Can an astrophysical black hole have a topologically non-trivial event horizon?

Cosimo Bambi^{1,2,*} and Leonardo Modesto^{3,†}

¹Institute for the Physics and Mathematics of the Universe, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan ²Arnold Sommerfeld Center for Theoretical Physics, Ludwig-Maximilians-Universität München, 80333 Munich, Germany ³Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada (Dated: September 19, 2018)

In 4-dimensional General Relativity, there are several theorems restricting the topology of the event horizon of a black hole. In the stationary case, black holes must have a spherical horizon, while a toroidal spatial topology is allowed only for a short time. In this paper, we consider spinning black holes inspired by Loop Quantum Gravity and by alternative theories of gravity. We show that the spatial topology of the event horizon of these objects changes when the spin parameter exceeds a critical value and we argue that the phenomenon may be quite common for non-Kerr black holes. Such a possibility may be relevant in astrophysics, as in some models the accretion process can induce the topology transition of the horizon.

PACS numbers: 04.50.Kd, 04.60.Bc, 97.60.Lf, 04.60.Pp

I. INTRODUCTION

The study of black hole (BH) uniqueness theorems started more than forty years ago and it is still a very active research field [1]. In 4-dimensional General Relativity, the Hawking's theorem ensures that the spatial topology of the event horizon must be a 2-sphere in the stationary case, under the main assumptions of asymptotically flat space-time and validity of the dominant energy condition [2]. Technically, the event horizon of a BH is defined as the boundary of the causal past of future null infinity. The spatial topology of the event horizon at a given time is the intersection of the Cauchy hypersurface at that time with the event horizon. According to the topological censorship theorem, in a globally hyperbolic and asymptotically flat space-time, any two causal curves extending from past to future infinity are homotopic [3]. As a BH with a toroidal spatial topology would violate this theorem, the hole must quickly close up, before a light ray can pass through [4]. Interestingly, numerical simulations find that toroidal horizons can form, but they exist for a short time, consistently with the topological censorship theorem [5].

In the vacuum, the only stationary and axisymmetric BH solution of the Einstein's equations in a 4-dimensional and asymptotically flat space-time is given by the Kerr geometry, which is completely specified by two parameters: the mass M and the spin angular momentum J – instead of J, it is more commonly used the spin parameter a = J/M or the dimensionless spin parameters $a_* = J/M^2$. While the Kerr metric may be a solution even in other theories of gravity, in general there is not a similar uniqueness theorem [6]. Unfortunately, for

the time being our knowledge of non-Kerr BH solutions in alternative theories of gravity is definitively limited. In most cases, analytic or numerical metrics are known only for non-rotating BHs. Approximated solutions in the slow-rotation limit have been obtained in Einstein-Gauss-Bonnet-dilaton (EGBd) gravity [7] and in Chern-Simons modified gravity [8]. The only 4-dimensional non-Kerr spinning BH example of a specific gravity theory is given by a numerical metric in EGBd gravity, recently found in [9]. On the other hand, from the observational point of view, fast-rotating BHs are the most interesting, as deviations from the Kerr metric are typically more evident and we would have more chances to test the model

with astrophysical data [10]. In this paper, we discuss two examples of spinning BHs in theories beyond General Relativity, proposed respectively in [11] and [12]. In both cases, the two metrics have not been obtained by solving specific field equations, but the fact they have an analytic form for arbitrary values of the spin parameter is very useful. The BH proposed in [11] is inspired by Loop Quantum Gravity and the deformations from the Kerr geometry are encoded in a polymeric function P and in a Plank scale parameter a_0 . The BH proposed in [12] is instead a phenomenological metric and may be seen as a BH solution in an unknown alternative theory of gravity¹. Both BHs have been obtained through a Newman-Janis transformation

¹ These metrics can be solutions of particular non-local generalizations of the Einstein's equations, as suggested in [13, 14]. The idea is to replace the Einstein's equations with the following set of equations of motion

 $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N \mathcal{O}(\Box/\Lambda^2)T_{\mu\nu} ,$

where $\mathcal{O}(\Box/\Lambda^2)$ is a generic non-local function of the covariant D'Alembertian operator and Λ is the energy scale of the modified gravity [14].

^{*} cosimo.bambi@ipmu.jp

[†] lmodesto@perimeterinstitute.ca

of a static solution. The key point of the two metrics is that there are no restrictions on the values of the spin parameter a_* . Interestingly, for high values of a_* they present similar features. If the BH is more prolate than the predictions of General Relativity, when it rotates fast the spatial topology of the event horizon changes from a 2-sphere to two disconnected 2-spheres. If the BH is more oblate, one finds a toroidal horizon or something very similar to a toroidal horizon. Our guess is that fast-rotating BHs with non-trivial topology are not peculiar predictions of these two solutions, but that they may be relatively common in the case of deviations from the Kerr geometry.

II. BLACK HOLES INSPIRED BY LOOP QUANTUM GRAVITY

Loop Quantum Gravity is a generally covariant and non-perturbative quantization of General Relativity [15]. BHs in Loop Quantum Gravity have been studied only very recently and it has been shown that the central singularity can be solved [16, 17]. Rotating BHs have been obtained in [11] through a Newman-Janis transformation. In Boyer-Lindquist coordinates, the loop-modified Kerr metric reads [11]

$$g_{tt} = -\frac{(\rho^2 + 2MPr)^2 \Delta}{\rho^4 \Sigma},$$

$$g_{t\phi} = -\frac{a \sin^2 \theta (\rho^2 + 2MPr)^2 (\Sigma - \Delta)}{\rho^4 \Sigma},$$

$$g_{\phi\phi} = \sin^2 \theta \left[\Sigma + \frac{a^2 \sin^2 \theta (\rho^2 + 2MPr)^2 (2\Sigma - \Delta)}{\rho^4 \Sigma} \right],$$

$$g_{rr} = \frac{(\rho^2 + 2MPr)^2 \Sigma}{\rho^4 \Delta + a^2 \sin^2 \theta (\rho^2 + 2MPr)^2},$$

$$g_{\theta\theta} = \Sigma,$$
(2.1)

where

$$\begin{split} \rho^2 &= r^2 + a^2 \cos^2 \theta \,, \\ \Delta &= r^2 - 2M(1+P^2)r + 4M^2P^2 + a^2 \cos^2 \theta \,, \end{split} \tag{2.2}$$

P is the polymeric function

$$P = \frac{\sqrt{1 + \gamma_I^2 \delta^2} - 1}{\sqrt{1 + \gamma_I^2 \delta^2} + 1},$$
 (2.3)

and γ_I and δ are respectively the Immirzi parameter and the "polymeric parameter". In principle, P can be either positive or negative, because γ_I may be complex. As for Σ , there is more than one possibility, because of some ambiguities in the procedure to get the metric. There are indeed at least two natural complexifications in the Newman-Janis construction procedure [11]:

Type I:
$$\Sigma = \rho^2 + B^2/r^2$$
, (2.4)

Type II:
$$\Sigma = \rho^2 + B^2/\rho^2$$
, (2.5)

where B is the "bounce constant" and has been fixed in two different ways in previous papers for the spherically symmetric solution [16, 18]:

Case a:
$$B = a_0$$
, (2.6)

Case b:
$$B = (2MP)^2$$
. (2.7)

Here $a_0 \sim L_{Pl}^2$ is the minimum area $(L_{Pl} \sim 10^{-33} \text{ cm})$ is the Planck length). Since in this paper we are interested in astrophysical BHs, we can neglect Planck-scale structures and we assume $a_0 = 0$. We have thus three slightly different solutions: Ia-IIa, Ib, and IIb. However, all our conclusions are independent of the ambiguities related to the Σ .

The event horizon is defined by the condition²

$$\rho^4 \Delta + a^2 \sin^2 \theta (\rho^2 + 2MPr)^2 = 0. \tag{2.8}$$

When P=0, one recovers the classical Kerr result

$$r_H = M \pm \sqrt{M^2 - a^2}$$
, (2.9)

where the sign + and - are respectively for the outer and inner horizon. The key-point of the Kerr metric is that the radial coordinate of the two horizons does not depend on θ and there are two topologically spherical horizons for a < M, one horizon in the extreme case a = M, and there is no horizon for a > M (naked singularity). For $P \neq 0$, Eq. (2.8) depends on θ and between the cases of two topologically spherical horizons for low values of a and no horizons for high values of a, we find that the two horizons merge into a horizon with non-trivial spatial topology. Let us call a_{\star}^{c} the lowest value of the dimensionless spin parameter for which the BH has a topologically non-trivial event horizon. For P > 0, there are two disconnected horizons with spherical topology, see the left panel in Fig. 1. For P < 0, there are two disconnected horizons with toroidal topology: the one coming from the merger of the outer and inner horizons and shown in the right panel of Fig. 1, and another small horizon near the origin. The latter is not shown in the figure and present some peculiar features. For instance, while the large toroidal horizon disappears above some critical spin parameter, like the horizons of the case P > 0,

Let us notice that there are a few definitions of horizon. Here we are interested in the event horizon, i.e. a boundary in the spacetime beyond which events cannot affect an outside observer. In a stationary space-time, the event horizon is also an apparent horizon, which is a surface of zero expansion for a congruence of outgoing null geodesics orthogonal to the surface. This means that at the apparent horizon null geodesics must have dr/dt=0, which implies $g^{rr}=0$, see e.g. Ref. [19]. The horizon relevant for the black hole thermodynamics is instead the Killing horizon, which is a null hyper-surface on which there is a null Killing vector field. For the metric in (2.1), the Killing horizon is defined by $g_{tt}g_{\phi\phi}-g_{t\phi}^2=0$. When the Hawking's rigidity theorem can be applied (like in the Kerr space-time), the event horizon and the Killing horizon coincide [2]. However, in general that is not true.

the small one does not and exists even when $a/M \gg 1$. Here topologically non-trivial event horizons are possible because Eq. (2.1) is not a solution of the Einstein's equations. Equivalently, the metric in Eq. (2.1) can be seen as a solution of the Einstein's equations in presence of a non-zero effective energy-momentum tensor violating the dominant energy condition [16].

III. BLACK HOLES IN ALTERNATIVE THEORIES OF GRAVITY

In Ref. [12], the authors have proposed a phenomenological metric to test gravity in the strong field regime. In other words, it is not a solution in any known gravity theory, but it is a simple parametrization of (hopefully generic) deviations from the Kerr geometry. The metric was obtained by starting from a deformed Schwarzschild solution and then by applying a Newman-Janis transformation. The non-zero metric coefficients in Boyer-Lindquist coordinates are [12]

$$g_{tt} = -\left(1 - \frac{2Mr}{\rho^2}\right) (1+h),$$

$$g_{t\phi} = -\frac{2aMr\sin^2\theta}{\rho^2} (1+h),$$

$$g_{\phi\phi} = \sin^2\theta \left[r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\rho^2}\right] + \frac{a^2(\rho^2 + 2Mr)\sin^4\theta}{\rho^2}h,$$

$$g_{rr} = \frac{\rho^2(1+h)}{\Delta + a^2h\sin^2\theta}, \quad g_{\theta\theta} = \rho^2,$$
(3.1)

where

$$\rho^{2} = r^{2} + a^{2} \cos^{2} \theta ,$$

$$\Delta = r^{2} - 2Mr + a^{2} ,$$

$$h = \sum_{k=0}^{\infty} \left(\epsilon_{2k} + \frac{Mr}{\rho^{2}} \epsilon_{2k+1} \right) \left(\frac{M^{2}}{\rho^{2}} \right)^{k} .$$
 (3.2)

The metric has an infinite number of free parameters ϵ_i and the Kerr solution is recovered when all these parameters are set to zero.

The event horizon of a BH described by the metric (3.1) is given by (please notice that in Ref. [12] the authors use an incorrect definition of event horizon and actually they compute the Killing horizon)

$$\Delta + a^2 h \sin^2 \theta = 0. \tag{3.3}$$

Like in (2.8), Eq. (3.3) depends on θ and in general, for high values of the spin parameter, the inner and the outer horizons merge together, with the result of forming a BH with a horizon with non-trivial topology. As a specific example, we can take the case discussed in [12] with a single free parameter, ϵ_3 , and $\epsilon_i = 0$ for $i \neq 3$. However, the picture is qualitatively the same if we take, for instance,

 ϵ_4 or ϵ_5 as free parameter instead of ϵ_3 . For $\epsilon_3 > 0$, the BH turns out to be more prolate than the Kerr one and, for high spin parameters, one finds two disconnected horizons with spherical topology, see the left panel in Fig. 2. If $a_* > 1$, there is no horizon, like in the Kerr metric. For $\epsilon_3 < 0$, the object is more oblate than a Kerr BH and one finds that fast-rotating objects have a horizon that looks toroidal. It is not really a torus because at r=0 there is a naked singularity which is still connected to the horizon. So, unlike a doughnut, here there is not a central hole. The case $a_* = 1.1$ and $\epsilon_3 = -1$ is shown on the right panel of Fig. 2. Unlike for the case $\epsilon_3 \geq 0$, when $\epsilon_3 < 0$ the horizon never disappears, even for $a/M \gg 1$. It just becomes more and more thin and looks like a disk.

IV. ASTROPHYSICAL BLACK HOLES

In the previous sections, we have discussed two specific examples of spinning BHs in four dimensions with topologically non-trivial event horizons. However, in both cases it is required that the value of spin parameter a exceeds some critical value. It is thus not clear if such a condition can be satisfied for astrophysical BHs. For instance, in the Kerr case, the solution describes a BH for $a \leq M$ and a naked singularity for a > M. However, Kerr naked singularities can unlikely be of astrophysical interest, as it is apparently impossible to overspin a Kerr BH up to a > M [20]. So, the purpose of this section is to show that BHs with topologically non-trivial event horizon could be created in the Universe.

A natural and very efficient mechanism to spin a compact object up is through the process of gas accretion from a disk. One can assume that the disk is on the equatorial plane of the object and that the disk's inner edge is located at a radius r_{in} . When the gas reaches the inner edge, it plunges to the BH with no further emission of radiation. If at the radius r_{in} the specific energy $E=-u_t$ and the specific angular momentum $L=u_\phi$ of a gas particle with 4-velocity u^μ are respectively E_{in} and L_{in} , the compact object changes its mass M and its spin angular momentum J by

$$\delta M = E_{in} \delta m ,$$

$$\delta J = L_{in} \delta m ,$$
(4.1)

where δm is the gas rest-mass. The evolution of the spin parameter is thus governed by the following equation [21]

$$\frac{da_*}{d\ln M} = \frac{1}{M} \frac{L_{in}}{E_{in}} - 2a_* \,. \tag{4.2}$$

 E_{in} and L_{in} depend on the metric of the space-time (see Eq. (4.7) below) and on the model of the accretion disk.

The simplest case is the geometrically thin and optically thick disk [22], whose inner edge is at the marginally stable circular orbit (also called innermost stable circular orbit, or ISCO): $r_{in} = r_{ms}$. For a generic stationary and axisymmetric space-time, the calculation of E_{ms} and

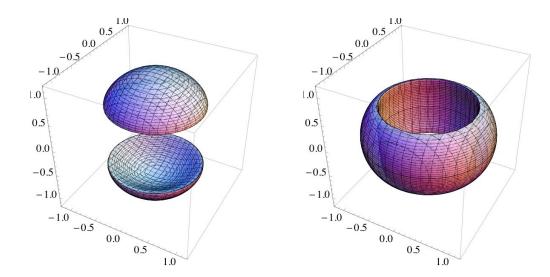


FIG. 1. Event horizons of loop-inspired black holes. Left panel: $a_* = 0.99$ and P = 0.01. Right panel: $a_* = 1.01$ and P = -0.01. See text for details.

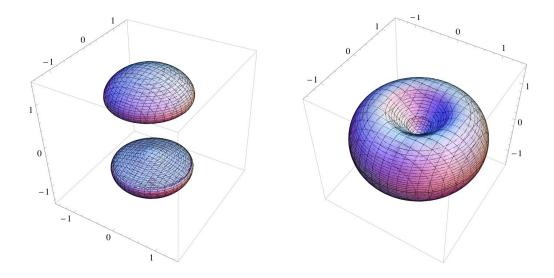


FIG. 2. Event horizons of black holes in possible alternative theories of gravity. Left panel: $a_* = 0.9$ and $\epsilon_3 = 1$. Right panel: $a_* = 1.1$ and $\epsilon_3 = -1$. See text for details.

 L_{ms} goes as follows (see e.g. Appendix B in Ref. [23] for more details). The disk's gas moves on nearly geodesic circular orbits and the equations of motion are

$$\dot{t} = \frac{Eg_{\phi\phi} + Lg_{t\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}, \qquad (4.3)$$

$$\dot{t} = \frac{Eg_{\phi\phi} + Lg_{t\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}},$$

$$\dot{\phi} = -\frac{Eg_{t\phi} + Lg_{tt}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}},$$
(4.3)

$$g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 = V_{\text{eff}}(E, L, r, \theta), \qquad (4.5)$$

where $V_{\rm eff}$ is the effective potential

$$V_{\text{eff}} = \frac{E^2 g_{\phi\phi} + 2E L g_{t\phi} + L^2 g_{tt}}{g_{t\phi}^2 - g_{tt} g_{\phi\phi}} - 1.$$
 (4.6)

Circular orbits in the equatorial plane are located at the zeros and the turning points of the effective potential: $\dot{r}=\dot{\theta}=0$ implies $V_{\rm eff}=0$, and $\ddot{r}=\ddot{\theta}=0$ requires $\partial_r V_{\rm eff}=\partial_\theta V_{\rm eff}=0$. E and L turn out to be

$$E = -\frac{g_{tt} + g_{t\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}},$$

$$L = \frac{g_{t\phi} + g_{\phi\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}},$$
(4.7)

where

$$\Omega = \frac{d\phi}{dt} = \frac{-\partial_r g_{t\phi} \pm \sqrt{(\partial_r g_{t\phi})^2 - (\partial_r g_{tt})(\partial_r g_{\phi\phi})}}{\partial_r g_{\phi\phi}} (4.8)$$

is the orbital angular velocity and the sign + (-) is for corotating (counterrotating) orbits. The orbits are stable under small perturbations if $\partial_r^2 V_{\rm eff} \leq 0$ and $\partial_\theta^2 V_{\rm eff} \leq 0$. At the ISCO, either $\partial_r^2 V_{\rm eff} = 0$ or $\partial_\theta^2 V_{\rm eff} = 0$. In this way, one determines $E_{in} = E_{ms}$ and $L_{in} = L_{ms}$ and can integrate Eq. (4.2) to get the equilibrium spin parameter a_*^{eq3} . For instance, an initially non-rotating BH in General Relativity reaches the equilibrium spin parameter $a_*^{eq} = 1$ after having increased its mass by a factor $\sqrt{6} \approx 2.4$ [21]. If the compact object is not a Kerr BH, the final value of the spin parameter may be larger than 1 and super-spinning compact objects may be created [25, 26].

In the case of non-Kerr BHs, topologically non-trivial horizons may have astrophysical relevance if the deviations from the Kerr geometry are not too small. For a loop BH and P = 0.01, the equilibrium spin parameter is $a_*^{eq} \approx 0.9831$, while the transition of the topology of the horizon happens at $a_*^c \approx 0.9805$. A few other cases are reported in Tab. I and we have checked that all the results do not depend on the ambiguities related to Σ . As $|P| \to 0$, a_*^{eq} and a_*^c go to 1 and even if a_*^{eq} is always larger than a_*^c , the difference is smaller and smaller. This fact forbids the possibility of changing the topology of the horizon for small values of |P|, as a_*^{eq} cannot be reached in the reality. In particular, the radiation emitted by the disk and captured by the BH reduces the value of a_*^{eq} computed from Eq. (4.2), as the radiation with angular momentum opposite to the BH spin has larger capture cross section. For example, in the case of the Kerr metric one finds the well-known "Thorne's limit" $a_*^{eq} \approx 0.998$ [27]. To get a rough estimate of the effect, we can assume that the radiation captured by the BH reduces a_*^{eq} by 0.002 (actually for loop BHs the effect is smaller, because for $P \neq 0$ the ISCO radius at a_*^{eq} is larger than the Kerr one). In this case, the accretion process from a thin disk can spin the BH up to the critical value a_*^c and induces the topology transition only if $P \gtrsim 0.01 \text{ or } P \lesssim -0.001.$

Tab. II shows the same quantities for the BHs described by the metric (3.1) (see also Fig. 2 in Ref. [12], where the authors show a_*^c as a function of ϵ_3 for $\epsilon_3 > 0$). Even here, $a_*^{eq} > a_*^c$, but the difference is smaller and smaller as $\epsilon_3 \to 0$. Like for the loop BHs, the topology transition of the event horizon occurs at $a_* < 1$ for more prolate objects, and at $a_* = 1$ for more oblate objects.

P	a_*^{eq}	a^c_*
0.01	0.9831	0.9805
0.001	0.9981	0.9980
0.0001	0.9998	0.9998
0.0	1.0	_
-0.0001	1.0002	1.0000
-0.001	1.0020	1.0000
-0.01	1.0192	1.0000
0.0 -0.0001 -0.001	1.0 1.0002 1.0020	1.0000 1.0000

TABLE I. Black holes inspired by Loop Quantum Gravity. Equilibrium spin parameter, a_s^{eq} , and critical spin parameter separating black holes with topologically different horizons, a_s^c , for some values of the polymeric function P. The case P = 0 corresponds to the classical Kerr metric.

ϵ_3	a_*^{eq}	a_*^c
10.0	0.5608	0.4355
1.0	0.8705	0.7910
0.1	0.9735	0.9596
0.0	1.0	_
-0.1	1.0334	1.0000
-1.0	1.1854	1.0000
-10.0	1.6531	1.0000

TABLE II. Black holes in possible alternative theories of gravity. Equilibrium spin parameter, a_*^{eq} , and critical spin parameter separating black holes with topologically different horizons, a_*^c , for some values of the parameter ϵ_3 . The case $\epsilon_3=0$ corresponds to the Kerr metric.

V. OBSERVATIONAL CONSTRAINTS

Up to now, we have considered arbitrary deviations from the Kerr geometry. However, either P for the metric (2.1) and ϵ_3 for (3.1) are subject to constraints from current observations.

For the loop BHs, one can apply the Birkhoff's theorem and constrain P by using current observational data in the Solar System. In the parametrized post-Newtonian (or PPN) framework [28], the line element of the asymptotic space-time in spherical coordinates is

$$ds^{2} = -A(r)dt^{2} + B(r)dr^{2} + r^{2}d\Omega^{2}, \qquad (5.1)$$

where

$$A(r) = 1 - \frac{M}{r} + 2(\beta_{PPN} - \gamma_{PPN}) \frac{M^2}{r^2} + \dots, (5.2)$$

$$B(r) = 1 + 2\gamma_{PPN} \frac{M}{r} + \dots,$$
 (5.3)

and β_{PPN} and γ_{PPN} are the PPN parameters. In classical General Relativity $\beta_{PPN}=\gamma_{PPN}=1$ and the

³ In the case of non-Kerr background, the picture may be more complicated and, in some cases, the gas may not be able to plunge from the ISCO to the BH, see Ref. [24]. If this is the case, accretion is possible only if the gas loses additional energy and angular momentum. However, such an effect occurs only for "extreme" objects, typically with spin parameter a_* well above the equilibrium value a_*^{eq} .

strongest constraints come from the Lunar Laser Ranging experiment [29] and the Cassini spacecraft [30]

$$|\beta_{PPN} - 1| < 2.3 \cdot 10^{-4} \quad \text{(LLR)},$$

 $|\gamma_{PPN} - 1| < 2.3 \cdot 10^{-5} \quad \text{(Cassini)}.$ (5.4)

The asymptotic form of the loop-modified Kerr metric reads

$$-g_{tt} = 1 - \frac{2m}{r} + 2\frac{4(P - P^2 + P^3)}{(1 - P)^4} \frac{m^2}{r^2} + \dots,$$

$$g_{rr} = 1 + 2\frac{(1 + P)^2}{(1 - P)^2} \frac{m}{r} + \dots,$$
(5.5)

where $m = (1 - P)^2 M$ is the gravitational mass as measured by a distant observer. From the Cassini spacecraft, we get

$$|P| < 0.6 \cdot 10^{-5} \,. \tag{5.6}$$

So, if P is really a constant, it must be so small that the accretion process can unlikely create loop BHs with non-trivial topology. However, the Immirzi parameter may be a running constant, approaching 0 at low energies/large distances and 1 at high energies/short distances, as discussed in [31]. If this is the case, the accretion process might still create BHs with non-trivial topology, as the constraint (5.6) would hold only far from the BH.

Let us now consider the metric (3.1). As discussed in Ref. [12], the Newtonian limit requires $\epsilon_0 = \epsilon_1 = 0$ and the Lunar Laser Ranging experiment demands $|\epsilon_2|$ $4.6 \cdot 10^{-4}$. On the other hand, there are no bounds on ϵ_i for i > 3 from the Solar System. Following the argument in Ref. [32], it is possible to constrain the deformation parameters with $i \geq 3$ from the estimate of the mean radiative efficiency of active galactic nuclei (AGN). Let us notice, however, that the final bound has to be taken with some caution. One can notice that the most luminous super-massive objects in galactic nuclei have a radiative efficiency $\eta > 0.15$ [33]. There are several sources of uncertainty to get this bound, but this value seems to be a reliable lower limit. Assuming that ϵ_i do not depend on M or J, we can constrain possible deviations from the Kerr metric, as $\eta \leq 1 - E_{ms}$. In the specific case $\epsilon_3 \neq 0$ and $\epsilon_i = 0$ for $i \neq 3$, one finds:

$$-1.1 < \epsilon_3 < 25. \tag{5.7}$$

Such a bound is weak and surely does not forbid the possibility of astrophysical BHs with non-trivial topology.

VI. CONCLUSIONS

In 4-dimensional General Relativity, a stationary BH must have a spherical horizon, while a toroidal horizon is allowed for a very short time. It is thus thought that the spatial topology of the horizon of astrophysical BHs is a 2-sphere. However, if the current BHs candidates are not the BHs predicted by General Relativity, this

conclusion may be wrong. Here we have discussed two examples of 4-dimensional non-Kerr spinning BHs and we have shown that for high values of the spin parameter the topology of the event horizon of these objects can change. Unfortunately, our current knowledge of these objects is definitively limited. Here we have considered the loop-improved Kerr metric found in Ref. [11] and the phenomenological metric proposed in [12] to perform tests of strong gravity. Interestingly, they present quite remarkable and qualitatively similar features in the case of fast-rotating objects and we guess that these properties may be common for non-Kerr spinning BHs. In particular, it seems that BHs more prolate than the Kerr one develop two disconnected topologically spherical horizons above some critical spin parameter. In the case of fastrotating objects more oblate than a Kerr BH, their horizon looks more like a torus, even if the central hole may be closed (like in the case of the BH in the right panel of Fig. 2, in which the event horizon extends up to the central singularity at r=0). In both this examples, the accretion process may overspin these objects above the critical spin parameter and induce the topology transition of the horizon. The topology change does not happen as a result of some jump or tunneling, but this fact should not be seen suspiciously: even in numerical simulations in General Relativity the gravitational collapse can produce a toroidal BH and then the hole quickly closes up continuously, as found for the first time in [5].

As final remark, let us notice that here we have not discussed the stability of these BHs. However, this issue cannot be addressed for the metrics (2.1) and (3.1): we do not know the field equations of the gravity theory having Eqs. (2.1) and (3.1) as solution, and therefore we cannot predict the evolution of small perturbations on these backgrounds. On very general grounds, we can simply say the event horizon of very fast-rotating objects becomes likely too small to prevent the ergoregion instability [34]. However, such an instability may occur only for BHs with spin parameters $a_* > a_*^{eq}$, which would be anyway unstable configurations. If, on the contrary, the instability appears at $a_* < a_*^{eq}$, as the spin-up due to the accretion process is an unavoidable phenomenon, these BHs would be a source of gravitational waves, potentially detectable by future experiments.

ACKNOWLEDGMENTS

We would like to thank Alexander Dolgov and Piero Nicolini for critically reading a preliminary version of this manuscript and providing useful feedback. We are also grateful to Francesco Caravelli for helpful comments. The work of C.B. was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan, and by the JSPS Grant-in-Aid for Young Scientists (B) No. 22740147. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of On-

tario through the Ministry of Research & Innovation. L.M. would like to acknowledge support and hospitality from the Institute for the Physics and Mathematics of the Universe at The University of Tokyo, where this

work was started. C.B. would like to acknowledge support and hospitality from the Perimeter Institute, where this work was finalized.

- D. Robinson, "Four decades of black holes uniqueness theorems" in Kerr Spacetime: Rotating Black Holes in General Relativity, eds. D. L. Wiltshire, M. Visser and S. M. Scott (Cambridge University Press, Cambridge, UK, 2009), pp. 115-143.
- [2] S. W. Hawking, Commun. Math. Phys. 25, 152-166 (1972).
- [3] J. L. Friedman, K. Schleich, D. M. Witt, Phys. Rev. Lett. 71, 1486-1489 (1993). [gr-qc/9305017].
- [4] T. Jacobson, S. Venkataramani, Class. Quant. Grav. 12, 1055-1062 (1995). [gr-qc/9410023].
- [5] S. A. Hughes, C. R. Keeton, P. Walker, K. T. Walsh,
 S. L. Shapiro, S. A. Teukolsky, Phys. Rev. **D49**, 4004-4015 (1994);
 S. L. Shapiro, S. A. Teukolsky, J. Winicour,
 Phys. Rev. **D52**, 6982-6987 (1995).
- [6] D. Psaltis, D. Perrodin, K. R. Dienes, I. Mocioiu, Phys. Rev. Lett. 100, 091101 (2008). [arXiv:0710.4564 [astro-ph]].
- [7] P. Pani, V. Cardoso, Phys. Rev. **D79**, 084031 (2009).[arXiv:0902.1569 [gr-qc]].
- [8] N. Yunes, F. Pretorius, Phys. Rev. **D79**, 084043 (2009).[arXiv:0902.4669 [gr-qc]].
- [9] B. Kleihaus, J. Kunz, E. Radu, Phys. Rev. Lett. 106, 151104 (2011). [arXiv:1101.2868 [gr-qc]].
- [10] C. Bambi, arXiv:1109.4256 [gr-qc].
- [11] F. Caravelli, L. Modesto, Class. Quant. Grav. 27, 245022 (2010). [arXiv:1006.0232 [gr-qc]].
- [12] T. Johannsen, D. Psaltis, Phys. Rev. **D83**, 124015 (2011). [arXiv:1105.3191 [gr-qc]].
- [13] L. Modesto, J. W. Moffat, P. Nicolini, Phys. Lett. B 695, 397-400 (2011) [arXiv:1010.0680 [gr-qc]]; A. Smailagic and E. Spallucci, Phys. Lett. B 688, 82 (2010) [arXiv:1003.3918 [hep-th]]; L. Modesto and P. Nicolini, Phys. Rev. D 82, 104035 (2010) [arXiv:1005.5605 [gr-qc]]; P. Nicolini, A. Smailagic and E. Spallucci, hole," Phys. Lett. B 632, 547 (2006) [arXiv:gr-qc/0510112]; P. Nicolini, Quantum Gravity: A Int. J. Mod. Phys. A 24, 1229 (2009) [arXiv:0807.1939 [hep-th]].
- [14] L. Modesto, arXiv:1107.2403 [hep-th].
- [15] C. Rovelli, "Quantum gravity" (Cambridge University Press, Cambridge, UK, 2004).
- [16] L. Modesto, Int. J. Theor. Phys. 49, 1649-1683 (2010). [arXiv:0811.2196 [gr-qc]]; L. Modesto, I. Premont-Schwarz, Phys. Rev. D80, 064041 (2009). [arXiv:0905.3170 [hep-th]]; L. Modesto, Adv. High Energy Phys. 2008, 459290 (2008). [gr-qc/0611043]; L. Modesto, Class. Quant. Grav. 23, 5587-5602 (2006). [gr-qc/0509078]; A. Ashtekar, M. Bojowald, Class. Quant. Grav. 23, 391-411 (2006). [gr-qc/0509075]; L. Modesto, Phys. Rev. D70, 124009 (2004) [gr-qc/0407097]; L. Modesto, Int. J. Theor. Phys. 45, 2235-2246 (2006); E. Alesci, L. Modesto, arXiv:1101.5792 [gr-qc].
- [17] S. Kloster, J. Brannlund, A. DeBenedictis, Class. Quant. Grav. 25, 065008 (2008). [gr-qc/0702036]; J. Brannlund,

- S. Kloster, A. DeBenedictis, Phys. Rev. **D79**, 084023 (2009). [arXiv:0901.0010 [gr-qc]]; A. DeBenedictis,
 S. Kloster, J. Brannlund, Class. Quant. Grav. **28**, 105023 (2011). [arXiv:1101.4631 [gr-qc]].
- [18] E. Brown, R. B. Mann, L. Modesto, Phys. Lett. B695, 376-383 (2011). [arXiv:1006.4164 [gr-qc]].
- [19] H. Ohanian, R. Ruffini, Gravitation and Spacetime, (W. W. Norton and Shin Won Agency Co., Seoul, South Korea, 2001); E. Poisson, A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics, (Cambridge University Press, Cambridge, UK, 2004).
- [20] E. Barausse, V. Cardoso, G. Khanna, Phys. Rev. Lett. 105, 261102 (2010). [arXiv:1008.5159 [gr-qc]].
- [21] J. M. Bardeen, Nature 226, 64 (1970).
- [22] I. D. Novikov, K. S. Thorne, "Astrophysics of Black Holes" in *Black Holes*, edited by C. De Witt and B. De Witt (Gordon and Breach, New York, New York, 1973), pp. 343-450.
- [23] C. Bambi, E. Barausse, Astrophys. J. 731, 121 (2011).
 [arXiv:1012.2007 [gr-qc]].
- [24] C. Bambi, E. Barausse, arXiv:1108.4740 [gr-qc].
- [25] C. Bambi, Europhys. Lett. 94, 50002 (2011).
 [arXiv:1101.1364 [gr-qc]]; C. Bambi, JCAP 1105, 009 (2011).
 [arXiv:1103.5135 [gr-qc]]; C. Bambi, arXiv:1104.2218 [gr-qc].
- [26] C. Bambi, K. Freese, Phys. Rev. D79, 043002 (2009).
 [arXiv:0812.1328 [astro-ph]]; C. Bambi, K. Freese, T. Harada, R. Takahashi, N. Yoshida, Phys. Rev. D80, 104023 (2009). [arXiv:0910.1634 [gr-qc]]; C. Bambi, T. Harada, R. Takahashi, N. Yoshida, Phys. Rev. D81, 104004 (2010). [arXiv:1003.4821 [gr-qc]]; C. Bambi, N. Yoshida, Phys. Rev. D82, 064002 (2010). [arXiv:1006.4296 [gr-qc]]; C. Bambi, N. Yoshida, Phys. Rev. D82, 124037 (2010). [arXiv:1009.5080 [gr-qc]].
- [27] K. S. Thorne, Astrophys. J. **191**, 507 (1974).
- [28] C. M. Will, Living Rev. Rel. 9, 3 (2005). [gr-qc/0510072].
- [29] J. G. Williams, S. G. Turyshev, D. H. Boggs, Phys. Rev. Lett. 93, 261101 (2004). [gr-qc/0411113].
- [30] B. Bertotti, L. Iess, P. Tortora, Nature **425**, 374 (2003).
- [31] J. -E. Daum, M. Reuter, PoS CNCFG2010, 003 (2010); J. -E. Daum, M. Reuter, arXiv:1012.4280 [hepth]; D. Benedetti, S. Speziale, JHEP 1106, 107 (2011). [arXiv:1104.4028 [hep-th]].
- [32] C. Bambi, Phys. Rev. **D83**, 103003 (2011). [arXiv:1102.0616 [gr-qc]].
- [33] M. Elvis, G. Risaliti, G. Zamorani, Astrophys. J. 565, L75-L77 (2002). [astro-ph/0112413]; J. -M. Wang, Y. -M. Chen, L. C. Ho, R. J. McLure, Astrophys. J. 642, L111-L114 (2006). [astro-ph/0603813]; S. W. Davis, A. Laor, Astrophys. J. 728, 98 (2011). [arXiv:1012.3213 [astro-ph.CO]].
- [34] V. Cardoso, P. Pani, M. Cadoni, M. Cavaglia, Phys. Rev. D77, 124044 (2008).
 [arXiv:0709.0532 [gr-qc]]; P. Pani, E. Barausse, E. Berti, V. Cardoso, Phys. Rev. D82,

044009 (2010). [arXiv:1006.1863 [gr-qc]].