of the six dynamical variables U_{pq} that are gauge-invariant under the transformations (43), namely

$$\psi(U_{pq}) = \psi(V_p U_{pq} V_q^{-1}). \tag{44}$$

These gauge transformations depend on four group elements, therefore $\mathcal{K} = \mathcal{L}_2[(SU(2))^6/(SU(2))^4]$ where the action of $(SU(2))^4$ on $(SU(2))^6$ is the one given in (43). We use the notation $\mathbf{U} = (U_{pq})$ for the 6-tuplet of group elements, and thus write states as $\psi(\mathbf{U})$. Similarly, we indicate an 6-tuplet of spins as $\mathbf{j} = (j_{pq})$. As well known (see for instance [1]), a basis in $\mathcal{K}$ is given by the spin-network states

$$\begin{aligned}
\psi_{\mathbf{j}}(\mathbf{U}) \;=\; & \langle \mathbf{U} | \mathbf{j} \rangle \\
=\; & R^{j_{12}}_{io}(U_{12})\, R^{j_{13}}_{jp}(U_{13})\, R^{j_{14}}_{kq}(U_{14})\, R^{j_{23}}_{lr}(U_{23})\, R^{j_{24}}_{ms}(U_{24})\, R^{j_{34}}_{nt}(U_{34})\; v^{ijk}\, v^{olm}\, v^{prn}\, v^{qst},
\end{aligned} \tag{45}$$

where the R^j_{kl} are the matrix elements of the $SU(2)$ representation j and v^{ikl} are the normalized invariant tensors. The index structure of equation (45) is determined by the geometry of the tetrahedron. The function $\psi_{\mathbf{j}}(\mathbf{U})$ is the spin-network function for a spin network having T^* as graph. (See [1] for details.)

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The left invariant vector field on each group can be identified as the operator associated to the triad field integrated along the edges of T. The integral of the $SU(2)$-norm of these gives the length of the edge; therefore the Casimir operators C_{pq} of the (pq)-th group

$$C_{pq} \, |\mathbf{j}\rangle = j_{pq}(j_{pq} + 1) \, |\mathbf{j}\rangle \tag{46}$$

can then be naturally identified as the operator giving the length square of the edge e^{pq} [21]. The tensor structure of the algebra of the $SU(2)$ representations implements the triangular relations satisfied by the length. The spectrum of the length of the edges e_{pq} is therefore given by

$$l_{pq} = \sqrt{j_{pq}(j_{pq} + 1)}. \tag{47}$$

The fact that the lengths have discrete spectrum is an immediate consequence of their conjugate variables being angles, and thus vary on a compact domain. Following [1], we can interpret the spectral properties of the partial observables as physical predictions of the quantum model.

4.2 Dynamics

The quantum dynamics is completely captured by the propagator [6]. In a general relativistic theory, the propagator is formally expressed as the function of the boundary variables given by (1). Recall, however, that in general the propagator is *not* a function of classical boundary variables; the reason is that the boundary quantities may fail to have continuous spectrum. If they have discrete spectrum, the propagator depends on the *quantum numbers* that label the discrete eigenvectors of the boundary quantities, and not on the corresponding continuous classical variables [1]. In our case, the propagator can be written in the basis (45), where it will be a function $W(j_{pq})$.

To find this function, recall that the classical dynamics requires three-dimensional space to be flat. This means that any parallel transport along a three-dimensional closed path must be trivial. Consider the four "elementary" closed paths γ_p on T^*, where γ_4 is defined by the sequence of edges $e_{12}e_{23}e_{31}$, that circle the vertex opposite to the face f^4, and so on. The flatness requirement is expressed by the parallel transport around each of these paths being trivial

$$U_{12}U_{23}U_{31} = \mathbf{1}, \tag{48}$$

and similarly for the other three. Therefore we can write the 3d flatness requirement (the Einstein equations) in the form

$$(U_{pq}U_{qr}U_{rp} - \mathbf{1}) = 0 \tag{49}$$

where $p \neq q \neq r$. We then write the main dynamical equation of the quantum theory in the form

$$(U_{pq}U_{qr}U_{rp} - \mathbf{1}) \, \psi_0(\mathbf{U}) = 0, \tag{50}$$

which can be interpreted as a Wheeler-D toeWitt equation. Its general solution is

$$\psi_0(\mathbf{U}) = f(\mathbf{U}) \prod_{pqr} \delta(U_{pq}U_{qr}U_{rp}), \tag{51}$$

where the delta function is the one on the group (for the Haar measure) and $f(\mathbf{U})$ is an arbitrary gauge invariant function. This equation defines the physical states ψ_0 that solve the dynamics of the theory. To express these states in the $\mathbf{j}$ basis, we simply project them on the basis states (45)

$$\psi_0(\mathbf{j}) = \int d\mathbf{U} \, \bar{\psi}_{\mathbf{j}}(\mathbf{U}) \, f(\mathbf{U}) \prod_{pqr} \delta(U_{pq}U_{qr}U_{rp}). \tag{52}$$

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It is easy to see that by gauge invariance, we can gauge fix *all* U_{pq} to unity in the integral, giving

$$
\begin{aligned}
\psi_0(\mathbf{j}) &= \int d\mathbf{U} \ \bar{\psi}_{\mathbf{j}}(\mathbf{U}) \ f(\mathbf{U}) \ \prod_{pq} \delta(U_{pq}) \\
&= c \ \psi_{\mathbf{j}}(1) \\
&= c \ \delta_{io}\delta_{jp}\delta_{kq}\delta_{lr}\delta_{ms}\delta_{nt} \ v^{ijk} \ v^{olm} \ v^{prn} \ v^{qst}.
\end{aligned}
\tag{53}
$$

The constant $c = f(\mathbf{1})$ can be absorbed in the normalization. The last line is the definition of the Wigner 6-j symbol, usually written as

$$
\psi_0(\mathbf{j}) = \left(\begin{array}{ccc} j_{12} & j_{13} & j_{14} \\ j_{34} & j_{24} & j_{23} \end{array} \right) \equiv \langle \mathbf{j}|0\rangle.
\tag{54}
$$

Thus, we conclude that there is a single state $|0\rangle$ (up to normalization) in $\mathcal{K}$ that solves the dynamics, and that this state is proportional to the Wigner 6-j symbol.

The physical amplitude of an arbitrary kinematical state $\psi \in \mathcal{H}$ is determined by its projection on the state that solve the dynamical equation, namely by its projection on the state $|0\rangle$

$$
A(\psi) = \langle 0|\psi\rangle.
\tag{55}
$$

The state $|0\rangle$ is called the "non-perturbative" vacuum state [1]. It expresses the dynamics of the theory. In other words, the physical amplitude for having the boundary configuration j_{pq} is the Wigner 6-j symbol. Namely the propagator of the theory is Wigner 6-j symbol

$$
W(\mathbf{j}) \equiv \langle \mathbf{j}|0\rangle = \left(\begin{array}{ccc} j_{12} & j_{13} & j_{14} \\ j_{34} & j_{24} & j_{23} \end{array} \right).
\tag{56}
$$

Now, this is precisely the result obtained by Ponzano and Regge on the basis of a physical ansatz on the discretization of the lengths, and a discretization of the Einstein-Hilbert action [22] ! (In our minimalist model, the functional integral (1) is trivial because there are no bulk degrees of freedom. Its result is therefore proportional to the exponential of the action. Ponzano and Regge found that the Wigner 6-j symbol (56) can in fact be viewed as a discretization of (the real part) of the exponential of the action.) The result is also equivalent (up to a phase) to the specialization to a single tetrahedron of the boundary amplitude computed in [14] and in [18]. In the present case, the discretization of the length is not introduced as an ansatz, but it is a standard quantum-mechanical consequence of the conjugate variable being an angle.

4.3   Quantum equilateral tetrahedron

So far, we have considered an arbitrary quantum tetrahedron. We now specialize the formalism to the case of an equilateral tetrahedron. The simplest way to do so is to restrict our attention to the states where four of the six edge lengths are equal. More precisely, we put

$$
\begin{aligned}
j_a &\equiv j_{13}, \\
j_b &\equiv j_{24}, \\
j_c &\equiv j_{12} = j_{23} = j_{34} = j_{41}
\end{aligned}
\tag{57}
$$

and we consider only the states

$$
|j_a, j_b, j_c\rangle = |j_c, j_a, j_c, j_c, j_b, j_c\rangle.
\tag{58}
$$

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Accordingly, we we restrict the states $\psi(U_{pq})$ to the subset of $(SU(2))^6$ determined by $U_{12} = U_{23} = U_{34} = U_{41}$. We write

$$\begin{aligned}
U_a &\equiv U_{13}, \\
U_b &\equiv U_{24}, \\
U_c &\equiv U_{12} = U_{23} = U_{34} = U_{41}.
\end{aligned} \tag{59}$$

The gauge transformations that preserve the resulting subspace are the ones for which

$$\begin{aligned}
V_1 = V_3 &\equiv V_a, \\
V_2 = V_4 &\equiv V_b,
\end{aligned} \tag{60}$$

under which the states $\psi(U_a, U_b, U_c)$ transform as

$$\psi(U_a, U_b, U_c) \rightarrow \psi(V_a U_a V_a^{-1}, V_b U_b V_b^{-1}, V_a U_c V_b^{-1}). \tag{61}$$

Using these gauge transformations, we can transform U_a, U_b, U_c to three rotation around three orthogonal axis, of three angles k_a, k_b, k_c. The interpretation of these angles is simple. Since spacetime is flat, we can choose the gauge in which the internal space is directly identified with flat spacetime. Then the rotation along the edge e_{pq} of T^* can be identified as the physical rotations that one undergoes in crossing the edge e^{pq} of T. These are precisely the external angles that were denoted k_a, k_b, k_c in the previous section. For an $SU(2)$ matrix, $\text{Tr}(U) = 2\cos(\phi/2)$, where ϕ is the rotation angle. Therefore we can consider the operator

$$T_a \equiv \text{Tr}(U_a) = 2\cos(k_a/2) = 2\cos(p_a/2), \tag{62}$$

which is now gauge invariant. The action of this operators is easily obtained from $SU(2)$ representation theory:

$$T_a \,|j_a, j_b, j_c\rangle = |j_a + 1/2, j_b, j_c\rangle + |j_a - 1/2, j_b, j_c\rangle, \tag{63}$$

and similarly for the other edges. In the next section we will show that the commutator between this operator and the length reproduces the classical Poisson brackets.

In summary, the boundary Hilbert state $\mathcal{K}$ is spanned by the states $|j_a, j_b, j_c\rangle$. The boundary observables a, b, c, p_a, p_b, p_c that measure the length of the edges of the tetrahedron and the external angles are represented by Casimir and trace operators, and the dynamics is given by the propagator

$$W(j_a, j_b, j_c) = \left(\begin{array}{ccc} j_a & j_c & j_c \\ j_b & j_c & j_c \end{array} \right), \tag{64}$$

which expresses the probability amplitude of measuring the lengths determined by j_a, j_b, j_c. This concludes the definition of the quantum theory. The predictions of the theory are given by the quantization of the lengths and by the relative probability amplitude (64).

5 Time evolution in the quantum theory

So far, we have viewed our system as a general relativistic system, in which predictions are expressed in terms of (probabilistic) relations between boundary partial observables, or probability amplitudes for boundary configurations. We now reinterpret the system as a system evolving in a time variable, as we did in the classical case. Thus, we see, say b and p_b as initial variables, a and p_a as final variables, and j_c at a time parameter. We must identify the Hilbert space of the system at fixed time.

Let us focus on the final state. This is described by the operators C_a and T_a that act on the variable U_a. The boundary Hilbert state $\mathcal{K}$, spanned by the states $|j_a, j_b, j_c\rangle$, can be decomposed as (a

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subspace, because of the Clebsch–Gordan relations of) the tensor product of three spaces $\mathcal{K}_a, \mathcal{K}_b, \mathcal{K}_c$ spanned by states $|j_a\rangle, |j_b\rangle, |j_c\rangle$ respectively. Let us focus on $\mathcal{K}_a$, which can be interpreted as the state space at fixed time.

$\mathcal{K}_a$ can be simply expressed as the space of the class functions $\psi(U_a)$, that is, the functions satisfying

$$\psi(U_a) = \psi(V_a U_a V_a^{-1}). \tag{65}$$

The basis $|j_a\rangle$ is defined by the characters

$$\langle U_a | j_a \rangle = \chi_{j_a}(U_a) = \frac{\sin((j_a + 1/2)U_a)}{\sin(U_a/2)}. \tag{66}$$

The Casimir and trace operators act as

$$C_a |j_a\rangle = j_a(j_a + 1)|j_a\rangle \tag{67}$$
$$T_a |j_a\rangle = |j_a + 1/2\rangle + |j_a - 1/2\rangle, \tag{68}$$

where the second relation is easily derived from the properties of the characters. It is convenient to define also the operator

$$S_a |j_a\rangle = i(|j_a + 1/2\rangle - |j_a - 1/2\rangle). \tag{69}$$

that satisfies $T_a^2 + S_a^2 = 4$, and is therefore a function of T_a

$$S_a = \sqrt{4 - T_a^2}. \tag{70}$$

Since we have identified T_a with $2\cos(p_a/2)$, it follows that we must identify S_a with $2\sin(p_a/2)$. The classical Poisson brackets

$$\{a, p_a\} = 1 \tag{71}$$

gives, for $T_a = 2\cos(p_a/2)$,

$$\{a, T_a\} = \sin(p_a/2) = 1/2\sqrt{4 - T_a^2} \tag{72}$$

Consider the operator J_a defined by $C_a = J_a(J_a + 1)$ and acting as

$$J_a |j_a\rangle = j_a |j_a\rangle. \tag{73}$$

A straightforward calculation gives

$$[J_a, T_a] = i/2S_a = i/2\sqrt{4 - T_a^2}. \tag{74}$$

Therefore we see that the operators J_a and T_a define a linear representation of the classical Poisson algebra defined by the observables a and $2\cos(p_a/2)$. We have then two options. The first is to identify the classical quantity a with the operator J_a. The second is to identify a with the square root of the Casimir. Both choices give the correct classical limit, since they become the same in the limit of large quantum numbers. The first gives a quantum theory in which the length is quantized in half-integers j_a; the second gives a quantum theory in which the length is quantized as $\sqrt{j_a(j_a + 1)}$. We identify the second choice with the quantization defined in the previous section.

A discrete time evolution is determined by the propagator (64), seen as a propagator from the state $|j_b\rangle$ to the state $|j_a\rangle$ in a (discrete) time j_c.

Recall that in the classical theory the long time evolution drives the system to the "Minkowski" configurations where $p_a = -\pi$. Let us study the quantum evolution for long times. For $j_c \to \infty$ we have [13]

$$W(j_a, j_b, j_c) = \begin{pmatrix} j_a & j_c & j_c \\ j_b & j_c & j_c \end{pmatrix} \to \frac{(-1)^{-(j_a + j_b + 2j_c)}}{2j_c}. \tag{75}$$

This can be written as

$$W(j_a, j_b, T) \to_{T \to \infty} \frac{(-1)^{-(j_a + j_b + 2T)}}{2T} = \frac{e^{-2\pi(iT)}}{2T} e^{-ij_b\pi} e^{-ij_a\pi} = \frac{e^{-iE_0T}}{2T} \psi_0(j_a)\,\psi_0(j_b). \qquad (76)$$

That is, for large T evolution projects on the (generalized) state

$$\psi_0(j_a) \equiv \langle j_a | 0_M \rangle = e^{-ij_a\pi}. \qquad (77)$$

It is easy to see that this is the generalized eigenstate of p_a with eigenvalue $-\pi$ (since p_a itself is not an operator in the theory, by this we mean, of course, a generalized eigenstate of $T_a = 2\cos(p_a/2)$ with eigenvalue $2\cos(-\pi/2) = 0$):

$$\begin{aligned} 2\cos(p_a/2)\psi_0(j_a) &= T_a e^{-ij_a\pi} = e^{-i(j_a+1/2)\pi} + e^{-i(j_a-1/2)\pi} \\ &= e^{-ij_a\pi}(e^{+i\pi/2} + e^{-i\pi/2}) = 2\cos(-\pi/2)\psi_0(j_a). \end{aligned} \qquad (78)$$

Therefore we have shown that the quantum dynamics converges to the classical dynamics on long times. It is appropriate to call $|0_M\rangle$ the "Minkowski" quantum state, since it minimizes the energy.

We have shown that the nonperturbative vacuum state $|0\rangle$ in $\mathcal{K}$ become a projector on $|0_M\rangle$ in the $T \to \infty$ limit. We can therefore write the suggestive expression

$$\lim_{j_c \to \infty} |0\rangle = |0_M\rangle\langle 0_M|. \qquad (79)$$

The bra/ket mismatch is only apparent: the l.h.s is a ket in $\mathcal{K}$, while the r.h.s. is an element of the tensor product between $\mathcal{K}_a$ and its dual, which can be identified with a subspace of $\mathcal{K}$ under

$$|j_a\rangle\langle j_b| \leftrightarrow |j_a, j_b, j_c\rangle. \qquad (80)$$

See [5] and [1] for details. Equation (79) is the expression proposed in [5] for computing the Minkowski vacuum state for spinfoam transition amplitudes. We see that in the present case this equation is correct. Notice, however, that in this euclidean context the limit is taken for *real* times.

Alternatively, we can study the continuous time evolution determined by quantizing the classical hamiltonian (32). Notice that (32) can be easily written in terms of the operators that we have defined

$$H(a, p_a, T) = 4\pi - 4\arccos\left(\sqrt{C_a}\,\frac{1}{\sqrt{4T^2 - (4T^2 - C_a)\,(T_a/2)^2}}\,(T_a/2)\right). \qquad (81)$$

Choosing this ordering (where the inverse and the arccos are defined by spectral decomposition) we have immediately that the eigenstate of the p_a with eigenvalues $-\pi$ is an eigenstates of the hamiltonian, with energy 2π, in accord with the corresponding classical result. The precise relation between the discrete time evolution defined by the propagator $W(j_a, j_b, j_c)$ and the continuum time evolution defined by the Hamiltonian will be studied elsewhere.

6 Conclusion

The model we have considered is obviously extremely simple, and we cannot derive general conclusions from its analysis. However, we think that the structure illustrated by this model does illustrate how a general covariant quantum field theory can be interpreted. Observables can be defined on a closed finite boundary. The classical dynamics can be expressed as a set of relations between these observables. In the model considered here, these are given in (20). The quantum theory can be defined in terms of a boundary Hilbert space $\mathcal{K}$, on which operators representing boundary observables are defined. The boundary observables are partial observables: they represent quantities whose measurements can

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be operationally defined in principle, but whose value cannot be predicted from the knowledge of the state, in general [16]. The spectral properties of these boundary operators are physical kinematical predictions of the theory. Dynamical predictions do not refer to values of partial observables, but rather to *relations* between these values. The quantum dynamics is captured by the nonperturbative vacuum state $|0\rangle$, or, equivalently, by the propagator (64), which is the expression of $|0\rangle$ on a basis that diagonalizes boundary observables: $W(j_a, j_b, j_c) = \langle j_a, j_b, j_c | 0 \rangle$. This state assigns a probability amplitude to any ensemble of boundary measurements.

A temporal interpretation of the model is not necessary, but it is possible [23]. By interpreting the "side" length c as a time variable, the propagator $W(j_a, j_b, j_c)$ can be interpreted as the transition probability amplitude from the initial state $|j_b\rangle$ to the final state $|j_a\rangle$ in a discrete time j_c. The energy that drives this evolution has minimum value on a state (that we have denote the "Minkowski" state) that can be obtained from the propagator: the propagator becomes proportional to a projector on this state in the large time limit.

The reader might wonder if this structure is tied to the fact that the theory we have considered here is topological. We do not think that this is the case. The fact that the theory is topological is only at the origin of the great simplicity that we have found in computing all features of the model explicitly. In particular, we have found no need of computing the functional integral (1) explicitly.

The model has some notable specific features. For instance, energy is bounded from below as well as a from above. This fact was first noticed in 3d quantum gravity by 't Hooft [24]. The consequence is that the proper time T itself is quantized, as in loop quantum cosmology [25].

We think that this simple model illustrates how quantum field theory can be defined and consistently interpreted in the absence of a background spacetime. In particular, the formalism and its interpretation are well-defined without the need of selecting a time variable. The relation with an approximate notion of time evolution is also illustrated by this model. The application of these ideas to full quantum general relativity in four dimensions is of course nontrivial, but this simple example indicates rather clearly a direction for defining observables and obtaining predictions in the general context.

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