

## **Outline Today's Lecture**

- finish Euler Equations and Transversality Condition
- Principle of Optimality: Bellman's Equation
- ullet Study of Bellman equation with bounded F
- contraction mapping and theorem of the maximum

#### Infinite Horizon $T=\infty$

$$V^*(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

subject to,

$$x_{t+1} \in \Gamma\left(x_t\right) \tag{1}$$

with  $x_0$  given

- sup {} instead of max {}
- $\bullet$  define  $\left\{x_{t+1}'\right\}_{t=0}^{\infty}$  as a plan
- define  $\Pi\left(x_{0}\right)\equiv\left\{ \left\{ x_{t+1}^{\prime}\right\} _{t=0}^{\infty}\left|x_{t+1}^{\prime}\in\Gamma\left(x_{t}^{\prime}\right)\right.$  and  $x_{0}^{\prime}=x_{0}\right\}$

# **Assumptions**

A1.  $\Gamma(x)$  is non-empty for all  $x \in X$ 

A2.  $\lim_{T\to\infty}\sum_{t=0}^{T}\beta^tF(x_t,x_{t+1})$  exists for all  $x\in\Pi(x_0)$  then problem is well defined

## Recursive Formulation: Bellman Equation

value function satisfies

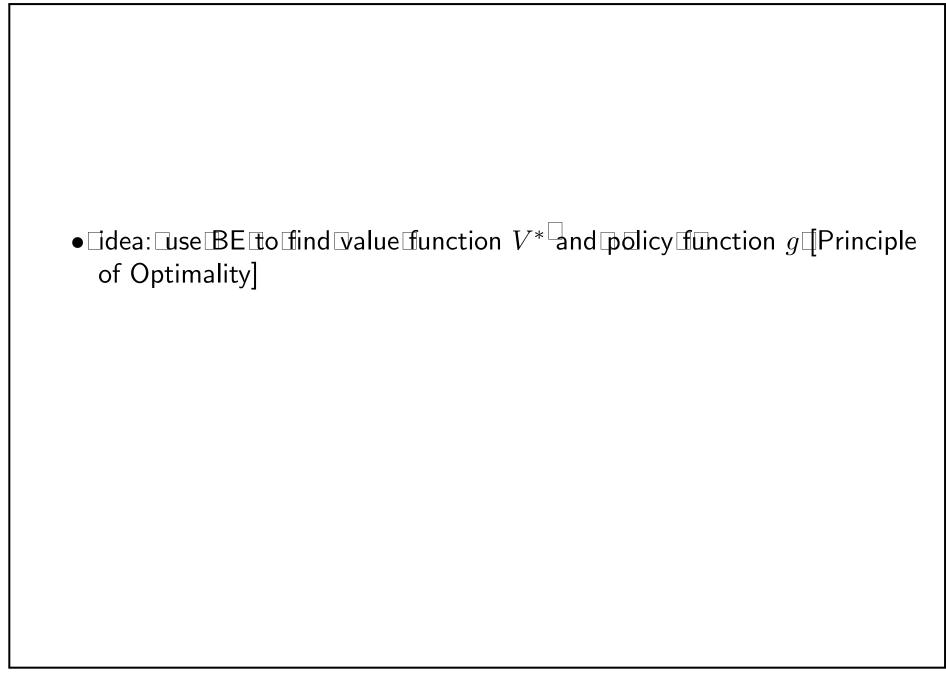
$$V^{*}(x_{0}) = \max_{\substack{\{x_{t+1}\}_{t=0}^{\infty} \\ x_{t+1} \in \Gamma(x_{t})}} \left\{ \sum_{t=0}^{\infty} \beta^{t} F(x_{t}, x_{t+1}) \right\}$$

$$= \max_{x_{1} \in \Gamma(x_{0})} \left\{ F(x_{0}, x_{1}) + \max_{\substack{\{x_{t+1}\}_{t=1}^{\infty} \\ x_{t+1} \in \Gamma(x_{t})}} \sum_{t=1}^{\infty} \beta^{t} F(x_{t}, x_{t+1}) \right\}$$

$$= \max_{x_{1} \in \Gamma(x_{0})} \left\{ F(x_{0}, x_{1}) + \beta \max_{\substack{\{x_{t+1}\}_{t=1}^{\infty} \\ x_{t+1} \in \Gamma(x_{t})}} \sum_{t=0}^{\infty} \beta^{t} F(x_{t+1}, x_{t+2}) \right\}$$

$$= \max_{x_{1} \in \Gamma(x_{0})} \left\{ F(x_{0}, x_{1}) + \beta V^{*}(x_{1}) \right\}$$

continued...



## Bellman Equation: Principle of Optimality

• Principle of Optimality idea: use the functional equation

$$V(x) = \max_{y \in \Gamma(x)} \left\{ F(x, y) + \beta V(y) \right\}$$

to find  $V^*$  and g

- note: nuisance subscripts t, t + 1, dropped
- ullet a solution is a function  $V\left(\cdot\right)$  the same on both sides
- **IF** BE has unique solution then  $V^* = V$
- ullet more generally the "right solution" to (BE) delivers  $V^*$