C. Barrabès P. A. Hogan

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Light-Like Signals from Violent Astrophysical Events

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C. Barrabès
University of Tours, France

P. A. Hogan
University College Dublin, Ireland

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SINGULAR NULL HYPERSURFACES IN GENERAL RELATIVITY
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For Werner Israel

Preface

Impulsive light–like signals in General Relativity are, like any models in theoretical physics, idealised mathematical models of objects that can arise in nature. In cataclysmic astrophysical events, such as supernovae and collisions of neutron stars, the final explosion produces a burst of matter travelling with the speed of light accompanied by a burst of gravitational radiation. These are the constituents of a general impulsive light–like signal. The gravitational fields of compact objects such as black–holes or neutron stars also resemble the fields of impulsive light–like signals when the objects are boosted to the speed of light. Spherically symmetric impulsive light–like signals are useful in the study of gravitational collapse and also in providing classical models of quantum phenomena in black–hole physics such as Hawking radiation and the entropy of a black-hole. Impulsive light–like signals play a central role in modeling black–hole production in high–energy collisions.

The space–time model of an impulsive light–like signal is a singular null hypersurface on which the Riemann curvature tensor exhibits a Dirac delta function singularity. This book is an exposition of the theory of such geometrical objects with a view to demonstrating its use in various physical scenarios. It is directed at readers who have a knowledge of General Relativity sufficient to carry out research in the subject. Thus we assume a knowledge of the tetrad formalism (in particular a familiarity with at least some aspects of the Newman–Penrose null tetrad formalism), the Cartan calculus of differential forms, extrinsic curvature and the Gauss–Codazzi equations, the Ehlers–Sachs theory of null geodesic congruences and the Petrov classification of gravitational fields. The notation and sign conventions we use are summarized in appendix A.

No man is an island and we both have developed our knowledge of General Relativity in collaboration with others. Among those with whom one

or other or both of us have collaborated are: G. F. R. Ellis, V. Frolov, T. Futamase, W. Israel, E. Poisson, I. Robinson, J. L. Synge and A. Trautman. During the writing of this book we have often consciously had these colleagues in mind as, in the words of Professor Synge, "an unofficial Board of Censors to eliminate obscurity and nonsense".

We were introduced to each other by Professor Werner Israel. He has been an inspiration to us and has shared with us his insight into the matters discussed in this book. It is natural and logical therefore to offer this work to him as a token of our gratitude.

We thank the Ministère des Affaires Étrangères for financially supporting our collaboration and in particular the French embassy in Dublin for their encouragement and support over the years. Our collaboration has involved many reciprocal visits over many years, which would not have been possible without the superb encouragement and cooperation of Annie and Pauline and our families, for which we are truly grateful.

C. Barrabès

P. A. Hogan

Contents

Chapter 1

Introduction

J. L. Synge constructed the earliest model in General Relativity of a light–like, spherical shell expanding into a vacuum, which he published under the provocative title [Synge (1957)] "A Model in General Relativity for the Instantaneous Transformation of a Massive Particle into Radiation". Of fundamental importance to the theory of impulsive light–like signals is Penrose's 'cut and paste' construction, illustrated most simply using his examples of a homogeneous plane impulsive gravitational wave and a 'spherical' wave, both propagating through a vacuum in which no gravitational field is present [Penrose (1972)].

1.1 Synge's Model

To see easily what Synge constructed and to get a perspective on it we start with the Kruskal form of the Schwarzschild line–element [Kruskal (1960)]

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \frac{32\, m^3}{r}\, \mathrm{e}^{-r/2m} dU\, dV \ , \tag{1.1}$$

with $r(U, V)$ given by

$$\left(\frac{r}{2m} - 1\right)\, \mathrm{e}^{r/2m} = -U\, V\ . \tag{1.2}$$

In figure 1.1 the region $\mathcal{M}^-$ is the subset $r \geq 0,\ U < -U_0$ $(U_0 > 0)$ of the white–hole part of the Kruskal extension of the Schwarzschild manifold. The future null–cone $U = -U_0$ is the history of Synge's spherical, light–like shell emerging from the singularity $r = 0$. In Synge's model $\mathcal{M}^+$ is Minkowskian space–time with line–element

$$ds_+^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - du\, dv\ , \tag{1.3}$$

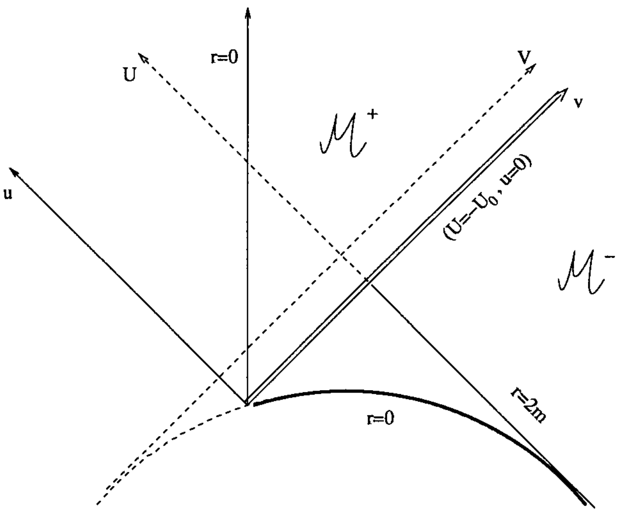

Fig. 1.1 $\mathcal{M}^-$ is the Schwarzschild white–hole space–time. $\mathcal{M}^+$ is Minkowskian space–time. The double–line is the history of the outgoing light–like shell.

with

$$r = \frac{1}{2}(v - u) , \tag{1.4}$$

and $u = 0$ corresponding to $U = -U_0$. The line–element of $\mathcal{M}^-$ is given by (1.1) with $r > 0$, $U < -U_0$. Synge required the line–elements in $\mathcal{M}^-$ and $\mathcal{M}^+$ to match continuously on $u = 0$ ($U = -U_0$). This means that we should have $r = \frac{v}{2}$ when $u = 0$ ($U = -U_0$) on account of (1.4). We see from (1.2) that when $U = -U_0$, r satisfies

$$\left(\frac{r}{2m} - 1\right) \mathrm{e}^{r/2m} = U_0 V . \tag{1.5}$$

This will be satisfied by $r = \frac{v}{2}$ if we express the Kruskal coordinate V in terms of v as

$$V = \frac{1}{U_0}\left(\frac{v}{4m} - 1\right) \mathrm{e}^{v/4m} . \tag{1.6}$$

Thus everywhere in $\mathcal{M}^-$ (1.2) now reads

$$\left(\frac{r}{2m} - 1\right) \mathrm{e}^{r/2m} = -\frac{U}{U_0}\left(\frac{v}{4m} - 1\right) \mathrm{e}^{v/4m} , \tag{1.7}$$

while the line–element (1.1) of $\mathcal{M}^-$ is given in coordinates (θ, ϕ, U, v) by

$$ds_-^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{4m}{U}\left(\frac{1-\frac{2m}{r}}{1-\frac{4m}{v}}\right) dv\, dU \ . \tag{1.8}$$

We can think of U as a function of u with $U(0) = -U_0$ and, intuitively, $dU/du > 0$. The coefficients of $du\, dv$ in (1.3) and (1.8) match on $u = 0$ if we take

$$U = -U_0 \mathrm{e}^{-u/4m} \ . \tag{1.9}$$

This puts the line-element (1.8) in the form

$$ds_-^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \left(\frac{1-\frac{2m}{r}}{1-\frac{4m}{v}}\right) du\, dv \ , \tag{1.10}$$

with $r(u, v)$ given by

$$\left(\frac{r}{2m} - 1\right) \mathrm{e}^{r/2m} = \left(\frac{v}{4m} - 1\right) \mathrm{e}^{(v-u)/4m} \ . \tag{1.11}$$

The line–element (1.10) with (1.11) is the form found by Synge [his equations (6.15) and (6.11) respectively] for the line–element of $\mathcal{M}^-$ while his form of the line–element of $\mathcal{M}^+$ is given by (1.3). This is consistent with $u = 0$ being the history of a light–like shell for which Einstein's field equations have the form (in units for which the gravitational constant and the speed of light are both unity)

$$R_{\mu\nu} = 8\pi\, \Omega(v)\, \delta(u)\, u_{,\mu}\, u_{,\nu} \ , \tag{1.12}$$

where $\delta(u)$ is the Dirac delta function which is singular on the null hypersurface $u = 0$. A comma denotes partial differentiation with respect to the coordinates $x^\mu = (\theta, \phi, u, v)$. The field equations (1.12) can be satisfied because whilst $r(u, v)$ is continuous across $u = 0$ its derivative $\partial r/\partial u$ is not. The jump in this quantity is easily seen to be

$$\left[\frac{\partial r}{\partial u}\right] = -\frac{m}{r} = -\frac{2m}{v} \ . \tag{1.13}$$

On account of (1.12), Ω is given by

$$-f\, \frac{\partial^2 r}{\partial u^2} + \frac{\partial r}{\partial u}\, \frac{\partial f}{\partial u} = 4\pi\, r\, f\, \Omega\, \delta(u) \ , \tag{1.14}$$

where f is the coefficient of $-2\,du\,dv$ in (1.3) and (1.10). From this we deduce that

$$\Omega(v) = \frac{m}{4\pi r^2} = \frac{m}{\pi v^2}\ . \tag{1.15}$$

In addition to (1.14) there are three non–trivial equations arising from (1.12), none of which contain second derivatives of f or r with respect to u. They therefore do not involve $\delta(u)$–terms and are satisfied by (1.3) and (1.10).

Synge constructed his model by solving Einstein's spherically symmetric field equations with a Ricci tensor of the form (1.12) which is concentrated on $U = -U_0$ $(u = 0)$ in the figure above and by requiring (a) the metric tensor components in $\mathcal{M}^-$ and $\mathcal{M}^+$ to match on $U = -U_0$ $(u = 0)$ and (b) elementary flatness on the time–like geodesic $r = 0$ in $\mathcal{M}^+$. It was unusual for the 1950's to take as his starting point a line–element of the form

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - 2\, f\, du\, dv\ , \tag{1.16}$$

with r, f each functions of the null coordinates u, v. The presentation of his results here shows how close he came to anticipating Kruskal. A simpler construction of Synge's model than that given here can be found in Chapter 3.

1.2 Penrose's Cut and Paste Approach

The next step in the study of singular null hypersurfaces in general relativity, in the way we wish to develop it here, was taken by Penrose in a classic paper [Penrose (1972)]. To illustrate his point of view we begin with the line–element of Minkowskian space–time in rectangular Cartesian coordinates and time (using units for which the speed of light is unity)

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2\ . \tag{1.17}$$

Putting $\sqrt{2}u = t - z\ ,\ \sqrt{2}v = t + z$ this takes the form

$$ds^2 = dx^2 + dy^2 - 2\,dudv\ . \tag{1.18}$$

Here in particular $u =$ constant are null hyperplanes generated by the null geodesic integral curves of the vector field $\partial/\partial v$. The coordinate v is an affine parameter along these null geodesics. A choice of (x, y) labels each such curve tangent to the hyperplane $u =$ constant. Also $u =$ constant is the history of a 2–plane in $\mathbb{R}^3$, parallel to the xy–plane, travelling with the speed of light in the positive z–direction. We can make one of the null hyperplanes $u = 0$ (say) the history of a homogeneous, impulsive (having profile $\delta(u)$, the Dirac delta function) plane gravitational wave by first subdividing Minkowskian space–time into two halves $\mathcal{M}^-(u \le 0)$ and $\mathcal{M}^+(u \ge 0)$ each with boundary $u = 0$. We shall write the line–element of $\mathcal{M}^-$ as

$$ds_-^2 = dx_-^2 + dy_-^2 - 2\,du_-dv_- \ , \tag{1.19}$$

and the line–element of $\mathcal{M}^+$ as

$$ds_+^2 = dx_+^2 + dy_+^2 - 2\,du_+dv_+ \ . \tag{1.20}$$

If we now re–attach $\mathcal{M}^-$ and $\mathcal{M}^+$ on $u = 0$ ($\Leftrightarrow u_- = 0 = u_+$) with *matching conditions* mapping points (x_-, y_-, v_-) on the minus side of $u = 0$ to points (x_+, y_+, v_+) on the plus side of $u = 0$ according to

$$x_+ = x_- \ , \qquad y_+ = y_- \ , \qquad v_+ = v_- + \frac{b}{2}(x_-^2 - y_-^2) + c\,x_-y_- \ , \tag{1.21}$$

where b, c are constants, then the resulting space–time $\mathcal{M}^- \cup \mathcal{M}^+$ has vanishing Ricci tensor everywhere, in particular on $u = 0$, and in Newman–Penrose notation (see Appendix) the only non–identically vanishing component of the Riemann curvature tensor is

$$\Psi_4 = (b - ic)\,\delta(u) \ . \tag{1.22}$$

This is a curvature tensor which is type N in the Petrov classification (the type associated with pure gravitational radiation) with $\partial/\partial v$ the degenerate principal null direction. The homogeneity of the wave with history $u = 0$ is due to the absence of a dependence of Ψ_4 on x, y. One way of checking these properties of $\mathcal{M}^- \cup \mathcal{M}^+$ is to write the line–element of $\mathcal{M}^- \cup \mathcal{M}^+$ in a coordinate system in which the metric tensor is continuous across $u = 0$ (this can be found using the information given in (1.19)–(1.21)) . The derivative of the metric tensor will jump across $u = 0$, as one would expect on account of (1.22). Then the curvature tensor can be directly calculated using this continuous metric tensor and shown to be given by (1.22), while the Ricci tensor of $\mathcal{M}^- \cup \mathcal{M}^+$ vanishes everywhere, in particular on $u = 0$. This is done in Chapter 2 for a general plane fronted impulsive light–like signal which includes the homogeneous impulsive gravitational wave described here. On the other hand these properties of $\mathcal{M}^- \cup \mathcal{M}^+$ can be verified by working directly in the coordinates $\{x_-, y_-, u_-, v_-\}$ in $\mathcal{M}^-$ and the coordinates $\{x_+, y_+, u_+, v_+\}$ in $\mathcal{M}^+$, with the matching conditions (1.21). We see from (1.19) and (1.20) that the line–element induced on the null hyperplane $u_+ = 0$ ($u = 0$) from its embedding in $\mathcal{M}^+$ is

$$dl_+^2 = dx_+^2 + dy_+^2 \ , \tag{1.23}$$

while the line–element induced on $u_- = 0$ ($u = 0$) from its embedding in $\mathcal{M}^-$ is

$$dl_-^2 = dx_-^2 + dy_-^2 \ . \tag{1.24}$$

Thus the matching conditions (1.21) ensure that $dl_+^2 = dl_-^2$. This is the most fundamental property of these matching conditions.

To exhibit Penrose's 'spherical' impulsive gravitational wave (it is not perfectly spherical as we shall see; a perfectly spherical gravitational wave propagating in a vacuum is impossible to construct in general relativity on account of the Birkhoff theorem [Birkhoff (1923)] which states that the only spherically symmetric vacuum gravitational field is the Schwarzschild field) we again start with the line–element of Minkowskian space–time (1.17) and this time write

$$x + iy = \sqrt{2}\, u\, \zeta \ , \quad z = -v + u\zeta\bar{\zeta} - \frac{1}{2}u \ , \quad t = -v + u\zeta\bar{\zeta} + \frac{1}{2}u \ , \tag{1.25}$$

where u, v are real and ζ is complex with complex conjugate $\bar{\zeta}$. Now (1.17) becomes

$$ds^2 = 2u^2 d\zeta\, d\bar{\zeta} + 2\, du\, dv \ . \tag{1.26}$$

The hypersurfaces v = constant are future null–cones with vertices on the null geodesic $u = 0$ and v is an affine parameter along this null geodesic, which is a common generator of each of the future null–cones. The other generators of the future null–cones are labelled by the complex coordinate ζ and u is an affine parameter along them. We wish to make one of these future null–cones $v = 0$ (say) the history of a spherical impulsive gravitational wave. Following Penrose we subdivide Minkowskian space–time into two halves $\mathcal{M}^-(v \leq 0)$ and $\mathcal{M}^+(v \geq 0)$ each having the null–cone $v = 0$ as boundary. We write the line–element of $\mathcal{M}^-$ as

$$ds_-^2 = 2u_-^2 d\zeta_- d\bar{\zeta}_- + 2\, du_- dv_- \ , \tag{1.27}$$

with $v_- = 0$ corresponding to $v = 0$, and we write the line–element of $\mathcal{M}^+$ as

$$ds_+^2 = 2u_+^2 d\zeta_+ d\bar{\zeta}_+ + 2\, du_+ dv_+ \ , \tag{1.28}$$

with $v_+ = 0$ corresponding to $v = 0$. Then the line–elements induced on the null–cone $v = 0$ by its embedding in $\mathcal{M}^-$ and $\mathcal{M}^+$ are

$$dl_-^2 = 2\,u_-^2\,d\zeta_-d\bar{\zeta}_- \qquad \text{and} \qquad dl_+^2 = 2\,u_+^2 d\zeta_+ d\bar{\zeta}_+ \ , \tag{1.29}$$

respectively. We now re–attach $\mathcal{M}^-$ and $\mathcal{M}^+$ on $v = 0$ ($\Leftrightarrow v_- = 0 = v_+$) mapping points $(\zeta_-, \bar{\zeta}_-, u_-)$ on the minus side of $v = 0$ to points $(\zeta_+, \bar{\zeta}_+, u_+)$ on the plus side of $v = 0$ so that $dl_-^2 = dl_+^2$. These *matching conditions* are

$$\zeta_+ = h(\zeta_-) \ , \ \bar{\zeta}_+ = \bar{h}(\bar{\zeta}_-) \ , \ u_+ = u_-/|h'(\zeta_-)| \ , \tag{1.30}$$

where $h(\zeta_-)$ is an analytic function of its argument and $h' = dh/d\zeta_-$. The mapping (1.30) in general takes generators of $v = 0$ on the minus side to generators on the plus side of $v = 0$ and at the same time rescales the affine parameter along them by different amounts on each generator leaving the vertex of the future null–cone invariant. Now the Ricci tensor of the re–attached space–time $\mathcal{M}^- \cup \mathcal{M}^+$ vanishes everywhere (in particular on $v = 0$) *automatically* and the Riemann curvature tensor is type N in the Petrov classification with $\partial/\partial u$ as degenerate principal null direction. Also the Riemann tensor is proportional to the Dirac delta function $\delta(v)$ which is singular on the future null–cone $v = 0$. The explanation for this miraculous result is given in Chapter 3. With the theory developed in Chapter 2 it can be verified using (1.28)–(1.30). We will however show here how it can be verified by writing the line–element of $\mathcal{M}^- \cup \mathcal{M}^+$ in a coordinate system in which the metric tensor is continuous across $v = 0$.

The line–element of the space–time $\mathcal{M}^- \cup \mathcal{M}^+$ described by (1.27)–(1.30) can be written in the form

$$ds^2 = 2\,U^2 \left| dZ + \frac{V\,\vartheta(V)}{2\,U}\,\bar{H}\,d\bar{Z}\right|^2 + 2\,dU\,dV \ , \tag{1.31}$$

which is continuous across $V = 0$ ($\Leftrightarrow v = 0$). Here Z is a complex coordinate with complex conjugate $\bar{Z}$ and U, V are real coordinates. Also $H(Z)$ is an analytic function of Z related to the analytic function h appearing in (1.30) and $\vartheta(V)$ is the Heaviside step function which is equal to unity if $V > 0$ and equal to zero if $V < 0$. We must confirm that (1.31) coincides with (1.27) when $V < 0$ and with (1.28) when $V > 0$ and also that (1.31) incorporates the matching conditions (1.30). To see this we first note that for $V < 0$ (and so $\vartheta(V) = 0$) the line–element (1.27) trivially agrees with (1.31) with

$$\zeta_- = Z \ , \ \bar{\zeta}_- = \bar{Z} \ , \ u_- = U \ , v_- = V \ . \tag{1.32}$$

For $V > 0$ (and thus $\vartheta(V) = 1$) we can put (1.28) in the form (1.31) with the transformation

$$\zeta_+ = h(Z) + \frac{V}{2U}\,\frac{h'\bar{h}''}{\bar{h}'}\left(1 - \frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right)^{-1} , \tag{1.33}$$

$$\bar{\zeta}_+ = \bar{h}(\bar{Z}) + \frac{V}{2U}\,\frac{\bar{h}'h''}{h'}\left(1 - \frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right)^{-1} , \tag{1.34}$$

$$u_+ = \frac{U}{|h'|}\left(1 - \frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right) , \tag{1.35}$$

$$v_+ = V\,|h'|\left(1 - \frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right)^{-1} . \tag{1.36}$$

Here $h(Z)$ is an analytic function of its argument, $h' = dh/dZ$, $h'' = d^2h/dZ^2$ and $H(Z)$ in (1.31) is obtained from $h(Z)$ by

$$H = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 . \tag{1.37}$$

Equations (1.31) and (1.37) were first published in [Nutku and Penrose (1992)]. These equations were independently derived in [Hogan (1993)] where the transformations (1.33)–(1.36) were first published. From (1.32)–(1.36) we see that *on* $V = 0$ ($\Leftrightarrow v_- = 0 = v_+$) we have

$$\zeta_+ = h(Z) = h(\zeta_-) \ , \ \bar{\zeta}_+ = \bar{h}(\bar{Z}) = \bar{h}(\bar{\zeta}_-) \ , \ u_+ = \frac{U}{|h'|} = \frac{u_-}{|h'|} , \tag{1.38}$$

which agrees with (1.30). *In this way the continuous form* (1.31) *of the line–element of* $\mathcal{M}^- \cup \mathcal{M}^+$ *is seen to have the very important property of incorporating the matching conditions* (1.30). A calculation of the Ricci tensor with the metric given via (1.31) shows that it vanishes everywhere in $\mathcal{M}^- \cup \mathcal{M}^+$, and in particular *on* $V = 0$. The Riemann tensor has only one non–vanishing Newman–Penrose component,

$$\Psi_4 = \frac{H(Z)}{2\,U}\,\delta(V) . \tag{1.39}$$

This is a Petrov type N Riemann tensor with $\partial/\partial U$ as degenerate principal null direction. It has a Dirac delta function singularity on $V = 0$. For these two reasons the future null–cone $V = 0$ is the history of an impulsive gravitational wave. The field (1.39) is also singular at $U = 0$ ($\Leftrightarrow u_+ = 0 = u_-$). This is not simply the vertex of the cone $V = 0$ but corresponds also to a null geodesic generator of the future null–cone $V = 0$. Hence there is a singular point on the expanding wave front. It is for this reason that we cannot consider the wave to be perfectly spherical. Finally we note from (1.35) and (1.36) that $u_+v_+ = U\,V$. This simple observation enables one to easily construct the Penrose wave propagating through the de Sitter or anti–de Sitter universes [Hogan (1992)].

1.3 Generalized Lorentz Transformations

It is natural to inquire into the origin of the coordinate transformation (1.33)–(1.36). One way of obtaining it, but not the only way, is as a generalization of a proper, orthochronous Lorentz transformation [Hogan (1994)]. It is easy to check, using (1.37), that $H(Z) = 0$ if and only if $h(Z)$ is fractional linear. In this case (1.30) is a proper, orthochronous Lorentz transformation of the future null–cone $V = 0$ to itself and there is no wave. One would have expected this. To derive (1.33)–(1.36) as a generalization of a proper, orthochronous Lorentz transformation we start in a standard way [Penrose and Rindler (1986)] by introducing the 2×2 Hermitian matrix

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} z-t & x-iy \\ x+iy & -(z+t)\end{pmatrix} . \tag{1.40}$$

For any $\mathcal{U}\in\mathrm{SL}(2,\mathbb{C})$ a proper, orthochronous Lorentz transformation can be written in the form

$$A \to \mathcal{U}\,A\,\mathcal{U}^{\dagger} , \tag{1.41}$$

where $\mathcal{U}^{\dagger}$ is the Hermitian conjugate of $\mathcal{U}$. If we write

$$\mathcal{U} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} , \tag{1.42}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ satisfying

$$\alpha\,\delta - \beta\,\gamma = 1 , \tag{1.43}$$

we can use (1.25) to express (1.41) as a coordinate transformation from $(\zeta, \bar\zeta, u, v)$ to new coordinates $(Z, \bar Z, U, V)$ or vice versa, where the line–element of Minkowskian space–time in these new coordinates has the same form as (1.26). Thus (1.41) is found to be equivalent to

$$u = (|\alpha|^2 - \sqrt{2}\,\alpha\,\bar\beta\,\bar Z - \sqrt{2}\,\bar\alpha\,\beta\,Z + 2|\beta|^2|Z|^2)\,U - 2\,|\beta|^2 V , \tag{1.44}$$

$$\zeta = \frac{(-\bar{\alpha}\,\gamma + \sqrt{2}\,\bar{\alpha}\,\delta\,Z + \sqrt{2}\,\bar{\beta}\,\gamma\,\bar{Z} - 2\bar{\beta}\,\delta\,|Z|^2)\,U + 2\bar{\beta}\,\delta\,V}{\sqrt{2}\,(|\alpha|^2 - \sqrt{2}\bar{\beta}\,\bar{Z} - \sqrt{2}\,\bar{\alpha}\,\beta\,Z + 2\,|\beta|^2|Z|^2)\,U - 2\sqrt{2}\,|\beta|^2 V}\,, \tag{1.45}$$

with v given by

$$v - u\,|\zeta|^2 = (-\frac{1}{2}|\gamma|^2 + \frac{1}{\sqrt{2}}\gamma\,\bar{\delta}\,\bar{Z} + \frac{1}{\sqrt{2}}\bar{\gamma}\,\delta\,Z - |\delta|^2|Z|^2)\,U + |\delta|^2 V\,. \tag{1.46}$$

Now take $h(Z)$ to be the fractional linear function

$$h(Z) = \frac{1}{\sqrt{2}}\left(\frac{\gamma - \sqrt{2}\,\delta\,Z}{-\alpha + \sqrt{2}\,\beta\,Z}\right)\,. \tag{1.47}$$

We first note that

$$|h'|^{-1} = |-\alpha + \sqrt{2}\,\beta\,Z|^2\,, \tag{1.48}$$

and

$$\frac{h''}{h'} = \frac{-2\sqrt{2}\beta}{-\alpha + \sqrt{2}\,\beta\,Z}\,. \tag{1.49}$$

Using these we can write (1.44) in the form

$$u = \frac{U}{|h'|}\left(1 - \frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right)\,. \tag{1.50}$$

Now rearranging terms in (1.45) we can express it as

$$\zeta = h(Z) + \frac{V\,\left(2\,|\beta|^2 h(Z) + \sqrt{2}\,\bar{\beta}\,\delta\right)}{\left|-\alpha + \sqrt{2}\,\beta\,Z\right|^2 U - 2|\beta|^2 V}\,. \tag{1.51}$$

With the help of (1.48) and (1.49) this can be written

$$\zeta = h(Z) + \frac{V}{U}\left(\frac{2\,|\beta|^2 h(Z) + \sqrt{2}\,\bar{\beta}\,\delta}{\left|-\alpha + \sqrt{2}\,\beta\,Z\right|^2}\right)\left(1 - \frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right)^{-1}\,. \tag{1.52}$$

We can write

$$\frac{2\,|\beta|^2 h(Z) + \sqrt{2}\,\bar{\beta}\,\delta}{\left|-\alpha + \sqrt{2}\,\beta\,Z\right|^2} = \frac{1}{2}\,\frac{h'\,\bar{h}''}{\bar{h}'}\,, \tag{1.53}$$

and thus (1.52) finally becomes

$$\zeta = h(Z) + \frac{V}{2U}\,\frac{h'\bar{h}''}{\bar{h}'}\left(1 - \frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right)^{-1}\,. \tag{1.54}$$

Now solving (1.46) for v we obtain

$$v = \frac{U\,V}{\left|-\alpha + \sqrt{2}\,\beta\,Z\right|^2 U - 2\,|\beta|^2 V} \ . \tag{1.55}$$

With the use of (1.48) and (1.49) this becomes

$$v = V\,|h'|\,\left(1 - \frac{V}{4U}\left|\frac{h''}{h'}\right|^2\right)^{-1} \ . \tag{1.56}$$

In (1.54) and its complex conjugate, along with (1.50) and (1.56), we have equations of the same form as (1.33)–(1.36) respectively. Thus (1.33)–(1.36) with $h(Z)$ given by (1.47) is the Lorentz transformation corresponding to the SL$(2,\mathbb{C})$ element (1.42) with (1.43). Hence we see that we obtain (1.33)–(1.36) by writing the Lorentz transformation in the form (1.50), (1.54) and (1.56) and then *generalizing these formulas by allowing $h(Z)$ in them to be an arbitrary analytic function.* If (1.50), (1.54) and (1.56) are substituted into the line–element (1.26), with $h(Z)$ arbitrary, then the line–element (1.31) emerges with $\vartheta(V) = 1$.

The procedure for generalization of the Lorentz group described here can be exploited to express the line–element of Penrose's space–time $\mathcal{M}^- \cup \mathcal{M}^+$ in different coordinates in which the metric tensor is continuous across the future null–cone $V = 0$. One can obtain this line–element in the interesting forms [Hogan (1994)]

$$ds^2 = 2U^2 p^{-2}\left|dZ + \frac{V\,\vartheta(V)}{2U}\,\bar{H}(\bar{Z})\,p^2 d\bar{Z}\right|^2 + 2dU\,dV - K\,dV^2 \ , \tag{1.57}$$

with

$$p = 1 + \frac{K}{2}\,|Z|^2 \ , \tag{1.58}$$

and $K = 0, \pm 1$. The case $K = 0$ is given by (1.31). Now instead of (1.39) we have

$$\Psi_4 = \frac{1}{2U}\,p^2 H(Z)\,\delta(V) \ . \tag{1.59}$$

For $K \neq 0$ the singularity $U = 0$ here corresponds to the vertex of the future null–cone $V = 0$ and the presence of the function $p \neq 1$ shows that the singular point on the wave front corresponds to $Z = \infty$. Further lines of development of the matters discussed in this section can be found in the cited works of Griffiths and Podolsky. The backscattered gravitational radiation left behind after a Penrose wave propagates radially through a Schwarzschild gravitational field is described in [Barrabès and Hogan (1994)]. Imploding–exploding Penrose waves are studied in [Hogan (1995)].

Chapter 2

General Description of an Impulsive Light–Like Signal

An impulsive light-like signal exists whenever the Riemann curvature tensor of the space–time manifold contains a Dirac δ-function with support on a null hypersurface. Such a signal can be an impulsive gravitational wave (a wave having a δ-function profile) or a thin shell of light-like matter (a neutrino fluid, for example) or a mixture of both, and its history is the null hypersurface on which the Riemann tensor is singular. Another equivalent way of describing an impulsive light-like signal is to say that there exists in the space–time a null hypersurface across which the metric tensor is only C^0. This second statement means that the metric is continuous across the null hypersurface but its first derivatives are not, and therefore a Dirac δ-function term appears in the curvature tensor.

The singular null hypersurface which represents the world-sheet or history of an impulsive light–like signal divides the space–time manifold into two domains with different metrics. Two equivalent ways of describing this situation will be considered [Barrabès and Israel (1991)]. One is based on the properties of distributions and it uses a common set of coordinates covering both sides of the null hypersurface. The other is a generalization to arbirary geometries of the 'cut and paste' approach of Penrose and it allows the space–time coordinates to be chosen freely and independently on the two sides of the singular null hypersurface. Each of these two descriptions has its own merits and both can be used alternatively. For comparison purposes the non–null case is summarized in appendix B.

As in Chapter 1 we call $\mathcal{M}^{\pm}$ the two domains of space–time $\mathcal{M}$ and $\mathcal{N}$ is their null hypersurface common boundary. The space–time $\mathcal{M} = \mathcal{M}^{+} \bigcup \mathcal{M}^{-}$ admits a pair of metric tensors g^{+} and g^{-}, defined on $\mathcal{M}^{+}$ and $\mathcal{M}^{-}$ respectively. Tensor fields of all types defined on $\mathcal{M}^{+}$ will be designated by an index + while those defined on $\mathcal{M}^{-}$ will be designated by an index −. To denote the jump across $\mathcal{N}$ of a tensor field $F^{\pm}$ defined on

$\mathcal{M}^{\pm}$ respectively, we will use the notation

$$[F] = F^{+}|_{\mathcal{N}} - F^{-}|_{\mathcal{N}} , \tag{2.1}$$

where $|_{\mathcal{N}}$ indicates that $F^{\pm}$ are to be evaluated *on* the $\pm$ sides of $\mathcal{N}$ respectively. Further notation and sign conventions can be found in appendix A.

2.1 Distributional Algorithm

Let $\{x^{\mu}\}$ be a local coordinate system covering both sides of the null hypersuface $\mathcal{N}$ and let $\Phi(x) = 0$ be the equation of $\mathcal{N}$ in these coordinates. The function Φ is chosen in such a way that $\Phi > 0$ in $\mathcal{M}^{+}$ and $\Phi < 0$ in $\mathcal{M}^{-}$. Greek indices take values $0, 1, 2, 3$ and the components of the metric tensor are $g^{\pm}_{\mu\nu}$ in $\mathcal{M}^{\pm}$ respectively.

If $F^{\pm}(x)$ are two tensor fields of the same type defined on $\mathcal{M}^{\pm}$ respectively, we define the hybrid tensor field $\tilde{F}$ by

$$\tilde{F}(x) = F^{+}\,\vartheta(\Phi) + F^{-}\,(1 - \vartheta(\Phi))\,, \tag{2.2}$$

where $\vartheta(\Phi)$ is the Heaviside step function introduced in (1.31). For the special case $F^{\pm} = g^{\pm}_{\mu\nu}$ we thus have $\tilde{g}_{\mu\nu}$ defined. However the metric is continuous across $\mathcal{N}$ and so for convenience we shall write

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} \qquad \text{with} \qquad [g_{\mu\nu}] = 0 \ , \tag{2.3}$$

in the notation introduced in (2.1). As a result of the definition (2.2) one has for two tensor fields of in general different types, $F^{\pm}$ and $G^{\pm}$, the product rule $\tilde{F}\tilde{G} = \tilde{FG}$. This is a consequence of the fact that $\vartheta(\Phi)(1 - \vartheta(\Phi))$ vanishes distributionally.

The normal vector field to the hypersurface $\mathcal{N}$ has components

$$n^{\mu} = \chi^{-1}(x)\, g^{\mu\nu}\, \Phi(x)_{,\nu} \ , \tag{2.4}$$

where, as always, the comma denotes partial differentiation with respect to x^{ν}. The function $\chi(x)$ defined on $\mathcal{N}$ is non–zero but is otherwise arbitrary. Throughout this book its sign will be chosen in order to have n^{μ} future–pointing. Since the hypersurface $\mathcal{N}$ is light–like its normal is light–like (or null) and thus satisfies

$$n \cdot n \equiv g_{\mu\nu} n^{\mu}\, n^{\nu}\,|_{\pm} = 0 \ . \tag{2.5}$$

Any vector orthogonal to n^μ is tangent to $\mathcal{N}$. Since n^μ is, by (2.5), orthogonal to itself it is tangent to $\mathcal{N}$. The integral curves of n^μ are the null geodesic generators of $\mathcal{N}$.

In order to describe the extrinsic properties of the null hypersurface $\mathcal{N}$ as imbedded in $\mathcal{M}$ we cannot use the normal to $\mathcal{N}$. We need to introduce a *transversal* vector field N defined on $\mathcal{N}$ and pointing *out* of $\mathcal{N}$. Thus N is chosen to satisfy

$$N \cdot n = g^{\mu\nu} N_\mu \, N_\nu \,|_\pm \;\equiv \eta^{-1} \neq 0 \;, \qquad N^+_\mu = N^-_\mu \equiv N_\mu . \tag{2.6}$$

The inequality in the first equation is the condition for N to be transverse to the hypersurface. The second equation ensures that the transversal is the same vector field on both sides of the hypersurface. If one takes $\eta < 0$ and if n is future-pointing then N is also future–pointing (it will often be convenient to take $\eta = -1$). We see that the transversal is not uniquely defined by the equations (2.6) because the scalar product $N \cdot n$ is invariant under the gauge transformation

$$N \rightarrow N' = N + v \,, \tag{2.7}$$

where v is an arbitrary vector tangent to the hypersurface $\mathcal{N}$. We also note that (2.4) and (2.6) lead to the relations $N^\mu \partial_\mu \Phi = \chi\,\eta^{-1}$ and $N = (\chi\,\eta^{-1})\partial/\partial\Phi$.

With these definitions the partial derivative of a quantity $\tilde{F}$ as defined above takes the following form

$$\partial_\mu \tilde{F} = \tilde{\partial_\mu F} \; + \; [F] \, \chi n_\mu \, \delta(\Phi) \, . \tag{2.8}$$

A singular term proportional to the Dirac δ-funtion appears whenever F is discontinuous across the hypersurface.

The metric and its tangential derivatives are continuous across $\mathcal{N}$, but its transverse derivatives are not. To characterize the discontinuities in the transverse derivatives of the metric tensor we define the symmetric tensor $\gamma_{\mu\nu}$ by

$$[\partial_\alpha g_{\mu\nu}] = \eta \, n_\alpha \, \gamma_{\mu\nu} \, . \tag{2.9}$$

This implies

$$N^\alpha [\partial_\alpha g_{\mu\nu}] = \gamma_{\mu\nu} \, . \tag{2.10}$$

The tensor $\gamma_{\mu\nu}$ is defined only on the null hypersurface $\mathcal{N}$ and, because of the non-uniqueness in the definition of the transversal, it is not uniquely defined by (2.9) or (2.10). We note that the last equation could have been replaced by

$$\gamma_{\mu\nu} = [\mathcal{L}_N \, g_{\mu\nu}] \ , \tag{2.11}$$

where $\mathcal{L}_N$ denotes the Lie derivative with respect to the vector field N. The only requirement on the choice of the tensor $\gamma_{\mu\nu}$ is that its projection onto the hypersurface $\mathcal{N}$ be unique. Such a condition leaves $\gamma_{\mu\nu}$ free up to the gauge transformation

$$\gamma_{\mu\nu} \to \gamma'_{\mu\nu} = \gamma_{\mu\nu} + v_\mu n_\nu + n_\mu v_\nu \ , \tag{2.12}$$

where v is an arbitrary vector field defined on $\mathcal{N}$.

Using (2.10) the Christoffel symbols are found to be given by

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} \ , \qquad [\Gamma^\lambda_{\mu\nu}] = \eta\{\gamma^\lambda_{(\mu} n_{\nu)} - \frac{1}{2}\gamma_{\mu\nu} n^\lambda\} \ . \tag{2.13}$$

Here round brackets around indices denote symmetrisation on those indices. Then using (2.8) the components of the Riemann curvature tensor $R_{\kappa\lambda\mu\nu}$, the Ricci tensor $R_{\mu\nu}$ and the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ each decompose into the sum of a tilded-term defined as in (2.2), and a term containing a Dirac δ-function:

$$R_{\kappa\lambda\mu\nu} = \tilde{R}_{\kappa\lambda\mu\nu} + \hat{R}_{\kappa\lambda\mu\nu} \, \eta \, \chi \, \delta(\Phi) \ , \tag{2.14}$$

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \hat{R}_{\mu\nu} \, \eta \, \chi \, \delta(\Phi) \ , \tag{2.15}$$

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} + \hat{G}_{\mu\nu} \, \eta \, \chi \, \delta(\Phi) \ . \tag{2.16}$$

The singular terms, indicated with a hat, are given by

$$\hat{R}_{\kappa\lambda\mu\nu} = 2 n_{[\kappa} \gamma_{\lambda][\mu} n_{\nu]} \ , \tag{2.17}$$

$$\hat{R}_{\mu\nu} = \gamma_{(\mu} n_{\nu)} - \frac{\gamma}{2} n_\mu n_\nu \ , \tag{2.18}$$

$$\hat{G}_{\mu\nu} = \gamma_{(\mu} n_{\nu)} - \frac{\gamma}{2} n_\mu n_\nu - \frac{\gamma^\dagger}{2} g_{\mu\nu} \ . \tag{2.19}$$

Square brackets around indices, as in (2.17), denote skew–symmetrisation on those indices and we have introduced the following quantities:

$$\gamma \equiv g^{\mu\nu}\gamma_{\mu\nu} \ , \qquad \gamma_\mu \equiv \gamma_{\mu\nu} n^\nu \ , \qquad \gamma^\dagger \equiv \gamma_{\mu\nu} n^\mu n^\nu = \gamma_\mu n^\mu \ . \tag{2.20}$$

With the Einstein tensor given by (2.16) we conclude from Einstein's field equations that the matter energy–momentum–stress tensor, if non–zero, will take the form

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + S_{\mu\nu} \, \eta\chi\delta(\Phi) \, . \tag{2.21}$$

The first term (with the tilde) corresponds to the matter content $T^{\pm}_{\mu\nu}$ of the exterior domains $\mathcal{M}^{\pm}$, if they are non–vacuum. The second term shows that there exists in general a shell of light–like matter, also called a null shell (with the null hypersurface $\mathcal{N}$ as its history in the space–time $\mathcal{M}$). The stress-energy tensor of the null shell is $T_{\mu\nu}|_{\mathcal{N}} = S_{\mu\nu}\,\eta\chi\delta(\Phi)$, where using (2.19), $S_{\mu\nu}$ is given by

$$16\pi S_{\mu\nu} = -\gamma n_\mu n_\nu - \gamma^\dagger g_{\mu\nu} + 2\gamma_{(\mu}n_{\nu)} \ . \tag{2.22}$$

We call $S_{\mu\nu}$ the surface stress-energy tensor of the null shell. It is symmetrical and satisfies the tangential property $S_{\mu\nu}n^\nu = 0$. We note from (2.22) that the condition for the non-existence of a null shell is

$$S_{\mu\nu} = 0 \ \Leftrightarrow \ \gamma_\mu = \frac{\gamma}{2}n_\mu \ . \tag{2.23}$$

It is tempting to interpret the three terms in (2.22) as being related to the energy density, the isotropic pressure and the energy current respectively of the null shell. However this is not directly possible as there is no rest–frame for a light–like shell. An operational definition of these quantities can be provided by the introduction of a family of free-falling observers whose world–lines intersect the null hypersurface $\mathcal{N}$, and by making measurements with respect to these observers [Poisson (2003)].

2.2 Transverse Curvature Algorithm

A second approach to modeling impulsive light–like signals involves choosing two local coordinate systems $\{x^\mu_\pm\}$ independently of each other in $\mathcal{M}^{\pm}$ respectively and introducing on $\mathcal{N}$ intrinsic coordinates $\{\xi^a\}$ with $a = 1, 2, 3$. The parametric equations of $\mathcal{N}$, considered embedded in $\mathcal{M}^{\pm}$, are $x^\mu_\pm = x^\mu_\pm(\xi^a)$ respectively. The three holonomic basis vectors $e_{(a)} = \partial/\partial\xi^a$ are tangent to $\mathcal{N}$ and have components

$$e^\mu_{(a)}|_\pm = \frac{\partial x^\mu_\pm}{\partial \xi^a} \ . \tag{2.24}$$

The fundamental matching condition between the geometries of $\mathcal{M}^{\pm}$ is that the induced metrics on $\mathcal{N}$ coincide. This reads

$$g_{ab} \equiv e_{(a)} \cdot e_{(b)} = g_{\mu\nu}\, e^\mu_{(a)}\, e^\nu_{(b)}|_\pm \ . \tag{2.25}$$

Since $\mathcal{N}$ is a null hypersurface this induced metric is degenerate and of rank 2. The normal vector n introduced in (2.4) and (2.5) satisfies

$$n \cdot e_{(a)}|_{\pm} = 0 \ . \tag{2.26}$$

If we choose the tangent vectors $e_{(a)}$ to $\mathcal{N}$ in such a way that one of them is the normal n then the other two are space–like vectors. In general we have

$$n^{\mu} = n^{a}\, e^{\mu}_{(a)} \ , \qquad g_{ab}\, n^{b} = 0 \ , \tag{2.27}$$

with the last equation a consequence of (2.26). As in the previous subsection we introduce a transversal N with the two conditions (2.6) now reading

$$N \cdot n|_{\pm} = n^{a}\, N_{a} = \eta^{-1} \neq 0 \ , \qquad N \cdot e_{(a)}|_{+} = N \cdot e_{(a)}|_{-} \equiv N_{a} \ . \tag{2.28}$$

The transversal is not unique and one has the transformation (2.7).

It is interesting here to recall the extrinsic curvature algorithm used for a time–like or space–like shell [Israel (1966)]. For a time–like shell the normal n is space–like. The extrinsic curvatures $K^{\pm}_{ab} = -n_{\mu}\, e^{\mu}_{(a)|\nu}\, e^{\nu}_{(b)}|_{\pm}$ (one for each side, with the subscript stroke denoting covariant differentiation) describe real transverse properties, and the surface stress-energy tensor of the shell is expressed solely in terms of the jump $[K_{ab}]$ of the extrinsic curvature (see appendix B). In our case the normal is tangential to the hypersurface. An expression like K_{ab} retains for a null hypersurface only tangential derivatives which are continuous across the surface, and therefore $[K_{ab}]$ vanishes identically. We thus replace the extrinsic curvature, calculated with n for a time–like shell, by the transverse curvatures $\mathcal{K}^{\pm}_{ab}$ defined by

$$\mathcal{K}^{\pm}_{ab} = -N_{\mu}\, e^{\mu}_{(a)|\nu}\, e^{\nu}_{(b)}|_{\pm} \ . \tag{2.29}$$

It is evident from this definition that $\mathcal{K}_{ab}$ is a symmetric three-tensor and that it depends on the choice of N. Under the transformation $N \longrightarrow N' = N + v^{a} e_{(a)}$ it transforms according to

$$\mathcal{K}_{ab} \rightarrow \mathcal{K}'_{ab} = \mathcal{K}_{ab} - v^{c}\, \Gamma_{cab} \ , \tag{2.30}$$

where $\Gamma_{cab} = e^{\mu}_{(c)}\, e_{(a)\mu|\nu}\, e^{\nu}_{(b)}$ are Ricci rotation coefficients. However since the last term in (2.30) only contains tangential derivatives of the metric it is continuous across $\mathcal{N}$ and the jump $[\mathcal{K}_{ab}]$ is invariant. We now define the symmetric three-tensor field on $\mathcal{N}$,

$$\gamma_{ab} \equiv 2[\mathcal{K}_{ab}] \ , \tag{2.31}$$

which is independent of the choice of the transversal N and is a true characterization of the imbedding of the null hypersurface.

The connection between γ_{ab} defined here and the four-tensor $\gamma_{\mu\nu}$ defined in the previous subsection is easily seen. Using (2.11) we have

$$\gamma_{\mu\nu}\, e^{\mu}_{(a)}\, e^{\nu}_{(b)} = \gamma_{ab}\ . \tag{2.32}$$

This relation gives a practical way to construct $\gamma_{\mu\nu}$. Having first obtained γ_{ab} from (2.29) and (2.31), $\gamma_{\mu\nu}$ is any four-tensor which has γ_{ab} as its projection onto $\mathcal{N}$. This is a manifestation of the indeterminacy of $\gamma_{\mu\nu}$ under the transformation (2.12).

Let us now consider the unique intrinsic contravariant components S^{ab} of the surface stress-energy tensor which appear in

$$S^{\mu\nu} \;=\; S^{ab} e^{\mu}_{(a)} e^{\nu}_{(b)}\,. \tag{2.33}$$

As the induced metric g_{ab} is degenerate it cannot be inverted in order to raise the Latin indices. Instead we use the fact that the four vectors $(N^{\mu}\ , e^{\mu}_{(a)})$ form a basis and write the completeness relation

$$g^{\mu\nu} = g_*^{ab}\, e^{\mu}_{(a)}\, e^{\nu}_{(b)} \;+\; 2\eta\, n^a\, e^{(\mu}_{(a)}\, N^{\nu)}\ . \tag{2.34}$$

This defines g_*^{ab} up to the following two conditions:

$$g_*^{ac} g_{cb} = \delta^a_b - \eta\, n^a\, N_b\ , \tag{2.35}$$

and

$$g_*^{ab}\, N_b + \eta\, n^a\, (N\cdot N) = 0\ . \tag{2.36}$$

One can see that $g_*^{ab} g_{ab} = 2$. Once the basis $\{N^{\mu},\ e^{\mu}_{(a)}\}$ is chosen then g_*^{ab} is obtained. Three of the six components are found from the three independent equations arising from (2.35) and (2.36). If the transversal N^{μ} is transformed according to (2.7) then $g_*^{ab} \to g_*^{ab} - \eta\,(v^a\, n^b + v^b\, n^a)$ with v^a defined by $v^{\mu} = v^a\, e^{\mu}_{(a)}$ and v^{μ} is the arbitrary tangent vector field to the hypersurface $\mathcal{N}$ appearing in (2.7). Introducing (2.20) and (2.34) into the expression (2.22) for $S^{\mu\nu}$ one obtains

$$16\,\pi\, S^{ab} = -(\gamma_{cd}\, g_*^{cd})\, n^a\, n^b - (\gamma_{cd}\, n^c\, n^d)\, g_*^{ab} + (g_*^{ac}\, n^b\, n^d + g_*^{bc}\, n^a\, n^d)\,\gamma_{cd}\ . \tag{2.37}$$

These contravariant components S^{ab} of the surface stress-energy tensor are independent of the freedom of choice of g_*^{ab} arising from the freedom (2.7) in the transversal. Thus S^{ab} are the intrinsic contravariant components of the surface stress-energy tensor. It is useful to note the following relations

$$\gamma^{\dagger} = \gamma_{\mu\nu}\, n^{\mu}\, n^{\nu} = \gamma_{ab}\, n^{a}\, n^{b} , \tag{2.38}$$

$$\gamma_{a} \equiv \gamma_{ab}\, n^{b} = \gamma_{\mu}\, e^{\mu}_{(a)} , \tag{2.39}$$

$$\gamma = \gamma_{\mu\nu}\, g^{\mu\nu} = \gamma_{ab}\, g_{*}^{ab} + 2\,\eta\, N^{\mu}\,\gamma_{\mu} . \tag{2.40}$$

One then sees that S^{ab} has the form

$$S^{ab} = \mu\, n^{a}\, n^{b} + P\, g_{*}^{ab} + Q^{a}\, n^{b} + Q^{b}\, n^{a} , \tag{2.41}$$

with

$$\mu = -\frac{1}{16\,\pi}\,\gamma_{cd}\, g_{*}^{cd} , \qquad P = -\frac{1}{16\,\pi}\,\gamma^{\dagger} , \qquad Q^{a} = \frac{1}{16\,\pi}\, g_{*}^{ac}\,\gamma_{c} . \tag{2.42}$$

These quantities can be operationally assigned the physical interpretations of surface energy density, surface isotropic pressure and surface energy current respectively, relative to a specified family of free–falling observers [Poisson (2003)]. Using (2.22), (2.27), (2.34) and (2.38) we note that $S_{ab} = S_{\mu\nu}\, e^{\mu}_{(a)}\, e^{\nu}_{(b)} = S^{cd}\, g_{ac}\, g_{db} = P\, g_{ab}$.

A choice of the intrinsic coordinates $\{\xi^{a}\}$, the normal n, the transversal N and the pseudo-inverse metric g_{*}^{ab} which is convenient in many applications is the following: Let λ be a parameter on the null geodesic generators of $\mathcal{N}$ which is common to both sides of the hypersurface (λ is not necessarily an affine parameter on the null generators) and let us assume that λ increases as one moves towards the future along the generators. We then take as intrinsic coordinates on $\mathcal{N}$ the set $\{\xi^{a}\} = \{\xi^{1} = \lambda, \xi^{A}\}$ where $A = 2, 3$. The normal n is chosen to coincide with the tangent basis vector associated with the parameter λ, thus $n = e_{(1)} = \partial/\partial\lambda$, and the transversal is (uniquely) defined by the four conditions $N \cdot n = -1$, $N \cdot e_{(A)} = 0$ and $N \cdot N = 0$. Hence both the normal and the transversal are future-pointing light–like vectors and are given by $n^{a} = \delta^{a}_{1}$, $N_{a} = -\delta^{1}_{a}$. Then one derives from (2.35) and (2.36) that $g_{*}^{1a} = 0$ and so the only non–vanishing components of g_{*}^{ab} are $g_{*}^{AB} = g^{AB}$, which is the inverse of g_{AB}. The completeness relation (2.34) then reads $g^{\mu\nu} = g^{AB}\, e^{\mu}_{(A)}\, e^{\nu}_{(B)} - 2\, n^{(\mu}\, N^{\nu)}$, and one has for the quantities (2.42),

$$\mu = -\frac{1}{16\,\pi}\,\gamma_{AB}\, g^{AB} , \qquad P = -\frac{1}{16\,\pi}\,\gamma_{11} , \qquad Q^{A} = \frac{1}{16\,\pi}\, g^{AB}\,\gamma_{B} , \tag{2.43}$$

with $\gamma_{A} = \gamma_{A1}$. We note the significant fact that the surface stress–energy tensor of a light–like shell has in general *four* independent components.

This is in contrast to the case of a time–like shell for which the surface stress–energy tensor has in general *six* independent components [Israel (1966)]. A computer algebra version of the theory presented here can be found in [Musgrave and Lake (1997)].

2.3 Splitting the Signal into a Shell and a Gravitational Wave

An impulsive light–like signal can be a null shell or an impulsive gravitational wave or a mixture of both. In the two previous sections we have exhibited the surface stress-energy tensor of the null shell, if it is non–zero, in a four dimensional form (2.22) and in an intrinsic way (2.37). In order to describe the wave part of the signal, if it exists, we must examine the Weyl tensor components $C_{\kappa\lambda\mu\nu}$ for the space–time $\mathcal{M}^- \cup \mathcal{M}^+$. It follows from (2.14)–(2.16) and (2.22) that the Weyl tensor components take the form

$$C_{\kappa\lambda\mu\nu} = \tilde{C}_{\kappa\lambda\mu\nu} + \hat{C}_{\kappa\lambda\mu\nu}\,\eta\,\chi\,\delta(\Phi)\,, \tag{2.44}$$

where $\hat{C}_{\kappa\lambda\mu\nu}$ is given by

$$\hat{C}_{\kappa\lambda\mu\nu} = 2\,n_{[\mu}\,\gamma_{\nu][\kappa}\,n_{\lambda]} - 8\,\pi\,\{S_{\kappa[\mu}\,g_{\nu]\lambda} - S_{\lambda[\mu}\,g_{\nu]\kappa}\} + \frac{16\,\pi\,S^{\alpha}_{\alpha}}{3}\,g_{\kappa[\mu}\,g_{\nu]\lambda}\,. \tag{2.45}$$

There is a part of the first term on the right hand side of (2.45) which is constructed from a part of $\gamma_{\mu\nu}$ which does not contribute to the surface stress–energy tensor. To see this it is best to consider the intrinsic form (2.37) of the surface stress–energy tensor. We see explicitly in (2.37) how S^{ab} is constructed algebraically from γ_{ab} given in (2.32). We note that γ_{ab} has in general six independent components. It is clear from (2.37) that a part of γ_{ab}, which we shall denote by $\hat{\gamma}_{ab}$, does not contribute to S^{ab}. This $\hat{\gamma}_{ab}$ satisfies

$$\hat{\gamma}_{ab}\,n^b = 0\;, \qquad g_*^{ab}\,\hat{\gamma}_{ab} = 0\;, \tag{2.46}$$

and so has only two independent components. These conditions on $\hat{\gamma}_{ab}$ are independent of the freedom in g_*^{ab} described following (2.36) above. We can write

$$\hat{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}g_*^{cd}\gamma_{cd}\,g_{ab} - 2\,\eta\,\gamma_{(a}N_{b)} + \eta^2\gamma^{\dagger}\{N_aN_b - \frac{1}{2}(N\cdot N)g_{ab}\}\;, \tag{2.47}$$

from which one easily checks that (2.46) are satisfied. Whilst $\hat{\gamma}_{ab}$ does not contribute to S^{ab} the remaining four components of γ_{ab} are needed to construct the four independent components of S^{ab}. The two components of $\hat{\gamma}_{ab}$ are required to construct the two degrees of freedom of polarization in general present in an impulsive gravitational wave. The precise decomposition of (2.45) into wave and matter parts is conveniently described by giving the components of these parts with respect to the basis $\{N^{\mu}\ , e^{\mu}_{(a)}\}$. Thus we can write

$$\hat{C}_{\kappa\lambda\mu\nu} = W_{\kappa\lambda\mu\nu} + M_{\kappa\lambda\mu\nu}\ , \tag{2.48}$$

with the components $W_{\kappa\lambda\mu\nu}e^{\kappa}_{(a)}\, e^{\lambda}_{(b)}\, e^{\mu}_{(c)}\, e^{\nu}_{(d)}$ and $W_{\kappa\lambda\mu\nu}e^{\kappa}_{(a)}\, e^{\lambda}_{(b)}\, e^{\mu}_{(c)}\, N^{\nu}$ of $W_{\kappa\lambda\mu\nu}$ on the basis vanishing identically and

$$W_{\kappa\lambda\mu\nu}e^{\kappa}_{(a)}\, N^{\lambda}\, e^{\mu}_{(b)}\, N^{\nu} = -\frac{1}{2}\,\eta^{-2}\hat{\gamma}_{ab}\ . \tag{2.49}$$

If we multiply the components $W_{\kappa\lambda\mu\nu}$ on the basis $\{N^{\mu}\ , e^{\mu}_{(a)}\}$ by n^{a} and use the first of (2.27) we find that

$$W_{\kappa\lambda\mu\nu}\, n^{\kappa} = 0\ , \tag{2.50}$$

so that this part of $\hat{C}_{\kappa\lambda\mu\nu}$ is type N in the Petrov classification with n^{μ} as fourfold degenerate principal null direction. Hence if this part of $\hat{C}_{\kappa\lambda\mu\nu}$ is non–zero then the signal with history $\mathcal{N}$ includes an impulsive gravitational wave having propagation direction n^{μ} in space–time.

The components of the matter part of $\hat{C}_{\kappa\lambda\mu\nu}$ in (2.48) on the basis $\{N^{\mu}\ , e^{\mu}_{(a)}\}$ are

$$M_{\kappa\lambda\mu\nu}e^{\kappa}_{(a)}e^{\lambda}_{(b)}e^{\mu}_{(c)}e^{\nu}_{d} = 8\,\pi\{g_{a[d}S_{c]b} - g_{b[d}S_{c]a}\} + \frac{16\,\pi}{3}S^{\alpha}_{\alpha}\, g_{a[c}g_{d]b}\ , \tag{2.51}$$

$$M_{\kappa\lambda\mu\nu}e^{\kappa}_{(a)}e^{\lambda}_{(b)}e^{\mu}_{(c)}N^{\nu} = 8\,\pi N_{[a}S_{b]c} - 8\pi g_{c[a}e^{\alpha}_{(b)]}S_{\alpha\beta}N^{\beta} + \frac{16\,\pi}{3}S^{\alpha}_{\alpha}\, g_{c[a}N_{b]}\ , \tag{2.52}$$

and

$$M_{\kappa\lambda\mu\nu}e^{\kappa}_{(a)}N^{\lambda}e^{\mu}_{(c)}N^{\nu} = -4\,\pi\{S_{ac}(N\cdot N) + g_{ac}\,S_{\alpha\beta}\,N^{\alpha}\,N^{\beta}\}$$
$$+8\,\pi\ S_{\alpha\beta}\,e^{\alpha}_{((a)}N_{c)}\,N^{\beta} + \frac{8\,\pi}{3}S^{\alpha}_{\alpha}\ \{g_{ac}\,N\cdot N - N_{a}\,N_{c}\}\ . \tag{2.53}$$

Returning now to (2.45) we see that contraction of $\hat{C}_{\kappa\lambda\mu\nu}$ with the normal gives

$$\hat{C}_{\kappa\lambda\mu\nu}\, n^{\nu} = -\frac{1}{2}\, n_{[\kappa}\gamma_{\lambda]}\, n_{\mu} + \frac{1}{6}\,\gamma^{\dagger}\, n_{[\kappa}\, g_{\lambda]\mu}\ . \tag{2.54}$$

The right hand side here vanishes if a light–like shell is not part of the signal, on account of (2.23). This is consistent with (2.50). A further contraction of (2.54) with the normal yields

$$\hat{C}_{\kappa\lambda\mu\nu}\,n^{\lambda}\,n^{\nu} = -\frac{1}{6}\,\gamma^{\dagger}\,n_{\kappa}\,n_{\mu}, \tag{2.55}$$

and this indicates that $M_{\kappa\lambda\mu\nu}$ is in general type II in the Petrov classification, with n^{μ} as a degenerate principal null direction, and specialises to type III if $\gamma^{\dagger}=0$. We note that we can have $M_{\kappa\lambda\mu\nu}=0$ even if a light–like shell is part of the signal with history $\mathcal{N}$ provided the surface stress–energy tensor is isotropic (with $P=0=Q^{a}$ in (2.41)). Hence the acid test for the existence of a light–like shell component to the signal remains the existence of a non–vanishing surface stress–energy tensor, whereas the acid test for the existence of an impulsive gravitational wave component to the signal is $W_{\kappa\lambda\mu\nu}\neq 0$.

2.4 An Example of a Plane Fronted Light–Like Signal

As an immediate illustration of the foregoing we consider a plane fronted impulsive light–like signal, which incorporates an impulsive gravitational wave and a light–like shell, propagating through a vacuum in which no gravitational field is present. In this case the history $\mathcal{N}$ of the signal is a null hyper*plane* in Minkowskian space–time. The continuous and the intrinsic techniques presented in the previous sections of this chapter will be used. We take the metrics of the two flat space–time domains $\mathcal{M}^{\pm}$ in the forms

$$ds^2_{\pm} = -2du_{\pm}dv_{\pm} + dx^2_{\pm} + dy^2_{\pm}\ , \tag{2.56}$$

and we label the coordinates $x^{\mu}_{\pm} = (u_{\pm}\ , v_{\pm}\ , x_{\pm}\ , y_{\pm})$ with $\mu = 0,1,2,3$. The equation of the null hypersurface $\mathcal{N}$ is $u_{+} = u_{-} = 0$ and $\mathcal{M}^{+}$ corresponds to $u_{+}\geq 0$ while $\mathcal{M}^{-}$ corresponds to $u_{-}\leq 0$. We take the matching conditions on $\mathcal{N}$ to be

$$x_{+} = x_{-}\ , \qquad y_{+} = y_{-}\ , \qquad v_{+} = F(v_{-}, x_{-}, y_{-})\ , \tag{2.57}$$

where F is arbitrary except for the assumption that $\partial F/\partial v_{-}\neq 0$. These conditions leave the induced line-element on $\mathcal{N}$, by its imbedding in $\mathcal{M}^{-}$ and in $\mathcal{M}^{+}$, invariant and so $ds^2|_{\mathcal{N}} = dx^2_{+} + dy^2_{+} = dx^2_{-} + dy^2_{-}$. This is the fundamental consideration in choosing (2.57).

A local coordinate system $x^\mu = (u, v, x, y)$ covering both sides of $\mathcal{N}$, in terms of which the metric tensor of $\mathcal{M}^- \cup \mathcal{M}^+$ is continuous across $\mathcal{N}$, can be obtained by making the trivial coordinate transformation in $\mathcal{M}^-$,

$$u_- = u\ , \qquad v_- = v\ , \qquad x_- = x\ , \qquad y_- = y\ , \tag{2.58}$$

and the non-trivial coordinate transformation in $\mathcal{M}^+$,

$$u_+ = \frac{u}{F_v}\ , \tag{2.59}$$

$$v_+ = F + \frac{u}{2\,F_v}\,(F_x^2 + F_y^2)\ , \tag{2.60}$$

$$x_+ = x + u\,\frac{F_x}{F_v}\ , \tag{2.61}$$

$$y_+ = y + u\,\frac{F_y}{F_v}\ . \tag{2.62}$$

The subscripts on $F(u, v, x, y)$ denote partial derivatives. In the coordinates (u, v, x, y) the equation of $\mathcal{N}$ is $u = 0$ and we see from (2.58) and (2.58)–(2.61) evaluated on $\mathcal{N}$ that the transformations (2.58) and (2.58)–(2.61) have the important property of incorporating the matching conditions (2.57). The continuous metric tensor of the space–time $\mathcal{M}^- \cup \mathcal{M}^+$ is given via the line–element [Barrabès and Hogan (2001)]

$$ds^2 = -2dv\left(du - \frac{u\vartheta(u)}{F_v}\,dF_v\right) + \left(dx + \frac{u\vartheta(u)}{F_v}\,dF_x\right)^2 + \left(dy + \frac{u\vartheta(u)}{F_v}\,dF_y\right)^2\ , \tag{2.63}$$

where, as always, $\vartheta(u)$ is the Heaviside step function.

To illustrate the distributional algorithm outlined in §2.1 we work in the local coordinates (u, v, x, y) with the metric tensor components $g_{\mu\nu}$ given via the line–element (2.63). We take as normal to $\mathcal{N}$, $n_\mu = -u_{,\mu}$ and for simplicity we select a transversal N^μ to satisfy $N^\mu\, n_\mu = -1$. Thus $\chi = -1$ in (2.4) and $\eta = -1$ in (2.6). The contravariant components of the normal are $n^\mu = \delta_1^\mu$ and a convenient choice for the transversal is $N^\mu = \delta_0^\mu$. Now the 4–tensor $\gamma_{\mu\nu}$ on $\mathcal{N}$ is found from (2.10) to be

$$\gamma_{\mu\nu} = \left[\frac{\partial g_{\mu\nu}}{\partial u}\right]\ , \tag{2.64}$$

which, using (2.63), results in

$$\gamma_\mu = \gamma_{\mu 2} = \left(0,\ \frac{2F_{vv}}{F_v},\ \frac{2F_{xv}}{F_v},\ \frac{2F_{yv}}{F_v}\right)\ , \tag{2.65}$$

$$\gamma = \frac{2}{F_v}\,(F_{xx} + F_{yy})\ , \qquad \gamma^\dagger = \gamma_{22} = \frac{2F_{vv}}{F_v}\ . \tag{2.66}$$

The surface stress-energy tensor is obtained from (2.22). We can also calculate the Riemann and Weyl tensors associated with the metric given via (2.63) and derive their singular parts. The singular part of the Weyl tensor has a Petrov Type N part and a Petrov Type II part given in Newman-Penrose notation by

$$\hat{\Psi}_4 = \frac{1}{2F_v}\,(F_{xx} - F_{yy} - 2iF_{xy})\ , \tag{2.67}$$

and

$$\hat{\Psi}_2 = \frac{1}{3}\,\frac{F_{vv}}{F_v}\ , \qquad \hat{\Psi}_3 = \frac{1}{\sqrt{2}F_v}\,(F_{xv} - iF_{yv})\ , \tag{2.68}$$

respectively. If $\hat{\Psi}_4 \neq 0$ then the impulsive light–like signal with history $u = 0$ includes an impulsive gravitational wave.

To study this plane fronted impulsive light–like signal using the transverse curvature algorithm of §2.2 we base our calculations on the metrics in $\mathcal{M}^\pm$ respectively, given via the line–element (2.56) in the local coordinates $\{x^\mu_\pm\}$, and on the matching conditions (2.57). The intrinsic coordinates on $\mathcal{N}(u_\pm = 0)$ may be chosen as $\xi^a = (v_-,\ x_-,\ y_-) = (v,\ x,\ y)$, using (2.58). The components (2.24) of the tangent basis vectors $e_{(a)} = \partial/\partial\xi^a$ are calculated using the matching conditions (2.57) along with (2.58). We find that on the minus side of $\mathcal{N}$ we have $e^\mu_{(a)}|_- = \delta^\mu_a$ and on the plus side of $\mathcal{N}$ we obtain $e^\mu_{(1)}|_+ = (0,\ F_v,\ 0,\ 0)$, $e^\mu_{(2)}|_+ = (0,\ F_x,\ 1,\ 0)$ and $e^\mu_{(3)}|_+ = (0,\ F_y,\ 0,\ 1)$. The normal is $n^\mu = e^\mu_{(1)}$ and the transversal N^μ is uniquely defined by $N^\mu\, n_\mu = -1$, $N^\mu\, N_\mu = 0$ and $N_\mu\, e^\mu_{(A)} = 0$ with $A = 2, 3$ and $\xi^A = (x, y)$. This results in $N^\mu|_- = \delta^\mu_0$ and $N^\mu|_+ = (F_v^{-1},\ 2\,F_v^{-1}(F_x^2 + F_y^2),\ F_v^{-1}F_x,\ F_v^{-1}F_y)$. We are now in a position to calculate the transverse curvatures $\mathcal{K}^\pm_{ab}$ from (2.29) and the 3–tensor γ_{ab} from (2.31). The result can be neatly summarised as

$$\gamma_{ab} = \frac{2\,F_{ab}}{F_v}\ , \tag{2.69}$$

where, as always, the subscripts on F denote partial derivatives. Since the induced metric (2.25) on $\mathcal{N}$ now has components $g_{1b} = 0$, $g_{AB} = \delta_{AB}$ $(A,\ B = 2,\ 3)$, we can take its pseudo–inverse g^{ab}_* to have components $g^{1b}_* = 0$, $g^{AB}_* = \delta_{AB}$. Now the intrinsic form of the surface stress–energy tensor is given by (2.41) with

$$\mu = -\frac{1}{8\,\pi\,F_v}\,(F_{xx}+F_{yy})\ , \qquad P = -\frac{F_{vv}}{8\,\pi F_v}\ , \qquad Q^A = -\frac{F_{Av}}{8\,\pi F_v}\ . \quad (2.70)$$

In addition $\hat{\gamma}_{ab}$ in (2.47), the part of γ_{ab} which describes the gravitational wave component of the signal if it exists, has the two non–vanishing components

$$\hat{\gamma}_{22} = -\hat{\gamma}_{33} = \frac{1}{F_v}\,(F_{xx}-F_{yy})\ , \qquad \hat{\gamma}_{23} = \hat{\gamma}_{32} = \frac{2\,F_{xy}}{F_v}\ . \qquad (2.71)$$

The compatibility between the results obtained using the distributional algorithm approach in the previous paragraph and using the transverse curvature algorithm here is now obvious. The example given by Penrose of a homogeneous plane fronted impulsive gravitational wave described in §1.2 is a particular case of the plane fronted impulsive light–like signal corresponding to $F = v + \frac{b}{2}(x^2 - y^2) + cxy$. It is easy to see from the calculations above that $\hat{\Psi}_2 = \hat{\Psi}_3 = 0$, $\hat{\Psi}_4 = (b - ic)$ and $\mu = P = Q^A = 0$ confirming that in this case the signal consists only of a gravitational wave. If one takes instead $F = v - \frac{a}{2}(x^2+y^2) + \frac{b}{2}(x^2-y^2) + cxy$ one gets the same value for $\hat{\Psi}_4$ but now there also exists a null shell with a surface energy density $\mu = a/4\pi$ and vanishing surface isotropic pressure and energy current.

The specific example at hand of a plane fronted impulsive light–like signal affords us the opportunity of a provocative illustration regarding the parametrisation of the null geodesic generators of $\mathcal{N}$. The choice of parameter is implicit in the choice of normal. We have chosen v as the parameter since the normal is $n = \partial/\partial v$ in our example. The question naturally arises: is v an affine parameter and is the answer identical for both sides of $\mathcal{N}$? We can immediately say that the answer to the second part of the question is 'no' because the components of the Riemannian connection jump across $\mathcal{N}$. In the above example one can check that

$$n_{\mu|\nu}\,n^{\nu}|_- = 0\ , \qquad n_{\mu|\nu}\,n^{\nu}|_+ = \frac{1}{2}\,\gamma^{\dagger}\,n_{\mu} = \frac{F_{vv}}{F_v}\,n_{\mu}\ , \qquad (2.72)$$

where, as always, the subscript stroke denotes covariant differentiation. Thus v is an affine parameter on the minus side of $\mathcal{N}$ but not on the plus side of $\mathcal{N}$. It is affine on both sides of $\mathcal{N}$ if $\gamma^{\dagger}$ vanishes or equivalently if F is linear in v, which requires the surface pressure P to vanish as one can see from (2.70). To put this in a general perspective we recall the Penrose classification [Penrose (1972)] of the geometry that the null hypersurface $\mathcal{N}$

inherits from $\mathcal{M}^-$ and $\mathcal{M}^+$. Since any null hypersurface $\mathcal{N}$ is generated by a uniquely defined two–parameter family of null geodesics one can consider a hierarchy of three types of intrinsic geometries in order of increasing structure:

Type I: The induced metrics match on $\mathcal{N}$.

Type II: Type I with parallel transport of the normal n^μ along the null generators matching.

Type III: Type II with parallel transport of any tangent vector to $\mathcal{N}$ along the null generators matching.

A type I geometry on $\mathcal{N}$ is the most general of the three types and is always assumed to hold in our considerations. A type II geometry requires that $n^\alpha{}_{|\mu}\, n^\mu|_+ = n^\alpha{}_{|\mu}\, n^\mu|_-$ or using (2.13) that $\gamma^\dagger = 0$. If we define the acceleration parameter κ by

$$n^\alpha{}_{|\mu}\, n^\mu = \kappa\, n^\alpha \ , \tag{2.73}$$

then $[\kappa] = \eta\, \gamma^\dagger/2$ and hence a type II geometry implies $[\kappa] = 0$. In particular this is realised when the null generators are affinely parametrised on both sides of $\mathcal{N}$. A type III geometry requires that $v^\alpha{}_{|\mu}\, n^\mu|_+ = v^\alpha{}_{|\mu}\, n^\mu|_-$ for any vector v^μ such that $n_\mu\, v^\mu = 0$, and using (2.13) this implies that $\gamma_\mu = A\, n_\mu$ where A is an arbitrary function on $\mathcal{N}$. *A physical interpretation in terms of the surface stress–energy of a null shell*, if such an object exists having history $\mathcal{N}$, *can be given to the type II and type III geometries.* The surface pressure is given by (2.42) and so a type II geometry corresponds to a pressure–free null shell. For a type III geometry the surface stress–energy tensor (2.22) reduces to $16\,\pi\, S_{\mu\nu} = (2A - \gamma)\, n_\mu\, n_\nu$. In this case there are no surface stresses, anisotropic or isotropic, and furthermore if $A = \gamma/2$ there is no shell and $\mathcal{N}$ is the history of an impulsive gravitational wave provided $\hat{\gamma}_{ab} \neq 0$.

Chapter 3

Illustrations and Implications of the Bianchi Identities

Impulsive light–like signals from *isolated* sources are of importance in astrophysics. To confirm the theory outlined in Chapter 2 we consider some examples using the Weyl fields of isolated multipole sources and the Schwarzschild and Kerr gravitational fields. In his classic work [Penrose (1972)] on impulsive gravitational waves Penrose introduced the hierarchical classification of intrinsic geometries (given in §2.4) which the null hypersurface history $\mathcal{N}$ of the wave front inherits from the space–times $\mathcal{M}^{\pm}$ it is embedded in. When the Penrose classification of intrinsic geometries is applied to impulsive light–like signals in general this classification can be related to the detailed physical characteristics of the signal. This is done in conjunction with the Bianchi identities and the twice–contracted Bianchi identities. As a further development of the theory we systematically deduce the consequences of these identities as we work through the hierarchy of induced geometries. Finally a study of the effect of impulsive light–like signals on the relative motion of neighbouring test particles leads to a theory for their detection.

3.1 Abrupt Changes in Multipole Moments

It has been known for a long time that when an isolated gravitating system looses energy in the form of gravitational waves the lowest multipole to radiate is the quadrupole [Landau and Lifshitz (1951)]. If an isolated gravitating body were to experience a sudden explosion (a supernova, for example) which could be described by a sudden change in its multipole moments then we would expect it to release a burst of light–like matter accompanied by a burst of gravitational radiation. This phenomenon could be modeled, at least in a first approximation, by an impulsive light–like signal. We would expect the impulsive gravitational wave part of the signal to owe its existence primarily to a sudden change in the quadrupole moment of the source and the matter part of the signal to exist primarily because of a change in the monopole moment (the mass) of the source.

As model of the gravitational field outside a static, axially symmetric, isolated source we take the Weyl static, axially symmetric, asymptotically flat solutions of Einstein's vacuum field equations [Kramer et al. (1980)]. These space–times have line–elements of the form

$$ds^2 = R^2 e^{-2U}(e^{2k} d\Theta^2 + \sin^2\Theta\, d\phi^2) + e^{2k-2U} dR^2 - e^{2U} dt^2 \ , \qquad (3.1)$$

with U, k functions of R, Θ given by

$$U = \sum_{n=0}^{\infty} \frac{A_n}{R^{n+1}} P_n(\cos\Theta) \ , \qquad (3.2)$$

and

$$k = \sum_{l,m=0}^{\infty} \frac{A_l\, A_m\, (l+1)\,(m+1)}{l+m+2} \left[\frac{P_{l+1}P_{m+1} - P_l\, P_m}{R^{l+m+2}}\right] \ , \qquad (3.3)$$

where A_n is a constant and $P_n(\cos\Theta)$ is the Legendre Polynomial of degree n in the variable $\cos\Theta$, for $n = 0, 1, 2, \dots$. Following [Bondi et al. (1962)] the relationship between the constants A_0, A_1, A_2 in (3.2) and the mass m of the source, its dipole moment D (which can be transformed away by a change of origin of R; we shall leave $D \neq 0$ since we wish to allow it to abruptly change) and its quadrupole moment Q are given by writing out the first few terms of the series (3.2) as

$$U = -\frac{m}{R} - \frac{D\,\cos\Theta}{R^2} - (Q + \frac{1}{3}m^3)\,\frac{P_2(\cos\Theta)}{R^3} + \dots \ . \qquad (3.4)$$

Similarly the first couple of terms in (3.3) read

$$k = -\frac{m^2}{2\,R^2}\sin^2\Theta - \frac{2\,m\,D}{R^3}\cos\Theta\,\sin^2\Theta + \dots \ . \qquad (3.5)$$

Since we need to identify a useful null hypersurface in the space–time with line–element (3.1) to use as the history of an impulsive light–like signal, the most convenient thing to do is to write (3.1) in Bondi form because this form is based on a family of null hypersurfaces in the space–time. The procedure for doing this can be found in [Bondi et al. (1962)]. The result is that (3.1) is transformed to

$$ds^2 = r^2\{f^{-1}d\theta^2 + f\,\sin^2\theta\, d\phi^2\} - 2\,g\,du\,dr - 2\,h\,du\,d\theta - c\,du^2 \ , \qquad (3.6)$$

with

$$f = 1 - \frac{Q}{r^3}\sin^2\theta + O(r^{-4})\ , \tag{3.7}$$
$$g = 1 + O(r^{-4})\ , \tag{3.8}$$
$$h = \frac{2\,D}{r}\sin\theta + \frac{3\,Q}{r^2}\sin\theta\,\cos\theta + O(r^{-3})\ , \tag{3.9}$$
$$c = 1 - \frac{2\,m}{r} - \frac{2\,D}{r^2}\cos\theta - \frac{Q}{r^3}\,(3\,\cos^2\theta - 1) + O(r^{-4})\ . \tag{3.10}$$

In the form (3.6) the hypersurfaces u = constant are *exactly* null (for all r and not just for large r; this is shown in the Appendix of [Barrabès et al. (1997a)]). Neglecting $O(r^{-4})$–terms the u = constant surfaces are generated by the geodesic integral curves of the future–pointing null vector field $\partial/\partial r$ with r an affine parameter along them. The expansion ρ and shear σ of this null geodesic congruence are

$$\rho = \frac{1}{r} + O(r^{-5})\ , \qquad \sigma = \frac{3\,Q}{2\,r^4}\sin^2\theta + O(r^{-5})\ , \tag{3.11}$$

respectively. Hence for large r the null hypersurfaces u = constant approximate future null–cones. To model the impulsive light–like signal produced when an abrupt change takes place in the multipole moments we subdivide this space–time into two halves $\mathcal{M}^-(u \le 0)$ and $\mathcal{M}^+(u \ge 0)$, both having the null hypersurface $u = 0$ as boundary. In $\mathcal{M}^-$ the line–element will take the form (3.6) in coordinates $x^\mu_- = (\theta, \phi, r, u)$. In $\mathcal{M}^+$ the line–element will have the same form but with the coordinates labeled $x^\mu_+ = (\theta_+, \phi_+, r_+, u)$ and the parameters labeled $m_+, D_+, Q_+, \ldots$. The line–elements dl^2_- and dl^2_+ on $u = 0$ induced by its embedding in $\mathcal{M}^-$ and $\mathcal{M}^+$ respectively are seen from (3.6) to be given by

$$dl^2_+ = r^2_+\{f^{-1}_+ d\theta^2_+ + f_+\sin^2\theta_+\,d\phi^2_+\}\ , \tag{3.12}$$

and

$$dl^2 = r^2\{f^{-1}d\theta^2 + f\sin^2\theta\,d\phi^2\}\ , \tag{3.13}$$

with

$$f_+ = 1 - \frac{Q_+}{r^3_+}\sin^2\theta_+ + O(r^{-4}_+)\ , \tag{3.14}$$

and f given by (3.7). Now (3.12) and (3.13) are the same line–element provided $\mathcal{M}^-$ and $\mathcal{M}^+$ are attached on $u = 0$ so that points (θ_+, ϕ_+, r_+)

on the plus side of $u = 0$ are mapped to points (θ, ϕ, r) on the minus side of $u = 0$ with the *matching conditions*

$$\theta_+ = \theta + \frac{[Q]}{r^3}\sin\theta\,\cos\theta + O(r^{-4})\ ,\ \phi_+ = \phi\ , \tag{3.15}$$

$$r_+ = r + \frac{[Q]}{2\,r^2}\,(1 - 3\,\cos^2\theta) + O(r^{-3})\ , \tag{3.16}$$

with $[Q] = Q_+ - Q$ the jump across $u = 0$ in the quadrupole moment of the isolated source. The physical properties of the signal with history $u = 0$ can now be worked out using the theory given in Chapter 2. In particular we shall use the transverse curvature algorithm of §2.1. As intrinsic coordinates on $\mathcal{N}$ we can use $\xi^a = (\theta, \phi, r)$. Now the tangent basis vectors $e_{(a)}$ to $\mathcal{N}$ have components (2.24). On the minus side of $\mathcal{N}$ these are now given by $e^\mu_{(a)}|_- = \delta^\mu_a$ while on the plus side, using (2.24) and the matching conditions (3.15) and (3.16), they are

$$e^\mu_{(1)}|_+ = (1 + \frac{[Q]}{r^3}\,\cos 2\,\theta + O(r^{-4}),\ 0,\ \frac{3\,[Q]}{2\,r^2}\,\sin 2\,\theta + O(r^{-3}),\ 0), \tag{3.17}$$

$$e^\mu_{(2)}|_+ = (0,\ 1,\ 0,\ 0)\ , \tag{3.18}$$

$$\begin{aligned} e^\mu_{(3)}|_+ = (&-\frac{3\,[Q]}{2\,r^4}\,\sin 2\,\theta + O(r^{-5}),\ 0,\ 1 - \frac{[Q]}{r^3}\,(1 - 3\,\cos^2\theta) \\ &+O(r^{-4}),\ 0)\ . \end{aligned} \tag{3.19}$$

The normal to $\mathcal{N}$ is given via the 1–form

$$n_\mu\,dx^\mu|_\pm = -du\ . \tag{3.20}$$

In view of the form of the line–element (3.6) a convenient transversal on the minus side of $\mathcal{N}$ is given by the null covariant vector

$${}^-N_\mu = (0, 0, -1, -\frac{1}{2} + \frac{m}{r} + \frac{D}{r^2}\,\cos\theta + \frac{Q}{2\,r^3}\,(3\,\cos^2\theta - 1) + O(r^{-4}))\ . \tag{3.21}$$

When this is calculated on the plus side of $\mathcal{N}$, using (3.17)–(3.19) and (2.28) together with the fact that it is null, its components are found to be

$${}^+N_1 = \frac{3\,[Q]}{2\,r^2}\,\sin 2\,\theta + O(r^{-3})\ , \tag{3.22}$$

$${}^+N_2 = 0\ , \tag{3.23}$$

$${}^+N_3 = -1 - \frac{[Q]}{r^3}\,(1 - 3\,\cos^2\theta) + O(r^{-4})\ , \tag{3.24}$$

$$\begin{aligned} {}^+N_4 = &-\frac{1}{2} + \frac{m_+}{r} + \frac{D_+}{r^2}\,\cos\theta - \frac{(2\,Q_+ - Q)}{2\,r^3}\,(1 - 3\,\cos^2\theta) \\ &+O(r^{-4})\ . \end{aligned} \tag{3.25}$$

Also η in (2.28) is now

$$\eta = -1 + O(r^{-3}) \ . \tag{3.26}$$

We calculate the transverse extrinsic curvature on the plus and minus sides of $\mathcal{N}$ using (2.29). On the minus side we obtain

$$\begin{aligned}
\mathcal{K}^-_{11} &= -\frac{r}{2} + m + \frac{3\,D}{r}\cos\theta - \frac{Q}{4\,r^2}\,(13 - 29\,\cos^2\theta) \\
&\quad + O(r^{-3}) \ , && (3.27)\\
\mathcal{K}^-_{12} &= 0 \ , && (3.28)\\
\mathcal{K}^-_{22} &= -\frac{r}{2}\,\sin^2\theta + m\,\sin^2\theta + \frac{3\,D}{r}\cos\theta\,\sin^2\theta \\
&\quad - \frac{Q}{4\,r^2}\,(3 - 19\,\cos^2\theta)\,\sin^2\theta + O(r^{-3}) \ , && (3.29)\\
\mathcal{K}^-_{13} &= -\frac{3\,D}{r^2}\,\sin\theta + O(r^{-3}) \ , && (3.30)\\
\mathcal{K}^-_{23} &= 0 \ , && (3.31)\\
\mathcal{K}^-_{33} &= O(r^{-5}) \ . && (3.32)
\end{aligned}$$

On the plus side these are

$$\begin{aligned}
\mathcal{K}^+_{11} &= -\frac{r}{2} + m_+ + \frac{3\,D_+}{r}\cos\theta - \frac{Q_+}{4\,r^2}\,(13 - 29\,\cos^2\theta) \\
&\quad - \frac{[Q]}{4\,r^2}\,(11 - 25\,\cos^2\theta) + O(r^{-3}) \ , && (3.33)\\
\mathcal{K}^+_{12} &= 0 \ , && (3.34)
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}^+_{22} &= -\frac{r}{2}\,\sin^2\theta + m_+\,\sin^2\theta + \frac{3\,D_+}{r}\cos\theta\,\sin^2\theta \\
&\quad - \frac{Q_+}{4\,r^2}\,(3 - 19\,\cos^2\theta)\,\sin^2\theta - \frac{[Q]}{4\,r^2}\,(3 + 7\,\cos^2\theta)\,\sin^2\theta \\
&\quad + O(r^{-3}) \ , && (3.35)\\
\mathcal{K}^+_{13} &= -\frac{3\,D_+}{r^2}\,\sin\theta + O(r^{-3}) \ , && (3.36)\\
\mathcal{K}^+_{23} &= 0 \ , && (3.37)\\
\mathcal{K}^+_{33} &= O(r^{-5}) \ . && (3.38)
\end{aligned}$$

Now by (2.31) the jump in the transverse extrinsic curvature is obtained. Clearly $\gamma_{12} = \gamma_{23} = 0$ and $\gamma_{33} = O(r^{-5})$ while

$$\gamma_{11} = 2\,[m] + \frac{6\,[D]}{r}\cos\theta - \frac{[Q]}{r^2}\,(12 - 27\,\cos^2\theta)$$

$$+O(r^{-3})\ , \tag{3.39}$$

$$\gamma_{22} = 2\,[m]\,\sin^2\theta + \frac{6\,[D]}{r}\,\cos\theta\,\sin^2\theta$$

$$-\frac{3\,[Q]}{r^2}\,(1-2\,\cos^2\theta)\,\sin^2\theta + O(r^{-3})\ , \tag{3.40}$$

$$\gamma_{12} = \gamma_{23} = 0\ , \tag{3.41}$$

$$\gamma_{13} = -\frac{3\,[D]}{r^2}\,\sin\theta + O(r^{-3})\ , \tag{3.42}$$

$$\gamma_{33} = O(r^{-5})\ , \tag{3.43}$$

with the square brackets denoting the jumps across $\mathcal{N}$ of the quantities within them. From this and (2.47) we see that $\hat{\gamma}_{ab}$ has two non–vanishing components given by

$$\hat{\gamma}_{11} = \frac{1}{2}\,(\gamma_{11} - \gamma_{22}\,\mathrm{cosec}^2\theta) + O(r^{-3})\ , \tag{3.44}$$

$$\hat{\gamma}_{22} = -\hat{\gamma}_{11}\,\sin^2\theta + O(r^{-3})\ . \tag{3.45}$$

Now the leading terms, for large r, in the stress–energy tensor of the light–like shell with history $\mathcal{N}$ are calculated from (2.33) and (2.37). We find that $S^{\mu 4} = S^{12} = S^{23} = 0,\ S^{11} = O(r^{-7}),\ S^{22} = O(r^{-7})$,

$$16\,\pi\,S^{13} = -\frac{3\,[D]}{r^4}\,\sin\theta + O(r^{-5}) \tag{3.46}$$

and

$$16\,\pi\,S^{33} = -\frac{4\,[m]}{r^2} - \frac{12\,[D]}{r^3}\,\cos\theta + \frac{3\,[Q]}{r^4}\,(5 - 11\,\cos^2\theta) + O(r^{-5})\ . \tag{3.47}$$

Hence we see that the stress in the light–like shell with history $\mathcal{N}$ is anisotropic primarily (for large r) due to the jump in the dipole moment of the source. The surface energy density μ defined in (2.42) is given by

$$\mu = -\frac{1}{4\,\pi\,r^2}\left\{[m] + \frac{3\,[D]}{r}\,\cos\theta - \frac{3\,[Q]}{4\,r^2}\,(5 - 11\,\cos^2\theta) + O(r^{-3})\right\}\ . \tag{3.48}$$

The principal contribution to this is the jump in the mass of the source and this is clearly also the principal contribution to the stress–energy tensor of the light–like shell. For $\mu > 0$ we must have $[m] < 0$ so that there is a mass loss as a result of the explosion.

In order to survey the wave and matter parts of the coefficient of the delta function in the Weyl tensor described in §2.3 it is convenient to introduce the null tetrad

$$m^\mu = (-\frac{1}{r\sqrt{2}}\,f^{1/2},\ -\frac{i}{r\sqrt{2}\,\sin\theta}\,f^{-1/2},\ 0,\ 0)\ , \tag{3.49}$$

$$\bar{m}^\mu = (-\frac{1}{r\sqrt{2}}\,f^{1/2},\ \frac{i}{r\sqrt{2}\,\sin\theta}\,f^{-1/2},\ 0,\ 0)\ , \tag{3.50}$$

$$n^\mu = (0,\ 0,\ 1+O(r^{-4}),\ 0)\ , \tag{3.51}$$

$$N^\mu = (O(r^{-3}),\ 0, -\frac{\mathrm{c}}{2},\ 1)\ , \tag{3.52}$$

with f, c given by (3.7) and (3.10). With respect to this tetrad the Newman–Penrose components of the matter part of (2.48), denoted ${}^M\hat{\Psi}_A$ with $A = 0, 1, 2, 3, 4$, are found to be

$${}^M\hat{\Psi}_0 = 0\,, \qquad {}^M\hat{\Psi}_1 = 0\,, \qquad {}^M\hat{\Psi}_2 = O(r^{-5})\ , \tag{3.53}$$

$${}^M\hat{\Psi}_3 = \frac{3\sqrt{2}\,[D]}{4\,r^3}\,\sin\theta + O(r^{-4})\ , \tag{3.54}$$

$${}^M\hat{\Psi}_4 = 0\ . \tag{3.55}$$

We see that ${}^M\hat{\Psi}_A$ is principally Petrov type III with the normal to $\mathcal{N}$ as degenerate principal null direction. The Newman–Penrose components ${}^W\hat{\Psi}_A$ of the wave part of (2.48) all vanish with the exception of

$${}^W\hat{\Psi}_4 = -\frac{3\,[Q]}{4\,r^4}\,(3 - 7\,\cos^2\theta) + O(r^{-5})\ . \tag{3.56}$$

We thus confirm that *the principal contribution to the gravitational wave part of the light–like signal with history $\mathcal{N}$ is the jump in the quadrupole moment of the source.* Since ${}^W\Psi_4 = O(r^{-4})$ we have here an unconventional r^{-1}–behaviour (so–called Peeling behavior) for the radiation part of the field [Sachs (1962)]. This is explained in [Bressange and Hogan (1999)].

A special case of the above example consists of a Schwarzschild source undergoing an abrupt change in its mass. In this case (3.6) is replaced by the Schwarzschild line–element

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\,d\phi^2) - 2\,du\,dr - \left(1 - \frac{2\,m}{r}\right)\,du^2\ . \tag{3.57}$$

Here u = constant are future–directed null–cones generated by the geodesic integral curves of the null vector field $\partial/\partial r$. This vector field is also the outward–pointing degenerate principal null direction of the Riemann tensor of the space–time. We consider a spontaneous abrupt change in the mass m of the source across the future–null cone $u = 0$ (say) and ask: what are the physical properties of $u = 0$ in this case? As before we imagine the Schwarzschild space–time divided into two halves $\mathcal{M}^-(u \leq 0)$ and $\mathcal{M}^+(u \geq 0)$ each with boundary $u = 0$ and then re–attaching the halves on $u = 0$ preserving, with the identity map, the induced line–element on $u = 0$:

$$dl^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ . \tag{3.58}$$

The calculation described above specialized to this Schwarzschild case shows that the re–attached space–time $\mathcal{M}^- \cup \mathcal{M}^+$ has a stress–energy tensor concentrated on $u = 0$ of the form

$$T^{\mu\nu} = S^{\mu\nu}\,\delta(u) \ , \tag{3.59}$$

and $S^{\mu\nu}$ are the components of the surface stress–energy tensor which are given by

$$16\,\pi\,S^{\mu\nu} = -\frac{4\,[m]}{r^2}\,n^\mu\,n^\nu \ . \tag{3.60}$$

Here $[m]$ is the jump in the mass across $u = 0$ and n^μ is the normal to $u = 0$ given via the 1–form $n_\mu\,dx^\mu = -du$. Thus there is in fact no stress in the outgoing light–like shell (as might be expected since the shell is spherical and expanding) and the surface energy density μ in (2.43) is given by

$$\mu = -\frac{[m]}{4\,\pi\,r^2} \ . \tag{3.61}$$

It is natural to assume that $[m] < 0$ for an expanding shell. Thus we can interpret the space–time $\mathcal{M}^- \cup \mathcal{M}^+$ as describing the vacuum gravitational field (modeled by the Schwarzschild space–time $\mathcal{M}^-$) due to a spherical source having mass m_- with an expanding spherical light–like shell propagating through it, leaving behind a vacuum gravitational field (modeled by the Schwarzschild space–time $\mathcal{M}^+$) due to a spherical source having mass $m_+ < m_-$. Specializing $\mathcal{M}^+$ to Minkowskian space–time by putting $m_+ = 0$ we obtain Synge's model described in §1.1 . In this case μ in (3.61) reduces to Ω in(1.15).

Another well–known vacuum gravitational field due to a source having multipole moments is described by the Kerr [Kerr (1963)] solution of Einstein's vacuum field equations. In this case the multipole moments are all constructed from the mass and the angular momentum of the source. Thus an abrupt change in the multipole moments can be effected by a sudden change in the mass and the angular momentum. The latter corresponds to a glitch or a sudden spin up or spin down of the source. If we also allow the direction of the angular momentum to change abruptly we can obtain an impulsive gravitational wave with the maximum two degrees of freedom of polarization. We thus require a form of the Kerr solution which involves the mass parameter and three components of the angular momentum per unit mass. One such form can easily be obtained starting with the Kerr solution with mass m and angular momentum per unit mass A written in Kerr's original coordinates $(\zeta, \bar{\zeta}, r, u)$ [apart from the simple replacement of polar angles (θ, ϕ) with the complex coordinate $\zeta = \sqrt{2}\,\mathrm{e}^{i\,\phi}\,\tan\theta/2$ and its complex conjugate] in the form

$$ds^2 = 2\,\frac{(r^2+P^2)}{[1+\frac{1}{2}\,\zeta\,\bar{\zeta}]^2}\,d\zeta\,d\bar{\zeta} - 2\,\Sigma\,(dr - i\,P_\zeta\,d\zeta + i\,P_{\bar{\zeta}}\,d\bar{\zeta} + S\,\Sigma)\ , \qquad (3.62)$$

with

$$P = A\left(\frac{1-\frac{1}{2}\,\zeta\,\bar{\zeta}}{1+\frac{1}{2}\,\zeta\,\bar{\zeta}}\right)\ , \qquad S = \frac{1}{2} - \frac{m\,r}{r^2+P^2}\ , \qquad (3.63)$$

and the 1–form Σ is given by

$$\Sigma = du + i\,P_\zeta\,d\zeta - i\,P_{\bar{\zeta}}\,d\bar{\zeta}\ , \qquad (3.64)$$

with $P(\zeta, \bar{\zeta})$ real–valued and $P_\zeta = \partial P/\partial\zeta$. The rotation

$$\zeta \longrightarrow \frac{\sqrt{2}\,\sin(\theta_1/2) - \zeta\,\mathrm{e}^{-i\,\phi_1}\,\cos(\theta_1/2)}{\mathrm{e}^{i\,\phi_1}\,\cos(\theta_1/2) + (\zeta/\sqrt{2})\,\sin(\theta_1/2)}\ , \qquad (3.65)$$

with θ_1, ϕ_1 constants, leaves the form of the line–element (3.62) invariant with P replaced by

$$P = \frac{a}{\sqrt{2}}\left(\frac{\zeta+\bar{\zeta}}{1+\frac{1}{2}\,\zeta\,\bar{\zeta}}\right) + \frac{b}{i\,\sqrt{2}}\left(\frac{\zeta-\bar{\zeta}}{1+\frac{1}{2}\,\zeta\,\bar{\zeta}}\right) + c\left(\frac{1-\frac{1}{2}\,\zeta\,\bar{\zeta}}{1+\frac{1}{2}\,\zeta\,\bar{\zeta}}\right)\ , \qquad (3.66)$$

and

$$a = A\,\sin\theta_1\cos\phi_1\ , \qquad b = A\,\sin\theta_1\sin\phi_1\ , \qquad c = A\,\cos\theta_1\ . \qquad (3.67)$$

We have here the Kerr solution with mass m and angular momentum three–vector $\mathbf{J} = (m\,a, m\,b, m\,c)$ having the same magnitude but a different direction than the original angular momentum three–vector $\mathbf{J} = (0, 0, 0, m\,A)$. Re–introducing the polar coordinates (θ, ϕ) as indicated above prior to (3.62) we arrive at the line–element [Hogan (1977)]

$$ds^2 = \rho^2(d\theta^2 + \sin^2\theta\,d\phi^2) - 2\,\Sigma\,(dr - N\,d\theta - M\,\sin\theta\,d\phi + S\,\Sigma)\ , \qquad (3.68)$$

with

$$\rho^2 = r^2 + P^2\,, \qquad P = (a\,\cos\phi + b\,\sin\phi)\,\sin\theta + c\,\cos\theta\,,$$
$$\Sigma = du + N\,d\theta + M\,\sin\theta\,d\phi\,, \qquad S = \frac{1}{2} - \frac{m\,r}{\rho^2}\,, \tag{3.69}$$

and

$$N = -a\,\sin\phi + b\,\cos\phi\,, \tag{3.70}$$
$$M = -(a\,\cos\phi + b\,\sin\phi)\,\cos\theta + c\,\sin\theta\,. \tag{3.71}$$

We see from (3.70) and (3.71) that

$$E^2 \equiv M^2 + N^2 = m^{-2}\{|\mathbf{J}|^2 - (\mathbf{n}\cdot\mathbf{J})^2\}\,, \tag{3.72}$$

where the unit three–vector $\mathbf{n}$ is given by $\mathbf{n} = (\sin\theta\,\cos\phi,\ \sin\theta\,\sin\phi, \cos\theta)$. In this case u = constant are not *null* hypersurfaces in general. However they are asymptotically null. Specifically if $O(r^{-2})$–terms are neglected then the vector field $\partial/\partial r$ is tangent to null hypersurfaces u = constant. The normal n^μ to u = constant, given by $n_\mu\,dx^\mu = -du$, satisfies $g_{\mu\nu}\,n^\mu\,n^\nu = O(r^{-2})$. In this approximation in which we neglect $O(r^{-2})$–terms, u = constant are portions of future null–cones generated by the integral curves of the vector field $\partial/\partial r$ with r an affine parameter along them. Thus for large r these generators are shear–free, null geodesics with expansion r^{-1} and the induced line–element on u = constant is given by

$$dl^2 = r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,. \tag{3.73}$$

We obtain this from (3.68) by putting u = constant and applying the approximation in which the ratio of the neglected metric tensor components to the retained metric tensor components is $O(r^{-2})$. Thus if a disturbance is propagating in the direction $\partial/\partial r$ then for sufficiently large r a front is formed (a null hypersurface is formed in space–time) with history $u = 0$ (say). We shall assume that across the null portion of $u = 0$ a spontaneous jump occurs in the parameters $\{m, a, b, c\}$ from values $\{m, a, b, \mathrm{c}\}$ to the past ($u < 0$) of this null portion of $u = 0$ to values $\{m_+, a_+, b_+, c_+\}$ to the future ($u > 0$) of this null portion of $u = 0$ and that the regions of space–time $\mathcal{M}^-(u \le 0)$ and $\mathcal{M}^+(u \le 0)$ on either side of the null part of $u = 0$ join on the null part of $u = 0$ with the identity map and thus the induced line–element (3.73) is trivially invariant under these matching conditions. We now use the theory of Chapter 2 to calculate the surface stress–energy tensor on the null part of $u = 0$ and also to calculate the matter and wave parts of the Weyl tensor associated with the impulsive light–like signal having as history the null part of $u = 0$. In coordinates $\{x^\mu\} = \{\theta, \phi, r, u\}$ the components of the surface stress–energy tensor $S^{\mu\nu}$ are found to be [the reader may wish to consult [Barrabès and Hogan (1997b)] for a guide through the calculations leading to these results]

$$16\,\pi\,S^{13} = \frac{[N]}{r^3} + O(r^{-4})\ , \tag{3.74}$$

$$16\,\pi\,S^{23} = \frac{[M]\,\mathrm{cosec}\theta}{r^3} + O(r^{-4})\ , \tag{3.75}$$

$$16\,\pi\,S^{33} = -\frac{4\,[m]}{r^2} + \frac{2\,[E^2]}{r^3} + O(r^{-4})\ , \tag{3.76}$$

with all other components small of order r^{-5}. Here as always the jump in a quantity across the null part of $u = 0$ is indicated by putting square brackets around that quantity. The quantities N, M, E given in (3.70)–(3.72) jump because the mass and angular momentum parameters jump. The existence of (3.74)–(3.76) confirms that the null part of $u = 0$ is the history of an impulsive light–like shell. The stress in the shell is anisotropic due to the jump in the Kerr angular momentum. The surface energy density (2.43) in this case reads

$$\mu = -\frac{1}{4\,\pi\,r^2}\,\left([m] - \frac{[E^2]}{2\,r} + O(r^{-2})\right)\ . \tag{3.77}$$

This generalizes (3.61) and $\mu > 0$ implies $[m] < 0$ once again. It is interesting to note that an expanding light–like shell sandwiched between two Reissner–Nordstrom space–times with different masses and charges (the charged version of the Schwarzschild example given above) has an exact μ given by (3.77) without the error term and with $[E^2]$ replaced by $[e^2]$, the jump in the square of the charge across the history of the light–like shell.

Next we examine the coefficient $\hat{C}_{\kappa\lambda\mu\nu}$ of $\delta(u)$ in the Weyl tensor for large r. This is given by (2.48). To display the matter and wave parts in this case we make use of an asymptotically null tetrad given by the 1–forms $du, dr + S\,du$ [with S given approximately by $S = \frac{1}{2} - \frac{m}{r}$] and $r\,(d\theta + i\,\sin\theta\,d\phi)/\sqrt{2}$ together with its complex conjugate. The Newman–Penrose components on this tetrad of the matter part of $\hat{C}_{\kappa\lambda\mu\nu}$ are given by

$$^M\hat{\Psi}_0 = O(r^{-5})\ ,\qquad ^M\hat{\Psi}_1 = O(r^{-4})\ ,\qquad ^M\hat{\Psi}_2 = O(r^{-3})\ , \tag{3.78}$$

$$^M\hat{\Psi}_3 = -\frac{[N - i\,M]}{4\,\sqrt{2}\,r^2} + O(r^{-3})\ ,\qquad ^M\hat{\Psi}_4 = O(r^{-3})\ . \tag{3.79}$$

The Newman–Penrose components on this tetrad of the wave part of $\hat{C}_{\kappa\lambda\mu\nu}$ all vanish except for

$$ {}^{W}\hat{\Psi}_4 = \frac{[m\,(N - i\,M)^2]}{4\,r^4} + O(r^{-5}) \ . \tag{3.80} $$

Thus if the angular momentum 3–vector were given by $\mathbf{J} = (0, 0, m\,c)$ for $u < 0$ and by $\mathbf{J} = (0, 0, m_+\,c_+)$ for $u > 0$ then $N = 0$ in (3.80) and the impulsive wave with amplitude (3.80) has one degree of freedom. Adding the change of direction to this change of magnitude of the angular momentum clearly adds the extra degree of freedom to the gravitational wave by making the leading term in (3.80) complex.

3.2 Impulsive Light–Like Signals as Recoil Effects: Electromagnetic Example

Perhaps the simplest example in electrodynamics of an impulsive electromagnetic wave is that produced when a point charge e receives an impulsive acceleration or deceleration. Specifically the charge e might be moving rectilinearly with constant 3–velocity v relative to the laboratory frame and is suddenly halted. Thus in Minkowskian space–time the history of the charge is a time–like geodesic which suddenly changes direction but remains a time–like geodesic. Hence there is a sudden change in the direction of the 4–velocity of the charge. This leads to a recoil effect which takes the form of the emission of a spherical impulsive electromagnetic wave [Penrose (1972)]. We begin by describing the construction of the electromagnetic field of this charge as a prelude to discussing the gravitational analogue of this physical situation.

We take $X^\mu = (x, y, z, t)$ to be rectangular Cartesian coordinates and time in Minkowskian space–time in terms of which the line–element reads (with $c = 1$)

$$ ds^2 = dx^2 + dy^2 + dz^2 - dt^2 = \eta_{\mu\nu}\,dX^\mu\,dX^\nu \ . \tag{3.81} $$

Let $X^\mu = x^\mu(u)$ be the parametric equations of the time–like world line of a charge e having u as proper time or arc length along it. The 4–velocity and 4–acceleration of the charge have components $v^\mu(u) = dx^\mu/du$ and $a^\mu(u) = dv^\mu/du$ respectively (with $\eta_{\mu\nu}\,v^\mu\,v^\nu \equiv v_\mu\,v^\mu = -1$ and consequently with $v_\mu\,a^\mu = 0$). Let (X^μ) be the coordinates of an event off the world–line of the charge and let $(x^\mu(u))$ be the coordinates of the event on the world–line of the charge where the past null cone with vertex (X^μ) intersects the world–line. The retarded distance r of (X^μ) from the world–line is given by [Synge (1965)]

$$r = -v_\mu\,(X^\mu - x^\mu(u))\ . \tag{3.82}$$

Here $X^\mu - x^\mu(u)$ is null and $r \geq 0$ with equality if and only if (X^μ) coincides with $(x^\mu(u))$. The Liénard–Wiechert field of the charge evaluated at (X^μ) is given by the Maxwell tensor

$$F_{\mu\nu}(X) = \frac{N_{\mu\nu}}{r} + \frac{III_{\mu\nu}}{r^2}\ , \tag{3.83}$$

with

$$N_{\mu\nu} = 2\,e\,q_{[\mu}\,k_{\nu]}\ , \qquad III_{\mu\nu} = 2\,e\,v_{[\mu}\,k_{\nu]}\ . \tag{3.84}$$

Here square brackets denote skew–symmetrisation, $k^\mu = r^{-1}\,(X^\mu - x^\mu(u))$ so that k^μ is null and, by (3.82), $v_\mu\,k^\mu = -1$. Also

$$q^\mu = a^\mu + (a^\nu\,k_\nu)\,v^\mu\ , \tag{3.85}$$

and this is a space–like 4–vector orthogonal to k^μ. The skew–symmetric tensor $N_{\mu\nu}$ in (3.84) is Petrov type N with degenerate principal null direction k^μ and thus the leading part for large r of the electromagnetic field of the charge (3.83) describes the radiation part of the field. The presence of this term is due entirely to the acceleration a^μ of the charge. Suppose now that at $u = 0$ (say) on the world–line of the charge, the charge receives an impulsive acceleration or deceleration (there is a sudden change in the 4–velocity of the charge at $u = 0$ leading to a Dirac delta function $\delta(u)$ in the 4–acceleration of the charge), then one would expect the resulting retarded radiation, described by $N_{\mu\nu}$ above, to take the form of an impulsive electromagnetic wave having the future null cone $u = 0$ as its history in Minkowskian space–time and having profile $\delta(u)$. However this information cannot be extracted from (3.83) and (3.84) because in particular r in (3.82) is not now defined at $u = 0$ since there is no unique tangent to the world line of the charge at $u = 0$. To see this clearly take the vertex of the future null cone $\mathcal{N}(u = 0)$ to be the origin of the coordinates (X^μ) (and so $x^\mu(0) = 0$). Thus if $P(X^\mu)$ is an event on $\mathcal{N}$ then (X^μ) satisfies

$$(X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = 0\ , \qquad X^4 > 0\ . \tag{3.86}$$

A useful parametric form of this is given by

$$X^1 + i\,X^2 = \frac{\sqrt{2}\,\zeta\,R_0}{1 + \frac{1}{2}\zeta\,\bar{\zeta}}\ , \qquad X^3 = \left(\frac{1 - \frac{1}{2}\zeta\,\bar{\zeta}}{1 + \frac{1}{2}\zeta\,\bar{\zeta}}\right) R_0\ , \qquad X^4 = R_0\ , \tag{3.87}$$

with ζ a complex variable with complex conjugate $\bar{\zeta}$ and R_0 is a real variable. Putting $p_0 = 1 + \frac{1}{2}\zeta\,\bar{\zeta}$ and substituting (3.87) into the Minkowskian line–element (3.81) yields the induced line-element on $\mathcal{N}$,

$$ds^2 = 2\,R_0^2 p_0^{-2}\,d\zeta\,d\bar{\zeta}\;. \tag{3.88}$$

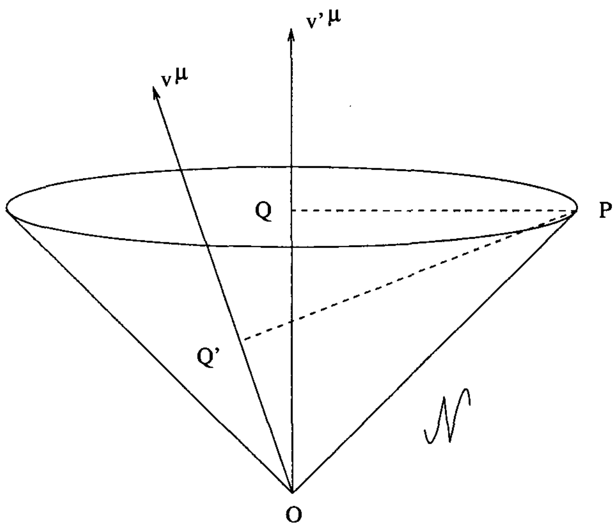

Fig. 3.1 The future null–cone $\mathcal{N}(u = 0)$ with $QP = r$ and $Q'P = r_+$. The region to the future (past) of the null-cone corresponds to $u > 0\,(u < 0)$.

In figure 3.1 QP is orthogonal to v^μ and $v_\mu = v^\mu = -1$ while $Q'P$ is orthogonal to v'^μ and $v'_\mu\,v'^\mu = -1$. The retarded distances $r = QP$ and $r_+ = Q'P$ are given by (using (3.82) with $x^\mu(u) = x^\mu(0) = 0$)

$$QP = r = -v^1\,X^1 - v^2\,X^2 - v^3\,X^3 + v^4\,X^4\;, \tag{3.89}$$

and

$$Q'P = r_+ = -v'^1\,X^1 - v'^2\,X^2 - v'^3\,X^3 + v'^4\,X^4\;. \tag{3.90}$$

Substituting from (3.87) into these we obtain

$$r = R_0\,p\,p_0^{-1} \qquad \text{and} \qquad r_+ = R_0\,p_+\,p_0^{-1}\;, \tag{3.91}$$

with

$$p = \frac{1}{2}\,\zeta\,\bar{\zeta}\,(v^4 + v^3) - \frac{1}{\sqrt{2}}\,\zeta\,(v^1 - i\,v^2) - \frac{1}{\sqrt{2}}\,\bar{\zeta}\,(v^1 + i\,v^2) + v^4 - v^3\;, \tag{3.92}$$

and p_+ is the same function as p but with v^μ in p replaced by v'^μ. Hence we see from (3.91) that we have the invariant statement

$$r_+ \, p_+^{-1} = r \, p^{-1} \ , \tag{3.93}$$

and the line–element (3.88) reads

$$ds^2 = 2 \, r^2 \, p^{-2} \, d\zeta \, d\bar{\zeta} \ . \tag{3.94}$$

Hence (3.93) is a change of affine parameter along the generator $\zeta =$ constant of $\mathcal{N}$ (OP in figure 2) which (a) leaves the vertex of $\mathcal{N}$ fixed and (b) leaves the induced metric on $\mathcal{N}$ invariant.

Now specialize to the following motion of the charge: In the frame of reference with respect to which the coordinates (X^μ) are measured (the laboratory frame) the charge moves with uniform 3–velocity v in the X^3–direction when $u < 0$ (to the past of the null cone $\mathcal{N}$). When $u > 0$ (to the future of the null cone $\mathcal{N}$) the charge is taken to be at rest in this frame. Thus for $u < 0$ the world–line of the charge is the time–like geodesic with unit tangent (4–velocity)

$$v^\mu = (0, 0, \gamma \, v, \gamma) \ , \tag{3.95}$$

with $\gamma = (1 - v^2)^{-1/2}$. For $u > 0$ the world–line of the charge is the time–like geodesic with unit tangent

$$v'^\mu = (0, 0, 0, 1) \ . \tag{3.96}$$

The line–element of Minkowskian space–time in the region $u < 0$, which we denote by $\mathcal{M}^-$, is given in cordinates $(\zeta, \bar{\zeta}, r, u)$ by

$$ds_-^2 = 2 \, r^2 \, p^{-2} \, d\zeta \, d\bar{\zeta} - 2 \, du \, dr - du^2 \ , \tag{3.97}$$

with

$$p = \gamma \left\{ \frac{1}{2} (1 + v) \, \zeta \, \bar{\zeta} + 1 - v \right\} \ . \tag{3.98}$$

This latter follows from (3.92) with the specialization (3.95). The line–element of Minkowskian space–time in the region $u > 0$ which we denote by $\mathcal{M}^+$ is given in coordinates $(\zeta_+, \bar{\zeta}_+, r_+, u_+)$ by

$$ds_+^2 = 2 \, r_+^2 \, p_+^{-2} \, d\zeta_+ \, d\bar{\zeta}_+ - 2 \, du_+ \, dr - du_+^2 \ , \tag{3.99}$$

with

$$p_+ = \frac{1}{2} \zeta_+ \, \bar{\zeta}_+ + 1 \ , \tag{3.100}$$

and $u_+ = 0$ corresponding to $u = 0$. We then attach the two halves of Minkowskian space–time $\mathcal{M}^-$ and $\mathcal{M}^+$ on $u = 0$ ($\Leftrightarrow u_+ = 0$) with the *matching conditions* (cf. (3.93))

$$\zeta_+ = \zeta\ , \qquad \bar{\zeta}_+ = \bar{\zeta}\ , \qquad r_+ = r\,p_+\,p^{-1}\ . \tag{3.101}$$

Thus in particular the line–elements induced on $\mathcal{N}$ by its embedding in $\mathcal{M}^-$ and $\mathcal{M}^+$, calculated from (3.97) with $u = 0$ and from (3.99) with $u_+ = 0$, are the same induced line–element. A more convenient form for (3.97) is obtained by introducing the coordinates ξ, ϕ via

$$\xi = \frac{1 - \frac{1}{2}\zeta\bar{\zeta}}{1 + \frac{1}{2}\zeta\bar{\zeta}}\ , \qquad \phi = \frac{1}{2}\log\left(\frac{\zeta}{\bar{\zeta}}\right)\ , \tag{3.102}$$

which results in (3.97) becoming

$$ds_-^2 = k^2\,r^2\left\{\frac{d\xi^2}{1 - \xi^2} + (1 - \xi^2)\,d\phi^2\right\} - 2\,du\,dr - du^2\ , \tag{3.103}$$

with $k^{-1} = \gamma\,(1 - v\,\xi)$. Also if ξ_+, ϕ_+ are given by (3.102) with $\zeta, \bar{\zeta}$ replaced by $\zeta_+, \bar{\zeta}_+$ we find that (3.99) becomes

$$ds_+^2 = r_+^2\left\{\frac{d\xi_+^2}{1 - \xi_+^2} + (1 - \xi_+^2)\,d\phi_+^2\right\} - 2\,du_+\,dr_+ - du_+^2\ . \tag{3.104}$$

The matching conditions (3.101) now read: on $u = 0\,(u_+ = 0)$

$$\xi_+ = \xi\ , \qquad \phi_+ = \phi\ , \qquad r_+ = k\,r\ . \tag{3.105}$$

To be clear about the physical interpretation of this geometrical construction we note the following: The line–elements (3.103) and (3.104) are two versions of the Minkowskian line–element which can be transformed into one another by the coordinate transformation

$$\xi_+ = \frac{\xi - v}{1 - v\,\xi}\ , \qquad \phi_+ = \phi\ , \qquad r_+ = r\ , \qquad u_+ = u\ . \tag{3.106}$$

Here v is a real parameter with $0 < v < 1$. The transformation (3.106) is a Lorentz transformation. To see this we first write the line–elements (3.103) and (3.104) in terms of rectangular Cartesian coordinates and time $\{x, y, z, t\}$ and $\{x_+, y_+, z_+, t_+\}$ respectively. The relations between these two sets of coordinates and the coordinates $\{\xi, \phi, r, u\}$ and $\{\xi_+, \phi_+, r_+, u_+\}$ are given by

$$\begin{aligned} &x = r\,k\sqrt{1 - \xi^2}\,\cos\phi\ , \qquad y = r\,k\sqrt{1 - \xi^2}\,\sin\phi\ , \\ &z = \gamma\,v\,u + r\,k\,\xi\ , \qquad t = \gamma\,u + r\,k\ , \end{aligned} \tag{3.107}$$

and

$$x_+ = r_+ \sqrt{1-\xi_+^2}\,\cos\phi_+ \ , \qquad y_+ = r_+ \sqrt{1-\xi_+^2}\,\sin\phi_+ \ ,$$
$$z_+ = r_+\,\xi_+ \ , \qquad t_+ = u_+ + r_+ \ . \tag{3.108}$$

Written in terms of the coordinates $\{x, y, z, t\}$ and $\{x_+, y_+, z_+, t_+\}$ the line–elements (3.103) and (3.104) take the usual form (3.81). The relationship between $\{x, y, z, t\}$ and $\{x_+, y_+, z_+, t_+\}$ corresponding to (3.106) is obtained by substituting (3.106) into (3.108) and using (3.107). The result is the Lorentz transformation

$$x_+ = x \ , \qquad y_+ = y \ , \qquad z_+ = \gamma\,(z - v\,t) \ , \qquad t_+ = \gamma\,(t - v\,z) \ . \tag{3.109}$$

In order to calculate the Maxwell field of the charge e performing the motion we are considering, it will be useful to express the line–element of the space–time $\mathcal{M}^- \cup \mathcal{M}^+$ in a coordinate system $\{\Xi, \Phi, R, U\}$ in which the metric tensor is continuous across $\mathcal{N}$ and *which incorporates the matching conditions* (3.105). In such a coordinate system we find that the line–element of $\mathcal{M}^- \cup \mathcal{M}^+$ is given by

$$ds^2 = K^2\,R^2\left\{\frac{d\Xi^2}{1-\Xi^2} + (1-\Xi^2)\,d\Phi^2\right\} - 2\,dU\,dR - dU^2 \ , \tag{3.110}$$

with $K^{-1} = \gamma\,(1 - v\,\Xi)$. The coordinate ranges are $-1 \le \Xi \le +1$, $0 \le \Phi < 2\,\pi$, $0 \le R < +\infty$, $-\infty < U < +\infty$. Here $\mathcal{N}$ is given by $U = 0$, $\mathcal{M}^-$ by $U < 0$ and $\mathcal{M}^+$ by $U > 0$. It is interesting to note that in these coordinates the metric tensor components are independent of U. To see how the matching conditions (3.105) come out of this formulation we note that when $U > 0$ (3.104) is transformed into (3.110) by the coordinate transformation

$$r_+ = K\,R\,\varphi \ , \qquad u_+ = \gamma\,U + K\,(1-\varphi)\,R \ ,$$
$$\xi_+ = \varphi^{-1}(\chi + \Xi) \ , \qquad \phi_+ = \Phi \ , \tag{3.111}$$

with

$$\varphi = (1 + 2\,\chi\,\Xi + \chi^2)^{1/2} \qquad \text{and} \qquad \chi = \frac{\gamma\,v\,U}{K\,R} \ . \tag{3.112}$$

Clearly when $U < 0$ the line–element (3.103) is transformed into (3.110) by the identity transformation

$$r = R \ , \qquad u = U \ , \qquad \xi = \Xi \ , \qquad \phi = \Phi \ . \tag{3.113}$$

Now we see from (3.111)–(3.113) that on $\mathcal{N}$ ($U = 0 \Leftrightarrow u = 0 \Leftrightarrow u_+ = 0$)

$$r_+ = K\,R = k\,r\ , \qquad \xi_+ = \xi\ , \qquad \phi_+ = \phi\ , \tag{3.114}$$

which agrees with the matching conditions (3.105). In addition it follows from (3.110) that the space–time $\mathcal{M}^-\cup\mathcal{M}^+$ is flat everywhere. In particular the Riemann tensor vanishes *on* $\mathcal{N}$ for the matching conditions (3.105) and thus $\mathcal{N}$ is not the history of an impulsive light–like signal (light–like shell or impulsive gravitational wave).

The electromagnetic field due to the charge e in $\mathcal{M}^-$ and in $\mathcal{M}^+$ is the Coulomb field. The electromagnetic 4–potential on $\mathcal{M}^-\cup\mathcal{M}^+$ is given by the 1–form field (in the coordinates $\{\Xi, \Phi, R, U\}$)

$$A = \frac{e}{r_+}\,(du_+ + dr_+)\,\vartheta(U) + \frac{e}{R}\,(dU + dR)\,(1 - \vartheta(U))\ , \tag{3.115}$$

with r_+, u_+ given in terms of Ξ, R, U by (3.111). Thus

$$A = A^+_{\rm Coul}\,\vartheta(U) + A^-_{\rm Coul}\,(1 - \vartheta(U))\ , \tag{3.116}$$

where $A^+_{\rm Coul}$ is the Coulomb potential 1–form in $\mathcal{M}^+$ due to a charge e with geodesic world–line $r_+ = 0$ and $A^-_{\rm Coul}$ is the Coulomb potential 1–form in $\mathcal{M}^-$ due to a charge e with geodesic world–line $r = 0$. We will denote the corresponding Coulomb field 2–forms (the exterior derivatives of $A^+_{\rm Coul}$ and $A^-_{\rm Coul}$) by $f^+_{\rm Coul}$ and $f^-_{\rm Coul}$ respectively. From (3.111) we have

$$\frac{1}{r_+}\,(du_+ + dr_+) = \varphi^{-1}\left(K\,\gamma\,v\,d\Xi + \frac{\gamma}{K\,R}\,dU + \frac{1}{R}\,dR\right)\ , \tag{3.117}$$

and thus the candidate for Maxwell 2–form on $\mathcal{M}^-\cup\mathcal{M}^+$ is

$$F = f^+_{\rm Coul}\,\vartheta(U) + f^-_{\rm Coul}\,(1 - \vartheta(U)) + e\,K\,\gamma\,v\,\delta(U)\,dU\wedge d\Xi\ . \tag{3.118}$$

The expressions for $f^\pm_{\rm Coul}$ can be found below in (3.126) and (3.127). In terms of the basis 1–forms on $\mathcal{M}^-\cup\mathcal{M}^+$,

$$\vartheta^1 = \frac{K\,R\,d\Xi}{\sqrt{1-\Xi^2}}\ , \qquad \vartheta^2 = K\,R\sqrt{1-\Xi^2}\,d\Phi\ ,$$
$$\vartheta^3 = dU\ , \qquad \vartheta^4 = dR + \frac{1}{2}\,dU\ , \tag{3.119}$$

we can write (3.118) as

$$F = f^+_{\rm Coul}\,\vartheta(U) + f^-_{\rm Coul}\,(1 - \vartheta(U)) - \frac{e\,\gamma\,v\,\sqrt{1-\Xi^2}}{R}\,\delta(U)\,\vartheta^1\wedge\vartheta^3\ . \tag{3.120}$$

Thus the delta function part of this is Petrov type N with degenerate principal null direction given via the 1–form ϑ^3. The vector field $\partial/\partial R$ is therefore the unique principal null direction. To show that (3.120) is a vacuum Maxwell field on $\mathcal{M}^- \cup \mathcal{M}^+$ excluding $R = 0$ but including the future null cone $\mathcal{N}\,(U = 0)$ we first calculate the Hodge dual of (3.118)

$$ {}^*F = {}^*f^+_{\rm Coul}\,\vartheta(U) + {}^*f^-_{\rm Coul}\,(1 - \vartheta(U)) + e\,K\,\gamma\,v\,\delta(U)\,{}^*(dU \wedge d\Xi)\ . \quad (3.121) $$

Since $d^*f^+_{\rm Coul} = 0 = d^*f^-_{\rm Coul}$, with d denoting the exterior derivative, we have

$$ d^*F = \delta(U)\,dU \wedge ({}^*f^+_{\rm Coul} - {}^*f^-_{\rm Coul}) + e\,\gamma\,v\,d\,[K\,\delta(U)\,{}^*(dU \wedge d\Xi)]\ . \quad (3.122) $$

Using

$$ {}^*(dU \wedge d\Xi) = -(1 - \Xi^2)\,dU \wedge d\Phi\ , \quad (3.123) $$

$$ {}^*(dU \wedge dR) = -K^2R^2\,d\Xi \wedge d\Phi\ , \quad (3.124) $$

$$ {}^*(dR \wedge d\Xi) = (1 - \Xi^2)\,dR \wedge d\Phi\ , \quad (3.125) $$

and the explicit expressions

$$ \begin{aligned} f^+_{\rm Coul} &= \frac{e\,\gamma}{K\,R^2\,\varphi^3}\,\{1 - v\,\Xi + \chi\,(\Xi - v)\}\,dU \wedge dR + \frac{e\,\chi}{R\,\varphi^3}\,dR \wedge d\Xi \\ &\quad + \frac{e}{R\,\varphi^3}\,\{\chi + \gamma^2 v\,(1 - v\,\Xi)\}\,dU \wedge d\Xi\ , \end{aligned} \quad (3.126) $$

and

$$ f^-_{\rm Coul} = \frac{e}{R^2}\,dU \wedge dR\ , \quad (3.127) $$

we find that

$$ \delta(U)\,dU \wedge ({}^*f^+_{\rm Coul} - {}^*f^-_{\rm Coul}) = e\,(K^2 - 1)\,\delta(U)\,dU \wedge d\Xi \wedge d\Phi\ , \quad (3.128) $$

and

$$ d\,[K\,\delta(U)\,{}^*(dU \wedge d\Xi)] = \delta(U)\,\frac{d}{d\Xi}\{(1 - \Xi^2)\,K\}\,dU \wedge d\Xi \wedge d\Phi\ . \quad (3.129) $$

Hence (3.122) reads

$$ d^*F = e\,\delta(U)\,\left\{K^2 - 1 + \gamma\,v\,\frac{d}{d\Xi}[(1 - \Xi^2)\,K]\right\}\,dU \wedge d\Xi \wedge d\Phi\ . \quad (3.130) $$

The right hand side of this equation vanishes since $K^{-1} = \gamma\,(1 - v\,\Xi)$. Hence (3.120) is a vacuum Maxwell field for all U and for $R > 0$. The final term in (3.120) thus represents a spherical impulsive electromagnetic wave having the future null cone $\mathcal{N}\,(U=0)$ as its history in Minkowskian space–time.

The tetrad defined by the 1–forms (3.119) on $\mathcal{M}^-\cup\mathcal{M}^+$ is a half–null tetrad. On the other hand the tetrad defined by the 1–forms $\{\vartheta^1, \vartheta^2, \omega^3, \omega^4\}$ with

$$\omega^3 = dR\ , \qquad \omega^4 = dU + dR\ , \tag{3.131}$$

is an orthonormal tetrad. The final term in the electromagnetic field (3.120) written on this orthonormal tetrad reads

$$\hat{F} = -\frac{e\,\gamma\,v\,\sqrt{1-\Xi^2}}{R}\,\delta(U)\,(\vartheta^1\wedge\omega^4 - \vartheta^1\wedge\omega^3)\ . \tag{3.132}$$

It thus follows that in the laboratory frame $\hat{F}$ corresponds to an electric 3–vector and a magnetic 3–vector given, respectively, by

$$\mathbf{E} = \mathbf{E}_0\,\delta(U)\ , \qquad \mathbf{H} = \mathbf{H}_0\,\delta(U)\ , \tag{3.133}$$

with $\mathbf{E}_0 = (\mathcal{E}, 0, 0)$ and $\mathbf{H}_0 = (0, \mathcal{E}, 0)$ where

$$\mathcal{E} = -\frac{e\,\gamma\,v\,\sqrt{1-\Xi^2}}{R}\ . \tag{3.134}$$

A measure of the intensity of this electromagnetic wave is given by

$$\mathcal{I} = \frac{1}{8\,\pi}\,(|\mathbf{E}_0|^2 + |\mathbf{H}_0|^2) = \frac{e^2\gamma^2(1-\Xi^2)}{4\,\pi\,R^2}\ . \tag{3.135}$$

A measure of the total intensity $\mathcal{I}_{\rm total}$ of this wave is obtained by integrating (3.135) over the spherical wave front $U=0$, $R=$ constant. The area element, obtained from (3.110), is

$$dA = K^2R^2d\Xi\,d\Phi\ , \tag{3.136}$$

with $K^{-1}=\gamma\,(1-v\,\Xi)$, $-1\le\Xi\le+1$ and $0\le\Phi<2\,\pi$. One readily verifies that the area of the wave front is $4\,\pi\,R^2$ and that

$$\mathcal{I}_{\rm total} = \frac{e^2}{2}\left\{\frac{1}{v}\log\left(\frac{1+v}{1-v}\right) - 2\right\}\ , \tag{3.137}$$

with $0<v<1$. This is typical of the 3–velocity dependence of the total intensity of electromagnetic bremsstrahlung (see, for example, Jackson's discussion of beta decay [Jackson (1967)]). For the charge e the deceleration from 3–velocity v to zero 3–velocity in the laboratory frame is instantaneous (at $U=0$) and since (3.137) has the characteristic 3–velocity dependence of electromagnetic bremsstrahlung it seems appropriate to refer to the radiation described by (3.132) as *instantaneous electromagnetic bremsstrahlung.* The collision of this electromagnetic wave with an impulsive gravitational wave, at large distance from the charge e, is worked out in [Barrabès and Hogan (2000)].

3.3 Impulsive Light–Like Signals as Recoil Effects: Gravitational Example

We consider here the analogous situation for a spherically symmetric mass m in general relativity to that of the charge e discussed in §3.2 . Hence in place of the Minkowskian line–element of $\mathcal{M}^-$ given by (3.103) we take the Schwarzschild line–element for $\mathcal{M}^-$ with a source of mass m,

$$ds_-^2 = k^2\,r^2\left\{\frac{d\xi^2}{1-\xi^2}+(1-\xi^2)\,d\phi^2\right\}-2\,du\,dr-\left(1-\frac{2\,m}{r}\right)\,du^2\ , \quad (3.138)$$

with $k^{-1}=\gamma\,(1-v\,\xi)$. Also in place of (3.104) we take $\mathcal{M}^+$ to be the Schwarzschild space–time with a source of mass m_+ with line–element

$$ds_+^2 = r_+^2\left\{\frac{d\xi_+^2}{1-\xi_+^2}+(1-\xi_+^2)\,d\phi_+^2\right\}-2\,du_+\,dr_+-\left(1-\frac{2\,m_+}{r_+}\right)\,du_+^2\ . \quad (3.139)$$

For greater generality we have assumed that the rest mass of the Schwarzschild source has changed from m in $\mathcal{M}^-$ to m_+ in $\mathcal{M}^+$. In (3.138) the constant v is the 3–velocity of the source which appears to be moving rectilinearly relative to a distant observer. By writing (3.138) in Bondi form [Bondi et al. (1962)] we can identify the Bondi mass or 'mass aspect' of the source. This turns out to be the mass aspect associated with a mass m moving along a symmetry axis with 3–velocity v (cf. [Bondi et al. (1962)], equation (72)) and thus confirms the physical interpretation of the parameter v in this case. To see this we begin by using the coordinate transformation (3.111) and (3.112), which for our present purpose we write as

$$r' = k\,r\,\varphi\ , \quad (3.140)$$

$$u' = \gamma\,u + k\,r\,(1-\varphi)\ , \quad (3.141)$$

$$\xi' = \varphi^{-1}(\chi+\xi)\ , \quad (3.142)$$

$$\phi' = \phi\ , \quad (3.143)$$

with

$$\varphi = (1 + 2\,\chi\,\xi + \chi^2)^{1/2}\ , \qquad \chi = \frac{\gamma\,v\,u}{k\,r}\ . \tag{3.144}$$

We use this to put (3.138) in the form

$$ds_-^2 = r'^2\left\{\frac{d\xi'^2}{1-\xi'^2} + (1-\xi'^2)\,d\phi'^2\right\} - 2\,du'\,dr' - du'^2 + \frac{2\,m\,k\,\varphi}{r'}\,du^2\ . \tag{3.145}$$

We can simplify the last term here assuming r' is large. First we note that

$$\varphi = 1 + \frac{\gamma\,v\,u\,\xi'}{r'} + O\left(\frac{1}{r'^2}\right)\ , \tag{3.146}$$

and so we obtain

$$\xi = \xi' - \frac{\gamma\,v\,k'\,u'\,(1-\xi'^2)}{r'} + O\left(\frac{1}{r'^2}\right)\ , \tag{3.147}$$

$$\phi = \phi'\ , \tag{3.148}$$

$$r = \frac{r'}{k'} + \gamma^2\,v\,k'\,u'\,(v - \xi') + O\left(\frac{1}{r'}\right)\ , \tag{3.149}$$

$$u = k'\,u' + O\left(\frac{1}{r'}\right)\ , \tag{3.150}$$

with $k'^{-1} = \gamma\,(1 - v\,\xi')$. Now (3.145) becomes

$$\begin{aligned} ds_-^2 = r'^2 &\left\{\frac{d\xi'^2}{1-\xi'^2} + (1-\xi'^2)\,d\phi'^2\right\} - 2\,du'\,dr' - du'^2 \\ &+ \frac{2\,m\,k'^3}{r'}\,(du' + k'\,u'\,\gamma\,v\,d\xi')^2 + O\left(\frac{1}{r'^2}\right)\ . \end{aligned} \tag{3.151}$$

A further transformation $r' \to r''$ given by

$$r'' = r' + \frac{m\,k'^5\,\gamma^2\,v^2\,u'^2\,(1-\xi'^2)}{2\,r'^2} + O\left(\frac{1}{r'^3}\right)\ , \tag{3.152}$$

puts the line–element (3.151) in Bondi form [Bondi et al. (1962)]

$$\begin{aligned} ds_-^2 = r''^2 &\left\{\frac{e^{2\,\lambda}}{(1-\xi'^2)}\,\left(d\xi' + U\,\sqrt{1-\xi'^2}\,du'\right)^2 + e^{-2\,\lambda}\,(1-\xi'^2)\,d\phi'^2\right\} \\ &-2\,e^{2\,\beta}\,du'\,dr'' - r''^{-1}\,V\,e^{2\,\beta}\,du'^2\ , \end{aligned} \tag{3.153}$$

with

$$\lambda = \frac{M\,\gamma^2\,v^2\,u'^2\,k'^2\,(1-\xi'^2)}{2\,r''^3} + O\left(\frac{1}{r''^4}\right)\ , \tag{3.154}$$

$$U = \frac{2\,M\,\gamma\,v\,u'\,k'\,\sqrt{1-\xi'^2}}{r''^3} + O\left(\frac{1}{r''^4}\right) , \tag{3.155}$$

$$\beta = O\left(\frac{1}{r''^2}\right) , \tag{3.156}$$

$$V = r'' - 2\,M + O\left(\frac{1}{r''}\right) , \tag{3.157}$$

and

$$M = m\,k'^3 = \frac{m\,\gamma^{-3}}{(1 - v\,\xi')^3} . \tag{3.158}$$

Here M is the 'mass aspect'. Given by (3.158) it has turned out to be precisely the mass aspect associated with a mass m moving with 3–velocity v along the axis of symmetry [Bondi et al. (1962)].

In (3.138) $u =$ constant are future null cones generated by the integral curves of the vector field $\partial/\partial r$, while $u_+ =$ constant in (3.139) are also future null cones generated by the integral curves of $\partial/\partial r_+$. In (3.138) we shall take $u \le 0$ and $\mathcal{M}^-$ as the region of space–time to the past of the null hypersurface $\mathcal{N}(u = 0)$. In (3.139) we take $u_+ \ge 0$, with $u_+ = 0 \Leftrightarrow u = 0$, and we take $\mathcal{M}^+$ to be the region of space–time to the future of $\mathcal{N}$. By analogy with the electromagnetic case in §3.2 we match $\mathcal{M}^-$ and $\mathcal{M}^+$ on $\mathcal{N}$ with the matching conditions (3.105) which ensure, by (3.138) and (3.139), that the line–elements induced on $\mathcal{N}$ by its embedding in $\mathcal{M}^-$ and in $\mathcal{M}^+$ agree. Thus we have $\{x^\mu_+\} = \{\xi_+, \phi_+, r_+, u_+\}$ and $\{x^\mu_-\} = \{\xi, \phi, r, u\}$ as independent local coordinate systems in $\mathcal{M}^+$ and $\mathcal{M}^-$ respectively. As normal to $\mathcal{N}$ we take the null vector n^μ given by the 1–form $n_\mu\, dx^\mu_\pm = -du$. To calculate the physical properties of $\mathcal{N}$ observed by the observer using the plus coordinates, we take $\{\xi^a\} = \{\xi_+, \phi_+, r_+\}$ with $a = 1, 2, 3$ as intrinsic coordinates on $\mathcal{N}$. The three holonomic basis vectors with components (2.24) are calculated using the matching conditions (3.105) relating $\{x^\mu_-\}$ and $\{\xi^a\}$. On the plus side of $\mathcal{N}$ these have components $e^\mu_{(a)}|_+ = \delta^\mu_a$ while on the minus side of $\mathcal{N}$ they are given by

$$e^\mu_{(1)}|_- = (1, 0, -r_+\gamma\,v, 0) , \tag{3.159}$$

$$e^\mu_{(2)}|_- = (0, 1, 0, 0) , \tag{3.160}$$

$$e^\mu_{(3)}|_- = (0, 0, \gamma\,(1 - v\,\xi_+), 0) . \tag{3.161}$$

As transversal on $\mathcal{N}$ we can take the covariant null vector in the coordinates $\{x^\mu_+\}$ with components ${}^+N_\mu = (0, 0, -1, -\frac{1}{2} + \frac{m_+}{r_+})$. Since $n^\mu = \delta^\mu_3$ we have ${}^+N_\mu\, n^\mu = -1$. This transversal on the minus side of $\mathcal{N}$ has components

${}^{-}N_\mu$. To ensure that this is the same covariant vector as ${}^{+}N_\mu$ we have to satisfy (2.28) together with ${}^{+}N_\mu\,{}^{+}N^\mu = 0$. This results in

$$ {}^{-}N_\mu = \left(-\frac{r_+ v}{1 - v\,\xi_+},\ 0,\ -\frac{1}{\gamma\,(1 - v\,\xi_+)},\ D\right) , \tag{3.162}$$

with

$$ D = -\frac{v^2\,\gamma\,(1-\xi_+^2)}{2\,(1 - v\,\xi_+)} - \frac{1}{2\,\gamma\,(1 - v\,\xi_+)} + \frac{m}{\gamma^2(1 - v\,\xi_+)^2 r_+} . \tag{3.163}$$

We can now calculate the extrinsic curvatures (2.29) and then evaluate the jumps (2.31). We find that $\gamma_{ab} = 0$ except for

$$ \gamma_{11} = (1 - \xi_+^2)^{-2}\,\gamma_{22} = \frac{2}{1 - \xi_+^2}\,(m_+ - m\,k^3) , \tag{3.164}$$

with $k^{-1} = \gamma\,(1 - v\,\xi)$. Now γ_{ab} is extended to a 4–tensor with components $\gamma_{\mu\nu}$ by 'padding–out' with zeros (the only requirement on $\gamma_{\mu\nu}$ being given by (2.32)).With our choice of a future–pointing normal and past–pointing transversal the surface stress–energy tensor is given by

$$ S_{\mu\nu} = \mu\, n_\mu\, n_\nu , \tag{3.165}$$

with

$$ \mu = \frac{1}{4\,\pi\,r_+^2}\,(m\,k^3 - m_+) . \tag{3.166}$$

Thus the null cone $\mathcal{N}$ is the history of a light–like shell with surface stress–energy given by (3.165). A calculation of the coefficient of the δ–function in the Weyl tensor, given by (2.45), reveals that it vanishes. Hence there is no possibility of the light–like signal with history $\mathcal{N}$ incorporating an impulsive gravitational wave. We note that μ in (3.166) is a momotonically increasing function of ξ_+. Thus on the interval $-1 \le \xi_+ \le +1$, σ is maximum at $\xi_+ = -1$ (in the direction of motion) and μ is minimum at $\xi_+ = -1$. This is as one would expect. A burst of null matter predominantly in the direction of motion is required to halt the mass. In this sense the model we have constructed here could be considered a limiting case of a Kinnersley rocket (see [Kinnersley (1969)] and also the papers by Bonnor cited in the bibliography).

By integrating (3.166) over the shell with area element $dA_+ = r_+^2\,d\xi_+\,d\phi_+$ with $-1 \le \xi_+ \le +1$ and $0 \le \phi_+ < 2\,\pi$ we obtain the total energy E_+ of the shell measured by the distant observer who sees the mass

m moving rectilearly with 3–velocity v in the direction $\xi_+ = +1$ suddenly halted. Thus

$$E_+ = \frac{1}{4\pi}\int_0^{2\pi} d\phi_+ \int_{-1}^{+1} (mk^3 - m_+)\, d\xi_+ \ . \tag{3.167}$$

This results in

$$E_+ = m\gamma - m_+ \ . \tag{3.168}$$

So the energy of the light–like shell is the difference between the relative masses before and after the emission of the light–like shell. To emphasize the conservation of energy we interpret (3.168) as saying that, in the reference frame relative to which the mass m_+ is at rest, the energy $m\gamma$ of the ingoing mass is transferred into the rest energy m_+ plus the energy E_+ of the emitted null shell.

When $v = 0\,(\gamma = 1)$ the energy of the shell is the difference in the rest masses (naturally taking $m_+ < m$) and this is the special case (3.61) above. On the other hand if $v \neq 0$ and $m = m_+$ then (3.168) becomes

$$E_+ = m\,(\gamma - 1) \ . \tag{3.169}$$

In this case all of the relative kinetic energy of the mass m before stopping is converted into the relativistic shell.

The simplest way of interpreting the model presented here is that it represents the gravitational field outside a moving spherical body (a star for instance) which, at a certain moment of retarded time, suddenly stops as a consequence of some internal process such as a laser–like nuclear reaction. The emission of a sharp burst of null matter (photons or neutrinos) predominantly in the forward direction provides the recoil momentum which is necessary to stop the body. As we are only interested in the exterior field we have not made any assumption about the structure of the body except that it is spherically symmetric and is initially moving with constant velocity. From the point of view of the global structure of space–time there are two possibilities to consider. The first would correspond to a point–like body and the maximal analytical extension of the space–time [Kruskal (1960)] would then show a light–like shell emerging from the white–hole region and propagating radially to future null infinity. The second possibility would be that the body has some finite radius, larger than its gravitational radius, and the light–like shell is directly emitted from the surface of the body. The idea of an extended body remaining rotationally symmetric when suddenly decelerated to rest is, of course, a strong idealization. It would be more realistic to expect the body to experience some deformation following the deceleration, which would lead to the emission of gravitational radiation, before settling back to a spherically symmetric state. This question is addressed in [Barrabès and Hogan (2002)].

The last stages of a binary neutron star collision would be expected to involve the emission of high frequency gravitational radiation followed by an impulsive light–like signal (incorporating gravitational radiation and light–like matter) carrying the news to distant observers that the collision has taken place. This would be followed by a ringing–down phase during which low frequency gravitational radiation would be emitted from the remnant of the collision until a black hole is formed. A qualitative description of this process can be found in [Futamase and Hogan (2000)].

3.4 Interaction of Impulsive Light–Like Signals with Matter

In order to include cosmological applications and as a prelude to the discussion of the detection of these signals in §3.6 we consider the space–time $\mathcal{M}^-\cup\mathcal{M}^+$ to contain a preferred congruence of time–like world–lines. These world–lines are the integral curves of a unit time–like vector field having components u^μ, with $u^\mu u_\mu = -1$, in terms of a local coordinate system $\{x^\mu\}$ covering both sides of the null hypersurface $\mathcal{N}$. This congruence can be arbitrarily chosen in each domain $\mathcal{M}^\pm$ but we will only consider the case in which u^μ is continuous across $\mathcal{N}$. This choice forbids $\mathcal{N}$ becoming the history of a shock wave propagating through the matter in the usual sense (see [Synge (1957)] and [Israel (1960)] for example). We choose u^μ to be continuous across $\mathcal{N}$ for the following reasons: (a) this is the minimal requirement consistent with a delta function appearing in the Weyl tensor (which can be seen by applying the Ricci identities to u^μ), (b) this allows finite jumps, with no delta function, to appear in the kinematical quantities associated with the integral curves of u^μ (4-acceleration, expansion, vorticity and shear– see Appendix) and (c) the jumps in the kinematical quantities are then simply related to the presence or otherwise of a light–like shell and/or a gravitational wave component in the signal with history $\mathcal{N}$. We allow the derivative of u^μ across $\mathcal{N}$ to jump across $\mathcal{N}$. This jump is described by a vector field λ^μ on $\mathcal{N}$ given by

$$[N^\mu\,\partial_\mu u^\alpha] = \lambda^\alpha\ . \tag{3.170}$$

It follows that the jump in the 4–acceleration $a^\mu = u^\nu\,\nabla_\nu u^\mu$ (it will be convenient to denote covariant differentiation, in this section, by ∇_μ rather than by a stroke) is given by

$$\eta^{-1}\,[a^{\mu}] = -s\,\lambda^{\mu} - s\,U^{\mu} - \frac{1}{2}\,(u_{\nu}\,U^{\nu})\,n^{\mu}\ , \tag{3.171}$$

where we have put $s = -u^{\mu}\,n_{\mu} > 0$ and $U^{\mu} = \gamma^{\mu\nu}\,u_{\nu}$ with $\gamma^{\mu\nu}$ given by (2.10). Using $u_{\mu}\,a^{\mu} = 0$ we have the following relation:

$$u^{\mu}\,U_{\mu} = -2\,u^{\mu}\,\lambda_{\mu}\ . \tag{3.172}$$

The expansion θ, the shear $\sigma_{\alpha\beta}$ and the vorticity $\omega_{\alpha\beta}$ of the time–like congruence are in general discontinuous across $\mathcal{N}$ and their jumps across $\mathcal{N}$ are given by

$$\eta^{-1}\,[\theta] = -\frac{1}{2}\,s\,\gamma + \lambda^{\mu}\,n_{\mu}\ , \tag{3.173}$$

with γ defined in (2.20),

$$\begin{aligned}\eta^{-1}\,[\sigma_{\alpha\beta}] = &-\frac{s}{2}\,\gamma_{\alpha\beta} - \lambda_{(\alpha}\,n_{\beta)} - s\,U_{(\alpha}\,u_{\beta)}\\ &-\frac{1}{2}\,(u_{\mu}\,U^{\mu})\,n_{(\alpha}\,u_{\beta)} - s\,\lambda_{(\alpha}\,u_{\beta)} - \frac{1}{3}\,\eta^{-1}\,[\theta]\,h_{\alpha\beta}\ ,\end{aligned} \tag{3.174}$$

and

$$\begin{aligned}\eta^{-1}\,[\omega_{\alpha\beta}] = &\,U_{[\alpha}\,n_{\beta]} + \lambda_{[\alpha}\,n_{\beta]} - s\,U_{[\alpha}\,u_{\beta]}\\ &-\frac{1}{2}\,(u_{\mu}\,U^{\mu})\,n_{[\alpha}\,u_{\beta]} - s\,\lambda_{[\alpha}\,u_{\beta]}\ ,\end{aligned} \tag{3.175}$$

where the projection tensor $h_{\alpha\beta} = g_{\alpha\beta} + u_{\alpha}\,u_{\beta}$ has been introduced. We can alternatively express (3.175) in terms of the (covariant) vorticity vector

$$\omega_{\alpha} = \frac{1}{2}\,\eta_{\alpha\beta\mu\nu}\,u^{\beta}\,\omega^{\mu\nu}\ , \tag{3.176}$$

where $\eta_{\alpha\beta\mu\nu} = \sqrt{-g}\,\epsilon_{\alpha\beta\mu\nu}$ with $g = \det(g_{\mu\nu})$ and $\epsilon_{\alpha\beta\mu\nu}$ is the four–dimensional Levi–Cività permutation symbol. The jump in the vorticity vector is given by

$$\eta^{-1}\,[\omega_{\alpha}] = \frac{1}{2}\,\eta_{\alpha\beta\mu\nu}\,u^{\beta}\,(U^{\mu} + \lambda^{\mu})\,n^{\nu}\ . \tag{3.177}$$

On the null hypersurface $\mathcal{N}$ we have the normal n^{μ} and the unit time–like vector field u^{μ} for which $u^{\mu}\,n_{\mu} = -s < 0$. In a local coordinate system $\{x^{\mu}\}$ covering both sides of $\mathcal{N}$ it is helpful to define on $\mathcal{N}$, but not tangent to $\mathcal{N}$, the null vector field

$$l^{\mu} = -\frac{1}{2s^{2}}\,n^{\mu} + \frac{1}{s}\,u^{\mu}\ , \tag{3.178}$$

with $l^\mu n_\nu = -1$. Now n^μ, l^μ can be supplemented by a complex null vector field m^μ tangent to $\mathcal{N}$ and also orthogonal to l^μ, and satisfying $m^\mu \bar{m}_\mu = +1$ (the bar denoting complex conjugation). The null tetrad $\{n^\mu, l^\mu, m^\mu, \bar{m}^\mu\}$ defined on $\mathcal{N}$ will be useful for the purposes of displaying formulas below. If the Newman–Penrose components $\hat{\Psi}_A$ $(A = 0, 1, 2, 3, 4)$ of the coefficient of the delta function in the Weyl tensor, $\hat{C}^{\mu\nu\rho\sigma}$ in (2.45), are calculated on this null tetrad they are given by

$$\hat{\Psi}_0 = 0, \qquad \hat{\Psi}_1 = 0, \qquad \hat{\Psi}_2 = -\frac{1}{6}\gamma^\dagger ,$$
$$\hat{\Psi}_3 = -\frac{1}{2}\gamma_\mu \bar{m}^\mu , \qquad \hat{\Psi}_4 = -\frac{1}{2}\gamma_{\mu\nu}\bar{m}^\mu \bar{m}^\nu . \tag{3.179}$$

This confirms (cf. §2.3) that the delta function in the Weyl tensor is in general Petrov Type II. If the induced geometry on $\mathcal{N}$ is Type II then $\gamma^\dagger = 0$ and $\hat{\Psi}_A$ is Petrov Type III whereas if the induced geometry on $\mathcal{N}$ is Type III then $\gamma^\dagger = 0$ and $\gamma_\mu \bar{m}^\mu = 0$ and $\hat{\Psi}_A$ is Petrov Type N. The signal with history $\mathcal{N}$ contains a gravitational wave if $\hat{\Psi}_4 \neq 0$.

Combining the jumps (3.171), (3.173)–(3.175) across $\mathcal{N}$, in the kinematical quantities associated with a time–like congruence intersecting $\mathcal{N}$, with the Newman–Penrose components $\hat{\Psi}_A$ (given by (3.179)) of the coefficients of the δ–function in the Weyl tensor, we obtain, with straightforward algebra:

Lemma 1:

(1) $[\sigma_{\mu\nu}\, m^\mu m^\nu] \neq 0 \Leftrightarrow \hat{\Psi}_4 \neq 0$;

(2) *If* $[\sigma_{\mu\nu}] = 0$ *then* $\hat{\Psi}_4 = 0$ *and*:

(a) $[a^\mu m_\mu] \neq 0 \Leftrightarrow \hat{\Psi}_3 \neq 0$;

(b) $[\omega^\mu] \neq 0 \Leftrightarrow \hat{\Psi}_3 \neq 0$;

(3) *If* $[\sigma_{\mu\nu}] = 0$ *and* $[a^\mu] = 0$ *then* $\hat{\Psi}_3 = \hat{\Psi}_4 = 0$ *and* $[\theta] \neq 0 \Leftrightarrow \hat{\Psi}_2 \neq 0$.

We note from (3.171) and (3.177) that $[a^\mu] = 0 \Rightarrow [\omega^\mu] = 0$. The converse is not true because again from (3.171) and (3.177) we find that if $[\omega^\mu] = 0$ then $[a^\mu] = s^{-2} n_\lambda [a^\lambda] (n^\mu - s\, u^\mu)$ and we can only conclude from this that in general $m_\mu [a^\mu] = 0$. This explains the appearance of all components of $[\omega^\mu]$ in part (2b) of the Lemma and of only one complex component of $[a^\mu]$

in part (2a) of the Lemma.

We are particularly interested in Lemma 1 when the time-like congruence of integral curves of u^μ are the world–lines of the cosmic fluid in a cosmological model. The first part of the Lemma says that if the signal with history $\mathcal{N}$ includes a gravitational wave then its effect on the cosmic fluid is to cause a jump across $\mathcal{N}$ in a complex component of the shear of the congruence and, if the passage of the signal through the fluid does not result in a jump in the fluid shear then the signal cannot contain a gravitational wave. In this latter case the signal is a light–like shell of matter with a Petrov Type II delta function in the Weyl tensor if the vorticity of the fluid jumps across $\mathcal{N}$ or if a complex component of the fluid 4–acceleration jumps across $\mathcal{N}$. If only the expansion of the fluid jumps across $\mathcal{N}$ then part (3) of the Lemma shows that the delta function in the Weyl tensor is Petrov Type III.

There is an interesting analogy between Lemma 1 and the usual decomposition of perturbations of cosmological models into scalar, vector and tensor parts with the tensor perturbations describing propagating gravitational waves and the other perturbations describing inhomogeneity in the matter distribution (see [Ellis and Bruni (1989)], [Hogan and Ellis (1997)] and [Hogan and O'Shea (2002)], for example). In Lemma 1 the analogue of the tensor perturbations is the jump in the shear of the time–like congruence which by part (1) is necessary for the signal with history $\mathcal{N}$ to include a gravitational impulse wave. The analogues of the vector and scalar perturbations are the jumps in the 4–acceleration and vorticity on the one hand and in the expansion on the other hand leading, by parts (2) – (3), to the possibility of the signal being a light–like shell of matter.

To illustrate Lemma 1 with an example of a signal consisting of a gravitational impulsive wave and a light–like shell propagating through the Einstein–de Sitter universe (say) we must choose a cosmological model left behind by the signal (the space–time $\mathcal{M}^+$ to the future of the null hypersurface $\mathcal{N}$) which has the properties: (a) its fluid 4–velocity joins continuously to that of the Einstein–de Sitter on $\mathcal{N}$ and (b) its fluid 4–velocity has shear. Thus the line–element of $\mathcal{M}^-$ is that of Einstein–de Sitter which, in coordinates $x^\mu_- = (t, r, \phi, z)$, reads

$$ds^2 = -dt^2 + t^{4/3\beta}\left(dr^2 + r^2 d\phi^2 + dz^2\right) , \qquad (3.180)$$

where β is a constant. Here the t–lines are the world– lines of the particles of a perfect fluid with isotropic pressure p and proper–density μ_0 satisfying the equation of state $p = (\beta - 1)\,\mu_0$. A simple example of a space–time $\mathcal{M}^+$

satisfying the requirements (a) and (b) above is the anisotropic Bianchi I space–time [Vajk and Eltgroth (1970)] with line–element, in coordinates $x_+^\mu = (t_+, r_+, \phi_+, z_+)$,

$$ds_+^2 = -dt_+^2 + A_+^2(dr_+^2 + r_+^2 d\phi_+^2) + B_+^2 dz_+^2 \ , \tag{3.181}$$

where

$$A_+ = t_+^{(3\beta-2)/6\beta} \ , \qquad B_+ = t_+^{2/3\beta} \ . \tag{3.182}$$

The t_+–lines are the world–lines of a perfect fluid with isotropic pressure p_+ and proper–density μ_+ satisfying $p_+ = \mu_+$. As boundary between $\mathcal{M}^-$ and $\mathcal{M}^+$ take the null hypersurface $\mathcal{N}$ to be given by

$$r_+ = T_+(t_+) \ , \qquad \frac{dT_+}{dt_+} = \frac{1}{A_+} \ , \tag{3.183}$$

in the plus coordinates and by

$$r = T(t) \ , \qquad \frac{dT}{dt} = t^{-2/3\beta} \ , \tag{3.184}$$

in the minus coordinates. As intrinsic coordinates on $\mathcal{N}$ we can use $\xi^a = (r, \phi, z)$. The induced line–elements on $\mathcal{N}$ from $\mathcal{M}^+$ and $\mathcal{M}^-$ match if

$$t_+ = t, \ r_+ = \frac{6\beta}{(3\beta+2)}\left[\frac{(3\beta-2)}{3\beta}r\right]^{\frac{(3\beta+2)}{(6\beta-4)}} \ , \ \phi_+ = \left(\frac{3\beta+2}{6\beta-4}\right)\phi, \ z_+ = z \ . \tag{3.185}$$

We must first check that the 4–velocities of the fluid particles with histories in $\mathcal{M}^+$ and $\mathcal{M}^-$ are continuous across $\mathcal{N}$. This has to be done with care as we now have two local coordinate systems $\{x_+^\mu\}$ and $\{x_-^\mu\}$ on either side of $\mathcal{N}$, overlapping on $\mathcal{N}$ according to (3.185). Let ${}^+v^\mu = (1,0,0,0)$ and ${}^-u^\mu = (1,0,0,0)$. Then ${}^+v^\mu$, ${}^-u^\mu$ are the fluid 4–velocities in $\mathcal{M}^+$ and $\mathcal{M}^-$ respectively. Let ${}^+u^\mu$ be the same vector as ${}^-u^\mu$ but calculated on the plus side of $\mathcal{N}$. We then compare (on $\mathcal{N}$) ${}^+v^\mu$ with ${}^+u^\mu$ and if they are equal then the fluid 4–velocity is continuous across $\mathcal{N}$. To do this we utilise the tangent basis vectors $e_{(a)} = \partial/\xi^a$, with $\xi^a = (r, \phi, z)$. Now ${}^+u^\mu$ is the same vector as ${}^-u^\mu$ if

$$[u_\mu \, u^\mu] = \left[u_\mu \, e^\mu_{(a)}\right] = 0 \ . \tag{3.186}$$

These are the same conditions (2.28) that a transversal N on $\mathcal{N}$ has to satisfy and indeed u^μ can be used as a transversal if desired. The four conditions (3.186) determine ${}^+u^\mu$ uniquely and for the example we are considering we obtain ${}^{+}u^{\mu} = (1, 0, 0, 0)$. Hence ${}^{+}u^{\mu} = {}^{+}v^{\mu}$ and the fluid 4–velocity is continuous across $\mathcal{N}$. Now using the theory outlined in Chapter 2 we find that $\gamma_{\mu\nu} = 0$ except for (quoting the non–vanishing components of $\gamma_{\mu\nu}$ in the coordinate system $\{x_{-}^{\mu}\}$ (say))

$$\gamma_{11} = \frac{(\beta - 2)}{\beta}\, t^{\frac{(2-\beta)}{2\beta}} \ , \quad \gamma_{22} = \frac{9\beta\,(\beta - 2)}{(3\beta - 2)^2}\, t^{\frac{(9\beta-2)}{6\beta}} \ . \tag{3.187}$$

With
$n_{-}^{\mu} = (t^{(-3\beta+2)/6\beta}, t^{-(3\beta+2)/6\beta}, 0, 0)$ and $m_{-}^{\mu} = 2^{-1/2}\, t^{-2/3\beta}\,(0, 0, i\,r^{-1}, 1)$ we find that

$$\gamma_{\mu} = \gamma_{\mu\nu}\, n_{-}^{\nu} = \delta_{\mu}^{1}\, \frac{(\beta - 2)}{\beta}\, t^{(2-3\beta)/3\beta} \ , \tag{3.188}$$

$$\gamma^{\dagger} = \gamma_{\mu}\, n_{-}^{\mu} = \frac{(\beta - 2)}{\beta}\, t^{(2-9\beta)/6\beta} \ , \tag{3.189}$$

$$\gamma_{\mu\nu}\, \bar{m}^{\mu}\, \bar{m}^{\nu} = -\frac{(\beta - 2)}{2\beta}\, t^{-(3\beta+2)/6\beta} \ , \tag{3.190}$$

where in (3.190) we have written r in terms of t following from (3.184). Comparison now with (3.179) shows that $\hat{\Psi}_3 = 0$ but $\hat{\Psi}_2 \neq 0$ and $\hat{\Psi}_4 \neq 0$. In addition we find that the vector field λ on $\mathcal{N}$ introduced in (3.170) vanishes, as does $U_{\alpha} = \gamma_{\alpha\beta}\, {}^{-}u^{\beta}$. We see from (3.188)–(3.190) that the geometry induced on $\mathcal{N}$ is a Type I geometry and that $\mathcal{N}$ is the history of both an impulsive gravitational wave and a light–like shell.

3.5 Implications of the Bianchi Identities

The main original references for our study of singular null hypersurfaces in general relativity are [Penrose (1972)] and [Barrabès and Israel (1991)]. In these the properties of the signal which can be obtained from the Bianchi identities are either implicit in the work or are only partially explicitly derived. A systematic approach to obtaining the consequences of the Bianchi identities, depending upon the type of induced geometry on $\mathcal{N}$ (in Penrose's sense), has been given in [Barrabès and Hogan (1998)]. This constitutes an extension of the theory outlined in Chapter 2. We recall from §2.1 that $\mathcal{N}$ has equation $\Phi(x^{\mu}) = 0$ and $n_{\mu} = \chi^{-1}\Phi_{,\mu}$ for some function χ defined on $\mathcal{N}$. By(2.2) and (2.16) the components of the Einstein tensor of the space–time $\mathcal{M}^- \cup \mathcal{M}^+$ have the form

$$G^{\mu\nu} = \hat{G}^{\mu\nu}\,\eta\chi\,\delta(\Phi) + \Theta(\Phi)\;{}^+G^{\mu\nu} + (1-\Theta(\Phi))\;{}^-G^{\mu\nu}\ , \tag{3.191}$$

where δ is the Dirac delta function, Θ is the Heaviside step function with $\Theta > 0$ in $\mathcal{M}^+$, $\Theta < 0$ in $\mathcal{M}^-$ and

$$\Theta_{,\mu} = \chi\, n_\mu\, \delta(\Phi)\ . \tag{3.192}$$

In (3.191) ${}^{\pm}G^{\mu\nu}$ are the components of the Einstein tensors in $\mathcal{M}^{\pm}$ respectively and can be written as $8\pi\,{}^{\pm}T^{\mu\nu}$ in terms of the respective energy–momentum–stress tensors. Also

$$\hat{G}^{\mu\nu} = 8\pi\, S^{\mu\nu}\ , \tag{3.193}$$

with $S^{\mu\nu}$ given by (2.22). Thus in particular

$$\hat{G}^{\mu\nu}\, n_\nu = 0\ . \tag{3.194}$$

We now apply the twice–contracted Bianchi identities $\nabla_\nu G^{\mu\nu} \equiv 0$ to (3.191) (it is convenient to denote the covariant derivative with respect to the Riemannian connection here by ∇_ν). On account of (3.194) the term in $\nabla_\nu G^{\mu\nu}$ involving the derivative of the delta function vanishes and we obtain

$$\nabla_\nu(\eta\,\chi\,\hat{G}^{\mu\nu}) + \chi\,[G^{\mu\nu}\, n_\nu] = 0. \tag{3.195}$$

Since $\hat{G}^{\mu\nu}$ in (3.191) is defined *on* $\mathcal{N}$ it only makes sense to calculate derivatives of $\hat{G}^{\mu\nu}$ tangential to $\mathcal{N}$. The number of such different tangential derivatives available to us depends upon the type of the induced geometry on $\mathcal{N}$. If the induced geometry is Type I then we obtain one meaningful equation from (3.195), namely,

$$-8\pi\,\eta^{-1}\,[T_{\mu\nu}\, n^\mu\, n^\nu] = \rho\,\gamma^\dagger\ . \tag{3.196}$$

Here square brackets as always denote the jump in the enclosed quantity across $\mathcal{N}$ and $\rho = m^\mu\, \bar{m}^\nu\, \nabla_\nu n_\mu$ is the expansion of the null geodesic integral curves of n^μ ($[\rho] = 0$ since ρ is intrinsic to $\mathcal{N}$). If the induced geometry is Type II then $\gamma^\dagger = 0$. Hence the acceleration parameter κ introduced in (2.73) is continuous across $\mathcal{N}$. Now the meaningful equations emerging from the twice contracted Bianchi identities are (3.196) with $\gamma^\dagger = 0$ and also

$$-16\pi\,\eta^{-1}\,[T_{\mu\nu}\, m^\mu\, n^\nu] = \gamma'_\mu\, m^\mu + 3\rho\,\gamma_\mu\, m^\mu + \sigma\,\bar{m}^\mu\,\gamma_\mu\ . \tag{3.197}$$

Here $\gamma'_\mu = n^\nu\,\nabla_\nu\,\gamma_\mu$ and $\sigma = m^\mu\, m^\nu\,\nabla_\nu n_\mu$ is the complex shear of the generators of $\mathcal{N}$, which is intrinsic even for a Type I geometry. Finally if the induced geometry is Type III then $\gamma_\mu = A\, n_\mu$ for some function A defined on $\mathcal{N}$. Now the right hand sides of (3.196) and (3.197) vanish and in addition we find that

$$\eta^{-1}\,[T_{\mu\nu}\, l^\mu\, n^\nu] = \mu' + (2\rho + \kappa)\,\mu\ , \tag{3.198}$$

where $8\pi\,\mu = A - \frac{1}{2}\gamma$ and $\mu' = \mu_{,\mu}\, n^\mu$.

To obtain the Bianchi identities we use the tensor representing the left and right duals of the Riemann curvature tensor [Robinson and Robinson (1972)]

$$G^{\mu\nu\rho\sigma} = \frac{1}{4}\,\eta^{\mu\nu\alpha\beta}\,\eta^{\rho\sigma\lambda\gamma}\,R_{\alpha\beta\lambda\gamma}\ , \tag{3.199}$$

where $R_{\alpha\beta\lambda\gamma}$ are the components of the Riemann curvature tensor and $\eta^{\alpha\beta\gamma\delta} = -(-g)^{-\frac{1}{2}}\,\epsilon_{\alpha\beta\gamma\delta}$. Then the Bianchi identities read

$$\nabla_\sigma\, G^{\mu\nu\rho\sigma} \equiv 0\ . \tag{3.200}$$

The right hand side of (3.199) can be written in terms of the Riemann and Ricci tensors and the Ricci scalar as

$$\begin{aligned} -G^{\mu\nu\rho\sigma} = {} & R^{\mu\nu\rho\sigma} - g^{\mu\rho}\,R^{\nu\sigma} - g^{\nu\sigma}\,R^{\mu\rho} \\ & + g^{\mu\sigma}\,R^{\nu\rho} + g^{\nu\rho}\,R^{\mu\sigma} + \frac{1}{2}\,R\,(g^{\mu\rho}\,g^{\nu\sigma} - g^{\mu\sigma}\,g^{\nu\rho})\ . \end{aligned} \tag{3.201}$$

It is then useful to substitute in (3.201) for the Riemann tensor in terms of the Weyl tensor. For the space–time $\mathcal{M}^- \cup \mathcal{M}^+$ the tensor (3.199) has a decomposition similar to that of the Einstein tensor (3.191),

$$G^{\mu\nu\rho\sigma} = \hat{G}^{\mu\nu\rho\sigma}\,\eta\chi\,\delta(\Phi) + \Theta(\Phi)\ {}^+G^{\mu\nu\rho\sigma} + (1 - \Theta(\Phi))\ {}^-G^{\mu\nu\rho\sigma}\ , \tag{3.202}$$

with

$$\hat{G}^{\mu\nu\rho\sigma} = -2n^{[\mu}\,\gamma^{\nu]\,[\rho}\,n^{\sigma]} - 2g^{\mu[\sigma}\,w^{\rho]\nu} + 2\,g^{\nu[\sigma}\,w^{\rho]\mu} - \gamma^\dagger\,g^{\mu[\rho}\,g^{\sigma]\nu}\ , \tag{3.203}$$

where $w^{\mu\nu} = \gamma^{(\mu}\,n^{\nu)} - \frac{1}{2}\,\gamma\,n^\mu\,n^\nu$. One readily sees that

$$\hat{G}^{\mu\nu\rho\sigma}\,n_\sigma = 0\ . \tag{3.204}$$

Thus when (3.202) is applied to (3.200) the term involving the derivative of the delta function vanishes and we obtain from (3.200)

$$\nabla_\sigma\,(\eta\,\chi\,\hat{G}^{\mu\nu\rho\sigma}) + \chi\ [G^{\mu\nu\rho\sigma}\,n_\sigma] = 0\ . \tag{3.205}$$

Now $\hat{G}^{\mu\nu\rho\sigma}$ is defined on $\mathcal{N}$ and so (3.205) only makes sense when it involves derivatives of $\hat{G}^{\mu\nu\rho\sigma}$ tangential to $\mathcal{N}$. This depends on the type of geometry induced on $\mathcal{N}$ by its embedding in $\mathcal{M}^+$ and in $\mathcal{M}^-$. If the geometry is Type I then we conclude from (3.205):

$$-8\pi\,\eta^{-1}\,[T_{\mu\nu}\,n^\mu\,n^\nu] = \rho\,\gamma^\dagger\ , \tag{3.206}$$

$$2\,\eta^{-1}\,[\Psi_0] = -\sigma\,\gamma^\dagger\ , \tag{3.207}$$

$$2\,\eta^{-1}\,[\Psi_1] - 8\pi\,\eta^{-1}\,[T_{\mu\nu}m^\mu\,n^\nu] = \rho\,m^\mu\gamma_\mu - \sigma\,\bar{m}^\mu\gamma_\mu\ , \tag{3.208}$$

Here (3.206) coincides with the equation (3.196) obtained from the twice–contracted Bianchi identities in the Type I case. $[\Psi_0]$ and $[\Psi_1]$ are the jumps in the Newman–Penrose components of the Weyl tensors of $\mathcal{M}^+$ and $\mathcal{M}^-$ across $\mathcal{N}$ (we use the components of the Weyl tensor, calculated on either side of $\mathcal{N}$, on the null tetrad $\{n^\mu\,l^\mu\,m^\mu\,\bar{m}^\mu\}$). If the geometry induced on $\mathcal{N}$ is Type II then (3.206-3.208) hold, with zeros on the right hand sides of (3.206) and (3.207) , and we have from (3.205)

$$2\,\eta^{-1}\,[\Psi_1]+8\pi\,\eta^{-1}\,[T_{\mu\nu}\,m^\mu\,n^\nu] = -2\,\sigma\,\bar{m}^\mu\,\gamma_\mu - 2\,\rho\,m^\mu\,\gamma_\mu - m^\mu\,\gamma'_\mu\ . \tag{3.209}$$

We note that the difference of equations (3.209) and (3.208) yields the equation (3.197) obtained already from the twice–contracted Bianchi identities in the Type II case. Finally if the geometry of $\mathcal{N}$ is Type III then (3.206-3.209) hold with the right hand sides all vanishing and in addition we have

$$\eta^{-1}\,[\Psi_2] - \frac{2\pi}{3}\,\eta^{-1}\,[T] = -\frac{1}{2}\,\sigma\,\gamma_{\mu\nu}\,\bar{m}^\mu\,\bar{m}^\nu + 4\pi\,\{\mu' + (\rho+\kappa)\,\mu\}\ , \tag{3.210}$$

$$-4\pi\,\eta^{-1}\,[T_{\mu\nu}\,\bar{m}^\mu\,\bar{m}^\nu] = \frac{1}{2}\,\gamma'_{\mu\nu}\bar{m}^\mu\,\bar{m}^\nu + \frac{1}{2}\,(\rho+\kappa)\,\gamma_{\mu\nu}\,\bar{m}^\mu\,\bar{m}^\nu - 4\pi\,\mu\,\bar{\sigma}\ , \tag{3.211}$$

where $\gamma'_{\mu\nu} = n^\sigma\,\nabla_\sigma\gamma_{\mu\nu}$.

In the general case of a Type I induced geometry on $\mathcal{N}$ we notice that the equations following from the Bianchi identities (3.206)– (3.208) are all algebraic relations between some components of $\gamma_{\mu\nu}$ and some of the jumps in the energy–momentum–stress tensors and the Weyl tensors of $\mathcal{M}^+$ and $\mathcal{M}^-$ across $\mathcal{N}$. The further consequences of the Bianchi identities when the induced geometry is Type II or III (equations (3.197), (3.198), (3.209)–(3.211)) can all be viewed as propagation equations for components of $\gamma_{\mu\nu}$ along the generators of $\mathcal{N}$ (derivatives along the generators being indicated by a prime). This is consistent because the Type I geometry by itself excludes the possibility of a unique parameter being assigned to the geodesic generators of $\mathcal{N}$ on both the plus and minus sides and hence unique propagation equations along these generators of quantities defined on $\mathcal{N}$ cannot exist.

We emphasise the algebraic nature of the Bianchi identities in the case of a Type I geometry by stating the following:
Lemma 2:
If the geometry induced on $\mathcal{N}$ is Type I then

(a) If $\rho \neq 0$ and/or $\sigma \neq 0$, $\hat{\Psi}_2$ satisfies

$$\rho\,\hat{\Psi}_2 = \frac{4\pi}{3}\,[T_{\mu\nu}\,n^\mu\,n^\nu]\ , \tag{3.212}$$

$$\sigma\,\hat{\Psi}_2 = \frac{1}{3}\,[\Psi_0]\ ; \tag{3.213}$$

(b) If $\rho^2 \neq |\sigma|^2 \neq 0$, $\hat{\Psi}_3$ is given by

$$[\Psi_1] - 4\,\pi\,[T_{\mu\nu}m^\mu\,n^\nu] = -\hat{\Psi}_3^*\,\rho + \hat{\Psi}_3\,\sigma\ , \tag{3.214}$$

and its complex conjugate (here $\hat{\Psi}_3^*$ is the complex conjugate of $\hat{\Psi}_3$).

We note again that ρ, σ are intrinsic to $\mathcal{N}$ ($[\rho] = 0 = [\sigma]$) for a Type I geometry. For the cosmological example given above the expansion ρ and shear σ of the generators of $\mathcal{N}$ are given by

$$\rho = \frac{1}{r}\,\left(\frac{3\beta+2}{3\beta-2}\right) \quad \text{and} \quad \sigma = \frac{1}{\sqrt{2}\,r}\ , \tag{3.215}$$

while $[T_{\mu\nu}\,n^\mu\,n^\mu] = s^2\,[\mu_0 + p]$, with s defined after (3.171), and since on $\mathcal{N}$ the continuous 4–velocity u^μ is orthogonal to the complex null vector m^μ tangent to $\mathcal{N}$, $[T_{\mu\nu}\,m^\mu\,n^\nu] = 0$ and one can readily verify that the algebraic equations in Lemma 2 are satisfied.

The richest induced geometry is of course Type III and in this case we can, with additional assumptions, deduce from the Bianchi identities some interesting conclusions which we summarise in the following:

Lemma 3:
If the geometry on $\mathcal{N}$ is Type III and if $\mathcal{M}^\pm$ are vacuum space–times then $[\Psi_0] = [\Psi_1] = 0$,

$$[\Psi_2] = \sigma\,\hat{\Psi}_4 - 4\,\pi\,\eta\,\rho\,\mu\ , \tag{3.216}$$

and thus if $[\Psi_2] = 0$ and $\hat{\Psi}_4 \neq 0$ then

(1) $\sigma = 0$ *and* $\rho \neq 0 \quad \Rightarrow \quad \mu = 0$,

(2) $\sigma = 0$ *and* $\rho = 0 \quad \Rightarrow \quad \mu \neq 0$ *is possible* ,

(3) $\sigma \neq 0 \quad \Rightarrow \quad \rho \neq 0$ *and* $\mu \neq 0$,

where the surface stress–energy tensor of the light–like shell now has the form $S_{\alpha\beta} = \mu\, n_\alpha\, n_\beta$.

We first note that part (1) of Lemma 3 explains the 'miracle' whereby the Penrose 'spherical' impulsive wave in §1.2 propagating through flat space–time *automatically* satisfies the vacuum field equations. The history $\mathcal{N}$ of the signal in this case is a future null–cone which is a shear–free ($\sigma = 0$) expanding ($\rho \neq 0$) null hypersurface. The induced geometry is Type III and thus by Lemma 3(1) the surface stress/energy tensor $S_{\alpha\beta}$ must vanish ($f = 0$). An example of part (1) of Lemma 3 in which $\mathcal{M}^\pm$ are not flat is provided by taking $\mathcal{M}^\pm$ to be two Petrov Type III Robinson–Trautman [Robinson and Trautman (1962)] vacuum space–times with line–elements of the form

$$ds^2_\pm = -2r^2_\pm p^{-2}_\pm d\zeta_\pm d\bar{\zeta}_\pm + 2du\, dr_\pm + K_\pm\, du^2 \ , \tag{3.217}$$

with $p_\pm = p_\pm(\zeta_\pm, \bar{\zeta}_\pm)$ and

$$K_\pm = \Delta_\pm \log p_\pm \quad , \quad \Delta_\pm K_\pm = 0 \ , \tag{3.218}$$

where $\Delta_\pm = 2p^2_\pm \partial^2/\partial\zeta_\pm\partial\bar{\zeta}_\pm$. These two space–times are joined together on the shear–free, expanding null hypersurface $\mathcal{N}$ with equation $u = 0$, with the matching conditions

$$\zeta_+ = h(\zeta_-) \quad \text{and} \quad r_+ = F(\zeta_-, \bar{\zeta}_-)\, r_- \ , \tag{3.219}$$

where h is an analytic function of ζ_- and $F(\zeta_-, \bar{\zeta}_-) = p_+/(|h'|p_-)$. In coordinates labelled $x^\mu_- = (\zeta_-, \bar{\zeta}_-, r_-, u)$ we find that $\gamma_{\mu\nu} = 0$ except for γ_{11} and $\gamma_{22} = \bar{\gamma}_{11}$ with

$$\gamma_{11} = -2r\frac{F'}{F}\frac{\partial}{\partial\zeta}\log\left(F'\, p^2_-\right) \ , \tag{3.220}$$

where $F' = \partial F/\partial\zeta_-$. Thus the induced geometry is Type III, there is no surface stress/energy tensor on $\mathcal{N}$ and, since $\hat{\Psi}_4 = \frac{1}{2}\gamma_{11}\, r^{-2}_- p^2_- \neq 0$, $\mathcal{N}$ is the history of an impulsive gravitational wave.

Part (2) of Lemma 3 shows that if $\mathcal{N}$ is a null hyper*plane* (with generators having vanishing shear and expansion) and if the matching of $\mathcal{M}^+$ and $\mathcal{M}^-$ on $\mathcal{N}$ is such that the induced geometry is Type III then $\mathcal{N}$ can be the history of a plane impulsive gravitational wave and/or a plane light–like shell of matter. For example take $\mathcal{M}^+$ to be a pp–wave space–time with line–element

$$ds_+^2 = dx_+^2 + dy_+^2 + 2du\, dv_+ + H(x_+, y_+, u)\, du^2 \ , \tag{3.221}$$

with $H_{x_+x_+} + H_{y_+y_+} = 0$ (subscripts here denoting partial derivatives). Take $\mathcal{M}^-$ to be flat space– time with line–element

$$ds_-^2 = dx^2 + dy^2 + 2du\, dv \ . \tag{3.222}$$

Now match $\mathcal{M}^+$ to $\mathcal{M}^-$ on the null hyperplane $\mathcal{N}$ $(u = 0)$ with (cf. (2.57)

$$x_+ = x \ , \qquad y_+ = y \ , \qquad v_+ = v + h(x, y) \ . \tag{3.223}$$

Using the theory of chapter 2 with $x_-^\mu = (\xi^a, u)$ and $\xi^a = (x, y, v)$ we find that $\gamma_{\mu 4} = 0$ and otherwise $\gamma_{ab} = -h_{,ab}$. Thus with $n_-^\mu = \delta_3^\mu$ we have $\gamma_\mu = \gamma_{\mu 3} = 0$ and so the geometry induced on $\mathcal{N}$ is Type III. We also find that

$$8\pi\, \mu = -\frac{1}{2}\gamma = h_{xx} + h_{yy} \ , \tag{3.224}$$

and

$$\hat{\Psi}_4 = -\frac{1}{2}\,(h_{xx} - h_{yy}) + i\, h_{xy} \ . \tag{3.225}$$

This shows explicitly that a light–like shell and a plane impulsive wave can co–exist, each with history $\mathcal{N}$.

A simple example of part (3) of Lemma 3 is a cylindrical fronted light–like signal with history $\mathcal{N}$ in flat space–time. Thus $\mathcal{M}^\pm$ have line–elements

$$ds_\pm^2 = (u + v_\pm)^2 d\phi_\pm^2 + dz_\pm^2 + 2du\, dv_\pm \ . \tag{3.226}$$

Now $\mathcal{N}$ $(u = 0)$ is a null hypersurface generated by shearing null geodesics $(\sigma \neq 0)$. We match the induced metrics on $\mathcal{N}$ with

$$\phi_+ = q(\phi) \ , \qquad z_+ = z \ , \qquad v_+ = v/q' \ , \tag{3.227}$$

with $q' = dq/d\phi$. In coordinates $x_-^\mu = (\phi, z, v, u)$ we find that $\gamma_{\mu\nu} = 0$ except for

$$\gamma_{11} = 2v \left\{ \frac{q''}{q'} - \frac{3}{2}\left(\frac{q''}{q'}\right)^2 + q'^2 - 1 \right\} \ . \tag{3.228}$$

Thus with $n_-^\mu = \delta_3^\mu$ we see that $\gamma_\mu = 0$ and the induced geometry on $\mathcal{N}$ is Type III. The shear σ and expansion ρ of the null geodesic generators of $\mathcal{N}$ satisfy

$$\rho = \sigma = \frac{1}{2v} \ , \tag{3.229}$$

while

$$4\pi\,\mu = \hat{\Psi}_4 = -\frac{1}{4v^2}\,\gamma_{11} \ . \tag{3.230}$$

Thus in general $\mu \neq 0$ and a shell and impulsive wave co–exist. We see that no signal exists with history $\mathcal{N}$ if and only if $\gamma_{11} = 0$. It is interesting to note that if no signal exists on $\mathcal{N}$ and the isometric transformations preserving this state form a *group* then they are given by (3.227) with $q(\phi) = \phi + c$, and $c =$ constant. There also exist other disconnected isometric transformations of $\mathcal{N}$, of the form (3.227), when no signal exists on $\mathcal{N}$, but these transformations do not form a group.

A corresponding Lemma to Lemma 3 which has applications to light–like signals propagating through a cosmic fluid is:

Lemma 4:
If the geometry on $\mathcal{N}$ is Type III and if $\mathcal{M}^\pm$ are perfect fluid space–times with u continuous across $\mathcal{N}$ then $[\Psi_0] = [\Psi_1] = [\mu_0 + p] = 0$ and

$$[\Psi_2] - \frac{4\pi}{3}\,[\mu_0] = \sigma\,\hat{\Psi}_4 - 4\,\pi\,\eta\,\rho\,\mu \ , \tag{3.231}$$

and thus if $[\Psi_2] = \frac{4\pi}{3}\,[\mu_0]$ and $\hat{\Psi}_4 \neq 0$ then the deductions are the same as (1) –(3) of Lemma 3.

The special case of the de Sitter universe is obtained by putting $-8\pi\,p = 8\pi\,\mu_0 = \Lambda$, where Λ is the cosmological constant. We will confine our observations here on Lemma 4 to the de Sitter case.

To illustrate (4.31) of Lemma 4 we let $\mathcal{M}^\pm$ both be de Sitter universes (with different cosmological constants $\Lambda_\pm$) having line–elements

$$ds_\pm^2 = \frac{2v_\pm^2\,d\zeta_\pm d\bar{\zeta}_\pm + 2du_\pm dv_\pm}{\left(1 + \frac{1}{6}\Lambda_\pm u_\pm v_\pm\right)^2} \ . \tag{3.232}$$

Here $\mathcal{N}$ $(u_+ = u_- = 0)$ is a future null–cone generated by expanding $(\rho \neq 0)$

shear–free ($\sigma = 0$) null geodesics. We match $\mathcal{M}^{\pm}$ on $\mathcal{N}$ with (cf. (1.30))

$$\zeta_+ = h(\zeta_-) \ , \qquad v_+ = \frac{v_-}{|h'|} \ , \tag{3.233}$$

where h is an analytic function of ζ_- and $h' = dh/d\zeta_-$. Now the induced geometry on $\mathcal{N}$ is Type III. In general $\hat{\Psi}_4 = -\chi_0/2v \neq 0$ with

$$\chi_0 = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 \ , \tag{3.234}$$

and

$$8\pi\,\rho\,\mu = \frac{1}{3}\,[\Lambda] \ . \tag{3.235}$$

This is the form taken by (4.31) for this example since now ${}^{\pm}\Psi_A = 0$ for $A = 0,1,2,3,4$, $8\pi\,[\mu_0] = [\Lambda]$, $\eta = +1$ and $\sigma = 0$. Thus if $[\Lambda] = 0$ then since $\rho = 1/v \neq 0$ we must have $\mu = 0$ and so $\mathcal{N}$ is the history of an impulsive gravitational wave [Hogan (1992)].

Finally as an illustration of conclusion (2) of Lemma 4 we consider $\mathcal{M}^+$ to be a Schwarzschild space– time with line–element in (slightly modified) Kruskal form

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\,d\phi^2) - \frac{64m^3}{r}\,\mathrm{e}^{(1-r/2m)}\,dU\,dV \ , \tag{3.236}$$

with $r = r\,(UV)$ given by

$$\left(\frac{r}{2m} - 1\right)\,\mathrm{e}^{\left(\frac{r}{2m}-1\right)} = 2U\,V \ , \tag{3.237}$$

and we take $\mathcal{M}^-$ to be de Sitter space–time (with $\Lambda > 0$) with line–element

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\,d\phi^2) - 2\frac{(1+\lambda\,r)^2}{\lambda^2}\,dU\,dV \ , \tag{3.238}$$

where $\lambda^2 = \Lambda/3$ and $r = r\,(UV)$ is given by

$$\frac{1-\lambda\,r}{1+\lambda\,r} = 2U\,V \ . \tag{3.239}$$

These match (cf. [Barrabès and Israel (1991)]) on the horizon $\mathcal{N}$ ($U = 0$) if $2m\,\lambda = 1$. We have rescaled one of the null coordinates in (3.236) and (3.237) to make the metric tensors given via the line–elements (3.236) and (3.238) continuous across $\mathcal{N}$. The horizon $U = 0$ is a null hyperplane generated by shear–free ($\sigma = 0$), expansion–free ($\rho = 0$) null geodesics. In the continuous coordinates (U, V, θ, ϕ) above we find using (2.10) that $\gamma_{\mu\nu} = 0$ except $\gamma_{22} = \gamma_{11}\,\sin^2\theta = -3V\,\sin^2\theta/4m^2$ and $f = 3V/32\pi m^2 \neq 0$.

If the situation above is reversed and $\mathcal{M}^+$ is de Sitter space–time (with $\Lambda > 0$) and $\mathcal{M}^-$ is Schwarzschild space–time then $\gamma_{\mu\nu} = 0$ except for $\gamma_{22} = \gamma_{11}\sin^2\theta = 3V\sin^2\theta/4m^2$ and $f = -3V/32\pi m^2$. In either case the induced geometry is type III. The equation $[\Psi_2] = \frac{4\pi}{3}[\mu_0]$ becomes $\frac{1}{4m^2} = \lambda^2$. There is no gravitational wave present ($\hat{\Psi}_4 = 0$) and $\mathcal{N}$ is the history of a light–like shell.

3.6 Theory of the Detection of Impulsive Light–Like Signals

The direct detection of gravitational waves will be realised in the near future [Will (1999), (2001)]. The detectors which are presently under construction are designed to observe signals in the frequency range from 10 Hz to 10 kHz. The main source of such signals is considered to be an inspiralling binary neutron star system. However a large variety of other sources of gravitational waves exists [Barish and Weiss (1999)] and among them are cataclysmic events such as supernovae, which are expected to produce bursts of gravitational radiation and of null matter. To detect such impulsive light–like signals we need to know their effect on the relative motion of neighbouring test particles. We consider here a congruence of time–like geodesics in a vacuum space–time crossing a singular null hypersurface and we focus attention on the relative motion of the test particles as their world lines cross the null hypersurface [Barrabès and Hogan (2001)].

The current design of gravity wave detectors is aimed at observing gravitational waves that establish oscillatory motion in the test particles (such as the waves from an inspiralling neutron star system). The light–like signals we are studying displace the test particles but do not establish oscillatory motion. A modification of the detectors would be necessary to observe these signals.

The behavior of geodesics in space–times of impulsive gravitational waves has been studied in the cited references to Balasin, Steinbauer and Kunzinger. Our study here differs from these because we include a light–like shell in the signal and we use a local coordinate system in which the metric tensor is continuous across the history of the signal in space–time whereas these authors make use of a distributional metric.

The principles upon which the detection of a light–like signal is carried out are based on the interaction of neighbouring test particles with the signal described by the geodesic deviation equation. In the space–time $\mathcal{M}^- \cup \mathcal{M}^+$ (which we shall consider to be a vacuum space–time except possibly on the null hypersurface $\mathcal{N}$) we consider a one–parameter family of integral curves of a vector field with components T^μ forming a 2–space M_2. We take T^μ to be a unit time–like vector field, so that

$$g_{\mu\nu}\,T^\mu\,T^\nu = -1\ , \tag{3.240}$$

and we take the integral curves of T^μ to be time–like geodesics with arc length as parameter along them. Thus

$$\dot{T}^\mu \equiv T^\mu{}_{|\nu}\,T^\nu = 0\ , \tag{3.241}$$

with the stroke denoting covariant differentiation with respect to the Levi–Civita connection associated with the metric tensor $g_{\mu\nu}$. The dot will denote covariant differentiation in the direction of T^μ of any tensor defined along the integral curves of T^μ. Let X^μ be an orthogonal connecting vector joining neighbouring integral curves of T^μ and tangent to M_2. Thus $g_{\mu\nu}\,T^\mu\,X^\nu = 0$ and

$$\dot{X}^\mu = T^\mu{}_{|\nu}\,X^\nu\ . \tag{3.242}$$

It follows that X^μ satisfies the geodesic deviation equation [Synge (1966)]

$$\ddot{X}^\mu = -R^\mu{}_{\lambda\sigma\rho}\,T^\lambda\,X^\sigma\,T^\rho\ , \tag{3.243}$$

where $R^\mu{}_{\lambda\sigma\rho}$ are the components of the Riemann tensor of the space–time $\mathcal{M}^-\cup\mathcal{M}^+$. For consistency with the assumed jumps (2.10) of the partial derivatives of the metric tensor components across $\mathcal{N}$, which as we know lead to a possible Dirac delta function in the Riemann tensor of $\mathcal{M}^-\cup\mathcal{M}^+$ (which is singular on $\mathcal{N}$) we find that we can assume that the partial derivatives of T^μ and X^μ jump across $\mathcal{N}$ and that these jumps have the forms

$$[T^\mu{}_{,\lambda}] = \eta\,P^\mu\,n_\lambda\ , \qquad [X^\mu{}_{,\lambda}] = \eta\,W^\mu\,n_\lambda\ , \tag{3.244}$$

for some vectors P^μ , W^μ defined on $\mathcal{N}$ (but not necessarily tangential to $\mathcal{N}$). Let $\{E_a\}$ be three vector fields defined along the time–like geodesics tangent to T^μ by parallel transporting $\{e_a\}$ along these geodesics. Thus

$$\dot{E}^\mu_a = 0\ , \tag{3.245}$$

and on $\mathcal{N}$ we take $E_a = e_a$. The jump in the partial derivatives of E^μ_a must take the form

$$\left[E^\mu_{a,\lambda}\right] = \eta\,F^\mu_a\,n_\lambda\ . \tag{3.246}$$

for some F^μ_a defined on $\mathcal{N}$.

The behavior of the orthogonal connecting vector X^μ as it crosses $\mathcal{N}(u=0)$ from the past $(u<0)$ to the future $(u>0)$ is important. Let $X^\mu_{(0)}$ denote X^μ evaluated on $\mathcal{N}$ (i.e. where the time–like geodesic, along which X^μ moves, intersects $\mathcal{N}$). Let $T^\mu_{(0)}$ denote T^μ evaluated on $\mathcal{N}$. It is convenient to write

$$X^\mu_{(0)} = X_{(0)}\, T^\mu_{(0)} + X^a_{(0)}\, e^\mu_a\ , \tag{3.247}$$

for some functions $X_{(0)}\ , X^a_{(0)}$ evaluated on $\mathcal{N}$. We obtain the following information on the vectors $P^\mu\ , W^\mu\ , F^\mu_a$ appearing in (3.244) and (3.247): Using (3.240) we find that

$$\gamma_{\mu\nu}\, T^\mu_{(0)}\, T^\nu_{(0)} + 2\, P_\mu\, T^\mu_{(0)} = 0\ , \tag{3.248}$$

and from (3.241) we derive

$$\gamma_{\alpha\beta}\, T^\alpha_{(0)}\, T^\beta_{(0)}\, n^\mu = 2\, n_\alpha\, T^\alpha_{(0)}\, \left\{P^\mu + \gamma^\mu_\beta\, T^\beta_{(0)}\right\}\ , \tag{3.249}$$

and this includes (3.248) as a special case. The orthogonality of $X^\mu\ , T^\mu$ leads to

$$\gamma_{\alpha\beta}\, X^\alpha_{(0)}\, T^\beta_{(0)} + T^\alpha_{(0)}\, W_\alpha + X^\alpha_{(0)}\, P_\alpha = 0\ . \tag{3.250}$$

The propagation equation (3.242) gives

$$W^\mu = X_{(0)}\, P^\mu\ . \tag{3.251}$$

If (3.251) is substituted in (3.250) then the resulting equation can be obtained from (3.249). The expression for W^μ obtained using (3.249) and (3.251) can be derived directly from the geodesic deviation equation (3.243). Finally (3.245) yields

$$(n_\alpha\, T^\alpha_{(0)})\, F^\mu_a = -\frac{1}{2}\, \gamma^\mu_\alpha\, e^\alpha_a\, (n_\beta\, T^\beta_{(0)}) + \frac{1}{2}\, n^\mu\, \left(\gamma_{\alpha\beta}\, e^\alpha_a\, T^\beta_{(0)}\right)\ . \tag{3.252}$$

As a consequence of (3.244) we deduce, for small u,

$$X^\mu = {}^-X^\mu + \eta\, \chi^{-1} u\, \vartheta(u)\, W^\mu\ , \tag{3.253}$$

with ${}^-X^\mu$ in general dependent on u and such that when $u=0$, ${}^-X^\mu = X^\mu_{(0)}$. Here as always $\vartheta(u)$ is the Heaviside step function which is equal to unity if $u>0$ and equal to zero if $u<0$. Similar equations to (3.253) can be obtained for $g_{\mu\nu}\ , T^\mu\ , E^\mu_a$ using (2.9), (3.244) and (3.247). It is convenient to calculate X^μ on the basis $\{E^\mu_a\ , T^\mu\}$. Its component in the direction of T^μ is, of course, zero and its components in the directions $\{E^\mu_a\}$ are

$$X_a = g_{\mu\nu}\, X^\mu\, E^\nu_a\ . \tag{3.254}$$

Using (3.247)–(3.253) we calculate (3.254) to read, for small $u > 0$,

$$X_a = \left(p_{ab} + \frac{1}{2}\,\eta\,\chi^{-1} u\,\gamma_{ab}\right) X^b_{(0)} + u\,{}^-V_{(0)a}\ , \qquad (3.255)$$

with γ_{ab} given by (2.32), ${}^-V_{(0)a} = d^-X_a/du$ evaluated at $u = 0$ and p_{ab} given by

$$p_{ab} = g_{ab} + \left(T_{(0)\mu}\, e^\mu_a\right)\,\left(T_{(0)\nu}\, e^\nu_b\right)\ , \qquad (3.256)$$

with g_{ab} found in (2.25). Although g_{ab} is degenerate we note that p_{ab} is non–degenerate. The final term in (3.255) is present due to the relative motion of the test particles before encountering the signal. It would represent the relative displacement for small $u > 0$ if no signal were present.

In parallel with (3.247) we can write

$$X^\mu = X\,T^\mu + X^a\,E^\mu_a\ . \qquad (3.257)$$

Since X^μ ,T^μ are orthogonal we have

$$X = X^a\,E^\mu_a\,T_\mu\ . \qquad (3.258)$$

Substituting (3.257) with (3.258) into (3.254) we obtain

$$X_a = P_{ab}\,X^b\ , \qquad (3.259)$$

with

$$P_{ab} = g_{\mu\nu}\,E^\mu_a\,E^\nu_b + E^\mu_a\,T_\mu\,E^\nu_b\,T_\nu\ . \qquad (3.260)$$

However on account of the parallel transport (3.241) and (3.245) of T^μ , E^μ_a we have $P_{ab} = p_{ab}$ and so we can write

$$X_a = p_{ab}\,X^a\ , \qquad (3.261)$$

and this equation can be inverted.

To see the separate effects of the wave and shell parts of the light–like signal on the relative position of the test particles we use in (3.255) the decomposition of γ_{ab} given by $\gamma_{ab} = \hat{\gamma}_{ab} + \bar{\gamma}_{ab}$ with $\hat{\gamma}_{ab}$ defined in (2.47). It is convenient to specialise the triad $\{e_a\}$ by choosing $e^\mu_1 = n^\mu$ and e_A, with $A = 2, 3$, orthogonal to $T^\mu_{(0)}$. It then follows from (2.27) that $n^a = \delta^a_1$ and that $g_{ab} = 0$ except for g_{AB} with $A, B = 2, 3$. We can specialise N^μ by taking $N^\mu = T^\mu_{(0)}$, $N \cdot e_{(A)} = 0$ and $N \cdot n = \eta^{-1}$. Hence $N \cdot N = -1$, $N_A = 0$ and $N_1 = \eta^{-1}$ respectively. It thus follows from (2.35) and (2.36) that $g^{ab}_* = 0$ except for $g^{11}_* = \eta^2$ and $g^{AB}_* = g^{AB}$. We shall take the (2,3)–2-surface in $\mathcal{N}$ to be the signal front and we shall assume that before the signal arrives the test particles are in this (2,3)–2-surface. The metric of this space–like 2-surface has components g_{AB} and since this is a Riemannian 2-surface we can choose coordinates $\{x^A\}$ such that $g_{AB} = p^{-2}\delta_{AB}$ with $p = p(x^A)$. Hence in these coordinates $\tilde{p}_{ab} = \mathrm{diag}(\eta^{-2}, p^{-2}, p^{-2})$.

Since $\hat{\gamma}_{ab}$ satisfies $\hat{\gamma}_{ab}\, n^b = 0 = g_*^{ab}\, \hat{\gamma}_{ab}$ we have now in our special frame $\hat{\gamma}_{a1} = 0$ for $a = 1, 2, 3$ and

$$(\hat{\gamma}_{AB}) = \begin{pmatrix} \hat{\gamma}_{22} & \hat{\gamma}_{23} \\ \hat{\gamma}_{23} & -\hat{\gamma}_{22} \end{pmatrix}$$

In this special frame S^{ab} is given by

$$16\,\pi\, S^{11} = -p^2\,(\gamma_{22} + \gamma_{33})\ , \tag{3.262}$$

$$16\,\pi\, S^{1B} = p^2\,\gamma_{1B}\ , \tag{3.263}$$

$$16\,\pi\, S^{AB} = -p^2\,\gamma_{11}\,\delta_{AB}\ . \tag{3.264}$$

Using (3.261) we can now write (3.255) as follows:

$$\eta^{-2}\, X^1 = X_1 = \eta^{-2}\, X^1_{(0)} + \frac{u}{2}\,\eta\,\chi^{-1}\,\gamma_{1b}\, X^b_{(0)} + u^-\, V_{(0)1}\ , \tag{3.265}$$

$$p^{-2}\, X^A = X_A = p^{-2}\, X^A_{(0)} + \frac{u}{2}\,\eta\,\chi^{-1}\,\gamma_{Ab}\, X^b_{(0)} + u^-\, V_{(0)A}\ . \tag{3.266}$$

We note that if there is a wave component to the signal it does not contribute to X^1. If initially in physical space the test particles are moving in the (2,3)–2-surface (the signal front) then $X^1_{(0)} = 0 = {}^-V_{(0)1}$. In this case if $S^{B1} \neq 0$ then $X^1 \neq 0$. Hence we can say that *if there is surface energy current* (cf. (2.42)) *in the light–like shell then test particles initially moving in the signal front are displaced out of this 2-surface after encountering the signal.*

Suppose that there is no surface energy current $\left(S^{1B} = 0\right)$ in the light–like shell and the test particles are initially at rest. Now $X^1_{(0)} = 0 = {}^-V_{(0)b}$ implies $X^1 = 0$ and we can rewrite (3.266) using (3.262) and the decomposition of γ_{ab} in the form

$$X^A = \left(1 - 8\pi\, u\,\eta\,\chi^{-1}\, S^{11}\right)\left(\delta_{AB} + \frac{u}{2}\,\eta\,\chi^{-1}\, p^2\hat{\gamma}_{AB}\right) X^B_{(0)}\ , \tag{3.267}$$

for small u. The factor $\delta_{AB} + \frac{u}{2}\,\eta\,\chi^{-1}\,p^2\hat{\gamma}_{AB}$, with $\hat{\gamma}_{AB}$ given in matrix form above, describes the usual distortion effect of the wave part of the signal on the test particles in the signal front (see [Schutz (1972)] and the explicit example given below) while the presence of the light–like shell has a focussing effect which leads to an overall diminution factor $1 - 8\pi\, u\,\eta\,\chi^{-1}\, S^{11} < 1$. This latter inequality arises as follows: Since we have chosen u to increase as one crosses $\mathcal{N}(u=0)$ from $\mathcal{M}^-(u<0)$ to $\mathcal{M}^+(u>0)$, we must have $\chi^{-1}\eta > 0$ on $\mathcal{N}$. The energy density μ, given in (2.42), is positive and so in our special frame we obtain $S^{11} = \mu > 0$. Thus we can conclude that *for a light–like shell with no surface energy current accompanying a gravitational wave the effect of the signal on test particles at rest in the signal front is to displace them relative to each other with the usual distortion due to the gravitational wave diminished by the presence of the light– like shell.*

A simple explicit example of (3.267) is provided by the plane fronted light–like signal described in §2.4 above specialised by taking F in (2.70) and (2.71) to be given by

$$F(x,y,v) = v - \frac{a}{2}\,(x^2+y^2) + \frac{b}{2}\,(x^2-y^2) + c\,x\,y\ , \tag{3.268}$$

with a, b, c constants and $a > 0$. This is a homogeneous signal with vanishing stress and energy density

$$\mu = \frac{a}{4\,\pi}\ , \tag{3.269}$$

in the light–like shell and with constant amplitude of each component of the gravitational wave since the entries in

$$(\hat{\gamma}_{AB}) = \begin{pmatrix} 2\,b & 2\,c \\ 2\,c & -2\,b \end{pmatrix}\ ,$$

are constants. Assuming zero relative velocity for the test particles before the signal arrives, (3.267) gives

$$X^2 = (1-2\,a\,u)\,(1+b\,u)\,\left(X^2_{(0)} + c\,u\,X^3_{(0)}\right)\ , \tag{3.270}$$

$$X^3 = (1-2\,a\,u)\,(1-b\,u)\,\left(c\,u\,X^2_{(0)} + X^3_{(0)}\right)\ . \tag{3.271}$$

Hence particles at rest on the circle $\left(X^2_{(0)}\right)^2 + \left(X^3_{(0)}\right)^2 =$ constant, before encountering the light–like signal, undergo a small displacement after encountering the signal which is composed of: (1) a rotation through the small angle $c\,u$ and (2) a deformation into an ellipse with semi axes of lengths $(1-2\,a\,u)\,(1+b\,u)$ and $(1-2\,a\,u)\,(1-b\,u)$. If the light–like shell is not part of the signal (i.e. if $a=0$) then the semi axes of the ellipse are *not* shortened by the factor $(1-2\,a\,u)$.

Chapter 4

Light–Like Boosts of Gravitating Bodies

In §3.3 a sudden reduction to zero in the 3–velocity of a moving spherical body was shown to produce a recoil effect which takes the form of an outgoing spherical impulsive light–like shell. A more extreme form of this is achieved by boosting a compact gravitating body to the speed of light. In this limit the gravitational field of the body resembles the gravitational field of a plane fronted impulsive gravitational wave. In our approach to the behaviour of a gravitational field under light–like boost we place the emphasis on boosting the space–time curvature with the metric playing a secondary role, because the Riemann curvature tensor represents unambiguously the gravitational field. In so doing we are strongly influenced by the classical works of Felix Pirani on the physical interpretation of the Riemann tensor [Pirani (1956), (1957)].

4.1 Introduction: Light-Like Boost of the Coulomb Field

It was pointed out many years ago by Bergmann [Bergmann (1947)] that to a fast moving observer travelling rectilinearly, the Coulomb field of a point charge e, with a time–like geodesic world line in Minkowskian space–time, resembles the electromagnetic field of a plane electromagnetic wave with a sharply peaked profile, the closer the speed v of the observer relative to the charge approaches the speed of light. In fact in the limit $v \rightarrow 1$ (we are using units in which the speed of light $c = 1$ throughout) the field of the charge seen by the observer is that of a plane fronted impulsive electromagnetic wave [Aichelburg and Embacher (1987)], [Aichelburg and Balasin (2000)]. The particle origin of this wave is reflected in the fact that the wave front contains a singular point. In other words the history of the wave front in Minkowskian space–time is a null hyperplane generated by parallel null geodesics and on one of these null geodesics the field amplitude is singular. To see this explicitly we begin with the line–element of Minkowskian space–time in rectangular Cartesian coordinates and time $\{\bar{x}, \bar{y}, \bar{z}, \bar{t}\}$,

$$ds^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 - d\bar{t}^2 \ . \tag{4.1}$$

For clarity in presentation we start with barred coordinates and then transform below to unbarred coordinates. Let the $\bar{t}$–axis ($\bar{x} = \bar{y} = \bar{z} = 0$) be the time–like geodesic world–line of a charge e. The Coulomb potential of this charge at $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ is given by the 1–form

$$A = -\frac{e}{\bar{r}}\, d\bar{t} \ , \qquad \bar{r} = (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{1/2} \ . \tag{4.2}$$

The corresponding electric field is given by the 2–form

$$F = dA = \frac{e}{\bar{r}^3}\ (\bar{x}\, d\bar{x} \wedge d\bar{t} + \bar{y}\, d\bar{y} \wedge d\bar{t} + \bar{z}\, d\bar{z} \wedge d\bar{t}) \ . \tag{4.3}$$

The field measured by an observer moving in the $-\bar{x}$ direction with speed v relative to the charge is obtained by making in (4.3) the Lorentz transformation

$$\bar{x} = \gamma\,(x - v\,t) \ , \qquad \bar{y} = y \ , \qquad \bar{z} = z \ , \qquad \bar{t} = \gamma\,(t - v\,x) \ , \tag{4.4}$$

with $\gamma = (1 - v^2)^{-1/2}$. Thus we can write

$$F = \frac{e\,\gamma^{-2}}{R^3}\ \{(x - v\,t)\, dx \wedge dt + (y\, dy + z\, dz) \wedge (dt - v\, dx)\} \ , \tag{4.5}$$

with

$$R = \left\{(x - v\,t)^2 + \gamma^{-2}(y^2 + z^2)\right\}^{1/2} \ . \tag{4.6}$$

We want the limit of (4.5) as $v \longrightarrow 1$. To this end we make the simple observation:

$$\frac{\gamma^{-2}}{R^3} = \frac{1}{y^2 + z^2}\,\frac{\partial}{\partial x}\left(\frac{x - v\,t}{R}\right) \ , \tag{4.7}$$

and so

$$\lim_{v \longrightarrow 1} \frac{\gamma^{-2}}{R^3} = (y^2 + z^2)^{-1}\,\frac{\partial}{\partial x}\left(\frac{x - t}{|x - t|}\right) \ . \tag{4.8}$$

Denoting as always by $\vartheta(u)$ the Heaviside step function we can write

$$\frac{x - t}{|x - t|} = 2\,\vartheta(x - t) - 1 \ , \tag{4.9}$$

and so (4.8) can be written

$$\lim_{v\longrightarrow 1}\frac{\gamma^{-2}}{R^3}=\frac{2\,\delta(x-t)}{y^2+z^2}\ . \tag{4.10}$$

Remembering that $(x-t)\,\delta(x-t)=0$ we see now from (4.5) that

$$\lim_{v\longrightarrow 1}F=\frac{2\,e\,\delta(x-t)}{y^2+z^2}\,(y\,dy+z\,dz)\wedge(dt-dx)=F_0\ (\text{say})\ . \tag{4.11}$$

Clearly we can write

$$F_0=dA_0\ ,\qquad \text{with}\qquad A_0=e\,\delta(x-t)\,\log(y^2+z^2)\,(dt-dx)\ , \tag{4.12}$$

and so F_0 is a solution of the source–free Maxwell equations which is singular on $x=t$ and also on the null geodesic generator of $x=t$ labelled by $y=z=0$. F_0 describes an impulsive electromagnetic wave (the 2–form F_0 is Type N in the Petrov classification with degenerate principal null direction given by the 1–form $dx-dt$) with profile $\delta(x-t)$.

Substituting the Lorentz transformation (4.4) into the potential 1–form (4.2) yields

$$A=-\frac{e}{R}\,(dt-v\,dx)\ , \tag{4.13}$$

with R given by (4.6). From this we arrive at

$$\lim_{v\longrightarrow 1}A=-e\,\frac{(dt-dx)}{|t-x|}\ , \tag{4.14}$$

which, of course, only makes sense if $x\neq t$. The limit (4.14), being a pure gauge term for $x>t$ and for $x<t$, is consistent with (4.11). To see what happens *on* $x=t$ in the limit $v\longrightarrow 1$ we first modify the Coulomb potential (4.2) by the addition of a gauge term. This gauge term is suggested by a clever analogous coordinate transformation [Aichelburg and Sexl (1971)] in the gravitational case. The gravitational case is discussed from our Riemann curvature centered point of view [Barrabès and Hogan (2001)] in §4.2 . The modified Coulomb potential 1–form is

$$A=-\frac{e}{\bar{r}}\,d\bar{t}-\frac{e\,d\bar{x}}{\sqrt{\bar{x}^2+1}}\ . \tag{4.15}$$

This potential 1–form obviously leads to the same electric field (4.3) as that derived from (4.7). The Lorentz transformation (4.4) applied to (4.15) yields

$$A = -\frac{e}{R}\,(dt - v\,dx) - \frac{e\,(dx - v\,dt)}{\{(x - v\,t)^2 + \gamma^{-2}\}^{1/2}} \ . \tag{4.16}$$

Now

$$dx - v\,dt = -dt + v\,dx + (1 - v)(dx + dt) \ , \tag{4.17}$$

and so for v near 1 we can write (4.16) as

$$A = -e\left[\frac{1}{R} - \frac{1}{\{(x - v\,t)^2 + \gamma^{-2}\}^{1/2}}\right](dt - v\,dx) + O\left((1 - v)\right) \ . \tag{4.18}$$

This can further be written

$$A = -e\,\frac{\partial}{\partial x}\left[\log\left(\frac{x - v\,t + R}{x - v\,t + \sqrt{(x - v\,t)^2 + \gamma^{-2}}}\right)\right](dt - v\,dx) + O\left((1 - v)\right) \ . \tag{4.19}$$

The logarithm term here appeared first in [Aichelburg and Sexl (1971)] and it is particularly useful when one observes that [Aichelburg and Sexl (1971)]

$$\lim_{v\longrightarrow 1}\log\left(\frac{x - v\,t + R}{x - v\,t + \sqrt{(x - v\,t)^2 + \gamma^{-2}}}\right) = (1 - \vartheta(x - t))\,\log(y^2 + z^2) \ . \tag{4.20}$$

Hence (4.19) gives

$$\begin{aligned}\lim_{v\longrightarrow 1} A &= e\,\frac{\partial}{\partial x}\left(\vartheta(x - t)\,\log(y^2 + z^2)\right)(dt - dx) \ ,\\ &= e\,\delta(x - t)\,\log(y^2 + z^2)\,(dt - dx) \ ,\end{aligned} \tag{4.21}$$

and so we have recovered A_0 in (4.12). It is interesting to note in this context that a comprehensive study of light–like boosts of electromagnetic multipole fields, although having little overlap in approach with our discussion in §4.3 below of gravitational multipole fields, has been carried out by Robinson and Rózga [Robinson and Rózga (1984)].

We regard the limit of the field (4.11) to be the important result from which the physical interpretation (that the boosted Coulomb field is the field of an impulsive electromagnetic wave in the limit $v \longrightarrow 1$) follows. The limit of the potential is in this sense of secondary importance. Thus *in the gravitational case we shall place an emphasis on boosting the space–time curvature while the metric will play a secondary role.*

4.2 Boosting an Isolated Gravitating Source

The first gravitational analogue of the boost discussed above is contained in an influential paper by Aichelburg and Sexl [Aichelburg and Sexl (1971)]. They have shown that to an observer moving rectilinearly relative to a sphere of mass m (the source of the Schwarzschild space–time) with speed v the space–time in the limit $v \to 1$ is a model of a plane impulsive gravitational wave. As in the electromagnetic case the history of this plane gravitational wave contains a null geodesic on which the field amplitude is singular. Many properties of this important result have been elucidated [Balasin and Nachbagauer (1995)], [Balasin (1997)], [Steinbauer (1998)], [Kunzinger and Steinbauer (1999)] and it is central to the description in general relativity of the high–speed collision of black–holes [D'Eath (1996)] and in modelling black–hole production in high–energy collisions [Eardley and Giddings (2002)]. We present here a derivation of the Aichelburg–Sexl result, emphasising the role of the curvature tensor. As in [Aichelburg and Sexl (1971)] we take the Schwarzschild line–element in isotropic coordinates as starting point:

$$ds^2 = (1+A)^4\,(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) - \frac{(1-A)^2}{(1+A)^2}\,d\bar{t}^2\ , \tag{4.22}$$

with

$$A = \frac{m}{2\,\bar{r}}\ , \qquad \bar{r} = \{\bar{x}^2 + \bar{y}^2 + \bar{z}^2\}^{1/2}\ . \tag{4.23}$$

The constant m is the mass of the source. Any other asymptotically flat form of the line–element in which the coordinates are asymptotically rectangular Cartesians and time will do. For example one could start with the Kerr–Schild form of (4.22) in coordinates $\{\bar{x}, \bar{y}, \bar{z}, \bar{t}\}$ in terms of which the flat background line–element is given by (4.1). Since we wish to emphasise the Riemann curvature tensor we require the curvature tensor components $\bar{R}_{ijkl}$ of the space–time with line–element (4.22) in coordinates $\{\bar{x}^i\} = \{\bar{x}, \bar{y}, \bar{z}, \bar{t}\}$. The non-identically vanishing components are

$$\begin{aligned}
\bar{R}_{1212} &= -\frac{m(\bar{r}^2 - 3\,\bar{z}^2)}{\bar{r}^5}\,(1+A)^2\ , \qquad & \bar{R}_{1313} &= -\frac{m(\bar{r}^2 - 3\,\bar{y}^2)}{\bar{r}^5}\,(1+A)^2\ ,\\
\bar{R}_{2323} &= -\frac{m(\bar{r}^2 - 3\,\bar{x}^2)}{\bar{r}^5}\,(1+A)^2\ , \qquad & \bar{R}_{1213} &= -\frac{3\,m\,\bar{y}\,\bar{z}}{\bar{r}^5}\,(1+A)^2\ ,\\
\bar{R}_{1223} &= \frac{3\,m\,\bar{x}\,\bar{z}}{\bar{r}^5}\,(1+A)^2\ , \qquad & \bar{R}_{1323} &= -\frac{3\,m\,\bar{x}\,\bar{y}}{\bar{r}^5}\,(1+A)^2\ ,
\end{aligned} \tag{4.24}$$

$$\bar{R}_{1414} = \frac{m\,(\bar{r}^2 - 3\,\bar{x}^2)}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ , \qquad \bar{R}_{2424} = \frac{m\,(\bar{r}^2 - 3\,\bar{y}^2)}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ ,$$
$$\bar{R}_{3434} = \frac{m\,(\bar{r}^2 - 3\,\bar{z}^2)}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ , \qquad \bar{R}_{1424} = -\frac{3\,m\,\bar{x}\,\bar{y}}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ ,$$
$$\bar{R}_{1434} = -\frac{3\,m\,\bar{x}\,\bar{z}}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ , \qquad \bar{R}_{2434} = -\frac{3\,m\,\bar{y}\,\bar{z}}{\bar{r}^5}\,\frac{(1-A)^2}{(1+A)^4}\ , \qquad (4.25)$$

Now make the Lorentz transformation (4.4). If $\{x^i\} = \{x, y, z, t\}$ then the non–identically vanishing components R_{ijkl} of the Riemann tensor in the unbarred coordinates are related to (4.24) and (4.25) via

$$\begin{aligned}
&R_{1212} = \gamma^2\left(\bar{R}_{1212} + v^2\bar{R}_{2424}\right)\ , \qquad R_{1313} = \gamma^2\left(\bar{R}_{1313} + v^2\bar{R}_{3434}\right)\ ,\\
&R_{2124} = -\gamma^2 v\left(\bar{R}_{2121} + \bar{R}_{2424}\right)\ , \qquad R_{3134} = -\gamma^2 v\left(\bar{R}_{3131} + \bar{R}_{3434}\right)\ ,\\
&R_{3124} = -\gamma^2 v\left(\bar{R}_{3121} + \bar{R}_{3424}\right)\ , \qquad R_{1234} = \gamma^2 v\left(\bar{R}_{3121} + \bar{R}_{3424}\right)\ ,\\
&R_{2434} = \gamma^2\left(\bar{R}_{2434} + v^2\bar{R}_{1213}\right)\ , \qquad R_{1213} = \gamma^2\left(\bar{R}_{1213} + v^2\bar{R}_{2434}\right)\ ,\\
&R_{1223} = \gamma\,\bar{R}_{1223}\ , \qquad R_{1323} = \gamma\bar{R}_{1323}\ , \qquad R_{1414} = \bar{R}_{1414}\ ,\\
&R_{2424} = \gamma^2\left(\bar{R}_{2424} + v^2\bar{R}_{1212}\right)\ , \qquad R_{3434} = \gamma^2\left(\bar{R}_{3434} + v^2\bar{R}_{1313}\right)\ ,\\
&R_{1424} = \gamma\,\bar{R}_{1424}\ , \qquad R_{1434} = \gamma\,\bar{R}_{1434}\ , \qquad R_{2323} = \bar{R}_{2323}\ . \qquad (4.26)
\end{aligned}$$

Of course when (4.24) and (4.25) are now substituted into (4.26) the barred coordinates in (4.24) and (4.25) are expressed in terms of the unbarred coordinates using (4.4). We now make the Aichelburg–Sexl boost of (4.26) by taking the limit $v \to 1$. In this limit the rest mass $m \to 0$ and $\gamma \to \infty$ in such a way that $m\,\gamma = p$ (say) remains finite and plays the role of the energy of the source. The calculation of the limit is straightforward and makes use of

$$\lim_{v \longrightarrow 1} \frac{\gamma^{-4}}{R^5} = \frac{4}{3}\,\frac{\delta(x-t)}{(y^2+z^2)^2}\ , \qquad (4.27)$$

which is obtained from (4.10) by differentiating with respect to y (or z) assuming y, z are non–zero. If we write

$$\tilde{R}_{ijkl} = \lim_{v \longrightarrow 1} R_{ijkl}, \qquad (4.28)$$

we find that $\tilde{R}_{ijkl} \equiv 0$ except for

$$\begin{aligned}
&\tilde{R}_{1212} = \tilde{R}_{2424} = -\tilde{R}_{1313} = -\tilde{R}_{3434} =\\
&\tilde{R}_{3134} = \tilde{R}_{2124} = -\frac{4\,p\,(y^2 - z^2)\,\delta(x-t)}{(y^2+z^2)^2}\ , \qquad (4.29)
\end{aligned}$$

$$\tilde{R}_{1213} = \tilde{R}_{2434} = \tilde{R}_{3124} = \tilde{R}_{2134}$$
$$= -\frac{8\,p\,y\,z\,\delta(x-t)}{(y^2+z^2)^2}\ , \tag{4.30}$$

We note that since $R_{jk} = 0$ we have $\tilde{R}_{jk} = \lim_{v\to 1} R_{jk} \equiv 0$. Substitution of the Lorentz transformation (4.4) into the line–element (4.22) and then taking the limit $v \to 1$ gives, in agreement with [Aichelburg and Sexl (1971)],

$$\lim_{v\longrightarrow 1} ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{4\,p}{|x-t|}\,(dt - dx)^2\ . \tag{4.31}$$

This holds for $x \neq t$ and is the analogue of (4.14). For $x > t$ and for $x < t$ this is the line–element of Minkowskian space– time (which of course is consistent with (4.29) and (4.30). For $x > t$ we can write (4.31) in the form

$$ds_+^2 = dy_+^2 + dz_+^2 - 2\,du\,dv_+\ , \tag{4.32}$$

with

$$y_+ = y\ , \qquad z_+ = z\ , \qquad u = t - x\ , \qquad v_+ = \frac{1}{2}(x+t) + 2p\,\log(x-t)\ . \tag{4.33}$$

For $x < t$ we can write (4.31) in the form

$$ds_-^2 = dy_-^2 + dz_-^2 - 2\,du\,dv_-\ , \tag{4.34}$$

with

$$y_- = y\ , \qquad z_- = z\ , \qquad u = t - x\ , \qquad v_- = \frac{1}{2}(x+t) - 2p\,\log(t-x)\ . \tag{4.35}$$

By (4.32) and (4.34) we see that $u = x - t = 0$ is a null hyperplane in Minkowskian space–time. The line–elements (4.32) and (4.34) are consistent with having a delta function in the Riemann curvature tensor which is singular on $x = t$ provided the two halves of Minkowskian space–time, $x > t$ and $x < t$, are attached on $x = t$ with (cf. §2.4)

$$y_+ = y_-\ , \qquad z_+ = z_-\ , \qquad v_+ = F(v_-, y_-, z_-)\ , \tag{4.36}$$

for some function F defined on $x = t$ for which $\partial F/\partial v_- \neq 0$. In the general case discussed in §2.4 , for any such F, this will make $x = t$ a model of the most general plane–fronted light–like signal propagating through flat space–time with a delta function in the curvature tensor singular on $x = t$ and with, in general, a delta function in the Ricci tensor aswell (so that the signal could be a plane–fronted impulsive gravitational wave accompanied by a plane–fronted light–like shell of null matter). The coefficients of the delta function in the Riemann tensor and the Einstein tensor are constructed from derivatives of the function F in (4.36). The explicit formulas are given in §2.4 . There it is shown that the line–element of the re–attached halves of Minkowskian space–time can be presented, once F is known, in a coordinate system in which the metric tensor is continuous across the singular hyperplane. This is given in (2.63). Thus the consistency of (4.36) with the calculated curvature tensor components (4.29) and (4.30) implies that in the present case

$$v_+ = F = v_- + 2p\,\log(y^2 + z^2)\ , \tag{4.37}$$

and that the signal in this case with history $x = t$ is an impulsive gravitational wave unaccompanied by a light–like shell (because $\tilde{R}_{jk} \equiv 0$). Moreover the function F in (4.37) is unique up to a (trivial) change of affine parameter v_+ along the generators of $x = t$. Finally we note from (4.29) and (4.30) that the amplitudes of the field components (the coefficients of the delta function) are singular on the generator $y = z = 0$ of the null hyperplane $x = t$. This is the remnant of the particle origin of the wave described by (4.29) and (4.30).

The re–working of the Aichelburg–Sexl boost of the Schwarzschild field given here serves to illustrate our point of view. We now apply our approach to boosting an isolated source having a gravitational field described by a Weyl, static, axially symmetric, asymptotically flat solution of Einstein's vacuum field equations. We have used these space–times already in §3.1 . It is convenient to begin here by writing the line–element as

$$ds^2 = \mathrm{e}^{2\sigma-2\psi}(d\bar{r}^2 + \bar{r}^2 d\bar{\theta}^2) + \mathrm{e}^{-2\psi}\bar{r}^2\sin^2\bar{\theta}\,d\bar{\phi}^2 - \mathrm{e}^{2\psi}\,d\bar{t}^2\ , \tag{4.38}$$

with

$$\psi = \sum_{l=0}^{\infty} \frac{A_l}{\bar{r}^{l+1}}\,P_l(\cos\bar{\theta})\ , \tag{4.39}$$

and

$$\sigma = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{A_l\,A_m\,(l+1)\,(m+1)}{(l+m+2)\,\bar{r}^{l+m+2}}\,(P_{l+1}\,P_{m+1} - P_l\,P_m)\ . \tag{4.40}$$

We can introduce coordinates $\{\bar{x}, \bar{y}, \bar{z}\}$ so that $\{\bar{x}, \bar{y}, \bar{z}, \bar{t}\}$ are asymptotically rectangular Cartesians and time simply by putting

$$\bar{x} = \bar{r}\,\sin\bar{\theta}\,\sin\bar{\phi}\ , \qquad \bar{y} = \bar{r}\,\sin\bar{\theta}\,\cos\bar{\phi}\ , \qquad \bar{z} = \bar{r}\,\cos\bar{\theta}\ . \tag{4.41}$$

Then we can write (4.38) in the form

$$ds^2 = g_{AB} d\bar{x}^A \, d\bar{x}^B + \mathrm{e}^{2\sigma - 2\psi} d\bar{z}^2 - \mathrm{e}^{2\psi} d\bar{t}^2 \ , \tag{4.42}$$

with $\{\bar{x}^A\} = \{\bar{x}, \bar{y}\}$ and

$$(g_{AB}) = \frac{\mathrm{e}^{-2\psi}}{(\bar{x}^2 + \bar{y}^2)} \begin{pmatrix} \mathrm{e}^{2\sigma}\bar{x}^2 + \bar{y}^2 & \left(\mathrm{e}^{2\sigma} - 1\right) \bar{x}\,\bar{y} \\ \left(\mathrm{e}^{2\sigma} - 1\right) \bar{x}\,\bar{y} & \mathrm{e}^{2\sigma}\bar{y}^2 + \bar{x}^2 \end{pmatrix} ,$$

and capital letters take values $1, 2$. The Legendre polynomials are now functions of $\bar{z}/\bar{r}$ with $\bar{r} = \{\bar{x}^2 + \bar{y}^2 + \bar{z}^2\}^{1/2}$.

We will consider a Lorentz boost in the $-\bar{x}$ direction given by (4.4) and also a Lorentz boost in the $-\bar{z}$ direction (along the symmetry axis) given by

$$\bar{x} = x \ , \qquad \bar{y} = y \ , \qquad \bar{z} = \gamma(z - v\,t) \ , \qquad \bar{t} = \gamma(t - v\,z) \ . \tag{4.43}$$

We will then follow this by taking the limit $v \to 1$ (which we are calling the Aichelburg–Sexl boost). In the monopole case considered by Aichelburg and Sexl the rest mass m of the source was taken to tend to zero in this limit in such a way that the energy $p = m\,\gamma$ remained finite. Now the behavior of all of the constants A_l $(l = 0, 1, 2, \ldots)$, related to the multipole moments of the source, in the limit $v \to 1$, has to be considered. A simple guide is the special case of the Curzon [Curzon (1924)] solution which is the subcase of (4.38)–(44.40) corresponding to two point masses having rest–masses m_1 , m_2 located on the $\bar{z}$–axis at $\bar{z} = \pm a$ respectively and held in position with a strut. In this case

$$A_l = -m_1\,a^l - m_2\,(-a)^l \ , \qquad l = 0, 1, 2, \ldots \ . \tag{4.44}$$

In any boost we will assume as before that m_1, m_2 tend to zero as $v \to 1$ like γ^{-1}, i.e.

$$m_1 = \gamma^{-1}\hat{p}_1 \ , \qquad m_2 = \gamma^{-1}\hat{p}_2 \ , \tag{4.45}$$

with $\hat{p}_1$, $\hat{p}_2$ independent of v. Thus for a Lorentz boost in the $-\bar{x}$ direction,

$$A_l = \gamma^{-1} p_l \ , \qquad l = 0, 1, 2, \ldots \ , \tag{4.46}$$

with

$$p_l = -\hat{p}_1\,a^l - \hat{p}_2\,(-a)^l \ . \tag{4.47}$$

On the other hand for a Lorentz boost in the $-\bar{z}$ direction we assume that a tends to zero as $v \to 1$ like γ^{-1}, i.e.

$$a = \gamma^{-1}\hat{a} \ , \tag{4.48}$$

with $\hat{a}$ independent of v. Hence in this case

$$A_l = \gamma^{-l-1}\, p_l \ , \qquad l = 0, 1, 2, \dots \ , \tag{4.49}$$

with now

$$p_l = -\hat{p}_1\, \hat{a}^l - \hat{p}_2(-\hat{a})^l \ . \tag{4.50}$$

The essential difference between (4.46) and (4.49) is due to the fact that (4.49) incorporates the Lorentz contraction (4.48). We shall in general consider only a class of vacuum gravitational fields described by (4.37)–(4.39) for which A_l scales with γ in the form of (4.46) for a Lorentz boost in the (transverse) $-\bar{x}$ direction and in the form (4.49) for a boost parallel to the axis of symmetry (the $-\bar{z}$ direction). Since we shall be considering later the limit $v \to 1$ and thus $\gamma^{-1} \to 0$ we can slightly weaken our assumptions by taking (4.46) and (4.49) to be the leading terms in A_l in the two cases for small γ^{-1} (thus allowing additional terms which go to zero faster than γ^{-1} in the limit $v \to 1$) without affecting the outcome of our calculations.

We begin by applying the Lorentz transformation (4.4) in the $-\bar{x}$ direction to the curvature tensor calculated from the line–element (4.42). The components $\bar{R}_{ijkl}$ of this curvature tensor, in the coordinates $\{\bar{x}\ , \bar{y}\ , \bar{z}\ , \bar{t}\ \}$ are now required. They are given by

$$\begin{aligned}
\bar{R}_{ABCD} &= {}^{(2)}\bar{R}_{ABCD} + \frac{1}{4}\mathrm{e}^{2\psi-2\sigma}\left(\frac{\partial g_{AD}}{\partial \bar{z}}\,\frac{\partial g_{BC}}{\partial \bar{z}} - \frac{\partial g_{AC}}{\partial \bar{z}}\,\frac{\partial g_{BD}}{\partial \bar{z}}\right) \ ,\\
\bar{R}_{ABC3} &= \frac{1}{2}\frac{\partial}{\partial \bar{z}}\left(\frac{\partial g_{BC}}{\partial \bar{x}^A} - \frac{\partial g_{AC}}{\partial \bar{x}^B}\right) + \frac{1}{2}\,\mathrm{e}^{2\psi-2\sigma}\left(\frac{\partial g_{33}}{\partial \bar{x}^B}\,\frac{\partial g_{AC}}{\partial \bar{z}} - \frac{\partial g_{33}}{\partial \bar{x}^A}\,\frac{\partial g_{BC}}{\partial \bar{z}}\right)\\
&\quad + \frac{1}{2}\,g^{EF}\left(\frac{\partial g_{AE}}{\partial \bar{z}}\,[BC, F] - \frac{\partial g_{BF}}{\partial \bar{z}}\,[AC, E]\right) \ ,\\
\bar{R}_{3434} &= -\frac{1}{2}\frac{\partial^2 g_{44}}{\partial \bar{z}^2} - \frac{1}{4}g^{EF}\,\frac{\partial g_{33}}{\partial \bar{x}^E}\,\frac{\partial g_{44}}{\partial \bar{x}^F} + \frac{1}{4}\mathrm{e}^{2\psi-2\sigma}\frac{\partial g_{33}}{\partial \bar{z}}\,\frac{\partial g_{44}}{\partial \bar{z}}\\
&\quad - \frac{1}{4}\mathrm{e}^{-2\psi}\left(\frac{\partial g_{44}}{\partial \bar{z}}\right)^2 \ ,\\
\bar{R}_{A3B3} &= -\frac{1}{2}\left(\frac{\partial^2 g_{AB}}{\partial \bar{z}^2} + \frac{\partial^2 g_{33}}{\partial \bar{x}^A \partial \bar{x}^B}\right) + \frac{1}{4}g^{EF}\,\frac{\partial g_{AE}}{\partial \bar{z}}\,\frac{\partial g_{BF}}{\partial \bar{z}}\\
&\quad + \frac{1}{2}g^{EF}\,[AB, E]\,\frac{\partial g_{33}}{\partial \bar{x}^F} + \frac{1}{4}\mathrm{e}^{2\psi-2\sigma}\left(\frac{\partial g_{33}}{\partial \bar{x}^A}\,\frac{\partial g_{33}}{\partial \bar{x}^B} + \frac{\partial g_{AB}}{\partial \bar{z}}\,\frac{\partial g_{33}}{\partial \bar{z}}\right) \ ,
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{A434} &= -\frac{1}{2}\frac{\partial^2 g_{44}}{\partial\bar{z}\partial\bar{x}^A} + \frac{1}{4}g^{EF}\,\frac{\partial g_{AE}}{\partial\bar{z}}\,\frac{\partial g_{44}}{\partial\bar{x}^F} + \frac{1}{4}\mathrm{e}^{2\psi-2\sigma}\frac{\partial g_{33}}{\partial\bar{x}^A}\,\frac{\partial g_{44}}{\partial\bar{z}} \\
&\quad -\frac{1}{4}\mathrm{e}^{-2\psi}\,\frac{\partial g_{44}}{\partial\bar{x}^A}\,\frac{\partial g_{44}}{\partial\bar{z}}\ , \\
\bar{R}_{A4B4} &= -\frac{1}{2}\frac{\partial^2 g_{44}}{\partial\bar{x}^A\partial\bar{x}^B} + \frac{1}{2}g^{EF}\,[AB,E]\,\frac{\partial g_{44}}{\partial\bar{x}^F} - \frac{1}{4}\mathrm{e}^{2\psi-2\sigma}\frac{\partial g_{44}}{\partial\bar{z}}\,\frac{\partial g_{AB}}{\partial\bar{z}} \\
&\quad -\frac{1}{4}\mathrm{e}^{-2\psi}\,\frac{\partial g_{44}}{\partial\bar{x}^A}\,\frac{\partial g_{44}}{\partial\bar{x}^B}\ .
\end{aligned} \tag{4.51}$$

Here $[AB,C]$ is the Christoffel symbol

$$[AB,C] = \frac{1}{2}\left(\frac{\partial g_{CA}}{\partial\bar{x}^B} + \frac{\partial g_{BC}}{\partial\bar{x}^A} - \frac{\partial g_{AB}}{\partial\bar{x}^C}\right)\ , \tag{4.52}$$

and

$$\begin{aligned}
{}^{(2)}\bar{R}_{ABCD} &= \frac{1}{2}\ (g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC}) \\
&\quad + g^{EF}\,([AD,E]\,[BC,F] - [AC,E]\,[BD,F])\ ,
\end{aligned} \tag{4.53}$$

with the comma denoting partial derivative with respect to $\bar{x}^A$. These are then transformed to R_{ijkl} in the coordinates $\{x\ ,y\ ,z\ ,t\}$ introduced in (4.4). The components R_{ijkl} are given by

$$\begin{aligned}
R_{abcd} &= \bar{R}_{abcd}\ , \qquad R_{ab14} = \bar{R}_{ab14}\ , \qquad R_{1414} = \bar{R}_{1414}\ , \\
R_{abc1} &= \gamma\,\left(\bar{R}_{abc1} - v\,\bar{R}_{abc4}\right)\ , \quad R_{abc4} = \gamma\,\left(\bar{R}_{abc4} - v\,\bar{R}_{abc1}\right)\ , \\
R_{a1b1} &= \gamma^2\left(\bar{R}_{a1b1} - v\,\bar{R}_{a1b4} - v\,\bar{R}_{a4b1} + v^2\bar{R}_{a4b4}\right)\ , \\
R_{a1b4} &= \gamma^2\left(\bar{R}_{a1b4} - v\,\bar{R}_{a1b1} - v\,\bar{R}_{a4b4} + v^2\bar{R}_{a4b1}\right)\ , \\
R_{a4b4} &= \gamma^2\left(\bar{R}_{a4b4} - v\,\bar{R}_{a1b4} - v\,\bar{R}_{a4b1} + v^2\bar{R}_{a1b1}\right)\ , \\
R_{a114} &= \gamma\,\left(\bar{R}_{a114} - v\,\bar{R}_{a414}\right)\ , \quad R_{a414} = \gamma\,\left(\bar{R}_{a414} + v\,\bar{R}_{a141}\right)\ .
\end{aligned} \tag{4.54}$$

We then finally calculate the limit of these components as $v \to 1$, which we denote by

$$\tilde{R}_{ijkl} = \lim_{v\longrightarrow 1} R_{ijkl}\ . \tag{4.55}$$

We note that since the space–time described by (4.37)–(4.39) is a vacuum space–time, the Ricci tensor R_{jk} vanishes and so $\tilde{R}_{jk} = \lim_{v\to 1} R_{jk} = 0$. In carrying out this programme of calculations everything hinges on the functions $\psi\ ,\sigma$ and their first and second derivatives with respect to $\bar{x}\ ,\bar{y}\ ,\bar{z}$ which must be evaluated in the coordinates $\{x\ ,y\ ,z\ ,t\}$ for substitution into (4B5). With $A_l = \gamma^{-1}p_l\ ,\ l = 0,1,2,\ldots$, as in (4.46), and

$$\bar{r} = \gamma\,R\ , \qquad R = \sqrt{(x-v\,t)^2 + \gamma^{-2}(y^2+z^2)}\ , \tag{4.56}$$

and starting with

$$\psi = \sum_{l=0}^{\infty} \frac{A_l}{\bar{r}^{l+1}} P_l\left(\frac{\bar{z}}{\bar{r}}\right) \ , \tag{4.57}$$

we can write

$$\psi = \sum_{l=0}^{\infty} p_l \, \frac{\gamma^{-l-2}}{R^{l+1}} \, P_l(w) \ , \tag{4.58}$$

with

$$w = \gamma^{-1} \frac{z}{R} \ . \tag{4.59}$$

It is clear from (4.58) that as $v \to 1$, $\psi \to 0$ like γ^{-2} for $x \neq t$ and $\psi \to 0$ like γ^{-1} for $x = t$. Starting with (4.57) we obtain

$$\frac{\partial \psi}{\partial \bar{x}} = -\sum_{l=0}^{\infty} p_l \, (x - v\, t) \, \frac{\gamma^{-l-3}}{R^{l+3}} \, P'_{l+1} \ , \tag{4.60}$$

$$\frac{\partial \psi}{\partial \bar{y}} = -\sum_{l=0}^{\infty} p_l \, y \, \frac{\gamma^{-l-4}}{R^{l+3}} \, P'_{l+1} \ , \tag{4.61}$$

$$\frac{\partial \psi}{\partial \bar{z}} = -\sum_{l=0}^{\infty} p_l \, (l+1) \, \frac{\gamma^{-l-3}}{R^{l+2}} \, P_{l+1} \ , \tag{4.62}$$

where the argument of the Legendre polynomials is w given in (4.59) and the prime denotes differentiation with respect to w. We have simplified (4.59) and (4.60) using

$$w \, P'_l + (l+1) \, P_l = P'_{l+1} \ , \tag{4.63}$$

for $l = 0, 1, 2, \ldots$, and (4.61) has been simplified using

$$(w^2 - 1) \, P'_l = (l+1) \ (P_{l+1} - w \, P_l) \ , \tag{4.64}$$

for $l = 0, 1, 2, \ldots$. We need the second derivatives of ψ and making use of the relations (4.63) and (4.64) satisfied by the Legendre polynomials we arrive at

$$\frac{\partial^2 \psi}{\partial \bar{x}^2} = -\sum_{l=0}^{\infty} p_l \, \frac{\partial}{\partial x} \left(\frac{(x - v\, t) \, \gamma^{-l-4} P'_{l+1}}{R^{l+3}} \right) \ , \tag{4.65}$$

$$\frac{\partial^2 \psi}{\partial \bar{x} \partial \bar{y}} = -\sum_{l=0}^{\infty} p_l \, (x - v\, t) \, \frac{\partial}{\partial y} \left(\frac{\gamma^{-l-3} P'_{l+1}}{R^{l+3}} \right) \ , \tag{4.66}$$

$$\frac{\partial^2\psi}{\partial\bar{y}^2} = -\sum_{l=0}^{\infty} p_l \frac{\partial}{\partial y}\left(\frac{y\,\gamma^{-l-4}P'_{l+1}}{R^{l+3}}\right) , \tag{4.67}$$

$$\frac{\partial^2\psi}{\partial\bar{z}^2} = \sum_{l=0}^{\infty} p_l\,(l+1)\,(l+2)\,\frac{\gamma^{-l-4}P_{l+2}}{R^{l+3}} , \tag{4.68}$$

$$\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{z}} = \sum_{l=0}^{\infty} p_l\,(l+1)\,(x-v\,t)\,\frac{\gamma^{-l-4}P'_{l+2}}{R^{l+4}} , \tag{4.69}$$

$$\frac{\partial^2\psi}{\partial\bar{y}\partial\bar{z}} = \sum_{l=0}^{\infty} p_l\,(l+1)\,y\,\frac{\gamma^{-l-5}P'_{l+2}}{R^{l+4}} . \tag{4.70}$$

To evaluate the limit $v \to 1$ of quantities involving the derivatives (4.59)–(4.61) and (4.64)–(4.69) we make use of the following:

$$\lim_{v\to 1} \frac{\gamma^{-l}P_l(w)}{R^{l+1}} = \frac{(-1)^{l+1}}{l!}\,g^{(l)}\,\delta(x-t) , \tag{4.71}$$

for $l = 1,2,3,\dots$, with $g = \log(y^2+z^2)$ and $g^{(l)} = \partial^l g/\partial z^l$. This is established by induction on l. It is true for $l = 1$ because

$$\frac{\gamma^{-1}P_1(w)}{R^2} = \frac{\gamma^{-1}w}{R^2} = \frac{\gamma^{-2}z}{R^3} , \tag{4.72}$$

and so by (4.10)

$$\lim_{v\to 1} \frac{\gamma^{-1}P_1(w)}{R^2} = \frac{2\,z}{y^2+z^2}\,\delta(x-t) = \frac{\partial g}{\partial z}\,\delta(x-t) . \tag{4.73}$$

Assume (4.71) holds for l and differentiate it with respect to z to obtain

$$\lim_{v\to 1} \frac{\gamma^{-l-1}}{R^{l+2}}\left\{-(l+1)\,w\,P_l + (1-w^2)\,P'_l\right\} = \frac{(-1)^{l+1}}{l!}g^{(l+1)}\,\delta(x-t) . \tag{4.74}$$

Using (4.64) this can be simplified to read

$$\lim_{v\to 1} \frac{\gamma^{-l-1}P_{l+1}(w)}{R^{l+2}} = \frac{(-1)^{l+2}}{(l+1)!}g^{(l+1)}\,\delta(x-t) , \tag{4.75}$$

and (4.71) holds for $(l+1)$ if it holds for l.

In addition to (4.71) we shall require one more result:

$$\lim_{v\to 1} \frac{\gamma^{-l-2}P'_{l+1}(w)}{R^{l+3}} = \frac{2\,(-1)^l}{l!}\,h^{(l)}\,\delta(x-t) , \tag{4.76}$$

for $l = 0,1,2,\dots$, with $h = (y^2+z^2)^{-1}$ and $h^{(l)} = \partial^l h/\partial z^l$. This can be established by induction on l or more directly by differentiating (4.71) with respect to y to obtain

$$-\lim_{v\to1}\frac{\gamma^{-l-2}y}{R^{l+3}}\,\{(l+1)\,P_l+w\,P_l'\}=\frac{(-1)^{l+1}}{l!}\,\frac{\partial}{\partial y}\left(g^{(l)}\right)\,\delta(x-t)\ ,\qquad(4.77)$$

for $l=1,2,3,\dots$. If we use (4.63) in the left hand side here and in the right hand side use

$$\frac{\partial}{\partial y}\left(g^{(l)}\right)=\frac{\partial^l}{\partial z^l}\left(\frac{\partial g}{\partial y}\right)=2\,y\,\frac{\partial^l}{\partial z^l}(y^2+z^2)^{-1}=2\,y\,h^{(l)}\ ,\qquad(4.78)$$

then, assuming $y\neq0$, (4.77) becomes (4.76) for $l=1,2,3,\dots$. That (4.76) also holds for $l=0$ follows from

$$\lim_{v\to1}\frac{\gamma^{-2}P_1'(w)}{R^3}=\lim_{v\to1}\frac{\gamma^{-2}}{R^3}=2\,h\,\delta(x-t)\ ,\qquad(4.79)$$

with the last equality coming from (4.10).

With the help of (4.71) and (4.76) limits of the derivatives (4.59)–(4.61) and (4.64)–(4.69) can be evaluated which are required for the calculation of $\tilde{R}_{ijkl}$ in (4.55). The reader can easily evaluate the limits needed with the information given above. We shall mention here, as a guide, some of the required limits. For example using (4.67) and (4.71) we obtain

$$\lim_{v\to1}\gamma^2\frac{\partial^2\psi}{\partial\bar{z}^2}=\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}g^{(l+2)}\,\delta(x-t)\ .\qquad(4.80)$$

By (4.68) and (4.76),

$$\lim_{v\to1}\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{z}}=0\ ,\qquad(4.81)$$

and we have here made use of $(x-t)\,\delta(x-t)=0$. Using (4.69) and (4.76) we find that

$$\lim_{v\to1}\gamma^2\frac{\partial^2\psi}{\partial\bar{y}\partial\bar{z}}=\sum_{l=0}^{\infty}p_l\,\frac{2\,(-1)^{l+1}}{l!}\,y\,h^{(l+1)}\,\delta(x-t)\ .\qquad(4.82)$$

We can write

$$2\,y\,h^{(l+1)}=\frac{\partial^{l+1}}{\partial z^{l+1}}\left(\frac{2\,y}{y^2+z^2}\right)=\frac{\partial^{l+1}}{\partial z^{l+1}}\left(\frac{\partial g}{\partial y}\right)=\frac{\partial^2}{\partial z\partial y}\,g^{(l)}\ ,\qquad(4.83)$$

and thus (4.82) becomes

$$\lim_{v\to1}\gamma^2\frac{\partial^2\psi}{\partial\bar{y}\partial\bar{z}}=\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}\,\frac{\partial^2 g^{(l)}}{\partial z\partial y}\,\delta(x-t)\ .\qquad(4.84)$$

Similarly from (4.64), (4.65) using (4.35) and $(x-t)\,\delta(x-t)=0$ we find that

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{x}^2}=0=\lim_{v\to 1}\gamma\,\frac{\partial^2\psi}{\partial\bar{x}\partial\bar{y}}\ . \tag{4.85}$$

Finally by (4.66) using (4.76),

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{y}^2}=\sum_{l=0}^{\infty}p_l\,\frac{2\,(-1)^{l+1}}{l!}\,\frac{\partial}{\partial y}\left(y\,h^{(l)}\right)\,\delta(x-t)\ . \tag{4.86}$$

But

$$\frac{\partial}{\partial y}\left(2\,y\,h^{(l)}\right)=\frac{\partial^{l+1}}{\partial y\partial z^l}\left(\frac{2\,y}{y^2+z^2}\right)=\frac{\partial^2 g^{(l)}}{\partial y^2}\ , \tag{4.87}$$

and so (4.86) reads

$$\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{y}^2}=\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}\,\frac{\partial^2 g^{(l)}}{\partial y^2}\,\delta(x-t)\ . \tag{4.88}$$

There remains to be considered the function σ given by (4.3). It is clear from (4.3) that σ is a linear combination with constant coefficients of ψ^2–terms. Hence as $v\to 1$ we have $\sigma\to 0$ like γ^{-4} for $x\neq t$ (compared with γ^{-2} for ψ) and $\sigma\to 0$ like γ^{-2} for $x=t$ (compared with γ^{-1} for ψ). In addition all first and second derivatives of σ *vanish* in the limit $v\to 1$, including the derivatives multiplied by γ or γ^2 (such as the left hand sides of (4.79)–(4.81) and (4.83)–(4.85) with ψ replaced by σ) which are required for the calculation of $\tilde{R}_{ijkl}$ in (4.54). We are now ready to calculate $\tilde{R}_{ijkl}$. We first find that the following components are non–zero:

$$\begin{aligned}\tilde{R}_{2434}=\tilde{R}_{2131}=\tilde{R}_{2134}=\tilde{R}_{3124}&=2\lim_{v\to 1}\gamma^2\frac{\partial^2\psi}{\partial\bar{z}\partial\bar{y}}\ ,\\ &=2\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}\,\frac{\partial^2 g^{(l)}}{\partial y\partial z}\,\delta(x-t)\ ,\end{aligned} \tag{4.89}$$

by (4.84). Next we find

$$\begin{aligned}\tilde{R}_{2424}=\tilde{R}_{2121}=\tilde{R}_{2124}&=\lim_{v\to 1}\gamma^2\left(2\,\frac{\partial^2\psi}{\partial\bar{y}^2}+\frac{\partial^2\psi}{\partial\bar{x}^2}\right)\ ,\\ &=2\sum_{l=0}^{\infty}p_l\,\frac{(-1)^{l+1}}{l!}\,\frac{\partial^2 g^{(l)}}{\partial y^2}\,\delta(x-t)\ ,\end{aligned} \tag{4.90}$$

by (4.85) and (4.88). And finally we obtain

$$\tilde{R}_{3434} = \tilde{R}_{3131} = \tilde{R}_{3134} = \lim_{v\to 1} \gamma^2 \left(2\,\frac{\partial^2\psi}{\partial\bar{z}^2} + \frac{\partial^2\psi}{\partial\bar{x}^2}\right) ,$$
$$= 2\sum_{l=0}^{\infty} p_l\,\frac{(-1)^{l+1}}{l!}\,\frac{\partial^2 g^{(l)}}{\partial z^2}\,\delta(x-t) , \tag{4.91}$$

by (4.80) and (4.85). Noting that $g = \log(y^2 + z^2)$ is a harmonic function for $y^2 + z^2 \neq 0$ so that

$$\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0 , \tag{4.92}$$

the right hand side of (4.91) is minus the right hand side of (4.90). We have now arrived at the analogue of (4.29) and (4.29) in the monopole case and we see that the singularity in (4.88)–(4.90) on $y = z = 0$ is more severe than in the monopole case, involving derivatives of g greater than the second.

Taking the limit $v \to 1$ of the line–element (4.42) following the coordinate transformation (4.4) results in

$$\lim_{v\to 1} ds^2 = dx^2 + dy^2 + dz^2 - dt^2 - \frac{4\,p_0}{|x-t|}\,(dt - dx)^2 , \tag{4.93}$$

for $x \neq t$ (we note that $p_0 = -p$ in (4.31)). We note that only p_0 appears in (4.93) and so (4.93) inherits information only from the monopole structure of the source. The curvature tensor (4.89)–(4.91) is influenced by all the multipole moments of the source and thus contains more information than (4.93). The discussion following (4.31) applies again here. The hyperplane $x = t$ is null and the Riemann tensor $\tilde{R}_{ijkl}$ given by (4.88)–(4.90) is consistent with re–attaching the two halves of Minkowskian space– time $x > t$ and $x < t$ with the matching (4.36) provided the function F in (4.36) is correctly chosen. The formulas for determining F are again given in §2.4 . The result is that now (4.37) is generalised to

$$v_+ = F = v_- - 2\sum_{l=0}^{\infty} p_l\,\frac{(-1)^l}{l!}\,\frac{\partial^l g}{\partial z^l} , \tag{4.94}$$

with $g = \log(y^2 + z^2)$. This agrees with (4.37) in the special monopole case when $p_0 = -p$, $p_l = 0$ for $l = 1, 2, 3, \ldots$.

We now consider an Aichelburg–Sexl boost in the parallel or $-\bar{z}$ direction. We begin again with the curvature tensor $\bar{R}_{ijkl}$ given in (4.54) and transform to the coordinates $\{x, y, z, t\}$ using the Lorentz boost (4.43) (instead of (4.4)). In this case starting with (4.57) with $A_l = \gamma^{-l-1} p_l$ $(l = 0, 1, 2, \dots)$, as in (4.49),

$$\psi = \sum_{l=0}^{\infty} p_l \, \frac{\gamma^{-2l-2}}{\hat{R}^{l+1}} \, P_l(\hat{w}) \ , \tag{4.95}$$

where

$$\bar{r} = \gamma \, \hat{R} \ , \qquad \hat{R} = \sqrt{\gamma^{-2}(x^2 + y^2) + (z - v\,t)^2} \ , \tag{4.96}$$

and

$$\hat{w} = \frac{\bar{z}}{\bar{r}} = \frac{z - v\,t}{\hat{R}} \ . \tag{4.97}$$

Once again we must calculate the first and second derivatives of ψ , σ with respect to $\{\bar{x}, \bar{y}, \bar{z}\}$ and express them in terms of $\{x, y, z, t\}$ via (4.43) for substitution into the curvature components. Then the limit $v \rightarrow 1$ of these components is taken to arrive at $\tilde{R}_{ijkl}$ in this case. The calculation parallels the one above but with some significant differences [see [Barrabès and Hogan (2001)] for details]. The result is that $\tilde{R}_{ijkl}$ in this case is the same curvature tensor as (4.28) and (4.29). The presence of the higher multipole moments than the monopole is lost in this boost. This is not surprising because the scaling of A_l in (4.49) reflects the Lorentz contraction of the source in the axial direction. The shape of the source remains the same but scaled down to the boosted observer [Rindler (1969)]. As $v \rightarrow 1$ it appears increasingly as a monopole source and hence we recover the original Aichelburg–Sexl result in this case.

4.3 Light–Like Boost of the Kerr Field

Working with the Riemann tensor is a guaranteed way of ensuring that the boosted field is unambiguously determined because the Riemann tensor is gauge–invariant. It is of some importance to apply this new point of view to the Kerr gravitational field [Barrabès and Hogan (2003a)]. It leads to results which are in general simpler and quite different to those to be found in the literature with the notable exception that we find ourselves in agreement with the limit calculated in [Balasin and Nachbagauer (1996)] (their equation (18) compared to our (4.125) below). These authors were the first to obtain this transverse light–like boosted Kerr gravitational field. They introduced a distributional energy–momentum tensor [Balasin and Nachbagauer (1995)] for the Kerr space–time. This energy–momentum tensor was then subjected to a Lorentz boost and the light–like limit was taken. The metric of the space–time after the light–like boost involves a single function which is obtained by solving a Poisson equation. The procedure is technically very impressive. However in the light of the existence of conflicting published results the limit calculated in [Balasin and Nachbagauer (1996)] benefits from being confirmed by the simpler derivation which we supply here.

We begin with the Kerr line–element [Kerr (1963)] in asymptotically rectangular Cartesian coordinates and time:

$$ds^2 = ds_0^2 + \frac{2m\bar{r}^3}{\bar{r}^4 + a^2\bar{z}^2}\left[\frac{\bar{z}}{\bar{r}}\,d\bar{z} + \frac{(\bar{r}\bar{x} + a\bar{y})}{\bar{r}^2 + a^2}\,d\bar{x} + \frac{(\bar{r}\bar{y} - a\bar{x})}{\bar{r}^2 + a^2}\,d\bar{y} - d\bar{t}\right]^2 , \tag{4.98}$$

with

$$ds_0^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 - d\bar{t}^2 . \tag{4.99}$$

The constants m, a are the mass and the angular momentum per unit mass respectively of the source and $\bar{r}$ is a function of $\bar{x}, \bar{y}, \bar{z}$ given by

$$\frac{\bar{x}^2 + \bar{y}^2}{\bar{r}^2 + a^2} + \frac{\bar{z}^2}{\bar{r}^2} = 1 . \tag{4.100}$$

Thus in coordinates $\bar{x}^\mu = (\bar{x}, \bar{y}, \bar{z}, \bar{t})$ the metric tensor components have the Kerr–Schild form [Kerr and Schild (1965)][Debney et al. (1969)]

$$\bar{g}_{\mu\nu} = \bar{\eta}_{\mu\nu} + 2\bar{H}\,\bar{k}_\mu\bar{k}_\nu , \tag{4.101}$$

with

$$\eta_{\mu\nu}\,d\bar{x}^\mu d\bar{x}^\nu = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 - d\bar{t}^2 , \tag{4.102}$$

$$\bar{k}_\mu d\bar{x}^\mu = \frac{\bar{z}}{\bar{r}}\,d\bar{z} + \frac{(\bar{r}\bar{x} + a\bar{y})}{\bar{r}^2 + a^2}\,d\bar{x} + \frac{(\bar{r}\bar{y} - a\bar{x})}{\bar{r}^2 + a^2}\,d\bar{y} - d\bar{t} , \tag{4.103}$$

$$\bar{H} = \frac{m\bar{r}^3}{\bar{r}^4 + a^2\bar{z}^2} , \tag{4.104}$$

with the covariant vector field $\bar{k}_\mu$ null with respect to $\bar{g}_{\mu\nu}$ and thus null with respect to the auxiliary Minkowskian metric tensor $\bar{\eta}_{\mu\nu}$. Later we shall be transforming to unbarred coordinates $\{x, y, z, t\}$. Since our approach is centered on the Riemann curvature tensor we shall require the coordinate components $\bar{R}_{\mu\nu\rho\sigma}$ of the Riemann curvature tensor calculated from the metric tensor (4.101). These are conveniently given in terms of the complex tensor

$$^{+}\bar{R}_{\mu\nu\rho\sigma} = \bar{R}_{\mu\nu\rho\sigma} + i\,^{*}\bar{R}_{\mu\nu\rho\sigma} , \tag{4.105}$$

where the left dual of the Riemann tensor (since the Kerr space–time is a vacuum space–time the left and right duals of the Riemann tensor are equal) is defined by ${}^*\bar{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\bar{\eta}_{\mu\nu\alpha\beta}\,\bar{R}^{\alpha\beta}{}_{\rho\sigma}$ with $\bar{\eta}_{\mu\nu\alpha\beta} = \sqrt{-\bar{g}}\,\epsilon_{\mu\nu\alpha\beta}$, $\bar{g} = \det(\bar{g}_{\mu\nu})$ and $\epsilon_{\mu\nu\alpha\beta}$ is the Levi–Civita permutation symbol. We find after a lengthy calculation, using many of the calculations in [Debney et al. (1969)], that for the Kerr space–time

$$ {}^+\bar{R}_{\mu\nu\rho\sigma} = -\frac{m\bar{r}^3}{(\bar{r}^2 + ia\bar{z})^3}\left\{\bar{g}_{\mu\nu\rho\sigma} + i\,\epsilon_{\mu\nu\rho\sigma} + 3\,\bar{W}_{\mu\nu}\,\bar{W}_{\rho\sigma}\right\}\ , \tag{4.106} $$

with

$$ \bar{g}_{\mu\nu\rho\sigma} = \bar{g}_{\mu\rho}\,\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\,\bar{g}_{\nu\rho}\ , \tag{4.107} $$

and the bivector $\bar{W}_{\mu\nu} = -\bar{W}_{\nu\mu}$ is given by the 2–form

$$ \begin{aligned} \frac{1}{2}\,\bar{W}_{\mu\nu}\,d\bar{x}^\mu \wedge d\bar{x}^\nu = \frac{\bar{r}}{(\bar{r}^2 + ia\bar{z})}\,&[\bar{x}(d\bar{x}\wedge d\bar{t} - i\,d\bar{y}\wedge d\bar{z}) + \bar{y}(d\bar{y}\wedge d\bar{t} \\ &- i\,d\bar{z}\wedge d\bar{x}) + (\bar{z} + ia)\,(d\bar{z}\wedge d\bar{t} - i\,d\bar{x}\wedge d\bar{y})]\,. \end{aligned} \tag{4.108} $$

We now consider a Lorentz boost in the transverse or $-\bar{x}$ direction with 3–velocity $v < 1$. Thus we transform from coordinates $\bar{x}^\mu = (\bar{x}, \bar{y}, \bar{z}, \bar{t})$ to coordinates $x^\mu = (x, y, z, t)$ given by

$$ \bar{x} = \gamma\,(x - vt)\ , \qquad \bar{y} = y\ , \qquad \bar{z} = z\ , \qquad \bar{t} = \gamma\,(t - vx)\ , \tag{4.109} $$

with $\gamma = (1-v^2)^{-1/2}$. The components ${}^+\bar{R}_{\mu\nu\rho\sigma}$ are transformed to ${}^+R_{\mu\nu\rho\sigma}$. The transformations naturally divide into three sets of equations. The first set has γ^2 as a factor on each right hand side and is

$$ \begin{aligned} {}^+R_{1A1B} &= \gamma^2({}^+\bar{R}_{1A1B} + v\,{}^+\bar{R}_{1AB4} + v\,{}^+\bar{R}_{1BA4} + v^{2+}\bar{R}_{A4B4})\ ,\\ {}^+R_{1A4B} &= \gamma^2({}^+\bar{R}_{1A4B} + v^+\bar{R}_{1AB1} + v^+\bar{R}_{4AB4} + v^{2+}R_{4A1B})\ ,\\ {}^+R_{4A4B} &= \gamma^2(+\bar{R}_{4A4B} + v\,{}^+\bar{R}_{4AB1} + v^+\bar{R}_{4BA1} + v^{2+}\bar{R}_{A1B1})\ , \end{aligned} \tag{4.110} $$

where the subscripts A, B take values $2, 3$. The second set has γ as a factor on each right side and reads

$$ \begin{aligned} {}^+R_{1A14} &= \gamma\,({}^+\bar{R}_{1A14} + v\,{}^+\bar{R}_{14A4})\ ,\\ {}^+R_{14A4} &= \gamma\,({}^+\bar{R}_{14A4} + v\,{}^+\bar{R}_{1A14})\ ,\\ {}^+R_{1ABC} &= \gamma\,({}^+\bar{R}_{1ABC} + v\,{}^+\bar{R}_{A4BC})\ ,\\ {}^+R_{4ABC} &= \gamma\,({}^+\bar{R}_{4ABC} + v\,{}^+\bar{R}_{A1BC})\ . \end{aligned} \tag{4.111} $$

Finally we have the set of transformed components which do not involve the γ factor:

$$^{+}R_{1414} = {^{+}\bar{R}_{1414}} \ , \qquad {^{+}R_{1423}} = {^{+}\bar{R}_{1423}} \ , \qquad {^{+}R_{2323}} = {^{+}\bar{R}_{2323}} \ . \quad (4.112)$$

In the right hand sides of (4.110)–(4.112) the components $^{+}\bar{R}_{\mu\nu\rho\sigma}$ are substituted from (4.106) and then the resulting quantities are written in terms of the coordinates $\{x, y, z, t\}$ using (4.109). There is one relation between these components:

$$^{+}R_{1234} + {^{+}R_{1423}} + {^{+}R_{1342}} = 0 \ . \qquad (4.113)$$

We now take the limit $v \to 1$. The gravitational field of the boosted source is given by

$$^{+}\tilde{R}_{\mu\nu\rho\sigma} = \lim_{v\to 1} {^{+}R_{\mu\nu\rho\sigma}} \ . \qquad (4.114)$$

Since the Kerr space–time is a vacuum space–time its Ricci tensor $\bar{R}_{\mu\nu} = 0$ *and hence the Lorentz transformed Ricci tensor* $R_{\mu\nu} = 0$ *and therefore* $\tilde{R}_{\mu\nu} = 0$. In this limit the rest–mass $m \to 0$ as $\gamma \to \infty$ in such a way that the relative energy $m\gamma = p$ (say) remains finite. For a source with multipole moments we have seen in §4.2 that all moments behave like the monopole moment m in the limit $v \to 1$ in the case of the transverse boost. Thus the parameter a is unaffected by the boost in this case. To evaluate the limit (4.114) explicitly we need the following:

$$\lim_{v\to 1} \frac{\gamma\,\bar{r}^3}{(\bar{r}^2 + ia\bar{z})^3} = \frac{2\,\delta(x-t)}{y^2 + (z+ia)^2} \ , \qquad (4.115)$$

and

$$\lim_{v\to 1} \frac{\gamma\,\bar{r}^5}{(\bar{r}^2 + ia\bar{z})^5} = \frac{4}{3}\,\frac{\delta(x-t)}{(y^2 + (z+ia)^2)^2} \ . \qquad (4.116)$$

To establish (4.115) we note that (4.100) and (4.109) can be used to write

$$\frac{\gamma\,\bar{r}^3}{(\bar{r}^2 + ia\bar{z})^3} = \frac{1}{(y^2 + (z+ia)^2)}\,\frac{\partial}{\partial x}\left(\frac{(x-vt)\,\bar{R}}{\bar{R}^2 + i\gamma^{-2}az}\right) \ . \qquad (4.117)$$

The quantity $\bar{R}$ is defined as follows: first we have from (4.100) that

$$\bar{r}^2 = \frac{1}{2}\{r^2 - a^2 + \sqrt{(r^2 - a^2)^2 + 4a^2z^2}\} \ , \qquad (4.118)$$

with $r^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 = \gamma^2(x - vt)^2 + y^2 + z^2$. Thus we have

$$r = \gamma R \ , \qquad R = \sqrt{(x - vt)^2 + \gamma^{-2}(y^2 + z^2)} \ , \qquad (4.119)$$

and so (4.118) yields

$$\bar{r} = \gamma\,\bar{R}\ , \qquad \bar{R}^2 = \frac{1}{2}\{R^2 - \gamma^{-2}a^2 + \sqrt{(R^2 - \gamma^{-2}a^2)^2 + 4\gamma^{-4}a^2z^2}\}\ , \tag{4.120}$$

giving us $\bar{R}$. With $\bar{R}$ and R now defined we see from (4.117) that

$$\lim_{v\to 1} \frac{\gamma\,\bar{r}^3}{(\bar{r}^2 + ia\bar{z})^3} = \frac{1}{(y^2 + (z + ia)^2)}\frac{\partial}{\partial x}\left(\frac{x-t}{|x-t|}\right)\ . \tag{4.121}$$

Using (4.9) we obtain (4.115). By (4.100) and (4.109) we have

$$\frac{\partial \bar{r}}{\partial y} = \frac{y\,\bar{r}^3}{\bar{r}^4 + a^2\bar{z}^2}\ , \tag{4.122}$$

and using this we find that (4.116) follows from (4.115) by differentiating (4.115) with respect to y and taking $y \neq 0$. When $a = 0$ we note that (4.115) and (4.116) reduce to (4.10) and (4.27) respectively.

We can now evaluate the limits (4.114). As an illustration we find

$$^+\tilde{R}_{1212} = \lim_{v\to 1}\left[\frac{3m\gamma^2\bar{r}^5}{(\bar{r}^2 + ia\bar{z})^5}\{z + i(a+y)\}^2\right] = 4p\left[\frac{z + i(a+y)}{y^2 + (z+ia)^2}\right]^2 \delta(x-t)\ , \tag{4.123}$$

by (4.116). After some algebra this can be rewritten as

$$^+\tilde{R}_{1212} = 4p\left[\frac{z + i(y-a)}{z^2 + (y-a)^2}\right]^2 \delta(x-t)\ , \tag{4.124}$$

and if

$$H = 2p\,\log\{(y-a)^2 + z^2\}\,\delta(x-t)\ , \tag{4.125}$$

we finally have

$$^+\tilde{R}_{1212} = (H_{yy} - iH_{yz})\ . \tag{4.126}$$

The subscripts on H denote second partial derivatives. In similar fashion we find that $\tilde{R}_{\mu\nu\rho\sigma} \equiv 0$ except for

$$\begin{aligned} ^+\tilde{R}_{1212} &= {}^+\tilde{R}_{2424} = -{}^+\tilde{R}_{1313} = -{}^+\tilde{R}_{3434} = -{}^+\tilde{R}_{3134} = {}^+\tilde{R}_{2124} \\ &= (H_{yy} - iH_{yz})\ , \end{aligned} \tag{4.127}$$

and

$$\begin{aligned} ^+\tilde{R}_{1213} &= {}^+\tilde{R}_{2434} = {}^+\tilde{R}_{3124} = {}^+\tilde{R}_{2134} \\ &= i\,(H_{yy} - iH_{yz})\ , \end{aligned} \tag{4.128}$$

When the Lorentz transformation (4.109) is applied to the line–element (4.98) and the limit $v \to 1$ taken we find that *if* $x - t > 0$ *then*

$$\lim_{v \to 1} ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{8\,p}{x-t}\,(dx - dt)^2\ , \tag{4.129}$$

and *if* $x - t < 0$ *then*

$$\lim_{v \to 1} ds^2 = dx^2 + dy^2 + dz^2 - dt^2\ . \tag{4.130}$$

For $x - t > 0$ we can write (4.129) in the form

$$ds_+^2 = dy_+^2 + dz_+^2 - 2\,du\,dv_+\ , \tag{4.131}$$

with

$$\begin{aligned} y_+ &= y \qquad z_+ = z\ , \qquad u = t - x\ ,\\ v_+ &= \frac{1}{2}\,(x+t) + 4\,p\,\log(x-t)\ . \end{aligned} \tag{4.132}$$

For $x - t < 0$ we can write (4.130) in the form

$$ds_-^2 = dy_-^2 + dz_-^2 - 2\,du\,dv_-\ , \tag{4.133}$$

with

$$\begin{aligned} y_- &= y \qquad z_- = z\ , \qquad u = t - x\ ,\\ v_- &= \frac{1}{2}\,(x+t)\ . \end{aligned} \tag{4.134}$$

By (4.131) and (4.133) we see that $u = 0$ is a null hyper*plane* in Minkowskian space–time. The line–elements (4.131) and (4.133) are consistent with having a delta function in the Riemann curvature tensor which is singular on $x = t$ provided the two halves of Minkowskian space–time , $x > t$ and $x < t$, are attached on $x = t$ with

$$y_+ = y_-\ , \qquad z_+ = z_-\ , \qquad v_+ = F(v_-, y_-, z_-)\ , \tag{4.135}$$

for some function F defined on $x = t$ for which $\partial F/\partial v_- \neq 0$. The particular function F which gives rise to the coefficients of the delta function in the Riemann tensor components listed in (4.127) and (4.128) can be calculated from the formulas given in §2.4 . It is easily found to be

$$v_+ = F = v_- + 2\,p\,\log\{(y-a)^2 + z^2\}\ . \tag{4.136}$$

The signal with history $x = t$ is an impulsive gravitational wave with a delta function profile and which is, in addition, singular on the null geodesic generator $y = a$, $z = 0$ of the null hyperplane $x = t$. This is identical to the light-like boosted Schwarzschild field [Aichelburg and Sexl (1971)] except that the singular generator is shifted from $y = 0$, $z = 0$ in that case to $y = a$, $z = 0$. The physical significance of this shifted generator is discussed in §4.4 below.

Once F in (4.136) is known the line-element of the space-time can be written in the form (2.63) in coordinates in which the metric tensor is continuous across $x = t$. For the light-like boosted Kerr field the resulting line-element can then be transformed into Kerr-Schild form at the expense of introducing the delta function into the metric tensor. This line-element is

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 - 2\,H\,(dx - dt)^2 \ , \tag{4.137}$$

with H given by (4.125). The corresponding Riemann tensor (4.127) and (4.128) is type N in the Petrov classification with degenerate principal null direction given via the 1-form $dx - dt$.

The light-like boost of the Kerr gravitational field described here involves a boost *transverse* to the symmetry axis of the Kerr source. To complete the picture we consider briefly the case of a light-like boost in the $-\bar{z}$-direction which is parallel to the symmetry axis. To motivate our approach to this we expand F given in (4.136) for the transverse light-like boost in powers of the parameter a. Thus we arrive at

$$F = v_- + 2\sum_{l=0}^{\infty}(-1)^l\,\frac{p_l}{l!}\,\frac{\partial^l}{\partial y^l}\left(\log(y^2+z^2)\right) \ , \tag{4.138}$$

with $p_l = p\,a^l$ for $l = 0, 1, 2, \ldots$. This is similar to the matching function F encountered in the transverse light-like boost of a static axially symmetric multipole field (4.94) whose original multipole moments, before taking the limit $v \longrightarrow 1$, were

$$A_l = m\,a^l \ , \qquad l = 0, 1, 2, \ldots \ , \tag{4.139}$$

and with $m = p\,\gamma^{-1}$. For the light-like boost in the parallel or $-\bar{z}$-direction (the direction of the symmetry axis) the analogy with the multipole field suggests that we take $a = \gamma^{-1}\hat{a}$ as $v \longrightarrow 1$ in this case, aswell as taking $m = p\,\gamma^{-1}$. Then (4.139) is replaced by

$$A_l = \gamma^{-l-1}p_l \ , \qquad p_l = p\,\hat{a}^l \ , \tag{4.140}$$

for $l = 0, 1, 2, \ldots$ as in the case of a light-like boost parallel to the symmetry axis in (4.49). Now make the Lorentz transformation

$$\bar{x} = x \ , \qquad \bar{y} = y \ , \qquad \bar{z} = \gamma\,(z - vt) \ , \qquad \bar{t} = \gamma\,(t - vz) \ . \qquad (4.141)$$

The components ${}^{+}\bar{R}_{\mu\nu\rho\sigma}$ in (4.106) are transformed to ${}^{+}R_{\mu\nu\rho\sigma}$. In this case the equations (4.110)and (4.111) are replaced by the same equations, with the subscript 1 replaced by the subscript 3 and with A, B, C each taking the values $1, 2$. Also the subscripts 1 and 3 are interchanged in equations (4.112). The relation (4.113) continues to hold. Now (4.116) is replaced by

$$\lim_{v\to 1} \frac{\gamma\,\bar{r}^5}{(\bar{r}^2 + ia\bar{z})^5} = \lim_{v\to 1} \frac{\gamma\,\bar{r}^5}{\{\bar{r}^2 + i\hat{a}(z - vt)\}^5} = \lim_{v\to 1} \frac{\gamma^{-4}}{\tilde{R}^5} = \frac{4}{3}\,\frac{\delta(z - t)}{(x^2 + y^2)^2} \ , \qquad (4.142)$$

where $\tilde{R} = \sqrt{(z - vt)^2 + \gamma^{-2}(x^2 + y^2)}$. The last equality in (4.142) is derived as (4.27). Calculating ${}^{+}\tilde{R}_{\mu\nu\rho\sigma}$ in (4.114) in this case we find that the non–identically vanishing components are

$$\begin{aligned} {}^{+}\tilde{R}_{2323} &= {}^{+}\tilde{R}_{2424} = -{}^{+}\tilde{R}_{2324} = -{}^{+}\tilde{R}_{3131} = -{}^{+}\tilde{R}_{1414} = {}^{+}\tilde{R}_{1314} \\ &= \frac{4\,p\,(x - iy)^2}{(x^2 + y^2)^2}\,\delta(z - t) \ , \end{aligned} \qquad (4.143)$$

and

$${}^{+}\tilde{R}_{3231} = {}^{+}\tilde{R}_{2414} = -{}^{+}\tilde{R}_{1324} = -{}^{+}\tilde{R}_{2314} = -\frac{4ip\,(x - iy)^2}{(x^2 + y^2)^2}\,\delta(z - t) \ . \qquad (4.144)$$

This is the same field as the light–like boosted Schwarzschild field derived in §4.2 . The generalisation of the calculations given here to a light–like boost of the Kerr gravitational field in an arbitrary direction involves tedious algebra but is straightforward and would result in an explicit limit for the Riemann tensor.

4.4 Deflection of Highly Relativistic Particles in the Kerr Field

To apply the transverse light–like boost calculated in §4.3 to study the deflection of highly relativistic particles in the Kerr field we proceed as follows: Let $\bar{S}$ be the distant inertial frame with respect to which the Kerr source is not in translational motion and let S be the inertial frame of a distant high speed test particle projected into the Kerr field. The relationship between $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ in $\bar{S}$ and (x, y, z, t) in S is taken to be given by the Lorentz transformation

$$\bar{x} = \gamma\,(x + v\,t)\ ,\ \bar{y} = y\ ,\ \bar{z} = z\ ,\ \bar{t} = \gamma\,(t + v\,x)\ , \tag{4.145}$$

with $\gamma = (1-v^2)^{-1/2}$. We shall work in an approximation in which v is close to unity. To the high speed observer in the frame $\bar{S}$ the Kerr gravitational field resembles, to a high degree of approximation, the field of a plane impulsive gravitational wave. The space–time model has line–element

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 - 2\,H\,(dx + dt)^2\ , \tag{4.146}$$

with

$$H = h(y,z)\,\delta(x+t)\ , \qquad h(y,z) = 2\,p\,\log\{(y-a)^2 + z^2\}\ . \tag{4.147}$$

The time–like world–line $x^\mu = x^\mu(s)$ of a test particle in the space–time with line–element given by (4.146) and (4.147) is given by the geodesic equations

$$\begin{aligned}
\ddot{x} &= H_x(\dot{x}+\dot{t})^2 + 2\,(h_y\,\dot{y} + h_z\,\dot{z})\,(\dot{x}+\dot{t})\,\delta(x+t)\ ,\\
\ddot{y} &= -h_y\,(\dot{x}+\dot{t})^2\delta(x+t)\ ,\\
\ddot{z} &= -h_z\,(\dot{x}+\dot{t})^2\delta(x+t)\ ,\\
\ddot{t} &= -H_x(\dot{x}+\dot{t})^2 - 2\,(h_y\,\dot{y} + h_z\,\dot{z})\,(\dot{x}+\dot{t})\,\delta(x+t)\ .
\end{aligned} \tag{4.148}$$

The dot indicates differentiation with respect to proper–time s. The subscripts on H and h indicate partial derivatives with, in particular, $H_x = h(y,z)\,\delta'(x+t)$ with the prime denoting differentiation of the delta function with respect to its argument. Following from (4.148) we have the first integral

$$(\dot{x}+\dot{t})\,\{\dot{x} - \dot{t} - 2\,H\,(\dot{x}+\dot{t})\} + \dot{y}^2 + \dot{z}^2 = -1\ . \tag{4.149}$$

Adding the first and last equations in (4.148) leads to $\ddot{x} + \ddot{t} = 0$ and so we have

$$x + t = C\,s\ , \tag{4.150}$$

where C is a constant of integration and we choose the point of intersection of the world–line of the test particle with the null hyperplane $x + t = 0$ to correspond to $s = 0$. We also take $s < 0$ to the past of $x + t = 0$ and $s > 0$ to the future of $x + t = 0$. We can now substitute into (4.148) $\delta(x+t) = C^{-1}\delta(s)$ and $\delta'(x+t) = C^{-2}\delta'(s)$. It follows from (4.148) that for $s > 0$ and for $s < 0$ the time–like world–line we are seeking is a goedesic of

Minkowskian space–time with line– element given by (4.146) with $H = 0$. As a result and in view of the form of the right hand sides of (4.148) we look for solutions of (4.148) and (4.149) which take the form

$$\begin{aligned} x &= x_0 + x_1 s + X_1 s\,\theta(s) + \hat{X}_1\theta(s) \ , \\ y &= y_0 + y_1 s + Y_1 s\,\theta(s) \ , \\ z &= z_0 + z_1 s + Z_1 s\,\theta(s) \ , \\ t &= -x_0 + t_1 s - X_1 s\,\theta(s) - \hat{X}_1\theta(s) \ , \end{aligned} \tag{4.151}$$

with the coefficients of $s^0, s, s\,\theta(s)$ and $\theta(s)$ here all constants. We note that when $s = 0$ we have $y = y_0$, $z = z_0$ and of course $x + t = 0$. In addition (4.150) and (4.151) imply that

$$C = x_1 + t_1 \ . \tag{4.152}$$

Substituting (4.151) into (4.149) gives

$$\begin{aligned} \hat{X}_1 &= h(y_0, z_0) \equiv (h)_0 \ , \\ x_1^2 + y_1^2 + z_1^2 - t_1^2 &= -1 \ , \\ 2\,C\,X_1 + 2\,y_1 Y_1 + 2\,z_1 Z_1 + Y_1^2 + Z_1^2 &= 0 \ . \end{aligned} \tag{4.153}$$

Now the first of (4.148) and also the fourth of (4.148) are satisfied (verification of this involves using $\theta(s)\,\delta(s) = \frac{1}{2}\delta(s)$, which is distributionally valid) while the second and third equations in (4.148) yield

$$Y_1 = -C\ (h_y)_0 \ , \qquad Z_1 = -C\ (h_z)_0 \ . \tag{4.154}$$

As in the first of (4.153) the brackets around a quantity followed by a subscript zero denote that the quantity is evaluated at $y = y_0, z = z_0$ which corresponds to $s = 0$. By (4.154) we can rewrite the last of (4.153) as

$$X_1 = (h_y)_0\, y_1 + (h_z)_0\, z_1 - \frac{C}{2}\left\{(h_y)_0^2 + (h_z)_0^2\right\} \ . \tag{4.155}$$

It is essential to have the discontinuous term present in (4.151). To see this we first note from (3.196) the simple limits

$$\begin{aligned} &\lim_{s\to 0^-} x = x_0 \ , \qquad \lim_{s\to 0^-} t = -x_0 \ , \\ &\lim_{s\to 0^+} x = x_0 + \hat{X}_1 \ , \qquad \lim_{s\to 0^+} t = -x_0 - \hat{X}_1 \ . \end{aligned} \tag{4.156}$$

Using these we have

$$\lim_{s\to 0^+}(x+t) = 0 = \lim_{s\to 0^-}(x+t)\ , \tag{4.157}$$

and

$$\lim_{s\to 0^+}\left(\frac{x-t}{2}\right) = x_0 + \hat{X}_1\ , \qquad \lim_{s\to 0^-}\left(\frac{x-t}{2}\right) = x_0\ . \tag{4.158}$$

Now $v_+ = \lim_{s\to 0^+}\left(\frac{x-t}{2}\right)$ and $v_- = \lim_{s\to 0^-}\left(\frac{x-t}{2}\right)$ are affine parameters along the generators of $x+t=0$ ($\Leftrightarrow s=0$) on the future (plus) side and on the past (minus) side respectively and by (4.158) they are related by

$$v_+ = v_- + \hat{X}_1 = v_- + (h)_0\ . \tag{4.159}$$

This is precisely the matching condition (4.136) required of these affine parameters on the null hyperplane $x+t=0$ for this null hyperplane to be the history of a gravitational wave. For this reason the presence of the discontinuous $\hat{X}_1$–terms in (4.151) is not surprising.

To set up the deflection problem for a high speed test particle projected with 3–velocity v in the positive $\bar{x}$–direction we take for $s<0$ in $\mathcal{S}$: $x_0 = 0\ , x_1 = 0\ , y_1 = 0\ , z_1 = 0\ , t_1 = 1$ and thus by (4.152) $C=1$. Hence (4.151) gives $x = 0\ , y = y_0\ , z = z_0\ , t = s$. Using the Lorentz transformation (4.145), with as always v assumed close to unity, we have in $\bar{\mathcal{S}}$ for $s<0$

$$\bar{x} = \gamma\, v\, s\ , \qquad \bar{y} = y_0\ , \qquad \bar{z} = z_0\ , \qquad \bar{t} = \gamma\, s\ . \tag{4.160}$$

Now for $s>0$ in $\mathcal{S}$ we have

$$x = X_1 s + \hat{X}_1\ , \qquad y = y_0 + Y_1 s\ , \qquad z = z_0 + Z_1 s\ , \qquad t = (1 - X_1)\, s - \hat{X}_1\ , \tag{4.161}$$

while in $\bar{\mathcal{S}}$ for $s>0$

$$\begin{aligned} \bar{x} &= \gamma\left\{(1-v)\hat{X}_1 + [v + (1-v)\,X_1]\,s\right\}\ , \\ \bar{y} &= y_0 + Y_1 s\ , \\ \bar{z} &= z_0 + Z_1 s\ , \\ \bar{t} &= \gamma\left\{(v-1)\hat{X}_1 + [1 - (1-v)\,X_1]\,s\right\}\ . \end{aligned} \tag{4.162}$$

If α is the angle of deflection of the high speed test particle out of the $\bar{x}\,\bar{z}$–plane for $s>0$ and if β is the angle of deflection out of the $\bar{x}\,\bar{y}$–plane for $s > 0$ then these angles are given by

$$\tan\alpha = \frac{\dot{\bar{y}}}{\sqrt{\dot{\bar{x}}^2 + \dot{\bar{z}}^2}} = \frac{Y_1}{\{Z_1^2 + \gamma^2[X_1\,(1-v) + v]^2\}^{1/2}}\ , \tag{4.163}$$

and

$$\tan\beta = \frac{\dot{\bar{z}}}{\dot{\bar{x}}} = \frac{Z_1}{\gamma\,[X_1\,(1-v) + v]}\ , \tag{4.164}$$

where now (4.154) and (4.155) reduce to

$$\begin{aligned} X_1 &= -\frac{1}{2}\left[(h_y)_0^2 + (h_z)_0^2\right]\ , \\ Y_1 &= -\,(h_y)_0\ , \qquad Z_1 = -\,(h_z)_0\ . \end{aligned} \tag{4.165}$$

The equations (4.163) and (4.164) hold for v close to unity. When (4.147) is substituted into (4.163) and (4.164) and we put $p = m\,\gamma$ (as indicated following (4.114)) we obtain the dominant terms by taking v close to unity. In this case the deflection angles α and β are given by

$$\tan\alpha = \frac{-4m\,(y_0 - a)}{\left\{\left[(y_0 - a)^2 + z_0^2 - 4m^2\right]^2 + 16m^2 z_0^2\right\}^{1/2}}\ , \tag{4.166}$$

and

$$\tan\beta = \frac{-4m\,z_0}{(y_0 - a)^2 + z_0^2 - 4m^2}\ . \tag{4.167}$$

If a particle is projected from a point in the equatorial plane ($z_0 = 0$) then (4.167) shows that $\beta = 0$ and thus the particle will remain in the equatorial plane. We see from (4.166) that $\alpha = \pi/2$ corresponds to $y_0 = \pm 2m + a$. This is the value of y_0 corresponding to the capture of the incoming particle in this case. If such a particle is projected from a point on the y–axis at which $y_0 = -\eta$ ($\eta > 0$) then the angle of deflection α is given by

$$\tan\alpha = \frac{4\,m\,(\eta + a)}{[(\eta + a)^2 - 4m^2]}\ . \tag{4.168}$$

This formula is most accurate for large values of the impact parameter η and in this case it gives the small deflection angle

$$\alpha = \frac{4\,m}{\eta} - \frac{4\,m\,a}{\eta^2}\ . \tag{4.169}$$

We note that the leading term in (4.169) is Einstein's small angle $\alpha_0 = 4m/\eta$ of deflection of light in the Schwarzschild field. The effect of the second term in (4.169) is to *decrease* the deflection angle from α_0 if $a > 0$ and to *increase* the deflection angle from α_0 if $a < 0$. For the original Kerr source $a > 0$ corresponds to angular momentum in the positive z–direction. If $a > 0$ then the singularity $y = a$, $z = 0$ in the impulsive gravitational wave front is further from the incoming particle than it would be in the Schwarzschild case ($a = 0$) and thus the deflection is decreased from its Schwarzschild value. If $a < 0$ the singularity is closer to the incoming particle and the deflection is increased.

4.5 The Exceptional Case of a Cosmic String

A cosmic string is a line source characterized by a conical singularity. The Riemann tensor of the space–time model vanishes and thus boosting a cosmic string to the speed of light has to be carried out on the metric tensor. However the important quantity in this case is the conical singularity and the effect of the light–like boost on the corresponding 'conical deficit' [Barrabès, Hogan and Israel (2002)].

For an infinite straight cosmic string at rest and extending along the $\bar{z}$-axis, the metric takes the form

$$ds^2 = -d\bar{t}^2 + d\bar{\rho}^2 + a^2\bar{\rho}^2 d\bar{\phi}^2 + d\bar{z}^2\,. \tag{4.170}$$

Here, $\bar{\phi}$ runs from 0 to 2π, and the parameter a takes account of the *conical deficit* $\bar{\Delta}\bar{\phi}$, given by

$$\frac{\bar{\Delta}\bar{\phi}}{2\pi} = 1 - a = 4\,\bar{\mu}\,, \tag{4.171}$$

where $\bar{\mu}$ is the mass per unit length of the string. In terms of the rectangular co-ordinates

$$\bar{x} = \bar{\rho}\cos\bar{\phi} \qquad , \qquad \bar{y} = \bar{\rho}\sin\bar{\phi}\,, \tag{4.172}$$

the metric (4.170) reads

$$ds^2 = -d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 - (1-a^2)\frac{(\bar{x}\,d\bar{y} - \bar{y}\,d\bar{x})^2}{\bar{x}^2 + \bar{y}^2}\,. \tag{4.173}$$

We now boost the string sideways along the $\bar{x}$-direction, by making the co-ordinate transformation

$$\bar{t} = t\,\cosh\chi - x\,\sinh\chi = \frac{u}{2}\,e^{\chi} + \frac{v}{2}\,e^{-\chi}\,,$$

$$\bar{x} = x \cosh\chi - t \sinh\chi = -\frac{u}{2}\,\mathrm{e}^{\chi} + \frac{v}{2}\,\mathrm{e}^{-\chi}\,,$$
$$\bar{y} = y, \qquad \bar{z} = z\ . \tag{4.174}$$

Here $u = t - x$, $v = t + x$ are the plane light–like co-ordinates. The flat part of the line element (4.173) (the first four terms) then becomes

$$-du\,dv + dy^2 + dz^2\,. \tag{4.175}$$

In the limit ($\chi \to +\infty$), the boost (4.174) becomes light–like. On noting that

$$x\,dy - y\,dx = \frac{e^{\chi}}{2}(y\,du - u\,dy) + O(e^{-\chi})\,,$$
$$x^2 + y^2 = \frac{e^{2\chi}}{4}(u^2 + b^2) + O(e^{-2\chi})\ , \tag{4.176}$$

where

$$b^2 = 4\,e^{-2\chi}(y^2 - \frac{1}{2}\,u\,v)\,, \tag{4.177}$$

the last term of (4.173) reduces to

$$-(1 - a^2)\frac{(y\,du - u\,dy)^2}{u^2 + b^2} + O(e^{-2\chi})\,. \tag{4.178}$$

The linear mass density of the string measured in the laboratory frame is

$$\mu = \bar{\mu}\cosh\chi = \frac{1}{4}\,(1 - a)\cosh\chi\,. \tag{4.179}$$

To obtain a regular limiting geometry we must require $\bar{\mu}$ to remain bounded as $\chi \to +\infty$. This introduces an additional factor $e^{-\chi}$ into (4.178) through the coefficient $(1 - a^2)$. From the identity

$$\lim_{b \longrightarrow 0} \frac{b}{u^2 + b^2} = \pi\delta(u)\,, \tag{4.180}$$

it follows by (4.177) that

$$\lim_{\chi \longrightarrow +\infty} \frac{e^{-\chi}}{u^2 + b^2} = \frac{\pi}{2|y|}\delta(u)\,. \tag{4.181}$$

Inserting this into (4.178) and combining with (4.175), we obtain the final limiting form of the boosted metric

$$d\tilde{s}^2 = \lim_{\chi \longrightarrow +\infty} ds^2 = -du\,dv + dy^2 + dz^2 - 8\,\pi\,\mu\,|y|\,\delta(u)\,du^2\,. \tag{4.182}$$

This represents the geometry of an infinite straight light–like string whose world-sheet is the 2-flat

$$u = t - x = 0 \qquad , \qquad y = 0\,, \tag{4.183}$$

and whose mass per unit length measured in the laboratory frame is μ.

We shall now show that (4.182) describes a conical singularity along the z-axis moving at the speed of light in the x- direction, with angular deficit $\Delta\phi$ related to the line density μ in the laboratory frame by an equation identical in form to (4.171). There are two ways of approach to this. We can start from the conical curvature singularity of metric (4.170) representing the original time–like string in its rest-frame and apply the boost (4.174). Alternatively we can work directly from (4.182). This is a metric of Kerr-Schild form. Both approaches are of interest and we shall consider them in turn.

In the first approach we begin with the static string line-element (4.170) or (4.173). The angular deficit $\bar{\Delta}\bar{\phi}$ of a 2-plane $\bar{\mathcal{S}}_2$ of constant $\bar{z}$ and $\bar{t}$ is associated with the distributional curvature

$${}^{(2)}R = 2\,\bar{\Delta}\bar{\phi}\,\delta_2\,, \tag{4.184}$$

where

$$\delta_2 = {}^{(2)}g^{-1/2}\,\delta(\bar{x})\,\delta(\bar{y}) = \frac{1}{a}\,\delta(\bar{x})\,\delta(\bar{y})\ , \tag{4.185}$$

is the invariant two-dimensional delta-function concentrated at the origin. To check (4.184), note that for Gaussian polar co-ordinates

$$ds^2_{(2)} = d\bar{\rho}^2 + f^2(\bar{\rho},\bar{\phi})d\bar{\phi}^2\ , \tag{4.186}$$

and the Gaussian curvature is

$${}^{(2)}R = -\frac{2}{f}\,\partial^2_{\bar{\rho}} f(\bar{\rho},\bar{\phi})\,. \tag{4.187}$$

We temporarily smooth out the conical singularity at $\bar{\rho} = 0$ by choosing f to be any smooth function $f_\epsilon(\bar{\rho})$ satisfying the conditions

$$f_\epsilon(\bar{\rho}) = a\bar{\rho} \qquad (\bar{\rho} \geq \epsilon) \qquad , \qquad \lim_{\bar{\rho}\longrightarrow 0} \frac{f_\epsilon(\bar{\rho})}{\bar{\rho}} = 1\,. \tag{4.188}$$

Then

$$\begin{aligned}\int\!\!\int {}^{(2)}R\,{}^{(2)}g^{1/2} d\bar{\rho}\,d\bar{\phi} &= -\,4\,\pi \int_0^\epsilon (\partial^2_{\bar{\rho}} f_\epsilon) d\bar{\rho} \\ &= 4\,\pi\,(1 - f'_\epsilon(\epsilon)) = 4\,\pi\,(1-a)\ . \end{aligned} \tag{4.189}$$

The conical deficit $\bar{\Delta}\bar{\phi}$ is defined by

$$2\pi - \bar{\Delta}\bar{\phi} = \lim_{\epsilon\longrightarrow 0}\left(\frac{\text{circumference}}{\text{radius}}\right)_{\bar{\rho}=\epsilon} = \lim_{\epsilon\longrightarrow 0}\frac{f_\epsilon(\epsilon)}{\epsilon} = 2\pi\, a\ . \qquad (4.190)$$

Combining these results leads to (4.184).

The Ricci tensor of the $\bar{x}\,\bar{y}$-plane $\bar{S}_2$ is, by (4.184),

$$^{(2)}R_{ab} = \bar{\Delta}\bar{\phi}\,^{(2)}g_{ab}\,\delta_2\ , \qquad (4.191)$$

in which the 2-metric $^{(2)}g_{ab}$ can be decomposed as

$$^{(2)}g_{ab} = \bar{\rho}_{,a}\,\bar{\rho}_{,b} + a^2\bar{\rho}^2\bar{\phi}_{,a}\,\bar{\phi}_{,b}\ . \qquad (4.192)$$

Since the extrinsic curvature of $\bar{S}_2$ is zero, the four-dimensional Ricci tensor of the metric (4.170) can be read off at once from (4.191). It is

$$^{(4)}R_{\alpha\beta} = \frac{\bar{\Delta}\bar{\phi}}{a}\,\delta(\bar{x})\,\delta(\bar{y})\,(\bar{\rho}_{,\alpha}\,\bar{\rho}_{,\beta} + a^2\bar{\rho}^2\bar{\phi}_{,\alpha}\,\bar{\phi}_{,\beta})\ , \qquad (4.193)$$

in which Greek indices are 4-dimensional and a comma denotes partial differentiation.

Under the boost (4.174), we have for large χ,

$$\bar{\rho} \simeq |\bar{x}| \simeq \frac{e^\chi}{2}|u|\ , \qquad \bar{\phi} = \tan^{-1}\frac{y}{x} = O(e^{-\chi})\ , \qquad \delta(\bar{x}) \simeq 2e^{-\chi}\delta(u)\ . \qquad (4.194)$$

We rescale $\bar{\Delta}\bar{\phi}$ to a new parameter $\Delta\phi$ by analogy with (4.179),

$$\Delta\phi = \bar{\Delta}\bar{\phi}\,\cosh\chi\ , \qquad (4.195)$$

and require $\Delta\phi$ to stay bounded as $\chi \to +\infty$. In this limit the second term of (4.193) becomes negligible, $a \to 1$ by (4.171), and we obtain

$$^{(4)}R_{\alpha\beta} = \Delta\phi\,\delta(u)\,\delta(y)\,u_{,\alpha}\,u_{,\beta}\ . \qquad (4.196)$$

Thus the stress-energy tensor

$$T_{\alpha\beta} = \frac{\Delta\phi}{8\pi}\,\delta(u)\,\delta(y)\,u_{,\alpha}\,u_{,\beta}\ , \qquad (4.197)$$

is indeed that of a distributional light–like source having as its history in space–time the null 2-flat $u = y = 0$, and with mass per unit length

$$\mu = \frac{\Delta\phi}{8\pi}\ , \qquad (4.198)$$

measured in the laboratory frame.

To establish the role of $\Delta\phi$ in (4.196) as an angular deficit in the laboratory frame, consider the 2-space $\mathcal{S}_2$ of constant z and t in the geometry (4.182). Recalling that $u = t - x$, we obtain from (4.196),

$$ {}^{(4)}R_{x\,x} = \Delta\phi\,\delta(x)\,\delta(y)\,. \tag{4.199}$$

Now, $u_{,\alpha}$ is light–like and geodesic for the flat background of the Kerr-Schild metric (4.182), and therefore retains these properties with respect to the full metric. From this geodesic property and the Gauss-Codazzi equations it follows that ${}^{(4)}R_{x\,x} = {}^{(2)}R_{x\,x}$. Thus

$$\frac{1}{2}\,{}^{(2)}R = {}^{(2)}R_{x\,x} = {}^{(4)}R_{x\,x} = \Delta\phi\,\delta(x)\,\delta(y)\,. \tag{4.200}$$

According to (4.184), this is indeed the distributional curvature in the $x\,y$-plane corresponding to an angular deficit $\Delta\phi$.

The alternative route to these results is to proceed directly from the Kerr-Schild metric (4.182) for the light–like string. Starting with the line–element

$$d\tilde{s}^2 = -du\,dv + dy^2 + dz^2 + 2H(u,v,y,z)\,du^2\ , \tag{4.201}$$

the Ricci tensor is

$${}^{(4)}R_{\alpha\beta} = (8HH_{,vv} - \nabla^2 H)u_{,\alpha}\,v_{\beta} - 4H_{,vv}\,u_{,(\alpha}\,v_{,\beta)} - 4H_{,va}\,e^{(a)}_{(\alpha}\,u_{,\beta)}\ , \tag{4.202}$$

where $\nabla^2 = \partial_y^2 + \partial_z^2$, $x^a = (y,z)$ and $e^{(a)}_{\alpha} = \partial x^a\,/\,\partial x^\alpha$. If we put $H = -4\,\pi\,\mu\,|y|\,\delta(u)$ from (4.182), so that $\nabla^2 H = -8\,\pi\,\mu\,\delta(y)\,\delta(u)$, we obtain

$${}^{(4)}R_{\alpha\beta} = 8\,\pi\,\mu\,\delta(y)\,\delta(u)\,u_{,\alpha}\,u_{,\beta}\ , \tag{4.203}$$

in agreement with (4.196) and (4.198).

Chapter 5

Spherically Symmetric Null Shells

The physical peculiarities of light–like shells and the efficacy of the method developed in Chapter 2 are well illustrated in the simple context of spherical symmetry. Spherically symmetric null shells yield models which can conveniently be used in the study of a number of phenomena, one of the most popular being gravitational collapse. They have also been utilised to provide classical models for quantum processes in the case of some aspects of black–hole physics such as Hawking radiation and the entropy of a black–hole. We give a general description, followed by a few illustrative examples, of non–stationary spherical null shells. Then we consider the case where the shell is located on an event or cosmological horizon which is common to two spherically symmetric space–times. Another interesting application of null shells concerns the interaction of matter and gravitational field. This problem, which is in general difficult for arbitrary sources, can be simplified by reducing it to the interaction (collision) between two null shells. It can be solved analytically with the assumption of spherical symmetry. Finally we show how some models of collapsing null shells can be used, on the one hand to obtain an upper bound on the amount of gravitational radiation emitted during the collapse of a convex body, and on the other hand to shed some light on the hoop conjecture

5.1 Properties and Examples of Non–Stationary Spherical Null Shells

We take the line–element of a spherically symmetric geometry in the form

$$ds^2 = -e^{\psi}\, du\left(f\, e^{\psi}\, du + 2\zeta\, dr\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\ , \qquad (5.1)$$

where u is the Eddington retarded or advanced time. The sign factor ζ is $+1\,(-1)$ if r increases (decreases) towards the future along a light ray on the light cone u = constant. The functions ψ and f both depend on u and r, and it is convenient to define the local mass function $m(u,r)$ by $f = 1 - 2m/r$. From Einstein's field equations we have

$$\frac{\partial m}{\partial u} = 4\pi r^2\, T^r_u\,, \qquad \frac{\partial m}{\partial r} = -4\pi r^2\, T^u_u\,, \qquad \frac{\partial \psi}{\partial r} = 4\pi r\, T_{rr}\,. \tag{5.2}$$

We consider a null shell whose history is the light cone $\mathcal{N}$ with equation $\Phi(x^\alpha) \equiv u =$ constant, and assume that $\mathcal{N}$ is non-stationary (the stationary case will be dealt with in the next section). The past and future domains $\mathcal{M}^-$ and $\mathcal{M}^+$, each with boundary $\mathcal{N}$, have each a line–element of the form (5.1) with different functions $\psi_\pm$, $f_\pm$ and $m_\pm$. The coordinates $x^\mu = (u, r, \theta, \phi)$ are common to the two sides of $\mathcal{N}$, and the induced metrics match on $\mathcal{N}$. Thus the unique induced line–element on $\mathcal{N}$ is

$$ds^2|_{\mathcal{N}} = r^2\,\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) = g_{ab}\, d\xi^a\, d\xi^b\,. \tag{5.3}$$

Here $\xi^a = (r, \xi^A) = (r, \theta, \phi)$ with $a = 1,2,3$ and $A = 2,3$, are three intrinsic coordinates on $\mathcal{N}$, and $e_{(a)} = \partial/\partial\xi^a$ is the associated holonomic basis of vectors tangent to $\mathcal{N}$. The future-directed light–like normal n is chosen as

$$n^\mu = \zeta e^\mu_{(1)} = \zeta \frac{\partial x^\mu}{\partial r}\,, \tag{5.4}$$

and for the transversal N we take the future-directed light–like vector satisfying $N\cdot N = 0$, $N \cdot n = \eta^{-1} = -1$ and $N \cdot e_{(A)} = 0$. The components of the transversal are thus

$$N^\mu = (e^{-\psi}, -\frac{\zeta}{2}\, f, 0, 0)\,. \tag{5.5}$$

With this choice of n and N their components with respect to the tangent basis $e_{(a)}$ are given by

$$n^a = \zeta\,\delta^a_1\,, \qquad N_a = N\cdot e_{(a)} = -\zeta\,\delta^1_a\,, \tag{5.6}$$

and this leads to an expression for the three-tensor g^{ab}_* introduced in §2.2 which is simply the matrix g^{AB} bordered by zeros. The non-vanishing components of the transverse curvature defined in (2.29) are now

$$\mathcal{K}^\pm_{rr} = \zeta\, \partial_r \psi_\pm\,, \qquad \mathcal{K}^{\theta\pm}_\theta = \mathcal{K}^{\phi\pm}_\phi = -\frac{\zeta}{2r}\, f_\pm\,. \tag{5.7}$$

Note that since r is a common coordinate to $\mathcal{M}^\pm$ the sign factor ζ must be identical on both sides of $\mathcal{N}$. By (2.31) and (2.37) we obtain a surface stress-energy tensor for the shell which has the perfect fluid form

$$S^{ab} = \mu\, n^a\, n^b + P\, g_*^{ab}\,, \tag{5.8}$$

with a surface energy density μ and pressure P given by

$$\mu = \frac{\zeta}{8\pi r}\,[f(r)] = -\frac{\zeta}{4\pi r^2}\,[m(r)]\,, \qquad P = -\frac{\zeta}{8\pi}\,[\partial_r\psi]\,. \tag{5.9}$$

Using (5.2) and introducing the expansion rate $\rho = 2\zeta/r$ of the null generators of $\mathcal{N}$, the last equation in (5.9) becomes $P\,\rho = [T_{\mu\nu}\, n^\mu\, n^\nu]$. This shows that the energy absorbed by the shell from its surroundings is transferred into work done by the surface pressure in dilating or contracting the shell.

Before describing some examples of non–stationary spherical null shells it will be useful to have an operational definition of the surface energy-density and of the surface pressure. We introduce an observer who is radially free-falling with 4–velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = (\dot{u}, \dot{r}, 0, 0)\,, \tag{5.10}$$

and suppose that the world line of the observer intersects the null shell at $\tau = 0$. The momentum normal to the shell $p_\perp \equiv u \cdot n = \dot{u}\, e^\psi$ is continuous across the shell (i.e. $[p_\perp] = 0$), but its tangential component is discontinuous. From (2.21), and noting that in the present case $\eta = -1$ and $\chi = -e^{-\psi}$, the full expression for the stress-energy tensor of the shell is

$$T^{\mu\nu}|_{\mathcal{N}} = S^{\mu\nu}\, e^{-\psi}\, \delta(u)\,. \tag{5.11}$$

Therefore the observed energy density μ_{obs} is defined by $u_\mu\, u_\nu\, T^{\mu\nu}|_{\mathcal{N}} = \mu_{obs}\,\delta(\tau)$ and is given by

$$\mu_{obs} = \mu\,|p_\perp|\,. \tag{5.12}$$

In similar fashion the observed surface pressure P_{obs} is given by

$$P_{obs} = P\,|p_\perp|\,. \tag{5.13}$$

Suppose now that the geometry is static. In this case, if there is no radial energy flow $T_{\mu\nu}\, n^\mu\, n^\nu = T_{rr} = 0$ and we have from (5.2) that $\psi = 0$, and the functions f and m only depend on r. It follows from (5.9) that the surface pressure vanishes in this case. For a Schwarzschild/Schwarzschild junction the energy-density is constant and satisfies $4\,\pi\, r^2 \mu = \zeta(m_- - m_+)$, where $m_\pm$ are the gravitational masses associated with the two domains $\mathcal{M}^\pm$. For an expanding shell ($\zeta = 1$) we then have $m_- > m_+$, with the reverse inequality holding for a contracting shell ($\zeta = -1$). If we now take for $\mathcal{N}$ an expanding light cone separating a Schwarzschild space–time $\mathcal{M}^-$ with $m_- = m$ from a de Sitter space–time $\mathcal{M}^+$ with $m_+(r) = \frac{4\pi}{3}\rho_0 r^3$, then the surface energy-density varies with r according to

$$4\pi r^2 \mu = m - \frac{4\pi}{3}\rho_0 r^3 \,. \tag{5.14}$$

As the shell expands from $r = 0$, we see that μ decreases and will vanish for some value of r. In order to avoid unphysical negative energies the shell negotiates a hairpin bend [Dray 1990], and starts contracting at the moment when $\mu = 0$ (see figure 5.1).

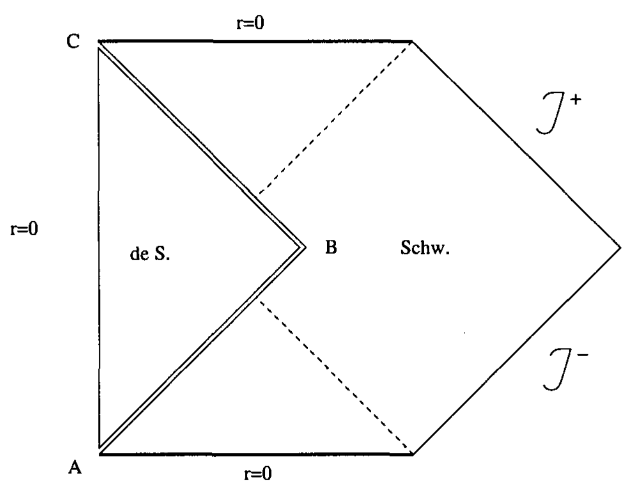

Fig. 5.1 Motion of a null shell in the Schwarzschild/de Sitter geometries. The double–line ABC represents the history of the null shell. Along AB the shell expands and along BC it contracts.

This discontinuous behavior can be understood by considering the case of a spherical time–like shell moving at a velocity close to the speed of light. The equation of such a shell is

$$[\mathrm{sgn}\{n^\alpha \,\partial_\alpha r\}\,(f(r) + \dot{r}^2)^{1/2}] = -\frac{M}{r}\,, \tag{5.15}$$

where $M = 4\,\pi\, r^2\,\mu$ is the proper inertial mass of the shell. For a dust shell M is a positive constant and the equation of motion (5.15) can be expressed in the form

$$\frac{M}{2}\{(f_+ + \dot{r}^2)^{1/2} + (f_- + \dot{r}^2)^{1/2}\} = m - \frac{4\pi}{3} r^3 \rho_0 \,. \tag{5.16}$$

It is clear that a shell which is initially expanding must reverse its motion before the right hand side of this equation becomes negative. For a rapidly moving shell with small proper mass ($\dot{r} >> 1$, $M << m$), (5.15) simplifies to

$$M|\dot{r}| \approx m - \frac{4\pi}{3} r^3 \rho_0 \,. \tag{5.17}$$

which is consistent with the equation (5.14) for the case of a null shell.

Examples are known for which the space–time is not static. A null shell radiated during the motion of a time–like shell and separating two Vaidya space–times with different parameters (mass and charge) has been described in [Frolov (1974)]. We take here a Friedmann-Robertson-Walker (FRW) geometry which is radiation-filled for the space–time $\mathcal{M}^-$, and for $\mathcal{M}^+$ a Vaidya geometry with or without an electrical charge. This situation has been considered to study the instability of a white– hole in a cosmological background [Barrabès, Brady and Poisson (1993)] and to describe the traversability of a Reissner–Nordström worm–hole connecting two FRW universes [Balbinot, Barrabès and Fabbri (1994)]. The example exhibits interesting properties of the null shell and its bordering space–times. For the FRW space–time $\mathcal{M}^-$ we use the coordinates $x^\mu_- = (v_-, \xi, \theta, \phi)$ where v_- is the advanced time, and write the metric $g^-_{\mu\nu}$ via the line–element

$$ds^2_- = -a^2\,\{dv_-(dv_- - 2d\xi) + f_k^2(\xi)\,(d\theta^2 + \sin^2\theta d\phi^2)\}\,. \tag{5.18}$$

The function $f_k(\xi) = (\xi, \sin\xi, \sinh\xi)$ for $k = 0, +1, -1$ respectively, and the scale-factor a is a function of v_- and ξ. For a radiation filled universe the stress-energy tensor is

$$T^{\mu\nu}_- = (\rho_- + p_-)u^\mu u^\nu + p_- g^{\mu\nu}_- \,, \tag{5.19}$$

with $p_- = \frac{\mu_-}{3}$, and the energy conservation implies

$$\mu_- a^4 = \text{constant} \equiv \mu_0\, a_0^4 \,. \tag{5.20}$$

The components of the four-velocity are $u^\mu = (a^{-1}, 0, 0, 0)$. It follows from the field equations that the scale-factor a is a solution of

$$(da)^2 = (C - ka^2)(dv - d\xi)^2 \,. \tag{5.21}$$

where we have introduced the constant $C \equiv 8\,\pi\,\mu_0\,a_0^4/3$.

The coordinates used for the Vaidya space–time $\mathcal{M}^+$ are $x_+^\mu = (v_+, r, \theta, \phi)$ and the metric $g_{\mu\nu}^+$ is given by the line–element

$$ds_+^2 = -dv_+\left(f(v_+, r)dv_+ - 2dv_+dr\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) . \tag{5.22}$$

For a charged Vaidya space–time we have

$$f(v_+, r) = 1 - \frac{2m(v_+)}{r} + \frac{e^2}{r^2} , \tag{5.23}$$

and the corresponding stress-energy tensor is

$$T_+^{\mu\nu} = \frac{1}{4\pi r^2}\frac{dm}{dv_+}\, l^\mu\, l^\nu + T_{\rm em}^{\mu\nu} , \tag{5.24}$$

where $l_\mu = -\partial_\mu v_+$ and $T^\mu_{{\rm em}\,\nu} = {\rm diag}\,(-1, -1, 1, 1) \times e^2/8\pi r^4$ is the electric part. The equations of the spherical null shell $\mathcal{N}$ in $\mathcal{M}^\pm$ are

$$2\, d\xi = dv_- , \qquad 2\, dr = f(v_+, r)\, dv_+ . \tag{5.25}$$

and the matching condition on $\mathcal{N}$ is

$$r = a\, f_k(\xi) . \tag{5.26}$$

The first equation in (5.25) gives $v_- = 2(\xi - \xi_0)$ where ξ_0 is a constant, and (5.21) shows that on $\mathcal{N}$ the scale factor a is a function of ξ only (or of v_- only). Let us take now r as parameter along the null generators of $\mathcal{N}$ (it can be checked that r is not an affine parameter). The normal $n = \zeta\, \partial/\partial r$ where $\zeta = \pm 1$ has components

$$n_+^\mu = \zeta\left(\frac{2}{f}, 1, 0, 0\right) , \qquad n_-^\mu = \zeta\,\frac{dv_-}{dr}\left(1, \frac{1}{2}, 0, 0\right) , \tag{5.27}$$

on the plus and minus sides of $\mathcal{N}$ respectively. We can now describe how the cosmological radiation interacts with the shell. In the simple case where there is no exchange of energy between the radiation and the shell we have from the discussion following (5.9) that $T_{\mu\nu}\, n^\mu\, n^\nu\,|_+ = T_{\mu\nu}\, n^\mu\, n^\nu\,|_-$. Substituting for $T_{\mu\nu}^\pm$ from (5.19) and (5.24) and using $l \cdot n = \zeta\, dv_+/dr$ and $u \cdot n = -a\zeta/2\,(dv_-/dr)$ we obtain

$$\frac{1}{4\,\pi\, r^2}\frac{dm}{dv_+}\left(\frac{dv_+}{dv_-}\right) = \frac{\mu_-\, a^2}{3} . \tag{5.28}$$

Then introducing the matching condition (5.26), the definition of the constant C (following (5.21)), and the relations (5.25) we arrive at

$$\frac{dm}{d\xi} = \left(1 - \frac{2m(v_+)}{r} + \frac{e^2}{r^2}\right) f_k^2(\xi) . \tag{5.29}$$

Since the advanced time v_+ is related to ξ via (5.25)–(5.26) this equation can be solved and the solution describes the evolution of the mass $m(v_+)$ of the Vaidya space–time under the action of the in-going cosmological radiation. In the application to the stability of a white–hole considered in [Barrabès, Brady and Poisson (1993)], $\mathcal{M}^+$ is an uncharged Vaidya space–time ($e = 0$), and $\mathcal{M}^-$ is a spatially flat , radiation–filled FRW universe ($k = 0$ and $f_k(\xi) = \xi$). The analysis of the solution of (5.29) shows that an infinite mass transfer takes place from the FRW universe. Thus all white–hole emissions are trapped and the white–hole disappears and is converted into a black–hole. In [Balbinot, Barrabès and Fabbri (1994)] $\mathcal{M}^+$ is a charged Vaidya space–time ($e \neq 0$) and $\mathcal{M}^-$ is a closed radiation–filled FRW universe ($k = 1$ and $f_k(\xi) = \sin\xi$) . The null shell contracts to the singularity $r = 0$ common to the two domains $\mathcal{M}^\pm$. Here again the accretion of energy from the FRW universe grows arbitrarily large on the Cauchy horizon of the worm–hole. This is analogous to the mass inflation phenomenon discussed in §5.3 .

Non–stationary spherical null shells are sometimes introduced as a simplified model of the gravitational collapse of a spherical body and also for studying the properties of black–holes. This can involve classical as well as quantum aspects of the physics of black–holes and in particular the creation of particles and Hawking radiation. For instance, if a Schwarzschild black–hole with mass m is formed by the collapse a null shell the metric of space–time can be written in the form

$$ds^2 = -\left(1 - \frac{2\,m\,\vartheta(v)}{r}\right) + 2dv\,dr + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right), \tag{5.30}$$

where $\vartheta(v)$ is the Heaviside step-function and we have assumed that the collapse of the shell takes place at the value $v = 0$ of advanced time. Space–time is flat inside the shell where $v < 0$ and has the Schwarzshild geometry outside where $v > 0$. An important equation is the relation $v = V(u)$ between the moment of retarded time u at which a null ray reaches $\mathcal{I}^+$ (future null infinity) and the moment of advanced time $v < 0$ at which it was sent from $\mathcal{I}^-$ (past null infinity). Using the null shell properties [Frolov and Novikov (1998), p. 362] one finds,

$$V(u) = -4\,m\,\left(1 + e^{-\kappa u}\right)\,, \tag{5.31}$$

where $\kappa = 1/4\,m$ is the surface gravity of the black–hole. Using this relation one obtains the mean energy flux of Hawking radiation at $\mathcal{I}^+$, and one shows that its asymptotic spectrum is that of a black body with temperature $T_H = (8\,\pi\,m)^{-1}$ or $T_H = \hbar\,c^3\,(8\,\pi\,G\,m)^{-1}$ in conventional units.

In fact the geometry of space–time near the horizon is not static but fluctuates. The fluctuations of the metric can be spontaneous and also induced by the quantum fluctuations of all other fields interacting with the gravitational field. A semi–classical study of the corrections to Hawking radiation due to metric fluctuations has been given in [Barrabès, Frolov and Parentani (1999), (2000)]. In this work the metric of space–time has the form (5.30) with m replaced by a fluctuating mass of the form $m(v) = m\,\{1 + m_0 \sin(\omega\, v + \varphi)\}\,\vartheta(v)$. Hence for $v < 0$ space–time is still flat but for $v > 0$ it has the Vaidya geometry. The modified relation (5.31) has been derived and its implications for the energy flux at $\mathcal{I}^+$ and the asymptotic spectrum of Hawking radiation have been calculated.

5.2 Horizon–Straddling Null Shells

For non–stationary spherical shells a natural choice for the parameter on the null generators of the history of the shell is the radial coordinate r. This parameter is common to the two sides of $\mathcal{N}$ and the requirement that induced geometries match determines how the two faces of $\mathcal{N}$ have to be soldered. We consider in this section the case where $\mathcal{N}$ is a stationary null hypersurface which arises when $\mathcal{N}$ is the common horizon $r = r_0$ between two spherically symmetric static space–times. Obviously r can no longer be used as a parameter along the null generators in this case. Another consequence of the stationary condition is that it opens up the possibility of having different ways of gluing the two space–times $\mathcal{M}^\pm$ on $\mathcal{N}$. We shall for instance consider here the two cases of a static and an affine soldering, but other possibilities could be considered as long as they satisfy the geometrical matching conditions. In the static case the surface energy density and pressure are time independent and the shell appears to a stationary observer to be in static equilibrium due to the balance between the surface pressure and external forces. For an affine soldering the surface pressure vanishes and a local inertial observer sees the shell in free fall, radially moving with the speed of light.

The line–elements of $\mathcal{M}^\pm$ are given by the expression (5.1) with different functions $f_\pm(u,r)$ and $\psi_\pm(u,r)$. The common horizon $r = r_0$ is such that $f_\pm(u,r_0) = 0$, or introducing the mass function $m(u,r)$, $r_0 = 2m_\pm(u,r_0)$. It then follows from the first equation in (5.2) that $T^r_u|_\pm = 0$.

As parameter along the null generators of $\mathcal{N}$ we take u which is continuous across the hypersurface and as intrinsic parameters on $\mathcal{N}$ we choose

$\xi^a = (u, \theta, \phi)$ with $a = 1, 2, 3$. The vectors

$$n^\mu = \frac{\partial x^\mu}{\partial u}\,, \qquad N^\mu = \zeta\, e^{-\psi}\, \frac{\partial x^\mu}{\partial r}\,, \tag{5.32}$$

are normal and transverse radial light–like vectors respectively, each identical on both faces of $\mathcal{N}$ and satisfying $N \cdot n = -1$. The surface gravity is defined as $\kappa_0 = \frac{1}{2}\frac{\partial f}{\partial r}|_{r=r_0}$. Using the second equation in (5.2) we find that the jump in the surface gravity across $\mathcal{N}$ is

$$[\kappa_0] = 4\,\pi\, r_0\, [T^u_u] = 4\,\pi\, r_0\, [T^r_r]\ . \tag{5.33}$$

Furthermore we have

$$T_{\mu\nu}\, n^\mu\, n^\nu|_\pm = -\zeta\, e^\psi\, T^r_u|_\pm = 0\,, \qquad T_{\mu\nu}\, n^\mu\, N^\nu|_\pm = -T^r_r|_\pm = -T^u_u|_\pm\ . \tag{5.34}$$

The null generators have equation $n^\mu_{|\nu}\, n^\nu = \kappa(u)\, n^\mu$ where the acceleration factor $\kappa(u)$ vanishes if the generators are affinely parametrized. It is easy to show that

$$\kappa(u)|_\pm = \Gamma^u_{uu}|_\pm = \partial_u \psi|_\pm - \zeta\, \kappa_0\, e^\psi|_\pm\ . \tag{5.35}$$

The transverse curvature $\mathcal{K}^\pm_{ab}$ defined by (2.29) has the non-vanishing components

$$\mathcal{K}^\pm_{uu} = \kappa(u)|_\pm\,, \qquad \mathcal{K}^{\pm\theta}_\theta = \mathcal{K}^{\pm\phi}_\phi = \frac{\zeta_\pm}{r}\, e^{-\psi_\pm}\ . \tag{5.36}$$

Thus the intrinsic expressions (2.37)-(2.42) show that the surface stress-energy tensor has the perfect fluid form with surface energy density μ and pressure P given by

$$4\,\pi\, r_0\, \mu = -[\zeta\, e^{-\psi}]\,, \qquad 8\,\pi\, P = -[\kappa(u)]\ . \tag{5.37}$$

We note that while the inertial mass of the shell $4\,\pi\, r_0^2\, \mu$ does not vanish in general, its gravitational mass satisfies $[m] = 0$ because we have a common horizon $r_0 = 2m_\pm$.

For simplicity we shall assume that the geometry is static without radial flow ($T_{rr} = 0$) so that, by (5.2), $\psi_\pm = \psi(u)$ and $f_\pm = f_\pm(r)$. We first consider the case when the two faces of $\mathcal{N}$ are statically soldered then $\psi = 0$ and u is the standard advanced or retarded time along the shell history. It follows immediately from (5.37) that

$$4\,\pi\, r_0\, \mu = -[\zeta]\,, \qquad 8\,\pi\, P = -[\zeta\, \kappa_0]\ . \tag{5.38}$$

If say $\zeta_- = -\zeta_+ = 1$ then the energy density $\mu > 0$ and an initially expanding bundle of light rays $u =$ constant is focused by the shell so as to become contracting. As an example of this situation consider a stationary light-like shell with a de Sitter interior and a Reissner-Nordström (RN) exterior located at the outer RN horizon (see figure 5.2). Along segment AB we have $\zeta_- = -\zeta_+ = 1$ and u plays the role of advanced time, while $\zeta_- = \zeta_+ = -1$ along segment BC and u plays the role of retarded time.

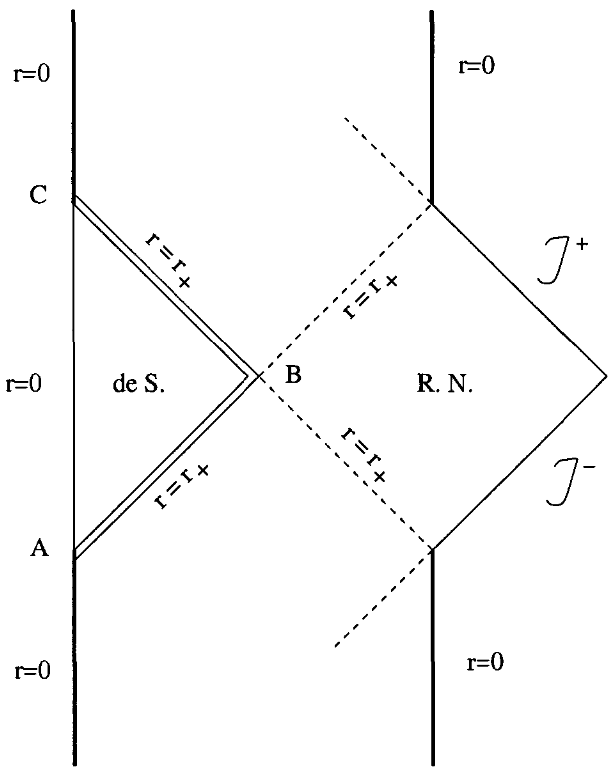

Fig. 5.2 Reissner-Nordström black-hole exterior space-time joined to an interior de Sitter geometry by a stationary null shell having history the double-line ABC. $r = r_+$ is the outer event horizon of the Reissner-Nordström black-hole.

For the Reissner-Nordström geometry we have

$$f_{\rm RN} = 1 - \frac{2m}{r} + \frac{e^2}{r^2}, \qquad -T^u_u = -T^r_r = T^\theta_\theta = T^\phi_\phi = \frac{e^2}{8\pi r^4}, \tag{5.39}$$

and for the de Sitter geometry

$$f_{\rm dS} = 1 - \frac{8\pi\rho_0 r^2}{3} = 1 - \frac{r^2}{a^2}, \qquad T^u_u = T^r_r = T^\theta_\theta = T^\phi_\phi = -\rho_0 . \tag{5.40}$$

The surface gravities are

$$\kappa_{0,\rm RN} = \frac{r_0 - m}{r_0^2} = \frac{1}{r_0^2}\left(m - \frac{e^2}{r_0}\right), \qquad \kappa_{0,\rm dS} = -\frac{1}{a}, \tag{5.41}$$

and as the soldering takes place on the outer RN horizon we have $a = m + \sqrt{m^2 - e^2}$ which implies that $2\,m\,a = a^2 + e^2$. Along the two segments AB and BC we have $\zeta_- = -\zeta_+ = 1$ and we find

$$\mu = \frac{1}{2\pi a}, \qquad P = \frac{m}{8\pi a^2} . \tag{5.42}$$

The energy density is positive and in–going expanding light rays are focused by the shell and start contracting. The abrupt change of direction of the shell cannot occur beyond point B otherwise the signs of ζ would be affected and the energy density would become negative.

Let us turn now to the case where the shell faces are affinely soldered. The parameter u on the null generators is affine and the acceleration factor κ vanishes. It then follows from (5.35) that

$$e^{-\psi} = -\zeta\,\kappa_0\,(u - u_0)\,, \tag{5.43}$$

and from (5.37) we obtain

$$4\,\pi\,r_0\,\mu = [\kappa_0\,(u - u_0)]\,, \qquad P = 0\;, \tag{5.44}$$

with $u_0 =$ constant and not necessarily identical on both faces. The energy density now varies in time and a kink in the direction of propagation of the shell must occur when $\mu = 0$ in order to avoid negative values for the energy. This situation resembles the one encountered in the non–stationary case.

Finally it is amusing to observe that for the particular choice of parameters $a^3 = e^2/3$ the surface gravities (5.41) of the de Sitter and Reissner-Nordström regions become equal provided that the shell is placed on the inner RN horizon. Hence for a static soldering we deduce from (5.38) that $\mu = P = 0$, and a perfectly smooth transition is now possible (no geometrical discontinuity and singularity). This yields a de Sitter-like model of a charged particle with a massless charge distribution balanced at the inner RN horizon if the charge fully occupies both sheets of the inner horizon. The gravitational mass of this "electron" is of the order of the Planck mass.

5.3 Colliding Spherical Null Shells

Interaction between matter and gravitational field is essentially nonlocal in character in addition to being non-linear. One of the useful features in a thin shell as a gravitational source is that it can effectively localize interactions that are ordinarily nonlocal by bringing into close proximity two regions space–time $\mathcal{M}^+$ and $\mathcal{M}^-$. A dramatic manifestation of this is the phenomenon of mass inflation, the conversion of arbitrarily large amounts of gravitational energy into material forms of energy. The essential non-locality of this process is clearly brought out when it is treated as continuous, as in [Poisson and Israel (1989), (1990)]. A continuum formulation is formidable, even under the restrictive assumption of spherical symmetry. However the interactions can be partially localized and their effects modeled in a simple setting by idealizing the sources as two massive null shells and considering their collision near a horizon. A simple relation connecting the gravitational masses in the vacuum region between two spherical light–like shells before and after the collision was first derived in [Dray and 't Hooft (1985)] and [Redmount (1985)]. It was later extended to rotating light–like shells in [Barrabès, Israel and Poisson (1990)] and to arbitrary light–like shells [Barrabès and Israel (1991)].

Let the space–like two-surface $\mathcal{S}$ be the intersection of two light–like shell histories, and introduce on $\mathcal{S}$ the parameters ξ^m, with $m = 2, 3$, and their associated tangent basis vectors $e^\mu_{(m)}$. Let $\mathcal{N}_i$ with $i = 1, 2, 3, 4$ be the null hypersurfaces corresponding to the histories of the light–like shells before and after the collision. $\mathcal{N}_3$ and $\mathcal{N}_4$ correspond to the incident shells, and $\mathcal{N}_1$ and $\mathcal{N}_2$ to the emergent ones (see figure 5.3).

The normal to $\mathcal{N}_i$ is denoted $n^\mu_{(i)} = \partial x^\mu / \partial \lambda_i$ with λ_i a (generally non-affine) parameter along the null generators, the same on both sides of $\mathcal{N}_i$. The following 2-tensor

$$K_{i,mn} = -n_{(i)\,\mu}\, e^\mu_{(m)|\nu}\, e^\nu_{(n)} \,, \tag{5.45}$$

is a measure of intrinsic properties of $\mathcal{N}_i$. Its trace K_i and the magnitude Σ_i of its trace-free part represent the dilation and the shear of the normal $n_{(i)}$ and each have the same value on both sides of $\mathcal{N}_i$.

We assume that each point of the surface of intersection $\mathcal{S}$ has a neighbourhood that can be covered by an admissible local coordinate system (for instance, Gaussian coordinates anchored to geodesics orthogonal to $\mathcal{S}$) in which the components of the four-metric are continuous and piecewise continuously differentiable. In other words, it is assumed that the points at

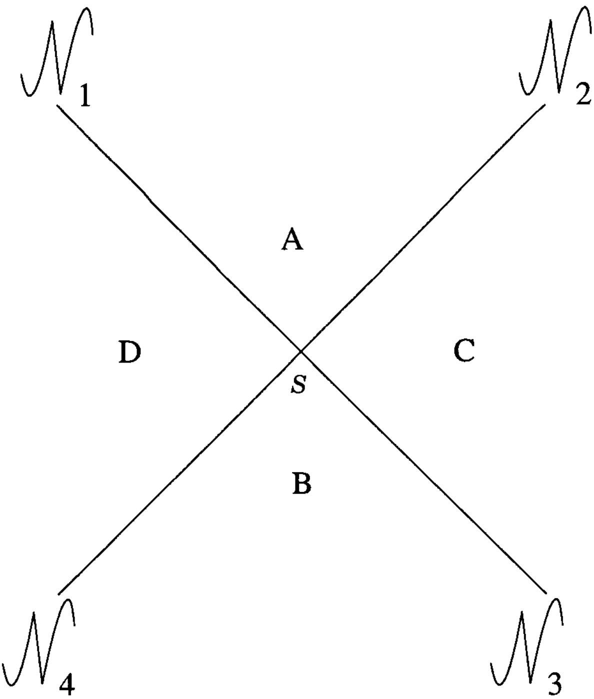

Fig. 5.3 Collision of two ingoing light–like shells $\mathcal{N}_3$ and $\mathcal{N}_4$ to form two outgoing light–like shells $\mathcal{N}_1$ and $\mathcal{N}_2$. $\mathcal{S}$ is the 2–surface of intersection.

which the shells interpenetrate constitute singularities which are not worse than those of a single shell. This excludes the possibility that points of $\mathcal{S}$ have conical or worse singularities of the four-geometry. This assumption gives unambiguous meaning to the equality or parallelism of a pair of vectors perpendicular to $\mathcal{S}$. Then we have at each point of $\mathcal{S}$

$$(n_{(1)} \cdot n_{(2)})(n_{(3)} \cdot n_{(4)}) = (n_{(1)} \cdot n_{(4)})(n_{(2)} \cdot n_{(3)}) , \tag{5.46}$$

since all four light light–like vectors $n_{(i)}$ are orthogonal to $\mathcal{S}$ and there are only two light–like directions orthogonal to a space–like two-surface.

The four hypersurfaces $\mathcal{N}_i$ divide the space–time near $\mathcal{S}$ into four sectors which we label A,C,B,D, clockwise from noon (see figure 5). In sector A we define the two scalars

$$F_A = \frac{K_1\, K_2}{n_{(1)} \cdot n_{(2)}} , \qquad \Sigma_A = \frac{\sigma_1\, \sigma_2}{n_{(1)} \cdot n_{(2)}} , \tag{5.47}$$

and use similar definitions for the six other scalars F_B, F_C, F_D and $\Sigma_B, \Sigma_C, \Sigma_D$. Using (5.46) we obtain the generalized collision laws

$$F_A\, F_B = F_C\, F_D\,, \tag{5.48}$$

$$|\Sigma_A\, \Sigma_B| = |\Sigma_C\, \Sigma_D|\,, \tag{5.49}$$

which are valid at any point on $\mathcal{S}$.

As an application of these relations we consider the case of spherical shells. The two shells are taken to be concentric before and after the collision, and in each in–going or out–going pair, one of the two shells is expanding and the other is contracting. The two-surface $\mathcal{S}$ at which the collision takes place is a sphere with radius r_0. The completeness relation takes the simple form in, say sector A,

$$g^{\mu\nu} = g^{mn}\, e^{\mu}_{(m)}\, e^{\nu}_{(n)} - 2\, \frac{n^{(\mu}_{(1)}\, n^{\nu)}_{(2)}}{n_{(1)} \cdot n_{(2)}}\,, \tag{5.50}$$

with similar relations holding in the other sectors B, C and D. Here g^{mn} is the inverse of the non-degenerate two-matrix $g_{mn} = e_{(m)} \cdot e_{(n)}$. For spherical shells, a natural choice for the parameters is the radial coordinate $\lambda_i = r$, and therefore $n^{\mu}_{(i)} = \zeta_i\, \partial x^{\mu}/\partial r$, with $\zeta_i = \pm 1$ and we use the same sign convention as for ζ in the metric (5.1). The shear vanishes ($\Sigma_i = 0$) and we have

$$K_i = \frac{2}{r}\, n^{\mu}_{(i)}\, \partial_{\mu} r = \frac{2\, \zeta_i}{r}\,. \tag{5.51}$$

Using the completeness relation we obtain in, say sector A

$$F_A = \frac{2}{r_0^2}\, f_A(r_0)\,, \tag{5.52}$$

with $f = g^{\mu\nu}\, \partial_{\mu} r\, \partial_{\mu} r = g^{rr}$ as in the metric (5.1). Hence the collision laws (5.48)–(5.49) reduce in the case of spherical shells to the single relation

$$f_A(r_0)\, f_B(r_0) = f_C(r_0)\, f_D(r_0)\,. \tag{5.53}$$

If we define as in §5.2 the mass function m then (5.53) relates the values of the gravitational masses m_A, m_B, m_C, m_D at the collision point $r = r_0$. In the weak field approximation and to linear order, it expresses conservation of gravitational mass in the collision. Let us call M_i the gravitational mass of the light–like shell with history $\mathcal{N}_i$, and assume for instance that $\mathcal{N}_4$ and

$\mathcal{N}_2$ are expanding and that $\mathcal{N}_3$ and $\mathcal{N}_1$ and are contracting. Using the sign convention for the ζ_i's and the junction condition (5.9) we have

$$M_4 = m_B - m_D\,, \;\; M_3 = m_C - m_B\,, \;\; M_2 = m_C - m_A\,, \;\; M_1 = m_A - m_D\,. \tag{5.54}$$

The validity of the conservation of the gravitational masses $M_3 + M_4 = M_1 + M_2$ is obvious from this result.

For non-spherical shells a quasi-local mass aspect m_Q was introduced by Hawking [Hawking (1968)]. For example in sector A the quasi–local mass aspect over $\mathcal{S}$ is

$$m_Q = \left(\frac{\mathcal{A}}{16\pi}\right)^{1/2}\left(1 - \frac{\mathcal{A}}{8\pi}F_A\right)\,, \tag{5.55}$$

with $\mathcal{A}$ denoting the area of $\mathcal{S}$. Then (5.53) gives a constraint on the in–going and out–going quasi–local mass aspects over $\mathcal{S}$.

The phenomenon of mass inflation mentioned at the beginning of this section occurs when shell 3, schematically representing in–fall from the gravitational wave tail of collapse, falls close to the inner horizon of the resulting black–hole. We consider here the case of a RN black–hole for which there exists an inner and an outer horizon, the inner horizon being a Cauchy horizon as well. The main features of the picture obtained with such a black–hole is valid for a rotating black–hole and mass-inflation is a generic process for black–holes. Solving (5.53) for the mass m_A of sector A we obtain

$$m_A = \frac{1}{f_B(r_0)}\left(m_C + m_D - m_B - \frac{2m_C m_D}{r_0}\right)\,. \tag{5.56}$$

We see immediately that as shell 3 is close to the inner horizon of sector B, $f_B(r_0)$ is arbitrarily small and correspondingly m_A is very large. In terms of the gravitational masses of the null shells we can also write

$$M_1 = M_3\left(1 - \frac{2M_4}{r_0}\right)^{-1}\,, \qquad M_2 = M_4\left(1 - \frac{2M_1}{r_0}\right)\,. \tag{5.57}$$

Hence on the one hand the imploding shell gains energy and the mass increase $M_1 - M_3$ can be made arbitrarily large. One the other hand the mass M_2 of the "outgoing" shell becomes negative and this shell is inside the black–hole and really contracting. It has been known for a long time [Penrose (1968)] that time-dependent perturbations are gravitationally blue–shifted as they propagate inwards near the Cauchy horizon which becomes unstable. In the presence of an outgoing energy flux, however small this flux might be, the local mass function diverges and the Cauchy horizon contracts and becomes the locus of a scalar curvature singularity. It has been shown that the null Cauchy horizon singularity is a precusor of a final space–like singularity along the $r = 0$ hypersurface.

Other interesting situations can be derived as particular cases of the general conservation laws (5.48), (5.49). We shall only consider spherically symmetric shells and therefore use the relation (5.53). As a first example a single shell exists before or after the collision. This yields a model for the fission of one shell into two shells, or conversely the fusion of two shells into a single one. If we take the case of fission then sectors B and D are identical, and if $f_B(r_0) \neq 0$ then (5.53) leads to

$$f_A(r_0) = f_C(r_0) \,. \tag{5.58}$$

If for instance we take for sector A the de Sitter geometry (see (5.40)) and for sector C the Schwarzschild geometry with mass m_C, then the above relation implies $r_0^3 = r_C\, a^2$, where $r_C = 2\, m_C$ is the horizon of the Schwarzschild black–hole. We thus have the two possibilities $a < r_0 < r_A$ or $a > r_0 > r_A$ for the place of the intersection.

We next consider the case when there is no in–going (out–going) shell and two out–going (in–going) shells. This is a model for the instantaneous creation (annihilation) of a pair of light shells with radius r_0. In the case of the creation of two shells the sectors B, C and D are identical and if this occurs outside the horizon of B≡C≡D then (5.53) gives the same relation as (5.58). This process has been used to mimic the creation of a de Sitter universe inside a Schwarzschild black–hole [Barrabès and Frolov (1996)]. Sector A is a de Sitter space–time with horizon $r = a$ and B≡C≡D is a Swharzshild space–time with mass m. The relation $r_0^3 = 2\, m\, a^2$ still applies with $a < r_0 < 2m$ so that the creation takes place inside the black–hole. It has been shown in the above reference that in fact a large number of such disconnected de Sitter universes can be created. This scenario gives a classical model corresponding to results in quantum theory to the effect that the vacuum fluctuation inside the horizon of a Schwarzschild black–hole can have a self-regulating effect on the rise of curvature, and that once the quantum fluctuations have died away the de Sitter state is the simplest possibility. A similar model of a transition from Schwarzschild to de Sitter geometry inside the horizon of the black–hole was initially proposed in [Frolov, Markhov and Mukhanov (1989), (1990)]. In this model the transition takes place on a space–like shell and is instantaneous.

The same situation of creation of two null shells from nothing can also be used as a classical model for the evaporation of a black–hole (see figure 5.4).

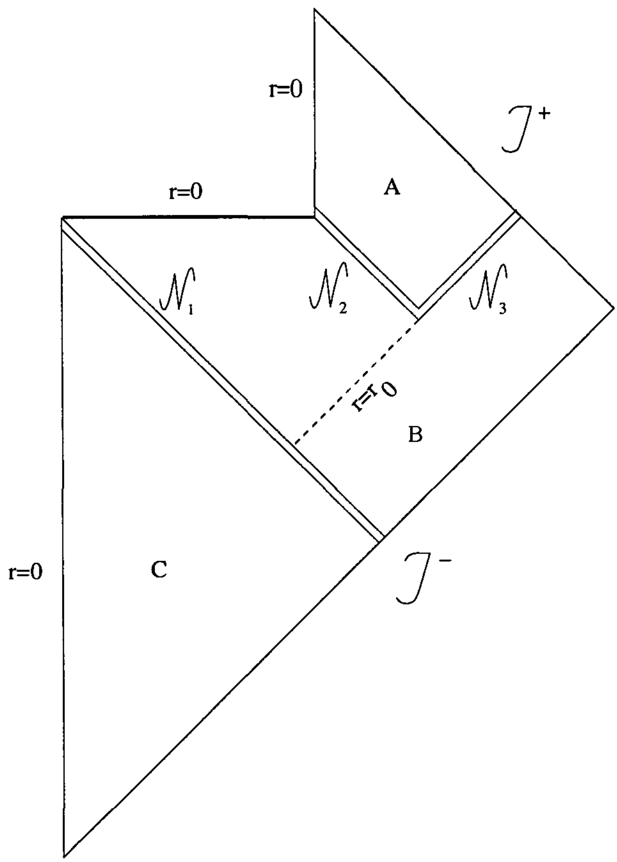

Fig. 5.4 Classical picture of the creation and evaporation of a Schwarzschild black–hole. Sector B has Schwarzschild geometry and sectors A and C are flat.

The first shell $\mathcal{N}_1$ forms a Schwarzschild black–hole with mass m and its interior space–time is flat. The two shells $\mathcal{N}_2$ and $\mathcal{N}_3$ (with $\mathcal{N}_2$ reaching future null infinity $\mathcal{I}^+$ and $\mathcal{N}_3$ falling into the singularity) represent the two partner photons which are created in the Hawking process on the horizon $r = r_0 = 2\,m$. The gravitational masses of the shells $\mathcal{N}_2$ and $\mathcal{N}_3$ are $M_2 = m$ and $M_3 = -m$. As a result sector A has flat geometry and the black–hole ceases to exist.

5.4 Gravitational Collapse, Gravitational Radiation and Singularities

The formation of a singularity in gravitational collapse and the avoidance of naked singularities is a central problem in general relativity. It is widely believed that *an event horizon forms whenever a matter distribution, whose stress-energy tensor satisfies appropriate physical conditions, collapses beyond a certain critical stage, generally taken to be the formation of a trapped surface.* This conjecture, known as the event horizon conjecture (EHC), has not yet been proved nor has it been given a mathematically rigorous formulation. The only rigorous results obtained so far concern necessary and sufficient conditions for the existence of a trapped surface (see for instance [Wald (1984)]) and a confinement theorem [Israel (1986)]. The EHC is closely related to what is commonly known as cosmic censorship first articulated in [Penrose (1969)] as: "Does there exist a cosmic censor who forbids the occurence of naked singularities, clothing each one in an absolute event horizon?".

The hoop conjecture (HC) proposed in [Thorne (1972)] states that *horizons form when and only when a mass M gets compacted into a region whose circumference C in every direction satisfies*

$$C \leq 2\pi(2M). \tag{5.59}$$

Although HC appears more specific than EHC, its formulation is still vague. The type of horizons which form is not specified and various definitions are possible for the mass and the circumference of the collapsing body. Furthermore, as noted in [Israel (1984)], "the significant difference between cosmic censorship and HC is that the latter avoids explicit reference to singularities and thus stops short of outlawing *all* naked singularities".

Spherical models of collapse –the only ones studied analytically so far– can produce naked singularities by shell–crossing or shell–focusing. These singularities may be transient or persistent. However all are weak, in the sense that the gravitational potential (i.e. the metric) remains bounded at the singularity, although the curvature becomes infinite. For instance for shell-focusing singularities the local mass–function $m = -r^3\Psi_2$, where Ψ_2 is the usual Newmann–Penrose coefficient, goes to zero at the singularity.

In this section we show how null shells (not necessarily spherically symmetric) can be used to justify the inequality stated in the hoop conjecture, and also to obtain an upper bound on the amount of gravitational radiation emitted during the gravitational collapse of a convex body. For this we use a model suggested in [Penrose (1973)] according to which the history of a shell collapsing with the speed of light acts as a causal boundary for the space–time region on its past side. We consider a convex thin shell which implodes from infinity with the speed of light in flat space–time. As long as the shell remains convex its history is a light–like convex hypersurface $\mathcal{N}$ whose interior is a flat space–time. Out–going light rays perpendicular to the shell and starting from the interior are more and more focused during the collapse. At some moment of time (which we assume to be the same on each point of the shell) the light rays become expansion–less and a trapped surface $\mathcal{S}$ forms.

It is convenient to take a non–uniform shell of photons coherently radially falling inwards. We introduce spherical coordinates $x^\mu = (t, r, \theta, \phi)$ for the flat interior region. In terms of the intrinsic coordinates $\xi^A = (\theta, \phi)$ the motion of the shell has the parametric equations

$$-t = r = h(\xi^A)\,. \tag{5.60}$$

The tangent vectors $e_{(A)} = \partial/\partial\xi^A$ have components

$$e^\mu_{(A)} = (-\partial_A h,\ \partial_A h,\ \delta^2_A,\ \delta^3_A)\,, \tag{5.61}$$

and the intrinsic metric is

$$g_{AB}\, d\xi^A\, d\xi^B = h^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{5.62}$$

The normal $n^\mu\,\partial_\mu = \partial_t - \partial_r$ is directed along the flow lines. The transverse vector N satisfying $N \cdot n = -1$ and $N \cdot N = N \cdot e_{(A)} = 0$ has components

$$N_\mu = (-\lambda,\ (1-\lambda),\ -\partial_\theta h,\ -\partial_\phi h)\ , \tag{5.63}$$

where $\lambda = (1 + g^{AB}\partial_A h \partial_B h)/2$. Note that all the above components are given on the inner face of $\mathcal{N}$. The transverse curvature $\mathcal{K}_{AB}$ associated with the transverse vector N is such that on the inner face we have

$$g^{AB}\mathcal{K}^-_{AB} = h^{-1}\,(1 - {}^2\nabla h)\,, \tag{5.64}$$

where ${}^2\nabla$ is the Laplacian calculated with the two-dimensional metric (5.62)

$${}^2\nabla = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,. \tag{5.65}$$

The exterior metric is unknown and a similar calculation for the outer face cannot be carried through in general. However, since $\mathcal{S}$ is a trapped surface we have for the outer transverse curvature $g^{AB}\mathcal{K}^+_{AB} = 0$.

The stress-energy tensor of the null shell has the simple form $S^{ab} = \mu\, n^a\, n^b$ as it is a dust of photons. Conservation of energy demands that $4\,\pi\, r^2\,\mu$ be conserved along the null generators tangent to n. Using the results of Chapter 2 for the intrinsic components of the surface stress–energy tensor we obtain

$$8\,\pi\, h\,\mu = 1 - {}^2\nabla h\,, \tag{5.66}$$

and the shell gravitational mass is

$$M = \int_S \mu h^2 d\Omega = \frac{1}{8\pi}\int_S h(1 - {}^2\nabla h)\, d\Omega\,. \tag{5.67}$$

Here, M (conserved in the collapse) is the advanced Bondi mass of the shell, equal in all these models to the ADM mass and the Hawking quasi–local mass.

A similar argument leads to a simple relation between the surface energy density μ and the mean curvature K of the surface $\mathcal{S}$ as defined in ordinary diffenetial geometry ($K = 2/r$ for a sphere of radius r). At the moment when $\mathcal{S}$ becomes trapped we have $K^+ = 0$ on its outer face and we can call $K^- = K$. The light rays perpendicular to $\mathcal{S}$ have tangent vector $N = d/d\lambda$, with λ some parameter. They also have expansion $\theta = K$, and modulus of shear and twist denoted by $|\sigma|$ and ω respectively. Then by integration of Raychaudhuri's equation [Wald (1984)]

$$\frac{d\theta}{d\lambda} + \frac{\theta^2}{2} + |\sigma|^2 + \omega^2 = R_{\mu\nu}\, N^\mu\, N^\nu\ , \tag{5.68}$$

across the null hypersurface $\mathcal{N}$ at the moment the trapped surface forms, we obtain

$$\mu = \frac{K}{16\,\pi}\,. \tag{5.69}$$

Minkowski's inequality of classical differential geometry

$$16\pi\mathcal{A} \leq \left(\int K\, dS\right)^2\,, \tag{5.70}$$

holds for any convex, closed surface of area $\mathcal{A}$. Using (5.69) it assures validity of the well–known Penrose–Gibbons "isoperimetric inequality" [Penrose (1973), Gibbons (1972)]

$$\mathcal{A} \leq 4\pi(2M)^2\ , \tag{5.71}$$

which is a necessary consequence of the assumption that EHC is valid.

We now first apply the above results to the calculation of the energy emitted as gravitational radiation during the collapse. If cosmic censorship is valid and a black–hole forms then the difference

$$E_{\max} = M - \frac{\mathcal{A}_H}{16\,\pi}\,, \tag{5.72}$$

must be non–negative. Here M is calculated from (5.67) and $\mathcal{A}_H = \int h^2 d\Omega$ is the area of the apparent horizon. Thus (5.72) represents an upper bound for the energy emitted as gravitational radiation in the collapse , since by the Hawking area theorem $\mathcal{A}_H$ cannot be larger than the area of the final stationary black–hole. This argument has been applied to the collapse of circular string loops [Hawking (1990)] and other non–spherical convex objects [Barrabès (1991); Barrabès, Israel and Letelier (1991)].

Before considering the hoop conjecture let us state the following theorem of classical differential geometry which has been demonstrated in [Barrabès et al. (1992)]: *Let D be a compact and convex domain of R^3 and K be the mean curvature at each point of its boundary ∂D. Let L be the maximum length of a plane curve drawn around ∂D and l the maximum circumference of an othogonal projection of D on a plane. K satisfies the following inequalities*

$$\pi L \le \int_{\partial D} K\, d\mathcal{A} \le 4l\,. \tag{5.73}$$

A similar result was independently derived in [Tod (1992)]. This theorem yields inequalities which are similar to the Minkowski inequality above but now apply to a closed curve instead a surface and will be more appropriate to HC.

Using the definition (5.67) of the mass M in the inequalities (5.73) we derive immediately

$$\pi\, L \le 16\,\pi\, M \le 4\, l\ . \tag{5.74}$$

These inequalities are in agreement with the spirit of the hoop conjecture since they put bounds on the mass enclosed by a trapped surface and the maximum circumference of two different closed curves associated with this surface. The following two conditions can thus be stated separately:

Condition 1: A necessary condition for the formation of an apparent horizon is that any matter or energy distribution with mass M gets compacted within a region whose plane section in every direction has a maximum perimeter $L \le 16\, M$.

Condition 2: A sufficient condition for the formation of an apparent horizon is that any matter or energy distribution with mass M gets compacted within a region whose orthogonal plane projection in every direction has a maximum perimeter $l \leq 4\pi M$.

Applications of these inequalities to particular distributions can be found in [Barrabès (1991); Barrabès, Israel and Letelier (1991)]. If the inequalities are not satisfied then these cannot be used as counterexamples to cosmic censorship principles. The singularities which might form are incidental as they are encountered en route to the formation of a much stronger final singularity which will be surrounded by an event horizon.

Chapter 6

Collisions of Plane Impulsive Light–Like Signals

There has been a tendency in the literature to use a variety of mathematical techniques to arrive at the space–time models of the gravitational fields following the collisions of plane gravitational waves and then to work backwards in time to discover what waves collided to produce such fields. Some of the incoming waves obtained in this way have bizarre profiles from a physical point of view. We explore the conventional approach of specifying the incoming impulsive light–like signals and looking for a key to unlock what is in space–time a boundary–value problem with prescribed data on intersecting null hypersurfaces.

6.1 Khan–Penrose Solution

The study of collisions of impulsive light–like signals began with the discovery by Khan and Penrose of the solution of Einstein's vacuum field equations which describes the gravitational field produced after two non–interacting, plane impulsive (linearly polarized) gravitational waves engage in a head–on collision. The resulting solution exhibits back–scattered gravitational radiation and a curvature singularity. The line–element of the space–time before and after the collision is given by [Khan and Penrose (1971)]

$$\begin{aligned} ds^2 &= t^2\left(\frac{r+q}{r-q}\right)\left(\frac{w+p}{w-p}\right)dx^2 + t^2\left(\frac{r-q}{r+q}\right)\left(\frac{w-p}{w+p}\right)dy^2 \\ &\quad -\frac{2\,t^3}{r\,w\,(p\,q+r\,w)^2}\,du\,dv\ , \end{aligned} \tag{6.1}$$

with $p = u\,\vartheta(u)$, $q = v\,\vartheta(v)$ and

$$\begin{aligned} r &= \sqrt{1-p^2}\ , \qquad w = \sqrt{1-q^2}\ , \\ t &= \sqrt{1-p^2-q^2} = \sqrt{r^2-q^2} = \sqrt{w^2-p^2}\ . \end{aligned} \tag{6.2}$$

When $v < 0$ this line–element coincides with (2.63) specialized by taking $F = v + \frac{1}{2}(x^2 - y^2)$. This is the space–time model of a plane impulsive gravitational wave, having as history the null hyperplane $u = 0$, propagating through a region in which no gravitational field is present. The propagation direction of the wave in space–time is that of the null vector field $\partial/\partial v$. When $u < 0$ (6.1) specialises to (2.63) with u and v interchanged. When $u > 0$ and $v > 0$ then (6.1) with $p = u$, $q = v$ is the space–time model of the gravitational field created after the head–on collision of the two plane impulsive gravitational waves. This line–element has the Rosen–Szekeres form

$$ds^2 = \mathrm{e}^{-U+V}\,dx^2 + \mathrm{e}^{-U-V}\,dy^2 - 2\,\mathrm{e}^{-M}\,du\,dv\ , \tag{6.3}$$

where U, V, M are each functions of (u, v) which can be read off by comparing (6.3) with (6.1) when $u > 0$ and $v > 0$. The functions U, V, M satisfy Einstein's vacuum field equations (see, for example [Griffiths (1991)], p. 39)

$$U_{uv} = U_u\,U_v\ , \tag{6.4}$$
$$2\,V_{uv} = U_u\,V_v + U_v\,V_u\ , \tag{6.5}$$
$$2\,U_u\,M_u = -2\,U_{uu} + U_u^2 + V_u^2\ , \tag{6.6}$$
$$2\,U_v\,M_v = -2\,U_{vv} + U_v^2 + V_v^2\ , \tag{6.7}$$
$$2\,M_{uv} = -U_{uv} + V_u\,V_v\ , \tag{6.8}$$

where the subscripts denote partial derivatives. The non–vanishing Newman–Penrose components of the Riemann curvature tensor calculated with the metric tensor given by (6.3) are

$$\Psi_0 = -\frac{1}{2}\,V_{vv} - \frac{1}{2}\,(M_v - U_v)\,V_v\ , \tag{6.9}$$
$$\Psi_2 = \frac{1}{2}\,M_{uv}\ , \tag{6.10}$$
$$\Psi_4 = -\frac{1}{2}\,V_{uu} - \frac{1}{2}\,(M_u - U_u)\,V_u\ . \tag{6.11}$$

The line–element (6.1) in the region $u > 0,\ v > 0$ is obtained by solving these field equations subject to the boundary conditions:

When $u > 0,\ v = 0$:

$$U = -\log(1 - u^2)\ , \qquad V = \log\left(\frac{1+u}{1-u}\right)\ , \qquad M = 0\ . \tag{6.12}$$

When $u = 0,\ v > 0$:

$$U = -\log(1 - v^2)\ , \qquad V = \log\left(\frac{1+v}{1-v}\right)\ , \qquad M = 0\ . \tag{6.13}$$

It is easy to survey the field equations. U is obtained by solving (6.4), then V is derived from (6.5) followed by M from (6.6) and (6.7) with (6.8) the integrability condition for (6.6) and (6.7). It is well–known that (6.4) can be integrated to yield

$$U = -\log(f(u) + g(v))\ , \tag{6.14}$$

for some functions $f(u)$, $g(v)$ determined by the boundary conditions (6.12) and (6.13). We easily see that for $u > 0\ ,\ v > 0$,

$$U = -\log(1 - u^2 - v^2)\ . \tag{6.15}$$

The paper by Khan and Penrose did not include a derivation of their solution and it is fun to speculate as to how they found it. In the text by Griffiths on colliding plane waves [Griffiths (1991)] no derivation is given either, just the re–casting of the solution in terms of the Ernst potential by Chandrasekhar and Ferrari [Chandrasekhar and Ferrari (1984)]. This latter work is important because it shows that the Ernst potential which leads to the Khan–Penrose solution has the same form as the Ernst potential which leads to the Schwarzschild solution for static space–times. We propose here a simple derivation of the Khan–Penrose solution [Barrabès, Bressange and Hogan (1999)]. The approach we use has led, as we point out later, to some further solutions of collision problems involving plane impulsive light–like signals.

Using (6.5) we can derive the equation

$$2\,\frac{\partial^2}{\partial u\,\partial v}\log\left(\frac{V_v}{V_u}\right) = \frac{\partial}{\partial u}\left(U_u\,\frac{V_v}{V_u}\right) - \frac{\partial}{\partial v}\left(U_v\,\frac{V_u}{V_v}\right)\ , \tag{6.16}$$

while from (6.9) and (6.11) we find

$$\frac{U_u\,V_v}{V_u^2}\,\Psi_4 - \frac{U_v\,V_u}{V_v^2}\,\Psi_0 = \frac{\partial^2}{\partial u\,\partial v}\log\left(\frac{V_u}{V_v}\right)\ . \tag{6.17}$$

These are interesting equations. They show that if we assume the separation of variables

$$\frac{V_u}{V_v} = \frac{P(u)}{Q(v)}\ , \tag{6.18}$$

for some functions $P(u), Q(v)$ then in principle there exist coordinates $\bar{u} = \bar{u}(u)$ and $\bar{v} = \bar{v}(v)$ such that

$$V_{\bar{u}} = V_{\bar{v}} \ . \tag{6.19}$$

Now (6.16) becomes, in this barred frame, the wave equation

$$U_{\bar{u}\bar{u}} = U_{\bar{v}\bar{v}} \ , \tag{6.20}$$

while (6.17) becomes

$$U_{\bar{u}}\,\Psi_4 = U_{\bar{v}}\,\Psi_0 \ . \tag{6.21}$$

If backscattered gravitational radiation exists after the collision then $\Psi_0 \neq 0$ and $\Psi_4 \neq 0$. There are two opposing classes of radiation here, one with propagation direction $\partial/\partial u$ and another with propagation direction $\partial/\partial v$. If we write (6.21) in the form

$$\frac{\Psi_4^2}{U_{\bar{v}}^2} = \frac{\Psi_0^2}{U_{\bar{u}}^2} \ , \tag{6.22}$$

then both sides of this equation have the dimensions of $(\text{length})^{-2}$ and thus the equation is open to the interpretation that in the barred frame the energy densities of the two classes of backscattered waves are equal. We note that field energy density is in general not an invariant quantity. For example in Maxwell's electromagnetic theory on Minkowskian space–time the energy density of the electromagnetic field is not Lorentz invariant.

With U given by (6.15) we can evaluate (6.5) on $v = 0$ to obtain

$$V_{vu} = \frac{u}{1-u^2}\,V_v \ , \tag{6.23}$$

giving V_v on $v = 0$ as $V_v = C\,(1-u^2)^{-1/2}$ where C is a constant of integration. From (6.13) we see that when $u = 0$, $V_v = 2/(1-v^2)$. This must agree with the solution of (6.23) when $u = v = 0$ and thus the constant of integration $C = 2$. A similar argument applies to V_u when $u = 0$ and when $v = 0$. Hence we may summarise by saying that when $v = 0$,

$$V_v = \frac{2}{\sqrt{1-u^2}} \qquad \text{and} \qquad V_u = \frac{2}{1-u^2} \ , \tag{6.24}$$

and when $u = 0$,

$$V_v = \frac{2}{1-v^2} \qquad \text{and} \qquad V_u = \frac{2}{\sqrt{1-v^2}} \ . \tag{6.25}$$

Using these the assumption (6.18) leads to

$$\sqrt{1-u^2}\,V_u = \sqrt{1-v^2}\,V_v \ . \tag{6.26}$$

If coordinates $(\bar{u},\bar{v})$ are introduced via

$$u = \sin\bar{u}\ , \qquad v = \sin\bar{v}\ , \tag{6.27}$$

then we can write (6.26) simply as (6.19) from which we conclude that

$$V = V(\bar{u}+\bar{v})\ . \tag{6.28}$$

In the barred coordinates (6.18) reads

$$U = -\log\{\cos(\bar{u}-\bar{v})\,\cos(\bar{u}+\bar{v})\}\ , \tag{6.29}$$

and the boundary conditions (6.15) and (6.16) to be satisfied by V and $\bar{M}$, with $e^{-\bar{M}} = e^{-M}\,\cos\bar{u}\,\cos\bar{v}$, are

when $\bar{v}=0$:

$$V = \log\left(\frac{1+\sin\bar{u}}{1-\sin\bar{u}}\right)\ , \qquad \bar{M} = -\log\cos\bar{u}\ ; \tag{6.30}$$

when $\bar{u}=0$:

$$V = \log\left(\frac{1+\sin\bar{v}}{1-\sin\bar{v}}\right)\ , \qquad \bar{M} = -\log\cos\bar{v}\ . \tag{6.31}$$

It thus follows from (6.28) that in $\bar{u}>0, \bar{v}>0$,

$$V = \log\left(\frac{1+\sin(\bar{u}+\bar{v})}{1-\sin(\bar{u}+\bar{v})}\right) = \log\left[\left(\frac{\cos\bar{u}+\sin\bar{v}}{\cos\bar{u}-\sin\bar{v}}\right)\left(\frac{\cos\bar{v}+\sin\bar{u}}{\cos\bar{v}-\sin\bar{u}}\right)\right]\ . \tag{6.32}$$

From this we see that

$$V_{\bar{u}} = V_{\bar{v}} = \frac{2}{\cos(\bar{u}+\bar{v})}\ . \tag{6.33}$$

The field equations (6.6) and (6.7) in terms of the barred variables read

$$2\,U_{\bar{v}}\,\bar{Q}_{\bar{v}} = -2\,(U_{\bar{v}\bar{v}} - U_{\bar{v}}^2) + V_{\bar{v}}^2\ , \tag{6.34}$$

$$2\,U_{\bar{u}}\,\bar{Q}_{\bar{u}} = -2\,(U_{\bar{u}\bar{u}} - U_{\bar{u}}^2) + V_{\bar{u}}^2\ , \tag{6.35}$$

with $\bar{Q} = \bar{M} + \frac{1}{2}\,U$. From (6.29) we have

$$U_{\bar{v}\bar{v}} - U_{\bar{v}}^2 = 2 + 2\,\tan(\bar{u}-\bar{v})\,\tan(\bar{u}+\bar{v})\ , \tag{6.36}$$

$$U_{\bar{u}\bar{u}} - U_{\bar{u}}^2 = 2 - 2\,\tan(\bar{u}-\bar{v})\,\tan(\bar{u}+\bar{v})\ . \tag{6.37}$$

Substituting (6.33), (6.36) and (6.37) into (6.34) and (6.35) yields

$$\bar{Q}_{\bar{u}} = \bar{Q}_{\bar{v}} = 2\,\tan(\bar{u}+\bar{v})\ . \tag{6.38}$$

The remaining field equation (6.8) expressed in the barred variables reads now

$$2\,\bar{Q}_{\bar{u}\bar{v}} = V_{\bar{u}}\,V_{\bar{v}}\ , \tag{6.39}$$

and it is clear from (6.33) and (6.39) that this equation is satisfied. The boundary conditions to be imposed on $\bar{Q} = \bar{M} + \frac{1}{2}\,U$ can be deduced from (6.31) and (6.32) and then (6.38) can be solved to give in $\bar{u} > 0, \bar{v} > 0$,

$$\bar{Q} = -2\,\log\cos(\bar{u}+\bar{v})\ . \tag{6.40}$$

Now the line–element in $\bar{u} > 0, \bar{v} > 0$ is given by (6.3) in the barred variables with U given by (6.29), V by (6.32) and $\bar{M}$ by

$$\mathrm{e}^{-\bar{M}} = \sqrt{\frac{\cos^3(\bar{u}+\bar{v})}{\cos(\bar{u}-\bar{v})}}\ . \tag{6.41}$$

Using the transformation (6.27) we return to the unbarred variables with U given by (6.15),

$$\mathrm{e}^{V} = \left(\frac{\sqrt{1-u^2}+v}{\sqrt{1-u^2}-v}\right)\left(\frac{\sqrt{1-v^2}+u}{\sqrt{1-v^2}-u}\right)\ , \tag{6.42}$$

$$\mathrm{e}^{-\bar{M}} = \frac{(1-u^2-v^2)^{3/2}}{(\sqrt{1-u^2}\sqrt{1-v^2}+u\,v)^2}\ , \tag{6.43}$$

$$d\bar{u} = \frac{du}{\sqrt{1-u^2}}\ , \qquad d\bar{v} = \frac{dv}{\sqrt{1-v^2}}\ , \tag{6.44}$$

from which we can reconstruct the Khan–Penrose solution (6.1) for $u > 0, v > 0$. We note that there is a curvature singularity in the region $u > 0, v > 0$ at $u^2 + v^2 = 1$.

The simple assumption upon which this derivation of the Khan–Penrose solution is based will be generalised in the next section. In the form in which it appears here it can yield the solution to other collision problems. As an illustration we consider the collision of a plane impulsive gravitational wave of the type considered here with another such wave sharing its wave front with a plane electromagnetic shock wave [Barrabès, Bressange and Hogan (1998)]. The line–element of the space–time containing the history of the latter signal can be put in the form

$$ds^2 = (1+b^2v_+^2)^{-1}\left\{-2du\,dv + (1+lv_+)^2dx^2 + (1-lv_+)^2dy^2\right\}\ , \tag{6.45}$$

where b, l are constants introduced for convenience to easily identify special cases and $v_+ = v\,\vartheta(v)$. For this space–time the only non–vanishing Newman–Penrose component of the Maxwell field is

$$\phi_0 = \frac{b\,\theta(v)}{1 + b^2 v_+^2} , \tag{6.46}$$

and the only non–vanishing Newman–Penrose component of the Weyl tensor is

$$\Psi_0 = -l\,\delta(v) . \tag{6.47}$$

Thus both the Maxwell and Weyl tensors are type N in the Petrov classification with $\partial/\partial u$ as degenerate principal null direction. The null hypersurface $v = 0$ is a null hyper*plane* and is the history of a plane electromagnetic shock wave on account of (6.46) and of a plane gravitational impulse wave on account of (6.47). We can remove the shock by putting $b = 0$ and we can remove the gravitational impulse wave by putting $l = 0$.

We consider now the head–on collision of a wave of the type described by (6.45) with a plane gravitational impulsive wave. This latter will be described by the space–time with line–element

$$ds^2 = -2du\,dv + (1 + ku_+)^2 dx^2 + (1 - ku_+)^2 dy^2 , \tag{6.48}$$

with k a useful constant and $u_+ = u\,\theta(u)$. The line–element in the region $u > 0, v > 0$ (after the collision) has the Rosen–Szekeres form (6.3)and the boundary conditions replacing (6.12) and (6.13) can be read off from the line–elements (6.45) and (6.48). They are given by:

When $v = 0, u > 0$:

$$\mathrm{e}^V = \frac{1 + ku}{1 - ku} , \qquad \mathrm{e}^M = 1 , \qquad \mathrm{e}^{-U} = 1 - k^2 u^2 , \tag{6.49}$$

When $u = 0, v > 0$:

$$\mathrm{e}^V = \frac{1 + lv}{1 - lv} , \qquad \mathrm{e}^M = 1 + b^2 v^2 , \qquad \mathrm{e}^{-U} = \frac{1 - l^2 v^2}{1 + b^2 v^2} . \tag{6.50}$$

In addition the Maxwell field in the region $u > 0, v > 0$ has two non–vanishing Newman–Penrose components ϕ_0, ϕ_2 which are both functions of (u, v) and satisfy the boundary conditions: when $v = 0, \phi_2 = 0$ and when $u = 0, \phi_0 = b(1 + b^2 v^2)^{-1}$. It is now a matter of solving the vacuum Einstein–Maxwell equations in the region $u > 0, v > 0$ (these can be found in [Griffiths (1991)] for example) for the unknown functions U, V, M, ϕ_2, ϕ_0 subject to the above boundary conditions. We find the following expressions for these functions:

$$\mathrm{e}^{-U} = \frac{F}{1+b^2v^2}\ , \tag{6.51}$$

$$\mathrm{e}^{V} = \frac{1+ku\sqrt{1-l^2v^2}+lv\sqrt{1-k^2u^2}}{1-ku\sqrt{1-l^2v^2}-lv\sqrt{1-k^2u^2}}\ , \tag{6.52}$$

$$\mathrm{e}^{-M} = \frac{H^2}{(1+b^2v^2)\left[(1-k^2u^2)(1-l^2v^2)F\right]^{1/2}}\ , \tag{6.53}$$

$$\phi_2 = \frac{-kbv\sqrt{1-l^2v^2}}{\left[(1-k^2u^2)F\right]^{1/2}H}\ , \tag{6.54}$$

$$\phi_0 = \frac{b\left\{(l^2+b^2)lkuv^3+\sqrt{1-k^2u^2}(1-l^2v^2)^{3/2}\right\}}{(1+b^2v^2)\left[(1-l^2v^2)F\right]^{1/2}H}\ , \tag{6.55}$$

where

$$F = 1-k^2u^2-l^2v^2-k^2b^2u^2v^2\ , \tag{6.56}$$

and

$$H = \sqrt{1-k^2u^2}\sqrt{1-l^2v^2}-kluv\ . \tag{6.57}$$

A calculation of the Weyl tensor components reveals the expected curvature singularity at $F=0$ for $u>0, v>0$. There are two important special cases: (1) if $b=0$ the solution above becomes the Penrose–Khan solution (6.1) and (2) if $l=0$ the solution becomes the Griffiths [Griffiths (1975)] solution describing the space–time following the collision of a plane gravitational impulse wave and a plane electromagnetic shock wave. Scalar field extensions of this solution have been constructed [Halilsoy and Sakalli (2003)].

6.2 Colliding Plane Impulsive Gravitational Waves: The General Case

The generalization of the Khan–Penrose solution to the collision of non-linearly polarized plane impulsive gravitational waves was found by Nutku and Halil [Nutku and Halil (1977)]. This solution was obtained using a harmonic mapping technique. It was later re-derived by Chandrasekhar and Ferrari [Chandrasekhar and Ferrari (1984)] ("This paper is addressed, principally, to a more standard derivation of the Nutku–Halil solution than the one sketched by the authors.") who, as mentioned in §6.1, developed an Ernst–type formulation for vacuum space–times admitting two space–like Killing vectors. They demonstrated that "in some sense, the Nutku–Halil solution occupies the same place in space–times with two space–like Killing vectors as the Kerr solution does in space–times with one time–like and one space–like Killing vector". Thus the origin of the solution is still quite mysterious. We will now discuss the assumption described above, which led to the solutions given in §6.1, in this more general context. The qualitative picture we have of the physics, at least up to the appearance of a curvature singularity, is quite simple: two non–interacting plane impulsive waves undergo a head–on collision and the interaction region afterwards contains backscattered radiation from both waves, neither of which remain plane. The backscattered radiation in the interaction region of space–time between the histories of the waves after the collision determines two intersecting congruences of null geodesics. These are the 'rays' associated with the two systems of backscattered radiation. Both congruences have expansion and shear. The ratio of the expansions of each congruence is immediately determined from Einstein's vacuum field equations and the boundary conditions. The shear of each congruence (the modulus of a complex variable in each case) depends in general in a simple way on the choice of parameter along the null geodesics (in the sense that a change of parameter induces a rescaling of the shear). *We assume that a parameter exists along each of the two families of null geodesics such that the shear (i.e. the modulus of the 'complex shear') of each congruence is equal.* This is the only assumption made apart from the usual assumption of analyticity of the solution of Einstein's field equations [Kramer et al. (1980)]. We shall see that it implies, together with the field equations, an equation that could be interpreted physically as saying that the energy density of the backscattered radiation from each wave after collision is the same. In addition this assumption leads to the complete integration of the vacuum field equations [Barrabès, Bressange and Hogan (1999)].

The line–element of the space–time describing the vacuum gravitational field of a single plane impulsive gravitational wave having the maximum two degrees of freedom of polarisation is obtained from (2.63) with $F = v + \frac{b}{2}(x^2 - y^2) + c\,x\,y$ where b, c are constants. This line–element takes the Rosen–Szekeres form (generalizing (6.3))

$$ds^2 = \mathrm{e}^{-U}\left(\mathrm{e}^{V}\cosh W\,dx^2 - 2\sinh W\,dx\,dy + \mathrm{e}^{-V}\cosh W\,dy^2\right) - 2\,\mathrm{e}^{-M}du\,dv\,, \tag{6.58}$$

with

$$e^{-U} = 1 - (b^2 + c^2)u_+^2 \ , \tag{6.59}$$

$$e^{-V} = \left[\frac{1 + (b^2 + c^2)u_+^2 - 2b\,u_+}{1 + (b^2 + c^2)u_+^2 + 2bu_+}\right]^{\frac{1}{2}} \ , \tag{6.60}$$

$$\sinh W = \frac{2c\,u_+}{1 - (b^2 + c^2)u_+^2} \ , \tag{6.61}$$

$$M = 0. \tag{6.62}$$

We consider the head–on collision of this wave with a wave of similar type. This latter wave is described by a space–time with line–element (6.58) but with

$$e^{-U} = 1 - (\alpha^2 + \beta^2)v_+^2 \ , \tag{6.63}$$

$$e^{-V} = \left[\frac{1 + (\alpha^2 + \beta^2)v_+^2 - 2\alpha v_+}{1 + (\alpha^2 + \beta^2)v_+^2 + 2\alpha v_+}\right]^{\frac{1}{2}} \ , \tag{6.64}$$

$$\sinh W = \frac{2\beta v_+}{1 - (\alpha^2 + \beta^2)v_+^2} \ , \tag{6.65}$$

$$M = 0, \tag{6.66}$$

with α, β real constants and $u_+ = u\,\vartheta(u)$, $v_+ = v\,\vartheta(v)$. The history of the wave front is the null hyperplane $v = 0$ in this case. Although it may seem awkward at first, it is actually very convenient to carry the constants b, c, α, β in order to illustrate special cases. For the collision we consider the space–time to have line–element (6.58) with U, V, W, M given by (6.59)– (6.62) in the region $v < 0, u > 0$ and given by (6.63)– (6.66) in the region $u < 0, v > 0$. The region $u < 0, v < 0$ has line–element (6.58) with $U = V = W = M = 0$ (which agrees with (6.59)–(6.66) when both $v < 0$ and $u < 0$). The line–element in the region $u > 0, v > 0$ (after the collision) has the form (6.58) with U, V, W, M functions of (u, v) satisfying the junction conditions (these are the O'Brien–Synge [O'Brien and Synge (1952)] junction conditions): If $v = 0, u > 0$ then U, V, W, M are given by (6.59)–(6.62) with $u_+ = u$ and if $u = 0, v > 0$ then U, V, W, M are given by (6.63)–(6.66) with $v_+ = v$. Einstein's vacuum field equations have to be solved for U, V, W, M in the interaction region $(u > 0, v > 0)$ after the collision subject to these boundary (junction) conditions. These equations are [Griffiths (1991)] :

$$U_{uv} = U_u\,U_v \ , \tag{6.67}$$

$$2V_{uv} = U_u\,V_v + U_v\,V_u - 2\,(V_u\,W_v + V_v\,W_u)\tanh W \ , \tag{6.68}$$

$$2W_{uv} = U_u\, W_v + U_v\, W_u + 2V_u\, V_v \sinh W \cosh W \ , \tag{6.69}$$

$$2U_u\, M_u = -2U_{uu} + U_u^2 + W_u^2 + V_u^2 \cosh^2 W \ , \tag{6.70}$$

$$2U_v\, M_v = -2U_{vv} + U_v^2 + W_v^2 + V_v^2 \cosh^2 W \ , \tag{6.71}$$

$$2M_{uv} = -U_{uv} + W_u\, W_v + V_u\, V_v \cosh^2 W \ . \tag{6.72}$$

The first of these equations can, as before, be immediately solved in conjuction with the boundary conditions to be satisfied by U on $u = 0$ and on $v = 0$ to yield, in $u > 0, v > 0$,

$$\mathrm{e}^{-U} = 1 - (b^2 + c^2)\, u^2 - (\alpha^2 + \beta^2)\, v^2 \ . \tag{6.73}$$

The problem is to solve (6.68)–(6.69) for V, W subject to the boundary conditions and then to solve (6.70) and (6.71) for M. Equation (6.72) is the integrability condition for (6.70) and (6.71).

We shall for the moment focus attention on the two field equations (6.68) and (6.69). All of our considerations from now on will apply to the interaction region of space–time $u > 0, v > 0$ after the collision. Introducing the complex variables

$$A = -V_u \cosh W + iW_u \ , \qquad B = -V_v \cosh W + iW_v \ , \tag{6.74}$$

we can rewrite the two real equations (6.68) and (6.69) as the single complex equation

$$2A_v = U_u\, B + U_v\, A - 2iA\, V_v \sinh W \ , \tag{6.75}$$

or equivalently as the single complex equation

$$2B_u = U_u\, B + U_v\, A - 2iB\, V_u \sinh W \ . \tag{6.76}$$

Given the form of the line–element (6.58) it is convenient to introduce a null tetrad $\{m, \bar{m}, l, n\}$ in the region $u > 0, v > 0$ defined by

$$\begin{aligned}\sqrt{2}\,\mathrm{e}^{-U/2}\, m = {} & \mathrm{e}^{-V/2}\left(\cosh\frac{W}{2} - i\sinh\frac{W}{2}\right)\frac{\partial}{\partial x} \\ & + \mathrm{e}^{V/2}\left(\sinh\frac{W}{2} - i\cosh\frac{W}{2}\right)\frac{\partial}{\partial y} \ ,\end{aligned} \tag{6.77}$$

$$l = \mathrm{e}^{M/2}\frac{\partial}{\partial v} \ , \tag{6.78}$$

$$n = \mathrm{e}^{M/2}\frac{\partial}{\partial u} \ , \tag{6.79}$$

with $\bar{m}$ the complex conjugate of m. The integral curves of the vector fields l and n are twist–free, null geodesics. The coordinate v is not an affine parameter along the integral curves of l and these curves have complex shear σ_l and real expansion ρ_l given by

$$\sigma_l = \frac{1}{2}\,\mathrm{e}^{M/2}\,B\ , \qquad \rho_l = \frac{1}{2}\,\mathrm{e}^{M/2}\,U_v\ , \tag{6.80}$$

with B as in (6.74). Likewise the coordinate u is not an affine parameter along the integral curves of n and these curves have complex shear σ_n and real expansion ρ_n given by

$$\sigma_n = \frac{1}{2}\,\mathrm{e}^{M/2}\,A\ , \qquad \rho_n = \frac{1}{2}\,\mathrm{e}^{M/2}\,U_u\ , \tag{6.81}$$

with A as in (6.74). We thus see from (6.73), (6.80) and (6.81) that the ratio ρ_l/ρ_n is now known in the region $u > 0, v > 0$. In terms of the variables introduced above the non–identically vanishing scale–invariant components [Griffiths (1991)] of the Riemann tensor in Newman–Penrose notation are Ψ_0, Ψ_2, Ψ_4 given by

$$2\,\Psi_0 = B_v + (M_v - U_v)\ B + iB\,V_v\sinh W\ , \tag{6.82}$$

$$2\,\Psi_2 = M_{uv} - \frac{1}{4}\ \left(A\,\bar{B} - \bar{A}\,B\right)\ , \tag{6.83}$$

$$2\,\bar{\Psi}_4 = A_u + (M_u - U_u)\ A + iA\,V_u\sinh W\ . \tag{6.84}$$

When these are non–zero we interpret Ψ_0 as describing radiation, having propagation direction n in space–time, backscattered from the wave with history $u = 0, v > 0$ and we interpret Ψ_4 as describing radiation, having propagation direction l in space–time, backscattered from the wave with history $v = 0, u > 0$. Thus the integral curves of the null vector fields n and l are the 'rays' associated with the backscattered radiation from the two separating waves after collision.

Let us write

$$A = |A|\ \mathrm{e}^{i\theta}\ , \qquad B = |B|\ \mathrm{e}^{i\phi}\ , \qquad f = \theta - \phi\ , \tag{6.85}$$

with θ and ϕ real. From (6.75) and (6.76) we can obtain the equations

$$\theta_v = -\frac{|B|}{2\,|A|}\,U_u\ \sin f - V_v\sinh W\ , \tag{6.86}$$

$$\phi_u = \frac{|A|}{2\,|B|}\,U_v\ \sin f - V_u\sinh W\ , \tag{6.87}$$

and

$$2f_{uv} + i\left(A\,\bar{B} - \bar{A}\,B\right) = -\left(U_u\,\frac{|B|}{|A|}\,\sin f\right)_u - \left(U_v\,\frac{|A|}{|B|}\,\sin f\right)_v \ . \tag{6.88}$$

If we now transform the tetrad $\{m, \bar{m}, l, n\}$ by the rotation

$$m \to \hat{m} = \mathrm{e}^{i\psi}\,m\ , \tag{6.89}$$

with ψ a real-valued function, then

$$\theta \to \hat{\theta} = \theta - 2\,\psi\ , \qquad \phi \to \hat{\phi} = \phi - 2\,\psi\ , \qquad f \to f\ , \tag{6.90}$$

and by (6.86) and (6.87)

$$\left(\hat{\theta} + \hat{\phi}\right)_u = \frac{|A|}{|B|}\,U_v\,\sin f + f_u - 2\,V_u \sinh W - 4\psi_u\ , \tag{6.91}$$

$$\left(\hat{\theta} + \hat{\phi}\right)_v = -\frac{|B|}{|A|}\,U_u\,\sin f - f_v - 2\,V_v \sinh W - 4\psi_v\ . \tag{6.92}$$

We are free to choose ψ to make the right hand sides of (6.91) and (6.92) vanish because the integrability condition for the resulting pair of first order partial differential equations for ψ is the field equation (6.88). Hence it is always possible to choose a tetrad so that θ and ϕ in (6.85) have the property that $\theta + \phi =$ constant. Thus in looking for an interesting assumption to make about the rays associated with the backscattered radiation fields we should consider the ratio

$$\frac{A}{B} = \frac{|A|}{|B|}\,\mathrm{e}^{if} = \frac{\sigma_n}{\sigma_l}\ , \tag{6.93}$$

with the last equality following from (6.80) and (6.81). We note that f satisfies the second order equation (6.88) to which we shall return later. It is clear from (6.74) that a change of parameters $u \to \bar{u} = \bar{u}(u)$ and $v \to \bar{v} = \bar{v}(v)$ along the integral curves of n and l rescales A and B by a function of u and a function of v respectively. This change of parameter obviously leaves the form of the line-element (6.58) invariant. Also from the field equation (6.75) we deduce that

$$\left(|A|^2\right)_v - U_v\,|A|^2 = \frac{1}{2}\,U_u\,\left(A\,\bar{B} + \bar{A}\,B\right)\ , \tag{6.94}$$

and from the equivalent equation (6.76) we find

$$\left(|B|^2\right)_u - U_u\,|B|^2 = \frac{1}{2}\,U_v\,\left(A\,\bar{B} + \bar{A}\,B\right)\ , \tag{6.95}$$

Using these two equations we obtain

$$2\left[\log\left(\frac{|A|^2}{|B|^2}\right)\right]_{uv}=\left(U_u\,\frac{(A\,\bar{B}+\bar{A}\,B)}{|A|^2}\right)_u-\left(U_v\,\frac{(A\,\bar{B}+\bar{A}\,B)}{|B|^2}\right)_v\ ,\tag{6.96}$$

which is a partner for the equation (6.88) for f. This suggests that it might be interesting to explore the following assumption concerning the rays associated with the backscattered radiation: *there exist parameters* $\bar{u},\bar{v}$ *along the integral curves of* n *and* l *respectively such that* $|A|^2=|B|^2$ *(or equivalently* $|\sigma_n|^2=|\sigma_l|^2$*).* This is equivalent to the assumption that there exist functions $C(u), D(v)$ such that

$$\frac{|A|^2}{|B|^2}=C(u)\,D(v)\ .\tag{6.97}$$

When $u=0$ it follows from (6.60) and (6.61) that

$$|B|^2=\frac{4\,(\alpha^2+\beta^2)}{(1-(\alpha^2+\beta^2)\,v^2)^2}\ ,\tag{6.98}$$

and when $v=0$ it follows from (6.64) and (6.65) that

$$|A|^2=\frac{4\,(b^2+c^2)}{(1-(b^2+c^2)\,u^2)^2}\ .\tag{6.99}$$

Also when $u=0$ we see from (6.73) that the right hand side of (6.94) vanishes and thus solving (6.94) for $|A|^2$ when $u=0$ we obtain

$$|A|^2=\frac{4\,(b^2+c^2)}{1-(\alpha^2+\beta^2)\,v^2}\ ,\tag{6.100}$$

with the constant numerator here (the constant of integration) chosen so that the two expressions (6.99) and (6.100) for $|A|^2$ agree when $u=0$ *and* $v=0$. Similarly when $v=0$ the right hand side of (6.95) vanishes and we readily obtain, when $v=0$,

$$|B|^2=\frac{4\,(\alpha^2+\beta^2)}{1-(b^2+c^2)\,u^2}\ .\tag{6.101}$$

Thus (6.97) together with the boundary conditions at $u=0$ and at $v=0$ results in

$$\frac{|A|^2}{|B|^2}=\left(\frac{b^2+c^2}{\alpha^2+\beta^2}\right)\left[\frac{1-(\alpha^2+\beta^2)\,v^2}{1-(b^2+c^2)\,u^2}\right]\ .\tag{6.102}$$

Hence there exist parameters $(\bar{u}, \bar{v})$ given by

$$\bar{u} = \sin^{-1}\left(u\sqrt{b^2+c^2}\right) , \qquad \bar{v} = \sin^{-1}\left(v\sqrt{\alpha^2+\beta^2}\right) , \tag{6.103}$$

such that

$$\frac{V_{\bar{u}}^2 \cosh^2 W + W_{\bar{u}}^2}{V_{\bar{v}}^2 \cosh^2 W + W_{\bar{v}}^2} = 1 . \tag{6.104}$$

We note that when $\bar{u} = 0$ we have $u = 0$ and when $\bar{v} = 0$ we have $v = 0$. We shall express (6.104) by saying that, in the coordinates $(\bar{u}, \bar{v})$, (6.102) reads $|A|^2 = |B|^2$. In the coordinates $(\bar{u}, \bar{v})$ the form of the field equations and the expressions for the Riemann tensor remain invariant, with the derivatives with respect to (u, v) being replaced by derivatives with respect to $(\bar{u}, \bar{v})$ and with M replaced by $\bar{M}$ according to

$$\mathrm{e}^{-\bar{M}} = \frac{\cos\bar{u}\,\cos\bar{v}}{\sqrt{\alpha^2+\beta^2}\,\sqrt{a^2+b^2}}\,\mathrm{e}^{-M} . \tag{6.105}$$

We note from (6.73) and (6.103) that in the barred coordinates

$$\mathrm{e}^{-U} = \cos(\bar{u}-\bar{v})\,\cos(\bar{u}+\bar{v}) . \tag{6.106}$$

Also in these coordinates (6.96) becomes

$$(U_{\bar{u}}\,\cos f)_{\bar{u}} = (U_{\bar{v}}\,\cos f)_{\bar{v}} \qquad \Longleftrightarrow \qquad U_{\bar{u}}\,f_{\bar{u}} = U_{\bar{v}}\,f_{\bar{v}} , \tag{6.107}$$

(the equivalence here following from (6.106) since now $U_{\bar{u}\bar{u}} = U_{\bar{v}\bar{v}}$) from which we conclude that

$$f = f(\lambda) \qquad \text{with} \qquad \lambda = \frac{\cos(\bar{u}-\bar{v})}{\cos(\bar{u}+\bar{v})} . \tag{6.108}$$

We see that $\lambda = 1$ when $\bar{u} = 0$ and/or when $\bar{v} = 0$. Also (6.94) and (6.95) become

$$\left(|A|^2\right)_{\bar{v}} - U_{\bar{v}}\,|A|^2 = U_{\bar{u}}\,|A|^2\,\cos f , \tag{6.109}$$

$$\left(|A|^2\right)_{\bar{u}} - U_{\bar{u}}\,|A|^2 = U_{\bar{v}}\,|A|^2\,\cos f . \tag{6.110}$$

It thus follows that, again in the barred coordinates,

$$|A|^2 = |B|^2 = \mathrm{e}^U\,g(\lambda) , \tag{6.111}$$

for some function $g(\lambda)$ satisfying

$$\lambda\,g' = g\,\cos f , \tag{6.112}$$

with the prime here and henceforth denoting differentiation with respect to λ.

In order to intepret physically the implications of our assumption that there exists $(\bar{u}, \bar{v})$ such that $|A|^2 = |B|^2$ we proceed as follows: using (6.93) and the field equations (6.70) and (6.71) in the expressions (6.82) and (6.84) for Ψ_0 and Ψ_4 we find that we can write

$$U_v \frac{\Psi_0}{B} = \frac{1}{2} \left\{ -iU_v\, f_v - U_{vv} + \frac{1}{2}\, |B|^2 + \frac{B}{2\,A}\, U_u\, U_v + U_v \left(\log \frac{|B|}{|A|} \right)_v \right\} , \tag{6.113}$$

and

$$U_u \frac{\bar{\Psi}_4}{A} = \frac{1}{2} \left\{ iU_u\, f_u - U_{uu} + \frac{1}{2}\, |A|^2 + \frac{A}{2\,B}\, U_u\, U_v + U_u \left(\log \frac{|A|}{|B|} \right)_u \right\} . \tag{6.114}$$

When these are expressed in the coordinates $(\bar{u}, \bar{v})$ we can put $|A|^2 = |B|^2$ and as above $U_{\bar{u}\bar{u}} = U_{\bar{v}\bar{v}}$ and $U_{\bar{u}}\, f_{\bar{u}} = U_{\bar{v}}\, f_{\bar{v}}$ and thus

$$U_{\bar{u}} \frac{\bar{\Psi}_4}{\bar{A}} = U_{\bar{v}} \frac{\Psi_0}{B} . \tag{6.115}$$

From this it follows, using the second of (6.80) and of (6.81) that

$$\rho_l^{-2}\, |\Psi_4|^2 = \rho_n^{-2}\, |\Psi_0|^2 . \tag{6.116}$$

With the coordinates carrying the dimensions of length both sides of this equation have the dimensions, $(\text{length})^{-2}$, of energy density. Both sides are positive definite expressions in terms of the backscattered radiation fields. Hence one might conclude that *in the coordinate system (the barred system) in which $|A|^2 = |B|^2$ the energy density of the backscattered radiation from each of the separating waves after the collision is the same.*

We begin by writing (6.88) in the barred system $(\bar{u}, \bar{v})$. Using (6.93) with $|A| = |B|$, (6.108) and (6.111) we find that

$$(1-\lambda^2)\, f'' - 2\,\lambda\, f' + \frac{g}{\lambda}\, \sin f = \frac{(1+\lambda^2)}{\lambda^2}\, \sin f - \frac{(1-\lambda^2)}{\lambda}\, f'\, \cos f , \tag{6.117}$$

with g given in terms of f by (6.112). We can simplify (6.117) to read

$$g = -\frac{\lambda}{\sin f}\, \frac{d}{d\lambda} \left[(1-\lambda^2) \left(f' + \frac{\sin f}{\lambda} \right) \right] . \tag{6.118}$$

We get a single third order equation for f by eliminating g (taken to be non-zero) between (6.112) and (6.118). Since we are working in the barred coordinate system (6.93) gives

$$\frac{A}{B} = \frac{-V_{\bar{u}} \cosh W + iW_{\bar{u}}}{-V_{\bar{v}} \cosh W + iW_{\bar{v}}} = e^{if} = \frac{1 - ih}{1 + ih} , \tag{6.119}$$

where, for convenience, we have introduced $h(\lambda)$ by the final equality. After eliminating g from (6.112) and (6.118) it is useful to write the resulting equation as a differential equation for $h(\lambda)$. Then defining

$$G = -\frac{2}{1+h^2} (1 - \lambda^2) \left(h' - \frac{h}{\lambda} \right) , \tag{6.120}$$

the equation for $h(\lambda)$ can be put in the form

$$\lambda G'' + G' - \frac{4}{\lambda} G = -\frac{GQ}{1 - \lambda^2} , \tag{6.121}$$

where

$$Q = \frac{\lambda}{2h} (1 - h^2) G' + \frac{1}{h} (1 + h^2) G . \tag{6.122}$$

We remark that if we define

$$P = \frac{\lambda}{2h} (1 + h^2) G' + \frac{1}{h} (1 - h^2) G , \tag{6.123}$$

then

$$\lambda P' = Q \qquad \text{and} \qquad P^2 - Q^2 = \lambda^2 (G')^2 - 4 G^2 . \tag{6.124}$$

In studying the differential equation (6.121) for h we found it helpful to write (6.121) and the second of (6.124) in the form

$$\lambda G'' + G' - \frac{4}{\lambda} G = -\frac{\lambda G P'}{1 - \lambda^2} , \tag{6.125}$$

$$P^2 - \lambda^2 (P')^2 = \lambda^2 (G')^2 - 4 G^2 , \tag{6.126}$$

and to work with these equations. Before proceeding further however we need to know h and h' when $\bar{u} = 0$ and/or when $\bar{v} = 0$, i.e. we require $h(1)$ and $h'(1)$. To find $h(1)$ start by writing (6.119), using (6.103), as

$$\frac{1 - i\, h(\lambda)}{1 + i\, h(\lambda)} = \frac{\sqrt{\alpha^2 + \beta^2}\, \sqrt{1 - (b^2 + c^2)\, u^2}}{\sqrt{b^2 + c^2}\, \sqrt{1 - (\alpha^2 + \beta^2)\, v^2}} \left(\frac{-V_u \cosh W + iW_u}{-V_v \cosh W + iW_v} \right) , \tag{6.127}$$

and evaluate this equation when $u = 0$ *and* $v = 0$. From the boundary conditions on V and W given by (6.60), (6.61), (6.64) and (6.65) we have that when $u = 0$ *and* $v = 0$:

$$V_u = 2\,b\ , V_v = 2\,\alpha\ , W_u = 2\,c\ , W_v = 2\,\beta\ , \tag{6.128}$$

and thus from (6.127),

$$\frac{1 - i\,h(1)}{1 + i\,h(1)} = \frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{b^2 + c^2}}\left(\frac{b - ic}{\alpha - i\beta}\right) = \mathrm{e}^{i(\hat{\alpha} - \hat{\beta})}\ , \tag{6.129}$$

where

$$\mathrm{e}^{i\hat{\alpha}} = \frac{b - ic}{\sqrt{b^2 + c^2}}\ , \qquad \mathrm{e}^{i\hat{\beta}} = \frac{\alpha - i\beta}{\sqrt{\alpha^2 + \beta^2}}\ . \tag{6.130}$$

It thus follows from (6.129) that

$$h(1) = -\tan\left(\frac{\hat{\alpha} - \hat{\beta}}{2}\right) = k\ (\text{say})\ . \tag{6.131}$$

Next to find $h'(1)$ we begin with (6.127) and by two differentiations obtain from it

$$-\frac{4i\,(b^2 + c^2)\,h'(1)}{(1 + i\,h(1))^2} = \left[\frac{\partial^2}{\partial u \partial v}\left(\frac{-V_u \cosh W + iW_u}{-V_v \cosh W + iW_v}\right)\right]_{(u=0, v=0)}\ . \tag{6.132}$$

To evaluate the right hand side here we first note from (6.73) that when $u = 0$ *and* $v = 0$, $U_u = 0$ and $U_v = 0$. Also from the boundary conditions on W we have $W = 0$ when $u = 0$ *and* $v = 0$. Now evaluating the field equations (6.68) and (6.69) when $u = 0$ *and* $v = 0$ we easily see that in this case

$$V_{uv} = 0\ , \qquad W_{uv} = 0\ . \tag{6.133}$$

From the boundary conditions satisfied by V and W we have, when $u = 0$ *and* $v = 0$:

$$V_{vv} = V_{uu} = W_{vv} = W_{uu} = 0\ . \tag{6.134}$$

Next differentiating (6.68) and (6.69) with respect to u we find that when $u = 0$ *and* $v = 0$:

$$V_{uvu} = 2\,b^2\alpha - 6\,c^2\alpha - 8\,\beta\,b\,c\ , \qquad W_{uvu} = 2\,\beta\,(b^2 + c^2) + 8\,\alpha\,b\,c\ . \tag{6.135}$$

Finally differentiating (6.68) and (6.69) with respect to v we find that when $u = 0$ *and* $v = 0$:

$$V_{uvv} = 2\,b\,\alpha^2 - 6\,b\,\beta^2 - 8\,\alpha\,\beta\,c\ , \qquad W_{uvv} = 2\,c\,(\alpha^2 + \beta^2) + 8\,b\,\alpha\,\beta\ . \tag{6.136}$$

Now substituting all of these results into the right hand side of (6.132) we obtain

$$h'(1) = k \;, \tag{6.137}$$

with k given by (6.131). Using (6.131) and (6.137) in (6.120) we see that

$$G(1) = 0 = G'(1) \;. \tag{6.138}$$

We can now set about solving (6.125) and (6.126) for G and then obtain $h(\lambda)$ from (6.120).

Differentiating (6.126) with respect to λ and using (6.125) we find that either (a) $P' = 0$ or (b) if $P' \neq 0$ then

$$\lambda\, P'' + P' - \frac{1}{\lambda}\, P = \frac{\lambda\, G\, G'}{1-\lambda^2} \;. \tag{6.139}$$

We can quickly dispose of case (a). If P = constant $\neq 0$ then (6.126) can be integrated to yield

$$G = \frac{P}{4}\left(c_0^2\lambda^{\pm 2} - \frac{1}{c_0^2\lambda^{\pm 2}}\right) \;, \tag{6.140}$$

where c_0 is a constant of integration. It is easy to see that this constant cannot be chosen to satisfy both boundary conditions (4'.22). Also if $P = 0$ then (6.126) integrates to

$$G = c_1\,\lambda^{\pm 2} \;, \tag{6.141}$$

with c_1 a constant of integration. Clearly we must have $c_1 = 0$ to satisfy (6.138). Thus the only acceptable solution in case (a) is $G = 0$. Turning now to case (b) with (6.139) holding we find that we can integrate this equation once (using (6.138)) to read

$$\lambda\, P' + \left(\frac{1+\lambda^2}{1-\lambda^2}\right) P = \frac{\lambda\, G^2}{2\,(1-\lambda^2)} \;. \tag{6.142}$$

By the first of (6.124) this can be written

$$(P+Q) + \lambda^2\,(P-Q) = \frac{\lambda}{2}\, G^2 \;, \tag{6.143}$$

and from (6.123) and (6.124) this reads

$$\lambda\, G' = \frac{\lambda\, h}{2\,(1+\lambda^2 h^2)}\, G^2 - 2\,\frac{(1-\lambda^2 h^2)}{(1+\lambda^2 h^2)}\, G \;. \tag{6.144}$$

A glance at (6.119) and (6.120) shows how h and thence G are constructed from the functions V and W appearing in the line–element (6.58) for $u > 0, v > 0$. On the boundaries of this region $\lambda = 1$ and within this region $\lambda > 1$ with λ becoming infinite when the right hand side of (6.73) vanishes. The $\lambda =$ constant > 1 curves densely fill the interior of the region $\mathcal{B}$(say) with boundaries (b_1) $u = 0, v > 0$, (b_2) $v = 0, u > 0$ and (b_3) the right hand side of (6.73) vanishing with $u > 0, v > 0$. Within $\mathcal{B}$ there is one curve $\lambda =$ constant > 1 passing through each point. When the field equations are completely integrated for $(u, v)\epsilon\mathcal{B}$ the boundary (b_3) turns out to be a curvature singularity. For G analytic in $\mathcal{B}$ we conclude from (6.138) and (6.144) that $G \equiv 0$ in $\mathcal{S}$. It thus follows from (6.120) with the boundary condition (6.131) that

$$h(\lambda) = k\,\lambda\ , \tag{6.145}$$

for $(u, v)\epsilon\mathcal{B}$.

We are now at the following stage in the integration of the field equations: the function U in the line–element (6.58) is given by (6.73) in coordinates (u, v) or by (6.106) in coordinates $(\bar{u}, \bar{v})$. Also on account of (6.119) and (6.145) the functions V and W in (6.58) satisfy the differential equation

$$\frac{A}{B} = \frac{-V_{\bar{u}}\cosh W + iW_{\bar{u}}}{-V_{\bar{v}}\cosh W + iW_{\bar{v}}} = \frac{1 - i\,k\,\lambda}{1 + i\,k\,\lambda}\ , \tag{6.146}$$

with λ given by (6.108) and k by (6.131). We shall now solve this complex equation for V and W in terms of the barred coordinates. First we need to note the boundary values of V and W in terms of the barred coordinates. By (6.60) and (6.103) we have when $\bar{u} = 0$,

$$\mathrm{e}^{-V} = \left[\frac{\left(\sqrt{\alpha^2+\beta^2} - \alpha\,\sin\bar{v}\right)^2 + \beta^2\sin^2\bar{v}}{\left(\sqrt{\alpha^2+\beta^2} + \alpha\,\sin\bar{v}\right)^2 + \beta^2\sin^2\bar{v}}\right]^{\frac{1}{2}} = \frac{\left|1 - \mathrm{e}^{i\hat{\beta}}\sin\bar{v}\right|}{\left|1 + \mathrm{e}^{i\hat{\beta}}\sin\bar{v}\right|}\ , \tag{6.147}$$

with the second equality following from (6.130). By (6.61) and (6.103) we have when $\bar{u} = 0$

$$\sinh W = \frac{2\,\beta\,\sin\bar{v}}{\sqrt{\alpha^2+\beta^2}\,\cos^2\bar{v}} = -i\,\frac{\left(\mathrm{e}^{-i\hat{\beta}}\sin\bar{v} - \mathrm{e}^{i\hat{\beta}}\sin\bar{v}\right)}{1 - \left|\mathrm{e}^{-i\hat{\beta}}\sin\bar{v}\right|^2}\ . \tag{6.148}$$

The corresponding boundary values on $\bar{v} = 0$ are obtained by replacing $\bar{v}$ by $\bar{u}$ and $\hat{\beta}$ by $\hat{\alpha}$ in the final expressions in (6.147) and (6.148). It is convenient to use a complex function E (the Ernst function) in place of the two real functions V and W defined (in a way that is suggested by the final expressions in (6.147) and (6.148)) by

$$e^{-V} = \left[\frac{(1-E)\,(1-\bar{E})}{(1+E)\,(1+\bar{E})}\right]^{\frac{1}{2}} , \qquad \sinh W = -i\,\frac{(E-\bar{E})}{1-|E|^2} , \tag{6.149}$$

or equivalently by

$$E = \frac{\sinh V\ \cosh W + i \sinh W}{1 + \cosh V\ \cosh W} . \tag{6.150}$$

Now (6.147) and (6.148) can be written neatly as:

$$\text{when}\quad \bar{u} = 0 , \qquad E = e^{-i\hat{\beta}} \sin \bar{v} , \tag{6.151}$$

and correspondingly

$$\text{when}\quad \bar{v} = 0 , \qquad E = e^{-i\hat{\alpha}} \sin \bar{u} . \tag{6.152}$$

In terms of E, the complex functions A, B can be written

$$A = -\frac{2\cosh W}{1-\bar{E}^2}\,\bar{E}_{\bar{u}} , \qquad B = -\frac{2\cosh W}{1-\bar{E}^2}\,\bar{E}_{\bar{v}} . \tag{6.153}$$

Substitution into (6.146) simplifies this equation to

$$E_{\bar{u}} - E_{\bar{v}} = i\,k\,\lambda\,(E_{\bar{u}} + E_{\bar{v}}) . \tag{6.154}$$

With λ given by (6.108) this equation establishes that

$$E = E(w) \qquad \text{with} \qquad w = \sin(\bar{u}+\bar{v}) + i\,k\,\sin(\bar{u}-\bar{v}) . \tag{6.155}$$

We can now determine E using the boundary conditions (6.151) and (6.152). To see this easily we first write k in (6.131) in the form

$$k = -i\,\frac{\left(e^{-i\hat{\alpha}} - e^{-i\hat{\beta}}\right)}{e^{-i\hat{\alpha}} + e^{-i\hat{\beta}}} . \tag{6.156}$$

Using this in (6.155) we see that we can consider E to have the functional dependence:

$$E = E\left(e^{-i\hat{\alpha}} \sin \bar{u}\, \cos \bar{v} + e^{-i\hat{\beta}} \cos \bar{u}\, \sin \bar{v}\right) . \tag{6.157}$$

Now the boundary conditions (6.151) and (6.152) establish that

$$E = e^{-i\hat{\alpha}} \sin \bar{u}\, \cos \bar{v} + e^{-i\hat{\beta}} \cos \bar{u}\, \sin \bar{v} , \tag{6.158}$$

and thus the functions V, W appearing in the line–element (6.58) are determined by (6.149) in the coordinates $(\bar{u}, \bar{v})$. They are then converted into the coordinates (u, v) using the transformations (6.103).

Finally in the barred system the field equations (6.70) and (6.71) for $\bar{M}$ read (using (6.149) and (6.153))

$$\bar{M}_{\bar{u}} = -\frac{U_{\bar{u}\bar{u}}}{U_{\bar{u}}} + \frac{1}{2}U_{\bar{u}} + \frac{2\,E_{\bar{u}}\bar{E}_{\bar{u}}}{U_{\bar{u}}\left(1-|E|^2\right)^2}\ , \tag{6.159}$$

$$\bar{M}_{\bar{v}} = -\frac{U_{\bar{v}\bar{v}}}{U_{\bar{v}}} + \frac{1}{2}U_{\bar{v}} + \frac{2\,E_{\bar{v}}\bar{E}_{\bar{v}}}{U_{\bar{v}}\left(1-|E|^2\right)^2}\ , \tag{6.160}$$

with $\bar{M}$ related to M by (6.105). Since we must have $M = 0$ when $u = 0$ and when $v = 0$ we see from (6.105) that

$$\text{when}\ \ \bar{u} = 0\ , \qquad \mathrm{e}^{-\bar{M}} = \frac{\cos\bar{v}}{\sqrt{b^2+c^2}\,\sqrt{\alpha^2+\beta^2}}\ , \tag{6.161}$$

and

$$\text{when}\ \ \bar{v} = 0\ , \qquad \mathrm{e}^{-\bar{M}} = \frac{\cos\bar{u}}{\sqrt{b^2+c^2}\,\sqrt{\alpha^2+\beta^2}}\ . \tag{6.162}$$

In (6.159) and (6.161), U is given by (6.106) and E by (6.158). Using (6.158) we find

$$E_{\bar{u}}\,\bar{E}_{\bar{u}} = 1-|E|^2 = E_{\bar{v}}\,\bar{E}_{\bar{v}}\ , \tag{6.163}$$

with

$$1-|E|^2 = \cos^2\left(\frac{\hat{\alpha}-\hat{\beta}}{2}\right)\cos^2(\bar{u}+\bar{v}) + \sin^2\left(\frac{\hat{\alpha}-\hat{\beta}}{2}\right)\cos^2(\bar{u}-\bar{v}). \tag{6.164}$$

The only complication in solving (6.159) and (6.161) is in dealing with the final term in each. In the case of (6.159) this now involves evaluating the integral

$$\int\frac{2\,d\bar{u}}{U_{\bar{u}}\left(1-|E|^2\right)} = \int\frac{2\,\lambda\,d\lambda}{(\lambda^2-1)\left\{\cos^2\left(\frac{\hat{\alpha}-\hat{\beta}}{2}\right)+\lambda^2\sin^2\left(\frac{\hat{\alpha}-\hat{\beta}}{2}\right)\right\}}\ , \tag{6.165}$$

where we have changed the variable of integration from $\bar{u}$ to λ, given in (6.108), with $\bar{v}$ held fixed. This integral is easy to evaluate and using

(6.164) again we obtain from (6.159)

$$\mathrm{e}^{-\bar{M}} = \frac{1 - |E|^2}{F(\bar{v})\,\sqrt{\cos(\bar{u} - \bar{v})\,\cos(\bar{u} + \bar{v})}}\,, \tag{6.166}$$

with $F(\bar{v})$ a function of integration. By (6.161) we find that in fact F is a constant given by

$$F = \sqrt{b^2 + c^2}\,\sqrt{\alpha^2 + \beta^2}\,. \tag{6.167}$$

It is straightforward to see that (6.166) with (6.167) also satisfies (6.161). The integration of Einstein's vacuum field equations is now complete.

We can summarise the calculations in this section as follows: the vacuum space–time in the region $\mathcal{B}$ has line– element of the form (6.58) with U given by (6.73), V and W given by (6.149) with the complex function E in (6.158) expressed in coordinates (u, v) as

$$E = (b + ic)\,u\,\sqrt{1 - (\alpha^2 + \beta^2)\,v^2} + (\alpha + i\beta)\,v\,\sqrt{1 - (b^2 + c^2)\,u^2}\,, \tag{6.168}$$

while M is reconstructed in coordinates (u, v) using (6.103), (6.105), (6.166) and (6.167). The result is

$$\mathrm{e}^{-M} = \frac{\left(1 - |E|^2\right)\,\mathrm{e}^{U/2}}{\sqrt{1 - (b^2 + c^2)\,u^2}\,\sqrt{1 - (\alpha^2 + \beta^2)\,v^2}}\,, \tag{6.169}$$

with e^U given in (6.73). If in (6.73), (6.168) and (6.169) we put $a^2 + b^2 = 1 = \alpha^2 + \beta^2$ we recover the original form of the Nutku and Halil [Nutku and Halil (1977)] solution. If in addition $\beta = c = 0$ (and thus $E = \bar{E}$ and so $W = 0$) we recover the original form (6.1) of the Khan–Penrose solution. We note that in the region $\mathcal{B}$ a curvature singularity [Khan and Penrose (1971)], [Nutku and Halil (1977)] is encountered on the boundary where $(\alpha^2 + \beta^2)\,v^2 + (b^2 + c^2)\,u^2 = 1$ and the solution above is valid only up to this space–like subspace.

6.3 Plane Impulsive Light–Like Signal Collisions: Some Non–Interacting Examples

As a further example we return to the general plane fronted impulsive light–like signal propagating through a region in which no gravitational field is present. The appropriate line–element is (2.63). We specialize this to the most general *homogeneous* such signal by choosing the function F to be

$$F(x,y,v) = v - \frac{a}{2}(x^2+y^2) + \frac{b}{2}(x^2-y^2) + c\,x\,y\ , \tag{6.170}$$

with a, b, c constants and $a \geq 0$. This signal has an isotropic stress–energy (vanishing isotropic pressure and energy current) with surface energy density

$$\mu = \frac{a}{4\pi}\ , \tag{6.171}$$

and is accompanied by an impulsive gravitational wave with

$$\hat{\Psi}_4 = b - i\,c\ , \tag{6.172}$$

having the maximum two degrees of freedom of polarization. The line–element (2.63) reduces in this case to

$$\begin{aligned} ds^2 = &\{(1-(a-b)\,u_+)\,dx + c\,u_+dy\}^2 + \{c\,u_+dx + (1-(a+b)\,u_+)\,dy\}^2 \\ &-2du\,dv\ . \end{aligned} \tag{6.173}$$

This line–element has the Rosen–Szekeres form (6.58) with

$$e^{-U} = 1 - 2a\,u_+ + (a^2-b^2-c^2)\,u_+^2\ , \tag{6.174}$$

$$e^{V} = \left[\frac{(1-(a-b)\,u_+)^2 + c^2u_+^2}{(1-(a+b)\,u_+)^2 + c^2u_+^2}\right]^{\frac{1}{2}}\ , \tag{6.175}$$

$$\sinh W = \frac{-2c\,u_+(1-a\,u_+)}{1-2a\,u_+ + (a^2-b^2-c^2)\,u_+^2}\ , \tag{6.176}$$

$$M = 0\ . \tag{6.177}$$

We shall consider the head–on collision of this plane–fronted light–like signal with a signal of similar type specialized to the case in which the signals are non–interacting after collision [Barrabès and Hogan (2002)]. This second signal is described by a space–time with line–element (6.58) with

$$e^{-U} = 1 - 2\alpha\,v_+ + (\alpha^2-\beta^2-\gamma^2)\,v_+^2\ , \tag{6.178}$$

$$e^{V} = \left[\frac{(1-(\alpha-\beta)\,v_+)^2 + \gamma^2v_+^2}{(1-(\alpha+\beta)\,v_+)^2 + \gamma^2v_+^2}\right]^{\frac{1}{2}}\ , \tag{6.179}$$

$$\sinh W = \frac{-2\gamma\,v_+(1-\alpha\,v_+)}{1-2\alpha\,v_+ + (\alpha^2-\beta^2-\gamma^2)\,v_+^2}\ , \tag{6.180}$$

$$M = 0\ , \tag{6.181}$$

where α, β, γ are real constants, $\alpha \geq 0$, and $v_+ = v\,\vartheta(v)$. Here the history of the light–like signal is the null hyperplane $v = 0$.

The space–time model of the head–on collision of two plane–fronted, homogeneous, impulsive, light–like signals will have a line–element of the form (6.58) with, in general, U, V, W and M functions of u, v. Einstein's vacuum field equations (6.67) are to be solved for U, V, W and M in the region $u > 0, v > 0$ (after the collision) subject to these boundary (junction) conditions: For $v = 0, u > 0$ the functions U, V, W and M are given by (2.22)–(2.25) with $u_+ = u$ and for $u = 0, v > 0$ the functions U, V, W and M are given by (2.26)–(2.29) with $v_+ = v$. The first of the field equations (6.67) can easily be solved for U subject to the boundary conditions to yield in $u > 0, v > 0$,

$$e^{-U} = 1 - 2\,a\,u - 2\,\alpha\,v + (a^2 - b^2 - c^2)\,u^2 + (\alpha^2 - \beta^2 - \gamma^2)\,v^2\ . \tag{6.182}$$

We can have non–interacting signals after collision by requiring that the region $u > 0, v > 0$ be, if possible, a portion of Minkowskian space–time. Necessary conditions for this to be possible can be obtained by examining approximate solutions of the vacuum field equations, subject to our boundary conditions, in the region $u > 0, v > 0$ near $u = 0$ (i.e. for small $v > 0$) and/or near $v = 0$ (i.e. for small $u > 0$). For this we shall use the Newman–Penrose components of the Riemann curvature tensor given by (6.82)–(6.84). From these necessary, but not necessarily sufficient, conditions for flatness in the region of space–time $u > 0, v > 0$ are

$$M_{uv} = 0\ , \tag{6.183}$$

and

$$A\,\bar{B} - \bar{A}\,B = 0\ . \tag{6.184}$$

The vacuum field equations (6.67)) and (3.6) imply that (6.183) is equivalent to

$$A\,\bar{B} + \bar{A}\,B = 2\,U_u U_v\ . \tag{6.185}$$

The conditions (6.184) and (6.185) are to hold throughout the region $u > 0, v > 0$ and in particular near $u = 0$ and/or $v = 0$ in this region. It will be sufficient for our immediate purpose to calculate (6.184) and (6.185) at $u = v = 0$ (strictly speaking in the limit $u \to 0^+$ and $v \to 0^+$). If *when* $v = 0$ we put

$$A = S\,e^{i\psi}\ , \qquad B = R\,e^{i\phi}\ , \tag{6.186}$$

we have immediately from our boundary conditions on $v=0$ $(u>0)$

$$S=\frac{2\sqrt{b^2+c^2}}{(1-a\,u)^2-(b^2+c^2)\,u^2}\ , \tag{6.187}$$

and $\psi=\theta+\pi$ with

$$\mathrm{e}^{i\theta}=\frac{b\{(1-a\,u)^2-(b^2+c^2)\,u^2\}+ic\,\{(1-a\,u)^2+(b^2+c^2)\,u^2\}}{\sqrt{b^2+c^2}\,\mathcal{F}\,\mathcal{G}}\ , \tag{6.188}$$

with

$$\mathcal{F}=\sqrt{(1-(a-b)\,u)^2+c^2u^2}\ ,\qquad \mathcal{G}=\sqrt{(1-(a+b)\,u)^2+c^2u^2}. \tag{6.189}$$

It is useful to note that

$$\theta_u=\frac{4\,b\,c\,u\,(1-a\,u)}{\mathcal{F}^2\mathcal{G}^2}\ . \tag{6.190}$$

The field equations (3.2) and (3.3) evaluated on $v=0$ give us the following differential equations for R,ϕ:

$$R_u=\frac{\{a-(a^2-b^2-c^2)\,u\}\,R}{(1-a\,u)^2-(b^2+c^2)\,u^2}+\frac{2\,\alpha\sqrt{b^2+c^2}\,\cos(\psi-\phi)}{\{(1-a\,u)^2-(b^2+c^2)\,u^2\}^2} \tag{6.191}$$

$$\phi_u=\frac{4\,b\,c\,u\,(1-a\,u)}{\mathcal{F}^2\mathcal{G}^2}+\frac{2\,\alpha\sqrt{b^2+c^2}\,\sin(\psi-\phi)}{R\,\{(1-a\,u)^2-(b^2+c^2)\,u^2\}^2}\ . \tag{6.192}$$

Writing $\hat{R}=R\sqrt{(1-a\,u)^2-(b^2+c^2)\,u^2}$ and using (6.190) these can be simplified to

$$\hat{R}_u=\frac{2\,\alpha\sqrt{b^2+c^2}\,\cos(\psi-\phi)}{\{(1-a\,u)^2-(b^2+c^2)\,u^2\}^{\frac{3}{2}}}\ , \tag{6.193}$$

$$(\psi-\phi)_u=\frac{-2\,\alpha\sqrt{b^2+c^2}\,\sin(\psi-\phi)}{\hat{R}\,\{(1-a\,u)^2-(b^2+c^2)\,u^2\}^{\frac{3}{2}}}\ . \tag{6.194}$$

These give us

$$\hat{R}\,\sin(\psi-\phi)=K\ , \tag{6.195}$$

$$\hat{R}\cos(\psi-\phi)=\frac{2\,\alpha\,\{a-(a^2-b^2-c^2)\,u\}}{\sqrt{b^2+c^2}\,\sqrt{(1-a\,u)^2-(b^2+c^2)\,u^2}}+C\ , \tag{6.196}$$

where K,C are constants of integration. From our boundary conditions we have, when $u=v=0$,

$$\mathrm{e}^{i\psi}=-\frac{(b+ic)}{a}\ ,\ \mathrm{e}^{i\phi}=-\frac{(\beta+i\gamma)}{\alpha}\ ,\ R=2\,\alpha\ . \tag{6.197}$$

This determines the constants K, C and writing $\hat{R}$ in terms of R we have from (3.24) and (3.25)

$$R\,\sin(\psi-\phi)=\frac{2\,(\beta\,c-\gamma\,b)}{\sqrt{b^2+c^2}\,\sqrt{(1-a\,u)^2-(b^2+c^2)\,u^2}}\ , \tag{6.198}$$

$$\begin{aligned}R\,\cos(\psi-\phi)&=\frac{2\,\alpha\,\{a-(a^2-b^2-c^2)\,u\}}{\sqrt{b^2+c^2}\,\{(1-a\,u)^2-(b^2+c^2)\,u^2\}}\\&\quad+\frac{2\,(\beta\,b+\gamma\,c-\alpha\,a)}{\sqrt{b^2+c^2}\,\sqrt{(1-a\,u)^2-(b^2+c^2)\,u^2}}\ .\end{aligned} \tag{6.199}$$

Now (63.13) evaluated at $v=0$ gives

$$\sin(\psi-\phi)=0\ , \tag{6.200}$$

and by (6.198) this is equivalent to

$$\beta\,c-\gamma\,b=0\ . \tag{6.201}$$

Next using U given by (6.182), S given by (6.187) and $R\,\cos(\psi-\phi)$ by (3.28), the equation (6.185) evaluated at $v=0$ is found to be equivalent to

$$\beta\,b+\gamma\,c-\alpha\,a=0\ . \tag{6.202}$$

Hence *the relations* (6.201) *and* (6.202) between the three real constants a, b, c associated with one in–coming light–like signal and the three real constants α, β, γ associated with the second incoming light–like signal *are necessary conditions for the region of space–time* $u>0, v>0$ *(after the collision) to be flat.* We note that (6.201) and (6.202) are invariant under the interchange of a, b, c with α, β, γ respectively and so would also emerge had we evaluated (6.184) and (6.185) at $u=0$ instead of at $v=0$. We note that (6.201) and (6.202) exclude the case of a collision of two plane impulsive gravitational waves [Khan and Penrose (1971)] (corresponding to $b\neq 0, \beta\neq 0, a=\alpha=\gamma=c=0$) and also a collision of two plane light–like shells [Dray and 't Hooft (1986)] (corresponding to $a\neq 0, \alpha\neq 0, b=c=\beta=\gamma=0$). In those two cases the space–time after the collision is not flat and contains a curvature singularity.

We shall henceforth assume that the conditions (6.201) and (6.202) are satisfied and we proceed to solve the vacuum field equations (3.2)–(3.6) in the region $u>0, v>0$ with the boundary conditions on U, V, W and M indicated in the opening paragraph of this section. We use the procedure described in §6.2. The preliminary calculations necessary to implement this procedure have been carried out above. These are the calculations determining S, ψ, R, ϕ in (6.186). The results are given by (6.187), (6.188), (6.198) and (6.199) with (6.201) and (6.202) now holding. Using these as starting point we find that, in addition to U already given in $u > 0, v > 0$ by (6.182), we obtain

$$e^V = \left[\frac{\{1-(a-b)\,u-(\alpha-\beta)\,v\}^2+(c\,u+\gamma\,v)^2}{\{1-(a+b)\,u-(\alpha+\beta)\,v\}^2+(c\,u+\gamma\,v)^2}\right]^{\frac{1}{2}} , \tag{6.203}$$

$$\sinh W = -2(c\,u+\gamma\,v)\,(1-a\,u-\alpha\,v)\,e^U , \tag{6.204}$$

$$M = 0 . \tag{6.205}$$

If in these functions we replace u by $u_+ = u\,\vartheta(u)$ and v by $v_+ = v\,\vartheta(v)$ then we can include in a single expression in each case the form of the function in the four different regions of the collision space–time. Having done this and having substituted the functions into the line–element (6.58) we arrive at the final form of our line–element:

$$\begin{aligned} ds^2 = {} & \{(1-(a-b)\,u_+-(\alpha-\beta)\,v_+)\,dx+(c\,u_++\gamma\,v_+)\,dy\}^2 \\ & +\{(c\,u_++\gamma\,v_+)\,dx+(1-(a+b)\,u_+-(\alpha+\beta)\,v_+)\,dy\}^2 \\ & -2\,du\,dv , \end{aligned} \tag{6.206}$$

with the six parameters $a, b, c, \alpha, \beta, \gamma$ satisfying (6.201) and (6.202). The non–identically vanishing Weyl tensor components for the space–time with line–element (6.206) are, in Newman–Penrose notation,

$$\Psi_0 = -\frac{\alpha\,\{a-(a^2-b^2-c^2)\,u_+\}\,\mathcal{P}}{(b^2+c^2)\,\mathcal{F}\,\mathcal{G}\,\mathcal{P}_1}\,\delta(v) , \tag{6.207}$$

$$\Psi_4 = -\frac{a\,\{\alpha-(\alpha^2-\beta^2-\gamma^2)\,v_+\}\,\mathcal{Q}}{(\beta^2+\gamma^2)\,\mathcal{F}'\,\mathcal{G}'\,\mathcal{Q}_1}\,\delta(u) . \tag{6.208}$$

Here

$$\mathcal{P} = b\,\mathcal{P}_1 + i\,c\,\mathcal{P}_2 , \tag{6.209}$$

$$\mathcal{Q} = \beta\,\mathcal{Q}_1 - i\,\gamma\,\mathcal{Q}_2 , \tag{6.210}$$

$$\mathcal{P}_1 = \{(1-a\,u_+)^2-(b^2+c^2)\,u_+^2\} , \tag{6.211}$$

$$\mathcal{P}_2 = \{(1-a\,u_+)^2+(b^2+c^2)\,u_+^2\} , \tag{6.212}$$

$$\mathcal{Q}_1 = \{(1-\alpha\,v_+)^2-(\beta^2+\gamma^2)\,v_+^2\} , \tag{6.213}$$

$$\mathcal{Q}_2 = \{(1-\alpha\,v_+)^2+(\beta^2+\gamma^2)\,v_+^2\} , \tag{6.214}$$

$$\mathcal{F}' = \sqrt{(1-(\alpha-\beta)\,v_+)^2+\gamma^2 v_+^2} , \tag{6.215}$$

$$\mathcal{G}' = \sqrt{(1-(\alpha+\beta)\,v_+)^2+\gamma^2 v_+^2} , \tag{6.216}$$

and $\mathcal{F}$, $\mathcal{G}$ are given by (6.189) with u replaced by u_+. Labelling the coordinates $x^\mu = (x, y, u, v)$ with $\mu = 1, 2, 3, 4$ the Ricci tensor components for the space–time with line–element (6.206) are

$$R_{\mu\nu} = -\frac{2\,\alpha}{\mathcal{P}_1}\,\delta(v)\,l_\mu\,l_\nu - \frac{2\,a}{\mathcal{Q}_1}\,\delta(u)\,n_\mu\,n_\nu\ , \tag{6.217}$$

with $l^\mu = \delta^\mu_3$ and $n^\mu = \delta^\mu_4$. We note that on $v = 0$ the (real) expansion ρ_l and the modulus of the complex shear $|\sigma_l|$ of the null geodesic integral curves of l^μ are given by

$$\rho_l = \frac{a\,\vartheta(u) - (a^2 - b^2 - c^2)\,u_+}{\mathcal{P}_1}\ , \qquad |\sigma_l| = \frac{\sqrt{b^2 + c^2}\,\vartheta(u)}{\mathcal{P}_1}\ , \tag{6.218}$$

while on $u = 0$ the (real) expansion ρ_n and the modulus of the complex shear $|\sigma_n|$ of the null geodesic integral curves of n^μ are given by

$$\rho_n = \frac{\alpha\,\vartheta(v) - (\alpha^2 - \beta^2 - \gamma^2)\,v_+}{\mathcal{Q}_1}\ , \qquad |\sigma_n| = \frac{\sqrt{\beta^2 + \gamma^2}\,\vartheta(v)}{\mathcal{Q}_1}\ . \tag{6.219}$$

We remark that for the space–time with line–element (6.206) it follows from (6.207), (6.208) and (6.217) that the region $u > 0, v > 0$ after the collision is flat. To see this explicitly we consider two cases separately.
I $(c = 0 = \gamma)$: for $u > 0, v > 0$ we have

$$ds^2 = \{1-(a-b)\,u-(\alpha-\beta)\,v\}^2dx^2+\{1-(a+b)\,u-(\alpha+\beta)\,v\}^2dy^2-2\,du\,dv\ , \tag{6.220}$$

with $a\,\alpha - b\,\beta = 0$. We first note that if $a = \pm b$ ($\Leftrightarrow \alpha \pm \beta$) then it is easy to transform (64.1) to Minkowskian form. Now assume $a \neq \pm b$ ($\Leftrightarrow \alpha \neq \pm\beta$) and make the transformation

$$u' = 1 - (a - b)\,u - (\alpha - \beta)\,v\ , \tag{6.221}$$

$$v' = 1 - (a + b)\,u - (\alpha + \beta)\,v\ . \tag{6.222}$$

This results in (64.1) taking the form

$$ds^2 = u'^2dx^2 + v'^2dy^2 + \lambda^2du'^2 - \mu^2dv'^2\ , \tag{6.223}$$

with $\lambda^{-2} = -2\,(\alpha - \beta)\,(a - b)$, $\mu^{-2} = 2\,(\alpha + \beta)\,(a + b)$. We note that $(\alpha - \beta)\,(a - b) < 0$ and also $(\alpha^2 - \beta^2)\,(a^2 - b^2) = -(\alpha\,b - \beta\,a)^2$ and so $(\alpha + \beta)\,(a + b) > 0$. Then with $\bar{u} = \lambda\,u', \bar{v} = \mu\,v', \bar{x} = \lambda^{-1}x, \bar{y} = \mu^{-1}y$ we have

$$ds^2 = \bar{u}^2d\bar{x}^2 + \bar{v}^2d\bar{y}^2 + d\bar{u}^2 - d\bar{v}^2\ . \tag{6.224}$$

Now put $X = \bar{u}\cos\bar{x}$, $Y = \bar{u}\sin\bar{x}$, $Z = \bar{v}\sinh\bar{y}$, $T = \bar{v}\cosh\bar{y}$ and we arrive at

$$ds^2 = dX^2 + dY^2 + dZ^2 - dT^2 \ . \tag{6.225}$$

We note that $0 \leq \bar{x} < 2\pi$ and so $0 \leq x < 2\pi\lambda$. This periodicity in the coordinate x has been noted already in a special case [Babala (1987)] of (6.206) which we draw attention to below.

II($c \neq 0 \neq \gamma$) : for $u > 0, v > 0$ in this general case we have from (6.206)

$$\begin{aligned} ds^2 = &\{(1 - (a-b)\,u - (\alpha - \beta)\,v)\,dx + (c\,u + \gamma\,v)\,dy\}^2 \\ &+\{(c\,u + \gamma\,v)\,dx + (1 - (a+b)\,u - (\alpha + \beta)\,v)\,dy\}^2 \\ &-2\,du\,dv \ , \end{aligned} \tag{6.226}$$

with $a\,\alpha - b\,\beta = c\,\gamma$ and $\beta\,c = b\,\gamma$. The rotation

$$x = \frac{x' + \Lambda\,y'}{\sqrt{1+\Lambda^2}} \ , \qquad y = \frac{-\Lambda\,x' + y'}{\sqrt{1+\Lambda^2}} \ , \tag{6.227}$$

with Λ satisfying

$$\Lambda^2 - 2\,\frac{b}{c}\,\Lambda - 1 = 0 \ , \tag{6.228}$$

results in (6.226) taking the form

$$\begin{aligned} ds^2 = &-2\,du\,dv + \{1 - (a' - b')\,u - (\alpha' - \beta')\,v\}^2 dx'^2 \\ &+\{1 - (a' + b')\,u - (\alpha' + \beta')\,v\}^2 dy'^2 \ , \end{aligned} \tag{6.229}$$

with

$$a' = a \ , \qquad b' = b - \Lambda\,c \ , \qquad \alpha' = \alpha \ , \qquad \beta' = \beta - \Lambda\,\gamma \ , \tag{6.230}$$

from which it follows that

$$a'\alpha' - b'\beta' = 0 \ . \tag{6.231}$$

Now (6.229) with (6.231) is identical to the case I considered above. It thus follows that (6.229) (and hence (6.226)) can be written in manifestly Minkowskian form and that in this case x' is a periodic coordinate.

The special case of (6.206) corresponding to $a = b, \alpha = \beta$ and $c = \gamma = 0$ gives a subset of a family of solutions originally found by Stoyanov [Stoyanov (1979)]. The solution (6.206) in this case describes the collision of two linearly polarized plane impulsive gravitational waves each sharing its wave front with a plane light–like shell with relative energy density proportional to the amplitude of the wave. The special case of (6.206) corresponding to $a \neq 0, b = 0, \alpha = 0, \beta \neq 0, c = \gamma = 0$, describes the collision of a linearly polarized plane impulsive gravitational wave with a plane light–like shell and was found by Babala [Babala (1987)]. The generalization of this for which $a \neq 0, b = c = 0, \alpha = 0, \beta \neq 0, \gamma \neq 0$ was found by Feinstein and Senovilla [Feinstein and Senovilla (1989)].

We see from (6.218) and (6.219) that the signals involved in the collision focus each other after collision on their signal fronts at $v = 0, \mathcal{P}_1 = 0$ and at $u = 0, \mathcal{Q}_1 = 0$, where it follows from (6.207) and (6.208) that the Weyl curvature is singular and from (6.217) that the surface stress–energy of the light–like shells (after collision) is singular in each case. On the signal front $u = 0, v > 0$ or $v = 0, u > 0$ the signals are focused on two cylinders with elliptic cross–sections in general. It is important to note in this regard that when $c = 0$ we have $\mathcal{F}\mathcal{G} = 0 \Leftrightarrow \mathcal{P}_1 = 0$ and similarly when $\gamma = 0$ we have $\mathcal{F}'\mathcal{G}' = 0 \Leftrightarrow \mathcal{Q}_1 = 0$. Also since the coordinates x in case I above and x' in case II are periodic the space–time region $u > 0, v > 0$ does not extend to infinity but fills a cylinder with, in general, elliptic cross–sections. A topological singularity occurs on $(a+\sqrt{b^2+c^2})u+(\alpha+\sqrt{\beta^2+\gamma^2})v = 1$, and the space– time cannot be uniquely extended beyond this singularity. For an exceptional choice of the parameters (see [Babala (1987)], for example) the two cylinders on which the signals focus each other degenerate into a point and a circle.

Chapter 7

Impulsive Light–Like Signals in Alternative Theories of Gravity

Scalar–tensor theories of gravity are motivated by the desire to quantise the gravitational field and scalar fields also arise in the dimensional reduction of Kaluza–Klein theories. The Einstein–Cartan theory of gravity is, from a geometrical point of view, a natural extension of general relativity while the Einstein–Gauss–Bonnet theory of gravity has been motivated by string theory and is of particular interest in the study of brane–world models. The discussion of singular null hypersurfaces in scalar–tensor theories, Einstein–Cartan theory and Einstein–Gauss– Bonnet theory provides an interesting perspective on the theory of these hypersurfaces outlined in Chapter 2.

7.1 Scalar-Tensor Theories of Gravitation

Most of the attempts to quantize the gravitational field have led to the conclusion that at Planckian energies the Einstein theory of gravity has to be extended in order to include scalar fields. In the low energy limit string theory gives back classical general relativity with a scalar field partner (the dilaton) and the effective action shows that the dilaton couples to the scalar curvature and to the other matter fields. Scalar fields (compactons) also arise in the process of dimensional reduction of Kaluza-Klein theories, and it has been shown that the presence in the action of high-order terms in the curvature and its derivative amounts to introducing scalar fields with appropriate potentials in the Einstein-Hilbert action of general relativity.

The importance of the role that scalar fields could play in a full theory of gravity has been known for a long time. The pioneering works of [Fierz (1956)], [Jordan (1959)], and [Brans and Dicke (1961)] have provided the first scalar-tensor theory of gravity (usually referred to as the Brans-Dicke theory) which includes, besides the metric tensor $g_{\mu\nu}$, a massless scalar field φ and a free parameter ω. This theory was later generalized by making the parameter field-dependent ($\omega = \omega(\varphi)$) and by introducing a potential term $V(\varphi)$. More recently, using nonlinear σ-models, multi-scalar-tensor theories have been considered and their predictions have been discussed and compared with general relativity in the weak-field and strong-field regimes [Damour and Esposito-Farese (1993) and (1996); Berkin and Hellings (1994)]. All of these alternative theories of gravity belong to the class of scalar-tensor theories in the sense that all the other fields (generically denoted by Ψ_m) exhibit a universal metric coupling to the gravitational field with the same metric tensor. On the other hand the scalar fields (dilatons, compactons) which appear in the string and Kaluza-Klein theories have a non-metric coupling with the fields Ψ_m, and induce a local space–time dependence of the coupling constants. This entails fundamental differences with general relativity, in particular with the equivalence principle [Damour (1995)].

It is known that in scalar-tensor and dilatonic theories two conformally related metrics can be used, the Jordan-Fierz or string metric and the Einstein metric. The Jordan-Fierz or string metric is usually referred to as the physical metric since the stress-energy tensor for the matter fields is conserved with respect to this metric and not with respect to the Einstein metric. However many of the mathematical properties of these theories (asymptotic behavior, Cauchy problem,etc.) are more conveniently investigated using the Einstein metric. For the study of junctions conditions in general the Einstein metric is the best suited (see [Barrabès and Bressange (1997)] and [Bressange (1998)]) as it leads to a simple set of equations which can be directly compared with the corresponding results in general relativity.

Scalar-tensor theories of gravitation are alternative theories of gravity which generalize in the most natural way the Brans-Dicke theory by introducing a finite number of scalar fields φ^i, $i = 1, 2, \ldots, n$ each characterized by a particular coupling constant to local matter (see for instance [Damour and Esposito-Farèse (1993) and (1996)] for a review of scalar-tensor theories). In a σ-model the scalar fields φ^i appear as internal local coordinates in an n-dimensional manifold (the target space) having line–element

$$d\sigma^2 = \gamma_{ij}(\varphi)\, d\varphi^i\, d\varphi^j \ . \tag{7.1}$$

Scalar-tensor theories are covariant field theories which coincide with general relativity in the post-Newtonian approximation. They are metric theories which means that the matter fields are minimally coupled to a universal covariant 2-tensor $\bar{g}_{\mu\nu}$, usually referred to as the physical metric or the Jordan-Fierz metric. The Einstein metric, denoted $g_{\mu\nu}$, is related to the Jordan-Fierz metric by

$$\bar{g}_{\mu\nu} = A^2(\varphi)\, g_{\mu\nu} \ , \tag{7.2}$$

where the conformal factor $A^2(\varphi)$ is a smooth function of the n scalar fields φ^i.

The general form of the action for scalar-tensor theories of gravity is, in the Einstein conformal-frame,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R - 2g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j\gamma_{ij} - 4B(\varphi)] + S_m[\psi_m, A^2(\varphi)g_{\mu\nu}] \, . \tag{7.3}$$

The final term S_m is the action for the matter fields (fermionic or bosonic) which are collectively denoted by ψ_m and, as indicated in S_m, couple directly to the Jordan-Fierz metric. The potential term B in the integrand is a smooth function (at least C^2) of the scalar-fields φ^i and may include a cosmological term. The field equations for the metric tensor $g_{\mu\nu}$ and the scalar fields φ^i which follow from the above action are respectively

$$G_{\mu\nu} = 8\pi G\,\{T_{\mu\nu} + T_{\mu\nu}(\varphi)\} \equiv 8\,\pi\, G\, \mathcal{T}_{\mu\nu} \ , \tag{7.4}$$

and

$$\Box\varphi^i + \gamma^i_{jk}\, g^{\mu\nu}\,\partial_\mu\varphi^j\,\partial_\nu\varphi^k - \beta^i(\varphi = -4\,\pi\, G\,\alpha^i(\varphi)\, T \ , \tag{7.5}$$

where $\Box$ is the covariant d'Alembertian calculated with the metric $g_{\mu\nu}$. We have denoted by $\mathcal{T}_{\mu\nu}$ the total stress-energy tensor, by $T_{\mu\nu}$ the stress-energy tensor of the matter fields,

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\,\frac{\delta S_m}{\delta g_{\mu\nu}} \ , \tag{7.6}$$

with $T = T^\mu_\mu$ and

$$T_{\mu\nu}(\varphi) \equiv \frac{\gamma_{ij}}{4\pi G}\,[\,\partial_\mu\,\varphi^i\,\partial_\nu\,\varphi^j - \frac{1}{2}\,g_{\mu\nu}\,(g^{\alpha\beta}\partial_\alpha\,\varphi^i\,\partial_\beta\,\varphi^j)\,] - \frac{B(\varphi)}{4\pi G}\,g_{\mu\nu} \, . \tag{7.7}$$

is the stress-energy tensor of the scalar-fields. In these equations the γ^i_{jk} 's are the Christoffel symbols associated with the σ-model metric γ_{ij} and the scalar-field indices are lowered or raised with this metric and its inverse. The quantities

$$\beta_i(\varphi) = \frac{\partial B(\varphi)}{\partial\varphi^i} \qquad , \qquad \alpha_i(\varphi) = \frac{\partial \ln A(\varphi)}{\partial\varphi^i} \, , \tag{7.8}$$

which appear in (7.5) are space–time scalars and represent coupling factors of the scalar fields to matter.

Another convenient form of (7.4) is

$$R_{\mu\nu} = 8\,\pi\,G\left(T_{\mu\nu} - \frac{1}{2}T\,g_{\mu\nu}\right) + \gamma_{ij}\partial_\mu\varphi^i\partial_\nu\varphi^j\;. \tag{7.9}$$

In the Einstein frame the stress–energy tensor of the matter fields is not conserved but instead satisfies

$$\nabla_\nu T^{\mu\nu} = \alpha_i(\varphi)T\nabla^\mu\varphi^i\;. \tag{7.10}$$

We use ∇_ν to denote covariant differentiation here with respect to the Riemannian connection calculated with the metric tensor $g_{\mu\nu}$ and $\bar{\nabla}_\nu$ to denote the same covariant derivative but calculated with the metric tensor $\bar{g}_{\mu\nu}$.

Let us briefly recall for the sake of completeness some properties of the Jordan-Fierz theory. In the Jordan-Fierz frame only one scalar field $\bar{\varphi}$ can be considered and the action takes the general form

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-\bar{g}}\,[\,\bar{\varphi}\bar{R} - \frac{\omega(\bar{\varphi})}{\bar{\varphi}}\,\bar{g}^{\mu\nu}\,\partial_\mu\bar{\varphi}\,\partial_\nu\bar{\varphi} + 2\bar{\varphi}\,\Lambda(\bar{\varphi})\,] + S_m[\psi_m, \bar{g}_{\mu\nu}]\,. \tag{7.11}$$

The field equations for $\bar{g}_{\mu\nu}$ and $\bar{\varphi}$ are respectively

$$\begin{aligned}\bar{R}_{\mu\nu} - \frac{\bar{R}}{2}\,\bar{g}_{\mu\nu} &= \frac{8\pi G}{\bar{\varphi}}\,\bar{T}_{\mu\nu} + \Lambda(\bar{\varphi})\,\bar{g}_{\mu\nu}\\ &+ \frac{\omega(\bar{\varphi})}{\bar{\varphi}^2}\left[\partial_\mu\bar{\varphi}\,\partial_\nu\bar{\varphi} - \frac{1}{2}\,\bar{g}_{\mu\nu}\,(\bar{g}^{\alpha\beta}\partial_\alpha\bar{\varphi}\,\partial_\beta\bar{\varphi})\right]\\ &+ \frac{1}{\bar{\varphi}}\,(\bar{\nabla}_\mu\bar{\nabla}_\nu\bar{\varphi} - \bar{g}_{\mu\nu}\,\bar{\Box}\,\bar{\varphi})\,,\end{aligned} \tag{7.12}$$

and

$$\bar{\Box}\,\bar{\varphi} + \frac{1}{2}\bar{g}^{\alpha\beta}\,\partial_\alpha\bar{\varphi}\,\partial_\beta\bar{\varphi}\frac{d}{d\bar{\varphi}}\ln(\frac{\omega(\bar{\varphi})}{\bar{\varphi}}) + \frac{\bar{\varphi}}{\omega(\bar{\varphi})}\left[\frac{\bar{R}}{2} + \frac{d}{d\bar{\varphi}}(\bar{\varphi}\Lambda(\bar{\varphi}))\right] = 0\,, \tag{7.13}$$

where $\bar{T}_{\mu\nu}$ is the stress-energy tensor of the matter fields

$$\bar{T}^{\mu\nu} \equiv \frac{2}{\sqrt{-\bar{g}}}\,\frac{\delta\,S_m}{\delta\,\bar{g}_{\mu\nu}}\,, \tag{7.14}$$

and where $\bar{\Box}$ is the d'Alembertian calculated with the metric $\bar{g}_{\mu\nu}$. Since the matter fields couple directly to the Jordan-Fierz metric their stress-energy tensor is conserved, thus $\bar{\nabla}_\nu \bar{T}^{\mu\nu} = 0$. We have the following relations between the two theories:

$$\bar{\varphi}^{-1} = A^2(\varphi)\,, \qquad 2\,\omega(\bar{\varphi}) + 3 = \alpha^{-2}(\varphi)\,, \qquad 2\,\Lambda(\bar{\varphi}) = -B(\varphi)\,A^{-2}(\varphi)\,. \tag{7.15}$$

The matter stress-energy tensors (7.6) and (7.14) are related by

$$T^{\mu\nu} = A^6(\varphi)\,\bar{T}^{\mu\nu}\,. \tag{7.16}$$

The Brans-Dicke theory corresponds to the particular case where $\omega(\bar{\varphi}) =$ constant and $\Lambda = 0$ in the Jordan-Fierz description, or to $\alpha =$ constant and $B = 0$ in the Einstein description. In this case we have

$$A(\varphi) \;=\; e^{\alpha\,\varphi}\,, \tag{7.17}$$

from (7.8).

Let $\mathcal{N}$ be the singular null hypersurface in space–time corresponding to the history of an impulsive light–like signal. Since the metric is only C^0 on $\mathcal{N}$ the total stress-energy tensor $\mathcal{T}_{\mu\nu}$ which appears on the right-hand side of (7.3) necessarily contains a δ-term with support on $\mathcal{N}$. By comparing the two fields equations (7.4) and (7.5) we easily see that the singular δ-term can only come from the matter part $T_{\mu\nu}$ of the total stress-energy tensor $\mathcal{T}_{\mu\nu}$. This a consequence of the presence of the trace term T in the right hand side of (2.5). Therefore the scalar fields φ^i are C^0 on $\mathcal{N}$ and C^3 elsewhere and the metric and the scalar fields have the same smoothness properties. This was to be expected on account of the metric coupling of the scalar fields to gravity. The metric $g_{\mu\nu}$, the scalar-fields φ^i and their tangential derivatives are thus continuous across $\mathcal{N}$ but their transverse derivatives are not. As in Chapter 2 we define their jumps across $\mathcal{N}$ by

$$[\,\partial_\mu\, g_{\alpha\beta}\,] = \eta\, n_\mu\, \gamma_{\alpha\beta} \qquad [\partial_\mu \varphi^i] \;=\; \eta\, n_\mu\, \zeta^i\,, \tag{7.18}$$

where n is the normal to $\mathcal{N}$, or equivalently since $N \cdot n = \eta^{-1}$,

$$\gamma_{\alpha\beta} = N^\mu\,[\,\partial_\mu g_{\alpha\beta}\,] \qquad \zeta^i = N^\mu\,[\,\partial_\mu \varphi^i\,]\,, \tag{7.19}$$

where N is the transverse vector.

Using the notation of Chapter 2 (in particular a quantity defined off the hypersurface $\mathcal{N}$ is denoted with a tilde and its singular part is denoted with a hat) we obtain for the Einstein tensor and for the d'Alembertian of the scalar fields

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} + \hat{G}_{\mu\nu}\,\eta\,\chi\,\delta(\Phi)\,, \tag{7.20}$$

and

$$\Box\varphi^i = (\Box\varphi^i)^\sim \ . \tag{7.21}$$

The singular part of the Einstein tensor has the same expression as in Chapter 2 namely

$$\hat{G}_{\mu\nu} = \gamma_{(\,\mu}\, n_{\nu\,)} - \frac{1}{2}\,(\gamma\, n_\mu\, n_\nu + \gamma^\dagger\, g_{\mu\nu}) \ , \tag{7.22}$$

while (7.21) shows that the d'Alembertian of the scalar fields has no singular part.The total stress-energy tensor $\mathcal{T}_{\mu\nu}$ can thus be decomposed as

$$\mathcal{T}_{\mu\nu} = \tilde{\mathcal{T}}_{\mu\nu} + S_{\mu\nu}\, \chi\, \eta\, \delta(\Phi) \ , \tag{7.23}$$

where the singular part $S_{\mu\nu}$, which is interpreted as the surface stress–energy tensor of the null shell, can only come from the stress-energy tensor $T_{\mu\nu}$ of the matter fields (see the discussion above). Introducing the above decompositions into the field equations (7.4) and (7.5) we obtain

$$16\pi\, S_{\mu\nu} = 2\,\gamma_{(\,\mu}\, n_{\nu\,)} - \gamma\, n_\mu\, n_\nu - \gamma^\dagger\, g_{\mu\nu} \ , \tag{7.24}$$

and

$$\alpha^i(\varphi)\, S = 0 \ . \tag{7.25}$$

Since the jumps α^i of the scalar fields do not vanish the surface stress–energy tensor is trace–free, which thus implies from (7.24) that $\gamma^\dagger = 0$ and $S_{\mu\nu}$ reduces to

$$16\,\pi\, S_{\mu\nu} = 2\,\gamma_{(\,\mu}\, n_{\nu\,)} - \gamma\, n_\mu\, n_\nu \ . \tag{7.26}$$

As shown in Chapter 2 the vanishing of the trace of the surface stress-energy tensor corresponds to a pressure–free null shell. The intrinsic components S^{ab} are easily constructed. They have the same form as (2.37) with the condition $\gamma^\dagger = 0$ reading $\gamma_{ab}\, n^a\, n^b = 0$. As in general relativity a null shell generally coexists with an impulsive gravitational wave, the latter being described by the part $\hat{\gamma}_{ab}$ of γ_{ab} defined in §2.3 .

Scalar–tensor theories usually include gauge fields and we briefly discuss the smoothness properties of these fields. For a non–Abelian gauge field of Yang–Mills type

$$F^a_{\mu\nu} = \nabla_\mu\, A^a_\nu - \nabla_\nu\, A^a_\mu - g\,(A_\mu \wedge A_\nu)^a \ . \tag{7.27}$$

The potential is C^0 on the singular null hypersurface $\mathcal{N}$ with the jump in its transverse first derivatives defined by

$$[\partial_\mu A^a_\nu] = \eta\, n_\mu\, \lambda^a_\nu \ . \tag{7.28}$$

This means that the field $F^a_{\mu\nu}$ is discontinuous across $\mathcal{N}$ with jumps given by

$$[F^a_{\mu\nu}] = \eta\,(n_\mu\, \lambda^a_\nu - n_\nu\, \lambda^a_\mu) \ . \tag{7.29}$$

The surface $\mathcal{N}$ is thus the history of a shock wave. The field equations satisfied by the gauge fields $\nabla_\nu F^{a\mu\nu} = 4\,\pi\, J^{a\mu}$ imply that the vector current $J^{a\mu}$ can be decomposed in the same way as the stress–energy tensor (7.23). Its singular part, denoted $j^{a\mu}$ and representing the non–Abelian surface current, is given by

$$4\,\pi\, j^{a\mu} = (\lambda^a \cdot n)\, n^\mu \ . \tag{7.30}$$

Thus only one component of λ^a_μ contributes to the surface current. Two of the remaining three components describe the shock wave.

Examples of singular null hypersurfaces in scalar–tensor theories are given in [Barrabès and Bressange (1997)] and [Bressange (1998)]. In the former no gauge fields are introduced and the case in which the hypersurface is time–like is also treated. In the latter work the electromagnetic gauge field is introduced and use is made of a generalization to the Brans–Dicke theory of a plane wave solution described in [Hogan (1999)].

Besides the scalar-tensor theories of gravity which have been described here there exist other theories of gravity which also introduce scalar fields such as the dilaton field. Although they look quite similar to scalar-tensor theories when no matter field is present, the dilatonic theories of gravity have an important difference which is due to the way in which the dilaton field couples to the other fields. We begin with the action incorporating the presence of a dilaton field [Damour (1995)]:

$$S = \int d^4x \frac{\sqrt{-g}}{4q}\left[R - 2\,(\nabla\varphi)^2\right] - \int d^4x \frac{\sqrt{-g}}{4} k_F\, e^{-2\,\kappa\,\varphi} F^2 + S_m\,[\Psi, \varphi, g] \ , \tag{7.31}$$

where φ is the dilaton field, Ψ is a matter field, F is a Maxwell field with $F = dA$ and A is the potential 1–form. Here q is the gravitational coupling constant ($q = 4\,\pi\, G$), κ is the coupling constant to the dilaton field and k_F is the coupling constant for the gauge field F. The action S_m for the matter fields Ψ can for instance be taken as

$$S_m\left[\Psi,\varphi,g^{\alpha\beta}\right]=\int d^4x\sqrt{-g}\left[-\frac{1}{2}\left(D_\alpha\Psi\right)\left(D^\alpha\Psi\right)^\star-e^{2\kappa\varphi}V(\Psi)\right]\ ,\quad(7.32)$$

where $D_\alpha=\partial_\alpha+ieA_\alpha$ is the gauge-covariant derivative, e is the associated charge and V is a potential. Such an expression for the total action S shows that the dilaton does not minimally couple to the different fields but that it induces space–time dependent coupling factors. The field equation for the dilaton which follows from this action is

$$\Box\varphi=-\frac{q\,\kappa\,k_F}{2}\,e^{-2\,\kappa\,\varphi}\,F^2+2\,q\,\kappa\,e^{2\,\kappa\,\varphi}\,V(\Psi)\ .\qquad(7.33)$$

Comparison with the analogous equation (7.5) of scalar-tensor theories, we see that only the potential $V(\Psi)$, and not the trace $T(\Psi)$ of the stress-energy tensor of the matter field, appears. Therefore since the matter field Ψ is discontinuous across the null hypersurface $\mathcal{N}$ and since the kinetic terms no longer appear in the field equation for the dilaton field, we conclude that the dilaton field φ is necessarily C^1 on $\mathcal{N}$ (we recall that the Maxwell field F is at worst discontinuous across $\mathcal{N}$). The jump in the second derivatives of φ across $\mathcal{N}$ is thus given by

$$[g^{\mu\nu}\partial_{\mu\nu}\varphi]=-\frac{q\,\kappa\,k_F}{2}\,e^{-2\,\kappa\,\varphi_{|\mathcal{N}}}\,[F^2]+2\,q\,\kappa\,e^{2\,\kappa\,\varphi_{|\mathcal{N}}}\,[V(\Psi)]\ .\qquad(7.34)$$

It thus follows from this brief description that the scalar fields in scalar-tensor theories behave quite differently to the dilaton field when crossing a singular null hypersurface: the former are C^0 while the latter is C^1 across the singular null hypersurface. This is a consequence of the difference in the way these fields couple to the matter field. Since the dilaton field is C^1 it will not contribute to the singular δ-term appearing in the field equations, and the expression for the surface stress-energy tensor of an arbitrary null shell is thus not affected by its presence and remains the same as in general relativity.

7.2 Einstein–Cartan Theory of Gravitation

The Einstein–Cartan theory of gravitation was proposed by Élie Cartan [Cartan (1922)] as a natural geometrical generalisation of general relativity in which the four dimensional pseudo–Riemannian space–time manifold has a linear connection which is compatible with the metric but is not symmetric (the torsion tensor of the connection is non–zero). Cartan took the torsion of the connection to be related to the intrinsic angular momentum of the matter. The theory has the property that for vacuum space–times it coincides with Einstein's theory. Perhaps for this reason it lay dormant for many years and was independently re-discovered by Sciama and Kibble [Sciama (1958)], [Kibble (1961)]. Its mathematical structure was analysed in terms of tensor–valued differential forms in a series of important papers by Trautman.

In a local coordinate system $\{x^\mu\}$ we write the components of the metric tensor as $g_{\mu\nu}$ and the components of the connection as $\Gamma^\lambda_{\mu\nu}$. The torsion tensor of this connection has components $Q^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} \neq 0$ and the compatibility of this connection with the metric tensor allows us to write the components of the connection in the form

$$\Gamma^\lambda_{\mu\nu} = {}^R\Gamma^\lambda_{\mu\nu} + \chi^\lambda{}_{\mu\nu} \ , \tag{7.35}$$

where ${}^R\Gamma^\lambda_{\mu\nu}$ are the components of the Riemannian connection calculated from the metric tensor $g_{\mu\nu}$ and $\chi_{\lambda\mu\nu} = g_{\lambda\sigma}\,\chi^\sigma{}_{\mu\nu}$ are the components of the 'contorsion' or 'defect' tensor of the connection and are given in terms of the components of the torsion tensor by

$$\chi_{\lambda\mu\nu} = -\frac{1}{2}\left(Q_{\lambda\mu\nu} + Q_{\mu\nu\lambda} + Q_{\nu\mu\lambda}\right) = -\chi_{\mu\lambda\nu} \ , \tag{7.36}$$

with $Q_{\lambda\mu\nu} = g_{\lambda\sigma}\,Q^\sigma{}_{\mu\nu} = -Q_{\lambda\nu\mu}$. The curvature tensor of the connection has components $R_{\mu\nu\lambda\rho}$ defined in the usual way. As a result it has the symmetry $R_{\mu\nu\lambda\rho} = -R_{\mu\nu\rho\lambda}$ and on account of the metric compatibility of the connection it also has the symmetry $R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho}$. The cyclic identity no longer holds because the torsion of the connection is non–zero. The Ricci tensor components are $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ and this is not a symmetric tensor. The (non–symmetric) Einstein tensor components $G_{\mu\nu}$ are constructed from the Ricci tensor and the Ricci scalar in the usual way. The so–called Einstein–Cartan field equations read

$$G_{\mu\nu} = 8\,\pi\,T_{\mu\nu} \ , \tag{7.37}$$

and

$$Q^\mu{}_{\nu\lambda} + \delta^\mu_\nu\,Q^\rho{}_{\lambda\rho} - \delta^\mu_\lambda\,Q^\rho{}_{\nu\rho} = 8\,\pi\,S^\mu{}_{\nu\lambda} \ . \tag{7.38}$$

Here $T_{\mu\nu}$ represents the energy–momentum–stress content of the matter distribution and $S^\mu{}_{\nu\lambda}$ is the spin tensor incorporating the intrinsic angular momentum in the matter distribution. In a vacuum in which $T_{\mu\nu} = 0 = S^\mu{}_{\nu\lambda}$ it follows from (7.38) that $Q^\mu{}_{\nu\lambda} = 0$ and the theory becomes general relativity. The Einstein–Cartan theory is a theory of non–propagating torsion

[Hehl et al. (1976)]. The Bianchi identities involve the torsion tensor algebraically and in particular the twice–contracted Bianchi identities read

$$G^{\nu}{}_{\rho|\nu} = R^{\mu}{}_{\lambda}\,Q^{\lambda}{}_{\rho\mu} + \frac{1}{2}\,R^{\mu\nu}{}_{\lambda\rho}\,Q^{\lambda}{}_{\mu\nu}\ . \tag{7.39}$$

The theory of singular null hypersurfaces described in Chapter 2 can be extended to this theory [Bressange (2000)].

With $\mathcal{N}$ the null hypersurface introduced in §2.1 , we assume as before that in a coordinate system $\{x^{\mu}\}$ covering both sides of $\mathcal{N}$ the metric tensor components $g_{\mu\nu}$ is continuous across $\mathcal{N}$. Thus $[g_{\mu\nu}] = 0$ as in §2.1 . With intrinsic coordinates $\{\xi^{a}\}$ $(a = 1,2,3)$ on $\mathcal{N}$ we have a basis of tangent vectors to $\mathcal{N}$, given by $e_{(a)} = \partial/\partial\xi^{a}$, having components $e^{\mu}_{(a)}$. Following [Bressange (2000)] we assume that *the tangential components of the torsion tensor are continuous across* $\mathcal{N}$. Thus $[e^{\mu}_{(a)}\,e^{\nu}_{(b)}\,e^{\lambda}_{(c)}\,Q_{\mu\nu\lambda}] \equiv [Q_{abc}] = 0$. We define the jump in the contorsion tensor across $\mathcal{N}$ by

$$[\chi_{\mu\nu\lambda}] = \eta\,x_{\mu\nu\lambda}\ , \tag{7.40}$$

with η given by (2.6). The jump in the Riemannian connection across $\mathcal{N}$ is due to the jump in the first derivatives of the metric tensor across $\mathcal{N}$ which we write in the form of (2.9) as

$$[\partial_{\alpha}\,g_{\mu\nu}] = \eta\,n_{\alpha}\,{}^{R}\gamma_{\mu\nu}\ , \tag{7.41}$$

where, as always, n_{α} are the covariant components of the normal to $\mathcal{N}$ and we write ${}^{R}\gamma_{\mu\nu}$ to distinguish this quantity from $\gamma_{\mu\nu}$ appearing in Chapter 2. Now $[\Gamma^{\lambda}_{\mu\nu}] = w^{\lambda}{}_{\mu\nu}$ with

$$w_{\mu\nu\lambda} = \frac{1}{2}\,\eta\,({}^{R}\gamma_{\mu\nu}\,n_{\lambda} + {}^{R}\gamma_{\lambda\mu}\,n_{\nu} - {}^{R}\gamma_{\lambda\nu}\,n_{\mu}) + \eta\,x_{\mu\nu\lambda}\ . \tag{7.42}$$

The singular part $\hat{R}_{\kappa\lambda\mu\nu}$ of the curvature tensor in (2.17) is modified in this case to read

$$\hat{R}_{\kappa\lambda\mu\nu} = n_{\mu}\,x_{\kappa\lambda\nu} - n_{\nu}\,x_{\kappa\lambda\mu} + 2\,n_{[\kappa}\,{}^{R}\gamma_{\lambda][\mu}\,n_{\nu]}\ . \tag{7.43}$$

Calculating the singular part of the Einstein tensor from this and using the field equations (7.37) the energy–momentum–stress tensor has the form (2.21) with the surface stress–energy tensor of the light–like shell with history $\mathcal{N}$ given by [Bressange (2000)]

$$\begin{aligned}16\,\pi\,S_{\mu\nu} = &-{}^{R}\gamma\,n_{\mu}\,n_{\nu} - {}^{R}\gamma^{\dagger}\,g_{\mu\nu} + 2\,{}^{R}\gamma_{(\mu}\,n_{\nu)}\\ &+2\,(n^{\sigma}\,x_{\sigma\mu\nu} - n_{\nu}\,x^{\sigma}{}_{\mu\sigma} + g_{\mu\nu}\,x^{\lambda}{}_{\sigma\lambda}\,n^{\sigma})\ .\end{aligned} \tag{7.44}$$

Since $x_{\lambda\mu\nu} = -x_{\mu\lambda\nu}$ we easily see from this that

$$S_{\mu\nu}\, n^{\mu} = 0 \ . \tag{7.45}$$

Now the twice–contracted Bianchi identities (7.39) involve a term containing the derivative of the delta function which vanishes on account of (7.45). Using $\vartheta(\Phi)\,\delta(\Phi) = \frac{1}{2}\,\delta(\Phi)$ further equations are obtained from (7.39) by equating the delta function terms on both sides. These have to be analysed in specific cases (cf. (3.195) and the discussion following). In addition to (7.45) the surface stress–energy tensor must also satisfy

$$S_{\mu\nu}\, n^{\nu} = 0 \ . \tag{7.46}$$

By (7.44) this is equivalent to

$$x_{\sigma\mu\rho}\, n^{\sigma}\, n^{\rho} + x^{\sigma}{}_{\rho\sigma}\, n^{\rho}\, n_{\mu} = 0 \ . \tag{7.47}$$

It is demonstrated in [Bressange (2000)] that this equation is satisfied as a result of the basic assumption that the components of the torsion tensor tangential to $\mathcal{N}$ are continuous across $\mathcal{N}$. In parallel with §2.2 one can give an intrinsic version of this theory of singular null hypersurfaces in Einstein–Cartan theory (see [Bressange (2000)] where the non– null case is also covered). In particular the intrinsic form of the surface stress–energy tensor turns out to be formally identical to (2.37) but with

$$\gamma_{ab} = {}^{R}\gamma_{ab} + \beta_{ab} \ , \tag{7.48}$$

where ${}^{R}\gamma_{ab}$ is the projection tangential to $\mathcal{N}$ of ${}^{R}\gamma_{\mu\nu}$ (thus ${}^{R}\gamma_{ab} = {}^{R}\gamma_{\mu\nu}\, e^{\mu}_{(a)}\, e^{\nu}_{(b)}$) while β_{ab} is the projection of $\beta_{\mu\nu} = -2\, N^{\rho}\, x_{\rho\mu\nu}$ tangential to $\mathcal{N}$ with N^{ρ} a transversal defined as in (2.28). Interested readers should consult [Bressange (2000)] for further details and also to see an explicit example of a null shell worked out in this theory.

7.3 Einstein–Gauss–Bonnet Theory of Gravitation

Space–times with noncompact extra dimensions are currently extensively studied, in particular in the context of brane–cosmology. These models, which are inspired by high-energy physics, string theory and M-theory, lead to a brane-world picture. In the most popular model [Randall and Sundrum]-[Binétruy et al. (2000)] our ordinary four-dimensional space–time is the history of a three-dimensional brane in a five-dimensional space–time (the bulk). All matter and gauge fields, except gravity, are confined to a 3-brane whose history is a time–like shell embedded with Z_2-symmetry in a 5-dimensional space–time having the anti-de Sitter geometry. An additional feature of string theory is that the low-energy effective action contains terms which are quadratic in the curvature. The Gauss-Bonnet term, $L_{GB} = R_{\alpha\beta\rho\sigma}\, R^{\alpha\beta\rho\sigma} - 4R_{\alpha\beta}\, R^{\alpha\beta} + R^2$, has then been introduced into the Einstein-Hilbert action. On the one hand it has been shown that the Gauss-Bonnet term is the only combination of such quadratic terms which does not produce any ghosts. On the other hand it has been known [Lovelock (1971)] for a long time that the Gauss- Bonnet term is the unique expression containing terms quadratic in the curvature which leads to field equations depending only on the first two derivatives of the metric and linear in the second derivatives (in four dimensions the introduction of the Gauss-Bonnet term is trivial and does not give any contribution to the field equations).

We show here how the theory presented in Chapter 2 can be extended to describe the junction-conditions on a singular null hypersurface when the Gauss-Bonnet term is present. This formalism places no restrictions on the matter content or the geometry of the outer space–time or on the matter content of the shell. The only restriction is the basic embedding condition that the induced metrics on the null hypersurface coincide. In spite of the presence of quadratic terms in the curvature no regularisation of the Dirac δ-function is required and a well–defined set of junction conditions is obtained. The junction conditions on an arbitrary hypersurface (time–like space–like or light–like) can be found in [Barrabès and Hogan (2003b)].

We consider a 5-dimensional space–time $\mathcal{M}$ with a system of local coordinates $\{x^\mu\}$, $\mu = 0,1,2,3,4$. The components of tensors on $\mathcal{M}$ in this coordinate system will be identified by an index 5. For example the metric tensor components will be denoted ${}^5g_{\mu\nu}$. The field equations are

$$ {}^5G_{\mu\nu} + \Lambda_5\ {}^5g_{\mu\nu} + 2\alpha H_{\mu\nu} = \kappa_5\ {}^5T_{\mu\nu}\,, \tag{7.49} $$

where ${}^5G_{\mu\nu}$ is the Einstein tensor calculated with the metric tensor ${}^5g_{\mu\nu}$. Also Λ_5 is the cosmological constant, α is a coupling constant and $H_{\mu\nu}$ is the Lovelock tensor[Lovelock (1971)], which originates in the Gauss-Bonnet term, and is given by

$$ \begin{aligned} H_{\mu\nu} = {}&{}^5R\,{}^5R_{\mu\nu} - 2\,{}^5R_\mu{}^\lambda\,{}^5R_{\lambda\nu} - 2\,{}^5R^{\alpha\beta}\,{}^5R_{\alpha\mu\beta\nu} + {}^5R_{\mu\rho\kappa\lambda}\,{}^5R_\nu{}^{\rho\kappa\lambda} \\ &-\frac{g_{\mu\nu}}{4}({}^5R_{\alpha\beta\rho\sigma}\,{}^5R^{\alpha\beta\rho\sigma} - 4\,{}^5R_{\alpha\beta}\,{}^5R^{\alpha\beta} + {}^5R^2)\,. \end{aligned} \tag{7.50} $$

Here ${}^5R_{\mu\rho\kappa\lambda}$, ${}^5R_{\mu\nu}$, ${}^5R$ are the components of the Riemann tensor, Ricci tensor and Ricci scalar respectively calculated with the metric tensor ${}^5g_{\mu\nu}$. In the right hand side of the field equations the coefficient κ_5 is the 5-D gravitational constant and ${}^5T_{\mu\nu}$ is the stress-energy tensor describing the matter content (of the bulk and the brane in the brane-world language).

We use the same notations as in chapter 2. All the equations (2.1) to (2.20) still apply with now values for the Greek indices ranging from 0 to 4. If we use the expression (2.17) for the Riemann tensor to calculate the Lovelock tensor $H_{\mu\nu}$ in (7.50), undefined δ^2-terms appear since $H_{\mu\nu}$ is quadratic in the Riemann tensor. However one can show that without having to introduce any regularisation of the δ-function these terms simply disappear. To see this we first note that for any tensor A having the form $A = \tilde{A} + \hat{A}\,\chi\,\epsilon\,\delta(\Phi)$ and any other tensor B having the same form their product AB contains a δ^2-term with coefficient $\hat{A}\hat{B}\chi^2\epsilon^2$. If this is applied to the calculation of $H_{\mu\nu}$ and use is made of the expressions for $\hat{R}$, $\hat{R}_{\mu\nu}$ and $\hat{R}_{\kappa\lambda\mu\nu}$ given or derived from (2.17) we find that the total contribution of all products of the type $\hat{A}\hat{B}$ is zero. This fortuitous result is a consequence of the quasi–linearity of the Gauss–Bonnet term. Therefore the coefficient of the δ^2 term in $H_{\mu\nu}$ vanishes and we can write

$$H_{\mu\nu} = \tilde{H}_{\mu\nu} + \hat{H}_{\mu\nu}\,\epsilon\chi\,\delta(\Phi)\,, \tag{7.51}$$

where $\hat{H}_{\mu\nu}$ is given by

$$\begin{aligned}\hat{H}_{\mu\nu} &= \hat{R}_{\mu\nu}\,{}^5\bar{R} + \hat{R}\,{}^5\bar{R}_{\mu\nu} - 2\,(\hat{R}^{\alpha\beta}\,{}^5\bar{R}_{\alpha\mu\beta\nu} + \hat{R}_{\alpha\mu\beta\nu}\,{}^5\bar{R}^{\alpha\beta})\\ &+\hat{R}_{\mu\kappa\rho\lambda}\,{}^5\bar{R}_{\nu}{}^{\kappa\rho\lambda} + \hat{R}_{\nu}{}^{\kappa\rho\lambda}\,{}^5\bar{R}_{\mu\kappa\rho\lambda} - 2\,(\hat{R}_{\mu}{}^{\lambda}\,{}^5\bar{R}_{\nu\lambda} + \hat{R}_{\nu}{}^{\lambda}\,{}^5\bar{R}_{\mu\lambda})\\ &-\frac{{}^5g_{\mu\nu}}{2}(\hat{R}_{\alpha\beta\rho\sigma}\,{}^5\bar{R}^{\alpha\beta\rho\sigma} - 4\,\hat{R}_{\alpha\beta}\,{}^5\bar{R}^{\alpha\beta} + \hat{R}\,{}^5\bar{R})\,.\end{aligned} \tag{7.52}$$

In this expression $\tilde{H}_{\mu\nu}$ has the form defined in (2.2), the bar denotes the 'average' of a quantity which is discontinuous across $\mathcal{N}$ (thus $\bar{A} = (A^+|_{\mathcal{N}} + A^-|_{\mathcal{N}})/2$ for any A for which $[A] \neq 0$) and the property $\Theta(x)\,\delta(x) = \frac{1}{2}\,\delta(x)$, which is distributionally valid, has been used. When these results are substituted into the field equations (7.49) we find that the stress-energy tensor on the right hand side of the equations is decomposed into a tilde term and a singular term with the latter indicating the presence, in general, of a thin shell with history $\mathcal{N}$. Thus

$${}^5T_{\mu\nu} = {}^5\tilde{T}_{\mu\nu} + S_{\mu\nu}\eta\chi\delta(\Phi)\,. \tag{7.53}$$

The tensor $S_{\mu\nu}$ is the surface stress-energy tensor of the shell. Identifying the singular terms on each side of the field equations we have

$$\kappa_5\,S_{\mu\nu} = \hat{G}_{\mu\nu} + 2\,\alpha\,\hat{H}_{\mu\nu}\,, \tag{7.54}$$

where $\hat{G}_{\mu\nu}$ is given by (2.19)-(2.20) and $\hat{H}_{\mu\nu}$ by (7.52). We already know from chapter 2 that $\hat{G}_{\mu\nu}n^\nu = 0$ and it can be checked that we also have $H_{\mu\nu}n^\nu = 0$. Therefore $S_{\mu\nu}n^\nu = 0$ and as expected the surface stress-energy tensor $S_{\mu\nu}$ is a tangential quantity.

In order to describe the singular null hypersurface in an intrinsic way we introduce on $\mathcal{N}$, as in §2.2 , four intrinsic coordinates $\{\xi^a\}$ with $a = 1, 2, 3, 4$ and the corresponding holonomic tangent vectors $e_{(a)} = \partial/\partial\xi^a$. We choose here $e_{(1)} = n$ future-directed and therefore the other vectors $e_{(A)}$ with $A = 2, 3, 4$ are space–like. We then define on $\mathcal{N}$ the basis $\{N, n, e_{(A)}\}$ where the transversal is chosen light–like, perpendicular to the $e_{(A)}$'s and oriented toward the future of $\mathcal{N}$. Thus N satisfies $N \cdot N = N \cdot e_{(A)} = 0$ and $N \cdot n = -1$. The completeness relation for this basis is

$$ {}^5g^{\alpha\beta} = g^{AB}\, e^{\alpha}_{(A)}\, e^{\beta}_{(B)} - 2\, n^{(\alpha}\, N^{\beta)}\,, \tag{7.55} $$

where g^{AB} is the inverse of $g_{AB} = e_{(A)} \cdot e_{(B)}$. We know that for a light–like hypersurface the induced metric $g_{ab} = e_{(a)} \cdot e_{(b)}$ is degenerate and therefore cannot be inverted.

Of the following two tensors

$$ K_{ab} = -n_\mu\, e^{\mu}_{(a)|\lambda}\, e^{\lambda}_{(b)}\,, \qquad \mathcal{K}_{ab} = -N_\mu\, e^{\mu}_{(a)|\lambda}\, e^{\lambda}_{(b)}\,, \tag{7.56} $$

where the stroke as before denotes covariant differentiation with respect to the Riemannian connection calculated with the five dimensional metric tensor on either side of $\mathcal{N}$, only the second tensor describes transverse properties and represents an extrinsic curvature. While K_{ab} is purely intrinsic and therefore continuous across $\mathcal{N}$, $\mathcal{K}_{ab}$ is discontinuous across $\mathcal{N}$ with a jump described by $\gamma_{ab} \equiv \gamma_{\mu\nu}\, e^{\mu}_{(a)}\, e^{\nu}_{(b)} = 2[\mathcal{K}_{ab}]$. If we express the tensors $S^{\mu\nu}$, $\hat{G}^{\mu\nu}$ and $\hat{H}^{\mu\nu}$ in terms of the tangent basis $\{e_{(a)}\}$, and thus write $S^{\mu\nu} = S^{ab}e^{\mu}_{(a)}e^{\nu}_{(b)}$, with similar expressions holding for $\hat{G}^{\mu\nu}$ and $\hat{H}^{\mu\nu}$, then the equation for $\hat{G}^{ab}$ is the analogue in five dimensions of the result given in (2.37)

$$ 2\,\hat{G}^{ab} = -(\gamma_{cd}\, g^{cd}_{*})\, n^a n^b \; - \gamma^{\dagger}\, g^{ab}_{*} \; + \{g^{ac}_{*}\, n^b n^d \; + g^{bc}_{*}\, n^a n^d\}\gamma_{cd}\,, \tag{7.57} $$

where one takes for g^{ab}_{*} the matrix g^{AB} bordered by zeros. The equation for $\hat{H}^{ab}$ is a intricate expression which contains the intrinsic curvature tensor R_{abcd}, the intrinsic three-dimensional curvature K_{AB}, and the extrinsic curvature $\mathcal{K}_{ab}$. This is more complicated than in the non–null case [Barrabès and Hogan (2003b)] because the Gauss-Codazzi equations written on the basis $\{N, n, e_{(A)}\}$ mix in a non simple way R_{abcd}, K_{AB}, and $\mathcal{K}_{ab}$. As in general relativity a null shell and an impulsive gravitational wave generally coexist with the hypersurface $\mathcal{N}$ representing their space–time history.

As an illustration of this theory we consider a spherical null shell propagating radially in a brane-cosmological model [Binétruy et al. (2000)]. It is convenient to write the line–element in terms of the Eddington retarded or advanced time coordinate u as

$$ds^2 = -du\,\{h(r)\,du + 2\,\zeta\,dr\} + r^2\,d\chi^2 + r^2\,f_k(\chi)^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,. \quad (7.58)$$

The constant factor ζ is $+1(-1)$ if the light cone $\mathcal{N}$ with equation $u = $ const. is expanding (contracting) towards the future. The functions $h(r)$ and $f_k(\chi)$ are given by

$$f_k(\chi) = (1,\,\sin\chi,\,\sinh\chi) \qquad \text{for} \qquad k = (0, 1, -1)\;, \quad (7.59)$$

respectively and

$$h(r) = k + \frac{r^2}{4\alpha}\,(1 - \sqrt{A(r)})\,. \quad (7.60)$$

In this expression the function $A(r)$ is given by [Boulware and Deser (1985)], [Cai (2002)]

$$A(r) = 1 + \frac{4}{3}\,\alpha\,\Lambda_5 + 8\,\alpha\,\frac{m}{r^4}\;, \quad (7.61)$$

with m playing the role of a mass parameter. The embedding requires that the function $f_k(\chi)$ be the same on both sides of $\mathcal{N}$, but the cosmological constant $\Lambda_5^{\pm}$ and the mass parameter $m_{\pm}$ can differ on both sides of $\mathcal{N}$. The components of the normal n and the transversal N are $n^\mu = \zeta\,\delta^\mu_r$ and $N^\mu = (1, -\zeta\,h(r)/2, 0, 0, 0)$. Now the only non–vanishing components of $\mathcal{K}_{ab}$ and K_{ab} are given by

$$\mathcal{K}^\chi_\chi = \mathcal{K}^\theta_\theta = \mathcal{K}^\phi_\phi = -\zeta\,h/2\,r\,, \qquad K^\chi_\chi = K^\theta_\theta = K^\phi_\phi = \zeta/r\,. \quad (7.62)$$

Because $\mathcal{N}$ is a null cone, and therefore the light–like signal is spherical–fronted, $\mathcal{N}$ cannot be the history of an impulsive gravitational wave. In fact $\mathcal{N}$ is the history of a spherical null shell which is simply characterized by its surface energy density ρ and surface pressure P (there is no heat current). In other words the junction conditions imply that the surface stress-energy tensor S^{ab} has components $S^{11} = \rho$, $S^{AB} = P\,g^{AB}$ and $S^{A1} = 0$ with

$$\kappa_5\,\rho = \frac{3\,\zeta}{2\,r}\,[h(r)] - \frac{2\,\alpha\,\zeta}{r^3}\,[h(r)]\left\{\frac{15}{2}\,\bar{h}(r) - \frac{(1 - 3k\,f_k(\chi)^2)}{f_k(\chi)^2}\right\}, \quad (7.63)$$

$$\kappa_5\,P=-\frac{9\,\alpha\,\zeta}{r^3}\,[h(r)]\,\left(1+\frac{\bar{h}(r)}{2}\right)\,. \tag{7.64}$$

When the Gauss-Bonnet term is absent ($\alpha=0$) the surface pressure vanishes ($P=0$) and the surface energy density of the null shell is given simply by

$$\kappa_5\rho=-\frac{3\,\zeta}{2\,r^3}\,[m]\,-\,\frac{\zeta\,r}{4}\,[\Lambda_5]\,. \tag{7.65}$$

A four dimensional analogue of this expression, in which a spherical light–like shell propagates radially with a Schwarzschild field outside and a de Sitter field inside, is given in §5.1 .

Appendix A

Notation

The signature of the metric tensor is taken to be +2. The components of the curvature tensor of a connection having components $\Gamma^{\mu}_{\lambda\sigma}$, in a local coordinate system $\{x^{\mu}\}$, are

$$R^{\mu}{}_{\kappa\rho\sigma} = \Gamma^{\mu}_{\kappa\sigma,\rho} - \Gamma^{\mu}_{\kappa\rho,\sigma} + \Gamma^{\lambda}_{\kappa\sigma}\,\Gamma^{\mu}_{\lambda\rho} - \Gamma^{\lambda}_{\kappa\rho}\,\Gamma^{\mu}_{\lambda\sigma} \ , \tag{A.1}$$

where a partial derivative is indicated by a comma. When $\Gamma^{\mu}_{\kappa\sigma} = \frac{1}{2}\, g^{\mu\lambda}\,(g_{\lambda\kappa,\sigma} + g_{\sigma\lambda,\kappa} - g_{\kappa\sigma,\lambda})$, where $g_{\mu\nu}$ are the components of the metric tensor (with inverse $g^{\mu\nu}$), is the Riemannian connection then (A.1) is the Riemann curvature tensor. The Ricci tensor has components $R_{\kappa\sigma} = R^{\mu}{}_{\kappa\mu\sigma}$ and the Ricci scalar is given by $R = g^{\mu\nu}\, R_{\mu\nu}$. The Einstein tensor has components $G_{\kappa\sigma} = R_{\kappa\sigma} - \frac{1}{2} g_{\kappa\sigma}\, R$ and with these conventions Einstein's field equations read

$$G_{\kappa\sigma} = 8\,\pi\, T_{\kappa\sigma} \ . \tag{A.2}$$

We use units in which c $= G = 1$ and $T_{\kappa\sigma}$ is the matter energy– momentum–stress tensor. The Weyl tensor components are

$$C_{\lambda\kappa\mu\sigma} = R_{\lambda\kappa\mu\sigma} - \frac{1}{2}\,(g_{\lambda\mu}\, R_{\kappa\sigma} + g_{\kappa\sigma}\, R_{\lambda\mu} - g_{\kappa\mu}\, R_{\lambda\sigma} - g_{\lambda\sigma}\, R_{\kappa\mu})$$
$$+\frac{1}{6} R\,(g_{\lambda\mu}\, g_{\kappa\sigma} - g_{\lambda\sigma}\, g_{\kappa\mu}) \ , \tag{A.3}$$

where $R_{\lambda\kappa\mu\sigma} = g_{\lambda\nu}\, R^{\nu}{}_{\kappa\mu\sigma}$ are the covariant components of the Riemann curvature tensor.

A null tetrad $\{m^{\mu}, \bar{m}^{\mu}, k^{\mu}, l^{\mu}\}$ consists of a complex null vector m^{μ} and its complex conjugate $\bar{m}^{\mu}$ and two real null vectors k^{μ}, l^{μ}. All scalar products of pairs of vectors of the tetrad vanish except $m_{\mu}\, \bar{m}^{\mu} = +1$, $k_{\mu}\, l^{\mu} = -1$. The Newman–Penrose components of the Maxwell tensor $F_{\mu\nu} = -F_{\nu\mu}$ on this null tetrad are defined by

$$\phi_0 = F_{\mu\nu}\, l^\mu\, m^\nu \ , \tag{A.4}$$

$$\phi_1 = \frac{1}{2}\, F_{\mu\nu}\, (l^\mu\, k^\nu + \bar{m}^\mu\, m^\nu) \ , \tag{A.5}$$

$$\phi_2 = F_{\mu\nu}\, \bar{m}^\mu\, k^\nu \ . \tag{A.6}$$

The Newman–Penrose components of the Weyl tensor on this null tetrad are defined by

$$\Psi_0 = C_{\mu\nu\rho\sigma}\, l^\mu\, m^\nu\, l^\rho\, m^\sigma \ , \tag{A.7}$$

$$\Psi_1 = C_{\mu\nu\rho\sigma}\, l^\mu\, k^\nu\, l^\rho\, m^\sigma \ , \tag{A.8}$$

$$\Psi_2 = C_{\mu\nu\rho\sigma}\, l^\mu\, m^\nu\, \bar{m}^\rho\, k^\sigma \ , \tag{A.9}$$

$$\Psi_3 = C_{\mu\nu\rho\sigma}\, l^\mu\, k^\nu\, \bar{m}^\rho\, k^\sigma \ , \tag{A.10}$$

$$\Psi_4 = C_{\mu\nu\rho\sigma}\, k^\mu\, \bar{m}^\nu\, k^\rho\, \bar{m}^\sigma \ . \tag{A.11}$$

Among the Newman–Penrose spin coefficients we require $\rho = m^\mu\, l_{\mu|\nu}\, \bar{m}^\nu$ and $\sigma = m^\mu\, l_{\mu|\nu}\, m^\nu$ where the stroke denotes covariant differentiation with respect to the Riemannian connection calculated with the metric tensor $g_{\mu\nu}$. If the integral curves of the vector field l^μ are affinely parametrised null geodesics then the real part of ρ is the expansion scalar and the imaginary part of ρ is the twist scalar associated with this geodesic congruence while σ is the complex shear ($|\sigma|$ is the shear) of this congruence.

If u^μ are the components of a unit time–like vector field (4– velocity) then $u^\mu\, u_\mu = -1$. The integral curves of u^μ have *expansion* scalar $\theta = u^\mu{}_{|\mu}$, *shear tensor*

$$\sigma_{\mu\nu} = u_{(\mu|\nu)} + \dot{u}_{(\mu}\, u_{\nu)} - \frac{1}{3}\,\theta\, h_{\mu\nu} \ , \tag{A.12}$$

where the round brackets denote symmetrisation, $\dot{u}_\mu = u_{\mu|\nu}\, u^\nu$ is the 4–acceleration and $h_{\mu\nu} = g_{\mu\nu} + u_\mu\, u_\nu$ is the projection tensor, and *vorticity tensor*

$$\omega_{\mu\nu} = u_{[\mu|\nu]} + \dot{u}_{[\mu}\, u_{\nu]} \ , \tag{A.13}$$

where square brackets denote skew–symmetrisation.

Appendix B

Singular Hypersurfaces: The Non–Null Case

Time–like shells have been used for a long time in general relativity as a convenient model to describe various phenomena such as for instance gravitational collapse. In this appendix we present the general theory of thin shells whose history in space–time is a singular time–like hypersurface. For completeness our formalism will include the case of singular space–like hypersurfaces as well. In order to facilitate comparison with the light–like case, which is the subject of the present book, a description similar to the one given in Chapter 2 will be used. The by now standard reference for the theory of thin shells in general relativity is [Israel (1966)].

The space–time manifold $\mathcal{M}$ is divided into two domains by a time–like or space–like hypersurface on which the metric tensor is only C^0. As a consequence of this the Riemann curvature tensor contains a Dirac δ-term with support on the hypersurface. We call $\mathcal{M}^{\pm}$ the two domains of space–time and $\mathcal{S}$ the non–null hypersurface on which they overlap, in the sense that $\mathcal{S} = \mathcal{M}^+ \bigcap \mathcal{M}^-$. The space–time manifold $\mathcal{M} = \mathcal{M}^+ \bigcup \mathcal{M}^-$ admits a pair of metric tensors g^+ and g^- defined on $\mathcal{M}^+$ and $\mathcal{M}^-$ respectively. All quantities referring either to $\mathcal{M}^+$ or to $\mathcal{M}^-$ will be denoted by an index + or − respectively. To denote the jumps across $\mathcal{S}$ we use the following notation: for $F^{\pm}$ a quantity (the components of a tensor for example) defined on $\mathcal{M}^{\pm}$ respectively we denote its jump across $\mathcal{S}$ by $[F] = F^+|_{\mathcal{S}} - F^-|_{\mathcal{S}}$, where $|_{\mathcal{S}}$ indicates that $F^{\pm}$ is to be evaluated on the $\pm$ sides of $\mathcal{S}$ respectively.

Distributional Algorithm: – Let $\{x^{\mu}\}$ be a local coordinate system covering both sides of the hypersuface $\mathcal{S}$ in terms of which the components of the metric tensor are continuous across $\mathcal{N}$. Let $\Phi(x) = 0$ be the equation of $\mathcal{S}$ in these coordinates. The function Φ is chosen in such a way that $\Phi > 0$ in $\mathcal{M}^+$ and $\Phi < 0$ in $\mathcal{M}^-$. Greek indices take values $0, 1, 2, 3$ and the components of the metric tensor are $g^{\pm}_{\mu\nu}$ in $\mathcal{M}^{\pm}$ respectively. If $F^{\pm}(x)$ are two quantities (the components of a tensor for example) defined on $\mathcal{M}^{\pm}$

respectively, we define the hybrid quantity $\tilde{F}$ by

$$\tilde{F}(x) = F^{+}\,\vartheta(\Phi) + F^{-}\,(1 - \vartheta(\Phi)) \ , \tag{B.1}$$

where $\vartheta(\Phi)$ is the Heaviside step function which is equal to unity when $\Phi > 0$ and equal to zero when $\Phi < 0$. Thus in particular for the metric tensor we have $\tilde{g}_{\mu\nu}$ defined and since the metric is continuous across $\mathcal{S}$ we can write $[g_{\mu\nu}] = 0$. As a result of the definition (B.1) we have for two quantities $F^{\pm}$ and $G^{\pm}$ the product rule, $\tilde{F}\tilde{G} = \tilde{FG}$, because $\vartheta(\Phi)(1-\vartheta(\Phi))$ vanishes distributionally. The normal to the hypersurface has components

$$n_{\mu} = \chi^{-1}(x)\,\partial_{\mu}\Phi(x) \ , \tag{B.2}$$

where $\chi(x)$ is a normalizing factor such that

$$n \cdot n = g^{\mu\nu} n_{\mu} n_{\nu}\,|_{\pm} = \epsilon \ , \tag{B.3}$$

with $\epsilon = +1\,(-1)$ if the hypersurface is time–like (space–like). With these definitions the partial derivative of $\tilde{F}$ takes the form

$$\partial_{\mu}\tilde{F} = \widetilde{\partial_{\mu}F} \ + \ [F]\,\chi n_{\mu}\,\delta(\Phi) \ . \tag{B.4}$$

A singular term proportional to the Dirac δ-funtion appears whenever F is discontinuous across the hypersurface $\mathcal{S}$. The metric and its tangential derivatives are continuous across $\mathcal{S}$ but its transverse derivatives are not. To characterize the discontinuities in the transverse derivatives of the metric tensor we define the symmetric tensor $\gamma_{\mu\nu}$ by

$$[\partial_{\alpha}\,g_{\mu\nu}] = \epsilon\,n_{\alpha}\,\gamma_{\mu\nu} \ . \tag{B.5}$$

The tensor $\gamma_{\mu\nu}$ is only defined *on* $\mathcal{S}$ and is such that its projection onto $\mathcal{S}$ is unique. Thus $\gamma_{\mu\nu}$ is free up to the gauge transformation

$$\gamma_{\mu\nu} \to \gamma'_{\mu\nu} = \gamma_{\mu\nu} + v_{\mu}n_{\nu} + n_{\mu}v_{\nu} \ , \tag{B.6}$$

where v is an arbitrary vector field on $\mathcal{S}$. This gauge freedom can always be used to have $\gamma_{\mu\nu}n^{\nu} = 0$. Using (B.5) we find that the Christoffel symbols satisfy

$$\Gamma^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu} \qquad \text{and} \qquad [\Gamma^{\lambda}_{\mu\nu}] = \epsilon\,\gamma^{\lambda}_{(\mu}n_{\nu)} - \frac{\epsilon}{2}\gamma_{\mu\nu}n^{\lambda} \ , \tag{B.7}$$

with round brackets around indices denoting symmetrisation. Then using (B.4) the Riemann curvature tensor $R_{\kappa\lambda\mu\nu}$ can be decomposed into the sum of a tilde-term defined as in (B.1) and a term containing a Dirac δ-function:

$$R_{\kappa\lambda\mu\nu} = \tilde{R}_{\kappa\lambda\mu\nu} + \hat{R}_{\kappa\lambda\mu\nu}\,\epsilon\chi\delta(\Phi) \ , \tag{B.8}$$

where

$$\hat{R}_{\kappa\lambda\mu\nu} = 2\, n_{[\kappa}\gamma_{\lambda][\mu}n_{\nu]} \; . \tag{B.9}$$

The square brackets here around indices denote skew–symmetrisation. A similar decomposition therefore exists for the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu}$. The singular part of the Einstein tensor is given by

$$\hat{G}_{\mu\nu} = \gamma_{(\mu}n_{\nu)} - \frac{\gamma}{2}\, n_\mu\, n_\nu - \frac{\gamma^\dagger}{2}\, g_{\mu\nu} - \frac{\epsilon}{2}\,(\gamma_{\mu\nu} - \gamma\, g_{\mu\nu}) \; , \tag{B.10}$$

where we have introduced the following quantities

$$\gamma \equiv g^{\mu\nu}\,\gamma_{\mu\nu} \, , \qquad \gamma_\mu \equiv \gamma_{\mu\nu}\, n^\nu \, , \qquad \gamma^\dagger \equiv \gamma_{\mu\nu}\, n^\mu\, n^\nu = \gamma_\mu\, n^\mu \; . \tag{B.11}$$

Recall that one can always choose the gauge such that $\gamma_\mu = \gamma^\dagger = 0$.

When these results are substituted into the field equations

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\,\pi\, T_{\mu\nu} \; , \tag{B.12}$$

where $G_{\mu\nu}$ is the Einstein tensor and Λ the cosmological constant, we find that the stress-energy tensor on the right hand side of the equations is decomposed into a tilde term and a singular term with the latter indicating the presence, in general, of a thin shell with history $\mathcal{S}$. Thus

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + S_{\mu\nu}\, \chi\, \delta(\Phi) \; . \tag{B.13}$$

The tensor $S_{\mu\nu}$ is the surface stress-energy tensor of the shell. Identifying the singular terms on each side of the field equations we have

$$8\,\pi\, S_{\mu\nu} = \epsilon\, \hat{G}_{\mu\nu} \; , \tag{B.14}$$

where $\hat{G}_{\mu\nu}$ is given by (B.10). The surface stress-energy tensor $S_{\mu\nu}$ is intrinsic to $\mathcal{S}$ since it satisfies $S_{\mu\nu}\, n^\nu = 0$ as one can check that $\hat{G}_{\mu\nu}\, n^\nu = 0$.

Extrinsic Curvature Algorithm: – Equation (B.14) represents the junction conditions on $\mathcal{S}$ in a system of coordinates covering both sides of the hypersurface. This presentation is closely related to the usual Israel junction conditions formalism [Israel (1966)] as there is a direct relation between the tensor $\gamma_{\mu\nu}$ and the jump in the extrinsic curvature. The formalism based on the extrinsic curvature has the advantage of allowing the space–time coordinates to be chosen freely and independently on each side of the hypersurface. Let $\{x^\mu_\pm\}$ be local coordinate systems for the domains $\mathcal{M}^\pm$ and introduce on the hypersurface $\mathcal{S}$ three intrinsic coordinates $\{\xi^a\}$ with

$a = 1, 2, 3$. The four holonomic basis vectors $e_{(a)} = \partial/\xi^a$ are tangent to $\mathcal{S}$ and have components

$$e^{\mu}_{(a)}|_{\pm} = \frac{\partial x^{\mu}_{\pm}}{\partial \xi^a} \ . \tag{B.15}$$

The completeness relation for the basis $\{n,\ e_{(a)}\}$ on the hypersurface reads

$$g^{\alpha\beta} = g^{ab}\, e^{\alpha}_{(a)}\, e^{\beta}_{(b)} + \epsilon\, n^{\alpha}\, n^{\beta} \ , \tag{B.16}$$

where g^{ab} is the inverse of the induced metric $g_{ab} = e_{(a)} \cdot e_{(b)}|_{\pm}$. Although the scalar product is taken with the two metrics $g^{\pm}_{\mu\nu}$ we assume, as the basic matching condition, that the induced metrics match on $\mathcal{S}$. The extrinsic curvature is defined by

$$K_{ab} = -n_{\mu}\, e^{\mu}_{(a)|\lambda}\, e^{\lambda}_{(b)} \ , \tag{B.17}$$

and takes different values $K^{\pm}_{ab}$ on each side of $\mathcal{S}$ as the Christoffel symbols in $\mathcal{M}^{\pm}$ differ – see (B.7). Here the stroke denotes covariant differentiation with respect to the Riemannian connection calculated with the metric tensor on either side of $\mathcal{S}$. It can be shown that

$$\gamma_{ab} \equiv 2[K_{ab}] \ , \tag{B.18}$$

is the projection on $\mathcal{S}$ of the tensor $\gamma_{\mu\nu}$ introduced in (B.5). Thus $\gamma_{ab} = \gamma_{\mu\nu}\, e^{\mu}_{(a)}\, e^{\nu}_{(b)}$. ¿From now on it will be convenient to use the gauge in which $\gamma_{\mu} = \gamma^{\dagger} = 0$. The projection of the surface stress–energy tensor onto $\mathcal{S}$ is given by $S_{ab} = S_{\mu\nu}\, e^{\mu}_{(a)}\, e^{\nu}_{(b)}$. In similar fashion the projection $\hat{G}_{ab}$ onto $\mathcal{S}$ of the singular parts of $G_{\mu\nu}$ is defined. The junction conditions (B.14) now read

$$8\,\pi\, S_{ab} = \epsilon\, \hat{G}_{ab} \ . \tag{B.19}$$

Using the Gauss-Codazzi equations and their contracted forms in (B.10) we find that

$$8\,\pi\, S_{ab} = -[K_{ab}] + [K]\, g_{ab} \ , \tag{B.20}$$

The equation (B.20) describes the evolution of the shell once a choice is made of the surface stress-energy tensor S_{ab} (dust or perfect fluid for example). In order to obtain the conservation law for the shell we consider the jump of the field equations (B.12) across $\mathcal{S}$ contracted with $n^{\mu}\, e^{\nu}_{(a)}$ and make use of the Gauss-Codazzi equations. The result is

$$S^{b}{}_{a;b} = -[T_{\mu\nu}\, n^{\mu}\, e^{\nu}_{(a)}] \ , \tag{B.21}$$

where the semicolon here denotes the covariant derivative associated with g_{ab}. For a time–like shell the $a = 0$ component of this equation gives the energy conservation equation. Another relation which can be derived from the Gauss-Codazzi equations is

$$S_{ab}\,\bar{K}^{ab} = [T_{\mu\nu}\,n^{\mu}\,n^{\nu}] - \frac{\epsilon}{8\pi}\,[\Lambda]\;, \tag{B.22}$$

where $\bar{K}_{ab} = (K^{+}_{ab} + K^{-}_{ab})/2$.

A comparison between the case of a time–like or space–like singular hypersurface described in this appendix and the light–like case described in Chapter 2 shows the following two important differences: i) The induced metric g_{ab} for light–like hypersurfaces is degenerate. Therefore its inverse g^{ab} cannot be defined and we have to introduce the pseudo–inverse metric g^{ab}_{*} – see equations (2.34-2.36). ii) The normal n in the light–like case is tangential to the hypersurface. Hence the quantity K_{ab} defined in (B.17) involves only tangential derivatives and cannot therefore describe extrinsic properties. For light–like hypersurfaces we need to introduce a transversal N and associate with this vector field a transverse curvature $\mathcal{K}_{ab}$ – see equations (2.28) and (2.29).

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