TULLIO LEVI-CIVITA

THE *n*-BODY PROBLEM IN GENERAL RELATIVITY

Volume I

SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

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LE PROBLÈME DES n CORPS EN RELATIVITÉ GÉNÉRALE

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Translator's note

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A. J. K.

Preface

IN THE MONOGRAPH SERIES directed by Henri Villat[1], several fascicules have been devoted to Relativity.

First there are the general presentations of Th. De Donder (nos. 8, 14, 43, 58), and then those more specifically devoted to Einsteinian gravitation – notably Georges Darmois's contribution (no. 25) and that of J. Haag (no. 46) on the Schwarzschild problem.

The present fascicule takes its place alongside the two latter monographs, but it has been conceived and composed in such a way that it may be read and understood by anyone with a knowledge of the principles of Absolute Differential Calculus and of Relativity – either from the original expositions of Einstein, Weyl, or Eddington, or, in French, from Cartan's excellent works[2] (for everything having to do with mathematical theories) and from Chazy's[3] (for Relativity and Celestial Mechanics), or naturally from Levi-Civita's *The Absolute Differential Calculus* (first edition, London and Glasgow, Blackie and Son, 1927) where the two original papers written in Italian are brought together: namely, *Calcolo differenziale assoluto* and *Fondamenti di meccanica relativistica* (Bologna, Zanichelli).

As for the present fascicule, it is hardly necessary to point out that, as its title indicates, we seek to establish in the simplest possible terms the relativistic aspect of what Newton and those who followed him regarded as the key to ordinary Celestial Mechanics.

Naturally, we assume a knowledge of the general principles of Relativity which unify all phenomena and, in particular – when we limit ourselves to a gravitating medium – which make it possible to set up partial differential equations governing the motion of that medium.

It will be noticed at once that this program (with the understanding that

[1] The French text of the present monograph first appeared as Fascicule CXVI of the *Mémorial des Sciences Mathématiques*. Paris, Gauthier-Villars, 1950.

[2] *Leçons sur la géométrie des espaces de Riemann*. Paris, Gauthier-Villars, 1928.

[3] *La Théorie de la Relativité et la Mécanique céleste*. Paris, Gauthier-Villars, tome 1, 1928; tome 2, 1930.

we will have recourse to whatever mathematical resources may serve our purpose) has as its guiding principle the very one on which ordinary Celestial Mechanics is constructed, with the aid of the Newtonian law of attraction between two particles and the law, likewise Newtonian, of the motion of each particle in the medium.

In keeping with this procedure, one can apply the principle of reaction to ordinary Mechanics and reduce the motion of several – let us say n- bodies, each subjected to mutual gravitational effects, to the motion of their centers of gravity: whence the ordinary differential equations of the n-body problem. The simplest case, $n = 2$, was not only stated but, to all intents and purposes, solved by Newton himself.

For $n > 2$, integration only very rarely succeeds, and it is necessary to have recourse to the methods of approximation (the theory of perturbations) that dominate the whole of Astronomy.

Because the modifications brought about by the theory of relativity, no matter how far-reaching and profound they may in principle be, give rise in ordinary cases to almost imperceptible quantitative divergences, one is certain, in advance, of being able to envisage the above-mentioned problems from the classical standpoint, but with numerous and complex corrections, generally very small, arising from the superior conception that serves as the starting point.

There is, of course, one celebrated case in which it has been possible to apply the new theory rigorously throughout. That is the case – to use ordinary Newtonian terms – of the two-body problem, considered under the limiting hypothesis that one of the two bodies is infinitesimally small and that the other is endowed with spherical symmetry. This is the Schwarzschild problem, justifiably regarded, because of its extreme importance, as the most valuable result of the Einstein theory, wherein one can find almost all the required 'touchstones' (see especially fascicule no. 46, by J. Haag, of the *Mémorial des Sciences mathématiques*).

But as soon as one goes beyond this so-called two-body problem (wherein, however, one body is infinitesimal and the other spherical), it seems that it is necessary to give up all hope of treating rigorously in terms of General Relativity the questions usually taken up in Classical Mechanics.

Even the two-body problem, solved so long ago by Newton, has very little chance of being successfully solved in terms of General Relativity.

This is because, in the relativistic scheme, the reaction principle no longer holds, and we do not even know how to begin going about the reduction of the partial differential equations (fourteen of them, with as many unknowns, in the form in which we put the problem – see further on, Chapter II) to ordinary differential equations – let alone integrate them.
So we are forced to give up seeking a rigorous solution and must rest content with a process of successive approximations.
It was EINSTEIN (1916) himself who took the first step toward integrating his equations by means of retarded potentials – provided, however, that only the linear terms in them are retained.[1] He revealed, among other things, the existence of gravitational waves. At almost the same time, DROSTE (1926) applied integration by successive approximations to the general equations of gravitating matter in motion.[2] The orders of magnitude are established with reference to the usual conditions of our planetary system, and the method of calculation is carried out for the n-body problem with remarks that simplify the task of obtaining, up to the first order, the relativistic corrections in the ordinary differential equations.
DE SITTER (1916) took up this important question again in his paper *On Einstein's theory of gravitation and its astronomical consequences.*[3] In the Newtonian theory the motion of the center of gravity of every body C is influenced by the attraction of the other bodies, but *not* by the body itself. But such is not the case even in an approximate relativistic scheme, although De Sitter expresses the conviction that C's effect on itself is simply expressed by a slight modification of the constant values of the masses and that no other effects from it would show up in the final equations.
We do well to recall further that MARCEL BRILLOUIN as early as 1922 pointed out that the reasons we have, in Newtonian gravitation, for ignoring whatever effects arise from the motion of the body itself in determining the field of force that moves it, are not nearly so evident when we consider a body or finite point mass in Einsteinian space.[4]

[1] 'Näherungsweise Integration der Feldgleichungen der Gravitation', *Preuß. Ak. der Wiss., Sitzungsberichte,* 1916, pp. 688–696.
[2] 'The Field of Moving Centres in Einstein's Theory of Gravitation', *Ak. van Wet. te Amsterdam,* **19** (1926) 447–455.
[3] *Monthly Notices of the R. A. S.* especially **67** (1916) 155–183.
[4] 'Gravitation einsteinienne. Statique. Point singulier. Le point matériel', *C. R. Acad. Sc.* **175** (1922) 1008–1012.

Nevertheless, De Sitter adopted that point of view. His formulas are valid for an infinitesimally small body in the given field of several other bodies in motion; but his formulas are not valid, unless special conditions are stipulated, for the prediction of the reciprocal influences within a system of celestial bodies.

Next (1937) there were brought out very clearly the additional hypotheses required to make possible the ultimate reduction of the n-body problem to the *a priori* conceivable type of classical differential equations with additional terms for the Einsteinian correction.[1]

For $n = 2$, the undisturbed problem is integrable in the classical fashion, and one has then only to calculate and interpret the perturbation. And that is exactly what will be done in the conclusion of the present exposition.

But before bringing this preface to a close, a few observations of an historical character are in order.

In my second article in *The American Journal of Mathematics*[2], (1937) it was announced that the Einsteinian correction of the two-body problem involved a secular acceleration of their center of gravity, which would move along the apsidal line common to the orbits (undisturbed) of the two bodies. This hasty conclusion resulted solely from a material error in calculation (the various steps of which – reproduced in the present paper – were, nevertheless, perfectly correct).

EINSTEIN, INFELD, and HOFFMANN[3] have approached the problem in an entirely different way by reducing the relativistic n-body problem to a special case of a new theory concerning the displacement of singularities – a theory which could affect the solution of certain partial differential equations.

The idea is a brilliant one, but at the present stage it requires extremely long and complicated calculations (see p. 88 of the paper just cited – footnote).

ROBERTSON (1938), notwithstanding, has applied the idea to the accelera-

[1] Levi-Civita, 'The Relativistic Problem of Several Bodies', *Am. Journal of Math.* **69** (1937).

[2] Levi-Civita, 'Astronomical Consequences of the Relativistic Two-Body problem', *Am. Journal of Math.* **69** (1937).

[3] 'The Gravitational Equations and the Problem of Motion', *Annals of Math.* **39** (1938) 65–100.

tion of the center of gravity and arrived at the wholly correct conclusion that, contrary to the contention put forward in my above-mentioned article (*Astronomical consequences of the relativistic two-body problem*) *there is no secular acceleration.*[1]

At almost the same time (1938) the existence of such an acceleration was ruled out by EDDINGTON and CLARK [2] by applying De Sitter's equations (duly corrected and simplified by an intuitive application of the principle of reaction).

We, on the contrary, will strive in the present paper to bring out each logical step and to develop the calculations in such a way that they may be followed without the least effort.

FOCK (1939) has recently achieved and published a painstaking analysis [3] of the circumstances that make it possible to approach a solution, through successive approximations, of the relativistic problem of continuous media subjected to gravitational equations. In certain respects he pushed approximation beyond our present goal, where we have in mind especially the problem of a finite number of bodies – in particular, of two separate bodies. Fock did not, however, take up the actual applications of his approximation.

[1] 'Note on the Preceding Paper: The Two-Body Problem in General Relativity', *Annals of Math.* **39** (1938) 101–104.

[2] 'The Problem of *n*-Bodies in General Relativity Theory', *Proc. Roy. Soc. London,* S. A., no. 927, **166** (1938) 465–475.

[3] 'Sur le mouvement des masses finies d'après la théorie de gravitation einsteinienne', *Journal of Physics, Academy of Sciences of the U.S.S.R.* **1** (1939).

Chapter I

EINSTEINIAN GRAVITATION

1. RIEMANN SPACES. SPACE-TIME

1. HISTORICAL BACKGROUND. In 1854 RIEMANN[1] introduced into Science the general concept of a space or metrical variety with n dimensions by expressing, *a priori*, the square of the distance between two infinitesimally separated points as a quadratic form of the differentials in the coordinates x^i

$$ds^2 = \sum_{ik}^{n} a_{ik}\, dx^i\, dx^k, \tag{I. 1}$$

whose coefficients $a_{ik} = a_{ki}$ are arbitrary functions of the variables x^i $(i = 1, 2, \ldots, n)$.[2]

GAUSS had adopted this point of view in his famous *Disquisitiones circa superficies curvas* in developing the differential geometry of a surface, independent of the properties of the space by which the surface is surrounded.

The geometry of the Riemannian space defined by the linear element (I. 1) is obviously invariant with respect to any arbitrary change of the x^i coordinates, if, as is well known, the change is applied to the ds^2 itself.

We take for granted a knowledge of the basic principles of the absolute differential calculus, which is indispensable for systematic (mathematical) operations on Riemannian spaces.

2. THE PARAMETERS AND MOMENTS OF A DIRECTION. GEODESICS. Given a point P in space (I. 1) having the coordinates x^i, let P′ be a point

[1] *Über die Hypothesen, welche der Geometrie zu Grunde liegen,* Habilitationsschrift.
[2] In modern litterature the summation sign for summation over repeated indices is often omitted. Here we maintain the author's notation. *(Note of the editor)*

infinitesimally close to P and having coordinates $x^i + dx^i$. The direction PP′ starting from P is characterized by the parameters

(I. 2) $$\lambda^i = \frac{dx^i}{ds} \qquad (i = 1, 2, \ldots, n)$$

or by the moments

(I. 2′) $$\lambda_i = \sum_{1\,k}^{n} a_{ik}\lambda^k \qquad (i = 1, 2, \ldots, n)$$

satisfying respectively the quadratic equations

(I. 3) $$\sum_{1\,ik}^{n} a_{ik}\lambda^i\lambda^k = 1\,,$$

(I. 3′) $$\sum_{1\,ik}^{n} a^{ik}\lambda_i\lambda_k = 1\,.$$

The λ_i or λ^i are the covariant or contravariant components of one and the same unit vector $\boldsymbol{\lambda}$ (tensor of the first rank) which is a function of point P in space.[1]

Let g be a curve in the space whose differential length is given by (I. 1) joining two points P_0 and P_1; and let g' be a curve infinitesimally separated from g, having the same extremities, P_0 and P_1 as g. The curve g' may be obtained by an infinitesimally small displacement of every point P $(x^1, \ldots, x'')$ of g so that P′ $(x^1 + \delta x^1, \ldots, x'' + \delta x'')$ is the position of P after the deformation. If the minimum value for the length of g is determined from among the aforementioned g' curves, g' has the same length as g, except for infinitesimals of a higher order, which means that the variation of the length g is zero

(I. 4) $$\delta \int ds = 0\,.$$

[1] A vector or tensor quantity will be indicated by boldface characters, in keeping with the current usage in ordinary Vector Calculus.

It will be noted that this variational equation may be satisfied by curves that do not follow the shortest path from P_0 to P_1. In any case, its integral curves are called geodesics.

3. SPACE-TIME ACCORDING TO EINSTEIN AND MINKOWSKI. An application of Riemann's geometrical ideas to the physical sciences which would never have occurred to Riemann and his immediate followers was achieved by EINSTEIN in his theory of Relativity, born, as is well known, from an interpretation of MICHELSON's famous experiment.
The theory of Relativity first leads us, as Einstein explicitly said, to represent mechanical phenomena as a kind of geometry of a four-dimensional space, with time as one of the coordinates. The metric of this space-time is conceived in Riemannian terms, but the quadratic form ds^2 is not definite, since the time differential plays, according to MINKOWSKI, a role that is algebraically different from that of the other coordinates.

4. THE DYNAMICS OF THE POINT-MASS AND THE GEODESIC PRINCIPLE. The famous principle of inertia of ordinary Mechanics postulates that the motions of a point-mass in a zero field, i.e., in the absence of any forces, are uniform and rectilinear. This law, unknown to pre-Galilean thinkers, has been the foundation of the entire structure of classical dynamics.
Well before he formulated the theory of General Relativity, Einstein [1] had already greatly extended the principle of inertia. In so doing, the geometrical domain of his construction was a space-time having *a priori* a metric of any sort and related (in an arbitrary way, but one which for the moment we leave unspecified) to the physical phenomena taking place in that domain.
If

$$ds^2 = \sum_{ik\,0}^{3} g_{ik} dx^i dx^k \tag{I. 5}$$

then, by designating the square of the distance between two infinitesimally

[1] Einstein and Großmann, *Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation.* Leipzig, Teubner, 1913. 38 pp.

separated points (the elementary space-time interval), with x^1, x^2, x^3 being the three spatial coordinates and x^0 being the time, the movements of a point-mass in a given (possibly zero) or unknown field are characterized by the variational equation

$$\delta \int ds = o, \tag{I. 4}$$

and are thus represented by the geodesics of the space-time under consideration (no. 2).

This is Einstein's geodesic principle. We are concerned with an equation that is invariant with respect to arbitrary transformations of coordinates, including the variable x^0; and this produces the widest possible general application of the principle with respect to the usual type of transformations of coordinates, not involving time.

But what are the laws that govern the construction of the interval ds^2?

The answer is of course supplied by the theory of General Relativity, which Einstein formulated somewhat later. In order to come to grips with this question more concretely, it may be well to make a preliminary review of a few concepts concerning the motion of a continuous medium.

2. CONTINUOUS MEDIA

1. REVIEW OF BASIC PRINCIPLES. The geometrical and kinematic bases of the Mechanics of continuous media, which assume, in the classical manner, a Euclidean space metric wherein the phenomenon of motion takes place, are easily extended to Riemannian spaces.

Here we will review those results that are of particular interest in the present monograph, starting with the very first elements: namely, the concept of a vector field in ordinary Euclidean space.

The vector **v** of the field is characterized by its Cartesian components v_x, v_y, v_z, which we assume to be functions of the coordinates x, y, z and of time t (non-stationary field). We obtain an intuitive picture of the field by supposing that **v** is the velocity of the material element of a continuous system which, at the instant t, occupies the position P (x, y, z).

The fact that we are concerned with a material system in the same field is expressed by the addition of a scalar function, density, which, multiplied

by the element of spatial volume dS, gives us the mass of the material element which at the instant t occupied the volume of dS.
We then say that in the region of space considered there is a permanent flux, if the distribution of the velocities $\mathbf{v}$ does not depend on t.
The lines of flow at the instant t are defined by the fact that along these lines the vector $\mathbf{v}$ is everywhere tangent. This condition is expressed by the differential system

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z},$$

for which the congruence of the integral curves is invariable in time if the flux is permanent.

2. THE ABSTRACT CONCEPT OF MATTER AND FLUX IN A RIEMANNIAN SPACE. Let us now assume an n-dimensional Riemannian space whose linear element is (I. 1).
To enlarge upon the foregoing picture, one has only to assume a congruence of lines given by a differential system

$$\text{(I. 6)} \qquad \frac{dx^1}{\lambda^1} = \frac{dx^2}{\lambda^2} = \ldots = \frac{dx^n}{\lambda^n},$$

where the λ^i, functions of the variables x^i, are the direction parameters of an arbitrary element, or more generally, where the λ^i are proportional to these parameters. We indicate by $d\tau$ the common value of the n relations (I. 6) by introducing in this way a parameter τ (defined to within an additive constant) which can serve to specify the position of any specific line of the congruence (I. 6). The ensemble of these lines depends on $n-1$ arbitrary constants $c_\alpha (\alpha = 1, \ldots, n-1)$; through each regular point [1] of the field under consideration there passes one and only one line.
We must now, by generalizing in a perfectly natural way what takes place in ordinary space, introduce still another scalar function ε(P) of the point P, or, if one wishes, of the coordinates x^i, which is to be interpreted as the

[1] In the sense usually adopted in Mechanics (and in Geometry), that is, considering the λ of the (I. 6) as differentiable so far as is required and not all of zero value in the domain under consideration.

density of matter, or of a similar entity, spread throughout the above-mentioned space, as will occur in a few of the cases to be subsequently considered.

The λ^i parameters and the function ε may depend not only on the x's, but also on another variable t (time); but in the cases of interest to us, the time t will be one of the x variables, and the role of the parameter that fixes position on the lines of the congruence will be played by another quantity, by the auxiliary τ for example, which has just been discussed.

Having assumed this, we will say that there is a flow of matter in the space under consideration if, where dV designates the (Riemannian) extension of an elementary domain surrounding the point P, we agree to attribute to dV the mass (or quantity of matter) $\varepsilon(\mathrm{P})d\mathrm{V}$. From that point on, one has only to apply to this space the same considerations brought into play in the very first principles of the Mechanics of continuous media, and one can thereby go all the way back to Euler's viewpoint and move from it to Lagrange's, and in this way recognize that we are, henceforward, in a position to follow the movement of any element of what we are wont to call matter.

Indeed, the integration of (I. 6) defines the motion of P in the form

$$\mathrm{P} = \mathrm{P}(c_\alpha, \tau)$$

and one has only to introduce this Lagrangian expression for P into the given $\varepsilon = \varepsilon(\mathrm{P})$ to obtain the expression, likewise Lagrangian, for ε.[1]

The permanent flux is characterized by the circumstances to which we referred above: namely, that the lines of congruence and the density ε are invariant (do not depend on t, considered as a variable over and above the x's).

3. THE WORLD OR TIME LINES. The term "world line" or "time line" designates the trajectories of the material elements in a permanent flux in space-time. The term extends to space-time the usual terminology in which the expression 'time diagram of the motion' is applied to the curve representing, in a Cartesian plane t, s, the variations in time of the curvilinear abscissa s of the trajectory, when the motion of a point in ordinary space is to be described.[2]

[1] Levi-Civita and Amaldi, *Compendio di meccanica razionale,* second edition. Bologna, Zanichelli, 1938. See also: Beltrami, 'Ricerche sulla cinematica dei fluidi' in *Opere* **2,** pp. 202–379.

[2] Cf. M. Lévy, *Eléments de Cinématique et de Mécanique.* Paris, E. Bernard, 1902.

We introduce the analytical concepts required to give a mathematical expression of the condition of conservation of mass for a permanent (or even non-permanent) flux in any given Riemannian space.

4. GREEN'S LEMMA.[1] Let $f(x^1, \ldots, x^n)$ be a continuous function together with its first-order derivatives in a domain $\mathfrak{V}$ whose surface is σ. Let us agree, in order to fix our ideas, to consider a Riemannian space endowed with a definite metric.
One has

$$\text{(I. 7)} \qquad \int_{\mathfrak{V}} \frac{\partial f}{dx^i} \frac{d\mathfrak{V}}{\sqrt{a}} = -\int_{\sigma} f n_i \frac{d\sigma}{\sqrt{a}} \qquad (i = 1, 2, \ldots, n),$$

where a is the discriminant of form (I. 1)[2] [the discriminant of such a form is in fact just equal to the determinant of the symmetric coefficients a_{ik} – Translator's note.], and n_i is the i^{th} direction cosine corresponding to x^i of the normal to σ, directed toward the interior of $\mathfrak{V}$.[3]
If we put

$$d\omega = dx^1\, dx^2 \ldots dx^n,$$

[1] The celebrated formula whereby an integral extended to a domain of space may be transformed into an integral extended to the boundary of the domain was first established by G. Green in his paper, 'An Essay on the application of mathematical analysis to the theories of electricity and magnetism' (Nottingham, 1828). But this work remained unnoticed until 1846, when Lord Kelvin pointed out its extraordinary interest at the very moment when mathematical and physical thought was turned toward the theory of the potential which Gauss, independently of Green, had just established on the Continent with his famous paper, *Allgemeine Lehrsätze in beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstoßungs-Kräfte,* published in 1840. Green's work was republished in tomes **39**, **44**, **47** of the *Journal de Crelle* (1850, 1852, 1854) and finally in Green's *Mathematical Papers* (1871). The formula

$$\int X\, dx + Y\, dy = \int\int \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dx\, dy$$

was demonstrated by Riemann in 1851 in his famous thesis. The illustrious author of that thesis seems to have been unaware of the papers of both Green and Gauss. That is why the theorem frequently bears Riemann's name.

[2] In the case of space-time, it must be replaced by its absolute value $-g$ in the analytic formulas that we are developing in this and the following numbers.

[3] The function f may also depend on any arbitrary number of parameters.

one has for the extent or volume of the element $d\mathfrak{V}$ of the domain

$$d\mathfrak{V} = \sqrt{a}\, d\omega .$$

The integral of the left-hand member is

$$\text{(I. 8)} \qquad \iint \cdots \int \frac{\partial f}{\partial x^i} dx^1\, dx^2 \ldots dx^n .$$

Let us first integrate with respect to the variable x^i by seeking the limits within which the integration may be effected. Let $d\sigma_i$ be an element of the surface $x^i =$ const. and let us consider the tube that would have this element as a base and whose lateral surface is made up of coordinate lines x^i (along which only the variable x^i varies). In general, this element, on entering the domain (at M_e) and on leaving it (at M_s), delimits several elements on σ. Since the domain is limited, the number of entries and exits is even, and we have

$$\int \frac{\partial f}{\partial x^i} dx^i = \sum (f_s - f_e),$$

the sum being extended over the differences of the values of f at the entry-points M_s and the exit-points M_e.
The integral of the volume (I. 8) then becomes

$$\text{(I. 9)} \qquad \int \sum (f_s - f_e)\, d\omega' ,$$

since we have put

$$d\omega' = \frac{d\omega}{dx^i} \cdot$$

Now, the direction normal to the cross-section of the tube being considered is the direction of a coordinate line x^i; so all its parameters are zero, except for the one with an i index, which is given by [1]

$$\frac{dx^i}{ds} = \frac{1}{\sqrt{a^{ii}}} \cdot$$

[1] It must be recalled that a^{jk} is the reciprocal element of a_{ik}, that is, the algebraic complement divided by the discriminant a.

If n_{ik} designates the direction cosines of the normal to σ_i, one has

$$n_{ik} = \frac{\delta_{ik}}{\sqrt{a^{ii}}},$$

where the well-known Kronecker symbol δ_{ik} is defined by

$$\delta_{ik} = \begin{cases} 0 & (i \neq k), \\ 1 & (i = k). \end{cases}$$

Moreover, by observing that $d\sigma$ and $d\sigma_i$ are defined by one and the same tube, we see that the relation

$$\text{(I. 10)} \qquad d\sigma\,|\cos(\mathbf{n}, i)| = d\sigma_i\,|\cos(\mathbf{n}_i, i)|,$$

still holds, in which the vectors $\mathbf{n}$ and $\mathbf{n}_i$ designate the directions normal to σ and σ_i.

On multiplying both members of (I. 10) by $\sqrt{a^{ii}}$, we have:

$$\text{(I. 11)} \qquad d\sigma\,|n_i| = \frac{d\sigma_i}{\sqrt{a^{ii}}},$$

where n_i designates the direction cosines of the normal to σ inside $\mathfrak{V}$.

In the formula

$$n_i = \sqrt{a^{ii}}\cos(\mathbf{n}, i),$$

the positive sign of the radical agrees with the sense of the x^i increasing along the coordinate lines.

But one has

$$\text{(I. 12)} \qquad d\sigma_i = \sqrt{aa^{ii}}\,d\omega',$$

and so

$$\text{(I. 13)} \qquad d\omega' = \frac{1}{\sqrt{a}}\,|n_i|\,d\sigma;$$

by substituting this last expression in the integral under consideration, one obtains the right-hand member of the formula stated at the outset.

5. THE DIVERGENCE THEOREM. If $\mathbf{n}$ is the unit vector of the arbitrarily oriented normal to σ at the point $M(x^1, \ldots, x^n)$, we say that the quantity

$$d\sigma \sum_{i}^{n} \lambda^i n_i = \boldsymbol{\lambda} \times \mathbf{n}\, d\sigma, \tag{I. 14}$$

(where the λ^i are the parameters of the congruence of a flux in space and n^i the direction cosines of the normal to σ) is the *flux through $d\sigma$* (per unit time). Since the surface σ is closed, if $\mathbf{n}$ is the normal inside the field $\mathscr{V}$ which it surrounds, the integral

$$\int_\sigma \boldsymbol{\lambda} \times \mathbf{n}\, d\sigma \tag{I. 15}$$

is the flow entering through σ.

Using these definitions, let us assume that $\mathbf{v}$ is a vector with v^i its contravariant components.

We know that, if $\mathbf{v}$ is a vector in Euclidean space with the Cartesian components $v_x(x, y, z)$, $v_y(x, y, z)$, $v_z(x, y, z)$, the differential expression

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

is an invariant with respect to a change of Cartesian coordinates and is referred to as "the divergence of the vector $\mathbf{v}$". This concept may be extended to any Riemannian space if we put

$$\operatorname{div} \mathbf{v} = \frac{1}{\sqrt{a}} \sum_{i}^{n} \frac{\partial}{\partial x^i} (\sqrt{a}\, v^i), \tag{I. 16}$$

the right-hand member being an invariant one in the general sense, that is, with respect to an arbitrary transformation of the x variables.

The concept of divergence may be similarly extended from the case of a vector in which the divergence is a scalar invariant, to the case of a tensor. In the latter case the divergence is a tensor but of one order lower than the original tensor.[1] SCHOUTEN and STRUIK – in their recent book *Einführung*

[1] Cf. Levi-Civita, T., *The Absolute Differential Calculus*. London and Glasgow, Blackie and Son, Ld., 1927, pp. 153–155.

in die neueren Methoden der Differentialgeometrie[1] – would use the term "valence" (Bd. I, p. 7).

Now we have, by virtue of Green's lemma,

$$\int_{\mathfrak{V}} \operatorname{div} \mathbf{v}\, d\mathfrak{V} = \sum_{i=1}^{n} \int_{\mathfrak{V}} \frac{\partial}{\partial x_i}(\sqrt{a}\, v^i) \frac{d\mathfrak{V}}{\sqrt{a}} = -\sum_{i=1}^{n} \int_{\sigma} v^i n_i \, d\sigma = -\int_{\sigma} \mathbf{v} \times \mathbf{n}\, d\sigma,$$

where $\mathbf{n}$ designates the normal inside $\mathfrak{V}$.

The formula

$$\text{(I. 17)} \qquad \int_{\mathfrak{V}} \operatorname{div} \mathbf{v}\, d\mathfrak{V} = -\int_{\sigma} \mathbf{v} \times \mathbf{n}\, d\sigma$$

is the famous theorem of divergence, which plays a basic part in the Mechanics of continuous media and in Mathematical Physics.

6. THE ANALYTIC EXPRESSION OF THE PRINCIPLE OF THE CONSERVATION OF MASS. We are here concerned with trying to express the fact that in any flux the mass remains invariable – that is to say, the mass which, at any given instant (that is, for any value of the parameter τ), occupies a region in space, is invariant during the motion of the matter. We assume that the motion is everywhere regular.

We will follow the Eulerian viewpoint by using two methods to calculate the total variation of the mass taking place in passage from τ to $\tau + d\tau$ in a fixed region $\mathfrak{V}$ of space, subsequently equating the results.

One first finds that this variation is

$$d\tau \int_{\mathfrak{V}} \frac{d\varepsilon}{d\tau} d\mathfrak{V}.$$

On the other hand, if there has been a variation in mass in $\mathfrak{V}$, that is because matter has passed through the surface σ surrounding $\mathfrak{V}$, and the total flow during the elementary $d\tau$ interval is

$$d\tau \int_{\sigma} \varepsilon \boldsymbol{\lambda} \times \mathbf{n}\, d\sigma = -d\tau \int_{\mathfrak{V}} \operatorname{div}(\varepsilon \boldsymbol{\lambda})\, d\mathfrak{V},$$

by virtue of the divergence theorem.

[1] Groningen, Noordhoff. Band I, 1935; Band II, 1938.

So one has

$$\int_{\mathfrak{V}} \left[\frac{\partial \varepsilon}{d\tau} + \operatorname{div}(\varepsilon\boldsymbol{\lambda})\right] d\mathfrak{V} = 0,$$

from which we deduce, by a well-known argument, the equation of continuity or of the conservation of mass

$$\text{(I. 18)} \qquad \frac{\partial \varepsilon}{\partial \tau} + \operatorname{div}(\varepsilon\boldsymbol{\lambda}) = 0.$$

7. THE APPLICATION OF THE PRECEDING CONCEPTS TO SPACE-TIME. The considerations that have just been set forth with respect to a Riemannian space of definite metric are easily extended to space-time.[1] Nevertheless, in this case, we assume that the vectors, tensors or scalar functions involved depend solely on the coordinates x^0, x^1, x^2, x^3, the variable t being replaced by x^0.[2] Consequently, it will here be a question of only permanent flows, and the equation of the conservation of mass for such a flow becomes

$$\text{(I. 19)} \qquad \operatorname{div}(\varepsilon\boldsymbol{\lambda}) = 0.$$

One must note that the congruence of the lines, which is, so to speak, the seat of the flow, is characterized by the differential system (I. 6) which, in this case, may be put into the form

$$\text{(I. 20)} \qquad \frac{dx^i}{ds} = \lambda^i(x^0, x^1, x^2, x^3) \qquad (i = 0, 1, 2, 3),$$

where the independent variable (generally called τ) is here identified with the arc length s (of an arbitrary integral curve) taken (on each curve) from an arbitrary point, and linked to the x's and their differentials dx by (I. 5) [or, in particular, (I. 1), if it is a question of a true Riemannian space with definite metric].

1 We will not have to consider the special circumstances that may arise for an indefinite metric when, in an exceptional case, one has to deal with isotropic varieties (of zero length).

2 In contemporary literature x^4 is frequently used instead of x^0 for the time coordinate. Here we have maintained the original notation of the author. *(Note of the editor)*

A very important special case for our pursuits is that of the geodesic congruences, which will concern us in the next sections.

8. A FEW OBSERVATIONS CONCERNING DIMENSIONS. The continuity equation (I. 19) is, as we will see, connected with the existence of an invariant integral that will be very important for the particular application we have in view.

We shall establish and develop a few considerations connected with it.

But first a preliminary observation must be made:

If a certain physical entity q (energy, matter, heat, etc.) moves in a variety (usually Riemannian) having the coordinates $x^0, x^1, \ldots, x^n$ as it describes a congruence of world lines, we will designate the parameters of the congruence by $\lambda^i (i = 0, 1, \ldots, n)$, and the corresponding vector of the field by $\boldsymbol{\lambda}$. Let $d\sigma$ be a surface element[1] normal, at an arbitrary point P, to the world line $\boldsymbol{\lambda}$ passing through P. We will call ε the "current density" (of the entity q) at a given instant, so that

$$\varepsilon \, d\sigma \, dt$$

represents the quantity of q that flows through $d\sigma$ between the instant t and the instant $t + dt$.

We must keep $\varepsilon d\sigma dt$ homogeneous with q, and the dimensions of $\varepsilon d\sigma$ will be $[q] t^{-1}$ and not $[q]$.

If $\varepsilon d\sigma$ is to represent an energy, the dimensions of q will have to be

$$ml^2 t^{-2} t = ml^2 t^{-1},$$

that is, the dimensions of an action. So we will thus be concerned with a flow of action.

If we wish to consider a space-time and employ the Römerian time x^0 with dimensions of length, instead of ordinary time, q would be homogeneous to an energy-length, that is, would have the dimensions

$$ml^3 t^{-2}.$$

9. THE FORM OF THE CONTINUITY EQUATION OF SPECIAL INTEREST IN RELATIVISTIC MECHANICS. Having established the foregoing,

[1] We say 'surface', as in ordinary space, even though we are concerned with an n-dimensional variety (ambient space being of $n+1$ dimensions).

let us consider a permanent flux in a space-time and equation (I. 19). As is well known [1], this equation means that the integral

$$\int \varepsilon \, d\mathfrak{V},$$

where

$$d\mathfrak{V} = \sqrt{-g}\, dx^0\, dx^1\, dx^2\, dx^3$$

and $-g$ is the discriminant of the linear element (I. 5), is an invariant with respect to the equations

$$\frac{dx^i}{ds} = \lambda^i \qquad (i = 0, 1, 2, 3) \tag{I. 20}$$

of the geodesic congruence or world lines.

This fact may be expressed in a form that will be especially useful in subsequent developments if we take as our domain of integration an elementary field selected in the following manner:

Let us consider an elementary section of space-time of thickness *ds*, initially orthogonal to any arbitrarily-chosen world line.

Let us first observe that, when we follow the four-dimensional flow, in which the elementary field of integration undergoes a special deformation, one can impose constant increments *ds* on the independent variable *s* – which means that the thickness of the elementary section (no longer necessarily perpendicular to the line) is invariable. This arises from the fact that, if we consider an infinitesimally small displacement on a given curve (in any Riemannian space) such that the modulus of the displacement of every point is the same for all the points on the curve, a variation, even a finite one, Δs of the variable *s* is an invariant of the deformation.

In fact, the condition imposed produces the equation

$$s' = s + c \qquad (c = \text{const.})$$

satisfied along the curve, whence

$$\text{arc}\, \mathrm{PP}_1 = \text{arc}\, \mathrm{P'P'_1}\,,$$

if P', P'_1 are the points P, P_1 after the shift along the world line.

[1] Cf. Levi-Civita and Amaldi, *Lezioni di meccanica razionale.* Bologna, Zanichelli; tomo II, capitolo X.

The foregoing considerations lead us to put the integral invariant in the form

$$\text{(I. 21)} \qquad \varepsilon\sqrt{-g}\, dS \frac{dx^0}{ds}\, ds = \text{const.},$$

where

$$dS = dx^1\, dx^2\, dx^3$$

and where we retain for the invariant the expression

$$\eta\, dS$$

with

$$\text{(I. 22)} \qquad \eta = \varepsilon\sqrt{-g}\,\frac{dx^0}{ds}$$

a quantity that, like ε, has the dimensions of an energy density.

10. GEODESIC MOTION. In developing the variational equation (I. 4) by the well-known processes of the Calculus of Variations and of Absolute Differential Calculus, one finds that the geodesics of a Riemannian space are defined by a differential system

$$\text{(I. 23)} \qquad \frac{d\lambda^i}{ds} + \sum_{kl}^{n} {}_{1}\, \Gamma^i_{kl}\lambda^k\lambda^l = 0 \qquad (i = 1, 2, \ldots, n),$$

where the Γ^i_{kl} are obviously the well-known Christoffel symbols of the second kind.

These equations express the property of the autoparallelism, according to Levi-Civita, of the geodesics – the left-hand members of these equations being the contravariant components p^i of the vector $\mathbf{p}$, a geodesic curve, that is

$$\text{(I. 24)} \qquad p^i = \sum_{k=1}^{n}\left(\frac{\partial\lambda^i}{\partial x^k} + \sum_{l=1}^{n} \Gamma^i_{kl}\lambda^l\right)\lambda^k = 0.$$

One may throw the covariant derivatives of the parameter λ^i into sharper relief by writing

$$p^i = \sum_{k=1}^{n} \lambda^i_{|k} \lambda^k, \tag{I. 25}$$

where $\lambda^i_{|k}$ is the covariant derivative of λ^i.

Thenceforward, the covariant components of the vector **p** are given by

$$p_r = \sum_{i=1}^{n} a_{ir} p^i = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} a_{ir} \lambda^i_{|k} \right) \lambda^k .$$

But, by virtue of Ricci's lemma, one has

$$\sum_{i=1}^{n} a_{ik} \lambda^i_{|k} = \sum_{i=1}^{n} (a_{ir} \lambda^i)_{|k} = \lambda_{r|k} .$$

Thus,

$$p_r = \sum_{k=1}^{n} \lambda_{r|k} \lambda^k . \tag{I. 25'}$$

We conclude that the geodesics of a space-time are characterized by parameters λ^i or by direction cosines λ_r defined respectively by the equations

$$p^i = \sum_{k=0}^{3} \lambda^i_{|k} \lambda^k = 0 \qquad (i = 1, 2, \ldots, n), \tag{I. 26}$$

or else

$$p_r = \sum_{k=0}^{3} \lambda_{r|k} \lambda^k = 0 \qquad (r = 1, 2, \ldots, n) \tag{I. 26'}$$

11. SUMMARY AND INTRODUCTION TO THE FOLLOWING SECTION. In the exposition that has been made in the preceding pages we have laid down, independent of the laws of gravity, the concepts of a metrical space-time and of flux, considering space-time to be filled with matter, the elements of which move along the lines of a given *a priori* congruence (geodesic, for example,) and which constitutes an invariable or fixed configuration.

The conservation of mass in this flux requires that the function ε, which is the density of matter (or possibly of some similar entity), and the vector $\boldsymbol{\lambda}$ be related to each other by the basic equation

$$\text{(I. 19)} \qquad \operatorname{div}(\varepsilon\boldsymbol{\lambda}) = 0 .$$

Before applying these generalizations to General Relativity, we must give special attention to the case where ε, the nature of which has been thus far left unspecified, may be, in particular, an energy-density.

3. THE GRAVITATIONAL EQUATIONS AND THE GEODESIC PRINCIPLE IN CONTINUOUS MEDIA WITHOUT INTERNAL STRESSES

1. THE EINSTEIN EQUATIONS. Thus far we have not fixed the law relating the metric of a space-time to the phenomena that take place within that space-time.

The great step forward was taken by EINSTEIN in 1916, when he expressed the general ideas in the paper[1] that is part of the short Einstein-Grossmann pamphlet. Einstein there discovered such a law, giving it in a form independent of whatever coordinate system might be involved.

The law's mathematical content, including Newton's law as a first approximation, is contained in the celebrated differential equations

$$\text{(I. 27)} \qquad G_{ik} - \frac{1}{2} G\, g_{ik} = -\kappa T_{ik} \qquad (i, k = 0, 1, 2, 3),$$

where the chief unknowns are the coefficients g_{ik} of the interval ds^2 of space-time (I. 5), which occur along with their first and second derivatives in the expression of the curvature tensor G_{ik} and its invariant G.

The constant κ is given by the relation

$$\text{(I. 28)} \qquad \kappa = \frac{8\pi f}{c^4} \sim 2{,}071 . 10^{-48}\, g^{-1}\, cm^{-1}\, sec^2 ,$$

[1] 'Grundlage der allgemeinen Relativitätstheorie', *Annalen der Physik,* ser. 4, **49** (1916) 769–833.

where

$$f \sim 6{,}675 . 10^{-8} \, g^{-1} \, cm^3 \, sec^{-2}$$

is the (Gauss) constant of universal gravitation.[1]

Those are the ten basic equations of General Relativity which establish a relation of interdependence between matter and the gravitational field whose differential metric structure is characterized by the tensor G_{ik} and which, to all intents and purposes, reduces to the single Poisson equation in the Newtonian approximation.

We still have to show how matter in motion exerts its influence upon these equations and, consequently, on the space-time metric.

Naturally, it is the energy tensor T_{ik} which, depending on the case, must be tied in with the other elements of the problem under consideration.

2. GRANULAR MEDIA. THE CORRESPONDING ENERGY TENSOR.

The simplest case is that of matter which is granular or, as will be said hereafter, perfect – that is, forming a discontinuous medium in which the internal stresses, classically defined (Cauchy) are negligible. A typical example is furnished by cosmic matter (dust particles) which is assumed by astronomers to have produced celestial bodies by a process of condensation and which a number of present-day scientists believe to be the basic substance of nebulae.

In this case, the covariant and contravariant components of the energy tensor T are, according to Einstein,

$$T_{ik} = \varepsilon \lambda_i \lambda_k \tag{I. 29}$$

or respectively

$$T^{ik} = \varepsilon \lambda^i \lambda^k , \tag{I. 29'}$$

where ε is the energy-density and λ_i the direction cosines (or λ^i respectively, the parameters) of the world lines of space-time, that is, of the lines of congruence of the flux corresponding, in space-time, to the motion of matter in physical space.

[1] The sign $\sim$ signifies "approximately".

It should be immediately pointed out that one could consider another[1], more general, form of the tensor **T** defined by the components

$$\mathrm{T}_{ik} = (\varepsilon + p)\,\lambda_i\lambda_k - p g_{ik}\,. \tag{I. 30}$$

This expression characterizes the case of a kind of matter wherein an isotropic pressure p is exerted, in addition, of course, to the inertial stresses which are also present in perfect matter.
But, for our purposes – as we will presently see – we may confine ourselves to the limiting case of a perfect matter (for which, consequently, the foregoing canonical form of the tensor **T** is valid) because the p terms are, in our case, of a negligible order of magnitude with respect to the reduced tensor (I. 29).
We now deduce the consequences resulting from this hypothesis and from the structure of the Einstein equations.

3. THE PRINCIPLES OF CONSERVATION. The tensor whose covariant components are the left-hand members of the Einstein equations is, as is well known, of zero divergence

$$\operatorname{div}\left(\mathbf{G} - \frac{1}{2}\mathrm{G}\mathbf{g}\right) = 0\,, \tag{I. 31}$$

where **g** and **G** are the tensors whose covariant components are g_{ik} and G_{ik}.
This property imposes the tensor equation that may justly be called the basic equation of the Relativistic Dynamics of a continuous medium.

$$\operatorname{div}\mathbf{T} = 0\,, \tag{I. 32}$$

according to which the divergence of the energy tensor is zero.
This tensor equation replaces, in a condensed form, the four equations that express the classical principles of the conservation of energy and of momentum: we deduce from this a remarkable consequence concerning the lines of flow, as we are going to see when we have recourse to the very first principles of absolute differential calculus.

[1] Cf. J. L. Synge, 'Relativistic Hydrodynamics', *Proc. London Math. Soc.*, second series, **43** (1937).

In fact, one can first write equation (I. 32) in the form

$$\sum_{0\,k}^{3} \mathrm{T}_{ik}^{|k} = 0 \qquad (i = 0, 1, 2, 3),$$

or even

$$\text{(I. 32')} \qquad \sum_{0\,kl}^{3} g^{kl}\mathrm{T}_{ik|l} = 0 \qquad (i = 0, 1, 2, 3).$$

where T_{ik}^{k}, $\mathrm{T}_{ik|l}$ designate respectively the contravariant and covariant derivatives of the tensor **T**.

By substituting in it the expression (I. 29) of T_{ik}, one has

$$\lambda_i \sum_{0\,kl}^{3} g^{kl}\lambda_k \frac{\partial \varepsilon}{\partial x^l} + \varepsilon\lambda_i \sum_{0\,k}^{3} \lambda_k^{|k} + \varepsilon \sum_{0\,l}^{3} \lambda_{i|l}\lambda^l = 0,$$

or finally,

$$\lambda_i \left(\sum_{0\,k}^{3} \frac{\partial \varepsilon}{\partial x^k}\lambda^k + \sum_{0\,k}^{3} \lambda_k^{|k} \right) + \varepsilon \sum_{0\,l}^{3} \lambda_{i|l}\lambda^l = 0,$$

and finally, putting

$$p_i = \sum_{0\,l}^{3} \lambda_{i|l}\lambda^l,$$

$$\text{(I. 33)} \qquad \lambda_i \operatorname{div}(\varepsilon\boldsymbol{\lambda}) + \varepsilon\, p_i = 0 \qquad (i = 0, 1, 2, 3),$$

where the p_i are the covariant components of the geodesic curvature **p** of the world lines.

This vector **p**, as is well known, and as results, moreover, from the formal expression of its covariant components p_i, is simply the vector derivative along the world line of its tangential vector $\boldsymbol{\lambda}$: it is normal to the line and has as its length the scalar curvature.[1]

The above equations express the fact that, wherever there is matter, the world lines are geodesics.

[1] Levi-Civita, *The Absolute Differential Calculus*, pp. 135, 139–140; also: De Donder, *Théorie des champs gravifiques*, *Mém. Sc. Math.*, **14**, Paris, Gauthier-Villars, 1926. pp. 12–16.

In fact, the vector **p** is normal to $\boldsymbol{\lambda}$ in any case; so, multiplying by λ^i and adding, one obtains from (I. 33)

$$\operatorname{div}(\varepsilon\boldsymbol{\lambda}) = 0,$$

which is the continuity equation for our gravitational flux. Then equations (I. 33) become

$$\text{(I. 26')} \qquad \varepsilon\, p_i = 0 \qquad (i = 0, 1, 2, 3),$$

from which we conclude, for $\varepsilon \neq 0$, that all the components of the vector **p** are zero simultaneously, that is, the geodesic property of the world lines which may likewise be expressed by the contravariant equations

$$\text{(I. 26)} \qquad p_i = 0 \qquad (i = 0, 1, 2, 3).$$

In empty space, one has $\varepsilon = 0$; so the world lines are devoid of meaning, and the latter equations are satisfied identically.

4. A RECONSIDERATION OF THE GEODESIC PRINCIPLE FOR EVERY ELEMENT OF THE MEDIUM. The principles of conservation and the canonical form of the tensor **T** thus involve the following consequences, when we consider a flux in space-time:

a) the world lines are geodesics or, more precisely, the congruence of the world lines is geodesic;

b) along these lines, there is a flow of perfect matter which is conserved.

It should be noted that the first conclusion depends on the canonical form (I. 29) or (I. 29′) of the tensor **T**.

These particular postulates – which are quite indicative and furnish, so to speak, a provisional theory of this category of phenomena when they are separated from all others – take on their full significance when they are linked with the theory of General Relativity.

One must then establish, or re-establish, complete interdependence between all the physical phenomena involved, and yet reduce to the barest minimum the category of the phenomena under consideration, interrelating the mathematical parameters that define them by means of equations sufficient in number to determine these phenomena within the framework supplied by the theory of General Relativity.

Chapter II

THE ANALYTICAL NATURE OF THE EQUATIONS OF THE GRAVITATIONAL MOTION OF A CONTINUOUS STRESS-FREE MEDIUM

1. UNKNOWN FUNCTIONS AND EQUATIONS

1. PREMISES. We are henceforward concerned with developing in a simple manner the program set forth in the last lines of the preceding section, limiting ourselves to the consideration of perfect matter in motion in what we would call a field (not given at the outset) in Newtonian Mechanics – a field maintained by the motion of the matter itself and, of course, reciprocally maintaining that motion.

In this way we have a simplified problem in General Relativity where – following the example of what is done much more easily in Celestial Mechanics, namely, neglecting a host of real circumstances – we content ourselves with laying down the basic theories of what may be considered the essential elements of the phenomenon.

From the mathematical point of view, the metric of space-time and the congruence of world lines are just such 'essential elements'.

The quantities that determine them are:

a) the ten coefficients g_{ik} of ds^2;

b) the four parameters λ^i of the congruence, connected to the g's by a quadratic equation, or the direction cosines λ_i (or any arbitrary combination of these quantities);

c) the density ε of the physical entity with which the flux is connected.

This makes a total of fourteen unknowns.

Naturally (in addition to the above-mentioned relation in finite terms) there are fourteen basic partial differential equations defining these fourteen quantities.

They are: the ten Einstein equations

(I. 27) $$G_{ik} - \frac{1}{2} G\, g_{ik} = -\kappa T_{ik} \qquad (i, k = 0, 1, 2, 3),$$

the equations (of conservation)

(I. 26) $$p^i = 0 \qquad (i = 0, 1, 2, 3)$$

which involve the geodesic property of the world lines and which may be reduced to three in number because of the identity

$$\sum_{i}^{3} {}_{0}\, p_i \lambda_i = 0$$

and the conservation equation

(I. 19) $$\operatorname{div}(\varepsilon \boldsymbol{\lambda}) = 0$$

The equation in finite terms, indicated at the outset, expresses, we may say, the fact that the vector $\boldsymbol{\lambda}$ is a unit vector written

(II. 1) $$\sum_{0\ ik}^{3} g_{ik} \lambda^i \lambda^k = 1 .$$

A slightly different form may be given to the Einstein equations by introducing the quantities

(II. 2) $$\tau_{ik} = T_{ik} - \frac{1}{2} T\, g_{ik}$$

where

(II. 3) $$T = \sum_{0\ ik}^{3} g^{ik} T_{ik}$$

is the linear invariant of the tensor **T**.
By multiplying equations (I. 27) by g^{ik} and summing the indices i, k, one has

(II. 4) $$G = \kappa T$$

and consequently

(II. 5) $$G_{ik} = -\kappa\tau_{ik} \qquad (i, k = 0, 1, 2, 3),$$

where, in our case,

(II. 6) $$\tau_{ik} = \varepsilon\left(\lambda_i\lambda_k - \frac{1}{2}g_{ik}\right)$$

because we have

$$T = \sum_{ik\,0}^{3} g^{ik}\,\varepsilon\lambda_i\lambda_k = \varepsilon\,.$$

2. NORMAL SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS. SOME BIBLIOGRAPHICAL DATA. Let us now turn our attention to the Einstein equations in which, as we have already pointed out, the unknown functions g_{ik} occur explicitly in the G_{ik} as combinations of their first and second derivatives. The G_{ik} are linear with respect to the second derivatives, although they are not in themselves linear.

Can we say that it is possible to find solutions of these equations by an appropriate choice of the other unknown functions involved in the problem?

In order to study this question, we must review[1] a few general concepts relative to partial differential equations of the form

(II. 6) $$\sum_{\nu\,1}^{m}\sum_{ik\,0}^{n} F_{\mu\nu}^{ik}\frac{\partial^2 \varphi_\nu}{\partial x^i\,\partial x^k} + \Phi_\mu = 0 \qquad (\mu = 1, 2, \ldots, m),$$

where the x^i $(i = 0, 1, 2, \ldots, n)$ play the role of independent variables, and the φ_ν $(\nu = 1, 2, \ldots, m)$ the role of unknown functions. The coefficients $F_{\mu\nu}^{ik}$ and the functions Φ_μ depend on the x's, φ's and their first derivatives.

By writing out the second derivatives with respect to a single one of these

[1] Cf. Levi-Civita, *Caratteristiche dei sistemi differenziali e propagazione ondosa,* Bologna, Zanichelli, 1931; also: Hadamard, *Leçons sur la propagation des ondes,* Paris, Hermann, 1903.

independent variables – let us say x^{n0}, omitting all the others – we may limit ourselves to writing

$$\text{(II. 6')} \qquad \sum_{1}^{m}{}_{\nu} F^{nn}_{\mu\nu} \frac{\partial^2 \varphi_\nu}{(\partial x^n)^2} + \ldots = 0 \qquad (\mu = 1, 2, \ldots, m);$$

and these equations are solvable with respect to

$$\frac{\partial^2 \varphi_\nu}{(\partial x^n)^2},$$

if the determinant Ω of the $F^{nn}_{\mu\nu}$ is different from zero. In that case, we say that the system is *normal with respect to* x^{n0}.

If we now go on to assume that everything is analytical, the Cauchy-Kowaleski theorem may be applied, and then there follows locally (in the well-known sense) the unique determination of the φ functions in the neighborhood of a particular value of the variable x^n or, in geometrical language, in the neighborhood of a plane

$$x^n = \bar{x}^n$$

referred to as the *carrier surface* of the data.

It is easy to see under what circumstances the normal character of the equations is preserved when a change of variables is effected, so that the carrier variety, instead of the plane x^n equalling a given constant, is an arbitrary surface of n dimensions

$$z(x^1, \ldots, x^n) = \text{const}.$$

Indeed, if we put

$$\text{(II. 7)} \qquad p_i = \frac{\partial z}{\partial x^i}$$

and

$$\text{(II. 8)} \qquad \omega_{\mu\nu} = \sum_{0}^{n}{}_{ik} F^{ik}_{\mu\nu} p_i p_k,$$

the system is normal relative to z, provided that the determinant

$$\text{(II. 9)} \qquad \Omega = \| \omega_{\mu\nu} \|$$

is not identically zero.

3. APPLICATION OF THE GENERAL THEORY TO THE PRESENT CASE. STATEMENT OF NON-NORMALITY NECESSARILY DUE TO THE GENERAL INVARIANCE OF THE SYSTEM. Now it can be shown that in the case we are considering, because of the tensor identity

$$\text{(II. 10)} \qquad \operatorname{div} \mathbf{E} = 0,$$

where

$$\text{(II. 11)} \qquad \mathbf{E} = \mathbf{G} - \frac{1}{2} \mathbf{G} \mathbf{g}$$

is the gravitational tensor, the Einstein system is not normal with respect to the time x^0, even in the simplest case where the functions $\mathbf{T}_{ik}$ are completely assigned by virtue of the x variables: and *a fortiori*, thus, in the case which interests us.

In order to demonstrate this, one has only to consider in the expression of the components E_{ik} of the tensor $\mathbf{E}$, the ensemble of linear terms in the second derivatives of the g_{ik} which is, by a calculation omitted here (but developed in the next chapter),

$$\text{(II. 12)} \qquad \mathrm{E}_{ik} = \mathrm{E}_{ki} = \frac{1}{2} \sum_{jh} g^{jh} \left(\frac{\partial^2 g_{jh}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^h} - \frac{\partial^2 g_{ih}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{ik}}{\partial x^j \partial x^h} \right)$$
$$- \frac{1}{2} g_{ik} \sum_{ljhr} g^{lr} g^{jh} \left(\frac{\partial^2 g_{jh}}{\partial x^l \partial x^r} - \frac{\partial^2 g_{jr}}{\partial x^l \partial x^h} \right) + \dots .$$

When we effect a change of variables, replacing x^0 by a combination

$$z(x^0\ x^1, x^2, x^3)$$

the expressions of the E_{ik} become

$$\text{(II. 12')} \qquad \mathrm{E}'_{ik} = \mathrm{E}'_{ki} = \mathrm{H} \frac{\partial^2 g_{ik}}{\partial z^2} + (p_i p_k - \mathrm{H} g_{ik}) \chi - (p_i \gamma_k + p_k \gamma_i) + g_{ik} \gamma,$$

having put

$$H = \frac{1}{2}\sum_{ik}^{3} g^{ik} p_i p_k ,$$
$$\chi = \frac{1}{2}\sum_{jh}^{3} g^{jh} \frac{\partial^2 g_{jh}}{\partial z^2} ,$$
$$\gamma_k = \frac{1}{2}\sum_{j}^{3} p^j \frac{\partial^2 g_{ik}}{\partial z^2} ,$$
$$\gamma = \frac{1}{2}\sum_{k}^{3} p^k \gamma_k .$$

Now, the tensor identity (II. 10) is expressed in the four identities

(II. 10′) $$\sum_{k}^{3} E_{ik}^{|k} = 0 ,$$

where the symbol of contravariant differentiation is an operator whose derivative part can be reduced to

$$\sum_{l}^{3} g^{kl} \frac{\partial}{\partial x^l} .$$

If we consider in particular the second derivatives with respect to z, the portion of the E_{ik} which depends on them is E'_{ik} and the operator is reduced to

$$\sum_{l}^{3} g^{kl} p_l \frac{\partial}{\partial z} = p^k \frac{\partial}{\partial z} .$$

Now the identities (II. 10′) become

(II. 10″) $$\sum_{0}^{3} p^k E''_{ik} + \ldots = 0 ,$$

where only the term shown contains the third derivatives of the g's, since

the expressions E''_{ik} are obtained from E'_{ik} by replacing the second derivatives of the g's (with respect to z) with the third derivatives.
In (II. 10″) the coefficients of the aforementioned third derivatives cancel out, because such is the case for the second derivatives in the expressions

$$\sum_{0}^{3}{}_{k}\, p^k \, E'_{ik}\,.$$

We conclude from this that

$$\sum_{0}^{3}{}_{h}\, p^k \, E'_{ik} = 0\,, \tag{II. 13}$$

which proves that the system under consideration is not normal, that is, is not solvable with respect to the second derivatives

$$\frac{\partial^2 g_{ik}}{\partial z^2}$$

of all the g's and z's.
More exactly, one could show that the determinant of the E'_{ik} with respect to the

$$\frac{\partial^2 g_{jh}}{\partial z^2}$$

allows, at the very most, the (algebraic) characteristic $10 - 4 = 6$.

2. ISOMETRIC COORDINATES. REDUCTION TO THE NORMAL FORM

1. THE SPECIAL EINSTEIN CASE. DE DONDER'S ANALYTICAL DEVICE AND LANCZÓS'S GEOMETRICAL INTERPRETATION. We must associate with the negative circumstance just established, the essential property of invariant existence equations with regard to an arbitrary change of the variables x or, if one prefers, existence equations valid for any choice of space-time coordinates whatsoever.
In keeping with this remark, there is reason to wonder if it might not be

possible to introduce variables x such that the system will become normal. If such coordinates do exist, we would be justified in concluding that the solutions exist.

EINSTEIN (1918) was the first to integrate his equations (for metrics infinitesimally close to Special Relativity) by using a special system of coordinates that were made to satisfy four supplementary equations whose meaning was not clear at the time.[1]

DE DONDER (1926) took up the question, formulating more exactly the special coordinates that render integration possible, but approaching the matter from a strictly analytical point of view.[2] LANCZÓS (1922) introduced the same coordinates as de Donder, but in a slightly different form; and he was able to give a remarkable geometrical explanation of them.

The following is the de Donder-Lanczós procedure:

We first recall the explicit expression of the Christoffel symbols of the first and second type

$$\begin{bmatrix} i\ h \\ j \end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^h} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ih}}{\partial x^j}\right),$$

$$\Gamma^l_{ik} = \sum_h^{0\ldots 3} g^{hl} \begin{bmatrix} i\ k \\ h \end{bmatrix}$$

and the divergence $\square\Phi$ of the simple tensor, whose covariant components are the first derivatives of the function Φ

$$\text{(II. 14)} \qquad \square\Phi = \sum_{ik}^{0\ldots 3} g^{ik}\Phi_{i|k} = \sum_{ik}^{0\ldots 3} g^{ik}\left(\frac{\partial^2\Phi}{\partial x^i\,\partial x^k} - \sum_l^{0\ldots 3}\Gamma^l_{ik}\frac{\partial\Phi}{\partial x^l}\right).$$

[1] Einstein, 'Näherungsweise Integration der Feldgleichungen der Gravitation', *Preuß. Ak. der Wiss., Sitzungsberichte*, 1916, 688–696; *Ibid.*, 1918, 154–167.

[2] De Donder, *La Gravifique einsteinienne et la Théorie des champs gravifiques. Mém. Sc. Math.* **14**, Paris, Gauthier-Villars, 1926.

[3] Lanczós, 'Ein vereinfachendes Koordinatensystem für die Einsteinschen Gravitationsgleichungen', *Physikalische Zeitschrift*, 1922; also: Finzi, 'Sulla riduzione a forma normale delle equazioni gravitazionali di Einstein', *Rend. Lincei*, 1938, 324–330.

If the function Φ is reduced to one of the independent variables x^j, one has

(II. 15) $$\Box x^j = -\sum_{ik=0}^{3} g^{ik} \Gamma_{ik}^{j}.$$

This is Lanczós's starting point.

Here we must also introduce the linear combinations of the $\Box x^j$ considered by de Donder

(II. 16) $$Z_i = \sum_{j=0}^{3} g_{ij} \Box x^j,$$

that is

(II. 16′) $$Z_i = -\sum_{j=0}^{3} g_{ij} \sum_{lk=0}^{3} g^{lk} \Gamma_{lk}^{i} = -\sum_{lk=0}^{3} g^{lk} \begin{bmatrix} l\ k \\ i \end{bmatrix}.$$

In order to put Z_i into a form better suited to our purposes, we note that

(II. 15′) $$\Box x^j = \frac{1}{\sqrt{-g}} \sum_{h=0}^{3} \frac{\partial}{\partial x^h} (\sqrt{-g}\, g^{hj})$$
$$= \sum_{h=0}^{3} \frac{\partial g^{hj}}{\partial x^h} + \sum_{h=0}^{3} g^{hj} \frac{\partial \log \sqrt{-g}}{\partial x^h}.$$

Thus

$$Z_i = \sum_{hj=0}^{3} g_{ij} \frac{\partial g^{hj}}{\partial x^h} + \sum_{hj=0}^{3} \frac{\partial \log \sqrt{-g}}{\partial x^h} g_{ij} g^{hj}.$$

Now we have

$$\sum_{j=0}^{3} g_{ij} g^{hj} = \delta_{hj} = \begin{cases} 0 & (h \neq j), \\ 1 & (h = j); \end{cases}$$

$$\sum_{j=0}^{3} g_{ij} \frac{\partial g^{hj}}{\partial x^h} = -\sum_{j=0}^{3} g^{hj} \frac{\partial g_{ij}}{\partial x^h}$$

and finally

$$\text{(II. 17)} \qquad Z_i = \frac{\partial \log \sqrt{-g}}{\partial x^i} + \sum_{hj}^{3} {}_0\, g_{ij} \frac{\partial g^{hj}}{\partial x^h} = \frac{\partial \log \sqrt{-g}}{\partial x^i} - \sum_{hj}^{3} {}_0\, g^{hj} \frac{\partial g_{ij}}{\partial x^h}.$$

Taking into account the identity

$$d \log \sqrt{-g} = \frac{1}{2} \sum_{ik}^{3} {}_0\, g^{ik}\, dg_{ik}$$

which is a result of the rule for the differentiation of the determinant of the form ds^2, we have

$$\text{(II. 18)} \qquad \frac{\partial \log \sqrt{-g}}{\partial x^i} = \frac{1}{2} \sum_{hj}^{3} {}_0\, g^{hj} \frac{\partial g_{hj}}{dx^i}.$$

Let us now form the expressions

$$\text{(II. 19)} \qquad \mathfrak{L}_{ik} = \frac{\partial Z_i}{\partial x^k} + \frac{\partial Z_k}{\partial x^i}.$$

One has

$$\frac{\partial Z_i}{\partial x^k} = \frac{\partial^2 \log \sqrt{-g}}{\partial x^i \partial x^k} - \sum_{hj}^{3} {}_0\, \frac{\partial}{\partial x^k}\left(g^{hj} \frac{\partial g_{ij}}{\partial x^h} \right),$$

so

$$\text{(II. 20)} \qquad \mathfrak{L}_{ik} = 2 \frac{\partial^2 \log \sqrt{-g}}{\partial x^i \partial x^k} - \sum_{hj}^{3} {}_0 \left(\frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} + \frac{\partial^2 g_{kj}}{\partial x^h \partial x^i} \right) g^{hj} + \ldots$$

by writing out the second derivatives of the g, which are the only ones that interest us.

We now propose to calculate the differences

$$\overline{\mathrm{G}}_{ik} = \mathrm{G}_{ik} - \frac{1}{2}\mathfrak{L}_{ik} \tag{II. 21}$$

or, more especially, the ensemble of the terms of these differences where the second derivatives of the g are involved.

One has

$$\sum_{h}^{\substack{3\\0}} \Gamma^h_{ih} = \sum_{hj}^{\substack{3\\0}} g^{hj}\begin{bmatrix} i\ h \\ j \end{bmatrix} = \frac{1}{2}\sum_{hj}^{\substack{3\\0}} g^{hj}\frac{\partial g_{hj}}{\partial x^i} = \frac{\partial \log\sqrt{-g}}{\partial x^i}$$

by virtue of formula (II. 18), then,

$$\sum_{h=0}^{3}\left(\frac{\partial \Gamma^h_{ih}}{\partial x^k} - \frac{\partial \Gamma^h_{ik}}{\partial x^h}\right)$$

$$= \frac{\partial^2 \log\sqrt{-g}}{\partial x^i\,\partial x^k} - \frac{1}{2}\sum_{hj=0}^{3}\frac{\partial}{\partial x^h}\left\{g^{hj}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right)\right\}$$

$$= \frac{\partial^2 \log\sqrt{-g}}{\partial x^i\,\partial x^k} + \frac{1}{2}\sum_{hj=0}^{3} g^{hj}\frac{\partial^2 g_{ik}}{\partial x^h\,\partial x^j}$$

$$- \frac{1}{2}\sum_{hj=0}^{3} g^{hj}\frac{\partial^2 g_{ij}}{\partial x^h\,\partial x^k} - \frac{1}{2}\sum_{hj=0}^{3} g^{hj}\frac{\partial^2 g_{kj}}{\partial x^h\,\partial x^i} + \ldots.$$

Consequently, there remains

$$\overline{\mathrm{G}}_{ik} = \frac{1}{2}\sum_{hj=0}^{3} g^{hj}\frac{\partial^2 g_{ik}}{\partial x^h\,\partial x^j} + \ldots. \tag{II. 22}$$

Moreover,

$$\Box g_{ik} = \sum_{hj}^{3} g^{hj}\left(\frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} - \sum_{l}^{3} \Gamma^l_{hj}\frac{\partial g_{ik}}{\partial x^l}\right) = \sum_{hj}^{3} g^{hj}\frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} + \dots .$$

Then

$$\text{(II. 23)} \qquad \mathrm{G}_{ik} - \frac{1}{2}\left(\frac{\partial \mathrm{Z}_i}{\partial x^k} + \frac{\partial \mathrm{Z}_k}{\partial x^i}\right) = \frac{1}{2}\Box g_{ik} + \dots ,$$

where the omitted terms contain no second derivative.
And so we have obtained the important result which we have, along with de Donder, set out to establish: the G_{ik} components of the Ricci-Einstein curvature tensor differ from the expressions

$$\frac{1}{2}\sum_{hj}^{3} g^{hj}\frac{\partial^2 g_{ik}}{\partial x^h \partial x^j}$$

only by those terms which contain solely the first derivatives of the g functions.
De Donder's conditions

$$\text{(II. 24)} \qquad \mathrm{Z}_i = 0 \qquad (i = 0, 1, 2, 3)$$

are equivalent to Lanczós's conditions

$$\text{(II. 25)} \qquad \Box x^i = 0 \qquad (i = 0, 1, 2, 3),$$

but the latter lend themselves to an important geometrical interpretation. To that end, let us consider the second-order partial differential equation

$$\text{(II. 26)} \qquad \Box \Phi = 0,$$

the operator $\Box$ being calculated by the space-time metric.
The integral surfaces of this equation are varieties analogous to the surfaces of constant time in ordinary Euclidean spaces, or, if one is concerned

with an unspecified metric where the coordinate x^0 indicates time, to the wave-surfaces of the propagation obeying the equation

$$\text{(II. 27)} \qquad \frac{1}{v^2}\frac{\partial^2 \mathbf{\Phi}}{\partial t^2} - \nabla^2 \mathbf{\Phi} = 0,$$

whose left-hand member is the operator $\square$.

Let there be four distinct families of integral surfaces $\mathbf{\Phi}_i = c_i$ of equation (II. 26): they can obviously be chosen in any of an infinite number of ways. Then it is perfectly clear that we will have

$$\square x^i = 0,$$

when we take as new x variables the $\mathbf{\Phi}$'s.

These conditions generalize the harmonic character of the Cartesian coordinates in a Euclidean metric; that is why we call the variables x satisfying them *isometric.*

2. THE EQUIVALENT FORM OF THE EINSTEIN EQUATIONS. We may thus conclude that it is permissible to substitute for the Einstein equations, the equations

$$\text{(II. 28)} \qquad \frac{1}{2}\square g_{ik} + \ldots = -\kappa\tau_{ik} \qquad (i, k = 0, 1, 2, 3),$$

provided that we add to them the four supplementary equations

$$\text{(II. 25)} \qquad \square x^i = 0,$$

or

$$\text{(II. 24)} \qquad Z_i = 0,$$

3. THE NORMALIZATION OF THE SYSTEM SIMULTANEOUSLY INCLUDING THE GRAVITATIONAL EQUATIONS AND THE GEODESIC PRINCIPLE. In short, by the use of isometric coordinates, we have, in

the preceding equations transposed our problem, into the following system employing isometric coordinates:

$$\text{(I)} \qquad \frac{1}{2}\Box g_{ik} + \ldots = -\kappa\varepsilon\left(\lambda_i\lambda_k - \frac{1}{2}g_{ik}\right) \qquad (i, k = 0, 1, 2, 3),$$

$$\text{(II)} \qquad p_i = 0 \qquad (i = 0, 1, 2, 3),$$

$$\text{(III)} \qquad \operatorname{div}(\varepsilon\boldsymbol{\lambda}) = 0,$$

$$\text{(IV)} \qquad Q = \sum_{ik\,0}^{3} g^{ik}\lambda_i\lambda_k = 1.$$

It is easy to see that this system is *normal* and, consequently, we may apply the Cauchy-Kowalewski existence theorem to it.

The system characterizes the evolution of the unknowns g, λ, ε, when the values of the functions themselves or of their first derivatives are given on a non-characteristic surface [1], or by giving the unknown functions, at the initial instant $x^0 = 0$, as arbitrary functions of the variables x^1, x^2, x^3.

It must, however, be pointed out that these arbitrary functions must satisfy the condition

$$\text{(II. 1)} \qquad \sum_{ik\,0}^{3} g^{ik}\lambda_i\lambda_k = 1$$

throughout the whole space.

This will become quite obvious from the identity

$$\text{(II. 29)} \qquad \frac{dQ}{ds} = 0$$

which we are going to prove.

In reality, we are to prove that the derivative

$$\frac{dQ}{ds}$$

[1] Levi-Civita, *loc. cit.*, p. 28.

is zero, when the equations of the geodesics are taken into account. One has

$$\frac{dQ}{ds} = \sum_{ik\,0}^{3} \lambda^k \left(\frac{dg_{ik}}{ds} \lambda^i + 2 g_{ik} \frac{d\lambda_i}{ds} \right),$$

or, by virtue of equations (I. 23)

$$\frac{dQ}{ds} = \sum_{ik\,0}^{3} \lambda^i \lambda^k \frac{dg_{ik}}{ds} - \sum_{ikjh\,0}^{3} 2 g_{ik} \Gamma^i_{jh} \lambda^h \lambda^k \lambda^j .$$

But one has

$$\sum_{0}^{3} g_{ik} \Gamma^i_{jh} = \begin{bmatrix} j & h \\ & k \end{bmatrix} = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^h} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^k} \right),$$

and so, likewise, the second summation of the expression that gives

$$\frac{dQ}{ds}$$

becomes

$$\sum_{hkj\,0}^{3} \lambda^h \lambda^k \lambda^j \left(\frac{\partial g_{jk}}{\partial x^h} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^k} \right) = \sum_{hkj\,0}^{3} \frac{\partial g_{hk}}{\partial x^j} \lambda^h \lambda^k \lambda^l$$

and we have

$$\frac{dQ}{ds} = \sum_{ik\,0}^{3} \frac{dg_{ik}}{ds} \lambda^i \lambda^k - \sum_{hkj\,0}^{3} \frac{\partial g_{hk}}{\partial x^j} \lambda^h \lambda^k \lambda^j = 0,$$

since

$$\frac{dg_{ik}}{ds} = \sum_{j\,0}^{3} \frac{\partial g_{ik}}{\partial x^j} \lambda^j .$$

The moment that

$$\frac{dQ}{ds}$$

is zero, Q remains constant along the world lines. From then on one has only to choose $Q = 1$ on the surface $x^0 = \bar{x}_0$ which carries the data, in order that $Q = 1$ shall be true throughout the space.

So we arrive at the fundamental conclusion that, *in isometric coordinates, the assigning of initial values to the unknowns, under the condition just set forth, determines in a unique way the metric and the flux having, as their lines of current, geodesics in space-time.*

Chapter III

CRITERIA OF APPROXIMATION AND REDUCED EQUATIONS

1. ORDERS OF MAGNITUDE AND MAXIMUM USE OF DIMENSIONLESS QUANTITIES

1. NUMERICAL EVALUATIONS. Since we have in mind the problem of n bodies, we would do well to follow the indications of Droste-De Sitter, but without eliminating *a priori* what, in the case of each body, arises intrinsically from that body. We will try to develop the logical transition (to the n-body case), while at the same time avoiding unnecessary complications – this by means of a few complementary qualitative hypotheses, to be used along with the principal approximation which results from the smallness of the velocities of celestial bodies as compared to the velocity of light – an approximation adopted by Einstein and his continuers

It is obviously of great importance to state as clearly as possible the circumstances concerning order of magnitude which are imposed in advance and in accord with which we will attack, simplify and solve the problem.

In the first place, as we have already stated in the Preface, we will be content, in the differential equations of motion, with achieving the second approximation.

Let us recall what we mean by this. In the problems that here interest us, the order of magnitude of the mechanical quantities, notably kinetic and potential energy, is that of our planetary system.

For the movements of this system, v^0 (the square of the velocity of the Sun, planet, or any satellite, i.e., twice the kinetic energy and assuming a unit-mass) is very small as compared to the square of the velocity of light c^2, and the order of magnitude of the relation

$$\beta^2 = \frac{v^2}{c^2}$$

is 10^{-8} in the case of the Earth and is of this order of magnitude – or smaller – for the other bodies in the solar system.
The same is true for the value of the Newtonian potential of the system, whether outside, or even in the interior, of the Sun, planets or satellites.
It may be helpful to illustrate this point with a calculation arising from the theory of the potential.
Let S be the field occupied by all the bodies C_h ($h = 0, 1, \ldots, n-1$) under consideration and let μ be the function representing the local density.
In what follows, we will frequently have occasion to consider the operator B which consists of forming the integral

$$\gamma = \mathrm{B}\mu = \int_S \frac{f\mu}{c^2 r} ds = \sum_{1}^{n}{}_h \int_{C_h} \frac{f\mu}{c^2 r} dC_h. \tag{III. 1}$$

This is a basic operation (and therefore indicated by B). It associates a magnitude γ, of zero dimensions (a pure number) with the density μ.
If μ_h is the maximum of μ in C_h, one has

$$\mathrm{B}\mu \leqslant \sum_{1}^{n}{}_h \frac{f\mu_h}{c^2} \int_{C_h} \frac{dC_h}{r}. \tag{III. 2}$$

Let us now recall the expression for the potential of a homogeneous sphere with a radius R and mass M.

$$\mathrm{U} = \begin{cases} \dfrac{f\mathrm{M}}{r} \text{ outside} \\ 2\pi f\mu \mathrm{R}^2 - \dfrac{2}{3}\pi f\mu r^2 \leqslant \dfrac{3}{2} f\dfrac{\mathrm{M}}{\mathrm{R}} \text{ inside} \end{cases} \tag{III. 3}$$

Then everywhere

$$\mathrm{U} \leqslant \frac{3}{2} f \frac{\mathrm{M}}{\mathrm{R}} \tag{III. 3}$$

and further

$$\text{(III. 4)} \qquad B\mu \leqslant \frac{3}{2}\sum_{1}^{n}{}_{h} \frac{f M_h}{c^2 R_h} = \frac{3}{2}\sum_{1}^{n}{}_{h} \frac{f M_\odot}{c^2 R_\odot} \frac{M_h}{M_\odot} : \frac{R_h}{R_\odot},$$

where $R_\odot$, $M_\odot$ designate the radius and mass, respectively, of the Sun. One could also write

$$B\mu \leqslant \frac{2f}{c^2}\sum_{1}^{n}{}_{h} \mu_h R_h^2 = \frac{2f}{c^2}\sum_{1}^{n}{}_{h} \frac{\mu_h}{\mu_\odot}\left(\frac{R_h}{R_\odot}\right)^2 : \frac{1}{\mu_\odot R_\odot^2}.$$

Taking into account the fact that, for our own planetary system, we can assume the μ_h to be of an order of magnitude at most comparable to $\mu_\odot$ and the R_h comparable to $R_\odot$, it follows that γ is of the order of 10^{-6}.

2. QUALITATIVE HYPOTHESES AND RESULTANT SIMPLIFICATIONS. DEFINITION OF THE ORDER OF APPROXIMATION. We will call 'of the first order' all terms that are pure numbers whose order of magnitude is that of a

$$\text{(III. 5)} \qquad \beta^2 = \frac{v^2}{c^2}$$

or a

$$\text{(III. 6)} \qquad \gamma.$$

A first approximation consists of neglecting the terms of a higher order.
In keeping with this, we will systematically neglect, in all relations, terms such as β^3 or $\beta\gamma$, etc., compared to one, with the stipulation that, if the preponderant terms in a formula are of a certain minimum order v, it will be necessary to retain – but only in conjunction with them – everything that is not of an order greater than $v + 1$.
In addition, so as to simplify as much as possible, it is naturally helpful to have recourse to isometric coordinates (cf. Chap. II, Section 2) such that

the element ds^2 remains very close to the ds^2 of Special Relativity, which has, as is well known, the pseudo-Euclidean form:

$$ds_0^2 = dx^{0^2} - (dx^{1^2} + dx^{2^2} + dx^{3^2}), \tag{III. 7}$$

where

$$x^0 = ct \tag{III. 8}$$

is simply what Levi-Civita calls *Römerian* time [1], and x^1, x^2, x^3 are orthogonal Cartesian coordinates.

We assume, then, that the space-time metric involves coordinates x^i very close to the pseudo-Euclidean type in the sense that the coefficients g_{ik} of the values

$$\overset{0}{g}_{ik} = \begin{cases} \pm 1 & (i = k), \\ 0 & (i \neq k), \end{cases} \tag{III. 9}$$

corresponding to ds_0^2 for quantities $-2\gamma_{ik}$ of at least the first order, even though the γ_{0i} are of an order not lower than $\frac{3}{2}$ [2].

In order to take into account, in the equations of motion, terms immediately higher than the Newtonian approximation, one has only – as Einstein was the first to point out – to calculate the preponderant part, of lowest order, of all the γ_{ik}, with the exception, however, of γ_{00}, for which it is necessary to retain not only the first order, but the second as well [3].

Following Eisenhart's example [4], we will use the symbol

$$e_i = \begin{cases} 1 & (i = 0), \\ -1 & (i \neq 0), \end{cases} \tag{III. 10}$$

[1] Cf. *The Absolute Differential Calculus*, p. 307.

[2] In fact, if γ_{0i} was of the first order, the difference of the velocities of propagation of light moving in two directions from a single point should also be of the first order. Optical experiments, however, have not shown this to be the case, and that eliminates the hypothesis. (Cf. Levi-Civita, *loc. cit.*, p. 369.)

[3] Cf. Levi-Civita, *loc. cit.*, p. 370.

[4] *Riemann Geometry*. Princeton University Press, 1926. Chap. II.

which allows us to write ds_0^2 in a condensed form

(III. 7′) $$ds_0^2 = \sum_i^{3}{}_0 \, e_i \, dx^{i^2} .$$

We can then retain

(III. 11) $$g_{ik} = \delta_{ik} e_i - 2\gamma_{ik} ,$$

by regarding γ_{ik} of the first order (at least) and γ_{0i} (i > 0) at least of the order $\frac{3}{2}$.

3. OTHER SOURCES OF REDUCTION. We will now introduce the approximation set forth above into the equations of system (I).
Let us first note that in the approximation of zero order, one has for the reciprocal elements $(g_{ik})^0$

(III. 12) $$(g^{ik})^0 = g^0_{ik} = \delta_{ik} e_i ,$$

which is quite simply proved by verifying the identities

$$\sum_{0\,j}^{3} (g^{ij})^0 g^0_{jk} = \delta_{ik}$$

by direct substitution of the indicated values.
Indeed, one has identically

$$\sum_{0\,j}^{3} \delta_{jk} e_k \delta_{ij} e_i = e_i e_k \sum_{0\,j}^{3} \delta_{jk} \delta_{ji} = \delta_{ik} e_i e_k = \delta_{ik} \qquad (i, k = 0, 1, 2, 3) .$$

As for the quantities

(III. 13) $$\beta_i = \frac{dx^i}{dx_0} \qquad (i = 1, 2, 3) ,$$

let us note that they are zero in the first approximation when we are concerned with very slow motions.

The same remark applies to the functions $\lambda (i = 1, 2, 3)$, whereas $\lambda^0 = 1$. In order to achieve the same approximation in all the equations of the problem, we must take into account the fact that our unknowns depend much less on x^0 than on the variables x^i $(i > 0)$, in the sense that if we admit (as is precisely the case in our problem) that a quantity f depends on time only through the movement of several points P_ν, of which f is a function, one has of necessity

$$\left|\frac{\partial f}{\partial x^0}\right| \leqslant \sum |\beta_{i\nu}| \left|\frac{\partial f}{\partial x^i}\right| \qquad (i > 0),$$

the symbol Σ referring obviously to the various spatial coordinates of each of these points.

If the velocity of these points is of the order of the β, the time derivatives are also of that order relative to the local derivatives.

As for the density μ, its initial distribution μ^0 (which, moreover, may be arbitrary) constitutes the zero-order approximation of that density.

Any subsequent approximation will naturally have to be expressed in finite terms by μ^0 and by the preceding approximations of the γ and β.

Throughout the remainder of this monograph we will denote an ensemble of terms of at least the order n by the sign $\bigcirc\ [n]$.

4. SIMILAR FORMULAS RELATIVE TO ds^2 AND TO THE MOMENTS OF THE WORLD LINES. With the foregoing established, and taking into account the fact that the $\beta_i (i > 0)$ are of the order $\frac{1}{2}$, let us consider the interval

$$\text{(I. 5)} \qquad ds^2 = \sum_{ik\,0}^{3} g_{ik}\, dx^i\, dx^k$$

which may be written:

$$\text{(I. 5')} \qquad ds^2 = dx^{0^2}\left(g_{00} + 2\sum_{i\,1}^{3} g_{0i}\frac{dx^i}{dx^0} + \sum_{ik\,1}^{3} g_{ik}\frac{dx^i}{dx^0}\frac{dx^k}{dx^0}\right)$$
$$= dx^{0^2}(1 - 2\gamma_{00} - \beta^2 + \bigcirc[2]),$$

because (Section 2 of this chapter), the g_{0i} are of the order $\frac{3}{2}$ at least, and for calculating the sum

$$\sum_{ik}^{3}{}_{1} g_{ik}\beta_i\beta_k\,;$$

one has only to substitute the values of the zero approximation g^0_{ik} for the g_{ik}, and the values indicated in Section 3 for the β.
One then has

$$\left(\frac{ds}{dx^0}\right)^2 = 1 - 2\gamma_{00} - \beta^2 + \bigcirc[2],$$

from which we deduce precisely

$$\text{(III. 14)} \qquad \frac{ds}{dx^0} = 1 - \gamma_{00} - \frac{1}{2}\beta^2 + \bigcirc[2].$$

Moreover, we have

$$\text{(III. 15)} \qquad \lambda_i = \sum_{j=0}^{3} g_i{}^j \frac{dx^j}{ds} = \sum_{j=0}^{3} \delta_{ij} e_i \frac{dx_j}{ds} + \bigcirc[1] = e_i \frac{dx^i}{ds} + \bigcirc[1].$$

For the successive calculations, we will have to evaluate the second-order term of γ_{00}, and that is why, in the formulas utilized to that end, we will put

$$\text{(III. 15')} \qquad \lambda_0 = \sum_j g_{0j}\lambda^j = g_{00}\frac{dx^0}{ds} + \bigcirc[2] =$$

$$= (1 - 2\gamma)\left(1 + \gamma + \frac{1}{2}\beta^2\right) + \bigcirc[2].$$

5. SIMILAR FORMULAS RELATIVE TO THE ENERGY TENSOR. One has

$$\tau_{ik} = \varepsilon\left(\lambda_i\lambda_k - \frac{1}{2}g_{ik}\right) = \varepsilon\left(e_i e_k \lambda^i \lambda^k - \frac{1}{2}\delta_{ik} e_i\right) + \bigcirc[1] \qquad (i > 0, k > 0).$$

Thus

(III. 16)
$$\begin{cases} \tau_{0i} = \varepsilon\, e_i \dfrac{dx^0}{ds}\dfrac{dx^i}{ds} = -\,\varepsilon \dfrac{dx^i}{dx^0}\left(\dfrac{dx^0}{ds}\right)^2 = \\ \quad = -\,\varepsilon\beta_i + \bigcirc[1] \qquad (i > 0), \\ \tau_{ik} = \dfrac{1}{2}\delta_{ik}\,\varepsilon + \bigcirc[1] \qquad (i > 0, k > 0). \end{cases}$$

More exactly

(III. 16′)
$$\tau_{ik} = \bigcirc[1] \qquad (i \neq k, i > 0, k > 0),$$

(III. 17)
$$\tau_{ii} = \frac{1}{2}\varepsilon + \bigcirc[1].$$

Now we must pursue the approximation, in order to obtain the expression of τ_{00}, through the first order (nevertheless neglecting all terms of an order higher than the first order).
In order to do this we must go back to the general expression

(III. 17′)
$$\tau_{00} = \varepsilon\left(\lambda_0^2 - \frac{1}{2}g_{00}\right)$$

and incorporate therein expression (III. 15′) for λ_0 and (III. 11) of g_{00}.
We find

(III. 18)
$$\tau_{00} = \varepsilon\left[(1-4\gamma)(1+2\gamma+\beta^2) - \frac{1}{2} + \gamma\right] =$$
$$= \varepsilon\left(\frac{1}{2} + \beta^2 - \gamma\right) + \bigcirc[2].$$

6. THE OPERATOR $\Box\,\Phi$ FOR A FUNCTION Φ OF THE FIRST ORDER.
We have

(III. 19)
$$\Box\,\Phi = \frac{1}{\sqrt{-g}}\sum_{0}^{3}{}_j \frac{\partial}{dx^j}(\sqrt{-g}\,\Phi^j),$$

where

$$\Phi^j = \sum_{k=0}^{3} g^{jk}\Phi_k = \sum_{k=0}^{3} (\delta_{jk} e_j + \bigcirc[1])\Phi_k = \sum_{k=0}^{3} \delta_{kj} e_j \Phi_k + \bigcirc[2] =$$
$$= e_j \Phi_j + \bigcirc[2],$$

Then

$$\text{(III. 20)} \qquad \Box\Phi = \sum_{j=0}^{3} \frac{\partial}{\partial x^j}(e_j \Phi_j) + \bigcirc[2] = \Box^0 \Phi + \bigcirc[2],$$

where

$$\text{(III. 21)} \qquad \Box^0 \Phi = \frac{\partial^2 \Phi}{\partial x^{0\,2}} - \sum_{i=1}^{3} \frac{\partial^2 \Phi}{\partial x^{i2}}$$

is the divergence of Φ calculated in terms of the Einstein-Minkowski metric.

2. REDUCTION OF THE EQUATIONS AND INTEGRATION BY MEANS OF POTENTIALS

1. THE EINSTEIN EQUATIONS. We have already noted that the portion of the G_{ik} components of the curvature tensor containing second derivatives of the g's reduces to

$$\text{(II. 22)} \qquad \overline{G}_{ik} = \frac{1}{2}\sum_{hj=0}^{3} g^{hj} \frac{\partial^2 g_{ik}}{\partial x^h \partial x^l} + \dots .$$

Moreover, since the expanded expression of the G_{ik} is, as is well known,

$$G_{ik} = G^{(1)}_{ik} + G^{(2)}_{ik},$$

where

$$G^{(1)}_{ik} = \sum_{h}^{3}{}_{9} \left(\frac{\partial \Gamma^h_{ih}}{\partial x^k} - \frac{\partial \Gamma^h_{ik}}{\partial x^k} \right),$$

$$G^{(2)}_{ik} = - \sum_{0}^{3}{}_{hl} \left(\Gamma^l_{ik} \Gamma^h_{hl} - \Gamma^l_{ih} \Gamma^h_{kl} \right),$$

and since we immediately recognize that the Christoffel symbols of the second type – according to the expressions (III. 11) of the g_{ik} – are of first order,

$$g_{ik} = \delta_{ik} e_i - 2\gamma_{ik}$$

together with the basic fact that the γ_{ik} must be regarded as at least of the first order – its follows that G^2_{ik} is a quantity of the second order.
Let us note likewise that since the omitted terms of expression (II. 20) of $\mathfrak{L}_{ik}$

$$- \sum_{0}^{3}{}_{hj} \left(\frac{\partial g^{hj}}{\partial x^k} \frac{\partial g_{ij}}{\partial x^h} + \frac{\partial g^{hj}}{\partial x^i} \frac{\partial g_{kj}}{\partial x^h} \right)$$

do not contain second derivatives, they are likewise at least of the second order.
Accordingly, one is assured that the terms omitted from the equality (II. 23) are at least of the second order, so that one can write

$$G_{ik} - \frac{1}{2} \left(\frac{\partial Z_i}{\partial x^k} + \frac{\partial Z_k}{\partial x^i} \right) = \frac{1}{2} \Box g_{ik} + \bigcirc [2].$$

Then, by virtue of the evaluation of the operator $\Box$ under consideration, the Einstein equations become in isometric coordinates – except for terms higher than the first order, and if the indices are not simultaneously zero,

(III. 22) $$- \overset{0}{\Box} \gamma_{ik} + \kappa \tau_{ik} = 0.$$

If we note that differentiation with respect to x_0 increases by $\frac{1}{2}$ the order of the quantity in question, we can substitute the Laplace operator $\overset{0}{\triangle}_2$

(evaluated according to a Euclidean metric) for $-\ \square^0$, and the equations, as a result, become (still neglecting anything beyond the first order)

$$\text{(III. 23)} \qquad \overset{0}{\triangle}_2\, \gamma_{ik} = -\,\kappa\tau_{ik} \qquad (i, k = 0, 1, 2, 3),$$

where the expressions of τ_{ik} are given by the preceding formulas.

Let us once more very clearly emphasize that the equation corresponding to the indices $i = k = 0$ must be treated separately, since one has also to take into account terms of the second order if one is to obtain from them γ_{00} *up to and including the second order.*

2. INTEGRATION OF THE EQUATIONS DEFINING THE γ_{ik} UP TO, BUT NOT INCLUDING, THE SECOND ORDER. We have just seen that one has

$$\text{(III. 16)} \qquad \tau_{0i} = -\,\varepsilon\beta_i + \bigcirc[1] \qquad (i > 0),$$

$$\text{(III. 16')} \qquad \tau_{ik} = \bigcirc[1] \qquad (i \neq k, i > 0, k > 0),$$

$$\text{(III. 17)} \qquad \tau_{ii} = \frac{1}{2}\varepsilon + \bigcirc[1] \qquad (i > 0).$$

The equations to be integrated are then

$$\text{(III. 24)} \qquad \overset{2}{\triangle}_0\, \gamma_{ik} = -\,\kappa\frac{1}{2}\delta_i^{\,k}\,\varepsilon + \bigcirc[1],$$

$$\text{(III. 25)} \qquad \overset{0}{\triangle}_2\, \gamma_{0i} = -\,\kappa\frac{\varepsilon}{2}(-\,2\beta_i) + \bigcirc[1].$$

Let us introduce the function γ by means of the equation

$$\text{(III. 26)} \qquad \overset{0}{\triangle}_2\, \gamma = -\,\frac{1}{2}\kappa\varepsilon = -\,\frac{1}{2}\kappa c^2\mu,$$

or, if one prefers

$$\text{(III. 26')} \qquad \overset{0}{\triangle}_2\, \gamma = -\,4\pi f\frac{\mu}{c^2},$$

That is, obviously, Poisson's famous equation.
The presence of the denominator c^2 reminds us that γ is the Newtonian potential for the distribution of matter being considered.
We deduce from this

(III. 27) $$\gamma = \frac{f}{c^2} \int_S \frac{\mu \, dS}{r}.$$

The comparison of the Poisson equation with the foregoing ones demonstrates the possibility of putting

(III. 28) $$\gamma_{ik} = \delta_{ik} \gamma + \bigcirc [2] \qquad (i, k = 1, 2, 3),$$

(III. 29) $$\gamma_{0i} = -2 \gamma_i + \bigcirc [\tfrac{3}{2}] \qquad (i = 1, 2, 3),$$

where γ is the dimensionless vector-potential, that is

(III. 30) $$\gamma_i = \frac{f}{c^2} \int_S \frac{\mu \beta_i}{r} \, dS.$$

In any case, one can retain

(III. 31) $$\gamma_{ik} = \delta_{ik} \gamma + \bigcirc [\tfrac{3}{2}]$$

3. THE EQUATION DEFINING γ_{00} UP TO THE SECOND APPROXIMATION. We need the approximate expression for the g_{ik} elements, which is given by

(III. 32) $$g^{ik} = \delta_{ik} e_i + 2 e_i e_k \gamma_{ik},$$

as is easily seen by considering the reciprocity relationship.
In addition, we notice that

$$g = -1 + 2 \sum_{0}^{3} {}_i \, e_i \gamma_{ii} + \bigcirc [2] = -1 + 2(\gamma - 3\gamma) + \bigcirc [2].$$

So

(III. 33) $$\sqrt{-g} = 1 + 2\gamma + \bigcirc[2]$$

and finally

(III. 34) $$\log\sqrt{-g} = 2\gamma + \bigcirc[2].$$

With these assumptions, the first Einstein equation is

(III. 35) $$G_{00} - \frac{\partial Z_0}{\partial x^0} = -\kappa\tau_{00}$$

and we proceed to the calculation of G_{00}, which we decompose as follows:

(III. 36) $$G_{00} = G_{00}^{(1)} + G_{00}^{(2)},$$

where

(III. 37) $$G_{00}^{(1)} = \frac{\partial}{\partial x^0}\sum_{h=0}^{3} \Gamma_{0h}^{h} - \sum_{h=0}^{3} \frac{\partial \Gamma_{00}^{h}}{\partial x^h}$$

(III. 38) $$G_{00}^{(2)} = -\sum_{hl=0}^{3} (\Gamma_{00}^{l}\Gamma_{hl}^{h} - \Gamma_{0h}^{l}\Gamma_{0l}^{h}).$$

One has moreover

$$\frac{\partial Z_0}{\partial x^0} = \frac{\partial^2 \log\sqrt{-g}}{\partial x^{02}} - \sum_{hj=0}^{3} \frac{\partial}{\partial x^0}\left(g^{hj}\frac{\partial g_{0j}}{\partial x^h}\right).$$

Now

$$\frac{\partial}{\partial x^0}\sum_{h=0}^{3} \Gamma_{0h}^{h} = \frac{\partial^2 \log\sqrt{-g}}{\partial x^{02}}$$

and, taking formulas (III. 11), (III. 12) and (III. 32) into account

$$\Gamma_{00}^{h} = \frac{1}{2}\sum_{j=0}^{3} g^{hj}\left(2\frac{\partial g_{0j}}{\partial x^0} - \frac{\partial g_{00}}{x^j}\right) = -2\sum_{j=0}^{3} g^{hj}\frac{\partial \gamma_{0j}}{\partial x^0} + \sum_{j=0}^{3} g^{hj}\frac{\partial \gamma_{00}}{\partial x^j}$$

$$= \sum_{j=0}^{3} g^{hj}\frac{\partial \gamma_{00}}{\partial x^j} - 2\sum_{j=0}^{3} g^{hj}\frac{\partial \gamma_{0j}}{\partial x^0},$$

$$\sum_{h=0}^{3}\frac{\partial \Gamma_{00}^{h}}{\partial x^h} = \sum_{hj=0}^{3} g^{hj}\frac{\partial^2 \gamma_{00}}{\partial x^h \partial x^j} - 2\sum_{hj=0}^{3} g^{hj}\frac{\partial^2 \gamma_{0j}}{\partial x^0 \partial x^h}$$

$$+ 2\sum_{j=0}^{3} e_h e_j \frac{\partial \gamma_{hj}}{\partial x_h}\left(\frac{\partial \gamma_{00}}{\partial x^j} - 2\frac{\partial \gamma_{0j}}{\partial x^0}\right) + \bigcirc[3].$$

One also has

$$\frac{\partial Z_0}{\partial x^0} - \frac{\partial^2 \log\sqrt{-g}}{\partial x^{02}} + 2\sum_{hj=0}^{3} g^{hj}\frac{\partial^2 \gamma_{0j}}{\partial x^0 \partial x^h} - \sum_{hj=0}^{3}\frac{\partial g^{hj}}{\partial x^0}\frac{\partial g_{0j}}{\partial x^h}$$

$$= \frac{\partial^2 \log\sqrt{-g}}{\partial x^{02}} + 2\sum_{hj=0}^{3} g^{hj}\frac{\partial^2 \gamma_{0j}}{\partial x^0 \partial x^h}.$$

Finally

$$G_{00}^{(1)} - \frac{\partial Z_0}{\partial x^0} = -\sum_{hj=0}^{3} g^{hj}\frac{\partial^2 \gamma_{00}}{\partial x^h \partial x^j} - 2\sum_{j=0}^{3} e_h e_j \frac{\partial \gamma_{hj}}{\partial x^h}\left(\frac{\partial \gamma_{00}}{\partial x^j} - 2\frac{\partial \gamma_{0j}}{\partial x^0}\right) + \bigcirc[3]$$

$$= -\sum_{hj=0}^{3}(\delta_{hj} e_j + 2 e_h e_j \gamma_{hj})\frac{\partial^2(\gamma + Z)}{\partial x^h \partial x^j}$$

$$- 2\sum_{j=0}^{3} e_h e_j \frac{\partial \gamma_{hj}}{\partial x^h}\left(\frac{\partial \gamma_{00}}{\partial x_j} - 2\frac{\partial \gamma_{0j}}{\partial x^0}\right),$$

having put

(III. 39) $$\gamma_{00} = \gamma + Z,$$

where Z is the second-order portion of γ_{00}. Continuing the calculations, we find

$$G^{(1)}_{00} - \frac{\partial Z_0}{\partial x^0} = -\square^0\gamma - \square^0 Z - 2\sum_{0}^{3}{}_{hj}\, e_h e_j \gamma_{hj} \frac{\partial^2 \gamma}{\partial x^h \partial x^j} \cdot$$

$$- 2\sum_{0}^{3}{}_{j}\, e_h e_j \frac{\partial \gamma_{hj}}{\partial x^h}\left(\frac{\partial \gamma_{00}}{\partial x^j} - 2\frac{\partial \gamma_{0j}}{\partial x^0}\right) + \bigcirc[3]$$

$$= -\square^0\gamma - \square^0 Z - 2\sum_{0}^{3}{}_{j}\, \gamma \frac{\partial^2 \gamma}{\partial x^{j2}}$$

$$- 2\sum_{0}^{3}{}_{j}\, e_h e_j \delta_{hj} \frac{\partial \gamma}{\partial x^h}\left(\frac{\partial \gamma}{\partial x^j} - 2\delta_{0j}\frac{\partial \gamma}{\partial x^0}\right) + \bigcirc[3],$$

where we have put

(III. 40) $$\gamma_{hj} = \delta_{hj}\gamma + \bigcirc[2],$$

by virtue of the considerations in the preceding section.

The final result is

(III. 41) $$G^{(1)}_{00} - \frac{\partial Z_0}{\partial x^0} = -\square^0\gamma - \square^0 Z - 2\gamma \overset{0}{\triangle}_2\gamma - 2\overset{0}{\triangle}_1\gamma + \bigcirc[3]$$

$$= -\square^0\gamma - \square^0 Z - \overset{0}{\triangle}_2\gamma^2 + \bigcirc[3],$$

where

(III. 42) $$\overset{0}{\triangle}_1\gamma = \sum_{1}^{3}{}_{i}\left(\frac{\partial \gamma}{\partial x^i}\right)^2$$

is the differential parameter of the first order, or Beltrami operator.

We still have to calculate $G^{(2)}_{00}$.
We have

$$G^{(2)}_{00} = \sum_{0}^{3}{}_{hj} (\Gamma^j_{0h}\Gamma^h_{0j} - \Gamma^j_{00}\Gamma^h_{hj}),$$

$$\Gamma^j_{0h} = \sum_{0}^{3}{}_{l} \delta_{lj} e_j \begin{bmatrix} o\ h \\ \hline l \end{bmatrix} + \ldots + \bigcirc[2] = e_j\left(\frac{\partial\gamma_{0h}}{\partial x^j} - \frac{\partial\gamma_{0j}}{\partial x^h}\right) + \bigcirc[2],$$

$$\Gamma^h_{0j} = \sum_{0}^{3}{}_{l} \delta_{lh} e_h\left(\frac{\partial\gamma_{0j}}{\partial x^l} - \frac{\partial\gamma_{0l}}{\partial x^j}\right) + \bigcirc[2] = e_h\left(\frac{\partial\gamma_{0j}}{\partial x^h} - \frac{\partial\gamma_{0h}}{\partial x^j}\right) + \bigcirc[2],$$

$$\sum_{0}^{3}{}_{hj} \Gamma^j_{0h}\Gamma^h_{0j} = \sum_{0}^{3}{}_{hj} e_h e_j\left(\frac{\partial\gamma_{0h}}{\partial x^j} - \frac{\partial\gamma_{0j}}{\partial x^h}\right)\left(\frac{\partial\gamma_{0j}}{\partial x^h} - \frac{\partial\gamma_{0h}}{\partial x^j}\right) + \bigcirc[3]$$

$$= -\sum_{0}^{3}{}_{hj} e_h e_j\left(\frac{\partial\gamma_{0h}}{\partial x^j}\right)^2 - \sum_{0}^{3}{}_{hj} e_h e_j\left(\frac{\partial\gamma_{0j}}{\partial x^h}\right)^2 + \bigcirc[3]$$

$$= 2\sum_{1}^{3}{}_{j} e_j\left(\frac{\partial\gamma}{\partial x^j}\right)^2 + \bigcirc[3] = 2\,\overset{0}{\triangle}_1\gamma + \bigcirc[3],$$

$$\sum_{0}^{3}{}_{hj} \Gamma^j_{00}\Gamma^h_{hj} = \sum_{0}^{3}{}_{j} \Gamma^j_{00}\sum_{0}^{3}{}_{h} \Gamma^h_{jh} = \sum_{0}^{3}{}_{j} \Gamma^j_{00}\frac{\partial \log\sqrt{-g}}{\partial x^j} = 2\sum_{0}^{3}{}_{j} \Gamma^j_{00}\frac{\partial\gamma}{\partial x^j} + \bigcirc[3].$$

But

$$\Gamma^j_{00} = \sum_{0}^{3}{}_{l} \delta_{lj} e_j \frac{1}{2}\left(2\frac{\partial g_{0l}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^l}\right) + \bigcirc[3] = \sum_{0}^{3}{}_{l} \delta_{lj} e_j\left(\frac{\partial g_{0l}}{\partial x^0} - \frac{1}{2}\frac{\partial g_{00}}{\partial x^l}\right) +$$

$$+ \bigcirc[3] = e_j\left(\frac{\partial g_{0j}}{\partial x^0} - \frac{1}{2}\frac{\partial g_{00}}{\partial x^j}\right) + \bigcirc[3] = -2e_j\delta_{0j}\frac{\partial\gamma}{\partial x^0} + e_j\frac{\partial\gamma}{\partial x^j} + \bigcirc[3].$$

So

$$\sum_{hj}^{3}{}_{0} \Gamma_{00}^{j} \Gamma_{hj}^{h} = -4\sum_{j}^{3}{}_{0} e_j \delta_{0j} \frac{\partial\gamma}{\partial x^0}\frac{\partial\gamma}{\partial x^j} + 2\sum_{j}^{3}{}_{0} e_j \left(\frac{\partial\gamma}{\partial x^i}\right)^2 + \bigcirc[3] =$$

$$= -2\mathop{\triangle}\limits^{0}{}_{1}\gamma + \bigcirc[3].$$

One finally has

(III. 43) $$\mathrm{G}_{0\,0}^{(2)} = 4\mathop{\triangle}\limits^{0}{}_{1}\gamma + \bigcirc[3].$$

So the first Einstein equation, at least up to the third order, is

(III. 44) $$-\Box^0\gamma - \Box^0\mathrm{Z} - \mathop{\triangle}\limits^{0}{}_{2}\gamma^2 + 4\mathop{\triangle}\limits^{0}{}_{1}\gamma = -\kappa\tau_{00},$$

and, for our purpose, must be further transformed.

4. THE TRANSFORMATION OF EQUATION (III. 44) BY APPLICATION OF THE PRINCIPLE OF THE CONSERVATION OF MASS AS INTERPRETED IN ORDINARY SPACE. To this end, we must recall our premise (see Section 1, no. 2, of the present chapter) to the effect that the space-time in which gravitational phenomena take place must be considered to be very close to a Euclidean variety, with the variables x_0, x_1, x_2, x_3, in their turn, differing very little from ordinary astronomical time multiplied by c and from Cartesian space-coordinates.

Under those conditions the relations between the coordinates x_1, x_2, x_3 of a given point, and possibly even the time x_0, resulting from the Einstein equations may be intuitively interpreted quite simply as if they were Cartesian coordinates in ordinary space and x_0 the Römerian time.

The slight inequalities introduced in this way by the relativistic schema – that is, formally, by the second-order terms in our equations – constitute precisely the touchstone that may permit us to verify, by means of astronomical observations, some of the consequences of the new Einsteinian Mechanics.

Let us now recall that $\varepsilon\sqrt{-g}\, dx^0\, d\mathrm{S}$ represents an elementary quantity of energy moving along its world line (see Chapter I, Section 2, no. 9). This world line has a very definite image in the abstract space x^0, x^1, x^2, x^3 or even in space, where x^0 is interpreted as time and x^1, x^2, x^3 as Cartesian coordinates. On the world line conceived in this way there is a spreading-out of a quantity of energy represented by $\varepsilon\sqrt{-g}\, dx^0\, dS$. By suppressing the fourth-dimensional element dx^0, or, more precisely, through division by the proper time element ds (which under ordinary circumstances is very close to dx^0), the energy

$$\varepsilon\sqrt{-g}\,\frac{dx^0}{ds}\,dS$$

is made to correspond to the Euclidean volume dS.
The ratio of this quantity to dS, namely

$$\eta = \varepsilon\sqrt{-g}\,\frac{dx^0}{ds}, \qquad \text{(I. 22)}$$

thus represents the density (three dimensional) of the energy distribution that is attached to any arbitrarily-chosen instant x^0, in Euclidean space S, wherein we have our intuitions and make our measurements.
We have just said that

$$\frac{dx^0}{ds}$$

is very close to unity. The same is true of $\sqrt{-g}$, so that the energy-density η (in the ordinary space into which, so to speak, we are projecting our theory) is not very different from the quantity ε. The density of the matter occupying that same space at a given instant is then

$$\mu = \frac{\eta}{c^2}, \qquad \text{(III. 45)}$$

according to Einstein's fundamental concept of the matter-and-energy proportion.
By recalling that, in more precise terms,

$$\frac{ds}{dx^0} = 1 - \gamma - \frac{1}{2}\beta^2 + \bigcirc[2], \qquad \text{(III. 14)}$$

$$\sqrt{-g} = 1 + 2\gamma + \bigcirc[2], \qquad \text{(III. 33)}$$

one has, to within the second order,

$$\text{(III. 46)}\qquad \varepsilon = (1 - 2\gamma)\left(1 - \gamma - \frac{1}{2}\beta^2\right)\eta = \left(1 - 3\gamma - \frac{1}{2}\beta^2\right)\eta,$$

whence we conclude that, in every term or combination of order 2 (or finally becoming of that order) we can quite safely replace ε by η, since their difference is of the first order.
This allows us to write

$$\text{(III. 47)}\qquad -\frac{1}{2}x\eta$$

and

$$\text{(III. 48)}\qquad \frac{1}{2}\varepsilon + \eta(\beta^2 - \gamma)$$

respectively, in place of

$$\text{(III. 47')}\qquad -\frac{1}{2}\kappa\varepsilon = \overset{0}{\triangle}_2\gamma$$

and of

$$\text{(III. 48')}\qquad \tau_{00}$$

in the Einstein equation that is the object of our analysis.
By noting that in this equation one can substitute $-\overset{0}{\triangle}_2 Z$ for $\Box^0 Z$ because Z is a quantity of the second order, one will be able to write the Einstein equation in question in the following form:

$$\overset{0}{\triangle}_2(Z - \gamma^2) = \frac{\partial^2\gamma}{\partial x^{02}} + \frac{1}{2}\kappa(\eta - \varepsilon) - \kappa\eta(\beta^2 - \gamma) - 4\overset{0}{\triangle}_1\gamma.$$

But, following relation (III. 46), one has

$$\eta - \varepsilon = \eta\left(3\gamma + \frac{1}{2}\beta^2\right),$$

so, the right-hand member is

$$\frac{\partial^2 \gamma}{\partial x^{02}} + \frac{1}{2}\kappa\eta\left(3\gamma + \frac{1}{2}\beta^2\right) + \kappa\eta\gamma - \kappa\eta\beta^2 - 4\,\overset{0}{\triangle}_1\gamma$$

$$= \frac{\partial^2 \gamma}{\partial x^{02}} + \frac{1}{2}\kappa\eta\gamma + \frac{1}{2}\kappa\eta\,4\gamma - 4\,\overset{0}{\triangle}_1\gamma - \frac{1}{2}\kappa\eta\frac{3}{2}\beta^2$$

$$= \frac{\partial^2 \gamma}{\partial x^{02}} + \frac{1}{2}\kappa\eta\gamma - 4(\overset{0}{\triangle}_1\gamma + \gamma\,\overset{0}{\triangle}_2\gamma) - \frac{1}{2}\kappa\eta\frac{3}{2}\beta^2$$

or finally, if we note that

$$\overset{0}{\triangle}_2\gamma^2 = 2(\overset{0}{\triangle}_1\gamma + \gamma\overset{0}{\triangle}_2\gamma),$$

the right-hand member in question reduces to

$$\frac{\partial^2 \gamma}{\partial x^{02}} + \frac{1}{2}\kappa\eta - 2\,\overset{0}{\triangle}_2\gamma^2 - \frac{1}{2}\kappa\eta\frac{3}{2}\beta^2,$$

and the corresponding equation, to

(III. 49) $$\overset{0}{\triangle}_2(Z + \gamma^2) = \frac{1}{2}\kappa\eta\gamma - \frac{1}{2}\kappa\eta\frac{3}{2}\beta^2 + \frac{\partial^2 \gamma}{\partial x^{02}}.$$

5. THE INTEGRATION OF EQUATION (III. 49). We put, further

(III. 50) $$Z = -\gamma^2 + \zeta$$

with the functions φ, ψ, υ having to satisfy the equations

(III. 51) $$\zeta = \varphi + \psi + \upsilon,$$

(III. 52) $$\overset{0}{\triangle}_2\,\varphi = \frac{1}{2}\kappa\eta\gamma,$$

(III. 53) $$\overset{0}{\triangle}_2\psi = -\frac{1}{2}\kappa\eta\frac{3}{2}\beta^2,$$

(III. 54) $$\overset{0}{\triangle}_2\upsilon = \frac{\partial^2 \gamma}{\partial x^{02}},$$

the integration of which, given the conditions we are after, is immediate. The right-hand member of the first one is

$$\frac{4\pi f}{c^4}\eta\gamma = -4\pi\frac{f}{c^2}\frac{\eta}{c^2}\gamma = -4\pi\frac{f}{c^2}\mu\gamma,$$

according to (III. 45), so one has only to recall Poisson's equation in order to express φ:

(III. 55) $$\varphi = -\frac{f}{c^2}\int_S \frac{\mu\gamma}{r}\,dS$$

where the integration can be extended to the whole of space S.
In a similar fashion one finds that

(III. 56) $$\psi = \frac{3}{2}\frac{f}{c^2}\int_S \frac{\mu\beta^2}{r}\,dS.$$

Finally, in order to integrate the third equation, let us recall that

$$\overset{0}{\Delta}_2 r = \frac{2}{r}.$$

And we then have

(III. 57) $$\overset{0}{\Delta}_2 \upsilon = \frac{f}{c^2}\frac{\partial^2}{\partial x^{02}}\int_S \frac{\mu\,dS}{r} = \frac{1}{2}\frac{f}{c^2}\frac{\partial^2}{\partial x^{02}}\int_S \mu\overset{0}{\Delta}_2 r\,dS.$$

Since the operators

$$\frac{\partial^2}{\partial x^{02}},\ \overset{0}{\Delta}_2$$

$\overset{0}{\Delta}_2$ are permutable, it follows that

(III. 58) $$\overset{0}{\Delta}_2\left(\upsilon - \frac{1}{2}\frac{f}{c^2}\frac{\partial^2}{\partial x^{02}}\int_S \mu r\,dS\right) = 0,$$

which furnishes adequate reason for stating that the quantity operated upon by $\triangle_2^0$ must be identically zero, because it is everywhere regular and zero at infinity.

So

(III. 59) $$\upsilon = \frac{1}{2}\frac{f}{c^2}\frac{\partial^2}{\partial x^{0^2}}\int_S \mu\, r\, dS\,.$$

6. SUMMARY OF THE FOREGOING CALCULATIONS. We can now bring together the results of the approximate integration of the Einstein equations, that is, the analytic expression of the ten gravitational potentials.

One has

(III. 60) $$\begin{cases} g_{ik} = 0 & (i \neq k, i, k = 1, 2, 3)\,, \\ g_{ii} = -(1 + 2\gamma) & (i > 0)\,, \\ g_{0i} = 4\gamma_i + \bigcirc[2] & (i > 0)\,, \\ g_{00} = 1 - 2\gamma + 2\gamma^2 - 2\zeta\,; & \end{cases}$$

(III. 51) $$\zeta = \varphi + \psi + \upsilon\,,$$

where

(III. 27) $$\gamma = \frac{f}{c^2}\int_S \frac{\mu\, dS}{r}\,,$$

(III. 30) $$\gamma_i = \frac{f}{c^2}\int_S \frac{\mu\beta_i}{r}\, dS \qquad (i = 1, 2, 3)\,;$$

(III. 55) $$\varphi = -\frac{f}{c^2}\int_S \frac{\mu\gamma}{r}\, dS\,,$$

(III. 56) $$\psi = \frac{3}{2}\frac{f}{c^2}\int_S \frac{\mu\beta^2}{r}\, dS\,,$$

(III. 59) $$\upsilon = \frac{1}{2}\frac{f}{c^2}\frac{\partial^2}{\partial x^{0^2}}\int_S \mu\, r\, dS\,.$$

Now, since the x^i coordinates are henceforth to be considered as Cartesian and rectangular, one has

$$\beta_i = \frac{dx^i}{dx^0} \quad \text{and} \quad \beta^2 = \sum_{i=1}^{3} \beta_i^2,$$

whereas the original ds^2 will be written

(III. 61) $$ds^2 = (1 - 2\gamma + 2\gamma^2 - 2\zeta)\, dx^{0^2} - (1 + 2\gamma)\, dl_0^2 + 8\, dx^0 \sum_{i=1}^{3} \gamma_i\, dx^i,$$

where

(III. 62) $$dl_0^2 = \sum_{i=1}^{3} dx^{i2}.$$

This expanded form of ds^2 will play a basic part in the formation of the differential equations of motion.
From this we can deduce the expression

$$\left(\frac{ds}{dx^0}\right)^2,$$

namely,

(III. 63) $$\left(\frac{ds}{dx^0}\right)^2 = 1 - 2\gamma + 2\gamma^2 - 2\zeta - (1 + 2\gamma)\,\beta^2 + 8 \sum_{i=1}^{3} \gamma_i \beta_i.$$

Chapter IV

INTERNAL STRESSES AND THEIR POSSIBLE OMISSION IN THE PRESENT APPROXIMATION

1. THE ENERGY TENSOR. PRESSURE AND ITS INVARIANCE ALONG THE WORLD LINES IN THE CASE OF CELESTIAL BODIES

1. GENERAL CONSIDERATIONS. In Chapter II we drew attention, conceptually, and consequently with regard to the existence theorem, to granular media (cosmic dust) for which we can put

(I. 29) $$T_{ik} = \varepsilon\lambda_i\lambda_k,$$

or, if we choose

(I. 29′) $$T^{ik} = \varepsilon\lambda^i\lambda^k.$$

Einstein, in a personal conversation, immediately raised the objection that celestial bodies are not at all the same thing as cosmic dust, and he expressed doubt as to the possibility of ever establishing in this way a theory that would include among its most important special cases the basic elements of what takes place in actual fact.

Since, in order to insure the individual identity and consistency of celestial bodies, internal stresses come into play in a very fundamental way, one must, as a first approximation, substitute for the positions given above, which do not take these stresses into account, a scheme in which internal stresses are introduced in the particular simple form of an isotropic pressure.

The energy tensor, in that case, takes on the form adopted by Synge, namely

(IV. 1) $$T_{ik} = (\varepsilon + p)\lambda_i\lambda_k - p\,g_{ik},$$

which obviously reduces to (I. 29) for $p = 0$.

2. RELATIVISTIC HYPOTHESIS WITH REGARD TO PRESSURE. As a basic postulate with regard to the pressure p, we will adopt the hypothesis that the pressure p exerted in a moving particle is preserved during the course of the motion, that is, all along its world line.

The mathematical expression of such a hypothesis is the formula

(IV. 2) $$\frac{dp}{ds} = 0,$$

the relativistic invariance of which is obvious.

From the physical point of view this hypothesis is most certainly valid for those celestial bodies that behave, to all intents and purposes, like solids.

2. RECONSIDERATION OF THE EQUATIONS OF MOTION

1. THE EFFECT OF p ON THE COMPONENTS τ_{ik}. THE SPEED OF SOUND INSIDE THE GRAVITATING MEDIUM. According to the expression

(IV. 1) $$\mathrm{T}_{ik} = (\varepsilon + p)\lambda_i\lambda_k - p\,g_{ik},$$

the term in p in the invariant

(II. 3) $$\mathrm{T} = \sum g^{ik}\mathrm{T}_{ik}$$

is

(IV. 3) $$\mathrm{T}^* = p\sum g^{ik}(\lambda_i\lambda_k - g_{ik}) = -3p.$$

Similarly, the term in p in

(II. 2) $$\tau_{ik} = \mathrm{T}_{ik} - \mathrm{T}\,g_{ik}$$

is

(IV. 4) $$\overset{*}{\tau}_{ik} = p(\lambda_i\lambda_k - g_{ik}) - \mathrm{T}^* g_{ik} = p(\lambda_i\lambda_k + 2g_{ik}).$$

There is no point in taking these additional terms into account in the differential equation

(III. 23) $$\overset{0}{\triangle}_2\gamma_{ik} = -\kappa\tau_{ik},$$

in the case where the indices are not simultaneously zero, since one can then neglect the terms higher than the first order and multiplied by ε. That is the immediate result of the fact that

$$\frac{p}{\varepsilon} = \frac{1}{c^2}\frac{p}{\mu} \tag{IV. 5}$$

may be regarded as a quantity of the first order.
In fact, if we assume that within the medium being considered the laws of ordinary Physics are at least roughly applicable, the ratio

$$\frac{p}{\mu}$$

of pressure to density may be interpreted (under isothermal conditions) as the square of the speed of sound, which is not too much higher–and often even lower–than the square of the velocity of celestial bodies. One is thus concerned with terms of the order of β^2.

2. THE MODIFICATION OF ε RESULTING FROM PRESSURE. On the other hand, it is of the utmost importance to take into account those terms singled out just a moment ago in the equation

$$\overset{0}{\triangle}_2 \gamma_{00} = -\kappa\tau_{00}, \tag{III. 23}$$

since we must calculate γ_{00} up to and including the second order.
According to what has already been said, the additional term in τ_{00} is (IV. 4)

$$\overset{*}{\tau}_{00} = p(\lambda_0^2 + 2g_{00}),$$

so that the additional term in the right-hand member of the equation (III 23) may be written

$$-\kappa p(\lambda_0^2 + 2g_{00}) = -\kappa p(3 + \bigcirc[2]) = -\kappa\varepsilon\frac{p}{\varepsilon}(3 + \bigcirc[2]) =$$

$$= -3\kappa\varepsilon\frac{p}{\varepsilon}(1 + \bigcirc[2]),$$

since it is impossible to replace either λ_0^2 or g_{00} by unity.

Operation B applied to the product

$$-3\kappa\varepsilon\frac{p}{\varepsilon}\bigcirc[2]$$

would give us, according to Chapter III, Section 1, a result belonging to the fourth order and hence negligible; but $-3\kappa p$, in the expression of γ_{00}, involves a dimensionless potential of density $-3\kappa p$ to within the factor

$$\frac{2f}{c^2}$$

By uniting this complementary term with the first-order potential γ arising from the density ε, one obtains a single potential for each of the bodies having the density

$$\varepsilon' = \varepsilon\left(1 - \frac{3p}{\varepsilon}\right)$$

to within the same factor.

If we assume that our bodies possess, with regard to the pressure p, the symmetry already assumed for the distribution of matter ε, ε can be replaced by ε' in our calculations.

As for the second-order terms in which ε still appears, one can obviously quite simply imagine, without troubling to put in the prime-mark everywhere, that ε' has been written in the place of every ε.

From this we deduce that the Poisson equations defining the ten potentials γ_{ik} $(i, k = 0, 1, 2, 3)$ are exactly the ones we had in the absence of all pressure, except for the somewhat less simple significance of the quantity ε.

3. THE EQUATIONS OF MOTION REPLACING THE GEODESIC PRINCIPLE AND THEIR INTERPRETATION IN THE CLASSICAL MANNER.

Let us also examine how the presence of pressure modifies the four conservation equations – that is, substantially, the equations of motion. Taking p into account, equations (I. 32′) become

$$\text{(IV. 6)}\quad (\varepsilon + p)\,k_i - \lambda_i\left[\frac{d(\varepsilon + p)}{ds} + (\varepsilon + p)\operatorname{div}\boldsymbol{\lambda}\right] - p_i = 0 \quad (i = 0, 1, 2, 3).$$

Since the curvature vector **k** is orthogonal to the world lines, i.e.,

$$\sum_{0}^{3}{}_{i}\, k_i \lambda^i = 0,$$

by multiplying the preceding equations by $-\lambda_i$ and adding, we obtain

$$\frac{d(\varepsilon + p)}{ds} + (\varepsilon + p)\operatorname{div}\boldsymbol{\lambda} + \sum_{0}^{3}{}_{i}\, p_i \lambda^i = 0.$$

The last term cancels out because of the assumption with regard to pressure, and there remains

(IV. 7) $$\frac{d\varepsilon}{ds} + (\varepsilon + p)\operatorname{div}\boldsymbol{\lambda} = 0.$$

According to this result, the equations of conservation (I. 32′) are reduced to

(IV. 6′) $$(\varepsilon + p)\, k_i - p_i = 0 \qquad (i = 0, 1, 2, 3).$$

One can obviously relate these equations simply to the space x_1, x_2, x_3, since the fourth equation, $i = 0$, is reduced to an identity, because of the aforementioned assumption and because of the property of the curve

$$\sum_{0}^{3}{}_{i}\, k_i \lambda^i = 0,$$

mentioned previously.

3. OVER-ALL CONSEQUENCES FOR EACH BODY

1. THE EQUATIONS OF MOTION FOR CENTERS OF GRAVITY. In the preceding section we have established the equations of motion (IV. 6′) for every material element in the medium under consideration. We see that the local influence of pressure is not in a general way negligible, since that influence is analytically expressed by the fact that, instead of

$$k_i = 0 \qquad (i = 0, 1, 2, 3)$$

(geodesic motion) we have, by (IV. 6′),

$$k_i = -\frac{p_i}{\varepsilon + p}$$

which are the covariant components of the vector

$$-\frac{1}{\varepsilon + p} \operatorname{grad} p .$$

2. THE SYMMETRICAL DISTRIBUTION OF PRESSURE. Although, as we have said, local influence does not disappear, it is permissible to conclude that, as a result of the ensemble of our hypotheses, the first approximation remains unchanged.

Indeed, if we consider precisely the center of gravity of one of our bodies, the pressure gradient – provided that the symmetry requirement is still met – becomes zero for the center of gravity.

Its motion is thus still subject to the equations

$$k_i = 0 ,$$

which express the geodesic principle.

3. JUSTIFICATION OF CONSIDERING ONLY GRANULAR MEDIA. It is therefore legitimate to limit our analysis of the first approximation, as we have several times anticipated, to granular media.

Chapter V

REDUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

1. THE LAGRANGE FUNCTION

1. REVIEW OF THE GEODESIC PRINCIPLE. The motion of every material element is characterized in space-time by an appropriate geodesic line of the ds^2 we have just found.
That is the well-known geodesic principle, formulated – as we have already stated in Chapter I – by Einstein even before the connections of the gravitational potentials with matter and its motion were recognized.
From the analytical point of view, this principle tells us that the proper motions (along which $ds^2 > 0$) are defined by the variational principle

$$\text{(I.)} \qquad \delta \int ds = 0 .$$

The vanishing of the variation is related, in the four-dimensional geometrical picture, to the passage of the world line concerned to any other line infinitesimally close and having the same extremities.
In (I) we must, therefore, attribute infinitesimally small arbitrary increments to the four coordinates x^0, x^1, x^2, x^3, with the additional condition that these increments vanish at the extremities. But it can be shown[1] that x^0 need not be made to vary, since, by that fact the left-hand member of (I) undergoes a variation that disappears by virtue of the conditions arising from the variation of the three space-coordinates x^1, x^2, x^3.
That being the case, let us attribute an equivalent form to (I), but a form that is better adapted to possible comparisons with the older Mechanics.

[1] Cf. Levi-Civita, *The Absolute Differential Calculus,* pp. 289–290.

One has

$$\left(\frac{ds}{dx^0}\right)^2 = 1 - 2\gamma + 2\gamma^2 - 2\zeta - (1 + 2\gamma)\beta^2 + 8\sum_{1}^{3}{}_i \gamma_i\beta_i$$

$$= 1 - 2\mathfrak{N} + 2\gamma^2 - 2\zeta - 2\gamma\beta^2 + 8\sum_{1}^{3}{}_i \gamma_i\beta_i,$$

where the function

(V. 1) $$\mathfrak{N} = \frac{1}{2}\beta^2 + \gamma$$

is obviously the Lagrangian function that one would obtain, other things being equal, in Newtonian Mechanics, defining the motion of a generic material element devoid of dimensions.

By extracting the square root, one obtains

(V. 2) $$\frac{ds}{dx^0} = 1 - \mathfrak{N} + \gamma^2 - \zeta - \gamma\beta^2 + 4\sum_{1}^{3}{}_i \gamma_i\beta_i - \frac{1}{2}\mathfrak{N}^2,$$

except for terms higher than the second order.

2. STRUCTURE OF THE LAGRANGIAN FUNCTION. Now the variational equation (I) may be written in the form

(V. 3) $$\delta\int\left(1 - \frac{ds}{dx^0}\right)dx^0 = 0,$$

since

$$\delta\int dx^0 = 0,$$

because of our observation regarding the possibility of not subjecting the variable x^0 to variation.

One can thus consider as the Lagrangian function

$$\mathfrak{L} = 1 - \frac{ds}{dx^0}$$

and one has explicitly

$$\mathfrak{L} = \mathfrak{N} + \mathfrak{D}, \tag{V. 4}$$

where

$$\mathfrak{D} = \frac{1}{2}\mathfrak{N}^2 - \gamma^2 + \zeta + \gamma\beta^2 - 4\sum_{i=1}^{3}\gamma_i\beta_i. \tag{V. 5}$$

It is the function $\mathfrak{D}$ that characterizes the slight Einsteinian modification of the motion of every material element and which we now intend to study more thoroughly.

2. THE IMMEDIATE SURROUNDINGS OF A POINT IN MOTION

1. CONVENTIONS OF NOTATION. Since it is our intention to make clear the various points of difference between the classical and relativistic approaches to the problem, we here make a few preliminary remarks about the circumstances (each one well known in itself, but here requiring to be considered simultaneously) that allow us to reduce the problem of the gravitational motion of several massive bodies to the problem of an equal number of point-masses.

In the expression of the Lagrangian function $\mathfrak{L}$, which we have just found, there occur, in a very fundamental way, a few potentials extended to the domain S (of Euclidean space x^1, x^2, x^3) occupied by these attracting bodies at a given instant.

Let us suppose in particular that the point P in question belongs to a body C (and C will likewise designate the field occupied by this body).

We can imagine the possibility of decomposing the potentials referred to into two parts, one extending out from C, the other to the residual portion S$'$ of S.

The potential γ is the only first-order term, and consequently the most important one.

However that may be, we will designate by the notation γ', φ, ... the portion of the potentials whose extension is in S$'$, and by the notation γ'', φ'', ... the portion pertaining to the domain C.

In keeping with this we will have

$$\gamma = \gamma' + \gamma'', \qquad \varphi = \varphi' + \varphi'', \qquad \ldots. \tag{V. 6}$$

In more general terms, a function of the same potentials, such as $\mathfrak{N}$, $\mathfrak{D}$, $\mathfrak{L}$, may be decomposed in the same fashion

$$\text{(V. 6')} \qquad \mathfrak{N} = \mathfrak{N}' + \mathfrak{N}'', \qquad \mathfrak{D} = \mathfrak{D}' + \mathfrak{D}'', \qquad \ldots,$$

where $\mathfrak{N}'$, $\mathfrak{D}''$, ... is the portion that one would have if the body C were eliminated, whereas $\mathfrak{N}'$, $\mathfrak{D}''$, ... characterizes the influence of the body C on the motion of P.

2. INTERNAL FORCES AND THE PRINCIPLE OF ELIMINATION IN CLASSICAL MECHANICS. AN OBSERVATION REGARDING THE NUMERICAL VALUES. Brillouin has most felicitously labeled *principle of elimination* the possibility of arriving, given certain appropriate conditions, at the conclusion that the term $\mathfrak{L}''$ exerts no influence at all on the motion of P.

Let us note here that in Classical Mechanics the conditions referred to above proceed substantially from the principle of reaction.

But let us examine the matter in the relativistic scheme referred to above. In a very precise way, one has

$$\text{(V. 7)} \qquad \mathfrak{N} = \mathfrak{N}' + \gamma'',$$

where

$$\text{(V. 8)} \qquad \mathfrak{N}' = \frac{1}{2}\beta^2 + \gamma',$$

$$\text{(V. 9)} \qquad \mathfrak{D} = \frac{1}{2}(\mathfrak{N}' + \gamma'')^2 - (\gamma' + \gamma'')^2 + \zeta' + \zeta'' + (\gamma' + \gamma'')\beta^2 - 4\sum_{1}^{3}{}_{i}(\gamma'_i + \gamma''_i)\beta_i,$$

having put

$$\text{(V. 10)} \qquad \mathfrak{D} = \mathfrak{D}' + \mathfrak{D}'',$$

$$\text{(V. 11)} \qquad \mathfrak{D}' = \frac{1}{2}\mathfrak{N}'^2 - \gamma'^2 + \zeta' + \gamma'\beta^2 - 4\sum_{1}^{3}{}'_{i}\gamma'_i\beta_i,$$

$$(\text{V. }12)\quad \mathfrak{D}'' = \mathfrak{N}'\gamma'' - \frac{1}{2}\gamma''^2 - 2\gamma'\gamma'' + \zeta'' + \gamma''\beta^2 - 4\sum_i^{3}{}_1\, \gamma_i''\beta_i$$

$$= \mathfrak{N}'\gamma'' - \frac{1}{2}\gamma''^2 - 2\gamma'\gamma'' + \varphi'' + \psi'' + \upsilon'' + \gamma''\beta^2 - 4\sum_{i=1}^{3} \gamma_i''\beta_i .$$

Thus

$$(\text{V. }13)\quad \mathfrak{L}'' = \gamma'' + \left(\frac{1}{2}\beta^2 + \gamma'\right)\gamma'' - \frac{1}{2}\gamma''^2 - 2\gamma'\gamma'' + \zeta'' + \gamma''\beta^2 - 4\sum_{i=1}^{3} \beta_i\gamma_i'' .$$

As a matter of fact, one can not really suppose that $\mathfrak{L}''$ is or even becomes, according to certain hypotheses, negligible with respect to $\mathfrak{L}'$, and this is immediately evident when we take into account only terms of the first order (both positive).

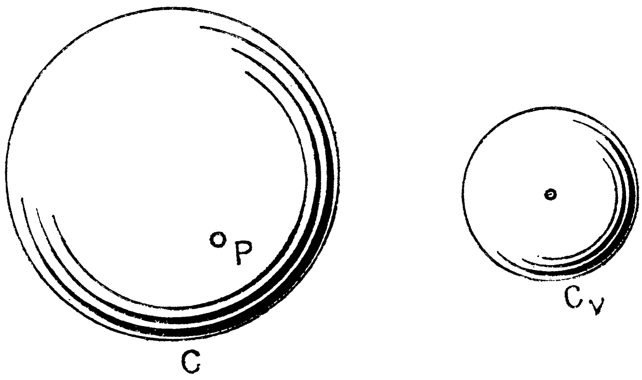

Fig. 1.

Indeed, *the potential γ'' dominates γ'*. One has only to consider the typical case of a system of attracting bodies C_ν having comparable masses and dimensions.

The contribution γ'' of the body C containing P is assuredly greater than that of any other body C_ν, because the distances from the various points in C are less than the analogous distances from points in any body C_ν.

Let us consider, in order to keep our ideas straight, the elementary case of several homogeneous spheres.

Let R and m be the radius and mass of C respectively; and R_ν and m_ν the analogous quantities for C_ν; let us suppose that P is the center of C.
The contribution of the body C_ν to γ' is

$$\frac{fm_\nu}{c^2}\frac{1}{r_\nu}.$$

The value at P (center of C) of the potential of this sphere is

$$\frac{3}{2}\frac{fm_\nu}{c^2}\frac{1}{R},$$

as is well known.
The order of magnitude of γ' with respect to γ_ν is then

$$\frac{\gamma}{\gamma_\nu} \sim \frac{r_\nu}{\mathrm{R}},$$

that is, in the case of celestial bodies, the order of the ratio of one of the mutual distances to the radius of a body (attracting sphere) – which is in no wise negligible.
But it must be pointed out that this preliminary difficulty also occurs in ordinary Mechanics where $\mathfrak{L}'$ is reduced to $\mathfrak{N}'$ and $\mathfrak{L}''$ to γ''.
In fact, in that case the Lagrangian equations of the motion of P are

(V. 14) $$\frac{d}{dx^0}\frac{\partial \mathfrak{L}}{\partial \beta_i} - \frac{\partial \mathfrak{L}}{\partial x^i} = 0 \qquad (i = 1, 2, 3),$$

that is

(V. 14′) $$\dot{\beta}_i = \frac{\partial \gamma'}{\partial x^i} + \frac{\partial \gamma''}{\partial x^i} \qquad \left(\cdot = \frac{d}{dx^0}\right),$$

where the derivatives that enter into the right-hand members obviously represent, to within the factor

$$\frac{1}{c^2},$$

the Newtonian forces outside and within the body C, respectively.
The foregoing considerations show that the internal forces arising from

γ'' are, considered individually, in general stronger than the external forces resulting from the potential γ'.

However, one must remember that there exists a classical combination of these equations: the equation governing the motion of the center of gravity, in which the contributions of γ'' (or, better, of its derivatives) cancel each other pairwise.

There we have, conceptually, the principle of elimination in Classical Mechanics.

In Relativistic Mechanics, one must consider not only γ'', but also all the terms of the second order, to achieve the next highest approximation to the Newtonian one.

So we must strive to eliminate above all else, in the equations of motion, the first-order terms that would result from γ'' and which disappear once and for all from the Newtonian equations of motion – a result obtained by once again introducing in an appropriate manner consideration of the center of gravity. But it is well to simplify at the same time, as much as possible, the second-order portion $\mathfrak{L}''$.

3. NUMERICAL CRITERIA IN MAKING THE CALCULATION. We now propose to introduce certain preliminary hypotheses about the bodies and their motions – hypotheses thats hould help to simplify the equations by means of an approximation that is more than adequate for emphasizing the fundamental differences with regard to ordinary Newtonian Mechanics.

To this end, we may content ourselves with the evaluation, in the second-order terms of the type

$$\beta_\mu^2, \quad \gamma_\nu \qquad (\mu, \nu = 0, 1, \ldots, n-1),$$

of each of the factors by means of a rather crude approximation, an approximation of barely one per cent, for example.

This criterion will lead us to conclusions that will be a good deal less exact than we might have deduced from the Lagrangian equations discussed a few pages back (where the Newtonian $\mathfrak{N}$-terms are of the first order and the terms included in $\mathfrak{D}$ are of the second order) without further modifications.

In fact, in that case, the approximation to be expected – that is, the ratio between second-order and first-order terms – is $10^{-12}: 10^{-6} = 10^{-6}$.

If on the other hand we also take it upon ourselves to evaluate some first-order term occurring in $\mathfrak{D}$ at most of order 10^{-n} (to simplify things we have put $n = 2$), we can then no longer count on an approximation of 10^{-6}, but of only 10^{-n} ($n < 6$).

4. CONSIDERATIONS FACILITATING REDUCTION TO A FINITE NUMBER OF POINT-MASSES. Having established our numerical criteria, we go on to consider the combination of the center of gravity (essential to the elimination of the contributions of γ'') and, at the same time, the structure of the various terms (of the second order) from which $\mathfrak{D}$ results.
To that end, we must consider the totality of point-masses P making up the body C. For every one of these the differential equations whose Lagrangian function $\mathfrak{L}$ is reduced to its Newtonian portion are valid:

$$\text{(V. 7)} \qquad \mathfrak{N} = \frac{1}{2}\beta^2 + \gamma' + \gamma''.$$

The center of gravity of all the points P behaves as if the internal forces were non-existent or – what amounts to the same thing – as if the center of gravity has as its Lagrangian function

$$\mathfrak{N}' = \frac{1}{2}\beta^2 + \gamma',$$

since the γ'' portion is self-eliminating in the differential equations.
It is on the basis of this result that we may go on to a further approximation for the Lagrangian function in our problem.
The differential equations that result from the geodesic principle are

$$\text{(V. 15)} \qquad \frac{d}{dx^0}\frac{\partial(\mathfrak{N}+\mathfrak{D})}{\partial \dot{x}^i} - \frac{\partial(\mathfrak{N}+\mathfrak{D})}{\partial x^i} = 0 \qquad (i = 1, 2, 3)$$

or, if we choose,

$$\text{(V. 15')} \qquad \dot{\beta}_i = \frac{\partial \gamma}{\partial x^i} + \frac{\partial \mathfrak{D}}{\partial x^i} - \frac{d}{dx^0}\frac{\partial \mathfrak{D}}{\partial \dot{x}^i} \qquad (i = 1, 2, 3).$$

The terms

$$\frac{\partial \gamma}{\partial x^i}$$

of the right-hand members are the Newtonian forces, and the residual terms represent the Einsteinian perturbations.
The three classical linear combinations expressing the principle of the motion of the center of gravity G are

$$\beta_i = \frac{\partial\gamma}{\partial x^i} + \frac{1}{m}\int_C \mu\left(\frac{\partial\mathfrak{D}}{\partial x^i} - \frac{d}{dx^0}\frac{\partial\mathfrak{D}}{\partial\dot{x}^i}\right)dC, \tag{V. 16}$$

where

$$m = \int_C \mu\, dC \tag{V. 17}$$

is the mass of C and the first members are nothing more nor less than the components of the acceleration of G and the right-hand members the components – reduced to mass-units – of the forces, which are separate in the Newtonian portion (which is, of course, of the first order) and in the binomials

$$\Pi_i = \frac{1}{m}\int_C \mu\frac{\partial\mathfrak{D}}{\partial x^i}dC - \frac{1}{m}\int_C \mu\frac{d}{\partial x^0}\frac{\partial\mathfrak{D}}{\partial\dot{x}^i}dC \tag{V. 18}$$

which characterize precisely the perturbation produced on the motion of the center of gravity.
It will be noted that the Π_i would be very complicated, either as a result of the structure of the function $\mathfrak{D}$, which depends simultaneously on the velocities and the accelerations of all the masses of the S system in question, or as a result of the integrations extended to the domain C, where G is the center of gravity.

3. SUPPLEMENTARY CONSIDERATIONS WITH RESPECT TO ORDINARY MECHANICS

1. CENTERS OF GRAVITATION. We now elaborate a few supplementary ideas essential to the formulation of the basic hypotheses upon which our theory rests.
We call the *center of gravity* of a body C the (or even every) point where

the resultant of the Newtonian attractions of the material elements of the body is zero.

It is easy to see that, for every body C, there exists at least one center of gravity.

In fact, the Newtonian potential U of every (limited) body C is a function that is limited throughout space (becoming zero at infinity).

Consequently there exists at least a point P_0 in space where the potential attains its maximum.

At that point P_0 the derivatives of U are necessarily zero, and the attraction is thus zero at P_0.

If we further recall that U is a harmonic function outside the body C and that such a function does not have a maximum nor a minimum within a regular field, we conclude that the point P_0 is not outside of C, and one can also state that it will not be on the contour of C, if the body is convex.

It is easy to prove that, if the body C has a center of symmetry O, the point O is a center of gravity.

One has, in fact,

$$\text{(V. 19)} \qquad U(x, y, z) = U(-x, -y, -z),$$

if x, y, z, are the coordinates of an arbitrarily chosen point in space relative to a system of axes with O as its origin.

From the identity just given we subsequently deduce (by differentiation and by putting $x = y = z = 0$) that the three derivatives

$$\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}$$

become zero at the point O.

Connected with the notion of a center of gravity is the idea of a *centrobaric body*. Lord KELVIN (1935) called barycentric the body C exerting upon every point in space an attraction directed toward a specific point O.[1]

Then O is at one and the same time a gravitational center and the center of gravity of the body C.

In such a case, the ellipsoid of inertia of the body is reduced to a sphere.

[1] Kelvin and Tait, *Treatise on Natural Philosophy*, **2**; see also: Fenici, 'Centri di gravitazione e corpi centrobarici', *Rend. Lincei*, 1935, 493–498.

2. THE NECESSARY CONDITION FOR THE MATERIAL REALITY OF THE CENTER OF GRAVITY. SYMMETRIES.[1] By definition the center of gravity G of a system S (continuous) which, at the instant it occupies a field C, is characterized by the equation

$$\text{(V. 20)} \qquad \mathrm{G} - \mathrm{O} = \frac{1}{m}\int_{\mathrm{C}} (\mathrm{P} - \mathrm{O})\, dm \qquad (m = \text{mass}),$$

where the point O is an arbitrary point, which we here put at the origin of a Cartesian system Oxyz. If $\mathbf{v}(\mathrm{P}|\,t)$ designates the velocity at the instant t of a point P of S, one has, by differentiating the preceding formula with respect to t

$$\text{(V. 21)} \qquad \dot{\mathrm{G}} = \frac{1}{m}\int_{\mathrm{C}} \mathbf{v}(\mathrm{P}\,|\,t)\, dm\,.$$

On the other hand, the velocity of the material element of S which at the instant t occupies the position G is $\mathbf{v}(\mathrm{G}|\,t)$.

Thus, if the point G is to be permanently bound to one material element, one must have

$$\text{(V. 22)} \qquad \mathbf{v}(\mathrm{G}\,|\,t) = \frac{1}{m}\int_{\mathrm{C}} \mathbf{v}(\mathrm{P}\,|\,t)\, dm\,.$$

It can be shown that this condition is also sufficient to assure the permanently barycentric character of an element bound to the center of gravity at the initial instant.

It is easy to see that the material character of the center of gravity is verified not only in rigid motions, but also in homographic motions, that is, in motions where the distribution of velocities at every instant is characterized by a linear law with respect to the coordinates of P

$$\text{(V. 23)} \qquad \mathbf{v}(\mathrm{P}\,|\,t) = \mathbf{v}_0(t) + \alpha_t(\mathrm{P} - \mathrm{O})\,,$$

where the symbol α_t designates a vector homography depending on the time t.

Indeed, one has

$$\mathbf{v}(\mathrm{G}\,|\,t) = \mathbf{v}_0(t) + \alpha_t(\mathrm{G} - \mathrm{O})$$

[1] Levi-Civita, 'Movimenti di un sistema continuo che rispettano l'invariabilità sostanziale del baricentro', *Acc. Pont. Nuovi Lincei*, **88** (1935) 151–155.

and

$$\int_C \mathbf{v}(P \mid t)\, dm = \int_C [\mathbf{v}_0(t) + \alpha_t(P - O)]\, dm = m\mathbf{v}_0(t) + \int_C \alpha_t(P - O)\, dm .$$

But the integration is commutable with the homography, so

$$\int_C \alpha_t(P - O)\, dm = \alpha_t \int_C (P - O)\, dm$$

and finally

$$\frac{1}{m}\int_C \mathbf{v}(P \mid t)\, dm = \mathbf{v}_0(t) + \alpha_t \frac{1}{m}\int_C (P - O)\, dm = \mathbf{v}_0(t) + \alpha_t(G - O) = \mathbf{v}(G \mid t) .$$

Q. E. D.

Let us note that for every body having at every instant a center of symmetry G, both geometrical and material, one can state, by virtue of the elementary properties of the center of gravity and by virtue of what has just been said concerning gravitational centers, that G is both a center of gravity and a gravitational center.

4. FURTHER HYPOTHESES LEADING TO ORDINARY DIFFERENTIAL EQUATIONS

1. GENERALITIES AND HYPOTHESIS I. We now seek to state as carefully and clearly as possible the additional hypotheses on which we construct the relativistic theory of gravitational motion.

As a matter of fact, the conditions that we seek should be the mathematical consequences of the differential equations of the motion of each point. Because of the insurmountable difficulties that this rigidly mathematical approach to the problem would involve, we find that it is convenient to postulate a few over-all characteristics of motion in keeping with astronomical observations.

Hypothesis I. We presuppose that the center of gravity G of every body C is materially real, that is, that it is always attached to the same material element.

But it is not necessary that this condition be rigorously verified; it will suffice to have it satisfied within the order of approximation 10^{-n}, which

we introduced in connection with the Einsteinian correction, $n \geqslant 2$ as we pointed out in Section 2, no. 3.
If G is materially real, its motion will be characterized in the same way as that of any other point-mass P (see No. 2) by a well-determined Lagrangian function $\mathfrak{L} = \mathfrak{N} + \mathfrak{D}$, and the Einsteinian perturbation will be the corresponding $\mathfrak{D}$ term.
We further assume that the center of gravity G is always a gravitational center.

2. HYPOTHESIS II. We assume that the body C is impelled by a quasi-translatory motion. Let us explain immediately the meaning that must be attributed to that term:
Let us begin with the definition of the translatory motion of a body C. That motion is one in which the points of the body are, at any given instant, all impelled with the same vector velocity – let us say with the velocity $\mathbf{v}_g$ of the center of gravity G.
In a practical sense, one can naturally regard as translatory any motion for which, with respect to v_g (the modulus of the vector $\mathbf{v}_g$), the absolute value of the vector difference $\Delta\mathbf{v}$ between the velocities at any two points of C at the same instant is negligible. Thus, the ratio

$$\frac{|\Delta\mathbf{v}|}{v_g}. \tag{V. 24}$$

will be negligible.
We do not say that this ratio is in itself negligible, but only that it never gets beyond a few hundredths (order of magnitude 10^{-2}), so that we may omit, as a quantity of an order higher than the first, every product of the type

$$\beta^2 \frac{|\Delta\mathbf{v}|}{v_g}, \quad \gamma \frac{|\Delta\mathbf{v}|}{v_g}, \quad \ldots.$$

We call a motion of this sort quasi-translatory.
And that is what happens in the case of the planets.
First of all, their deformations are negligible, and they behave, consequently, like bodies that may be regarded as rigid.
In actual fact their motion is not purely translatory: it is made up of translation and rotation.
However, for any arbitrary point in the body, the velocity due to rotation

attains only a few hundredths of the translatory velocity common to the whole body.

For example, in the case of the earth, the velocity due to rotation (one complete rotation per day) has the maximum value of one half a kilometer per second, whereas the translatory velocity is 30 km/sec. Thus

$$\frac{|\Delta \mathbf{v}|}{v_g} \sim 2 \times \frac{1}{60} = 0.03 .$$

Clearly an order of magnitude falling within the limits that have just been indicated.

3. HYPOTHESIS III. We assume that the quantity

$$\left(\frac{d}{r}\right)^2,$$

– the square of the ratio of the maximum dimension d of C to the minimum distance r between the points of the body C and those of S′ – is negligible.

It will be noted that this circumstance is usually allowed, since the time of Clairaut, in Newtonian Celestial Mechanics in order to reduce the classical n-body problem to a problem of n point-masses.

We are not here concerned with a crude approximation, of the order of 10^{-2}, but of a much more precise one – for example, of the order 10^{-4} in the typical Sun-Earth case. Were that not the case, one would have to take corrective terms into account, as one actually does in the study of the perturbations resulting from the satellites of Jupiter.

4. A FEW CONSEQUENCES OF THE FOREGOING HYPOTHESES. Let us first note that, as a result of Hypothesis I, the derivatives of γ'' are zero at G, and since this potential is involved in an additive way in $\mathfrak{L}$; and consequently enters into the equations of motion only through its derivatives, it follows that γ'' behaves as a constant.

According to Hypothesis II, the β_i components of the velocity of every point of C and the square β^2 of that velocity are, like γ'', invariable inside C.

So we may put, as a first step,

(V. 25) $$\gamma'' = \omega = \text{const},$$

which results in the following expressions for φ'', ψ'', ψ_i'':

(V. 26) $$\varphi'' = \frac{f}{c^2}\int_C \frac{\mu\gamma''}{r}\,dC = \omega^2,$$

(V. 27) $$\psi'' = \frac{3}{2}\frac{f}{c^2}\int_C \frac{\mu\beta^2}{r}\,dC = \frac{3}{2}\omega\beta^2,$$

(V. 28) $$\gamma_i'' = \frac{f}{c^2}\int_C \frac{\mu\beta_i}{r}\,dC = \omega\beta_i.$$

Finally, one has

(V. 29) $$\upsilon'' = \frac{1}{2}\frac{f}{c^2}\frac{\partial^2}{\partial x^{0 2}}\int_C \mu\, r\, dC = 0,$$

because the integral

$$\int_C \mu\, r\, dC,$$

(where r designates the distance between any point of C and the center of gravity) is, like μdC, a constant during actual motion.
According to the preceding results, one has

$$\mathfrak{N}'\gamma'' = \left(\frac{1}{2}\beta^2 + \gamma'\right)\gamma'' = \omega\left(\frac{1}{2}\beta^2 + \gamma'\right),$$

(V. 30) $$\left\{\begin{aligned} -\frac{1}{2}\gamma''^2 &= -\frac{1}{2}\varpi^2, \\ -2\gamma'\gamma'' &= -2\omega\gamma, \\ \psi'' &= \frac{3}{2}\omega\beta^2, \\ \gamma_i'' &= \omega\beta_i. \end{aligned}\right.$$

These expressions involve simplifications that are essential to the structure of the Einsteinian perturbation, but in order to take them into account it will be necessary to get an over-all picture of the question by considering the ensemble of bodies in motion.
And that is our object in the next section.

5. THE n-BODY PROBLEM

1. GRAVITATIONAL RADII. NOTATION. We designate by $C_\nu (\nu = 0, 1, \ldots, n-1)$ the bodies, and by P_ν the center of gravity of C_ν, m_ν and

$$l_\nu = \frac{fm_\nu}{c^2} \tag{V. 31}$$

the mass and gravitational radii of C_ν.

It should be recalled that this radius is a length that, for the Sun, is of about one and a half kilometers.

In addition, the attraction of a celestial body C_ν on a very distant point is almost completely reducible to the attraction of its entire mass at its center of gravity.

Thus, by virtue of Hypothesis III, the potential of the body C_ν on the center of gravity P_h of the body C_h is given, just as if the mass of C_ν were entirely concentrated at the center of gravity P_ν, by

$$\frac{fm_\nu}{r(P_\nu, P_h)} \tag{V. 32}$$

and the corresponding dimensionless potential is, consequently,

$$\frac{f}{c^2}\frac{m_\nu}{r(P_\nu, P_h)} = \frac{l_\nu}{r(P_\nu, P_h)}. \tag{V. 33}$$

Further, and with the same degree of approximation, one can maintain constant and equal to

$$\frac{l_\nu}{r(P_\nu, P_h)}$$

the unitary potential due to the attraction of the body C_ν, inside the body C_h.

In our planetary system, the gravitational radii l do not exceed that of the sun, as we have just recalled.

Henceforward, we will attach the index h to every quantity related to the center of gravity P_h of the body C_h; for example, β_h^2 is the square of the velocity of P_h; $\beta_{k|i}$ is the component along the x_i axis of the velocity of

P_h, γ_h is the potential at P_h due to all the bodies designated by an index other than h; etc.
So we will have

$$\gamma_h' = \sum_{\nu\,0}^{n-1\;\prime(h)} \frac{l}{r(P_\nu, P_h)}, \tag{V. 34}$$

where the notation $'^{(h)}$ used in the sum expresses the fact that the value h is excluded as a value for the summation index ν.
If we need to fix in particular a point P inside C_h, we will naturally write

$$\sum_{\nu\,0}^{n-1\;\prime(h)} \frac{l_\nu}{r(P_\nu, P)}, \tag{V. 34'}$$

the *numerical* value still being, of course γ_h', within the order of approximation that has been adopted.
According to the preceding considerations, we may regard as constant the integral

$$\gamma_h' = \frac{f}{c^2}\int_{C_h} \frac{\mu(Q)}{r(Q, P)}\, dG_h = \omega_h, \tag{V. 25}$$

which gives the potential of the body C_h at any point P of the body itself, for we are interested precisely in considering the value of γ_h'' at P_h, and we know that the derivations of γ_h'' at P_h are zero.
It goes without saying that ω_h, as in fact the dimensionless potential γ is at any point, a first-order quantity.[1]

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[1] One has

$$\omega_h = \frac{f}{c^2}\int_{C_h} \frac{\mu\, dC_h}{r(Q, P_h)} \leqq \frac{f}{c^2}\bar{\mu}.\, 4\pi \int_0^{\bar{r}} \rho\, d\rho = \frac{f}{c^2}\, 2\pi\, \bar{r}^2\, \bar{\mu} = \frac{3}{2}\frac{f\bar{m}}{c^2}\frac{1}{\bar{r}} = \frac{\bar{l}}{\bar{r}},$$

where $\bar{\mu}$ is an upper limit of μ inside C_h, $\bar{r}$ is the maximum distance from the center of gravity P_h to the surface of the body C_h, $\bar{m}$ is the mass enclosed in a homogeneous sphere of density $\bar{\mu}$ and radius $\bar{r}$, $\bar{l}$ being the corresponding gravitational radius.

2. THE CALCULATION OF THE POTENTIALS φ_h. QUANTITIES THAT MAY BE TREATED AS CONSTANTS.

One has at the outset

$$\varphi_h = \varphi'_h + \varphi''_h, \tag{V. 26}$$

where

$$\varphi'_h = -\frac{f}{c^2}\int_{S'} \frac{\mu\gamma}{r(Q,P)}\, dS', \tag{V. 27}$$

if Q designates a point of integration belonging to S′ and P an arbitrary parametric point of C_h, especially P_h. But, if S′ is the ensemble of the bodies C_ν, with the exception of C_h, one has almost exactly, according to Hypothesis III

$$\begin{aligned}
\text{(V. 28)}\quad \varphi'_h &= -\frac{f}{c^2}\sum_{\nu=0}^{n-1}{}'^{(h)} \frac{1}{r(P_\nu, P)}\int_{C_\nu} \mu(Q)\gamma_\nu(Q)\, dC_\nu \\
&= -\frac{f}{c^2}\sum_{\nu=0}^{n-1}{}'^{(h)} \frac{1}{r(P_\nu, P)}\left[\int_{C_\nu} \mu(Q)\gamma'_\nu(Q)\, dC_\nu + \int_{C_\nu} \mu(Q)\gamma''_\nu(Q)\, dC_\nu\right] \\
&= -\frac{f}{c^2}\sum_{\nu=0}^{n-1}{}'^{(h)} \frac{1}{r(P_\nu, P)}\sum_{\rho=0}^{n-1}{}'^{(\nu)} \frac{l_\rho m_\nu}{r(P_\nu, P_\rho)} \\
&\quad -\left(\frac{f}{c^2}\right)^2 \sum_{\nu=0}^{n-1}{}'^{(h)} \frac{1}{r(P_\nu, P)}\int_{C_\nu} \mu(Q)\, dC_\nu \int_{C_\nu} \frac{\mu(Q')\, dC_\nu)}{r(Q', Q)} \\
&= -\sum_{\nu=0}^{n-1}{}'^{(h)} \frac{l_\nu}{r(P_\nu, P)}\sum_{\rho=0}^{n-1}{}'^{(\nu)} \frac{l_\rho}{r(P_\nu, P_\rho)} - \sum_{\nu=0}^{n-1}{}'^{(h)} \frac{l_\nu \chi_\nu}{r(P_\nu, P)},
\end{aligned}$$

having put

$$l_\nu \chi_\nu = \left(\frac{f}{c^2}\right)^2 \int_{C_\nu} \mu(Q)\, dC_\nu \int \frac{\mu(Q')\, dC_\nu}{r(Q', Q)}. \tag{V. 29}$$

It is convenient to write

$$\text{(V. 30)} \qquad \varphi'_h = \varphi'^{*}_h - \sum_{0}^{n-1}{}'^{(h)}_{\nu} \frac{l_\nu \chi_\nu}{r(\mathrm{P}_\nu, \mathrm{P})},$$

where

$$\text{(V. 31)} \qquad \varphi'^{*}_h = -\sum_{0}^{n-1}{}'^{(n)}_{\nu} \frac{l_\nu}{r(\mathrm{P}_\nu, \mathrm{P})} \sum_{0}^{n-1}{}'^{(\nu)}_{\rho} \frac{l_\rho}{r(\mathrm{P}_\nu, \mathrm{P}_\rho)}.$$

Let us now turn our attention to

$$\begin{aligned} \text{(V. 32)} \quad \varphi''_h &= -\frac{f}{c^2}\int_{\mathrm{C}_h} \frac{\mu(\gamma'_h + \gamma''_h)}{r(\mathrm{Q}, \mathrm{P})} d\mathrm{C}_h \\ &= -\frac{f}{c^2}\int_{\mathrm{C}_h} \frac{\mu}{r(\mathrm{Q}, \mathrm{P})} \sum_{0}^{n-1}{}'^{(h)}_{\nu} \frac{l_\nu}{r(\mathrm{P}_\nu, \mathrm{P})} d\mathrm{C}_\nu - \frac{f}{c^2}\int_{\mathrm{C}_h} \frac{\mu\gamma''_h}{r(\mathrm{Q}, \mathrm{P})} d\mathrm{C}_h \\ &= -\sum_{0}^{n-1}{}'^{(h)}_{\nu} \frac{l_\nu}{r(\mathrm{P}_\nu, \mathrm{P})} \int_{\mathrm{C}_h} \frac{\mu\, d\mathrm{C}_h}{r(\mathrm{Q}, \mathrm{P})} - \frac{f}{c^2}\int_{\mathrm{C}_h} \frac{\mu\gamma''_h}{r(\mathrm{Q}, \mathrm{P})} d\mathrm{C}_h, \end{aligned}$$

The integral

$$\int_{\mathrm{C}_h} \frac{\mu\, d\mathrm{C}_h}{r(\mathrm{Q}, \mathrm{P})}$$

must be calculated at P_h, so it may be assimilated with the constant ω_h, and the first sum reduces to

$$-\omega_h \sum_{0}^{n-1}{}'^{(h)}_{\nu} \frac{l_\nu}{r(\mathrm{P}_\nu, \mathrm{P})}.$$

As for the second term

$$-\frac{f}{c^2}\int_{C_h}\frac{\mu\gamma_h''}{r(Q,P)}\,dC_h,$$

we note that:
the Body C_h is symmetrical with respect to its center of gravity P_h, so

$$\mu(Q)=\mu(Q'),$$

if Q, Q′ are symmetrical with respect to P_h, which means that

$$\gamma''(Q)=\gamma''(Q').$$

Then

$$(\mu\gamma_h'')_Q=(\mu\gamma_h'')_{Q'}.$$

It follows that the quantity under consideration has zero derivatives at P_h and may be assimilated with another constant ω_h' of the body C_h, which is additive and, therefore, negligible.
So we have definitively

$$\varphi_h''=-\omega_h\sum_{\nu\,0}^{n-1}{}'^{(h)}\frac{l_\nu}{r(P_\nu,P)} \tag{V.33}$$

and

$$\varphi_h=\varphi_h'^{*}+\varphi_h''^{*}, \tag{V.34}$$

having put

$$\varphi_h''^{*}=-\sum_{\nu\,0}^{n-1}{}'^{(h)}\frac{l_\nu(\chi_\nu+\omega_h)}{r(P_\nu,P)}. \tag{V.35}$$

3. THE CALCULATION OF THE TERMS ψ_h', v_h', γ_i. One obviously has

$$\psi_h'=\frac{3}{2}\sum_{\nu\,0}^{n-1}{}'^{(h)}\frac{l_\nu\beta_\nu^2}{r(P_\nu,P_h)} \tag{V.36}$$

and

$$(V.37) \qquad \upsilon'_h = \frac{1}{2}\frac{\partial^2}{\partial x^{02}} \sum_{\nu}^{\prime(h)}{}_{0}^{n-1} l_\nu\, r(P_\nu, P_h)\,.$$

And here we apply the observation that the value of the integral

$$(V.38) \qquad \int_D \mu\, r\, dC$$

at a very distant point P, reduces to very nearly zero, that is, to terms close to the order $(d/r)^2$ to the product of the mass of C multiplied by the distance from P to the center of gravity of C – as is the case for attraction. The expression of the γ_i quantities is

$$(V.39) \qquad \gamma_{h\,|\,i} = \gamma'_{h\,|\,i} + \gamma''_{r\,|\,i}\,,$$

where

$$(V.40) \qquad \begin{cases} \gamma'_{h\,|\,i} = \displaystyle\sum_{0}^{n-1}{}'^{(h)}_{\nu} \frac{l_\nu \beta_{\nu\,|\,i}}{r(P_\nu, P)}\,, \\ \gamma''_{h\,|\,i} = \omega_h \beta_{h\,|\,i}\,. \end{cases}$$

4. MASS COEFFICIENTS (RELATIVE MASSES). Hereafter we will find it expedient to introduce the ratios (proper fractions)

$$(V.41) \qquad \lambda_\nu = \frac{m_\nu}{m} \qquad (\nu = 0, 1, \ldots, n-1)\,,$$

such that

$$(V.42) \qquad \sum_{0}^{n-1}{}_{\nu} \lambda_\nu = 1\,.$$

Then, if l is the gravitational radius of the total mass

$$(V.43) \qquad l = \frac{fm}{c^2}\,,$$

one has

$$(\text{V.}44) \qquad l_\nu = \lambda_\nu l \qquad (\nu = 0, 1, \ldots, n-1).$$

5. THE THREE TERMS OF THE LAGRANGE FUNCTION $\mathfrak{L}_h$: $\mathfrak{N}_h$ (THE NEWTONIAN PART); $\mathfrak{D}'_h$ (THE EINSTEIN "POINT" PERTURBATION); $\mathfrak{D}''_h$ (THE EINSTEIN "EXTENSION-IN-SPACE" PERTURBATION). The preceding developments are sufficient to allow us to write the expanded expressions of the functions $\mathfrak{D}'_h \mathfrak{D}''_h$. They are

$$(\text{V.}45) \quad \mathfrak{D}'_h = \frac{1}{2}\,\mathfrak{N}'^2_h - \gamma'^2_h + \gamma'_h \beta^2_h + l \sum_{0\ \nu}^{n-1}{}'^{(h)} \frac{\lambda_\nu}{r(\mathrm{P}_\nu, \mathrm{P}_h)}\left[\frac{3}{2}\beta^2_h - \gamma'_\nu - 4\,\underline{\beta_h} \times \underline{\beta_\nu}\right]$$

$$+ \frac{1}{2}\, l \frac{\partial^2}{\partial x^{0\,2}} \sum_{0\ \nu}^{n-1}{}'^{(h)} \lambda_\nu\, r(\mathrm{P}_\nu, \mathrm{P}_h).$$

As for $\mathfrak{D}'''_h$, let us note that

$$(\text{V. }46) \qquad \zeta''_h = \varphi''^*_h + \psi''_h.$$

But (V. 12)

$$\mathfrak{D}''_h = \mathfrak{N}'_h \omega_h - 2\,\omega_h \gamma'_h + \zeta''_h + \omega_h \beta^2_h - 4 \sum_{3\ i}^{1} \gamma''_{h\,|\,i} \beta_{h\,|\,i},$$

so, by substituting therein the expressions calculated for φ''^*_h, ψ''_h, $\gamma''_{h|i}$ [(V. 35) and V. 40)], one has

$$(\text{V. }47) \qquad \mathfrak{D}''_h = \omega_h\left(\frac{1}{2}\beta^2_h + \gamma'_h\right) - 2\,\omega_h \gamma'_h + \omega_h \beta^2_h - 4\,\omega_h \beta^2_h$$

$$+ \frac{3}{2}\omega_h \beta^2_h - l \sum_{0\ \nu}^{n-1}{}'^{(h)} \frac{\lambda_\nu(\chi_\nu + \omega_h)}{r(\mathrm{P}_\nu, \mathrm{P})}$$

$$= -\,\omega_h \beta^2_h - l \sum_{0\ \nu}^{n-1}{}'^{(h)} \frac{\lambda_\nu(\chi_\nu + 2\,\omega_h)}{r(\mathrm{P}_\nu, \mathrm{P})}.$$

The Lagrangian function $\mathfrak{L}_h$ of the n-body problem is then

(V. 48) $$\mathfrak{L}_h = \mathfrak{N}_h + \mathfrak{D}'_h + \mathfrak{D}''_h,$$

where

$$\mathfrak{N}_h = \frac{1}{2}\beta_h^2 + \gamma'_h$$

is the Newtonian term;

(V. 49) $$\mathfrak{D}'_h = \frac{1}{2}\mathfrak{N}'^2_h - \gamma'^2_h + \zeta'_h + \gamma'_h\beta_h^2 - 4\sum_i^{1\ldots 3} \gamma'_{h\,|\,i}\beta_{h\,|\,i}$$

is the Einstein correction term for the masses reduced to points, a term which, by and large, goes back to Droste and De Sitter (see Preface); let us recall that

(V. 50) $$\begin{cases} \zeta'_h &= \varphi'_h + \psi'_h + \upsilon'_h, \\ \varphi'^*_h &= -\,l^2 \sum_\nu^{0\ldots n-1}{}'^{(h)} \dfrac{\lambda_\nu}{r(P_\nu, P)} \sum_\rho^{0\ldots n-1}{}'^{(\nu)} \dfrac{\lambda_\rho}{r(P_\nu, P)}, \\ \psi'_h &= \dfrac{3}{2} l \sum_\nu^{0\ldots n-1}{}'^{(h)} \dfrac{\lambda_\nu \beta_\nu^2}{r(P_\nu, P)}, \\ \upsilon'_h &= \dfrac{1}{2} l \dfrac{\partial^2}{\partial x^{0^2}} \sum_\nu^{0\ldots n-1}{}'^{(h)} \lambda_\nu r(P_\nu, P), \\ \gamma'_{h\,|\,i} &= l \sum_\nu^{0\ldots n-1}{}'^{(h)} \dfrac{\lambda_\nu \beta_{\nu\,|\,i}}{r(P_\nu, P)}; \end{cases}$$

finally

$$\mathfrak{D}''_h = -\,\omega_h\beta_h^2 - l \sum_\nu^{0\ldots n-1}{}'^{(h)} \frac{\lambda_\nu(\chi_\nu + 2\omega_h)}{r(P_\nu, P_h)}$$

is the correction term due to the extension of the bodies.

6. THE CONSTANTS APPEARING IN THE DIFFERENTIAL EQUATIONS. In the three terms of the Lagrangian function just now given in its expanded form there appear, in addition to the coefficients of mass λ_h, $2n$ gravitational parameters ω_h, χ_h, each of which depends upon the material structure of the body C_h and which, according to our hypotheses, behave like constants during the course of motion.

These parameters appear, nevertheless, only in the terms $\mathfrak{D}''_h$ arising from the Einsteinian correction resulting from extension.

Moreover, in our approximation it is possible to eliminate all these parameters by means of a suitable artifice (see Chapter VI, Section 1, no. 2) combined with a slight modification of the λ coefficients – that is, by a slight alteration of the constants that characterize the masses.

7. PRECAUTIONS TO BE OBSERVED IN THE CONSTRUCTION OF THE DIFFERENTIAL EQUATIONS. The Lagrangian function $\mathfrak{L}_h$ is related to the x^i coordinates of the point P and to the $\beta_{h|i}$ components of its velocity, if we take special care to note that, after differentiation, the x^i coordinates must be replaced by the x^i_h coordinates of the center of gravity of the body C_h.

The point P_h itself and all the other $P_\nu (\nu \neq h)$, as well as the components $\beta_{\nu|i}$ of their velocities, will be parameters when differentiation with respect to the x^i is applied.

A special observation about differentiation with respect to x^0, which occurs in the expression of υ'_h, must be made. This partial derivative is independent of the coordinates x^i ($i = 1, 2, 3$) of the point P, whereas it directly affects the variability of the function with respect to time, because of the motion of the other points (parametric) $P_\nu (\nu \neq h)$.

That is why we will specifically designate the operation

$$\frac{\partial}{\partial x^0}$$

by the notation

$$\frac{\partial'^{(h)}}{\partial' x^0},$$

writing in conformity with this

$$\upsilon'_h = \frac{1}{2} l \frac{\partial'^{(h)}}{\partial' x^0} \sum_{\nu}^{n-1}{}'^{(h)} \lambda_\nu r(P_\nu, P). \tag{V. 51}$$

Chapter VI

THE TWO-BODY PROBLEM. EXTENSION OF THE ELIMINATION PRINCIPLE TO THE GENERAL n-BODY CASE

1. THE LAGRANGE FUNCTION FOR THE ABSOLUTE MOTION OF EACH OF TWO BODIES

1. MAXIMUM EXTENSION OF THE USE OF CLASSICAL NOTATION. We now propose to apply the general theory set forth in Chapter V to the especially important case of two celestial bodies.

The conclusions at which we will finally arrive will be valid, not only for the classical cases of Sun-and-planet or planet-and-satellite, but likewise for double stars, for which astronomy has, in several hundred cases, been able to determine orbits, either by direct observation or by spectroscopic methods.[1]

We designate by the index h all the quantities (scalar or vector) related to any one of the two bodies and by the index $h + 1$ all the quantities related to the other of the two bodies, regarding, naturally, the indices of equal parity 0, 2; 1, 3 as identifiable. It follows that the sums of Chapter V, Section 5:

$$\sum_{0}^{n-1}{}'^{(h)}_{\nu}, \qquad \sum_{0}^{n-1}{}'^{(\nu)}_{\rho},$$

are reduced to a single term bearing the index $h + 1$, $\nu + 1$, ... In this way we have

$$\gamma'_h = \frac{l\lambda_{h+1}}{r(\mathrm{P}_{h+1}, \mathrm{P})}, \tag{VI. 1}$$

[1] Cf. Armellini, *Trattato di Astronomia siderale*, Bologna, Zanichelli, 1931, 2

(VI. 2) $$\varphi_h'^{*} = - l^2 \frac{\lambda_{h+1}}{r(\mathrm{P}_{h+1}, \mathrm{P})} \frac{\lambda_h}{r(\mathrm{P}_{h+1}, \mathrm{P}_h)}.$$

To form the equations of the motion of the point P_h, which is the center of gravity of C_h, we must derive the function $\varphi_h'^{|*}$ with regard to the coordinates of the point P, putting, after differentiation $\mathrm{P} \equiv \mathrm{P}_h$, whereas the distance $r(\mathrm{P}_{h+1}, \mathrm{P}_h)$ must be considered as a constant.

To avoid any confusion, we introduce for the distance $\mathrm{P}_h\mathrm{P}_{h+1}$ the two notations r and r^*, with the convention that r^* must not be differentiated and that $r \equiv r(\mathrm{P}_{h+1}, \mathrm{P}_h)$ may replace what we have thus far designated by $r(\mathrm{P}_{h+1}, \mathrm{P})$, and also when it is a question of differentiating and substituting P_h for P.

So we put

(VI. 2′) $$\varphi_h'^{*} = - l^2 \frac{\lambda_{h+1}}{r} \frac{\lambda_h}{r^*}$$

and

(VI. 1) $$\gamma_h' = \frac{l\lambda_{h+1}}{r}.$$

One still has

(VI. 3) $$\psi_h' = \frac{3}{2} l \frac{\lambda_{h+1} \beta_{h+1}^2}{r},$$

(VI. 4) $$\upsilon_h' = \frac{1}{2} l \lambda_{h+1} \frac{\partial'^{(h)2} r}{\partial' x^{02}},$$

(VI. 5) $$\gamma_{h|i}' = \frac{l\lambda_{h+1}\beta_{h+1|i}}{r}.$$

Let us now write the expression of the functions in the two-body case.

$$\mathfrak{N}_h = \mathfrak{N}_h', \quad \mathfrak{D}_h', \quad \mathfrak{D}_h''$$

One has

$$\mathfrak{N}_h = \mathfrak{N}'_h = \frac{1}{2}\beta_h^2 + \frac{l\lambda_{h+1}}{r} \tag{VI. 6}$$

$$\mathfrak{D}'_h = \frac{1}{2}\mathfrak{N}'^2_h - \gamma'^2_h - l^2\frac{\lambda_h\lambda_{h+1}}{r^* r} + \frac{3}{2}\frac{l\lambda_{h+1}\beta^2_{h+1}}{r} + \frac{1}{2}l\lambda_{h+1}\frac{\partial'^{(h)2} r}{\partial' x^{02}} + \gamma'_h\beta^2_h - 4\frac{l\lambda_{h+1}}{r}\sum_{i=1}^{3}\beta_{h|i}\beta_{h+1|i}\,, \tag{VI. 7}$$

where, we repeat, r replaces $r(\mathrm{P}_{h+1}, \mathrm{P}_h)$, while r^* has the same value, but must not be differentiated with respect to the coordinates of P_h.

The expression of $\mathfrak{D}''_h$ is

$$\mathfrak{D}''_h = -\omega_h\beta^2_h - \frac{l\lambda_{h+1}(\chi_{h+1} + 2\omega_h)}{r}, \tag{VI. 8}$$

2. SIMPLIFICATIONS PERMITTED BY OUR APPROXIMATION. A DEVICE FOR EMPHASIZING THE ELIMINATION PRINCIPLE. One can simplify the Lagrange function by the following means, as we anticipated in Chapter V, section 4, subsection 6 for any arbitrary n:

Let us introduce a constant σ_h (of the first order) unspecified *a priori*, and let us consider the function

$$(1 + \sigma_h)\mathfrak{L}_h, \tag{VI. 9}$$

which, by neglecting the $\sigma_h\mathfrak{D}'_h$ and $\sigma_h\mathfrak{D}''_h$ terms (which would be at least of the third order) reduces to

$$(1 + \sigma_h)\mathfrak{N}'_h + \mathfrak{D}'_h + \mathfrak{D}''_h$$

producing the same equations as those encompassed in the variational formula

$$\delta\int\mathfrak{L}_h\,dt = 0\,.$$

We will be able to take advantage of the indeterminacy of σ_h to simplify the form of the function, slightly modifying the two constants λ_h–a pro-

cedure which permits us to eliminate the four constants ω_h, χ'_h from the differential equations.
One has, as a matter of fact, momentarily neglecting $\mathfrak{D}'_h$,

$$(1 + \sigma_h)\mathfrak{N}'_h + \mathfrak{D}''_h = \frac{1}{2}\beta_h^2 + \frac{l\lambda_{h+1}}{r}[1 + \sigma_h - (\chi_{h+1} + 2\omega_h)] + \frac{1}{2}(\sigma_h - \omega_h)\beta_h^2 . \tag{VI. 10}$$

So, we have only to choose

$$\sigma_h = 2\omega_h \tag{VI. 11}$$

and to put

$$\lambda'_h = \lambda_h(1 - \chi_h) \qquad (h = 0, 1), \tag{VI. 12}$$

which represents another slight alteration of the masses in order to obtain the first part of the Lagrangian function in the form

$$\frac{1}{2}\beta_h^2 + \frac{l\lambda'_{h+1}}{r} . \tag{VI. 13}$$

In reality, in the expression of $\mathfrak{D}'_h$ the old mass-coefficients λ still occur, but since the terms of $\mathfrak{D}'_h$ are all of a high order, we can, with confidence, replace the λ's by the corresponding λ''s that we have just introduced by means of (VI. 12).
We are thus, in defining the motion of point P'_h, ultimately led – still writing λ (a constant coefficient playing the part of mass) instead of λ' – to the Lagrange function

$$\mathfrak{L}_h = \mathfrak{N}_h + \mathfrak{D}_h , \tag{VI. 14}$$

where

$$\mathfrak{N}_h = \frac{1}{2}\beta_h^2 + \frac{l\lambda_{h+1}}{r} \tag{VI. 15}$$

and

$$\mathfrak{D}_h = \frac{1}{2}\mathfrak{N}_h^2 - \gamma_h^2 - \gamma_h\overset{*}{\gamma}_{h+1} + \mathfrak{D}_h^{(1)} + \mathfrak{D}_h^{(2)} , \tag{VI. 16}$$

having put

$$\gamma_h = \frac{l\lambda_{h+1}}{r} \qquad \overset{*}{\gamma}_h = \frac{l\lambda_{h+1}}{\overset{*}{r}}$$

$$\text{(VI. 17)} \qquad \begin{cases} \mathfrak{D}_h^{(1)} = \gamma_h\left(\beta_h^2 - 4\underline{\beta_h} \times \underline{\beta_{h+1}} + \frac{3}{2}\beta_{h+1}^2\right), \\ \mathfrak{D}_h^{(2)} = \frac{1}{2}\, l\lambda_{h+1}\, \frac{\partial'^{(h)2} r}{\partial' x^{02}}. \end{cases}$$

It is important to note that, in keeping with the transformations that have just been presented, the $\mathfrak{D}_h''$ terms, which depended in addition on the four gravitational constants ω_h, χ_h, have completely disappeared. In this way we have achieved the elimination of everything which, in the differential equations, still depended on the extension of the bodies, by assimilating them in this respect to simple point-masses.

2. RELATIVE MOTION

1. FORMULAS PERTAINING TO THE TWO-BODY PROBLEM IN CLASSICAL MECHANICS CONSIDERED AS PREPARATION FOR THE RELATIVISTIC TREATMENT. In seeking the left-hand member of one of the second-order equations defining both the absolute movement of the two bodies and their integration, let us first note that in the calculation of the classical Lagrangian binomials

$$\text{(VI. 18)} \qquad \beta_{h\,|\,i} = \frac{d}{dx^0}\frac{\partial}{\partial\beta_{h\,|\,i}} - \frac{\partial}{\partial x_h^i} \qquad (h = 0, 1\,;\ i = 1, 2, 3)\,,$$

it is convenient to introduce, after differentiation, the relative coordinates of one of the two bodies – let us say P_1 – with respect to the other, P_0, i.e., the differences

$$\text{(VI. 19)} \qquad x^i = x_1^i - x_0^i = (-1)^h(x_{h+1}^i - x_h^i) = (-1)^{h+1}(x_h^i - x_{h+1}^i)\,,$$

and the components of the absolute velocity as a function of the components of the corresponding relative velocity β.

To do this we have to utilize the definition of the relative velocity

$$\text{(VI. 20)} \qquad \underline{\beta} = \underline{\beta}_1 - \underline{\beta}_0 = (-1)^h (\underline{\beta}_{h+1} - \underline{\beta}_h) = (-1)^{h+1} (\underline{\beta}_h - \underline{\beta}_{h+1}) .$$

It is most important to note that, the moment we are concerned with terms of a higher order, we may legitimately associate with (VI. 20) the classical simplification concerning the immobility of the center of gravity. In this regard, one does well to bear in mind the following points:

The space-time that forms the basis of our theory differs very little from a pseudo-Euclidean or Einstein-Minkowski space. That being so, we can go on to choose this space – for which the principal part of ds^2 is ds_0^2 – in such a way that the origin is the center of gravity of the two points P_0, P_1, whose velocity is constant and may be considered zero in the Newtonian approximation.

So we have as a first approximation

$$\text{(VI. 21)} \qquad \lambda_h \underline{\beta}_h + \lambda_{h+1} \underline{\beta}_{h+1} = 0 .$$

This equation permits us to express the absolute velocities as a function of $\underline{\beta}$. One has, by combination with relation (IV. 20):

$$\text{(VI. 22)} \qquad \begin{cases} \underline{\beta}_h = (-1)^{h+1} \lambda_{h+1} \underline{\beta} + O[2] , \\ \underline{\beta}_{h+1} = (-1)^h \lambda_h \underline{\beta} + O[2] . \end{cases}$$

These approximate expressions of the absolute velocities will obviously suffice for calculations of the terms higher than the first order.

Similarly, in the calculation of the Einsteinian correction of a Keplerian motion[1], all other quantities may be replaced by the determination attached to each one in Keplerian motion.

For the convenience of the reader we reprint herewith a few classical formulas in this connection, but using modern notation.

In the calculations to follow, we will have only to cite the numbers of the formulas in the table

$$\text{(VI. 23)} \qquad \dot{\beta}_i = \frac{\partial}{\partial x^i}\left(\frac{l}{r}\right) \qquad (i = 1, 2, 3) \qquad \text{(equations of motion),}$$

[1] We mean, starting from initial conditions that, in the ordinary theory, would give rise to a rigorously Keplerian motion.

(VI. 24) $$\frac{1}{2}\beta^2 - \gamma = e \qquad \text{(integral of effective forces)},$$

(VI. 25) $$r^2\dot{\theta} = \frac{C}{c} \qquad \text{(area integral in the plane of the relative orbit)},$$

we designate by

(VI. 26) $$e = \frac{E}{c^2}$$

the energy-constant E divided by c^2 and by the constant C of the areas; differentiation with respect to x^0 is designated by a dot [e.g., $\dot{D}$.]. In addition, θ designates the true angle in the plane of the relative orbit.
The constant

$$\frac{C}{c}$$

is a length. As to its order of magnitude, let us note that in the planetary motions under consideration, $r\theta$ is the circular velocity divided by c, and hence of the order β, that is $\frac{1}{2}$, so everywhere $r^2\theta$ is the product of a length (of the order of magnitude of planetary distances) and a pure number of the order $\frac{1}{2}$.
Remembering that

$$l = \frac{fm}{c^2}$$

is a length, it is appropriate to introduce a new numerical constant of order $\frac{1}{2}$, namely a, related to C by the relation

(VI. 27) $$\frac{l}{a} = \frac{C}{c}.$$

Then the integral of the areas is

(VI. 25′) $$r^2\dot{\theta} = \frac{l}{a}.$$

The quantity

(VI. 27′)
$$a = \frac{1}{r\theta}\left(\frac{l}{r}\right)$$

is of the order $\frac{1}{2}$, because

$$\frac{1}{r}$$

is of the first order.

2. TRANSITION TO RELATIVE MOTION. Having established the foregoing, let us point out that the formal application of the operators $B_{h|i}$ ($h = 0, 1$; $i = 1, 2, 3$) to the Lagrangian functions of the two bodies – functions which are henceforth completely explicit – would lead us to the six second-order differential equations of their absolute motion.

Since we are seeking the relative motion, it suffices conceptually to obtain three second-order equations having as their unknowns the relative coordinates (VI. 19)

$$x^i = x_1^i - x_0^i.$$

It would clearly be desirable in this connection to avoid making the six above-mentioned equations explicit, but rather to form directly a single Lagrangian function containing exclusively the elements of relative motion; unfortunately, we did not succeed in achieving that calculation synthetically.

Lacking that, we had to be content with the successive calculation of the various terms of the equations of absolute motion, limiting ourselves to the appearance, in place of the $B_{h|i}$, of the B^i operators, where only the relative coordinates and velocities are involved in the higher-order terms. In this way one can, within the indicated approximation, obtain from the six original equations, six combinations, three of which result from a Lagrangian function $\mathfrak{L}$, exclusively involving relative motion, while the others define the absolute motion of the center of gravity.

3. APPLICATION OF THE $B_{h|i}$ OPERATOR TO $\mathfrak{N}_h$. We have rigorously

$$B_{h\,|\,i}\mathfrak{N}_h = \dot{\beta}_{h\,|\,i} + l\,\lambda_{h+1}\frac{\partial\frac{l}{r}}{\partial x_h^i}.$$

So, by taking into account the approximate values (VI. 22)

$$\mathrm{B}_{h\,|\,i}\,\mathfrak{N}_h = (-1)^{h+1}\lambda_{h+1}\left(\dot{\beta}_i + l\frac{\partial\frac{l}{r}}{\partial x^i}\right) + \bigcirc[2].$$

Naturally, in the terms of $\mathfrak{D}_h$ where $\mathrm{B}_{h|i}\ \mathfrak{N}_h$ is multiplied by a quantity of the first order, one can neglect the terms $\bigcirc$ [2].
And this is the appropriate time to bring in (with modern notation, of course) the classical Lagrangian function of relative Newtonian motion

(VI. 28) $$\mathfrak{N} = \frac{1}{2}\beta^2 + \gamma,$$

where

(VI. 29) $$\gamma = \gamma_h + \gamma_{h+1} = \frac{l}{r}.$$

One then has

(VI. 30) $$\mathrm{B}_{h\,|\,i}\,\mathfrak{N}_h = (-1)^{h+1}\lambda_{h+1}\,\mathrm{B}_i\,\mathfrak{N},$$

where B_i designates the operator

(VI. 31) $$\mathrm{B}_i = \frac{d}{dx^0}\frac{\partial}{\partial\beta_i} - \frac{\partial}{\partial x^i}.$$

4. APPLICATION OF THE OPERATOR $\mathrm{B}_{h|i}$ TO $\frac{1}{2}\ \mathfrak{N}_h^2$. One has

$$\mathrm{B}_{h\,|\,i}\left(\frac{1}{2}\,\mathfrak{N}_h^2\right) = \frac{d}{dx^0}\frac{\partial}{\partial\beta_{h\,|\,i}}\left(\frac{1}{2}\,\mathfrak{N}_h^2\right) - \frac{\partial}{\partial x_h^i}\left(\frac{1}{2}\,\mathfrak{N}_h^2\right)$$
$$= \frac{d}{dx^0}\left(\mathfrak{N}_h\frac{\partial\,\mathfrak{N}_h}{\partial\beta_{h\,|\,i}}\right) - \mathfrak{N}_h\,\frac{\partial\mathfrak{N}_h}{\partial x_h^i} = \mathfrak{N}_h\,\mathrm{B}_{h\,|\,i}\,\mathfrak{N}_h + \beta_{h\,|\,i}\,\dot{\mathfrak{N}}_h.$$

But

$$\mathrm{B}_{h\,|\,i}\,\mathfrak{N}_h = (-1)^{h+1}\,\lambda_{h+1}\left[\frac{d}{dx^0}\left(\frac{1}{2}\beta^2+\gamma\right)-\frac{\partial\gamma}{\partial x^i}\right]$$

and

$$\dot{\mathfrak{N}}_h = \frac{d}{dx^0}\left(\frac{1}{2}\lambda_{h+1}^2\,\beta^2+\lambda_{h+1}\,\gamma\right),$$

thus, excluding terms of a higher order, one has

$$\beta_{h\,|\,i}\,\dot{\mathfrak{N}}_h = (-1)^{h+1}\,\lambda_{h+1}\,\beta_i\frac{d}{dx^0}\left(\frac{1}{2}\lambda_{h+1}^2\,\beta^2+\lambda_{h+1}\,\gamma\right).$$

By noting that the term

$$\mathfrak{N}_h\,\mathrm{B}_{h\,|\,i}\,\mathfrak{N}_h$$

is negligible because it is of an order higher than two, and taking into account the integral of the active forces (VI. 24), one has

(VI. 32)

$$\begin{aligned}\mathrm{B}_{h\,|\,i}\left(\frac{1}{2}\mathfrak{N}_h^2\right) &= (-1)^{h+1}(\lambda_{h+1}^2+\lambda_{h+1}^3\,\beta_i\dot{\gamma})\\ &= (-1)^{h+1}(\lambda_{h+1}^2+\lambda_{h+1}^3)\{(\beta_i\gamma)^{\cdot}-\dot{\beta}_i\gamma\}\\ &= (-1)^{h+1}(\lambda_{h+1}^2+\lambda_{h+1}^3)\left\{\mathrm{B}_i\left(\frac{1}{2}\beta_2\gamma\right)+\frac{1}{2}\beta_2\frac{\partial\gamma}{\partial x^i}-\gamma\frac{\partial\gamma}{\partial x^i}\right\}\\ &= (-1)^{h+1}(\lambda_{h+1}^2+\lambda_{h+1}^3)\,\mathrm{B}_i\left(\frac{1}{2}\beta_2\gamma-e\gamma\right).\end{aligned}$$

5. APPLICATION OF THE OPERATOR $\mathrm{B}_{h|i}$ TO $(-\gamma_h^2-\gamma_h\overset{*}{\gamma}_{h+1})$. We note that the function in question does not depend on the $\beta_{h|i}$. So the operator reduces to

$$-\frac{\partial}{\partial x_h^i} = -(-1)^{h+1}\,\frac{\partial}{\partial x^i}.$$

One has

$$\text{(VI. 33)}\qquad \mathrm{B}_{h\,|\,i}(-\gamma_h^2 - \gamma_h\gamma_{h+1}^*) = (-1)^{h+1}\frac{\partial}{\partial x^i}(\lambda_{h+1}^2\gamma^2 + \lambda_h\lambda_{h+1}\gamma\gamma^*)$$

$$= (-1)^{h+1}\left(2\lambda_{h+1}^2\gamma\frac{\partial\gamma}{\partial x^i} + \lambda_h\lambda_{h+1}\gamma^*\frac{\partial\gamma}{\partial x^i}\right)$$

$$= (-1)^{h+1}\lambda_{h+1}(2\lambda_{h+1} + \lambda_h)\gamma\frac{\partial\gamma}{\partial x^i}$$

$$= (-1)^{h+1}\lambda_{h+1}(\lambda_{h+1} + 1)\gamma\frac{\partial\gamma}{\partial x^i}$$

$$= (-1)^{h+1}\lambda_{h+1}(\lambda_{h+1} + 1)\mathrm{B}_i\left(-\frac{1}{2}\gamma^2\right).$$

In order to understand the various steps in this calculation, it is necessary to recall the remark concerning r^* (Section 1, no. 1.).

6. APPLICATION OF THE OPERATOR $\mathrm{B}_{h|i}$ TO $\mathfrak{D}_h^{(1)}$. One has

$$\mathrm{B}_{h\,|\,i}\mathfrak{D}_h^{(1)} = \mathrm{B}_{h\,|\,i}[\gamma_h(\beta_h^2 - 4\underline{\beta}_h \times \underline{\beta}_{h+1} + \tfrac{3}{2}\beta_{h+1}^2)]$$

$$= \frac{d}{dx_0}(2\gamma_h\beta_{h\,|\,i}) - 4\frac{d}{dx^0}(\beta_{h+1\,|\,i}\gamma_h) - \beta_h^2\frac{\partial\gamma_h}{\partial x_h^i}$$

$$+ 4\beta_h \times \underline{\beta}_{h+1}\frac{\partial\gamma_h}{\partial x_h^i} - \tfrac{3}{2}\beta_{h+1}^2\frac{\partial\gamma_h}{\partial x_h^i}$$

$$= (-1)^{h+1}2\lambda_{h+1}^2(\beta_i\gamma)^\cdot - (-1)^{h+1}\lambda_{h+1}^3\beta^2\frac{\partial\gamma}{\partial x^i}$$

$$-(-1)^h 4\lambda_h\lambda_{h+1}(\beta_i\gamma)^\cdot$$

$$-(-1)^{h+1}4\lambda_h\lambda_{h+1}^2\beta^2\frac{\partial\gamma}{\partial x^i} - (-1)^{h+1}\tfrac{3}{2}\lambda_h^2\lambda_{h+1}\beta^2\frac{\partial\gamma}{\partial x^i}$$

$$= (-1)^{h+1}2\lambda_{h+1}(\lambda_h + 1)(\beta_i\gamma)^\cdot$$

$$-(1)^{h+1}(\lambda_{h+1}^3 + 4\lambda_h\lambda_{h+1}^2 + \tfrac{3}{2}\lambda_h^2\lambda_{h+1})\beta^2\frac{\partial x^i}{\partial\gamma}$$

$$= (-1)^{h+1}2\lambda_{h+1}(\lambda_{h+1})(\beta_i\gamma)^\cdot$$

$$-(-1)^{h+1}\lambda_{h+1}(\lambda_{h+1}^2 + 4\lambda_h\lambda_{h+1} + \tfrac{3}{2}\lambda_h^2)\beta^2\frac{\partial x^i}{\partial\gamma}$$

$$= (-1)^{h+1}\, 2\,\lambda_{h+1}\,(\lambda_{h+1}) \left\{ \mathrm{B}_i(\tfrac{1}{2}\beta^2\gamma) + \tfrac{1}{2}\beta^2 \frac{\partial\gamma}{\partial x^i} \right\}$$

$$- (-1)^{h+1}\, \lambda_{h+1}\,(\lambda_{h+1}^2 + 4\,\lambda_h\lambda_{h+1} + \tfrac{3}{2}\,\lambda_h^2)\,\beta^2 \frac{\partial\gamma}{\partial x^i}$$

$$= (-1)^{h+1}\, 2\,\lambda_{h+1}\,(\lambda_{h+1})\, \mathrm{B}_i(\tfrac{1}{2}\beta^2\gamma)$$

$$+ (-1)^{h+1}\, \lambda_{h+1}\, \{\lambda_{h+1} - \lambda_{h+1}^2 - 4\,\lambda_h\lambda_{h+1} - \tfrac{3}{2}\,\lambda_h^2\}\, \beta^2 \frac{\partial\gamma}{\partial x^i}.$$

Remembering that

$$\lambda_h + \lambda_{h+1} = 1,$$

one has

$$1 + \lambda_h - \lambda_{h+1}^2 - 4\,\lambda_h\lambda_{h+1} - \tfrac{3}{2}\,\lambda_h^2$$
$$= \lambda_h(3 - 4\,\lambda_{h+1} - \tfrac{5}{2}\,\lambda_h) = \lambda_h(\tfrac{1}{2}\,\lambda_h - \lambda_{h+1}).$$

Moreover

$$\beta^2 \frac{\partial\gamma}{\partial x^i} = 2\gamma \frac{\partial\gamma}{\partial x^i} + 2e \frac{\partial\gamma}{\partial x^i} = \frac{\partial}{\partial x^i}(\gamma^2 + 2e\gamma) = -\,\mathrm{B}_i(\gamma^2 + 2e\gamma).$$

Thus, if one introduces the product

(VI.34) $$p = \lambda_h\,\lambda_{h+1}$$

of the expressions

$$\frac{m_0}{m_0 + m_1}, \frac{m_1}{m_0 + m_1};$$

where the m's are constants that are very nearly the masses of the two bodies (VI. 12) one has

(VI. 35) $$\mathrm{B}_{h\,|\,i}\mathfrak{D}_k^{(n)} = (-1)^{h+1}(\lambda_{h+1} + p)\,\mathrm{B}_i(\beta^2\gamma)$$
$$+ (-1)^{h+1}\, p\,(\lambda_{h+1} - \tfrac{1}{2}\,\lambda_h)\,\mathrm{B}_i(\gamma^2 + 2e\gamma).$$

7. APPLICATION OF THE OPERATOR $B_{h|i}$ TO $\mathfrak{D}_h^{(2)}$. It will be useful, first of all, to give an expression of r as a function of γ and of the constants of integration e and a – an expression deduced from the integrals of the active forces and the areas [(VI. 24) and (VI. 25)].
It is finally obtained by noting that

$$\beta^2 = \dot{r}^2 + r^2\dot{\theta}^2 .$$

Then

$$\dot{r}^2 = \beta^2 - r^2\dot{\theta}^2 = \beta^2 - \left(\frac{\gamma}{a}\right)^2 = 2e + 2\gamma - \frac{\gamma^2}{a^2}. \tag{VI. 36}$$

With that established, let us first of all calculate the derivative

$$\frac{\partial'^{(h)2} r}{\partial' x^{02}} .$$

One has, following the remark in Section 1,

$$\frac{\partial'^{(h)} r}{\partial' x^0} = \sum_{j=1}^{3} \frac{x_{h+1}^j - x_h^j}{r} \beta_{h+1}^j .$$

Thus

$$\begin{aligned} \frac{\partial'^{(h)2} r}{\partial' x^{02}} &= \sum_{j=1}^{3} \frac{x_{k+1}^j - x_h^j}{r} \dot{\beta}_{h+1}^j + \frac{1}{r}\beta_{h+1}^2 \\ &\quad - \frac{1}{r^2}\frac{\partial'^{(h)} r}{\partial' x^0} \sum_{j=1}^{3} (x_{h+1}^j - x_h^j)\,\beta_{h+1}^j \\ &= \sum_{j=1}^{3} \frac{x_{h+1}^j - x_h^j}{r} \dot{\beta}_{h+1}^j + \frac{1}{r}\beta_{h+1}^2 \\ &\quad - \frac{1}{r}\sum_{kj=1}^{3} \frac{\partial r}{\partial x_{h+1}^k}\frac{\partial r}{\partial x_{h+1}^j} \beta_{h+1}^k \beta_{h+1}^j . \end{aligned} \tag{VI. 37}$$

Now we will apply the operator $B_{h|i}$ to $\mathfrak{D}_h^{(2)}$ noting that this function does not depend on the $\beta_{h|i}$'s.
Moreover, if one expresses the various terms as a function of the coordinates and velocity of P_i with respect to P_0, one has

$$\sum_{j=1}^{3} \frac{x_{h+1}^j - x_h^j}{r} \dot{\beta}_{h+1\,|\,j} = \sum_{j=1}^{3} (-1)^{h+1} \frac{x_j}{r} (-1)^h \lambda_h \dot{\beta}_j = -\lambda_h \sum_{j=1}^{3} \frac{x^j}{r} \dot{\beta}_j ,$$

$$\frac{1}{r} \beta_{h+1}^2 = \frac{\lambda_h^2}{r} \beta^2 ,$$

$$\sum_{kj=1}^{3} \frac{\partial r}{\partial x_{h+1}^k} \frac{\partial r}{\partial x_{h+1}^j} \beta_{h+4}^k \beta_{h+1}^j = \lambda_h^2 \sum_{kj=1}^{3} \frac{\partial r}{\partial x^k} \frac{\partial r}{\partial x^j} \beta_k \beta_j = \lambda_h^2 \beta_2^2 ,$$

where β_r designates the radial velocity (relative) of P_1, that is, $\dot{r}$.
Thus, one has

$$B_{h\,|\,i} \mathfrak{D}_h^{(2)} = -\tfrac{1}{2} l \lambda_{h+1} \frac{\partial}{\partial x_h^l} \frac{\partial'^{(h)2} r}{\partial' x^{02}}$$

$$= -(-1)^{h+1} \tfrac{1}{2} l \lambda_{h+1} \frac{\partial}{\partial x^i} \left\{ -\lambda_h \sum_{j=1}^{3} \frac{x^j}{r} \dot{\beta}_j + \frac{\lambda_h^2}{r} \beta^2 - \frac{\lambda_h^2}{r} \beta_r^2 \right\}$$

$$= -(-1)^{h+1} \tfrac{1}{2} \lambda_{h+1}$$

$$\times \left\{ -l \lambda_h \frac{\partial}{\partial x^i} \sum_{j=1}^{3} \dot{\beta}_j \frac{\partial r}{\partial x^j} + \lambda_h^2 \beta^2 \frac{\partial \gamma}{\partial x^i} - \lambda_h^2 \beta_r^2 \frac{\partial \gamma}{\partial x^i} \right\}$$

$$= -(-1)^{h+1} \tfrac{1}{2} \lambda_{h+1}$$

$$\times \left\{ -l \lambda_h \sum_{ij=1}^{3} \dot{\beta}_j \frac{\partial_r^2}{\partial x^i \partial x^j} + \lambda_h^2 \beta^2 \frac{\partial \gamma}{\partial x^i} - \lambda_h^2 \beta_r^2 \frac{\partial x^i}{\partial \gamma} \right\} .$$

By virtue of equations (VI. 23) one has

$$\sum_{ij}^{3} \beta_j \frac{\partial^2 r}{\partial x^i \partial x^j} = -\frac{l}{r^2} \sum_{ij}^{3} \frac{\partial r}{\partial x^j} \frac{\partial^2 r}{\partial x^i \partial x^j} = -\frac{1}{2} \frac{l}{r^2} \frac{\partial}{\partial x^i} \sum_{j}^{3} \left(\frac{\partial r}{\partial x^j}\right)^2 = 0.$$

Because of (VI. 36), we thus obtain

$$\text{(VI. 38)} \quad \mathrm{B}_{h\,|\,i} \mathfrak{D}_h^{(2)} = -(-1)^{h+1} \lambda_{h+1} \times \left\{ \lambda_h^2 (\gamma + e) \frac{\partial \gamma}{\partial x^i} - \lambda_h^2 \left(e + \gamma - \frac{\gamma^2}{2a^2} \right) \frac{\partial \gamma}{\partial x^i} \right\}$$

$$= -(-1)^{h+1} p \lambda_h \frac{1}{2a^2} \gamma^2 \frac{\partial \gamma}{\partial x^i}$$

$$= -(-1)^{h+1} p \lambda_h \frac{\partial}{\partial x^i} \left(\frac{1}{6} \frac{\gamma^3}{a^2} \right) = (-1)^{h+1} p \lambda_h \mathrm{B}_i \left(\frac{1}{6} \frac{\gamma^3}{a^2} \right).$$

8. CONCLUSION OF THE FOREGOING CALCULATIONS AND DEDUCTION OF THE LAGRANGE FUNCTION GOVERNING RELATIVE MOTION. Here, then, is the final and definitive result of the Lagrangian operator applied to the Einsteinian perturbation $\mathfrak{D}_h$ expressed in terms of the operator of the relative motion

$$\text{(VI. 39)} \quad \mathrm{B}_{h\,|\,i} \mathfrak{D}_h = (-1)^{h+1} \lambda_{h+1}^2 (\lambda_{h+1} + 1) \mathrm{B}_i (\tfrac{1}{2} \beta^2 \gamma - e\gamma)$$
$$+ (-1)^{h+1} \lambda_{h+1} (\lambda_{h+1} + 1) \mathrm{B}_i (-\tfrac{1}{2} \gamma^2)$$
$$+ (-1)^{h+1} (\lambda_{h+1} + p) \mathrm{B}_i (\beta^2 \gamma)$$
$$+ (-1)^{h+1} p (\lambda_{h+1} - \tfrac{1}{2} \lambda_h) \mathrm{B}_i (\gamma^2 + 2e\gamma)$$
$$+ (-1)^{h+1} p \lambda_h \mathrm{B}_i \left(\frac{1}{6} \frac{\gamma^3}{a^2} \right).$$

Now, it will be useful to recall that, since

$$\lambda_0 + \lambda_1 = 1,$$

one has

$$\lambda_0^2 + \lambda_1^2 = 1 - 2p, \quad \lambda_0^3 + \lambda_1^5 = 1 - 3p, \quad p = \lambda_0 \lambda_1 .$$

To end up with the Lagrangian function $\mathfrak{L}$ of the relative motion, we must calculate the expression

$$\begin{aligned}
\text{(VI.40)} \quad & (-1)^{h+1} \mathrm{B}_{h\,|\,i} \mathfrak{L}_h + (-1)^h \mathrm{B}_{h+1\,|\,i} \mathfrak{L}_{h+1} \\
&= (-1)^h \{\mathrm{B}_{h+1\,|\,i} \mathfrak{L}_{h+1} - \mathrm{B}_{h\,|\,i} \mathfrak{L}_h\} \\
&= (-1)^h \{\mathrm{B}_{h+1\,|\,i} \mathfrak{N}_{h+1} - \mathrm{B}_{h\,|\,i} \mathfrak{N}_h\} \\
&\quad + (-1)^h \{\mathrm{B}_{h+1\,|\,i} (\tfrac{1}{2} \mathfrak{N}_{h+1}^2) - \mathrm{B}_{h\,|\,i} (\tfrac{1}{2} \mathfrak{N}_h^2)\} \\
&\quad + (-1)^h \{\mathrm{B}_{h+1\,|\,i} (-\gamma_{h+1}^2 - \gamma_{h+1} \overset{*}{\gamma}_h) - \mathrm{B}_{h\,|\,i} (-\gamma_h^2 - \gamma_h \overset{*}{\gamma}_{h+1})\} \\
&\quad + (-1)^h \{\mathrm{B}_{h+1\,|\,i} \mathfrak{D}_{h+1}^{(1)} - \mathrm{B}_{h\,|\,i} \mathfrak{D}_h^{(1)}\} \\
&\quad + (-1)^h \{\mathrm{B}_{h+1\,|\,i} \mathfrak{D}_{h+1}^{(2)} - \mathrm{B}_{h\,|\,i} \mathfrak{D}_h^{(2)}\} .
\end{aligned}$$

One has

$$\begin{aligned}
& (-1)^h \{\mathrm{B}_{h+1\,|\,i} \mathfrak{N}_{h+1} - \mathrm{B}_{h\,|\,i} \mathfrak{N}_h\} \\
&= (-1)^h \{(-1)^h \lambda_h \mathrm{B}_i \mathfrak{N} + (-1)^h \lambda_{h+1} \mathrm{B}_i \mathfrak{N}\} = \mathrm{B}_i \mathfrak{N} , \\
& (-1)^h \{\mathrm{B}_{h+1\,|\,i} (\tfrac{1}{2} \mathfrak{N}_{h+1}^2) - \mathrm{B}_{h\,|\,i} (\tfrac{1}{2} \mathfrak{N}_h^2)\} \\
&= (-1)^h \{(-1)^h (\lambda_h^2 + \lambda_h^3) \mathrm{B}_i (\tfrac{1}{2} \beta^2 \gamma - e\gamma) \\
&\qquad + (-1)^h (\lambda_{h+1}^2 + \lambda_{h+1}^3) \mathrm{B}_i (\tfrac{1}{2} \beta^2 \gamma - e\gamma)\} \\
&= (1 - \tfrac{5}{2} p) \mathrm{B}_i (\beta^2 \gamma - 2 e\gamma) , \\
& (-1)^h \{\mathrm{B}_{h+1\,|\,i} (-\gamma_{h+1}^2 - \gamma_{h+1} \overset{*}{\gamma}_h) - \mathrm{B}_{h\,|\,i} (-\gamma_h^2 - \gamma_h \overset{*}{\gamma}_{h+1})\} \\
&= (-1)^h \{(-1)^h (\lambda_h + \lambda_h^2) \mathrm{B}_i (-\tfrac{1}{2} \gamma^2) \\
&\qquad + (-1)^h (\lambda_{h+1} + \lambda_{h+1}^2) \mathrm{B}_i (-\tfrac{1}{2} \gamma^2)\} = \mathrm{B}_i [(p-1) \gamma^2] ,
\end{aligned}$$

$$(-1)^h\{\mathrm{B}_{h+1\,|\,i}\mathfrak{D}^{(1)}_{h+1} - \mathrm{B}_{h\,|\,i}\mathfrak{D}^{(1)}_{h}\}$$
$$= (-1)^h\{(-1)^h(\lambda_h + p)\,\mathrm{B}_i(\beta^2\gamma) + (-1)^h p(\lambda_h - \tfrac{1}{2}\lambda_{h+1})\,\mathrm{B}_i(\gamma^2 + 2e\gamma)$$
$$+ (-1)^h(\lambda_{h+1} + p)\,\mathrm{B}_i(\beta^2\gamma)$$
$$+ (-1)^h p(\lambda_{h+1} - \tfrac{1}{2}\lambda_h)\,\mathrm{B}_i(\gamma^2 + 2e\gamma)\}$$
$$= \mathrm{B}_i[(1 + 2p)\beta^2\gamma + \tfrac{1}{2}p(\gamma^2 + 2e\gamma)],$$
$$(-1)^h\{\mathrm{B}_{h+1\,|\,i}\mathfrak{D}^{(2)}_{h+1} - \mathrm{B}_{h\,|\,i}\mathfrak{D}^{(2)}_{h}\}$$
$$= (-1)^h\left\{(-1)^{h+1}p\lambda_{h+1}\frac{\partial}{\partial x^i}\left(\frac{1}{6}\frac{\gamma^3}{a^2}\right) - (-1)^h p\lambda_h\frac{\partial}{\partial x^i}\left(\frac{1}{6}\frac{\gamma^3}{a^2}\right)\right\}$$
$$= -p\frac{\partial}{\partial x^i}\left(\frac{1}{6}\frac{\gamma^3}{a^2}\right) = \mathrm{B}_i\left(\frac{1}{6}p\frac{\gamma^3}{a^2}\right).$$

The preceding calculations give us the Lagrange function

$$\mathfrak{L} = \mathfrak{N} + \mathfrak{D}, \tag{VI.41}$$

where

$$\mathfrak{N} = \tfrac{1}{2}\beta^2 + \gamma \tag{VI.28}$$

and

$$\mathfrak{D} = \tfrac{1}{2}(4 - p)\beta^2\gamma + 6e(p - \tfrac{1}{3})\gamma + (\tfrac{3}{2}p - 1)\gamma^2 + \frac{1}{6}\frac{p}{a^2}\gamma^3, \tag{VI.42}$$

for the motion of the body P_1 relative to P_0.
Or it may be written in another form

$$\mathfrak{L} = \tfrac{1}{2}\Phi\beta^2 + \psi, \tag{VI.43}$$

where Φ and ψ designate the functions

$$\Phi = 1 + (4 - p)\gamma, \tag{VI.44}$$

$$\psi = \gamma + 6e(p - \tfrac{1}{3})\gamma + (\tfrac{3}{2}p - 1)\gamma^2 + \frac{1}{6}\frac{p}{a^2}\gamma^3. \tag{VI.45}$$

9. AN EQUIVALENCE THEOREM. Our Lagrangian problem allows the energy integral

$$\frac{1}{2}\Phi\beta^2 - \psi = e, \tag{VI.46}$$

where the constant differs from that of the Keplerian motion by a quantity of an order higher than the first (in our approximation we are permitted to replace this constant by the last-mentioned quantity).

According to a well-known theorem in Analytical Mechanics,[1] a bundle of trajectories of the above-mentioned two-body problem corresponding to a specific value of the constant e is given by the geodesics using

$$ds^2 = 2\,\Phi(\psi + e)\,dl_0^2, \tag{VI.47}$$

where the linear element of Euclidean space is designated by dl_0.

It is likewise a bundle of trajectories of zero energy in a motion in ordinary space where the potential of the forces would be

$$\text{(VI.48)}\quad \mathrm{F} = \Phi(\psi + e) = \{1 + (4 - p)\gamma\}$$

$$\times\left[\left(e + \gamma + 6e(p - \tfrac{1}{3})\gamma + (\tfrac{3}{2}p - 1)\gamma^2 + \frac{1}{6}\frac{p}{a^2}\gamma^3\right)\right],$$

again equal, except for quantities of a higher order, to

$$\mathrm{F} = e + \gamma + (3 + \tfrac{1}{2}p)\,\gamma^2 + \frac{1}{6}\frac{p}{a^2}\,\gamma^3. \tag{VI.48$'$}$$

This auxiliary problem allows the two integrals of the active forces (with zero constant) and of the areas that are written

$$\beta^2 = 2\,\mathrm{F}, \tag{VI.49}$$

[1] Levi-Civita and Amaldi, *Lezioni di meccanica razionale,* Bologna, Zanichelli, 1927, **2**, capitolo X.

that is

(VI. 49′) $$\dot{r}'^2 + r^2\dot{\theta}'^2 = 2\,\mathrm{F}$$

and

(VI. 25′) $$r^2\dot{\theta}' = \frac{l}{a}.$$

By eliminating Römerian time we get

(VI. 50) $$\gamma^2 + \left(\frac{d\gamma}{d\theta}\right)^2 = 2\,a^2\,\mathrm{F},$$

whence, as a final result,

(VI. 50′) $$\left(\frac{d\gamma}{d\theta}\right)^2 = 2\,a^2\,e + 2\,a^2\,\gamma - \gamma^2 + 2\,a^2(3 + \tfrac{1}{2}\,p)\,\gamma^2 + \tfrac{1}{3}\,p\,\gamma^3,$$

This equation has the form

(VI. 51) $$\left(\frac{d\gamma}{d\theta}\right)^2 = f(\gamma) + g(\gamma),$$

where

(VI. 52) $$f(\gamma) = -\,\gamma^2 + 2\,a^2\,\gamma + 2\,a^2\,e$$

is the quadratic function that one would have in Keplerian motion, which is – as we have seen – of the second order in our notation, and where

(VI. 53) $$g(\gamma) = 2\,a^2(3 + \tfrac{1}{2}\,p)\,\gamma^2 + \tfrac{1}{3}\,p\,\gamma^3$$

is the term resulting from the Einsteinian correction (which is of the third order). At this point it is appropriate to insert a general proposition on the calculus of periods.

10. DIGRESSION CONCERNING CALCULATION OF THE PERIOD OF A SOLUTION OF WEIERSTRASS'S DIFFERENTIAL EQUATION:

$$\left(\frac{dq}{dt}\right)^2 = \mathfrak{F}(q). \tag{VI. 54}$$

It is a well-known result, due to Weierstrass, that if the function $\mathfrak{F}(q)$ is continuous within the interval of $a \leqslant q \leqslant b$, where it is positive and simply reduces to zero only for $q = a$, $q = b$, q (t) is a periodic function of t, provided that its initial value be included between a and b, and the period is given by

$$\tau = 2\int_a^b \frac{dq}{\sqrt{\mathfrak{F}(q)}}. \tag{VI. 55}$$

In the elementary case

$$\mathfrak{F}(q) = \omega^2 (q - a)(b - q), \tag{VI. 56}$$

one has, as a result, the period

$$\tau_0 = \frac{2\pi}{\omega}. \tag{VI. 57}$$

Now, in pursuing our objective it is of interest to consider the function

$$\mathfrak{F}(q) = \omega^2 (q - a)(b - q) + g(q) \tag{VI. 58}$$

treating g (q) as being of a higher order with respect to the first term throughout the interval within a, b, and, naturally, treating g (q) as a continuous function within the interval a', b', which contains within it the interval a, b.

In that case, one can give, following Levi-Civita[1], a remarkable expression of the period by means of the following formula:

$$\tau = \tau_0 + \frac{1}{\omega^3}\int_0^\pi \frac{\mathrm{G}(q, a, b)}{\mathfrak{B}(q, a, b)}\, d\theta, \tag{VI. 59}$$

[1] 'Sul calcolo effettivo del periode in un caso tipico di prima approssimazione', *Revista de Ciencia*, Lima (Peru), 1937, p. 421.

where the variables g and θ are connected by the relation

$$q = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b)\cos\theta \tag{VI. 60}$$

and

$$\mathrm{G}(q,a,b) = \begin{vmatrix} g(q) & q & 1 \\ g(a) & a & 1 \\ g(b) & b & 1 \end{vmatrix}, \quad \mathfrak{B}(q,a,b) = (b-a)(q-a)(b-q). \tag{VI. 61}$$

This formula (where the function under the integration sign has the appearance of $\frac{0}{0}$ at the extremities of the interval of integration) may be transformed[1] in such a way that it will be far handier in our calculations.

In exact terms, we have

$$\tau = \tau_0 + \frac{1}{\omega^3}\int_0^{\pi} g''(q)\sin^2\theta\, d\theta, \tag{VI. 62}$$

assuming, of course, that $g\ (q)$ possesses an integrable second derivative. That is the form in which we will apply the formula in what follows.

11. ADVANCE OF THE PERIHELION IN THE CASE OF TWO FINITE MASSES: A SUPPLEMENTARY TERM IN THE EINSTEIN FORMULA FOR THE ADVANCE OF THE PERIHELION. Observations that had long been on record persistently indicated a slight displacement in the perihelion of certain planets, especially Mercury, in the plane of the orbit and in the direction of the motion, and this displacement could not be explained by the classical theory of planetary perturbations. According to Leverrier and the most authoritative successive evaluations, this residual displacement, as yet unknown, might be estimated at 42″ per century.

Einstein was the first to recognize that the advance of the perihelion of

[1] Cartovitch, 'Sul calcolo effettivo del periodo del moto perturbato in un caso tipico di prima approssimazione', *Rend. Lincei,* 1938.
A noteworthy expression for the remaining elements in these formulas has recently been developed by Sansone. See his Note.

each planet around the Sun appeared as a mathematical consequence of his new law of gravitation, which gives for this shift almost exactly the value that is actually observed. But his analysis is valid only in the case of a body of negligible mass gravitating in the field of a body (spherical) of finite mass.

Our theory leads us to perfect this basic result, within the indicated order of approximation, in such a way that the more refined form will be valid for bodies of comparable mass and will be everywhere verifiable by astronomical observations of double stars after a great many periods.

We are going to demonstrate that the advance of the perihelion can be increased by a maximum of one-twelfth of the value given by Einstein.

For the projected application of formula (VI. 62), it is necessary to recall that the classical expression of γ as a function of θ, that is, in Keplerian motion, is

(VI. 63) $$\gamma = \frac{l}{r} = \frac{l}{p} + \frac{el}{p}\cos\theta,$$

where p designates the parameter of the Keplerian ellipse and e its eccentricity.

The period with respect to the angle (real) θ is in conformity with

$$\tau_0 = 2\pi.$$

In this case of disturbed motion, one has

$$g''(\gamma) = 4a^2(3 + \tfrac{1}{2}p) + 2p\gamma,$$

as results from the double differentiation of the function $g(\gamma)$ (VI. 53). Then

$$g''(\gamma)\sin^2\theta = 4a^2(1 + \tfrac{1}{2}p)\sin^2\theta + 2p\left(\frac{l}{p} + \frac{el}{p}\cos\theta\right)\sin^2\theta.$$

Integrating for the interval $0, \pi$, and noting that

$$a^2 = \frac{l}{p},$$

we find, by applying formula (VI. 62),

$$\tau = 2\pi + \sigma, \tag{VI. 64}$$

$$\sigma = \sigma_e \left(1 + \frac{p}{3}\right), \tag{VI. 65}$$

where we have designated by

$$\sigma_e = 6\pi a^2 = 6\pi \frac{l}{p} \tag{VI. 66}$$

the advance in the perihelion discovered by Einstein.

Recalling that $p = \lambda_0 \lambda_1$ and that $\lambda_0 + \lambda_1 = 1$, one sees that the maximum of $\frac{1}{3}p$ is $\frac{1}{12}$, Q. E. D.

12. THE APSIDAL PERIOD AND SIDEREAL NOTATION. In the foregoing sections we have, within the approximation adopted, solved the two-body problem in two steps:

1. determining the trajectory;
2. determination of the law of time.

The result of the first step has led us to recognize the spiraloid nature of the orbit as being almost exactly assimilable with a slowly turning ellipse, with its most salient geometrical characteristic being the slight advance of the perihelion, which is rigorously constant, even within the second approximation.

In that approximation the time law may be reduced to the calculation of the apsidal period, that is, of the constant time-interval T which elapses between two consecutive passages of one (any one) of the bodies from one perihelion to another.

If, however, we take into account the fact that the distance r of the two bodies is a rigorously periodic function of the real angle θ, having the period

$$\tau = 2\pi + \sigma,$$

where σ designates the advance of the perihelion – that is, double the apsidal angle – and if we consider that according to formula (VI. 25′) (integral of the areas) θ is a periodic function of x^0, there is then reason to fix our attention on the duration of a sidereal revolution, that is, on the time (likewise constant within the limits of our approximation) that it takes the body to resume the same orientation in space.

To determine the relation between the sidereal period T and the apsidal period T_k, let us take as our starting point the first integral

$$\frac{\partial \mathfrak{L}}{\partial \dot{\theta}} = \text{const} \tag{VI. 67}$$

deduced from the function

$$\mathfrak{L} = \tfrac{1}{2}\Phi\beta^2 + \psi,$$

because of the cyclic character of the variable θ.

It must here be noted that the constant may be replaced by a, which differs from it by a higher-order quantity, and one has

$$\Phi r^2 \frac{d\theta}{dx^0} = \frac{l}{a}, \tag{VI. 67'}$$

whence one deduces that

$$dt = \frac{a}{lc}(1 + 4\gamma) r^2 \, d\theta. \tag{VI. 68}$$

Extending the integration to a period of θ gives us

$$\mathrm{T} = \frac{a}{lc}\int_0^{2\pi+\sigma} (1 + 4\gamma) r^2 \, d\theta = \mathrm{T}_k + \frac{a}{lc}\sigma (r^2)_{\theta=0} + \frac{4a}{lc}\int_0^{2\pi} \gamma r^2 \, d\theta, \tag{VI. 69}$$

where

$$\mathrm{T}_k = \frac{n}{2\pi} \tag{VI. 70}$$

is the sidereal period.

As we know, one has

(VI. 71) $$r_{\theta=0} = a(1-e).$$

In addition, by virtue of Kepler's equation one has

(VI. 72) $$n\,dt = (1 - e\cos u)\,du,$$

and hence

$$\frac{4a}{lc}\int_0^{2\pi} \gamma r^2\, d\theta = \frac{4}{n}\int_0^{\mathrm{T}_k} \gamma n\, dt = \frac{4}{n}\frac{l}{r}\frac{r}{a}\int_0^{2\pi} du = \frac{8\pi l}{an}.$$

Thus

(VI. 73) $$\mathrm{T} = \mathrm{T}_k + \frac{8\pi l}{an} + \frac{\sigma}{\mathrm{C}} a^2 (1-e)^2.$$

Let us then transform the last term, remembering that

(VI. 74) $$\frac{a^2}{\mathrm{C}} = \frac{1}{n\sqrt{1-e^2}}.$$

One has

$$\sigma\frac{(1-e)^2}{n\sqrt{1-e^2}} = 6\pi l(1+\tfrac{1}{3}p)\frac{(1-e)^2}{an(1-e^2)\sqrt{1-e^2}} = \mathrm{T}_k\frac{3l}{a}(1+\tfrac{1}{3}p)\frac{(1-e)^2}{(1-e^2)^{\frac{3}{2}}}.$$

Finally

(VI. 75) $$\mathrm{T} = \mathrm{T}_k(1+b),$$

having put

(VI. 76) $$b = \frac{l}{a}\left[4 + 3(1+\tfrac{1}{3}p)\frac{\sqrt{1-e}}{(1-e)^{\frac{3}{2}}}\right],$$

a small numerical quantity defined as a function of the elements a, e of the osculating Keplerian orbit, 1 being the gravitational radius and p the product $\lambda_0 \lambda_1$.

3. THE MOTION OF THE CENTER OF GRAVITY

1. THE ACCELERATION OF THE CENTER OF GRAVITY IN THE SECOND APPROXIMATION. If one recalls that

$$\lambda_h = \frac{m_h}{m} \qquad (h = 0, 1),$$

and designating the absolute velocity of the center of gravity by w, one has

$$\text{(VI.77)} \qquad \dot{w}_i = \lambda_h \dot{\beta}_{h\,|\,i} + \lambda_{h+1} \dot{\beta}_{h+1\,|\,i}.$$

As we have already repeatedly remarked, the right-hand member is zero in the Newtonian approximation. In order to evaluate the Einsteinian correction, it is obviously sufficient to pursue the calculation further, taking the immediately higher-order terms into account. According to our relativistic equations (Section 1, no. 2) of the absolute motion of the two bodies, one has first of all

$$\text{(VI.78)} \qquad \begin{aligned} \dot{w}_i &= \mathrm{B}_{h\,|\,i}(\lambda_h \mathfrak{N}_h) + \mathrm{B}_{h+1\,|\,i}(\lambda_{h+1} \mathfrak{N}_{h+1} \\ &= -\lambda_h \mathrm{B}_{i\,|\,h} \mathfrak{D}_h - \lambda_{p+1} \mathrm{B}_{h+1\,|\,i} \mathfrak{D}_{h+1}. \end{aligned}$$

Now let us put

$$\text{(VI.79)} \qquad \delta = (-1)^h (\lambda_h - \lambda_{h+1}) = \lambda_0 - \lambda_1$$

We find

(VI.80)

$$\begin{aligned} \dot{w}_i = {} & (-1)^h p(\lambda_{h+1}^2 + \lambda_{h+1}) \mathrm{B}_i(\tfrac{1}{2}\gamma\beta^2 - e\gamma) \\ & + (-1)^{h+1} p(\lambda_h^2 + \lambda_h) \mathrm{B}_i(\tfrac{1}{2}\gamma\beta^2 - e\gamma) \\ & + (-1)^h p(1 + \lambda_{h+1}) \mathrm{B}_i(-\tfrac{1}{2}\gamma^2) \\ & + (-1)^{h+1} p(1 + \lambda_h) \mathrm{B}_i(-\tfrac{1}{2}\gamma^2) + (-1)^h (1 + \lambda_h) \mathrm{B}_i(\beta^2\gamma) \\ & + (-1)^h p(p - \tfrac{1}{2}\lambda_h^2) \mathrm{B}_i(\gamma^2 + 2e\gamma) + (-1)^{h+1} p(1 + \lambda_{h+1}) \mathrm{B}_i(\beta^2\gamma) \\ & + (-1)^{h+1} p(p - \tfrac{1}{2}\lambda_{h+1}^2) \mathrm{B}_i(\gamma^2 + 2e\gamma) + (-1)^h \mathrm{B}_i\left(\frac{1}{6}\frac{\gamma^3}{a^2}\right) \end{aligned}$$

$$+ (-1)^{h+1}\, p\, \mathrm{B}_i\left(\frac{1}{6}\frac{\gamma^3}{a^2}\right)$$

$$= -2\, p\,\delta \mathrm{B}_i(\tfrac{1}{2}\beta^2\gamma - e\gamma) - p\,\delta\mathrm{B}_i(-\tfrac{1}{2}\gamma^2)$$

$$+ p\,\delta\mathrm{B}_i(\beta^2\gamma) - \tfrac{1}{2} p\,\delta\mathrm{B}_i(\gamma^2 + 2e)$$

$$= p\,\delta\mathrm{B}_i(e\gamma) = -ep\,\delta\frac{\partial\gamma}{\partial x^i}.$$

2. THE ABSENCE OF A SECULAR TERM. Now we have

$$\frac{d\theta}{dx^0} = \frac{l}{a}\frac{1}{r^2},$$

so

(VI. 81) $$\dot{w}_i = \frac{dw_i}{d\theta}\frac{d\theta}{dx^0} = \frac{l}{a}\frac{1}{r^2}\frac{dw_i}{d\theta} = -ep\,\delta l\frac{\partial\frac{1}{r}}{\partial x^i}$$

and consequently

(VI. 82) $$\frac{dw_i}{d\theta} = eap\,\delta\frac{\partial r}{\partial x_i} \qquad (i = 1,2,3),$$

that is

(VI. 83) $$\frac{dw_1}{d\theta} = h\cos\theta, \quad \frac{dw_2}{d\theta} = h\sin\theta, \quad \frac{dw_3}{d\theta} = 0,$$

having put

(VI. 84) $$h = eap\,\delta.$$

We conclude that the acceleration of the center of gravity is still within the plane of the orbits and depends periodically on θ, which, in its turn, is a periodic function of time – thus bringing about the disappearance of the secular term. And that result is in keeping with the conclusions that have been established by various means – on one hand by Robertson, following a new theory of Einstein's, and on the other by Eddington and Clark.

4. AN EXTENSION OF THE DEVICE OF SECTION 1 EMPHASIZING THE ELIMINATION PRINCIPLE IN THE n-BODY PROBLEM

We stated (Chapter V, Section 4, no. 6) that it is possible, by a slight alteration of the λ constants characterizing the masses, in our approximation, to reduce the relativistic problem of several gravitating bodies to that of the motion of point-masses, which is dominated by an appropriate Lagrange function.

In other words, one can come up with an effective expression of what – as we previously stated – Brillouin has called the principle of elimination, according to which, even in Relativistic Mechanics, everything takes place as if each body in the system exerted no influence on the motion of its center of gravity, under certain conditions about to be explained.

We have demonstrated that this is possible in the case of two bodies – the case set forth in Chapter VI.

Now we propose to demonstrate that the principle of elimination may in general be achieved by the same artifice that was employed in the two-body problem.

For every body C_h one introduces a constant σ_h of the first order. The function

$$(1 + \sigma_h)\,\mathfrak{L}_h$$

which reduces to

$$(1 + \sigma_h)\,\mathfrak{N}_h + \mathfrak{D}_h' + \mathfrak{D}_h''.$$

except for terms of a higher order, produces the same equations as those of our problem.

Now one can determine (and in this possibility resides the importance of the device) the constants σ_h in such a way that the $2\,n$ constants ω_h, χ_h, which depend upon the extension and structure of the bodies, likewise do not enter into the differential equations of the motion.

In order to see this one has only to consider the expression

$$(1 + \sigma_h)\,\mathfrak{N}_p + \mathfrak{D}_h'',$$

since $\mathfrak{D}_h'$ is independent of the aforementioned constants.

By virtue of our hypotheses, one can substitute

$$\tfrac{1}{2}\beta_h^2 + \gamma_h'$$

for $\mathfrak{N}_h$, which differs from that expression only by an additive constant. Then, if we take into account the expressions of γ_h' and of $\mathfrak{D}_h''$, one has

$$(1+\sigma_h)\mathfrak{N}_h + \mathfrak{D}_h'' = \tfrac{1}{2}\beta_h^2 + \tfrac{1}{2}\sigma_h\beta_h^2 + l(1+\sigma_h)\sum_{0}^{n-1}{}'^{(h)}_{\nu} \frac{\lambda_\nu}{r(P_\nu, P_h)}$$

$$- l\sum_{0}^{n-1}{}'^{(h)}_{\nu} \frac{\lambda_\nu(\chi_\nu + 2\omega_h)}{r(P_\nu, P_h)} - \omega_h\beta_h^2$$

$$= \tfrac{1}{2}\beta_h^2 + (\tfrac{1}{2}\sigma_h - \omega_h)\beta_h^2$$

$$+ l\sum_{0}^{n-1}{}'^{(h)}_{\nu} \frac{\lambda_\nu(1-\chi_\nu) + \lambda_\nu(\sigma_h - 2\omega_h)}{r(P_\nu, P_h)}.$$

Obviously one has only to put

$$\sigma_h = 2\omega_h \qquad (h = 0, 1, \ldots, n-1)$$

in order to obtain

$$(1+\sigma_h)\mathfrak{N}_h + \mathfrak{D}_h'' = \tfrac{1}{2}\beta_h^2 + l\sum_{0}^{n-1}{}'^{(h)}_{\nu} \frac{\lambda_\nu'}{r(P_\nu, P_h)},$$

with

$$\lambda_\nu' = \lambda_\nu(1-\chi_\nu) \qquad (\nu = 0, 1, \ldots, \nu - 1).$$

As we know, the constants ω_h, χ_h are definitely of the first order; while the terms of $\mathfrak{D}_h'$ are all of a higher order; and that permits us to replace the λ's by λ''s. And thereafter the prime may be eliminated throughout. Thus, we end up with the following basic result:

In the relativistic problem of several (let us say n) gravitating bodies, every body P_h of the system can be made to depend on a Lagrange function $\bar{\mathfrak{L}}_h$ ($h = 0, 1, \ldots, n-1$) having the following structure:

$$\bar{\mathfrak{L}}_h = \mathfrak{N}_h + \mathfrak{D}_h,$$

where

$$\mathfrak{N}_h = \tfrac{1}{2}\beta_h^2 + \gamma_h ,$$

$$\gamma_h = l \sum_{\nu\,0}^{n-1}{}'^{(h)} \frac{\lambda_\nu}{r(\mathrm{P}_\nu, \mathrm{P}_h)} \; ;$$

$$\mathfrak{D}_h = \tfrac{1}{2}\mathfrak{N}_h^2 - \gamma_h^2 + \zeta_h + \gamma_h \beta_h^2 - 4\textstyle\sum_i \gamma_{h\,|\,i}\beta_{h\,|\,i} ,$$

$$\zeta_h = \varphi_h + \psi_h + \upsilon_h ,$$

$$\varphi_h = -\,l^2 \sum_{\nu\,0}^{n-1}{}^{(h)} \frac{\lambda_\nu}{r(\mathrm{P}_\nu, \mathrm{P}_h)} \sum_{\rho\,0}^{n-1}{}'^{(\nu)} \frac{\lambda_\rho}{r(\mathrm{P}_\nu, \mathrm{P}_\rho)} ,$$

$$\psi_h = \tfrac{3}{2}\, l \sum_{\nu\,0}^{n-1}{}^{(h)} \frac{\lambda_\nu \beta_\nu^2}{r(\mathrm{P}_\nu, \mathrm{P}_h)} ,$$

$$\upsilon_h = \tfrac{1}{2}\, l \frac{\partial^2}{\partial x^{0\,2}} \sum_{\nu\,0}^{n-1}{}'^{(h)} \lambda_\nu\, r(\mathrm{P}_\nu, \mathrm{P}_h) ,$$

$$\gamma_{h\,|\,i} = l \sum_{\nu\,0}^{n-1}{}'^{(h)} \frac{\lambda_\nu \beta_{h\,|\,i}}{r(\mathrm{P}_\nu, \mathrm{P}_h)} ,$$

the extension and the structure of the bodies enter the picture only through the λ parameters, which differ very little from the masses (compared to the total mass of the system, so that

$$\sum_{h\,0}^{n-1} \lambda_h = 1) .$$