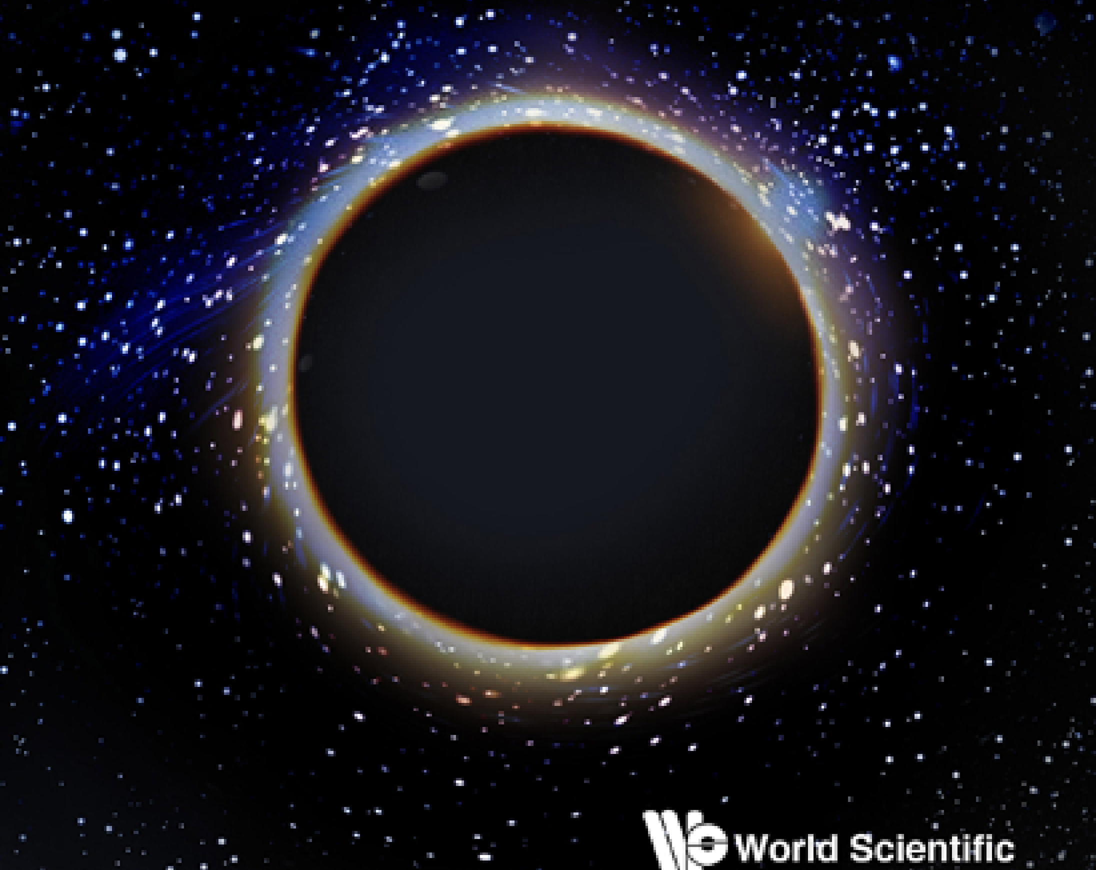
GENERAL RELATIVITY

A First Examination

Marvin Blecher



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Virginia Tech, USA

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I dedicate this work to Freda, the love of my life, the fount of my happiness and contentment, and to my family whose love sustains me.

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Preface

Upon retirement, I sought a learning project. My career was spent as an experimental physicist studying nuclei and particles. Throughout, I was deeply impressed by the connection between my area and general relativity and cosmology. Now, I had the time to dig into the details of those subjects. When I asked, the Virginia Tech Physics Department granted my request to teach the introductory semester course in general relativity. I did so three times and found that nothing reduces knowledge deficiency like teaching smart students.

Many excellent texts were available for my studies. However, there was too much material in them for students to cover in a single semester. So I developed my own set of notes, that explained the essentials of the subject in the course time period. This text evolved from those notes. I'm grateful to my students because their questions pushed me to explain difficult concepts as transparently as possible. I'm also grateful to many of my colleagues who participated in discussions, helped with proofs and critiqued some of my material.

Einstein's theory is now a century in age. It is well tested, but still inspires significant theoretical and experimental work. And it definitely intrigues students. Advanced undergraduate physics majors, first year physics graduate and engineering students have taken the course. Physics Department faculty and faculty from other departments have sat in on many lectures. I thank them all for their comments and questions. They helped me acquire a deeper understanding of the subject.

Most of the material in this book is found in various forms in other texts, but here, there is much that is novel and many more steps than usual are included in proofs. In chapter two, the twin problem with acceleration for the traveling twin is worked out. The way gravity affects time is first discussed for weak gravity via conservation of energy using a Newtonian formulation with relativistic mass. In chapter five, weak gravity is discussed and the way gravity affects time is rigorously covered. The Schwarzschild metric is obtained and in problem five of that chapter, students are asked to resolve the Schwarzschild problem with the cosmological constant included. Then they are asked to show that in weak gravity a very small repulsive Newtonian force arises. In chapter seven on black holes, an example, based on the film "Interstellar" is presented. The example discusses why a large gravitational time dilation is possible near a spinning, but not a static black hole. In chapter 8, gravitational waves are discussed. Just as this book was going to press the LIGO experiment announced the first direct detection of such a wave. Luckily, I was able to discuss this in the text. The theory behind the Nobel prize winning, gravitational wave indirect detection results, from the Hulse-Taylor binary pulsar, is worked out with elliptical orbits. Other texts that discuss this experiment have used circular orbits. As the eccentricity is large the latter orbits disagree with experiment. In chapter nine on cosmology, the results of numerical integrations using the current data for all the energy densities, are discussed. This allows the students to peer into both the past and the future for values of the universal scale factor, the Hubble parameter, the age of the universe and the horizon distance.

Marvin Blecher

Acknowledgments

I'm grateful to colleagues who participated in discussions with me or read some of my material. I especially thank Tatsu Takeuchi who helped with proofs, read and critiqued many of my chapters and John Simonetti who cleared up many concepts and explained details of the astronomical experiments discussed in the text. Eric Sharpe and James Gray also welcomed my questions. I'm indebted to Ms Samantha Spytek, an undergraduate physics major at Virginia Tech, for preparing many of the figures. Thanks are also due to Robin Blecher for some of the art work.

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Chapter 1

Review of Special Relativity

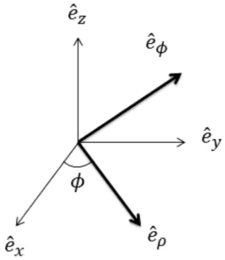
1.1 Introduction

The theory of Special Relativity (SR) was introduced by A. Einstein in 1905. It deals with the observations of inertial observers in the absence of gravity. The theory of General Relativity (GR) that includes gravitation (and thus acceleration) was published in 1915. For English translations see [Einstein (1905)]. The latter theory predicted the deflection of light near a massive body like the sun. When a British team led by A. S. Eddington confirmed this prediction near the end of the first world war, Einstein became world famous, even among people who had no particular interest in science.

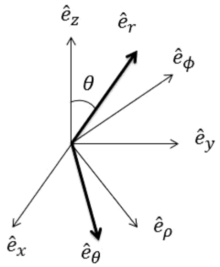
In relativity an observation is the assignment of coordinates, x^μ, $\mu = 0, 1, 2, 3$, for the time and space location of an event. Space is continuous and functions of the coordinates can be differentiated. Upon partial differentiation with respect to one of the coordinates, the others are held constant. This insures that the coordinates are independent,

$$x^\mu{}_{,\nu} \equiv \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu = \delta_\nu{}^\mu = 1,\ \mu = \nu,\ \delta^\mu{}_\nu = 0,\ \mu \neq \nu. \tag{1.1}$$

As will be seen, $\delta^\mu{}_\nu$, is the Kronecker delta tensor. Note the shorthand notation for the partial derivative by use of a comma. Such a shorthand will keep some of the formulas of GR, with many partial derivatives, to a reasonable length. In rectangular coordinates, $x^\mu = (t, x, y, z)$. If curvilinear coordinates are used, the coordinates, $x^{\mu'}$, are different and a rotation carries you from one set of coordinates to the other. In cylindrical coordinates, $x^{\mu'} = (t, \rho, \phi, z)$, because as illustrated in Fig. 1.1, the rotation changes the direction indicating unit vectors, $\hat{e}_x,\ \hat{e}_y \to \hat{e}_\rho,\ \hat{e}_\phi$. Similarly for spherical coordinates, $x^{\mu'} = (t, \theta, \phi, r)$, since, $\hat{e}_\rho,\ \hat{e}_z \to \hat{e}_\theta,\ \hat{e}_r$. Other texts employ



$$\begin{pmatrix} \hat{e}_\rho \\ \hat{e}_\phi \\ \hat{e}_z \end{pmatrix} = \begin{pmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix}$$



$$\begin{pmatrix} \hat{e}_\theta \\ \hat{e}_\phi \\ \hat{e}_r \end{pmatrix} = \begin{pmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{pmatrix} \begin{pmatrix} \hat{e}_\rho \\ \hat{e}_\phi \\ \hat{e}_z \end{pmatrix}$$

Fig. 1.1 Rotation relations for changing unit vectors from one coordinate system to another.

an extra renaming and take $x^{\mu'=0-3} = t, r, \theta, \phi$, but a rose by any name would smell sweet. Note, the spatial components of vectors change in the same way as the unit vectors.

The time of the event is read on a clock at rest with respect to the observer, at the spatial coordinates of the event. In the inertial frames of SR, an observer may suppose that there are synchronized clocks at rest at every point in space. This would not be the case when gravity is taken into account because such clocks would run at different rates in a varying gravitational field. Simultaneous events for a given observer are those occuring at the same time on the clocks nearest them, that are at rest with respect to the observer.

Einstein developed SR from two postulates: (1) the laws of physics are the same for all inertial observers no matter their relative velocities; (2) all inertial observers measure the same speed of light in vacuum, $c = 3 \times 10^8\,\mathrm{m/s}$. It is the second postulate that causes conflict with the Newtonian concept of time flowing independent of everything else. This leads to the observation that events simultaneous to one observer may not be so to another. Also c becomes the limiting speed in order to preserve causality. In GR the the word "inertial" is removed and the principle of equivalence, no gravitational effect is experienced when freely falling in a uniform gravitational field, must be taken into account.

In hindsight it is easy to see where the postulates come from. Various inertial observers in relative motion do electromagnetic experiments in their own rest frames. They find that the equations of Maxwell for the electric, magnetic fields $(\vec{E}, \vec{B})$ explain the results. Further, in vacuum and using MKS units, each finds they lead to a wave equation with a unique velocity,

$$\nabla \cdot \vec{E} = 0 = \nabla \cdot \vec{B}, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t},$$

$$0 = \nabla^2 \vec{B} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}, \quad v = (\mu_0 \epsilon_0)^{-1/2} = c.$$

As c is so special in SR and GR, it is convenient to work in a system of units where velocities are dimensionless and, $c = 1$. Then time is expressed in meters like the other coordinates and acceleration is expressed in inverse meters:

$$\begin{aligned} c &= 1 = 3 \times 10^8 \text{ m s}^{-1}, \\ s &= 3 \times 10^8 \text{ m}, \\ a &= \text{m s}^{-2} = 0.111 \times 10^{-16} \text{ m}^{-1}. \end{aligned} \tag{1.2}$$

Similarly, in GR, Newton's gravitational constant, G, is so special that it is convenient to also use, $G = 1$. This leads to the natural system of units. Here other mechanical quantities like mass, energy, momentum and angular momentum can be expressed in meters to the correct power:

$$\begin{aligned} 1 &= \frac{G}{c^2} = \frac{6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}}{[3 \times 10^8]^2 \text{ m}^2 \text{ s}^{-2}}, \\ &= 0.742 \times 10^{-27} \text{ kg}^{-1} \text{ m}, \\ M &= \text{kg} = 0.742 \times 10^{-27} \text{ m}, \\ E &= \text{kg m}^2 \text{ s}^{-2} = 0.824 \times 10^{-44} \text{ m}. \end{aligned} \tag{1.3}$$

Suppose a result is obtained in naturalized units for say, $\hbar = h/(2\pi) = 2.612 \times 10^{-70}\,\text{m}^2$, where h is Planck's constant. One can calculate the value in MKS units by noting that in this system the units are those of angular momentum, $\text{kg}\,\text{m}^2\,\text{s}^{-1}$. Then, multiply the value in natural units by unity with a quantity that expressed in MKS units will give the desired units,

$$
\begin{aligned}
\hbar &= 2.612 \times 10^{-70}\ \mathrm{m}^2[c/(G/c^2)], \\
&= \frac{(2.612 \times 10^{-70})(3 \times 10^8)}{0.742 \times 10^{-27}}\ \mathrm{m}^2(\mathrm{m/s})/(\mathrm{m/kg}), \\
&= 1.056 \times 10^{-34}\ \mathrm{kg\ m}^2\ \mathrm{s}^{-1}.
\end{aligned}
$$

1.2 Lorentz Transform

Two observers O and O′ are considered. They use parallel axes and rectangular coordinates. Rotations, like those in Fig. 1.1, allow them to align their, z axes along the relative velocity. Thus, O uses, x^μ, and says O′ is moving in the, z direction with speed, $V(< 1)$, while O′ uses, $x^{\mu'}$, and says O is moving in the, $-z$ direction with speed, V.

When their origins overlapped the clocks were synchronized, $t = x^0 = t' = x^{0'} = 0$. In this geometry, $(x, y) = (x', y')$, or $x^{1,2} = x^{1',2'}$, because there is no relative motion in these directions. However, because, $c = 1$, for both observers, space and time are inter-connected and now termed spacetime. if O′ says that events led to changes in coordinates, $dz' = dx^{3'}$, $dt' = dx^{0'}$, the components of the displacement vector, $dr^{\mu'}$, then O would calculate from the chain rule of differential calculus,

$$
\begin{aligned}
dx^3 = dz &= \frac{\partial z}{\partial z'}dz' + \frac{\partial z}{\partial t'}dt' + \frac{\partial z}{\partial x'}dx' + \frac{\partial z}{\partial y'}dy' \equiv x^3{}_{,\mu'}\, dx^{\mu'}, \\
&= x^3{}_{,3'}\, dx^{3'} + x^3{}_{,0'}\, dx^{0'} \qquad (1.4)
\end{aligned}
$$

$$
\begin{aligned}
dx^0 = dt &= \frac{\partial t}{\partial z'}dz' + \frac{\partial t}{\partial t'}dt' + \frac{\partial t}{\partial x'}dx' + \frac{\partial t}{\partial y'}dy' \equiv x^0{}_{,\mu'}\, dx^{\mu'}, \\
&= x^0{}_{,3'}\, dx^{3'} + x^0{}_{,0'}\, dx^{0'}. \qquad (1.5)
\end{aligned}
$$

This is a linear transform because the vector components appear to the power unity. The coefficients, the partial derivatives, multiplying the vector components are relations between the coordinates of the different frames and are independent of the vectors. Thus such a transform must work, not only for the displacement vector, but for all vectors. If a set of four quantities, V^μ, do not transform as above, then they are not components of a vector.

For the Lorentz transform, the partial derivatives will soon be obtained. If the transform was a rotation, say from rectangular to cylindrical coordinates, the partial derivatives, as will be seen, have different values.

Note the summation over an index definition, in Eqs. (1.4) and (1.5), requires the same index repeated as both super-script (contravariant) and sub-script (covariant). In the case of the partial derivative a covariant index results from the contravariant index in the denominator. Coordinates, by tradition, are always written with contravariant indexes as opposed to tensors, vectors are tensors of rank one, that have both types of indexes. Covariant vectors are also called dual vectors or one-forms.

Let both observers O and O′ concentrate on a light ray,

$$
\begin{aligned}
1 &= \left(\frac{dx^1}{dx^0}\right)^2 + \left(\frac{dx^2}{dx^0}\right)^2 + \left(\frac{dx^3}{dx^0}\right)^2, \\
1 &= \left(\frac{dx^{1'}}{dx^{0'}}\right)^2 + \left(\frac{dx^{2'}}{dx^{0'}}\right)^2 + \left(\frac{dx^{3'}}{dx^{0'}}\right)^2, \\
(d\tau)^2 &\equiv (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = 0, \qquad (1.6) \\
(d\tau')^2 &\equiv (dx^{0'})^2 - (dx^{1'})^2 - (dx^{2'})^2 - (dx^{3'})^2 = 0. \qquad (1.7)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(d\tau)^2 &= (d\tau')^2, \qquad (1.8) \\
(dx^0)^2 - (dx^3)^2 &= (dx^{0'})^2 - (dx^{3'})^2.
\end{aligned}
$$

Applying Eqs. (1.4) and (1.5) to the above equation,

$$
\begin{aligned}
(dx^0)^2 &= (x^0{}_{,0'}\, dx^{0'})^2 + (x^0{}_{,3'}\, dx^{3'})^2 + 2x^0{}_{,0'}\, x^0{}_{,3'}\, dx^{0'} dx^{3'}, \\
(dx^3)^2 &= (x^3{}_{,0'}\, dx^{0'})^2 + (x^3{}_{,3'}\, dx^{3'})^2 + 2x^3{}_{,0'}\, x^3{}_{,3'}\, dx^{0'} dx^{3'}, \\
1 &= (x^0{}_{,0'})^2 - (x^3{}_{,0'})^2 \equiv \cosh^2\alpha - \sinh^2\alpha, \\
1 &= -(x^0{}_{,3'})^2 + (x^3{}_{,3'})^2 \equiv \cosh^2\beta - \sinh^2\beta, \\
0 &= \cosh\alpha \sinh\beta - \sinh\alpha \cosh\beta = \sinh(\beta - \alpha), \quad \alpha = \beta.
\end{aligned}
$$

The above results force Eq. (1.8) to hold even for other than light travel. So, $d\tau$, is an invariant, a tensor of rank zero, numerically the same in all frames. The invariant, $d\tau$, is the proper time, that read on a clock at rest with respect to the observer. For light travel, assumed in vacuum unless otherwise noted, $d\tau = 0$. The interval, dS, is related to, $d\tau$, via, $(dS)^2 = -(d\tau)^2$. Although, τ, is a member of the Greek alphabet and could be used to indicate a coordinate or tensor element, it is reserved for the proper time.

In order to calculate, $\sinh\alpha$ and $\cosh\alpha$, O concentrates on the position of O′. In time period, dt, O′ changes position so that, $dz = V dt$. However, O′ says, "I am at rest while my clock has advanced by, dt'". Using Eqs. (1.4) and (1.5),

$$\begin{aligned} dt &= dt' \cosh\alpha,\ \ V dt = dt' \sinh\alpha, \\ V &= \tanh\alpha, \\ \gamma &\equiv \cosh\alpha = (1 - V^2)^{-1/2} = x^0{}_{,0'} = x^3{}_{,3'}\,, \qquad (1.9) \\ \gamma V &= \sinh\alpha = x^0{}_{,3'} = x^3{}_{,0'}\,. \qquad (1.10) \end{aligned}$$

The reverse transform from unprimed to primed coordinates just requires,

$$\begin{aligned} & t \leftrightarrow t',\ \ z \leftrightarrow z',\ \ V \to -V, \\ & \gamma = x^{0'}{}_{,0} = x^{3'}{}_{,3}\,,\ \ -\gamma V = x^{0'}{}_{,3} = x^{3'}{}_{,0}\,. \qquad (1.11) \end{aligned}$$

Note that for low speeds, $(1 - V^2)^{-1/2} \to 1$, and the Galilean transform is recovered.

1.3 Physics Consequences

First, consider simultaneity. O and O′ are coincident, O′ says two events occur at the same time so, $dt' = 0$, but at positions from the origin, $\pm dz'$. O says that the time differences from zero of the two events are,

$$dt_\pm = \pm\gamma V dz', \quad dt_+ - dt_- \neq 0.$$

So in general observers in relative motion do not agree on simultaneity. Only if events are spatially coincident will observers in relative motion agree on simultaneity. Thus to compare times, clocks at the same spatial position have to be compared. Also note, the present of O′, when all clocks are synchronized, is connected with all points in the past, present and future of O. In spacetime all space and time points are available. Time isn't a special quantity, it's just a one dimensional projection of spacetime.

Next consider causality. In the unprimed frame, observer O fires a bullet at, $t = 0$, from the origin that hits a target at time, dt, and position, $dz = v_b dt$, where, v_b, is the speed of the bullet. O′ also says the bullet was fired from the origin at, $t' = 0$, as that's where the clocks were synchronized and says the target was hit at,

$$dt' = \gamma(dt - V[v_b dt]) = dt\gamma(1 - V v_b).$$

According to O′, if, $dt' < 0$, the target is hit before the bullet is fired. That would violate causality and can happen only if, $v_b > 1$. So, $c = 1$, is the limiting speed.

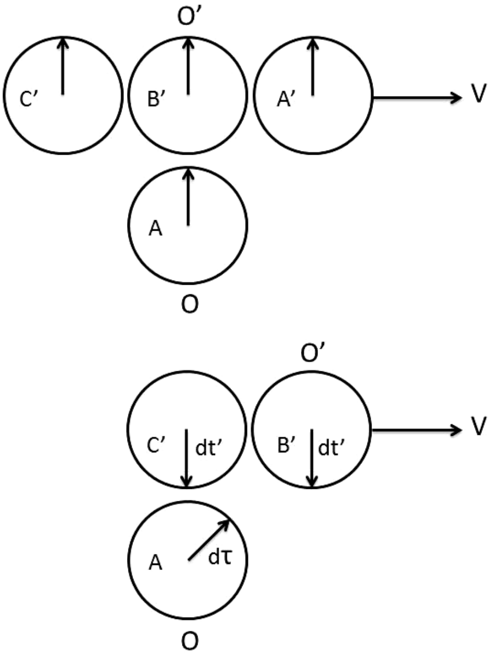


Fig. 1.2 Comparison of clocks, the view of O.

Now consider the comparison of clocks. Observer O and clock A are spatially coincident. Other clocks, B, C, ... are at rest with respect to O and read the same time as A. Observer O′ and clock B′ are spatially coincident. Other clocks, A′, C′, ... are at rest with respect to O′ and read the same time as B′. All the clocks are synchronized when A and B′ are spatially coincident. Since O says A is at rest, when it has ticked off a time period, $dt = d\tau$, as in Fig. 1.2, what will the clocks of O′ read? O concludes that clock C′ is now spatially coincident with clock A. Thus the times on these two clocks can be compared. O says A has ticked off, $d\tau$, while remaining at rest. From the Lorentz transform, the time ticked off on C′ is,

$$dt' = \gamma d\tau > d\tau.$$

This result is called time dilation. Time and all processes related to it run slower for O who considers herself at rest as compared with those of O′ who O sees moving. So A ticks slower. Also the heart rate and other biological rates of O are slower.

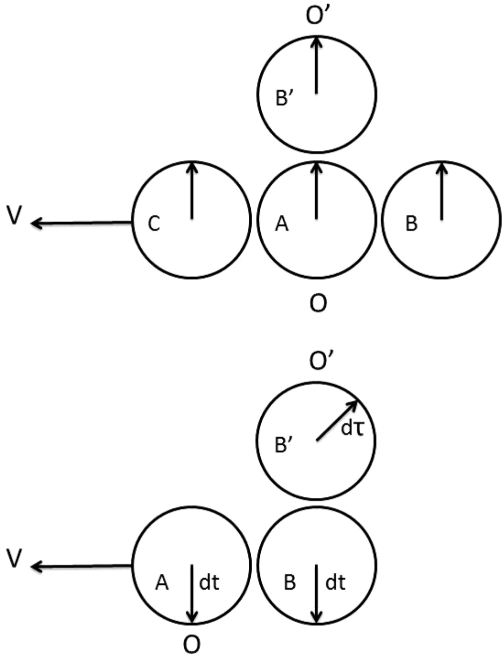


Fig. 1.3 Comparison of clocks, the view of O′.

Alternately, O′ considers B′ at rest, as in Fig. 1.3. When it has ticked off time, $dt' = d\tau$, what will the clocks at rest with respect to O read? According to O′, clock B has moved into spatial coincidence with clock B′. Thus the times on these two clocks can be compared. From the Lorentz transform,

$$dt = \gamma d\tau > d\tau.$$

Now time and all processes related to it run slower for O′ who considers himself at rest as compared with those of O who O′ sees moving. So B′ ticks slower and the biological rates of O′ are slower.

Strange as it seems, both observers are correct about these particular measurements. There is no third observer who could decide the case in favor of O or O′.

Time dilation has been experimentally verified in the laboratory with particles called muons. Muons at rest have a mean lifetime of about, 660 m. Thus if you begin with a large number, N_0, of muons at rest, the number, N, left after time, t m, is, $N/N_0 = \exp(-t/660)$. Then the half life for muons,

$t_{1/2} = 660 \ln 2\,\text{m}$, is the time when half of the starting muons remain. This result must be interpreted as the probability of muon survival. After a time, $t = nt_{1/2}$, experimenters expect, $N/N_0 = (1/2)^n$.

In the laboratory muons, can be created with speed, $V \approx 1$, and almost all are observed to travel far longer distances than, 660 m. This means that the clock attached to the muon (B′) has not yet ticked off even one mean lifetime, while the clocks (A, B, C) ...), attached to the laboratory, that the muon sees moving, have ticked off a much longer time period.

One feature of SR that hasn't been directly confirmed is length contraction. Suppose a rod of length, dz', is at rest in O′. That length could be measured by O′ by measuring the coordinate of each end at arbitrary times. However, O sees the rod moving and so must measure the coordinates of the two ends at the same time, $dt = 0$, to get the moving length, dz. Thus,

$$\begin{aligned} dz' &= \gamma dz, \\ L(rest) &= \gamma L(moving) > L(moving). \end{aligned} \tag{1.12}$$

By considering the muons, confirmation of this result can be inferred. In the laboratory the speedy muons easily travel much more than 660 m, the distance between points i and j in Fig. 1.4. However, according to the muons, these points are moving and the distance between them is length contracted to a much smaller value. Thus the clock attached to the muon has not ticked off a mean lifetime in the time these points travel past the muon.

These considerations lead to what is called the "twin paradox". Identical twins are separated at birth. Each one, O or O′, says her sister travels away in a rapidly moving space ship and then returns to the position of birth at a later time. Due to time dilation you might think that each twin is correct in saying, "I am younger", because when time comparison was considered above, if O′ considered clock B′, the time on B′ was less than on A, while if O considered clock A, the time on A was less than on B′.

However, if one was moving rapidly in the positive, z direction, acceleration would be experienced in order to turn around. An accelerating observer is not an inertial observer and cannot always use the Lorentz transform of SR. The trips are asymmetric. The point of view of the twin who hasn't experienced acceleration and who can use SR, is correct. That twin, as shown in an example in chapter two, is older.

As will be seen, the accelerated twin cannot distinguish between a mechcanical force or oppositely directed gravitational force. In stronger gravity

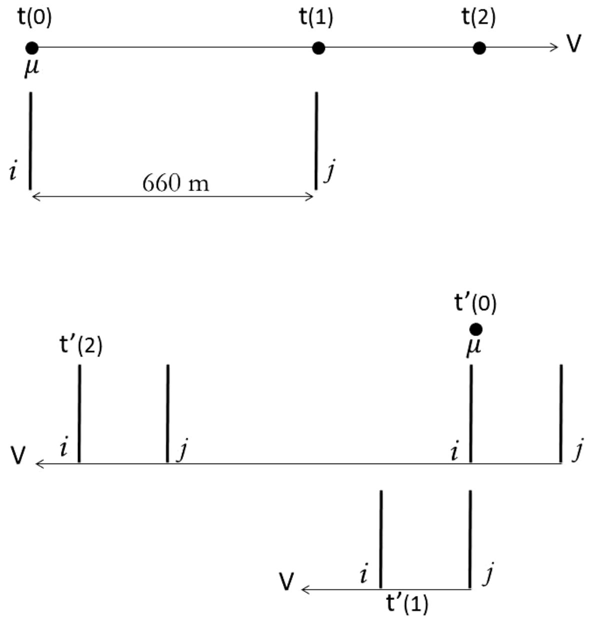


Fig. 1.4 Travel distance of speedy muons: top, bottom are the lab, muon views.

time runs slower. So of course clocks tick at a slower rate, hearts beat at a slower rate, and so on.

1.4 Spacetime Diagrams

Many GR texts stress the concept of spacetime diagrams. Analytic calculations are favored by this author. However, for completeness, the former are briefly discussed in this section. We draw on a flat sheet and must be concerned with four coordinates, thus one dimensional motion in the, z direction is considered.

Draw a set of axes for frame O as shown on the top of Fig. 1.5. The upward vertical axis represents increasing time, t, while the rightward horizontal axis represents increasing, z. At some time, t_1, particle A is at, z_1. If particle A has rest mass, it can remain at rest with respect to O. Its world line is just a vertical line upward from, (z_1, t_1), and on the clock attached to A the coordinate time duration, dt, is just the proper time duration, $d\tau$. Photons, B and C, starting from the same position

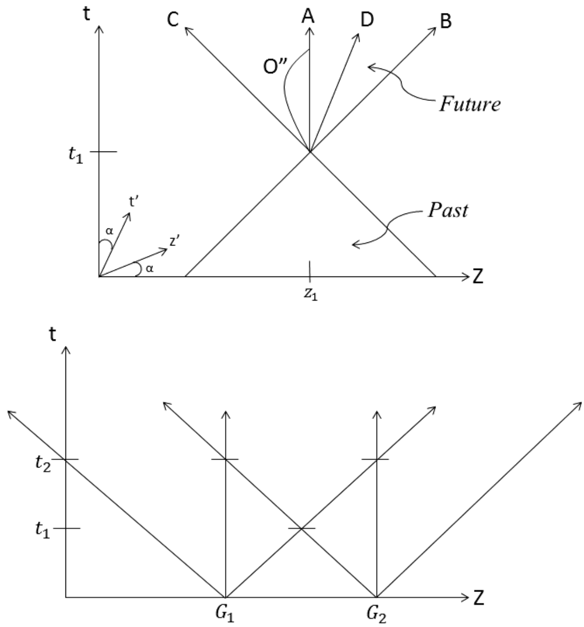


Fig. 1.5 Light cone physics, top for various travelers, bottom for separated galaxies.

and time must move with unit speed, so that, $|\frac{dz}{dt}| = 1$. Events outside of the triangle defined by world lines B and C are inaccessible starting from (z_1, t_1). This triangle is called the future triangle. If motion in two spatial directions is considered the triangle becomes a cone. A massive particle, D, that moves with constant speed, $V < 1$, relative to O has a world line with, $|\frac{dz}{dt}| < 1$. Its proper time duration, measured relative to, t_1, is, $d\tau = dt/\gamma$. Here, dt, is the time on a clock at rest with respect to A that, according to D, is coincident with D's position. Let twins O and O$''$ start at this spacetime point. O notes that her twin, moving with accelerated motion, always stays within the future triangle, at first moving away from O, but later returning. The gist of the twin "paradox" is that it is only O$''$ that experiences acceleration and she would have to use GR to compare ages. If not created at, (z_1, t_1), then A, B, C, D and O$''$ can get there only from the past triangle, obtained by extending B and C backwards in time.

As seen from O, the time and space axes, (t', z') of frame O$'$, in which D is at rest, can be drawn with hyperbolic angle, $\alpha = \tanh^{-1} V$, relative to the axes of O. This is because the Lorentz transform requires,

$$t = t' \cosh\alpha + z' \sinh\alpha = \gamma[t' + Vz'],$$
$$z = z' \cosh\alpha + t' \sinh\alpha = \gamma[z' + Vt'].$$

The light cones of two observers, in separated galaxies at, $z_{1,2} = G_{1,2}$, are shown on the bottom of Fig. 1.5. Initially their light is non-interacting. Only after time, t_1, will some light signals from the two observers be able to interact. Only after time, t_2, will light signals originating from, $G_{1,2}$, be able to form one side of a future triangle at, $G_{2,1}$. Actually, the situation is much more complicated because the universe is expanding. Space is being created so that the distance between any pair of typical particles is increasing with time. Moreover the expansion is accelerating so that there may be an observer, G_2, that observer, G_1, says is separating faster than the speed of light. That's a phenomenon, discussed in a later chapter, that is not built into relativity.

Problems

1. Convert from MKS units to natural units, $G = c = 1$: (a) luminosity flux, $= 10^{10}\,\mathrm{J\,s^{-1}\,m^{-2}}$; (b) density of water, $= 10^3\,\mathrm{kg\,m^{-3}}$. Convert from natural units to MKS units: (a) rest energy density of a proton, $0.3 \times 10^{-9}\,\mathrm{m^{-2}}$.
2. Consider the rotations that transform the unit vectors, $\hat{e}_{x,y,z} \leftrightarrow \hat{e}_{\rho,\phi,z} \leftrightarrow \hat{e}_{\theta,\phi,r}$, into one another. Show that the spatial components of an arbitrary vector transform in the same way as the unit vectors.
3. O and O$'$ have parallel axes. According to O, O$'$ is moving with velocity,

$$\vec{V} = (V^1, V^2, V^3) = |\vec{V}|(\sin a \cos b, \sin a \sin b, \cos a).$$

What is the Lorentz transform, $x^{\mu}{}_{,\nu'}$, between these reference frames? What is the relationship between, $x^{\mu}{}_{,\nu'}$ and $x^{\nu'}{}_{,\mu}$? Prove that the proper time, Eq. (1.8), is an invariant. If O looks at a rod that formally had rest length, L, that now is aligned with the velocity vector and moving with velocity, $\vec{V}$, what will O consider the length to be?

Do this problem as follows: find the rotation that takes O$'$ to O$''$ such that, $\hat{e}^{3''}$, is aligned with, $\vec{V}$. Then apply the Lorentz tranform. This takes you to system $\bar{O}$, where $\bar{O}$ is obtained from O by the same rotation that took O$'$ to O$''$. Now apply the inverse rotation to get from $\bar{O}$ to O.

4. Consider three inertial observers O, O$'$, and O$''$. These observers could be considered the origins of three frames of reference with parallel rectangular axes. The clocks at rest relative to themselves are synchronized when they overlap. When the clock attached to O$''$ has ticked off a time, $dx^{0''} = dt''$, it is desired to calculate the time, $dx^0 = dt$ and position information, $dx^{i=2,3} = (dy, dz)$ of O$''$ according to O for the following cases:

   Case I, O$''$ moves with speed, V^2, in the, y direction with respect to O$'$ and O$'$ moves with speed, V^3, in the, z direction with respect to O.

   Case II, O$''$ moves with speed, V^3, in the z direction with respect to O$'$ and O$'$ moves with speed, V^2, in the y direction with respect to O.

   (a) Calculate, $\Delta f \equiv f_I - f_{II}$, where, $f = dt,\ dz,\ dy,\ (dt)^2 - (dy)^2 - (dz)^2$ and $(dy)^2 + (dz)^2$.

   (b) Show that these results agree with those of problem three.

5. Cosmic rays are composed mainly of high energy protons. Neglect gravity and assume the velocities of the protons are along a line passing through the center of earth. Let protons (1,2) move towards the earth's center with speeds, $(V, \bar{V})$, relative to earth's center. What is the speed of proton 2 relative to proton 1 if the velocities relative to earth's center are in the same or opposite direction? Determine these speeds if, $V = 0.99$, $\bar{V} = 0.98$.
6. According to O, O$'$ is moving with velocity, $\vec{V} = 0.9\hat{e}_3$. In O$'$ there are mirrors at, $x^{3'} = \pm 1\,\text{m}$. At, $t = t' = 0$, light rays are emitted in O$'$ heading for each of the mirrors. According to O, what are the positions and times that each of the mirrors reflect the rays and what is the time and position of O$'$ when the rays return to O$'$? Can O$'$ place the mirror at, $x^{3'} = -1\,\text{m}$, somewhere else at negative, $x^{3'}$, so that O says both mirrors reflect the light at the same time? Then will both rays return to, $x^{3'} = 0$, at the same time, according to O?
7. Consider a runner holding a 10 m pole at its midpoint. The runner is at rest with respect to a barn that is, 5 m, long. The runner now gets to a speed, V, to run through the barn whose (rear, front) door is (closed, open). The speed is such that the pole appears to be, 5 m, long according to a barn observer at the barn's midpoint. The barn observer says at the instant the runner is at the midpoint of the barn the pole just fits within the barn. Thus the rear door can be opened and the front door shut at this instant and the pole passes through. The runner says the barn is

only, 2.5 m, long. According to the runner how does the pole avoid being hit by the doors? In this question, assume the doors open and close instantaneously.

8. High energy muons are a component of cosmic rays. Suppose these muons are moving towards the earth's center with speed, $v = 0.999$. According to earth clocks how long does it take for a trip of, 1300 m? This is a trip down a moderately high mountain. How long does the trip take according to clocks attached to the muons? What fraction of the starting muons complete the trip? According to these muons, what is the distance traveled? Measurements at the top and bottom of such a mountain gave the first confirmation of time dilation.

Chapter 2

Vectors and Tensors in Spacetime

2.1 Metric Tensor

In chapter one the contravariant displacement vector, dr^μ, and the invariant proper time element, $d\tau$, were discussed. It was noted that, $(d\tau)^2 \neq \sum_{\mu=0}^{3} dr^\mu dr^\mu$, but rather, from Eqs. (1.4) and (1.5) and the discussion following, summation occurs when the same index appears as both covariant and contravariant. Thus,

$$\begin{aligned}(d\tau)^2 &= (dt)^2 - [(dx)^2 + (dy)^2 + (dz)^2], &(2.1)\\ &\equiv -dr_\mu dr^\mu \equiv -g_{\mu\nu} dr^\nu dr^\mu, &(2.2)\\ &= -g_{\nu\mu} dr^\mu dr^\nu = -g_{\nu\mu} dr^\nu dr^\mu. &(2.3)\end{aligned}$$

A vector necessarily has covariant and contravariant components. The quantity, $g_{\mu\nu}$, with two covariant indexes is a tensor of rank two called the covariant metric tensor. When summing over an index that appears as both contravariant and covariant, the sum is over 0-3 for a Greek index and over 1-3 for a Roman index. For example, $\delta^\mu{}_\mu = 4$, but, $\delta^i{}_i = 3$. Eqs. (2.2) and (2.3) yield, $g_{\mu\nu} = g_{\nu\mu}$, and so it is a symmetric tensor. In an inertial frame using rectangular coordinates, the metric tensor is particularly simple and given a special symbol, $g_{\mu\nu} = \eta_{\mu\nu}$. From Eqs. (2.1) and (2.2),

$$1 = \eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00}, \quad \eta_{\mu\nu} = 0, \ \mu \neq \nu. \qquad (2.4)$$

From Eqs. (2.2) and (2.3) it is seen that the covariant metric tensor lowers indices or converts a contravariant index to a covariant one. As the determinant of, $g_{\mu\nu} \neq 0$, it must have an inverse, the contravariant metric tensor. The latter, $g^{\mu\nu} = g^{\nu\mu}$, converts a covariant to a contravariant index.

Thus,

$$
\begin{aligned}
dr^\mu dr_\mu &= g_{\mu\nu} dr^\mu dr^\nu = g^{\mu\nu} dr_\mu dr_\nu, \\
dr^\nu &= g^{\mu\nu} dr_\mu = g^{\mu\nu} g_{\mu\beta} dr^\beta, \\
g^{\mu\nu} g_{\mu\beta} &= \delta^\nu{}_\beta = \delta_\beta{}^\nu. \qquad (2.5)
\end{aligned}
$$

In GR the metric tensor elements may be complicated functions of position and time. Once they are obtained the problem is effectively solved. However, it is easy enough to show, as is done in the next section, that this tensor acquires complexity, even in an inertial frame, just by going to cylindrical coordinates.

2.2 Vector Transforms

So far we have seen that there are quantities that don't depend on an index that are invariants, like $d\tau$. In view of the discussion following Eq. (1.5), quantities that depend on one index and transform, as in Eq. (1.4), between different reference frames or coordinate systems are contravariant vectors or tensors of rank one. The transforms of tensors of higher rank are discussed in section 2.3. If there are two reference frames or sets of coordinates, O and O$'$, with coordinates, x^μ and $x^{\mu'}$, then components of contravariant vectors transform like,

$$
\begin{aligned}
V^\mu &= x^\mu{}_{,\nu'} V^{\nu'}, \text{ or,} \qquad (2.6) \\
g^{\mu\alpha} V_\alpha &= x^\mu{}_{,\nu'} g^{\beta'\nu'} V_{\beta'}, \text{ and,} \\
V_\sigma &= \delta_\sigma{}^\alpha V_\alpha = g_{\sigma\mu} g^{\mu\alpha} V_\alpha, \\
&= g_{\sigma\mu} g^{\beta'\nu'} x^\mu{}_{,\nu'} V_{\beta'} = x^{\beta'}{}_{,\sigma} V_{\beta'}, \qquad (2.7)
\end{aligned}
$$

Eq. (2.7) is the rule for transforming the components of covariant vectors.

This can be illustrated for O$'$ being cylindrical and O being rectangular coordinates. The relations between coordinates are,

$$
\begin{aligned}
x &= \rho \cos\phi, \; y = \rho \sin\phi, \\
\rho &= (x^2 + y^2)^{1/2}, \; \phi = \tan^{-1}(y/x), \\
dx &= x_{,\rho}\, d\rho + x_{,\phi}\, d\phi, \\
&= \cos\phi d\rho - \rho \sin\phi d\phi, \\
dy &= y_{,\rho}\, d\rho + y_{,\phi}\, d\phi, \\
&= \sin\phi d\rho + \rho\cos\phi d\phi, \text{ thus,} \\
(dx)^2 + (dy)^2 &= (d\rho)^2 + \rho^2 (d\phi)^2.
\end{aligned}
$$

The position vector, r^μ, has components, (t, x, y, z), while, $r^{\mu'}$, has components, $(t, \rho, 0, z)$. Let's see that this works out. It is obvious that the, t and z, components are the same, thus, with $\mu' = (1.2)$,

$$\begin{aligned}
r^1 &= r^\rho = x^1{}_{,\mu}\, r^\mu = \rho_{,x}\, x + \rho_{,y}\, y = [(x)^2 + (y)^2]^{1/2} = \rho, \\
r^2 &= r^\phi = x^2{}_{,\mu}\, r^\mu = \phi_{,x}\, x + \phi_{,y}\, y = \rho^{-2}(-yx + xy) = 0, \\
r_1 &= r_\rho = x^\mu{}_{,1}\, r_\mu = x_{,\rho}\, x + y_{,\rho}\, y = \rho^{-1}[(x)^2 + (y)^2] = \rho, \\
r_2 &= r_\phi = x^\mu{}_{,2}\, r_\mu = x_{,\phi}\, x + y_{,\phi}\, y = -yx + xy = 0.
\end{aligned}$$

This calculation is repeated for the displacement vector, dr^μ, with components, (dt, dx, dy, dz), in rectangular coordinates. In cylindrical coordinates the, ρ, ϕ, components are,

$$\begin{aligned}
dr^1 &= dr^\rho = \rho_{,x}\, dx + \rho_{,y}\, dy = \rho^{-1}(xdx + ydy) = d\rho, \\
dr_1 &= dr_\rho = x_{,\rho}\, dx + y_{,\rho}\, dy = \rho^{-1}(xdx + ydy) = d\rho = dr^1, \\
dr^2 &= dr^\phi = \phi_{,x}\, dx + \phi_{,y}\, dy = \rho^{-2}(-ydx + xdy) = d\phi, \\
dr_2 &= dr_\phi = x_{,\phi}\, dx + y_{,\phi}\, dy = (-ydx + xdy) = \rho^2 d\phi = \rho^2 dr^2.
\end{aligned}$$

Since, $(d\tau)^2$, is an invariant,

$$\begin{aligned}
g_{ij} dr^i dr^j &= g_{i'j'} dr^{i'} dr^{j'},\ i, j, i', j' = 1, 2, \\
(dx)^2 + (dy)^2 &= (d\rho)^2 + \rho^2 (d\phi)^2, \\
&= g_{\rho\rho}(d\rho)^2 + g_{\phi\phi}(d\phi)^2 + 2g_{\rho\phi} d\rho d\phi, \\
g_{11} = g_{\rho\rho} = 1,\ g_{22} &= g_{\phi\phi} = \rho^2,\ g_{12} = 0,\ g^{i'i'} = 1/g_{i'i'}.
\end{aligned}$$

One also notes the following required relations hold:

$$\begin{aligned}
(r_{\mu'}, dr_{\mu'}) &= g_{\mu'\nu'}(r^{\nu'}, dr^{\nu'}), \\
(r^{\mu'}, dr^{\mu'}) &= g^{\mu'\nu'}(r_{\nu'}, dr_{\nu'}), \\
g^{\mu'\nu'} dr_{\mu'} dr_{\nu'} &= g_{\mu'\nu'} dr^{\mu'} dr^{\nu'}.
\end{aligned}$$

The invariant volume is given by,

$$d^4V = (-det(g_{\mu\nu}))^{1/2} dx^0 dx^1 dx^2 dx^3. \tag{2.8}$$

as the value in rectangular coordinates, of its spatial part is, $dxdydz$, and in cylindrical coordinates is, $\rho d\rho d\phi dz$. The case of spherical coordinates is left as an exercise. Then for one dimensional distance, say between radial coordinates (a, b), the proper distance is, L_p, where,

$$L_p = \int_a^b dr (g_{rr})^{1/2}. \tag{2.9}$$

It depends on the metric. Generally, it would be larger than $|b-a|$ in a gravitational field because there space is curved. The speed of light is still unity because the proper time, that on a clock at rest with respect to an observer, runs slower in a stronger field.

The spatial parts of vectors are usually written in terms of components, V^i, and orthogonal basis vectors, $\vec{e}_i$. In 2D for rectangular coordinates, the latter can be taken as constant unit vectors, $\hat{e}_x$ and $\hat{e}_y$, while for cylindrical coordinates they are, $\hat{e}_\rho$ and $\vec{e}_\phi = \rho\hat{e}_\phi$. The latter are not constant, but vary with, ϕ. The correct distance squared between points in the, ρ, ϕ, plane is obtained from, $|d\rho\hat{e}_\rho + d\phi\rho\hat{e}_\phi|^2$. However, it is not necessary to use basis vectors when we work with the metric tensor.

In chapter one, the Lorentz transform between frames O and O′, in relative motion, in rectangular coordinate systems, with parallel axes, was studied. The transform for, dr^μ and dr_μ, was just Eqs. (2.6) and (2.7). So quantities depending on one index and transforming via the Lorentz transform are vectors in spacetime. This property will be used below to construct additional vectors.

2.3 Tensor Transforms

If a quantity is a tensor of rank higher than one, it will transform such that for each contravariant index there will be a partial derivative factor as in Eq. (2.6) and for each covariant index a partial derivative factor as in Eq. (2.7). The product of the factors multiplies an element of the tensor. To see how this comes about, consider the quantity, $V^\mu W_\nu$, where V, W are contravariant, covariant vectors. The product depends on two indexes and each factor in the product transforms like a vector. The total transform is,

$$\begin{aligned} T^\mu{}_\nu \equiv V^\mu W_\nu &= x^\mu{}_{,\alpha'}\, V^{\alpha'} x^{\beta'}{}_{,\nu}\, W_{\beta'}, \\ &= x^\mu{}_{,\alpha'}\, x^{\beta'}{}_{,\nu}\, V^{\alpha'} W_{\beta'} = x^\mu{}_{,\alpha'}\, x^{\beta'}{}_{,\nu}\, T^{\alpha'}{}_{\beta'}. \end{aligned} \tag{2.10}$$

Since it transforms in the prescribed manner, $T^\mu{}_\nu$ is a mixed tensor of rank two. Thus, for the metric tensor,

$$g_{\mu\nu} = x^{\alpha'}{}_{,\mu}\, x^{\beta'}{}_{,\nu}\, g_{\alpha'\beta'}, \tag{2.11}$$

$$g^{\mu\nu} = x^\mu{}_{,\alpha'}\, x^\nu{}_{,\beta'}\, g^{\alpha'\beta'}. \tag{2.12}$$

Checking that this holds for the Lorentz transform derived in chapter one, where $g^{\mu\nu} = \eta^{\mu\nu}$, yields using Eqs. (1.9)–(1.11),

$$\begin{aligned}\eta^{00} &= x^0{}_{,\alpha'}\, x^0{}_{,\beta'}\, \eta^{\alpha'\beta'}, \\ &= (x^0{}_{,0'})^2\eta^{0'0'} + (x^0{}_{,3'})^2\eta^{3'3'} = \gamma^2(-1+V^2) = -1.\end{aligned}$$

The other elements can similarly be shown to work out. Try it. It is obvious how a tensor of any rank is now transformed. If of rank integer, n, there will be a product of, n, appropriate partial derivatives multiplying a tensor element.

2.4 Forming Other Vectors

Since, dr^μ, is a vector and, $d\tau$, is an invariant, the quantity, $U^\mu = \frac{dr^\mu}{d\tau}$, is another vector with units of velocity (not so for photons because there, $d\tau = 0$). Suppose SR observer O′ says an object has 3-velocity components, $\frac{dr^{i'}}{dt'} = \bar{v}^{i'} = \bar{v}_{i'}$. An observer moving with the object says that $d\tau$ has elapsed, but from chapter one, $d\tau = dt'/\bar{\gamma}'$, where, dt', is the time elapsed according to O′, $\bar{\gamma}' = (1-(\bar{v}')^2)^{-1/2}$, and $(\bar{v}')^2 = \bar{v}^{i'}\bar{v}_{i'}$. Then,

$$U^{i'} = \bar{\gamma}'\frac{dr^{i'}}{dt'} = \bar{\gamma}'\bar{v}^{i'} = U_{i'}, \tag{2.13}$$

$$U^{0'} = \bar{\gamma}'\frac{dr^{0'}}{dt'} = \bar{\gamma}' = -U_{0'}, \tag{2.14}$$

$$-\eta_{\mu'\nu'}U^{\mu'}U^{\nu'} = -\bar{\gamma}'^2(-1+(\bar{v}')^2) = 1, \quad \text{invariant.} \tag{2.15}$$

There are similar equations for the U^μ in O, just remove the primes.

The 3-velocity transforms come from applying the Lorentz transform to these vectors, for a relative speed between frames of V in the z direction,

$$U^0 = \bar{\gamma} = \gamma(U^{0'} + VU^{3'}) = \gamma\bar{\gamma}'(1+V\bar{v}^{3'}), \tag{2.16}$$

$$U^3 = \bar{\gamma}\bar{v}^3 = \gamma(U^{3'} + VU^{0'}) = \gamma\bar{\gamma}'(\bar{v}^{3'} + V),$$

$$\bar{v}^3 = \frac{\bar{v}^{3'} + V}{1+V\bar{v}^{3'}} = \frac{\gamma(dx^{3'} + Vdx^{0'})}{\gamma(dx^{0'} + Vdx^{3'})} = \frac{dx^3}{dt}, \tag{2.17}$$

$$\bar{\gamma}\bar{v}^{1,2} = \bar{\gamma}'\bar{v}^{1',2'},$$

$$\bar{v}^{1,2} = \frac{\bar{v}^{1',2'}}{\gamma(1+V\bar{v}^{3'})} = \frac{dx^{1',2'}}{\gamma(dx^{0'} + Vdx^{3'})} = \frac{dx^{1,2}}{dt}. \tag{2.18}$$

The quantity $d\tau$ has vanished from Eqs. (2.17) and (2.18). Thus, they also hold for photons. For example, if, $\bar{v}^{1'} = 1$, $\bar{v}^{2'} = \bar{v}^{3'} = 0$, $\bar{v}_1\bar{v}^1 = \gamma^{-2}$, $\bar{v}^2 = 0$, $\bar{v}_3\bar{v}^3 = V^2$ and $(\bar{v})^2 = \bar{v}_i\bar{v}^i = 1$. All observers see the same speed for light. For a slowly moving object like the earth in the gravitational field of the sun, $\gamma \approx 1$, so, $U^0 \approx 1$, $U^i \approx 0$.

2.5 Twin Problem Revisited

The twin problem with a given, simple acceleration can now be considered. Twin O is at rest, while twin O′ is strapped into a chair in a rocket, as shown in Fig. 2.1. The rocket accelerates in the z direction so that O′ experiences an acceleration, $g = 9.807/(9 \times 10^{16})$ m^{-1}, and is quite comfortable. On the clock of O′, that measures the proper time of O′ the acceleration lasts for a time, T. The acceleration reverses direction for a time, $2T$, and then reverses direction again for a time, T, to bring the rocket back to rest and O′ to the position of O. What is the time on the clock of O? Only O can interpret the problem with SR.

The 3-acceleration magnitude of O′ is, g, according to O′. The 3-acceleration of O′ according to O must be calculated. The latter as usual is, $\frac{V_R}{dt}$, where, V_R, is the speed of the rocket with respect to O. The

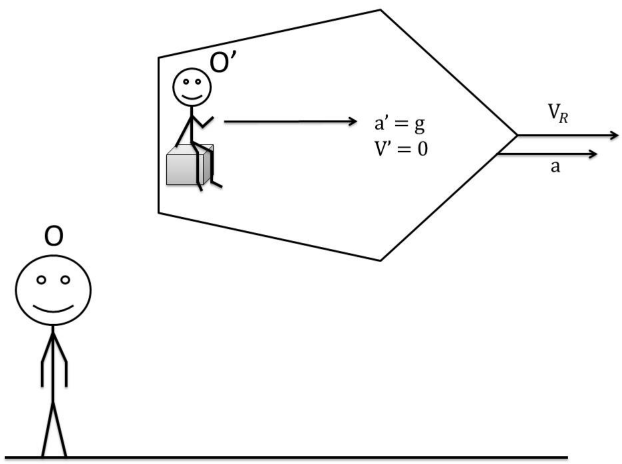


Fig. 2.1 The twin problem: O′, at rest in the rocket, experiences acceleration g on going from and returning to twin O.

3-acceleration enables the speed of the rocket with respect to O to be calculated and that allows comparison of clocks. The 3-acceleration is obtained from an acceleration vector, $A^\mu \equiv \frac{dU^\mu}{d\tau}$, similar to the way the 3-velocity was obtained from, U^μ. For motion in SR frame O in the 3 direction with speed, V and $a = \frac{dV}{dt}$,

$$A^\mu \equiv \frac{dU^\mu}{d\tau} = \gamma \frac{dU^\mu}{dt}, \tag{2.19}$$

$$\begin{aligned} A^0 &= \gamma \frac{d\gamma}{dt}, \\ &= \gamma \frac{-1}{2(1-V^2)^{-3/2}} \left(-2V \frac{dV}{dt} \right), \\ &= \gamma^4 a V, \end{aligned} \tag{2.20}$$

$$\begin{aligned} A^3 &= \gamma \frac{d}{dt} \frac{dx^3}{d\tau} = \gamma \frac{d}{dt} \left(\gamma \frac{dx^3}{dt} \right) = \gamma \frac{d(\gamma V)}{dt}, \\ &= \gamma \left(\gamma a + \frac{d\gamma}{dt} V \right) = \gamma^2 a + \gamma^4 V^2 a, \\ &= \gamma^2 a (1 + V^2 \gamma^2), \\ &= \gamma^4 a. \end{aligned} \tag{2.21}$$

The rocket moves with speed, V_R, that changes with time with respect to the earth frame. In the rocket frame O′ is at rest, but has acceleration, $a' = \pm g$, depending on the direction. O can use the Lorentz transform to transform vectors from the O′ frame to the O frame,

$$\begin{aligned} A^0 &= \gamma_R [A^{0'} + V_R A^{3'}], \qquad \gamma_R = (1 - V_R^2)^{-1/2}, \\ V_R a \gamma_R^4 &= \gamma_R [v' a' \gamma'^4 + V_R \gamma'^4 a'] = \gamma_R V_R (\pm g), \\ a &= \frac{dV_R}{dt} = \pm g \gamma_R^{-3}, \text{ so,} \\ \pm g dt &= dV_R \gamma_R^3 = dV_R (1 - V_R^2)^{-3/2}. \end{aligned} \tag{2.22}$$

Using the relationship between coordinate time and proper time yields,

$$\begin{aligned} (d\tau)^2 &= (dt)^2 - (dz)^2 = (dt)^2 (1 - V_R^2), \\ d\tau &= dt (1 - V_R^2)^{1/2}, \\ &= (\pm 1/g) dV_R / (1 - V_R^2). \end{aligned}$$

On the first leg, $\tau = T - 0 = T$, take the plus sign and integrate from, $V_R = 0$ to $V_R(F)$. On the second leg, $\tau = 2T - T = T$, take the minus sign, but integrate from, $V_R = V_R(F)$ to 0. Thus the two legs yield the same result. For the third leg, $\tau = 3T - 2T = T$, take the minus sign, but integrate from, $V_R = 0$ to $-V_R(F)$, and for the last leg, $\tau = 4T - 3T = T$, take the plus sign, but integrate from, $V_R = -V_R(F)$ to 0. Thus all legs yield the same result. So the calculation below will be done for the first leg:

$$T = (1/g) \int_0^{V_R(F)} dV_R/(1 - V_R^2) = (1/2g) \ln \frac{1 + V_R(F)}{1 - V_R(F)},$$

$$\exp(2gT) = \frac{1 + V_R(F)}{1 - V_R(F)},$$

$$V_R(F) = \frac{\exp(2gT) - 1}{\exp(2gT) + 1}.$$

If, $T = 1$ y $= 0.316 \times 10^8$ s $= 0.948 \times 10^{16}$ m, then, $gT = 1.033$, and, $V_R(F) = (\exp(2.066) - 1)/(\exp(2.066) + 1) = 0.775$. Then from Eq. (2.22),

$$gt = \int_0^{V_R(F)} dV_R(1 - V_R^2)^{-3/2} = V_R(F)(1 - V_R(F)^2)^{-1/2} = 1.226,$$

$$t = (0.918 \times 10^{16})(1.226)\ m = 0.375 \times 10^8\ s = 1.187\ y.$$

So for the entire trip, O′ ages, 4 y while O ages, 4.748 y. Even more striking, if $T = 10$ y, $t = 1.483 \times 10^4$ y!

2.6 Momentum and Energy

Another important vector comes from multiplying the velocity vector by an invariant quantity with units of mass, $mU^\mu \equiv P^\mu$, where, P, is the momentum. In the system where, $c = 1$, it has units of kg. In the natural units, where, $c = G = 1$, it has units of m. By going to the rest frame of the object we see, m, is the rest mass. The spatial components of the momentum vector are,

$$P^i = (1 - |\vec{V}|^2)^{-1/2} m V^i = \gamma m V^i, \quad |\vec{V}|^2 = V^i V_i, \tag{2.23}$$

These reduce to their usual non-relativistic values when, $|\vec{V}| \ll 1$. The time component is,

$$P^0 = \gamma m \equiv E, \tag{2.24}$$
$$= m(1 + |\vec{V}|^2/2 + ...).$$

At low speeds the last equation is just the rest energy plus the non-relativistic kinetic energy or the total mechanical energy. So for any speed the time component of the momentum vector is the mechanical energy, E. A non-invariant relativistic mass, $M \equiv E/c^2 = E$, is a useful construct for the example in section 2.8. A little manipulation yields other important relations:

$$m^2 = -g_{\mu\nu}P^\mu P^\nu = (\gamma m)^2 - P_i P^i = E^2 - |\vec{P}|^2,$$
$$E = [|\vec{P}|^2 + m^2]^{1/2} \equiv KE + m, \tag{2.25}$$

where, KE, is the relativistic kinetic energy. Photons have, $m = 0$, so for them,

$$E = KE = |\vec{P}| = M = h\nu, \tag{2.26}$$

where, ν, is the frequency.

2.7 Doppler Shift

The Doppler shift for photons in SR can be obtained from the Lorentz transform of photon momentum. The situation is illustrated in Fig. 2.2. O′ is moving relative to O with speed, V, in the, z, direction. A laser, at rest in O′, sends out photons with, $P^{0'} = E' = h\nu'$, $P^{3'} = -h\nu'$. If the photons don't travel in the $-z$ direction, they won't get to O. From the Lorentz transform,

$$E = \gamma(E' + VP^{3'}) = \frac{E' + VP^{3'}}{(1 - V^2)^{1/2}},$$

$$h\nu = h\nu' \frac{1 - V}{(1 - V^2)^{1/2}} = h\nu' \left(\frac{1 - V}{1 + V}\right)^{1/2},$$

$$\frac{\nu}{\nu'} = \left(\frac{1 - V}{1 + V}\right)^{1/2}, \ \frac{\lambda}{\lambda'} = \left(\frac{1 + V}{1 - V}\right)^{1/2}. \tag{2.27}$$

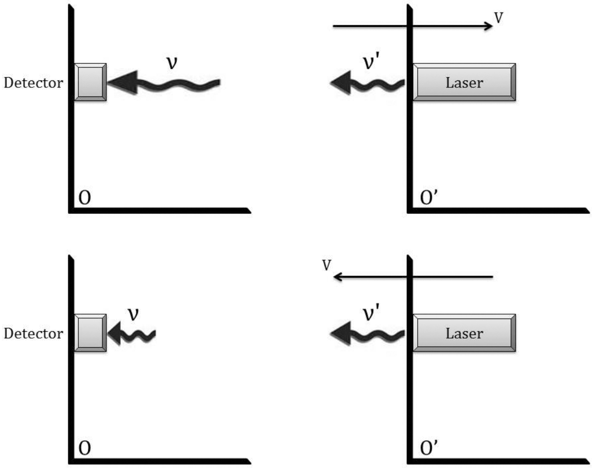


Fig. 2.2 The laser at rest in O′ emits light of frequency, ν'. The light received by O has frequency, ν.

For a source receding from O the received frequency is shorter, the wavelength, λ, is longer or a red shift. If you change the sign of V so that the source is advancing towards O there is a blue shift. Examination of the line spectra from very far away galaxies and knowledge of the Doppler shift leads to the conclusion that they are all separating from us, producing an expanding universe. One might think that the expansion is taking place in an already existing space, but we now know the expansion is due to more space being created between any pair of objects.

Another way to get this result is, perhaps more satisfying, as it uses invariants. In O′ the light source is at rest and the light has, $P^{0'} = h\nu'$, $P^{3'} = -h\nu'$. Let O be the observer that O′ says is moving in the, $-z$ direction with speed, V. The velocity of O according to O′ is,

$$U^{0'}(O) = \gamma(U^0(O) - VU^3(O)) = \gamma U^0(O) = \gamma,$$

$$U^{3'}(O) = \gamma(U^3(O) - VU^0(O)) = -\gamma V U^0(O) = -\gamma V,$$

$$g_{\mu'\nu'}U^{\mu'}(O)P^{\nu'}(light) = \gamma(-h\nu')(1-V) = -h\nu'\left(\frac{1-V}{1+V}\right)^{1/2} = -E,$$

$$E(obs) = -U_{\mu'}(obs)P^{\mu'} = -U^{\mu'}(obs)P_{\mu'}. \tag{2.28}$$

Here, $E(obs)$, is the energy measured by the observer and, $U_{\mu'}(obs)$, is the velocity of the observer determined a second party, here O$'$. $P^{\mu'}$, is the momentum of the object whose energy is desired by the observer, in this case, light, also determined by the second party. In the above case, if the observer and second party are at rest in O$'$, then the velocity of the observer is, $U_{\mu'} = (-1, 0, 0, 0)$ and, $E(obs) = -(-1)P^{0'} = h\nu'$. For massive particles you can easily show that Eq. (2.28) still holds.

2.8 Gravity Affects Time

Using the relativistic mass, it is easy to show that gravity affects time. Consider a photon with, $KE = M = h\nu$, emitted at the surface of the sun, of mass, M_s, and radius, R_s. When it gets far from the sun, say to the earth, of mass, M_e, and radius, R_e, a distance, D, from the sun, the frequency changes so that total energy is conserved. At earth, $KE' = M' = h\nu'$. The setup is illustrated in Fig. 2.3. The frequency of the sunlight at earth can be calculated from energy conservation with Newtonian gravitational potential energy included. This calculation is not relativistically rigorous as it mixes non-relativistic and relativistic concepts. However, it gives you the correct result when gravity is a very weak effect,

$$\begin{aligned} KE + PE &= KE' + PE', \\ M - MM_s/R_s &= M' - M'M_s/D - M'M_e/R_e, \\ h\nu(1 - M_s/R_s) &= h\nu'(1 - [M_s/R_s][(R_s/D) + (M_e/M_s)(R_s/R_e)]), \end{aligned}$$

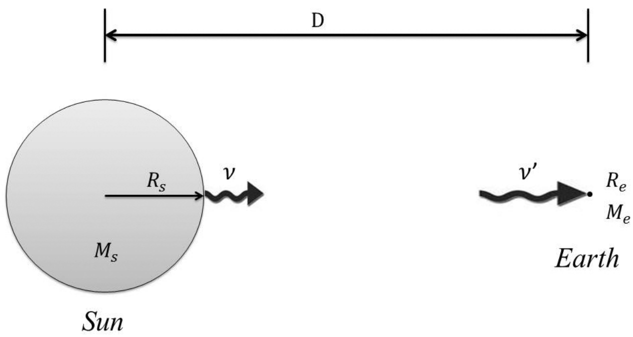


Fig. 2.3 Light of frequency ν is emitted at the surface of the sun. The light received at earth, where gravity is weaker, has frequency ν'.

$$R_s/D = 4.64 \times 10^{-3}, \quad (M_e/M_s)(R_s/R_e) = 0.33 \times 10^{-3},$$
$$1 - \nu'/\nu \approx M_s/R_s = 1.476 \times 10^3/6.96 \times 10^8 = 2.1 \times 10^{-6}.$$

So at the sun, light from a physical process like the line spectrum of helium is emitted, and travels to earth. At earth observers compare this light with the frequency of light from the line spectrum of helium on earth. The light from the sun has a lower frequency than the comparison light of frequency, ν, and so a larger wavelength. This is the gravitational red shift. Frequency is inversely related to time, in this case, the time between successive wave crests. So the proper time earth clock indicates the time between crests of the light from the sun is longer than the similar time from the earth light. This means it is longer than the time on a proper time clock at the sun. That's because, if a source is to be considered helium, it has to be helium everywhere. Otherwise the first postulate of relativity is violated. The time between crests measured on a clock at the same position as the source doesn't depend on the position, regardless of gravity. Source process times and clock times are affected equally. Thus, proper time clocks in stronger gravity are ticking at a slower rate than those in weaker gravity.

An observer far from a black hole would say it takes a very long time for a foolish astronaut to get to the hole's horizon, whereas the astronaut doesn't think anything is amiss time-wise.

You also see that GR is able to explain the twin paradox. When one twin experiences acceleration, she can't distinguish it from a gravitational field. When she is in a field, the other twin's clock runs at a faster rate. Thus the twin experiencing gravity returns younger.

2.9 The Pound Rebka Experiment

These predictions have been confirmed for light from stars, but was first confirmed in the earth's gravity by, [Pound and Rebka (1959)]. The experiment required use of the Mössbauer effect and a wide variety of other physical processes.

As an introduction to this important experiment, the emission and absorption of photons are reviewed. Consider a free atom at rest, such as one in a gas, with a nucleus in an excited state of energy, E_H. It will emit a photon and undergo a transition to a lower energy state, E_L. In order to conserve energy and momentum the atom must recoil,

$$E_H - E_L \equiv \Delta E = h\nu + |\vec{P}|^2/(2M_A) = h\nu + (h\nu)^2/(2M_A),$$

where ν is the photon frequency and M_A is the atomic mass. If a similar atom's nucleus was excited by absorbing a photon,

$$\Delta E = h\nu' - |\vec{P}'|^2/(2M_A) = h\nu' - (h\nu')^2/(2M_A).$$

Thus it is seen that, $h\nu < \Delta E < h\nu'$, so that the emitted photon cannot be reabsorbed by another free nucleus, if, as is the case here, the intrinsic line-width of the transition is small enough. The intrinsic line-width is due to the time-energy uncertainty relation. If the lifetime of the excited state is long, so that there is ample time to make an energy measurement, then the error in the energy will be very small.

In 1958 R. Mössbauer discovered that when such atoms were part of a crystal lattice, the crystal as a whole, with zero phonon excitation, recoiled in a large fraction of the transitions. For this discovery Mössbauer was awarded the Nobel prize. In this case, $M_A \to \infty$, and $h\nu = h\nu'$. Thus the emitted photon could be reabsorbed. A very good source with which to see this effect is, ^{57}Fe. It results in an excited nuclear state, 14.4 KeV, above the ground state, from electron capture in, ^{57}Co, (lifetime 272 d). You can get the cobalt at a cyclotron using the reaction ^{56}Fe(d,n)^{57}Co. The cobalt diffuses into the lattice when the iron target slab is heated. As natural iron has only, 2.17% of ^{57}Fe, the slab can be enriched in isotope 57.

In a simple experiment an enriched, thin source slab is placed close to a thin absorber slab. Directly behind the latter is a photon detector, perhaps a NaI crystal mounted on a phototube. If the absorbing slab is ordinary iron almost no reduction in counting rate is observed, but if it is enriched in, ^{57}Fe, an obvious reduction is seen because much more emitted photons are absorbed. This holds whether the experiment is conducted at the bottom or top of a high tower, $H = 22.55$ m, in the case of the Pound Rebka experiment. That is, if source and detector are at the same place in a gravitational field the measured frequency is independent of where they are.

In the Pound Rebka experiment, the tower was located on the Harvard campus. The source was carefully prepared with a very narrow intrinsic line width. Let it be at the bottom, $r_2 = R_e$, while a similarly carefully prepared absorber is at the top, $r_1 = R_e + H$, where gravity is weaker. After solid angle effects are accounted for, a reduction in count rate is not observed nor is one expected. The gravitational red shift reduces the photons' frequency so thay cannot be reabsorbed. This, at least confirms that the frequency at the top is not that at the bottom, but hasn't confirmed whether or not the frequency has decreased and by how much.

In order to measure this and see if it agrees with the GR prediction, the source was moved so that in addition to the above effect, there is also a Doppler shift. If the source moves towards the absorber with speed, V, then the frequency at the absorber is, using the Doppler shift and the gravitational red shift,

$$\frac{\nu_{2,1}}{\nu_{2,2}} = \left(\frac{1+V}{1-V}\right)^{1/2} \left[1 - M_e\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right],$$
$$\approx (1+V)\left[1 - M_e\left(\frac{1}{R_e} - \frac{1}{R_e + H}\right)\right].$$

The approximation is due to, $V \ll 1$. Since, $H \ll R_e$ the above can be approximated as,

$$\frac{\nu_{2,1}}{\nu_{2,2}} \approx (1+V)\left[1 - \frac{M_e}{R_e}\left(1 - \frac{1}{1 + H/R_e}\right)\right],$$
$$\approx (1+V)\left[1 - \frac{M_e}{R_e}\left(1 - (1 - H/R_e)\right)\right],$$
$$= (1+V)\left[1 - \frac{M_e H}{R_e^2}\right].$$

For maximum absorption one desires the above ratio to be unity, so that, $1 + V = \left(1 - M_e H/R_e^2\right)^{-1} \approx 1 + M_e H/R_e^2$, or, $V = 2.46 \times 10^{-15}$. In MKS units, this is, 0.74×10^{-6} m/s. Moving the source so slowly would be extremely difficult. However, Pound and Rebka moved it in a sinusoidal manner so that, $V = V_0 \cos\omega t$. The speed, V_0, could be much larger than the above value, but by observing the way the detector intensity varied with, $\cos\omega t$, the red shift value could be deduced. The original experiment found agreement with GR to the, 10%, level. It has since been improved to less than, 1%.

2.10 Global Positioning System

The global positioning system is in wide use and GR provides an important correction. Before worrying about the role played by GR, let's look at a simple minded system where only SR needs to be worried about. Imagine our observer does not know where he is, though he is actually at, dz_0. Overhead at height, $H \ll R_e$, fly many planes with speed, V, in the, z, direction. The planes are at rest with respect to each other. At high frequency, ν, they broadcast the time on their own clocks and their position, (dt', dz'). These are connected to times and positions in the observer's frame by the Lorentz transform,

$$\gamma = (1 - V^2)^{-1/2} \approx 1 + V^2/2,$$
$$dt = \gamma(dt' + V dz'), \qquad dz = \gamma(dz' + V dt').$$

The time of flight, dT, of the electromagnetic wave to the observer is, $dT = (h^2 + [dz - dz_0]^2)^{1/2}$, and the time of arrival at the observer is, $dt_0 = dT + dt$. The observer has a system that interprets the signals so that two signals from different planes that arrive simultaneously are deduced. Of course this means that the two signals were not emitted simultaneously by the planes. One emitted at, n/ν, and the other at, m/ν, where m, n, are integers. Also simultaneous arrival really means arrival within a narrow time window that depends on the accuracy of your device. If two planes have simultaneous arrivals,

$$dT_A + dt_A = dT_B + dt_B,$$
$$(h^2 + [dz_A - dz_0]^2)^{1/2} + dt_A = (h^2 + [dz_B - dz_0]^2)^{1/2} + dt_B.$$

The above quadratic equation allows calculation of, dz_0. The accuracy of the position depends on the accuracy of the time measurements. For extreme accuracy the correction due to the relativistic factor, γ, needs to be included.

Now the real GPS, illustrated in Figs. 2.4 and 2.5, is a system of twenty-four satellites. They orbit earth with a twelve hour period, T. Newton's theory will give us an idea of the accuracy and the corrections needed. Circular motion is considered,

$$mV^2/R = mM_e/R^2,$$
$$V = (M_e/R)^{1/2} = 2\pi R/T, \qquad R = [TM_e^{1/2}/(2\pi)]^{2/3},$$
$$T = (12)(3600)(3 \times 10^8) \text{ m},$$
$$M_e = (5.97 \times 10^{24})(0.742 \times 10^{-27}) \text{ m},$$
$$R = 0.265 \times 10^5 \text{ km} = 4.15 R_e,$$
$$V = 0.128 \times 10^{-4}.$$

Thus the time correction has two parts: first the factor, γ, from SR and second from GR because the clocks of the receiver and satellite experience

Fig. 2.4 Twenty-four GPS satellites orbit earth and broadcast their positions and times at high frequency.

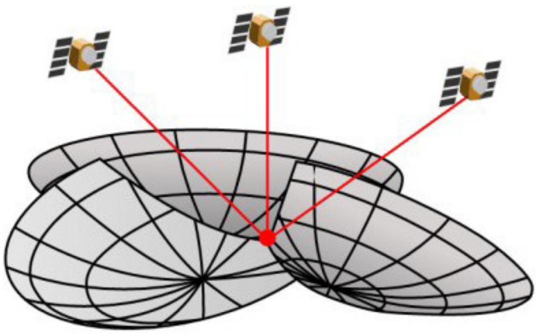


Fig. 2.5 Light speed signals from three satellites, that arrive at the same time, provide a unique earth position.

gravity of differing strength. The sizes of the corrections are,

$$\gamma = (1 - V^2)^{-1/2} \approx 1 + V^2/2 = 1 + 1.6 \times 10^{-10},$$

$$(1 - M_e/R_e)/dt_E = (1 - M_e/(4.15R_e))/dt_S,$$

$$\begin{aligned} dt_e/dt_S &= [1 - M_e/R_e]/[1 - M_e/(4.15R_e)], \\ &\approx [1 - M_e/R_e][(1 + M_e/(4.15R_e)], \\ &= 1 - 0.76(M_e/R_e) = 1 - 4 \times 10^{-10}. \end{aligned}$$

The magnitude of the GR correction is larger than that from SR.

Motion about the sun affects earth and satellite equally, but you might think that the formula for, γ, could only be used at the poles, where an earth observer is at rest with respect to the earth's center, while at other latitudes an earth observer is moving in a circle about an axis through the earth's poles and center. However, the earth is not a sphere and the polar radius is less than the equatorial radius. There is a latitude effect for the gravitational potential at the surface. You also must add to the brew the fact that the earth is spinning, but one cannot tell the difference between the centripetal force and an additional, oppositely directed gravitational force. The total gravitational potential at earth's surface turns out to be independent of latitude, so that clocks anywhere on the surface run at the same rates. For a more complete discussion, see [Drake (2006)].

Due to the large distance between the surface of earth and the satellite, exquisite timing accuracy is required to get a location accuracy of, 2 m, at earth's surface. This was the accuracy required by the military and is likely outdated. For example, in 1 ns, light travels 0.3 m. So approximately 6 ns accuracy is needed. Modern atomic clocks can easily keep time to this accuracy. However, if both SR and GR corrections were not included the errors would quickly compound and the system wouldn't work. This alone makes GR very practical. It may be that a very clever person would have noticed that, without the GR correction, the GPS could be made to work by an arbitrary scaling of time. So, I have heard it argued, that one does not have to know about GR to make the GPS work. However, without knowledge of GR, it would have taken a much longer time to get the GPS running correctly. Also, it is always better to understand a complex system rather than making use of non-understood fixes.

There are other corrections to be mindful of: the relativistic Doppler effect, the varying of the index of refraction along the path of the electromagnetic wave, and that the orbits are not circular.

2.11 Tensor Equations

In a fully relativistic theory, energy is just one component of the momentum vector. Momentum and energy conservation is just conservation of each component of that vector. Suppose it holds in one frame where, for example, a muon, as iillustrated in Fig. 2.6, decays into three lighter particles, $\mu \to \nu\bar{\nu}e$. In the μ rest frame, O, conservation of momentum is,

$$0 = P^{\beta}(I) - P^{\beta}(F) = P^{\beta}(\mu) - P^{\beta}(\nu) - P^{\beta}(\bar{\nu}) - P^{\beta}(e). \quad (2.29)$$

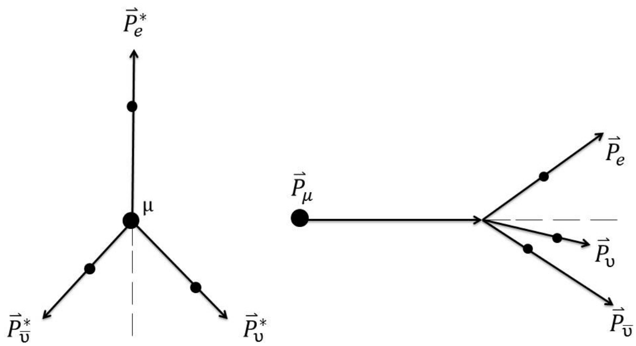


Fig. 2.6 Muon decay: On the left is the view in the muon's rest frame (O). On the right is the view in the laboratory, frame O′.

If the Lorentz transform is applied to every term in the above so that the quantities in a frame, O′, where the muon is moving, are obtained, it is obvious that momentum conservation holds in O′,

$$0 = x^{\alpha'}{}_{,\beta}\,(P^\beta(I) - P^\beta(F)) = P^{\alpha'}(I) - P^{\alpha'}(F).$$

Thus we strive to write physical law as a tensor equation as in Eq. (2.29). Then if the law holds for one observer, it holds for all.

Other vectors are easy to define. For example, one could define a force vector, $F^\mu \equiv \frac{dP^\mu}{d\tau}$. You can calculate the force components on each of two charged particles at rest. Then you Lorentz transform to a frame in which the particles are moving. In the latter frame, one notes, magnetic effects enter the scene. In this way, the laws of electromagnetism can be derived from Coulomb's law and SR. That material is useful for an electrodynamics course, but not pertinent for further study of GR.

Problems

1. In an inertial frame of SR and in rectangular coordinates the metric for the spatial coordinates is η_{ij}. Obtain the metric and determinant of the metric in spherical coordinates. Also obtain all, $(r^i,\ r_i,\ dr^i,\ dr_i)$ and the basis vectors in terms of unit vectors, where,

$$x = x^1 = r\sin\theta\cos\phi = x^{3'}\sin x^{1'}\cos x^{2'},$$
$$y = x^2 = r\sin\theta\sin\phi = x^{3'}\sin x^{1'}\sin x^{2'},$$
$$z = x^3 = r\cos\theta = x^{3'}\sin x^{1'},$$

$$r = x^{3'} = [(x)^2 + (y)^2 + (z)^2]^{1/2} = [\eta_{ij} x^i x^j]^{1/2},$$
$$\phi = x^{2'} = \tan^{-1} y/x = \tan^{-1} x^2/x^1,$$
$$\theta = x^{1'} = \cos^{-1}(z/[(x)^2 + (y)^2 + (z)^2]^{1/2}) = \cos^{-1}(x^3/[\eta_{ij} x^i x^j]^{1/2}).$$

2. If, x^μ, are spherical coordinates and, $x^{\mu'}$, are cylindrical coordinates, use knowledge of, $g_{\mu'\nu'}$, and the tensor transform rule to obtain, $g_{\mu\nu}$. Also explicitly show, $x^\mu{}_{,\nu'} = g_{\nu'\alpha'} g^{\mu\beta} x^{\alpha'}{}_{,\beta}$.
3. Show that Eq. (2.6) leads to Eq. (2.7) for the Lorentz transform discussed in chapter one. Show explicitly for this transform that, $x^\mu{}_{,\nu'} = \eta^{\mu\alpha}\eta_{\nu'\beta'} x^{\beta'}{}_{,\alpha}$.
4. Show from the transform equations that, $g_{\mu\nu} T^\nu{}_\beta$, where, $T^\nu{}_\beta$, is a mixed tensor of rank two, is a covariant tensor of rank two. Show, $U_\mu \frac{dU^\mu}{d\tau} = 0$, where, U^μ, is the velocity, if, $\frac{dg_{\alpha\beta}}{d\tau} = 0$.
5. Show from the transform equations that, $\delta^\mu{}_\nu$, is a mixed tensor of rank two. What are, $g^{\mu\nu} g_{\xi\chi}$, $g^{\mu\mu} g_{\nu\nu}$, $g^{\mu\nu} g_{\mu\nu}$, $g^{\mu\mu} g_{\mu\mu}$ and $g^{\mu\nu} V_\xi$, where, V_ξ, is a vector.
6. In the twin problem, discussed in the chapter, the rocket moved relative to frame O with spatial components of velocity and acceleration in the, z, direction only.
   (a) Suppose the rocket had nonzero spatial components of 3-velocity and 3-acceleration in all directions relative to O′ of, $\vec{v'}$, $\vec{a'}$, with components, $v^{i'} = \frac{dx^{i'}}{dt'}$, $a^{i'} = \frac{dv^{i'}}{dt'}$. If O′ moves with speed, V, in the, z direction relative to O, what are, a^j, in terms of, V, $v^{i'}$, and $a^{i'}$? You already know, v^j, in terms of, V and $v^{i'}$.
   (b) Show that Eq. 2.22 can be obtained from the invariant, $(\eta_{\mu\nu} A^\mu A^\nu)^{1/2}$, and thus if one inertial observer sees an object accelerating, all inertial observers see it accelerating.
7. A person on the surface of the earth lives, 80 y, as reckoned by his/her proper time clock. What would the lifetime be in a space station a distance of, $2.5R_e$, from the center of the earth on his/her proper time clock? What would the earth clock read at time of death in the space station? Repeat the above calculation for lifetime in a mine, a distance of, $0.99R_e$, from earth's center, assuming a spherically symmetric earth with uniform density. In the discussion of the GPS system, it was noted that the time dilation correction due to the different velocities was smaller than that due to gravitation by a factor of, 0.4, when the satellite was at, $4.15R_e$. So neglect this effect in this problem.

8. Use the data for the Pound-Rebka experiment. For fixed source and detector, what is the difference in energy, in eV, between photons emitted at the bottom and photons arriving at the top of the tower.
9. A laser emits light of frequency, ν, in the $-z$ direction. The laser is mounted to a land rocket that also moves in the $-z$ direction, with speed, V with respect to the earth. A second such rocket has a mirror mounted on it and reflects the light received from the laser. It moves with speed, V' relative to the earth. What frequency is measured at the laser rocket for the reflected light. Consider all possible cases, $V > V'$, $V < V'$ and V' moving in the $\pm z$ direction relative to earth. The visible spectrum lies between, $(0.43 - 0.75) \times 10^{13}$ Hz. Suppose, $V = 500$ mph $\ll c$, what is the magnitude of, V', such that mirror rocket travel in the, $\mp z$, directions yield reflected light at the laser with the lowest and highest visible frequencies.
10. The most efficient rocket one can conceive of would convert its rest mass to photons. The photons would part from the remaining mass in one direction and propel the remaining mass in the reverse direction. What fraction of the original mass remains when its speed is, 0.9, 0.925, 0.95, 0.975?
11. At present high energy proton accelerators new particles and their antiparticles can be produced by the reactions, $p + p \to X + \bar{X} + p + p$, $M_X \gg M_p$. There are two types of accelerators: one beam and a fixed target, liquid hydrogen, or colliding beams with equal and opposite momenta. In accelerating protons to the highest energies, it is the kinetic energy of the beam(s) that determines cost. Calculate the minimum kinetic energy to produce the final state for both types of accelerators and determine which is cheaper to construct? In the case of Higgs boson production, $M_X = 0.125$ TeV/c^2, all known conservation laws would be satisfied without also requiring the production of the anti Higgs. In this case, what is the minimum collider kinetic energy? Could the Higgs be produced at any current fixed target facility?
12. A particle of rest mass M decays into two particles of rest masses, M_1 and M_2. If, M, is traveling with momentum, $P\hat{e}_z$, in the laboratory, calculate the cosine of the opening angle, $\cos\theta_o$, between the two particles in the laboratory in terms of the cosine of the emission angle, $\cos\theta^*$, with respect to, $\hat{e}_z$, of, M_1, in the rest frame of, M. Graph this function for, $M = 1$ m, $M_{1,2} = 0.3,\ 0.5$ m and $P = 1, 1.5$ m. Note that

there is a maximum opening angle between the decay particles in the laboratory that decreases as, P, increases.

13. In the last chapter, cosmology is dealt with. There, the cosmic microwave background (CMB) radiation has a Planck distribution with a temperature, $T_0 = 2.73$ K. In the early universe the temperature was much higher and the energy of the photons, $E_\gamma = h\nu = k_B T$, much greater. Here, k_B, is Boltzman's constant and T is the absolute temperature. When the energy was high enough particle, anti-particle pairs could be produced via, $\gamma + \gamma \rightarrow X + \bar{X}$. What is the minimum photon energy required to produce electron, positron, ($X = e$), pairs? The scale factor of the universe compared to what it is now, is, $Q/Q_0 = T_0/T$. What was this ratio when, $e^\pm$, pairs could no longer be produced because the universe was expanding and cooling? The reaction considered here explains why an upper limit for gamma ray energy in cosmic rays is observed. Very energetic gamma rays travel long distances to reach earth. They are traveling through a sea of, 2.73 K, photons or higher as they started out long ago, and are quite likely to collide with one and produce electron positron pairs, before reaching us. Calculate the observable maximum gamma ray energy, in MeV. There is evidence for a diffuse xray background with energy, $\approx 2 \times 10^6 k_B T_0$, that would lower the above observable maximum energy considerably. Recalculate that energy for this diffuse xray background.
14. The Robertson-Walker metric describes the development of the universe. Its elements are, $g_{00} = -1$, $g_{33} = g_{rr} = [Q(t)]^2$, $g_{22} = g_{\phi\phi} = (Q(t) r \sin\theta)^2$, $g_{11} = g_{\theta\theta} = (Q(t)r)^2$ and $g_{\mu\nu} = 0, (\mu \neq \nu)$. Here, r, is the radial coordinate from an origin, say the center of the sun. Consider a galaxy at radial coordinate, R. The galaxy can be considered to be on the surface of a sphere of radial coordinate, R. What is the circumference and area of the sphere? What is the proper distance from the origin? As, $Q(t)$, is getting bigger with time, one sees that the proper distance between any origin and a radial coordinate is expanding. Space is being created between the two. Experiment indicates that there is a non-understood "dark energy" driving the expansion and that it will not be reversed by the gravity of the existing mass.

On the scale of the solar system, the universal expansion is much too small to be observed. The metric that describes a planet's orbit about the sun is the static Schwarzschild metric. Here, $g_{00} = -(g_{rr})^{-1} = -(1 - 2M_s/r)$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = \sin^2\theta g_{\theta\theta}$, where, M_s, is the sun's mass and, r, is the planet's radial coordinate relative to the sun's center. For earth, what is the value of, L_p/r, correct to 0.1 percent?

15. Show that even in a complicated metric like the Schwarzschild metric, see problem fourteen, an observer still measures unity for the speed of light.

Chapter 3

Covariant Differentiation, Equations of Motion

3.1 Differentiation of Invariants and Vectors

In the previous chapters, the importance of tensors in spacetime was stressed. Any time a new quantity is encountered, it will have to be checked to see if it is a tensor. If it isn't, its transformation properties are not obvious. Construction of new tensors has, so far, taken the form of products of known tensors, or total differentiation with respect to, τ. For example, $(d\tau)^2 = dr_\mu dr^\mu$, $g^{\mu\nu}g_{\xi\nu} = \delta^\mu{}_\xi = \delta_\xi{}^\mu$, or $U^\mu = \frac{dr^\mu}{d\tau}$. From studies of the calculus of 3-vectors, one recalls that partial differentiation with respect to the coordinates, produces new 3-vectors and scalars through the gradient and divergence operations. In spacetime such partial differentiation also leads to important new tensors.

Consider an invariant that is a function of position, $\Phi = \Phi(x^\mu) = \Phi(x^{\mu'})$, for example, $d\tau$. It has no index associated with it. Taking the partial derivative with respect to a coordinate yields,

$$\Phi_{,\nu} = x^{\xi'}{}_{,\nu}\, \Phi_{,\xi'}\,. \tag{3.1}$$

However, this is the rule for the transformation of a covariant vector and so another vector is added to our arsenal.

Throughout, a special rectangular coordinate system, $x^{\bar{\mu}}$, is used. It has the property that, $(d\tau)^2 = -g_{\bar{\mu}\bar{\nu}}dx^{\bar{\mu}}dx^{\bar{\nu}} = -\eta_{\bar{\mu}\bar{\nu}}dx^{\bar{\mu}}dx^{\bar{\nu}}$. If we are in an inertial frame of SR there is no restriction on the size of the frame. If gravity is included, it will be seen later in the chapter, that at any point in space, such a set of coordinates can be found. However, the frame that includes this point, may necessarily be arbitrarily small. So the frame is called a **locally inertial frame**. Curvilinear coordinates in this frame are labeled, $x^{\bar{\mu}'}$.

The gradient of a scalar field, Φ, is given by, $g^{\mu\nu}\Phi_{,\nu}$, because in an inertial frame the expected results for the spatial components are obtained,

$$\begin{aligned}\nabla^{\bar{\mu}}\Phi &\equiv g^{\bar{\mu}\bar{\nu}}\Phi_{,\bar{\nu}} = \eta^{\bar{\mu}\bar{\nu}}\Phi_{,\bar{\nu}}\,,\\ \vec{\nabla}\Phi &= \Phi_{,x}\,\hat{e}_x + \Phi_{,y}\,\hat{e}_y + \Phi_{,z}\,\hat{e}_z.\end{aligned} \tag{3.2}$$

If cylindrical coordinates were chosen, the components of the gradient in terms of the basis vectors have the expected form,

$$\begin{gathered}\nabla^{\rho}\Phi = g^{\rho\rho}\Phi_{,\rho} = \Phi_{,\rho}\,, \qquad \nabla^{\phi}\Phi = g^{\phi\phi}\Phi_{,\phi} = \Phi_{,\phi}/\rho^2,\\ \vec{\nabla}\Phi - \Phi_{,z}\,\hat{e}_z = \Phi_{,\rho}\,\vec{e}_\rho + (\Phi_{,\phi}/\rho^2)\vec{e}_\phi = \Phi_{,\rho}\,\hat{e}_\rho + (\Phi_{,\phi}/\rho)\hat{e}_\phi.\end{gathered} \tag{3.3}$$

The case of spherical coordinates is left as an exercise.

A similar partial differentiation of a vector can be performed. For example, for the position vector,

$$\begin{aligned}dr^{\bar{\mu}} &= dx^{\bar{\mu}}, \text{ thus } \partial r^{\bar{\mu}} = \partial x^{\bar{\mu}},\\ r^{\bar{\mu}}{}_{,\bar{\nu}} &= x^{\bar{\mu}}{}_{,\bar{\nu}} = \delta^{\bar{\mu}}{}_{\bar{\nu}}.\end{aligned}$$

So the partial derivative of the component of a vector with respect to a coordinate appears to be an element of a tensor of rank two, see chapter two, problem five. However, in cylindrical or spherical coordinates, none of the, $r^{\bar{\mu}'}$, depend on coordinate, $\phi = x^2$, so that, $r^2{}_{,2} = 0$. In general, terms in addition to the partial derivative of a vector with respect to a coordinate are needed to obtain an object that transforms as a tensor of rank two.

In order to see how to handle this, return to the calculus of 3-vectors. For the position vector in the plane perpendicular to, $\hat{e}_z$, in cylindrical coordinates, using the results of the rotation indicated in Fig. 1.1,

$$\begin{aligned}\vec{r} &= r^{\bar{i}'}\vec{e}_{\bar{i}'} = r^1\vec{e}_1 = r^\rho\vec{e}_\rho,\\ \vec{r}_{,1} &= \vec{r}_{,\rho} = \rho_{,\rho}\,\vec{e}_\rho + \rho\vec{e}_{\rho,\rho}\,,\\ &= \vec{e}_\rho = \hat{e}_\rho = \vec{e}_1,\\ \vec{r}_{,2} &= \vec{r}_{,\phi} = \rho_{,\phi}\,\vec{e}_\rho + \rho\vec{e}_{\rho,\phi}\,,\\ &= \rho(\cos\phi\hat{e}_x + \sin\phi\hat{e}_y)_{,\phi}\,,\\ &= \rho(-\sin\phi\hat{e}_x + \cos\phi\hat{e}_y),\\ &= \rho\hat{e}_\phi = \vec{e}_2, \text{ thus,}\\ \delta_{\bar{j}'}^{\bar{i}'} &= (\vec{r}_{,\bar{j}'})^{\bar{i}'}.\end{aligned} \tag{3.4}$$

the same as the rectangular coordinate result.

In tensor notation, the terms needed, in addition to the partial derivative, lead to the covariant derivative,

$$V^{\mu}{}_{;\nu} = V^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\beta\nu} V^{\beta}. \tag{3.5}$$

Note the use of the semi colon to indicate covariant differentiation. The left side is the $\binom{\mu}{\nu}$ element of the mixed tensor that is the covariant derivative of vector, V. The quantity, $\Gamma^{\mu}_{\beta\nu}$, is called the Christoffel (C) symbol and it can be calculated solely from the, $g_{\mu\nu}$. The first term is all that is needed in, $x^{\bar{\mu}}$, coordinates, because, $\Gamma^{\bar{\mu}}_{\bar{\beta}\bar{\nu}} = 0$. This derivative obeys the same algebra as regards products as does the partial derivative, because it does so in, $x^{\bar{\mu}}$, coordinates, where it is just the partial derivative. The other terms take into account the way the unit vectors, or in tensor notation the way the, $g_{\mu\nu}$, depend on the coordinates.

None of the terms on the right side of Eq. (3.5) transforms like a tensor of rank two, but the sum does. The reason the sum does is that in the inertial frame, in any coordinates,

$$\delta^{\bar{\mu}}_{\bar{\nu}} = r^{\bar{\mu}}{}_{;\bar{\nu}} \quad \delta^{\bar{\mu}'}_{\bar{\nu}'} = r^{\bar{\mu}'}{}_{;\bar{\nu}'} \tag{3.6}$$

so it is a general result concerning the transformation properties. Changing the vector changes the resulting tensor. For an arbitrary vector, a straight forward, but lengthy proof, using the tensor transformation rule, is left as problem two.

In order to illustrate that the covariant derivative yields expected results, consider cylindrical coordinates of an inertial frame. As shall be seen, the only nonzero C symbols with indexes, $(\bar{i}', \bar{j}', \bar{k}') = (1, 2)$, are, $\Gamma^{2}_{12} = \Gamma^{\phi}_{\rho\phi} = 1/\rho$ and $\Gamma^{1}_{22} = \Gamma^{\rho}_{\phi\phi} = -\rho$. The covariant derivative of the position vector in the plane perpendicular to. $\hat{e}_z$, is,

$$\begin{aligned}
r^{\bar{i}'}{}_{;\bar{j}'} &= r^{\bar{i}'}{}_{,\bar{j}'} + \Gamma^{\bar{i}'}_{\bar{j}'\bar{\alpha}'} r^{\bar{\alpha}'}, \text{ with components,} \\
r^{\rho}{}_{;\rho} &= r^{\rho}{}_{,\rho} + \Gamma^{\rho}_{\rho\bar{\alpha}} r^{\bar{\alpha}} = 1, \\
r^{\rho}{}_{;\phi} &= r^{\rho}{}_{,\phi} + \Gamma^{\rho}_{\phi\bar{\alpha}} r^{\bar{\alpha}} = \Gamma^{\rho}_{\phi\phi} r^{\phi} = 0, \\
r^{\phi}{}_{;\phi} &= r^{\phi}{}_{,\phi} + \Gamma^{\phi}_{\phi\bar{\alpha}} r^{\bar{\alpha}} = \Gamma^{\phi}_{\phi\rho} r^{\rho} = 1, \\
r^{\phi}{}_{;\rho} &= r^{\phi}{}_{,\rho} + \Gamma^{\phi}_{\rho\bar{\alpha}} r^{\bar{\alpha}} = \Gamma^{\phi}_{\rho\phi} r^{\phi} = 0, \\
\delta^{\bar{i}'}_{\bar{j}'} &= r^{\bar{i}'}{}_{;\bar{j}'}\,.
\end{aligned} \tag{3.7}$$

Note that Eq. (3.7) agrees with Eq. (3.6) and the 3-vector results of Eq. (3.4).

The scalar divergence of a vector is defined, Div $V = V^{\mu};_{\mu}$, because in an inertial frame the expected results are obtained for the spatial part, shown first in rectangular and then in cylindrical coordinates for the (ρ, ϕ) terms,

$$V^{\bar{\mu}},_{\bar{\mu}} = V^t,_t + V^x,_x + V^y,_y + V^z,_z \,.$$

$$V^{\rho};_{\rho} = V^{\rho},_{\rho} + \Gamma^{\rho}_{\alpha\rho} V^{\alpha} = V^{\rho},_{\rho} \,,$$

$$V^{\phi};_{\phi} = V^{\phi},_{\phi} + \Gamma^{\phi}_{\alpha\phi} V^{\alpha} = V^{\phi},_{\phi} + \Gamma^{\phi}_{\rho\phi} V^{\rho} = V^{\phi},_{\phi} + V^{\rho}/\rho,$$

$$V^{\rho};_{\rho} + V^{\phi};_{\phi} = V^{\rho},_{\rho} + V^{\rho}/\rho + V^{\phi},_{\phi} = (\rho V^{\rho}),_{\rho}/\rho + V^{\phi},_{\phi} \,. \tag{3.8}$$

If the vector is the gradient of a scalar, Φ, then in an inertial frame, the expected wave equation for, Φ, is obtained. As above, first in rectangular and then in cylindrical coordinates. Using Eqs. (3.2) and (3.3),

$$(\nabla^{\bar{\mu}}\Phi);_{\bar{\mu}} = -\Phi,_t,_t + \Phi,_x,_x + \Phi,_y,_y + \Phi,_z,_z \,.$$

$$\begin{aligned}
(\nabla^{\rho}\Phi);_{\rho} + (\nabla^{\phi}\Phi);_{\phi} &= (\nabla^{\rho}\Phi),_{\rho} + \Gamma^{\rho}_{\alpha\rho}\nabla^{\alpha}\Phi + (\nabla^{\phi}\Phi),_{\phi} + \Gamma^{\phi}_{\alpha\phi}\nabla^{\alpha}\Phi,\\
&= (\nabla^{\rho}\Phi),_{\rho} + \Gamma^{\phi}_{\rho\phi}\nabla^{\rho}\Phi + (\nabla^{\phi}\Phi),_{\phi} \,,\\
&= \Phi,_{\rho},_{\rho} + \Phi,_{\rho}/\rho + \Phi,_{\phi},_{\phi}/\rho^2,\\
&= (\rho\Phi,_{\rho}),_{\rho}/\rho + \Phi,_{\phi},_{\phi}/\rho^2.
\end{aligned} \tag{3.9}$$

We can figure out the covariant derivative of a covariant vector by considering an invariant, such as the scalar product of two vectors. Using Eq. (3.1) and renaming summed over indexes, yields,

$$\begin{aligned}
V_{\mu};_{\nu} U^{\mu} + V_{\mu} U^{\mu};_{\nu} &= (V_{\mu}U^{\mu});_{\nu} = (V_{\mu}U^{\mu}),_{\nu} \,,\\
V_{\mu};_{\nu} U^{\mu} + V_{\mu}(U^{\mu},_{\nu} + \Gamma^{\mu}_{\alpha\nu}U^{\alpha}) &= V_{\mu},_{\nu} U^{\mu} + V_{\mu}U^{\mu},_{\nu} \,,\\
V_{\mu};_{\nu} U^{\mu} &= V_{\mu},_{\nu} U^{\mu} - V_{\mu}\Gamma^{\mu}_{\alpha\nu}U^{\alpha},\\
&= V_{\mu},_{\nu} U^{\mu} - V_{\alpha}\Gamma^{\alpha}_{\mu\nu}U^{\mu},\\
V_{\mu};_{\nu} &= V_{\mu},_{\nu} - \Gamma^{\alpha}_{\mu\nu}V_{\alpha}.
\end{aligned} \tag{3.10}$$

The C symbol can be shown to be symmetric in its covariant indexes, because the order of partial differentiation doesn't matter,

$$\Phi;_{\bar{\mu}};_{\bar{\nu}} = \Phi,_{\bar{\mu}},_{\bar{\nu}} = \Phi,_{\bar{\nu}},_{\bar{\mu}} = \Phi;_{\bar{\nu}};_{\bar{\mu}} \,. \tag{3.11}$$

This is a tensor equation and so it holds in general, thus,

$$\begin{aligned}
\Phi,_{\mu};_{\nu} &= \Phi;_{\mu};_{\nu} = \Phi;_{\nu};_{\mu} = \Phi,_{\nu};_{\mu} \,,\\
\Phi,_{\mu},_{\nu} - \Gamma^{\alpha}_{\mu\nu}\Phi,_{\alpha} &= \Phi,_{\nu},_{\mu} - \Gamma^{\alpha}_{\nu\mu}\Phi,_{\alpha} \,,\\
\Gamma^{\alpha}_{\nu\mu} &= \Gamma^{\alpha}_{\mu\nu}.
\end{aligned} \tag{3.12}$$

Due to this property, $\Gamma^{\alpha}_{\mu\nu}$, has a term like, $x^{\bar{\beta}}{}_{,\mu}{}_{,\nu}$. This term satisfies Eq. (3.12) and, if in the inertial frame, $x^{\bar{\beta}}{}_{,\mu}{}_{,\nu} \to x^{\bar{\beta}}{}_{,\bar{\mu}}{}_{,\bar{\nu}} = 0$, as required. To take care of the contravariant index and get rid of the index $\bar{\beta}$, one could try,

$$\Gamma^{\alpha}_{\mu\nu} = x^{\alpha}{}_{,\bar{\beta}}\, x^{\bar{\beta}}{}_{,\mu}{}_{,\nu}\,. \tag{3.13}$$

It turns out that Eq. (3.13) holds. However, a more useful form, see below, in terms of the metric tensor and its derivatives, will be used. The proof of Eq. (3.13) starts with the more useful form and is left as problem ten.

3.2 Differentiation of Tensors

Given two vectors V and W, the product, $V^{\mu}W_{\nu}$, transforms like a mixed tensor of rank two, and its covariant derivative yields,

$$\begin{aligned}
T^{\mu}{}_{\nu};_{\alpha} &= (V^{\mu}W_{\nu});_{\alpha} = V^{\mu};_{\alpha} W_{\nu} + V^{\mu}W_{\nu};_{\alpha}\,,\\
&= (V^{\mu}{}_{,\alpha} + \Gamma^{\mu}_{\beta\alpha}V^{\beta})W_{\nu} + V^{\mu}(W_{\nu}{}_{,\alpha} - \Gamma^{\beta}_{\nu\alpha}W_{\beta}),\\
&= (V^{\mu}W_{\nu})_{,\alpha} + \Gamma^{\mu}_{\beta\alpha}V^{\beta}W_{\nu} - \Gamma^{\beta}_{\nu\alpha}V^{\mu}W_{\beta},\\
&= T^{\mu}{}_{\nu,\alpha} + \Gamma^{\mu}_{\beta\alpha}T^{\beta}{}_{\nu} - \Gamma^{\beta}_{\nu\alpha}T^{\mu}{}_{\beta},
\end{aligned} \tag{3.14}$$

yielding a mixed tensor of rank three. The contravariant, covariant index requires a positive, negative sign for the C symbol. In a similar manner one obtains the covariant derivatives of a covariant or contravariant tensor of rank two. If the rank is higher, say, n, then, n, C symbols with appropriate signs are needed. In the case of the metric tensor,

$$g^{\mu\nu};_{\alpha} = g^{\mu\nu}{}_{,\alpha} + \Gamma^{\mu}_{\beta\alpha}g^{\beta\nu} + \Gamma^{\nu}_{\alpha\beta}g^{\mu\beta} = 0, \tag{3.15}$$

$$g_{\mu\nu};_{\alpha} = g_{\mu\nu}{}_{,\alpha} - \Gamma^{\beta}_{\mu\alpha}g_{\beta\nu} - \Gamma^{\beta}_{\alpha\nu}g_{\mu\beta} = 0. \tag{3.16}$$

The reason the above tensors are zero is that in an inertial frame, $g_{\bar{\mu}\bar{\nu}};_{\bar{\alpha}} = \eta_{\bar{\mu}\bar{\nu}};_{\bar{\alpha}} = \eta_{\bar{\mu}\bar{\nu}}{}_{,\bar{\alpha}} = 0$. As this is a tensor equation, it holds in all frames and leads to the more useful form for, $\Gamma^{\lambda}_{\mu\nu}$,

$$\begin{aligned}
0 &= g_{\mu\nu};_{\alpha} + g_{\mu\alpha};_{\nu} - g_{\alpha\nu};_{\mu}\,,\\
&= g_{\mu\nu}{}_{,\alpha} + g_{\mu\alpha}{}_{,\nu} - g_{\alpha\nu}{}_{,\mu}\\
&\quad - \Gamma^{\beta}_{\mu\alpha}g_{\beta\nu} - \Gamma^{\beta}_{\alpha\nu}g_{\mu\beta} - \Gamma^{\beta}_{\mu\nu}g_{\beta\alpha} - \Gamma^{\beta}_{\alpha\nu}g_{\mu\beta} + \Gamma^{\beta}_{\mu\alpha}g_{\beta\nu} + \Gamma^{\beta}_{\mu\nu}g_{\alpha\beta},\\
2g_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} &= (g_{\mu\nu}{}_{,\alpha} + g_{\mu\alpha}{}_{,\nu} - g_{\alpha\nu}{}_{,\mu}),\\
2g^{\mu\lambda}g_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} &= 2\delta^{\lambda}{}_{\beta}\Gamma^{\beta}_{\alpha\nu} = g^{\mu\lambda}(g_{\mu\nu}{}_{,\alpha} + g_{\mu\alpha}{}_{,\nu} - g_{\alpha\nu}{}_{,\mu}),\\
\Gamma^{\lambda}_{\alpha\nu} &= g^{\mu\lambda}(g_{\mu\nu}{}_{,\alpha} + g_{\mu\alpha}{}_{,\nu} - g_{\alpha\nu}{}_{,\mu})/2.
\end{aligned} \tag{3.17}$$

Once the metric tensor is obtained, the C symbols are easily calculated. For example, in cylindrical coordinates, $x^{\bar{\mu}'}$, the previous quoted results for the nonzero C symbols are substantiated. Let, $(\bar{i}', \bar{j}', \bar{k}')$ take on the values, $(1, 2)$. Note that nonzero metric elements are, $g_{11} = 1$ and $g_{22} = \rho^2$,

$$\begin{aligned}
\Gamma^{\bar{i}'}_{\bar{j}'\bar{k}'} &= g^{\bar{i}'\bar{m}'}(g_{\bar{j}'\bar{m}'},_{\bar{k}'} + g_{\bar{k}'\bar{m}'},_{\bar{j}'} - g_{\bar{j}'\bar{k}'},_{\bar{m}'})/2,\\
&= g^{\bar{i}'\bar{i}'}(g_{\bar{i}'\bar{j}'},_{\bar{k}'} + g_{\bar{i}'\bar{k}'},_{\bar{j}'} - g_{\bar{j}'\bar{k}'},_{\bar{i}'})/2,\\
&= g^{\bar{i}'\bar{i}'}(\delta^{\bar{i}'}_{\bar{j}'} g_{\bar{i}'\bar{i}'},_{\bar{k}'} + \delta^{\bar{i}'}_{\bar{k}'} g_{\bar{i}'\bar{i}'},_{\bar{j}'} - \delta^{\bar{j}'}_{\bar{k}'} g_{\bar{k}'\bar{k}'},_{\bar{i}'})/2,\\
\Gamma^1_{11} &= g^{11}g_{11},_1/2 = 0, \quad \Gamma^2_{22} = g^{22}g_{22},_2/2 = 0,\\
\Gamma^1_{12} &= g^{11}g_{11},_2/2 = 0, \quad \Gamma^2_{11} = -g^{22}g_{11},_2/2 = 0,\\
\Gamma^2_{12} &= \Gamma^{\phi}_{\rho\phi} = g^{22}g_{22},_1/2 = \rho^{-2}\rho^2,_\rho/2 = 1/\rho,\\
\Gamma^1_{22} &= \Gamma^{\rho}_{\phi\phi} = -g^{11}g_{22},_1/2 = -\rho.
\end{aligned}$$

3.3 Gravity and the Locally Inertial Frame

Liberal use is made of the thought experiments proposed by Einstein, illustrated in Figs. 3.1 and 3.2, to crystallize his ideas concerning GR. When gravity is included, there is a gravitational field everywhere. It is at first not clear that an inertial frame can be found. Standing in an elevator, gravity is experienced by the floor pushing up on our feet. There is no upward acceleration as gravity counters this force. If the elevator started accelerating upward, a stronger upward push would be experienced. However, an elevator observer couldn't tell whether the effect was due to an additional localized gravitational force pushing the feet to the elevator floor or a force due to a machine capable of lifting the elevator.

If the elevator cable broke and the elevator observer released some objects from rest, a camera fixed to the elevator would show all objects remaining at rest with respect to each other. It wouldn't matter whether some objects were more or less massive or if they were made of different materials. A camera fixed to the earth would show all objects falling with the same acceleration. This is because inertial mass and gravitational mass are the same. As Einstein put it, the elevator observer is free falling and doesn't experience gravity. The results of all non-gravity experiments in the elevator will be the same as the results obtained in an inertial frame. This is the "weak" principle of equivalence. The "strong" principle says the

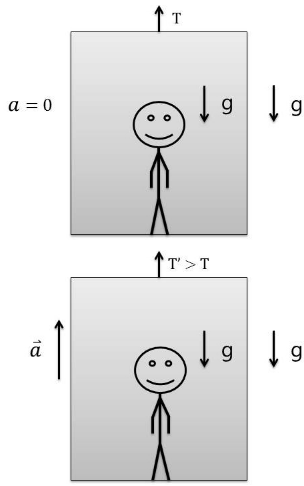


Fig. 3.1 Top, observer at rest in an elevator is also rest with respect to earth. Bottom, elevator and observer accelerating upward with respect to earth. The accelerating observer feels a stronger upward force from the floor and could conclude gravity is stronger than when at rest.

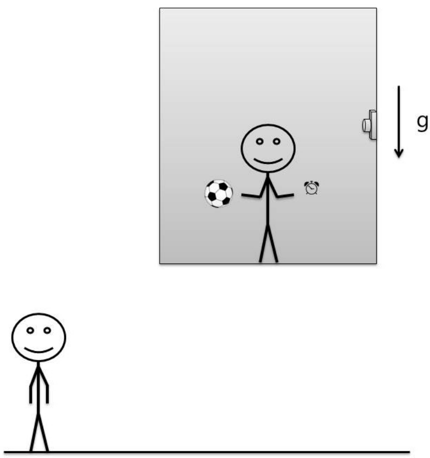


Fig. 3.2 Freely falling elevator observer releases various objects and notes they do not fall; concluding that there is no gravity. Earth observer, not freely falling, notes all objects in the elevator are falling, with the same acceleration, because of gravity.

results of any experiment will be the same as those in an inertial frame. It is somewhat ironic, gravity is a complicated subject, but its distinguishing feature is, if you are freely falling, it is not experienced.

Gravitational strength varies with position. Thus, one would experience gravity through tidal forces. However, in an arbitrarily small volume a free falling observer will, to first order, not experience gravity. For example, no one is conscious of freely falling in the gravitational field of the sun. This is the essence of a locally inertial frame. In this arbitrarily small frame coordinates, $x^{\bar{\mu}}$, can be found with origin at arbitrary point, P, such that,

$$g_{\bar{\alpha}\bar{\beta}}(x^{\bar{\mu}}) = \eta_{\bar{\alpha}\bar{\beta}} + O(x^{\bar{\mu}})^2,$$

$$g_{\bar{\alpha}\bar{\beta}}|_P = \eta_{\bar{\alpha}\bar{\beta}}, \tag{3.18}$$

$$g_{\bar{\alpha}\bar{\beta}},_{\bar{\chi}}|_P = g_{\bar{\alpha}\bar{\beta}};_{\bar{\chi}}|_P = 0, \tag{3.19}$$

$$g_{\bar{\alpha}\bar{\beta}},_{\bar{\chi}},_{\bar{\xi}}|_P \neq 0. \tag{3.20}$$

Another way to see the locally inertial frame in our lives is easy. People live on the curved surface of a spherical earth. However, because at any position, a person occupies such a small portion of this curved surface, that person is not conscious of the curvature. As will be seen gravity causes space curvature.

3.4 Local Flatness Theorem

The existence of a locally inertial frame is the statement that curved space has a flat space tangent to it at any point. Above, it was noted that at any point, P, in spacetime, one can find a rectangular coordinate system, $x^{\bar{\mu}}$, such that at, P, the metric is the metric of SR and its first partial derivative vanishes, but not necessarily its second partial. That is the metric near, P, is approximately that of SR, the differences being second order in the coordinates.

The proof begins by noting that there is some relation between an arbitrary coordinate system and the locally inertial system, $x^{\mu} = x^{\mu}(x^{\bar{\nu}})$, and all orders of the partial derivatives, $x^{\mu},_{\bar{\nu}},\ldots$, exist. The latter and the metric are expanded in a Taylor series, shown below to second order. The expansion is about point, P, for coordinates close to, P,

$$
\begin{aligned}
x^{\mu}{}_{,\bar{\alpha}} &= x^{\mu}{}_{,\bar{\alpha}}|_P + x^{\mu}{}_{,\bar{\alpha}}{}_{,\bar{\chi}}|_P \Delta^{\bar{\chi}} + x^{\mu}{}_{,\bar{\alpha}}{}_{,\bar{\chi}}{}_{,\bar{\xi}}|_P \Delta^{\bar{\chi}}\Delta^{\bar{\xi}}/2, \\
x^{\mu}{}_{,\bar{\alpha}}\, x^{\nu}{}_{,\bar{\beta}}\, g_{\mu\nu} &= g_{\bar{\alpha}\bar{\beta}} = g_{\bar{\alpha}\bar{\beta}}|_P + g_{\bar{\alpha}\bar{\beta}}{}_{,\bar{\chi}}|_P \Delta^{\bar{\chi}} \\
&\quad + g_{\bar{\alpha}\bar{\beta}}{}_{,\bar{\chi}}{}_{,\bar{\xi}}|_P \Delta^{\bar{\chi}}\Delta^{\bar{\xi}}/2, \qquad (3.21) \\
\Delta^{\bar{\chi}} &= x^{\bar{\chi}} - x^{\bar{\chi}}|_P, \text{ thus, } \Delta^{\bar{\chi}}|_P = 0, \\
\Delta^{\bar{\chi}}{}_{,\bar{\lambda}}|_P &= \delta^{\bar{\chi}}_{\bar{\lambda}}, \ (\Delta^{\bar{\chi}}\Delta^{\bar{\xi}})_{,\bar{\lambda}}{}_{,\bar{\gamma}}|_P = \delta^{\bar{\chi}}_{\bar{\lambda}}\delta^{\bar{\xi}}_{\bar{\gamma}} + \delta^{\bar{\chi}}_{\bar{\gamma}}\delta^{\bar{\xi}}_{\bar{\lambda}}.
\end{aligned}
$$

At point P the differences in the coordinates, $\Delta^{\bar{\chi}}$, vanish and Eq. (3.21) becomes,

$$
g_{\bar{\alpha}\bar{\beta}}|_P = [x^{\mu}{}_{,\bar{\alpha}}\, x^{\nu}{}_{,\bar{\beta}}\, g_{\mu\nu}]|_P = \eta_{\bar{\alpha}\bar{\beta}}.
$$

This equality is a set of ten independent equations, because the metric tensor, $g_{\mu\nu}$, is symmetric and has ten independent values. However, there are sixteen independent first order partial derivatives available, $x^{\mu}{}_{,\bar{\alpha}}$, more than enough to satisfy these equations.

If Eq. (3.21) is partially differentiated with respect to, $x^{\bar{\lambda}}$, and evaluated at point, P, the leading term, second order and all the higher order terms with remaining coordinate differences vanish and

$$
\begin{aligned}
g_{\bar{\alpha}\bar{\beta}}{}_{,\bar{\lambda}}|_P &= [(x^{\mu}{}_{,\bar{\alpha}}\, x^{\nu}{}_{,\bar{\beta}}\, g_{\mu\nu})_{,\bar{\lambda}}]|_P, \\
&= [x^{\mu}{}_{,\bar{\alpha}}\, x^{\nu}{}_{,\bar{\beta}}\, g_{\mu\nu}{}_{,\bar{\lambda}} + x^{\mu}{}_{,\bar{\alpha}}{}_{,\bar{\lambda}}\, x^{\nu}{}_{,\bar{\beta}}\, g_{\mu\nu} + x^{\mu}{}_{,\bar{\alpha}}\, x^{\nu}{}_{,\bar{\beta}}{}_{,\bar{\lambda}}\, g_{\mu\nu}]|_P = 0.
\end{aligned}
$$

This equality is a set of forty independent equations, because the metric tensor is symmetric and there are forty independent values of, $g_{\mu\nu}{}_{,\bar{\lambda}}$, ten for each, $\bar{\lambda}$. As the first order partial derivatives have been used, these equations can be satisfied if there are at least forty independent second order partial derivatives, $x^{\mu}{}_{,\bar{\alpha}}{}_{,\bar{\lambda}}$ and $x^{\nu}{}_{,\bar{\beta}}{}_{,\bar{\lambda}}$. Since the order of partial differentiation is unimportant, there are exactly forty such derivatives, ten for each, μ or ν, just enough to satisfy the equations.

If Eq. (3.21) is partially differentiated twice with respect to, $x^{\bar{\lambda}}$ and $x^{\bar{\gamma}}$, and evaluated at point, P, the leading term, first order term and all the terms higher than second order with remaining coordinate differences vanish and

$$
\begin{aligned}
g_{\bar{\alpha}\bar{\beta}},_{\bar{\lambda}},_{\bar{\gamma}}|_P &= [(x^\mu,_{\bar{\alpha}}\, x^\nu,_{\bar{\beta}}\, g_{\mu\nu}),_{\bar{\lambda}},_{\bar{\gamma}}]|_P, \\
&= [(x^\mu,_{\bar{\alpha}}\, x^\nu,_{\bar{\beta}}\, g_{\mu\nu},_{\bar{\gamma}},_{\bar{\lambda}} \\
&\quad +(x^\mu,_{\bar{\alpha}},_{\bar{\gamma}}\, x^\nu,_{\bar{\beta}},_{\bar{\lambda}} + x^\mu,_{\bar{\alpha}},_{\bar{\gamma}},_{\bar{\lambda}}\, x^\nu,_{\bar{\beta}} + x^\mu,_{\bar{\alpha}}\, x^\nu,_{\bar{\beta}},_{\bar{\gamma}},_{\bar{\lambda}})g_{\mu\nu}]|_P.
\end{aligned}
$$

This equality is a set of one hundred independent equations, because there are one hundred (10×10) independent values of, $g_{\mu\nu},_{\bar{\gamma}},_{\bar{\lambda}}$, since the metric tensor is symmetric and the order of partial differentiation is unimportant. As the first and second order partial derivatives have been used, at least one hundred independent third order partial derivatives, $x^\mu,_{\bar{\alpha}},_{\bar{\gamma}},_{\bar{\lambda}}$, are required for a solution. However, as the order of partial differentiation is unimportant there are only eighty such derivatives, twenty for each, μ: $\bar{\alpha}\bar{\gamma}\bar{\lambda} = 000$, 001, 002, 003, 011 012, 013, 022, 023, 033, 013, 022, 023, 033, 111, 112, 113, 123, 133, 222, 223, 233, 333. Thus, there aren't enough parameters to satisfy the one hundred equations. The twenty lacking parameters are needed, as shall be seen, to describe the curvature of space.

At each point, P, the locally inertial rectangular coordinates, $x^{\bar{\mu}}$, may be very complicated functions of the rectangular coordinates of a non-inertial observer. The metric tensor elements in these coordinates are $\eta_{\bar{\alpha}\bar{\beta}}$ and the C symbols are zero, but derivatives of the C symbols don't, in general, vanish. Transforming to other frames, even if rectangular coordinates are used, will lead to non-constant metric tensors and C symbols. However, the covariant derivative of the metric tensor remains $g_{\mu\nu};_\xi = 0$ and the result for the C symbols in terms of the metric tensor and its derivatives, Eq. (3.17), remains valid.

3.5 GR Equations of Motion

In the locally inertial frame specified by, $x^{\bar{\mu}}$, all objects travel in a straight line without acceleration, so that the equation of motion of a particle with rest mass is,

$$\frac{d^2x^{\bar{\mu}}}{d\tau^2} = \frac{dU^{\bar{\mu}}}{d\tau} = 0. \tag{3.22}$$

Using, Eq. (3.13), the above, in another frame, becomes,

$$
\begin{aligned}
0 = \frac{dU^{\bar{\alpha}}}{d\tau} &= \frac{d(x^{\bar{\alpha}},_\mu U^\mu)}{d\tau} = x^{\bar{\alpha}},_\mu \frac{dU^\mu}{d\tau} + U^\mu \frac{d}{d\tau} x^{\bar{\alpha}},_\mu, \\
&= x^{\bar{\alpha}},_\mu \frac{dU^\mu}{d\tau} + U^\mu x^{\bar{\alpha}},_\mu,_\nu \frac{dx^\nu}{d\tau},
\end{aligned}
$$

$$= x^{\bar{\alpha}}{}_{,\mu} \frac{dU^\mu}{d\tau} + U^\mu x^{\bar{\alpha}}{}_{,\mu}{}_{,\nu} U^\nu,$$
$$= x^\beta{}_{,\bar{\alpha}} (x^{\bar{\alpha}}{}_{,\mu} \frac{dU^\mu}{d\tau} + U^\mu x^{\bar{\alpha}}{}_{,\mu}{}_{,\nu} U^\nu) = \delta^\beta{}_\mu \frac{dU^\mu}{d\tau} + U^\mu U^\nu \Gamma^\beta_{\mu\nu},$$
$$= \frac{dU^\beta}{d\tau} + U^\mu U^\nu \Gamma^\beta_{\mu\nu}. \tag{3.23}$$

The equation of motion involves the C symbol and thus the metric tensor.

Since a photon has, $d\tau = 0$, the above cannot be used as its equation of motion. One must substitute another parameter, say, dq. Allowable parameters are called affine parameters. For example, $d\tau = kdq$, yields an allowable, dq, and for photons, constant, $k = 0$. The parameter, q, describes the path, such that as the photon moves, $\frac{dx^\mu}{dq} = W^\mu$, is the tangent vector, with the property,

$$W_\mu W^\mu = g_{\mu\nu} W^\mu W^\nu = g_{\mu\nu} \frac{dx^\mu}{dq}\frac{dx^\nu}{dq} = \frac{g_{\mu\nu} dx^\mu dx^\nu}{dqdq},$$
$$= \left(\frac{d\tau}{dq}\right)^2 = 0. \tag{3.24}$$

In the inertial frame the geodesic is a straight line and the tangent vector doesn't change. The result in another frame is, following the procedure after Eq. (3.22),

$$0 = \frac{dW^{\bar{\alpha}}}{dq} = \frac{d(x^{\bar{\alpha}}{}_{,\mu} W^\mu)}{dq} = x^{\bar{\alpha}}{}_{,\mu}{}_{,\nu} W^\mu W^\nu + x^{\bar{\alpha}}{}_{,\mu} \frac{dW^\mu}{dq},$$
$$= x^\beta{}_{,\bar{\alpha}} (x^{\bar{\alpha}}{}_{,\mu}{}_{,\nu} W^\mu W^\nu + x^{\bar{\alpha}}{}_{,\mu} \frac{dW^\mu}{dq}),$$
$$= \Gamma^\beta_{\mu\nu} W^\mu W^\nu + \frac{dW^\beta}{dq},$$
$$= \frac{d^2 x^\beta}{dq^2} + \Gamma^\beta_{\mu\nu} \frac{dx^\mu}{dq}\frac{dx^\nu}{dq}. \tag{3.25}$$

Problems

1. What is the covariant derivative, $\delta^\mu{}_{\beta;\nu}$, in a frame where gravity must be taken into account. Evaluate the partial derivative, $\delta^\mu{}_{\beta,\nu}$, and show, $x^\beta{}_{,\alpha'}\, x^\mu{}_{,\rho'}\, x^{\rho'}{}_{,\nu}{}_{,\beta} = -x^\mu{}_{,\alpha'}{}_{,\nu}$.
2. Show that in general, the partial derivative of a contravariant vector, $V^\mu{}_{,\nu}$, doesn't transform like a tensor. Show that the C symbol, $\Gamma^\mu_{\nu\xi}$,

doesn't transform like a tensor. Show that the covariant derivative of a vector, $V^\mu{}_{;\nu}$, transforms like a mixed tensor of rank two.

3. Show explicitly that, $(V^\mu V^\nu)_{;\chi} = V^\mu{}_{;\chi} V^\nu + V^\mu V^\nu{}_{;\chi}$, where V is a vector. Don't use the argument that it is a tensor equation that holds in an inertial frame and thus holds in all frames. Show that if, $U^\mu V^\nu{}_{;\mu} = W^\nu$, then, $U^\mu V_\nu{}_{;\mu} = W_\nu$, where $U,\ V,\ W$ are vectors.
4. The importance of writing the laws of physics as tensor equations has been discussed. However, the laws of electromagnetism, conservation of charge and Maxwell's equations, are usually written in 3-vector notation. In vacuum and in an SR inertial rectangular frame these equations are, in naturalized units,

$$\vec{\nabla} \cdot \vec{J} + \rho_{,t} = 0,$$
$$\vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{B} = \vec{\nabla} \times \vec{A},$$
$$\vec{\nabla} \times \vec{E} + \vec{B}_{,t} = 0, \qquad \vec{E} = -\vec{\nabla}\Psi - \vec{A}_{,t}\,,$$
$$\vec{\nabla} \cdot \vec{E} - \rho/\epsilon_0 = 0 = \vec{\nabla} \cdot \vec{E} - \mu_0 \rho,$$
$$\vec{\nabla} \times \vec{B} - \vec{E}_{,t} = \mu_0 \vec{J},$$

where $(\rho, \vec{J})$ is the current, charge density, $(\vec{E}, \vec{B})$ is the (electric, magnetic) field, $(\Psi, \vec{A})$ is the (vector, scalar) potential and (ϵ_0, μ_0) is the (permittivity, permeability) of free space.

Show that the divergence of the vector, $J^\mu = (\rho, \vec{J})$, leads to charge conservation as a tensor equation. Show that the components of the 3-vectors $(\vec{E},\ \vec{B})$ can be written as elements of a tensor of rank two,

$$F_{\nu\mu} = A_{\mu,\nu} - A_{\nu,\mu} = -F_{\mu\nu}, \text{ where,}$$
$$A_\mu = (-\Psi, \vec{A}).$$

and that the four Maxwell equations can be written as two tensor equations,

$$F^{\mu\nu}{}_{;\nu} = \mu_0 J^\mu = J^\mu/\epsilon_0,$$
$$0 \equiv F_{\mu\nu;\xi} + F_{\xi\mu;\nu} + F_{\nu\xi;\mu}\,.$$

5. Consider the following useful, as we shall see, metric, written in spherical coordinates, $x^{0,1,2,3} = t, \theta, \phi, r$:

$$(d\tau)^2 = -g_{\mu\nu} dx^\mu dx^\nu,$$
$$= \exp[2\Phi(r)](dt)^2 - (\exp[2\Delta(r)](dr)^2 + (r)^2[(\sin\theta d\phi)^2 + (d\theta)^2]).$$

With the correct forms for $\Phi(r)$ and $\Delta(r)$ this is either the metric of an inertial frame in spherical coordinates or the Schwarzschild metric used to discuss the motion of light and planets due to the sun's gravity. Find all the nonzero C symbols. If confined to the surface of a sphere, so that, $(r) = a =$ constant, one would be conscious of curvature, even in SR. In this case, what are the nonzero C symbols?

6. Show explicitly, for the metric of problem five that, $g_{\mu\nu};_{\chi} = 0$. Now specialize to an inertial frame. In this frame what are the nonzero C symbols. You have already found the result if, $r = a$.
7. For the metric of problem five in an inertial frame, use the C symbols calculated in problem five to find the gradient and Laplacian of a scalar function of position.
8. What does the quantity, $V^{\mu};_{\lambda};_{\gamma}$, transform like? In the presence of a gravitational field,

$$V^{\mu};_{\lambda};_{\gamma} - V^{\mu};_{\gamma};_{\lambda} = -R^{\mu}{}_{\xi\lambda\gamma}V^{\xi}.$$

Evaluate, $R^{\mu}{}_{\xi\lambda\gamma}$, in terms of the C symbols and their partial derivatives. Then, argue that,

$$V_{\mu};_{\lambda};_{\gamma} - V_{\mu};_{\gamma};_{\lambda} = R^{\xi}{}_{\mu\lambda\gamma}V_{\xi}.$$

Show $R^{\mu}{}_{\xi\lambda\gamma} = -R^{\mu}{}_{\xi\gamma\lambda}$. What does the quantity, $R^{\mu}{}_{\xi\lambda\gamma}$, transform like? This quantity is the Riemann curvature tensor. It plays an important role in GR and will be discussed in much more detail in a later chapter.
9. If, $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}$, is a general mixed tensor, use the results of problem eight to evaluate, $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots};_{\lambda};_{\gamma} - T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots};_{\gamma};_{\lambda}$.
10. Derive the alternate form for the C symbol, $\Gamma^{\xi}_{\mu\nu} = x^{\xi},_{\bar{\alpha}}\, x^{\bar{\alpha}},_{\mu},_{\nu}$, from, $\Gamma^{\xi}_{\mu\nu} = g^{\xi\beta}(g_{\mu\beta},_{\nu} + g_{\nu\beta},_{\mu} - g_{\mu\nu},_{\beta})/2$.
11. A point particle of finite rest mass moves in a gravity free region of empty space. What are the Newtonian equations of motion for the cylindrical coordinates, (t, ρ, ϕ, z). Show the the GR equations of motion give the same results.
12. For the metric of problem five, find the equations of motion of GR for a particle with rest mass? Show that a planar solution is possible. In this case, find another constant of the motion.

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Chapter 4

Curvature

4.1 Geodesics

In the geometry of flat space, geodesics are the paths of minimum distance between two points for motion with constant velocity or the paths that minimize the travel time. Light in empty space certainly fits this case and often the phrase, "light travels in straight lines", has been heard. This is because gravity is so weak that the deviation from a straight line path is extremely small.

The thought experiments of Einstein, illustrated in Fig. 4.1, show that in the presence of gravity light moves in a curved path. The top of the figure shows two equivalent observers, O, one in gravity free space, the other freely falling in a constant gravitational field. They observe that a horizontally traveling light ray enters and exits their capsule a distance, L, above the floor. The bottom of the figure shows an observer, O$'$, not freely falling in the gravitational field. O$'$ also observes, that for the freely falling capsule, light entered and exited the same distance, above the floor. However, according to, O$'$, the exit point will have fallen in the time light crossed the capsule. Thus, the light also must have fallen or moved in a downward curved path. The conclusion to be drawn is that gravity affects light, a break from Newtonian physics.

Observers in a gravitational field and not freely falling, know that nothing can make the trip between two points faster than light. Thus, for gravity, the "straight lines" or geodesics are actually curved paths. Mathematically, the GR equations of motion also lead to geodesics for the motion of a particle with rest mass. Making use of Eqs. (3.16)–(3.18) and (3.24) and

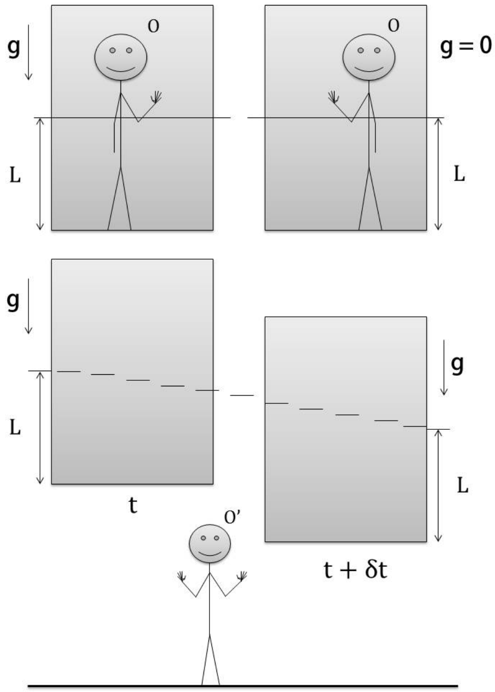

Fig. 4.1 Top: equivalent observers, one in gravity free space, the other freely falling in a gravitational field. They both observe horizontally traveling light enter and leave their capsule a distance, L, above the floor. Bottom: an observer, not freely falling in the field, observes light travels in a curved trajectory to enter and leave the same distance above the floor.

renaming summed over indexes when necessary yields,

$$
\begin{aligned}
\frac{d(g_{\mu\nu}U^\mu U^\nu)}{d\tau} &= \frac{dg_{\mu\nu}}{d\tau}U^\mu U^\nu + g_{\mu\nu}\left(\frac{dU^\mu}{d\tau}U^\nu + U^\mu\frac{dU^\nu}{d\tau}\right), \\
&= g_{\mu\nu},_\gamma \frac{dx^\gamma}{d\tau}U^\mu U^\nu + g_{\mu\nu}\left(\frac{dU^\mu}{d\tau}U^\nu + U^\mu\frac{dU^\nu}{d\tau}\right), \\
&= g_{\mu\nu},_\gamma U^\gamma U^\mu U^\nu - g_{\mu\nu}(\Gamma^\mu_{\alpha\beta}U^\nu + U^\mu\Gamma^\nu_{\alpha\beta})U^\alpha U^\beta, \\
&= (g_{\alpha\beta},_\gamma - g_{\mu\gamma}\Gamma^\mu_{\alpha\beta} - g_{\gamma\nu}\Gamma^\nu_{\alpha\beta})U^\alpha U^\beta U^\gamma, \\
&= (g_{\lambda\beta}\Gamma^\lambda_{\gamma\alpha} + g_{\alpha\lambda}\Gamma^\lambda_{\beta\gamma} - g_{\mu\gamma}\Gamma^\mu_{\alpha\beta} - g_{\gamma\nu}\Gamma^\nu_{\alpha\beta})U^\alpha U^\beta U^\gamma, \\
&= (g_{\mu\gamma}\Gamma^\mu_{\beta\alpha} + g_{\gamma\nu}\Gamma^\nu_{\beta\alpha} - g_{\mu\gamma}\Gamma^\mu_{\alpha\beta} - g_{\gamma\nu}\Gamma^\nu_{\alpha\beta})U^\alpha U^\beta U^\gamma = 0, \\
g_{\mu\nu}U^\mu U^\nu &= K = -1.
\end{aligned}
$$

This is true as the initial condition is $(d\tau)^2 = -g_{\mu\nu}dx^\mu dx^\nu$. Then $-1 = g_{\mu\nu}U^\mu U^\nu$ and the expression for $(d\tau)^2$ always holds along the path.

One can introduce a parameter, q, to describe the path of a body under the influence of gravity. Given the value of that parameter, the position on the path is determined. However, algebra allows manipulations so that the final result is independent of, q. The proper time elapsed when the body moves from A to B is,

$$\begin{aligned}
T_{BA} &= \int_A^B \frac{d\tau}{dq}dq, \\
&= \int_A^B dq \frac{(-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}}{dq}, \\
&= \int_A^B dq \left(-g_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}\right)^{1/2}.
\end{aligned}$$

Vary the path from $x^\mu(q)$ to $x^\mu(q)+\delta x^\mu(q)$ while keeping the end points fixed. The change in the elapsed time is,

$$\delta T_{BA} = -\int_A^B dq\frac{N}{D}, \text{ where,}$$

$$N = \left(g_{\mu\nu},_\beta\, \delta x^\beta \frac{dx^\mu}{dq}\frac{dx^\nu}{dq} + g_{\mu\nu}\frac{d(\delta x^\mu)}{dq}\frac{dx^\nu}{dq} + g_{\mu\nu}\frac{dx^\mu}{dq}\frac{d(\delta x^\nu)}{dq}\right),$$

$$D = 2\left(-g_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq}\right)^{1/2} = \frac{2d\tau}{dq},$$

$$\begin{aligned}
\delta T_{BA} &= -\int_A^B \frac{dqdq}{2d\tau}\left(g_{\mu\nu},_\beta\, \delta x^\beta \frac{dx^\mu}{dq}\frac{dx^\nu}{dq} + g_{\mu\nu}\frac{d(\delta x^\mu)}{dq}\frac{dx^\nu}{dq} + g_{\mu\nu}\frac{dx^\mu}{dq}\frac{d(\delta x^\nu)}{dq}\right), \\
&= -\int_A^B \frac{dqdqd\tau}{d\tau d\tau}\left(g_{\mu\nu},_\beta\, \delta x^\beta \frac{dx^\mu}{dq}\frac{dx^\nu}{dq} + 2g_{\mu\nu}\frac{d(\delta x^\mu)}{dq}\frac{dx^\nu}{dq}\right)/2, \\
&= -\int_A^B d\tau\left(g_{\mu\nu},_\beta\, \delta x^\beta U^\mu U^\nu/2 + g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}U^\nu\right).
\end{aligned}$$

The second term, $T2$, can be integrated by parts with the proviso that $\delta x^\mu = 0$ at the end points. After such integration, use of the expression for the C symbol simplifies the result,

$$
\begin{aligned}
T2 &= \int_A^B d\tau [g_{\mu\nu} U^\nu] \frac{d(\delta x^\mu)}{d\tau} = \int_A^B [g_{\mu\nu} U^\nu] d(\delta x^\mu) \equiv \int_A^B [W] d(V), \\
&= (WV)_{endpoints} - \int_A^B V dW = -\int_A^B V dW = -\int_A^B d\tau V \frac{dW}{d\tau}, \\
&= -\int_A^B d\tau \delta x^\mu \frac{d(g_{\mu\nu} U^\nu)}{d\tau}, \\
&= -\int_A^B d\tau \delta x^\mu \left(g_{\mu\nu} \frac{dU^\nu}{d\tau} + U^\nu U^\lambda g_{\mu\nu,\lambda} \right), \\
&= -\int_A^B d\tau \delta x^\beta \left(g_{\beta\nu} \frac{dU^\nu}{d\tau} + U^\mu U^\lambda g_{\beta\mu,\lambda} \right), \\
\delta T_{BA} &= -\int_A^B d\tau \delta x^\beta F, \text{ where,} \\
F &= g_{\mu\lambda,\beta} U^\mu U^\lambda/2 - U^\mu U^\lambda g_{\beta\mu,\lambda} - g_{\beta\nu} \frac{dU^\nu}{d\tau}, \\
&= g_{\mu\lambda,\beta} U^\mu U^\lambda/2 - \left(U^\mu U^\lambda g_{\beta\mu,\lambda} + U^\mu U^\lambda g_{\beta\mu,\lambda} \right)/2 - g_{\beta\nu} \frac{dU^\nu}{d\tau}, \\
&= g_{\mu\lambda,\beta} U^\mu U^\lambda/2 - \left(U^\mu U^\lambda g_{\beta\mu,\lambda} + U^\lambda U^\mu g_{\beta\lambda,\mu} \right)/2 - g_{\beta\nu} \frac{dU^\nu}{d\tau}, \\
&= U^\mu U^\lambda (g_{\mu\lambda,\beta} - g_{\beta\mu,\lambda} - g_{\beta\lambda,\mu})/2 - g_{\beta\nu} \frac{dU^\nu}{d\tau}, \text{ but,} \\
\Gamma^\nu_{\mu\lambda} &= -g^{\alpha\nu} (g_{\mu\lambda,\alpha} - g_{\alpha\mu,\lambda} - g_{\alpha\lambda,\mu})/2 \\
g_{\beta\nu} \Gamma^\nu_{\mu\lambda} &= -\delta^\alpha{}_\beta (g_{\mu\lambda,\alpha} - g_{\alpha\mu,\lambda} - g_{\alpha\lambda,\mu})/2 = -(g_{\mu\lambda,\beta} - g_{\beta\mu,\lambda} - g_{\beta\lambda,\mu})/2, \\
F &= -g_{\beta\nu} \left[\Gamma^\nu_{\mu\lambda} U^\mu U^\lambda + \frac{dU^\nu}{d\tau} \right] = 0, \quad \delta T_{BA} = 0. \qquad (4.1)
\end{aligned}
$$

The last line follows from the equations of motion. Thus, the time for the trip is an extremum, usually a minimum, so the path is a geodesic. As the object moves from point to point, the clock attached to the object ticks at the rate appropriate for the gravitational field at the point. For a time extremum, a straight line path is unlikely, and so the object, like light, moves in a curved path so that it takes an extremum of proper time to make the trip.

However, for light the element of proper time always vanishes, $d\tau = 0$. Thus another explanation for the geodesic that light must travel along is required. The one deduced by Einstein is that gravity curves space and all objects, even photons, must travel along that curvature. It's similar to us traveling on the surface of a spherical earth. Empty space or vacuum is no longer seen to be just volume where objects can position themselves. It is rather like a fabric that can be pulled and contorted by the gravitational field of far away objects.

On a cosmological scale, things are even more complicated. Space is always being created between any two points because the universe is filled with an unknown form of energy, "dark energy", dependent on the size of the universe. When we get to cosmology, it will be seen that this is now the dominant energy because the universe has expanded to such a large volume. It's a really strange universe we inhabit and just common sense obtained from every day observation could never lead you to its inner workings.

4.2 Parallel Transport

The characterization of curvature starts with the concept of parallel transport. On a flat surface, as in Fig. 4.2, draw an arbitrary closed path $ABCA$. Here a circle is used so that at various places along the path, some tangent vectors, $\vec{W}$, that can point in all directions, are shown. For parallel transport start at, A, and draw on the surface, a small parallel transport vector, $\vec{V}$, in any direction. Proceed to a neighboring point on the path and draw on the surface a small transport vector as parallel as possible to the, $\vec{V}$, previously drawn. On a flat surface it is possible to draw the vector exactly parallel. When once again at, A, the identical vectors would be redrawn because the surface is flat. In this sense a flat surface has no intrinsic curvature. A cylinder can be constructed by rolling a flat sheet and so has no intrinsic curvature.

A sphere cannot be made from a flat sheet and so has intrinsic curvature. One can find at least one path on the sphere's surface, as in Fig. 4.3, for which the vectors, $\vec{V}$, would not repeat. Pick the path, $ABCA$, such that, B and C, are on the equator and, A, is at a pole. At, A, start with a vector, $\vec{V}$, on the sphere's surface, that is tangent to an arc of longitude. As one proceeds to, B, along the longitude, a new parallel transport vector cannot be drawn on the surface exactly parallel to, $\vec{V}$. The best one can do is draw that vector along the tangent vector. At, B, that vector is perpendicular to the equator and remains so as one proceeds to, C. From there the return to,

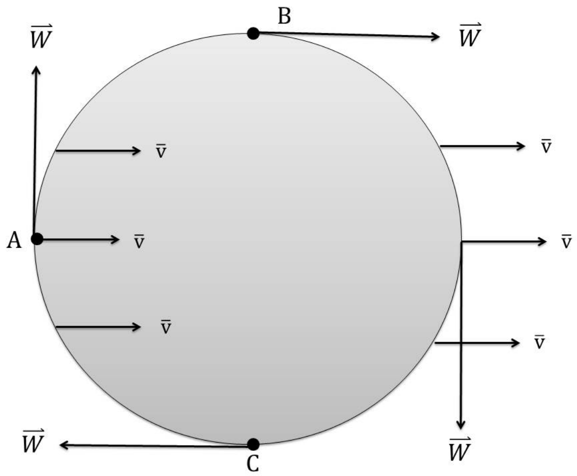

Fig. 4.2 An arbitrary closed curve, here a circle, on a flat surface, with tangent vectors, $\vec{W}$. Parallel transport vectors $\vec{V}$, at any point on the curve, can be drawn on the surface parallel to each other.

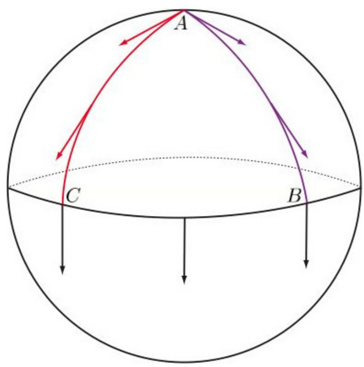

Fig. 4.3 A sphere has intrinsic curvature so that there are paths for which the parallel transport vectors do not line up after traversing a closed curve on the surface.

A, is again along a longitude. The parallel transport vectors on the surface, will be opposite the tangent vectors. Upon reaching, A, the final parallel transport vector is different from the initial one.

In spacetime these vectors have four components, V^μ, W^μ. At any point, P, one can go to a locally inertial frame, so that in a small enough neighborhood of, P, as you proceed along the curve specified by affine parameter, q, with tangent vector, $W^{\bar{\nu}} = \frac{dx^{\bar{\nu}}}{dq}$, $V^{\bar{\mu}}$, is constant. This leads to a tensor equation that is taken as the frame invariant definition of parallel transport of V^μ along, W^ν,

$$0 = \frac{dV^{\bar{\mu}}}{dq}|_P = V^{\bar{\mu}},_{\bar{\nu}}\frac{dx^{\bar{\nu}}}{dq} = W^{\bar{\nu}}V^{\bar{\mu}},_{\bar{\nu}} = W^{\bar{\nu}}V^{\bar{\mu}};_{\bar{\nu}} = W^\nu V^\mu;_\nu\,. \tag{4.2}$$

The last equality occurs because the next to last equality established the above as a tensor equation.

In flat space the geodesics are straight lines. These are the only curves that parallel transport their tangent vector. In curved space, the geodesics are drawn as "straight" as possible by demanding parallel transport of the tangent vector. This leads to the equation of motion in terms of q. From Eq. (4.2), with, $V^{\bar{\nu}} = W^{\bar{\nu}}$,

$$\begin{aligned} 0 &= W^\nu W^\mu;_\nu = W^\nu(W^\mu,_\nu + W^\beta\Gamma^\mu_{\beta\nu}) = W^\nu W^\mu,_\nu + W^\nu W^\beta\Gamma^\mu_{\beta\nu}, \\ &= \frac{dx^\nu}{dq}\left(\frac{dx^\mu}{dq}\right),_\nu + \frac{dx^\nu}{dq}\frac{dx^\beta}{dq}\Gamma^\mu_{\beta\nu} = \frac{d^2x^\mu}{dq^2} + \frac{dx^\nu}{dq}\frac{dx^\beta}{dq}\Gamma^\mu_{\beta\nu}. \end{aligned} \tag{4.3}$$

The above equations can be used to quantify curvature. Consider travel along the elemental closed curve, $ABCDA$, that bounds area on a spherical surface, such as that blown up in Fig. 4.4. The curvature is not noticeable for such a small area. The angles, $(\theta,\ \phi) = (x^1,\ x^2)$, vary such that along element AB, $x^1 = a$, x^2 varies; along BC, $x^2 = b+\delta b$, x^1 varies; along CD, $x^1 = a+\delta a$, x^2 varies, and along AD, $x^2 = b$, x^1 varies. Vector, V^μ, defined at A, is parallel transported around the curve. The tangent vector is finite and constant for each element and using Eq. (4.2), the contributions for the elements are,

$$0 = V^\mu;_i\,, \quad V^\mu,_i = -V^\beta\Gamma^\mu_{i\beta}, \quad i = 1, 2, \tag{4.4}$$

$$\begin{aligned} V^\mu(AB) &\equiv V^\mu(B) - V^\mu(A), \\ V^\mu(AB) &= \int_A^B dx^2 V^\mu,_2|_{x^1=a}, \\ &= -\int_A^B dx^2(V^\beta\Gamma^\mu_{2\beta})|_{x^1=a}, \\ &= -\int_b^{b+\delta b} dx^2(V^\beta\Gamma^\mu_{2\beta})|_{x^1=a}, \text{ similarly,} \end{aligned}$$

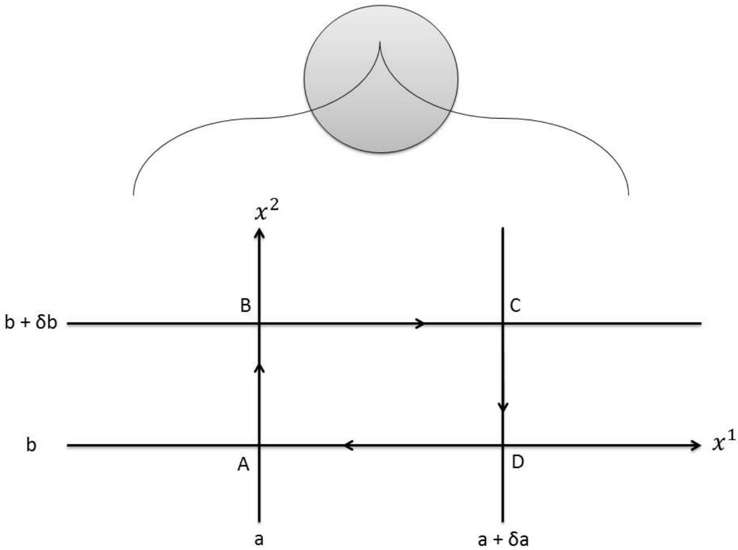

Fig. 4.4 Blowup of an elemental area on a spherical surface that is bound by closed curve, $ABCDA$.

$$
\begin{aligned}
V^{\mu}(BC) &= -\int_{B}^{C} dx^1 (V^{\beta}\Gamma^{\mu}_{1\beta})|_{x^2=b+\delta b}, \\
&= -\int_{a}^{a+\delta a} dx^1 (V^{\beta}\Gamma^{\mu}_{1\beta})|_{x^2=b+\delta b}, \\
V^{\mu}(CD) &= -\int_{C}^{D} dx^2 (V^{\beta}\Gamma^{\mu}_{2\beta})|_{x^1=a+\delta a}, \\
&= -\int_{b+\delta b}^{b} dx^2 (V^{\beta}\Gamma^{\mu}_{2\beta})|_{x^1=a+\delta a}, \\
V^{\mu}(DA) &= -\int_{D}^{A} dx^1 (V^{\beta}\Gamma^{\mu}_{1\beta})|_{x^2=b}, \\
&= -\int_{a+\delta a}^{a} dx^2 (V^{\beta}\Gamma^{\mu}_{1\beta})|_{x^2=b}.
\end{aligned}
$$

The minus sign can be used to flip the limits in any of the above integrals.

Add the element contributions, make use of the definition of the partial derivative and find that the change of the vector when once again at A, is

$$\begin{aligned}
\delta V^\mu(A) &= \int_b^{b+\delta b} dx^2 \left[(V^\beta\Gamma^\mu_{2\beta})|_{x^1=a+\delta a} - (V^\beta\Gamma^\mu_{2\beta})|_{x^1=a}\right] \\
&\quad - \int_a^{a+\delta a} dx^1 \left[(V^\beta\Gamma^\mu_{1\beta})|_{x^2=b+\delta b} - (V^\beta\Gamma^\mu_{1\beta})|_{x^2=b}\right], \\
&= -\left[\int_a^{a+\delta a} \delta b dx^1 (V^\beta\Gamma^\mu_{1\beta})_{,2} - \int_b^{b+\delta b} \delta a dx^2 (V^\beta\Gamma^\mu_{2\beta})_{,1}\right], \\
&= -\delta a\delta b\left[(V^\beta\Gamma^\mu_{1\beta})_{,2} - (V^\beta\Gamma^\mu_{2\beta})_{,1}\right], \\
&= -\delta a\delta b\left(V^\beta{}_{,2}\,\Gamma^\mu_{1\beta} + V^\beta\Gamma^\mu_{1\beta,2} - V^\beta{}_{,1}\,\Gamma^\mu_{2\beta} - V^\beta\Gamma^\mu_{2\beta,1}\right), \\
&= -\delta a\delta b\left(-V^\nu\Gamma^\beta_{2\nu}\Gamma^\mu_{1\beta} + V^\beta\Gamma^\mu_{1\beta,2} + V^\nu\Gamma^\beta_{1\nu}\Gamma^\mu_{2\beta} - V^\beta\Gamma^\mu_{2\beta,1}\right), \\
&= \delta a\delta b V^\beta\left(\Gamma^\nu_{2\beta}\Gamma^\mu_{1\nu} - \Gamma^\mu_{1\beta,2} - \Gamma^\nu_{1\beta}\Gamma^\mu_{2\nu} + \Gamma^\mu_{2\beta,1}\right).
\end{aligned}$$

The last line results from using Eq. (4.4) to rewrite the partial derivatives, $V^\beta{}_{,(1,2)}$. The quantity in parentheses is nonzero for this curve because a sphere has intrinsic curvature. In problem three, one calculates, in the absence of gravity, a finite curvature on a spherical surface and none on a cylindrical surface, where $g_{33} = \eta_{33} = 1$, and only varying $\phi = x^2$ matters.

4.3 Curvature Tensors

When gravity is present and there are no boundary surfaces, there is no reason to allow just $x^{1,2}$ to vary, so let them be replaced by generalized coordinates, $x^{\gamma,\lambda}$. Then the above quantity in parentheses is defined as the Riemann curvature tensor, that applies to spacetime,

$$R^\mu{}_{\beta\gamma\lambda} = \Gamma^\nu_{\lambda\beta}\Gamma^\mu_{\gamma\nu} - \Gamma^\mu_{\gamma\beta,\lambda} - \Gamma^\nu_{\gamma\beta}\Gamma^\mu_{\lambda\nu} + \Gamma^\mu_{\lambda\beta,\gamma}\,. \tag{4.5}$$

The proof that $R^\mu_{\beta\gamma\lambda}$ is a tensor was carried out in chapter three, problem eight where the following results were obtained for vector, V^μ,

$$V^\mu{}_{;\lambda\,;\gamma} - V^\mu{}_{;\gamma\,;\lambda} = R^\mu{}_{\beta\lambda\gamma}V^\beta,\; R^\mu{}_{\beta\lambda\gamma} = -R^\mu{}_{\beta\gamma\lambda}. \tag{4.6}$$

Since the covariant derivative of a tensor is a tensor, the left side of Eq. (4.6) is a tensor. Thus the right side of the first equality must be a tensor. Since, V^β, is a tensor of rank one, $R^\mu{}_{\beta\lambda\gamma}$, must be a tensor of rank four.

In a locally inertial frame, this tensor simplifies because the C symbols, but not their partial derivatives vanish and, $g^{\bar{\mu}\bar{\nu}};_{\bar{\chi}} = g^{\bar{\mu}\bar{\nu}},_{\bar{\chi}} = 0$,

$$
\begin{aligned}
\Gamma^{\bar{\mu}}_{\bar{\lambda}\bar{\beta}},_{\bar{\gamma}} &= [(g^{\bar{\alpha}\bar{\mu}}[g_{\bar{\beta}\bar{\alpha}},_{\bar{\lambda}} + g_{\bar{\lambda}\bar{\alpha}},_{\bar{\beta}} - g_{\bar{\beta}\bar{\lambda}},_{\bar{\alpha}}]),_{\bar{\gamma}}]/2,\\
&= (g^{\bar{\alpha}\bar{\mu}},_{\bar{\gamma}}[g_{\bar{\beta}\bar{\alpha}},_{\bar{\lambda}} + g_{\bar{\lambda}\bar{\alpha}},_{\bar{\beta}} - g_{\bar{\beta}\bar{\lambda}},_{\bar{\alpha}}] + g^{\bar{\alpha}\bar{\mu}}[g_{\bar{\beta}\bar{\alpha}},_{\bar{\lambda}} + g_{\bar{\lambda}\bar{\alpha}},_{\bar{\beta}} - g_{\bar{\beta}\bar{\lambda}},_{\bar{\alpha}}],_{\bar{\gamma}})/2,\\
&= (g^{\bar{\alpha}\bar{\mu}}[g_{\bar{\beta}\bar{\alpha}},_{\bar{\lambda}} + g_{\bar{\lambda}\bar{\alpha}},_{\bar{\beta}} - g_{\bar{\beta}\bar{\lambda}},_{\bar{\alpha}}],_{\bar{\gamma}})/2,\\
&= g^{\bar{\alpha}\bar{\mu}}(g_{\bar{\beta}\bar{\alpha}},_{\bar{\lambda}},_{\bar{\gamma}} + g_{\bar{\lambda}\bar{\alpha}},_{\bar{\beta}},_{\bar{\gamma}} - g_{\bar{\beta}\bar{\lambda}},_{\bar{\alpha}},_{\bar{\gamma}})/2.
\end{aligned}
$$

Similarly, and noting the metric tensor is symmetric and the order of partial differentiation is immaterial,

$$
\begin{aligned}
\Gamma^{\bar{\mu}}_{\bar{\gamma}\bar{\beta}},_{\bar{\lambda}} &= g^{\bar{\alpha}\bar{\mu}}(g_{\bar{\beta}\bar{\alpha}},_{\bar{\gamma}},_{\bar{\lambda}} + g_{\bar{\gamma}\bar{\alpha}},_{\bar{\beta}},_{\bar{\lambda}} - g_{\bar{\beta}\bar{\gamma}},_{\bar{\alpha}},_{\bar{\lambda}})/2,\\
R^{\bar{\mu}}{}_{\bar{\beta}\bar{\gamma}\bar{\lambda}} &= g^{\bar{\alpha}\bar{\mu}}(g_{\bar{\lambda}\bar{\alpha}},_{\bar{\beta}},_{\bar{\gamma}} - g_{\bar{\beta}\bar{\lambda}},_{\bar{\alpha}},_{\bar{\gamma}} - g_{\bar{\gamma}\bar{\alpha}},_{\bar{\beta}},_{\bar{\lambda}} + g_{\bar{\beta}\bar{\gamma}},_{\bar{\alpha}},_{\bar{\lambda}})/2, \qquad (4.7)\\
g_{\bar{\nu}\bar{\mu}}R^{\bar{\mu}}{}_{\bar{\beta}\bar{\gamma}\bar{\lambda}} &= g_{\bar{\nu}\bar{\mu}}g^{\bar{\alpha}\bar{\mu}}(g_{\bar{\lambda}\bar{\alpha}},_{\bar{\beta}},_{\bar{\gamma}} - g_{\bar{\beta}\bar{\lambda}},_{\bar{\alpha}},_{\bar{\gamma}} - g_{\bar{\gamma}\bar{\alpha}},_{\bar{\beta}},_{\bar{\lambda}} + g_{\bar{\beta}\bar{\gamma}},_{\bar{\alpha}},_{\bar{\lambda}})/2,\\
R_{\bar{\nu}\bar{\beta}\bar{\gamma}\bar{\lambda}} &= (g_{\bar{\lambda}\bar{\nu}},_{\bar{\beta}},_{\bar{\gamma}} - g_{\bar{\beta}\bar{\lambda}},_{\bar{\nu}},_{\bar{\gamma}} - g_{\bar{\gamma}\bar{\nu}},_{\bar{\beta}},_{\bar{\lambda}} + g_{\bar{\beta}\bar{\gamma}},_{\bar{\nu}},_{\bar{\lambda}})/2. \qquad (4.8)
\end{aligned}
$$

The above is the completely covariant curvature tensor. One then observes the following results follow trivially,

$$R_{\bar{\nu}\bar{\beta}\bar{\gamma}\bar{\lambda}} = -R_{\bar{\beta}\bar{\nu}\bar{\gamma}\bar{\lambda}} = -R_{\bar{\nu}\bar{\beta}\bar{\lambda}\bar{\gamma}} = R_{\bar{\gamma}\bar{\lambda}\bar{\nu}\bar{\beta}}, \tag{4.9}$$

$$0 = R_{\bar{\nu}\bar{\beta}\bar{\gamma}\bar{\lambda}} + R_{\bar{\nu}\bar{\lambda}\bar{\beta}\bar{\gamma}} + R_{\bar{\nu}\bar{\gamma}\bar{\lambda}\bar{\beta}}. \tag{4.10}$$

Equations (4.9) and (4.10) are tensor equations and so hold in all frames,

$$R_{\nu\beta\gamma\lambda} = -R_{\beta\nu\gamma\lambda} = -R_{\nu\beta\lambda\gamma} = R_{\gamma\lambda\nu\beta}, \tag{4.11}$$

$$0 = R_{\nu\beta\gamma\lambda} + R_{\nu\lambda\beta\gamma} + R_{\nu\gamma\lambda\beta}. \tag{4.12}$$

4.4 Ricci Tensor, Bianchi Identity, Einstein Tensor

The Ricci tensor is defined as follows:

$$R_{\beta\lambda} \equiv R^{\mu}{}_{\beta\mu\lambda}. \tag{4.13}$$

Note the contraction keeps the first and third covariant indexes. Other contractions yield zero or no additional information. It is easy to show, by starting in an inertial frame, that due to the symmetry properties of the curvature tensor,

$$R^{\mu}{}_{\beta\lambda\mu} = -R^{\mu}{}_{\beta\mu\lambda}, \qquad 0 = R^{\mu}{}_{\mu\beta\lambda}. \tag{4.14}$$

As the metric tensor is symmetric and the order of partial differentiation is unimportant, one can use Eqs. (4.7) and (4.11), rename summed over indexes and show that this tensor is symmetric. Begin in a locally invariant frame,

$$\begin{aligned}
R^{\bar{\mu}}{}_{\bar{\beta}\bar{\mu}\bar{\lambda}} &= g^{\bar{\alpha}\bar{\mu}}(g_{\bar{\lambda}\bar{\alpha}},_{\bar{\beta}},_{\bar{\mu}} - (g_{\bar{\beta}\bar{\lambda}},_{\bar{\alpha}},_{\bar{\mu}} + g_{\bar{\mu}\bar{\alpha}},_{\bar{\beta}},_{\bar{\lambda}}) + g_{\bar{\beta}\bar{\mu}},_{\bar{\alpha}},_{\bar{\lambda}})/2, \\
R^{\bar{\mu}}{}_{\bar{\lambda}\bar{\mu}\bar{\beta}} &= g^{\bar{\alpha}\bar{\mu}}(g_{\bar{\beta}\bar{\alpha}},_{\bar{\lambda}},_{\bar{\mu}} - (g_{\bar{\lambda}\bar{\beta}},_{\bar{\alpha}},_{\bar{\mu}} + g_{\bar{\mu}\bar{\alpha}},_{\bar{\lambda}},_{\bar{\beta}}) + g_{\bar{\lambda}\bar{\mu}},_{\bar{\alpha}},_{\bar{\beta}})/2, \\
R_{\bar{\beta}\bar{\lambda}} - R_{\bar{\lambda}\bar{\beta}} &= g^{\bar{\alpha}\bar{\mu}}(g_{\bar{\lambda}\bar{\alpha}},_{\bar{\beta}},_{\bar{\mu}} + g_{\bar{\beta}\bar{\mu}},_{\bar{\alpha}},_{\bar{\lambda}} - g_{\bar{\beta}\bar{\alpha}},_{\bar{\lambda}},_{\bar{\mu}} - g_{\bar{\lambda}\bar{\mu}},_{\bar{\alpha}},_{\bar{\beta}})/2, \\
&= g^{\bar{\alpha}\bar{\mu}}(g_{\bar{\lambda}\bar{\mu}},_{\bar{\beta}},_{\bar{\alpha}} + g_{\bar{\beta}\bar{\alpha}},_{\bar{\mu}},_{\bar{\lambda}} - g_{\bar{\beta}\bar{\alpha}},_{\bar{\lambda}},_{\bar{\mu}} - g_{\bar{\lambda}\bar{\mu}},_{\bar{\alpha}},_{\bar{\beta}})/2 = 0.
\end{aligned}$$

The last equation is a tensor equation and holds in all frames.

$$R_{\beta\lambda} = R_{\lambda\beta}. \tag{4.15}$$

From the Ricci tensor an invariant, the Ricci scalar may be formed,

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\xi}{}_{\mu\xi\nu} = g^{\mu\nu}g^{\chi\xi}R_{\chi\mu\xi\nu}. \tag{4.16}$$

The following contraction may also prove useful,

$$g^{\mu\nu}R^{\chi}{}_{\mu\beta\nu} = R^{\chi}{}_{\beta} \tag{4.17}$$

In an inertial frame the Riemann curvature tensor is zero even if the metric tensor has non-constant elements. For example, in the absence of gravity and in cylindrical coordinates, the nonzero C symbols have been seen to be, $\Gamma^2_{12} = 1/\rho$, $\Gamma^1_{22} = -\rho$. Thus,

$$\begin{aligned}
R &= g^{\bar{\mu}'\bar{\nu}'}R_{\bar{\mu}'\bar{\nu}'} = g^{\bar{\mu}'\bar{\mu}'}R_{\bar{\mu}'\bar{\mu}'}, \\
R_{\bar{\mu}'\bar{\mu}'} &= R^{\bar{\xi}'}{}_{\bar{\mu}'\bar{\xi}'\bar{\mu}'} = \Gamma^{\bar{\chi}'}_{\bar{\mu}'\bar{\mu}'}\Gamma^{\bar{\xi}'}_{\bar{\xi}'\bar{\chi}'} - \Gamma^{\bar{\chi}'}_{\bar{\mu}'\bar{\xi}'}\Gamma^{\bar{\xi}'}_{\bar{\mu}'\bar{\chi}'} + \Gamma^{\bar{\xi}'}_{\bar{\mu}'\bar{\mu}'},_{\bar{\xi}'} - \Gamma^{\bar{\xi}'}_{\bar{\mu}'\bar{\xi}'},_{\bar{\mu}'}, \\
0 &= R_{00} = R_{33}, \\
R_{11} &= -\Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{12},_1 = -\rho^{-2} - \rho^{-1},_\rho = 0, \\
R_{22} &= \Gamma^1_{22}\Gamma^2_{21} - \Gamma^1_{22}\Gamma^2_{21} - \Gamma^2_{21}\Gamma^1_{22} + \Gamma^1_{22},_1 = 1 - 1 = 0.
\end{aligned}$$

In problem three, it is shown that if constrained to a cylindrical surface where ρ=constant, the C symbols need recalculating, but there is still no curvature. In the case of spherical coordinates the curvature vanishes in the unconstrained free space of an inertial frame, but not if constrained to a spherical surface.

Return to the expression for the completely covariant curvature tensor in a locally inertial frame and take a partial derivative,

$$R_{\bar{\nu}\bar{\beta}\bar{\gamma}\bar{\lambda}} = (g_{\bar{\lambda}\bar{\nu}},_{\bar{\beta}},_{\bar{\gamma}} - g_{\bar{\beta}\bar{\lambda}},_{\bar{\nu}},_{\bar{\gamma}} - g_{\bar{\gamma}\bar{\nu}},_{\bar{\beta}},_{\bar{\lambda}} + g_{\bar{\beta}\bar{\gamma}},_{\bar{\nu}},_{\bar{\lambda}})/2,$$
$$R_{\bar{\nu}\bar{\beta}\bar{\gamma}\bar{\lambda}},_{\bar{\mu}} = (g_{\bar{\lambda}\bar{\nu}},_{\bar{\beta}},_{\bar{\gamma}},_{\bar{\mu}} - g_{\bar{\beta}\bar{\lambda}},_{\bar{\nu}},_{\bar{\gamma}},_{\bar{\mu}} - g_{\bar{\gamma}\bar{\nu}},_{\bar{\beta}},_{\bar{\lambda}},_{\bar{\mu}} + g_{\bar{\beta}\bar{\gamma}},_{\bar{\nu}},_{\bar{\lambda}},_{\bar{\mu}})/2,$$
$$R_{\bar{\nu}\bar{\beta}\bar{\mu}\bar{\gamma}},_{\bar{\lambda}} = (g_{\bar{\gamma}\bar{\nu}},_{\bar{\beta}},_{\bar{\mu}},_{\bar{\lambda}} - g_{\bar{\beta}\bar{\gamma}},_{\bar{\nu}},_{\bar{\mu}},_{\bar{\lambda}} - g_{\bar{\mu}\bar{\nu}},_{\bar{\beta}},_{\bar{\gamma}},_{\bar{\lambda}} + g_{\bar{\beta}\bar{\mu}},_{\bar{\nu}},_{\bar{\gamma}},_{\bar{\lambda}})/2,$$
$$R_{\bar{\nu}\bar{\beta}\bar{\lambda}\bar{\mu}},_{\bar{\gamma}} = (g_{\bar{\mu}\bar{\nu}},_{\bar{\beta}},_{\bar{\lambda}},_{\bar{\gamma}} - g_{\bar{\beta}\bar{\mu}},_{\bar{\nu}},_{\bar{\lambda}},_{\bar{\gamma}} - g_{\bar{\lambda}\bar{\nu}},_{\bar{\beta}},_{\bar{\mu}},_{\bar{\gamma}} + g_{\bar{\beta}\bar{\lambda}},_{\bar{\nu}},_{\bar{\mu}},_{\bar{\gamma}})/2,$$
$$0 = R_{\bar{\nu}\bar{\beta}\bar{\gamma}\bar{\lambda}},_{\bar{\mu}} + R_{\bar{\nu}\bar{\beta}\bar{\mu}\bar{\gamma}},_{\bar{\lambda}} + R_{\bar{\nu}\bar{\beta}\bar{\lambda}\bar{\mu}},_{\bar{\gamma}} = R_{\bar{\nu}\bar{\beta}\bar{\gamma}\bar{\lambda}};_{\bar{\mu}} + R_{\bar{\nu}\bar{\beta}\bar{\mu}\bar{\gamma}};_{\bar{\lambda}} + R_{\bar{\nu}\bar{\beta}\bar{\lambda}\bar{\mu}};_{\bar{\gamma}}\,.$$

The last step follows because in the locally inertial frame the partial derivative of a tensor is the covariant derivative. Thus the above holds in any frame and is known as the Bianchi identity,

$$0 = R_{\nu\beta\gamma\lambda};_{\mu} + R_{\nu\beta\mu\gamma};_{\lambda} + R_{\nu\beta\lambda\mu};_{\gamma}\,. \tag{4.18}$$

Since the covariant derivative of the metric tensor is zero, we have from the Bianchi identity,

$$\begin{aligned}0 &= g^{\nu\gamma}(R_{\nu\beta\gamma\lambda};_{\mu} + R_{\nu\beta\mu\gamma};_{\lambda} + R_{\nu\beta\lambda\mu};_{\gamma}),\\ &= (g^{\nu\gamma}R_{\nu\beta\gamma\lambda});_{\mu} + (g^{\nu\gamma}R_{\nu\beta\mu\gamma});_{\lambda} + (g^{\nu\gamma}R_{\nu\beta\lambda\mu});_{\gamma}\end{aligned}$$

Using Eqs. (4.11) and (4.13)–(4.18), this leads to,

$$\begin{aligned}0 &= R^{\gamma}{}_{\beta\gamma\lambda};_{\mu} + R^{\gamma}{}_{\beta\mu\gamma};_{\lambda} + R^{\gamma}{}_{\beta\lambda\mu};_{\gamma} = R_{\beta\lambda};_{\mu} - R^{\gamma}{}_{\beta\gamma\mu};_{\lambda} + R^{\gamma}{}_{\beta\lambda\mu};_{\gamma}\,,\\ &= R_{\beta\lambda};_{\mu} - R_{\beta\mu};_{\lambda} + R^{\gamma}{}_{\beta\lambda\mu};_{\gamma} = g^{\beta\lambda}(R_{\beta\lambda};_{\mu} - R_{\beta\mu};_{\lambda} - R^{\gamma}{}_{\beta\mu\lambda};_{\gamma}),\\ &= R;_{\mu} - R^{\lambda}{}_{\mu};_{\lambda} - R^{\gamma}{}_{\mu};_{\gamma} = \delta^{\gamma}{}_{\mu}R;_{\gamma} - 2R^{\gamma}{}_{\mu};_{\gamma}\,,\\ &= (\delta^{\gamma}{}_{\mu}R - 2R^{\gamma}{}_{\mu});_{\gamma} = g^{\mu\nu}(\delta^{\gamma}{}_{\mu}R - 2R^{\gamma}{}_{\mu});_{\gamma} = (g^{\gamma\nu}R - 2R^{\gamma\nu});_{\gamma}\,.\end{aligned}$$

The Einstein tensor, $G^{\gamma\nu}$, is defined

$$G^{\gamma\nu} \equiv R^{\gamma\nu} - g^{\gamma\nu}R/2 = G^{\nu\gamma}, \tag{4.19}$$
$$G^{\gamma\nu};_{\gamma} = 0. \tag{4.20}$$

This tensor is symmetric and is an essential quantity for solving problems with gravity.

Problems

1. Show that, $g^{\mu\nu}g_{\nu\chi},_{\beta} = -g^{\mu\nu},_{\beta}\,g_{\nu\chi}$, $g^{\mu\nu},_{\beta} = -(\Gamma^{\mu}_{\chi\beta}g^{\chi\nu} + \Gamma^{\nu}_{\chi\beta}g^{\chi\mu})$, and $\Gamma^{\nu}_{\mu\nu} = 0.5g^{\nu\chi}g_{\nu\chi},_{\mu}$.

2. Show that there are only twenty independent elements of, $R_{\alpha\beta\mu\nu}$, that can be nonzero. Show that the contraction, $R_{\mu\nu} = R^{\chi}{}_{\mu\chi\nu}$, is the only independent contraction of the Riemann curvature tensor.
3. Calculate the C symbols and curvature, R, in cylindrical coordinates in an inertial frame, when, $x^1 = \rho = a$, constant. Repeat the calculation for spherical coordinates. In that case show that in unconstrained space the curvature vanishes, but on the surface of a sphere, where, $x^3 = r = a$, constant, the curvature is finite.
4. In spherical coordinates, the Wormhole metric has nonzero elements: $g_{00} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = a^2 + (r)^2$ and $g_{\phi\phi} = \sin^2\theta g_{\theta\theta}$, where, a, is a constant. Calculate the curvature, R, and show that it is negative.
5. Evaluate, $G^{\mu\nu}{}_{,\mu}$ and $G_{\mu\nu}{}_{,\mu}$, in terms of the metric, Ricci tensor, $R_{\mu\nu}$, and $R^{\mu\nu}$, and the C symbols.
6. It shall be seen that Einstein's equation for the metric is, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$, where, $G_{\mu\nu}$, is the Einstein tensor and, $T_{\mu\nu}$, is the energy momentum tensor. The factor, 8π, is required for agreement with Newtonian physics, where the latter is valid. The constant, Λ, is called the cosmological constant. It can be neglected for solar system problems. Einstein introduced it to make a static universe. When data showed an expanding universe, it was set to zero and Einstein called it his greatest error. As shall be seen, the universe is expanding too rapidly for, $\Lambda = 0$. It must be positive and is now the dominant contribution to the energy in the universe. In vacuum, $T_{\mu\nu} = 0$, what is, R? In cosmology, a useful model, is to describe the universe as a perfect fluid, for which, $T_{\mu\nu} \neq 0$. On a solar system scale, earth can be thought to be moving in vacuum under the influence of the metric set up by the sun and, $T_{\mu\nu} = 0$. Show that it is impossible for the worm hole metric of problem four to exist in vacuum? Calculate all the elements of, $T_{\mu\nu}$, for that metric including the cosmological constant.
7. The Robertson-Walker metric provides a good description of the universe,

$$(d\tau)^2 = (dt)^2 - Q^2(t)\left[\frac{(dr)^2}{1 - k(r)^2} + (r)^2\left([d\theta]^2 + \sin^2\theta\,[d\phi]^2\right)\right],$$

where, $Q(t)$, the universal scale factor and the constant, $k = 0, \pm 1$, are obtained from experimental data. All observers use the same cosmic time, t, and any origin yields the same physics. Find the curvature if we live in an expanding universe, $Q(t) > 0$ and $\frac{dQ(t)}{dt} > 0$.

8. For the metric of problem seven, calculate the elements of the Einstein tensor, $G_{\mu\nu}$.
9. For the metric of chapter three, problem five, the nonzero C symbols were calculated. It will be seen that such a metric is that found by Schwarzschild for a planet moving in the sun's gravity. He derived the proper forms for $\Phi(r)$ and $\Delta(r)$,

$$(d\tau)^2 = \exp[2\Phi(r)](dt)^2 \\ - \left[\exp[2\Delta(r)](dr)^2 + (r)^2 \left([d\theta]^2 + \sin^2\theta\,[d\phi]^2\right)\right].$$

   Show that for this metric there are only six independent nonzero values of, $R_{\mu\xi\chi\nu}$, and calculate them.
10. Calculate the elements of the Ricci tensor, $R_{\mu\nu}$, from the results of problem nine. Then calculate the curvature, R and $G_{\mu\nu}$.

Chapter 5

Gravity and General Relativity

5.1 Review of Newtonian Gravity

Gravitation, the weakest force, is due to, and acts on all forms of energy. In Newtonian gravity, the relativistic mass, M is the pertinent variable. The other known forces are explained with quantum mechanics as due to the exchange of particles called force carriers. The weak force is experienced by all particles with rest mass. The force carriers are the $W^{\pm}, Z^0$ bosons. Photons are the electromagnetic force carriers and particles with charge or higher order moments experience this interaction. Particles made up of quarks like pions, protons, and neutrons experience the strong force whose carriers are gluons. It should come as no surprise that when gravity and quantum mechanics are connected another force carrier, the graviton, appears. Of course Newton's theory contains none of such concepts.

Newton's theory of gravity is very similar to the classical theory of electrostatics. As students are probably more familiar with mathematics like the Gauss law and the divergence theorem from electrostatics, these two theories are compared so that the desired gravitational result is simply obtained. For this short section, MKS units are used so the equations will be familiar.

Both theories postulate an action at a distance force on each of two point particles, (1,2). If, $\vec{r}$, is the 3-vector that points from body 2 to body 1 with, r, being the distance between the bodies, the forces on the bodies are,

$$\vec{F}_{E,G}(1) = M(1)\vec{a}(1) = N_{E,G} f_{E,G}(1) f_{E,G}(2)\vec{r}/r^3 = -\vec{F}(2) = -M(2)\vec{a}(2),$$
$$N_E = (4\pi\epsilon_0)^{-1}, \quad f_E = Q, \quad N_G = -G, \quad f_G = M.$$

In electrostatics, charges, Q, can be positive or negative and like charges repel while unlike charges attract. Mass, M, is always positive and the gravitational force is attractive. The normalization constants, N_E and N_G, reflect the strength of the forces.

In electrostatics an electric field, $\vec{E} = -\vec{\nabla}\Psi_E$, can be defined as the negative gradient of the electrostatic potential, Ψ_E. As the gravitational force has the same spatial form, the same may be done for gravity, $\vec{G} = -\vec{\nabla}\Psi_G$, where,

$$\begin{aligned}
\vec{E}(1) &= \vec{F}_E(1)/Q(1) = (4\pi\epsilon_0)^{-1}Q(2)\vec{r}/r^3 = -\vec{\nabla}\Psi_E(1),\\
\Psi_E(1) &= (4\pi\epsilon_0)^{-1}Q(2)/r,\\
\vec{G}(1) &= \vec{F}_G(1)/M(1) = -GM(2)\vec{r}/r^3 = -\vec{\nabla}\Psi_G(1),\\
\Psi_G(1) &= -GM(2)/r.
\end{aligned}$$

where $\Psi_{E,G}(1)$ is the electrostatic, gravitational potential at $\vec{r}_1$, because of the presence of the body at $\vec{r}_2$. If there are many bodies, $\Psi_{E,G}$ is the potential at a point in space due to charges or masses at other points.

Due to the inverse square nature of the force, there is a Gauss law for both forces, and making use of the divergence theorem,

$$\begin{aligned}
\int_V \vec{\nabla}\cdot\vec{\nabla}\Psi_{E,G}dV &= \int_S \vec{\nabla}\Psi_{E,G}\cdot\hat{n}dS = -\int_S -\vec{\nabla}\Psi_{E,G}\cdot\hat{n}dS,\\
&= -4\pi N_{E,G}f_{E,G(tot)} = -4\pi N_{E,G}\int_V \rho_{Q,M}dV,\\
\nabla^2\Psi_E &= -\rho_Q/\epsilon_0,\\
\nabla^2\Psi_G &= 4\pi G\rho_M = 4\pi\rho_M \quad \text{(in natural units).} \qquad (5.1)
\end{aligned}$$

This result in natural units where $\rho_M = \mathrm{m}^{-2}$, will be useful in obtaining the metric element, g_{00}, when gravity is weak. In the above equations, $\hat{n}$ is the unit 3-vector normal to surface, S, bounding volume, V, and, ρ_Q and ρ_M, are the charge and mass densities within V, the source of the field. The quantities, $f_{E(tot)}$ and $f_{G(tot)}$, are the total charge and mass inside the volume. For the volume integral, $\nabla^2\Psi$ is evaluated at all points inside V, and for the surface integral, $\vec{\nabla}\Psi$ is evaluated at all points on the surface, S. Note that for point charges or masses $f_{E,G}(i)$, the charge or mass density can be expressed in terms of the Dirac delta function,

$$\rho_{E,G} = \sum_i f_{E,G}(i)\delta(\vec{r} - \vec{r}(i)).$$

An item of importance is the weakness of the gravitational interaction. At the surface of the sun, $(M/R)_s = 1.484\times10^3/0.696\times10^9 = 2.13\times10^{-6}$. At the surface of earth $(M/R)_e = 3\times10^{-4}(M/R)_s$. For a white dwarf, with the mass of the sun, but a radius one hundred times smaller than that of the sun, $(M/R)_{wd} \approx 2 \times 10^{-4}$. For a neutron star, $M_n = 1.4 M_s$, $R_n \approx 14$ km $= 20 \times 10^{-6} R_s$, $(M/R)_n = 0.15$ and here GR is needed for accuracy. Aside from such compact objects and black holes, GR effects are minute.

5.2 Weak Gravity in GR

The language used in Newtonian mechanics is inappropriate. One should not say that mass, M, is moving in the gravitational potential produced by other masses, but rather, all the energy in the universe has caused there to be a non-flat metric in which M moves. For the earth, the sun is the main cause of the non-flatness and the metric is stationary, $g_{\mu\nu},_0 = 0$. The earth moves slowly compared with the speed of light and the GR equations of motion yield,

$$\begin{aligned}
0 &= \frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu \qquad U^i \approx 0, \\
0 &= \frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{00} U^0 U^0, \\
&= \frac{dU^\alpha}{d\tau} + g^{\alpha\chi}[g_{0\chi},_0 + g_{0\chi},_0 - g_{00},_\chi] \left(\frac{dx^0}{d\tau}\right)^2 \Big/ 2, \\
&= \frac{dU^\alpha}{d\tau} - g^{\alpha\chi} g_{00},_\chi \left(\frac{dt}{d\tau}\right)^2 \Big/ 2. \qquad (5.2)
\end{aligned}$$

Gravity is a weak force unless you are in the vicinity of truly massive, compact object. So whatever $g_{\mu\nu}$ is, it is very close to $\eta_{\mu\nu}$. When the term **weak gravity** is specifically used, it means one is not seeking an exact solution, but rather an approximate one where second order deviations from $\eta_{\mu\nu}$ are neglected. As there is only a slight change from $\eta_{\mu\nu}$, rectangular coordinates are required and the following definitions hold,

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1, \qquad (5.3)$$

$$g^{\mu\nu} \equiv \eta^{\mu\nu} + f^{\mu\nu}, \qquad |f^{\mu\nu}| \ll 1, \text{ and solving for } f^{\mu\nu}, \qquad (5.4)$$

$$\begin{aligned}
g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_{\ \nu} &= (\eta^{\mu\alpha} + f^{\mu\alpha})(\eta_{\alpha\nu} + h_{\alpha\nu}), \\
&\approx \eta^{\mu\alpha}\eta_{\alpha\nu} + \eta^{\mu\alpha} h_{\alpha\nu} + \eta_{\alpha\nu} f^{\mu\alpha},
\end{aligned}$$

$$\delta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \eta^{\mu\alpha}h_{\alpha\nu} + \eta_{\alpha\nu}f^{\mu\alpha},$$
$$f^{\mu\chi} = \eta^{\chi\nu}\eta_{\alpha\nu}f^{\mu\alpha} = -\eta^{\chi\nu}\eta^{\mu\alpha}h_{\alpha\nu} = -h^{\mu\chi}. \tag{5.5}$$

For the motion of the earth about the sun, Newtonian mechanics provides an adequate description that the GR prediction must agree with. In Eq. (5.2), each value of α has to be examined,

$$\begin{aligned}
0 &\approx \frac{dU^0}{d\tau} - \eta^{0\chi}h_{00},_{\chi}\left(\frac{dt}{d\tau}\right)^2\Big/2,\\
&= \frac{dU^0}{d\tau} + h_{00},_0\left(\frac{dt}{d\tau}\right)^2\Big/2 = \frac{dU^0}{d\tau},\\
U^0 &= \frac{dx^0}{d\tau} = \frac{dt}{d\tau} = K = 1, \quad t = \tau,\\
0 &= \frac{dU^i}{d\tau} - \eta^{i\chi}h_{00},_{\chi}/2,\\
&\approx \frac{d^2x^i}{dt^2} - h_{00},_i/2 = a^i - h_{00},_i/2.
\end{aligned}$$

The constant choice, $K = 1$, leads to agreement with Newtonian gravity. Let M be the mass of moving object and M' be the source mass that provides the metric. The last equation yields,

$$\begin{aligned}
Mh_{00},_i/2 &= Ma^i = -M\Psi_G,_i,\\
h_{00}/2 &= -\Psi_G + K' = -\Psi_G,\\
h_{00} &= -2\Psi_G = 2M'/r.
\end{aligned} \tag{5.6}$$

Taking $K' = 0$ makes $-2\Psi_G = h_{00} = 0$, at $r = \infty$, and is the obvious choice to provide agreement with Newtonian gravity.

5.3 Gravitational Red Shift

Now that g_{00} has been obtained in the context of GR for a stationary field, the question of how gravity affects clocks can be reconsidered in a completely GR context. A review of the material in section 2.8 will prove worthwhile. The present discussion works not only for the weak gravity metric under consideration, but for future exact metrics.

So suppose, as in Fig. 5.1, a Helium source at, r_2, emits photons that are observed by experimenters at rest at, r_1 and r_2, in the metric set up by

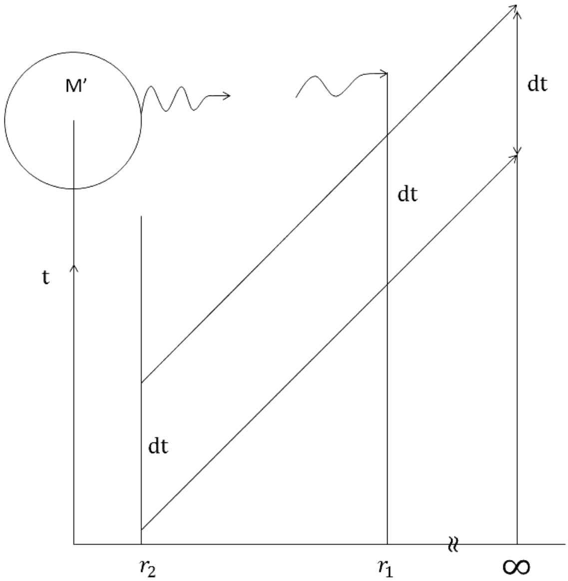

Fig. 5.1 Identical sources at, $r_{1,2}$, in the metric set up by mass, M', have their frequencies measured by observers at rest at those positions. The frequency of the light that traveled from, r_2 to r_1, is also determined.

source mass, M'. The observer at, r_1, in weaker gravity, also has a Helium source. The desired prediction that can be compared with measurement is the ratio, $\nu_{2,1}/\nu_{2,2} = \nu_{2,1}/\nu_{1,1}$, where the numerator is the frequency of the light from r_2, measured at r_1, and the denominator is the frequency from identical sources measured at the positions of the sources. The first method of predicting this ratio makes use of invariants,

$$
\begin{aligned}
[P^{\mu}]_{photon}[U_{\mu}]_{obs.} &= [P^{\mu}]_{photon}\left[g_{\mu\nu}\frac{dx^{\nu}}{d\tau}\right]_{obs.}, \\
&= [P^{0}]_{photon}\left[g_{00}\frac{dt}{d\tau}\right]_{obs.} = [P^{0}]_{photon}([-g_{00}]_{obs.})^{1/2}, \\
&= h\nu_{2,2}(-g_{00}(2))^{1/2} = h\nu_{2,1}(-g_{00}(1))^{1/2}, \\
&= h\nu_{2,2}(1-2M'/r_2)^{1/2} = h\nu_{2,1}(1-2M'/r_1)^{1/2}, \\
\nu_{2,1}/\nu_{1,1} &= \nu_{2,1}/\nu_{2,2} = (1-2M'/r_2)^{1/2}(1-2M'/r_1)^{-1/2}, \\
&\approx 1 - M'(1/r_2 - 1/r_1). \qquad (5.7)
\end{aligned}
$$

A second way to get this result is to note, from Fig. 5.1, that the world lines of successive wave crests traveling from r_2 to r_1, and then on to a far away point, traveled along identical geodesics with constant speed. Thus, the journey times are the same. The far away, at rest observer, is not experiencing gravity. That observer measures the proper time period between crests to be, $d\tau_\infty = dt$, the coordinate time period. Working backwards along the world lines of the crests, one can see that at $r(1,2)$, the coordinate time periods between crests remains, dt. However, the proper time periods are the inverses of the frequencies, thus,

$$\frac{d\tau_{2,2}}{d\tau_{2,1}} = \frac{dt_{2,2}}{dt_{2,1}}\left(\frac{-g_{00}(2)}{-g_{00}(1)}\right)^{1/2} = \left(\frac{-g_{00}(2)}{-g_{00}(1)}\right)^{1/2},$$

$$\frac{\nu_{2,1}}{\nu_{2,2}} = \frac{\nu_{2,1}}{\nu_{1,1}} = \left(\frac{-g_{00}(2)}{-g_{00}(1)}\right)^{1/2} \approx 1 - M'(1/r_2 - 1/r_1).$$

Exactly as in Eq. (5.7).

Since, $r_1 > r_2$, gravity is weaker at r_1. Then, $\nu_{2,1}/\nu_{1,1} < 1$, and $\lambda_{2,1}/\lambda_{1,1} > 1$. This is known as the gravitational red shift. It is the same result obtained in section 2.8 using energy conservation and relativistic mass. That worked because weak gravity is operational. In addition, $d\tau_{2,2}/d\tau_{2,1} < 1$. Thus, the proper time tick rate for a clock depends on the position. It is greater for the clock experiencing weaker gravity.

5.4 Einstein's Field Equations

In Newton's theory, the gravitational potential, Ψ, is related to the source, the mass density, ρ, via Eq. (5.1). For a source point mass, M', the solution is, $\Psi = -M'/r$, where, r, is the distance from the mass. Einstein's theory is based on curved spacetime. Therefore, one must determine the metric. The equation that relates the metric and the source must be covariant. That is it must be a tensor equation so all observers would write the same equation, using their own coordinates. Instead of a mass density, that is related to the, T^{00}, element of the stress-energy or energy-momentum tensor, seen by a particular observer, a covariant theory requires the entire tensor to be the source. So the correct equation would look like,

$$O(g^{\mu\nu}) = aT^{\mu\nu}, \tag{5.8}$$

where a is a constant and O is a differential operator such that the general properties of both sides of the above equation are the same. Those are: each side is a symmetric tensor, and since momentum is conserved, each tensor has vanishing divergence, $T^{\mu\nu};_{\nu} = 0$. The tensor, $T^{\mu\nu}$, is discussed in detail later, when it is needed. In the solar system, planets move in vacuum in the metric set up by the sun, so that $T^{\mu\nu} = 0$, and the constant, a, doesn't matter.

For curved spacetime, O will at least contain terms with the metric tensor and its first and second partial derivatives. That's just what is found in the Ricci tensor, $R^{\mu\nu}$. A nonzero $R^{\mu\nu}$ means finite curvature and the presence of gravity. Einstein struggled with this problem for many years before finally finding the simplest form,

$$O(g^{\mu\nu}) = R^{\mu\nu} + a' g^{\mu\nu} R + \Lambda g^{\mu\nu} = aT^{\mu\nu},$$

where a' and Λ are constants.

The constant, Λ, is known as the cosmological constant. It must be determined by observation. Einstein originally included it so that the universe would be static. When an expanding universe was discovered, he discarded it, considering its inclusion his greatest error. Present day cosmology requires it, and in fact, it is positive and the driver of an accelerating expansion that will never cease. That is as the universe gets bigger its effect gets stronger. However, for solar system predictions it is extremely small and may be neglected.

From momentum conservation,

$$0 = T^{\mu\nu};_{\nu} = [(R^{\mu\nu} + a' g^{\mu\nu} R + \Lambda g^{\mu\nu});_{\nu}] = [(R^{\mu\nu} + a' g^{\mu\nu} R);_{\nu}].$$

However, when the Einstein tensor, $G^{\mu\nu}$, was derived, Eqs. (4.19) and (4.20) yielded,

$$G^{\mu\nu} = R^{\mu\nu} - g^{\mu\nu} R/2, \quad G^{\mu\nu};_{\nu} = 0.$$

So take $a' = -1/2$, then $O(g^{\mu\nu}) = G^{\mu\nu} + \Lambda g^{\mu\nu}$. The Einstein field equations are then

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = aT^{\mu\nu} = 8\pi T^{\mu\nu}. \tag{5.9}$$

The value $a = 8\pi$ is required in order that the predictions of GR agree with the Newtonian predictions, where the latter are applicable, see problem three. In the empty space of our solar system, there is no energy-momentum tensor, so

$$0 = T^{\mu\nu} = G^{\mu\nu} = R^{\mu\nu}. \tag{5.10}$$

More complicated field equations have been proposed, but the one above, selected by Einstein for its simplicity and beauty has withstood every experimental test.

5.5 Schwarzschild Solution

Almost immediately after Einstein introduced the field equations, K. Schwarzschild found an exact solution. An English translation of his paper is available [Schwarzschild (1916)]. The case in point was the one that was considered with weak gravity, the metric produced by a static, spherically symmetric, massive object in vacuum. The metric tensor won't depend on t, but will depend on $\vec{r},\ d\vec{r}$, such that it has rotational invariance. Time independence leads to energy conservation and rotational invariance leads to conservation of certain angular momentum components. Thus, it pays to work in spherical coordinates. So for the rest of this chapter, r^p means $(r)^p$, and not the pth component of the position vector.

Try the most general rotationally invariant form for the proper time element,

$$\begin{aligned}(d\tau)^2 &= -g_{\mu\nu}dx^\mu dx^\nu,\\ &\equiv A(r)(dt)^2 - 2B(r)(\vec{r}\cdot d\vec{r})dt - C(r)(\vec{r}\cdot d\vec{r})^2 - D(r)d\vec{r}\cdot d\vec{r},\\ &= A(r)(dt)^2 - 2B(r)rdrdt - [C(r)r^2 + D(r)](dr)^2\\ &\quad -D(r)r^2[(d\theta)^2 + (\sin\theta d\phi)^2]. \end{aligned} \tag{5.11}$$

One can eliminate the $dtdr$ term with the following transformation,

$$t \equiv t' - E(r), \qquad dE(r) \equiv -rdrB(r)/A(r). \tag{5.12}$$

It is then easy to show, see problem four, that this leads to,

$$\begin{aligned}(d\tau)^2 &= A(r)(dt')^2 - [(rB(r))^2/A(r) + C(r)r^2 + D(r)](dr)^2\\ &\quad -D(r)r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2],\\ &\equiv A(r)(dt')^2 - F(r)(dr)^2 - r^2D(r)[(d\theta)^2 + (\sin\theta d\phi)^2]. \end{aligned} \tag{5.13}$$

A final transform redefines r and allows the proper time to be cast in a form where the metric tensor is diagonal, see problem four,

$$r'^2 \equiv r^2D(r), \tag{5.14}$$

$$\begin{aligned}(d\tau)^2 &\equiv \exp[2\Phi(r')](dt')^2 - \exp[2\Delta(r')](dr')^2\\ &\quad -r'^2[(d\theta)^2 + (\sin\theta d\phi)^2]. \end{aligned} \tag{5.15}$$

From now on the primes will be dropped. The last form has been encountered in many of the problems. Since the metric tensor is diagonal,

$g^{\mu\mu} = 1/g_{\mu\mu}$. In chapter three, problem five the C symbols were calculated and the nonzero ones are:

$$\begin{aligned}
\Gamma^r_{rr} &= \Delta,_r\,, \\
\sin^2\theta\Gamma^r_{\theta\theta} &= -r\exp[-2\Delta]\sin^2\theta = \Gamma^r_{\phi\phi}, \\
\Gamma^r_{tt} &= \exp[-2\Delta]\exp[2\Phi]\Phi,_r\,, \\
\Gamma^\theta_{r\theta} &= \Gamma^\theta_{\theta r} = r^{-1}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\
\Gamma^\phi_{r\phi} &= \Gamma^\phi_{\phi r} = r^{-1}, \qquad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta, \\
\Gamma^t_{rt} &= \Gamma^t_{tr} = \Phi,_r\,.
\end{aligned} \tag{5.16}$$

In chapter four, problem ten the nonzero Einstein tensor elements were found:

$$\begin{aligned}
G_{tt} &= r^{-2}\exp[2\Phi][r(1-\exp[-2\Delta])],_r\,, \\
G_{\theta\theta} &= (r)^2\exp[-2\Delta][\Phi,_r,_r + (\Phi,_r)^2 - \Phi,_r\Delta,_r + (r)^{-1}(\Phi,_r - \Delta,_r)], \\
G_{\phi\phi} &= \sin^2\theta G_{\theta\theta}, \\
G_{rr} &= -r^{-2}(\exp[2\Delta]-1) + 2r^{-1}\Phi,_r\,.
\end{aligned} \tag{5.17}$$

In empty space, $G_{\mu\nu} = 0$,

$$\begin{aligned}
0 = G_{tt} &= r^{-2}\exp[2\Phi][r(1-\exp[-2\Delta])],_r\,, \text{ so,} \\
0 &= [r(1-\exp[-2\Delta])],_r\,, \\
b &= r(1-\exp[-2\Delta]), \\
\exp[-2\Delta] &= 1 - b/r, \\
0 = G_{rr} &= -r^{-2}(\exp[2\Delta]-1) + 2(r)^{-1}\Phi,_r\,, \text{ so,} \\
2r^{-1}\Phi,_r &= r^{-2}(1/(1-b/r)-1) = r^{-2}(b/r)/(1-b/r), \\
2\Phi,_r &= r^{-2}b/(1-b/r) = (\ln[1-b/r]),_r\,, \\
2\Phi &= \ln[(1-b/r)b'], \qquad \exp[2\Phi] = b'(1-b/r).
\end{aligned}$$

The last equation means, $g_{00} = -b'(1-b/r) = b'(-1+b/r)$. However, the weak gravity case yielded, $g_{00} = -1 + 2M'/r$. The Schwarzschild and weak gravity results must agree in the case that gravity is weak, so the constants are $b' = 1$ and $b = 2M'$, where M' is the relativistic mass of the metric source. Then,

$$\exp[2\Phi] = \exp[-2\Delta] = (1 - 2M'/r). \tag{5.18}$$

If the cosmological constant, Λ, was included, see problem five, it is still simple to calculate the functions $\exp[2\Phi]$ and $\exp[2\Delta]$. Then one can show that in the weak gravity approximation, $M'/r \ll 1$, $\Lambda \ll 1$; there is a repulsive Newtonian force due to Λ. This makes sense as the cosmological constant is driving the universal expansion. However, as will be seen in the problem, its contribution on a solar system scale cannot be observed.

5.6 Conserved Quantities — Massive Particles

Knowledge of the metric tells us what quantities, if any, are conserved. Such knowledge is very helpful in solving the equations of motion. Light has one constant of the motion in a varying gravitational field, its speed. Eq. (4.3) was obtained from parallel transport and for a particle of rest mass, m, $d\tau \neq 0$. Using $g_{\mu\alpha};_\nu = 0$, and renaming summed over indexes, this equation leads to

$$\begin{aligned}
0 &= W^\nu W^\mu;_\nu\,, \\
&= m^2 U^\nu U^\mu;_\nu = mU^\nu (mU^\mu);_\nu = P^\nu P^\mu;_\nu\,, \\
&= g_{\mu\alpha} P^\nu P^\mu;_\nu = P^\nu (g_{\mu\alpha} P^\mu);_\nu\,, \\
&= P^\nu P_\alpha;_\nu = P^\nu (P_{\alpha},_\nu - P_\beta \Gamma^\beta_{\alpha\nu}), \\
&= m\frac{dx^\nu}{d\tau} P_{\alpha},_\nu - P^\nu P_\beta \Gamma^\beta_{\alpha\nu} = m\frac{dP_\alpha}{d\tau} - P^\nu P_\beta \Gamma^\beta_{\alpha\nu}, \\
\frac{dP_\alpha}{d\tau} &= P^\nu P_\beta \Gamma^\beta_{\alpha\nu}/m = P^\nu P_\beta g^{\beta\chi}(g_{\alpha\chi},_\nu + g_{\chi\nu},_\alpha - g_{\alpha\nu},_\chi)/(2m), \\
&= (P^\nu P^\chi g_{\alpha\chi},_\nu + P^\chi P^\nu [g_{\chi\nu},_\alpha - g_{\alpha\nu},_\chi])/(2m), \\
&= P^\chi P^\nu (g_{\alpha\nu},_\chi + g_{\chi\nu},_\alpha - g_{\alpha\nu},_\chi)/(2m), \\
&= P^\chi P^\nu g_{\chi\nu},_\alpha/(2m).
\end{aligned} \tag{5.19}$$

So if $g_{\chi\nu},_\alpha = 0$, then P_α is constant along the geodesic. In a stationary metric, $g_{\chi\nu},_0 = 0$, so P_0 is constant. In the case of weak gravity and low speeds this means the total energy is constant.

The constancy of energy can be illustrated to lowest order in small quantities. Use the fact that $|\vec{P}| \ll m$ so that $h_{ij}P^iP^j/m^2$ and $h_{ii}|\vec{P}|^2/m^2$

can be neglected,

$$
\begin{aligned}
m^2 &= -g_{\mu\nu}P^\mu P^\nu, \\
&= -(-1+h_{00})(P^0)^2 - [(1+h_{ii})|\vec{P}|^2 + 2h_{ij}P^iP^j], \\
1 &\approx (1-h_{00})(P^0/m)^2 - (|\vec{P}|/m)^2, \\
&= (1+2M'/r)(P^0/m)^2 - (|\vec{P}|/m)^2, \\
P^0/m &= (1+(|\vec{P}|/m)^2)^{1/2}(1+2M'/r)^{-1/2}, \\
&\approx (1+(|\vec{P}|/m)^2/2)(1-M'/r), \\
P^0 &\approx m - mM'/r + |\vec{P}|^2/(2m) = RE + PE + KE = E, \\
P_0 &= g_{00}P^0 \approx -P^0 = -E, \text{ constant.}
\end{aligned}
$$

Outside a spherically symmetric massive body there will be axial symmetry. One can find coordinates such that the $g_{\mu\nu}$ are independent of the angle about that axis. One can take the axis such that ϕ is that angle. Using the Schwarzschild metric for a slowly moving particle,

$$
\begin{aligned}
P_\phi &= g_{\phi\phi}P^\phi, \\
&= (r\sin\theta)^2 m\frac{d\phi}{dt},
\end{aligned}
$$

is constant. You recognize this relation as the conservation of angular momentum in Newtonian theory or Kepler's rule that planets sweep out equal areas in equal times.

Problems

1. In the case of weak gravity, find, $h^{00},\ h^{0i},\ h^{ij}$, in terms of, $h_{00},\ h_{0i},\ h_{ij}$.
2. For the static case of weak gravity, and a spherically symmetric source, it was found,

$$
\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \\
g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu}, \quad |h^{\mu\nu}| \ll 1, \\
h_{00} &= -2\Psi_G = 2M'/r,
\end{aligned}
$$

where, r, is the radial coordinate from the gravitational source mass, M'. The result, $h_{0i} = h^{0i} = h_{ij} = h^{ij} = 0$, also holds because otherwise the metric would be direction dependent. In this case, all the h_{ii} have to be equal and nonzero, as no direction is favored,

$$(d\tau)^2_{weak} = (1 - 2M'/r)(dt)^2 - (1 + h_{ii})[(dx)^2 + (dy)^2 + (dz)^2].$$

Compare $(d\tau)^2_{weak}$ with the Schwarzschild result for large r, and obtain, h_{ii}.

3. In the weak gravity case, GR and Newtonian gravity agree. Here one can take a stationary state such that, $\Lambda = U^i = 0$, and, $T_{00} = \rho_M$, as the only nonzero element of, $T_{\mu\nu}$, where, ρ_M is small and spherically symmetric. Show that, $R = a\rho_M$, where, a, is the constant in Eq. (5.9). Show, $R_{00} = a\rho_M/2 = R^i{}_{0i0} = -\nabla^2 h_{00}/2$. Thus, $a = 8\pi$.
4. Start with Eq. (5.11). Apply Eq. (5.12) and show that Eq. (5.13) is obtained. Then apply Eq. (5.14) and show that Eq. (5.15) is obtained.
5. Repeat the calculation of the Schwarzschild metric with the cosmological constant, Λ, included. Find, $\exp[2\Phi]$ and $\exp[-2\Delta]$. You should find that the term with, Λ, is multiplied by, r^2. Thus, even though the data of chapter nine yield, $\Lambda \approx 5 \times 10^{-54}$ m^{-2}, one cannot say that for very large, r, this term is small. However, on a solar system scale, it is negligible. In this region weak gravity is satisfied. Find, h_{00}, Ψ_G, and, $\vec{F}_{Newton}$. Interpret the, Λ, term.
6. Transform the metric of SR to a frame with coordinates, $x^{\mu'}$, where

$$x^0 = [a^{-1} + x^{3'}]\sinh(ax^{0'}) \qquad x^3 = [a^{-1} + x^{3'}]\cosh(ax^{0'}) - a^{-1},$$
$$x^i = x^{i'} \quad i = i' = 1, 2.$$

For small, a, such that, $ax^{0'} = at' \ll 1$, show the transform is to a nonrelativistic uniformly accelerating frame. What are the constants of the motion? Calculate, $(d\tau)^2$, in the accelerating frame without approximation. Find the constants of the motion. In this frame clocks, at, $x^{3'} = 0, h$, are at rest and measure proper times. What is, $d\tau_h/d\tau_0$, and explain the result?

7. Consider the following Schwarzschild-like metric from the expression for, $(d\tau)^2$,

$$(d\tau)^2 = \exp[2\Phi(r)](dt)^2 - (\exp[2\Lambda(r)](dr)^2 + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]).$$

Find all the conserved components of a freely falling particle's covariant momentum vector. Show that if the geodesic begins with, $x^1 = \theta = \pi/2$ and $P^1 = 0$, these values never change.

8. Repeat the calculations of problem seven for the Robertson-Walker metric,

$$(d\tau)^2 = (dt)^2 - Q^2(t)((1-kr^2)^{-1}(dr)^2 + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]).$$

If, $k = 0$, in addition to, $\theta = \pi/2$, and, $P^1(0) = 0$, what can be said about, $P_3 = P_r$?

9. At earth's surface, electrons and positrons of rest energy, 0.511 MeV, annihilate at rest into two photons, $e^- + e^+ \rightarrow \gamma + \gamma$. One photon makes it to a far away static, spherically symmetric compact star that has, $M = 1.5M_s$, and surface radial coordinate, $\bar{R} = 10$ km. Explain why a single gamma ray in the final state is impossible. At the star's surface what is the photon energy in MeV? If the decay occurred at the star, what photon energy would be measured at Earth?

10. For radial motion, show that the derivatives with respect to, τ, of the Schwarzschild coordinates are not the energy and momentum per unit rest mass of a particle, that is, what they are in a locally inertial frame. Show how they are related to those quantities.

11. Calculate the magnitude of the acceleration, $a = (\frac{dU_\mu}{d\tau}\frac{U^\mu}{d\tau})^{1/2}$, in the Schwarzschild metric, for the case, $U^i = 0$. At the surfaces of the earth and sun, show the result is very close to the expected Newtonian result. Suppose you are just outside a static Schwarzschild black hole where, $2M' = R$ and $r = R + \delta$, $\delta/R \ll 1$. Show that in order to have an acceleration, equal to say, g, that, R, would have to be enormous.

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Chapter 6

Solar System Tests of General Relativity

6.1 Equations of Motion

The metric in the solar system is due mainly to the relativistic mass of the sun, $M' = M_s$. The general two body problem in GR has not been solved analytically, so the sun's mass will be taken as much larger than that of the planet considered and the gravitational effects of other planets will be neglected. Also the sun's rotation will be neglected. Under these conditions the metric is that obtained by Schwarzschild. In this chapter, spherical coordinates will be used and $r^p \equiv (r)^p$. The equations of motion involve the C symbols and from Eqs. (5.16)–(5.18),

$$\begin{aligned}
\exp[-2\Delta] &= 1 - 2M'/r, \qquad -2\Delta = \ln[1 - 2M'/r], \\
\exp[2\Phi] &= 1 - 2M'/r, \qquad 2\Phi = \ln[1 - 2M'/r], \\
\Gamma^t_{rt} &= \Phi,_r = (1 - 2M'/r)^{-1}(M'/r^2), \\
\Gamma^r_{rr} &= \Delta,_r = -(1 - 2M'/r)^{-1}(M'/r^2), \\
\Gamma^r_{tt} &= \exp[2\Phi]\exp[-2\Delta]\Phi,_r = (1 - 2M'/r)(M'/r^2), \\
\sin^2\theta\Gamma^r_{\theta\theta} &= -r\sin^2\theta(1 - 2M'/r) = \Gamma^r_{\phi\phi}.
\end{aligned} \tag{6.1}$$

Two constants of the motion are expected for massive particles since $g_{\mu\nu},_0$ and $g_{\mu\nu},_2 = 0$. However, the motion of both photons and massive particles is desired. So the equations of motion, Eq. (3.25), are written in terms of an affine parameter, q,

$$0 = \frac{d^2x^\mu}{dq^2} + \Gamma^\mu_{\chi\nu}\frac{dx^\chi}{dq}\frac{dx^\nu}{dq}.$$

The value of this parameter will emerge from the solution of the equations.

In chapter three, problem twelve, it was found that a planar solution, $\theta = \pi/2$, is allowed. Thus, for the coordinate, $x^2 = \phi$,

$$
\begin{aligned}
0 &= \frac{d^2\phi}{dq^2} + 2\Gamma^{\phi}_{r\phi}\frac{dr}{dq}\frac{d\phi}{dq} + 2\Gamma^{\phi}_{\phi\theta}\frac{d\phi}{dq}\frac{d\theta}{dq}, \\
&= \frac{d^2\phi}{dq^2} + \frac{2}{r}\frac{dr}{dq}\frac{d\phi}{dq}, \\
&= \frac{d^2\phi}{dq^2}\Big/\frac{d\phi}{dq} + \frac{2}{r}\frac{dr}{dq}, \\
&= \frac{d[\ln(\frac{d\phi}{dq}) + \ln r^2]}{dq}, \\
&= \frac{d[\ln r^2 \frac{d\phi}{dq}]}{dq}, \text{ so,} \\
J &= r^2\frac{d\phi}{dq}, \qquad d\phi = (J/r^2)dq, \qquad (6.2)
\end{aligned}
$$

where J is the constant of the motion due to $g_{\mu\nu},_2 = 0$.

For the coordinate $x^0 = t$,

$$
\begin{aligned}
0 &= \frac{d^2t}{dq^2} + 2\Gamma^{t}_{rt}\frac{dr}{dq}\frac{dt}{dq}, \\
&= \frac{d^2t}{dq^2} + \frac{1}{1-2M'/r}\frac{2M'}{r^2}\frac{dr}{dq}\frac{dt}{dq}, \\
&= \frac{d^2t}{dq^2}\Big/\frac{dt}{dq} + \frac{1}{1-2M'/r}\frac{2M'}{r^2}\frac{dr}{dq}, \\
&= \frac{d(\ln\frac{dt}{dq} + \ln[1-2M'/r])}{dq}, \\
&= \frac{d\ln[(1-2M'/r)\frac{dt}{dq}]}{dq}, \\
J' &= (1-2M'/r)\frac{dt}{dq} \equiv 1, \\
\frac{dt}{dq} &= \frac{1}{1-2M'/r}, \; dt = \frac{dq}{1-2M'/r}. \qquad (6.3)
\end{aligned}
$$

To first order in terms of M'/r, $dt \approx dq(1+2M'/r)$. Setting constant, $J' = 1$, is a normalization choice. It makes sense because when,

$r \to \infty$, $dt = dq$. Since $M'/r \ll 1$, Eq. (6.2) gives, $r^2 \frac{d\phi}{dq} \approx r^2 \frac{d\phi}{dt} = J$, and leads to conservation of the ϕ component of angular momentum per unit mass, of the particle moving in the metric.

The above information is inserted into the equation for $x^3 = r$. After multiplication by $\frac{1}{1-2M'/r}\frac{dr}{dq}$, the solution is transparent;

$$
\begin{aligned}
0 &= \frac{d^2r}{dq^2} + \Gamma^r_{rr}\left(\frac{dr}{dq}\right)^2 + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{dq}\right)^2 + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{dq}\right)^2 + \Gamma^r_{tt}\left(\frac{dt}{dq}\right)^2, \\
&= \frac{d^2r}{dq^2} - \frac{(M'/r^2)\left(\frac{dr}{dq}\right)^2}{1-2M'/r} \\
&\quad + (1-2M'/r)\left[-r\left(\frac{d\phi}{dq}\right)^2 + \left(M'/r^2\right)\left(\frac{dt}{dq}\right)^2\right], \\
&= \frac{d^2r}{dq^2} - \frac{(M'/r^2)(\frac{dr}{dq})^2}{1-2M'/r} - \frac{(1-2M'/r)J^2}{r^3} + \frac{(M'/r^2)}{1-2M'/r}, \\
&= \frac{\frac{d^2r}{dq^2}\frac{dr}{dq}}{1-2M'/r} - \frac{\left(M'/r^2\right)\left(\frac{dr}{dq}\right)^3}{(1-2M'/r)^2} - \frac{J^2\frac{dr}{dq}}{r^3} + \frac{\left(M'/r^2\right)\frac{dr}{dq}}{(1-2M'/r)^2}, \\
&= \frac{1}{2}\frac{d}{dq}\left[\frac{\left(\frac{dr}{dq}\right)^2}{1-2M'/r} + \frac{J^2}{r^2} - \frac{1}{1-2M'/r}\right],
\end{aligned}
$$

$$
-E' = \frac{\left(\frac{dr}{dq}\right)^2}{1-2M'/r} + \frac{J^2}{r^2} - \frac{1}{1-2M'/r}, \tag{6.4}
$$

where E' is the second constant of the motion, due to $g_{\mu\nu},_0$.

Eq. (6.4) allows determination of q and the interpretation of E';

$$
\left(\frac{dr}{dq}\right)^2 = (1-2M'/r)(-E' - (J/r)^2 + (1-2M'/r)^{-1}), \tag{6.5}
$$

$$
(dr)^2 = (dq)^2[-E' - (J/r)^2 + (1-2M'/r)^{-1}](1-2M'/r),
$$

$$
(d\tau)^2 = (1-2M'/r)(dt)^2 - (1-2M'/r)^{-1}(dr)^2 - r^2(d\phi)^2,
$$

$$
\begin{aligned}
&= (1-2M'/r)^{-1}(dq)^2 - (Jdq/r)^2 \\
&\quad -(dq)^2[-E' - (J/r)^2 + (1-2M'/r)^{-1}] \\
&= E'(dq)^2, \qquad d\tau = E'^{1/2}dq.
\end{aligned} \tag{6.6}
$$

So for photons, $E' = 0$, while for massive particles, $E' > 0$, and $dq \propto d\tau$. Thus, q is obviously an affine parameter. For a slowly moving particle of rest mass, m, with $M' \ll r$, Eqs. (6.3) and (6.5) yield,

$$
\begin{aligned}
-E' - (J/r)^2 + (1-2M'/r)^{-1} &= (1-2M'/r)^{-3}\left(\frac{dr}{dt}\right)^2, \\
&\approx \left(\frac{dr}{dt}\right)^2, \\
-E' &\approx \left(\frac{dr}{dt}\right)^2 + (J/r)^2 - (1+2M'/r), \\
(1-E')/2 &= \left[\left(\frac{dr}{dt}\right)^2 + (J/r)^2\right]/2 - M'/r, \\
m[1+(1-E')/2] &= RE + KE + PE = E.
\end{aligned} \tag{6.7}
$$

In general, the quantity E can be interpreted as the total energy or relativistic mass, M, of the particle in the gravitational field. When $E' > 1$, the total energy is less than the rest energy, and may even be negative. This phenomena occurs when the particle is in a strong gravitational field.

6.2 Orbit Equations

The orbit equations are obtained from Eqs. (6.2), (6.3), (6.5) and (6.6). For example,

$$
\frac{d\phi}{dr} = \frac{d\phi}{dq}\Big/\frac{dr}{dq} = \pm\frac{J}{r^2(1-2M'/r)^{1/2}}\frac{1}{([(1-2M'/r)^{-1} - E'] - [J/r]^2)^{1/2}},
$$

$$
D[r] \equiv \frac{1}{([(1-2M'/r)^{-1} - E']/J^2 - 1/r^2)^{1/2}}. \tag{6.8}
$$

Thus,

$$d\phi = \pm \frac{dr}{r^2(1-2M'/r)^{1/2}} D[r], \tag{6.9}$$

$$dt = \pm \frac{dr}{J(1-2M'/r)^{3/2}} D[r], \tag{6.10}$$

$$d\tau = \pm \frac{drE'^{1/2}}{J(1-2M'/r)^{1/2}} D[r]. \tag{6.11}$$

It should be noted that t is the time according to a far away, at rest observer, while τ is the time on a clock attached to the particle. Integration of the above equations will give r as a function of ϕ, t or τ. The correct sign is determined by the result of the integration. Each integral is an elliptic integral and could be worked out numerically. However, if $M'/r \ll 1$ everywhere then to first order in M'/r, the integration can be done analytically. In order to accomplish this, one expands $(1-2M'/r)^n = 1 - n(2M'/r) + [n(n-1)/2](2M'/r)^2 + \ldots$ and keeps the lowest order term. In some of the mathematical manipulations below, I have followed [Weinberg (1972)] and filled in some steps.

6.3 Light Deflection

Solar system tests have shown the correctness of GR, even if equations for the metric, more complicated than Einstein's, cannot be ruled out. Einstein's fame was established by the positive result of close to the predicted value for the deflection of light passing near the surface of the sun. The first positive result experiment was carried out in 1919 by a British team [Dyson (1920)] led by Eddington.

In the case of light deflection by the sun's gravity, one looks at stars whose lines of sight come as close as possible to the edge of the sun's disk. Of course, there must be a complete solar eclipse, as in Fig. 6.1, to make such a measurement. Six months previous to the eclipse, the same stars were viewed. At that time, the sun is no longer between the earth and the stars. In practice stars closer than twice the sun's radius, $2R_s$, cannot be observed. Light from the star initially travels in almost zero gravity along the line of sight, given by angle $\phi(I)$. It's path deflects as it gets close to the sun's surface where the minimum distance from the sun's center is, r_0. The light finally winds up very far away along the line of sight, given by angle

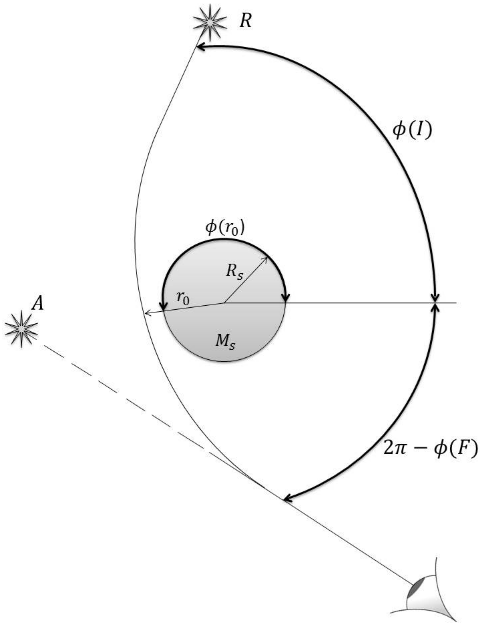

Fig. 6.1 when the sun is in total eclipse, light from stars passing close to the sun's disk appear at apparent position A. When the sun is not between the earth and star, the latter appears at its real position, R.

$\phi(F)$. The earth–sun distance is $> 200R_s$, and so the metric, at earth, is again very close to $\eta_{\mu\nu}$. The angle ϕ is very well measured and the angular deflection from the initial direction is $\delta\phi = (\phi(F) - \phi(I)) - \pi$. However, the orbit is symmetric about the line along r_0, the distance of closest approach, so that $\delta\phi = 2(\phi(r_0) - \phi(I)) - \pi$.

For light use Eq. (6.8) with $E' = 0$, and $M'/r_0 \ll 1$. At $r = r_0$, use Eq. (6.4) with, $\frac{dr}{dq} \propto \frac{dr}{d\phi} = 0$,

$$J^{-2} = (1 - 2M'/r_0)r_0^{-2}, \tag{6.12}$$

$$\delta\phi + \pi = \pm 2\int_{\infty}^{r_0} dr(1 - 2M'/r)^{-1/2} r^{-2} D[r], \tag{6.13}$$

$$D[r] = \left(J^{-2}(1 - 2M'/r)^{-1} - r^{-2}\right)^{-1/2},$$

then,

$$D[r] = \left(r_0^{-2}\frac{1 - 2M'/r_0}{1 - 2M'/r} - r^{-2}\right)^{-1/2}. \tag{6.14}$$

On expanding the terms with M'/r, or M'/r_0, and keeping only factors linear in, M'/r or M'/r_0, the integral becomes, see problem one,

$$\delta\phi + \pi = \pm 2 \int_{\infty}^{r_0} dr \frac{1}{r[(r/r_0)^2 - 1]^{1/2}}$$
$$\times \left(1 + \frac{M'}{r_0}\frac{r_0}{r} + \frac{M'r}{r_0^2}\left[\frac{r}{r_0} + 1\right]^{-1}\right).$$

Now change variables such that $u = r/r_0$, $dr = r_0 du$,

$$\delta\phi + \pi = \pm 2 \int_{\infty}^{1} du \frac{1}{u[u^2 - 1]^{1/2}} \left(1 + \frac{M'}{r_0}\left[u^{-1} + \frac{u}{u+1}\right]\right),$$
$$= -2(-\pi/2 - 2M'/r_0) = \pi + 4M'/r_0,$$
$$\delta\phi = 4M'/r_0. \tag{6.15}$$

Since $4M'/R_s = 8.53 \times 10^{-6}$ r(adians) $= 1.75''$, the effect is very small.

The 1919 data were pictures of stars stored on photographic plates and the distances were measured with calipers. Over a period of six months there is bound to be a change of scale due to changes in temperature and the position of the telescope on the ground. The data were compared with the predicted value by calibrating a scale constant, S,

$$\delta\phi = (4M'/R_s)(R_s/r_0) + S(r_0/R_s).$$

Stars with $r_0 \gg R_s$, would not change position and these give the best fit for S. There were two observing stations. At Sobral, a island near the Brazilian coast, seven stars with $r_0/R_s = 2 - 6$, gave, $\delta\phi = (1.98 \pm 0.16)''$, while at Principe, an island near the coast of Guinea, five stars with $r_0/R_s = 2 - 6$, gave, $\delta\phi = (1.61 \pm 0.40)''$. This was sufficient to confirm that light is affected by gravity. The numerical results are in reasonable agreement with the Einstein prediction.

The effect is small, and the accuracy is such that even including more modern optical measurements, metric equations more complicated than Einstein's cannot be ruled out. However, recent radio telescope measurements made at the VLBI facility [Lebach (1995)], have confirmed Einstein's prediction to the one percent level. That's what modern instrumentation can do for you.

Due to gravitational deflection of light, the universe offers interesting "illusions" termed gravitational lensing, as illustrated in Fig. 6.2, There

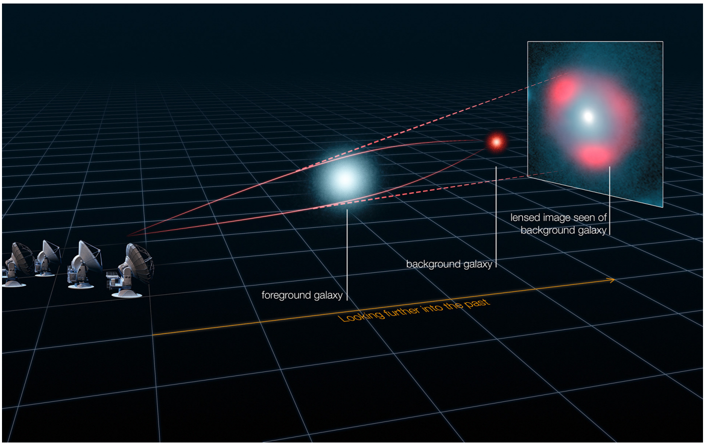

Fig. 6.2 Mass in the form of galaxies and galaxy clusters between earth and a very far away source can deflect light so that multiple images are seen.

may be mass, in the form of a galaxy or cluster of galaxies, between earth and a very distant source. Light rays from the source are deflected by the intervening mass and follow different paths to earth. The earth observer can then see, in different directions, multiple images of the same source. If the masses line up just right, rings called "Einstein Rings" are observed. Use Google or some other world wide web (WWW) search program to find "gravitational lensing" and you'll find a host of fascinating images.

6.4 Perihelia Advance

The second test, is the precession of perhelia of planets close to the sun. This means Mercury for our solar system. The Newtonian orbit for the two body problem indicates no precession. Mercury's precession is illustrated in Fig. 6.3. While precessing, the orbit will still have maximum, minimum distances from the sun, r_+, r_-, where, $\frac{dr}{d\phi} = 0$. So Eq. (6.5) is used to obtain J, E' in terms of $r_\pm$,

$$\begin{aligned} 0 &= (1 - 2M'/r_+)^{-1} - E' - J^2/r_+^2, \\ &= (1 - 2M'/r_-)^{-1} - E' - J^2/r_-^2, \\ J^2 &= [r_-^{-2} - r_+^{-2}]^{-1}[(1 - 2M'/r_-)^{-1} - (1 - 2M'/r_+)^{-1}], \end{aligned} \tag{6.16}$$

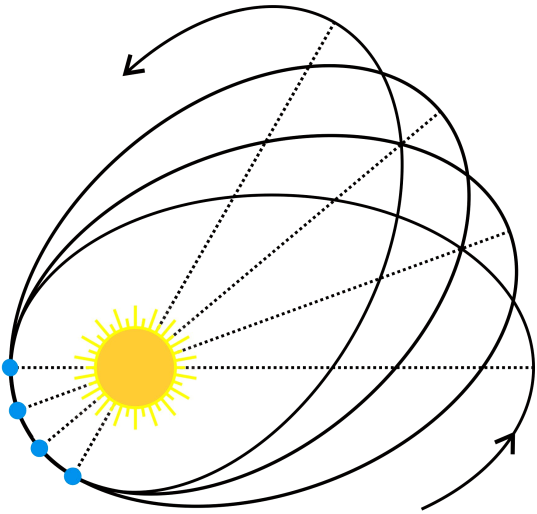

Fig. 6.3 The perihelion of Mercury precessing as it makes successive orbits about the sun.

$$
\begin{aligned}
E' &= (1 - 2M'/r_-)^{-1} - J^2 r_-^{-2}, \\
&= (r_+^{-2} - r_-^{-2})^{-1}[r_+^2(1 - 2M'/r_+)^{-1} - r_-^2(1 - 2M'/r_-)^{-1}].
\end{aligned}
\tag{6.17}
$$

The integral for the orbit, Eqs. (6.8) and (6.9), requires evaluation of the function $D[r]$, here written in terms of $r_\pm$,

$$
\begin{aligned}
f &\equiv (D[r])^{-2} + r^{-2}, \\
f &\equiv J^{-2}[(1 - 2M'/r)^{-1} - E']. \qquad (6.18)\\
&= (r_+ r_-)^{-2}[(1 - 2M'/r_-)^{-1} - (1 - 2M'/r_+)^{-1}]^{-1} \\
&\quad \times [r_-^2((1 - 2M'/r_-)^{-1} - (1 - 2M'/r)^{-1}) \\
&\quad + r_+^2((1 - 2M'/r)^{-1} - (1 - 2M'/r_+)^{-1})]. \qquad (6.19)
\end{aligned}
$$

The above is a positive quantity as each difference is positive. The steps leading from Eq. (6.18) to Eq. (6.19) are explored in problem two.

Once again the integrand must be expanded in powers of M'/r, and the leading correction kept in order to do the integral analytically. However, in each difference in the last equation, the leading term is $\propto M'$. So unless the expansion is carried out to second order the mass, M' will cancel. This expansion is,

$$(1-2M'/r)^{-1} \approx 1 + 2M'/r + (2M'/r)^2 = 1 + (2M'/r)(1+2M'/r).$$

It yields,

$$\begin{aligned} f &= (r_+r_-)^{-2}[(2M'/r_-)(1+2M'/r_-) - (2M'/r_+)(1+2M'/r_+)]^{-1} \\ &\quad \times (r_-^2[(2M'/r_-)(1+2M'/r_-) - (2M'/r)(1+2M'/r)] \\ &\quad + r_+^2[(2M'/r)(1+2M'/r) - (2M'/r_+)(1+2M'/r_+)]), \end{aligned} \tag{6.20}$$

$$(D[r])^{-2} = K(r_-^{-1} - r^{-1})(r^{-1} - r_+^{-1}). \tag{6.21}$$

Note that Eq. (6.20) yields $(D[r_\pm])^{-2} = 0$, and indicates $(D[r])^{-2}$ is a function of r^{-1} and r^{-2}, so that Eq. (6.21) follows. To find the constant, K, equate the constant terms in both forms of $(D[r])^{-2}$. This is mathematically equivalent to evaluation at $r = \infty$,

$$\begin{aligned} K &= (r_+r_-)^{-1} \\ &\quad \times [2M'/r_+(1+2M'/r_+) - 2M'/r_-(1+2M'/r_-)]^{-1} \\ &\quad \times (2M'r_-(1+2M'/r_-) - 2M'r_+(1+2M'/r_+)] \\ &= [r_-(1+2M'/r_+) - r_+(1+2M'/r_-)]^{-1}(r_- - r_+), \\ &= [1 + 2M'(r_-^{-1} + r_+^{-1})]^{-1} \approx 1 - 2M'(r_-^{-1} + r_+^{-1}), \end{aligned}$$

$$K^{-1/2} \approx 1 + M'(r_-^{-1} + r_+^{-1}). \tag{6.22}$$

When Mercury goes from r_- to r_+, $\delta\phi/2$, is defined to be the change in, ϕ. As the orbit is symmetric, the total change in ϕ in one revolution is $\delta\phi$, where

$$\begin{aligned} \delta\phi &= \pm 2K^{-1/2} \int_{r_-}^{r_+} \frac{dr}{r^2[(1-2M'/r)(r_-^{-1} - r^{-1})(r^{-1} - r_+^{-1})]^{1/2}}, \\ &\approx \pm 2K^{-1/2} \int_{r_-}^{r_+} \frac{dr(1+M'/r)}{r^2[(r_-^{-1} - r^{-1})(r^{-1} - r_+^{-1})]^{1/2}}. \end{aligned} \tag{6.23}$$

If one makes the change of variable,

$$u = a/r + b, \qquad u(r_\pm) \equiv \pm 1, \tag{6.24}$$

the integral becomes, see problem two,

$$\begin{aligned}
\delta\phi &= \pm 2K^{-1/2} \int_{-1}^{1} du \frac{1 + \frac{M'}{2}[(r_+^{-1} + r_-^{-1}) + (r_+^{-1} - r_-^{-1})u]}{(1-u^2)^{1/2}}, \\
\delta\phi &= \pi[1 + M'(r_-^{-1} + r_+^{-1})][2 + M'(r_+^{-1} + r_-^{-1})], \\
&\approx 2\pi + 3\pi M'(r_-^{-1} + r_+^{-1}).
\end{aligned} \tag{6.25}$$

If the orbit is closed, the above should equal 2π. So the perihelion has advanced by $3\pi M'(r_-^{-1} + r_+^{-1})$. For the orbit of Mercury, $r_+ = 6.98 \times 10^{10}$ m and $r_- = 4.60 \times 10^{10}$ m. Thus per revolution the advance of perihelion is

$$\delta\phi - 2\pi = 3\pi(1.484 \times 10^3)(0.360 \times 10^{-10})\ \mathrm{r} = 0.104''.$$

Mercury's orbital period is 87.96 d, so that in a century it makes 415 revolutions. The cumulative effect is $\delta\phi - 2\pi = 43''$ per century. Current experimental results are in excellent agreement with Einstein's theory. The deviation is less than one percent. The other planets being farther from the sun would yield much smaller values than Mercury.

As opposed to light deflection, where light starts and ends in essentially zero gravity, Mercury is always in a gravitational field. So when it is observed with light, the light is traveling in a gravitational field. For a single revolution, one would have to worry about such small corrections. However, the effect is cumulative and for many revolutions, this correction is not a worry. Think about measuring the period of a simple pendulum. The small error one might make for one period is negligible if the total time for many periods is measured.

Newtonian physics predicts a much larger advance per century due to other perturbations of $5557''$, of which $5025''$ is due to the precession of the rotation axis of the earth. The observation is $5600''$, and the extra $43''$ agrees with the prediction of GR [Clemence (1947)]. Before GR, Newtonian physics was thought adequate and for the Mercury problem, there was speculation that perhaps there was unseen mass between the sun and Mercury or perhaps the sun was non-spherical, etc. Einstein was not working on GR to solve the Mercury problem, but realized that the theory could be applied to it. His biographers tell us that he was delirious with joy upon finding those 43″.

6.5 Radar Signal Delay

When Mercury is near superior conjunction, radar from earth is reflected back with a delay predicted by GR. This effect is also known as Shapiro delay. The situation is shown in Fig. 6.4, where r_0 is the distance of closest approach to the sun. Here the straight line, Newtonian path just grazes the sun's disk. As in the case of light deflection, $E' = 0$, and at r_0, $0 = \frac{dr}{dq} = \frac{dr}{dt}$. Thus Eqs. (6.12) and (6.14) give, J^2 and $D[r]$, while Eq. (6.10) yields the integrand of the desired integral, with,

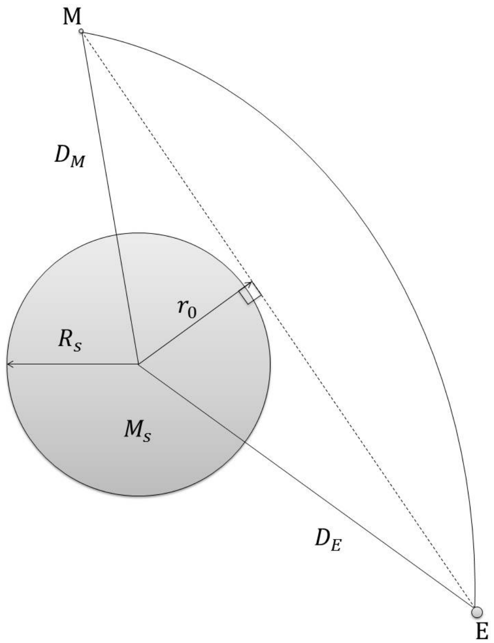

Fig. 6.4 When Mercury is near superior conjunction, radar signals sent from and reflected back to earth travel for a longer time along the solid GR geodesic than along the dashed straight line, Newtonian path.

$$
\begin{aligned}
M' &= M_s, \\
J^2 &= r_0^2(1-2M'/r_0)^{-1}, \\
dt &= \pm\frac{dr(1-2M'/r_0)^{1/2}}{r_0(1-2M'/r)^{3/2}}\left(r_0^{-2}\frac{(1-2M'/r_0)}{(1-2M'/r)}-r^{-2}\right)^{-1/2}, \\
&= \pm dr(1-2M'/r)^{-1}\left(1-\frac{1-2M'/r}{1-2M'/r_0}\left(\frac{r_0}{r}\right)^2\right)^{-1/2}.
\end{aligned}
$$

In order to avoid elliptic integrals, expand the above and keep the leading correction, see problem three, to obtain,

$$
t_{E,M} \approx \pm\int_{r_0}^{D_{E,M}} \frac{rdr[1+M'(2/r+(r_0/r)(r+r_0)^{-1})]}{[r^2-r_0^2]^{1/2}}, \tag{6.26}
$$

where $D_{E,M}$ is the distance from the center of the sun to the planets. The result is,

$$
\begin{aligned}
t_{E,M} &= M'\left[2\ln\frac{D_{E,M}+[D_{E,M}^2-r_0^2]^{1/2}}{r_0}+\left(\frac{D_{E,M}-r_0}{D_{E,M}+r_0}\right)^{1/2}\right] \\
&\quad + \left[D_{E,M}^2-r_0^2\right]^{1/2}.
\end{aligned} \tag{6.27}
$$

The prediction is compared with radio waves just grazing the sun, $r_0 = R_s$. Since the radio waves go to Mercury and are reflected back to earth,

$$
\begin{aligned}
t_{total} &= 2(t_E+t_M) \equiv 2(T+T'), \\
T &= [D_E^2-R_s^2]^{1/2}+[D_M^2-R_s^2]^{1/2},
\end{aligned} \tag{6.28}
$$

$$
\begin{aligned}
T'/M' &= 2\ln\frac{(D_E+[D_E^2-R_s^2]^{1/2})([D_M+[D_M^2-R_s^2]^{1/2})}{R_s^2} \\
&\quad + \left(\frac{D_E-R_s}{D_E+R_s}\right)^{1/2}+\left(\frac{D_M-R_s}{D_M+R_s}\right)^{1/2}, \\
&= 2\ln\frac{D_E(1+[1-(R_s/D_E)^2]^{1/2})D_M(1+[1-(R_s/D_M)^2]^{1/2})}{R_s^2} \\
&\quad + \left(\frac{1-R_s/D_E}{1+R_s/D_E}\right)^{1/2}+\left(\frac{1-R_s/D_M}{1+R_s/D_M}\right)^{1/2},
\end{aligned} \tag{6.29}
$$

where $2T$ is what you would expect if gravity did not deflect radar and, $0 < 2T' \propto M'$, is the lowest order GR correction. As T' is positive, it is said that the radar signals are delayed.

To see the size of the effect, the following approximate distances and solar mass, in meters, are used:

$$D_{E,M} = 0.5(r_+ + r_-)_{E,M} = 1.49 \times 10^{11},\ 5.79 \times 10^{10},$$

$$R_s = 6.96 \times 10^8,\ M' = 1.48 \times 10^3,\ \text{thus},$$

$$2T = 4.14 \times 10^{11}\ \text{m} = 1.05 \times 10^3\ \text{s},$$

$$2T' \approx 4M' \left[1 + \ln 4 \frac{D_E}{R_s} \frac{D_M}{R_s} \right] = 7.2 \times 10^4\ \text{m} = 240\ \mu\text{s}. \quad (6.30)$$

The delay is a very small effect, $T'/T = 2.3 \times 10^{-7}$. Even so, by 1971 the team led by Shapiro achieved a result in agreement with GR at the five percent level, [Shapiro (1971)]. A similar experiment making use of the Cassini space craft obtained a result in agreement with GR at a level of less than one percent, [Berlotti (2003)].

A measurement of such exquisite accuracy is fraught with difficulties. Modern atomic clocks can easily measure t_{total} with sufficient accuracy. However, one must subtract from this $2T$, a quantity almost as large. Optical measurements of the distances are nowhere near accurate enough. There are other critical sources of systematic error that must be considered: The solar corona is an effective index of refraction for radar propagation. Reflection at Mercury's surface occurs, not from a single point, but from a large area, each element of which has its own reflection properties and orbital and rotational motion. The reflection properties are studied at inferior conjunction and the times of the Doppler shifted frequency distribution allow one to obtain the reflection time from the closest point to earth. Using a large set of astronomical data, the distances were obtained from GR predictions in terms of fits for many parameters. See [Shapiro (1964)] for a description of the original experiment.

Problems

1. Consider the integrand ($\equiv Int$) of Eqs. (6.13) and (6.14). Expand the terms involving $M'/r \ll 1$, and keep only terms linear in M'/r. Fill in the steps to obtain the equation following Eq. (6.14).

2. Start with Eqs. (6.16)–(6.18) and fill in the steps leading to Eq. (6.19). Then start with the integral of Eq. (6.23), make the variable change of Eq. (6.24) and fill in the steps leading to Eq. (6.25).
3. Start with the equation above Eq. (6.26) and show how Eq. (6.26) is obtained.
4. Using the Schwarzschild metric, suppose a particle with finite rest mass is launched from the surface of a sphere of area, A', in the direction of increasing radial coordinate. The stationary curvature is provided by the sphere's mass, M'. Assume $M'/R' \ll 1$, where R' is the radial coordinate at the sphere's surface. What is the radial coordinate at the surface in terms of A'? Find the escape speed as measured by an observer at the surface of the sphere, and show that this speed is the same as the Newtonian result. What proper distance is traveled by the particle when it reaches a radial coordinate where an observer measures that the escape speed is reduced by half?
5. Using the data of problem four, calculate the time on a clock attached to the moving mass for the trip between the two radial coordinates. Then calculate the time on a clock at rest at, $r = \infty$.
6. Use the data of problem four. Suppose a mass is released from rest at a radial coordinate, $r = 100R'$. At the surface of the sphere what is the speed? What is the proper distance traveled and the time on a clock attached to the moving mass?
7. The Schwarzschild metric permits circular orbits, where, r, is constant. For massive particles, show that two such orbits are possible if, $12(M'^2/\bar{J}^2) < 1$, where, $\bar{J}^2 = J^2/E'$. Show that the larger orbit is stable, but not the smaller. For the latter, what is the smallest possible radial coordinate? Start by showing,

$$1/E' = \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2M'}{r}\right)\left(1 + \frac{\bar{J}^2}{r^2}\right) \equiv \left(\frac{dr}{d\tau}\right)^2 + f[r],$$

   The function $f[r]$ is an effective potential and radial coordinates of circular orbits exist where the potential is an extremum, $\frac{df[r]}{dr} = 0$.
8. For the conditions of problem seven with, $r = 10M'$, find, J and E'. Determine whether there is another circular orbit possible. If another is possible find its radial coordinate and determine which orbit is stable.
9. For the data of problem eight find the time on a clock attached to the particle and on a clock at rest very far away when one revolution is

completed. Compare $\frac{d\phi}{dt}$ with the Newtonian value for the same circular orbit.

10. Using the data and results of problems seven, eight and nine, find the speed, $|\vec{v}|$, of the object in the circular orbit, as determined on a clock at rest at the radial coordinate of the orbit. Determine the speed for, $r = (10, 4.29, 3)M'$. What can be said because of the speed at $r = 3M'$? Is there a stable circular orbit for light?
11. A neutron star has a mass, $M' = 1.5M_s$, and a radius, $r_0 = 15$ km. If the star's angular momentum is neglected, the metric in its vicinity is the Schwarzschild metric. If light from far away just skims its surface, what is the deflection? Compare this result with that obtained by expanding and keeping M'/r terms to lowest order.

Chapter 7

Black Holes

7.1 Static Black Holes

In the era before SR and GR there was speculation about possible compact, spherically symmetric, objects of large mass M' and radius R, from which even light could not escape. The incorrect argument was made on the basis of the escape speed, v_E, of an object of mass M, using Newtonian mechanics

$$M(v_E)^2/2 - MM'/R = 0, \qquad v_E = (2M'/R)^{1/2}.$$

So when $v_E = 1$, $R = 2M'$, light would be bound to the compact object.

The connection to GR is easy to see. It is just where the Schwarzschild metric has a singularity, other than, $r = 0$. Continuing to use $r^p \equiv (r)^p$, the metric is,

$$\begin{aligned}(d\tau)^2 &= (1 - 2M'/r)(dt)^2 - (1 - 2M'/r)^{-1}(dr)^2 \\ &\quad -r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], \\ &= (1 - R/r)(dt)^2 - (1 - R/r)^{-1}(dr)^2 - r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2].\end{aligned}$$

Thus,

$$1 - R/r = 0,\ (1 - R/r)^{-1} = \infty \text{ when } r = R,$$

where, R is called the Schwarzschild radius. For an object with the mass of the sun it has a very small value, $R = 2M_s = 2.968 \times 10^3$ m. In the case of the sun, such a radius is well within the sun's radius and wouldn't contain much of the sun's mass. A black hole however is a real singularity at $r = 0$, and R is external to it. In the region accessible to observation, $R/r < 1$, the applications of the Schwarzschild metric found in chapter six apply. However, to emphasize that black holes are spoken of, $2M'$ will be replaced by R for the rest of this chapter.

The apparent singularity at R is not real and is due to the choice of coordinates. This can be seen by recalling that in deriving the Schwarzschild metric, the Ricci tensor, $R_{\mu\nu}$, vanished in vacuum. Thus there can't be a real singularity at the vacuum point, $r = R$. One can seek other coordinates that make the apparent singularity disappear. The following ones, known as the Kruskal coordinates, (r', t'), do the trick

$$r'^2 - t'^2 = K^2(r/R - 1)\exp[r/R], \tag{7.1}$$

$$2r't'/(r'^2 + t'^2) = \tanh[t/R],\ t = R\tanh^{-1}[2r't'/(r'^2 + t'^2)]. \tag{7.2}$$

For given r Eq. (7.1) yields hyperbolas in the (r', t') plane, as illustrated in Fig. 7.1. If $r > R$ the hyperbolas are symmetric about the r' axis and cross that axis at values of positive r' that increase as $r \to \infty$. If $r < R$, they are symmetric about the t' axis and cross that axis at positive values of t', that increase as, $r \to 0$. The singularity at, $r = 0$, in these coordinates appears as a hyperbola, but since it is a singularity it isn't completely

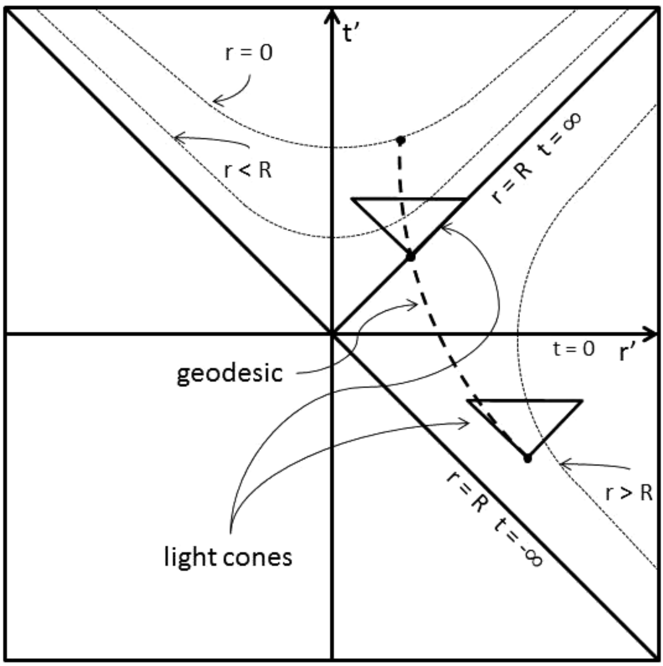

Fig. 7.1 Kruskal coordinates, (r', t'), for a black hole. In these coordinates, the only singularity is at $r = 0$. The light cones are erect and as wide open as in SR, so once light or any massive particle goes to, $r < R$, it can never get to $r > R$ again.

understood by current physics. When, $r = R$, $t' = \pm r'$. These are lines with slopes ± 1 that define the asymptotic behavior of the hyperbolas.

For given, t, Eq. (7.2) allows introduction of constant K', where,

$$K' = \tanh[t/R], \quad 0 \leq |K'| \leq 1, \tag{7.3}$$

$$\begin{aligned}
0 &= K'r'^2 - 2r't' + K't'^2, \\
t' &= (2r' \pm [(2r')^2 - (2K'r')^2]^{1/2})/(2K'), \\
&= (r'/K')[1 \pm (1 - K'^2)^{1/2}].
\end{aligned} \tag{7.4}$$

These are lines that start at the origin and go in the positive, r' direction with slope, $[1 \pm (1 - K'^2)^{1/2}]/K'$. Note that when, $t = \pm\infty$, $K' = \pm 1$. Thus, these lines overlap the lines of, $r = R$. A massive particle or photon traveling along the world line indicated by the solid curve in Fig. 7.1, from large r heading for $r = R$, will cut through all the time lines and get there at $t = \infty$. This time is measured on the clock of a far away, at rest observer. However, the time on a clock moving with the particle, the particle's proper time, proceeds as if nothing unusual was happening. As the light cones in these coordinates are as erect and open as in SR, once the object gets to $r < R$ it must go to the singularity.

In order to see this, the metric must written in terms of r', t'. A little manipulation of Eq. (7.2) yields,

$$\begin{aligned}
dt &= R\frac{(r'^2 + t'^2)2(t'dr' + r'dt') - 2r't'[2(r'dr' + t'dt')]}{[1 - (2r't'/[r'^2 + t'^2])^2](r'^2 + t'^2)^2}, \\
&= 2R\frac{dr't'((r'^2 + t'^2 - 2r'^2) + dt'r'(r'^2 + t'^2 - 2t'^2)}{(r'^2 + t'^2)^2 - (2r't')^2}, \\
&= 2R(r'^2 - t'^2)^{-1}[-t'dr' + r'dt'], \\
(dt)^2 &= (2R)^2\frac{t'^2(dr')^2 + r'^2(dt')^2 - 2r't'dr'dt'}{(r'^2 - t'^2)^2}, \\
&= (2R)^2\frac{t'^2(dr')^2 + r'^2(dt')^2 - 2r't'dr'dt'}{K^4(r/R - 1)^2\exp[2r/R]}, \\
&= 4R^4\frac{t'^2(dr')^2 + r'^2(dt')^2 - 2r't'dr'dt'}{K^4r^2(1 - R/r)^2\exp[2r/R]}, \\
(1 - R/r)(dt)^2 &= 4R^4\frac{t'^2(dr')^2 + r'^2(dt')^2 - 2r't'dr'dt'}{K^4r^2(1 - R/r)\exp[2r/R]}.
\end{aligned} \tag{7.5}$$

Similarly manipulation of Eq. (7.1) yields,

$$
\begin{aligned}
2(r'dr' - t'dt') &= (K^2/R)\exp[r/R](1 + r/R - 1)dr, \\
&= (K/R)^2 r \exp[r/R]dr, \\
(dr)^2 &= 4R^4 \frac{(r'dr')^2 + (t'dt')^2 - 2r't'dr'dt'}{K^4 r^2 \exp[2r/R]}, \\
\frac{(dr)^2}{1 - R/r} &= 4R^4 \frac{(r'dr')^2 + (t'dt')^2 - 2r't'dr'dt'}{K^4 r^2 (1 - R/r)\exp[2r/R]}.
\end{aligned} \tag{7.6}
$$

Combining Eqs. (7.5) and (7.6) gives,

$$
\begin{aligned}
(1 - R/r)(dt)^2 - \frac{(dr)^2}{1 - R/r} &= 4R^4 \frac{[r'^2 - t'^2][(dt')^2 - (dr')^2]}{K^4 r^2 (1 - R/r)\exp[2r/R]}, \\
&= \frac{4R^3[(dt')^2 - (dr')^2]}{K^2 r \exp[r/R]}.
\end{aligned}
$$

This yields the relation for the element of proper time,

$$
d\tau^2 = \frac{4R^3[(dt')^2 - (dr')^2]}{K^2 r \exp[r/R]} - r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]. \tag{7.7}
$$

In Eq. (7.7), r is not regarded as a coordinate, but as a function of r' and t'. One can see that there is no singularity at, $r = R$. The metric is well defined as long as r^2 is positive definite. There is a singularity at, $r = 0$, as expected.

The condition for light signals is, $d\tau = 0$. Thus for radial travel, $d\theta = d\phi = 0$, one obtains $dt' = \pm dr'$. This means the light cones shown in the figure are as vertically erect and fully open as they are in SR. Massive object world lines would move inside the cones because their speed is less than that of light. So once inside $r = R$, objects and light move towards increasing t', in and on the sides of the cones and must get to the singularity. The surface at R is a one way membrane called an event horizon. As the object moves towards $r = R$, it may send out light signals to a far away observer. The light may possess a given frequency, so that the crests are emitted at regular intervals of the object's proper time. As the proper time is running slower and slower compared to the time on a far away clock, the time periods between intervals on the far away clock tend to be longer and longer. The light is red shifted to the extreme and such signals disappear from the view of the far away observer before the object reaches the horizon.

Black holes are discussed because observation has confirmed their existence. Stars near the center of our galaxy have been observed for more than sixteen years. Their orbits indicate rotation about a huge, but unseen mass. Other sightings include the accretion of visible matter into a dark area. Such is often accompanied by the emission of x rays and gamma rays as the gravitational pulls on the matter are so violent. Black holes arise from the supernova of massive stars. If the remnant mass is greater than the mass that could be supported by neutron degeneracy pressure, a purely quantum effect, nothing can prevent a complete collapse to a singularity.

It would be good to know the nature of the singularity. In the literature it is spoken of as a point of infinite density. We have other experience with infinite density. For example, electrons and muons have no structure, but from their motion in electromagnetic fields, have finite mass and charge. However, K. Thorne writes that there is no matter in a black hole [Thorne (2014)]. There is only the infinite warping of spacetime, the in-falling matter being crunched out of existence, with an event horizon left behind. Our present physics just cannot handle a singularity, a quantum theory of gravity is required.

7.2 Black Holes With Angular Momentum

R. P. Kerr [Kerr (1967)] found the metric for the space outside a spinning sphere. This occurred almost fifty years after Einstein published GR theory, so one gets a feel for the difficulty of the problem. The paper's length is a single page. No doubt an elegant proof for a mathematician. However, it is not transparent for mere mortals like this author. Nor have I found a proof that is sufficiently transparent and brief to justify presention in this text. Lengthy proofs are available, see [Adler (1965)] and [Chandrasekhar (1983)]. For our purposes I'll ask you to accept that the metric written in terms of spherical coordinates has the following form,

$$(d\tau)^2 = \left(1 - Rr/\sum\right)(dt)^2 + \left(2Rra\sin^2\theta/\sum\right)dtd\phi - \left(\sum/\Lambda\right)(dr)^2$$
$$-\sum(d\theta)^2 - \sin^2\theta\left(r^2 + a^2 + Rra^2\sin^2\theta/\sum\right)(d\phi)^2, \tag{7.8}$$

$$\Lambda = r^2 + a^2 - Rr, \qquad \sum = r^2 + a^2\cos^2\theta. \tag{7.9}$$

When, $a = 0$, the metric becomes that of a static, Schwarzschild black hole. One observes that the angular momentum forces an additional constant, a, into the metric. That constant has units of meters in our naturalized units, but that is also the units of angular momentum per unit mass, and it is interpreted as such.

Since, $(d\tau)^2 = -g_{\mu\nu}dx^\mu dx^\nu$, here the element of area, $[g_{11}g_{22}]^{1/2}d\theta d\phi$, does not have the form of the element of area for a sphere, $r^2 \sin\theta d\theta d\phi$, and so, r cannot be interpreted as the radial coordinate of a sphere. The metric, $g_{\mu\nu}$, has off diagonal elements, but it is easy to find, $g^{\mu\nu}$, using, $g^{\xi\mu}g_{\xi\nu} = \delta^\mu_\nu$. Problem five, asks for the proof of the following results,

$$g^{ii} = 1/g_{ii},\ i = 1, 3, \tag{7.10}$$

$$g^{02} = g^{20} = g_{02}/\Psi,\ g^{00} = -g_{22}/\Psi,\ g^{22} = -g_{00}/\Psi, \tag{7.11}$$

$$\Psi = \Lambda \sin^2\theta. \tag{7.12}$$

The metric elements are independent of, x^0 and x^2, so that, P_0 and P_2, are constants of the motion for particles with rest mass. So even if a particle is placed into the field with $P_2 = 0$ but $P_0 \neq 0$, subsequently the angular velocity is nonzero,

$$\begin{aligned}
P^0 &= m\frac{dt}{d\tau} = g^{0\nu}P_\nu = g^{00}P_0,\\
P^2 &= m\frac{d\phi}{d\tau} = g^{2\nu}P_\nu = g^{20}P_0,\\
\frac{d\phi}{dt} &= \frac{d\phi}{d\tau}\Big/\frac{dt}{d\tau} = \frac{g^{20}}{g^{00}} = -\frac{g_{20}}{g_{22}},\\
&= \frac{Rra}{\Sigma[r^2 + a^2] + Rra^2\sin^2\theta} > 0.
\end{aligned}$$

Thus the particle will develop angular momentum and angular velocity because of the metric. The angular velocity is in the direction of the spinning mass. This effect is called frame-drag.

GR predicts that a gyroscope orbiting a non-spinning spherically symmetric mass will precess (Geodetic precession) in its plane of motion. Frame-drag causes gyroscope precession perpendicular to the plane of motion, the Lense-Thirring effect. In order to confirm these effects, the Gravity Probe B experiment, was started about fifty years ago by NASA. The experiment consists of four extremely sensitive gyros in a polar orbit about earth. The experimental results [Everitt (2011)], in milliarcseconds/year are: Geodetic precession, 6602 ± 18, and frame-drag, 37.2 ± 7.2. These are to be compared with the GR predictions: 6606 and 39.2. The Geodetic result is in excellent agreement with GR. Frame-drag precession is a much smaller effect and the large error is due to consideration of numerous sources of systematic error. So while in agreement with GR, it can't rule out competing theories.

An interesting effect for light is predicted when $dr = d\theta = 0$, $\theta = \pi/2$, $r = R$,

$$\begin{aligned}
(d\tau)^2 = 0 &= (1 - R/r)(dt)^2 + (2Ra/r)dtd\phi - (r^2 + a^2 + Ra^2/r)(d\phi)^2, \\
0 &= 1 - R/r + (2Ra/r)\frac{d\phi}{dt} - (r^2 + a^2 + Ra^2/r)\left(\frac{d\phi}{dt}\right)^2, \\
\frac{d\phi}{dt} &= \frac{-2Ra/r \pm [(2Ra/r)^2 + 4(1 - R/r)(r^2 + a^2 + Ra^2/r]^{1/2}}{-2[r^2 + a^2 + Ra^2/r)]}, \\
&= \frac{a \mp a}{(R^2 + 2a^2)}.
\end{aligned}$$

The solution, $\frac{d\phi}{dt} = 2a/(R^2 + 2a^2)$, represents light going in the same direction as the rotating black hole. As dt is the proper time of a far away observer, the angular velocity according to that observer is constrained by the properties of the black hole. The other solution is zero, a rather unexpected result, if light is emitted in the direction opposite that of the black hole's rotation. A material particle that moves slower than light would soon find itself rotating in the same direction as the black hole even if it started out with arbitrarily large kinetic energy rotating in the opposite direction.

However, one must realize that the far away observer can neither confirm nor falsify this prediction for light speed as that observer cannot observe these photons. If an effect can neither be confirmed nor falsified, then we are not speaking of a scientific prediction. As noted in chapter two, problem fifteen, the measurement of light speed requires a local observer, if there could be one, armed with a proper time clock and proper distance measuring device. That observer would experience severe frame-drag, move in the same direction as the black hole, and would find the photon traveling with $c = \frac{dL_p}{d\tau} = 1$ in the opposite direction.

The nature of the singularities for a black hole with angular momentum are more complicated than that of a Schwarzschild black hole. There is a true singularity at $\Sigma = 0,\ (r,\ \theta) = (0,\ \pi/2)$, as every term in $(d\tau)^2$ blows up there except for the $(dr)^2$ and $(d\theta)^2$ terms both of which vanish. You could also form the scalar, $R^{\xi\mu\nu\chi}R_{\xi\mu\nu\chi}$, to see that it has the above singularity condition.

In the case of the Schwarzschild black hole there was a coordinate singularity because the $(dr)^2$ term blew up at $r = R$. In this case, $g_{33} = \frac{\Sigma}{\Lambda} \to \infty$, when, $\Lambda \to 0$,

$$0 = r^2 + a^2 - Rr, \quad R_{\pm} = R/2 \pm ([R/2]^2 - a^2)^{1/2}. \tag{7.13}$$

One observes that the maximum limit for a is $R/2$. This is called the static limit and here, $R_+ = R_-$. The inner horizon, given by R_-, is also called the Cauchy horizon. Both horizons are one way, you can cross only going in. The outside or event horizon is the boundary between the black hole and the outside world and if $a = 0$ reduces to R, but in this case it is smaller.

For a Schwarzschild black hole, $g_{00} = 0$ when $g_{33} = \infty$. However, for a rotating black hole the surface where $g_{00} = 0$ occurs when,

$$\begin{aligned} 0 &= \Sigma - Rr = r^2 - Rr + a^2 \cos^2\theta, \\ r_\pm &= R/2 \pm ([R/2]^2 - a^2\cos^2\theta)^{1/2}. \end{aligned} \tag{7.14}$$

The values of $r_\pm$ depend on θ. In general, $r_- \leq R_+$, and is within the event horizon. It is also within the Cauchy horizon except at $|\cos\theta| = 1$, where they coincide. The region between, R_+ and r_+, is called the ergosphere. In this region, all objects experience frame-drag. Due to the, $\cos^2\theta$, term, the outer extremity of the ergosphere is not a true sphere. It is flattened and equal to the event horizon at $|\cos\theta| = 1$, but otherwise extends beyond the event horizon and so, is accessible to observation.

Black holes are predicted to have a finite life span. They evaporate with the emission of Hawking radiation that has a very low temperature black body spectrum. It is a quantum effect in the region of extreme gravity, a still unsolved problem in general. Here the process is very briefly described. Near the event horizon quantum fluctuations produce particle anti-particle pairs. Normally these particles would immediately recombine, before energy and momentum non-conservation could be observed. However, the tremendous gravity at the horizon may capture one of the particles before recombination. The net effect to an outside observer, could such low temperatures be detected, is that a particle has emerged from — and thus mass has been lost by — the black hole, or R has decreased.

7.3 An Interstellar Example

The movie "Interstellar" illustrates some wonderful effects concerning GR and black holes. Here, one such effect, showing a distinct difference between a spinning and a static black hole, is discussed. Astronauts, from a dying earth, punch through a worm hole. They find themselves in a different part of the universe, that otherwise is unreachable, because of the large proper distance from earth. They explore the planets of a solar system, whose star is a huge, rotating black hole, aptly named Gargantua. They hope to find a planet that can serve as a new home. One planet is in a circular orbit very close to the event horizon. This is possible because Gargantua is rotating extremely close to its maximum value, $a = (1 - 1.3 \times 10^{-14})R/2$. An enormous time dilation factor of $\approx 6 \times 10^4$, as required by the movie director, is experienced, [Thorne (2014)]. The pilot astronaut spends a short time — hour(s) — on the planet, and after the entire trip, has hardly aged. However, he returns to find the young daughter he left behind, is now an aged woman on her death bed.

This example shows how such phenomena can be possible with a spinning, but not a static, black hole. Solve problem seven and find that orbits with $\theta = \pi/2$ are possible, just as in the static case. In problem eight, the velocity components, U^0 and U^3, are calculated,

$$U^0 = \frac{dt}{d\tau} = -E'^{-1/2}\Lambda^{-1}\left(g_{22} - g_{02}J\right), \tag{7.15}$$

$$(U^3)^2 = \left(\frac{dr}{d\tau}\right)^2 = r^{-2}\left(-\Lambda + E'^{-1}\left[g_{22} - 2g_{02}J + g_{00}J^2\right]\right), \tag{7.16}$$

where the constants of the motion are $U_0 \equiv E'^{-1/2}$ and $U_2 \equiv JE'^{-1/2} \equiv \bar{J}$. The former constant is dimensionless and the latter has units of meters or angular momentum per unit mass. The time period dt is that read on an at rest, far away clock and $d\tau$ is the time period on a clock attached to the planet.

The solution proceeds as for chapter six problem seven, where circular orbits in the Schwarzschild metric were studied. The expression for $(U^3)^2$ is, per unit mass, a radial kinetic energy term expressed as a function of a constant energy and an effective potential energy $f[r, J, E']$. For stable circular orbits $U^3 = 0$, $\frac{df}{dr} = 0$, and $\frac{d^2 f}{dr^2} > 0$. These conditions yield,

$$E'^{-1} = r^{-2}\left(r^2 + a^2 - rR + \frac{J^2 - a^2}{E'} - \frac{R\left[J + a\right]^2}{rE'}\right),$$

$$\begin{aligned} 0 &= -2r^{-3}\left(r^2 + a^2 - rR + \frac{J^2 - a^2}{E'} - \frac{R\left[J + a\right]^2}{rE'}\right) \\ &\quad + r^{-2}\left(2r - R + \frac{R\left[J + a\right]^2}{r^2 E'}\right), \\ &= r^{-4}\left[Rr^2 - 2r\left(a^2 + \frac{J^2 - a^2}{E'}\right) + \frac{3R\left[J + a\right]^2}{E'}\right]. \end{aligned}$$

Thus,

$$r = \frac{1}{R}\left(a^2 + \frac{J^2 - a^2}{E'}\right) \times \left[1 \pm \left(1 - \frac{3R^2\left[J + a\right]^2}{E'\left[a^2 + \frac{J^2-a^2}{E'}\right]^2}\right)^{1/2}\right]. \tag{7.17}$$

When $a = 0$ the solution is,

$$= \frac{\bar{J}^2}{R}\left[1 \pm \left(1 - \frac{3R^2}{\bar{J}^2}\right)^{1/2}\right]. \tag{7.18}$$

The static result agrees with the solution to chapter six problem seven where it was found that only the positive sign led to stable orbits. The requirement that the argument of the square root isn't negative yields $3R^2/\bar{J}^2 \leq 1$. The time dilation is greatest for small values of r so take $\bar{J}^2 = 3R^2$. Then Eqs. (7.18), (7.16), and (7.15) become,

$$\begin{aligned} r &= 3R, \\ E'^{-1} &= (1 - R/r)(1 + \bar{J}^2/r^2) = 8/9, \\ \frac{dt}{d\tau} &= g^{00}U_0 = U_0/g_{00} = -(1 - R/r)^{-1}E'^{-1/2}, \\ &\approx (3/2)(8/9)^{1/2} = 1.4. \end{aligned}$$

So from a static black hole, a time dilation factor ≤ 1.4 is possible.

In Gargantua's case, you might expect a larger time dilation, as the radial coordinate of the event horizon is half what it would be in the static case. However, a time dilation factor of 6×10^4 is truly impressive. The circular orbit solutions, using Eq. (7.17), are far more complicated. In the case of a spinning black hole, the two constants of the motion do not group together as in the static case, where one constant $\bar{J}$ determines the stable orbit radial coordinate. One cannot pick any arbitrary pair of values for these constants, because they are intimately tied to the radial coordinate through $\frac{df}{dr} = 0$, a quadratic equation, and Eq. (7.16), a cubic equation, that must be simultaneously satisfied.

See [Chandrasekhar (1983)] for the full flavor of the problem's complexity. In order to show that Eqs. (7.15) and (7.16) coincide with those of Chandrasekhar, rename the constants of the motion: $E'^{-1/2} = -E$, $JE'^{-1/2} = L$. To start the solution for the closest in stable circular orbits, Chandrsekhar first shows that E and L, can be obtained once r is known,

$$Q = 1 - \frac{R}{2}\frac{3}{r} + 2a\left(\frac{R}{2r^3}\right)^{1/2},$$

$$E = \frac{1}{Q^{1/2}}\left(1 - \frac{R}{r} + a\left(\frac{R}{2r^3}\right)^{1/2}\right),$$

$$L = \left(\frac{rR}{2Q}\right)^{1/2}\left[\left(\frac{a}{r}\right)^2 + 1 - 2a\left(\frac{R}{2r^3}\right)^{1/2}\right].$$

Then Chandrsekhar quotes, unfortunately without proof, the following result that is very important for this example,

$$r = \frac{R}{2}\left(1 + (4\delta)^{1/3}\right), \quad \text{when } a = \frac{R}{2}\left(1-\delta\right), \ \delta \to 0.$$

If one now uses $\delta = 1.3 \times 10^{-14}$ the value quoted in [Thorne (2014)], the following results are obtained,

$$r = 0.500019R, \quad E,\ L = 0.577372 \times (1,\ R), \quad \frac{dt}{d\tau} = 61874.$$

So the movie illustrates a valid scientific effect.

Problems

1. A massive, spherically symmetric star goes supernova. The remains of the star are a mass three times that of the sun, $M' = 3M_s$, that will collapse radially to a static black hole. If the remains start with a surface radial coordinate twice that of the sun, $r = 2R_s$, what is R/r, where R is the Schwarzschild radius? The surface starts its collapse from rest. As the star collapses, excited, ${}^4He$, atoms at the surface emit, radially, photons of characteristic frequency, ν', while the atoms continue to freely fall with the rest of the star. What frequency, ν, will be observed by a stationary observer at, $2R_s$, in terms of the photon emission radial coordinate, r? When the collapse goes through, R, show that the photons are red shifted to the extreme. As a hint, recall Eqs. (2.27), (2.28) and (6.2)–(6.6).
2. Other metrics can be used to illustrate that light cannot escape from a singularity. The following metric from the element of proper time will

also do the trick,

$$(d\tau)^2 = (1 - M'/r)^2(dt)^2 - (1 - M'/r)^{-2}(dr)^2 \\ - r^2((d\theta)^2 + \sin^2\theta(d\phi)^2).$$

There are singularities at $r = M'$ and $r = 0$. Change coordinates to (v, r, θ, ϕ) with the transformation, $t = v - f(r)$, so that $g_{rr} = 0$. Then the singularity at $r = M'$ is removed. Find $f_{,r}$ the new metric, and f such that, $v = 0$ at $r = t = 0$. Does this metric describe a black hole? That is can light originating at $r < M'$ get to $r > M'$ while light originating at $r > M'$ can get to both $r < M'$ and $r = \infty$.

3. A particle crosses the event horizon, $r = R$, of a Schwarzschild black hole. What is the maximum proper time to get to the singularity at $r = 0$? Suppose the particle starts from rest at, $r = 5R$, how much proper time passes before it reaches $r = R$?
4. In problem three the proper time for travel to the horizon of the black hole is finite. Here the time, t, on a clock of an at rest, far away observer, is considered. Take a particle at rest far from the black hole so that, $E' \approx 1$. Calculate how long it takes to get to the horizon of the black hole?
5. Show Eqs. (7.8) and (7.9) lead to Eqs. (7.10)–(7.12).
6. Compare the ratios of the radial coordinates and areas of the horizon of a static Schwarzschild black hole to a rotating black hole.
7. For the static Schwarzschild black hole, a solution, $\theta = \pi/2$, is possible. Write out the equation of motion for the coordinate, θ, and show it also a solution for a rotating black hole?
8. For a rotating black hole and the solution $\theta = \pi/2$, find $\frac{dr}{d\tau}$ for a particle with rest mass, m. If you use the fact that $P_0 = P_t$ and $P_2 = P_\phi$ are constants of the motion, you'll get the result faster than if you write out, $\frac{d^2x^\xi}{d\tau^2} + \Gamma^\xi_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$. Use the following definitions: $E'^{-1/2} \equiv P_0/m$ and $JE'^{-1/2} \equiv P_2/m$.
9. Using the results of problems seven and eight, find the orbit equation, $\frac{dr}{d\phi}$, the angular velocity, $\frac{d\phi}{dt}$ and $\frac{dr}{dt}$.
10. Use the results of problems seven, eight and nine. Suppose a black hole has maximum spin, $a = R/2$. A massive particle is released from rest at, $r = 5R$, what is, $\frac{d\phi}{dt}$ and $\frac{dr}{dt}$, at the outer edge of the ergosphere and at the horizon?

Chapter 8

Gravitational Waves

8.1 The Weak Gravity Wave Equation

Einstein's equations predict gravitational waves. The sources of the waves typically involve very strong gravitational effects, for example, the onset of a supernova. However, such sources are usually so far away that the wave amplitude at earth is extremely small and the wave can be treated in the weak gravity approximation. At this time it would be beneficial to review section two and problem one of chapter five, where, in rectangular coordinates, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$ and $|h_{\mu\nu,\xi}| \ll 1$. In deriving the wave equation, some important properties of $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$, the trace inverse, defined below, are of importance,

$$\begin{aligned}
h_{\mu\nu} &= h_{\nu\mu}, \\
h^{00} &= h_{00},\ \ h^{ij} = h_{ij},\ \ h^{0i} = -h_{0i}, \\
h^{\chi}{}_{\nu} &= g^{\chi\mu} h_{\mu\nu}, \\
&= (\eta^{\chi\mu} - h^{\chi\mu}) h_{\mu\nu} \approx \eta^{\chi\mu} h_{\mu\nu}, \\
h &\equiv h^{\nu}{}_{\nu}, \\
\bar{h}_{\mu\nu} &\equiv h_{\mu\nu} - \eta_{\mu\nu} h/2.
\end{aligned} \tag{8.1}$$

In problem one, you are asked to show the steps that lead from here to,

$$\bar{h}^{\beta}{}_{\nu} = h^{\beta}{}_{\nu} - \delta^{\beta}{}_{\nu} h/2, \quad \bar{h} = -h. \tag{8.2}$$

In order to get to a wave equation, a gauge transformation is needed. Such transforms are encountered in electrodynamics, where, for example, the vector potentials, $\vec{A}$ and $\vec{A}\,' = \vec{A} + \vec{\nabla}\Psi$, yield the same magnetic field. However, if $\nabla \cdot \vec{A} \neq 0$, Ψ can be chosen so that $\nabla \cdot \vec{A}' = 0$. Here, each coordinate is slightly changed, such that, a new metric, also approximately that of SR, is obtained. The gauge vectors are functions of the coordinates, $\xi^\mu[x^\nu]$. They are taken to be small as are their derivatives, $|\xi^\mu| \ll 1$, and $|\xi^\mu{}_{,\beta}| \ll 1$. To first order in small quantities,

$$\begin{aligned}
x^{\alpha'} &\equiv x^\alpha + \xi^\alpha[x^\beta],\\
x^\alpha &= x^{\alpha'} - \xi^\alpha[x^\beta] = x^{\alpha'} - \xi^\alpha[x^{\beta'} - \xi^\beta] \approx x^{\alpha'} - \xi^\alpha[x^{\beta'}],\\
\xi^\alpha{}_{,\beta} &= \xi^\alpha{}_{,\beta'}\, x^{\beta'}{}_{,\beta} = \xi^\alpha{}_{,\beta'}\,(x^\beta + \xi^\beta)_{,\beta}\,,\\
&\approx \xi^\alpha{}_{,\beta'}\,,\\
\delta^{\alpha'}_{\beta'} &= x^{\alpha'}{}_{,\beta'} = (x^\alpha + \xi^\alpha)_{,\beta}\, x^\beta{}_{,\beta'}\,,\\
&= (\delta^\alpha_\beta + \xi^\alpha{}_{,\beta})(x^{\beta'} - \xi^\beta)_{,\beta'}\,,\\
&= (\delta^\alpha_\beta + \xi^\alpha{}_{,\beta})(1 - \xi^\beta{}_{,\beta'}),\\
&= \delta^\alpha_\beta + \xi^\alpha{}_{,\beta} - \delta^\alpha_\beta \xi^\beta{}_{,\beta'}\,,\\
&= \delta^\alpha_\beta + \xi^\alpha{}_{,\beta} - \xi^\alpha{}_{,\beta'}\,,\\
&= \delta^\alpha_\beta = x^\alpha{}_{,\beta}\,,\ \text{ergo } \eta_{\mu'\nu'} = \eta_{\mu\nu},\\
x^\alpha{}_{,\beta'} &= x^{\alpha'}{}_{,\beta'} - \xi^\alpha{}_{,\beta'} = \delta^\alpha_\beta - \xi^\alpha{}_{,\beta}\,,\\
x^{\alpha'}{}_{,\beta} &= x^\alpha{}_{,\beta} + \xi^\alpha{}_{,\beta} = \delta^\alpha_\beta + \xi^\alpha{}_{,\beta}\,,\\
g_{\alpha'\beta'} &= \eta_{\alpha'\beta'} + h_{\alpha'\beta'} = \eta_{\alpha\beta} + h_{\alpha'\beta'}.
\end{aligned} \tag{8.3}$$

In problem one, you are asked to show the steps that lead from here to,

$$\begin{aligned}
h_{\alpha\beta} &= h_{\alpha'\beta'} + \xi_{\alpha,\beta} + \xi_{\beta,\alpha}\,,\\
h_{\alpha'\beta'} &= h_{\alpha\beta} - (\xi_{\alpha,\beta} + \xi_{\beta,\alpha}).
\end{aligned} \tag{8.4}$$

The gist of all this is that $h_{\mu\nu}$ is not a unique tensor, but it remains small when it is changed by a gauge transformation. The new form, obtained from $g_{\mu'\nu'} = x^\xi{}_{,\mu'}\, x^\chi_{\nu'} g_{\xi\chi}$, may be more advantageous to solving a particular problem.

To first order the Einstein tensor becomes a wave equation proportional to the energy-momentum tensor that is interpreted as the source. The calculation begins with the curvature tensor,

$$\begin{aligned}
R_{\alpha\beta\mu\nu} &= g_{\alpha\chi} R^\chi{}_{\beta\mu\nu} = g_{\alpha\chi}[\Gamma^\chi_{\beta\nu}{}_{,\mu} - \Gamma^\chi_{\beta\mu}{}_{,\nu} + \Gamma^\chi_{\gamma\mu}\Gamma^\gamma_{\beta\nu} - \Gamma^\chi_{\gamma\nu}\Gamma^\gamma_{\beta\mu}],\\
\Gamma^\chi_{\beta\mu} &= g^{\sigma\chi}[g_{\beta\sigma,\mu} + g_{\mu\sigma,\beta} - g_{\beta\mu,\sigma}]/2 = \eta^{\sigma\chi}[h_{\beta\sigma,\mu} + h_{\mu\sigma,\beta} - h_{\beta\mu,\sigma}]/2.
\end{aligned}$$

The, $\Gamma\Gamma$, terms in the curvature tensor are at least second order in, $h_{\mu\nu},_{\xi}$, and can be neglected. The derivative terms become,

$$\begin{aligned}\Gamma^{\chi}_{\beta\mu},_{\nu} &= \eta^{\sigma\chi}[h_{\beta\sigma},_{\mu},_{\nu} + h_{\mu\sigma},_{\beta},_{\nu} - h_{\beta\mu},_{\sigma},_{\nu}]/2,\\ \Gamma^{\chi}_{\beta\nu},_{\mu} &= \eta^{\sigma\chi}[h_{\beta\sigma},_{\nu},_{\mu} + h_{\nu\sigma},_{\beta},_{\mu} - h_{\beta\nu},_{\sigma},_{\mu}]/2,\\ &= \eta^{\sigma\chi}[h_{\beta\sigma},_{\mu},_{\nu} + h_{\nu\sigma},_{\beta},_{\mu} - h_{\beta\nu},_{\sigma},_{\mu}]/2.\end{aligned}$$

The curvature tensor is,

$$\begin{aligned}R_{\alpha\beta\mu\nu} &= \eta_{\alpha\chi}\eta^{\sigma\chi}[h_{\nu\sigma},_{\beta},_{\mu} - h_{\beta\nu},_{\sigma},_{\mu} - h_{\mu\sigma},_{\beta},_{\nu} + h_{\beta\mu},_{\sigma},_{\nu}]/2,\\ &= [h_{\nu\alpha},_{\beta},_{\mu} - h_{\beta\nu},_{\alpha},_{\mu} - h_{\mu\alpha},_{\beta},_{\nu} + h_{\beta\mu},_{\alpha},_{\nu}]/2.\end{aligned}$$

Using the metric, the Ricci tensor can be calculated,

$$\begin{aligned}R_{\beta\nu} &= g^{\alpha\mu}R_{\alpha\beta\mu\nu} = \eta^{\alpha\mu}R_{\alpha\beta\mu\nu},\\ &= [h_{\nu}{}^{\mu},_{\beta},_{\mu} - \eta^{\alpha\mu}h_{\beta\nu},_{\alpha},_{\mu} - h_{\mu}{}^{\mu},_{\beta},_{\nu} + h_{\beta}{}^{\alpha},_{\alpha},_{\nu}]/2,\\ &= [h_{\nu}{}^{\mu},_{\beta},_{\mu} - \eta^{\alpha\mu}h_{\beta\nu},_{\alpha},_{\mu} - h,_{\beta},_{\nu} + h_{\beta}{}^{\alpha},_{\alpha},_{\nu}]/2.\end{aligned} \tag{8.5}$$

In problem one you will be asked to show that the second term in the above equation is,

$$-(\nabla^2 - \frac{\partial^2}{\partial t^2})h_{\beta\nu} \equiv -\Box h_{\beta\nu}.$$

So one gets an inkling of how the wave equation comes about.

The Ricci scalar and the Einstein tensor are now evaluated. Renaming some summed over indexes helps carry out the calculation, the scalar is,

$$\begin{aligned}R &= g^{\beta\nu}R_{\beta\nu} = \eta^{\beta\nu}R_{\beta\nu},\\ &= \eta^{\beta\nu}[h_{\nu}{}^{\mu},_{\beta},_{\mu} - \eta^{\alpha\mu}h_{\beta\nu},_{\alpha},_{\mu} - h,_{\beta},_{\nu} + h_{\beta}{}^{\alpha},_{\alpha},_{\nu}]/2,\\ &= [h^{\beta\mu},_{\mu},_{\beta} - \eta^{\alpha\mu}h^{\beta}{}_{\beta},_{\mu},_{\alpha} - \eta^{\beta\nu}h,_{\beta},_{\nu} + h^{\nu\alpha},_{\alpha},_{\nu}]/2,\\ &= h^{\mu\alpha},_{\mu},_{\alpha} - \eta^{\alpha\mu}h,_{\mu},_{\alpha}.\end{aligned}$$

This yields the Einstein tensor,

$$\begin{aligned}G_{\beta\nu} &= R_{\beta\nu} - g_{\beta\nu}R/2 \approx R_{\beta\nu} - \eta_{\beta\nu}R/2,\\ &= [h_{\nu}{}^{\mu},_{\beta},_{\mu} - \eta^{\alpha\mu}h_{\beta\nu},_{\alpha},_{\mu} - h,_{\beta},_{\nu} + h_{\beta}{}^{\alpha},_{\alpha},_{\nu}\\ &\quad - \eta_{\beta\nu}(h^{\mu\alpha},_{\mu},_{\alpha} - \eta^{\alpha\mu}h,_{\mu},_{\alpha})]/2.\end{aligned}$$

If the above is written in terms of the variable $\bar{h}$, the trace inverse, instead of h, and a gauge transform is applied, out pops the wave equation.

$$\begin{aligned}
G_{\beta\nu} &= (1/2)[(\bar{h}_\nu{}^\mu - \delta_\nu{}^\mu \bar{h}/2)_{,\beta}{}_{,\mu} - \eta^{\alpha\mu}(\bar{h}_{\beta\nu} - \eta_{\beta\nu}\bar{h}/2)_{,\alpha}{}_{,\mu} \\
&\quad + \bar{h}_{,\beta}{}_{,\nu} + (\bar{h}_\beta{}^\alpha - \delta_\beta{}^\alpha \bar{h}/2)_{,\alpha}{}_{,\nu} \\
&\quad - \eta_{\beta\nu}((\bar{h}^{\mu\alpha} - \eta^{\mu\alpha}\bar{h}/2)_{,\mu}{}_{,\alpha} + \eta^{\alpha\mu}\bar{h}_{,\mu}{}_{,\alpha})], \\
&= (1/2)(\bar{h}_\nu{}^\mu{}_{,\beta}{}_{,\mu} - \eta^{\alpha\mu}\bar{h}_{\beta\nu,\alpha}{}_{,\mu} + \bar{h}_\beta{}^\alpha{}_{,\alpha}{}_{,\nu} - \eta_{\beta\nu}\bar{h}^{\mu\alpha}{}_{,\mu}{}_{,\alpha} \\
&\quad - [\delta_\nu{}^\mu \bar{h}_{,\beta}{}_{,\mu} - 2\bar{h}_{,\beta}{}_{,\nu} + \delta_\beta{}^\alpha \bar{h}_{,\alpha}{}_{,\nu} \\
&\quad - (\eta_{\beta\nu}\eta^{\alpha\mu}\bar{h}_{,\alpha}{}_{,\mu} + \eta_{\beta\nu}\eta^{\mu\alpha}\bar{h}_{,\mu}{}_{,\alpha} - 2\eta_{\beta\nu}\eta^{\mu\alpha}\bar{h}_{,\mu}{}_{,\alpha})]/2), \\
&= [\bar{h}_\nu{}^\mu{}_{,\beta}{}_{,\mu} - \eta^{\alpha\mu}\bar{h}_{\beta\nu,\alpha}{}_{,\mu} + \bar{h}_\beta{}^\alpha{}_{,\alpha}{}_{,\nu} - \eta_{\beta\nu}\bar{h}^{\mu\alpha}{}_{,\mu}{}_{,\alpha}]/2, \\
&= [(\bar{h}_\nu{}^\mu{}_{,\mu})_{,\beta} - \eta^{\alpha\mu}\bar{h}_{\beta\nu,\alpha}{}_{,\mu} + (\bar{h}_\beta{}^\alpha{}_{,\alpha})_{,\nu} - \eta_{\beta\nu}\eta^{\mu\xi}(\bar{h}_\xi{}^\alpha{}_{,\alpha})_{,\mu}]/2.
\end{aligned}$$

The Lorentz gauge, in analogy with electrodynamics, is $\bar{h}_\sigma{}^\chi{}_{,\chi} = 0$. Then the above equation simplifies to a single term. There are four equations, $\bar{h}_\sigma{}^\chi{}_{,\chi} = 0$, one for each σ. There are four free gauge functions, ξ^σ. So it is possible to find a gauge, using Eq. (8.4), for which the simplification holds. The steps are traced out in problem two. Then, the Einstein tensor becomes,

$$\begin{aligned}
2G_{\beta\nu} &= 16\pi T_{\beta\nu} = -\eta^{\alpha\mu}\bar{h}_{\beta\nu,\alpha}{}_{,\mu} = -\Box\bar{h}_{\beta\nu}, \\
\Box\bar{h}_{\beta\nu} &= -16\pi T_{\beta\nu}, \quad \Box\bar{h}^{\beta\nu} = -16\pi T^{\beta\nu}.
\end{aligned} \tag{8.6}$$

This is a wave equation with a source proportional to the energy-momentum tensor. The wave propagates with the speed of light and all frequencies are allowed.

8.2 Plane Waves

In empty space there is no source, so plane wave solutions are possible. With enough plane waves of different wave vectors and accompanying amplitudes, any wave shape can be accommodated by superposition. In the case of a single wave vector, k_χ, with amplitude, $A_{\mu\nu}$, a complex constant, the wave function in rectangular coordinates is the real part of $\bar{h}_{\mu\nu} = A_{\mu\nu}\exp(ik_\chi x^\chi)$. The phase factor is an invariant. It is easily shown that,

$$\bar{h}_{\mu\nu},_{\beta} = ik_{\beta}\bar{h}_{\mu\nu}, \tag{8.7}$$

$$\Box\bar{h}_{\mu\nu} = -(k_{\beta}k^{\beta})\bar{h}_{\mu\nu} = 0, \quad k_{\beta}k^{\beta} = 0. \tag{8.8}$$

As with an electromagnetic wave, the relation between frequency, $k^0 \equiv \omega$, and wave 3-vector, $\vec{k}$, is identical. In free space there is no dispersion and the phase and group velocities are unity. The direction of $\vec{k}$ at a particular point is the direction of wave travel at that point. The gauge condition forced $\bar{h}_{\mu}{}^{\nu},_{\nu} = 0$. Thus,

$$k_{\nu}A_{\mu}{}^{\nu} = 0. \tag{8.9}$$

This is another restriction, an orthogonality restriction, on, $A_{\mu}{}^{\nu}$.

A more useful solution can be obtained by again applying a gauge transformation, with vector, ξ_{α}, that satisfies $\Box\xi_{\alpha} = 0$. It can produce a solution, ${}^{TT}\bar{h}_{\mu\nu}$, with amplitude, $\bar{A}_{\mu\nu}$, that is traceless, ${}^{TT}\bar{h}_{\mu}{}^{\mu} = \bar{A}_{\mu}{}^{\mu} = 0$. Using $\xi_{\mu} = B_{\mu}\exp(ik_{\chi}r^{\chi})$, and results from problem two,

$$\begin{aligned}
\bar{h}_{\mu'\nu'} &\equiv {}^{TT}\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu} - \xi_{\mu},_{\nu} - \xi_{\nu},_{\mu} + \eta_{\mu\nu}\xi^{\chi},_{\chi}, \\
\bar{A}_{\mu\nu} &= A_{\mu\nu} - i(B_{\mu}k_{\nu} + k_{\mu}B_{\nu} - \eta_{\mu\nu}B^{\chi}k_{\chi}), \qquad (8.10) \\
\bar{A}_{\mu}{}^{\alpha} &= A_{\mu}{}^{\alpha} - i(B_{\mu}k^{\alpha} + k_{\mu}B^{\alpha} - \delta_{\mu}{}^{\alpha}B^{\chi}k_{\chi}), \\
0 = \bar{A}_{\mu}{}^{\mu} &= A_{\mu}{}^{\mu} - i(B_{\mu}k^{\mu} + k_{\mu}B^{\mu} - 4B^{\chi}k_{\chi}), \\
&= A_{\mu}{}^{\mu} + 2iB_{\mu}k^{\mu} = A_{\mu}^{\mu} + 2iB^{\mu}k_{\mu}, \quad iB^{\mu}k_{\mu} = -A_{\mu}{}^{\mu}/2. \qquad (8.11)
\end{aligned}$$

Thus, if $A_{\mu}{}^{\mu} \neq 0$, one can choose the B_{μ} to make $\bar{A}_{\mu}{}^{\mu} = 0 = {}^{TT}\bar{h}_{\mu}{}^{\mu}$. This gives one requirement for the four B_{μ}. The other three can be used to impose the condition $\bar{A}_{\mu\nu}U^{\nu} = 0$, because from Eqs. (8.9) and (8.10),

$$\begin{aligned}
k^{\mu}\bar{A}_{\mu\nu} &= k^{\mu}A_{\mu\nu} - i(B_{\mu}k^{\mu}k_{\nu} + B_{\nu}k^{\mu}k_{\mu} - k_{\nu}B^{\chi}k_{\chi}) = k^{\mu}A_{\mu\nu} = 0, \\
0 &= (\bar{A}_{\mu\nu}k^{\mu})U^{\nu} = k^{\mu}(\bar{A}_{\mu\nu}U^{\nu}). \qquad (8.12)
\end{aligned}$$

The remaining three requirements on the four B_{μ} can force,

$$\bar{A}_{j\nu}U^{\nu} = 0. \tag{8.13}$$

Then Eqs. (8.12) and (8.13) force,

$$\bar{A}_{0\nu}U^{\nu} = 0. \tag{8.14}$$

In the language of the subject, the superscript, TT, stands for transverse traceless gauge. In order to see how it is transverse, let the wave be moving in the $z = 3$ direction and pick $U^\nu = \delta^\nu{}_0$. Then,

$$0 = U^\nu \bar{A}_{\mu\nu} = \delta^\nu{}_0 \bar{A}_{\mu\nu} = \bar{A}_{\mu 0} = \bar{A}^{\mu 0},$$
$$= k_\nu \bar{A}^{\mu\nu} = \omega(-\bar{A}^{\mu 0} + \bar{A}^{\mu 3}) = \omega \bar{A}^{\mu 3}, \; \bar{A}^{\mu 3} = 0,$$

In this case only, $\bar{A}_{(11),(22),(12),(21)}$ are nonzero and the symmetry and traceless conditions yield,

$$\bar{A}_{12} = \bar{A}_{21},$$
$$0 = \bar{A}_\mu{}^\mu = \bar{A}_1{}^1 + \bar{A}_2{}^2,$$
$$\bar{A}_1{}^1 = -\bar{A}_2{}^2, \qquad \bar{A}_{11} = -\bar{A}_{22}. \tag{8.15}$$

In this gauge the following also holds,

$$-({}^{TT}h) = {}^{TT}\bar{h} \; = \; {}^{TT}\bar{h}_\mu{}^\mu \; = \; 0, \text{ so,}$$
$${}^{TT}\bar{h}_{\mu\nu} = {}^{TT}h_{\mu\nu} - \eta_{\mu\nu}({}^{TT}h)/2 \; = \; {}^{TT}h_{\mu\nu}. \tag{8.16}$$

8.3 The Graviton

For electromagnetic waves, the wave function of the vector field, A_μ, can describe all of the physics. When this field is quantized, the quanta are photons with spin, $s = 1$. In quantum electrodynamics, the interactions to lowest order are the exchange of virtual photons. In GR, the wave function of the field describing the physics is a tensor of rank two, $\bar{h}_{\mu\nu}$. Thus a quantum theory of gravity has a exchange particle of spin, $s = 2$, with zero rest mass called the graviton.

A transparent way to see this is to consider what happens to a transverse electromagnetic or transverse, traceless gravitational plane wave under rotations. If the plane wave is traveling in the 3 direction, the only nonzero amplitudes are A_j for the electromagnetic vector potential and A_{jk} for the gravitational wave, where $(j,k) \neq 3$. If you rotate these wave functions by angle ϕ about the axis along the propagation direction, then from Fig. 1.1, the nonzero elements of the rotation matrix are: $R^1{}_1 = R^2{}_2 = \cos\phi$, $R^1{}_2 = -R^2{}_1 = \sin\phi$, $R^3{}_3 = 1$.

Using the rotation, $A'_j \; = \; R^k{}_j A_k$, the electromagnetic amplitudes become,

$$A'_1 = R^1{}_1 A_1 + R^2{}_1 A_2 = \cos\phi A_1 - \sin\phi A_2,$$
$$A'_2 = R^1{}_2 A_1 + R^2{}_2 A_2 = \sin\phi A_1 + \cos\phi A_2,$$

$$
\begin{aligned}
A'_1 \pm iA'_2 &= (\cos\phi \pm i\sin\phi)A_1 + (-\sin\phi \pm i\cos\phi)A_2,\\
&= \exp(\pm i\phi)A_1 \pm (\cos\phi \pm i\sin\phi)iA_2 = \exp(\pm i\phi)[A_1 \pm iA_2].
\end{aligned}
$$

Using the rotation, $A'_{jk} = R^l{}_j R^n{}_k \bar{A}_{ln}$, the gravitational amplitudes become,

$$
\begin{aligned}
\bar{A}'_{11} &= R^1{}_1R^1{}_1\bar{A}_{11} + R^1{}_1R^2{}_1\bar{A}_{12} + R^2{}_1R^1{}_1\bar{A}_{21} + R^2{}_1R^2{}_1\bar{A}_{22},\\
&= \cos^2\phi\bar{A}_{11} - 2\sin\phi\cos\phi\bar{A}_{12} - \sin^2\phi\bar{A}_{11},\\
&= \cos 2\phi\bar{A}_{11} - \sin 2\phi\bar{A}_{12},\\
\bar{A}'_{22} &= R^1{}_2R^1{}_2\bar{A}_{11} + R^1{}_2R^2{}_2\bar{A}_{12} + R^2{}_2R^1{}_2\bar{A}_{21} + R^2{}_2R^2{}_2\bar{A}_{22},\\
&= \sin^2\phi\bar{A}_{11} + 2\sin\phi\cos\phi\bar{A}_{12} - \cos^2\phi\bar{A}_{11},\\
&= -\cos 2\phi\bar{A}_{11} + \sin 2\phi\bar{A}_{12},\\
\bar{A}'_{12} &= R^1{}_1R^1{}_2\bar{A}_{11} + R^1{}_1R^2{}_2\bar{A}_{12} + R^2{}_1R^1{}_2\bar{A}_{21} + R^2{}_1R^2{}_2\bar{A}_{22},\\
&= \sin\phi\cos\phi\bar{A}_{11} + (\cos^2\phi - \sin^2\phi)\bar{A}_{12} + \sin\phi\cos\phi\bar{A}_{11},\\
&= \sin 2\phi\bar{A}_{11} + \cos 2\phi\bar{A}_{12},\\
\bar{A}'_{11} \pm i\bar{A}'_{12} &= -\bar{A}'_{22} \pm i\bar{A}'_{12},\\
&= \exp(\pm 2i\phi)(\bar{A}_{11} \pm i\bar{A}_{12}),\\
&= \exp(\pm 2i\phi)(-\bar{A}_{22} \pm i\bar{A}_{12}).
\end{aligned}
$$

In quantum mechanics a plane wave is an eigenfunction of linear momentum. If under rotation by angle ϕ about the direction of propagation, the eigenfunction is transformed, $\Psi \to \exp(in\phi)\Psi$, then it is also an eigenfunction of helicity, $h = n$. In our case $n = \pm 1$ for electrodynamics and $n = \pm 2$ for gravitation. When the fields are quantized, this leads to spin $s = 1$ photons with only the projections $s_z = \pm 1$ along the direction of propagation. For gravity it leads to spin $s = 2$ gravitons with only the projections $s_z \pm 2$. Since string theory naturally allows for a spin $s = 2$ particle, a great many talented theorists study string theory hoping to unify gravity and the other interactions, as all of the latter are due to particle exchange.

8.4 Gravitational Waves Affect Particles

In order to see how the wave affects free particles, consider a particle initially at rest in a Lorentz frame with velocity components, $U^0 = 1$ and $U^i = 0$.

As previously shown, in the TT gauge this velocity forced, ${}^{TT}h_{\mu 0} = 0$. The following equation of motion for the particle's coordinates is obtained,

$$\begin{aligned}
0 &= \frac{dU^{\mu}}{d\tau} + \Gamma^{\mu}_{\nu\chi} U^{\nu} U^{\chi}, \\
\frac{dU^{\mu}}{d\tau}(0) &= -\Gamma^{\mu}_{00}, \\
&= -\eta^{\mu\sigma}[{}^{TT}h_{\sigma 0,0} + {}^{TT}h_{0\sigma,0} - {}^{TT}h_{00,\sigma}]/2 = 0.
\end{aligned}$$

So initially the acceleration vanishes. That means the particle will be at rest at a later time and by the same argument the acceleration will be zero at a later time. Thus the particle can remain at its coordinate in this gauge. This has no invariant geometrical meaning. Suppose particles at $x = 0, \epsilon$ experience a wave. The proper distance between them changes to,

$$\begin{aligned}
l &= \int_0^{\epsilon} (g_{xx})^{1/2} dx \approx \epsilon (g_{xx}[x=0])^{1/2}, \\
&= \epsilon(1 + {}^{TT}h_{xx}[x=0])^{1/2}, \\
&\approx \epsilon(1 + {}^{TT}h_{xx}[x=0]/2). \qquad (8.17)
\end{aligned}$$

Thus the wave can be detected because the proper distance between two objects will wiggle in the presence of the wave.

Observations of pulsing periods of a pulsar in a binary neutron star system indicate that energy is being carried away. This is an indirect detection of a wave, that is continuously generated, and is described in a later section. The hope is that waves from sufficiently close, violent, short-lived, astronomical events will be observable on earth. This is hopefully accomplished by the LIGO experiment[LIGO (2015)] via construction of identical interferometers in Washington state and Louisiana.

In Fig. 8.1, one interferometer is illustrated. A laser beam is split and travels down the arms of a long "L". The closest mirrors to the beam splitter are slightly transmitting. The splitter and mirrors are hung in vacuum like pendulums so that they are essentially free particles. The light travels back and forth about one hundred times in light storage arms, 4 km long. The light from the arms interferes destructively so that no light gets to the photo-detector, but heads back towards the laser where it is recirculated by a power recycling mirror. The latter is not shown in the figure.

If there was a gravitational wave of sufficient power, it would alter a huge region of space and the laser beams wouldn't return to the beam-splitter out of phase. A signal would appear in the photo-detector. This would happen in both interferometers with a separation time of 7 ms, the light

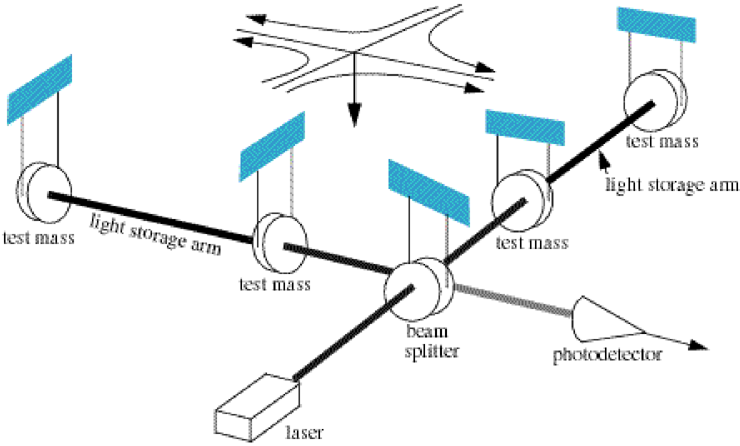

Fig. 8.1 One of two LIGO interferometers. See text for description.

travel time between them. Such a coincidence greatly limits the background noise. The interferometers are updated all the time and their signal to noise resolution has improved so that, $h \approx 10^{-22}$, is close to detectable. That's quite an achievement. However, many expected signals are still in the noise. In order to confirm a wave detection, a coincidence of like signals by the interferometers plus knowledge of what the signal should look like is required. The latter is where theoreticians make their presence felt. For example, the signals from coalescing black holes, or from super nova, yield very distinctive time structures.

In February, 2016, close to when this book went to press, LIGO announced observation, as illustrated in Fig. 8.2, of a robust signal, $h \approx 10^{-21}$. It was produced by two black holes about 1 Bly away. In < 0.5 s they merged into a single black hole. A new age of astronomy, of Galilean significance, has begun!

8.5 Wave Equation Solution With Sources

If there are no sources, plane wave solutions are allowed. However, without knowledge of the source, the wave amplitudes and the power carried by the wave cannot be calculated. When scientists build a detector like LIGO, they know the minimum signals that can be observed. Before many hundreds of millions of dollars are spent, one would like to be confident that the sought

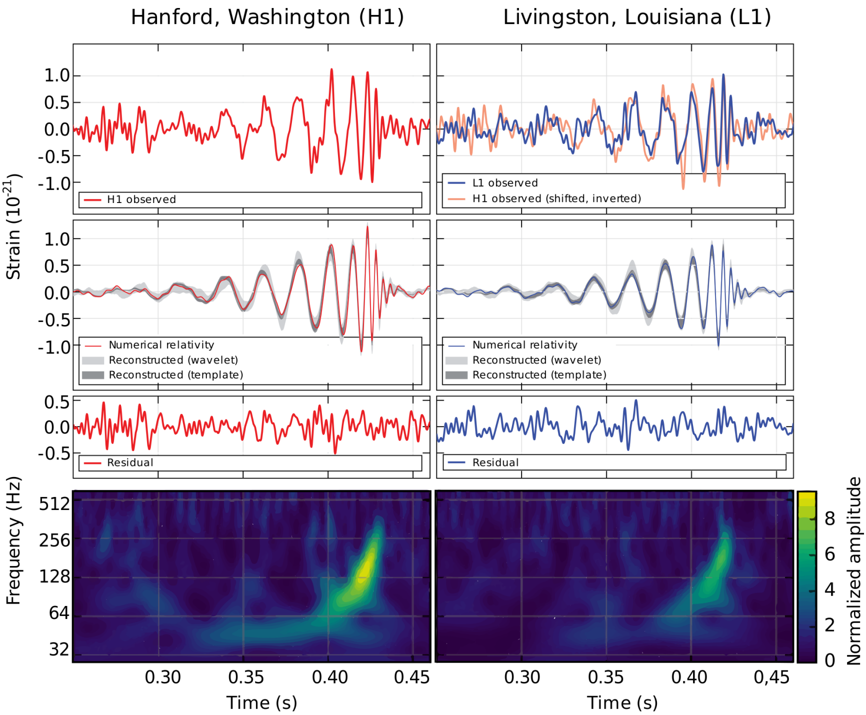

Fig. 8.2 LIGO evidence for the gravitational wave detection. See text for description.

for gravitational waves carry enough power to be observed by the detector. Thus solution of the wave equation with sources is a necessary part of the program because such a signal is distinctive from the background noise even though the two may have the same power level.

Most readers have solved the wave equation with sources in electrodynamics courses, so they may skip this section without loss. It is included for completeness and as a handy reference. The method of Fourier analysis is used to solve Eq. (8.6). This analysis makes use of the following theorem concerning the Dirac delta function,

$$\delta(t'-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \exp[i\omega(t'-t)], \tag{8.18}$$

that one is asked to prove in problem three.

One sees that Eq. (8.6) looks like,

$$(\nabla^2 - \frac{\partial^2}{\partial t^2})\Psi[\vec{r},t] = -4\pi S[\vec{r},t],$$

where $\Psi = \bar{h}^{\mu\nu}$ is the wave function and $S = 4T^{\mu\nu}$ is the source. Using Fourier transforms, one can write,

$$\Psi[\vec{r},t] = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \Psi[\vec{r},\omega]\exp[-i\omega t], \text{ where,}$$

$$\Psi[\vec{r},\omega] = \int_{-\infty}^{\infty} dt'\Psi[\vec{r},t']\exp[i\omega t'], \text{ because,}$$

$$\Psi[\vec{r},t] = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt'\Psi[\vec{r},t']\exp[i\omega(t'-t)],$$

$$= \int_{-\infty}^{\infty} dt'\Psi[\vec{r},t']\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \exp[i\omega(t'-t)]),$$

$$= \int_{-\infty}^{\infty} dt'\Psi[\vec{r},t']\delta(t'-t) = \Psi[\vec{r},t].$$

The formal mathematical solution to the wave equation can be expressed as an integral over all space and time,

$$\Psi[\vec{r},t] \equiv \int_{-\infty}^{\infty} dt' \int_{\infty} dV' S[\vec{r}\,',t']G[\vec{r},t,\vec{r}\,',t'], \text{where,}$$

$$\Box\Psi[\vec{r},t] = (\nabla^2 - \frac{\partial^2}{\partial t^2})\Psi[\vec{r},t] = -4\pi S[\vec{r},t],$$

$$= \int_{-\infty}^{\infty} dt' \int_{\infty} dV' S[\vec{r}\,',t']\Box G[\vec{r},t,\vec{r}\,',t'],$$

$$\Box G[\vec{r},t,\vec{r}\,',t'] = -4\pi\delta(t'-t)\delta(\vec{r}\,'-\vec{r}) \equiv -4\pi\delta(t'-t)\delta[\vec{R}],$$

$$= -4\pi\delta[\vec{R}]\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \exp[i\omega(t'-t)].$$

The function G is called a Green function. It is easily solved for using Fourier transforms,

$$G[\vec{r},t,\vec{r}\,',t'] = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega G[\vec{r},\omega,\vec{r}\,',t']\exp[-i\omega t],$$

$$\Box G[\vec{r},t,\vec{r}\,',t'] = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \Box(G[\vec{r},\omega,\vec{r}\,',t']\exp[-i\omega t]),$$

$$\Box(G[\vec{r},\omega,\vec{r}\,',t']\exp[-i\omega t]) = -4\pi\delta[\vec{R}]\frac{1}{2\pi}\exp[i\omega(t'-t)],$$

$$(\nabla^2 + \omega^2)G[\vec{r},\omega,\vec{r}\,',t'] = -4\pi\delta[\vec{R}]\exp[i\omega t'],$$

$$G[\vec{r},\omega,\vec{r}\,',t'] \equiv g[R,\omega]\exp[i\omega t'].$$

The last form results because we are in unbounded space and the source for $G[\vec{r},\omega,\vec{r}\,',t']$ is only nonzero at $\vec{R}=0$. Thus the spatial dependence can only be radial, a function of R. When $R\neq 0$, the differential equation to solve is,

$$\begin{aligned}
0 &= (\nabla^2+\omega^2)g[R,\omega] = (\nabla_R^2+\omega^2)g[R,\omega],\\
&= R^{-1}(Rg[R,\omega]),_R,_R + \omega^2 g[R,\omega],\\
&= (Rg[R,\omega]),_R,_R + \omega^2(Rg[R,\omega]),\\
Rg[R,\omega] &= \exp[\pm i\omega R],\\
RG[\vec{r},\omega,\vec{r}\,',t'] &= \exp[i(\pm\omega R+\omega t')] = \exp[i\omega(\pm R+t')],\\
G[\vec{r},\omega,\vec{r}\,',t'] &= \exp[i\omega(\pm R+t')]/R,\\
G[\vec{r},t,\vec{r}\,',t'] &= \frac{1}{2\pi}\int_{\infty}^{\infty} d\omega\frac{\exp[i\omega(\pm R+t'-t)]}{R},\\
&= |\vec{r}-\vec{r}\,'|^{-1}\delta[t'-(t\mp(|\vec{r}-\vec{r}\,'|)],\\
\bar{h}^{\mu\nu}[\vec{r},t] &= \int_{\infty} dV'\int_{-\infty}^{\infty} dt' S[\vec{r}\,',t']\frac{\delta[t'-(t\mp(|\vec{r}-\vec{r}\,'|)]}{|\vec{r}-\vec{r}\,'|},\\
\bar{h}^{\mu\nu}[\vec{r},t] &= \int_{\infty} dV'\frac{S[\vec{r}\,',t-|\vec{r}-\vec{r}\,'|]}{|\vec{r}-\vec{r}\,'|},\\
&= 4\int_{\infty} dV'\frac{T^{\mu\nu}[\vec{r}\,',t-|\vec{r}-\vec{r}\,'|]}{|\vec{r}-\vec{r}\,'|}. \qquad (8.19)
\end{aligned}$$

In the last form you note that the minus sign is chosen. Then the wave function at $\vec{r},t$, depends on the source at $\vec{r}\,'$ at an earlier time. Earlier by $|\vec{r}-\vec{r}\,'|$, because, $c=1$. This is physical as the wave must travel from the source to the observation point. If the plus sign was chosen, then what happens later than t at the source would dictate the wave function at $\vec{r},t$. That is why the solution is called the retarded time solution. Whatever the t' dependence of the source, just substitute $t-|\vec{r}-\vec{r}\,'|$ for t'.

8.6 The Energy-Momentum Tensor

Now is the appropriate time to consider the energy-momentum tensor. It is easiest to begin in a SR frame. Consider a system of n free particles at

${x_n}^{\bar{\chi}}(t)$, with momenta, ${P_n}^{\bar{\chi}}(t)$, at time, t. The system has a momentum density,

$$T^{\bar{\mu}0}(x^{\bar{\chi}}) \equiv \sum_n {P_n}^{\bar{\mu}}(t)\delta(\vec{r}-\vec{r}_n(t)) = \sum_n {P_n}^{\bar{\mu}}(t)\frac{d{x_n}^0(t)}{dt}\delta(\vec{r}-\vec{r}_n(t)).$$

Its current is defined by,

$$T^{\bar{\mu}\bar{i}}(x^{\bar{\chi}}) \equiv \sum_n {P_n}^{\bar{\mu}}(t)\frac{d{x_n}^{\bar{i}}(t)}{dt}\delta(\vec{r}-\vec{r}_n(t)), \text{thus,}$$

$$T^{\bar{\mu}\bar{\nu}}(x^{\bar{\chi}}) \equiv \sum_n {P_n}^{\bar{\mu}}(t)\frac{d{x_n}^{\bar{\nu}}(t)}{dt}\delta(\vec{r}-\vec{r}_n(t)), \tag{8.20}$$

$$\begin{aligned}
{P_n}^{\bar{\nu}}(t) &= m_n {U_n}^{\bar{\nu}}(t) = m_n\frac{d{x_n}^{\bar{\nu}}(t)}{d\tau}, \\
&= m_n(1-(v_n)^2)^{-1/2}\frac{d{x_n}^{\bar{\nu}}(t)}{dt} = E_n\frac{d{x_n}^{\bar{\nu}}(t)}{dt}, \\
T^{\bar{\mu}\bar{\nu}}(x^{\bar{\chi}}) &= \sum_n \frac{{P_n}^{\bar{\mu}}(t){P_n}^{\bar{\nu}}(t)}{E_n(t)}\delta(\vec{r}-\vec{r}_n(t)), \qquad so,
\end{aligned}$$

$$T^{\bar{\mu}\bar{\nu}} = T^{\bar{\nu}\bar{\mu}}, \text{ symmetric.} \tag{8.21}$$

The next proof shows that $T^{\bar{\mu}\bar{\nu}}$ is a tensor of rank two. Write Eq. (8.20) as an integral over the 4D delta function,

$$\begin{aligned}
T^{\bar{\mu}\bar{\nu}}(x^{\bar{\chi}}) &= \sum_n \int dt' {P_n}^{\bar{\mu}}(t')\frac{d{x_n}^{\bar{\nu}}(t')}{dt'}\delta(\vec{r}-\vec{r}_n(t'))\delta(t'-t), \\
&= \sum_n \int d\tau {P_n}^{\bar{\mu}}(\tau)\frac{d{x_n}^{\bar{\nu}}(\tau)}{d\tau}\delta(\vec{r}-\vec{r}_n(\tau))\delta(\tau-t), \\
&= \sum_n \int d\tau {P_n}^{\bar{\mu}}(\tau){U_n}^{\bar{\nu}}(\tau)\delta(\vec{r}-\vec{r}_n(\tau))\delta(\tau-t).
\end{aligned}$$

On the right side of the first line dt' cancels. Thus, t' can be replaced by the invariant, τ, when one changes variables. The right side of the last equation is a contravariant tensor of rank two, because on the right side such a tensor is multiplied by the invariant, $d\tau$, and also by the invariant 4D delta function. As this is a tensor equation, $T^{\mu\nu}$ is a symmetric tensor of rank two in any frame.

Take a partial derivative of this tensor with respect to a spatial coordinate. From Eq. (8.20),

$$
\begin{aligned}
T^{\bar{\mu}\bar{i}}(x^{\bar{\chi}})_{,\bar{i}} &= \sum_n P_n{}^{\bar{\mu}}(t)\frac{dx_n{}^{\bar{i}}(t)}{dt}\delta(\vec{r}-\vec{r}_n(t))_{,\bar{i}} \\
&= \sum_n P_n{}^{\bar{\mu}}(t)\frac{dx_n{}^{\bar{i}}(t)}{dt}\frac{\partial}{\partial x^{\bar{i}}}\delta(\vec{r}-\vec{r}_n(t)), \\
&= -\sum_n P_n{}^{\bar{\mu}}(t)\frac{dx_n{}^{\bar{i}}(t)}{dt}\frac{\partial}{\partial x_n{}^{\bar{i}}(t)}\delta(\vec{r}-\vec{r}_n(t)), \\
&= -\sum_n P_n{}^{\bar{\mu}}(t)\delta(\vec{r}-\vec{r}_n(t))_{,t}\,, \text{ however,} \\
(f(t)g)_{,t} &= \frac{df(t)}{dt}g + f(t)g_{,t}\,, \quad -f(t)g_{,t} = -(f(t)g)_{,t} + \frac{df(t)}{dt}g, \\
T^{\bar{\mu}\bar{i}}(x^{\bar{\chi}})_{,\bar{i}} &= -T^{\bar{\mu}0}(x^{\bar{\chi}})_{,0} + \sum_n \frac{dP_n{}^{\bar{\mu}}(t)}{dt}\delta(\vec{r}-\vec{r}_n(t)), \\
T^{\bar{\mu}\bar{\nu}}(x^{\bar{\chi}})_{,\bar{\nu}} &= \sum_n \frac{dP_n{}^{\bar{\mu}}(t)}{dt}\delta(\vec{r}-\vec{r}_n(t)).
\end{aligned}
$$

Since $\frac{dP_n{}^{\bar{\mu}}(t)}{dt}$ is proportional to the force on the particle, the right side of the above equation is just the density of the force. One sees from the above, if the particles are free, $T^{\bar{\mu}\bar{\nu}}{}_{,\bar{\nu}} = 0$ and that is conservation of momentum. However, in this inertial frame,

$$T^{\bar{\mu}\bar{\nu}}{}_{,\bar{\nu}} = T^{\bar{\mu}\bar{\nu}}{}_{;\bar{\nu}} = 0 = T^{\mu\nu}{}_{;\nu} \tag{8.22}$$

The above is a tensor equation and holds in any frame. Thus the energy-momentum tensor has the properties enumerated when the Einstein equation, Eq. (5.9), was discussed. If the particles were charged, they would interact via long range forces other than gravity. An additional term, $T^{\mu\nu}_{E\&M}$, could be added to our tensor and the sum of these tensors would have a zero divergence.

8.7 Quadrupole Radiation

An important case occurs when the source varies harmonically in time, $S = 4T^{\mu\nu} = j_\omega[\vec{r}\,']\exp[-i\omega t']$. With the source near the origin, the observation point, $\vec{r}$, may be in one of three zones, near, intermediate and far. Each zone allows for different approximations. The far zone where $d \ll \lambda \ll r$ is of interest to us as a violent event triggering a gravitational wave is likely to occur far from us. Here, d is the source size and is much less than the wavelength of the radiation, that in turn, is much less than the radial coordinate of the observation point. Then,

$$\begin{aligned}
|\vec{r}-\vec{r}\,'| &= (r^2+r'^2-2\vec{r}\cdot\vec{r}\,')^{1/2} = r(1-2\vec{r}\cdot\vec{r}\,'/r^2+(r'/r)^2)^{1/2}, \\
&\approx r(1-\vec{r}\cdot\vec{r}\,'/r^2) = r-\hat{e_r}\cdot\vec{r}\,', \\
\bar{h}^{\mu\nu}[\vec{r},t] &= \int_\infty dV' j_\omega[\vec{r}\,'] \frac{\exp[-i\omega(t-[r-\hat{e_r}\cdot\vec{r}\,'])]}{r-\hat{e_r}\cdot\vec{r}\,'}, \\
&\approx \frac{\exp[i(kr-\omega t)]}{r}\int_\infty dV' j_\omega[\vec{r}\,']\exp[-ik\hat{e_r}\cdot\vec{r}\,'], \\
&\approx \frac{\exp[i(kr-\omega t)]}{r}\int_\infty dV' j_\omega[\vec{r}\,'], \text{ to lowest order,} \qquad (8.23) \\
&= \frac{4}{r}\int_\infty dV' T^{\mu\nu}[\vec{r}\,', t-r]. \qquad (8.24)
\end{aligned}$$

The far zone is approximately locally inertial because it is so far from the source. In our units, $2\pi/\lambda = k = \omega$, but kr and ωt are written so that the equations look familiar. For the harmonic dependence, the solution, Eq. (8.23), looks like an out going spherical wave with amplitude given by the integral. Recall that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ thus the solution for $h^{\mu\nu}$ and $\bar{h}^{\mu\nu}$ are expressed in terms of the rectangular coordinates and raising and lowering of indices is done by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$. After these manipulations are carried out the amplitudes can be expressed in other coordinate systems.

From energy conservation, the lowest order approximation yields,

$$\begin{aligned}
0 = T^{\mu\nu}{}_{;\nu} &= T^{\mu\nu}{}_{,\nu} + \Gamma^\mu_{\xi\nu}T^{\xi\nu} + \Gamma^\nu_{\mu\xi}T^{\mu\xi}, \\
&\approx T^{\mu\nu}{}_{,\nu} = T^{00}{}_{,0} + T^{0k}{}_{,k}\,, \\
&= (T^{00}{}_{,0} + T^{0k}{}_{,k})_{,0} = T^{00}{}_{,0\,,0} + T^{0k}{}_{,k\,,0}\,, \\
T^{00}{}_{,0\,,0} &= (-T^{0k}{}_{,0})_{,k} = -(-T^{jk}{}_{,j})_{,k} = T^{jk}{}_{,j\,,k}\,, \\
\int_\infty dV x^i x^n T^{00}{}_{,0\,,0} &= \int_\infty dV x^i x^n T^{jk}{}_{,j\,,k}\,.
\end{aligned}$$

Integrate the integral on the right side of the last equation by parts twice. The surface terms, far from the origin, vanish because $T^{jk} = 0$

there. Thus,

$$
\begin{aligned}
\int_\infty dV x^i x^n T^{00}{}_{,0}{}_{,0} &= \int_\infty dV x^i x^n [T^{jk}{}_{,j}]_{,k}\,, \\
&= -\int_\infty dV [T^{jk}{}_{,j}][x^i x^n]_{,k}\,, \\
&= -\int_\infty dV [T^{jk}{}_{,j}][x^i{}_{,k}\, x^n + x^i x^n{}_{,k}], \\
&= -\int_\infty dV [T^{jk}{}_{,j}][\delta^i{}_k x^n + \delta^n{}_k x^i], \\
&= \int_\infty dV T^{jk}[\delta^i{}_k x^n{}_{,j} + \delta^n{}_k x^i{}_{,j}], \\
&= \int_\infty dV T^{jk}(\delta^i{}_k \delta^n{}_j + \delta^n{}_k \delta^i{}_j), \\
&= \int_\infty dV (T^{ni} + T^{ni}) = 2\int_\infty dV T^{in}.
\end{aligned}
$$

Using this result in Eq. (8.24) yields,

$$
\begin{aligned}
r\bar{h}^{in}[\vec{r}, t]/2 &= \int_\infty dV' x'^i x'^n (T^{00}[\vec{r}\,', t-r])_{,0}{}_{,0}\,, \\
r\bar{h}^{in}[\vec{r}, t]/2 &= \frac{d^2 \int_\infty dV' x'^i x'^n T^{00}[\vec{r}\,', t-r]}{dt^2} \equiv \frac{d^2 I^{in}}{dt^2} = \frac{d^2 I_{in}}{dt^2}. \quad (8.25)
\end{aligned}
$$

Since the wave function is a real quantity, the **real part** of Eq. (8.25) is the solution. This leads to quadrupole radiation because $\bar{h}_{\mu\nu}$ is a tensor of rank two. In electromagnetism, in lowest order, dipole radiation is possible, because A_μ is a tensor of rank one.

8.8 Gravity Wave Flux and Power

Flux is energy/area/time. In natural units it is just m^{-2}. This quantity is integrated over the area of a sphere to determine how much power is carried away from the source by the wave. The result from electromagnetic waves cannot be taken over directly as their amplitudes are tensors of rank one, while a gravity wave amplitude is a tensor of rank two. Thus while the flux is still proportional to the absolute square of the amplitude, the all important, proportionality factor is different. In order to calculate it, the approach of B.Schutz [Schutz (2009)], is followed.

Consider a plane wave moving in the z direction, possibly that between earth and a distant source. The flux given to an approximately continuous array of oscillators, elemental springs, aligned along the x direction, in the plane $z = 0$ is calculated. The springs have unstretched length, l_0, and equal masses, m, at their ends. The small spring and damping constants are, $m\omega_0^2/2$ and $m\gamma$. The number of springs per unit area, $\frac{dn}{dA} = \alpha$. As the oscillators acquire energy, the wave loses energy and its amplitude decreases. The relationship between flux and amplitude is then found, not to depend on the springs, but is a property of the wave. The springs are just used as calculation facilitators.

In flat space the equations of motion for the masses of a given spring at $x_{1,2}$ are,

$$\frac{d^2x_2}{dt^2} = -\omega_0^2(x_2 - x_1 - l_0)/2 - \gamma\frac{d(x_2 - x_1)}{dt},$$

$$\frac{d^2x_1}{dt^2} = \omega_0^2(x_2 - x_1 - l_0)/2 + \gamma\frac{d(x_2 - x_1)}{dt},$$

$$\frac{d^2(x_2 - x_1 - l_0)}{dt^2} = -\omega_0^2(x_2 - x_1 - l_0) - 2\gamma\frac{d(x_2 - x_1 - l_0)}{dt}. \tag{8.26}$$

When the wave is encountered, Eq. (8.17) yields the proper length between the masses,

$$\begin{aligned} l &= (x_2 - x_1)(1 +^{TT} h_{xx}/2) \\ &\approx x_2 - x_1 + l_0\ {}^{TT}\bar{h}_{xx}/2, \\ x_2 - x_1 &= l - l_0\ {}^{TT}\bar{h}_{xx}/2. \end{aligned}$$

The terms on the right side of Eq. (8.26) are unmodified by ${}^{TT}\bar{h}_{xx}$ because they are already expressed in terms of small quantities. This leads to,

$$\begin{aligned} 0 &= \frac{d^2(l - l_0\ {}^{TT}\bar{h}_{xx}/2)}{dt^2} + \omega_0^2(l - l_0) + 2\gamma\frac{d(l - l_0)}{dt}, \\ &\equiv \frac{d^2\xi}{dt^2} + \omega_0^2\xi + 2\gamma\frac{d\xi}{dt} - \frac{l_0}{2}\frac{d^2\ {}^{TT}\bar{h}_{xx}}{dt^2}. \end{aligned} \tag{8.27}$$

One can take a wave with a single amplitude element, ${}^{TT}\bar{h}_{xx}$,

$${}^{TT}\bar{h}_{xx} = \bar{A}\cos\Omega(t - z). \tag{8.28}$$

Insertion of Eq. (8.28) into Eq. (8.27) yields a differential equation with a sinusoidal solution,

$$0 = \frac{d^2\xi}{dt^2} + \omega_0^2\xi + 2\gamma\frac{d\xi}{dt} + (l_0\bar{A}\Omega^2/2)\cos\Omega(t-z),$$
$$\xi \equiv B\cos[\Omega(t-z)-\epsilon].$$

In order to obtain the constants, B, ϵ, plug the solution into the differential equation and note that what multiplies both $\cos\Omega(t-z)$ and $\sin\Omega(t-z)$ must vanish,

$$\begin{aligned}0 &= B(\omega_0^2-\Omega^2)[\cos\Omega(t-z)\cos\epsilon + \sin\Omega(t-z)\sin\epsilon] \\ &- 2B\gamma\Omega[\sin\Omega(t-z)\cos\epsilon - \cos\Omega(t-z)\sin\epsilon] \\ &+(l_0\bar{A}\Omega^2/2)\cos\Omega(t-z), \\ &= B\sin\Omega(t-z)[\sin\epsilon(\omega_0^2-\Omega^2) - 2\gamma\Omega\cos\epsilon], \\ &= \cos\Omega(t-z)(B[(\omega_0^2-\Omega^2)\cos\epsilon + 2\gamma\Omega\sin\epsilon] + l_0\bar{A}\Omega^2/2),\end{aligned}$$

$$\tan\epsilon = \frac{2\gamma\Omega}{\omega_0^2-\Omega^2}, \quad \sin\epsilon,\ \cos\epsilon = \frac{2\gamma\Omega, \quad \omega_0^2-\Omega^2}{[(2\gamma\Omega)^2+(\omega_0^2-\Omega^2)^2]^{1/2}}, \tag{8.29}$$

$$B = -(l_0\bar{A}\Omega^2/2)[(2\gamma\Omega)^2+(\omega_0^2-\Omega^2)^2]^{-1/2},\ B/\bar{A} \ll 1. \tag{8.30}$$

Each oscillator is responding with a steady oscillation, $B\cos(\Omega t-\epsilon)$, because the energy dissipated by friction is compensated by the work done on the spring by the gravitational forces of the wave. Multiplying Eq. (8.27) by $md\xi/2$ yields,

$$0 = md\xi\left[\frac{d^2\xi}{dt^2}/2 + \omega_0^2\xi\right] + m\gamma d\xi\frac{d\xi}{dt} - (ml_0/4)d\xi\frac{d^2\ {}^{TT}\bar{h}_{xx}}{dt^2}.$$

Each term in the equation has units of energy. The energy dissipated by friction is $dE = m\gamma d\xi\frac{d\xi}{dt}$, so that the time rate of that energy change, averaged over a period, T, is,

$$\begin{aligned}\left\langle\frac{dE}{dt}\right\rangle &= (m\gamma/T)\int_0^T dt(\frac{d\xi}{dt})^2, \\ &= (m\gamma\Omega^2B^2/T)\int_{-z}^{T-z} d(t-z)\sin^2(\Omega(t-z)-\epsilon), \\ &= m\gamma\Omega^2B^2/2.\end{aligned}$$

As the wave passes through the $z = 0$ plane, all the oscillators take part, and the time averaged decrease in the flux carried by the wave is,

$$\langle \delta F \rangle = -m\gamma\alpha\Omega^2 B^2/2. \tag{8.31}$$

It is small as, $|B| \ll 1$.

Next, the change in the wave amplitude is required. The oscillations of the oscillator masses are also a source of a gravitational wave. The energy density is dominated by the mass term. For an oscillator at a distance $r = (\rho^2 + z^2)^{1/2}$ from the observation point on the z axis,

$$\begin{aligned}
T_{00} &= \sum_{i=1}^{2} m_i \delta[\vec{r}\,' - \vec{r}_i] = m([\delta(x' - x_1) + \delta(x' - x_2)]\delta(y')\delta(z')), \\
&= m([\delta(x' - [-l_0 - (l - l_0)]/2) + \delta(x' - [l_0 + (l - l_0)]/2)]\delta(y')\delta(z')), \\
&= m([\delta(x' - [-l_0 - B\cos(\Omega(t - r) - \epsilon)]/2) \\
&\quad + \delta(x' - [+l_0 + B\cos(\Omega(t - r) - \epsilon)]/2)]\delta(y')\delta(z')).
\end{aligned}$$

From Eq. (8.24), $\bar{h}^{00} = constant/r$, because the integral of a one dimensional delta function is zero or one. This is not a wave that takes energy to very far away points. The amplitude, Eq. (8.25), can be calculated by noting the y and z positions of the masses are always zero. So that the only nonzero term, from one oscillator is,

$$\begin{aligned}
\delta\bar{h}^{xx} &= \frac{2}{r}\frac{d^2 I^{xx}}{dt^2} = (m/r)\frac{d^2[l_0 + B\cos(\Omega(t - r) - \epsilon)]^2}{dt^2}, \\
&\approx -\frac{2ml_0 B\Omega^2}{r}\cos(\Omega(t - r) - \epsilon).
\end{aligned} \tag{8.32}$$

In the last equation the, B^2, term is neglected in lowest order.

At a point on the z axis, the contribution from all of the oscillators is,

$$\begin{aligned}
\bar{h}_{xx} &= -2Bl_0 m\Omega^2 \int_0^\infty \alpha 2\pi\rho d\rho \cos[\Omega(t - r) - \epsilon]/r, \\
\rho d\rho &= (r^2 - z^2)^{1/2} d(r^2 - z^2)^{1/2} = r dr, \\
\bar{h}_{xx} &= -4\pi\alpha Bl_0 m\Omega^2 \int_z^\infty \cos[\Omega(t - r) - \epsilon] dr \\
&= 4\pi\alpha Bl_0 m\Omega \sin[\Omega(t - r) - \epsilon]|_z^\infty \\
&= -4\pi\alpha Bl_0 m\Omega \sin[\Omega(t - z) - \epsilon].
\end{aligned}$$

In the last line, $\sin\infty = 0$, has been used.

Since the incident wave was in the TT gauge, the above must be put in that gauge. Then it can be combined with the original wave to see what the reduction in amplitude is when passing through the plane of the oscillators. The only nonzero spatial component of k_μ is $k_3 = \Omega$. Thus from Eqs. (8.10) and (8.11), using $B/\bar{A} \ll 1$, and that there is a single amplitude,

$$\begin{aligned} {}^{TT}\bar{h}_{xx} &= \bar{h}_{xx} - \bar{h}_i{}^i/2 = \bar{h}_{xx}/2, \\ {}^{TT}\bar{h}_{xx}(\text{net}) &= \bar{A}\cos(\Omega(t-z)) - 2\pi\alpha B l_0 m\Omega \sin(\Omega(t-z)-\epsilon), \\ &= \bar{A}[(1 + 2\pi\alpha(B/\bar{A})l_0 m\Omega \sin\epsilon)\cos(\Omega(t-z)) \\ &\quad -2\pi\alpha(B/\bar{A})l_0 m\Omega \sin(\Omega(t-z))\cos\epsilon]. \end{aligned} \tag{8.33}$$

Now consider the quantity, f,

$$\begin{aligned} f &\equiv (\bar{A} + 2\pi\alpha B l_0 m\Omega \sin\epsilon)[\cos(\Omega(t-z)+\Psi)], \\ &= (\bar{A} + 2\pi\alpha B l_0 m\Omega \sin\epsilon)[\cos(\Omega(t-z))\cos\Psi - \sin(\Omega(t-z))\sin\Psi], \\ &= \bar{A}\cos\Psi(1 + 2\pi\alpha(B/\bar{A})l_0 m\Omega \sin\epsilon)[\cos(\Omega(t-z)) - \sin(\Omega(t-z))\tan\Psi]. \end{aligned}$$

The above is approximately the same as Eq. (8.33) because you can take,

$$\begin{aligned} \tan\Psi &= \frac{(2\pi\alpha(B/\bar{A})l_0 m\Omega \cos\epsilon}{1 + 2\pi\alpha(B/\bar{A})l_0 m\Omega \sin\epsilon} \ll 1, \text{ thus, } \cos\Psi \approx 1, \\ {}^{TT}\bar{h}_{xx}(net) &\approx (\bar{A} + 2\pi\alpha B l_0 m\Omega \sin\epsilon)\cos(\Omega(t-z)-\Psi). \end{aligned}$$

So apart from a small phase shift Ψ, the change in the wave amplitude is $\delta\bar{A} = 2\pi\alpha B l_0 m\Omega \sin\epsilon$. Using Eqs. (8.29) - (8.31) gives,

$$\begin{aligned} \frac{\delta\langle F\rangle}{\delta\bar{A}} &= -\frac{m\gamma\alpha\Omega^2 B^2/2}{2\pi\alpha B l_0 m\Omega \sin\epsilon} = -\frac{\gamma\Omega B}{4\pi l_0 \sin\epsilon}, \\ &= \frac{\gamma\Omega(l_0/2)\bar{A}\Omega^2/[(2\gamma\Omega)^2 + (\omega_0^2 - \Omega^2)^2]^{1/2}}{4\pi l_0(2\gamma\Omega)/[(2\gamma\Omega)^2 + (\omega_0^2 - \Omega^2)^2]^{1/2}} = \frac{1}{16\pi}\Omega^2\bar{A}, \\ \langle F\rangle &= \frac{\Omega^2\bar{A}^2}{32\pi} = \frac{\Omega^2}{32\pi}\left\langle {}^{TT}\bar{h}_{xx}\ {}^{TT}\bar{h}^{xx}\right\rangle = \left\langle \frac{d^{TT}\bar{h}_{xx}}{dt}\frac{d^{TT}\bar{h}^{xx}}{dt}\right\rangle. \end{aligned}$$

Though oscillators were used, the last equation has only wave properties; the oscillators have disappeared. They were just used as facilitators.

In general a wave traveling in the z direction has more than just an $\bar{h}^{xx}$ amplitude. From Eqs. (8.11), (8.15),

$$
\begin{aligned}
{}^{TT}\bar{h}_{xx} = -\,{}^{TT}\bar{h}_{yy} &= \bar{h}_{xx} - \bar{h}/2 = \bar{h}_{xx} - (\bar{h}_{xx} + \bar{h}_{yy})/2, \\
&= \frac{1}{2}(\bar{h}_{xx} - \bar{h}_{yy}) = \frac{1}{r}\frac{d^2(I_{xx} - I_{yy})}{dt^2}, \\
{}^{TT}\bar{h}_{xy} = {}^{TT}\bar{h}_{yx} &= \bar{h}_{xy} = \bar{h}_{yx} = \frac{2}{r}\frac{d^2 I_{xy}}{dt^2}.
\end{aligned}
$$

So, for a wave traveling in the z direction, the general form for the flux is

$$
\langle F\rangle = \frac{1}{32\pi}\left\langle \frac{d\,{}^{TT}\bar{h}_{ij}}{dt}\frac{d\,{}^{TT}\bar{h}^{ij}}{dt}\right\rangle. \tag{8.34}
$$

Now define,

$$
\bar{I}_{ij} \equiv I_{ij} - \delta_{ij} I_k{}^k/3. \tag{8.35}
$$

One can prove, see problem four, that $\bar{I}$ is traceless and $I_{ij} = \bar{I}_{ij}$ if the amplitudes are in the transverse traceless gauge. The above equations can then be written,

$$
\begin{aligned}
{}^{TT}\bar{h}_{xx} &= -\,{}^{TT}\bar{h}_{yy} = \frac{1}{r}\frac{d^2(\bar{I}_{xx} - \bar{I}_{yy})}{dt^2}, \\
{}^{TT}\bar{h}_{xy} &= {}^{TT}\bar{h}_{yx} = \frac{2}{r}\frac{d^2\bar{I}_{xy}}{dt^2}, \\
32\pi\langle F\rangle &= \left\langle \frac{d\,{}^{TT}\bar{h}_{xx}}{dt}\frac{d\,{}^{TT}\bar{h}^{xx}}{dt} + \frac{d\,{}^{TT}\bar{h}_{xy}}{dt}\frac{d\,{}^{TT}\bar{h}^{xy}}{dt}\right\rangle \\
&\quad + \left\langle \frac{d\,{}^{TT}\bar{h}_{yx}}{dt}\frac{d\,{}^{TT}\bar{h}^{yx}}{dt} + \frac{d\,{}^{TT}\bar{h}_{yy}}{dt}\frac{d\,{}^{TT}\bar{h}^{yy}}{dt}\right\rangle, \\
\langle F\rangle &= \frac{\left\langle \frac{d^3(\bar{I}_{xx} - \bar{I}_{yy})}{dt^3}\frac{d^3(\bar{I}^{xx} - \bar{I}^{yy})}{dt^3} + 4\frac{d^3\bar{I}_{xy}}{dt^3}\frac{d^3\bar{I}^{xy}}{dt^3}\right\rangle}{16\pi r^2}.
\end{aligned} \tag{8.36}
$$

Obtaining Eq. (8.36) is also a part of problem four. The reason $\bar{I}_{ij}$ has been used is because the power radiated is easiest to calculate with it. Also it doesn't lead to radiation if a spherically symmetric mass distribution oscillates radially. This is the Birkhoff theorem that shows that in empty space an object giving rise to a Schwarzschild metric will still yield a static metric if it oscillates radially. Problem eleven leads one through the proof presented in [Weinberg (1972)].

Waves typically travel radially outward from the source and not just in the, z, direction. The luminosity, L, or the power radiated, is the normally outward wave flux integrated over the surface of a sphere of radius, r, with center at the source. Eq. (8.36), is not yet writtten in a form, that makes the outward wave flux, at any point on the sphere, obvious. However, it can be put into such a form with a little manipulation. On the, z, axis at the sphere's surface, the normal outward wave is in the, z, direction. There one can rewrite Eq. (8.36), using $0 = \bar{I}_{xx} + \bar{I}_{yy} + \bar{I}_{zz}$ as,

$$\langle F \rangle = \frac{\left\langle 2\frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{ij}}{dt^3} - 4\frac{d^3\bar{I}_{zj}}{dt^3}\frac{d^3\bar{I}^{zj}}{dt^3} + \frac{d^3\bar{I}_{zz}}{dt^3}\frac{d^3\bar{I}^{zz}}{dt^3}\right\rangle}{16\pi r^2}. \tag{8.37}$$

The first product in Eq. (8.37) is independent of the integration point, (θ, ϕ), on the surface. The products with the index, z, are particular to the integration point. They arose because at an arbitrary point, the outward normal unit vector, $\hat{n}$, has components $n^i = n_i = x^i/r$. If the point is on the z axis, then $n^{1,2,3} = (0, 0, 1)$, and, $\frac{d^3\bar{I}_{jz}}{dt^3} = n^i\frac{d^3\bar{I}_{ji}}{dt^3}$. In general then, Eq. (8.36) can be written,

$$16\pi r^2\langle F \rangle = \left\langle 2\frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{ij}}{dt^3} - 4n^i n_k\frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{kj}}{dt^3}\right\rangle + \left\langle n^i n^j n_l n_m\frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{lm}}{dt^3}\right\rangle,$$

$$L = \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi r^2\langle F \rangle. \tag{8.38}$$

As an illustration, if the point was on the x axis, the normally outward wave in the x direction is required and $n^{1,2,3} = (1, 0, 0)$. Then, $n^i\frac{d^3\bar{I}_{ij}}{dt^3} = \frac{d^3\bar{I}_{xj}}{dt^3}$, and,

$$16\pi r^2\langle F \rangle = \left\langle 2\frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{ij}}{dt^3} - 4\frac{d^3\bar{I}_{xj}}{dt^3}\frac{d^3\bar{I}^{xj}}{dt^3}\right\rangle + \left\langle \frac{d^3\bar{I}_{xx}}{dt^3}\frac{d^3\bar{I}^{xx}}{dt^3}\right\rangle.$$

As expected, the above has the same form as Eq. (8.37) with x replacing, z.

When integrating over the solid angle, only the unit vectors need to be integrated as the $\bar{I}$ terms are independent of the integration point. The integrals are easily calculated, see problem five,

$$n^{1,2,3} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi = 4\pi,$$

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi n^i n_k = \frac{4\pi}{3}\delta^i{}_k,$$

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi n^i n^j n_l n_m = \frac{4\pi}{15}(\delta^{ij}\delta_{lm} + \delta^i{}_m\delta^j{}_l + \delta^i{}_l\delta^j{}_m).$$

Using these results the luminosity is,

$$\begin{aligned} 4L &= \left\langle 2\frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{ij}}{dt^3} - \frac{4}{3}\frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{ij}}{dt^3}\right\rangle \\ &\quad + \left\langle \frac{1}{15}\left(\frac{d^3\bar{I}^i{}_i}{dt^3}\frac{d^3\bar{I}^l{}_l}{dt^3} + 2\frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{ij}}{dt^3}\right)\right\rangle, \\ L &= \frac{1}{4}\left(2 - \frac{4}{3} + \frac{2}{15}\right)\left\langle \frac{d^3\bar{I}_{ij}}{dt^3}\frac{\bar{I}^{ij}}{dt^3}\right\rangle \\ &= \frac{1}{5}\left\langle \frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{ij}}{dt^3}\right\rangle. \end{aligned} \tag{8.39}$$

8.9 Binary Neutron Star System Radiation

The binary pulsar, PSR B1913+16, shown schematically in Fig. 8.3, was discovered by R. Hulse and J. H. Taylor [Hulse (1975)] in 1973 at the South American (Arecibo) radio telescope. It is in a gravitational bound state with an unseen neutron star. The pulsar is a magnetized neutron star whose rapid rotation generates a plasma that is the source of beamed radio waves, seen at earth as periodic pulses, every 0.059 s. This is because the radio waves are beamed along the magnetic axis, but that axis rotates about the spin axis of the star. The rotation period of such a massive compact body is very stable against external perturbations and so comprises a very accurate clock, that modern timing devices can measure with high precision. Search the WWW for "pulsar" and you'll find some wonderful images. Neutron stars are highly compact, massive objects supported by neutron degeneracy, a purely quantum effect. For a mass of $1.4M_s$, the neutron star surface radial coordinate is $\approx 10 - 20$ km.

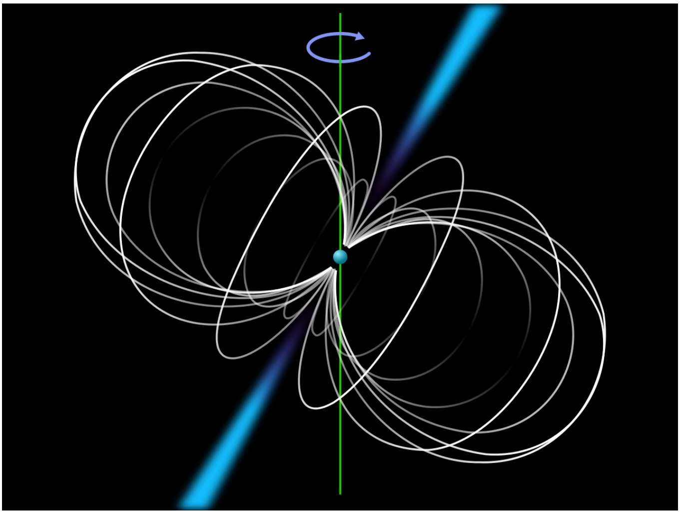

Fig. 8.3 A pulsar, the axis along which radio waves are beamed is rotating about the pulsar rotation axis. The beam is periodically in the line of sight of an earth observer.

Upon discovery, it was noted that the pulsing rate varied. This was interpreted as due to the pulsar traveling in a changing gravitational field as it orbited an unseen neighbor. The pulsar was tracked for decades and the parameters of the orbit were obtained from the slight changes in the pulsing rate. The orbiting is a source of a gravitational wave. The wave carries away energy that would be reflected in changes in the orbit. Since the pulsar is a radio emitter, the experimenters have to remove the distortion due to the index of refraction of the intergalactic medium. An optical pulsar would have been simpler.

The discoverers were joined by J. M. Weisberg who performed much of the data analysis. Their paper [Weisberg (2010)] and references therein, describe the intricacies of extracting the orbit parameters. They found that this is a wonderful system with which to test GR. For example, the advance of the periastron is $\approx 35,000$ times that of the perihelion of Mercury, the periastron is the distance of closest approach to its unseen neighbor. However, the prize here is the detection of a gravitational wave carrying energy away from the system.

The pulsar orbital period was measured at periastron, over the course of decades. The best fit parameters extracted from the data are: masses, $m_1 \approx m_2 \approx 1.4M_s$; distance from earth, $r = 6400$ parsec (pc), where, 1 pc $= 3.3$ ly$=3.1\times10^{16}$ m; period, $T = 7.75\ h$; eccentricity, $e = 0.617$; and

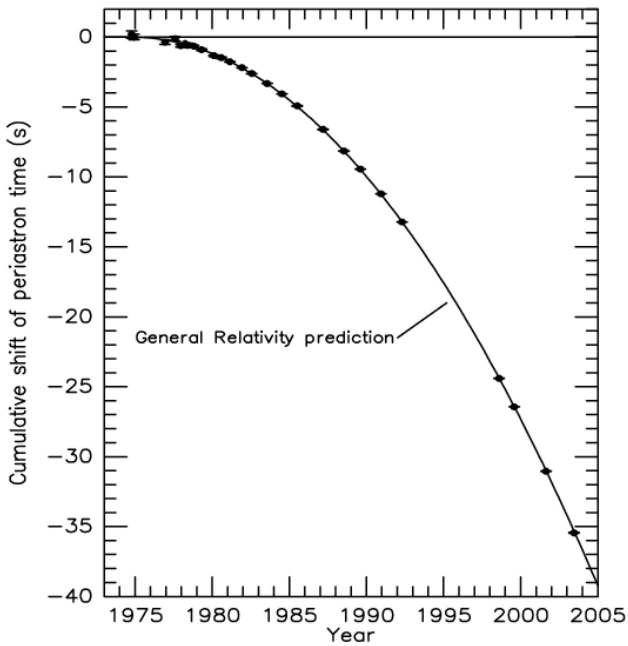

Fig. 8.4 The cumulative change of the time of periastron, the closest approach distance between the neutron stars, of the Hulse-Taylor pulsar PSR B1913+16 binary system. The data points agree with the solid curve GR calculation to within 0.33%

semi major axis, $a = 1.95 \times 10^9$ m. The prize was obtained through the detection of a decrease in the period, $\frac{dT}{dt} = -(2.4056 \pm 0.0051) \times 10^{-12}$, because a gravitational wave is radiating energy. The excellent agreement with the GR calculation is shown in Fig. 8.4. This "indirect" confirmation of gravity waves led to the 1993 Nobel Prize for the discoverers.

Some texts that call attention to this system, treat the orbits as circular. This gives a result off by an order of magnitude from the correct result because the eccentricity is large. Though more calculation is required, the Newtonian elliptical orbit calculation will be used in this text. It was first worked out by P. C. Peters [Peters (1964)], although in a different notation and with many steps left to the reader.

Before beginning the calculation, one should review the properties of the two body Newtonian system, see problem six. The gravitational wave amplitude is expected to be very small and so Newtonian orbits specify the star positions for T^{00} adequately. The reduced mass, μ, travels in an elliptical orbit in the $\theta = \pi/2$ plane, with focus at the center of mass, taken at rest. Its position, $\vec{r} = \rho[\phi]\hat{e}_\rho$, period, T, and energy of the system, E, are obtained from,

$$m \equiv m_1 + m_2, \qquad \mu \equiv m_1 m_2/m,$$

$$\rho^2 \frac{d\phi}{dt} = \frac{J}{\mu} = \left[ma\left(1 - e^2\right)\right]^{1/2}, \tag{8.40}$$

$$\rho = \left[a\left(1 - e^2\right)\right] / \left(1 + e\cos\phi\right), \tag{8.41}$$

$$T = \frac{2\pi a^{3/2}}{m^{1/2}}, \qquad a = \left(\frac{m^{1/2} T}{2\pi}\right)^{2/3}, \tag{8.42}$$

$$E = -\frac{1}{2}\frac{\mu m}{a} = -\frac{\mu}{2}\left(\frac{2\pi m}{T}\right)^{2/3}, \tag{8.43}$$

$$\frac{dT}{dt} = \frac{3\pi}{m^{1/2}} a^{1/2} \frac{da}{dt} = \frac{3}{(2\pi)^{2/3}} \frac{T^{5/3}}{\mu m^{2/3}} \frac{dE}{dt}, \tag{8.44}$$

$$\frac{dE}{dt} = \frac{1}{2}\frac{\mu m}{a^2}\frac{da}{dt}, \tag{8.45}$$

where J is a constant angular momentum, a is the semi major axis, and e is the eccentricity. In Newton's theory of the two body system, $\frac{da}{dt} = 0$, but in GR the gravitational waves, generated by the motion of the masses, take power out of the system. The energy decreases, becoming more negative, with time. So a decreases with time and thus so does T. Such changes can be compared with observation to test Einstein's theory.

The wave power calculation needs the energy-momentum tensor, assumed dominated by the mass density. The latter is used to get the integrals, I_{ij},

$$T_{00} = \sum_{i=1}^{2} m_i \delta\left(x' - \vec{r}_{ix}^{\,R}\right) \delta\left(y' - \vec{r}_{iy}^{\,R}\right) \delta\left(z'\right),$$

$$I_{xy,xx,yy} = \sum_{i=1}^{2} m_i \left(\vec{r}_{ix}^{\,R} \vec{r}_{iy}^{\,R},\ \left(\vec{r}_{ix}^{\,R}\right)^2,\ \left(\vec{r}_{iy}^{\,R}\right)^2\right). \tag{8.46}$$

The super script R means the retarded value, that, when the wave left the binary. From now on, all the variables a, e, ϕ have retarded values and the super script is dropped. The mass positions are obtained from that of the reduced mass,

$$\vec{r}_{i(x,y)} = (-1)^i \frac{\mu}{m_i} \vec{r}_{(x,y)} = (-1)^i \frac{\mu}{m_i} \rho(\cos\phi, \sin\phi). \tag{8.47}$$

The second derivatives with respect to time of the, I's, are needed for the wave amplitude and the third derivatives for the power carried by the wave. However, the time derivatives of a and e are due to the gravitational wave radiation and so are $\ll \frac{d\phi}{dt}$. Thus, the latter is the only time derivative that counts and it is given by Eq. (8.40) .

From Eqs. (8.40) and (8.41), the xx terms are,

$$\begin{aligned}
I_{xx} &= \sum_{i=1}^{2} m_i \left(\frac{\mu}{m_i}\right)^2 \left[a\left(1-e^2\right)\right]^2 \left(\frac{\cos\phi}{1+e\cos\phi}\right)^2, \\
&= \mu^2 \left(\sum_{i=1}^{2} \frac{1}{m_i}\right) \left[a\left(1-e^2\right)\right]^2 \left(\frac{\cos\phi}{1+e\cos\phi}\right)^2, \\
&= \mu \left[a\left(1-e^2\right)\right]^2 \left(\frac{\cos\phi}{1+e\cos\phi}\right)^2, \\
\frac{dI_{xx}}{dt} &= \mu \left[a\left(1-e^2\right)\right]^2 \left(\frac{-2\cos\phi\sin\phi}{\left(1+e\cos\phi\right)^2} + \frac{2e\cos^2\phi\sin\phi}{\left(1+e\cos\phi\right)^3}\right) \frac{d\phi}{dt}, \\
&= -2\mu \left[ma\left(1-e^2\right)\right]^{1/2} \frac{\cos\phi\sin\phi}{1+e\cos\phi} \\
&= -\mu \left[ma\left(1-e^2\right)\right]^{1/2} \frac{\sin 2\phi}{1+e\cos\phi}, \\
\frac{d^2 I_{xx}}{dt^2} &= \frac{2\mu m}{a\left(1-e^2\right)} \left[\sin^2\phi - \cos^2\phi - e\cos^3\phi\right], \qquad (8.48) \\
\frac{d^3 I_{xx}}{dt^3} &= \frac{2\mu m^{3/2}}{\left[a\left(1-e^2\right)\right]^{5/2}} \left(1+e\cos\phi\right)^2 \\
&\quad \times \left[2\sin 2\phi + 3e\sin\phi\cos^2\phi\right]. \qquad (8.49)
\end{aligned}$$

The yy terms are,

$$\begin{aligned}
I_{yy} &= \mu \left[a\left(1-e^2\right)\right]^2 \left(\frac{\sin\phi}{1+e\cos\phi}\right)^2 = \frac{\mu\left[a\left(1-e^2\right)\right]^2}{\left(1+e\cos\phi\right)^2} - I_{xx}, \\
\frac{dI_{yy}}{dt} &= 2\mu \left[ma\left(1-e^2\right)\right]^{1/2} \frac{e\sin\phi}{1+e\cos\phi} - \frac{dI_{xx}}{dt},
\end{aligned}$$

$$\frac{d^2 I_{yy}}{dt^2} = \frac{2\mu m}{a\left(1-e^2\right)}\left(e^2 + e\cos\phi + \cos 2\phi + e\cos^3\phi\right), \tag{8.50}$$

$$\begin{aligned}\frac{d^3 I_{yy}}{dt^3} &= \frac{-2\mu m^{3/2}}{\left[a\left(1-e^2\right)\right]^{5/2}}\left(1+e\cos\phi\right)^2 \\ &\quad \times\left[2\sin 2\phi + e\sin\phi\left(1+3\cos^2\phi\right)\right].\end{aligned} \tag{8.51}$$

The xy terms are,

$$\begin{aligned} I_{xy} &= \mu\left[a\left(1-e^2\right)\right]^2 \frac{\sin\phi\cos\phi}{\left(1+e\cos\phi\right)^2}, \\ &= \mu\left[a\left(1-e^2\right)\right]^2 \frac{\sin 2\phi}{2\left(1+e\cos\phi\right)^2}, \\ \frac{dI_{xy}}{dt} &= \mu\left[ma\left(1-e^2\right)\right]^{1/2}\frac{\cos 2\phi + e\cos\phi}{1+e\cos\phi}, \end{aligned}$$

$$\frac{d^2 I_{xy}}{dt^2} = -\frac{\mu m}{a\left(1-e^2\right)}\sin\phi\left[4\cos\phi + e\left(1+\sin^2\phi + 3\cos^2\phi\right)\right], \tag{8.52}$$

$$\begin{aligned}\frac{d^3 I_{xy}}{dt^3} &= \frac{-2\mu m^{3/2}}{\left[a\left(1-e^2\right)\right]^{5/2}}\left(1+e\cos\phi\right)^2 \\ &\quad \times\left[2\cos 2\phi - e\cos\phi\left(1-3\cos^2\phi\right)\right].\end{aligned} \tag{8.53}$$

The wave equation solutions are,

$$\bar{h}_{ij} = \frac{2}{r}\frac{d^2 I_{ij}}{dt^2} = \frac{4\mu m}{a\left(1-e^2\right)r} f_{ij}\left[\phi\right], \tag{8.54}$$

where f_{xx}, f_{yy}, f_{xy} are functions of ϕ and are given in Eqs. (8.48), (8.50), and (8.52). Using the best fit orbit parameters, the factor multiplying, f_{ij}, is $\approx 0.7\times 10^{-22}$. This shows where the small amplitudes expected by LIGO come from.

The luminosity can be calculated using Eqs. (8.35) and (8.39),

$$\begin{aligned}\bar{I}_{ij} &\equiv I_{ij} - \frac{1}{3}\delta_{ij} I_k{}^k = I_{ij} - \frac{1}{3}\delta_{ij}\left(I_{xx} + I_{yy}\right), \\ \bar{I}_{xx} &= I_{xx} - \frac{1}{3}\left(I_{xx} + I_{yy}\right) = \frac{1}{3}\left(2I_{xx} - I_{yy}\right),\end{aligned}$$

$$
\begin{aligned}
\bar{I}_{yy} &= I_{yy} - \frac{1}{3}\left(I_{xx} + I_{yy}\right) = -\frac{1}{3}\left(I_{xx} - 2I_{yy}\right), \\
\bar{I}_{zz} &= -\frac{1}{3}\left(I_{xx} + I_{yy}\right), \\
\bar{I}_{xy} &= I_{xy}, \\
L &= \frac{1}{5}\left\langle \frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{ij}}{dt^3}\right\rangle, \\
9\frac{d^3\bar{I}_{ij}}{dt^3}\frac{d^3\bar{I}^{ij}}{dt^3} &= 18\frac{d^3 I_{xy}}{dt^3}\frac{d^3 I^{xy}}{dt^3} \\
&\quad + \left(2\frac{d^3 I_{xx}}{dt^3} - \frac{d^3 I_{yy}}{dt^3}\right)\left(2\frac{d^3 I^{xx}}{dt^3} - \frac{d^3 I^{yy}}{dt^3}\right) \\
&\quad + \left(2\frac{d^3 I_{yy}}{dt^3} - \frac{d^3 I_{xx}}{dt^3}\right)\left(2\frac{d^3 I^{yy}}{dt^3} - \frac{d^3 I^{xx}}{dt^3}\right) \\
&\quad + \left(\frac{d^3 I_{xx}}{dt^3} + \frac{d^3 I_{yy}}{dt^3}\right)\left(\frac{d^3 I^{xx}}{dt^3} + \frac{d^3 I^{yy}}{dt^3}\right), \\
L &= \frac{8\mu^2 m^3}{15\left[a\left(1-e^2\right)\right]^5} \\
&\quad \times \left\langle (1 + e\cos\phi)^4 (e^2\sin^2\phi + 12[1 + e\cos\phi]^2)\right\rangle. \qquad (8.55)
\end{aligned}
$$

In order to average over a period, Eq. (8.40) is used to change the integration variable from, t to ϕ,

$$
\begin{aligned}
L &= \frac{8\mu^2 m^3}{15\left[a\left(1-e^2\right)\right]^5}\frac{1}{T} \\
&\quad \times \int_0^T dt (1 + e\cos\phi)^4 (e^2\sin^2\phi + 12[1 + e\cos\phi]^2), \\
dt &= \frac{\rho^2}{\left[ma\left(1-e^2\right)\right]^{1/2}} d\phi, \\
L &= \frac{8\mu^2 m^3}{15\left[a\left(1-e^2\right)\right]^5}\frac{m^{1/2}}{2\pi a^{3/2}}\frac{\left[a\left(1-e^2\right)\right]^2}{\left[ma\left(1-e^2\right)\right]^{1/2}} \\
&\quad \times \int_0^{2\pi} \left(1 + e\cos\phi\right)^2\left[e^2\sin^2\phi + 12\left(1 + e\cos\phi\right)^2\right] d\phi,
\end{aligned}
$$

$$= \frac{4\mu^2 m^3}{15\pi a^5 \left(1-e^2\right)^{7/2}}$$

$$\times \int_0^{2\pi} d\phi \left(1 + e\cos\phi\right)^2 \left[12 + 24e\cos\phi + e^2\left(1 + 11\cos^2\phi\right)\right].$$

However,

$$\int_0^{2\pi} d\phi \cos^{0,1,2,3,4}\phi = 2\pi, 0, \pi, 0, \frac{3}{4}\pi.$$

So the final answer is,

$$L = \frac{32}{5}\frac{\mu^2 m^3}{a^5}\frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{\left(1-e^2\right)^{7/2}} \equiv \frac{32}{5}\frac{\mu^2 m^3}{a^5}F[e],$$

$$= \frac{32}{5}\left(\frac{2\pi}{T}\right)^{10/3}\mu^2 m^{4/3} F[e] = \frac{2}{5}\left(\frac{2\pi m}{T}\right)^{10/3} F[e]. \qquad (8.56)$$

Since $e = 0.617$, the enhancement factor due to the eccentricity is $F[e] = 11.84$. Thus the luminosity and the period change are a factor, 11.84, times greater in magnitude than would be calculated for the circular orbit case.

The change in the period, obtained using Eqs. (8.44) and (8.56) yields,

$$-\frac{dE}{dt} = \langle L \rangle,$$

$$\frac{dT}{dt} = -\frac{3}{(2\pi)^{2/3}}\frac{T^{5/3}}{\mu m^{2/3}}\frac{32}{5}(2\pi)^{10/3}\frac{\mu^2 m^{4/3}}{T^{10/3}}F[e],$$

$$= -3(2\pi)^{8/3}\frac{32}{5}\frac{\mu m^{2/3}}{T^{5/3}}F[e] = -\frac{48\pi}{5}\left(\frac{2\pi m}{T}\right)^{5/3} F[e], \qquad (8.57)$$

$$= -30.16 \times 11.84\left(\frac{2\pi 2 \times 1.4 \times 1.484 \times 10^3}{7.75 \times 3.6 \times 10^3 \times 3 \times 10^8}\right)^{5/3},$$

$$= -2.4 \times 10^{-12} = 76\ \mu\text{s/y}.$$

This result agrees with the data within 0.33%. The above expression can be used to find, T and a, after any elapsed time since $\frac{dT}{dt} \propto -T^{-5/3}$.

Problems

1. Start with Eq. (8.1) and show the steps that lead to Eq. (8.2). Starting with Eq. (8.3) show the steps leading to Eq. (8.4). Show that the second term in Eq. (8.5) $= -\Box h_{\beta\nu}$.
2. In showing that GR admits a wave equation solution, the utility of a gauge transformation and use of the trace inverse was noted,

$$h_{\mu'\nu'} = h_{\mu\nu} - \xi_{\mu},_{\nu} - \xi_{\nu},_{\mu}\,,$$
$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h/2 = h_{\mu\nu} - \eta_{\mu\nu}\eta^{\chi\beta}h_{\beta\chi}/2.$$

   (a) Show, $\bar{h}^{\mu\nu} = \bar{h}^{\mu'\nu'} + \eta^{\chi\nu}\xi^{\mu},_{\chi} + \eta^{\chi\mu}\xi^{\nu},_{\chi} - \eta^{\mu\nu}\xi^{\alpha},_{\alpha}$.
   (b) Show that if, $\bar{h}^{\mu\nu};_{\nu} = \bar{h}^{\mu\nu},_{\nu} \neq 0$, the above gauge transform makes $\bar{h}^{\mu'\nu'};_{\nu'} = 0$, with the correct gauge vector choice. What is the choice?
3. Prove $\delta(t'-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \exp[i\omega(t'-t)]$.
4. Prove that if $\bar{I}_{ij} \equiv I_{ij} - \delta_{ij}I_k{}^k/3$, then $\mathrm{Trace}(\bar{I}) = \sum_i \bar{I}_{ii} = \bar{I}_i{}^i = 0$. Then prove that for a wave traveling in the z direction, Eq. (8.36) and Eq. (8.37) are the same.
5. Prove,

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi n^i n_k = 4\pi\delta^i{}_k/3, \quad \text{and,}$$
$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi n^i n^j n_l n_m = 4\pi(\delta^{ij}\delta_{lm} + \delta^i{}_m\delta^j{}_l + \delta^i{}_l\delta^j{}_m)/15.$$

6. Obtain Eqs. (8.40)- (8.45) for the orbit of the Hulse-Taylor pulsar from Newtonian physics.
7. Show for the Hulse-Taylor pulsar that the wavelength of the gravitational wave is much larger than the source size and much smaller than the distance to the observation point. This justifies use of quadrupole radiation for the power, in the far zone .
8. Calculate the period as a function of time for the Hulse-Taylor pulsar. How many years will it take for the semi major axis, a, to be reduced to half of the observed value and for a to fall to one tenth of its observed value?
9. Suppose the stars of the Hulse-Taylor binary system traveled in the same circular orbit of radius, R, being at opposite points of the

diameter. In this case, calculate the amplitudes, $\bar{h}_{ij}$, and show that the frequency of the gravitational wave is twice that of the star rotation.

10. Use the results from problem nine and carry out the calculation of the power output. Show that you obtain the last form of Eq. (8.56) with $F[e] = 1$.
11. To prove the Birkhoff theorem, consider a spherically symmetric, time varying gravitational field in empty space. It has a metric in the form of Eq. (5.11) with the functions $A,\ B,\ C,\ D$ now functions of r and t. Show that a transform to new coordinates,

$$r' = r[D(r,t)]^{1/2},\ dt' \text{ is of the form } F(r',t)dt - G(r',t)dr',$$

where $F(r',t) = t'_{,t}$, $-G(r',t) = t'_{,r'}$, and $F_{,r'} = -G_{,t'}$, are such as to get rid of the $dr'dt'$ term. The metric, after dropping the primes, is then, given by Eq. (5.15), except now, $\Phi = \Phi(r,t)$ and $\Delta = \Delta(r,t)$. Find the nonzero C symbols that, due to the time dependence, did not appear in Eq. (5.16). Next, show that $R_{03} \propto \Delta_{,t}$ and so in free space, $\Delta = \Delta(r)$, and $g_{33} = 1/(1 - 2M'/r)$. Next calculate the terms in R_{00} and R_{33} that have time derivatives. Show they are $\propto$ ($\Delta_{,t}$ or $\Delta_{,t,t}$) and so vanish in free space. Thus as in the stationary case, $0 = R_{00} = R_{33}$ yields $(g_{00}g_{33})_{,r} = 0$, or, $g_{00} = -f(t)(1 - 2M'/r)$. Finally, show that if a new time, $dt'' = dt[f(t)]^{1/2}$, is defined, the Schwarzschild metric is obtained. Ergo, a spherically symmetric pulsating gravitational field does not produce a gravitational wave that can carry energy infinitely far away.

Chapter 9

Cosmology

9.1 Robertson-Walker Metric

There are a large number of particles that group into various structures in the universe. In order to make headway, the cosmological principle is used. That principle states that no position is favored or every observer's view is the same. At first this seems crazy, as our position is certain to give a view of the solar system that is different from that of the sun or another star. So to the principle, we have to add the proviso, **when averaged over a large enough distance**, $\approx 10^9$ pc. Such a distance includes many galaxy clusters. On this scale, as shown in Fig. 9.1, the universe is spherically symmetric about any point or isotropic and looks the same in all directions or is homogeneous. Also, when a far away source is observed, the observer is looking back in time, because the radiation travels with finite velocity.

The best model of the universe has it starting in a hot, dense state about, 13.8, Bya. It has since been expanding and thus cooling, review chapter two problem thirteen. There is a relic signal from the past, from about 0.37 My after the start, called the cosmic microwave background (CMB). This background, observed in all directions, yields the most ideal Planck black body spectrum yet found. The CMB originates from the time when the radiation was much hotter, but still cool enough so that atoms could form. At that point the electrons could no longer scatter the radiation. The universe became transparent to the CMB photons, and they too cooled with the universal expansion. The decreasing temperature is a scalar providing a time marker for the expansion. In Fig. 9.2 the data of the COBE experiment [COBE (2015)] is compared with a Planck spectrum. A summary of all the data [Fixsen (2009)] shows a $T = 2.726$ K spectrum is observed everywhere you look and substantiates a homogeneous, isotropic universe.

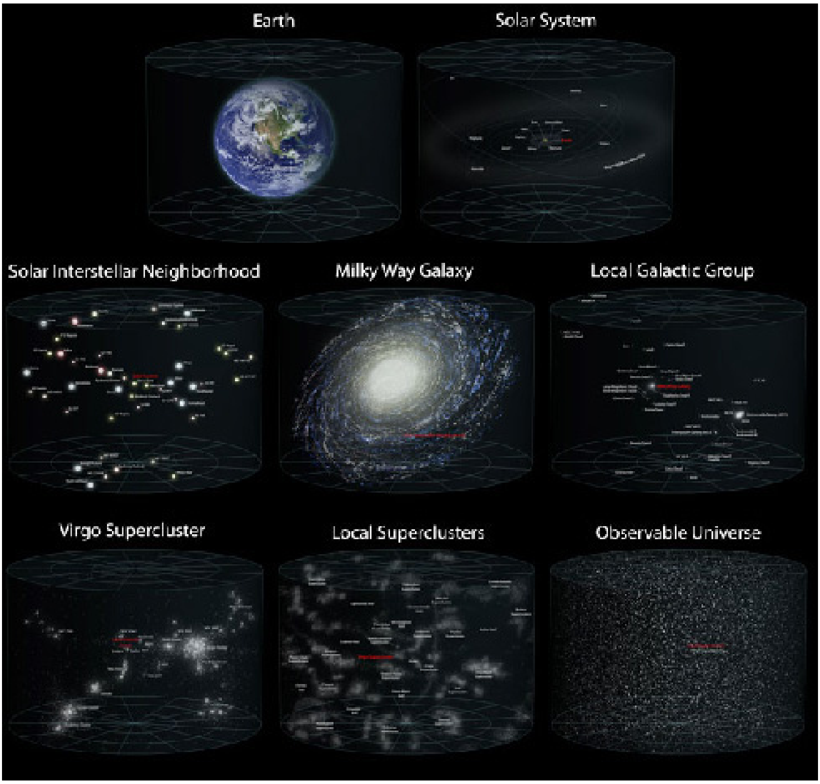

Fig. 9.1 Views of the universe at various distance scales. At the largest scale the universe appears homogeneous and isotropic.

The theoretical, solid curve, prediction is wider than the error bars on the data points. The COBE, WMAP [WMAP (2015)] and PLANCK [PLANCK (2015)] experiments have observed subtle structure in different directions in the sky, on a much finer temperature scale, $\approx 10^{-5}$ K. This structure is thought to be due to random fluctuations in the extremely young universe and led to formation of structures observed today.

Experiments are carried out and they yield $g_{\mu\nu}(x^\chi)$, $T_{\mu\nu}(x^\chi)$. A different coordinate system, $x^{\chi'}$, is considered equivalent if all of universal history appears the same when expressed in terms of these coordinates. That is at any place and time, expressed in terms of X^χ, the relations between the two systems are,

$$g_{\mu\nu}(X^\chi) = g_{\mu'\nu'}(X^\chi),$$
$$T_{\mu\nu}(X^\chi) = T_{\mu'\nu'}(X^\chi).$$

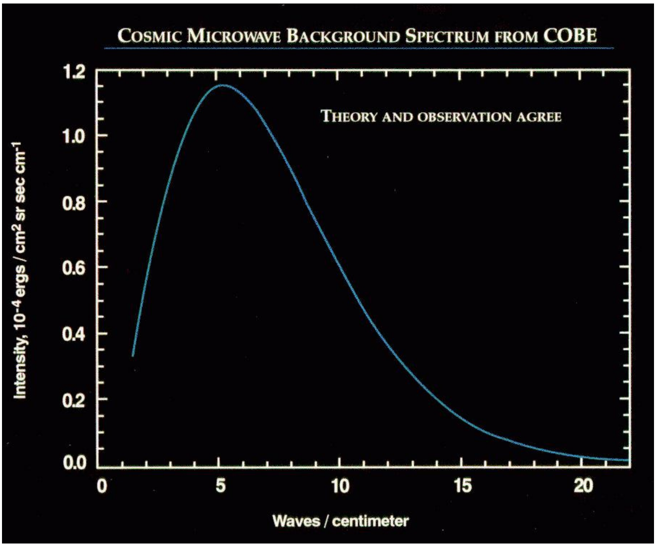

Fig. 9.2 The COBE Project [COBE (2015)] CMB data compared with a Planck spectrum.

Then as cosmic time has been determined from a scalar, that scalar is only a function of time,

$$S(x^0) = S(x^{0'}),$$
$$x^0 = x^{0'}.$$

So all "good" coordinate systems use the same "cosmic" time. These assumptions allow a suitable metric to be obtained.

The expanding universe, or cause of the decrease in the universal black-body temperature requires a time dependent metric of the form,

$$d\tau^2 \equiv dt^2 - Q^2 C_{ij} dx^i dx^j, \; Q = Q[t]. \tag{9.1}$$

Here, $\vec{r} = x^i \hat{e}_i$, $d\vec{r} = dx^i \hat{e}_i$ and the scale factor, Q, is only a function of cosmic time. The reason $g_{0i} = 0$, is that it prevents a preferred direction. The reason $g_{00} = -1$, is because cosmic time is the same for all observers. The quantity, C_{ij}, is required to make the universe spherically symmetric and seen to be the same if one translates from one origin to any other. The following, with $r^p = (r)^p$, is the simplest form that will do the trick,

$$\begin{aligned} C_{ij} dx^i dx^j &= \eta_{ij} dx^i dx^j + \frac{k(\eta_{mn} x^m dx^n)^2}{1 - k\eta_{st} x^s x^t} \\ &= d\vec{r} \cdot d\vec{r} + \frac{k(\vec{r} \cdot d\vec{r})^2}{1 - k\vec{r} \cdot \vec{r}}, \end{aligned} \tag{9.2}$$

$$
\begin{aligned}
d\vec{r}\cdot d\vec{r} &= (dr)^2 + r^2((d\theta)^2 + \sin^2\theta(d\phi)^2),\ \ \vec{r}\cdot d\vec{r} = rdr,\\
C_{ij}dx^i dx^j &= \frac{(dr)^2}{1-kr^2} + r^2((d\theta)^2 + \sin^2\theta(d\phi)^2), \qquad (9.3)\\
k &= 1, 0, -1. \qquad (9.4)
\end{aligned}
$$

It is clear that any other coordinate system that is just a rotation of coordinates will give the same form because the metric is expressed in scalar products. The curvature constant, k, is needed to account for possible, flat, open or closed universes. The scale factor, Q, allows for a universe that evolves in time.

This equation yields the Robertson-Walker metric [Robertson (1935)], [Walker (1936)]. It will soon be seen that the curvature constant, k, can take on only the values enumerated above. The quasi translation below, takes the origin to, $\vec{a}$. It is just an ordinary translation if $k = 0$, and it is obvious that in this case, $C_{ij}dx^i dx^j$ and $C_{i'j'}dx^{i'}dx^{j'}$ have the same form. However, this is also true for the other values of k, see problem one.

$$
\begin{aligned}
x^{i'} &= x^i + a^i\left[(1-kr^2)^{1/2} - (1-(1-k(a)^2)^{1/2})\frac{\vec{a}\cdot\vec{r}}{(a)^2}\right], \qquad (9.5)\\
C_{i'j'}dx^{i'}dx^{j'} &= \eta_{i'j'}dx^{i'}dx^{j'} + \frac{k\eta_{l'm'}x^{l'}dx^{m'}}{1-k\eta_{s't'}x^{s'}x^{t'}}.
\end{aligned}
$$

The reason constant k can take on only one of the above three values is because r and Q can be renormalized,

$$
\begin{aligned}
k > 0, &\qquad \text{choose}\quad \bar{r} = k^{1/2}r,\quad \bar{Q} = Qk^{-1/2},\\
(1-kr^2)^{-1} &= (1-\bar{r}^2)^{-1},\\
(dr)^2 &= k^{-1}(d\bar{r})^2,\\
d\tau^2 &= dt^2 - \bar{Q}^2\left[\frac{(d\bar{r})^2}{1-\bar{r}^2} + \bar{r}^2((d\theta)^2 + \sin^2\theta(d\phi)^2)\right],\\
k < 0, &\qquad \text{choose}\quad \bar{r} = |k|^{1/2}r,\qquad \bar{Q} = Q|k|^{-1/2},\\
(1-kr^2)^{-1} &= (1+|k|r^2)^{-1} = (1+\bar{r}^2)^{-1},\\
(dr)^2 &= |k|^{-1}(d\bar{r})^2,\\
d\tau^2 &= dt^2 - \bar{Q}^2\left[\frac{(d\bar{r})^2}{1+\bar{r}^2} + \bar{r}^2((d\theta)^2 + \sin^2\theta(d\phi)^2)\right].
\end{aligned}
$$

So from now on the bars over r and Q, can be dropped.

To get an idea of how the curvature is affected by k, consider the three possible cases. When $k = 0$, the quasi translation becomes an ordinary translation so that the universe is flat. The metric is diagonal, so the determinant of the metric and the invariant volume element are,

$$(-det(g_{\mu\nu}))^{1/2} = Q^3 \frac{r^2}{(1-kr^2)^{1/2}} \sin\theta,$$
$$d^4V = Q^3 dt \frac{r^2}{(1-kr^2)^{1/2}} dr \sin\theta d\theta d\phi.$$

If $k = 1$, choose $\sin u = r$,

$$u = \sin^{-1} r,$$
$$du = (1-r^2)^{-1/2} dr,$$
$$d^4V = Q^3 dt \sin^2 u du \sin\theta d\theta d\phi.$$

As u increases, it increases faster than $\sin u$. So for large, u, the areas of spheres do not increase as fast as in a flat universe. Such a universe is closed. If $k = -1$, choose $\sinh u = r$,

$$u = \sinh^{-1} r,$$
$$du = (1+r^2)^{-1/2} dr,$$
$$d^4V = Q^3 dt \sinh^2 u du \sin\theta d\theta d\phi.$$

As u increases, it increases slower than $\sinh u$. So for large u, the areas of spheres increase faster as compared to a flat universe. Such a universe is open.

The problem of cosmology is to find $Q(t)$ and k. The sections that follow indicate what observations are made and what they lead to. It turns out that $k = 0$, but for historical reasons and to indicate what cosmologists had to go through, it will be written as k for a while longer.

9.2 The Red Shift

As elements are de-excited they emit characteristic photons, the line spectra of the elements. By comparing light from distant sources with what is observed in the laboratory, it is found that the characteristic wavelengths are always shifted to the red in very distant sources and the farther away, the more the shift. This doesn't work for close objects like the Andromeda galaxy at a distance of $\approx 2.6 \times 10^6$ ly, because local gravitational effects and other local motion mask the expanding universe effect. This red shift is different from that encountered in chapter five, where the light was emitted in stronger or weaker gravity than the observed light.

Assume a light wave crest leaves a typical galaxy at radial coordinate $r = d$, time $t = t_1$, when $Q = Q_1$. The light travels in the $-\hat{r}$ direction reaching us at the origin at $t = t_0$, when $Q = Q_0$. Many red shifts are observed at t_0, so the red shift is labeled by t_1. The next crest leaves at $t_1 + \delta t_1$, and arrives at $t_0 + \delta t_0$. During the small period of time between crests, Q is essentially constant. As t increases, r decreases and the metric yields for $d\theta = d\phi = 0$,

$$\begin{aligned}
0 = (d\tau)^2 &= (dt)^2 - Q^2(dr)^2(1-kr^2)^{-1}, \\
dt/Q &= -dr(1-kr^2)^{-1/2}, \\
\int_{t_1}^{t_0} dt/Q &= -\int_d^0 dr(1-kr^2)^{-1/2} = f(d), \\
f[d] &= (\sin^{-1} d, d, \sinh^{-1} d),\ k = (1, 0, -1), \\
&\approx d,\ \text{if } d \ll 1 \text{ or } k = 0,
\end{aligned} \tag{9.6}$$

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} dt/Q = f(d) = \int_{t_1}^{t_0} dt/Q. \tag{9.7}$$

Call $\frac{1}{Q} = \frac{dW(t)}{dt} \equiv \frac{dW}{dt}$, then the last equation yields,

$$\begin{aligned}
W(t_0) - W(t_1) &= W(t_0 + \delta t_0) - W(t_1 + \delta t_1), \\
W(t_0 + \delta t_0) - W(t_0) &= W(t_1 + \delta t_1) - W(t_1), \\
\delta t_0 \frac{dW}{dt}|_{t_0} &= \delta t_1 \frac{dW}{dt}|_{t_1}, \\
\frac{\delta t_0}{Q_0} &= \frac{\delta t_1}{Q_1},\ Q_{0,1} \equiv Q(t_0), Q(t_1).
\end{aligned}$$

The above equation leads to a simple relation between the frequencies of the emitted and received radiation,

$$\frac{\delta t_0}{\delta t_1} = \frac{\nu_1}{\nu_0} = \frac{\lambda_0}{\lambda_1} = \frac{Q_0}{Q_1} \equiv 1 + z. \tag{9.8}$$

If $z > 0$, $\lambda_0/\lambda_1 > 1$ — and there is a red shift, while if $z < 0$, $\lambda_0/\lambda_1 < 1$ — there is a blue shift. If, for far away sources, there is always a red shift, $Q_0/Q_1 > 1$ — the universe is expanding. This is what the observations indicate and z is called the red shift.

In elementary physics classes and indeed when SR was studied, one can show such shifts find a natural interpretation in terms of the Doppler effect. This is not strictly correct as frequencies are affected by the gravitational field of the universe. It is approximately correct, as shown below, for close in sources with small outward radial speeds. From Eq. (2.26),

$$\nu_1/\nu_0 = [(1+v)/(1-v)]^{1/2} \approx 1 + v = 1 + z, \quad v = z.$$

The same result is obtained from Eq. (9.8). However, to relate that result to v, the proper distance, L_p to the light source is needed. For a light source at radial coordinate, d, L_p is given by the metric and Eq. (9.6),

$$L_p = \int_0^d (g_{rr})^{1/2} dr = Q_0 \int_0^d dr(1-kr^2)^{-1/2} = Q_0 f(d), \tag{9.9}$$

$$\approx Q_0 d, \text{ if } d \ll 1 \text{ or } k = 0. \tag{9.10}$$

In texts the proper distance is also called distance now, proper motion distance, co-moving distance, or co-moving radial distance. In this case, the proper velocity of the source is,

$$\begin{aligned} v_p &\equiv \frac{dL_p}{dt} = d\frac{dQ}{dt}|_{t_0} \equiv d\frac{dQ_0}{dt}, \\ \frac{dQ_0}{dt} &= \frac{Q_0 - Q_1}{t_0 - t_1} \qquad t_1 \to t_0, \\ \frac{v_p}{d} &= \frac{Q_1}{t_0 - t_1}\left[\frac{Q_0}{Q_1} - 1\right], \\ &= \frac{Q_1}{t_0 - t_1} z \approx \frac{Q_0}{t_0 - t_1} z = \frac{L_p}{t_0 - t_1}\frac{z}{d}, \\ z &\approx v_p. \end{aligned}$$

This is the same result as obtained using the Doppler shift. However, it only works for close sources, where $1 = c \approx L_p/(t_0 - t_1)$, and the fact that the universal expansion has increased the proper distance from emission to arrival time can be neglected.

In 1929, E. Hubble observed, what he thought were far away sources, as he had no idea of the size of the universe. We now know they are rather close in objects. He noticed that essentially all the sources showed red shifted light, indicating an expanding universe. He measured their non-relativistic speeds from the red shift using the Doppler effect. He also obtained an estimate of the distances to the sources using Cepheid variable stars, discussed below. He severely underestimated the distances because many of his cepheids were very dim and because the observed stars were so close,

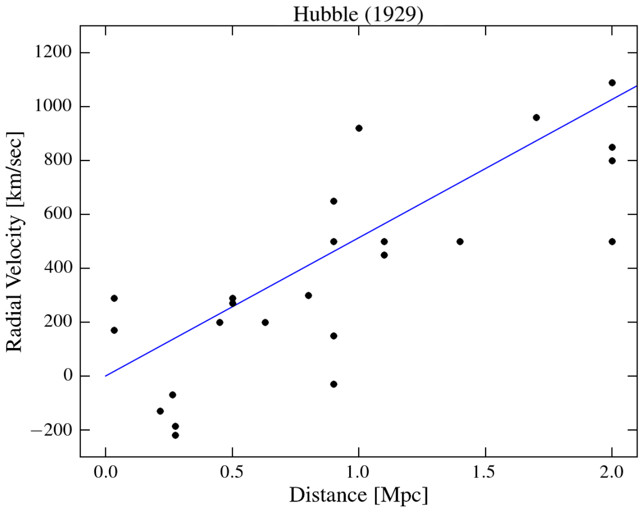

Fig. 9.3 Modern replotting of Hubble's 1929 data Figure 1 by Richard Pogge, The Ohio State University, using the original data from Hubble, E., 1929, PNAS, 15, 168, Tables 1 and 2.

local motion was hiding the universal expansion. Though his data had large spread, he inferred a linear relation between velocity and distance, $z \approx v_p = Hd$. This is the only such relation that would be the same for all observers, independent of position, as illustrated in [Weinberg (1977)] Fig. 1. This data gave the first value for the Hubble parameter, at times mistakenly called the Hubble constant, H. He reasoned, that if far away objects were all rushing away from us, then at an earlier time they must have been on top of each other. This allowed him to arrive at an expansion age for the universe: $v = d/t_H = Hd,\ t_H = H^{-1}$. The determined age was much lower than the current value, but showed that it was to be measured in billions of years. This announcement [Hubble (1929)], of a linear relationship between velocity and distance ignited the field. Such measurements still continue, with ever improving accuracy.

9.3 Determining Distance

For very close stars, the distance can be determined by parallax. As the earth orbits the sun, the star, whose distance is to be measured, appears to move against the background of fixed, distant stars, as indicated in Fig. 9.4. From the extreme wanderings that occur half a year apart, the parallax angle, p, can be determined from the straight line light paths,

$$b/d_{\|} = b/d = \tan p \approx p, \quad d_{\|} = b/p,$$

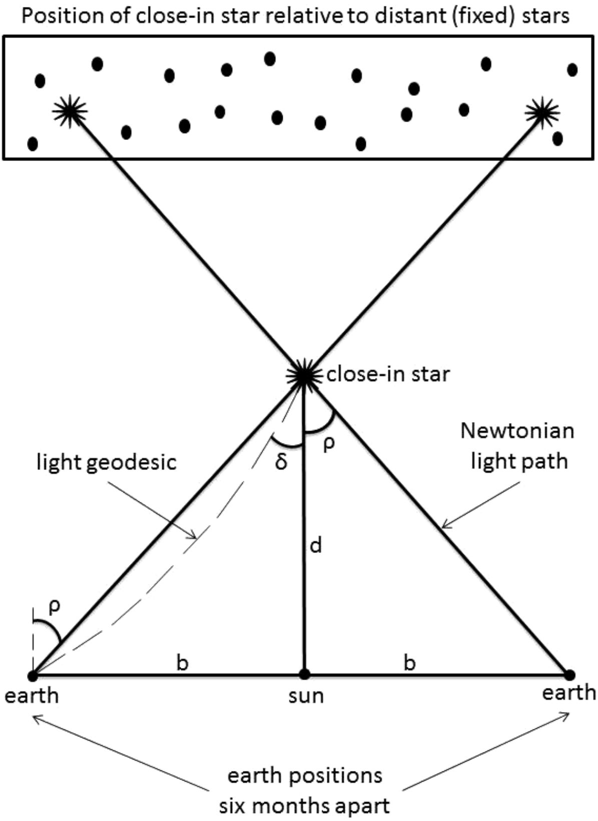

Fig. 9.4 Parallax is the wandering of a close star against the background of fixed, distant stars as the earth orbits the sun. From the extremes of the wanderings the parallax angle is determined and geometry yields the distance to the star.

where, taking the origin at the sun's center, $b = 1.496 \times 10^{11}$ m is the radial coordinate of earth's center in its nearly circular orbit about the sun, and d is the radial coordinate of the star. In Newtonian physics, these coordinates are the distances. This formula is good enough because gravity is so weak and only close stars are considered.

GR requires taking gravity into consideration. The light paths are geodesics as indicated by the dashed curve in Fig. 9.4. Following the development of [Weinberg (1972)], light leaves the source at position $\vec{d}$, and eventually reaches us. In the coordinate system $x^{\mu'}$, where the origin is at the light source, the tip of the ray path is at $\vec{r}\,' = \hat{n}r'$, where $\hat{n}$ is a fixed unit vector and r' is a parameter describing positions along the path relative to the source. In order to translate to the coordinate system in which the light source is at $\vec{d}$ and the origin is at the center of the sun, the quasi translation, Eq. (9.5), must be used so that the same metric holds for both observers. However, in this case, use $\vec{a} = \vec{d}$ and $x^i \leftrightarrow x^{i'}$. Thus,

$$\vec{r}\,(r') = r'\hat{n} + \vec{d}\left[\left(1 - kr'^2\right)^{1/2} - \left[1 - \left(1 - kd^2\right)^{1/2}\right]\frac{r'\hat{n}\cdot\hat{d}}{d}\right]. \qquad (9.11)$$

Only light rays passing close to the origin are observed, so that $\hat{n}$ must nearly point in the $-\hat{d}$ direction. Thus, $\hat{n} \approx -\hat{d} + \vec{\delta}$, where, $\delta \ll 1$, and, $\hat{d}\cdot\vec{\delta} = 0$. In what follows only first order terms in δ are kept. In Fig. 9.4, it is the small angle between the light path and $-\hat{d}$, as measured in the $x^{\mu'}$ system. If the relation for $\hat{n}$ is inserted into Eq. (9.11),

$$\begin{aligned}
\vec{r}\,[r'] &= r'\left(-\hat{d} + \vec{\delta}\right) + \vec{d}\left[\left(1 - kr'^2\right)^{1/2} - \left[1 - \left(1 - kd^2\right)^{1/2}\right]\frac{-r'}{d}\right],\\
&= r'\vec{\delta} + \hat{d}\left[-r' + d\left(\left(1 - kr'^2\right)^{1/2} - \left[1 - \left(1 - kd^2\right)^{1/2}\right]\frac{-r'}{d}\right)\right],\\
&= r'\vec{\delta} + \hat{d}\left[d\left(1 - kr'^2\right)^{1/2} - r'\left(1 - kd^2\right)^{1/2}\right],\\
&\approx d\vec{\delta},\ \ at\ r' = d.
\end{aligned}$$

It is seen that the light ray comes closest to the origin when $r' = d$, as only the very small vector, $\vec{\delta}$, is involved.

The impact parameter, b, is really the proper distance of the light from the origin at this point. Using the above equation and Eqs. (9.6), (9.7) and (9.9), $b = Q_0 d\delta$. Measurements of parallax amount to measurements of the light direction as a function of b. The light ray has a direction, at this point, given by the derivative of the above vector,

$$\begin{aligned}
\frac{d\vec{r}(r')}{dr}|_d &= \vec{\delta} + \hat{d}\left[d\left[-kr'\right]\left(1 - kr'^2\right)^{-1/2} - \left(1 - kd^2\right)^{1/2}\right]|_d,\\
&= \vec{\delta} + \hat{d}\left(1 - kd^2\right)^{-1/2}\left[-kd^2 - \left(1 - kd^2\right)\right],\\
&= \vec{\delta} - \hat{d}\left(1 - kd^2\right)^{-1/2},\\
\vec{e} &\equiv -\left(1 - kd^2\right)^{1/2}\frac{d\vec{r}(r')}{dr'}|_d,\\
&= \hat{d} - \left(1 - kd^2\right)^{1/2}\vec{\delta} \approx \hat{e}.
\end{aligned}$$

To first order in $\vec{\delta}$, the observer's line of sight, at this point, is in the direction of $\vec{e}$. This direction is opposite the light ray direction at this point. Then the angle between the line of sight and $\hat{d}$ is, noting that the small angle approximation holds,

$$\begin{aligned} p &\approx |\hat{e} - \hat{d}|, \\ &= \left(1 - kd^2\right)^{1/2} \delta, \\ &= b(1 - kd^2)^{1/2}/(Q_0 d), \\ d_{||} \equiv b/p &= Q_0 d/(1 - kd^2)^{1/2} \approx Q_0 d = L_p. \end{aligned} \tag{9.12}$$

The above is an exact result if $k = 0$. From Euclidean geometry, a source at distance $d_{||}$, with impact parameter $b \ll d_{||}$ has a parallax angle, $p = b/d_{||}$, so that the above expression is general. These distances can only be measured for stars close to us. There, $d_{||}$ is the proper distance to the star.

Take $d_{||}$ for the closest stars to be 4 ly, or 3.8×10^{16} m. In this case, $p = 0.83''$. The best satellite telescope, Hipparcos [HIPP (2015)], can measure angles to a precision of $0.002''$. So individual stars up to about 1600 ly can be measured by parallax. If you can measure lots of stars, n, in the same vicinity, for example, a globular cluster or nearby galaxy, the error can be beaten down by a factor, $n^{1/2}$. Millions of stars in the Magellanic Clouds, have been measured, so that parallax can reach there. The GAIA experiment [Soszynski (2012)] hopes to reach a sensitivity of $20 \times 10^{-6}\ ''$, allowing stars tens of thousands of light years from earth to be measured by parallax.

To go to larger distances, the comparison of absolute L_{ab}, and apparent luminosity L_{ap}, is exploited, where absolute means the actual power output and apparent is the measured power. Put the origin at the center of the observing mirror, instead the sun. Let the mirror have radius, b, area, A, and normal along the line of sight. The light that reaches the mirror surface lies in a cone, in the $x^{\mu'}$ system with half angle, δ. Thus the fractional solid angle of this cone is $\Delta\Omega/4\pi$, where $\Delta\Omega = \pi\delta^2 = \pi[b/(Q_0 d)]^2 = A/(Q_0 d)^2$.

This quantity is just the fraction of all isotropically emitted photons that reach the mirror. It is just the inverse square law. However, due to the red shift, each photon emitted with frequency, $\nu = E/h$, at Q_1, is shifted to a lower frequency, $\nu(Q_1/Q_0) = \nu/(1 + z)$, when observed at Q_0. Also, the photons that were emitted at time intervals, δt_1, are received at intervals, $\delta t_1 Q_0/Q_1$. Thus the absolute luminosity is decreased on traveling the requisite distance in all directions such that,

$$\frac{dE}{dt}\,(\text{received}) = L_{ab} \left(\frac{Q_1}{Q_0}\right)^2 \frac{A}{4\pi Q_0^2 d^2},$$

$$L_{ap} = \frac{dE}{dt}\,(\text{received})\,/A = \frac{L_{ab} Q_1^2}{4\pi Q_0^4 d^2} \equiv \frac{L_{ab}}{4\pi d_L^2},$$

$$d_L \equiv \left(\frac{L_{ab}}{4\pi L_{ap}}\right)^{1/2} = \frac{Q_0^2 d}{Q_1}, \tag{9.13}$$

$$= [Q_0 d]\,\frac{Q_0}{Q_1} = d_{\|}\,(1+z) = L_p(1+z). \tag{9.14}$$

That is, the apparent luminosity is decreased via the inverse square law from which the luminosity distance, d_L, is defined as in Euclidean geometry. The luminosity distance is larger than the proper distance because after the light is emitted the source and earth separate because of the expanding universe and the light must travel a longer distance than just the proper distance when it was emitted. And the light itself is red shifted. For objects with small red shifts, this doesn't amount to much, however, as we shall see there are many objects with large red shifts. For close objects, $z \approx 0$, and $d_{\|} = d_L$. So for sources that allow a parallax distance measurement, the apparent luminosity leads to the source's absolute luminosity.

There are corrections that experimenters must apply. For example, detectors are sensitive to only part of the electromagnetic spectrum. Some light originally leaving the source is red shifted out of the sensitive region, while other light is red shifted in. Corrections must be made for our rotation about the Milky Way center and the absorption of light on its journey.

Once a large number of stars have their distances measured by parallax, their L_{ab} is determined from L_{ap}. Some stars are variable in their luminosity. For example, Cepheid variables have periods of intensity variation that are closely linked to L_{ab}. The trends in the data are shown in Fig. 9.5 The observation of variable stars with this characteristic in far away galaxies, gives their L_{ab} from their period. Observation of L_{ap} yields d_L for the galaxy. Such methods get us out past a number of galaxy clusters. These results yield the L_{ab} of the brightest stars in galaxies and brightest galaxies in clusters. Looking out to even farther away galaxies and clusters, it can be assumed that their brightest stars and galaxies have the same L_{ab} and the distance to them is determined by L_{ap}.

Nowadays, type 1 supernovae, that have no hydrogen lines in the visible spectrum, are used as standard candles. Their L_{ab} is calibrated from close in explosions and used for the most remote sources. They are standard candles because their explosions arise from the same cause. These relatively rare sources were white dwarfs, supported from collapse by electron degeneracy. The latter is a purely quantum effect due to the statistics of identical fermions. They add mass by accretion from nearby objects until carbon fusion is ignited and the star blows up. It emits as much visible energy as the entire galaxy for a very short time. Their observation gets us out to Bly

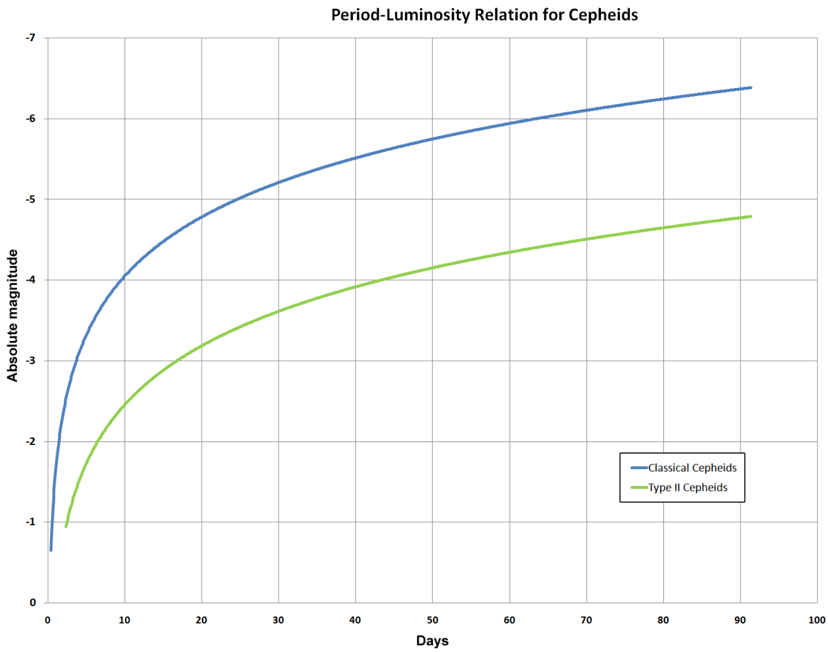

Fig. 9.5 The period, absolute luminosity relation for Cepheid variable stars. The curves indicate the trends of the data. Type II stars are metal rich and brighter that the metal poor type I stars.

distances. The WWW has many figures for visualizing the topics in this section.

9.4 Red Shift Versus Distance Relation

Return to the conditions of section two where the red shift was defined. An expansion of Q, about t_0, can provide a relation between the radiation travel time, $u_1 \equiv t_0 - t_1$, or the luminosity distance, d_L, and the red shift, z. Assume, the expansion can neglect terms of third order or higher, in the expansion variable, u,

$$\begin{aligned} Q &= Q_0 - u\frac{dQ_0}{dt} + \frac{u^2}{2}\frac{d^2Q_0}{dt^2}, \qquad \frac{d^2Q_0}{dt^2} \equiv \frac{d^2Q}{dt^2}|_{t_0}, \\ &= Q_0\left[1 - \frac{u}{Q_0}\frac{dQ_0}{dt} + \frac{u^2}{2Q_0}\frac{d^2Q_0}{dt^2}\right], \\ &\equiv Q_0\left[1 - H_0 u - q_0 H_0^2 u^2/2\right]. \end{aligned} \tag{9.15}$$

In the above equation,

$$H = \frac{1}{Q}\frac{dQ}{dt}, \ -q = \frac{1}{H^2 Q}\frac{d^2Q}{dt^2} = 1 + \frac{1}{H^2}\frac{dH}{dt}, \tag{9.16}$$

where H is the Hubble parameter with present value H_0 and $-q$ is the acceleration parameter with present value $-q_0$.

Evaluate Eq. (9.15) at time t_1, where $u = u_1$ and find the relation between the travel time and red shift,

$$\begin{aligned}
1 &= \frac{Q_0}{Q_1}\left[1 - H_0 u_1 - H_0^2 q_0 u_1^2/2\right], \\
&= (1+z)\left[1 - H_0 u_1 - H_0^2 q_0 u_1^2/2\right], \\
z &= \frac{H_0 u_1 + H_0^2 q_0 u_1^2/2}{1 - H_0 u_1 - H_0^2 q_0 u_1^2/2}, \\
&\approx \left[H_0 u_1 + H_0^2 q_0 u_1^2/2\right]\left[1 + H_0 u_1\right], \\
&\approx H_0 u_1 + H_0^2 u_1^2\left(1 + q_0/2\right).
\end{aligned}$$

This is a quadratic equation that is easily solved for u_1,

$$\begin{aligned}
u_1 &= \frac{-H_0 + H_0\left[1 + 4z\left(1 + q_0/2\right)\right]^{1/2}}{2H_0^2\left(1 + q_0/2\right)}, \\
&= -H_0\frac{1 - \left(1 + 2z\left(1 + q_0/2\right) - \left[4z\left(1 + q_0/2\right)\right]^2/8\right)}{2H_0^2\left(1 + q_0/2\right)}, \\
&= \frac{z}{H_0}\left[1 - z\left(1 + q_0/2\right)\right].
\end{aligned} \tag{9.17}$$

The relation between d_L and z is due to Eq. (9.7), where $f(d) = d$ and Eqs. (9.14)–(9.17) are used. Since $du = -dt$,

$$\begin{aligned}
d &= \int_{t_1}^{t_0} dt/Q = Q_0^{-1}\int_{t_1}^{t_0} dt\left[1 - H_0 u - H_0^2 q_0 u^2/2\right]^{-1}, \\
Q_0 d &\approx -\int_{u_1}^{0} du\left[1 + H_0 u + q_0 H_0^2 u^2/2 + H_0^2 u^2\right], \\
&= \int_{0}^{u_1} du\left[1 + H_0 u + \left(1 + q_0/2\right) H_0^2 u^2\right] \approx u_1(1 + H_0 u_1/2),
\end{aligned}$$

$$
\begin{aligned}
&= \frac{z}{H_0}\left[1 - z\left(1 + q_0/2\right)\right]\left[1 + z\left(1 - z\left(1 + q_0/2\right)\right)/2\right], \\
&\approx \frac{z}{H_0}\left[1 - z\left(1 + q_0\right)/2\right], \\
d_L = Q_0 d\left(1+z\right) &= \frac{z\left(1+z\right)}{H_0}\left[1 - z\left(1 + q_0\right)/2\right], \\
&\approx \frac{z}{H_0}\left[1 + z\left(1 - q_0\right)/2\right]. \qquad (9.18)
\end{aligned}
$$

To lowest order in z, the above is the Hubble relation, $z = H_0 d_L$. So the program is to measure z and d_L as accurately as possible, for many objects, to determine H_0 and q_0. To measure the latter you need to measure at very large distances, where $z > 0.5$. Here type 1 supernovae serve as standard candles. A number of them, at various d_L are needed to determine the shape of the curve. For H_0, objects with $0.1 \leq z \leq 0.25$ are needed to confirm the approximate linearity of the d_L versus z relation. For smaller red shifts, you could be measuring a local velocity rather than the universal expansion, because even for the Virgo cluster, that was beyond Hubble's reach, $d_L \approx 16.5$ Mpc, and $z \approx 0.04$.

A sample of the modern data, using supernovae as standard candles is shown in Fig. 9.6. Hubble's data region is a very small area near the origin. The two groups providing the data are the HighZ Supernova Search Team[Riess (1998)], and the Supernova Cosmology Project[Perlmutter (1999)]. Shown schematically as rectangular bands the data, especially at the highest red shift values, indicate that a straight line fit fails and an accelerating expansion is a better explanation. The data are fit to Robertson-Walker dynamic models for the matter and dark energy densities as discussed in Section 6. The latest analysis from the PLANCK Collaboration [PLANCK (2015)] yields the following,

$$H_0 = (67.8 \pm 1.7\%) \quad \text{km s}^{-1}\ \text{Mpc}^{-1}, \qquad (9.19)$$

$$t_{H_0} = H_0^{-1} = (14.4 \pm 1.7\%) \times 10^9 \text{ y}, \qquad (9.20)$$

$$q_0 \approx -0.54. \qquad (9.21)$$

The result for q_0 is based on a calculation using the best fits for the matter and dark energy densities, see problem twelve. However, just fitting would have found $-q_0 > 0$. Thus, $\frac{d^2Q_0}{dt^2} > 0$, the expansion is accelerating. The universe is open and will expand forever. As shall be seen, a

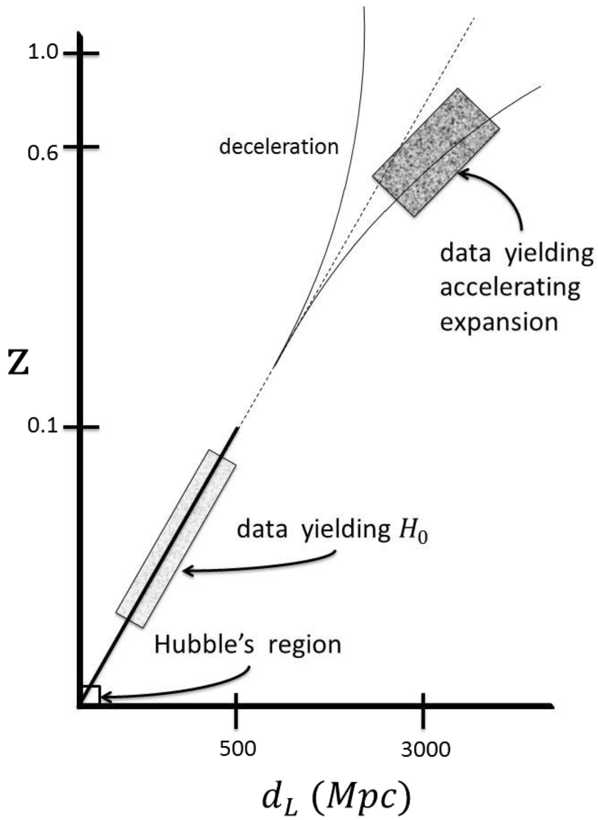

Fig. 9.6 Supernova data for the luminosity distance versus red shift. The data are illustrated schematically by rectangular bands and indicate that the universe is undergoing an accelerating expansion.

way to explain this in GR is to keep a nonzero cosmological constant, Λ, in Eq. (5.9). The contribution from the cosmological constant is a form of non-understood "dark" energy that provides a negative pressure. In a younger universe, dark energy was not so important, however, it has now taken over.

This leads to a note worthy item. The proper velocity, $v_p = d\frac{dQ}{dt}$, is going to get larger and larger as time increases. A future observer may see, $v_p > 1$. This doesn't conflict with relativity. The photons emitted by an object still have $c = 1$, however, space is being created between objects. This is absent from relativity theory, and that's what's making $v_p > 1$.

9.5 Fluids

In order to solve Einstein's equation, Eq. (5.9), the energy-momentum tensor, $T_{\mu\nu}$, is needed. Also, even though gravity is weak, GR is required. For example, if the universe had a uniform mass density $\bar{\rho}$, the quantity $M/r = 4\pi\bar{\rho}r^2/3 > 1$, at some r. In order to make headway, the many bodies making up the universe are subject to a simplifying assumption, namely that they constitute a perfect fluid. Then, an momentum-energy tensor that makes testable predictions, can be obtained.

In the SR frame O where all the particles are at rest the fluid is called dust. In this frame the number density of dust particles is, $N/V = n\ \mathrm{m}^{-3}$. In another SR frame O$'$ where the particles are moving with velocity, $\vec{v}\,'$, the volume element is contracted by a factor, $1/\gamma = (1 - |\vec{v}'|^2)^{1/2}$, so that $n' = \gamma n$.

The flux of particles across a surface is the number crossing the surface in the direction of the normal to the surface per unit area per unit time. Thus all particles within a distance $v^{\bar{i}'}\Delta t'$ of the surface and within area ΔA — that defines the size of the surface, where $v^{\bar{i}'}$ is the speed in the normal direction — will cross the surface in time period, $\Delta t'$,

$$\begin{aligned} f &= \left(n'v^{\bar{i}'}\Delta t'\Delta A\right) / \left(\Delta t'\Delta A\right), \\ &= \gamma n v^{\bar{i}'}. \end{aligned}$$

Note that the vector, $N^{\bar{\mu}'} = nU^{\bar{\mu}'}$, combines both the flux and number density,

$$\begin{aligned} N^{0'} &= nU^{0'} = \gamma n, \\ N^{\bar{i}'} &= nU^{\bar{i}'} = \gamma n v^{\bar{i}'}, \\ N^{\bar{\mu}'}N_{\bar{\mu}'} &= \gamma^2 n^2 \left(-1 + (v')^2\right) = -n^2. \end{aligned} \tag{9.22}$$

This result shows that n is an invariant, just as a particle's momentum vector, $P^\mu = mU^\mu$, yields $P^\mu P_\mu = -m^2$ for the particle's invariant rest mass.

In frame O, the energy of each particle is just the rest energy. If all the particles had the same rest mass, m, the total energy density would be, $\rho = nm$. In the frame O$'$, the energy would be m increased by a factor, γ, and the number density would be similarly increased, thus $\rho' = \gamma^2\rho$. The energy-momentum tensor is the mathematical way to express these things,

$$T^{\bar{\mu}'\bar{\nu}'} = P^{\bar{\mu}'}N^{\bar{\nu}'} = \rho U^{\bar{\mu}'}U^{\bar{\nu}'}. \tag{9.23}$$

It's obvious that this tensor is symmetric and its divergence is zero. Physically, it represents the flux of $P^{\bar{\mu}'}$ across a surface of constant, $x^{\bar{\nu}'}$.

For example, dust has only one nonzero element, $T^{00} = \rho$. Using Eqs. (1.9)–(1.11) and (2.10), the tensor in O′ is,

$$\begin{aligned}
T^{\bar{\mu}'\bar{\nu}'} &= x^{\bar{\mu}'}{}_{,0}\, x^{\bar{\nu}'}{}_{,0}\, \rho, \\
T^{\bar{0}'\bar{0}'} &= x^{\bar{0}'}{}_{,0}\, x^{\bar{0}'}{}_{,0}\, \rho = \gamma^2 \rho, \\
T^{\bar{0}'\bar{i}'} &= x^{\bar{0}'}{}_{,0}\, x^{\bar{i}'}{}_{,0}\, \rho = \gamma^2 \rho v^{\bar{i}'}, \\
T^{\bar{i}'\bar{j}'} &= x^{\bar{i}'}{}_{,0}\, x^{\bar{j}'}{}_{,0}\, \rho = \gamma^2 \rho v^{\bar{i}'} v^{\bar{j}'}.
\end{aligned}$$

The element, $T^{\bar{i}'\bar{j}'}$, being a momentum change across a surface, is proportional to the force on the surface. In a liquid this is equivalent to the pressure, $p =$ Energy/Volume $=$ Force/Area.

To get to the properties of a perfect fluid, consider a general fluid and its thermodynamic properties. In this case, one can consider the rest frame of each fluid element. This is the element's momentarily co-moving reference frame (MCRF). Since fluids can be accelerated, this frame may not remain the MCRF at all times. Other elements have different MCRF's. In this frame there is no bulk flow and no spatial momentum in the particles. All scalar quantities associated with a fluid element are defined to be the values in the MCRF. These include: rest energy, Nm; rest energy density, mn; energy density, ρ, that includes all energies; number density, n; internal energy per particle, $\rho/n - m$; temperature, T; pressure, p; and entropy per particle, S. In addition there are the vectors, $U^{\bar{\mu}}$ and $N^{\bar{\mu}}$.

In the element's MCRF energy, E can be exchanged by the fluid absorbing or emitting heat, $\pm dQ$, or by doing work or by having work done on it, $\pm dW = \pm p dV$. When this is allowed other elements of $T^{\mu\nu}$ become finite. For example, the first law of thermodynamics yields, for an element with N particles,

$$\begin{aligned}
dE &= dQ - p dV, \\
V &= N/n, \qquad dV = -N dn/n^2, \\
E &= \rho V, \qquad dE = V d\rho + \rho dV, \\
dQ &= V d\rho + \rho dV + p dV, \\
&= \frac{N}{n}\left(d\rho - (\rho + p)\,\frac{dn}{n}\right),
\end{aligned}$$

$$dq \equiv dQ/N,$$

$$ndq = d\rho - (\rho + p)\frac{dn}{n} = nTdS. \tag{9.24}$$

For the energy-momentum tensor this means $T^{00} = \rho$, the energy density. Since none of the particles in the element have spatial momentum, $T^{0\bar{i}}$, the energy flux is a heat conduction term as is $T^{\bar{i}0}$. In the case of a perfect fluid, there is no heat conduction in the MCRF, so that $T^{0\bar{i}} = T^{\bar{i}0} = 0$. Viscosity is a force parallel to the interface between elements. Its absence, in a perfect fluid, means that the force is perpendicular to the interface and $T^{\bar{i}\bar{j}} = 0, \bar{i} \neq \bar{j}$. This makes the energy-momentum tensor for a perfect fluid diagonal. Also there is no preferred direction so all $T^{\bar{i}\bar{i}}$ have the same value. The spatial matrix is just a multiplication of the identity matrix. This feature will be preserved for the fluid as a whole as it is true for each element. This approximation yields,

$$\begin{aligned} T^{00} &= \rho, \\ T^{\bar{i}\bar{i}} &= p, \qquad \text{thus,} \\ T^{\bar{\mu}\bar{\nu}} &= (\rho + p)\, U^{\bar{\mu}} U^{\bar{\nu}} + p\eta^{\bar{\mu}\bar{\nu}}, \qquad U^{\bar{i}} = 0. \end{aligned} \tag{9.25}$$

In the above equation, $U^{\bar{i}} \to 0$, otherwise there is a preferred direction. The above equation is a tensor equation and is the equation in a locally inertial frame. For GR in general,

$$T^{\mu\nu} = (\rho + p)\, U^{\mu} U^{\nu} + pg^{\mu\nu}, \qquad U^{i} = 0. \tag{9.26}$$

As previously discussed, this tensor and the vector that forms it have zero divergence,

$$0 = T^{\nu\mu}{}_{;\mu}\,, \tag{9.27}$$

$$0 = U^{\mu}{}_{;\mu} = N^{\mu}{}_{;\mu}\,. \tag{9.28}$$

9.6 Robertson-Walker Einstein Dynamics

The nonzero C symbols, curvature tensors and scalar, and Einstein tensor for the Robertson-Walker metric were evaluated in chapter four, problems seven and eight. The following results were obtained for the C symbols,

$$\Gamma^{0}_{ii} = g_{ii}\frac{1}{Q}\frac{dQ}{dt}, \quad \Gamma^{i}_{0i} = \frac{1}{Q}\frac{dQ}{dt},$$

$$\Gamma^r_{rr} = \frac{kr}{1-kr^2},\ \sin^2\theta\Gamma^r_{\theta\theta} = -r\left(1-kr^2\right)\sin^2\theta = \Gamma^r_{\phi\phi},$$

$$\Gamma^\theta_{\theta r} = r^{-1},\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,\quad \Gamma^\phi_{\phi r} = r^{-1},\ \Gamma^\phi_{\phi\theta} = \cot\theta. \tag{9.29}$$

These allowed the calculation of the Ricci scalar and tensor elements,

$$R_{00} = -3\frac{1}{Q}\frac{d^2Q}{dt^2} = R^{00},$$

$$R_{ii} = g_{ii}\frac{1}{Q^2}\left[2\left(k+\left(\frac{dQ}{dt}\right)^2\right)+Q\frac{d^2Q}{dt^2}\right] = g^2_{ii}R^{ii} = g_{ii}R^{ii}/g^{ii},$$

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\mu}R_{\mu\mu} = 6\frac{1}{Q^2}\left[Q\frac{d^2Q}{dt^2}+\left(\frac{dQ}{dt}\right)^2+k\right]. \tag{9.30}$$

and the Einstein tensor elements,

$$G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2,\quad G^{\mu\nu} = R^{\mu\nu} - g^{\mu\nu}R/2,$$

$$G_{00} = 3\frac{1}{Q^2}\left[\left(\frac{dQ}{dt}\right)^2+k\right],$$

$$G_{ii} = -g_{ii}\frac{1}{Q^2}\left[2Q\frac{d^2Q}{dt^2}+\left(\frac{dQ}{dt}\right)^2+k\right]. \tag{9.31}$$

The Einstein equation for the, 00, element is,

$$G^{00} + \Lambda g^{00} = 8\pi T^{00} = 8\pi\left[(\rho+p)\,U^0U^0 + pg^{00}\right],$$

$$G^{00} = 8\pi\left[(\rho+p)\,U^0U^0 + pg^{00} - g^{00}\frac{\Lambda}{8\pi}\right].$$

Writing the cosmological constant term in the form of a perfect fluid is revealing,

$$G^{00} \equiv 8\pi\left[(\rho+p)\,U^0U^0 + pg^{00} + (\rho_\Lambda + p_\Lambda)\,U^0U^0 + p_\Lambda g^{00}\right],$$

$$\rho_\Lambda + p_\Lambda = 0,\ p_\Lambda = -\frac{\Lambda}{8\pi} = -\rho_\Lambda,$$

$$G^{00} = 8\pi\left[\rho+\rho_\Lambda\right]. \tag{9.32}$$

In these equations, ρ is the energy density due to matter and radiation, so that $\rho = \rho_R + \rho_M$, and p is the ordinary pressure associated with these energies. They can only be functions of time. However, positive Λ yields a positive "dark energy density" and a negative "dark pressure". These stay constant as the universe expands and because the universe is now so large, Λ drives the expansion. It is interesting to recall that in chapter five problem four, the Schwarzschild problem with Λ included was solved. In weak gravity, Λ led to a repulsive Newtonian force.

Using, Eq. (9.16), $g^{00} = -1$, $\left(U^0\right)^2 = 1$, and inserting the explicit form for G^{00} into Eq. (9.32), yields,

$$3\frac{1}{Q^2}\left[\left(\frac{dQ}{dt}\right)^2 + k\right] = 8\pi\left[\rho + \rho_\Lambda\right],$$

$$\left(\frac{dQ}{dt}\right)^2 = -k + 8\pi Q^2\left[\rho + \rho_\Lambda\right]/3, \text{ or,} \tag{9.33}$$

$$\frac{3H^2}{8\pi} = -\frac{3k}{8\pi Q^2} + \left[\rho + \rho_\Lambda\right]. \tag{9.34}$$

The present day critical energy density, ρ_c, can be defined,

$$\rho_c = \frac{3H_0^2}{8\pi}, \quad \rho_c\,(MKS) = \frac{3H_0^2c^2}{8\pi G} = 7.66 \times 10^{-10}\ \mathrm{Jm^{-3}}. \tag{9.35}$$

Eq. (9.34), can at the present time, be rewritten as,

$$1 = \frac{1}{\rho_c}\left[-\frac{3k}{8\pi Q_0^2} + \rho_0 + \rho_\Lambda\right] \equiv \Omega_k + \Omega_{R0} + \Omega_{M0} + \Omega_\Lambda. \tag{9.36}$$

The data of the Planck Collaboration [PLANCK (2015)] yields,

$$\Omega_k = 0,\ \Omega_{R0} \approx 5.5 \times 10^{-5},\ \Omega_{M0} = 0.308,\ \Omega_\Lambda = 0.692. \tag{9.37}$$

These results are best fits to lots of data including red shift versus distance and the very small measurable non-uniformity of the CMB. However, the observed density ratio for the ordinary luminous matter is $\Omega_{OM0} \approx 0.05$, so that there is a form of “dark” matter, that is neutral and does not interact with ordinary matter, except via gravitation, such that $\Omega_{DM0} \approx 0.243$. It’s an interesting universe, the stuff that the stars and ourselves are made of accounts for only five percent of its energy density. The rest is now made up of non-understood dark matter and energy.

Experimenters in deep underground laboratories have mounted experiments to directly detect the rare interactions of dark and ordinary matter. In space, they look for gamma rays from dark matter, anti-matter annihilation, but so far there is no positive result. There is good indirect evidence for dark matter. For example, much stronger than expected gravitational lensing by galaxies of distant objects is observed. Another example concerns stars far from the center of galaxies close to us. As the galaxies are close, red shifts due to the universal expansion are extremely small. The stars far from the galaxy center are rotating about the galaxy center of mass, located near the luminous center of the galaxy. The velocity of these stars can be measured by observing Doppler shifted spectral lines of the elements. Some stars will have blue shifted lines, while stars on the other side of the center will have red shifted lines.

In order to keep things simple, suppose the bulk of the galactic luminous mass, M, is spherically distributed and the stars at distance, R, from the center are well outside of this mass distribution. Let the velocity relative to the galactic center be V. Then from elementary Newtonian considerations, the centripetal acceleration is due to gravity,

$$V^2/R = M/R^2, \ V \propto R^{-1/2}.$$

However, the observations on Andromeda [Rubin (1970)], are in conflict with this expectation. If only the luminous matter was available, the stars should be flying off from the galaxy. They found that the velocity varied from being flat to $V \propto R$. The latter behavior would be expected for stars moving inside an unseen uniform mass density, ρ,

$$V^2/R = \left[(4\pi/3)\,\rho R^3\right]/R^2 = (4\pi/3)\,\rho R,$$
$$V^2 = (4\pi/3)\,\rho R^2, \ V \propto R.$$

This enormous sea of unseen matter in which the galaxy is contained is termed "dark matter".

You might ask how the luminous mass of stars is determined? Many stars exist as binary pairs. By determining their orbits about the common center of mass, the masses may be obtained. The mass, luminosity and temperature of the stars form a profile of these stars. It is found that all stars follow a sort of universal distribution on a plot of luminosity versus temperature and so the masses of stars that are not part of a binary pair are are deduced from their position on the plot, called a Hertzsprung-Russell diagram, see [Kaufman (1985)].

Returning to the Einstein field equations, since Q gives the size of the universe, the quantity $Q^2\rho_{OM+DM} \to Q^2M/Q^3 = M/Q$ decreases as Q increases. However, $Q^2\rho_\Lambda$ increases as Q^2 increases because ρ_Λ is constant. This term has taken over as the dominant energy density as, Q, has increased so much. The radiation energy density is small now because the universe has expanded and cooled, but as seen below was once the dominant term. Since $k = 0$, $H_0 > 0$ and $\frac{dQ_0}{dt} > 0$. From Eqs. (9.26), (9.31)–(9.33), the Einstein equation for G^{rr} is,

$$G^{rr} + 8\pi\frac{\Lambda}{8\pi}g^{rr} = 8\pi T^{rr} = 8\pi p g^{rr},$$

$$\begin{aligned}
0 &= g^{rr}\left[-\frac{1}{Q^2}\left(k + \left(\frac{dQ}{dt}\right)^2 + 2Q\frac{d^2Q}{dt^2}\right) + 8\pi\left(\frac{\Lambda}{8\pi} - p\right)\right], \\
&= g^{rr}\left[-\frac{8\pi}{3}\left(\rho + \frac{\Lambda}{8\pi}\right) - \frac{2}{Q}\frac{d^2Q}{dt^2} + 8\pi\left(\frac{\Lambda}{8\pi} - p\right)\right], \\
&= -\frac{8\pi}{3}\left(\rho + \frac{\Lambda}{8\pi}\right) - \frac{2}{Q}\frac{d^2Q}{dt^2} + 8\pi\left(\frac{\Lambda}{8\pi} - p\right), \\
\frac{2}{Q}\frac{d^2Q}{dt^2} &= 8\pi\left[-\left(\rho/3 + p\right) + (2/3)\frac{\Lambda}{8\pi}\right] > 0.
\end{aligned}$$

The second derivative is positive now, because $p \approx 0$, and the dark energy density is dominant. Thus the expansion will continue at an ever faster rate. If there was no dark energy content a positive $\frac{d^2Q}{dt^2}$ would not be observed and the universe would eventually collapse with a big crunch.

9.7 The Early Universe

In the very young, smaller universe, the energy density was dominated by the radiation. At present the universal CMB presents us with the most perfect blackbody spectrum, Planck distribution, available. Penzias and Wilson, radio astronomers at Bell Labs, discovered this spectrum in 1964-65. The interesting story of that discovery is described by S. Weinberg [Weinberg (1977)]. In the Planck distribution, the energy per unit volume in the range of wavelengths between λ and $\lambda + d\lambda$ is given by,

$$\begin{aligned}
du &= 16\pi^2\hbar c\frac{d\lambda}{\lambda^5}\left(\exp\left[\frac{2\pi\hbar c}{\lambda k_B T_0}\right] - 1\right)^{-1}, \\
u &= \int_0^\infty du = aT^4, \\
&= \frac{8\pi^5\left(k_B T\right)^4}{15\left(2\pi\hbar c\right)^3} = 7.565\times 10^{-16}T^4\ \mathrm{Jm}^{-3}.
\end{aligned} \tag{9.38}$$

The above formula is written in MKS units so it appears as you have learned it in elementary thermodynamics. Problem eight explores the conversion to natural units. The temperature must be given in Kelvin. At present,

$T_0 = 2.726$ K. The distribution vanishes at both $\lambda = 0, \infty$, and reaches a maximum at $\lambda \approx 1.263\, \hbar c/(k_B T)$.

In the past, the size of the universe was smaller by the cube of the scale ratio, $f = Q/Q_0$. Also the wavelength of each photon would be smaller by a factor of f, because as time decreases, the space between typical particles decreases and the photons' wavelengths are compressed by the same factor. This is just the same as saying the photons have been red shifted to the values we now measure. As a photon's energy goes inversely as its wavelength, the energy density is larger by the factor f^{-4}. In the past,

$$
\begin{aligned}
du' &= \frac{du}{f^4} = 16\pi^2 \hbar c \frac{d\lambda}{f^4 \lambda^5} \left(\exp\left[\frac{2\pi\hbar c}{\lambda k_B T_0} \right] - 1 \right)^{-1}, \\
\lambda' &\equiv f\lambda \qquad d\lambda' = f d\lambda, \\
du' &= 16\pi^2 \hbar c \frac{d\lambda'/f}{\lambda'^5/f} \left(\exp\left[\frac{2\pi\hbar c f}{\lambda' k_B T_0} \right] - 1 \right)^{-1}, \\
&= 16\pi^2 \hbar c \frac{d\lambda'}{\lambda'^5} \left(\exp\left[\frac{2\pi\hbar c}{\lambda' k_B T'} \right] - 1 \right)^{-1}.
\end{aligned}
$$

So in the early universe the Planck distribution also described the radiation, but with a higher temperature $T' = T_0/f$. As the radiation energy density is $\propto T^4$, in that epoch the radiation dominates, and the Einstein equation for G^{00} yields,

$$
\begin{aligned}
G^{00} &= 3\left(\frac{1}{Q}\frac{dQ}{dt} \right)^2 = 8\pi\rho_R, \\
\left(\frac{dQ}{dt} \right)^2 &= H_0^2 Q^2 \frac{\rho_R}{\rho_c} \\
&= H_0^2 Q^2 \frac{\rho_{R0}}{\rho_c} \left(\frac{Q_0}{Q} \right)^4, \\
&= H_0^2 Q_0^4 \Omega_{R0} Q^{-2} \equiv b^2 Q^{-2}, \\
\frac{dQ}{dt} &= b\frac{1}{Q}, \text{ so } Q dQ = b dt, \\
Q^2/2 &= b\,(t - t').
\end{aligned} \tag{9.39}
$$

So, $Q = 0$ was achieved at some finite time in the past. Taking $t' = 0$, leads to $Q = 0$ when $t = 0$ and $Q \to t^{1/2}$. So there had to be a start to the expansion, a "Big Bang", and time began then.

The very early universe, way before 0.01 s is not as complicated as now. Then there were just elementary particles and anti-particles in thermal equilibrium with the photons and the temperature was very high. There were equal numbers of all types of particles as the energetic photon-photon collisions could produce all types of particle anti-particle pairs and the pairs rapidly annihilated into photons. However, expansion is a cooling process and the density was decreasing. When T became so low that even electron-positron pairs could no longer be created in photon-photon collisions, matter particles and anti-particles went through their final annihilation. In chapter two, problems twelve and thirteen, it was found that the photon energy was then 0.511 MeV, the temperature was 0.59×10^{10} K, and $Q/Q_0 = 4.6 \times 10^{-10}$. As we are here now and there is no observed evidence for primordial anti-matter, some matter was left over. Using the energy density data given in the previous section and the most simplistic model for the nucleon and photon energies, a nucleon to CMB photon fraction of $\approx 0.23 \times 10^{-9}$ is obtained in problem nine. More refined models obtain a fraction of $\approx 0.61 \times 10^{-9}$. Thus, for every order 10^9 anti-matter particles, there must have been order $10^9 + 1$, matter particles. No one knows why.

All the free matter particles except the protons and electrons are unstable and decay away. For example, except for the neutron, the muon has the longest mean life, 660 m or 2.2 μs. So all of these exotic particles are gone well before the universe age is 0.01 s. They are now produced at accelerators or by astrophysical processes. The free neutron has a mean life of, 880 s, so many of them stick around much longer, but as some decay into protons, $n \to pe\bar{\nu}$, the number of protons increases and that of neutrons decreases. The universe is still too hot for nuclei more complicated than ^1H to form and not be blasted apart by the radiation. However, at a universe age of between 180–240 s, the nucleons could freeze into stable complex nuclei. Calculations predicting 76% ^{1}H and 24% ^{4}He are in agreement with observation.

There are only minute amounts of other hydrogen, helium and lithium isotopes, because their binding energies/nucleon are so much smaller than that of ^{4}He. Essentially all the neutrons froze into ^{4}He. You may ask why weren't there heavier nuclei, with even larger binding energies per nucleon formed? Well, in her wisdom, Mother Nature did not make any stable nuclei with five or eight nucleons. So it wasn't possible for ^{4}He to pick up another nucleon, nor was it possible for two ^{4}He nuclei to combine. Later on very massive stars formed and died, the heavier elements being created in fusion reactions in the stars, reactions between stars and during supernova. The latter causes the star's contents to be spewed out into space and to become the material of a later generation of stars, like our sun.

It is still too hot for atoms to form because their binding energies are in the eV range while nuclear binding energies are in the MeV range. So the photons continue scattering from the electrons. At ≈0.37 My, atoms formed and the universe became transparent to the photons. As the universe expanded, their wavelengths stretched and their temperature decreased, while maintaining their Planck distribution. This is the present day CMB. The universe became transparent to neutrinos before photons as their probability for interaction is smaller. So the former have a lower temperature. However, such times are so small, they don't effect these calculations. It would be nice to see the neutrino black body spectrum and measure a lower T than we do for the photons, but that seems an impossibility at present.

In this early time era, Eq. (9.39) yields,

$$Q = at^{1/2} \qquad H = \frac{1}{Q}\frac{dQ}{dt} = \frac{1}{2t}.$$

As you go back in time, H is increasing rapidly. Also, if $t = 0$ is the beginning, then radiation that left then, that is observed at time t, defines the horizon distance, $D_H(t)$. This is the proper distance to the radiation source that emitted photons at $t = 0$, that are observed at t. For that radiation,

$$\begin{aligned}
(d\tau)^2 &= (dt)^2 - Q^2(t)(dr)^2 = 0, \\
dr &= \frac{dt}{Q}, \qquad \int_0^{r_{max}(t)} dr = \int_0^t \frac{dt'}{Q'}, \\
D_H(t) &= Q\int_0^{r_{max}(t)} dr = Q\int_0^t \frac{dt'}{Q'}, \qquad (9.40) \\
&= t^{1/2}\int_0^t dt' t'^{-1/2} = 2t.
\end{aligned}$$

So as you go back to very small times, the size is decreasing like $t^{1/2}$, but the horizon is decreasing like t. This means the horizon is getting smaller faster than the size of the universe. So less of the universe could be observed. For example, for $t = (0.0001,\ 0.01)$ s, $Q/[aD_H] = 0.5t^{-1/2} = 0.5\,(100,\ 10)$ $\mathrm{s}^{-1/2}$.

If there was no causal connection between particles at the beginning, how can one account for the flatness and the observation that the background radiation is the same in all directions. This is especially true for opposite directions. A model called "Inflation" [Guth (1998)], postulates that before the Planck time $t_P = \left(\hbar G/c^5\right)^{1/2} \approx 0.5 \times 10^{-43}$ s, the universe was causally connected and went through a vast, extremely rapid inflation in size before it settled down to the expansion path outlined in this chapter. The Planck time is constructed from the three basic constants, G, $\hbar$ and c. A Planck length, $L_P = ct_P \approx 2 \times 10^{-35}$ m and a Planck mass, $M_P = (\hbar c/G)^{1/2} \approx 1 \times 10^{22}$ MeV$/c^2$ may similarly be constructed. Since $\hbar$ is present, a quantum theory of gravity is required to understand the beginning.

9.8 Matter and Dark Energy Domination

For most of the history of the universe, the dominant energies have been a combination of matter and dark energy, Λ. Dark energy is now taking over, but the period where matter was dominant, and dark energy was neglected can be examined,

$$\begin{aligned}
\left(\frac{dQ}{dt}\right)^2 &= H_0^2 Q^2 \frac{\rho_{OM+DM}}{\rho_c} \equiv H_0^2 Q^2 \frac{\rho_M}{\rho_c}, \\
&= H_0^2 Q^2 \frac{\rho_{M0}}{\rho_c} \left(\frac{Q_0}{Q}\right)^3, \\
&= H_0^2 Q_0^3 \Omega_{M0} Q^{-1} \equiv b^2 Q^{-1}, \\
Q^{1/2} dQ &= b dt,\ 2Q^{3/2}/3 = b\left(t - t''\right),\ t'' \approx 0, \\
Q &\propto t^{2/3},\ \frac{dQ}{dt} \propto 2t^{-1/3}/3, \\
H &= \frac{1}{Q}\frac{dQ}{dt} = \frac{2}{3t},\ t = \frac{2}{3}H^{-1}.
\end{aligned}$$

Here, $t'' \approx 0$ — compared to billions of years — is the time when radiation was no longer important. In this case, $t_{H_0} = (2/3)\, H_0^{-1} = 9.7$ By. The factor, 2/3, due to GR, was not included in Hubble's evaluation of the age of the universe. This was the expectation before dark energy was added to the energy-momentum tensor. The calculation above really jammed up the works for awhile because some of the globular clusters in our galaxy are measured to be older. It takes the cosmological constant or dark energy to make an older universe. If dark energy becomes completely dominant, then you get $H(\text{future}) = \text{constant}$. So the critical density will not change in the future.

This leads to a perplexing problem. One assumes that the dark energy has something to do with the vacuum. However, if one worked out vacuum energy (all those virtual particles) it would be seen that the measured dark energy is smaller by more than seventy orders of magnitude than the calculated value.

If all energy densities are kept in the calculation and a change of variable is employed, such that $u \equiv Q/Q_0$, $dQ/Q = du/u$,

$$\left(\frac{1}{Q}\frac{dQ}{dt}\right)^2 = H_0^2\left(\rho_R + \rho_M + \rho_\Lambda\right)/p_c,$$

$$= H_0^2\left(\Omega_\Lambda + \Omega_{M0}\left(Q_0/Q\right)^3 + \Omega_{R0}\left(Q_0/Q\right)^4\right),$$

$$= H_0^2\left(\Omega_\Lambda + \Omega_{M0}u^{-3} + \Omega_{R0}u^{-4}\right),$$

$$\frac{1}{Q}\frac{dQ}{dt} = H = H_0\left(\Omega_\Lambda + \Omega_{M0}u^{-3} + \Omega_{R0}^{-4}\right)^{1/2}, \tag{9.41}$$

$$dt = \frac{du}{H_0 u\left(\Omega_\Lambda + \Omega_{M0}u^{-3} + \Omega_{R0}u^{-4}\right)^{1/2}}. \tag{9.42}$$

Initially $u = 0$ while at the present time $u = 1$. Thus,

$$t_0 = H_0^{-1}\int_0^1 \frac{du}{u}\left(\Omega_\Lambda + \Omega_{M0}u^{-3} + \Omega_{R0}u^{-4}\right)^{-1/2} = 13.8 \text{ By},$$

when the integral is performed numerically. This indicates the expansion time is close to the Hubble expansion time, H_0^{-1}. One can differentiate Eq. (9.41) with respect to time and calculate the present value for q_0, see problem twelve.

The proper distance to a source that emitted light at $t \approx 0$ that is just received today is the distance to the horizon. From Eqs. (9.40) and (9.42) this is,

$$D_H(t) = Q\int \frac{dt'}{Q'} = \frac{Q}{Q_0}\int dt'\left(\frac{Q'}{Q_0}\right)^{-1}, \tag{9.43}$$

$$D_{H0} = H_0^{-1}\int_0^1 \frac{du}{u^2}\left(\Omega_\Lambda + \Omega_{M0}u^{-3} + \Omega_{R0}u^{-4}\right)^{-1/2} = 46 \text{ Bly},$$

when the integral is done numerically. However, there are objects farther away so that their light will be viewed in the future. There are also objects that will never be viewed. This is because the speed of light, once emitted, is unity, but the space between objects is expanding more and more rapidly so no observer will see the light from such separated objects. Thus we can't tell how large the universe is at any time. This also explains why the night sky is dark, as every line of sight does not end on a star.

One can see what the past was and what the future holds by changing the integration upper limit. With increasing time, H has decreased rapidly, turned sharply, and is entering its ultimate fate of constancy. When the universe is twice as large as now, $H \approx 0.85 H_0$. At that size, the age will be $\approx 1.8\, t_0$. When the universe was ten times smaller than it is now, its age was $\approx 0.25\, t_0$, but when the universe is ten times as large as now, the age will be $\approx 3.6\, t_0$. This is a potent illustration of the accelerating expansion. By the time that size is reached, $H \approx 0.8\, H_0$, and will remain constant; the ratio, D_H/Q, will have risen rapidly, turned and become approximately constant, at a value of $\approx 1.3 D_{H_0}/Q_0$; and the horizon distance will be, $D_H \approx 13 D_{H0}$, and rising linearly. It's intriguing to imagine what the view would be.

Problems

1. Consider the quasi translation of coordinates to new origin at $\vec{a}$,

$$\vec{r}\,' = \vec{r} + \vec{a}[(1 - kr^2)^{1/2} - [1 - (1 - ka^2)^{1/2}]\frac{\vec{r} \cdot \vec{a}}{a^2}],$$

where $r^2 = \vec{r} \cdot \vec{r}$ and $a^2 = \vec{a} \cdot \vec{a}$. Show that this relation leads to,

$$\eta_{i'j'} dx^{i'} dx^{j'} + \frac{k(\eta_{m'n'} x^{m'} dx^{n'})^2}{1 - k\eta_{s't'} x^{s'} x^{t'}} = \eta_{ij} dx^i dx^j + \frac{k(\eta_{mn} x^m dx^n)^2}{1 - k\eta_{st} x^s x^t}, \text{ or,}$$

$$d\vec{r}\,' \cdot d\vec{r}\,' + \frac{k(\vec{r}\,' \cdot d\vec{r}\,')^2}{1 - k\vec{r}\,' \cdot \vec{r}\,'} = d\vec{r} \cdot d\vec{r} + \frac{k(\vec{r} \cdot d\vec{r})^2}{1 - k\vec{r} \cdot \vec{r}}, \text{ or,}$$

$$\frac{d\vec{r}\,' \cdot d\vec{r}\,'(1 - kr'^2) + k(\vec{r}\,' \cdot d\vec{r}\,')^2}{1 - kr'^2} = \frac{d\vec{r} \cdot d\vec{r}(1 - kr^2) + k(\vec{r} \cdot d\vec{r})^2}{1 - kr^2}.$$

This can be proved with straight forward but lengthy algebra. Derive, $\vec{r}\,' \cdot \vec{r}\,'$, $\vec{r}\,' \cdot d\vec{r}\,'$, and, $d\vec{r}\,' \cdot d\vec{r}\,'$, in terms of, $\vec{r}$, $d\vec{r}$ and $\vec{a}$, to begin the calculation.

2. A distant galaxy has a red shift, $z = 0.25$. Using the red shift versus distance relation, what is the luminosity distance, d_L, and the proper distance now? What is the approximate proper velocity now as given by the Hubble relation? What was the proper distance when light left the galaxy? Use the data for H_0 and q_0 in the text.
3. Show that $H(z) \approx H_0(1 - \frac{dH_0}{dt} z \frac{1}{H_0^2} + \ldots)$.

4. Using $\frac{dQ}{dt}/Q = H$, show, $1 + z = \exp[\int_{t_1}^{t_0} H dt]$, where t_1, is the time light left the source at d, and t_0, is the time observed at the origin. Also show that $H_1 = -\frac{dz}{dt}/(1+z)$. Note many red shifts are observed at t_0, so the red shift, z, is labeled by t_1, the time light was emitted.
5. Consider the conservation of energy and momentum, $T^{\nu\mu};_\mu = 0$, for a perfect fluid in a locally inertial frame. Show this leads to conservation of entropy, $\frac{dS}{d\tau} = 0$, or adiabatic flow. Make use of Eqs. (9.24)–(9.28).
6. Assume the perfect fluid that represents the universe. Do not assume $k = 0$. Here $(\rho(t),\ p(t))$ are the (energy density, pressure) of the universe and are functions of time. Apply energy and momentum conservation and show that for the Robertson-Walker metric,

$$0 = \frac{d\rho}{dt} + 3(\rho + p)\frac{1}{Q}\frac{dQ}{dt} = \frac{d(\rho Q^3)}{dt} + p\frac{dQ^3}{dt}.$$

7. Use the results of problem six to show that in the present universe, $\rho \propto Q^{-3}$. Show that in the early, radiation dominated universe, the equation of state is $p = \rho/3$.
8. In cosmology and astronomy, black body radiation plays an important role. In this problem the Planck spectrum is written in MKS units,

$$\frac{du}{d\lambda} = \frac{16\pi^2\hbar c}{\lambda^5}\frac{1}{exp[\frac{2\pi\hbar c}{\lambda k_B T}] - 1},$$

$$u = \frac{8\pi^5 k_B^4}{15(2\pi\hbar c)^3}T^4,$$

$$\lambda_{max} T = \frac{0.2014(2\pi)\hbar c}{k_B},$$

where du in Jm^{-3} is the energy density of the radiation in the region between wavelengths $(\lambda,\ \lambda + d\lambda)$ in m; T is the temperature in K; and $k_B = 1.381 \times 10^{-23}$ in JK^{-1} is Boltzman's constant. The constants c and $\hbar$, are given in the text. λ_{max} is the wavelength for which $\frac{du}{d\lambda}$ is a maximum. Evaluate the numerical values in the above equations in MKS units. Transform the equations to natural units and evaluate the numerical values. Note that T is the same in both sets of units.
9. Use the present values of the ordinary matter energy density, the radiation energy density and the temperature to calculate approximately the number of photons/baryon. The proton rest energy is 938 MeV, and $k_B = 8.62 \times 10^{-11}$ MeV/K.

10. Hydrogen atoms could begin to form when the typical energy of a photon was about $0.04E_I$, where E_I is the atom's ionization energy, 13.6 eV. From this time to the present, how much have the photons been red shifted? At that time, what was the ratio of the radiation energy density to the mass energy density, including dark matter? Use results from problem 8, if needed.
11. In the text it was shown that for a radiation energy density dominated universe, $Q \propto t^{1/2}$. Now show that the typical photon energy is given by $k_B T = \alpha t^{-1/2}$, and evaluate α. Find the times in seconds when $k_B T = (1,\ 10^3,\ 10^6)$ MeV. Take $k = 0$ for the curvature constant. Hint: work in MKS units, use the Plank formula for the radiation energy density, express α in terms of $G, \hbar, c$ and then numerically evaluate α.
12. Using the data provided, calculate q_0, the present day value of the acceleration parameter.

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