Fractal Dimension in 3d Spin-Foams

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In this paper we perform the calculation of the spectral dimension of the space-time in 3d quantum gravity using the dynamics of the Ponzano-Regge vertex (PR) and its quantum group generalization (Turaev-Viro model (TV) [10]). We realize this considering a very simple decomposition of the 3d space-time and introducing a boundary state which selects a classical geometry on the boundary. We obtain that the spectral dimension of the space-time runs from ≈ 2 to 3, across a ≈ 1.5 phase, when the energy of a probe scalar field decreases from high $E \lesssim E_P$ to low energy. For the TV model the spectral dimension at hight energy increase with the value of the cosmological constant Λ. At low energy the presence of Λ does not change the spectral dimension.

Introduction. In past years many approaches to quantum gravity studied the fractal properties of the quantum space-time. In particular in *causal dynamical triangulation* (CDT) [1] and *asymptotically safe quantum gravity* (ASQG) [2], a fractal analysis of the space-time gives a two dimensional effective manifold at high energy. In both approaches the spectral dimension is $\mathcal{D}_s = 2$ at small scales and $\mathcal{D}_s = 4$ at large scales. The previous ideas have been applied in the context of *non commutativity* to a quantum sphere and κ-Minkowski [3] and in Loop Quantum Gravity [4]. The spectral dimension has been studied also in the cosmology of a Lifshitz universe [5] and in Causal Sets [6]. Spectral analysis is a useful tool to understand the *effective form* of the space at small and large scales. We believe that the fractal analysis could be also a useful tool to predict the behaviour of the 2-point and n-point functions at small scales [7] and to attack the singularity problems of general relativity in a full theory of quantum gravity [8].

In this paper we apply to the Ponzano-Regge (PR) model [9] and to the Turaev-Viro model (TV) [10], [11], [12] the analysis introduced in [4]. We consider the appropriate spinfoam model and we use the very simple decomposition of the $3d$ space-time introduced by Speziale in [13]. The other ingredient is the general boundary formalism useful to define the boundary geometry [14]. All the space-time is approximated by a single tetrahedron and the boundary state is peaked on the boundary geometry of it.

The paper is organized as follows. In the first section we define the framework and we recall the definition of spectral dimension in quantum gravity. The analysis in this section is general and not strongly related to the PR or TV models. The analysis is correct for any spin-foam model. In the second section we calculate explicitly the spectral dimension for the PR and the TV theories using the general boundary formalism to define the 3d quantum gravity path integral.

The Spectral Dimension. The following definition of a fractal dimension is borrowed from the theory of diffusion processes on fractals [15] and is easily adapted to the quantum gravity context. Let us study the diffusion of a scalar test (probe) particle on a d-dimensional classical Euclidean manifold with a fixed smooth metric $g_{\mu\nu}(x)$. The corresponding heat-kernel $K_g(x, x'; T)$ giving the probability for the particle to diffuse from x' to x during the fictitious diffusion time T (this is just a fictitious time and the scalar field in general is just a tool to understand the fractal properties of the space-time) satisfies the heat equation

$$\partial_T K_g(x, x'; T) = \Delta_g K_g(x, x'; T) \tag{1}$$

where Δ_g denotes the scalar Laplacian: $\Delta_g \phi \equiv g^{-1/2} \partial_\mu (g^{1/2} g^{\mu\nu} \partial_\nu \phi)$. The heat-kernel is a matrix element of the operator $\exp(T \Delta_g)$,

$$K_g(x, x'; T) = \langle x' | \exp(T \Delta_g) | x \rangle. \tag{2}$$

In the random walk picture its trace per unit volume,

$$P_g(T) \equiv V^{-1} \int d^d x \sqrt{g(x)} \, K_g(x, x; T)$$
$$\equiv V^{-1} \operatorname{Tr} \exp(T \Delta_g), \tag{3}$$

has the interpretation of an average return probability. (Here $V \equiv \int d^d x \sqrt{g}$ denotes the total volume.) It is well known that P_g possesses an asymptotic expansion (for $T \to 0$) of the form $P_g(T) = (4\pi T)^{-d/2} \sum_{n=0}^{\infty} A_n T^n$. For an infinite flat space, for instance, it reads $P_g(T) = (4\pi T)^{-d/2}$ for all T. Thus from the knowledge of the function P_g one can recover the dimensionality of the target manifold as the T-independent logarithmic derivative

$$d = -2 \frac{d \ln P_g(T)}{d \ln T} \tag{4}$$

This formula can also be used for curved spacetimes and spacetimes with finite volume V provided that T is not taken too large.

In quantum gravity it is natural to replace $P_g(T)$ by its expectation value on a state $|\Psi\rangle$. Symbolically,

$$P(T) := \langle \hat{P}_g(T) \rangle = \int_\Psi Dg P(T) e^{iS(g)}. \tag{5}$$

Given $P(T)$, the spectral dimension of the quantum spacetime is defined in analogy with (4):

$$\mathcal{D}_s = -2 \frac{d \ln P(T)}{d \ln T}. \tag{6}$$

We can formally also to replace the equation (1) with the correspondent expectation value

$$\partial_T \langle K_{\hat{g}}(x, x'; T) \rangle = \langle \Delta_{\hat{g}} K_{\hat{g}}(x, x'; T) \rangle. \quad (7)$$

The Spectral Dimension in Quantum Gravity. In quantum gravity we define (5) the spectral dimension in the general boundary formalism. We introduce a gaussian state $|\psi_{\mathbf{q}}\rangle$ peaked on the boundary geometry $\mathbf{q} = (q, p)$ defined by the metric and the conjugate momentum. We can think the boundary geometry to be the boundary of a d-dimensional ball. The state is symbolically given by:

$$\Psi_{\mathbf{q}}(s) \sim e^{-(s-q)^2 + ips}. \quad (8)$$

The amplitude (5) can be defined for a general spin-foam model

$$\frac{\langle W | \hat{P}_g(T) | \Psi_{\mathbf{q}} \rangle}{\langle W | \Psi_{\mathbf{q}} \rangle} = \frac{\sum_{s_1, s_2} W(s_1) \langle s_1 | \hat{P}_g | s_2 \rangle \psi_{\mathbf{q}}(s_2)}{\sum_s W(s) \Psi_{\mathbf{q}}(s)}. \quad (9)$$

Where $W(s)$ codifies the spin-foam dynamics [16]. For the purpose of the paper we will consider the PR model (TV model); the vertex amplitude is encoded in the $\{6j\}$-symbol, $W(s) \propto \{6j\}$ ($W(s) \propto \{6j\}_q$ for TV and q, the quantum deformation of the $SU(2)$ group, is related to the cosmological constant Λ by $q = \exp(2i\sqrt{\Lambda}l_P)$. Since we are interested in the scaling of the Laplacian to analyze the fractal properties of the space-time, we can approximate the metric in the Laplacian with the inverse of the $SU(2)$ Casimir operator. We recall that in $3d$ quantum gravity the Casimir operator is related to the length spectrum of a link e in the simplycial decomposition by the relation [17]

$$L_e^2 = l_P^2 C^2(j_e) = l_P^2 [j_e(j_e + 1) + c], \quad (10)$$

where the constant is chosen to be $c = 1/4$ in line with [13]. In $3d$ gravity we approximate the 3-ball with a single tetrahedron and the boundary S^2 sphere by the surface of the tetrahedron given by the six triangles. We consider fixed four of the six representations (j) and we call the other two free representations by j_e ($e = 1, 2$). Following the ideas and notation above we define the operator $\hat{P}_g(T)$ in the following way,

$$\hat{P}_{j_e}(T) := V^{-1} \text{Tr } e^{T \frac{C_0^2}{C_e^2} \Delta_0} := V^{-1} \text{Tr } \hat{\mathcal{O}}_e. \quad (11)$$

Where Δ_0 is the Laplacian at a lower infrared scale, j_e is fixed (for example) to $j_e = j_1$ and

$$\hat{\mathcal{O}}_e := e^{T \frac{C_0^2}{C_e^2} \Delta_0}. \quad (12)$$

The boundary state in the notation above is

$$\Psi_j(j_e) = \mathcal{N}^{-1} e^{-\frac{2}{3j} \sum_e^2 (j_e - j)^2 + i\theta \sum_e^2 (j_e + 1/2)}. \quad (13)$$

where $\mathcal{N}$ is a normalization factor. The dihedral angles $\theta = \arccos(-1/3)$ define the boundary extrinsic geometry for an equilateral tetrahedron. Now we have all the
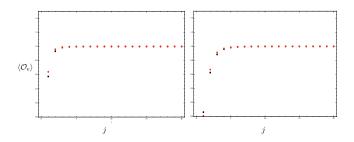

FIG. 1: This is the plot of the modulus of the expectation value (14) $|\eta \langle W | \hat{\mathcal{O}}_{j_1} | \Psi_j \rangle|$ (black dots) compared with the exponential $\exp(-TC_0^2|\Delta_0|/[j(j+1) + c])$ in the case $c = 1/4$ (red dots). The expectation value is calculated for $T = 1$ and $T = 10$, $0 \leqslant j_1, j_2 \leqslant 2j$.
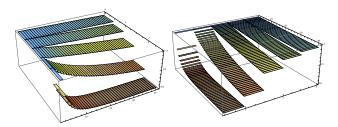

FIG. 2: Plot of the amplitude (14) for $1 \lesssim j \lesssim 6$ and $1 \lesssim TC_0^2 \lesssim 40$.

ingredients to calculate the expectation value (9) using (12) and (13), In particular, since the geometry appears only in the operator $\hat{\mathcal{O}}_e$, we can calculate the expectation value of this operator,

$$\eta \langle W | \hat{\mathcal{O}}_{j_1} | \Psi_j \rangle = \eta \sum_{j_{1,2}=0}^{2j} W(j_1, j_2, j) \mathcal{O}_{j_1} \Psi_j(j_{1,2})$$

$$= \eta \sum_{j_{1,2}=0}^{2j} \prod_{e=1}^{6} (2j_e + 1) \{6j\} e^{-T \frac{C_0^2 |\Delta_0|}{[j_1(j_1+1)+c]}} \Psi_j(j_{1,2}), \quad (14)$$

where we introduced the following notation for the normalization, $\eta^{-1} := \langle W | \Psi_j \rangle$. We also replaced the Laplacian Δ_0 with $-|\Delta_0|$. Before to calculate the amplitude (14) we replace Δ_0 with $|\Delta_0| \propto 1/j^2$, this assumption will be clear later in the paper.

The result of the calculation (14) is given in Fig.1 and compared with the exponential $\exp[-TC_0^2|\Delta_0|/(j(j+1)+c)]$ in the case $c = 1/4$, $C_0^2 = 1$ and $|\Delta_0| \propto 1/j^2$. The plots in Fig.1 are for $T = 1$ and $T = 10$. We can observe a perfect agreement for $j \gtrsim 4$. This agreement is supported by the plots in Fig.2 and Fig.3 where the amplitude (14) on the left and the function $\exp(-X|\Delta_0|/[j(j+1)+1/4])$ ($X = TC_0^2$) on the right coincide for $j \gtrsim 4$. In Fig.4 we plotted a section of (14) for $j = 6$ and $X \in [0, 100]$. This section coincides with the function $\exp(-X|\Delta_0|/[j(j+1) + 1/4])$ evaluated on $j = 6$.

In the range $1 \lesssim j \lesssim 12$ we have interpolated the exact result (14) numerically and obtained a different exponential form of the amplitude. The points data and
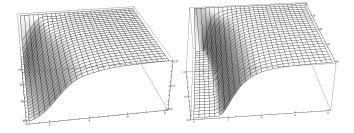
FIG. 3: Plot of the amplitude (14) for $1 \lesssim j \lesssim 8$ and $1 \lesssim X \lesssim 40$ on the left and of the function $\exp(-X|\Delta_0|/[j(j+1)+1/4])$ on the right, where $X = TC_0^2$ This plots show there is good agreement for $j \gtrsim 4$.
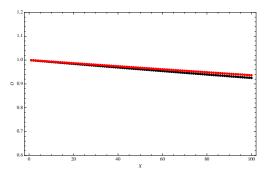

FIG. 4: Plot of the amplitude (14) for $j = 6$ as function of $X = TC_0^2$ on the left (black points) and of the function $\exp(-X|\Delta_0|/[j(j+1)+1/4])$ $1 \lesssim TC_0^2 \lesssim 40$.

the fit are given in Fig.5. The points are fitted by the function $a \exp(b/j^\alpha)$, where $a \approx 1.00$, $b \approx 0.55$ and $\alpha \approx 3.03$ for $T = 1$. Recalling that $\Delta \propto 1/j^2$ we conclude that at the Planck scale,

$$\eta \langle W|\hat{\mathcal{O}}_{j_1}|\Psi_j \rangle \approx \mathrm{e}^{-T\, 0.55 \, \frac{|\Delta_0|}{j^{1.03}}}. \qquad (15)$$

We will use this result to calculate the spectral dimension at the Planck scale then for $T \approx 1$ in Planck units, this is the reason why we fixed $T = 1$ in the expectation value. We can reproduce the behavior of (14) for $j \gtrsim 4$ (in Fig.5 the function $\exp(-TC_0^2\Delta_0/[j(j+1)+c])$ coincides perfectly with the exact expectation value (14) from $j \gtrsim 4$) also analytically using the asymptotic large j limit of the $\{6j\}$ symbol. For large j we have: $\{6j\} \propto \exp(iS_R(j_e) + i\pi/4) + c.c.$ Using this property of the symbol and replacing the sum in (14) with an integral on $\delta j_{1,2} := j_{1,2} - j$ (for $j \gg 1$) we obtain $\exp(TC_0^2\Delta_0/[j(j+1)+c])$.

What we learnt from the explicit calculation of (9) can be summarized as follows,

$$\langle \hat{\mathcal{O}}_e \rangle \approx \begin{cases} e^{-T \frac{C_0^2}{j(j+1)+c} \Delta_0} & \text{for } j \gg 1 \; (j \gtrsim 4), \\ a\, e^{-T \frac{b\, C_0^2}{j^\alpha} \Delta_0} & \text{for } j \approx 1 \; (1 \lesssim j \lesssim 4). \end{cases} \qquad (16)$$

Where $\alpha \approx 1.03$. We introduce a Diff-invariant scale defined by $\ell := j_e l_P$ The result of the (9) can be summarized in the scaling property of the Laplacian operator
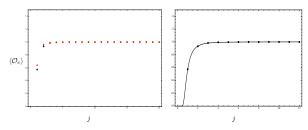

FIG. 5: The plot on the left represents the points obtained from the evaluation of the amplitude(14) for $1 \lesssim j \lesssim 20$ (black points). The red points refer to the function $\exp(-TC_0^2\Delta_0/[j(j+1)+c])$ for $C_0^2 = 1$ (it is an irrelevant constant) and $T = 1$; it is evident that for small values of the representation j the two function are different. The plot on the right represents an interpolation of the black points in the picture on the left for $1 \lesssim j \lesssim 12$.

with the scale ℓ (or with the energy scale $k \approx 1/\ell$),

$$\Delta_j \approx \begin{cases} \frac{C_{j_0}^2}{j(j+1)+c}\Delta_{j_0} & \text{for } j \gg 1 \; (j \gtrsim 4), \\ \frac{c'}{j^\alpha}\Delta_{j'_0} & \text{for } j \approx 1 \; (1 \lesssim j \lesssim 4). \end{cases} \qquad (17)$$

Where we introduced the infrared scales $0 \to j_0, j'_0 \; j_0 \gg 1$, $j'_0 \gtrsim 4$), $c' = bC_0^2$ and, by definition, $C_{j_0}^2 = j_0(j_0+1)+c$.

We denote the scaling of the Laplacian operator suggested by (17) by a general function in the momentum space. We introduce here a physical input to put the momentum k in our analysis. If we want to observe the space-time with a microscope of resolution $l = l_P j$ (the infrared length is $\ell_0 := l_P j_0$) we must use a (fictitious) probing scalar field of momentum $k \sim 1/l$. The scaling property of the Laplacian in terms of k can be obtained by replacing: $l \sim 1/k$, $l_0 \sim 1/k_0$ and $l_P \sim 1/E_P$, where k_0 is an infrared energy cutoff and E_P is the Planck energy. We define the covariant Laplacian at the scale k introducing the function $\mathbb{S}_k$,

$$\Delta_k = \mathbb{S}_k(k,k_0)\,\Delta_{k_0}. \qquad (18)$$

It is straightforward to derive the scaling function from (17) and using the arguments above

$$\mathbb{S}_k(k,k_0) \approx \begin{cases} \frac{k^2[\,E_P(E_P+k_0)+ck_0^2\,]}{k_0^2[\,E_P(E_P+k)+ck^2\,]} + 1 & \text{for } k \lesssim \frac{E_P}{4}, \\ c'\, k^\alpha & \text{for } \frac{E_P}{4} \lesssim k \lesssim E_P. \end{cases} \qquad (19)$$

We added a factor one in the infrared limit to facilitate the spectral dimension calculations. The scaling function $\mathbb{S}_k(k,k_0)$ represents also , using the definition of the Laplacian, the scaling of the inverse of the metric, $\langle g^{\mu\nu} \rangle_k = \mathbb{S}_k(k,k_0)\langle g^{\mu\nu} \rangle_{k_0}$.

We suppose that the diffusion process involves (approximately) only a small interval of scales near k then the corresponding heat kernel contains the Δ_k for this specific and fixed value of the momentum scale k. Denoting the eigenvalues of $-\Delta_{k_0}$ by E_n and the corresponding

eigenfunctions by $\phi_n(x) = \langle x|E_n\rangle$, we have the following eigenvalue equation for the Laplacian

$$\Delta_{k_0}|E_n\rangle = -E_n|E_n\rangle,$$

$$\langle E_n|E_m\rangle = \int d^4x' \sqrt{g_0(x')}\, \phi_n^*(x')\phi_n(x') = \delta_{n,m}. \quad (20)$$

Using (20) and the definition (2) we can calculate explicitly the heat kernel $K_k(x,x';T) = \langle x'|\langle \hat{\mathcal{O}}_e\rangle|x\rangle$. By using (16), (17), (18) and (20) we have

$$K_k(x,x';T) = \langle x'|e^{T\Delta_k}|x\rangle$$
$$= \sum_n \phi_n^*(x')\phi_n(x)\, e^{-T\,\mathbb{S}_k(k,k_0)E_n}. \quad (21)$$

From the knowledge of the propagation kernel (21) we can time-evolve any initial probability distribution $p(x;0)$ according to $p(x;T) = \int d^4x' \sqrt{g_0(x')}\, K(x,x';T)\, p(x';0)$, where g_0 the determinant of $\langle g_{\mu\nu}\rangle_{k_0}$. If the initial distribution has an eigenfunction expansion of the form $p(x;0) = \sum_n C_n\,\phi_n(x)$ we obtain for arbitrary x,

$$p(x;T) = \int d^4x' \sqrt{g_0(x')}\, K(x,x';T)\, p(x';0) =$$
$$= \sum_n C_n\,\phi_n(x)\, e^{-\mathbb{S}(k,k_0)\,E_n\,T} \quad (22)$$

where we used the wave function normalization (20). If the C_n's are significantly different from zero only for a single eigenvalue E_n, we are dealing with a single-scale problem and then we can identify $k^2 = E_n$. However, in general the C_n's are different from zero over a wide range of eigenvalues. In this case we face a multiscale problem where different modes ϕ_n probe the spacetime on different length scales.

If $\Delta(k_0)$ is the Laplacian on the flat space, the eigenfunctions $\phi_n \equiv \phi_p$ are plane waves with momentum p^μ, and they resolve structures on a length scale ℓ of order $1/|p|$. Hence, in terms of the eigenvalue $E_n \equiv E_p = p^2$ the resolution is $\ell \approx 1/\sqrt{E_n}$. This suggests that when the manifold is probed by a mode with eigenvalue E_n it "sees" the metric $\langle g_{\mu\nu}\rangle_k$ for the scale $k = \sqrt{E_n}$. Actually the identification $k = \sqrt{E_n}$ is correct also for a curved spacetime because the parameter k just identifies the scale we are probing. Therefore we can conclude that under the spectral sum of (22) we must use the scale $k^2 = E_n$ which depends explicitly on the resolving power of the corresponding mode. In eq. (22), $\mathbb{S}_k(k,k_0)$ can be interpreted as $\mathbb{S}(E_n)$. Thus we obtain the traced propagation kernel,

$$P(T) = \sum_n \frac{e^{-T\,\mathbb{S}(E_n)E_n}}{V_{\langle g\rangle_{k_0}}} = V^{-1}\mathrm{Tr}\left(e^{\mathbb{S}(-\Delta_{k_0})\,\Delta_{k_0}\,T}\right). \quad (23)$$

It is convenient to choose k_0 as a macroscopic scale in a regime where there are not strong quantum gravity effects.

We assume for a moment that $\langle g_{\mu\nu}\rangle_{k_0}$ is an approximately flat metric. In this case the trace in eq. (23) is easily evaluated in a plane wave basis:

$$P(T) = \int \frac{d^dp}{(2\pi)^d}\, e^{-T\,\mathbb{S}(p)p^2}. \quad (24)$$

where we used the flat metric $\langle g_{\mu\nu}\rangle_{k_0} = \delta_{\mu\nu}$ and $\Delta_{k_0}|p\rangle = -p^2|p\rangle$.

The dependence from T in (24) determines the fractal dimensionality of spacetime via (6). In the limits $T \to \infty$ and $T \to 0$ where we are probing very large and small distances, respectively, we obtain the dimensionalities corresponding to the largest and smallest length scales possible. The limits $T \to \infty$ and $T \to 0$ of $P(T)$ are determined by the behaviour of $\mathbb{S}(p)$ for $p \to 0$ and $p \to \infty$, respectively.

The above assumption that $\langle g_{\mu\nu}\rangle_{k_0}$ is flat was not necessary to obtain the spectral dimension at any fixed scale. This follows from the fact that even for a curved metric the spectral sum (23) can be represented by an Euler-Mac Laurin series which always implies (23) as the leading term for $T \to 0$.

Now we have all the ingredients to calculate the spectral dimension using (24) inside the definition (6). For the PR model the scaling function $\mathbb{S}(p)$ is obtained from (19) replacing k with p. The spectral dimension for $j \gtrsim 4$ or $k \lesssim E_P/4$ increases from $\mathcal{D}_s \approx 1.5$ to $\mathcal{D}_s \approx 3$ at low energy as it is evident from the plot in Fig.6. For $1 \lesssim j \lesssim 4$ or $E_P \lesssim k \lesssim E_P/4$ using the proper scaling we find $\mathcal{D}_s \approx 1.98$. We conclude that the fractal dimension decrease from the Planck energy to an intermediate scale where take the value ≈ 1.5 (for $k \approx E_P/4$) and increase again to ≈ 3 at low energy (Fig.6). For the TV model we have differences only for $j \lesssim 4$ and the result is plotted in Fig.6 on the right. That plot gives the spectral dimension as a function of the cosmological constant. The spectral dimension is in the range $2.00 \lesssim \mathcal{D}_s \lesssim 2.059$ for $0.001 \lesssim \Lambda \lesssim 0.009$ in Planck units. In other words the spectral dimension increases with the increase of the cosmological constant at the Planck scale.

Conclusions and Discussion. In this paper we calculated explicitly the spectral dimension ($\mathcal{D}_s$) for the 3d quantum spacetime using the Ponzano Regge spinfoam model. We considered the simplest decomposition of the spacetime and we used the general boundary formalism to characterize the scaling properties of the expectation value for the traced propagation kernel. Using the technical simplifications repeatedly used in the graviton propagator calculations we have evaluated the nonperturbative expectation value of the heat kernel.

In the PR model and for $k \lesssim E_P/4$ we have plotted $\mathcal{D}_s$ as a function of a fictitious diffusion time T or equivalently as a function of the length scale. We obtained three phases: a short scale phase $l_P \lesssim l \lesssim 4l_P$ of spectral dimension $\mathcal{D}_s \approx 2$, an intermediate scale phase $\ell \gtrsim 4l_P$ of spectral dimension $\mathcal{D}_s = 1.5$ and a large scale phase with $\mathcal{D}_s = 3$.
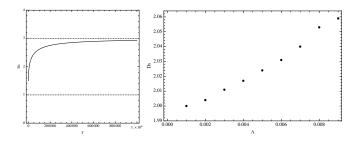
For the TV model the results are equal for $k \lesssim E_P/4$ and to feel the effect of the cosmological constant we must goes beyond that energy. The spectral dimension depends on Λ as it is evident from the plot in Fig.6.

We interpret the results in the following way. At high energy the spectral dimension is $\mathcal{D}_s < 3$ because the manifold presents holes typical of an atomic structure. The cosmological constant basically decreases the number of holes increasing the spectral dimension.

FIG. 6: The plot on the left represents the spectral dimension for $j \gtrsim 4$ or $k \lesssim E_P/4$ as function of the fictitious time T. The dimension at hight energy is 1.5. We have plotted $T \in [0.005, 10^6]$ and used $E_p = 1000$, $k_0 = 0.01$. The plot on the right represents the spectral dimension as function of the cosmological constant Λ in the Turaev-Viro model at the Planck scale ($1 \lesssim j \lesssim 9$).

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