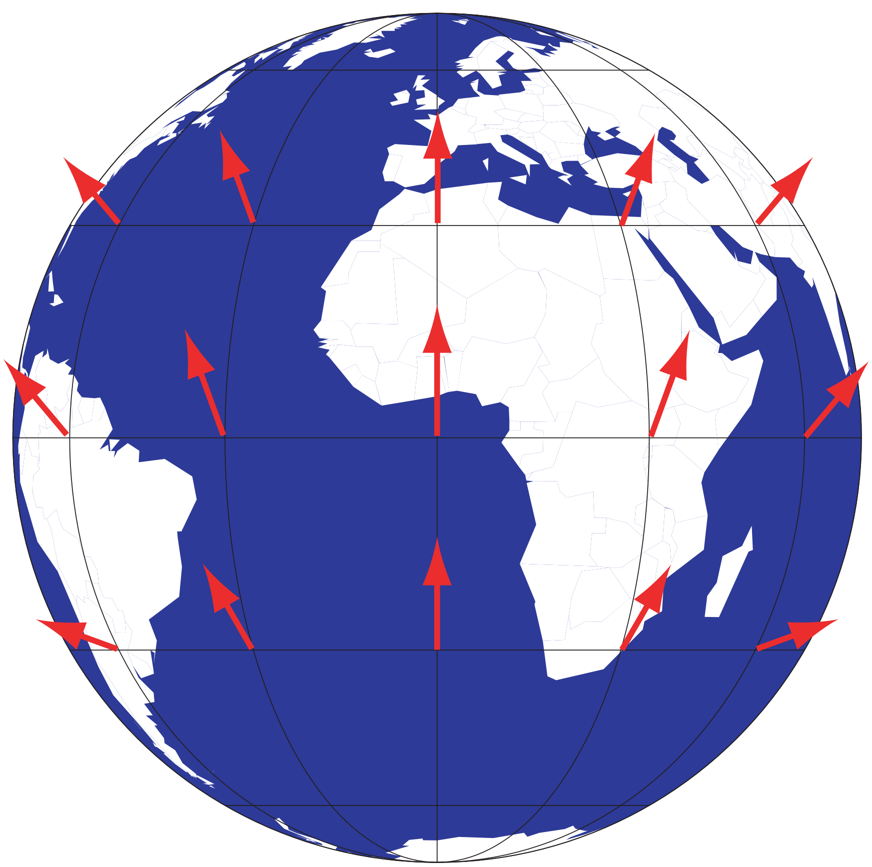
The Mathematics of Continuity

From General Relativity to Classical Dynamics



Albert Tarantola

WARNING!

This is an unfinished text

It was a project for a book, but,
as my interests shifted towards a different project,
I decided to stop this, and start writing my
Elements for Physics.

The occasional reader is invited to have a look
at the Preface and Table of Contents.

It may well be that she/he finds
useful some parts of this text.

The Mathematics of Continuity:

from General Relativity
to Classical Dynamics

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Preface

The theory of continuous media is not of universal validity but it is applicable in numerous domains. Not only we can analyze the dynamics of fluids or solids, but we know that even a good model of the space-time itself can be based on the theory.

The notions necessary to describe general coordinate systems in Euclidean spaces —or curved surfaces embedded in them— are the same as those needed to describe general (non Euclidean) spaces, and are the object of the *differential geometry*. As our physical space is not Euclidean, the study of the differential geometry is essential. General spaces may have curvature and torsion, and the tensors needed to describe them satisfy some important identities: the *Bianchi identities*.

As demonstrated by Einstein, the geometrical analysis of space-time introduces some tensors that, when identified with material properties, give —because of mathematical properties valid in any space— not only the fundamental equations of conservation of mass, linear and angular momentum, but also describe the gravitational interactions.

The equations so obtained describe the dynamics of continuous media, but leave some degrees of freedom: those that are locked by the *constitutive equations* that allow the conservation equations to describe different types of physical media with different rheologies.

Einstein's gravitation theory identifies mass with space-time curvature. But, originally, the theory assumed that space-time had no torsion. This is because when the theory was developed (1915) [note: give here a historically accurate data] it was not yet realized that, in addition to mass, particles have another intrinsic property: spin (Stern and Gerlach's paper dates from 1922). There has been so many failed attemps to "geometrize" other particle properties, like the electric charge, that I see as a social mystery the fact that it is not widely recognized how the simple dropping of the assumption *no torsion* leads to a consistent theory where, while mass produces curvature of space-time, spin produces space-time torsion. This theory is not only more general, but more equilibrated that the traditional one (treating in an equivalent way curvature and torsion). An interesting result is that if mass and space-time curvature are related through a fundamental constant (the gravitational constant G), spin and space-time torsion are related through the same constant. This is enough for trying to put this theory forward.

An interesting by-product of the theory is that it naturally leads, in the classical limit, to the dynamics of media where it is not assumed that stresses are symmetric, and where the "microtorques" existing, for instance, in magnetized media, appear naturally.

Why the theory presented in this book is not widely accepted as *the* gravitation theory (or, equivalently, as *the* theory of continuous media)? This is possibly due to the hiatus existing between 4-D and 3-D (plus time) formulations. It is true that exact 3-D formulations are so cumbersome that one may give up any development before the understanding of the results, but it is also true that only 3-D formulations allow the introduction of the notion of "force", that gives an intuitive sense to the gravitational interactions. Also, how could one realize the importance of the fact mentioned above (the same constant G couples mass to curvature and spin to torsion) if, as in usual formulations, a system of units is taken where $G = c = 1$?

This book will start describing the geometry of a general manifold (any dimension, with curvature and torsion, with or without metric). Then the 4-D physics of space-time will be introduced. It will then be acknowledged that one of these 4 dimensions is special (the time), and an exact 3-D formulation will be deduced. We will then see how complex can be the gravitational interactions: tens of different forces exist in a continuous medium submitted to gravitational interactions only. No one, to our knowledge, has explored in depth this part of the theory.

I accept in full the viewpoint expressed by John Maddox in a leading article of the journal Nature (October 22, 1992):

> *A model is an approximate description of reality [...] and a good model is a model which, on the one hand, is comprehensible in the sense of providing an image for the mind and which, at the same time, is calculable. [...] It is difficult to think [...] of a more generally applicable model of the real world unless it is the tautologous continuum model of three-dimensional reality. Formally, at least, you can calculate almost anything that way. The deformation of a solid object under the influence of an external field of force? No problem. [...] But a little reflection will show that these elegant ways of talking, which brought great fame to late-Victorian Cambridge, are not much more than ways of doing calculus in three dimensions (whence the term "tautologous"). They have little to say about physics.*

In this book, I take more seriosly the definition and interpretation of the geometry (what is curvature, torsion, parallel transportation, true derivatives) than the definition of mechanical quantities, that will be introduced "by analogy" to classical physics. Our problem here will not be to analyze the experimental limits of the theory, but rather to give a complete formal development of it.

I have been helped in this work by the nice environment existing at the Institut de Physique du Globe de Paris. Also, my friend Bartolomé Coll has always had time to listen to my (repetitive) questions, and has always given me the good advice for a safe advance through the conceptual jungle of non directly intuitive space-times. My wife, Maria Zamora, has always actively protected my working hours against the assaults of the exterior world.

Paris, November 1995
A.T.

Contents

Part I

First Part: Geometry

Chapter 1

Tensor fields

The first part of this book recalls some of the mathematical tools developed to describe the geometric properties of a space. By "geometric properties" one understands those properties that Pythagoras (6th century B.C.) or Euclid (3rd century B.C.) were interested on. The only major conceptual progress since those times has been the recognition that the *physical* space may not be Euclidean, but may have curvature and torsion, and that the behaviour of clocks depends on their space displacements.

Still these representations of the space accept the notion of continuity (or, equivalently, of differentiability). New theories are being developed dropping that condition (e.g. Nottale, 1993). They will not be examined here.

A mathematical structure can describe very different physical phenomena. For instance, the structure "3-D vector space" may describe the combination of forces being applied to a particle, as well as the combination of colors. The same holds for the mathematical structure "differential manifold". It may describe the 3-D physical space, any 2-D surface, or, more importantly, the 4-dimensional space-time space brought into physics by Minkowski and Einstein. The same theorem, when applied to the physical 3-D space, will have a geometrical interpretation (stricto sensu), while when applied to the 4-D space-time will have a dynamical interpretation.

The aim of this first chapter is to introduce the fundamental concepts necessary to describe geometrical properties: those of tensor calculus. Many books on tensor calculus exist. Then, why this chapter here? Essentially because no uniform system of notations exist (indices at different places, different signs…). It is then not possible to start any serious work without fixig the notations first. This chapter does not aim to give a complete discussion on tensor calculus. Among the many books that do that, the best are (of course) in French, and Brillouin (1960) is the best among them. Many other books contain introductory discussions on tensor calculus. Weinberg (1972) is particularly lucid. I do not pretend to give a complete set of demonstrations, but to give a complete description of interesting properties, some of which are not easily found elsewhere.

Perhaps original is a notation proposed to distinguish between densities and capacities. While the trick of using indices in upper or lower position to distinguish between tensors or forms (or, in metric spaces, to distinguish between "contravariant" or "covariant" components) makes formulas intuitive, I propose to use a bar (in upper or lower position) to distinguish between densities (like a probability density) or capacities (like a volume element), this also leading to intuitive results. In particular the bijection existing between these objects in metric spaces becomes as "natural" as the one just mentioned between contravariant and covariant components.

1.1 Chapter's overview

A vector at a point of an space can intuitively be imagined as an "arrow". As soon as we can introduce vectors, we can introduce other objects, the *forms*. A form at a point of an space can intuitively be imagined as a series of parallel planes… At any point of a space we may have tensors, of which the vectors of elementary texts are a particular case. Those tensors may describe the properties of the space itself (metric, curvature, torsion…) or the properties of something that the space "contains", like the stress at a point of a continuous medium.

If the space into consideration has a metric (i.e., if the notion of distance between two points has a sense), only tensors have to be considered. If there is not a metric, then, we have to simultaneously consider tensors and forms.

It is well known that in a transformation of coordinates, the value of a probability density $\overline{f}$ at any point of the space is multiplied by the Jacobian of the transformation. In fact, a probability density is a scalar field that has well defined tensor properties. This suggests to introduce two different notions where sometimes only one is found: for instance, in addition to the notion of mass density, $\overline{\rho}$, we will also consider the notion of volumetric mass ρ , identical to the former only in Cartesian coordinates. If $\overline{\rho}(\mathbf{x})$ is a mass density, and $v^i(\mathbf{x})$ a true vector, like a velocity. Their product $\overline{p}^i(\mathbf{x}) = \overline{\rho}(\mathbf{x})\, v^i(\mathbf{x})$ will not transform like a true vector: there will be an extra multiplication by the Jacobian. $\overline{p}^i(\mathbf{x})$ is a density too (of linear momentum).

In addition to tensors and to densities, the concept of "capacity" will be introduced. Under a transformation of coordinates, a capacity is *divided* by the Jacobian of the trasformation. An example is the capacity element $d\underline{V} = dx^0 dx^1 \dots,$ not to be assimilated to the volume element dV . The product of a capacity by a density gives a true scalar, like in $dM = \overline{\rho}\, d\underline{V}$.

It is well known that if there is a metric, we can define a bijection between forms and vectors (we can "raise and lower indices") through $V_i = g_{ij} V^j$. The square root of the determinant of $\{g_{ij}\}$ will be denoted $\overline{g}$ and we will see that it defines a natural bijection between capacities, tensors, and densities, like in $\overline{p}^i = \overline{g}\, p^i,$ so, in addition to the rules concerning the indices, we will have rules concerning the "bars".

Without a clear understanding of the concept of densities and capacities, some properties remain obscure. We can, for instance, easily introduce a Levi-Civita capacity $\underline{\varepsilon}_{ijk\dots}$, or a Levi-Civita density (the components of both take only the values -1, +1 or 0). A Levi-Civita pure tensor can be defined, but it does not have that simple property. The lack of clear understanding of the need to work simultaneously with densities, pure tensors, and capacities, forces some authors to juggle with "pseudo-things" like the pseudo-vector corresponding to the vector product of two vectors, or to the curl of a vector field.

Many of the properties of tensor spaces arte not dependent on the fact that the space may have a metric (i.e., a notion of distance). We will only assume that we have a metric when the property to be demonstrated will require it. In particular, the definition of "covariant" derivative, in the next chapter, will not depend on that assumption.

In this chapter, the dimension of the differentiable manifold (i.e., space) into consideration, is arbitrary (but finite). We will use Latin indices $\{i, j, k, \dots\}$ to denote the components of tensors.

In the second part of the book, as we will specifically deal with the physical space and space-time, the Latin indices $\{i, j, k, \dots\}$ will be reserved for the 3-D physical space, while the Greek indices $\{\alpha, \beta, \gamma, \dots\}$ will be reserved for the 4-D space-time.

1.2 A small introduction to tensors

This small introduction is for readers not well acquainted with tensor language. It is independent of the rest of the book, and can be skipped without harm.

The velocity of the wind at the top of Eiffel's tower, at a given moment, can be represented by a *vector* $\mathbf{v}$ with components, in some local, given, basis, $\{v^i\}$ $(i = 1, 2, 3)$. The velocity of the wind is defined at any point $\mathbf{x}$ of the atmosphere at any time t : we have a *vector field* $v^i(\mathbf{x}, t)$.

The water's temperature at some point in the ocean, at a given moment, can be represented by a *scalar* T . The field $T(\mathbf{x}, t)$ is a *scalar field*.

The state of stress at a given point of the Earth's crust, at a given moment, is represented by a *second order tensor* $\boldsymbol{\sigma}$ with components $\{\sigma^{ij}\}$ $(i = 1, 2, 3;\ j = 1, 2, 3)$. In a general model of continuous media, where it is not assumed that the stress tensor is symmetric, this means that we need 9 scalar quantities to characterize the state of stress. In more particular models, the stress tensor is symmetric, $\sigma^{ij} = \sigma^{ji}$, and only six scalar quantities are needed. The stress field $\sigma^{ij}(\mathbf{x}, t)$ is a *second order tensor field*. Another example of second order tensor arises in the analysis of deformation of an elastic medium. If $u_i(\mathbf{x}, t)$ is the displacement field (displacement at time t of the point that, in the undeformed state, was at point $\mathbf{x}$), the strain field is defined by

$$\varepsilon_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i - \nabla_i u_k \, \nabla_j u^k) \qquad (i = 1, 2, 3;\ j = 1, 2, 3)\,, \tag{1.1}$$

where ∇_i means true derivative along the i-th direction (the difference between true — or "covariant" — derivatives and partial derivatives will be seen below).

Tensor fields can be combined, to give other fields. One example has just been given, that defines the deformation field ε_{ij} as a function of the displacement field $u_i(\mathbf{x}, t)$. Another example is the scalar field

$$\mathcal{E}(\mathbf{x}, t) = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma^{ij}(\mathbf{x}, t)\, \varepsilon_{ij}(\mathbf{x}, t) = \frac{1}{2}\, \sigma^{ij}(\mathbf{x}, t)\, \varepsilon_{ij}(\mathbf{x}, t)\,, \tag{1.2}$$

representing the elastic energy density stored at a point of an elastic medium, due to the elastic deformation. Also, if n_i is a unit vector considered at a point inside a medium, the vector

$$\tau^i(\mathbf{x}, t) = \sum_{j=1}^{3} \sigma^{ij}(\mathbf{x}, t)\, n_j(\mathbf{x}) = \sigma^{ij}(\mathbf{x}, t)\, n_j(\mathbf{x}) \qquad (i = 1, 2, 3) \tag{1.3}$$

represents the traction that the medium at one side of the surface defined by the normal n_i exerts the medium at the other side, at the considered point.

As a further example, if the deformations of an elastic solid are small enough, the stress tensor is related linearly to the strain tensor (Hooke's law). A linear relation between two second order tensors means that each component of one tensor can be computed as a linear combination of all the components of the other tensor:

$$\sigma^{ij}(\mathbf{x}, t) = \sum_{k=1}^{3} \sum_{\ell=1}^{3} c^{ijk\ell}(\mathbf{x})\, \varepsilon_{k\ell}(\mathbf{x}, t) = c^{ijk\ell}(\mathbf{x})\, \varepsilon_{k\ell}(\mathbf{x}, t) \qquad (i = 1, 2, 3;\ j = 1, 2, 3)\,. \tag{1.4}$$

The *fourth order tensor* c^{ijkl} represents a property of an elastic medium: its elastic stiffness. As each index takes 3 values, there are $3 \times 3 \times 3 \times 3 = 81$ scalars to define the elastic stiffness of a solid at a point (assuming some symmetries we may reduce this number to 21, and asuming isotropy of the medium, to 2).

We are not here interested in the physical meaning of equations 1.1 to 1.4, but in their structure. First, tensor notations are such that they are independent on the coordinates being used. This is not obvious, as changing the coordinates implies changing the local basis where the components of vectors and tensors are expressed. That the equalities 1.1 to 1.4 hold for any coordinate system, means that all the components of all tensors will change if we change the coordinate system being used (for instance, from Cartesian to spherical coordinates), but still the two sides of the expression will take equal values.

The mechanics of the notation, once understood, are such that it is only possible to write expressions that make sense (see a list of rules at the end of this section).

For reasons about to be discussed, indices may come in upper or lower positions, like in v^i , f_i or $T_i{}^j$. The definitions will be such that in all tensor expression (i.e., in all expressions that will be valid for all coordinate systems), the sums over indices will always concern an index in lower position an one index on upper position. For instance, we may encounter the expressions $\varphi = \sum_{i=1}^{3} A_i B^i$ or $A_i = \sum_{j=1}^{3} \sum_{k=1}^{3} D_{ijk} E^{jk}$. These expressions are simplified by not writing explicitly the sums, like in $\varphi = A_i B^i$ and $A_i = D_{ijk} E^{jk}$. This notation is useful as one easily forgets that one is dealing with sums, and that it happens that, with respect to the usual tensor operations (sum with another tensor field, multiplication with another tensor field, and derivation), a sum of such terms is handled as one single term of the sum could be handled.

In an expression like $A_i = D_{ijk} E^{jk}$ it is said that the indices j and k have been *contracted* (or are "dummy indices"), while the index i is a *free index*. A tensor equation is assumed to hold for all possible values of the free indices.

In some spaces, like our physical 3-D space, it is posible to define the distance between two points. These are called *metric spaces*. A mathematically convenient manner to introduce a metric is by defining the length of an arc Γ by $S = \int_\Gamma ds$, where, for instance, in Cartesian coordinates, $ds^2 = dx^2 + dy^2 + dz^2$ or, in spherical coordinates, $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$. In general, we write $ds^2 = g_{ij} dx^i dx^j$, and we call $g_{ij}(\mathbf{x})$ the *metric field* or, simply, the *metric*.

The components of a vector $\mathbf{v}$ are associated to a given basis (the vector will have different components on different basis). If a basis $\mathbf{e}_i$ is given, then, the components v^i are defined through $\mathbf{v} = v^i \mathbf{e}_i$ (implicit sum). The *dual basis* of the basis $\{\mathbf{e}_i\}$ is denoted $\{\mathbf{e}^i\}$ and is defined by the equation $\mathbf{e}_i \mathbf{e}^j = \delta_i{}^j$ (equal to 1 if i are the same index and to 0 if not). When there is a metric, this equation can be interpreted as a scalar vector product, and the dual basis is just another basis (identical to the first one when working with Cartesian coordinates in Euclidena spaces, but different in general). The properties of the dual basis will be analyzed later in the chapter. Here we just need to recall that if v^i are the components of the vector $\mathbf{v}$ on the basis $\{\mathbf{e}_i\}$ (remember the expression $\mathbf{v} = v^i \mathbf{e}_i$), we will denote by v_i are the components of the vector $\mathbf{v}$ on the basis $\{\mathbf{e}^i\}$: $\mathbf{v} = v_i \mathbf{e}^i$. In that case (metric spaces) the components on the two basis are related by $v_i = g_{ij} v^i$: It is said that "the metric tensor ascends (or descends) the indices".

Here is a list with some rules helping to recognize tensor equations:

- A tensor expression must have the same *free* indices, at the top and at the bottom, of the two sides of an equality. For instance, the expressions

$$\varphi = A_i \, B^i \tag{1.5}$$

$$\varphi = g_{ij} \, B^i \, C^j \tag{1.6}$$

$$A_i = D_{ijk} \, E^{jk} \tag{1.7}$$

$$D_{ijk} = \nabla_i F_{jk} \tag{1.8}$$

are valid, but the expressions

$$A_i = F_{ij} \, B^i \tag{1.9}$$

$$B^i = A_j \, C^j \tag{1.10}$$

$$A_i = B^i \tag{1.11}$$

are not valid.

- Sum and multiplication of tensors (with eventual "contraction" of indices) gives tensors. For instance, if D_{ijk} , G_{ijk} and $H_i{}^j$ are tensors,

$$J_{ijk} = D_{ijk} + G_{ijk} \tag{1.12}$$

$$K_{ijk\ell}{}^m = D_{ijk}\, H_\ell{}^m \tag{1.13}$$

and

$$L_{ik\ell} = D_{ijk}\, H_\ell{}^j \tag{1.14}$$

are tensors.

- True (or "covariant") derivatives of tensor felds give tensor fields. For instance, if E^{ij} is a tensor field,

$$M_i{}^{jk} = \nabla_i E^{jk} \tag{1.15}$$

and

$$B^j = \nabla_i E^{ij} \tag{1.16}$$

are tensor fields. But partial derivatives of tensors do not define, in general, tensors. For instance, if E^{ij} is a tensor field,

$$M_i{}^{jk} = \partial_i V^{jk} \tag{1.17}$$

and

$$B^j = \partial_i V^{ij} \tag{1.18}$$

are not tensors, in general.

- All "objects with indices" that are normally introduced are tensors, with four notable exceptions. The first exception are the coordinates $\{x^i\}$ (to see that it makes no sense to add coordinates, think, for instance, in adding the spherical coordinates of two points). But the differentials dx^i appearing in an expression like $ds^2 = g_{ij}\, dx^i\, dx^j$ do correspond to the components on a vector $d\mathbf{r} = dx^i \mathbf{e}_i$. Another notable exception is the "symbol" ∂_i mentioned above. The third exception is the "connection" $\Gamma_{ij}{}^k$ to be introduced later in the chapter. In fact, it is because both of the symbols ∂_i and $\Gamma_{ij}{}^k$ are not tensors than an expression like

$$\nabla_i V^j = \partial_i V^j + \Gamma_{ik}{}^j\, V^k \tag{1.19}$$

can have a tensorial sense: if one of the terms at right was a tensor and not the other, their sum could never give a tensor. The objects ∂_i and $\Gamma_{ij}{}^k$ are both non tensors, and "what one term misses, the other term has". The fourth and last case of "objects with indices" which are not tensors are the Jacobian matrices arising in coordinate changes

$$J_i{}^I = \frac{\partial x'^I}{\partial x^i} \,. \tag{1.20}$$

That this is not a tensor is obvious when considering that, contrarily to a tensor, the Jacobian matrix is not defined per se, but it is only defined when two different coordinate systems have been chosen. A tensor exists even if no coordinate system at all has been defined.

1.3 Tensors, in general

This section introduces the very basic tensor concepts (vectors, forms, bases, changes of coordinates. . .). It also introduces the notions of densities and capacities.

It is not assumed that the manifold under consideration has a metric. The particular properties of metric spaces are analyzed in section 1.4.

1.3.1 Differentiable manifolds

A manifold is a continuous space of points. In an n-dimensional manifold it is always possible to "draw" *coordinate lines* in such a way that to any point $\mathcal{P}$ of the manifold correspond coordinates $\{x^1, x^2, \dots, x^n\}$ and vice versa.

Saying that the manifold is a continuous space of points is equivalent to say that the coordinates themselves are "continuous", i.e., if they are, in fact, a part of $\mathcal{R}^n$. On such manifolds we define physical fields, and the continuity of the manifold will allow to define the derivatives of the considered fields. When derivatives of fields on a manifold can be defined, the manifold is then called a *differentiable manifold*.

Obvious examples of differentiable manifolds are the lines and surfaces of ordinary geometry. Our 3-D physical space (with, possibly, curvature and torsion) is also represented by a differentiable manifold. The space-time of general relativity is a four dimensional differentiable manifold.

A coordinate system may not "cover" all the manifold. For instance, the poles of a sphere are as ordinary as any other point in the sphere, but the coordinates are singular there (the coordinate φ is not defined). Changing the coordinate system around the poles will make any problem related to the coordinate choice to vanish there. A more serious difficulty appears when at some point, not the coordinates, but the manifold itself is singular (the linear tangent space is not defined at this point). Those ane named "essential singularities". No effort will be made on this book to classify them.

1.3.2 Tangent Linear Space. Tensors.

Consider, for instance, in classical dynamics, a trajectory $x^i(t)$ on a space which may not be flat, as the surface of a sphere. The trajectory is "on" the sphere. If we define now the velocity at some point,

$$v^i = \frac{dx^i}{dt} \, , \tag{1.21}$$

we get a vector which is not "on" the sphere, but *tangent* to it. It belongs to what is called the *tangent linear space* to the considered point. At that point, we will have a basis for vectors. At another point, we will another tangent linear space, and another vector basis.

More generally, at every point of a differential manifold, we can consider different vector or tensor quantities, like the *forces*, *velocities*, or *stresses* of mechanics of continuous media. As suggested by figure 1.1, those tensorial objects do not belong to the nonlinear manifold, but to the *tangent linear space* to the manifold at the considered point (that will only be introduced intuitively here).

At every point of an space, tensors can be added, multiplied by scalars, contracted, etc. This means that at every point of the manifold we have to consider a different vector space (in general, a tensor space). It is important to understand that two tensors at two different points of the space belong to two different tangent spaces, and can not be added as such (see figure 1.1). This is why we will later need to introduce the concept of "parallel transport of tensors".

All through this book, the two names *linear space* and *vector space* will be used as completely equivalent.

The structure of vector space is too narrow to be of any use in physics. What is needed is the structure where equations like

$$\begin{aligned} \lambda &= R_i \, S^i \\ T^j &= U_i \, V^{ij} + \mu \, W^j \\ X^{ij} &= Y^i \, Z^j \end{aligned} \tag{1.22}$$

make sense. This structure is that of a *tensor space*. In short, a tensor space is a collection of vector spaces and rules of multiplication and differentiation that use elements of the vector spaces considered to get other elements of other vector spaces.

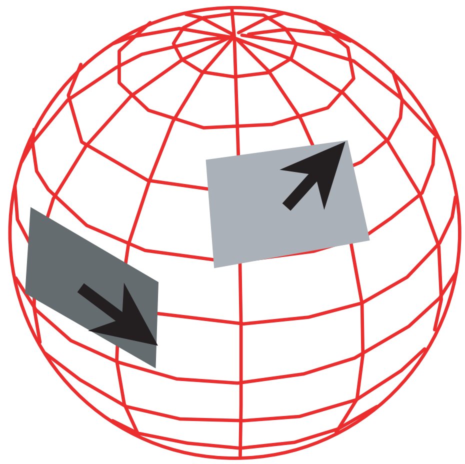

Figure 1.1: Surface with two planes tangent at two points, and a vector drawn at each point. As the vectors belong to two different vector spaces, their sum is not defined. Should we need to add them, for instance, to define true (or "covariant") derivatives of the vector field, then, we would need to transport them (by "parallel transportation") to a common point.

1.3.3 Vectors and Forms

When we introduce some vector space, with elements denoted, for instance, $\mathbf{V}$, $\mathbf{V}' \dots$, it often happens that a new, different, vector space is needed, with elements denoted, for instance $\mathbf{F}$, $\mathbf{F}' \dots$, and such that when taking an element of each space, we can "multiply" them and get a scalar,

$$\lambda = \langle\, \mathbf{F} \,,\; \mathbf{V} \,\rangle \, . \tag{1.23}$$

In terms of components, this will be written

$$\lambda = F_i \; V^i \, . \tag{1.24}$$

The product in 1.23–1.24, is called a *duality product*, and it has to be clearly distinguished from an inner (or scalar) product: in an inner product, we multiply two elements of a vector space; in a duality product, we multiply an element of a vector space by an element of a "dual space".

This operation can always be defined, including the case where the do not have a metric (and, therefore, a scalar product). As an example, imagine that we work with pieces of metal and we need to consider the two parameters "electric conductivity" σ and "temperature" T. We may need to consider some (possibly nonlinear) function of σ and T, say $S(\sigma,T)$. For instance, $S(\sigma,T)$ may represent a "misfit function" on the (σ,T) space of those encountered when solving inverse problems in physics if we are measuring the parameters σ and T using indirect means. In this case, S is adimensional[1]. We may wish to know by which amont will S change when passing from point (σ_0,T_0) to a neighbouring point $(\sigma_0+\Delta\sigma, T_0+\Delta T)$. Writing only the first order term, and using matrix notations,

$$S(\sigma_0+\Delta\sigma, T_0+\Delta T) = S(\sigma_0,T_0) + \begin{pmatrix} \frac{\partial S}{\partial \sigma} \\ \frac{\partial S}{\partial T} \end{pmatrix}^T \begin{pmatrix} \Delta\sigma \\ \Delta T \end{pmatrix} + \dots \, , \tag{1.25}$$

where the partial derivatives are taken at point (σ_0,T_0). Using tensor notations, setting $\mathbf{x} = (x^1,x^2) = (\sigma,T)$, we can write

$$S(\mathbf{x}+\Delta\mathbf{x}) = S(\mathbf{x}) + \sum_i \frac{\partial S}{\partial x^i}\, \Delta x^i$$

[1] For instance, one could have the simple expression $S(\sigma,T) = \frac{|\sigma-\sigma_0|}{s_P} + \frac{|T-T_0|}{s_T}$, where s_P and s_T are standard deviations (or mean deviations) of some probability distribution.

Box 1.1 Definition of vectors

Consider the 3-D physical space, with coordinates $\{x^i\} = \{x^1, x^2, x^3\}$. In classical mechanics, the trajectory of a particle is described by the three functions of time $x^i(t)$. Obviously the three values $\{x^1, x^2, x^3\}$ are not the components of a vector, as an expression like $x^i(t) = x^i_I(t) + x^i_{II}(t)$ has, in general, no sense (think, for instance, in the case where we use spherical coordinates).

Define now the velocity of the particle at time t_0 :

$$v^i(t_0) = \left(\frac{\mathrm{d}x^i}{\mathrm{d}t} \right)_{t=t_0} \quad .$$

If two particles coincide at some point of the space $\{x^1_0, x^2_0, x^3_0\}$, it makes sense to define, for instance, their relative velocity by $v^i(x^1_0, x^2_0, x^3_0, t_0) = v^i_I(x^1_0, x^2_0, x^3_0, t_0) - v^i_{II}(x^1_0, x^2_0, x^3_0, t_0)$. The v^i are the components of a vector.

If we change coordinates, $x'^I = x'^I(x^j)$, then the velocity is defined, in the new coordinate system, $v'^I = \mathrm{d}x'^I/\mathrm{d}t$, and we have $v'^I = \mathrm{d}x'^I/\mathrm{d}t = \partial x'^I/\partial x^i \; \mathrm{d}x^i/\mathrm{d}t$, i.e.,

$$v'^I = \frac{\partial x'^I}{\partial x^i} \; v^i \; ,$$

which is the standard rule for transformation of the components of a vector when the coordinates (and, so, the natural basis) change.

Objects with upper or lower indices not always are *tensors*. The four classical objects which do not have necessarily tensorial character are:

- the coordinates $\{x^i\}$,
- the partial differential operator ∂_i ,
- the Connection Coefficients $\Gamma_{ij}{}^k$,
- the elements of the Jacobian matrix $J_i{}^I = \partial x'^I/\partial x^i$.

$$\begin{aligned} &= S(\mathbf{x}) + \gamma_i \ \Delta x^i \\ &= S(\mathbf{x}) + \langle \boldsymbol{\gamma} \, , \, \Delta\mathbf{x} \rangle \ , \end{aligned} \tag{1.26}$$

where the notation introduced in equations 1.23–1.24 is used. As above, the partial derivatives are taken at point $\mathbf{x}_0 = (x_0^1, x_0^2) = (\sigma_0, T_0)$.

Note: say that figure 1.2 illustrates the definition of gradient as a tangent linear application. Say that the "mille-feuilles" are the "level-lines" of that tangent linear application.

Note: I have to explain somewhere the reason for putting an index in lower position to represent $\partial/\partial x^i$, i.e., to use the notation

$$\partial_i = \frac{\partial}{\partial x^i} \ .$$

Note: I have also to explain in spite of the fact that we have here partial derivatives, we have defined a tensorial object: the partial derivative of a scalar equals its true (covariant) derivative.

It is important that we realize that there is no "scalar product" involved in equations 1.26. Here are the arguments:

- The components of γ_i are **not** the components of a vector in the (σ, T) space. This can directly be seen by an inspection of their physical dimensions. As the function S is adimensional (see footnote 1), the components of $\boldsymbol{\gamma}$ have as dimensions the **inverse** of the physical dimensions of the components of the vector $\Delta\mathbf{x} = (\Delta x^1, \Delta x^2) = (\Delta\sigma, \Delta T)$. This clearly means that $\Delta\mathbf{x}$ and $\boldsymbol{\gamma}$ are "objects" that do not belong to the same space.

- If equations 1.26 involved a scalar product we could define the norm of $\mathbf{x}$, the norm of $\boldsymbol{\gamma}$ and the angle between $\mathbf{x}$ and $\boldsymbol{\gamma}$. But these norms and angle are not defined. For instance, what could be the norm of $\mathbf{x} = (\Delta\sigma, \Delta T)$? Should we choose an L_2 norm? Or, as suggested by footnote 1, an L_1 norm? And, in any case, how could we make consistent such a definition of a norm with a change of variables where, instead of electric conductivity we use electric resistivity? (Note: make an appendix where the solution to this problem is given).

The product in equations 1.26 is not a scalar product (i.e., it is not the "product" of two elements belonging to the same space): it is a "duality product", multiplying an element of a vector space and one element of a "dual space".

Why this discussion is needed? Because of the tendency of imagining the gradient of a function $S(\sigma, T)$ as a vector (an "arrow") in the $S(\sigma, T)$ space. If the gradient is not an arrow, then, what it is? Note: say here that figures 1.3 and 1.4 answer this by showing that an element of a dual space can be represented as a "mille-feuilles".

Up to here we have only considered a vector space and its dual. But the notion generalizes to more general tensor spaces, i.e., to the case where "we have more than one index". For instance, instead of equation 1.24 we could use an equation like

$$\lambda = F_{ij}{}^{k} \ V^{ij}{}_{k} \tag{1.27}$$

to define scalars, consider that we are doing a duality product, and also use the notation of equation 1.23 to denote it. But this is not very useful, as, from a given "tensor" $F_{ij}{}^{k}$ we can obtain scalar by operations like

$$\lambda = F_{ij}{}^{k} \ V^i \ W^j{}_k \ . \tag{1.28}$$

It is better, in general, to just write explicitly the indices to indocate which sort of "product" we consider.

Sometimes (like in quantum mechanics), a "bra-ket" notation is used, where the name stands for the *bra* "$\langle \ |$" and the *ket* "$| \ \rangle$". Then, instead of $\lambda = \langle \ \mathbf{F} \ , \ \mathbf{V} \ \rangle$ one writes

$$\lambda = \langle \ \mathbf{F} \mid \mathbf{V} \ \rangle = F_i \ V^i \ . \tag{1.29}$$

Then, the bra-ket notation is also used for the expression

$$\lambda = \langle \mathbf{V} \mid \mathbf{H} \mid \mathbf{W} \rangle = H_{ij} \, V^i \, W^j \, . \tag{1.30}$$

Note: say that the general rules for the change of component values in a change of coordinates, allow us to talk about "tensors" for "generalized vectors" as well as for "generalized forms".

The "number of indices" that have to be used to represent the components of a tensor is called the *rank*, or the *order* of the tensor. Thus the tensors $\mathbf{F}$ and $\mathbf{V}$ just introduced are second rank, or second order. A tensor object with components $R_{ijk}{}^{\ell}$ could be called, in all rigor, a "(third-rank-form)-(first-rank-vector)" will we wil not try to usew this heavy terminology, the simple writing of the indices being explicit.

Note: say that if there is a metric, there is a trivial identification between a vector space and its dual, through equations like $F_i = g_{ij} V^j$, or $S^{ijk}{}_{\ell} = g^{ip} g^{jq} g^{kr} g_{\ell s} R_{pqr}{}^{s}$, and in that case, the same letter is used to designate one vector and *its* dual element, as in $V_i = g_{ij} V^j$, and $R^{ijk}{}_{\ell} = g^{ip} g^{jq} g^{kr} g_{\ell s} R_{pqr}{}^{s}$. But in *non metric* spaces (i.e., spaces without metric), there is usually a big difference between an space and its dual.

Gradient and Hessian Explain somewhere that if $\phi(\mathbf{x})$ is a scalar function, the Taylor development

$$\phi(\mathbf{x} + \Delta\mathbf{x}) = \phi(\mathbf{x}) + \langle \, \mathbf{g} \, | \Delta\mathbf{x} \, \rangle + \frac{1}{2!} \, \langle \, \Delta\mathbf{x} \, | \, \mathbf{H} \, | \, \Delta\mathbf{x} \, \rangle \tag{1.31}$$

defines the gradien $\mathbf{g}$ and the Hessian $\mathbf{H}$.

Old text We may want the gradient to be "perpendicular" at the level lines of φ at $\mathcal{O}$, but there is no *natural* way to define a scalar product in the $\{P, T\}$ space, so we can not naturally define what "perpendicularity" is. That there is no natural way to define a scalar product does not mean that we can not define one: we can define many. For any symmetric, positive-definite matrix with the right physical dimensions (i.e., for any covariance matrix), the expression

$$\left(\begin{bmatrix} \delta P_1 \\ \delta T_1 \end{bmatrix} , \begin{bmatrix} \delta P_2 \\ \delta T_2 \end{bmatrix} \right) = \begin{bmatrix} \delta P_1 \\ \delta T_1 \end{bmatrix}^T \begin{bmatrix} C_{PP} & C_{PT} \\ C_{TP} & C_{TT} \end{bmatrix}^{-1} \begin{bmatrix} \delta P_2 \\ \delta T_2 \end{bmatrix}$$

defines a scalar product. By an appropriate choice of the covariance matrix, we can make any of the two lines in figure 1.3 (or any other line) to be perpendicular to the level lines at the considered point: the gradient at a given point is something univocally defined, even in the absence of any scalar product; the "direction of steepest descent" is not, and there are as many as we may choose different scalar products. The gradient is not an arrow, i.e, it is not a *vector*. So, then, how to draw the gradient? Roughly speaking, the gradient is the *linear tangent application* at the considered point. It is represented in figure 1.4. As, by definition, it is a linear application, the level lines are straight lines, and the spacing of the level lines in the tangent linear application corresponds to the spacing of the level lines in the original function around the point where the gradient is computed. Speaking more technically, it is the development

$$\begin{aligned} \varphi(\mathbf{x} + \delta\mathbf{x}) &= \varphi(\mathbf{x}) + \langle \, \mathbf{g} \, , \, \delta\mathbf{x} \, \rangle + \ldots \\ &= \varphi(\mathbf{x}) + g_i \, \delta x^i + \ldots \, , \end{aligned}$$

when limited to its first order, that defines the tangent linear application. The *gradient* of φ is then $\mathbf{g}$. The gradient $\mathbf{g} = \{g_i\}$ at $\mathcal{O}$ allows to associate a scalar to any vector $\mathbf{V} = \{V^i\}$ (also at $\mathcal{O}$): $\lambda = g_i \, V^i = \langle \, \mathbf{g} \, , \, \mathbf{V} \, \rangle$. This scalar is the difference of the values at the top and the bottom of the arrow representing the vector $\mathbf{V}$ on the local tangent linear application to φ at $\mathcal{O}$. The index on the gradient can be a lower index, as the gradient is not a vector.

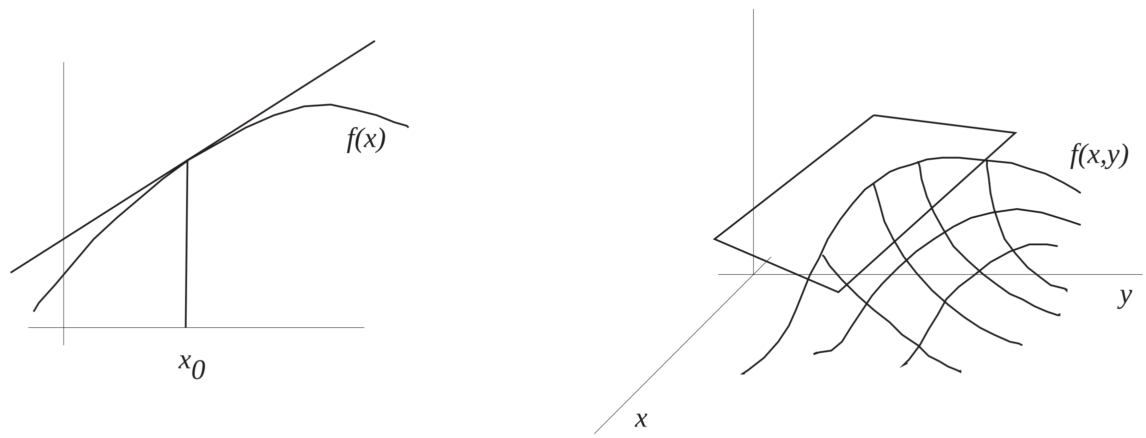

Figure 1.2: The gradient of a function (i.e., of an application) at a point x_0 is the tangent linear application at the given point. Let $x \mapsto f(x)$ represent the original (possibly nonlinear) application. The tangent linear application could be considered as mapping x into the values given by the linearized approximation fo $f(x)$: $x \mapsto F(x) = \alpha + \beta\, x$. (Note: explain better). Rather, it is mathematically simpler to consider that the gradient maps *increments* of the independent variable x , $\Delta x = x - x_0$ into increments of the linearized dependent variable: $\Delta y = y - f(x_0)$: $\Delta x \mapsto \Delta y = \beta\, \Delta x$. (Note: explain this MUCH better).

Note: say that figure 1.5 illustrates the fact that an element of the dual space can be represented as a "mille-feuilles" in the "primal" space or a an "arrow" un the dual space. And reciprocally.

Note: say that figure 1.6 illustrates the sum of arrows and the sum of "mille-feuilles".

Note: say that figure 1.7 illutrstes the sum of "mille-feuilles" in 3-D.

1.3.4 Natural Basis

A coordinate system associates to any point of the space, its coordinates. Each individual coordinate can be seen as a function associating, to any point of the space, the particular coordinate. We can define the gradient of this scalar function. We will have as many gradients f^i as coordinates x^i . As a gradient, we have seen, is a form, we will have as many forms as coordinates. The usual requirements that coordinate systems have to fulfill (different points of the space have different coordinates, and vice versa) gives n linearly independent forms (we can not obtain one of them by linear combination of the others), i.e., a *basis* for the forms.

If we have a basis $\mathbf{f}^i$ of forms, then we can introduce a basis $\mathbf{e}_i$ of vectors, through

$$\langle\, \mathbf{f}^i \,,\, \mathbf{e}_j \,\rangle = \delta^j{}_i \ . \tag{1.32}$$

This is illustrated in figure 1.8

If we define the components V^i of a vector $\mathbf{V}$ by

$$\mathbf{V} = V^i\, \mathbf{e}_i \ , \tag{1.33}$$

then, we can compute the components V^i by the formula

$$V^i = \langle\, \mathbf{f}^i \,,\, \mathbf{V} \,\rangle \ , \tag{1.34}$$

as we have

$$\langle\, \mathbf{f}^i \,,\, \mathbf{V} \,\rangle = \langle\, \mathbf{f}^i \,,\, V^j\, \mathbf{e}_j \,\rangle = \langle\, \mathbf{f}^i \,,\, \sum_j V^j\, \mathbf{e}_j \,\rangle = \sum_i V^j\, \langle\, \mathbf{f}^i \,,\, \mathbf{e}_j \,\rangle = \sum_i V^j\, \delta^i{}_j = V^j\, \delta^i{}_j = V^i \ . \tag{1.35}$$

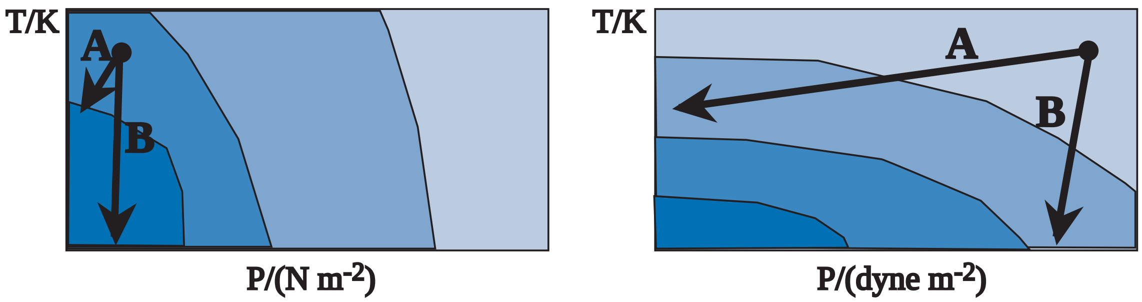

Figure 1.3: A scalar function $\varphi(P,T)$ depends on pressure and temperature. From a given point, two directions in the $\{P,T\}$ space are drawn. Which one corresponds to the gradient of $\varphi(P,T)$? In the figure at left, the pressure is indicated in International Units (m, kg, s), while in the figure at right, the c.g.s. units (cm, g, s) are used (remember that 1 Pa $=$ 10 dyne/cm^{-2}). From the left figure, we may think that the gradient is direction A , while from the figure at right we may think it is B . It is none: the right definition of gradient (see text) only allows, as graphic representation, the result shown in figure 1.4.

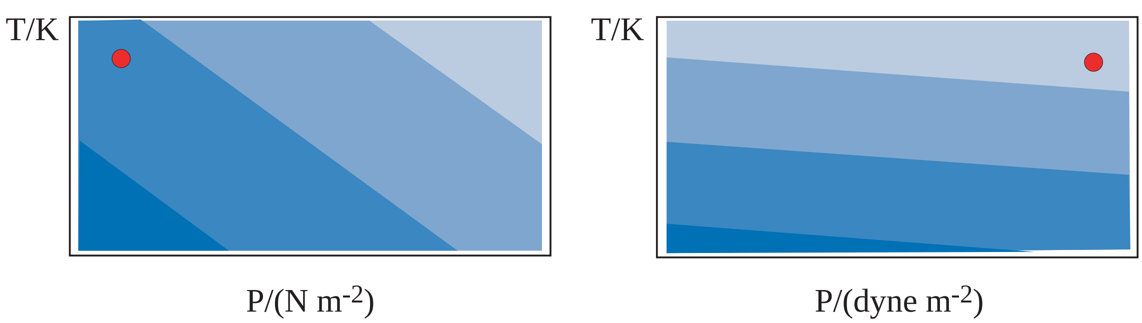

Figure 1.4: Gradient of the function displayed in figure 1.3, at the considered point. As the gradient is the linear tangent application at the given point, it is a linear application, and its level lines are stright lines. The value of the gradient at the considered point equals the value of the original function at that point. The spacing of the level lines in the gradient corresponds to the spacing of the level lines in the original function around the point where the gradient is computed. The two figures shown here are perflectly equivalent, as it should.

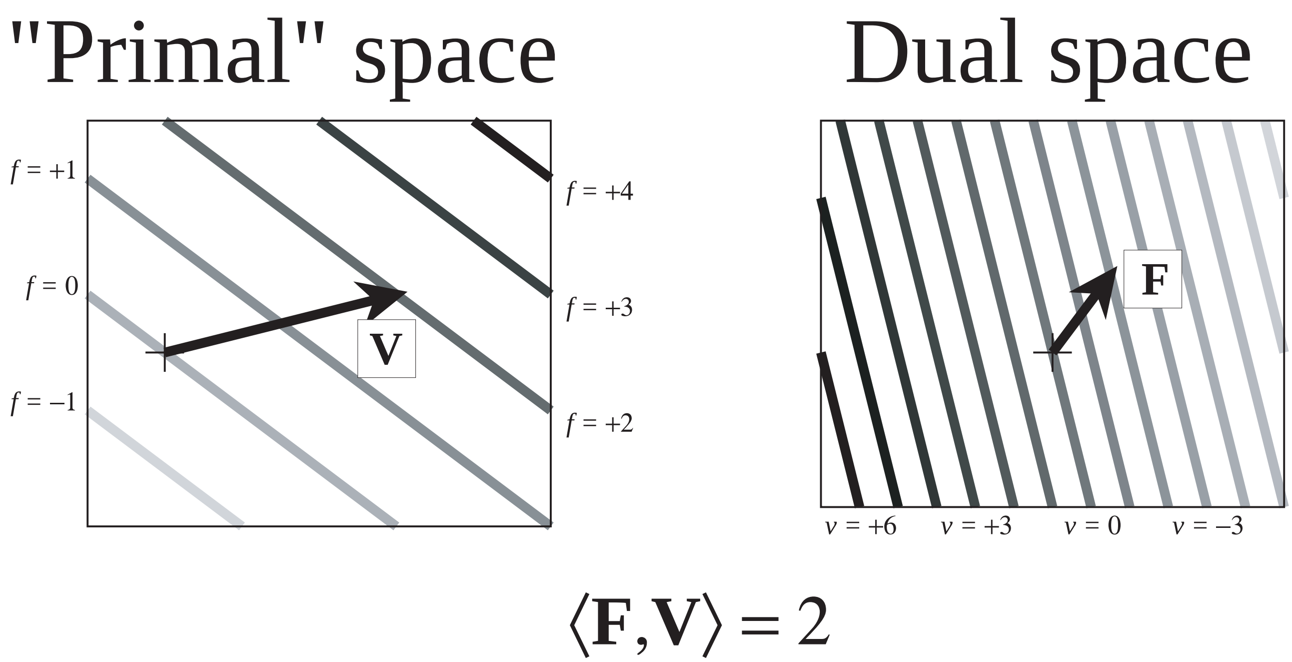

Figure 1.5: A point, at the left of the figure, may serve as the origin point for any vector we may want to represent. As usual, we may represent a vector $\mathbf{V}$ by an arrow. Then, a form $\mathbf{F}$ is represented by an oriented pattern of lines (or by an oriented pattern of surfaces in 3-D) with the line of zero value passing through the origin point. Each line has a value, that is the number that the form associates to any vector whose end point is on the line. Here, $\mathbf{V}$ and $\mathbf{F}$ are such that $\langle \mathbf{F}, \mathbf{V} \rangle = 2$. But a form is an element of the dual space, wich is also a linear space. In the dual space, then, the form $\mathbf{F}$ can be represented by an arrow (figure at right). In turn, V is represented, in the dual space, by a pattern of lines.

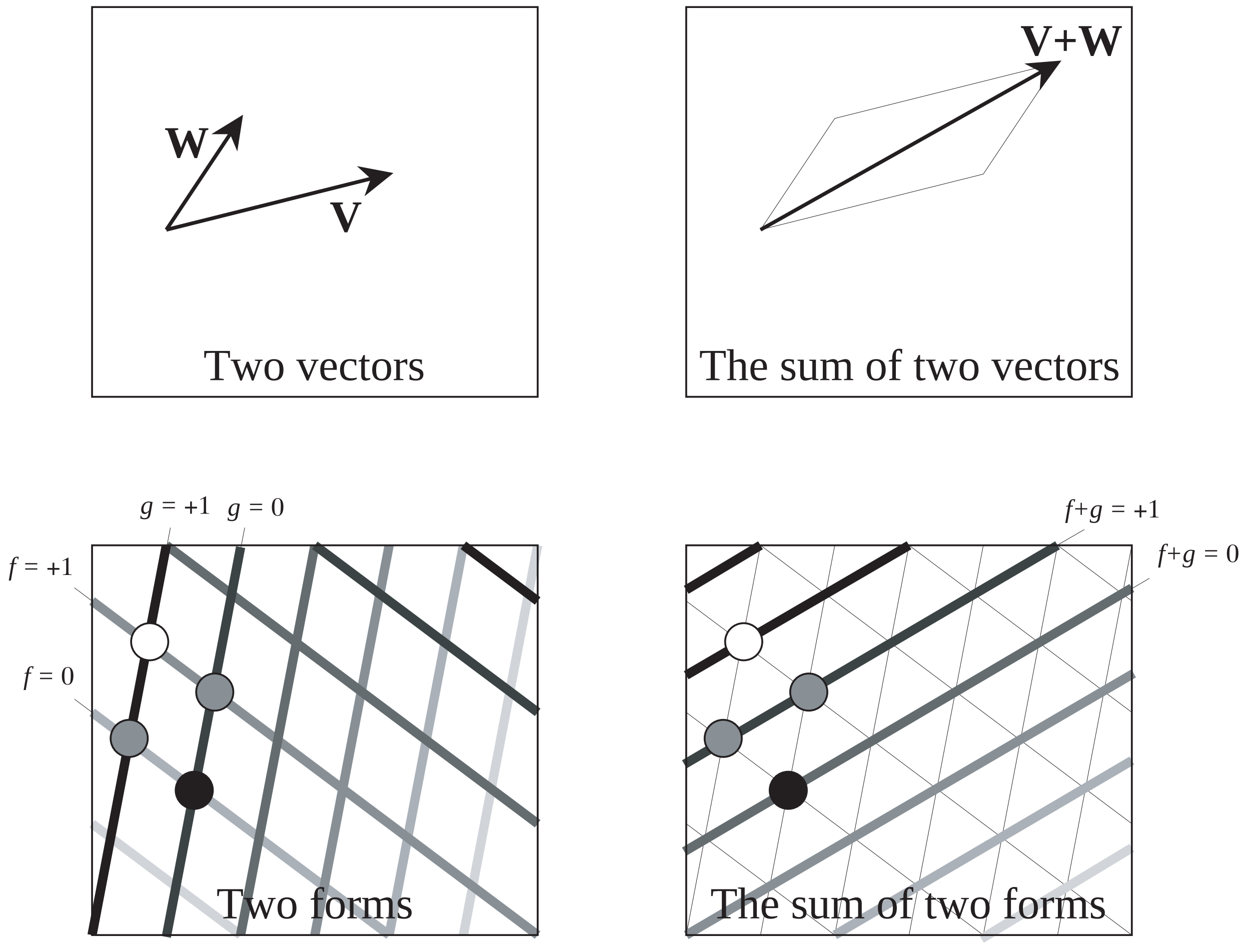

Figure 1.6: When representing vectors by arrows, the sum of two vectors is given by the main diagonal of the "parallelogram" drawn by two arrows. Then, a form is represented by a pattern of lines. The sum of two forms can be geometrically obtained using the "parallelogram" defined by the principal lozenge (containing the origin and with positive sense for both forms): the secondary diagonal of the lozenge is a line of the sum of the two forms. Note: explain this better.

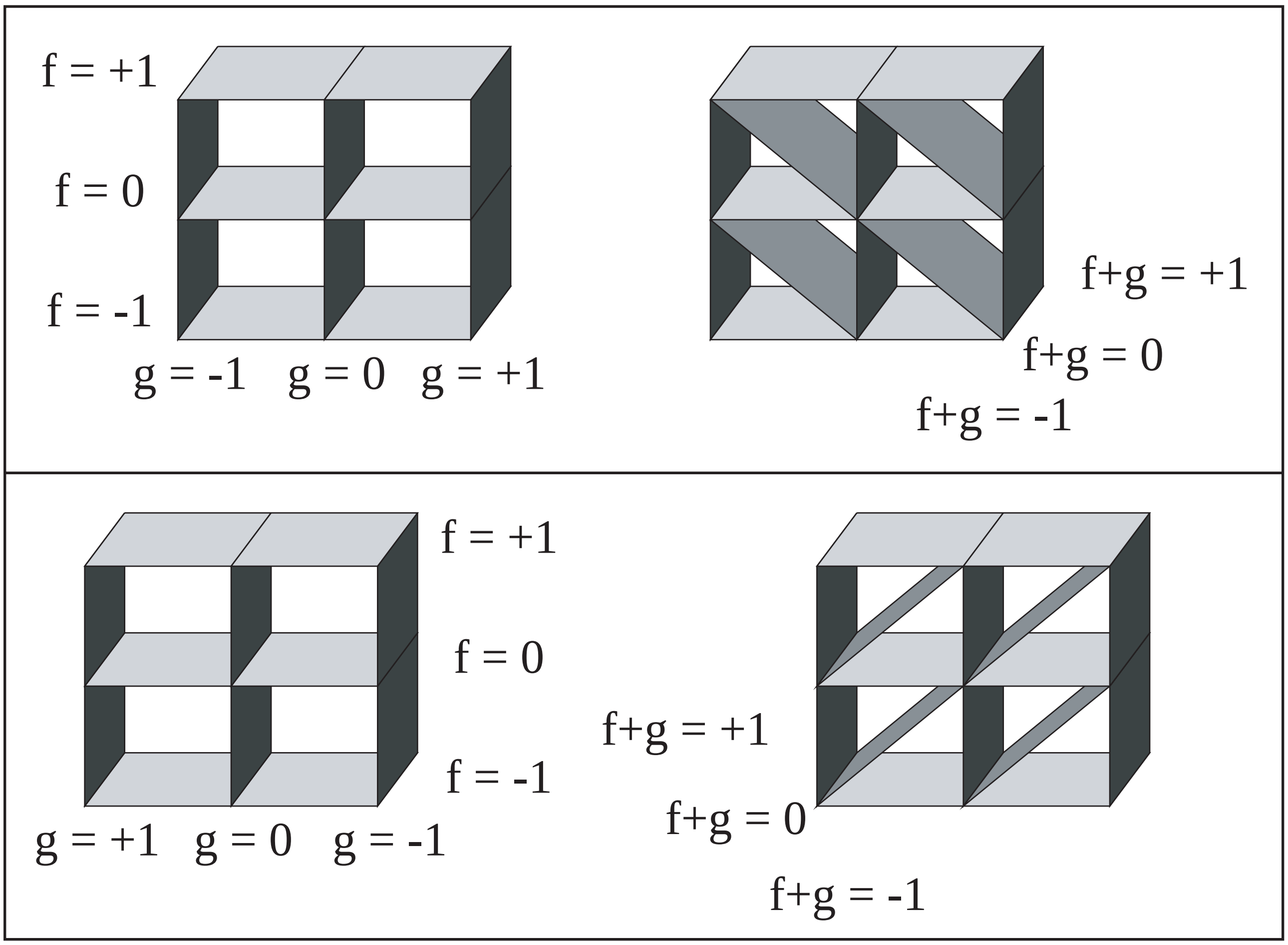

Figure 1.7: Sum of two forms, like in the previous figure, but here in 3-D. Note: explain that this figure can be "sheared" an one wants (we do not need to have a metric). Note: explain this better.

Note that the computation of the components of a vector does not involve a scalar product, but a duality product.

To find the equivalent of equations 1.33 and 1.34 for forms, one defines the components F_i of a form $\mathbf{F}$ by

$$\mathbf{F} = F_i \, \mathbf{f}^i \ , \tag{1.36}$$

and one easily gets

$$F_i = \langle \, \mathbf{F} \, , \, \mathbf{e}_i \, \rangle \ . \tag{1.37}$$

The notation $\mathbf{e}_i$ for the basis of vectors is quite universal. Although the notation $\mathbf{f}^i$ seems well adapted for a basis of forms, it is quite common to use the same letter for the basis of forms and for the basis of vectors. In what follows, we will use the notation

$$\mathbf{e}^i \equiv \mathbf{f}^i \ . \tag{1.38}$$

whose dangerousness vanishes only if we have a metric, i.e., when we can give sense to an expression like $\mathbf{e}_i = g_{ij} \mathbf{e}^j$. Using this notation the expressions

$$\mathbf{V} = V^i \, \mathbf{e}_i \quad \Longleftrightarrow \quad V^i = \langle \, \mathbf{f}^i \, , \, \mathbf{V} \, \rangle \qquad ; \qquad \mathbf{F} = F_i \, \mathbf{f}^i \ , \quad \Longleftrightarrow \quad F_i = \langle \, \mathbf{F} \, , \, \mathbf{e}_i \, \rangle \tag{1.39}$$

become

$$\mathbf{V} = V^i \, \mathbf{e}_i \quad \Longleftrightarrow \quad V^i = \langle \, \mathbf{e}^i \, , \, \mathbf{V} \, \rangle \qquad ; \qquad \mathbf{F} = F_i \, \mathbf{e}^i \ , \quad \Longleftrightarrow \quad F_i = \langle \, \mathbf{F} \, , \, \mathbf{e}_i \, \rangle \ . \tag{1.40}$$

We have now basis for vectors and forms, so wa can write expressions like $\mathbf{V} = V^i \, \mathbf{e}_i$ and $\mathbf{F} = F_i \, \mathbf{e}^i$. We need basis for objects "with more than one index", so we can write expressions like

$$\mathbf{B} = B^{ij} \, \mathbf{e}_{ij} \qquad ; \qquad \mathbf{C} = C_{ij} \, \mathbf{e}^{ij} \qquad ; \qquad \mathbf{D} = C_i{}^j \, \mathbf{e}^i{}_j \qquad ; \qquad \mathbf{E} = E_{ijk\ldots}{}^{\ell mn\ldots} \, \mathbf{e}^{ijk\ldots}{}_{\ell mn\ldots} \tag{1.41}$$

The introduction of these basis raises a difficulty. While we have an immediate intuitive representation for vectors (as "arrows") and for forms (as "millefeuilles"), tensor objects of higher rank are more difficult to represent. If a symmetric 2-tensor, like the stress tensor σ^{ij} of mechanics, can be viewed as an ellipsoid, how could we view a tensor $T_{ijk}{}^{\ell m}$? It is the power of mathematics to suggest analogies, so we can work even without geometric interpretations. But this absence of intuitive interpreation of high-rank tensors tells us that we will have to introduce the basis for these objects in a non-intuitive way. Essentially, what we want is that the basis for high rank tensors is not independent for the basis of vectors and forms. We want, in fact, more than this. Given two vectors U^i and V^i , we understand what we mean when we define a 2-tensor $\mathbf{W}$ by $W^{ij} = U^i \, V^j$. The basis for 2-tensors is perfectly defined by the condition that we wish that the components of $\mathbf{W}$ are precisely $U^i \, V^j$ and not, for instance, the values obtained after some rotation or change of coordinates.

This is enough, and we could directly use the notations introduced by equations 1.42. Instead, common mathematical developments introduce the notion of "tensor product", and, instead of notations like $\mathbf{e}_{ij}$, $\mathbf{e}^{ij}$, $\mathbf{e}_i{}^j$, or $\mathbf{e}^{ijk\ldots}{}_{\ell mn\ldots}$, introduce the notations $\mathbf{e}_i \otimes \mathbf{e}_i$, $\mathbf{e}^i \otimes \mathbf{e}^j$, $\mathbf{e}_i \otimes \mathbf{e}^j$, or $\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \ldots \mathbf{e}_\ell \otimes \mathbf{e}_m \otimes \mathbf{e}_n \otimes \ldots$. Then, equations 1.42 are written

$$\mathbf{B} = B^{ij} \, \mathbf{e}_i \otimes \mathbf{e}_i \quad ; \quad \mathbf{C} = C_{ij} \, \mathbf{e}^i \otimes \mathbf{e}^j \quad ; \quad \mathbf{D} = C_i{}^j \, \mathbf{e}^i \otimes \mathbf{e}^j$$

$$\mathbf{E} = E_{ijk\ldots}{}^{\ell mn\ldots} \, \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \ldots \mathbf{e}_\ell \otimes \mathbf{e}_m \otimes \mathbf{e}_n \otimes \ldots \ . \tag{1.42}$$

What follows is an old text, to be updated.

In section 1.4.1 we will properly introduce the metric tensor. Let us show here that if the space into consideration has a scalar product, then, the metric can be computed. Here, the scalar product of two vectors $\mathbf{V}$ and $\mathbf{W}$ is denoted $\mathbf{V} \cdot \mathbf{W}$. Then, defining

$$d\mathbf{r} = dx^i \ \mathbf{e}_i \tag{1.43}$$

and

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} \tag{1.44}$$

gives

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = (dx^i \, \mathbf{e}_i) \cdot (dx^j \, \mathbf{e}_j) = (\mathbf{e}_i \cdot \mathbf{e}_j) \; dx^i \; dx^j \, . \tag{1.45}$$

Defining the metric tensor

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{1.46}$$

gives then

$$ds^2 = g_{ij} \; dx^i \; dx^j \, . \tag{1.47}$$

To emphasize that at every point of the manifold we have a different tensor space, and a different basis, we can always write explicitly the dependence of the basis vectors on the coordinates, as in $\mathbf{e}_i(\mathbf{x})$. Equation 1.33 is then just a short notation for

$$\mathbf{V}(\mathbf{x}) = \; V^i(\mathbf{x}) \; \mathbf{e}_i(\mathbf{x}) \, , \tag{1.48}$$

while equation 1.36 is a short notation for

$$\mathbf{F}(\mathbf{x}) = F_i(\mathbf{x}) \; \mathbf{e}^i(\mathbf{x}) \, . \tag{1.49}$$

Here and in most places of the book, the notation $\mathbf{x}$ is a short-cut notation for $\{x^1, x^2, \ldots\}$. The reader should just remember that $\mathbf{x}$ represents a point in the space, but it is not a vector.

It is important to realize that, when dealing with tensor mathematics, a single basis is a basis for all the vector spaces at the considered point. For instance, the vector $\mathbf{V}$ may be a velocity, and the vector $\mathbf{E}$ may be an electric field. The two vectors belong to different vector spaces, but the are obtained as "linear combinations" of the same basis vectors:

$$\begin{aligned} \mathbf{V} &= V^i \; \mathbf{e}_i \\ \mathbf{E} &= E^i \; \mathbf{e}_i \, , \end{aligned} \tag{1.50}$$

but, of course, the components are not pure real numbers: they have dimensions. Box 1.2 recalls what the dimensions of components are.

Let us examine the components of the basis vectors (on the basis they define). Obviously,

$$(\mathbf{e}_i)^j = \delta_i{}^j \qquad (\mathbf{e}^j)_i = \delta_i{}^j \, , \tag{1.51}$$

or, explicitly,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \qquad \ldots \, . \tag{1.52}$$

Equivalently, for the basis of 2-tensors we have

$$(\mathbf{e}_i \otimes \mathbf{e}_j)^{kl} = \; \delta_i{}^k \; \delta_j{}^l \tag{1.53}$$

$$\begin{gathered} \mathbf{e}_1 \otimes \mathbf{e}_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix} \qquad \mathbf{e}_1 \otimes \mathbf{e}_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix} \qquad \cdots \\ \mathbf{e}_2 \otimes \mathbf{e}_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix} \qquad \mathbf{e}_2 \otimes \mathbf{e}_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix} \qquad \cdots \end{gathered} \tag{1.54}$$

$$\dots \qquad \dots \qquad \dots$$

and similar formular for other basis.

Note: say somewhere that the definition of basis vectors given above imposes that the vectors of the natural basis are, at any point, tangent to the coordinate lines at that point. The notion of tangency is independent of the existence, or not, of a metric, i.e., of the possibility of measuring distances in the space. This is not so for the notion of perpendicularity, that makes sense only if we can measure distances (and, therefore, angles). In, general, then, the vectors of the natural basis are tangent to the coordinate lines. When a metric has been introduced, the vectors in the natural basis at a given point will be mutually perpendicular only if the coordinate lines themselves are mutually perpendicular at that point. Ordinary coordinates in the Euclidean 3-D space (Cartesian, cylindrical, spherical,...) define coordinate lines that are orthogonal at every point. Then, the vectors of the natural basis will also be mutually orthogonal at all points. *But the vectors of the natural basis are* **not***, in general, normed to 1*. For instance, figure XXX illustrates the fact that the norm of the vectors of the natural basis in polar coordinates are, at point (r, φ) , $\|\mathbf{e}_r\| = 1$ and $\|\mathbf{e}_\varphi\| = r$.

1.3.5 Change of Coordinates

Let us consider two different coordinate systems, $\{x^i\}$ and $\{x'^I\}$. The two following matrices are introduced, called *Jacobian matrices*,

$$J'_I{}^i(\{x'^J\}) = \frac{\partial x^i}{\partial x'^I}(\{x'^J\}) \, , \tag{1.55}$$

$$J_i{}^I(\{x^j\}) = \frac{\partial x'^I}{\partial x^i}(\{x^j\}) \, . \tag{1.56}$$

For short, we simply write

$$J'_I{}^i = \frac{\partial x^i}{\partial x'^I} \tag{1.57}$$

and

$$J_i{}^I = \frac{\partial x'^I}{\partial x^i} \, . \tag{1.58}$$

Sometimes, the same letter J is used to designate both $J'_I{}^i$ and $J_i{}^I$, as the position of indices indicates exactly the matrix we consider. We do not follow that convention here, as it complicates some expressions we will find below.

To remember which matrix has an index, think that, for instance, $J'_I{}^i$ is obtained taking the partial derivative of the functions $x^i(x'^I)$, and, as such, we obtain a function of x'^I , from where the index.

As we will see in section 1.3.6, the partial derivatives $J'_I{}^i$ and $J_i{}^I$, are important because they allow, in a coordinate change, to obtain the components of tensors in the new coordinate system. For instance, if we consider, at a given point in the space, a tensor with components $T_{ij\dots}{}^{k\ell\dots}$ in the natural basis associated to some coordinate system $\{x^i\}$, and we change to another coordinate system $\{x'^I\}$, then, the components of the same tensor in the natural basis associated to the new coordinate system will be

$$T'_{IJ\dots}{}^{KL\dots} = J'_I{}^i \; J'_J{}^j \dots \; T_{ij\dots}{}^{k\ell\dots} \; J_k{}^K \; J_\ell{}^L \dots \; . \tag{1.59}$$

The determinants of the Jacobian matrices, are called *Jacobian determinants*, or, simply, *Jacobians*,

$$\mathcal{J}' = \det(\{J'_I{}^i\}) = \frac{1}{n!} \, \overline{\varepsilon}^{IJK\dots} \; J'_I{}^i \; J'_J{}^j \; J'_K{}^k \dots \; \underline{\varepsilon}_{ijk\dots} \tag{1.60}$$

Box 1.2 Which dimensions have the components of a vector?

Contrarily to the basis of elementary calculus, the vectors defining the natural basis are not normed to one. Rather, it follows from $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ that the length (i.e., the norm) of the basis vector $\mathbf{e}_i$ is

$$\|\mathbf{e}_i\| = \sqrt{g_{ii}} \, .$$

For instance, if in the Euclidean 3-D space with Cartesian coordinates

$$\|\mathbf{e}_x\| = \|\mathbf{e}_y\| = \|\mathbf{e}_z\| = 1 \, ,$$

the use of spherical coordinates gives

$$\|\mathbf{e}_r\| = 1 \qquad \|\mathbf{e}_\theta\| = r \qquad \|\mathbf{e}_\varphi\| = r \sin\theta \, .$$

Denoting by $[\|\mathbf{V}\|]$ the physical dimension of (the norm of) a vector, this gives

$$[\|\mathbf{e}_i\|] = [\sqrt{g_{ii}}] \; .$$

For instance, in Cartesian coordinates,

$$[\|\mathbf{e}_x\|] = [\|\mathbf{e}_y\|] = [\|\mathbf{e}_z\|] = 1 \, ,$$

and in spherical coordinates,

$$[\|\mathbf{e}_r\|] = 1 \qquad [\|\mathbf{e}_\theta\|] = L \qquad [\|\mathbf{e}_\varphi\|] = L \, ,$$

where L represents the dimension of a *length*. A vector $\mathbf{V} = V^i \mathbf{e}_i$ has components with dimensions

$$\left[V^i \right] = \frac{[\|\mathbf{V}\|]}{[\|\mathbf{e}_i\|]} = \frac{[\|\mathbf{V}\|]}{[\sqrt{g_{ii}}]} \, .$$

For instance, in Cartesian coordinates,

$$[V^x] = [V^y] = [V^y] = [\|\mathbf{V}\|]$$

and in spherical coordinates,

$$[V^r] = [\|\mathbf{V}\|] \qquad \left[V^\theta \right] = \frac{[\|\mathbf{V}\|]}{L} \qquad [V^\varphi] = \frac{[\|\mathbf{V}\|]}{L} \, .$$

In general, the physical dimension of the component $T_{ij\dots}{}^{k\ell\dots}$ of a tensor $\mathbf{T}$ is

$$\begin{aligned} \left[T_{ij\dots}{}^{k\ell\dots} \right] &= [\|\mathbf{T}\|] \, [\|\mathbf{e}_i\|] \, [\|\mathbf{e}_j\|] \dots \frac{1}{[\|\mathbf{e}_k\|]} \frac{1}{[\|\mathbf{e}_\ell\|]} \dots \\ &= [\|\mathbf{T}\|] \, [\sqrt{g_{ii}}] \, [\sqrt{g_{jj}}] \dots \frac{1}{[\sqrt{g_{kk}}]} \frac{1}{[\sqrt{g_{\ell\ell}}]} \dots \, . \end{aligned}$$

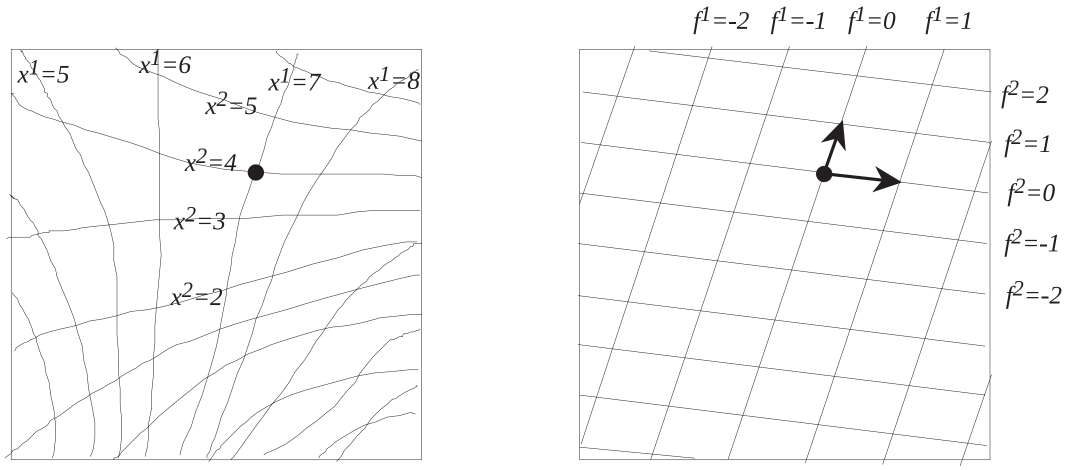

Figure 1.8: A system of coordinates, at left, and their gradients, at right. These gradient are forms. When in an n-dimensional space we have n forms, we can define n associate vertors by $\langle\, \mathbf{f}^i \,,\, \mathbf{e}_j \,\rangle = \delta^j{}_i$.

$$\mathcal{J} = \det(\{J_i{}^I\}) = \frac{1}{n!}\,\overline{\varepsilon}^{ijk\dots}\; J_i{}^I\; J_j{}^J\; J_k{}^K \dots\; \underline{\varepsilon}_{IJK\dots}\,, \tag{1.61}$$

where the Levi-Civita's "symbols" $\underline{\varepsilon}_{ijk\dots}$ take the value $+1$ if $\{i,j,k,\dots\}$ is an *even* permutation of $\{1,2,3,\dots\}$, the value -1 if $\{i,j,k,\dots\}$ is an *odd* permutation of $\{1,2,3,\dots\}$, and the value 0 if some indices are identical. The Levi-Civita's tensors will properly be defined in Section 1.6.1.

It is easy to see that the matrices $J'_I{}^i$ and $J_i{}^I$ are mutually inverses:

$$J_i{}^J J'_J{}^k = \delta_i{}^k\,, \tag{1.62}$$

and

$$J'_I{}^j J_j{}^K = \delta_I{}^K\,, \tag{1.63}$$

which implies that the Jacobians are also mutually inverses:

$$\mathcal{J} = \frac{1}{\mathcal{J}'}\,. \tag{1.64}$$

The second order partial derivatives will also be useful. As there is no risk of confusion, as they have three indices, the same symbols J' and J can be used:

$$J'_{JK}{}^i(\{x'^L\}) = \frac{\partial^2 x^i}{\partial x'^J \partial x'^K}(\{x'^L\})\,, \tag{1.65}$$

and

$$J_{jk}{}^I(\{x^l\}) = \frac{\partial^2 x'^I}{\partial x^j \partial x^k}(\{x^l\})\,, \tag{1.66}$$

or, for short,

$$J'_{JK}{}^i = \frac{\partial^2 x^i}{\partial x'^J \partial x'^K}\,, \tag{1.67}$$

and

$$J_{jk}{}^I = \frac{\partial^2 x'^I}{\partial x^j \partial x^k}\,. \tag{1.68}$$

In Euclidean spaces, the connection coefficients (see below) vanish if we use Cartesian coordinates $\{y^I\}$. In any other coordinate system $\{x^i\}$, they are given by (see demonstration below)

$$\Gamma_{ij}{}^k = J_{ij}{}^S J'_S{}^k\,, \tag{1.69}$$

i.e.,

$$\Gamma_{ij}{}^{k} = \frac{\partial^2 y^S}{\partial x^i \partial x^j} \, \frac{\partial x^k}{\partial y^S} \, . \tag{1.70}$$

When the coordinate system $\{x^i\}$ is called "old", and the coordinate system $\{x'^I\}$ is called "new", then, what is usually called "the" Jacobian determinant is that of $J'_I{}^i$.

1.3.6 Tensors, Densities, and Capacities

We are familiar with two sorts of "scalar" objects (i.e., objects "without indices"): one class has its values at every point of the space unchanged when we change coordinates, like a temperature field; the other class has its values multiplied by the Jacobian of the transformation, like a probability density.

It is often less obvious but the Jacobian of the transformation also appears when transforming the components of some "tensor" objects (i.e., objects "with indices"): when it is said that a vector product of two vectors is a vector, *excepted that it changes sign if we reverse the sense of one of the coordinate lines*, we are, in fact, multiplying the components of the "vector" by the Jacobian of the transformation.

Let us start with the crude, general definition. An object with components $\mathcal{Q}_{ij\dots}{}^{kl\dots}$ is called a *density of order* p if, in a change of coordinates

$$x'^I = x'^I(\{x^j\}) \, , \tag{1.71}$$

the components transform as

$$\mathcal{Q}'_{IJ\dots}{}^{KL\dots} = (\mathcal{J}')^p \; J'_I{}^i \; J'_J{}^j \dots \; \mathcal{Q}_{ij\dots}{}^{k\ell\dots} \; J_k{}^K \; J_\ell{}^L \dots \tag{1.72}$$

where $\{J'_I{}^i\}$ and $\{J_i{}^I\}$ are the Jacobian matrices

$$J'_I{}^i(\{x'^J\}) = \frac{\partial x^i}{\partial x'^I}(\{x'^J\}) \, , \tag{1.73}$$

and

$$J_i{}^I(\{x^j\}) = \frac{\partial x'^I}{\partial x^i}(\{x^j\}) \, , \tag{1.74}$$

and $\mathcal{J}'$ is the Jacobian determinant, i.e, the determinant of the matrix $\{J'_I{}^i\}$.

If $p = 0$, that is, if the Jacobian is not in equation 1.72, we say that $\mathcal{Q}_{ij\dots}{}^{kl\dots}$ is a (pure) *tensor*, and we use the simple notation $Q_{ij\dots}{}^{kl\dots}$.

If $p = 1$ we say that $\mathcal{Q}_{ij\dots}{}^{kl\dots}$ is a (tensor) *density*, and we use the notation $\overline{Q}_{ij\dots}{}^{kl\dots}$.

If $p = -1$ we say that $\mathcal{Q}_{ij\dots}{}^{kl\dots}$ is a (tensor) *capacity*, and we use the notation $\underline{Q}_{ij\dots}{}^{kl\dots}$.

To be explicit, let us write how change the components of tensors with rank 0 , 1 , and 2 .

Comment: Say somewhere that the "bars" have the same sort of rules as the indices: one upper bar multiplied by a lower bar gives no bar, the total number of bars at each side has to be "homogeneous", etc.

As we are still dealing with spaces that may not have a metric, we cannot transform forms into vectors, or densities and capacities into pure tensors. This will be done in Section 1.4.1. Comment: But the structure of the book has changed, and now I have to give all this here.

Comment: I have to say somewhere that, as in a change of variables, a density is multiplied by the Jacobian determinant, this, gives, when applied to a change of the fundamental density $\overline{g}$, from Cartesian coordinates (where it takes the numerical value of 1), to an arbitrary system of coordinates, that $\overline{g}$ equals the value of the Jacobian determinant (between the given coordinates and the Cartesian ones). I have checked this in dimension 2. It is difficult to give a valid expression of the property, as the fundamental density is a density and the Jacobian determinant a pure scalar. Let us try nevertheless.

The Jacobian is

$$J_I{}^i = \frac{\partial x^i}{\partial X^I} , \tag{1.75}$$

and the Jacobian determinant is

$$\mathcal{J} = \frac{1}{n!} \overline{\varepsilon}^{ijk\dots} J_i{}^I J_j{}^J J_k{}^K \dots \underline{\varepsilon}_{IJK\dots} . \tag{1.76}$$

We see that $\mathcal{J}$ is a true scalar. If the coordinates $\{X^I\}$ are Cartesian coordinates in an Euclidean space, we have the property

$$\overline{g} = \mathcal{J} . \tag{1.77}$$

But how can we have an identity between a density and a true scalar? Assuredly, this is not a tensor equation. For it makes only sense in the particular case when the $\{X^I\}$ are Cartesian coordinates. (Comment: more fundamentally, $\overline{g}$ is a tensor field, and $\mathcal{J}$ is not). In that case, the distinction between true scalars, densities and capacities, disappears and, for instance, we have

$$\underline{\varepsilon}_{IJK\dots} = \varepsilon_{IJK\dots} = \overline{\varepsilon}_{IJK\dots} . \tag{1.78}$$

Then, we can write

$$\overline{\mathcal{J}} = \frac{1}{n!} \overline{\varepsilon}^{ijk\dots} J_i{}^I J_j{}^J J_k{}^K \dots \varepsilon_{IJK\dots} , \tag{1.79}$$

which shows that, in the special circumstance that the $\{X^I\}$ are Cartesian coordinates, the Jacobian determinant $\mathcal{J}$ can be considered a density as well.

Comment: what precedes seems bizarre...

Comment: Give somewhere the formula $\partial_i \overline{g} = \overline{g} \Gamma_i$. It can be justified by the fact that, for any density, $\overline{s}$, $\nabla_k \overline{s} = \partial_k \overline{s} - \Gamma_k \overline{s}$, and the result follows by using $\overline{s} = \overline{g}$ and remembering that $\nabla_k \overline{g} = 0$.

1.4 Tensors in metric spaces

1.4.1 The metric tensor

Comment: explain here that it is possible to give a lot of structure to a manifold (tangent linear space, (covariant) derivation, etc.) without the need of a metric. It is introduced here to simplify the text, as, if not, we would have need to come bak to most of the results to add the particular properties arising when there is a metric. But, in all rigor, it would be preferable to introduce the metric after, for instance, the definition of covariant differentiaition, that does not need it.

Having a metric in a differential manifold means being able to define the length of a line. This will then imply that we can define a scalar product at every local tangent linear space (and, thus, the angle between two crossing lines).

The metric will also allow to define a natural bijection between vectors and forms, and between tensors densities and capacities.

A metric is defined when a second rank symetric form $\mathbf{g}$ with components g_{ij} is given. The length L of a line $x^i(\lambda)$ is then defined by the line integral

$$L = \int_\lambda ds , \tag{1.80}$$

where

$$ds^2 = g_{ij} dx^i dx^j . \tag{1.81}$$

Once we have a metric, it is possible to define a bijection between forms and vectors. For, to the vector $\mathbf{V}$ with components V^i we can associate the form $\mathbf{F}$ with components

$$F_i = g_{ij} \, V^j . \tag{1.82}$$

Then, it is customary to use the same letter to designate a vector and a form that are linked by this natural bijection, as in

$$V_i = g_{ij}\ V^j \ . \tag{1.83}$$

The inverse of the previous equation is written

$$V^i = g^{ij}\ V_j \ , \tag{1.84}$$

where

$$g_{ij} g^{jk} = \delta_i{}^k \ . \tag{1.85}$$

The reader will easily give sense to the expression

$$\mathbf{e}_i = g_{ij} \mathbf{e}^j \ . \tag{1.86}$$

The equations above, and equations like

$$T_{ij\ldots}{}^{kl\ldots} = g_{ip}\ g_{jq} \ldots g^{kr} g^{ls} \ldots T^{pq\ldots}{}_{rs\ldots} \ , \tag{1.87}$$

are summarized by saying that “the metric tensor allows to raise and lower indices”.

The value of the metric at a particular point of the manifold allows to define a scalar product for the vectors in the local tangent linear space. Denoting the scalar product of two vectors $\mathbf{V}$ and $\mathbf{W}$ by $\mathbf{V} \cdot \mathbf{W}$, we can use any of the definitions

$$\mathbf{V} \cdot \mathbf{W} = g_{ij}\ V^i\ W^j = V_i\ W^j = V^i\ W_j \ . \tag{1.88}$$

To define parallel transportation of tensors, we have introduced a connection $\Gamma_{ij}{}^k$. Now that we have a metric we may wonder if when parallel-transporting a vector, it conserves constant length. It is easy to show (see demonstration in [Comment: where?]) that this is true if we have the *compatibility condition*

$$\nabla_i g_{jk} = 0 \, , \tag{1.89}$$

i.e.,

$$\partial_i g_{jk} = g_{sk} \Gamma_{ij}{}^s + g_{js} \Gamma_{ik}{}^s \ . \tag{1.90}$$

The compatibility condition 1.89 implies that the metric tensor and the nabla symbol commute:

$$\nabla_i (g_{jk} T^{pq\ldots}{}_{rs\ldots}) = g_{jk} (\nabla_i T^{pq\ldots}{}_{rs\ldots}), \tag{1.91}$$

which, in fact, means that it is equivalent to take a covariant derivative, then raise or lower an index, or first raise or lower an index, then take the covariant derivative.

Note: introduce somewhere the notation

$$\Gamma_{ijk} = g_{ks} \Gamma_{ij}{}^s \, , \tag{1.92}$$

warn the reader that this is just a *notation*: the connection coefficients are not the components of a tensor. and say that if the condition 1.89 holds, then, it is possible to compute the connection coefficients from the metric and the torsion:

$$\Gamma_{ijk} = \frac{1}{2} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right) + \frac{1}{2} \left(S_{ijk} + S_{kij} + S_{kji} \right) \ . \tag{1.93}$$

As the basis vectors have components

$$(\mathbf{e}_i)^j = \delta_i{}^j \ , \tag{1.94}$$

we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = g_{ij} \, . \tag{1.95}$$

Defining

$$d\mathbf{r} = dx^i \; \mathbf{e}_i \tag{1.96}$$

gives then

$$d\mathbf{r} \cdot d\mathbf{r} = ds^2 \, . \tag{1.97}$$

We have seen that the metric can be used to define a natural bijection between forms and vectors. Let us now see that it can also be used to define a natural bijection between tensors, densities, and capacities.

We denote by $\overline{\overline{g}}$ the determinant of g_{ij} :

$$\overline{\overline{g}} = \det(\{g_{ij}\}) = \frac{1}{n!} \; \overline{\varepsilon}^{ijk\dots} \; \overline{\varepsilon}^{pqr\dots} \; g_{ip} \; g_{jq} \; g_{kr} \dots \, . \tag{1.98}$$

The two upper bars recall that $\overline{\overline{g}}$ is a second order density, as there is the product of two densities at the right-hand side.

For a reason that will become obvious soon, the square root of $\overline{\overline{g}}$ is denoted $\overline{g}$:

$$\overline{\overline{g}} = \overline{g} \; \overline{g} \, . \tag{1.99}$$

In (Comment: where?) we demonstrate that we have

$$\partial_i \; \overline{g} = \overline{g} \; \Gamma_{is}{}^s \, . \tag{1.100}$$

Using expression (Comment: which one?) for the (covariant) derivative of a scalar density, this simply gives

$$\nabla_i \; \overline{g} = \partial_i \; \overline{g} - \overline{g} \; \Gamma_{is}{}^s = 0 \, , \tag{1.101}$$

which is consistent with the fact that

$$\nabla_i \; g_{jk} = 0 \, . \tag{1.102}$$

We can also define the determinant of g^{ij} :

$$\underline{\underline{g}} = \det(\{g^{ij}\}) = \frac{1}{n!} \; \underline{\varepsilon}_{ijk\dots} \; \underline{\varepsilon}_{pqr\dots} \; g^{ip} \; g^{jq} \; g^{kr} \dots \, , \tag{1.103}$$

and its square root $\underline{g}$:

$$\underline{\underline{g}} = \underline{g} \; \underline{g} \, . \tag{1.104}$$

As the matrices g_{ij} and g^{ij} are mutually inverses, we have

$$\overline{g} \; \underline{g} = 1 \, . \tag{1.105}$$

Using the scalar density $\overline{g}$ and the scalar capacity $\underline{g}$ we can associate tensor densities, pure tensors, and tensor capacities. Using the same letter to designate the objects related through this natural bijection, we will write expressions like

$$\overline{\rho} = \overline{g} \; \rho \, , \tag{1.106}$$

$$\overline{V}^i = \overline{g} \; V^i \, , \tag{1.107}$$

or

$$T_{ij\dots}{}^{kl\dots} = \underline{g} \; \overline{T}_{ij\dots}{}^{kl\dots} \, . \tag{1.108}$$

So, if g_{ij} and g^{ij} can be used to "lower and raise indices", $\overline{g}$ and $\underline{g}$ can be used to "put and remove bars".

Comment: say somewhere that $\overline{g}$ is the *density of volumetric content*, as the volume element of a metric space is given by

$$dV = \overline{g}\; d\underline{\tau} \,, \tag{1.109}$$

where $d\underline{\tau}$ is the *capacity element* defined in (Comment: where?), and which, when we take an element along the coordinate lines, equals $dx^1 dx^2 dx^3 \dots$.

Comment: Say that we can demonstrate that, in an Euclidean space, the matrix representing the metric equals the product of the Jacobian matrix times the transposed matrix:

$$\{g_{ij}\} = \begin{pmatrix} g_{11} & g_{12} & \cdots \\ g_{21} & g_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\partial X^1}{\partial x^1} & \frac{\partial X^1}{\partial x^2} & \cdots \\ \frac{\partial X^2}{\partial x^1} & \frac{\partial X^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \times \begin{pmatrix} \frac{\partial X^1}{\partial x^1} & \frac{\partial X^2}{\partial x^1} & \cdots \\ \frac{\partial X^1}{\partial x^2} & \frac{\partial X^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} . \tag{1.110}$$

In short,

$$g_{ij} = \sum_K \frac{\partial X^K}{\partial x^i} \frac{\partial X^K}{\partial x^j} \,. \tag{1.111}$$

This follows directly from the general equation

$$g_{ij} = \frac{\partial X^I}{\partial x^i} \frac{\partial X^J}{\partial x^j}\, g_{IJ} \tag{1.112}$$

using the fact that, if the $\{X^I\}$ are Cartesian coordinates,

$$\{g_{IJ}\} = \begin{pmatrix} g_{11} & g_{12} & \cdots \\ g_{21} & g_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ 0 & 0 & \dots \end{pmatrix} . \tag{1.113}$$

1.4.2 The scalar product

Comment: explain here that the metric introduces a bijection between forms and vectors:

$$V_i = g_{ij} V^j \,. \tag{1.114}$$

Comment: introduce here the notation

$$(\mathbf{V}, \mathbf{W}) = g_{ij} V^i W^j = V_i W^i = W_i V^i \,. \tag{1.115}$$

1.5 Integration, estimation of densities

Comment: explain here what the "capacity element"is. Explain that, in polar coordinates, it is given by $dr d\varphi$, to be compared with the "surface element" $r dr d\varphi$. Comment figure 1.9.

Bijection between densities, tensors, and capacities

Comment: the text below has already been written in the section where the metric was introduced.

We have seen that the metric can be used to define a natural bijection between forms and vectors. Let us now see that it can also be used to define a natural bijection between tensors, densities, and capacities.

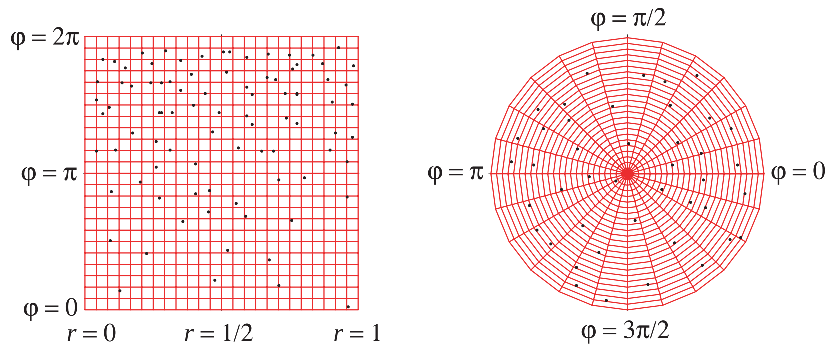

Figure 1.9: We consider, in an Euclidean space, a cylinder with a circular basis of radius 1, and cylindrical coordinates (r, φ, z). Only a section of the cylinder is represented in the figure, with all its thickness, dz, projected on the drawing plane. At left, we have represented a "map" of the corresponding circle, and, at right, the coordinate lines on the circle itself. All the "cells" at left have the same capacity $d\underline{V} = dr d\varphi dz$, while the cells at right have the volume $dV(r, \varphi, z) = r dr d\varphi dz$. The points represent particles with given masses. If, at left, at point with coordinates (r, φ, z) the sum of all the masses inside the local cell is denoted, dM, then, the mass *density* at this point is estimated by $\overline{\rho}(r, \varphi, z) = dM/d\underline{V}$, i.e., $\overline{\rho}(r, \varphi) = dM/(dr d\varphi dz)$. If, at right, at point (r, φ, z) the total mass inside the local cell is dM, the *volumetric* mass at this point is estimated by $\rho(r, \varphi, z) = dM/dV(r, \varphi, z)$, i.e., $\rho(r, \varphi, z) = dM/(r dr d\varphi dz)$. By definition, then, the total mass inside a volume V will be found by $M = \int_V d\underline{V}\, \overline{\rho}(r, \varphi, z) = \int_V dr d\varphi dz \overline{\rho}(r, \varphi, z)$ or by $M = \int_V dV(r, \varphi, z) \rho(r, \varphi, z) = \int_V r dr d\varphi dz \rho(r, \varphi, z)$.

We denote by $\overline{\overline{g}}$ the determinant of g_{ij}:

$$\overline{\overline{g}} = \det(\{g_{ij}\}) = \frac{1}{n!}\, \overline{\varepsilon}^{ijk\ldots}\, \overline{\varepsilon}^{pqr\ldots}\, g_{ip}\, g_{jq}\, g_{kr} \ldots . \tag{1.116}$$

The two upper bars recall that $\overline{\overline{g}}$ is a second order density, as there is the product of two densities at the right-hand side.

For a reason that will become obvious soon, the square root of $\overline{\overline{g}}$ is denoted $\overline{g}$:

$$\overline{\overline{g}} = \overline{g}\, \overline{g} . \tag{1.117}$$

In (Comment: where?) we demonstrate that we have

$$\partial_i\, \overline{g} = \overline{g}\, \Gamma_{is}{}^{s} . \tag{1.118}$$

Using expression (Comment: which one?) for the (covariant) derivative of a scalar density, this simply gives

$$\nabla_i\, \overline{g} = \partial_i\, \overline{g} - \overline{g}\, \Gamma_{is}{}^{s} = 0 , \tag{1.119}$$

which is consistent with the fact that

$$\nabla_i\, g_{jk} = 0 . \tag{1.120}$$

We can also define the determinant of g^{ij}:

$$\underline{\underline{g}} = \det(\{g^{ij}\}) = \frac{1}{n!}\, \underline{\varepsilon}_{ijk\ldots}\, \underline{\varepsilon}_{pqr\ldots}\, g^{ip}\, g^{jq}\, g^{kr} \ldots , \tag{1.121}$$

and its square root $\underline{g}$:

$$\underline{\underline{g}} = \underline{g}\, \underline{g} . \tag{1.122}$$

	capacity	tensor	density
0-rank	$\underline{s}' = \mathcal{J}\ \underline{s}$	$s' = s$	$\overline{s}' = \mathcal{J}'\ \overline{s}$
1-form	$\underline{F}'_I = \mathcal{J}\ J'_I{}^i \underline{F}_i$	$F'_I = J'_I{}^i F_i$	$\overline{F}'_I = \mathcal{J}'\ J'_I{}^i \overline{F}_i$
1-vector	$\underline{V}'^I = \mathcal{J}\ \underline{V}^i J_i{}^I$	$V'^I = V^i J_i{}^I$	$\overline{V}'^I = \mathcal{J}'\ \overline{V}^i J_i{}^I$
2-form	$\underline{Q}'_{IJ} = \mathcal{J}\ J'_I{}^i J'_J{}^j \underline{Q}_{ij}$	$Q'_{IJ} = J'_I{}^i J'_J{}^j Q_{ij}$	$\overline{Q}'_{IJ} = \mathcal{J}'\ J'_I{}^i J'_J{}^j \overline{Q}_{ij}$
(1-form)-(1-vector)	$\underline{R}'_I{}^J = \mathcal{J}\ J'_I{}^i \underline{R}_i{}^j J_j{}^J$	$R'_I{}^J = J'_I{}^i R_i{}^j J_j{}^J$	$\overline{R}'_I{}^J = \mathcal{J}' J'_I{}^i \overline{R}_i{}^j J_j{}^J$
(1-vector)-(1-form)	$\underline{S}'^I{}_J = \mathcal{J}\ J_i{}^I \underline{S}^i{}_j J'_J{}^j$	$S'^I{}_J = J_i{}^I S^i{}_j J'_J{}^j$	$\overline{S}'^I{}_J = \mathcal{J}'\ J_i{}^I \overline{S}^i{}_j J'_J{}^j$
2-vector	$\underline{T}'^{IJ} = \mathcal{J}\ \underline{T}^{ij} J_i{}^I J_j{}^J$	$T'^{IJ} = T^{ij} J_i{}^I J_j{}^J$	$\overline{T}'^{IJ} = \mathcal{J}'\ \overline{T}^{ij} J_i{}^I J_j{}^J$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1.1: Comment: write here the legend

As the matrices g_{ij} and g^{ij} are mutually inverses, we have

$$\overline{g}\ \underline{g} = 1\,. \tag{1.123}$$

Using the scalar density $\overline{g}$ and the scalar capacity $\underline{g}$ we can associate tensor densities, pure tensors, and tensor capacities. Using the same letter to designate the objects related through this natural bijection, we will write expressions like

$$\overline{\rho} = \overline{g}\ \rho\,, \tag{1.124}$$

$$\overline{V}^i = \overline{g}\ V^i\,, \tag{1.125}$$

or

$$T_{ij\dots}{}^{kl\dots} = \underline{g}\ \overline{T}_{ij\dots}{}^{kl\dots}\,. \tag{1.126}$$

So, if g_{ij} and g^{ij} can be used to "lower and raise indices", $\overline{g}$ and $\underline{g}$ can be used to "put and remove bars".

Comment: say somewhere that $\overline{g}$ is the *density of volumetric content*, as the volume element of a metric space is given by

$$dV = \overline{g}\ d\underline{\tau}\,, \tag{1.127}$$

where $d\underline{\tau}$ is the *capacity element* defined in (Comment: where?), and which, when we take an element along the coordinate lines, equals $dx^1 dx^2 dx^3 \dots$.

1.6 The Levi-Civita's and the Kronecker's tensors

1.6.1 The Levi-Civita's tensor

There are two different Levi-Civita's symbols: $\underline{\varepsilon}_{ijk\dots}$ and $\overline{\varepsilon}^{ijk\dots}$. The number of indices they contain equals the dimension of the space. For instance, in spaces of dimension 1, 2, 3, and 4 , one of the Levi-Civita's tensors will be written, respectively, $\underline{\varepsilon}_i$, $\underline{\varepsilon}_{ij}$, $\underline{\varepsilon}_{ijk}$, and $\underline{\varepsilon}_{ijkl}$.

They are defined by:

$$\underline{\varepsilon}_{ijk\dots} = \begin{cases} +\ 1 & \text{if } ijk\dots \text{ is an } \textit{even} \text{ permutation of } 12\dots n \\ \ 0 & \text{if some indices are identical} \\ -\ 1 & \text{if } ijk\dots \text{ is an } \textit{odd} \text{ permutation of } 12\dots n\,, \end{cases} \tag{1.128}$$

and, equivalently,

$$\overline{\varepsilon}^{ijk\dots} = \begin{cases} +\ 1 & \text{if } ijk\dots \text{ is an } \textit{even} \text{ permutation of } 12\dots n \\ \quad 0 & \text{if some indices are identical} \\ -\ 1 & \text{if } ijk\dots \text{ is an } \textit{odd} \text{ permutation of } 12\dots n\,. \end{cases} \tag{1.129}$$

In fact, $\underline{\varepsilon}_{ijk\dots}$ and $\overline{\varepsilon}^{ijk\dots}$ are more than “symbols”: they are respectively a *capacity* and a *density*, in the sense that, if when changing the coordinates, we *compute* the new components of the Levi-Civita's capacity and density using the rules applying to all capacities and densities, the properties 1.128–1.129 remain satisfied.

Comment: be more explicit. Say that if $\underline{\varepsilon}_{ijk\dots}$ satisfies the property 1.128, then, in a change of coordinates, the transformed object

$$\underline{\varepsilon}'_{IJK\dots} = \mathcal{J}'\ J'_I{}^i\ J'_J{}^j\ J'_K{}^k \dots \underline{\varepsilon}_{ijk\dots} \tag{1.130}$$

also satisfies 1.128.

Comment: say that an exercice gives the direct demonstration in 2-D.

Commment: give somewhere the general demonstration.

Notice that the Levi-Civita's capacity $\underline{\varepsilon}_{ijk\dots}$ has been defined with lower indices, and the Levi-Civita's density $\overline{\varepsilon}^{ijk\cdots}$ with upper indices. Raising the indices of $\underline{\varepsilon}_{ijk\dots}$ gives $\underline{\varepsilon}^{ijk\cdots}$, but **does not** give $\overline{\varepsilon}^{ijk\cdots}$: there are two different objects.

We have just defined the Levi-Civita's density and capacity. Both are simple in what they take only the values $+1, 0, -1$ in any coordinate system. A pure tensor can be defined through the canonical bijection:

$$\varepsilon_{ijk\dots} = \overline{g}\ \underline{\varepsilon}_{ijk\dots}\,. \tag{1.131}$$

The reader should remember that this pure tensor, instead of the values $\{+1, 0, -1\}$ it takes the values $\{+\overline{g}, 0, -\overline{g}\}$.

1.6.2 Determinants

The Levi-Civita's tensors can be used to define determinants. For instance, the determinants of the tensors $Q_{ij}, R_i{}^j, S^i{}_j$, and T^{ij} are defined by

$$Q = \frac{1}{n!}\ \varepsilon^{ijk\dots}\ \varepsilon^{mnr\dots}\ Q_{im}\ Q_{jn}\ Q_{kr}\dots\ , \tag{1.132}$$

$$\begin{aligned} R &= \frac{1}{n!}\ \varepsilon^{ijk\dots}\ \varepsilon_{mnr\dots}\ R_i{}^m\ R_j{}^n\ R_k{}^r\dots\ , \\ &= \frac{1}{n!}\ \overline{\varepsilon}^{ijk\dots}\ \underline{\varepsilon}_{mnr\dots}\ R_i{}^m\ R_j{}^n\ R_k{}^r\dots\ , \end{aligned} \tag{1.133}$$

$$\begin{aligned} S &= \frac{1}{n!}\ \varepsilon_{ijk\dots}\ \varepsilon^{mnr\dots}\ S^i{}_m\ S^j{}_n\ S^k{}_r\dots\ , \\ &= \frac{1}{n!}\ \underline{\varepsilon}_{ijk\dots}\ \overline{\varepsilon}^{mnr\dots}\ S^i{}_m\ S^j{}_n\ S^k{}_r\dots\ , \end{aligned} \tag{1.134}$$

and

$$T = \frac{1}{n!}\ \varepsilon_{ijk\dots}\ \varepsilon_{mnr\dots}\ T^{im}\ T^{jn}\ T^{kr}\dots\ , \tag{1.135}$$

where the Levi-Civita's tensors $\underline{\varepsilon}_{ijk\dots}$, $\varepsilon_{ijk\dots}$, $\overline{\varepsilon}^{ijk\dots}$ and $\varepsilon^{ijk\dots}$ have as many indices as the space under consideration has dimensions.

1.6.3 The Kronecker's tensor

There are two Kronecker's "symbols", ${g_i}^j$ and ${g^i}_j$. They are defined similarly:

$$ {g_i}^j = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are the same index} \\ 0 & \text{if } i \text{ and } j \text{ are different indices}\, , \end{cases} \tag{1.136} $$

and

$$ {g^i}_j = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are the same index} \\ 0 & \text{if } i \text{ and } j \text{ are different indices}\, . \end{cases} \tag{1.137} $$

Comment: I should be avoid this last notation.

It can easily be seen

(Comment: how?)

that ${g^i}_j$ are more than 'symbols': they are *tensors*, in the sense that, if when changing the coordinates, we *compute* the new components of the Kronecker's tensors using the rules applying to all tensors, the property (Comment: which equation?) remains satisfied.

The Kronecker's tensors are defined even if the space has not a metric defined on it. Note that, sometimes, instead of using the symbols ${g_i}^j$ and ${g^j}_j$ to represent the Kronecker's tensors, the symbols ${\delta_i}^j$ and ${\delta^j}_j$ are used. But then, using the metric g_{ij} to "lower an index" of ${\delta_i}^j$ gives

$$ \delta_{ij} = g_{jk} {\delta_i}^k = g_{ij} \ , \tag{1.138} $$

which means that, if the space has a metric, the Kronecker's tensor and the metric tensor are the same object. Why, then, use a different symbol? The use of the symbol ${\delta_i}^j$ may lead, by inadvertence, after lowering an index, to assing to δ_{ij} the value 1 when i and j are the same index. This is obviously wrong: if there is not a metric, δ_{ij} is not defined, and if there is a metric, δ_{ij} equals g_{ij} , which is only 1 in Euclidean spaces using Cartesian coordinates.

There is only one Kronecker's tensor, and ${g_i}^j$ and ${g^i}_j$ can be deduced one from the other raising and lowering indices. But, even in that case, we dislike the notation g^i_j, where the place of each index is not indicated, and we will not use it sistematically.

Warning: a common error in beginners is to give the value 1 to the symbol ${g_i}^i$ (or to ${g^i}_i$) . In fact, the right value is n , the dimension of the space, as there is an implicit sum assumed: ${g_i}^i = {g_0}^0 + {g_1}^1 + \ldots = 1 + 1 + \ldots = n$.

1.6.4 The Kronecker's determinants

Let us denote by n the dimension of the space into consideration. The Levi-Civita's tensor has then n indices. For any (non-negative) integer p satisfying $p \leq n$, consider the integer q such that $p + q = n$. The following property holds:

$$ \underline{\varepsilon}_{i_1 \ldots i_p s_1 \ldots s_q} \ \overline{\varepsilon}^{j_1 \ldots j_p s_1 \ldots s_q} = q! \ \det \begin{pmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \ldots & \delta^{j_p}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \ldots & \delta^{j_p}_{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{j_1}_{i_p} & \delta^{j_2}_{i_p} & \ldots & \delta^{j_p}_{i_p} \end{pmatrix} \ , \tag{1.139} $$

where ${\delta_i}^j$ stands for the Kronecker's tensor. The determinant at the right-hand side is called the *Kronecker's determinant*, and is denoted $\delta^{j_1 j_2 \ldots j_p}_{i_1 i_2 \ldots i_p}$:

$$ \delta^{j_1 j_2 \ldots j_p}_{i_1 i_2 \ldots i_p} = \det \begin{pmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \ldots & \delta^{j_p}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \ldots & \delta^{j_p}_{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{j_1}_{i_p} & \delta^{j_2}_{i_p} & \ldots & \delta^{j_p}_{i_p} \end{pmatrix} \ . \tag{1.140} $$

As the Kronecker's determinant is defined as a product of Levi-Civita's tensors, it is itself a tensor. It generalizes the definition of the Kronecker's tensor $\delta_i{}^j$, as it has the properties

$$\delta^{j_1 j_2 \ldots j_m}_{i_1 i_2 \ldots i_m} = \begin{cases} +1 & \text{if } (j_1, j_2, \ldots, j_m) \text{ is an even permutation of } (i_1, i_2, \ldots, i_m) \\ -1 & \text{if } (j_1, j_2, \ldots, j_m) \text{ is an odd permutation of } (i_1, i_2, \ldots, i_m) \\ 0 & \text{if two of the } i'\text{s or two of the } j'\text{s are the same index} \\ 0 & \text{if } (i_1, i_2, \ldots, i_m) \text{ and } (j_1, j_2, \ldots, j_m) \text{ are different sets of indices.} \end{cases} \tag{1.141}$$

As applying the same permutation to the indices of the two Levi-Civita's tensors of equation 1.139 will not change the total sign of the expression, we have

$$\begin{aligned} \underline{\varepsilon}_{i_1 \ldots i_p s_1 \ldots s_q} \, \overline{\varepsilon}^{j_1 \ldots j_p s_1 \ldots s_q} &= \\ \underline{\varepsilon}_{s_1 \ldots s_q i_1 \ldots i_p} \, \overline{\varepsilon}^{s_1 \ldots s_q j_1 \ldots j_p} &= q! \; \delta^{j_1 j_2 \ldots j_p}_{i_1 i_2 \ldots i_p} \, , \end{aligned} \tag{1.142}$$

but we only perform a permutation in one of the Levi-Civita's tensors, then we must care about the sign of the permutation, and we obtain

$$\begin{aligned} \underline{\varepsilon}_{i_1 \ldots i_p s_1 \ldots s_q} \, \overline{\varepsilon}^{s_1 \ldots s_q j_1 \ldots j_p} &= \\ \underline{\varepsilon}_{s_1 \ldots s_q i_1 \ldots i_p} \, \overline{\varepsilon}^{j_1 \ldots j_p s_1 \ldots s_q} &= (-1)^{pq} \, q! \; \delta^{j_1 j_2 \ldots j_p}_{i_1 i_2 \ldots i_p} \, . \end{aligned} \tag{1.143}$$

This possible change of sign has only effect in spaces with even dimension $(n = 2, 4, \ldots)$, as in spaces with odd dimension $(n = 3, 5, \ldots)$ the condition $p + q = n$ implies that pq is an even number, and $(-1)^{pq} = +1$.

Remark that a multiplication and a division by $\overline{g}$ will not change the value of an expression, so that, instead of using Levi-Civita's density and capacity we can use Levi-Civita's true tensors. For instance,

$$\underline{\varepsilon}_{i_1 \ldots i_p s_1 \ldots s_q} \, \overline{\varepsilon}^{j_1 \ldots j_p s_1 \ldots s_q} = \varepsilon_{i_1 \ldots i_p s_1 \ldots s_q} \, \varepsilon^{j_1 \ldots j_p s_1 \ldots s_q} \, . \tag{1.144}$$

Comment: explain better.

In Boxes 1.3 to 1.5 we give special formulas to spaces with dimension 2 , 3 , and 4 . As shown in appendix XXX, these formulas replace more elementary identities between grad, div, rot, ...

As an example, a well known identity like

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \tag{1.145}$$

is obvious, as the three formulas correspond to the expression $\varepsilon_{ijk} a^i b^j c^k$. The identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \; \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \; \mathbf{c} \tag{1.146}$$

is easily demonstrated, as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \varepsilon_{ijk} a^j (\mathbf{b} \times \mathbf{c})^k = \varepsilon_{ijk} a^j \varepsilon^{k\ell m} b_\ell c_m \, , \tag{1.147}$$

which, using 1.157, gives

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (a^m c_m) b_i - (a^m b_m) c_i = (\mathbf{a} \cdot \mathbf{c}) \; \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \; \mathbf{c} \, . \tag{1.148}$$

Comment: I should clearly say here that we have the identity

$$\underline{\varepsilon}_{ijk\ldots} \, \overline{\varepsilon}^{\ell mn\ldots} = \varepsilon_{ijk\ldots} \, \varepsilon^{\ell mn\ldots} \, . \tag{1.149}$$

Comment: say somewhere that if $B_{i_1 \ldots i_p}$ is a totally antisymmetric tensor, then

$$\frac{1}{p!} \, \delta^{\ell_1 \ldots \ell_p}_{i_1 \ldots i_p} \, B_{\ell_1 \ldots \ell_p} = B_{i_1 \ldots i_p} \tag{1.150}$$

Box 1.3 The Kronecker's determinants in 2-D

$$\delta^{k\ell}_{ij} = (1/0!)\,\varepsilon_{ij}\,\varepsilon^{k\ell} = \delta^k_i\delta^\ell_j - \delta^\ell_i\delta^k_j \tag{1.153}$$

$$\delta^{k}_{j} = (1/1!)\,\varepsilon_{ij}\,\varepsilon^{ik} = \delta^k_j \tag{1.154}$$

$$\delta = (1/2!)\,\varepsilon_{ij}\,\varepsilon^{ij} = 1 \tag{1.155}$$

Box 1.4 The Kronecker's determinants in 3-D

$$\delta^{\ell mn}_{ijk} = (1/0!)\,\varepsilon_{ijk}\,\varepsilon^{\ell mn} = \delta^\ell_i\delta^m_j\delta^n_k + \delta^m_i\delta^n_j\delta^\ell_k + \delta^n_i\delta^\ell_j\delta^m_k - \delta^\ell_i\delta^n_j\delta^m_k - \delta^n_i\delta^m_j\delta^\ell_k - \delta^m_i\delta^\ell_j\delta^n_k \tag{1.156}$$

$$\delta^{\ell m}_{jk} = (1/1!)\,\varepsilon_{ijk}\,\varepsilon^{i\ell m} = \delta^\ell_j\delta^m_k - \delta^m_j\delta^\ell_k \tag{1.157}$$

$$\delta^{\ell}_{k} = (1/2!)\,\varepsilon_{ijk}\,\varepsilon^{ij\ell} = \delta^\ell_k \tag{1.158}$$

$$\delta = (1/3!)\,\varepsilon_{ijk}\,\varepsilon^{ijk} = 1 \tag{1.159}$$

Comment: give somewhere the property

$$\frac{1}{q!}\,\delta^{k_1\ldots k_p\ell_1\ldots\ell_q}_{i_1\ldots i_p j_1\ldots j_q}\,\delta^{j_1\ldots j_q}_{m_1\ldots m_q} = \delta^{k_1\ldots k_p\ell_1\ldots\ell_q}_{i_1\ldots i_p m_1\ldots m_q}\,. \tag{1.151}$$

Comment: give somewhere the property

$$\frac{1}{q!}\,\underline{\varepsilon}_{i_1\ldots i_p j_1\ldots j_q}\,\delta^{j_1\ldots j_q}_{k_1\ldots k_q} = \underline{\varepsilon}_{i_1\ldots i_p k_1\ldots k_q}\,. \tag{1.152}$$

Note: Check if there are not factors $(-1)^{pq}$ missing.

Box 1.5 The Kronecker's determinants in 4-D

$$
\begin{aligned}
\delta^{mnpq}_{ijk\ell} &= (1/0!)\ \varepsilon_{ijk\ell}\,\varepsilon^{mnpq} \\
&= +\delta_i^m\delta_j^n\delta_k^p\delta_\ell^q + \delta_i^m\delta_j^p\delta_k^q\delta_\ell^n + \delta_i^m\delta_j^q\delta_k^n\delta_\ell^p + \delta_i^n\delta_j^q\delta_k^p\delta_\ell^m + \delta_i^n\delta_j^p\delta_k^m\delta_\ell^q + \delta_i^n\delta_j^m\delta_k^q\delta_\ell^p \\
&\quad +\delta_i^p\delta_j^q\delta_k^m\delta_\ell^n + \delta_i^p\delta_j^m\delta_k^n\delta_\ell^q + \delta_i^p\delta_j^n\delta_k^q\delta_\ell^m + \delta_i^q\delta_j^m\delta_k^p\delta_\ell^n + \delta_i^q\delta_j^n\delta_k^m\delta_\ell^p + \delta_i^q\delta_j^p\delta_k^n\delta_\ell^m \\
&\quad -\delta_i^m\delta_j^n\delta_k^q\delta_\ell^p - \delta_i^m\delta_j^p\delta_k^n\delta_\ell^q - \delta_i^m\delta_j^q\delta_k^p\delta_\ell^n - \delta_i^n\delta_j^p\delta_k^q\delta_\ell^m - \delta_i^n\delta_j^q\delta_k^m\delta_\ell^p - \delta_i^n\delta_j^m\delta_k^p\delta_\ell^q \\
&\quad -\delta_i^p\delta_j^q\delta_k^n\delta_\ell^m - \delta_i^p\delta_j^m\delta_k^q\delta_\ell^n - \delta_i^p\delta_j^n\delta_k^m\delta_\ell^q - \delta_i^q\delta_j^m\delta_k^n\delta_\ell^p - \delta_i^q\delta_j^n\delta_k^p\delta_\ell^m - \delta_i^q\delta_j^p\delta_k^m\delta_\ell^n \qquad (1.160) \\
\delta^{mnp}_{jk\ell} &= (1/1!)\ \varepsilon_{ijk\ell}\,\varepsilon^{imnp} \\
&= \delta_j^m\delta_k^n\delta_\ell^p + \delta_j^n\delta_k^p\delta_\ell^m + \delta_j^p\delta_k^m\delta_\ell^n - \delta_j^m\delta_k^p\delta_\ell^n - \delta_j^n\delta_k^m\delta_\ell^p - \delta_j^p\delta_k^n\delta_\ell^m \qquad (1.161) \\
\delta^{mn}_{k\ell} &= (1/2!)\ \varepsilon_{ijk\ell}\,\varepsilon^{ijmn} \\
&= (\delta_k^m\delta_\ell^n - \delta_k^n\delta_\ell^m) \qquad (1.162) \\
\delta^{m}_{\ell} &= (1/3!)\ \varepsilon_{ijk\ell}\,\varepsilon^{ijkm} \\
&= \delta_\ell^m \qquad (1.163) \\
\delta &= (1/4!)\ \varepsilon_{ijk\ell}\,\varepsilon^{ijk\ell} \\
&= 1 \qquad (1.164)
\end{aligned}
$$

Chapter 2

Derivatives of tensors

Note: rewrite this text...

Besides the necessary introduction of the notations, the aim of the chapter is to demonstrate that the tensors describing curvature and torsion satisfy, by definition, certain identities—the Bianchi identities.— In the 4-D space-time, after postulating a relationship between the tensors describing space-time curvature and torsion and the tensors describing mass and spin—the Einstein-Cartan equations,— the Bianchi identities lead to the differential equations governing the dynamics of continuous media.

2.1 Parallel transportation

Transporting a vector parallel to itself is trivial in an Euclidean space, but not so in a general space. The disturbing property of non-Euclidean spaces is that the vector we obtain by parallel transportation between two points depends on the path followed. This is obvious, for instance, when transporting vectors at the surface of a sphere (figure 2.1 shows an example).

2.1.1 Example of parallel transportation in a metric space

In a metric space, i.e., in a space where the notion of *length of a line* exists, it is not difficult to define the notion of parallel transport of a vector.

First, we need to introduce the notion of *geodesic line*: a line joigning two points is a geodesic line (for short: a *geodesic*) if all other neighbouring lines joining the two points have greater length. For instance, the great circle joigning two points in a sphere is the shortest of all neighbouring lines joining the two points: great circles on a sphere are geodesics.

Second, we need to remark that if the notion of length of a line exists, then we also have the notion of *angle*, as the angle (in *radians*) betwen two directions is just a ratio of two lengths. We will later see more precise characterizations of lengths and angles.

Then, the transportation of a vector parallel to itself along a geodesic gives, by definition, the vector with constant "length" that makes a constant angle with the geodesic.

We will see below how the parallel transportation of a vector along a line which is not a geodesic may be defined. If the path is not specified, the parallel transportation of a vector between two points is made along the geodesic joining the two points.

In a space with curvature, the result of parallel transport of a vector between two points depends on the path followed: in a sphere, transporting a vector parallel to itself from the Equator to the Pole, then from the Pole to another point in the Equator, does not give the same vector as when transporting the original vector directly along the Equator (note that a path made by two geodesics that do not join straightly is not a geodesic path).

Let us familiarize more with the parallel transport of a vector at the surface of a sphere.

Consider a vector at the North geographic pole of the Earth, and let us transport it, parallel to itself, to all other points of the Earth. As no particular path is specified, let us transport it along the geodesics of the sphere (the the great circles). The great circles joining the North pole to any other point of the sphere are the Meridians. If the given vector at the North pole points towards, say, the Meridian with longitude $\lambda = \pm 180^\circ$ (the line of change of date), it makes with any Meridian (at the Pole) an angle equal to the longitude of the Meridian. Then, the vector obtained by parallel transport along any Meridian will conserve this angle. This gives the vector field represented in figure 2.2. All the vectors of the figure are tangent to the sphere and have the same length (the orthographic projection used here gives, of course, different projected lengths). Notice that the vector obtained on a sphere by parallel transport of a vector at a given point on the sphere is uniquely defined everywhere excepted at the antipodal point: it is possible to go from one point to the antipodal point along different great circles, each one giving a different parallel-transported vector.

The components of this vector field are, in the natural basis,

$$V^\theta(\theta, \varphi) = -\frac{V_0}{R}\cos\varphi \qquad V^\varphi(\theta, \varphi) = \frac{V_0}{R}\frac{\sin\varphi}{\sin\theta} \,. \tag{2.1}$$

It is easy to verify (using the notion of metric tensor to be introduced below) that this corresponds to a vector field with constant norm $\|\mathbf{V}\| = V_0$, and making with any Meridian an angle equal to the longitude of the Meridian.

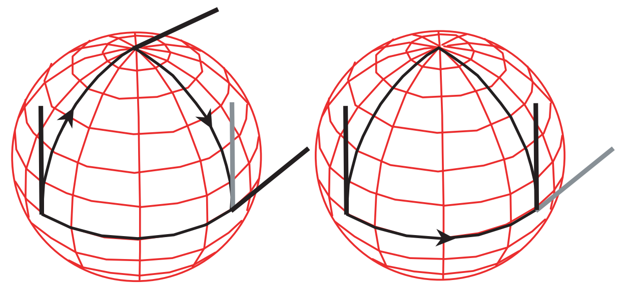

Figure 2.1: An illustration of a parallel transport of vectors. Left: a vector at the Equator, pointing to the North Pole, is transported, parallel to itself, to the Pole, then it is transported, parallel to itself, to the equator. Right: the same vector is transported, parallel to itself, to the same point, following a different path. The two vectors so obtained are not parallel to each other: the result of parallel transportation between two points depends, in general, on the path followed. Here, if the two Meridians followed during the parallel transportation are distant of 90 degrees, the two vectors obtained by parallel transportation of the same vector are perpendicular to each other.

Even if this field is defined by paralel transportation on the sphere from a single vector and, this should not be called a "constant" field. For its divergence (to be defined below) is nonzero,

$$\mathrm{div}\mathbf{V} = \frac{V_0}{R}\cos\varphi\frac{2\sin^2\frac{\theta}{2}}{\sin\theta} , \tag{2.2}$$

excepted at the pole itself (as it should, as very close to the pole we can replace the sphere by the tangent plane, and the parallel transport of a vector on an Euclidean plane gives a constant vector field).

The definition of parallel transportat of a vector along an arbitrary line, not necessarily a geodesic, requires "geodetic" notions, i.e., the notions that schools of physical geography teach to geodesists.

Consider that we have two directions defined at a point, i.e., two lines (not necessarily geodesics) intersecting at a point. Point O in figure 2.3 represents the intersecting point, and points A and B are points along each of the lines (the limit will be taken when these points tend to point O). If ACB is a geodesic with the length of AC equal to the length of CB , and OCD is a geodesic with the length of OC equal to the length of CD , then, by definition, in the limit where points A and B tend to point O , the direction BD is parallel to the direction OA , and the direction AD is parallel to the direction OB .

Comment: explain why I have drawn small "straight" lines in figure 2.3.

Comment: figures 2.4 and **??** just happen to be there.

We will see below that if we know how to transport a vector, then we know how to transport a tensor of any order.

So far, for the paralle transport of a vector in metric spaces. Let us now turn to an example of parallel transport in spaces where the notion of distance is not introduced.

2.1.2 Example of parallel transportation in a space without metric

If the manifold where we work is the physical space (or a surface embedded on the physical space), we may introduce the notion of metric (only very nonintuitive theories of the space-time do not assume the possibility of measuring distances or durations).

But we may need to develop methematics in a space which is not the physical space, and in which the notion of distance may not be defined. Let us see an example.

Assume an electrical conductor whose state we characterize by two parameters: its electrical conductivity, c , and the velocity v of some acoustic waves traveling inside the conductor. Assume

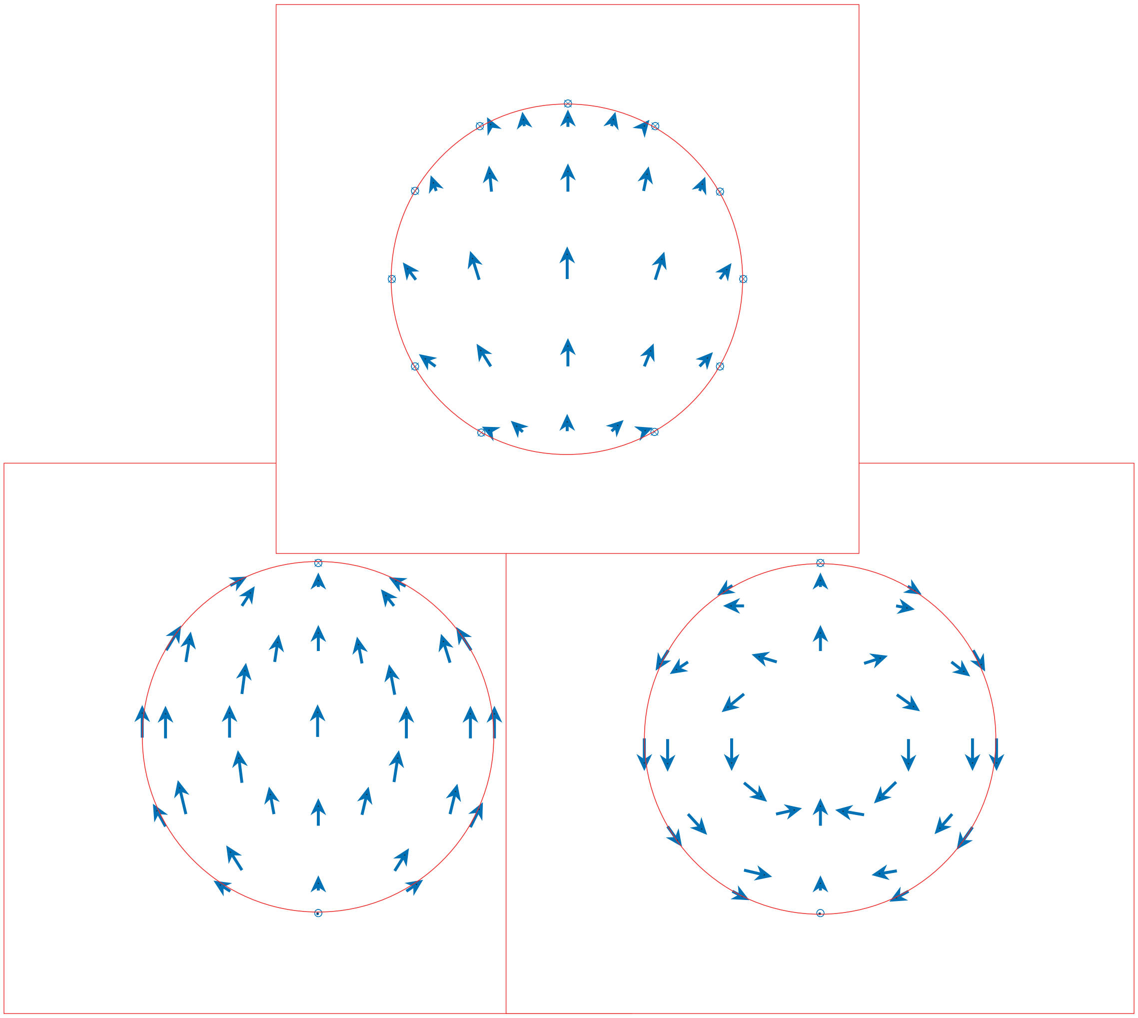

Figure 2.2: A vector at the North geographic pole of the Earth is transported, parallel to itself, to all other points of the Earth. Unless otherwise specified, the parallel transportation from one point to another point is made along a geodesic (here a great circle). The geodesics joining the North pole to any other point are the Meridians. The given vector at the North pole pointing towards the Meridian with longitude $\lambda = \pm 180°$, it makes, at the North pole, with any Meridian an angle equal to the longitude of the Meridian. Then, the vector obtained by parallel transport along any Meridian will conserve this angle, and this gives the vector field represented in the figure. All the vectors are tangent to the sphere and have the same length (the orthographic projection used here gives, of course, different projected lengths). Notice that the vector obtained on a sphere by parallel transport of a vector at a given point on the sphere is uniquely defined everywhere excepted at the antipodal point: it is possible to go from one point to the antipodal point along different great circles, each one giving a different parallel-transported vector.

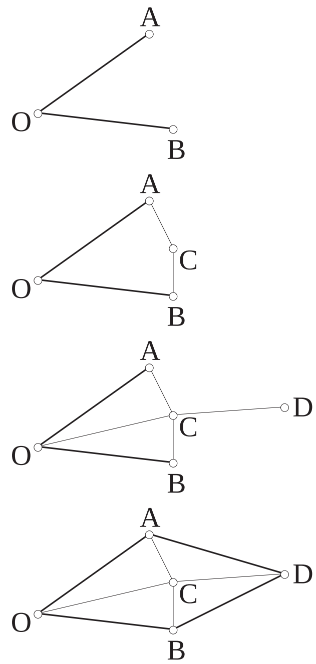

Figure 2.3: Consider three points O, A, B. In the limit where points A and B tend to point O, the lines OA and OB define two directions. We wish to transport the direction OA, parallel to itself, from point O to point B. This is accomplished by, first, locating the unique point C such that the distance AC equals the distance CB and that the line ACB is geodesic (i.e., no other point would give a smaller value for the distance AC plus the distance CB). Once that point is located, the unique point D is defined similarly: the distance OC equals the distance CD and the line OCD is geodesic. In the limit whwre all the distances tend to zero, the direction BD is, by definition, the direction obtained by parallel transport of the direction OA (and the direction AD is the direction line obtained by parallel transport of the direction OB). We have simply built a parallelogram. (Comment: is the length BD equal to the length OA? If yes, we do not need the point C and we can locate directly the point D)

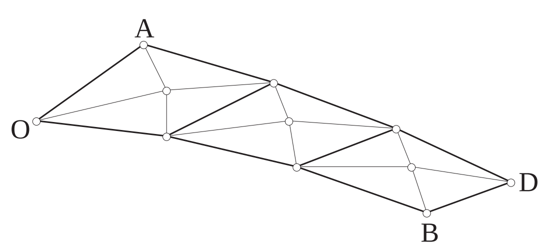

Figure 2.4: The method defined in 2.3 can be iterated to transport a direction along a finite arbitrary line OB. Comment: how can I demonstrate that the paralle transport along a geodesic conserves the angles?

that all the other parameters we are interested in are simply functions of the state of the system, i.e., functions of c , and v .

For instance, when the system is at point (c_0, v_0) , we may measure the rate of increase of conductivity, $\dot{c}$, and the rate of increase of the velocity of the acoustic waves, $\dot{v}$. The values $(\dot{c}, \dot{v})$ can be seen as the components of a vector at point (c_0, v_0) , of the conductivity-velocity space (see figure 2.5).

If we want to develop some mathematics in the conductivity-velocity space, we may need to define the ("covariant") derivatives of some parameter with respect to the conductivity or the velocity. But, as we will see below, there is no proper definition of derivatives without definition of parallel transportation of vectors. What then may be the vector obtained at any point by parallel transport of the vector $(\dot{c}, \dot{v})$ at point (c_0, v_0) ? The question is graphically illustrated at the bottom of figure 2.5. The suggested answer, that the vector translated keeps its components constant, can not be the answer. Let us see why.

There are in physics many physical parameters with this double property:

1. The parameter may only take positive values.

2. One indifferently uses the parameter or its inverse.

Here are some examples:

- Conductivity c and resistivity $\rho = 1/c$.
- Velocity v and slowness $s = 1/v$.
- Period (of a periodical phenomenon) T and frequency $\nu = 1/T$.
- Half-life of a radioactive nucleus, T and the desintegration rate $\lambda = \dots$.
- Mass density ρ and mass lightness $\ell = 1/\rho$.
- Temperature T and thermodynamic parameter $\beta = 1/kT$ (k is the Boltzmann's constant).

Comment: mention here Jaynes (1968).

Thus, if we have chosen as parameters the conductivity and the velocity (of acoustic waves), another physicist may choose the resistivity and the slowness, and there is no way of deciding which choice is "more natural."

But the definition of parallel transportation guessed at the bottom of figure 2.5 is not consistent with that freedom of choice for the parameters: Should the physicist working with resistivity and slowness also attempt to define the parallel transportation of a vector by keeping the components constant, she/he would afound another vector at the final point.

Let us be more precise. The top of figure 2.6 shows some vectors (with constant components) on the plane conductivity-velocity. Using the relation between resistivity and conductivity, $\rho = 1/c$, and between slowness and velocity, $s = 1/v$, we can map this figure into the figure at bottom. Should one be tempted to consider that the vectors at the top figure are all parallel, this would not be true for the vectors at the bottom figure.

Of course, all this is reciprocal for the alternate choice of parameters. The assumption that a set of vectors in the resistivity-slowness plane with constant components is a set of parallel vectors (top of figure 2.7) is not consistent with the figure obtained by coordinate transformation(bottom of figure 2.7).

Comment: explain here the Jaynes argument. It leads to the conclusion that the set of positive parameters described above have some fundamental properties. For instance, the null information probability density is not constant, but of the form $1/x$, they do not usually show normal probability densities but log-normal ones, etc.

Much simpler, then, that those parameters, are their logarithms. While, for instance, the null information probability density for a period T is

$$f(T) = \frac{\text{const.}}{T} \; , \tag{2.3}$$

the null information probability density for the logarithmic period, $T^* = \log(T/T_0)$ (T_0 arbitrary), is (as deduced from the rules of change of variables in probability theory) constant:

$$f^*(T^*) = \text{const.} \tag{2.4}$$

Also, if a period shows often log-normal normal probability densities, the logarithmic period shows normal ones.

Comment: say somewhere that the form of a probability density is not changed when passing from, say, a logarithmic period, to a logarithmic frequency.

The simplicity of the logarithmic parameters (for positive parameters) with respect to probability density functions is kept for the problem of parallel trabsport.

If physicist A, who prefers to work with conductivity and velocity, and physicist B, who prefers to work with resisitivity and slowness, both agree that parallel transportation can only be defined with simplicity in logarithmic parameters, then all problems disappear. Figure 2.8 illustrates this: The set of (truly) paralle vectors at top, in the logarithmic plane conductivity-velocity, maps, through the transformation $\rho = \frac{1}{c}$, and $s = \frac{1}{v}$, into the set of (truly) parallel vectors at bottom, in the logarithmic plane resistivity-slowness, as the logarithmic parameters are related by $\rho^* = -c^*$ and $s^* = -v^*$. Comment: explain all this much better.

Thus we have seen a set of nontrivial rules defining the parallel transportation of vectors in a space where the notion of metric does not exist (for what is the distance between two points?).

Comment: mention somewhere that all this is compatible with the choice of metric

$$\begin{pmatrix} g_{cc} & g_{cv} \\ g_{vc} & g_{vv} \end{pmatrix} = \begin{pmatrix} 1/c^2 & 0 \\ 0 & 1/v^2 \end{pmatrix} \; . \tag{2.5}$$

2.1.3 Important Notation

Comment: say what follows elsewhere.

In this section we will pave the way for the definition of "covariant" derivatives, i.e., proper definitions of derivatives that, when applied to tensor fields, give tensor fields. The fundamental concept at the base of the definition of a tensor derivative is that of parallel transport of vectors. I will introduce very explicit notations, as there are some technical details which are nontrivial. In particular, we will see that, when manipulating tensor fields, we may wish to introduce some fields that greatly simplify calculations, but which are not tensor fields (the "connection" is one of them).

Comment: in one of the previous sections, I should make explicit that, to any point of a curve, we can associate the tangent geodesic. Then, when I consider to neighbouring points along a curve, it is unimportant to consider that the points are along the given curve or along the tangent geodesic.

Let $\mathbf{x} \to \mathbf{V}(\mathbf{x})$ be a tensor field associating to any point $\mathbf{x}$ of the space the vector $\mathbf{V}(\mathbf{x})$. In this section, we will not simplify the discourse to just say "the vector field $\mathbf{V}(\mathbf{x})$." (Or yes?).

As discussed above (section XXX), unless another choice is explicitly stated, "the parallel transportation of the vector $\mathbf{V}$ from point $\mathbf{x}_A$ to point $\mathbf{x}_B$ " will mean along the geodesic joining the points $\mathbf{x}_A$ and $\mathbf{x}_B$.

The following notation is not common, and is very important for the definition of derivatives. Let $\mathbf{x} \to \mathbf{V}(\mathbf{x})$ be a vector field, and let $\mathbf{y}$ be one particular point of the space. The value of the vector

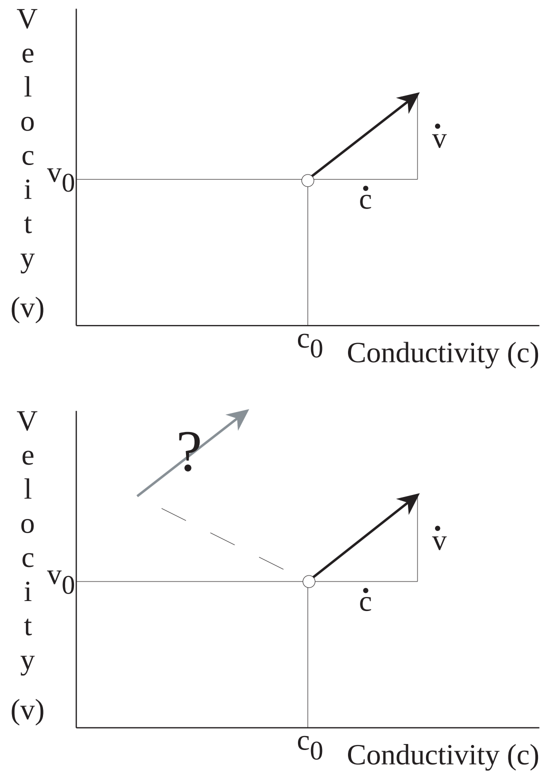

Figure 2.5: To be written.

field at point $\mathbf{y}$ is $\mathbf{V}(\mathbf{y})$. When transporting the vector $\mathbf{V}(\mathbf{y})$, parallel to itself, from point $\mathbf{y}$ to another point $\mathbf{x}$, we will, by definition, obtain a vector at point $\mathbf{x}$, and there is no reason that the vector so obtained at point $\mathbf{x}$ equals $\mathbf{V}(\mathbf{x})$, unless the vector field is constant. This is illustrated in figure 2.9. The vector obtained at point $\mathbf{x}$ by parallel transport from point $\mathbf{y}$ to point $\mathbf{x}$ of the vector $\mathbf{V}(\mathbf{y})$ will be denoted $\mathbf{V}(\mathbf{x} \parallel \mathbf{y})$.

For given point $\mathbf{y}$, the expression $\mathbf{x} \to \mathbf{V}(\mathbf{x} \parallel \mathbf{y})$ defines a vector field which depends on the value of the vector field $\mathbf{x} \to \mathbf{V}(\mathbf{x})$ at the single point $\mathbf{y}$, but that contains no information about the values of this vector field at other points (comment: very confusing...). See figure 2.9

2.1.4 Parallel transport of the basis vectors

The (natural) basis vectors at any point $\mathbf{x}$ have been defined in section XXX, where they have been denoted by $\mathbf{e}_i(\mathbf{x})$. In the preceding section we have introduced the notation $\mathbf{V}(\mathbf{x} \parallel \mathbf{y})$ to denote the vector obtained at point $\mathbf{x}$ by parallel transport of the vector $\mathbf{V}(\mathbf{y})$. Then, by $\mathbf{e}_i(\mathbf{x} \parallel \mathbf{y})$ we will denote the vectors obtained at point $\mathbf{x}$ by parallel transport of the basis $\mathbf{e}_i(\mathbf{y})$. Of course, the vectors obtained by transporting the basis from point $\mathbf{y}$ to point $\mathbf{x}$, $\mathbf{e}_i(\mathbf{x} \parallel \mathbf{y})$, will not, in general, equal the basis vectors at $\mathbf{x}$, $\mathbf{e}_i(\mathbf{x})$.

At point $\mathbf{x}$, the transported basis vectors, $\mathbf{e}_i(\mathbf{x} \parallel \mathbf{y})$, will have some components on the local basis $\mathbf{e}_i(\mathbf{x})$, that will be denoted $\gamma_i{}^j(\mathbf{x}|\mathbf{y})$:

$$\mathbf{e}_i(\mathbf{x} \parallel \mathbf{y}) = \gamma_i{}^j(\mathbf{x}|\mathbf{y})\, \mathbf{e}_j(\mathbf{x}) \,. \tag{2.6}$$

The coefficients for the parallel transportation of the dual basis will be denoted $\ell_i{}^j(\mathbf{x}|\mathbf{y})$:

$$\mathbf{e}^i(\mathbf{x} \parallel \mathbf{y}) = \mathbf{e}^j(\mathbf{x})\ \ell_j{}^i(\mathbf{x}|\mathbf{y}) \,. \tag{2.7}$$

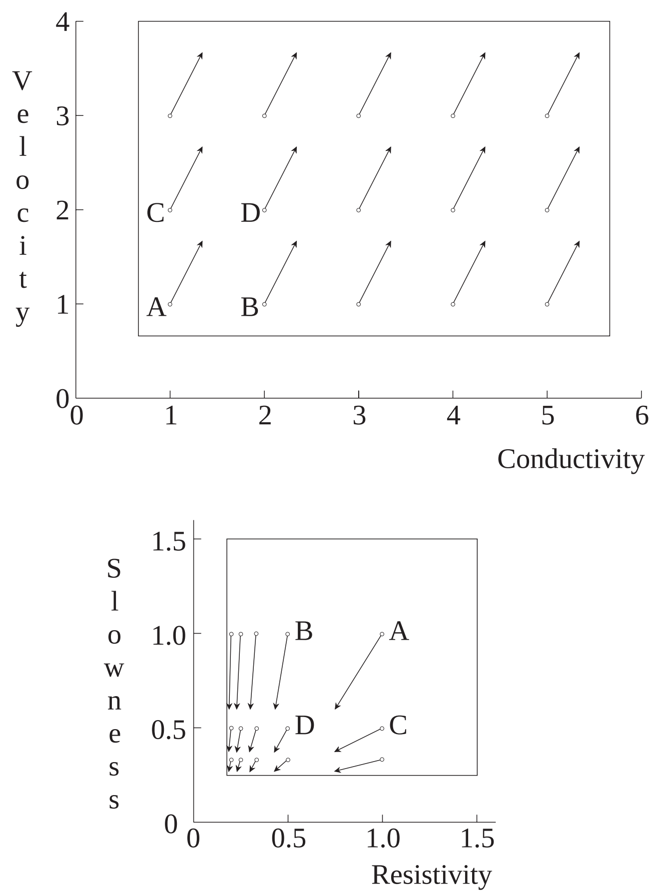

Figure 2.6: To be written. Comment: explain here what are the points A, B, C, and D. Comment: explain that there are units, not written.

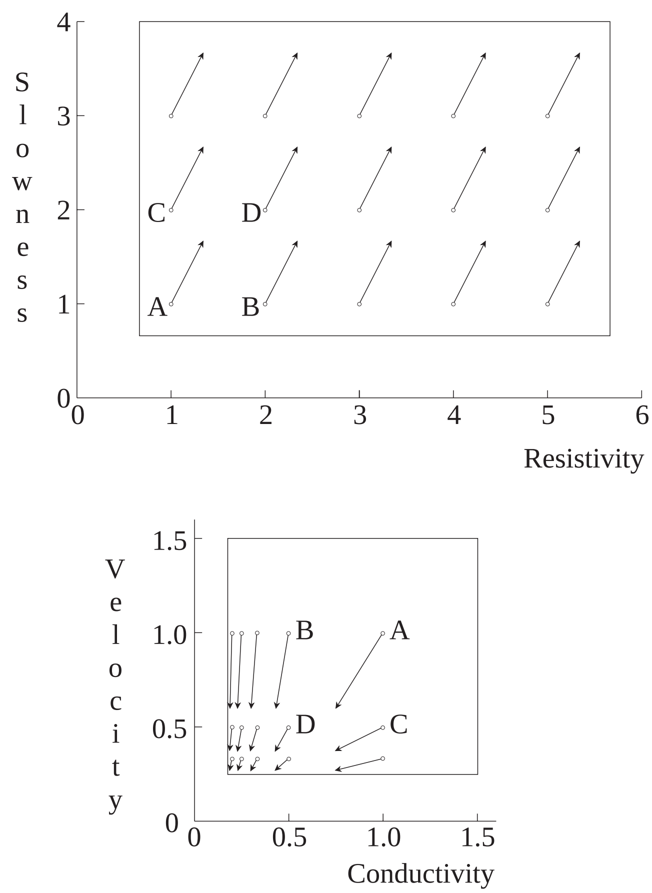

Figure 2.7: To be written.

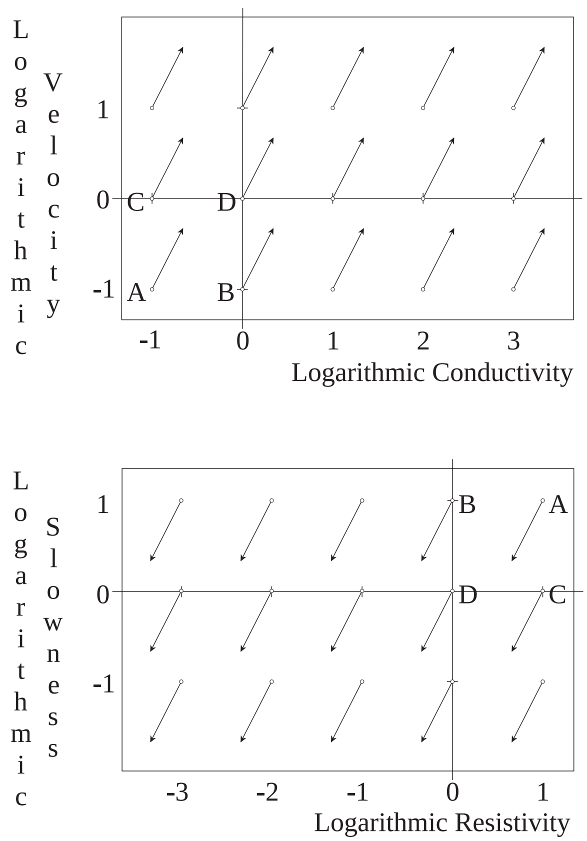

Figure 2.8: To be written.

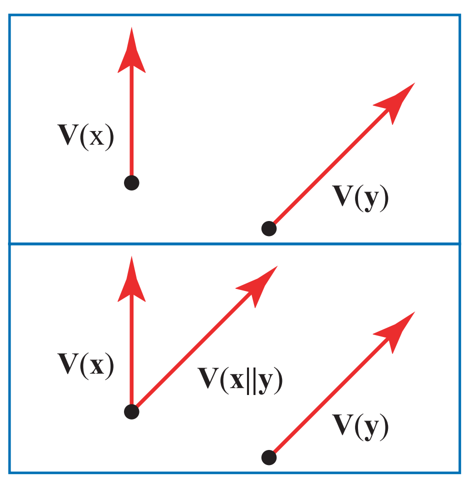

Figure 2.9: The notation described here is not common, and is very important for the definition of derivatives. Let $\mathbf{x} \to \mathbf{V}(\mathbf{x})$ be a vector field, and let $\mathbf{y}$ be one particular point of the space. The value of the vector field at point $\mathbf{y}$ is $\mathbf{V}(\mathbf{y})$. When transporting the vector $\mathbf{V}(\mathbf{y})$, parallel to itself, from point $\mathbf{y}$ to another point $\mathbf{x}$, we will, by definition, obtain a vector at point $\mathbf{x}$, and there is no reason that the vector so obtained at point $\mathbf{x}$ equals $\mathbf{V}(\mathbf{x})$, unless the vector field is constant. The vector obtained at point $\mathbf{x}$ by parallel transport from point $\mathbf{y}$ to point $\mathbf{x}$ of the vector $\mathbf{V}(\mathbf{y})$ will be denoted $\mathbf{V}(\mathbf{x} \parallel \mathbf{y})$. For a given point $\mathbf{y}$, the expression $\mathbf{x} \to \mathbf{V}(\mathbf{x} \parallel \mathbf{y})$ defines a vector field which depends on the value of the vector field $\mathbf{x} \to \mathbf{V}(\mathbf{x})$ at the single point $\mathbf{y}$, but that contains no information about the values of this vector field at other points.

As transporting a vector or a form from point $\mathbf{x}$ to point $\mathbf{x}$ along the geodesic means no transportation at all, we have

$$\gamma_i{}^j(\mathbf{x}|\mathbf{x}) = g_i{}^j = \delta_i{}^j \qquad \ell_i{}^j(\mathbf{x}|\mathbf{x}) = g_i{}^j = \delta_i{}^j \ . \tag{2.8}$$

As the transportation of a basis and of the dual basis must preserve their duality character, we will have

$$\mathbf{e}_i(\mathbf{x} \parallel \mathbf{y})\, \mathbf{e}^j(\mathbf{x} \parallel \mathbf{y}) = g_i{}^j = \delta_i{}^j \ , \tag{2.9}$$

i.e.,

$$\gamma_i{}^j(\mathbf{x}|\mathbf{y})\, \ell_j{}^k(\mathbf{x}|\mathbf{y}) = g_i{}^j = \delta_i{}^j \qquad \ell_i{}^j(\mathbf{x}|\mathbf{y})\, \gamma_j{}^k(\mathbf{x}|\mathbf{y}) = g_i{}^j = \delta_i{}^j \tag{2.10}$$

(the matrix representing $\gamma_i{}^j$ and the matrix representing $\ell_i{}^j$ are mutually inverses). Comment: say somewhere that, in fact, these equations simply mean that that the parallel transportation of the Kronecker's tensor gives the Kronecker's tensor.

Comment: I have to find how to espress $\gamma_i{}^j(\mathbf{x}|\mathbf{y})$ as a function of the metric, when there is one.

Comment: Old demonstration. Using

$$\mathbf{e}^i(\mathbf{x} \parallel \mathbf{y})\, \mathbf{e}_k(\mathbf{x} \parallel \mathbf{y}) = \ell_j{}^i(\mathbf{x}|\mathbf{y})\, \gamma_k{}^m(\mathbf{x}|\mathbf{y})\, \mathbf{e}^j(\mathbf{x})\, \mathbf{e}_m(\mathbf{x}) \ , \tag{2.11}$$

$$\mathbf{e}^i(\mathbf{x} \parallel \mathbf{y})\, \mathbf{e}_k(\mathbf{x} \parallel \mathbf{y}) = g^i{}_k \tag{2.12}$$

and

$$\mathbf{e}^j(\mathbf{x})\, \mathbf{e}_m(\mathbf{x}) = g^j{}_m \ , \tag{2.13}$$

we deduce

$$\gamma_k{}^j(\mathbf{x} \parallel \mathbf{y})\, \ell_j{}^i(\mathbf{x} \parallel \mathbf{y}) = g_k{}^i \ . \tag{2.14}$$

In fact, we have simply transported the Kronecker's tensor.

Example: In an Euclidean space with Cartesian coordinates, transporting a vector parallel to itself gives, on the local basis at the final point, the same components than on the local basis at the original point. Then,

$$\gamma_i{}^j(\mathbf{x}|\mathbf{y}) = g_i{}^j = \delta_i{}^j \ . \tag{2.15}$$

•

Example: In polar coordinates, let $\{\mathbf{e}_r(r_0,\varphi_0), \mathbf{e}_\varphi(r_0,\varphi_0)\}$ denote the natural basis at point (r_0,φ_0) , and let $\{\mathbf{e}_r(r,\varphi \parallel r_0,\varphi_0), \mathbf{e}_\varphi(r,\varphi \parallel r_0,\varphi_0)\}$ denote the vectors obtained by parallel transport of $\{\mathbf{e}_r(r_0,\varphi_0), \mathbf{e}_\varphi(r_0,\varphi_0)\}$ from point (r_0,φ_0) to point (r,φ) . It is quite easy to express the transported vectors on the local basis, as there is only a rotation and a change of length of the vector $\mathbf{e}_\varphi$. We have (see figure 2.10)

$$\begin{pmatrix} \mathbf{e}_r(r,\varphi \parallel r_0,\varphi_0) \\ \mathbf{e}_\varphi(r,\varphi \parallel r_0,\varphi_0) \end{pmatrix} = \begin{pmatrix} \cos(\varphi-\varphi_0) & -\frac{1}{r}\sin(\varphi-\varphi_0) \\ r_0\sin(\varphi-\varphi_0) & \frac{r_0}{r}\cos(\varphi-\varphi_0) \end{pmatrix} \begin{pmatrix} \mathbf{e}_r(r,\varphi) \\ \mathbf{e}_\varphi(r,\varphi) \end{pmatrix} \ . \tag{2.16}$$

This gives, for the coefficients $\gamma_i{}^j$,

$$\begin{pmatrix} \gamma_r{}^r(r,\varphi|r_0,\varphi_0) & \gamma_r{}^\varphi(r,\varphi|r_0,\varphi_0) \\ \gamma_\varphi{}^r(r,\varphi|r_0,\varphi_0) & \gamma_\varphi{}^\varphi(r,\varphi|r_0,\varphi_0) \end{pmatrix} = \begin{pmatrix} \cos(\varphi-\varphi_0) & -\frac{1}{r}\sin(\varphi-\varphi_0) \\ r_0\sin(\varphi-\varphi_0) & \frac{r_0}{r}\cos(\varphi-\varphi_0) \end{pmatrix} \ , \tag{2.17}$$

and the coefficients $\ell_i{}^j$ are obtained by inverting the matrix:

$$\begin{pmatrix} \ell_r{}^r(r,\varphi|r_0,\varphi_0) & \ell_r{}^\varphi(r,\varphi|r_0,\varphi_0) \\ \ell_\varphi{}^r(r,\varphi|r_0,\varphi_0) & \ell_\varphi{}^\varphi(r,\varphi|r_0,\varphi_0) \end{pmatrix} = \begin{pmatrix} \cos(\varphi-\varphi_0) & \frac{1}{r_0}\sin(\varphi-\varphi_0) \\ -r\sin(\varphi-\varphi_0) & \frac{r}{r_0}\cos(\varphi-\varphi_0) \end{pmatrix} \ . \tag{2.18}$$

•

Note: we have there

$$\ell_i^j(\mathbf{x}|\mathbf{y}) = \gamma_i^j(\mathbf{y}|\mathbf{x}) \ . \tag{2.19}$$

Check if this is a general property and, if yes, give the corresponding theorem.

Example: The surface of the sphere being a curved (2-D) space, the result of parallel transportation of a vector between two points depends on the path followed (see for instance figure 2.1). The transportation of vectors along geodesics (great circles) has been illustrated by figure 2.2. Let us now examine the parallel transportation along the coordinate lines: if the Meridians are geodesics, the Parallels are not. We show in appendix XXX that, for the parallel transport along a Meridian, we have

$$\begin{aligned}\mathbf{e}_\theta(\theta,\varphi \parallel \theta',\varphi) &= \mathbf{e}_\theta(\theta,\varphi)\\ \mathbf{e}_\varphi(\theta,\varphi \parallel \theta',\varphi) &= \frac{\sin\theta'}{\sin\theta}\,\mathbf{e}_\theta(\theta,\varphi)\,,\end{aligned}\tag{2.20}$$

i.e.,

$$\begin{pmatrix}\gamma_\theta{}^\theta(\theta,\varphi|\theta',\varphi) & \gamma_\theta{}^\varphi(\theta,\varphi|\theta',\varphi)\\ \gamma_\varphi{}^\theta(\theta,\varphi|\theta',\varphi) & \gamma_\varphi{}^\varphi(\theta,\varphi|\theta',\varphi)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \sin\theta'/\sin\theta\end{pmatrix}\,,\tag{2.21}$$

while, for the parallel transportation along a Parallel, we have

$$\begin{aligned}\mathbf{e}_\theta(\theta,\varphi \parallel \theta,\varphi') &= \cos\left[(\varphi-\varphi')\cos\theta\right]\mathbf{e}_\theta(\theta,\varphi) - \frac{1}{\sin\theta}\,\sin\left[(\varphi-\varphi')\cos\theta\right]\mathbf{e}_\varphi(\theta,\varphi)\\ \mathbf{e}_\varphi(\theta,\varphi \parallel \theta,\varphi') &= \sin\theta\sin\left[(\varphi-\varphi')\cos\theta\right]\mathbf{e}_\theta(\theta,\varphi) + \cos\left(\left[\varphi-\varphi'\right)\cos\theta\right]\mathbf{e}_\varphi(\theta,\varphi)\,,\end{aligned}\tag{2.22}$$

i.e.,

$$\begin{pmatrix}\gamma_\theta{}^\theta(\theta,\varphi|\theta,\varphi') & \gamma_\theta{}^\varphi(\theta,\varphi|\theta,\varphi')\\ \gamma_\varphi{}^\theta(\theta,\varphi|\theta,\varphi') & \gamma_\varphi{}^\varphi(\theta,\varphi|\theta,\varphi')\end{pmatrix} = \begin{pmatrix}\cos\left[(\varphi-\varphi')\cos\theta\right] & -\sin\left[(\varphi-\varphi')\cos\theta\right]/\sin\theta\\ \sin\theta\sin\left[(\varphi-\varphi')\cos\theta\right] & \cos\left(\left[\varphi-\varphi'\right)\cos\theta\right]\end{pmatrix}.\tag{2.23}$$

•

2.1.5 Parallel transport of vectors

Let us naturally denote $V^i(\mathbf{x}\parallel\mathbf{y})$ the components on the local basis at point $\mathbf{x}$ of the vector $\mathbf{V}(\mathbf{x}\parallel\mathbf{y})$:

$$\mathbf{V}(\mathbf{x}\parallel\mathbf{y}) = V^i(\mathbf{x}\parallel\mathbf{y})\,\mathbf{e}_i(\mathbf{x})\,.\tag{2.24}$$

How can we express them as a function of the original components of the vector?

At point $\mathbf{y}$ we have

$$\mathbf{V}(\mathbf{y}) = V^i(\mathbf{y})\,\mathbf{e}_i(\mathbf{y})\,.\tag{2.25}$$

If we transport to point $\mathbf{x}$ both the vector $\mathbf{V}(\mathbf{y})$ and the basis $\mathbf{e}_i(\mathbf{y})$, the components of the transported vector on the transported basis will remain the same they were at the original point:

$$\mathbf{V}(\mathbf{x}\parallel\mathbf{y}) = V^i(\mathbf{y})\,\mathbf{e}_i(\mathbf{x}\parallel\mathbf{y})\,,\tag{2.26}$$

or, using equation 2.6,

$$\mathbf{V}(\mathbf{x}\parallel\mathbf{y}) = V^i(\mathbf{y})\,\gamma_i{}^j(\mathbf{x}|\mathbf{y})\,\mathbf{e}_j(\mathbf{x})\,,\tag{2.27}$$

i.e.,

$$V^j(\mathbf{x}\parallel\mathbf{y}) = V^i(\mathbf{y})\,\gamma_i{}^j(\mathbf{x}|\mathbf{y})\,.\tag{2.28}$$

This important equation solves formally the problem of parallel transport of vectors: if the functions $\gamma_i{}^j(\mathbf{x}|\mathbf{y})$ are given, then we know the result of parallel transportation of any vector from any point to any other point.

For a form (or a "covariant vector") we will have

$$\mathbf{F}(\mathbf{x}\parallel\mathbf{y}) = \mathbf{e}^i(\mathbf{x}\parallel\mathbf{y})\,F_i(\mathbf{y}) = \mathbf{e}^j(\mathbf{x})\,\ell_j{}^i(\mathbf{x}|\mathbf{y})\,F_i(\mathbf{y})\,,\tag{2.29}$$

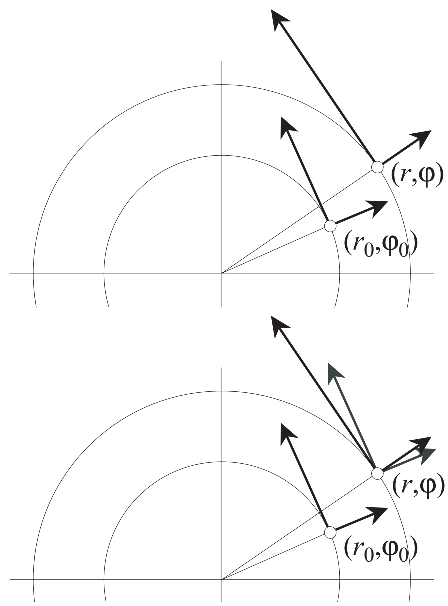

Figure 2.10: Top: the local basis at two points. Bottom: in addition, the vectors obtained at point XXX by parallel transport of the basis at XXX.

i.e.,

$$F_j(\mathbf{x} \parallel \mathbf{y}) = \ell_j{}^i(\mathbf{x}|\mathbf{y})\, F_i(\mathbf{y}) \,. \tag{2.30}$$

For a general tensor

$$\mathbf{T}(\mathbf{y}) = T_{k\ell\ldots}{}^{ij\cdots}(\mathbf{y})\ \mathbf{e}^k(\mathbf{y}) \otimes \mathbf{e}^\ell(\mathbf{y}) \otimes \ldots \mathbf{e}_i(\mathbf{y}) \otimes \mathbf{e}_j(\mathbf{y}) \otimes \ldots \tag{2.31}$$

we obtain

$$T_{pq\ldots}{}^{mn\cdots}(\mathbf{x} \parallel \mathbf{y}) = T_{k\ell\ldots}{}^{ij\cdots}(\mathbf{y})\, \ell_p{}^k(\mathbf{x}|\mathbf{y})\, \ell_q{}^\ell(\mathbf{x}|\mathbf{y})\ \ldots\ \gamma_i{}^m(\mathbf{x}|\mathbf{y})\, \gamma_j{}^n(\mathbf{x}|\mathbf{y})\ \ldots \,. \tag{2.32}$$

Let us look with some detail at equation 2.27. It is a perfectly valid tensor equation at point $\mathbf{x}$, in the sense that we have one tensor at each side of the equality sign, and that the two tensors are equal in the standard sense. But, besides the field variable $\mathbf{x}$, the equation has, at both sides, the variable $\mathbf{y}$. We have to interpret properly the difference between the variables $\mathbf{x}$ and $\mathbf{y}$ in this equation. For any value of $\mathbf{y}$, we have an ordinary tensor equation at point $\mathbf{x}$, so, in this sense, $\mathbf{x}$ is a field variable, but $\mathbf{y}$ is not. Comment: explain here that the variable $\mathbf{y}$ plays here a role very similar to the variable y in a conditional probability density $f(x|y)$: while x is a *random variable*, y is not.

2.1.6 Parallel transport of densities and capacities

We have just seen that during the parallel transportation of a vector, its components on the local basis may change (because the local basis may be different at every point).

We have introduced above, three kinds of scalar fields: true scalars, $\varphi(\mathbf{x})$, densities $\overline{\varphi}(\mathbf{x})$ and capacities $\underline{\varphi}(\mathbf{x})$. As these are tensor fields, they must have their parallel transportation defined.

We are going to justify here the following result: The parallel transportation of a true scalar $\varphi(\mathbf{y})$, from point $\mathbf{y}$ to point $\mathbf{x}$, leaves its value unchanged:

$$\varphi(\mathbf{x} \parallel \mathbf{y}) = \varphi(\mathbf{y}) \,. \tag{2.33}$$

The parallel transportation of a scalar capacity $\underline{\varphi}(\mathbf{y})$ gives

$$\underline{\varphi}(\mathbf{x} \parallel \mathbf{y}) = \frac{\overline{g}(\mathbf{y})}{\overline{g}(\mathbf{x})}\, \underline{\varphi}(\mathbf{y}) \,. \tag{2.34}$$

The parallel transportation of a scalar density $\overline{\varphi}(\mathbf{y})$ gives

$$\overline{\varphi}(\mathbf{x} \parallel \mathbf{y}) = \frac{\overline{g}(\mathbf{x})}{\overline{g}(\mathbf{y})}\, \overline{\varphi}(\mathbf{y}) \,. \tag{2.35}$$

Here, $\overline{g}(\mathbf{x})$ denotes the value at point $\mathbf{x}$ of the "fundamental density", i.e, the square root of the determinant of the metric tensor (see section xxx). Comment: say that, in non metric spaces, a function $\overline{g}(\mathbf{x})$ has to be given that, even it is not related to a metric, defines, in an ad-hoc way, the parallel transportation of densities and capacities.

We have

$$T_p{}^m(\mathbf{x} \parallel \mathbf{y}) = T_k{}^i(\mathbf{y})\, \ell_p{}^k(\mathbf{x}|\mathbf{y})\, \gamma_i{}^m(\mathbf{x}|\mathbf{y}) \tag{2.36}$$

which gives

$$T_s{}^s(\mathbf{x} \parallel \mathbf{y}) = T_k{}^i(\mathbf{y})\, \ell_s{}^k(\mathbf{x}|\mathbf{y})\, \gamma_i{}^s(\mathbf{x}|\mathbf{y}) \tag{2.37}$$

and using

$$\ell_s{}^k(\mathbf{x}|\mathbf{y})\, \gamma_i{}^s(\mathbf{x}|\mathbf{y}) = \delta_i{}^k \,, \tag{2.38}$$

$$T_s{}^s(\mathbf{x} \parallel \mathbf{y}) = T_s{}^s(\mathbf{y}) \,. \tag{2.39}$$

As the trace of a (true) tensor is a true scalar, this justifies the definition 2.33. Equations 2.34 and 2.35 follow from it when considering that the product (resp. the ratio) of a capacity (resp. a density) field by the fundamental density gives a pure tensor field.

2.1.7 The Connection

In what follows we will need the definition

$$\Gamma_{ki}{}^{j}(\mathbf{x}|\mathbf{y}) = \frac{\partial \gamma_i{}^j}{\partial x^k}(\mathbf{x}|\mathbf{y}) \,. \tag{2.40}$$

where the loose notations indicate derivation with respect to the second variable in the function $\gamma_i{}^j(\mathbf{x}|\mathbf{y})$. As there is no risk of confusion, and to economize symbols, we will also use the letter Γ to denote the value of the expression 2.40 for $\mathbf{y} = \mathbf{x}$:

$$\Gamma_{ki}{}^{j}(\mathbf{x}) = \Gamma_{ki}{}^{j}(\mathbf{x}|\mathbf{x}) \,. \tag{2.41}$$

Here we are taking derivatives with respect the variable $\mathbf{y}$ which, as mentioned in section 2.1.4 is not a "field" variable. In addition, as we take partial derivatives, and not "covariant" derivatives, the index k in $\Gamma_{ki}{}^{j}$ is not a tensor index: $\Gamma_{ki}{}^{j}$ is not a tensor.

Similarly, for the coefficients defining the parallel transport of forms we define

$$\mathrm{L}_{ij}{}^{k}(\mathbf{x}|\mathbf{y}) = \frac{\partial \ell_j{}^k}{\partial y^i}(\mathbf{x}|\mathbf{y}) \tag{2.42}$$

and

$$\Gamma_{ki}{}^{j}(\mathbf{x}) = \Gamma_{ki}{}^{j}(\mathbf{x}|\mathbf{x}) \,. \tag{2.43}$$

Taking the derivative of equation 2.14 gives

$$\frac{\partial \gamma_k{}^j}{\partial y^m}(\mathbf{x}|\mathbf{y})\, \ell_j{}^i(\mathbf{x}|\mathbf{y}) + \gamma_k{}^j(\mathbf{x}|\mathbf{y})\, \frac{\partial \ell_j{}^i}{\partial y^m}(\mathbf{x}|\mathbf{y}) = 0 \,, \tag{2.44}$$

and, making $\mathbf{y} = \mathbf{x}$,

$$\Gamma_{mk}{}^{j}(\mathbf{x}|\mathbf{x}) g_j{}^i + g_k{}^j \mathrm{L}_{mj}{}^{i}(\mathbf{x}|\mathbf{x}) = 0 \,, \tag{2.45}$$

i.e.,

$$\Gamma_{mk}{}^{i}(\mathbf{x}) = -\mathrm{L}_{mk}{}^{i}(\mathbf{x}) \,. \tag{2.46}$$

Comment: this is a very important equation.

We have introduced the coefficients $\gamma_i{}^j(\mathbf{x}|\mathbf{y})$ and $\Gamma^k_{ij}(\mathbf{x})$, corresponding the vector basis, and the coefficients $\ell_i{}^j(\mathbf{x}|\mathbf{y})$ and $\mathrm{L}^k_{ij}(\mathbf{x})$, corresponding the dual basis. We could well introduce different names for those fields (that have all in common of not being ordinary tensor fields). Rather we will simply say "the connection coefficients $\ell_i{}^j(\mathbf{x}|\mathbf{y})$ ", or "the connection coefficients $\Gamma_{ij}{}^k(\mathbf{x})$ ", the symbol indicating the field being used.

Example: Polar coordinates. We compute here the connection directly from the coefficients defining the parallel transportation of tensors, without using the metric. We have seen alsewhere that the coefficients are given, for polar coordinates, by

$$\begin{pmatrix} \gamma_r{}^r(r_0,\varphi_0|r,\varphi) & \gamma_r{}^\varphi(r_0,\varphi_0|r,\varphi) \\ \gamma_\varphi{}^r(r_0,\varphi_0|r,\varphi) & \gamma_\varphi{}^\varphi(r_0,\varphi_0|r,\varphi) \end{pmatrix} = \begin{pmatrix} \cos(\varphi-\varphi_0) & \frac{1}{r_0}\sin(\varphi-\varphi_0) \\ -r\sin(\varphi-\varphi_0) & \frac{r}{r_0}\cos(\varphi-\varphi_0) \end{pmatrix} \,. \tag{2.47}$$

The definition 2.40 then gives

$$\begin{pmatrix} \Gamma_{rr}{}^r(r_0,\varphi_0|r,\varphi) & \Gamma_{rr}{}^\varphi(r_0,\varphi_0|r,\varphi) \\ \Gamma_{r\varphi}{}^r(r_0,\varphi_0|r,\varphi) & \Gamma_{r\varphi}{}^\varphi(r_0,\varphi_0|r,\varphi) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\sin(\varphi-\varphi_0) & \frac{1}{r_0}\cos(\varphi-\varphi_0) \end{pmatrix} \tag{2.48}$$

$$\begin{pmatrix} \Gamma_{\varphi r}{}^r(r_0,\varphi_0|r,\varphi) & \Gamma_{\varphi r}{}^\varphi(r_0,\varphi_0|r,\varphi) \\ \Gamma_{\varphi\varphi}{}^r(r_0,\varphi_0|r,\varphi) & \Gamma_{\varphi\varphi}{}^\varphi(r_0,\varphi_0|r,\varphi) \end{pmatrix} = \begin{pmatrix} -\sin(\varphi-\varphi_0) & \frac{1}{r_0}\cos(\varphi-\varphi_0) \\ -r\cos(\varphi-\varphi_0) & -\frac{r}{r_0}\sin(\varphi-\varphi_0) \end{pmatrix} \,. \tag{2.49}$$

Finally, definition 2.41 gives

$$\begin{pmatrix} \Gamma_{rr}{}^{r}(r,\varphi) & \Gamma_{rr}{}^{\varphi}(r,\varphi) \\ \Gamma_{r\varphi}{}^{r}(r,\varphi) & \Gamma_{r\varphi}{}^{\varphi}(r,\varphi) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \tag{2.50}$$

$$\begin{pmatrix} \Gamma_{\varphi r}{}^{r}(r,\varphi) & \Gamma_{\varphi r}{}^{\varphi}(r,\varphi) \\ \Gamma_{\varphi\varphi}{}^{r}(r,\varphi) & \Gamma_{\varphi\varphi}{}^{\varphi}(r,\varphi) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{r} \\ -r & 0 \end{pmatrix} . \tag{2.51}$$

•

2.1.8 Parallel transport of the metric tensor

The notions of parallel transport defined above need rules that may not be based on any notion of metric (see the example conductivity-velocity). If there is a metric, it is also possible to define the parallel transport, as if we can measure distances and angles, we can perform parallel transportation.

Then, if the have the symbols $\gamma_i{}^j(\mathbf{x} \parallel \mathbf{y})$ defining the parallel transportation and we have a metric $g_{ij}(\mathbf{x})$, there must be some relations between them that ensure compatibility (comment: explain better).

Comment: I have to explain here why “the parallel transportation of the metric gives the metric”. In any case, this writes

$$\mathbf{g}(\mathbf{x} \parallel \mathbf{y}) = \mathbf{g}(\mathbf{x}) \, , \tag{2.52}$$

i.e.,

$$g_{ij}(\mathbf{x} \parallel \mathbf{y}) = g_{ij}(\mathbf{x}) \, . \tag{2.53}$$

The general rule of transportation of tensors (equation 2.32) gives

$$g_{ij}(\mathbf{x} \parallel \mathbf{y}) = \ell_i{}^p(\mathbf{x}|\mathbf{y}) \, \ell_j{}^q(\mathbf{x}|\mathbf{y}) \, g_{pq}(\mathbf{y}) \, , \tag{2.54}$$

or, using 2.53,

$$g_{ij}(\mathbf{x}) = \ell_i{}^p(\mathbf{x}|\mathbf{y}) \, \ell_j{}^q(\mathbf{x}|\mathbf{y}) \, g_{pq}(\mathbf{y}) \, . \tag{2.55}$$

This is the constraint linking the notion of parallel transport and the metric (when it exists). (Comment: yes, but how can I *compute* the coefficients $\ell_i{}^j(\mathbf{x} \parallel \mathbf{y})$ from the metric $g_{ij}(\mathbf{x})$?)

Taking partial derivatives in equation 2.55 with respect to the (non-field) variable $\mathbf{y}$ and using the definition 2.42 of the connection gives

$$\mathrm{L}_{mi}{}^p(\mathbf{x}|\mathbf{y}) \, \ell_j{}^q(\mathbf{x}|\mathbf{y}) \, g_{pq}(\mathbf{y}) + \ell_i{}^p(\mathbf{x}|\mathbf{y}) \, \mathrm{L}_{mj}{}^q(\mathbf{x}|\mathbf{y}) \, g_{pq}(\mathbf{y}) + \ell_i{}^p(\mathbf{x}|\mathbf{y}) \, \ell_j{}^q(\mathbf{x}|\mathbf{y}) \, \partial_m g_{pq}(\mathbf{y}) = 0 \, . \tag{2.56}$$

Taking there $\mathbf{x} = \mathbf{y}$, and using the notation 2.43 gives

$$\mathrm{L}_{mi}{}^p(\mathbf{x}) \, \delta_j{}^q \, g_{pq}(\mathbf{x}) + \delta_i{}^p \, \mathrm{L}_{mj}{}^q(\mathbf{x}) \, g_{pq}(\mathbf{x}) + \delta_i{}^p \, \delta_j{}^q \, \partial_m g_{pq}(\mathbf{x}) = 0 \, , \tag{2.57}$$

i.e.,

$$\mathrm{L}_{mij}(\mathbf{x}) + \mathrm{L}_{mji}(\mathbf{x}) + \partial_m g_{ij}(\mathbf{x}) = 0 \, , \tag{2.58}$$

where I have introduced the notation

$$\mathrm{L}_{ijk}(\mathbf{x}) = g_{ks}(\mathbf{x}) \, \mathrm{L}_{ij}{}^s(\mathbf{x}) \tag{2.59}$$

that toguether with

$$\Gamma_{ijk}(\mathbf{x}) = g_{ks}(\mathbf{x}) \, \Gamma_{ij}{}^s(\mathbf{x}) \tag{2.60}$$

are common, but we should not forget thet the connections $\Gamma_{ij}{}^k$ and $\mathrm{L}_{ij}{}^k$ are not tensors. Using finally the property 2.46, equation 2.58 gives a simple and important relationship between the connection and the metric:

$$\partial_k g_{ij} = \Gamma_{kij} + \Gamma_{kji} \, . \tag{2.61}$$

The torsion tensor will be defined below as

$$S_{ij}{}^{k} = \Gamma_{ij}{}^{k} - \Gamma_{ji}{}^{k} \,. \tag{2.62}$$

With this definition, a simple substitution shows that equation 2.61 implies the property

$$\Gamma_{ijk} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right) + \frac{1}{2}\left(S_{ijk} + S_{kij} + S_{kji}\right) \,. \tag{2.63}$$

It shows that if the metric and the torsion are given, then we can compute the connection. This, in particular, shows that the metric alone is not enough to compute the connection.

Comment: I have a problem here. Apparently, the metric perfectly defines the parallel transportation, so it should define the coefficients $\gamma_i{}^j(\mathbf{x}|\mathbf{y})$ and, thus, the connection. This implies that **my previous definition of parallel transportation from the metric is only valid for spaces without torsion**.

Comment: how can I compute $\gamma_i{}^j(\mathbf{x} \parallel \mathbf{y})$ from $g_{ij}(\mathbf{x})$?

Example: In the Euclidean plane with polar coordinates, the connection coefficients $\ell_i{}^j$ are given by (equation 2.18)

$$\begin{pmatrix} \ell_r{}^r(r,\varphi|r_0,\varphi_0) & \ell_r{}^\varphi(r,\varphi|r_0,\varphi_0) \\ \ell_\varphi{}^r(r,\varphi|r_0,\varphi_0) & \ell_\varphi{}^\varphi(r,\varphi|r_0,\varphi_0) \end{pmatrix} = \begin{pmatrix} \cos(\varphi-\varphi_0) & \frac{1}{r_0}\sin(\varphi-\varphi_0) \\ -r\sin(\varphi-\varphi_0) & \frac{r}{r_0}\cos(\varphi-\varphi_0) \end{pmatrix} \,. \tag{2.64}$$

Them the parallel transportation of a tensor $F_{ij}(r_0,\varphi_0)$ from the point (r_0,φ_0) to a point (r,φ) is given, using equation 2.32, by

$$F_{ij}(\mathbf{x} \parallel \mathbf{x}_0) = \ell_i{}^p(\mathbf{x}|\mathbf{x}_0)\, \ell_j{}^q(\mathbf{x}|\mathbf{x}_0)\, F_{pq}(\mathbf{x}_0) \,. \tag{2.65}$$

As an example, this gives, for the component $\varphi\varphi$:

$$\begin{aligned} F_{\varphi\varphi}(r,\varphi \parallel r_0,\varphi_0) &= \ell_\varphi{}^r(r,\varphi|r_0,\varphi_0)\, \ell_\varphi{}^r(r,\varphi|r_0,\varphi_0)\, F_{rr}(r_0,\varphi_0) \\ &+ \ell_\varphi{}^r(r,\varphi|r_0,\varphi_0)\, \ell_\varphi{}^\varphi(r,\varphi|r_0,\varphi_0)\, F_{r\varphi}(r_0,\varphi_0) \\ &+ \ell_\varphi{}^\varphi(r,\varphi|r_0,\varphi_0)\, \ell_\varphi{}^r(r,\varphi|r_0,\varphi_0)\, F_{\varphi r}(r_0,\varphi_0) \\ &+ \ell_\varphi{}^\varphi(r,\varphi|r_0,\varphi_0)\, \ell_\varphi{}^\varphi(r,\varphi|r_0,\varphi_0)\, F_{\varphi\varphi}(r_0,\varphi_0) \,, \end{aligned} \tag{2.66}$$

i.e.,

$$\begin{aligned} F_{\varphi\varphi}(r,\varphi \parallel r_0,\varphi_0) &= r^2\sin^2(\varphi-\varphi_0)\, F_{rr}(r_0,\varphi_0) \\ &- (r^2/r_0)\, \sin(\varphi-\varphi_0)\, \cos(\varphi-\varphi_0)\, F_{r\varphi}(r_0,\varphi_0) \\ &- (r^2/r_0)\, \sin(\varphi-\varphi_0)\, \cos(\varphi-\varphi_0)\, F_{\varphi r}(r_0,\varphi_0) \\ &+ (r^2/r_0^2)\, \cos^2(\varphi-\varphi_0)\, F_{\varphi\varphi}(r_0,\varphi_0) \,. \end{aligned} \tag{2.67}$$

So far for a general tensor. The metric tensor in polar coordinates is, at any point (r,φ) ,

$$\begin{pmatrix} g_{rr}(r,\varphi) & g_{r\varphi}(r,\varphi) \\ g_{\varphi r}(r,\varphi) & g_{\varphi\varphi}(r,\varphi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \,. \tag{2.68}$$

As it is symmetric, equation 2.67 simplifies first to

$$\begin{aligned} g_{\varphi\varphi}(r,\varphi \parallel r_0,\varphi_0) &= r^2\sin^2(\varphi-\varphi_0)\, g_{rr}(r_0,\varphi_0) \\ &+ (r^2/r_0^2)\, \cos^2(\varphi-\varphi_0)\, g_{\varphi\varphi}(r_0,\varphi_0) \,, \end{aligned} \tag{2.69}$$

and, using the values $g_{rr}(r_0,\varphi_0) = 1$ and $g_{\varphi\varphi}(r_0,\varphi_0) = r_0^2$, finally gives

$$g_{\varphi\varphi}(r,\varphi \parallel r_0,\varphi_0) = r^2 \left(\sin^2(\varphi-\varphi_0) + \cos^2(\varphi-\varphi_0)\right) = r^2 \,, \tag{2.70}$$

which is simply the value of the metric at the new point (r,φ) : the parallel transportation of the metric from point $\mathbf{x}_0$ to point $\mathbf{x}$ gives the metric at point $\mathbf{x}$. •

2.2 Derivatives

2.2.1 Derivatives

Let us first consider an ordinary function $x \to f(x)$ of the ordinary variable x (here "ordinary" means that we do not introduce tensorial concepts). The *derivative* of the function $f(\cdot)$ at point x is the function $f'(\cdot)$ defined by the condition that the first order development

$$f(x+\delta x) = f(x) + f'(x)\,\delta x \tag{2.71}$$

must hold for any δx .

For a vector field $\mathbf{x} \to \mathbf{V}(\mathbf{x})$, the *partial derivatives* $\partial_j V^i$ of the vector field at point $\mathbf{x}$ are defined by the condition that the first order development

$$V^i(\mathbf{x}+\delta\mathbf{x}) = V^i(\mathbf{x}) + \Big(\partial_j V^i\Big)(\mathbf{x})\ \delta x^j \tag{2.72}$$

must hold for any $\delta\mathbf{x}$. Although equation 2.72 makes perfect sense, *it is not a tensor equation*. For we know (see section XXX) that tensors defined at a point of a manifold belong to the linear space tangent to the manifold at that point. At a given point, we can add and multiply tensors, but equation 2.72 has, at his left hand side, a vector at point $\mathbf{x}+\delta\mathbf{x}$ and, at his right hand side, a vector at point $\mathbf{x}$. So, even if the functions

$$\Big(\partial_j V^i\Big)(\mathbf{x})\ \delta x^j = V^i(\mathbf{x}+\delta\mathbf{x}) - V^i(\mathbf{x}) \tag{2.73}$$

are defined, they are not the components of a tensor field. This implies that, in a change of coordinates in the space, these functions will not transform according to the rule XXX that tansforms the components of a tensor field. The transformation rules of these functions are complicated and uninteresting. (Comment: say that these functions describe how the components of the tensor field change when we change the point but not the basis).

By opposition to what precedes the first order development

$$V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) = V^i(\mathbf{x}) + \Big(\nabla_j V^i\Big)(\mathbf{x})\ \delta x^j \tag{2.74}$$

is a perfectly valid tensor equation. The expression in the left hand side describes the vector at point $\mathbf{x}$ obtained by parallel transportation of the vector $\mathbf{V}(\mathbf{x}+\delta x)$ from point $\mathbf{x}+\delta x$ to point $\mathbf{x}$. We have at both sides tensors at the same point of the space. Then, $(\nabla_j V^i)(\mathbf{x})$ is a tensor field, named the *tensor derivative* of the field $\mathbf{V}(\mathbf{x})$. Comment: say that equivalent names are "covariant derivative" or, simply, "derivative".

Note: we can write equation 2.74 as

$$V^i(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}) = V^i(\mathbf{x}) - \Big(\nabla_j V^i\Big)(\mathbf{x})\ \delta x^j\ , \tag{2.75}$$

and the diference between equations 2.74 and 2.75 gives a symmetric definition for the covariant derivative:

$$V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) - V^i(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}) = 2\ \Big(\nabla_j V^i\Big)(\mathbf{x})\ \delta x^j\ . \tag{2.76}$$

Taylor's series: We know the properties

$$f(x+\delta x) = f(x) + f'(x)\ \delta x + \frac{1}{2!} f''(x)\ (\delta x)^2 + \dots\ . \tag{2.77}$$

and

$$V^i(\mathbf{x}+\delta\mathbf{x}) = V^i(\mathbf{x}) + (\partial_j V^i)(\mathbf{x})\ \delta x^j + \frac{1}{2!}(\partial_k\partial_j V^i)(\mathbf{x})\ \delta x^j\ \delta x^k + \dots\ . \tag{2.78}$$

I conjecture that the development

$$V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) = V^i(\mathbf{x}) + (\nabla_j V^i)(\mathbf{x})\ \delta x^j + \frac{1}{2!}(\nabla_k \nabla_j V^i)(\mathbf{x})\ \delta x^j\ \delta x^k + \dots\ , \tag{2.79}$$

holds, but I have yet to prove it. Comment: find the proof. •

From

$$\mathbf{V}(\mathbf{x} \parallel \mathbf{y}) = V^i(\mathbf{y})\, \mathbf{e}_i(\mathbf{x} \parallel \mathbf{y}) \tag{2.80}$$

and

$$\mathbf{e}_i(\mathbf{x} \parallel \mathbf{y}) = \gamma_i{}^j(\mathbf{x}|\mathbf{y})\, \mathbf{e}_j(\mathbf{x}) \tag{2.81}$$

we have

$$\begin{aligned}
\mathbf{V}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) &= V^i(\mathbf{x}+\delta\mathbf{x})\, \gamma_i{}^j(\mathbf{x}|\mathbf{x}+\delta\mathbf{x})\, \mathbf{e}_j(\mathbf{x}) \\
&= \left(V^i(\mathbf{x}) + \frac{\partial V^i}{\partial x^k}(\mathbf{x})\, \delta x^k + \dots \right) \left(\gamma_i{}^j(\mathbf{x}|\mathbf{x}) + \Gamma_{ki}{}^j(\mathbf{x}|\mathbf{x})\, \delta x^k + \dots \right) \mathbf{e}_j(\mathbf{x}) \\
&= \left(V^i(\mathbf{x}) + \frac{\partial V^i}{\partial x^k}(\mathbf{x})\, \delta x^k + \dots \right) \left(g_i{}^j + \Gamma_{ki}{}^j(\mathbf{x})\, \delta x^k + \dots \right) \mathbf{e}_j(\mathbf{x}) \\
&= \left(V^j(\mathbf{x}) + \left(\frac{\partial V^j}{\partial x^k}(\mathbf{x}) + \Gamma_{ki}{}^j(\mathbf{x})\, V^i(\mathbf{x}) \right) \delta x^k + \dots \right) \mathbf{e}_j(\mathbf{x})\,.
\end{aligned} \tag{2.82}$$

Then, from the definition

$$V^j(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) = V^j(\mathbf{x}) + \left(\nabla_k V^j \right)(\mathbf{x})\, \delta x^k + \dots \tag{2.83}$$

it follows

$$\left(\nabla_k V^j \right)(\mathbf{x}) = \left(\partial_k V^j \right)(\mathbf{x}) + \Gamma_{ki}{}^j(\mathbf{x})\, V^i(\mathbf{x})\,. \tag{2.84}$$

Comment: say, in the equations above, which symbols "=" and "+" correspond the tensor equations and which do not.

The general formula for the covariant derivative of a tensorial object (density of order p) is

$$\begin{aligned}
\nabla_i T^{jk\dots}{}_{lm\dots} \;=\;& \partial_i T^{jk\dots}{}_{lm\dots} - \ p\ \Gamma_{is}{}^s T^{jk\dots}{}_{lm\dots} \\
&+ \Gamma_{is}{}^j T^{sk\dots}{}_{lm\dots}\ +\ \Gamma_{is}{}^k T^{js\dots}{}_{lm\dots} + \dots \\
&- \Gamma_{il}{}^s T^{jk\dots}{}_{sm\dots}\ -\ \Gamma_{im}{}^s T^{jk\dots}{}_{ls\dots} - \dots
\end{aligned} \tag{2.85}$$

As the term $\Gamma_{is}{}^s$ appears quite often, the following notation is useful:

$$\Gamma_i = \Gamma_{is}{}^s\,. \tag{2.86}$$

(Comment: say somewhere that, in all rigor, if $\nabla\mathbf{V}$ denotes the covariant derivative in intrinsic notation, we should write $(\nabla\mathbf{V})_i{}^j$ instead of $\nabla_i V^j$ but the last notation is much easier to handle.)

To be explicit, we write below the covariant derivatives of tensor objects with rank 0 , 1 , and 2 .

Derivatives in Euclidean spaces

We consider here an Euclidean space, and explain how covariant derivatives can be computed in an arbitrary system of coordinates.

The equation

$$\left(\nabla_k V^j \right)(\mathbf{x}) = \left(\partial_k V^j \right)(\mathbf{x}) + \Gamma_{ki}{}^j(\mathbf{x})\, V^i(\mathbf{x})\,, \tag{2.87}$$

taken “au pied de la lettre” seems to imply that covariant differentiation will impose, when using a finite-difference computational grid, that we consider at the same point $\mathbf{x}$, a vector $\mathbf{V}(\mathbf{x})$ and its derivative $(\nabla\mathbf{V})(\mathbf{x})$. This is not so.

Let us see, for instance, how a practical computation should proceed in an Euclidean space with arbitrary coordinates.

In what follows, $\{X^I\}$ denotes a Cartesian system of coordinates, while $\{x^i\}$ denotes another, arbitrary, system of coordinates.

Using the Cartesian coordinates, we can compute the covariant derivative of a tensor field using simple partial derivatives:

$$\nabla_M T_{IJ\dots}{}^{KL\dots} = \frac{\partial T_{IJ\dots}{}^{KL\dots}}{\partial X^M} \, . \tag{2.88}$$

From this we can deduce the components of the same tensor in the general system of coordinates:

$$\begin{aligned} \nabla_m T_{ij\dots}{}^{kl\dots} &= \frac{\partial X^M}{\partial x^m}\frac{\partial X^I}{\partial x^i}\frac{\partial X^J}{\partial x^j}\cdots\frac{\partial x^k}{\partial X^K}\frac{\partial x^\ell}{\partial X^L}\cdots\nabla_M T_{IJ\dots}{}^{KL\dots} \\ &= \frac{\partial X^M}{\partial x^m}\frac{\partial X^I}{\partial x^i}\frac{\partial X^J}{\partial x^j}\cdots\frac{\partial x^k}{\partial X^K}\frac{\partial x^\ell}{\partial X^L}\cdots\frac{\partial\, T_{IJ\dots}{}^{KL\dots}}{\partial X^M} \, . \end{aligned} \tag{2.89}$$

Using

$$\frac{\partial T_{IJ\dots}{}^{KL\dots}}{\partial X^M} = \frac{\partial x^n}{\partial X^M}\frac{\partial\, T_{IJ\dots}{}^{KL\dots}}{\partial x^n} \tag{2.90}$$

and

$$\frac{\partial X^M}{\partial x^m}\frac{\partial x^n}{\partial X^M} = \delta_m{}^n \tag{2.91}$$

gives

$$\nabla_m T_{ij\dots}{}^{kl\dots} = \frac{\partial X^I}{\partial x^i}\frac{\partial X^J}{\partial x^j}\cdots\frac{\partial x^k}{\partial X^K}\frac{\partial x^\ell}{\partial X^L}\cdots\frac{\partial\, T_{IJ\dots}{}^{KL\dots}}{\partial x^m} \, . \tag{2.92}$$

Expresing the Cartesian components $T_{IJ\dots}{}^{KL\dots}$ in terms of the working coordinates,

$$T_{IJ\dots}{}^{KL\dots} = \frac{\partial x^p}{\partial X^I}\frac{\partial x^q}{\partial X^J}\cdots\frac{\partial X^K}{\partial x^r}\frac{\partial X^L}{\partial x^s}\cdots T_{pq\dots}{}^{rs\dots} \, , \tag{2.93}$$

finally gives

$$\nabla_m T_{ij\dots}{}^{kl\dots} = \frac{\partial X^I}{\partial x^i}\frac{\partial X^J}{\partial x^j}\cdots\frac{\partial x^k}{\partial X^K}\frac{\partial x^\ell}{\partial X^L}\cdots\frac{\partial}{\partial x^m}\left(\frac{\partial x^p}{\partial X^I}\frac{\partial x^q}{\partial X^J}\cdots\frac{\partial X^K}{\partial x^r}\frac{\partial X^L}{\partial x^s}\cdots T_{pq\dots}{}^{rs\dots}\right) \, . \tag{2.94}$$

In particular this gives, for a vector field,

$$\nabla_m T^k = \frac{\partial x^k}{\partial X^K}\frac{\partial}{\partial x^m}\left(\frac{\partial X^K}{\partial x^r}T^r\right) \, . \tag{2.95}$$

These equations have two important uses. First, they give compact analytic formulas. For instance, when computing the divergence of a vector field using spherical coordinates, it directly gives the compact expression

$$\nabla_i V^i = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V^r) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\; V^\theta) + \frac{\partial V^\varphi}{\partial\varphi} \tag{2.96}$$

and not an expression with all derivatives developed, as we would have obtained from the formula

$$\left(\nabla_k V^j\right)(\mathbf{x}) = \left(\partial_k V^j\right)(\mathbf{x}) + \Gamma_{ki}{}^j(\mathbf{x})\, V^i(\mathbf{x}) \, . \tag{2.97}$$

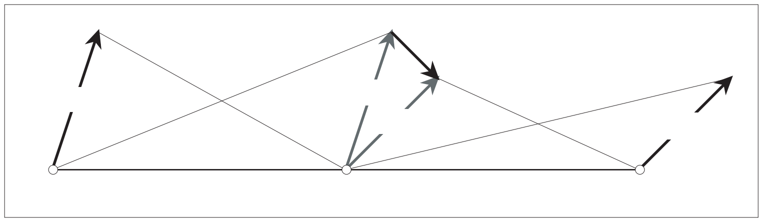

Figure 2.11: Finite-difference (centered) approximation to covariant derivatives. In the limit when $\delta\mathbf{x} \to 0$, this defines the covariant derivative through the equation $V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) - V^i(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}) = 2\ (\nabla_k V^i)(\mathbf{x})\ \delta\mathbf{x}^k$ or, to speak properly, $\mathbf{V}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) - \mathbf{V}(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}) = 2\ (\nabla\mathbf{V})_k{}^i(\mathbf{x})\ \delta\mathbf{x}^k\ \mathbf{e}_i(\mathbf{x})$. (Comment: explain that the thin lines are geodesics crossing at their mid points). (Comment: explain that if N is the dimension of the space, we need N different directions $\delta\mathbf{x}_1, \delta\mathbf{x}_2, \dots, \delta\mathbf{x}_N$ to compute all the values $\nabla_i V^k$).

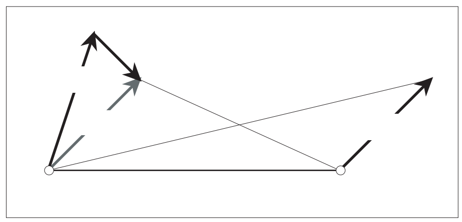

Figure 2.12: Finite-difference (non-centered) approximation to covariant derivatives. In the limit when $\delta\mathbf{x} \to 0$, this defines the covariant derivative through the equation $V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) - V^i(\mathbf{x}) = (\nabla_k V^i)(\mathbf{x})\ \delta\mathbf{x}^k$ or, to speak properly, $\mathbf{V}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) - \mathbf{V}(\mathbf{x}) = (\nabla\mathbf{V})_k{}^i(\mathbf{x})\ \delta\mathbf{x}^k\ \mathbf{e}_i(\mathbf{x})$.

Second, it can be directly applied to a numerical computation of covariant derivatives. If the partial derivatives $\partial x^i/\partial X^I$ are known (analytically or numerically), it is a very convenient formula to use (see chapter XXX) for more details).

Comment: Explain somewhere that equations 2.94 and 2.95 are, of course compatible with the expression giving the connection in terms of Cartesian coordinates. For instance, equation 2.95 gives

$$\begin{aligned} \nabla_m T^k &= \frac{\partial x^k}{\partial X^K}\frac{\partial}{\partial x^m}\left(\frac{\partial X^K}{\partial x^r}T^r\right) \\ &= \frac{\partial x^k}{\partial X^K}\left(\frac{\partial X^K}{\partial x^r}\frac{\partial T^r}{\partial x^m} + \frac{\partial^2 X^K}{\partial x^m \partial x^r}T^r\right) \\ &= \frac{\partial x^k}{\partial X^K}\frac{\partial X^K}{\partial x^r}\frac{\partial T^r}{\partial x^m} + \frac{\partial x^k}{\partial X^K}\frac{\partial^2 X^K}{\partial x^m \partial x^r}T^r \\ &= \delta^k_r\,\frac{\partial T^r}{\partial x^m} + \Gamma_{mr}{}^k T^r \\ &= \frac{\partial T^k}{\partial x^m} + \Gamma_{mr}{}^k T^r\,, \end{aligned} \tag{2.98}$$

where we have used the expression (demonstrated somewhere)

$$\frac{\partial x^k}{\partial X^K}\frac{\partial^2 X^K}{\partial x^m \partial x^r} = \Gamma_{mr}{}^k\,. \tag{2.99}$$

Comment: The following text seems too big to be considered as a caption for the previous table.

Equations 1 give the components of the transported basis on the local basis at the arriving point. Equation 2 expresses that the matrix with the coefficients $\gamma_i{}^k$ is the inverse of the matrix with the

Basic formulas for parallel transportation		
$\mathbf{e}_i(\mathbf{x} \parallel \mathbf{y}) = \gamma_i{}^j(\mathbf{x}\vert\mathbf{y})\, \mathbf{e}_j(\mathbf{x})$	$\mathbf{e}^i(\mathbf{x} \parallel \mathbf{y}) = \mathbf{e}^j(\mathbf{x})\, \ell_j{}^i(\mathbf{x}\vert\mathbf{y})$	(1)
$\gamma_i{}^j(\mathbf{x}\vert\mathbf{y})\, \ell_j{}^k(\mathbf{x}\vert\mathbf{y}) = \ell_i{}^j(\mathbf{x}\vert\mathbf{y})\, \gamma_j{}^k(\mathbf{x}\vert\mathbf{y}) = \delta_i{}^k$		(2)
$\gamma_i{}^j(\mathbf{x}\vert\mathbf{x}) = \delta_i{}^j$	$\ell_i{}^j(\mathbf{x}\vert\mathbf{x}) = \delta_i{}^j$	(3)
$\gamma_i{}^j(\mathbf{x}\vert\mathbf{y})\, \gamma_j{}^k(\mathbf{y}\vert\mathbf{x}) = \delta_i{}^k$	$\ell_i{}^j(\mathbf{x}\vert\mathbf{y})\, \ell_j{}^k(\mathbf{y}\vert\mathbf{x}) = \delta_i{}^k$	(4)
$\ell_i{}^j(\mathbf{x}\vert\mathbf{y}) = \gamma_i{}^j(\mathbf{y}\vert\mathbf{x})$		(5)
$V^i(\mathbf{x} \parallel \mathbf{y}) = V^j(\mathbf{y})\, \gamma_j{}^i(\mathbf{x}\vert\mathbf{y})$	$F_i(\mathbf{x} \parallel \mathbf{y}) = \ell_i{}^j(\mathbf{x}\vert\mathbf{y})\, F_j(\mathbf{y})$	(6)
$T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{y}) = \ell_i{}^p(\mathbf{x}\vert\mathbf{y})\, \ell_j{}^q(\mathbf{x}\vert\mathbf{y})\, \dots\, T_{pq\dots}{}^{mn\dots}(\mathbf{y})\, \gamma_m{}^k(\mathbf{x}\vert\mathbf{y})\, \gamma_n{}^\ell(\mathbf{x}\vert\mathbf{y})\, \dots$		(7)
$\Gamma_{ij}{}^k(\mathbf{x}\vert\mathbf{y}) = \frac{\partial \gamma_j{}^k}{\partial y^i}(\mathbf{x}\vert\mathbf{y})$	$\mathrm{L}_{ij}{}^k(\mathbf{x}\vert\mathbf{y}) = \frac{\partial \ell_j{}^k}{\partial y^i}(\mathbf{x}\vert\mathbf{y})$	(8)
$\Gamma_{ij}{}^k(\mathbf{x}) = \Gamma_{ij}{}^k(\mathbf{x}\vert\mathbf{x})$	$\mathrm{L}_{ij}{}^k(\mathbf{x}) = \mathrm{L}_{ij}{}^k(\mathbf{x}\vert\mathbf{x})$	(9)
$\mathrm{L}_{ij}{}^k(\mathbf{x}) = -\Gamma_{ij}{}^k(\mathbf{x})$		(10)

Definition of covariant derivatives	
$T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = T_{ij\dots}{}^{k\ell\dots}(\mathbf{x}) + \left(\nabla_p T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\, \delta x^p + \dots$	(11)

Expression of covariant derivatives	
$\nabla_p T_{ij\dots}{}^{k\ell\dots} = \partial_p T_{ij\dots}{}^{k\ell\dots} + \mathrm{L}_{pi}{}^s T_{sj\dots}{}^{k\ell\dots} + \mathrm{L}_{pj}{}^s T_{is\dots}{}^{k\ell\dots} + \dots + \Gamma_{ps}{}^k T_{ij\dots}{}^{s\ell\dots} + \Gamma_{ps}{}^\ell T_{ij\dots}{}^{ks\dots} + \dots$	(12)

Metric spaces		
$g_{ij}(\mathbf{x}) = \ell_i{}^p(\mathbf{x}\vert\mathbf{y})\, \ell_j{}^q(\mathbf{x}\vert\mathbf{y})\, g_{pq}(\mathbf{y})$		(13)
$\partial_k g_{ij} = \Gamma_{kij} + \Gamma_{kji}$		(14)
$\Gamma_{ijk} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right) + \frac{1}{2}\left(S_{ijk} + S_{kij} + S_{kji}\right)$	$\boxed{S_{ij}{}^k = \Gamma_{ij}{}^k - \Gamma_{ji}{}^k}$	(15)

Euclidean spaces	
$\Gamma_{ij}{}^k = \frac{\partial^2 X^K}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial X^K} = -\frac{\partial X^I}{\partial x^i} \frac{\partial X^J}{\partial x^j} \frac{\partial^2 x^k}{\partial X^I \partial X^J}$	(16)

Table 2.1: Intended text too large (see below).

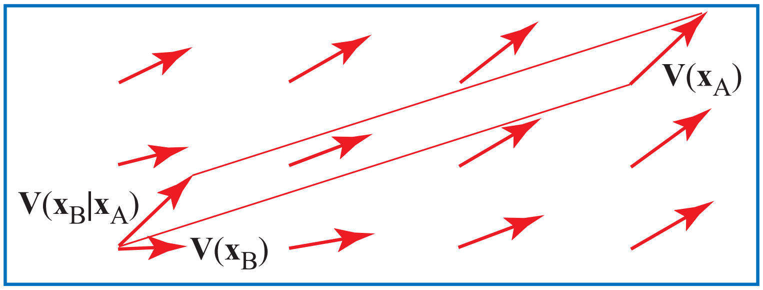

Figure 2.13: Parallel transportation of a vector. This figure schematically represents a vector field $\mathbf{V}(\mathbf{x})$. The two vectors $\mathbf{V}(\mathbf{x}_A)$ and $\mathbf{V}(\mathbf{x}_B)$ represent the values of the vector fields at points $\mathbf{x}_A$ and $\mathbf{x}_A$ respectively. Also indicated is the vector $\mathbf{V}(\mathbf{x}_B|\mathbf{x}_A)$, the vector at point $\mathbf{x}_B$ obtained by parallel transport of $\mathbf{V}(\mathbf{x}_A)$. If, as in left of figure XXX, the space is Euclidean and we introduce, as an intermediary, Cartesian coordinates, then, we have the simple relationship (see text): $\mathbf{V}^i(\mathbf{x}_B|\mathbf{x}_A) = \frac{\partial x^i}{\partial y^K}(\mathbf{y}_B)\,\frac{\partial y^K}{\partial x^j}(\mathbf{x}_A)\;\mathbf{V}^j(\mathbf{x}_A)$. If the space is not assumed to be Euclidean, Cartesian coordinates may not exist. The only way then to perform parallel transportation of a vector is by assuming the connection coefficients $\{\Gamma_{ij}{}^k\}$ given and to use the first order approximation (see text) $\mathbf{V}^j(\mathbf{x}+\delta\mathbf{x} \mid \mathbf{x}) = \mathbf{V}^j(\mathbf{x}) - \Gamma_{ki}{}^j(\mathbf{x})\,\delta x^k\;\mathbf{V}^i$ to … at neighbouring points (or at distant points through analytic or numerical integration).

coefficients $\ell_i{}^j$. Equations 3 say that transporting a tensor from a point $\mathbf{x}$ to the same point (along a geodesic) means no transportation at all. Equations 4 express the fact that transporting a tensor from point $\mathbf{y}$ to point $\mathbf{x}$, then back to point $\mathbf{y}$ (always along the geodesic) gives the original tensor. Equation 5 says that the coefficients transporting the basis from $\mathbf{x}$ to $\mathbf{y}$ are the same that the coefficients transporting the *dual* basis from $\mathbf{y}$ to $\mathbf{x}$. Equations 6 expresses how the components of a vector or a form change in a parallel transportation, while equation 7 gives the general result for an arbitrary tensor. Equations 8 define the two point connection, while equations 9 define the one point connection. Equation 10 expresses that the connection associated to vectors is opposite to the connection associated to forms. Equation 11 gives the definition of covariant derivatives (the "…" stand for terms of high order [proportional to $\delta x^p \delta x^q$, to $\delta x^p \delta x^q \delta x^r$, etc.]. Equation 12 gives the general formula for covariant derivatives using the connection. Equation 13 gives the relation between the coefficients defining the parallel transportation and the metric (wheren there is one): the parallel transportation of the metric from point $\mathbf{y}$ to point $\mathbf{x}$ gives the metric at point $\mathbf{x}$. Equation 14 gives the relation between the metric and the connection, from which it is possible to obtain (equation 15) the formula expressing the connection from the metric and the torsion (the torsion is to be introduced in section XXX). Equation 16 shows that, in Euclidean spaces, in addition to arbitrary working coordinates $\{x^i\}$, one can always introduce Cartesian coordinates $\{X^I\}$, and then the connection coefficients can be computed from the partial derivatives of one coordinate system with respect to the other.

2.2.2 Second derivatives

For an arbitrary tensor field $T_{jk\ldots}{}^{pq\cdots}(\mathbf{x})$, we have defined its derivative $(\nabla_i T_{jk\ldots}{}^{pq\cdots})\,(\mathbf{x})$, by the first order development

$$T_{jk\ldots}{}^{pq\cdots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) = T_{jk\ldots}{}^{pq\cdots}(\mathbf{x}) + (\nabla_i T_{jk\ldots}{}^{pq\cdots})\,(\mathbf{x})\,\delta x^i \; . \tag{2.100}$$

If the tensor field $T_{jk\ldots}{}^{pq\cdots}(\mathbf{x})$, is itself the derivative of some other field,

$$T_{jk\ell\ldots}{}^{pq\cdots} = \nabla_j S_{k\ell\ldots}{}^{pq\cdots} \; , \tag{2.101}$$

Capacity	**Tensor**	**Density**
$\nabla_k \underline{s} = \partial_k \underline{s} + \Gamma_k \underline{s}$	$\nabla_k s = \partial_k s$	$\nabla_k \overline{s} = \partial_k \overline{s} - \Gamma_k \overline{s}$
$\nabla_k \underline{F}_i = \partial_k \underline{F}_i + \Gamma_k \underline{F}_i - \Gamma_{ki}{}^s \underline{F}_s$ $\nabla_k \underline{V}^i = \partial_k \underline{V}^i + \Gamma_k \underline{V}^i + \Gamma_{ks}{}^i \underline{V}^s$	$\nabla_k F_i = \partial_k F_i - \Gamma_{ki}{}^s F_s$ $\nabla_k V^i = \partial_k V^i + \Gamma_{ks}{}^i V^s$	$\nabla_k \overline{F}_i = \partial_k \overline{F}_i - \Gamma_k \overline{F}_i - \Gamma_{ki}{}^s \overline{F}_s$ $\nabla_k \overline{V}^i = \partial_k \overline{V}^i - \Gamma_k \overline{V}^i + \Gamma_{ks}{}^i \overline{V}^s$
$\nabla_k \underline{Q}_{ij} = \partial_k \underline{Q}_{ij} + \Gamma_k \underline{Q}_{ij} - \Gamma_{ki}{}^s \underline{Q}_{sj} - \Gamma_{kj}{}^s \underline{Q}_{is}$ $\nabla_k \underline{R}_i{}^j = \partial_k \underline{R}_i{}^j + \Gamma_k \underline{R}_i{}^j - \Gamma_{ki}{}^s \underline{R}_s{}^j + \Gamma_{ks}{}^j \underline{R}_i{}^s$ $\nabla_k \underline{S}^i{}_j = \partial_k \underline{S}^i{}_j + \Gamma_k \underline{S}^i{}_j + \Gamma_{ks}{}^i \underline{S}^s{}_j - \Gamma_{kj}{}^s \underline{S}^i{}_s$ $\nabla_k \underline{T}^{ij} = \partial_k \underline{T}^{ij} + \Gamma_k \underline{T}^{ij} + \Gamma_{ks}{}^i \underline{T}^{sj} + \Gamma_{ks}{}^j \underline{T}^{is}$	$\nabla_k Q_{ij} = \partial_k Q_{ij} - \Gamma_{ki}{}^s Q_{sj} - \Gamma_{kj}{}^s Q_{is}$ $\nabla_k R_i{}^j = \partial_k R_i{}^j - \Gamma_{ki}{}^s R_s{}^j + \Gamma_{ks}{}^j R_i{}^s$ $\nabla_k S^i{}_j = \partial_k S^i{}_j + \Gamma_{ks}{}^i S^s{}_j - \Gamma_{kj}{}^s S^i{}_s$ $\nabla_k T^{ij} = \partial_k T^{ij} + \Gamma_{ks}{}^i T^{sj} + \Gamma_{ks}{}^j T^{is}$	$\nabla_k \overline{Q}_{ij} = \partial_k \overline{Q}_{ij} - \Gamma_k \overline{Q}_{ij} - \Gamma_{ki}{}^s \overline{Q}_{sj} - \Gamma_{kj}{}^s \overline{Q}_{is}$ $\nabla_k \overline{R}_i{}^j = \partial_k \overline{R}_i{}^j - \Gamma_k \overline{R}_i{}^j - \Gamma_{ki}{}^s \overline{R}_s{}^j + \Gamma_{ks}{}^j \overline{R}_i{}^s$ $\nabla_k \overline{S}^i{}_j = \partial_k \overline{S}^i{}_j - \Gamma_k \overline{S}^i{}_j + \Gamma_{ks}{}^i \overline{S}^s{}_j - \Gamma_{kj}{}^s \overline{S}^i{}_s$ $\nabla_k \overline{T}^{ij} = \partial_k \overline{T}^{ij} - \Gamma_k \overline{T}^{ij} + \Gamma_{ks}{}^i \overline{T}^{sj} + \Gamma_{ks}{}^j \overline{T}^{is}$
$\vdots$	$\vdots$	$\vdots$

Table 2.2: Note: write here the caption.

Divergence of densities
$\nabla_k \overline{V}^k = \partial_k \overline{V}^k$
$\nabla_k \overline{T}^{ik} = \partial_k \overline{T}^{ik} + \Gamma_{ks}{}^i \overline{T}^{sk}$
$\nabla_k \overline{T}_i{}^k = \partial_k \overline{T}_i{}^k - \Gamma_{ki}{}^s \overline{T}_s{}^k$
If $\overline{T}^{ij} = -\overline{T}^{ji}$ and $\Gamma_{ij}{}^k = \Gamma_{ji}{}^k$, then, $\nabla_k \overline{T}^{ik} = \partial_k \overline{T}^{ik}$.

Table 2.3: Note: explain that the divergence of densities has one less term than the divergence of a tensor, as two of the terms of the previous table compensate.

equation 2.100 gives the first order development

$$(\nabla_j S_{k\ell\ldots}{}^{pq\ldots})(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = (\nabla_j S_{k\ell\ldots}{}^{pq\ldots})(\mathbf{x}) + (\nabla_i \nabla_j S_{k\ell\ldots}{}^{pq\ldots})\,(\mathbf{x})\,\delta x^i\,, \tag{2.102}$$

that defines the second order derivative of a tensor field.

In section xxx we will see that the operator $\nabla_i \nabla_j$ is not identical to $\nabla_j \nabla_i$: contrarily to the partial derivatives, the covariant derivatives do not conmute.

Figure 2.14 gives a pictorial representation of the finite-difference computation of (noncentered) second derivatives.

Comment: all this is probably false. I should only say that the derivative of a tensor field is a tensor field, so we can take derivatives of it.

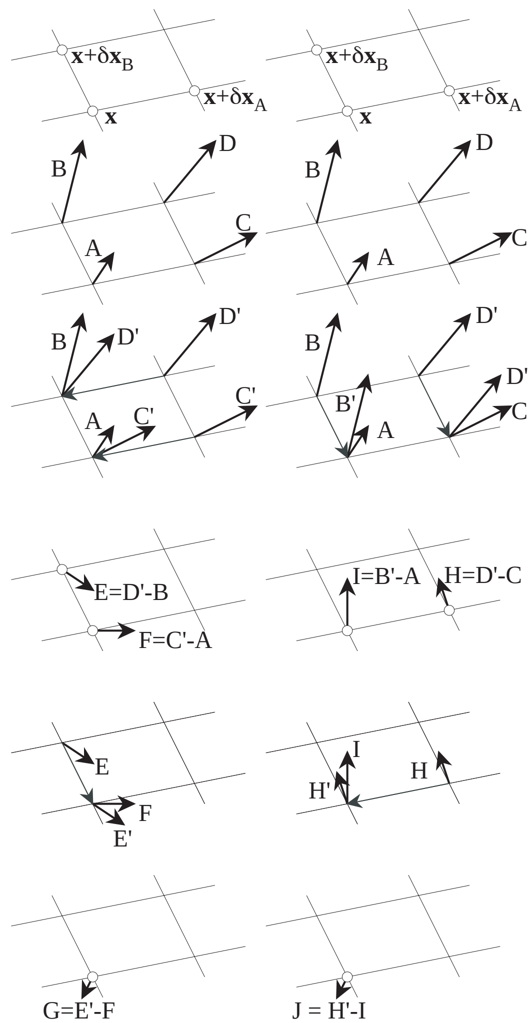

Figure 2.14: See caption here below (LaTeX error when put here).

Caption of the figure above: Comment: this figure has been redrawn, and the caption has to be rewritten accordingly. At a point $\mathbf{x}$ of the space, consider the two coordinate perturbations $\delta\mathbf{x}_A$ and $\delta\mathbf{x}_B$. Consider also a vector field $V^i(\mathbf{x})$ defined at all points of the space. Let us describe the column at left. Using parallel transportation, from the expressions $V^i(\mathbf{x} + \delta\mathbf{x}_B \parallel \mathbf{x} + \delta\mathbf{x}_B + \delta\mathbf{x}_A) = V^i(\mathbf{x} + \delta\mathbf{x}_B) + \left(\nabla_j V^i\right)(\mathbf{x} + \delta\mathbf{x}_B)\, \delta x^j_A$ and $V^i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_A) = V^i(\mathbf{x}) + \left(\nabla_j V^i\right)(\mathbf{x})\, \delta x^j_A$ we can easily deduce the vectors $\left(\nabla_j V^i\right)(\mathbf{x} + \delta\mathbf{x}_B)\, \delta x^j_A$ and $\left(\nabla_j V^i\right)(\mathbf{x})\, \delta x^j_A$, while from the expression $\left(\nabla_j V^i\right)(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_B)\, \delta x^j_A = \left(\nabla_j V^i\right)(\mathbf{x})\, \delta x^j_A + \left(\nabla_k \nabla_j V^i\right)(\mathbf{x})\, \delta x^j_A\, \delta x^k_B$ we easily deduce the vector $G = \left(\nabla_k \nabla_j V^i\right)(\mathbf{x})\, \delta x^j_A\, \delta x^k_B$. Similarly, in the right column, we deduce the vector $J = \left(\nabla_k \nabla_j V^i\right)(\mathbf{x})\, \delta x^j_B\, \delta x^k_A$. The difference between these two vectors, namely $\left(\nabla_k \nabla_j - \nabla_j \nabla_k V^i\right)(\mathbf{x})\, \delta x^j_A\, \delta x^k_B$, is, in general, not zero, if the space has curvature. Comment: explain better that, in the left column, we move first left, then down, while in the right column, we first move down, then left.

2.2.3 Derivatives of the metric tensor

To be written.

2.2.4 Old text: Ordinary derivatives

Let $f(x)$ be a real function of the real variable x . The expression

$$f(x+\delta x) = f(x) + f'(x)\,\delta x \tag{2.103}$$

defines the *secant* of the function $f(x)$ at points x and $x+\delta x$ as the linear function taking the value $f(x)$ at point x and with slope $f'(x)$. The secant is, of course, a function of δx . The limit of the secant when $\delta x \to 0$ is called the *tangent* of $f(x)$ at point x . The slope of the tangent is the *derivative*.

Let now $f(\mathbf{x}) = f(x,y)$ be a real function of the two-dimensional variable $\mathbf{x} = \{x,y\}$. Setting

$$\delta\mathbf{x}_1 = \begin{pmatrix} \delta x_1 \\ \delta y_1 \end{pmatrix} \qquad \delta\mathbf{x}_2 = \begin{pmatrix} \delta x_2 \\ \delta y_2 \end{pmatrix} , \tag{2.104}$$

the secant of the function $f(\mathbf{x})$ at points $\mathbf{x}$, $\mathbf{x}\ \delta\mathbf{x}_1$ and $\mathbf{x}\ \delta\mathbf{x}_2$ is defined as being the linear application taking the value $f(x,y)$ at point (x,y) and with slopes (along each axis) $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ defined the the set of two equations with two unknowns

$$\begin{aligned} f(x+\delta x_1, y+\delta y_1) &= f(x,y) + \frac{\partial f}{\partial x}(x,y)\,\delta x_1 + \frac{\partial f}{\partial y}(x,y)\,\delta y_1 \\ f(x+\delta x_2, y+\delta y_2) &= f(x,y) + \frac{\partial f}{\partial x}(x,y)\,\delta x_2 + \frac{\partial f}{\partial y}(x,y)\,\delta y_2 \,. \end{aligned} \tag{2.105}$$

Again, the limit of the secant when $\delta\mathbf{x}_1 \to 0$ and $\delta\mathbf{x}_2 \to 0$ is the tangent to $f(x,y)$ at (x,y) .

If in the 1-D case, equation 2.103 can be solved for δx , and the limit defining the derivative can be explicitly written as

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} , \tag{2.106}$$

equation 2.105 involves the solution of a linear system, and the explicit expression is less simple, although this is not important. Of course, when choosing the "perturbed points" $\mathbf{x}+\delta\mathbf{x}_1$ and $\mathbf{x}+\delta\mathbf{x}_2$ along the coordinate lines,

$$\delta\mathbf{x}_1 = \begin{pmatrix} \delta x \\ 0 \end{pmatrix} \qquad \delta\mathbf{x}_2 = \begin{pmatrix} 0 \\ \delta y \end{pmatrix} , \tag{2.107}$$

then, equation 2.105 becomes

$$\begin{aligned} f(x+\delta x, y) &= f(x,y) + \frac{\partial f}{\partial x}(x,y)\,\delta x \\ f(x, y+\delta y) &= f(x,y) + \frac{\partial f}{\partial y}(x,y)\,\delta y \,, \end{aligned} \tag{2.108}$$

and, in the limit, we can write

$$\frac{\partial f'}{\partial x}(x,y) = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x)}{\delta x} \qquad \frac{\partial f'}{\partial y}(x,y) = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x)}{\delta y} . \tag{2.109}$$

It is geometrically clear that the tangent function so defined is independent, in the limit, of the particular "vectors" $\delta\mathbf{x}_1$ and $\delta\mathbf{x}_2$ chosen.

2.2.5 Stars and Crosses

Here we introduce the practical? geometrical? aspects of the computation of covariant derivatives. We will not use explicitly the connection coefficients Comment: but I must also explain how the computation is to be made if we use them).

Comment: explain that all this will be ready for a finite difference computation.

Comment: I have to write somewhere the three equations

$$T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) - T_{ij\dots}{}^{k\ell\dots}(\mathbf{x}) = \left(\nabla_m T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ \delta x^m \tag{2.110}$$

$$T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) - T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}) = 2\ \left(\nabla_m T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ \delta x^m \tag{2.111}$$

and

$$T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_A) - T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_B) = \left(\nabla_m T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ (\delta x^m_A - \delta x^m_B)\,, \tag{2.112}$$

and say that, independently of any star or cross, one of these equations defines the covariant derivative (if for any. . . in the limit. . .).

There are two fundamental ways to compute finite differences. For instance, in 2-D one may take two point along each direction or, alternatively, one can take a “Mercedes star”. Comment: explain better.

Stars:

Let us first examine the special cases in 2 and 3 dimensions, then, the general, n-dimensional case.

2-D: Consider the points indicated in figure 2.15, and assume that a vector field $V^i(\mathbf{x})$ has been defined on the space. Using the definition of tensorial (or covariant) derivative (equation XXX), we can write, at first order, the set of equations

$$\begin{aligned}
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1) - V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2) &= \left(\nabla_j V^i\right)(\mathbf{x})\ \left(\delta x^j_1 - \delta x^j_2\right)\\
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2) - V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_3) &= \left(\nabla_j V^i\right)(\mathbf{x})\ \left(\delta x^j_2 - \delta x^j_3\right)\\
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_3) - V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1) &= \left(\nabla_j V^i\right)(\mathbf{x})\ \left(\delta x^j_3 - \delta x^j_1\right)\,.
\end{aligned} \tag{2.113}$$

These three equations are not independent (we can obtain any of them as [minus] the sum of the other two): it is enough to take two of them.

Each of the two selected equations being valid for $i=1$ and $i=2$ we have, in all, four equations, exactly the number needed to estimate the four components of $\nabla_j V^i(\mathbf{x})$. For any finite value of $\delta\mathbf{x}_1$, $\delta\mathbf{x}_2$, and $\delta\mathbf{x}_3$, this will give a finite-difference approximation to the covariant derivative of the vector field. In the limit when $\delta\mathbf{x}_1$, $\delta\mathbf{x}_2$, and $\delta\mathbf{x}_3$, tend to zero, we obtain the exact value of the covariant derivative.

For short, equation 2.113 can be written

$$\begin{aligned}
\mathbf{V}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1) - \mathbf{V}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2) &= \left(\nabla_j \mathbf{V}\right)(\mathbf{x})\ \left(\delta x^j_1 - \delta x^j_2\right)\\
\mathbf{V}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2) - \mathbf{V}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_3) &= \left(\nabla_j \mathbf{V}\right)(\mathbf{x})\ \left(\delta x^j_2 - \delta x^j_3\right)\\
\mathbf{V}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_3) - \mathbf{V}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1) &= \left(\nabla_j \mathbf{V}\right)(\mathbf{x})\ \left(\delta x^j_3 - \delta x^j_1\right)\,.
\end{aligned} \tag{2.114}$$

More generally, if a tensor field

$$\mathbf{T}(\mathbf{x}) = T^{ij\dots}{}_{k\ell\dots}(\mathbf{x})\ \mathbf{e}_i(\mathbf{x}) \otimes \mathbf{e}_j(\mathbf{x}) \otimes \dots \mathbf{e}^k(\mathbf{x}) \otimes \mathbf{e}^\ell(\mathbf{x}) \otimes \dots \tag{2.115}$$

has been defined on the space the following set of equations perfectly defines the finite-difference approximation the the covariant derivatives of the tensor field:

$$\begin{aligned}
\mathbf{T}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_1^j - \delta x_2^j\right) \\
\mathbf{T}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_3) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_2^j - \delta x_3^j\right) \\
\mathbf{T}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_3) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_3^j - \delta x_1^j\right) .
\end{aligned} \tag{2.116}$$

Example: Consider the Euclidean plane, with Cartesian coordinates, and let us evaluate the gradient of a scalar field.

Figure 2.16 shows the "Mercedes star" chosen to evaluate derivatives. We have

$$\begin{pmatrix} \delta x_1^1 \\ \delta x_1^2 \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \delta x_2^1 \\ \delta x_2^2 \end{pmatrix} = \begin{pmatrix} -\delta/2 \\ +\sqrt{3}\delta/2 \end{pmatrix} \qquad \begin{pmatrix} \delta x_3^1 \\ \delta x_3^2 \end{pmatrix} = \begin{pmatrix} -\delta/2 \\ -\sqrt{3}\delta/2 \end{pmatrix} . \tag{2.117}$$

Applying equation 2.116 to a scalar field $\Phi(x,y)$ gives

$$\begin{aligned}
\Phi(x+\delta, y) - \Phi(x - \tfrac{1}{2}\delta, y + \tfrac{\sqrt{3}}{2}\delta) &= (\nabla_x \Phi)(x,y)\, (\tfrac{3}{2}\delta) + (\nabla_y \Phi)(x,y)\, (-\tfrac{\sqrt{3}}{2}\delta) \\
\Phi(x - \tfrac{1}{2}\delta, y + \tfrac{\sqrt{3}}{2}\delta) - \Phi(x - \tfrac{1}{2}\delta, y - \tfrac{\sqrt{3}}{2}\delta) &= (\nabla_x \Phi)(x,y)\, (0) + (\nabla_y \Phi)(x,y)\, (\sqrt{3}\delta) \\
\Phi(x - \tfrac{1}{2}\delta, y - \tfrac{\sqrt{3}}{2}\delta) - \Phi(x+\delta, y) &= (\nabla_x \Phi)(x,y)\, (-\tfrac{3}{2}\delta) + (\nabla_y \Phi)(x,y)\, (-\tfrac{\sqrt{3}}{2}\delta)
\end{aligned} \tag{2.118}$$

From these three (nonindependent) equations we get

$$(\nabla_y \Phi)(x,y) = \frac{\Phi(x - \frac{1}{2}\delta, y + \frac{\sqrt{3}}{2}\delta) - \Phi(x+\delta, y)}{\sqrt{3}\delta} \tag{2.119}$$

$$(\nabla_x \Phi)(x,y) = \frac{2}{3\delta}\left[\Phi(x+\delta,y) - \frac{1}{2}\left(\Phi(x - \frac{1}{2}\delta, y + \frac{\sqrt{3}}{2}\delta) + \Phi(x - \frac{1}{2}\delta, y - \frac{\sqrt{3}}{2}\delta)\right)\right] . \tag{2.120}$$

Comment: these notations are too complicated. I should name the points P , P_1 , P_2 , and P_3 , and give the equations

$$(\nabla_y \Phi)(P) = \frac{\Phi(P_2) - \Phi(P_1)}{\sqrt{3}} \tag{2.121}$$

$$(\nabla_x \Phi)(P) = \frac{2}{3}\left[\Phi(P_1) - \frac{1}{2}\left(\Phi(P_2) + \Phi(P_3)\right)\right] . \tag{2.122}$$

Comment: say somewhere that for the derivative along y this gives the simple difference between the values of Φ at points P_2 and P_3 , while for the derivative along x this gives the simple difference between the value of Φ at point P_1 and the average of the values of Φ at points P_2 , and P_3 . Comment: say also somewhere the this generalizes to arbitrary dimension, and give the corresponding formulas.

3-D: Comment: say here that we have a tetrahedron. For a vector field in 3-D (see figure 2.17), equation XXX gives, at first order,

$$\begin{aligned}
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1) - V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2) &= \left(\nabla_j V^i\right)(\mathbf{x}) \left(\delta x_1^j - \delta x_2^j\right) \\
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2) - V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_3) &= \left(\nabla_j V^i\right)(\mathbf{x}) \left(\delta x_2^j - \delta x_3^j\right) \\
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_3) - V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_4) &= \left(\nabla_j V^i\right)(\mathbf{x}) \left(\delta x_3^j - \delta x_4^j\right) \\
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_4) - V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1) &= \left(\nabla_j V^i\right)(\mathbf{x}) \left(\delta x_4^j - \delta x_1^j\right) ,
\end{aligned} \tag{2.123}$$

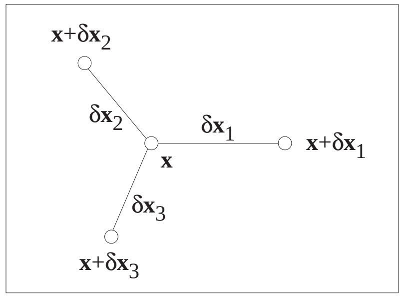

Figure 2.15: To be written.

and, again, any set of three equations among the four can be taken as the set used to define the components of the covariant derivatives of the vector field (the three equation split, for the three values of the index $i = \{1, 2, 3\}$) into nine equations, which is the number of components to be evaluated for $\nabla_i V^j(\mathbf{x})$ in three dimensions.

For a general tensor,

$$\begin{aligned}
\mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_1) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_2) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_1^j - \delta x_2^j\right) \\
\mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_2) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_3) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_2^j - \delta x_3^j\right) \\
\mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_3) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_4) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_3^j - \delta x_4^j\right) \\
\mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_4) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_1) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_4^j - \delta x_1^j\right) .
\end{aligned} \tag{2.124}$$

N-D: Comment: say here that we take N+1 points. We have

$$\begin{aligned}
\mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_1) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_2) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_1^j - \delta x_2^j\right) \\
\mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_2) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_3) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_2^j - \delta x_3^j\right) \\
\dots &= \dots \\
\mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_n) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_{n+1}) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_n^j - \delta x_{n+1}^j\right) \\
\mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_{n+1}) - \mathbf{T}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_1) &= (\nabla_j \mathbf{T})(\mathbf{x}) \left(\delta x_{n+1}^j - \delta x_1^j\right) .
\end{aligned} \tag{2.125}$$

From these $n+1$ equations, we can arbitrarily take n of them. Whatever the number of components of the tensor $\mathbf{T}(bfx)$ may be we will have three times as many equations, and this is the number of components of the *gradient* of the tensor.

1-D: Give some sense to what follows. See figure 2.18.

$$\begin{aligned}
\mathbf{T}_{11\dots}{}^{11\dots}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_1) - \mathbf{T}_{11\dots}{}^{11\dots}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_2) &= \left(\nabla_1 \mathbf{T}_{11\dots}{}^{11\dots}\right)(\mathbf{x}) \left(\delta x_1^j - \delta x_2^j\right) \\
\mathbf{T}_{11\dots}{}^{11\dots}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_2) - \mathbf{T}_{11\dots}{}^{11\dots}(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_1) &= \left(\nabla_1 \mathbf{T}_{11\dots}{}^{11\dots}\right)(\mathbf{x}) \left(\delta x_2^j - \delta x_1^j\right) .
\end{aligned} \tag{2.126}$$

This is the same equation.

Comment: what is a tensor in 1-D (only the index $i = 1$ is possible)?

Crosses:

Let us first examine the special cases in 2 and 3 dimensions, then, the general, n-dimensional case.

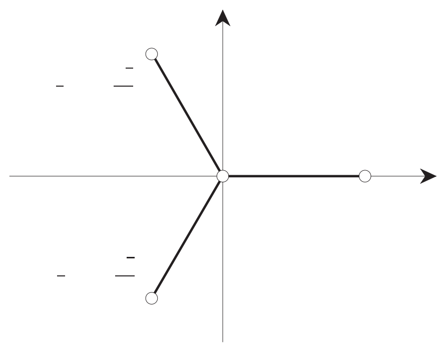

Figure 2.16: To be written.

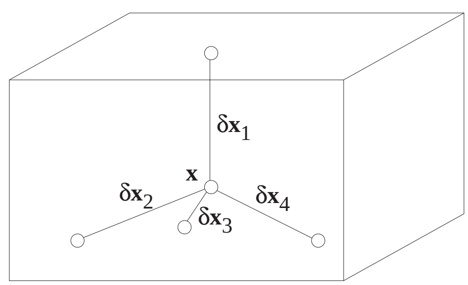

Figure 2.17: To be written.

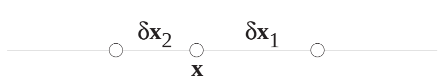

Figure 2.18: To be written.

2-D: Consider the points indicated in figure 2.19, and assume that a vector field $V^i(\mathbf{x})$ has been defined on the space. Using the definition of tensorial (or covariant) derivative (equation XXX), we can write, at first order, the set of equations

$$\begin{aligned} V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1/2) - V^i(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}_1/2) &= \left(\nabla_j V^i\right)(\mathbf{x})\ \delta x_1^j \\ V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2/2) - V^i(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}_2/2) &= \left(\nabla_j V^i\right)(\mathbf{x})\ \delta x_2^j\,. \end{aligned} \tag{2.127}$$

Each of the two equations being valid for $i=1$ and $i=2$ we have, in all, four equations, exactly the number needed to estimate the four components of $(\nabla_j V^i)(\mathbf{x})$. For any finite value of $\delta\mathbf{x}_1$, and $\delta\mathbf{x}_2$, this will give a finite-difference approximation to the covariant derivative of the vector field. In the limit when $\delta\mathbf{x}_1$ and $\delta\mathbf{x}_2$ tend to zero, we obtain the exact value of the covariant derivative.

For a general tensor field, equation 2.127 becomes

$$\begin{aligned} T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1/2) - T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}_1/2) &= \left(\nabla_m T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ \delta x_1^m \\ T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2/2) - T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}_2/2) &= \left(\nabla_m T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ \delta x_2^m\,. \end{aligned} \tag{2.128}$$

and, again, we have as much equations as unknowns to solve for the finite-difference approximation to the covariant derivative and, in the limit, to the derivative itself.

Comment: I have to say somewhere why I can get the exact derivative having selected to take the limit along two specially selected directlions only.

Example: Consider the Euclidean plane, with Cartesian coordinates, and let us evaluate the gradient of a scalar field.

Figure 2.20 shows the star chosen to evaluate derivatives. Applying equation 2.128 to a scalar field $\Phi(x,y)$ gives

$$\begin{aligned} \Phi(x+\delta/2,y) - \Phi(x-\delta/2,y) &= (\nabla_x\Phi)(x,y)\ \delta \\ \Phi(x,y+\delta/2) - \Phi(x,y-\delta/2) &= (\nabla_y\Phi)(x,y)\ \delta\,, \end{aligned} \tag{2.129}$$

i.e.,

$$(\nabla_x\Phi)(x,y) = \frac{\Phi(x+\delta/2,y) - \Phi(x-\delta/2,y)}{\delta} \tag{2.130}$$

$$(\nabla_y\Phi)(x,y) = \frac{\Phi(x,y+\delta/2) - \Phi(x,y-\delta/2)}{\delta}\,. \tag{2.131}$$

3-D: For a vector tensor in 3-D (see figure 2.21), equation XXX gives, at first order,

$$\begin{aligned} T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1/2) - T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}_1/2) &= \left(\nabla_m T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ \delta x_1^m \\ T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_2/2) - T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}_2/2) &= \left(\nabla_m T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ \delta x_2^m \\ T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_3/2) - T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}_3/2) &= \left(\nabla_m T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ \delta x_3^m\,. \end{aligned} \tag{2.132}$$

and, again, we have as many equations as unknowns.

N-D: Comment: say here that we take N+1 points. We have

$$\begin{aligned} T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_1/2) - T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}_1/2) &= \left(\nabla_j T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ \delta x_1^j \\ \dots &= \dots \\ T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_n/2) - T_{ij\dots}{}^{k\ell\dots}(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}_n/2) &= \left(\nabla_j T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ \delta x_n^j\,. \end{aligned} \tag{2.133}$$

We have as many equations as unknowns.

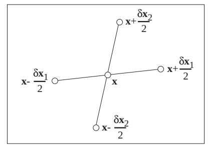

Figure 2.19: To be written.

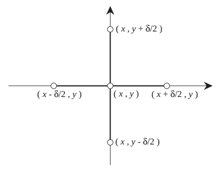

Figure 2.20: To be written.

1-D: Comment: say here that we take N+1=2 points. See figure 2.22. We have

$$T_{11\dots}{}^{11\dots}(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}/2) - T_{11\dots}{}^{11\dots}(\mathbf{x} \parallel \mathbf{x}-\delta\mathbf{x}/2) \quad = \quad \left(\nabla_j T_{ij\dots}{}^{k\ell\dots}\right)(\mathbf{x})\ \delta x^j \,. \tag{2.134}$$

We have one equation and one unknown.

2.3 The finite volume method

This section is to become an important section of the chapter, for this is the main numerical menthod we are going to use to estimate (covariant) derivatives.

Assume we want to numerically estimate the divergence of a vector field ϕ^i at a given point. We consider a volume around the point, and write the divergence theorem (see section 5.2)

$$\int_{3D} dV\ \nabla_i \phi^i = \int_{2D} dS_i\ \phi^i \,, \tag{2.135}$$

If the vector field is smooth enough, and the volume small enough, we can write

$$\nabla_i \phi^i = \frac{1}{\Delta V} \int_{2D} dS_i\ \phi^i \,. \tag{2.136}$$

Dividing the surface into small pieces on each of which the field can be approximately taken as constant gives

$$\nabla_i \phi^i = \frac{1}{\Delta V} \sum_{\text{All facets}} \Delta S_i\, \phi^i \,. \tag{2.137}$$

Figure 2.23 illustrates this in a 2-D context.

Note: explain here how to compute all types of derivatives for all types of tensors. For instance, if σ^{ij} is the stress tensor of dynamics of continuous media, how to estimate $\nabla_i \sigma^{ij}$?

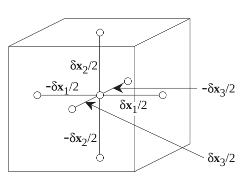

Figure 2.21: To be written.

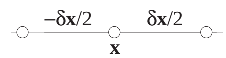

Figure 2.22: To be written.

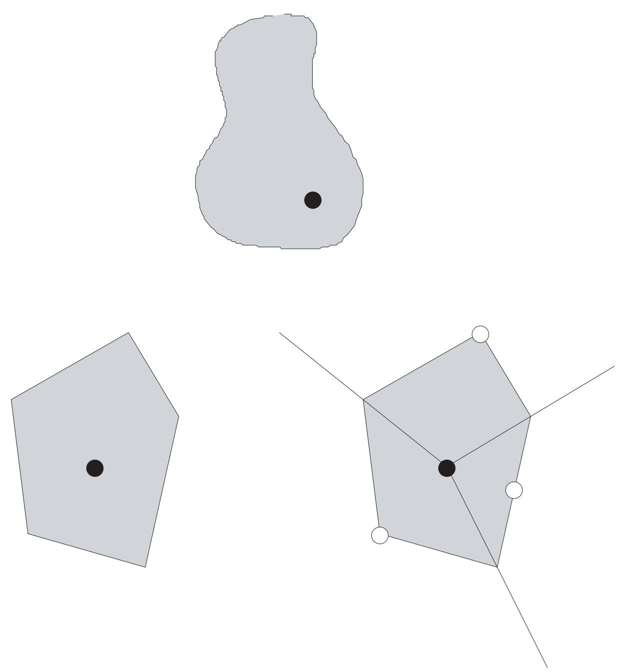

Figure 2.23: Top: The gradient of a tensor field at the black point can be estimated, through the use of the divergence theorem, by integrating the field along the boundary, i.e, by estimating the flux of the tensor field. Bottom left: Usually, a polygon is chose as finite volume. Bottom right: Approximating the values of the tensor field at the boundary by a few points (here, the three white points), the gradient of the tensor field is obtained as a discrete sum (here with three terms). Note: This has to be much better explained, as we can integrate only scalars. We have to use “test” vector fields.

2.4 Appendices

2.4.1 Appendix: Parallel transportation

We consider here an Euclidean space with cartesian coordinates $\{X^I\}$, and a “working” system of coordinates $\{x^i\}$ that may not be cartesian (think, for instance, in the spherical coordinates for the 3-D physical space).

In Section xxx we have denoted by $\mathbf{V}(\mathbf{x}_B \parallel \mathbf{x}_A)$ the vector (or tensor) at point $\mathbf{x}_B$ obtained by parallel transportation of $\mathbf{V}(\mathbf{x}_A)$ from point A to point B .

Using the working coordinates $\{x^i\}$ there is no obvious way of performing that parallel transportation. Instead, let us transform, at point A , the components of $\mathbf{V}(\mathbf{x}_A)$ into the components in the basis associated with the cartesian coordinates:

$$V'^I(\mathbf{X}_A) = \frac{\partial X^I}{\partial x^i}(\mathbf{x}_A)\; V^i(\mathbf{x}_A)\,. \tag{2.138}$$

Now, in cartesian coordinates, the parallel transportation is trivial, as tensors keep constant components:

$$V'^I(\mathbf{X}_B \parallel \mathbf{X}_A) = V'^I(\mathbf{X}_A)\,. \tag{2.139}$$

We can now transform back, at point B , the components $V'^I(\mathbf{X}_B \parallel \mathbf{X}_A)$ into the components corresponding to the working coordinates $\{x^i\}$:

$$V^i(\mathbf{x}_B \parallel \mathbf{x}_A) = \frac{\partial x^i}{\partial X^I}(\mathbf{X}_B)\; V'^I(\mathbf{X}_B \parallel \mathbf{X}_A)\,. \tag{2.140}$$

This gives the final result:

$$V^i(\mathbf{x}_B \parallel \mathbf{x}_A) = \frac{\partial x^i}{\partial X^M}(\mathbf{X}_B)\; \frac{\partial X^M}{\partial x^j}(\mathbf{x}_A)\; V^j(\mathbf{x}_A)\,. \tag{2.141}$$

As the coefficients of parallel transportation were defined (see XXX) by

$$V^i(\mathbf{x}_B \parallel \mathbf{x}_A) = \gamma_j{}^i\,(\mathbf{x}_B|\mathbf{x}_A) V^j(\mathbf{x}_A)\,, \tag{2.142}$$

this gives

$$\boxed{\gamma_j{}^i(\mathbf{x}_B|\mathbf{x}_A) = \frac{\partial x^i}{\partial X^M}(\mathbf{X}(\mathbf{x}_B))\; \frac{\partial X^M}{\partial x^j}(\mathbf{x}_A)\,.} \tag{2.143}$$

As

$$\Gamma_{kj}{}^i(\mathbf{x}_B|\mathbf{x}_A) = \frac{\partial \gamma_j{}^i}{\partial x^k_A}(\mathbf{x}_B|\mathbf{x}_A) \tag{2.144}$$

(the index “A” in the derivative is to recall that it concerns the second variable), this gives

$$\Gamma_{kj}{}^i(\mathbf{x}_B|\mathbf{x}_A) = \frac{\partial x^i}{\partial X^M}(\mathbf{X}(\mathbf{x}_B))\; \frac{\partial^2 X^M}{\partial x^j \partial x^k}(\mathbf{x}_A)\,, \tag{2.145}$$

i.e., recalling that we use the notation $\Gamma_{kj}{}^i(\mathbf{x})$ for $\Gamma_{kj}{}^i(\mathbf{x}|\mathbf{x})$,

$$\boxed{\Gamma_{kj}{}^i(\mathbf{x}) = \frac{\partial x^i}{\partial X^M}(\mathbf{X}(\mathbf{x}))\; \frac{\partial^2 X^M}{\partial x^j \partial x^k}(\mathbf{x})\,.} \tag{2.146}$$

Example: We have somewhere calculated the coefficients for the parallel transportation of vectors in the Euclidean plane with polar coordinates using direct geometrical considerations. We could also have used the result here above.

En vrac:

$$\{x^2, x^2\} = \{r, \varphi\}$$
$$\{X^1, X^2\} = \{x, y\}$$
$$r^2 = x^2 + y^2$$
$$\operatorname{tg} \varphi = \frac{y}{x}$$
$$x = r \cos \varphi$$
$$y = r \sin \varphi$$
$$\begin{pmatrix} \partial x/\partial r & \partial x/\partial \varphi \\ \partial y/\partial r & \partial y/\partial \varphi \end{pmatrix} = \begin{pmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{pmatrix}$$
$$\begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial \varphi/\partial x & \partial \varphi/\partial y \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\frac{1}{r} \sin \varphi & \frac{1}{r} \cos \varphi \end{pmatrix}$$

For instance, we have

$$\gamma_r{}^r(r_A, \varphi_A | r_B, \varphi_B) = \frac{\partial r}{\partial x}(r_A, \varphi_A) \frac{\partial x}{\partial r}(r_B, \varphi_B) + \frac{\partial r}{\partial y}(r_A, \varphi_A) \frac{\partial y}{\partial r}(r_B, \varphi_B)$$
$$= \cos \varphi_A \cos \varphi_B + \sin \varphi_A \sin \varphi_B = \cos(\varphi_B - \varphi_A) \, .$$

In total:

$$\begin{pmatrix} \gamma_r{}^r(r_A, \varphi_A | r_B, \varphi_B) & \gamma_r{}^\varphi(r_A, \varphi_A | r_B, \varphi_B) \\ \gamma_\varphi{}^r(r_A, \varphi_A | r_B, \varphi_B) & \gamma_\varphi{}^\varphi(r_A, \varphi_A | r_B, \varphi_B) \end{pmatrix} = \begin{pmatrix} \cos(\varphi_B - \varphi_A) & \frac{1}{r_A} \sin(\varphi_B - \varphi_A) \\ -r_B \sin(\varphi_B - \varphi_A) & \frac{r_B}{r_A} \cos(\varphi_B - \varphi_A) \end{pmatrix}$$

From this, the connection coefficients can be obtained by a simple derivation.

2.4.2 Appendix: Old text: Parallel transportation

We consider here an euclidean space with cartesian coordinates $\{y^I\}$, and a "working" system of coordinates $\{x^i\}$ that may not be cartesian (think, for instance, in the spherical coordinates for the 3-D physical space).

In Section xxx we have denoted by $\mathbf{V}(\mathbf{x}_B \parallel \mathbf{x}_A)$ the vector (or tensor) at point $\mathbf{x}_B$ obtained by parallel transportation of $\mathbf{V}(\mathbf{x}_A)$ from point A to point B .

Using the working coordinates $\{x^i\}$ there is no obvious way of performing that parallel transportation. Instead, let us transform, at point A , the components of $\mathbf{V}(\mathbf{x}_A)$ into the components in the basis associated with the cartesian coordinates:

$$V'^I(\mathbf{y}_A) = \frac{\partial y^I}{\partial x^i}(\mathbf{x}_A) \; V^i(\mathbf{x}_A) \, . \tag{2.147}$$

Now, in cartesian coordinates, the parallel transportation is trivial, as tensors keep constant components:

$$V'^I(\mathbf{y}_B \parallel \mathbf{y}_A) = V'^I(\mathbf{y}_A) \, . \tag{2.148}$$

We can now transform back, at point B , the components $V'^I(\mathbf{y}_B \parallel \mathbf{y}_A)$ into the components corresponding to the working coordinates $\{x^i\}$:

$$V^i(\mathbf{x}_B \parallel \mathbf{x}_A) = \frac{\partial x^i}{\partial y^I}(\mathbf{y}_B) \; V'^I(\mathbf{y}_B \parallel \mathbf{y}_A) \, . \tag{2.149}$$

This gives the final result:

$$\boxed{V^i(\mathbf{x}_B \parallel \mathbf{x}_A) = \frac{\partial x^i}{\partial y^I}(\mathbf{y}_B)\ \frac{\partial y^I}{\partial x^j}(\mathbf{x}_A)\ V^j(\mathbf{x}_A)\,.} \tag{2.150}$$

In Section xxx the partial derivatives of a vector field where defined by the first order development

$$V^i(\mathbf{x} + \delta\mathbf{x}) = V^i(\mathbf{x}) + (\partial_j V^i)(\mathbf{x})\ \delta x^j + \dots\ , \tag{2.151}$$

while the (covariant) derivatives where defined by

$$V^i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = V^i(\mathbf{x}) + (\nabla_j V^i)(\mathbf{x})\ \delta x^j + \dots\,. \tag{2.152}$$

Using equation 2.141 we have

$$V^i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = \frac{\partial x^i}{\partial y^I}(\mathbf{y}(\mathbf{x}))\ \frac{\partial y^I}{\partial x^j}(\mathbf{x} + \delta\mathbf{x})\ V^j(\mathbf{x} + \delta\mathbf{x})\,. \tag{2.153}$$

Using equation 2.151 and the first order development

$$\frac{\partial y^I}{\partial x^j}(\mathbf{x} + \delta\mathbf{x}) = \frac{\partial y^I}{\partial x^j}(\mathbf{x}) + \frac{\partial^2 y^I}{\partial x^j \partial x^k}(\mathbf{x})\ \delta x^k\ , \tag{2.154}$$

the last equation can be written, up to the first order,

$$V^i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = \frac{\partial x^i}{\partial y^I}(\mathbf{y}(\mathbf{x}))\ \left(\frac{\partial y^I}{\partial x^j}(\mathbf{x}) + \frac{\partial^2 y^I}{\partial x^j \partial x^k}(\mathbf{x})\ \delta x^k\right)\left(V^i(\mathbf{x}) + (\partial_j V^i)(\mathbf{x})\ \delta x^j\right)\ , \tag{2.155}$$

or, using

$$\frac{\partial x^i}{\partial y^I}(\mathbf{y}(\mathbf{x}))\ \frac{\partial y^I}{\partial x^j}(\mathbf{x}) = \delta^i{}_j\ , \tag{2.156}$$

we get, up to the first order,

$$V^i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = V^i(\mathbf{x}) + \left((\partial_k V^i)(\mathbf{x}) + \frac{\partial x^i}{\partial y^I}(\mathbf{y}(\mathbf{x}))\ \frac{\partial^2 y^I}{\partial x^j \partial x^k}(\mathbf{x})\ V^j(\mathbf{x})\right)\ \delta x^k + \dots \tag{2.157}$$

A direct comparison with 2.152 directly gives

$$(\nabla_k V^i)(\mathbf{x}) = (\partial_k V^i)(\mathbf{x}) + \frac{\partial x^i}{\partial y^I}(\mathbf{y}(\mathbf{x}))\ \frac{\partial^2 y^I}{\partial x^j \partial x^k}(\mathbf{x})\ V^j(\mathbf{x})\ , \tag{2.158}$$

or, for short,

$$\nabla_k V^i = \partial_k V^i + \Gamma_{kj}{}^i\ V^j\ , \tag{2.159}$$

where the connection coefficients $\Gamma_{jk}{}^i$ are defined by

$$\Gamma_{jk}{}^i(\mathbf{x}) = \frac{\partial x^i}{\partial y^I}(\mathbf{y}(\mathbf{x}))\ \frac{\partial^2 y^I}{\partial x^j \partial x^k}(\mathbf{x})\,. \tag{2.160}$$

For a density, equation 2.141 becomes

$$\overline{V}^i(\mathbf{x}_B \parallel \mathbf{x}_A) = \mathcal{J}^{-1}(\mathbf{y}_B)\ \frac{\partial x^i}{\partial y^I}(\mathbf{y}_B)\ \mathcal{J}(\mathbf{x}_A)\ \frac{\partial y^I}{\partial x^j}(\mathbf{x}_A)\ \overline{V}^j(\mathbf{x}_A)\ , \tag{2.161}$$

and, developing the demonstration along the same lines as before (WARNING: This has to be done), we obtain, for the derivative of a vector density,

$$\nabla_k \overline{V}^i = \partial_k V^i + \Gamma_{kj}{}^i\ \overline{V}^j + \Gamma_{kj}{}^j \overline{V}^i\,. \tag{2.162}$$

2.4.3 Appendix: Taylor's series

The Taylor's series:

Note: this appendix is, for the moment being, just an essay (i.e., a set of conjectures, to be worked out).

$$
\begin{aligned}
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) &= V^i(\mathbf{x}) \\
&+ \left(\nabla_j V^i\right)(\mathbf{x})\ \delta x^j \\
&+ \frac{1}{2}\left(\nabla_j \nabla_k V^i\right)(\mathbf{x})\ \delta x^j\ \delta x^k \\
&+ \frac{1}{3!}\left(\nabla_j \nabla_k \nabla_\ell V^i\right)(\mathbf{x})\ \delta x^j\ \delta x^k\ \delta x^\ell \\
&+ \ \dots
\end{aligned} \tag{2.163}
$$

The "two step" Taylor's series:

$$
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_\mathrm{A} \parallel \mathbf{x}+\delta\mathbf{x}_\mathrm{A}+\delta\mathbf{x}_\mathrm{B}) =
$$

$$
\begin{aligned}
&= V^i(\mathbf{x}) \\
&+ \left(\nabla_j V^i\right)(\mathbf{x})\ \left(\delta x^j_\mathrm{A}+\delta x^j_\mathrm{B}\right) \\
&+ \frac{1}{2}\Big[\left(\nabla_k \nabla_j V^i\right)(\mathbf{x})\ \left(\delta x^j_\mathrm{A}+\delta x^j_\mathrm{B}\right)\ \left(\delta x^k_\mathrm{A}+\delta x^k_\mathrm{B}\right) + \left(\{\nabla_k\nabla_j - \nabla_j\nabla_k\}V^i\right)(\mathbf{x})\ \delta x^j_\mathrm{B}\ \delta x^k_\mathrm{A}\Big] \\
&+ \frac{1}{3!}\Big[\left(\nabla_\ell\nabla_k\nabla_j V^i\right)(\mathbf{x})\ \left(\delta x^j_\mathrm{A}+\delta x^j_\mathrm{B}\right)\ \left(\delta x^k_\mathrm{A}+\delta x^k_\mathrm{B}\right)\ \left(\delta x^\ell_\mathrm{A}+\delta x^\ell_\mathrm{B}\right) \\
&\qquad + \Big[\left(\nabla_\ell\{\nabla_k\nabla_j - \nabla_j\nabla_k\} + \{\nabla_\ell\nabla_k\nabla_j - \nabla_j\nabla_k\nabla_\ell\}\right)V^i\Big](\mathbf{x})\ \delta x^j_\mathrm{B}\ \delta x^k_\mathrm{A}\ \delta x^\ell_\mathrm{A} \\
&\qquad + \Big[\left(\{\nabla_\ell\nabla_k - \nabla_k\nabla_\ell\}\nabla_j + \{\nabla_\ell\nabla_k\nabla_j - \nabla_j\nabla_k\nabla_\ell\}\right)V^i\Big](\mathbf{x})\ \delta x^j_\mathrm{B}\ \delta x^k_\mathrm{B}\ \delta x^\ell_\mathrm{A}\Big] \\
&+ \ \dots
\end{aligned} \tag{2.164}
$$

Going from one point to the other by two different paths:

$$
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_\mathrm{A} \parallel \mathbf{x}+\delta\mathbf{x}_\mathrm{A}+\delta\mathbf{x}_\mathrm{B}) - V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_\mathrm{B} \parallel \mathbf{x}+\delta\mathbf{x}_\mathrm{B}+\delta\mathbf{x}_\mathrm{A}) =
$$

$$
\begin{aligned}
&= \left(\{\nabla_k\nabla_j - \nabla_j\nabla_k\}V^i\right)(\mathbf{x})\ \delta x^j_\mathrm{B}\ \delta x^k_\mathrm{A} \\
&+ \frac{1}{3!}\Big[\Big[\left(\nabla_\ell\{\nabla_k\nabla_j - \nabla_j\nabla_k\} + \{\nabla_k\nabla_j - \nabla_j\nabla_k\}\nabla_\ell + 2\{\nabla_\ell\nabla_k\nabla_j - \nabla_j\nabla_k\nabla_\ell\}\right)V^i\Big](\mathbf{x})\ \delta x^j_\mathrm{B}\ \delta x^k_\mathrm{A}\ \delta x^\ell_\mathrm{A} \\
&\qquad + \Big[\left(\nabla_j\{\nabla_\ell\nabla_k - \nabla_k\nabla_\ell\} + \{\nabla_\ell\nabla_k - \nabla_k\nabla_\ell\}\nabla_j + 2\{\nabla_\ell\nabla_k\nabla_j - \nabla_j\nabla_k\nabla_\ell\}\right)V^i\Big](\mathbf{x})\ \delta x^j_\mathrm{B}\ \delta x^k_\mathrm{B}\ \delta x^\ell_\mathrm{A}\Big] \\
&+ \ \dots
\end{aligned} \tag{2.165}
$$

Demonstration: Taylor's series

To demonstrate equation 2.163, we will assume that a development

$$
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) = V^i(\mathbf{x}) + \left(A^1_j V^i\right)(\mathbf{x})\ \delta x^j + \left(A^2_{jk} V^i\right)(\mathbf{x})\ \delta x^j\ \delta x^k + \left(A^3_{jk\ell} V^i\right)(\mathbf{x})\ \delta x^j\ \delta x^k\ \delta x^\ell + \dots \tag{2.166}
$$

exists, where the operators $\mathbf{A}^n$ are linear, and then show that it implies

$$
A^2_{jk} = \frac{1}{2} A^1_j A^1_k \qquad A^3_{jk\ell} = \frac{1}{3!} A^1_j A^1_k A^1_\ell \qquad \dots \qquad A^n_{i^1 i^2 \dots i^{n-1} i^n} = \frac{1}{n!} A^1_{i^1} A^1_{i^2} \dots A^1_{i^{n-1}} A^1_{i^n}\ . \tag{2.167}
$$

As we know that, at first order,

$$V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) = V^i(\mathbf{x}) + \left(\nabla_j V^i\right)(\mathbf{x})\ \delta x^j + \dots\ , \tag{2.168}$$

then

$$A^1_j = \nabla_j\ , \tag{2.169}$$

and equation 2.163 will be demonstrated.

First, let us write equation 2.166 using the variable $\mathbf{y}$:

$$V^i(\mathbf{y} \parallel \mathbf{y}+\delta\mathbf{y}) = V^i(\mathbf{y}) + \left(A^1_j V^i\right)(\mathbf{y})\ \delta y^j + \left(A^2_{jk} V^i\right)(\mathbf{y})\ \delta y^j\ \delta y^k + \left(A^3_{jk\ell} V^i\right)(\mathbf{y})\ \delta y^j\ \delta y^k\ \delta y^\ell + \dots\ . \tag{2.170}$$

Setting there $\mathbf{y} = \mathbf{x}+\delta\mathbf{x}/2$ and $\delta\mathbf{y} = \delta\mathbf{x}/2$ gives

$$\begin{aligned} V^i(\mathbf{x}+\frac{\delta\mathbf{x}}{2} \parallel \mathbf{x}+\delta\mathbf{x}) \ &= \ V^i(\mathbf{x}+\frac{\delta\mathbf{x}}{2}) \\ &+ \ \left(A^1_j V^i\right)(\mathbf{x}+\frac{\delta\mathbf{x}}{2})\ \frac{\delta x^j}{2} \\ &+ \ \left(A^2_{jk} V^i\right)(\mathbf{x}+\frac{\delta\mathbf{x}}{2})\ \frac{\delta x^j}{2}\ \frac{\delta x^k}{2} \\ &+ \ \left(A^3_{jk\ell} V^i\right)(\mathbf{x}+\frac{\delta\mathbf{x}}{2})\ \frac{\delta x^j}{2}\ \frac{\delta x^k}{2}\ \frac{\delta x^\ell}{2} \\ &+ \ \dots\ . \end{aligned} \tag{2.171}$$

Transporting this equality from point $\mathbf{x}+\delta\mathbf{x}/2$ to point $\mathbf{x}$ gives

$$\begin{aligned} V^i(\mathbf{x} \parallel \mathbf{x}+\frac{\delta\mathbf{x}}{2} \parallel \mathbf{x}+\delta\mathbf{x}) \ &= \ V^i(\mathbf{x} \parallel \mathbf{x}+\frac{\delta\mathbf{x}}{2}) \\ &+ \ \left(A^1_j V^i\right)(\mathbf{x} \parallel \mathbf{x}+\frac{\delta\mathbf{x}}{2})\ \frac{\delta x^j}{2} \\ &+ \ \left(A^2_{jk} V^i\right)(\mathbf{x} \parallel \mathbf{x}+\frac{\delta\mathbf{x}}{2})\ \frac{\delta x^j}{2}\ \frac{\delta x^k}{2} \\ &+ \ \left(A^3_{jk\ell} V^i\right)(\mathbf{x} \parallel \mathbf{x}+\frac{\delta\mathbf{x}}{2})\ \frac{\delta x^j}{2}\ \frac{\delta x^k}{2}\ \frac{\delta x^\ell}{2} \\ &+ \ \dots\ , \end{aligned} \tag{2.172}$$

i.e., as $V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}/2 \parallel \mathbf{x}+\delta\mathbf{x}) = V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x})$ (transportation along a geodesic),

$$\begin{aligned} V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) \ &= \ V^i(\mathbf{x} \parallel \mathbf{x}+\frac{\delta\mathbf{x}}{2}) \\ &+ \ \left(A^1_j V^i\right)(\mathbf{x} \parallel \mathbf{x}+\frac{\delta\mathbf{x}}{2})\ \frac{\delta x^j}{2} \\ &+ \ \left(A^2_{jk} V^i\right)(\mathbf{x} \parallel \mathbf{x}+\frac{\delta\mathbf{x}}{2})\ \frac{\delta x^j}{2}\ \frac{\delta x^k}{2} \\ &+ \ \left(A^3_{jk\ell} V^i\right)(\mathbf{x} \parallel \mathbf{x}+\frac{\delta\mathbf{x}}{2})\ \frac{\delta x^j}{2}\ \frac{\delta x^k}{2}\ \frac{\delta x^\ell}{2} \\ &+ \ \dots\ . \end{aligned} \tag{2.173}$$

Developing each term gives

$$V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) \ = \ V^i(\mathbf{x}) + \left(A^1_j V^i\right)(\mathbf{x})\frac{\delta x^j}{2} + \left(A^2_{jk} V^i\right)(\mathbf{x})\frac{\delta x^j}{2}\frac{\delta x^k}{2} + \left(A^3_{jk\ell} V^i\right)(\mathbf{x})\frac{\delta x^j}{2}\frac{\delta x^k}{2}\frac{\delta x^\ell}{2} + \dots$$

$$
\begin{aligned}
&+ \quad A^1_j \left(V^i(\mathbf{x}) + \left(A^1_k V^i \right)(\mathbf{x}) \frac{\delta x^k}{2} + \left(A^2_{k\ell} V^i \right)(\mathbf{x}) \frac{\delta x^k}{2} \frac{\delta x^\ell}{2} + \dots \right) \frac{\delta x^j}{2} \\
&+ \quad A^2_{jk} \left(V^i(\mathbf{x}) + \left(A^1_\ell V^i \right)(\mathbf{x}) \frac{\delta x^\ell}{2} + \dots \right) \frac{\delta x^j}{2} \frac{\delta x^k}{2} \\
&+ \quad A^3_{jk\ell} \left(V^i(\mathbf{x}) + \dots \right) \frac{\delta x^j}{2} \frac{\delta x^k}{2} \frac{\delta x^\ell}{2} \\
&+ \quad \dots , \qquad (2.174)
\end{aligned}
$$

i.e., grouping the terms of same order,

$$
\begin{aligned}
V^i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) \quad = \quad & V^i(\mathbf{x}) \\
+ \quad & \frac{1}{2} \left(A^1_j + A^1_j \right) V^i(\mathbf{x})\, \delta x^j \\
+ \quad & \frac{1}{4} \left(A^2_{jk} + A^1_j A^1_k + A^2_{jk} \right) V^i(\mathbf{x})\, \delta x^j\, \delta x^k \\
+ \quad & \frac{1}{8} \left(A^3_{jk\ell} + A^1_j A^2_{k\ell} + A^2_{jk} A^1_\ell + A^3_{jk\ell} \right) V^i(\mathbf{x})\, \delta x^j\, \delta x^k\, \delta x^\ell \\
+ \quad & \dots . \qquad (2.175)
\end{aligned}
$$

A comparison of this with equation 2.166 directly leads to equation 2.167.

Demonstration: The "two step" Taylor's series

Comment: In what follows, I do not think there is any need in having the *norm* $\|\delta\mathbf{x}\|$ defined, but then, I do not know which notation should I use to replace $O(\|\delta\mathbf{x}\|^n)$.

We start with

$$V^i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = V^i(\mathbf{x}) + \left(\nabla_j V^i \right)(\mathbf{x})\, \delta x^j + \frac{1}{2} \left(\nabla_k \nabla_j V^i \right)(\mathbf{x})\, \delta x^j\, \delta x^k + O(\|\delta\mathbf{x}\|^3) \qquad (2.176)$$

and

$$\left(\nabla_j V^i \right)(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = \left(\nabla_j V^i \right)(\mathbf{x}) + \left(\nabla_k \nabla_j V^i \right)(\mathbf{x})\, \delta x^k + O(\|\delta\mathbf{x}\|^2) \, . \qquad (2.177)$$

Replacing $\mathbf{x}$ by $\mathbf{x} + \delta\mathbf{x}_A$ and $\delta\mathbf{x}$ by $\delta\mathbf{x}_B$ in equation 2.176 gives

$$
\begin{gathered}
V^i(\mathbf{x} + \delta\mathbf{x}_A \parallel \mathbf{x} + \delta\mathbf{x}_A + \delta\mathbf{x}_B) = \\
V^i(\mathbf{x} + \delta\mathbf{x}_A) + \left(\nabla_j V^i \right)(\mathbf{x} + \delta\mathbf{x}_A)\, \delta x^j_B + \frac{1}{2} \left(\nabla_k \nabla_j V^i \right)(\mathbf{x} + \delta\mathbf{x}_A)\, \delta x^j_B\, \delta x^k_B + O(\|\delta\mathbf{x}\|^3) \, , \qquad (2.178)
\end{gathered}
$$

and a parallel transportation of this equation to point $\mathbf{x}$ gives

$$
\begin{gathered}
V^i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_A \parallel \mathbf{x} + \delta\mathbf{x}_A + \delta\mathbf{x}_B) = \\
V^i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_A) + \left(\nabla_j V^i \right)(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_A)\, \delta x^j_B + \frac{1}{2} \left(\nabla_k \nabla_j V^i \right)(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}_A)\, \delta x^j_B\, \delta x^k_B + O(\|\delta\mathbf{x}\|^3) \, . \qquad (2.179)
\end{gathered}
$$

Replacing there the developments 2.176 and 2.177 gives

$$
\begin{aligned}
V^i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_A \parallel \mathbf{x} + \delta\mathbf{x}_A + \delta\mathbf{x}_B) \quad = \quad & V^i(\mathbf{x}) + \left(\nabla_j V^i \right)(\mathbf{x})\, \delta x^j_A + \frac{1}{2} \left(\nabla_k \nabla_j V^i \right)(\mathbf{x})\, \delta x^j_A\, \delta x^k_A + O(\|\delta\mathbf{x}\|^3) \\
+ \quad & \left(\nabla_j V^i \right)(\mathbf{x})\, \delta^j_B + \left(\nabla_k \nabla_j V^i \right)(\mathbf{x})\, \delta x^j_B\, \delta x^k_A + O(\|\delta\mathbf{x}\|^3) \\
+ \quad & \frac{1}{2} \left(\nabla_k \nabla_j V^i \right)(\mathbf{x})\, \delta x^j_B\, \delta x^k_B + O(\|\delta\mathbf{x}\|^3) \\
+ \quad & O(\|\delta\mathbf{x}\|^3) \, , \qquad (2.180)
\end{aligned}
$$

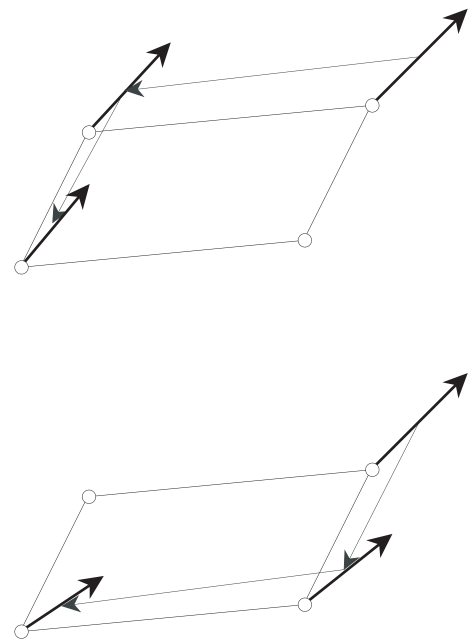

Figure 2.24: A vector is transported, parallel to itself, from one point to another point, following two different paths. At the first order of approximation, the two vectors thus obtained are equal. At the seond order of approximation, the difference between these two vectors equals $= (\nabla_k \nabla_j - \nabla_j \nabla_k)\, V^i \; \delta x^j_{\mathrm{B}} \; \delta x^k_{\mathrm{A}}$. Note: See text for an explanation. Note: explain better.

which can be written, after reordering, as

$$\begin{aligned} V^i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}_A \parallel \mathbf{x} + \delta\mathbf{x}_A + \delta\mathbf{x}_B) &= V^i(\mathbf{x}) \\ &+ \left(\nabla_j V^i\right)(\mathbf{x}) \left(\delta x^j_A + \delta^j_B\right) \\ &+ \frac{1}{2}\Big[\left(\nabla_k \nabla_j V^i\right)(\mathbf{x}) \left(\delta x^j_A + \delta^j_B\right) \left(\delta x^k_A + \delta^k_B\right) \\ &+ \left(\{\nabla_k \nabla_j - \nabla_j \nabla_k\} V^i\right)(\mathbf{x})\, \delta x^j_B \delta x^k_A\Big] \\ &+ O(\|\delta\mathbf{x}\|^3)\,. \end{aligned} \tag{2.181}$$

This third order terms in 2.164 can be found similarly, by just keeping them in the developments.

Equation 2.165 is a direct consequence of 2.164 and can be demonstrated by direct substitution.

2.4.4 Appendix: Some usual formulas of vector analysis

Let be $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ vector fields, φ a scalar field, and $\Delta\mathbf{a}$ the vector Laplacian (the Laplacian applied to each component of the vector). The following list of identities holds:

$$\operatorname{div} \mathbf{rot}\, \mathbf{a} = 0 \tag{2.182}$$

$$\mathbf{rot}\, \mathbf{grad}\, \varphi = 0 \tag{2.183}$$

$$\operatorname{div}(\varphi\mathbf{a}) = (\mathbf{grad}\, \varphi) \cdot \mathbf{a} + \varphi(\operatorname{div} \mathbf{a}) \tag{2.184}$$

$$\mathbf{rot}(\varphi\mathbf{a}) = (\mathbf{grad}\, \varphi) \times \mathbf{a} + \varphi(\mathbf{rot}\, \mathbf{a}) \tag{2.185}$$

$$\mathbf{grad}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\mathbf{rot}\,\mathbf{b}) + \mathbf{b}\times(\mathbf{rot}\,\mathbf{a}) \tag{2.186}$$

$$\mathrm{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{rot}\,\mathbf{a}) - \mathbf{a}\cdot(\mathbf{rot}\,\mathbf{b}) \tag{2.187}$$

$$\mathbf{rot}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\mathrm{div}\,\mathbf{b}) - \mathbf{b}(\mathrm{div}\,\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} \tag{2.188}$$

$$\mathbf{rot}\,\mathbf{rot}\,\mathbf{a} = \mathbf{grad}(\mathrm{div}\,\mathbf{a}) - \Delta\mathbf{a}\ . \tag{2.189}$$

Using the nabla symbol everywhere, these equations become:

$$\nabla\cdot(\varphi\mathbf{a}) = (\nabla\varphi)\cdot\mathbf{a} + \varphi(\nabla\cdot\mathbf{a}) \tag{2.190}$$

$$\nabla\times(\varphi\mathbf{a}) = (\nabla\varphi)\times\mathbf{a} + \varphi(\nabla\times\mathbf{a}) \tag{2.191}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}) \tag{2.192}$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b}) \tag{2.193}$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} \tag{2.194}$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \Delta\mathbf{a}\ . \tag{2.195}$$

The following three vector equations are also often useful:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \tag{2.196}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c} \tag{2.197}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{2.198}$$

As, in tensor notations, the scalar product of two vectors is $\mathbf{a}\cdot\mathbf{b} = a_i\,b^i$, and the vector product has components $(\mathbf{a}\times\mathbf{b})_i = \varepsilon_{ijk}\ a^j\ b^k$ (see section XXX), the identities 2.196–2.198 correspond respectively to:

$$\nabla_i\varepsilon^{ijk}\nabla_j\,a_k = 0 \tag{2.199}$$

$$\varepsilon^{ijk}\nabla_j\nabla_k\,\varphi = 0 \tag{2.200}$$

$$a_i\ \varepsilon^{ijk}\ b_j\ c_k = b_i\ \varepsilon^{ijk}\ c_j\ a_k = c_i\ \varepsilon^{ijk}\ a_j\ b_k \tag{2.201}$$

$$\varepsilon^{ijk}\ a_j\ (\varepsilon_{k\ell m}\ b^\ell\ c^m) = (a_j\ c^j)\ b^i - (a_j\ b^j)\ c^i \tag{2.202}$$

$$(\varepsilon^{ijk}\ a_j\ b_k)(\varepsilon_{i\ell m}\ c^\ell\ d^m) = a_i\ (\varepsilon^{ijk}\ b_j(\varepsilon_{k\ell m}\ c^\ell\ d^m))\ , \tag{2.203}$$

while the identities 2.190–2.195 correspond respectively to

$$\nabla_i(\varphi\, a^i) = (\nabla_i\, \varphi) a^i + \varphi(\nabla_i\, a^i) \tag{2.204}$$

$$\varepsilon^{ijk}\nabla_j(\varphi\, a_k) = \varepsilon^{ijk}(\nabla_j\, \varphi) a_k + \varphi\varepsilon^{ijk}\nabla_j\, a_k \tag{2.205}$$

$$\nabla_i(a_j\, b^j) = (a^j\nabla_j) b_i + (b^j\nabla_j) a_i + \varepsilon_{ijk} a^j(\varepsilon^{k\ell m}\nabla_\ell\, b_m) + \varepsilon_{ijk} b^j(\varepsilon^{k\ell m}\nabla_\ell\, a_m) \tag{2.206}$$

$$\nabla_i(\varepsilon^{ijk} a_j b_k) = b_k\varepsilon^{kij}\nabla_i a_j - a_j\varepsilon^{jik}\nabla_i b_k \tag{2.207}$$

$$\varepsilon^{ijk}\nabla_j(\varepsilon_{k\ell m} a^\ell b^m) a^i\nabla_j b^j - b^i\nabla_j a^j + b^j\nabla_j a^i - a^j\nabla_j b^i \tag{2.208}$$

$$\varepsilon^{ijk}\nabla_j(\varepsilon_{k\ell m}\nabla^\ell a^m) = \nabla^i(\nabla_j a^j) - \nabla^j\nabla_j a^i\,, \tag{2.209}$$

where the (inelegant) notation ∇^i represents $g^{ij}\nabla_j$.

The truth of the set of equations 2.199–2.209, when not obvious, is easily demonstrated by the simple use of the property (see section XXX)

$$\varepsilon_{ijk}\, \varepsilon^{k\ell m} = \delta_i{}^\ell\ \delta_j{}^m - \delta_i{}^m\ \delta_j{}^\ell \tag{2.210}$$

Comment: I am assuming that the nabla symbol and the ε_{ijk} symbol conmute. Is that true?

2.4.5 Appendix: Transporting the basis along Meridians and Parallels

Let us find here the formulas for the parallel transportation of the basis vectors at the surface of the sphere.

The metric:

$$\begin{pmatrix} g_{\theta\theta}(\theta,\varphi) & g_{\theta\varphi}(\theta,\varphi) \\ g_{\varphi\theta}(\theta,\varphi) & g_{\varphi\varphi}(\theta,\varphi) \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \tag{2.211}$$

Notice thet the length of the basis vectors is, at point (θ,φ) , $\|\mathbf{e}_\theta\| = R$ and $\|\mathbf{e}_\varphi\| = R\sin\theta$.

The connection:

$$\begin{pmatrix} \Gamma_{\theta\theta}{}^\theta(\theta,\varphi) & \Gamma_{\theta\varphi}{}^\theta(\theta,\varphi) \\ \Gamma_{\varphi\theta}{}^\theta(\theta,\varphi) & \Gamma_{\varphi\varphi}{}^\theta(\theta,\varphi) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\sin\theta\cos\theta \end{pmatrix} \tag{2.212}$$

$$\begin{pmatrix} \Gamma_{\theta\theta}{}^\varphi(\theta,\varphi) & \Gamma_{\theta\varphi}{}^\varphi(\theta,\varphi) \\ \Gamma_{\varphi\theta}{}^\varphi(\theta,\varphi) & \Gamma_{\varphi\varphi}{}^\varphi(\theta,\varphi) \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{cotg}\theta \\ \operatorname{cotg}\theta & 0 \end{pmatrix} \tag{2.213}$$

General first order transportation formula:

$$\mathbf{e}_i(\mathbf{x} \parallel \mathbf{x}+\delta\mathbf{x}) = \mathbf{e}_i(\mathbf{x}) + \Gamma_{ki}{}^j(\mathbf{x})\,\delta x^k\,\mathbf{e}_j(\mathbf{x}) + \dots \tag{2.214}$$

Special case for the sphere:

$$\mathbf{e}_\theta(\theta,\varphi \parallel \theta+\delta\theta,\varphi+\delta\varphi) = \mathbf{e}_\theta(\theta,\varphi) + \operatorname{cotg}\theta\,\delta\varphi\,\mathbf{e}_\varphi(\theta,\varphi) \tag{2.215}$$

$$\mathbf{e}_\varphi(\theta,\varphi \parallel \theta+\delta\theta,\varphi+\delta\varphi) = \mathbf{e}_\varphi(\theta,\varphi) + \operatorname{cotg}\theta\,\delta\theta\,\mathbf{e}_\varphi(\theta,\varphi) - \sin\theta\cos\theta\,\delta\varphi\,\mathbf{e}_\theta(\theta,\varphi) \tag{2.216}$$

Along a Meridian:

$$\mathbf{e}_\theta(\theta,\varphi \parallel \theta+\delta\theta,\varphi) = \mathbf{e}_\theta(\theta,\varphi) \tag{2.217}$$

$$\mathbf{e}_\varphi(\theta,\varphi \parallel \theta+\delta\theta,\varphi) = \mathbf{e}_\varphi(\theta,\varphi) + \operatorname{cotg}\theta\,\delta\theta\,\mathbf{e}_\varphi(\theta,\varphi) \tag{2.218}$$

This integrates to

$$\mathbf{e}_\theta(\theta,\varphi \parallel \theta',\varphi) = \mathbf{e}_\theta(\theta,\varphi) \tag{2.219}$$

$$\mathbf{e}_\varphi(\theta,\varphi \parallel \theta',\varphi) = \frac{\sin\theta'}{\sin\theta}\,\mathbf{e}_\varphi(\theta,\varphi) \tag{2.220}$$

An easy computation shows that the length of the transported vectors is

$$\begin{aligned} \|\mathbf{e}_\theta(\theta,\varphi \parallel \theta',\varphi)\| &= R \\ \|\mathbf{e}_\varphi(\theta,\varphi \parallel \theta',\varphi)\| &= R\sin\theta' \,, \end{aligned}$$

i.e., the length of the basis vectors at the original point.

Along a Parallel:

$$\mathbf{e}_\theta(\theta,\varphi \parallel \theta,\varphi+\delta\varphi) = \mathbf{e}_\theta(\theta,\varphi) + \operatorname{cotg}\theta\,\delta\varphi\,\mathbf{e}_\varphi(\theta,\varphi) \tag{2.221}$$

$$\mathbf{e}_\varphi(\theta,\varphi \parallel \theta,\varphi+\delta\varphi) = \mathbf{e}_\varphi(\theta,\varphi) - \sin\theta\cos\theta\,\delta\varphi\,\mathbf{e}_\theta(\theta,\varphi) \tag{2.222}$$

This integrates to

$$\begin{aligned} \mathbf{e}_\theta(\theta,\varphi \parallel \theta,\varphi') &= \cos\left[(\varphi-\varphi')\cos\theta\right]\mathbf{e}_\theta(\theta,\varphi) - \frac{1}{\sin\theta}\,\sin\left[(\varphi-\varphi')\cos\theta\right]\mathbf{e}_\varphi(\theta,\varphi) \\ \mathbf{e}_\varphi(\theta,\varphi \parallel \theta,\varphi') &= \sin\theta\sin\left[(\varphi-\varphi')\cos\theta\right]\mathbf{e}_\theta(\theta,\varphi) + \cos\left[(\varphi-\varphi')\cos\theta\right]\mathbf{e}_\varphi(\theta,\varphi)\,. \end{aligned} \tag{2.223}$$

An easy computation shows that the length of the transported vectors is

$$\begin{aligned} \|\mathbf{e}_\theta(\theta,\varphi \parallel \theta,\varphi')\| &= R \\ \|\mathbf{e}_\varphi(\theta,\varphi \parallel \theta,\varphi')\| &= R\sin\theta \,, \end{aligned}$$

i.e., the length of the basis vectors at the original point.

2.4.6 Appendix: Turning around the Pole

Let us use the results of the previous appendix to transport a basis a whole turn around the pole, following a parallel (figure 2.25).

Using equation 2.223 we obtain

$$\begin{aligned} \mathbf{e}_\theta(\theta,\varphi \parallel \theta,\varphi-2\pi) &= \cos(2\pi\cos\theta)\,\mathbf{e}_\theta(\theta,\varphi) - \frac{1}{\sin\theta}\,\sin(2\pi\cos\theta)\,\mathbf{e}_\varphi(\theta,\varphi) \\ \mathbf{e}_\varphi(\theta,\varphi \parallel \theta,\varphi-2\pi) &= \sin\theta\sin(2\pi\cos\theta)\,\mathbf{e}_\theta(\theta,\varphi) + \cos(2\pi\cos\theta)\,\mathbf{e}_\varphi(\theta,\varphi)\,. \end{aligned} \tag{2.224}$$

This shows that the vectors of the transported basis turn, with respect to the local natural basis, with period $2\pi\cos\theta$. Near the Pole, $\cos\theta \approx 1$, and the period is close to 2π . But if the transported natural basis turns with respect to the local basis by approximately one turn per turn, this is mainly due to the fact that the natural basis itself turns, not so the transported basis (see figure 2.26. That the parallel transportation of the basis vectors around the Pole will only produce, at most, a small turn, is understandable, as near the Pole, we can assimilate the surface of the sphere to its tangent

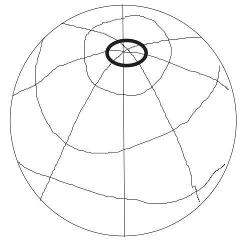

Figure 2.25: To be written.

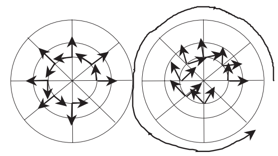

Figure 2.26: Left: The natural basis at some points near the Pole. Right: The parallel transport of a basis around the Pole, along a Parallel. Near the Pole, the basis only performs a slow turn (in the limit, for a Parallel tending to the Pole, no turn at all). See text.

(Euclidean) plane, and that the parallel transport of vectors in an Euclidean space does not provoke any turn.

Comment: too much "turn", probably not well used.

If the local basis turns the angle 2π , and the transported basis turns the angle $2\pi\cos\theta$ with respect to the local basis, the absolute turn of the transported basis is

$$\Delta\alpha = 2\pi(1-\cos\theta) \tag{2.225}$$

(θ is the colatitude of the Parallel). A Taylor's development gives

$$\Delta\alpha = \pi\left(\theta^2 - \frac{\theta^4}{12} + \dots\right) . \tag{2.226}$$

Near the Pole, $\theta \to 0$, and $\Delta\alpha \to 0$. Let $\delta\alpha$ be the second order approximation to $\Delta\alpha$. We have

$$\delta\alpha = \pi\theta^2 . \tag{2.227}$$

As the "radius" of the small circle around the Pole is $\varepsilon = R\theta$,

$$\delta\alpha = \frac{\pi\varepsilon^2}{R^2} . \tag{2.228}$$

This rotation undergone (?) by the basis vectors (or by any vector) in a trip around the Pole (or around any point at the surface of a sphere) attests the curvature of the space. It is a second order effect.

2.4.7 Appendix: Derivatives of the basis vectors

By definition of dual basis,

$$\mathbf{e}_i \mathbf{e}^j = \delta_i{}^j . \tag{2.229}$$

Therefore, the components of the basis vectors are

$$(\mathbf{e}_i)^j = \mathbf{e}_i \mathbf{e}^j = \delta_i{}^j . \tag{2.230}$$

The parallel transport of the Kronecker's tensor gives

$$\begin{aligned} \delta_i{}^k(\mathbf{x} \parallel \mathbf{y}) &= \ell_i{}^p(\mathbf{x}|\mathbf{y})\, \delta_p{}^m(\mathbf{y})\, \gamma_m{}^k(\mathbf{x}|\mathbf{y}) \\ &= \ell_i{}^p(\mathbf{x}|\mathbf{y})\, \delta_p{}^m\, \gamma_m{}^k(\mathbf{x}|\mathbf{y}) \\ &= \ell_i{}^j(\mathbf{x}|\mathbf{y})\, \gamma_j{}^k(\mathbf{x}|\mathbf{y}) \\ &= \delta_i{}^k\,. \end{aligned} \tag{2.231}$$

The definition of derivative then leads to

$$\nabla_i \delta_j{}^k = 0\,. \tag{2.232}$$

As $\delta_j{}^k$ is the $k-$th component of $\mathbf{e}_j$, we can write

$$\nabla_i\, (\mathbf{e}_j)^k = 0 \tag{2.233}$$

or, for short,

$$\nabla_i \mathbf{e}_j = 0\,. \tag{2.234}$$

This equation is consistent with the introduction of a metric, as, then

$$g_{ij} = \mathbf{e}_i\, \mathbf{e}_j\,, \tag{2.235}$$

and the standard derivation rules lead to

$$\nabla_i g_{jk} = \nabla_i(\mathbf{e}_j \mathbf{e}_k) = \mathbf{e}_j(\nabla_i \mathbf{e}_k) + (\nabla_i \mathbf{e}_j)\mathbf{e}_k = 0\,. \tag{2.236}$$

It is also consistent with the blind application of the derivation formula (demonstrated for components of tensors only) to a vector:

$$\nabla_i \mathbf{e}_j = \partial_i \mathbf{e}_j - \Gamma_{ij}{}^k \mathbf{e}_k\,, \tag{2.237}$$

as

$$\partial_i \mathbf{e}_j = \Gamma_{ij}{}^k \mathbf{e}_k\,. \tag{2.238}$$

The danger comes from the following fallacious argument. Equation

$$(\mathbf{e}_i)^j(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = (\mathbf{e}_i)^j(\mathbf{x}) + \left[\nabla_m (\mathbf{e}_i)^j\right](\mathbf{x})\, \delta x^m + \dots \tag{2.239}$$

if interpreted as defining the derivative of the Kronecker's tensor,

$$\delta_i{}^j(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = \delta_i{}^j(\mathbf{x}) + \left(\nabla_m \delta_i{}^j\right)(\mathbf{x})\, \delta x^m + \dots \tag{2.240}$$

(which is zero) is perfectly valid, as the parallel transport of the Kronecker's tensor gives the Kronecker's tensor (see just above). If interpreted as if we could "drop the component j ":

$$\mathbf{e}_i(\mathbf{x} \parallel \mathbf{x} + \delta\mathbf{x}) = \mathbf{e}_i(\mathbf{x}) + (\nabla_m \mathbf{e}_i)\,(\mathbf{x})\, \delta x^m + \dots \tag{2.241}$$

makes no sense. For, the parallel transport of a basis vector does not give the basis vector.

A small demonstration

Comment: I have not found a direct demonstration of $\partial_i \mathbf{e}_j = \Gamma_{ij}{}^k \mathbf{e}_k$. Rather, I can demonstrate that if an expression of the type $\partial_i \mathbf{e}_j = A_{ij}{}^k \mathbf{e}_k$ exists, then $A_{ij}{}^k = \Gamma_{ij}{}^k$.

For from

$$\partial_m g_{ij} = \Gamma_{mij} + \Gamma_{mji} \tag{2.242}$$

and

$$g_{ij} = \mathbf{e}_i \, \mathbf{e}_j \tag{2.243}$$

it follows

$$\partial_m (\mathbf{e}_i \mathbf{e}_j) = \Gamma_{mij} + \Gamma_{mji} \, , \tag{2.244}$$

$$\mathbf{e}_j \partial_m \mathbf{e}_i + \mathbf{e}_i \partial_m \mathbf{e}_j = \Gamma_{mij} + \Gamma_{mji} \, , \tag{2.245}$$

$$\mathbf{e}_j A_{mi}{}^k \mathbf{e}_k + \mathbf{e}_i A_{mj}{}^k \mathbf{e}_k = \Gamma_{mij} + \Gamma_{mji} \, , \tag{2.246}$$

$$g_{jk} A_{mi}{}^k + g_{ik} A_{mj}{}^k = \Gamma_{mij} + \Gamma_{mji} \, , \tag{2.247}$$

$$A_{mij} + A_{mji} = \Gamma_{mij} + \Gamma_{mji} \, . \tag{2.248}$$

Warning: this only demonstrates the equality between the symmetric part of $A_{ij}{}^k$ and that of $\Gamma_{ij}{}^k$. We still need to demonstrate the equality of the antisymmetric parts. Comment: The antisymetric part of $\Gamma_{ij}{}^k$ is a tensor: the torsion tensor.

2.4.8 Appendix: Parallel transport at the surface of a sphere

Consider the surface of a sphere, with the spherical coordinates $\{\theta, \varphi\}$, and let $\mathbf{V}(\theta, \varphi)$ be a vector field defined on it.

Demonstrate that the vector $\mathbf{V}(\theta + \delta\theta, \varphi + \delta\varphi \parallel \theta, \varphi)$, i.e., the vector obtained at point $(\theta + \delta\theta, \varphi + \delta\varphi)$ by parallel transport of the vector $\mathbf{V}(\theta, \varphi)$, has components

$$V^\theta(\theta + \delta\theta, \varphi + \delta\varphi \parallel \theta, \varphi) = V^\theta(\theta, \varphi) + \sin\theta \;\cos\theta \; V^\varphi(\theta, \varphi) \; \delta\varphi \tag{2.249}$$

and

$$V^\varphi(\theta + \delta\theta, \varphi + \delta\varphi \parallel \theta, \varphi) = V^\varphi(\theta, \varphi) - \operatorname{cotg}\theta \left(V^\varphi(\theta, \varphi) \; \delta\theta + V^\theta(\theta, \varphi) \; \delta\varphi \right) \, . \tag{2.250}$$

Demonstrate also that, along a meridian (θ, φ_0) , these finite-difference expressions give, for a finite displacement from point (θ_0, φ_0) ,

$$V^\theta(\theta, \varphi_0 \parallel \theta_0, \varphi_0) = V^\theta(\theta_0, \varphi_0) \tag{2.251}$$

and

$$V^\varphi(\theta, \varphi_0 \parallel \theta_0, \varphi_0) = \frac{\sin\theta_0}{\sin\theta} \; V^\varphi(\theta_0, \varphi_0) \, . \tag{2.252}$$

Demonstrate finally that, along a parallel (θ_0, φ) , this gives

$$V^\theta(\theta_0, \varphi \parallel \theta_0, \varphi_0) = A_0 \;\sin\left(\cos\theta_0 \; (\varphi - \Phi_0)\right) \tag{2.253}$$

$$V^\varphi(\theta_0, \varphi \parallel \theta_0, \varphi_0) = \frac{A_0}{\sin\theta_0} \;\cos\left(\cos\theta_0 \; (\varphi - \Phi_0)\right) \, , \tag{2.254}$$

where

$$A_0 = \sin \frac{V^\theta(\theta_0, \varphi_0)}{\sin\left(\operatorname{tg}^{-1}\left(\frac{1}{\sin\theta_0} \, \frac{V^\theta(\theta_0,\varphi_0)}{V^\varphi(\theta_0,\varphi_0)} \right) \right)} \tag{2.255}$$

and

$$\Phi_0 = \varphi_0 - \frac{1}{\cos\theta_0} \,\mathrm{tg}^{-1}\left(\frac{1}{\sin\theta_0} \, \frac{V^\theta(\theta_0,\varphi_0)}{V^\varphi(\theta_0,\varphi_0)} \right) . \tag{2.256}$$

(Comment: Say somewhere that this means that for a parallel transport along a parallel of the sphere, a vector transported along the equator “does not turn”, while a vector transported along a parallel of the sphere near the poles “makes one turn per turn”. For any other parallel of the sphere, the vector “turns with a velocity between 0 and one turn per turn”.

Solution. (Comment: this demonstration uses the connection coefficients. I should make one without them).

As we have seen in xxx, the parallel transport of a vector is defined by

$$V^i(\mathbf{x} + \delta\mathbf{x} \parallel \mathbf{x}) = V^i(\mathbf{x}) - \Gamma_{kj}{}^i(\mathbf{x}) \; V^j(\mathbf{x}) \; \delta\mathbf{x} \, . \tag{2.257}$$

We have also seen in equations 2.329 that the only non-vanishing connection coefficients are

$$\Gamma_{\theta\varphi}{}^\varphi = \mathrm{cotg}\,\theta \; ; \quad \Gamma_{\varphi\theta}{}^\varphi = \mathrm{cotg}\,\theta \; ; \quad \Gamma_{\varphi\varphi}{}^\theta = -\sin\theta \;\cos\theta \; . \tag{2.258}$$

Equations 2.249–2.250 follow directly from this.

Obtaining equations 2.251–2.256 is just a matter of very classical integration of differential equations. Its truth can simply be demonstrated by direct differentiation and verification that, at point (θ_0, φ_0) , the expressions degenerate into $V^\theta(\theta_0,\varphi_0)$ and $V^\varphi(\theta_0,\varphi_0)$.

(Comment: formulas checked February 19, 1993).

2.4.9 Appendix: Cartesian coordinates: Metric, Connection...

Line element

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{2.259}$$

Metric

$$\begin{pmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{yx} & g_{yy} & g_{yz} \\ g_{zx} & g_{zy} & g_{zz} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.260}$$

Fundamental density

$$\overline{g} = 1 \tag{2.261}$$

Connection

$$\begin{aligned}
\begin{pmatrix} \Gamma_{xx}{}^x & \Gamma_{xy}{}^x & \Gamma_{xz}{}^x \\ \Gamma_{yx}{}^x & \Gamma_{yy}{}^x & \Gamma_{yz}{}^x \\ \Gamma_{zx}{}^x & \Gamma_{zy}{}^x & \Gamma_{zz}{}^x \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} \Gamma_{xx}{}^y & \Gamma_{xy}{}^y & \Gamma_{xz}{}^y \\ \Gamma_{yx}{}^y & \Gamma_{yy}{}^y & \Gamma_{yz}{}^y \\ \Gamma_{zx}{}^y & \Gamma_{zy}{}^y & \Gamma_{zz}{}^y \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} \Gamma_{xx}{}^z & \Gamma_{xy}{}^z & \Gamma_{xz}{}^z \\ \Gamma_{yx}{}^z & \Gamma_{yy}{}^z & \Gamma_{yz}{}^z \\ \Gamma_{zx}{}^z & \Gamma_{zy}{}^z & \Gamma_{zz}{}^z \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{2.262}$$

Contracted connection

$$\begin{pmatrix} \Gamma_x \\ \Gamma_y \\ \Gamma_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.263}$$

Relationship between covariant and contravariant components for first order tensors

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V^x \\ V^y \\ V^z \end{pmatrix} \tag{2.264}$$

Relationship between covariant and contravariant components for second order tensors

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} = \begin{pmatrix} T_x{}^x & T_x{}^y & T_x{}^z \\ T_y{}^x & T_y{}^y & T_y{}^z \\ T_z{}^x & T_z{}^y & T_z{}^z \end{pmatrix} = \begin{pmatrix} T^{xx} & T^{xy} & T^{xz} \\ T^{yx} & T^{yy} & T^{yz} \\ T^{zx} & T^{zy} & T^{zz} \end{pmatrix} \tag{2.265}$$

Norm of the vectors of the natural basis

$$\|\mathbf{e}_x\| = \|\mathbf{e}_y\| = \|\mathbf{e}_z\| = 1 \tag{2.266}$$

Norm of the vectors of the normed basis

$$\|\widehat{\mathbf{e}}_x\| = \|\widehat{\mathbf{e}}_y\| = \|\widehat{\mathbf{e}}_z\| = 1 \tag{2.267}$$

Missing Comment: give also the norms of the vectors of the dual basis.

Relations between components on the natural and the normed basis for first order tensors

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \widehat{V}_x \\ \widehat{V}_y \\ \widehat{V}_z \end{pmatrix} \quad ; \quad \begin{pmatrix} V^x \\ V^y \\ V^z \end{pmatrix} = \begin{pmatrix} \widehat{V}^x \\ \widehat{V}^y \\ \widehat{V}^z \end{pmatrix} \tag{2.268}$$

Relations between components on the natural and the normed basis for second order tensors

$$\begin{aligned}
\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} &= \begin{pmatrix} \widehat{T}_{xx} & \widehat{T}_{xy} & \widehat{T}_{xz} \\ \widehat{T}_{yx} & \widehat{T}_{yy} & \widehat{T}_{yz} \\ \widehat{T}_{zx} & \widehat{T}_{zy} & \widehat{T}_{zz} \end{pmatrix} \\
\begin{pmatrix} T_x{}^x & T_x{}^y & T_x{}^z \\ T_y{}^x & T_y{}^y & T_y{}^z \\ T_z{}^x & T_z{}^y & T_z{}^z \end{pmatrix} &= \begin{pmatrix} \widehat{T}_x{}^x & \widehat{T}_x{}^y & \widehat{T}_x{}^z \\ \widehat{T}_y{}^x & \widehat{T}_y{}^y & \widehat{T}_y{}^z \\ \widehat{T}_z{}^x & \widehat{T}_z{}^y & \widehat{T}_z{}^z \end{pmatrix} \\
\begin{pmatrix} T^{xx} & T^{xy} & T^{xz} \\ T^{yx} & T^{yy} & T^{yz} \\ T^{zx} & T^{zy} & T^{zz} \end{pmatrix} &= \begin{pmatrix} \widehat{T}^{xx} & \widehat{T}^{xy} & \widehat{T}^{xz} \\ \widehat{T}^{yx} & \widehat{T}^{yy} & \widehat{T}^{yz} \\ \widehat{T}^{zx} & \widehat{T}^{zy} & \widehat{T}^{zz} \end{pmatrix}
\end{aligned} \tag{2.269}$$

2.4.10 Appendix: Spherical coordinates: Metric, Connection…

Line element

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \tag{2.270}$$

Metric

$$\begin{pmatrix} g_{rr} & g_{r\theta} & g_{r\varphi} \\ g_{\theta r} & g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi r} & g_{\varphi\theta} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2.271}$$

Fundamental density

$$\overline{g} = r^2 \sin\theta \tag{2.272}$$

Connection

$$\begin{aligned}
\begin{pmatrix} \Gamma_{rr}{}^{r} & \Gamma_{r\theta}{}^{r} & \Gamma_{r\varphi}{}^{r} \\ \Gamma_{\theta r}{}^{r} & \Gamma_{\theta\theta}{}^{r} & \Gamma_{\theta\varphi}{}^{r} \\ \Gamma_{\varphi r}{}^{r} & \Gamma_{\varphi\theta}{}^{r} & \Gamma_{\varphi\varphi}{}^{r} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{pmatrix} \\
\begin{pmatrix} \Gamma_{rr}{}^{\theta} & \Gamma_{r\theta}{}^{\theta} & \Gamma_{r\varphi}{}^{\theta} \\ \Gamma_{\theta r}{}^{\theta} & \Gamma_{\theta\theta}{}^{\theta} & \Gamma_{\theta\varphi}{}^{\theta} \\ \Gamma_{\varphi r}{}^{\theta} & \Gamma_{\varphi\theta}{}^{\theta} & \Gamma_{\varphi\varphi}{}^{\theta} \end{pmatrix} &= \begin{pmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} \\
\begin{pmatrix} \Gamma_{rr}{}^{\varphi} & \Gamma_{r\theta}{}^{\varphi} & \Gamma_{r\varphi}{}^{\varphi} \\ \Gamma_{\theta r}{}^{\varphi} & \Gamma_{\theta\theta}{}^{\varphi} & \Gamma_{\theta\varphi}{}^{\varphi} \\ \Gamma_{\varphi r}{}^{\varphi} & \Gamma_{\varphi\theta}{}^{\varphi} & \Gamma_{\varphi\varphi}{}^{\varphi} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1/r \\ 0 & 0 & \operatorname{cotg}\theta \\ 1/r & \operatorname{cotg}\theta & 0 \end{pmatrix}
\end{aligned} \tag{2.273}$$

Contracted connection

$$\begin{pmatrix} \Gamma_r \\ \Gamma_\theta \\ \Gamma_\varphi \end{pmatrix} = \begin{pmatrix} 2/r \\ \operatorname{cotg}\theta \\ 0 \end{pmatrix} \tag{2.274}$$

Relationship between covariant and contravariant components for first order tensors

$$\begin{pmatrix} V_r \\ V_\theta \\ V_\varphi \end{pmatrix} = \begin{pmatrix} V^r \\ r^2 V^\theta \\ r^2\sin^2\theta\, V^\varphi \end{pmatrix} \tag{2.275}$$

Relationship between covariant and contravariant components for second order tensors

$$\begin{pmatrix} T_{rr} & \frac{1}{r^2}T_{r\theta} & \frac{1}{r^2\sin^2\theta}T_{r\varphi} \\ T_{\theta r} & \frac{1}{r^2}T_{\theta\theta} & \frac{1}{r^2\sin^2\theta}T_{\theta\varphi} \\ T_{\varphi r} & \frac{1}{r^2}T_{\varphi\theta} & \frac{1}{r^2\sin^2\theta}T_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} T_r{}^r & T_r{}^\theta & T_r{}^\varphi \\ T_\theta{}^r & T_\theta{}^\theta & T_\theta{}^\varphi \\ T_\varphi{}^r & T_\varphi{}^\theta & T_\varphi{}^\varphi \end{pmatrix} = \begin{pmatrix} T^{rr} & T^{r\theta} & T^{r\varphi} \\ r^2 T^{\theta r} & r^2 T^{\theta\theta} & r^2 T^{\theta\varphi} \\ r^2\sin^2\theta\, T^{\varphi r} & r^2\sin^2\theta\, T^{\varphi\theta} & r^2\sin^2\theta\, T^{\varphi\varphi} \end{pmatrix} \tag{2.276}$$

Norm of the vectors of the natural basis

$$\|\mathbf{e}_r\| = 1 \qquad ; \qquad \|\mathbf{e}_\theta\| = r \qquad ; \qquad \|\mathbf{e}_\varphi\| = r\sin\theta \tag{2.277}$$

Norm of the vectors of the normed basis

$$\|\widehat{\mathbf{e}}_r\| = \|\widehat{\mathbf{e}}_\theta\| = \|\widehat{\mathbf{e}}_\varphi\| = 1 \tag{2.278}$$

Missing Comment: give also the norms of the vectors of the dual basis.

Relations between components on the natural and the normed basis for first order tensors

$$\begin{pmatrix} V_r \\ V_\theta \\ V_\varphi \end{pmatrix} = \begin{pmatrix} \widehat{V}_r \\ r\,\widehat{V}_\theta \\ r\sin\theta\,\widehat{V}_\varphi \end{pmatrix} \qquad ; \qquad \begin{pmatrix} V^r \\ V^\theta \\ V^\varphi \end{pmatrix} = \begin{pmatrix} \widehat{V}^r \\ \frac{1}{r}\widehat{V}^\theta \\ \frac{1}{r\sin\theta}\widehat{V}^\varphi \end{pmatrix} \tag{2.279}$$

Relations between components on the natural and the normed basis for second order tensors

$$\begin{aligned}
\begin{pmatrix} T_{rr} & T_{r\theta} & T_{r\varphi} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta\varphi} \\ T_{\varphi r} & T_{\varphi\theta} & T_{\varphi\varphi} \end{pmatrix} &= \begin{pmatrix} \widehat{T}_{rr} & r\widehat{T}_{r\theta} & r\sin\theta\,\widehat{T}_{r\varphi} \\ r\widehat{T}_{\theta r} & r^2\widehat{T}_{\theta\theta} & r^2\sin\theta\,\widehat{T}_{\theta\varphi} \\ r\sin\theta\,\widehat{T}_{\varphi r} & r^2\sin\theta\,\widehat{T}_{\varphi\theta} & r^2\sin^2\theta\,\widehat{T}_{\varphi\varphi} \end{pmatrix} \\
\begin{pmatrix} {T_r}^r & {T_r}^\theta & {T_r}^\varphi \\ {T_\theta}^r & {T_\theta}^\theta & {T_\theta}^\varphi \\ {T_\varphi}^r & {T_\varphi}^\theta & {T_\varphi}^\varphi \end{pmatrix} &= \begin{pmatrix} {\widehat{T}_r}{}^r & \frac{1}{r}{\widehat{T}_r}{}^\theta & \frac{1}{r\sin\theta}{\widehat{T}_r}{}^\varphi \\ r{\widehat{T}_\theta}{}^r & {\widehat{T}_\theta}{}^\theta & \frac{1}{\sin\theta}{\widehat{T}_\theta}{}^\varphi \\ r\sin\theta\,{\widehat{T}_\varphi}{}^r & \sin\theta\,{\widehat{T}_\varphi}{}^\theta & {\widehat{T}_\varphi}{}^\varphi \end{pmatrix} \\
\begin{pmatrix} T^{rr} & T^{r\theta} & T^{r\varphi} \\ T^{\theta r} & T^{\theta\theta} & T^{\theta\varphi} \\ T^{\varphi r} & T^{\varphi\theta} & T^{\varphi\varphi} \end{pmatrix} &= \begin{pmatrix} \widehat{T}^{rr} & \frac{1}{r}\widehat{T}^{r\theta} & \frac{1}{r\sin\theta}\widehat{T}^{r\varphi} \\ \frac{1}{r}\widehat{T}^{\theta r} & \frac{1}{r^2}\widehat{T}^{\theta\theta} & \frac{1}{r^2\sin\theta}\widehat{T}^{\theta\varphi} \\ \frac{1}{r\sin\theta}\widehat{T}^{\varphi r} & \frac{1}{r^2\sin\theta}\widehat{T}^{\varphi\theta} & \frac{1}{r^2\sin^2\theta}\widehat{T}^{\varphi\varphi} \end{pmatrix}
\end{aligned} \tag{2.280}$$

Note: say somewhere in this appendix that the two following formulas are quite useful in deriving the formulas above.

$$\frac{1}{r^n}\frac{\partial}{\partial r}(r^n\psi) = \frac{\partial\psi}{\partial r} + \frac{n}{r}\psi \tag{2.281}$$

$$\frac{1}{\sin^n\vartheta}\frac{\partial}{\partial\vartheta}(\sin^n\vartheta\,\psi) = \frac{\partial\psi}{\partial\vartheta} + n\,\mathrm{cotg}\vartheta\,\psi\,. \tag{2.282}$$

2.4.11 Appendix: Cylindrical coordinates: Metric, Connection...

Line element

$$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2 \tag{2.283}$$

Metric

$$\begin{pmatrix} g_{rr} & g_{r\varphi} & g_{rz} \\ g_{\varphi r} & g_{\varphi\varphi} & g_{\varphi z} \\ g_{zr} & g_{z\varphi} & g_{zz} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.284}$$

Fundamental density

$$\overline{g} = r \tag{2.285}$$

Connection

$$\begin{aligned}
\begin{pmatrix} {\Gamma_{rr}}^r & {\Gamma_{r\varphi}}^r & {\Gamma_{rz}}^r \\ {\Gamma_{\varphi r}}^r & {\Gamma_{\varphi\varphi}}^r & {\Gamma_{\varphi z}}^r \\ {\Gamma_{zr}}^r & {\Gamma_{z\varphi}}^r & {\Gamma_{zz}}^r \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} {\Gamma_{rr}}^\varphi & {\Gamma_{r\varphi}}^\varphi & {\Gamma_{rz}}^\varphi \\ {\Gamma_{\varphi r}}^\varphi & {\Gamma_{\varphi\varphi}}^\varphi & {\Gamma_{\varphi z}}^\varphi \\ {\Gamma_{zr}}^\varphi & {\Gamma_{z\varphi}}^\varphi & {\Gamma_{zz}}^\varphi \end{pmatrix} &= \begin{pmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} {\Gamma_{rr}}^z & {\Gamma_{r\varphi}}^z & {\Gamma_{rz}}^z \\ {\Gamma_{\varphi r}}^z & {\Gamma_{\varphi\varphi}}^z & {\Gamma_{\varphi z}}^z \\ {\Gamma_{zr}}^z & {\Gamma_{z\varphi}}^z & {\Gamma_{zz}}^z \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{2.286}$$

Contracted connection

$$\begin{pmatrix} \Gamma_r \\ \Gamma_\varphi \\ \Gamma_z \end{pmatrix} = \begin{pmatrix} 1/r \\ 0 \\ 0 \end{pmatrix} \tag{2.287}$$

Relationship between covariant and contravariant components for first order tensors

$$\begin{pmatrix} V_r \\ V_\varphi \\ V_z \end{pmatrix} = \begin{pmatrix} V^r \\ r^2 V^\varphi \\ V^z \end{pmatrix} \tag{2.288}$$

Relationship between covariant and contravariant components for second order tensors

$$\begin{pmatrix} T_{rr} & \frac{1}{r^2} T_{r\varphi} & T_{rz} \\ T_{\varphi r} & \frac{1}{r^2} T_{\varphi\varphi} & T_{\varphi z} \\ T_{zr} & \frac{1}{r^2} T_{z\varphi} & T_{zz} \end{pmatrix} = \begin{pmatrix} T_r{}^r & T_r{}^\varphi & T_r{}^z \\ T_\varphi{}^r & T_\varphi{}^\varphi & T_\varphi{}^z \\ T_z{}^r & T_z{}^\varphi & T_z{}^z \end{pmatrix} = \begin{pmatrix} T^{rr} & T^{r\varphi} & T^{rz} \\ r^2 T^{\varphi r} & r^2 T^{\varphi\varphi} & r^2 T^{\varphi z} \\ T^{zr} & T^{z\theta} & T^{zz} \end{pmatrix} \tag{2.289}$$

Norm of the vectors of the natural basis

$$\|\mathbf{e}_r\| = 1 \qquad ; \qquad \|\mathbf{e}_\varphi\| = r \qquad ; \qquad \|\mathbf{e}_z\| = 1 \tag{2.290}$$

Norm of the vectors of the normed basis

$$\|\widehat{\mathbf{e}}_r\| = \|\widehat{\mathbf{e}}_\varphi\| = \|\widehat{\mathbf{e}}_z\| = 1 \tag{2.291}$$

Missing Comment: give also the norms of the vectors of the dual basis.

Relations between components on the natural and the normed basis for first order tensors

$$\begin{pmatrix} V_r \\ V_\varphi \\ V_z \end{pmatrix} = \begin{pmatrix} \widehat{V}_r \\ r\,\widehat{V}_\varphi \\ \widehat{V}_z \end{pmatrix} \qquad ; \qquad \begin{pmatrix} V^r \\ V^\varphi \\ V^z \end{pmatrix} = \begin{pmatrix} \widehat{V}^r \\ \frac{1}{r}\widehat{V}^\varphi \\ \widehat{V}^z \end{pmatrix} \tag{2.292}$$

Relations between components on the natural and the normed basis for second order tensors

$$\begin{aligned}
\begin{pmatrix} T_{rr} & T_{r\varphi} & T_{rz} \\ T_{\varphi r} & T_{\varphi\varphi} & T_{\varphi z} \\ T_{zr} & T_{z\varphi} & T_{zz} \end{pmatrix} &= \begin{pmatrix} \widehat{T}_{rr} & r\widehat{T}_{r\varphi} & \widehat{T}_{rz} \\ r\widehat{T}_{\varphi r} & r^2\widehat{T}_{\varphi\varphi} & r\widehat{T}_{\varphi z} \\ \widehat{T}_{zr} & r\widehat{T}_{z\varphi} & \widehat{T}_{zz} \end{pmatrix} \\
\begin{pmatrix} T_r{}^r & T_r{}^\varphi & T_r{}^z \\ T_\varphi{}^r & T_\varphi{}^\varphi & T_\varphi{}^z \\ T_z{}^r & T_z{}^\varphi & T_z{}^z \end{pmatrix} &= \begin{pmatrix} \widehat{T}_r{}^r & \frac{1}{r}\widehat{T}_r{}^\varphi & \widehat{T}_r{}^z \\ r\widehat{T}_\varphi{}^r & \widehat{T}_\varphi{}^\varphi & r\widehat{T}_\varphi{}^z \\ \widehat{T}_z{}^r & \frac{1}{r}\widehat{T}_z{}^\varphi & \widehat{T}_z{}^z \end{pmatrix} \\
\begin{pmatrix} T^{rr} & T^{r\varphi} & T^{rz} \\ T^{\varphi r} & T^{\varphi\varphi} & T^{\varphi z} \\ T^{zr} & T^{z\varphi} & T^{zz} \end{pmatrix} &= \begin{pmatrix} \widehat{T}^{rr} & \frac{1}{r}\widehat{T}^{r\varphi} & \widehat{T}^{rz} \\ \frac{1}{r}\widehat{T}^{\varphi r} & \frac{1}{r^2}\widehat{T}^{\varphi\varphi} & \frac{1}{r}\widehat{T}^{\varphi z} \\ \widehat{T}^{zr} & \frac{1}{r}\widehat{T}^{z\varphi} & \widehat{T}^{zz} \end{pmatrix}
\end{aligned} \tag{2.293}$$

2.4.12 Appendix: Gradient, Divergence and Curl in usual coordinate systems

Here we analyze the 3-D Euclidean space, using Cartesian, spherical or cylindrical coordinates. The words scalar, vector, and tensor mean “true” scalars, vectors and tensors, respectively. The scalar densities, vector densities and tensor densities (see section XXX) are named explicitly.

Definitions

If $\mathbf{x} \to \phi(\mathbf{x})$ is a scalar field, its *gradient* is the form defined by

$$G_i = \nabla_i \phi \,. \tag{2.294}$$

If $\mathbf{x} \to \overline{V}^i(\mathbf{x})$ is a vector density field, its *divergence* is the scalar density defined by

$$\overline{D} = \nabla_i \overline{V}^i \,. \tag{2.295}$$

If $\mathbf{x} \to F_i(\mathbf{x})$ is a form field, its *curl* (or *rotational*) is the vector density defined by

$$\overline{R}^i = \overline{\varepsilon}^{ijk} \nabla_j F_k \,. \tag{2.296}$$

Properties

These definitions are such that we can replace everywhere true ("covariant") derivatives by partial derivatives (see exercise XXX). This gives, for the gradient of a density,

$$G_i = \nabla_i \phi = \partial_i \phi \,, \tag{2.297}$$

for the divergence of a vector density,

$$\overline{D} = \nabla_i \overline{V}^i = \partial_i \overline{V}^i \,, \tag{2.298}$$

and for the curl of a form,

$$\overline{R}^i = \overline{\varepsilon}^{ijk} \nabla_j F_k = \overline{\varepsilon}^{ijk} \partial_j F_k \tag{2.299}$$

[this equation is only valid for spaces without torsion; the general formula is $\overline{R}^i = \overline{\varepsilon}^{ijk} \nabla_j F_k = \overline{\varepsilon}^{ijk} (\partial_j F_k - \frac{1}{2} S_{jk}{}^{\ell} V_\ell)$].

These equations lead to particularly simple expressions. For instance, the following table shows that the explicit expressions have the same form for Cartesian, spherical and cylindrical coordinates (or for whatever coordinate system).

	Cartesian	Spherical	Cylindrical
Gradient	$G_x = \partial_x \phi$ $G_y = \partial_y \phi$ $G_z = \partial_z \phi$	$G_r = \partial_r \phi$ $G_\theta = \partial_\theta \phi$ $G_\varphi = \partial_\varphi \phi$	$G_r = \partial_r \phi$ $G_\varphi = \partial_\varphi \phi$ $G_z = \partial_z \phi$
Divergence	$\overline{D} = \partial_x \overline{V}^x + \partial_y \overline{V}^y + \partial_z \overline{V}^z$	$\overline{D} = \partial_r \overline{V}^r + \partial_\theta \overline{V}^\theta + \partial_\varphi \overline{V}^\varphi$	$\overline{D} = \partial_r \overline{V}^r + \partial_\varphi \overline{V}^\varphi + \partial_z \overline{V}^z$
Curl	$\overline{R}^x = \partial_y F_z - \partial_z F_y$ $\overline{R}^y = \partial_z F_x - \partial_x F_z$ $\overline{R}^z = \partial_x F_y - \partial_y F_x$	$\overline{R}^r = \partial_\theta F_\varphi - \partial_\varphi F_\theta$ $\overline{R}^\theta = \partial_\varphi F_r - \partial_r F_\varphi$ $\overline{R}^\varphi = \partial_r F_\theta - \partial_\theta F_r$	$\overline{R}^r = \partial_\varphi F_z - \partial_z F_\varphi$ $\overline{R}^\varphi = \partial_z F_r - \partial_r F_z$ $\overline{R}^z = \partial_r F_\varphi - \partial_\varphi F_r$

Remarks

Although we have only defined the gradient of a true scalar, the divergence of a vector density, and the curl of a form, the definitions can be immediately be extended by "putting bars on" and "taking bars off" (see section XXX).

As an example, from equation 2.294, we can immediately write the definition of the gradient of a scalar density,

$$\overline{G}_i = \nabla_i \overline{\phi} \,, \tag{2.300}$$

from equation 2.295 we can write the definition of the divergence of a (true) vector field,

$$D = \nabla_i V^i \, , \tag{2.301}$$

and from equation 2.296 we can write the definition of the curl of a form as a true vector,

$$R^i = \varepsilon^{ijk} \nabla_j F_k \, , \tag{2.302}$$

or a true form,

$$R_\ell = g_{\ell i} \, \varepsilon^{ijk} \nabla_j F_k \, . \tag{2.303}$$

Although equation 2.301 seems well adapted to the practical computation of the divergence of a true vector, it is better to use 2.298 instead. For we have successively

$$\overline{D} = \partial_i \overline{V}^i \qquad \Longleftrightarrow \qquad \overline{g} \, D = \partial_i (\overline{g} \, V^i) \qquad \Longleftrightarrow \qquad D = \frac{1}{\overline{g}} \partial_i (\overline{g} \, V^i) \, . \tag{2.304}$$

This last expression provides directly compact expressions for the divergence of a vector. For instance, as the fundamental density $\overline{g}$ takes, in Cartesian, spherical and cylindrical coordinates, respectively the values 1 , $r^2 \sin\theta$ and r , this leads to the results of the following table.

$$\text{Divergence, Cartesian coordinates} \; : \; D = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} + \frac{\partial V^z}{\partial z} \tag{2.305}$$

$$\text{Divergence, Spherical coordinates} \; : \; D = \frac{1}{r^2} \frac{\partial (r^2 V^r)}{\partial r} + \frac{1}{\sin\theta} \frac{\partial (\sin\theta \, V^\theta)}{\partial \theta} + \frac{\partial V^\varphi}{\partial \varphi} \tag{2.306}$$

$$\text{Divergence, Cylindrical coordinates} \; : \; D = \frac{1}{r} \frac{\partial (r V^r)}{\partial r} + \frac{\partial V^\varphi}{\partial \varphi} + \frac{\partial V^z}{\partial z} \tag{2.307}$$

Replacing the components on the natural basis by the components on the normed basis (see section XXX) gives

$$\text{Divergence, Cartesian coordinates} \; : \; D = \frac{\partial \widehat{V}^x}{\partial x} + \frac{\partial \widehat{V}^y}{\partial y} + \frac{\partial \widehat{V}^z}{\partial z} \tag{2.308}$$

$$\text{Divergence, Spherical coordinates} \; : \; D = \frac{1}{r^2} \frac{\partial (r^2 \widehat{V}^r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\sin\theta \, \widehat{V}^\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \widehat{V}^\varphi}{\partial \varphi} \tag{2.309}$$

$$\text{Divergence, Cylindrical coordinates} \; : \; D = \frac{1}{r} \frac{\partial (r \widehat{V}^r)}{\partial r} + \frac{1}{r} \frac{\partial \widehat{V}^\varphi}{\partial \varphi} + \frac{\partial \widehat{V}^z}{\partial z} \tag{2.310}$$

These are the formulas given in elementary texts (not using tensor concepts).

Similarly, although 2.303 seems well adapted to a practical computation of the curl, it is better to go back to equation 2.299. We have, successively,

$$\overline{R}^i = \overline{\varepsilon}^{ijk} \partial_j F_k \quad \Longleftrightarrow \quad \overline{g} \, R^i = \overline{\varepsilon}^{ijk} \partial_j F_k \quad \Longleftrightarrow \quad R^i = \frac{1}{\overline{g}} \overline{\varepsilon}^{ijk} \partial_j F_k \quad \Longleftrightarrow \quad R_\ell = \frac{1}{\overline{g}} g_{\ell i} \, \overline{\varepsilon}^{ijk} \partial_j F_k \, . \tag{2.311}$$

This last expression provides directly compact expressions for the curl. For instance, as the fundamental density $\overline{g}$ takes, in Cartesian, spherical and cylindrical coordinates, respectively the values 1 , $r^2 \sin\theta$ and r , this leads to the results of the following table.

$$\text{Curl, Cartesian coordinates} \; : \; \begin{aligned} R_x &= \partial_y F_z - \partial_z F_y \\ R_y &= \partial_z F_x - \partial_x F_z \end{aligned} \tag{2.312}$$

$$R_z = \partial_x F_y - \partial_y F_x$$

$$\text{Curl, Spherical coordinates} \quad : \quad \begin{aligned} R_r &= \frac{1}{r^2 \sin\theta}(\partial_\theta F_\varphi - \partial_\varphi F_\theta) \\ R_\theta &= \frac{1}{\sin\theta}(\partial_\varphi F_r - \partial_r F_\varphi) \\ R_\varphi &= \sin\theta\,(\partial_r F_\theta - \partial_\theta F_r) \end{aligned} \tag{2.313}$$

$$\text{Curl, Cylindrical coordinates} \quad : \quad \begin{aligned} R_r &= \frac{1}{r}(\partial_\varphi F_z - \partial_z F_\varphi) \\ R_\varphi &= r(\partial_z F_r - \partial_r F_z) \\ R_z &= \frac{1}{r}(\partial_r F_\varphi - \partial_\varphi F_r) \end{aligned} \tag{2.314}$$

Replacing the components on the natural basis by the components on the normed basis (see section XXX) gives

$$\text{Curl, Cartesian coordinates} \quad : \quad \begin{aligned} \widehat{R}_x &= \partial_y \widehat{F}_z - \partial_z \widehat{F}_y \\ \widehat{R}_y &= \partial_z \widehat{F}_x - \partial_x \widehat{F}_z \\ \widehat{R}_z &= \partial_x \widehat{F}_y - \partial_y \widehat{F}_x \end{aligned} \tag{2.315}$$

$$\text{Curl, Spherical coordinates} \quad : \quad \begin{aligned} \widehat{R}_r &= \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,\widehat{F}_\varphi)}{\partial\theta} - \frac{\partial\widehat{F}_\theta}{\partial\varphi}\right) \\ \widehat{R}_\theta &= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial\widehat{F}_r}{\partial\varphi} - \frac{\partial(r\widehat{F}_\varphi)}{\partial r}\right) \\ \widehat{R}_\varphi &= \frac{1}{r}\left(\frac{\partial(r\widehat{F}_\theta)}{\partial r} - \frac{\partial\widehat{F}_r}{\partial\theta}\right) \end{aligned} \tag{2.316}$$

$$\text{Curl, Cylindrical coordinates} \quad : \quad \begin{aligned} \widehat{R}_r &= \frac{1}{r}\left(\frac{\partial\widehat{F}_z}{\partial\varphi} - \frac{\partial(r\widehat{F}_\varphi)}{\partial z}\right) \\ \widehat{R}_\varphi &= \frac{\partial\widehat{F}_r}{\partial z} - \frac{\partial\widehat{F}_z}{\partial r} \\ \widehat{R}_z &= \frac{1}{r}\left(\frac{\partial(r\widehat{F}_\varphi)}{\partial r} - \frac{\partial\widehat{F}_r}{\partial\varphi}\right) \end{aligned} \tag{2.317}$$

These are the formulas given in elementary texts (not using tensor concepts).

Comment: I should remember not to put this back in a table, as it is not very readable:

Box 2.1 Polar coordinates

(Two-dimensional Euclidean space with non-Cartesian coordinates).

$$ds^2 = r^2 + r^2 d\varphi^2 \tag{2.320}$$

$$\Gamma_{r\varphi}{}^{\varphi} = 1/r \ ; \quad \Gamma_{\varphi r}{}^{\varphi} = 1/r \ ; \quad \Gamma_{\varphi\varphi}{}^{r} = -r \ ; \quad \text{(the others vanish)} \tag{2.321}$$

$$R_{ij} = 0 \tag{2.322}$$

$$\nabla_i V^i = \frac{1}{r}\frac{\partial}{\partial r}(r V^r) + \frac{\partial V^\varphi}{\partial \varphi} \tag{2.323}$$

	Curl
Cartesian	$\widehat{R}_x = \partial_y \widehat{F}_z - \partial_z \widehat{F}_y$ $\widehat{R}_y = \partial_z \widehat{F}_x - \partial_x \widehat{F}_z$ $\widehat{R}_z = \partial_x \widehat{F}_y - \partial_y \widehat{F}_x$
Spherical	$\widehat{R}_r = \frac{1}{r\sin\theta}\left(\frac{\partial \sin\theta \widehat{F}_\varphi}{\partial\theta} - \frac{\partial \widehat{F}_\theta}{\partial\varphi}\right)$ $\widehat{R}_\theta = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \widehat{F}_r}{\partial\varphi} - \frac{\partial r \widehat{F}_\varphi}{\partial r}\right)$ $\widehat{R}_\varphi = \frac{1}{r}(\frac{\partial r \widehat{F}_\theta}{\partial r} - \frac{\partial \widehat{F}_\varphi}{\partial\theta})$
Cylindrical	$\widehat{R}_r = \frac{\partial \widehat{F}_z}{\partial\varphi} - \frac{\partial r \widehat{F}_\varphi}{\partial z}$ $\widehat{R}_\varphi = \frac{\partial \widehat{F}_r}{\partial z} - \frac{\partial \widehat{F}_z}{\partial r}$ $\widehat{R}_z = \frac{1}{r}\left(\frac{\partial r \widehat{F}_\varphi}{\partial r} - \frac{\partial \widehat{F}_r}{\partial\varphi}\right)$

Comment: What follows is not very interesting and should be suppresed.

From 2.300 we can write

$$\widehat{g}\, G_i = \nabla_i(\widehat{g}\,\phi)\,, \tag{2.318}$$

which leads to the formula

$$G_i = \frac{1}{\widehat{g}}\ \nabla_i(\widehat{g}\,\phi)\,. \tag{2.319}$$

For instance, as the fundamental density $\widehat{g}$ takes, in Cartesian, spherical and cylindrical coordinates, respectively the values 1 , $r^2 \sin\theta$ and r , this leads to the results of the following table.

	Cartesian	Spherical	Cylindrical
Gradient	$\widehat{G}_x = \frac{\partial\overline{\phi}}{\partial x}$ $\widehat{G}_y = \frac{\partial\overline{\phi}}{\partial y}$ $\widehat{G}_z = \frac{\partial\overline{\phi}}{\partial z}$	$\widehat{G}_r = r^2 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\overline{\phi}\right)$ $\widehat{G}_\theta = \sin\theta \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\overline{\phi}\right)$ $\widehat{G}_\varphi = \frac{\partial\overline{\phi}}{\partial\varphi}$	$\widehat{G}_r = r\frac{\partial}{\partial r}\left(\frac{1}{r}\overline{\phi}\right)$ $\widehat{G}_\varphi = \frac{\partial\overline{\phi}}{\partial\varphi}$ $\widehat{G}_z = \frac{\partial\overline{\phi}}{\partial z}$

2.4.13 Appendix: Some boxes

(Comment: mention here the boxes with different coordinate systems).

Box 2.2 Cylindrical coordinates

(Three-dimensional Euclidean space with non-Cartesian coordinates).

$$ds^2 = r^2 + r^2 d\varphi^2 + dz^2 \tag{2.324}$$

$$\Gamma_{r\varphi}{}^{\varphi} = 1/r \ ; \quad \Gamma_{\varphi r}{}^{\varphi} = 1/r \ ; \quad \Gamma_{\varphi\varphi}{}^{r} = -r \ ; \quad \text{(the others vanish)} \tag{2.325}$$

$$R_{ij} = 0 \tag{2.326}$$

$$\nabla_i V^i = \frac{1}{r}\frac{\partial}{\partial r}(r V^r) + \frac{\partial V^\varphi}{\partial \varphi} + \frac{\partial V^z}{\partial z} \tag{2.327}$$

Box 2.3 Geographical coordinates

(Two-dimensional non-Euclidean space).

$$ds^2 = R^2(d\theta^2 + \sin^2\theta \ d\varphi^2) \tag{2.328}$$

$$\Gamma_{\theta\varphi}{}^{\varphi} = \operatorname{cotg}\theta \ ; \quad \Gamma_{\varphi\theta}{}^{\varphi} = \operatorname{cotg}\theta \ ; \quad \Gamma_{\varphi\varphi}{}^{\theta} = -\sin\theta \ \cos\theta \ ; \quad \text{(the others vanish)} \tag{2.329}$$

$$R_{\theta\theta} = 1/R^2 \ ; \quad R_{\varphi\varphi} = 1/R^2 \ ; \quad \text{(the others vanish)} \ ; \quad R = 2/R^2 \tag{2.330}$$

$$\nabla_i V^i = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \ V^\theta) + \frac{\partial V^\varphi}{\partial\varphi} \tag{2.331}$$

Box 2.4 Spherical coordinates

(Three-dimensional Euclidean space).

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \ d\varphi^2 \tag{2.332}$$

$$\begin{array}{ccc} \Gamma_{r\theta}{}^{\theta} = 1/r \ ; & \Gamma_{r\varphi}{}^{\varphi} = 1/r \ ; & \Gamma_{\theta r}{}^{\theta} = 1/r \ ; \\ \Gamma_{\theta\theta}{}^{r} = -r \ ; & \Gamma_{\theta\varphi}{}^{\varphi} = \operatorname{cotg}\theta \ ; & \Gamma_{\varphi r}{}^{\varphi} = 1/r \ ; \\ \Gamma_{\varphi\theta}{}^{\varphi} = \operatorname{cotg}\theta \ ; & \Gamma_{\varphi\varphi}{}^{r} = -r\sin^2\theta \ ; & \Gamma_{\varphi\varphi}{}^{\theta} = -\sin\theta\cos\theta \ ; \\ & \text{(the others vanish)} & \end{array} \tag{2.333}$$

$$R_{ij} = 0 \tag{2.334}$$

$$\nabla_i V^i = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V^r) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \ V^\theta) + \frac{\partial V^\varphi}{\partial\varphi} \tag{2.335}$$

2.4.14 Appendix: Computing in polar coordinates

General formula

Simple-minded computation From

$$div\ \mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV^r) + \frac{\partial V^\varphi}{\partial \varphi}\ , \tag{2.336}$$

we obtain, using a simple-minded discretisation, at

$$\begin{aligned}(div\ \mathbf{V})(r,\varphi) = \frac{1}{r}\ &\frac{(r+\delta r)V^r(r+\delta r,\varphi) - (r-\delta r)V^r(r-\delta r,\varphi)}{2\ \delta r} \\ &+ \frac{V^\varphi(r,\varphi+\delta\varphi) - V^\varphi(r,\varphi-\delta\varphi)}{2\ \delta\varphi}\ .\end{aligned} \tag{2.337}$$

Computation through parallel transport The notion of parallel transport leads to

$$\begin{aligned}(div\ \mathbf{V})(r,\varphi) = &\frac{V^r(r,\varphi \parallel r+\delta r,\varphi) - V^r(r,\varphi \parallel r-\delta r,\varphi)}{2\ \delta r} \\ &+ \frac{V^\varphi(r,\varphi \parallel r,\varphi+\delta\varphi) - V^\varphi(r,\varphi \parallel r,\varphi-\delta\varphi)}{2\ \delta\varphi}\ ,\end{aligned} \tag{2.338}$$

which gives

$$\begin{aligned}(div\ \mathbf{V})(r,\varphi) = &\frac{V^r(r+\delta r,\varphi) - V^r(r-\delta r,\varphi)}{2\ \delta r} \\ &+ \cos(\delta\varphi)\ \frac{V^\varphi(r,\varphi+\delta\varphi) - V^\varphi(r,\varphi-\delta\varphi)}{2\ \delta\varphi} \\ &+ \frac{\sin(\delta\varphi)}{\delta\varphi}\ \frac{1}{r}\ \frac{V^r(r+\delta r,\varphi) + V^r(r-\delta r,\varphi)}{2}\ .\end{aligned} \tag{2.339}$$

Note: Natural basis and "normed" basis The components on the natural basis V^r et V^φ are related with the components on the normed basis $\widehat{V}^r$ and $\widehat{V}^\varphi$ through

$$\widehat{V}^r = V^r \tag{2.340}$$

and

$$\widehat{V}^\varphi = r\ V^\varphi\ . \tag{2.341}$$

Divergence of a constant field

A constant vector field (oriented "as the x axis") has components

$$V^r(r,\varphi) = k\cos\varphi \tag{2.342}$$

and

$$V^\varphi(r,\varphi) = -\frac{k}{r}\sin\varphi\ . \tag{2.343}$$

Simple-minded computation An exact evaluation of approximation 2.337 gives

$$(div\ \mathbf{V})(r,\varphi) = \frac{k}{r}\cos\varphi\left(1 - \frac{\sin(\delta\varphi)}{\delta\varphi}\right)\ , \tag{2.344}$$

expression with an error of order $(\delta\varphi)^2$.

Computation through parallel transport An exact evaluation of approximation 2.339 gives

$$(div\ \mathbf{V})(r, \varphi) = 0 \ , \tag{2.345}$$

as it should.

Chapter 3

Curvature and Torsion

Note: rewrite this text…

We will be interested in spaces with curvature and torsion. All geometrical properties of such spaces are perfectly defined when a torsion $S_{ij}{}^{k}$ and a metric g_{ij} are given. From them, the "connection coefficients" $\{\Gamma_{ij}{}^{k}\}$ can be computed that allow the "parallel transportation" of the basis vectors: $\partial_j \mathbf{e}_k = \mathbf{e}_i \, \Gamma_{jk}{}^{i}$.

3.1 Curvature and Torsion

Covariant derivatives do not commute in general. Rather, for an arbitrary vector field with components V^i a direct computation shows that we have

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) V^l = R_{ijk}{}^l \, V^k - (\nabla_k V^l) \, S_{ij}{}^k \, , \tag{3.1}$$

where

$$R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l - \partial_j \Gamma_{ik}{}^l + \Gamma_{is}{}^l \Gamma_{jk}{}^s - \Gamma_{js}{}^l \Gamma_{ik}{}^s \tag{3.2}$$

and

$$S_{ij}{}^k = \Gamma_{ij}{}^k - \Gamma_{ji}{}^k \, . \tag{3.3}$$

The tensor $R_{ijk}{}^l$ so defined is termed the curvature (Riemann) tensor. The contraction defined by

$$R_{ij} = R_{sij}{}^s \tag{3.4}$$

is named the Ricci tensor.

As we have not yet introduced a metric, we can not yet “raise and lower indices” of tensors. When this is done, it is possible to show that, in addition to the obvious symmetry

$$R_{ijkl} = -R_{jikl} \, , \tag{3.5}$$

the curvature tensor has also the symmetries (Hehl, 19XX)

$$R_{ijkl} = -R_{ijlk} \tag{3.6}$$

and

$$R_{ijkl} = R_{klij} \, . \tag{3.7}$$

Then, the *curvature scalar* is defined by

$$R = R_i{}^i . \tag{3.8}$$

(Comment: give here some examples illustrating the notion of curvature.)

The tensor $S_{ij}{}^k$ defined above is termed the torsion tensor. In spite of the fact that $\Gamma_{ij}{}^k$ are not the components of a tensor, those of $S_{ij}{}^k$ are .

(Comment: demonstrate that.)

The following contraction is introduced:

$$S_i = S_{si}{}^i . \tag{3.9}$$

Notice that, by definition, if curvature and torsion vanish, then covariant derivatives commute.

(Comment: give here some examples illustrating the notion of torsion.)

3.2 The Bianchi Identities

It is easy to show (see problem [Comment: which problem?]) that, by definition, the curvature and the torsion tensors, satisfy the following identities, termed the *Bianchi identities*:

$$\nabla_{[i} R_{jk]q}{}^p = S_{[ij}{}^s R_{k]sq}{}^p \tag{3.10}$$

and

$$\nabla_{[i} S_{jk]}{}^p = R_{[ijk]}{}^p + S_{[ij}{}^s S_{k]s}{}^p \, . \tag{3.11}$$

where $[ijk]$ means a sum with circular permutation: $ijk + jki + kij$. Explicitly,

$$\nabla_i R_{jkq}{}^p + \nabla_j R_{kiq}{}^p + \nabla_k R_{ijq}{}^p = S_{ij}{}^s R_{ksq}{}^p + S_{jk}{}^s R_{isq}{}^p + S_{ki}{}^s R_{jsq}{}^p \tag{3.12}$$

and

$$\nabla_i S_{jk}{}^p + \nabla_j S_{ki}{}^p + \nabla_k S_{ij}{}^p = R_{ijk}{}^p + R_{jki}{}^p + R_{kij}{}^p + S_{ij}{}^s S_{ks}{}^p + S_{jk}{}^s S_{is}{}^p + S_{ki}{}^s S_{js}{}^p \, . \tag{3.13}$$

3.2.1 The Contracted Bianchi identities

If we have introduced a metric we can define the contraction of the Ricci tensor,

$$R = g^{ij} R_{ij} = R_i{}^i \ , \tag{3.14}$$

called the *curvature*.

In the Bianchi identities 3.10–3.11 we can now contract the index of the covariant derivatives with the indexes in upper position, in order to get an expression involving the divergence of some tensors. This gives (see problem [Comment: which problem?]):

$$\nabla_k E_j{}^k = F_j \ , \tag{3.15}$$

and

$$\nabla_i C_{jk}{}^i = H_{jk} \ , \tag{3.16}$$

where

$$E_j{}^k = R_j{}^k - \frac{1}{2}\delta_j{}^k R \tag{3.17}$$

is called the *Einstein Tensor*,

$$C_{ij}{}^k = S_{ij}{}^k + S_i \delta_j{}^k - S_j \delta_i{}^k \tag{3.18}$$

is called the *Cartan tensor*, and where

$$F_j = S_{lr}{}^s (\frac{1}{2} R_{js}{}^{lr} + \delta_j{}^l R_s{}^r) \tag{3.19}$$

and

$$H_{jk} = (R_{jk} - R_{kj}) + S_s S_{jk}{}^s \ . \tag{3.20}$$

We will see in the next chapter that the contracted Bianchi identities are fundamental to make the link between space-time geometry and physics.

3.2.2 The Contracted Bianchi identities: global form

An integration per parts easily shows that the equality

$$\int_{\mathcal{V}_4} dV_4 \, \left(\nabla_k \psi^j\right) \, E_j{}^k + \int_{\mathcal{V}_4} dV_4 \, \psi^j \, F_j = \int_{\mathcal{V}_3} dV_3 \, \psi^j \, A_j \tag{3.21}$$

is equivalent to

$$\int_{\mathcal{V}_4} dV_4 \, \psi^j \, \left(F_j - \nabla_k E_j{}^k\right) = \int_{\mathcal{V}_3} dV_3 \, \psi^j \, \left(A_j - n_k \, E_j{}^k\right) \ . \tag{3.22}$$

If the condition 3.21 is satisfied for any field $\psi^j(\mathbf{x})$, then, for any point $\mathbf{x} \in \mathcal{V}_4$,

$$\nabla_k E_j{}^k = F_j \ , \tag{3.23}$$

and, for any point $\mathbf{x} \in \mathcal{V}_3$,

$$n_k E_j{}^k = A_j \ . \tag{3.24}$$

Note: explain what this means and why this is useful.

Similarly, an integration per parts easily shows that the equality

$$\int_{\mathcal{V}_4} dV_4 \, \left(\nabla_i \psi^{jk}\right) \, C_{jk}{}^i + \int_{\mathcal{V}_4} dV_4 \, \psi^{jk} \, H_{jk} = \int_{\mathcal{V}_3} dV_3 \, \psi^{jk} \, B_{jk} \tag{3.25}$$

is equivalent to

$$\int_{\mathcal{V}_4} dV_4\, \psi^{jk} \left(H_{jk} - \nabla_i C_{jk}{}^i \right) = \int_{\mathcal{V}_3} dV_3\, \psi^{jk} \left(B_{jk} - n_i\, C_{jk}{}^i \right) . \tag{3.26}$$

If the condition 3.25 is satisfied for any field $\psi^{jk}(\mathbf{x})$, then, for any point $\mathbf{x} \in \mathcal{V}_4$,

$$\nabla_i C_{jk}{}^i = H_{jk} \, , \tag{3.27}$$

and, for any point $\mathbf{x} \in \mathcal{V}_3$,

$$n_i C_{jk}{}^i = B_{jk} \, . \tag{3.28}$$

Note: explain what this means and why this is useful.

Note: the symbols used here above (ψ and H) may not be clever. Change?

Note: explain here the sense of these global equations, and explain that they can be used numerically to force some fuzzy knowledge to satisfy the theoretical constraints.

Note: explain here to what correspond these integrals in the 4-D space-time (actions?).

3.3 Derivatives of the Levi-Civita's tensors?

To be written?

Chapter 4

More tensors

4.1 Totally Antisymmetric tensors

A tensor is completely antisymmetric if any *even* permutation of indices does not change the value of the components, and if any *odd* permutation of indices changes the sign of the value of the components:

$$t_{pqr\dots} = \begin{cases} + \quad t_{ijk\dots} & \text{if } ijk\dots \text{ is an } \textit{even} \text{ permutation of } pqr\dots \\ - \quad t_{ijk\dots} & \text{if } ijk\dots \text{ is an } \textit{odd} \text{ permutation of } pqr\dots\,, \end{cases} \tag{4.1}$$

For instance, a fourth rank tensor t_{ijkl} is totally antisymmetric if

$$\begin{aligned} t_{ijkl} &= t_{iklj} = t_{iljk} = t_{jilk} = t_{jkil} = t_{jlki} \\ &= t_{kijl} = t_{kjli} = t_{klij} = t_{likj} = t_{ljik} = t_{lkij} \\ &= -t_{ijlk} = -t_{ikjl} = -t_{ilkj} = -t_{jikl} = -t_{jkli} = -t_{jlik} \\ &= -t_{kilj} = -t_{kjil} = -t_{klji} = -t_{lijk} = -t_{ljki} = -t_{lkij}\,, \end{aligned} \tag{4.2}$$

a third rank tensor t_{ijk} is totally antisymmetric if

$$t_{ijk} = t_{jki} = t_{kji} = -t_{ikj} = -t_{jik} = -t_{kji}\,, \tag{4.3}$$

a second rank tensor t_{ij} is totally antisymmetric if

$$t_{ij} = -t_{ji}\,, \tag{4.4}$$

and a first rank tensor t_i can always be considered totally antisymmetric.

Well known examples of totally antisymmetric tensors are the Levi-Civita's tensors of any rank, the rank-two electromagnetic tensors, the "vector product" of two vectors:

$$c_{ij} = a_i b_j - a_j b_i\,, \tag{4.5}$$

etc.

Comment: say somewhere that the Kronecker's tensors and determinants are totally antisymmetric.

4.2 Dual tensors

In a space with n dimensions, let p and q be two (nonnegative) integers such that $p+q=n$. To any totally antisymmetric tensor of rank p, $B^{i_1\dots i_p}$, we can associate a totally antisymmetric tensor of rank q, $b_{i_1\dots i_q}$, defined by

$$b_{i_1\dots i_q} = \frac{1}{p!}\,\varepsilon_{i_1\dots i_q j_1\dots j_p}\,B^{j_1\dots j_p}\,. \tag{4.6}$$

The tensor $\mathbf{b}$ is called the *dual* of $\mathbf{B}$, and we write

$$\mathbf{b} = \mathrm{Dual}[\mathbf{B}] \tag{4.7}$$

or

$$\mathbf{b} =^* \mathbf{B} \tag{4.8}$$

From the properties of the product of Levi-Civita's tensors it follows that the dual of the dual gives the original tensor, excepted for a sign:

$$^*(^*\mathbf{B}) = \mathrm{Dual}[\mathrm{Dual}[\mathbf{B}]] = (-1)^{p(n-p)}\,\mathbf{B}\,. \tag{4.9}$$

For spaces with odd dimension $(n = 1, 3, 5, \dots)$, the product $p(n-p)$ is even, and

$$^*(^*\mathbf{B}) = \mathbf{B} \qquad \text{(spaces with odd dimension)}\,. \tag{4.10}$$

For spaces with even dimension $(n = 2, 4, 6, \dots)$, we have

$$^*(^*\mathbf{B}) = (-1)^p\mathbf{B} \qquad \text{(spaces with even dimension)}\,. \tag{4.11}$$

Although definition 4.6 has been written for pure tensors, it can obviously be written for densities and capacities,

$$\begin{aligned} b_{i_1\dots i_q} &= \frac{1}{p!}\,\underline{\varepsilon}_{i_1\dots i_q j_1\dots j_p}\,\overline{B}^{j_1\dots j_p} \\ \underline{b}_{i_1\dots i_q} &= \frac{1}{p!}\,\underline{\varepsilon}_{i_1\dots i_q j_1\dots j_p}\,B^{j_1\dots j_p}\,, \end{aligned} \tag{4.12}$$

or for tensor where covariant and contravariant indices have replaced each other:

$$\begin{aligned} d^{i_1\dots i_q} &= \frac{1}{p!}\,\varepsilon^{i_1\dots i_q j_1\dots j_p}\,D_{j_1\dots j_p} \\ d^{i_1\dots i_q} &= \frac{1}{p!}\,\overline{\varepsilon}^{i_1\dots i_q j_1\dots j_p}\,\underline{D}_{j_1\dots j_p} \\ \overline{d}^{i_1\dots i_q} &= \frac{1}{p!}\,\overline{\varepsilon}^{i_1\dots i_q j_1\dots j_p}\,D_{j_1\dots j_p}\,, \end{aligned} \tag{4.13}$$

Boxes 4.1 to 4.3 give explicitly the dual tensor relations in spaces with 2, 3, and 4 dimensions.

Example: Consider an antisymmetric tensor E_{ij} in three dimensions. It has components

$$\begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{12} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} = \begin{pmatrix} 0 & E_{12} & E_{13} \\ E_{21} & 0 & E_{23} \\ E_{31} & E_{32} & 0 \end{pmatrix}\,, \tag{4.14}$$

with $E_{ij} = -E_{ji}$. The definition

$$\overline{e}^i = \frac{1}{2!}\,\overline{\varepsilon}^{ijk}\,E_{jk} \tag{4.15}$$

gives

$$\begin{pmatrix} 0 & E_{12} & E_{13} \\ E_{21} & 0 & E_{23} \\ E_{31} & E_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \overline{e}^3 & -\overline{e}^2 \\ -\overline{e}^3 & 0 & \overline{e}^1 \\ \overline{e}^2 & -\overline{e}^1 & 0 \end{pmatrix}\,, \tag{4.16}$$

which is the classical relation between the three independent components of a 3-D antisymmetric tensor and the components of a vector density.

Box 4.1 Dual tensors in 2-D

In 2-D, we may need to take the following duals of contravariant (antisymmetric) tensors:

$$ {}^*B_{ij} = \frac{1}{0!}\,\varepsilon_{ij}\,B \qquad {}^*B_{ij} = \frac{1}{0!}\,\underline{\varepsilon}_{ij}\,\overline{B} \qquad {}^*\underline{B}_{ij} = \frac{1}{0!}\,\underline{\varepsilon}_{ij}\,B \tag{4.19} $$

$$ {}^*B_{i} = \frac{1}{1!}\,\varepsilon_{ij}\,B^j \qquad {}^*B_{i} = \frac{1}{1!}\,\underline{\varepsilon}_{ij}\,\overline{B}^j \qquad {}^*\underline{B}_{i} = \frac{1}{1!}\,\underline{\varepsilon}_{ij}\,B^j \tag{4.20} $$

$$ {}^*B = \frac{1}{2!}\,\varepsilon_{ij}\,B^{ij} \qquad {}^*B = \frac{1}{2!}\,\underline{\varepsilon}_{ij}\,\overline{B}^{ij} \qquad {}^*\underline{B} = \frac{1}{2!}\,\underline{\varepsilon}_{ij}\,B^{ij} \tag{4.21} $$

We may also need to take duals of covariant tensors:

$$ {}^*B^{ij} = \frac{1}{0!}\,\varepsilon^{ij}\,B \qquad {}^*B^{ij} = \frac{1}{0!}\,\overline{\varepsilon}^{ij}\,\underline{B} \qquad {}^*\overline{B}^{ij} = \frac{1}{0!}\,\overline{\varepsilon}^{ij}\,B \tag{4.22} $$

$$ {}^*B^{i} = \frac{1}{1!}\,\varepsilon^{ij}\,B_j \qquad {}^*B^{i} = \frac{1}{1!}\,\overline{\varepsilon}^{ij}\,\underline{B}_j \qquad {}^*\overline{B}^{i} = \frac{1}{1!}\,\overline{\varepsilon}^{ij}\,B_j \tag{4.23} $$

$$ {}^*B = \frac{1}{2!}\,\varepsilon^{ij}\,B_{ij} \qquad {}^*B = \frac{1}{2!}\,\overline{\varepsilon}^{ij}\,\underline{B}_{ij} \qquad {}^*\overline{B} = \frac{1}{2!}\,\overline{\varepsilon}^{ij}\,B_{ij} \tag{4.24} $$

As in a space with an even number of dimensions the dual of the dual of a tensor of rank p equals $(-1)^p$ the original tensor (see text), we have, in 2-D, that for a tensor with 0 or 2 indices, ${}^*({}^*\mathbf{B}) = \mathbf{B}$, while for a tensor with 1 index, ${}^*({}^*\mathbf{B}) = -\mathbf{B}$.

Example: The *vector product* of two vectors U_i and V_i can be either defined as the antisymmetric tensor

$$ W_{ij} = U_i V_j - V_j U_i \, , \tag{4.17} $$

or as the vector density

$$ \overline{w}^i = \frac{1}{2!}\,\overline{\varepsilon}^{ijk}\,U_j\,V_k \, . \tag{4.18} $$

The two definitions are equivalent, as W_{ij} and $\overline{w}^i$ are mutually duals.

Definition 4.18 shows that the vector product of two vectors is not a pure vector, but a vector density. Changing the sense of one axis gives a Jacobian equal to -1 , thus *changing the sign of the vector product* $\overline{w}^i$.

4.3 Exterior Product of tensors

In a space of dimension n , let $A_{i_1 i_2 \dots i_p}$ and $B_{i_1 i_2 \dots i_q}$, be two totally antisymmetric tensors with ranks p and q such that $p + q \le n$. Note: check that total antisymmetry has been defined. The *exterior product* of the two tensors is denoted

$$ \mathbf{C} = \mathbf{A} \wedge \mathbf{B} \tag{4.43} $$

and is the totally antisymmetric tensor of rank $p+q$ defined by

$$ C_{i_1 \dots i_p j_1 \dots j_q} = \frac{1}{(p+q)!}\,\delta^{k_1 \dots k_p \ell_1 \dots \ell_q}_{i_1 \dots i_p j_1 \dots j_q}\,A_{k_1 i_2 \dots k_p}\,B_{\ell_1 i_2 \dots \ell_q} \, . \tag{4.44} $$

Box 4.2 Dual tensors in 3-D

In 3-D, we may need to take the following duals of contravariant (totally antisymmetric) tensors:

$$ {}^*B_{ijk} = \frac{1}{0!}\,\varepsilon_{ijk}\,B \qquad {}^*B_{ijk} = \frac{1}{0!}\,\underline{\varepsilon}_{ijk}\,\overline{B} \qquad {}^*\underline{B}_{ijk} = \frac{1}{0!}\,\underline{\varepsilon}_{ijk}\,B \tag{4.25}$$

$$ {}^*B_{ij} = \frac{1}{1!}\,\varepsilon_{ijk}\,B^k \qquad {}^*B_{ij} = \frac{1}{1!}\,\underline{\varepsilon}_{ijk}\,\overline{B}^k \qquad {}^*\underline{B}_{ij} = \frac{1}{1!}\,\underline{\varepsilon}_{ijk}\,B^k \tag{4.26}$$

$$ {}^*B_{i} = \frac{1}{2!}\,\varepsilon_{ijk}\,B^{jk} \qquad {}^*B_{i} = \frac{1}{2!}\,\underline{\varepsilon}_{ijk}\,\overline{B}^{jk} \qquad {}^*\underline{B}_{i} = \frac{1}{2!}\,\underline{\varepsilon}_{ijk}\,B^{jk} \tag{4.27}$$

$$ {}^*B = \frac{1}{3!}\,\varepsilon_{ijk}\,B^{ijk} \qquad {}^*B = \frac{1}{3!}\,\underline{\varepsilon}_{ijk}\,\overline{B}^{ijk} \qquad {}^*\underline{B} = \frac{1}{3!}\,\underline{\varepsilon}_{ijk}\,B^{ijk} \tag{4.28}$$

We may also need to take duals of covariant tensors:

$$ {}^*B^{ijk} = \frac{1}{0!}\,\varepsilon^{ijk}\,B \qquad {}^*B^{ijk} = \frac{1}{0!}\,\overline{\varepsilon}^{ijk}\,\underline{B} \qquad {}^*\overline{B}^{ijk} = \frac{1}{0!}\,\overline{\varepsilon}^{ijk}\,B \tag{4.29}$$

$$ {}^*B^{ij} = \frac{1}{1!}\,\varepsilon^{ijk}\,B_k \qquad {}^*B^{ij} = \frac{1}{1!}\,\overline{\varepsilon}^{ijk}\,\underline{B}_k \qquad {}^*\overline{B}^{ij} = \frac{1}{1!}\,\overline{\varepsilon}^{ijk}\,B_k \tag{4.30}$$

$$ {}^*B^{i} = \frac{1}{2!}\,\varepsilon^{ijk}\,B_{jk} \qquad {}^*B^{i} = \frac{1}{2!}\,\overline{\varepsilon}^{ijk}\,\underline{B}_{jk} \qquad {}^*\overline{B}^{i} = \frac{1}{2!}\,\overline{\varepsilon}^{ijk}\,B_{jk} \tag{4.31}$$

$$ {}^*B = \frac{1}{3!}\,\varepsilon^{ijk}\,B_{ijk} \qquad {}^*B = \frac{1}{3!}\,\overline{\varepsilon}^{ijk}\,\underline{B}_{ijk} \qquad {}^*\overline{B} = \frac{1}{3!}\,\overline{\varepsilon}^{ijk}\,B_{ijk} \tag{4.32}$$

As in a space with an odd number of dimensions the dual of the dual of a tensor always equals the original tensor (see text), we have, in 3-D, that for all tensors above, ${}^*({}^*\mathbf{B}) = \mathbf{B}$.

Box 4.3 Dual tensors in 4-D

In 4-D, we may need to take the following duals of contravariant (totally antisymmetric) tensors:

$$ {}^*B_{ijk\ell} = \frac{1}{0!}\,\varepsilon_{ijk\ell}\,B \qquad {}^*B_{ijk\ell} = \frac{1}{0!}\,\underline{\varepsilon}_{ijk\ell}\,\overline{B} \qquad {}^*\underline{B}_{ijk\ell} = \frac{1}{0!}\,\underline{\varepsilon}_{ijk\ell}\,B \tag{4.33} $$

$$ {}^*B_{ijk} = \frac{1}{1!}\,\varepsilon_{ijk\ell}\,B^\ell \qquad {}^*B_{ijk} = \frac{1}{1!}\,\underline{\varepsilon}_{ijk\ell}\,\overline{B}^\ell \qquad {}^*\underline{B}_{ijk} = \frac{1}{1!}\,\underline{\varepsilon}_{ijk\ell}\,B^\ell \tag{4.34} $$

$$ {}^*B_{ij} = \frac{1}{2!}\,\varepsilon_{ijk\ell}\,B^{k\ell} \qquad {}^*B_{ij} = \frac{1}{2!}\,\underline{\varepsilon}_{ijk\ell}\,\overline{B}^{k\ell} \qquad {}^*\underline{B}_{ij} = \frac{1}{2!}\,\underline{\varepsilon}_{ijk\ell}\,B^{k\ell} \tag{4.35} $$

$$ {}^*B_{i} = \frac{1}{3!}\,\varepsilon_{ijk\ell}\,B^{jk\ell} \qquad {}^*B_{i} = \frac{1}{3!}\,\underline{\varepsilon}_{ijk\ell}\,\overline{B}^{jk\ell} \qquad {}^*\underline{B}_{i} = \frac{1}{3!}\,\underline{\varepsilon}_{ijk\ell}\,B^{jk\ell} \tag{4.36} $$

$$ {}^*B = \frac{1}{4!}\,\varepsilon_{ijk\ell}\,B^{ijk\ell} \qquad {}^*B = \frac{1}{4!}\,\underline{\varepsilon}_{ijk\ell}\,\overline{B}^{ijk\ell} \qquad {}^*\underline{B} = \frac{1}{4!}\,\underline{\varepsilon}_{ijk\ell}\,B^{ijk\ell} \tag{4.37} $$

We may also need to take duals of covariant tensors:

$$ {}^*B^{ijk\ell} = \frac{1}{0!}\,\varepsilon^{ijk\ell}\,B \qquad {}^*B^{ijk\ell} = \frac{1}{0!}\,\overline{\varepsilon}^{ijk\ell}\,\underline{B} \qquad {}^*\overline{B}^{ijk\ell} = \frac{1}{0!}\,\overline{\varepsilon}^{ijk\ell}\,B \tag{4.38} $$

$$ {}^*B^{ijk} = \frac{1}{1!}\,\varepsilon^{ijk\ell}\,B_\ell \qquad {}^*B^{ijk} = \frac{1}{1!}\,\overline{\varepsilon}^{ijk\ell}\,\underline{B}_\ell \qquad {}^*\overline{B}^{ijk} = \frac{1}{1!}\,\overline{\varepsilon}^{ijk\ell}\,B_\ell \tag{4.39} $$

$$ {}^*B^{ij} = \frac{1}{2!}\,\varepsilon^{ijk\ell}\,B_{k\ell} \qquad {}^*B^{ij} = \frac{1}{2!}\,\overline{\varepsilon}^{ijk\ell}\,\underline{B}_{k\ell} \qquad {}^*\overline{B}^{ij} = \frac{1}{2!}\,\overline{\varepsilon}^{ijk\ell}\,B_{k\ell} \tag{4.40} $$

$$ {}^*B^{i} = \frac{1}{3!}\,\varepsilon^{ijk\ell}\,B_{jk\ell} \qquad {}^*B^{i} = \frac{1}{3!}\,\overline{\varepsilon}^{ijk\ell}\,\underline{B}_{jk\ell} \qquad {}^*\overline{B}^{i} = \frac{1}{3!}\,\overline{\varepsilon}^{ijk\ell}\,B_{jk\ell} \tag{4.41} $$

$$ {}^*B = \frac{1}{4!}\,\varepsilon^{ijk\ell}\,B_{ijk\ell} \qquad {}^*B = \frac{1}{4!}\,\overline{\varepsilon}^{ijk\ell}\,\underline{B}_{ijk\ell} \qquad {}^*\overline{B} = \frac{1}{4!}\,\overline{\varepsilon}^{ijk\ell}\,B_{ijk\ell} \tag{4.42} $$

As in a space with an even number of dimensions the dual of the dual of a tensor of rank p equals $(-1)^p$ the original tensor (see text), we have, in 4-D, that for a tensor with 0, 2 or 4 indices, ${}^*({}^*\mathbf{B}) = \mathbf{B}$, while for a tensor with 1 or 3 indices, ${}^*({}^*\mathbf{B}) = -\mathbf{B}$.

Permuting the set of indices $\{k_1 \dots k_p\}$ by the set $\{\ell_1 \dots \ell_q\}$ in the above definition gives the property

$$(\mathbf{A} \wedge \mathbf{B}) = (-1)^{pq} \, (\mathbf{B} \wedge \mathbf{A}) \, . \tag{4.45}$$

It is also easy to see that the associativity property holds:

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C} \, . \tag{4.46}$$

Comment: say that $\delta^{j_1 j_2 \dots}_{i_1 i_2 \dots}$ are the components of the Kronecker's determinant defined in Section 1.6.4.

Say that it equation 1.151 gives the property

$$(\mathbf{A1} \wedge \mathbf{A2} \wedge \dots \mathbf{AP})_{i_1 i_2 \dots i_p} = \frac{1}{p!} \delta^{j_1 j_2 \dots j_p}_{i_1 i_2 \dots i_p} A1_{j_1} A2_{j_2} \dots AP_{j_p} \, . \tag{4.47}$$

4.3.1 Particular cases:

It follows from equation 1.150 that the exterior product of a tensor of rank zero (a scalar) by a totally antisymmetric tensor of any order is the simple product of the scalar by the tensor:

$$(A \quad , \quad B_{i_1 \dots i_q}) \qquad \rightarrow \qquad (\mathbf{A} \wedge \mathbf{B})_{i_1 \dots i_q} = A \, B_{i_1 \dots i_q} \, . \tag{4.48}$$

For the exterior product of two vectors we easily obtain (independently of the dimension of the space into consideration)

$$(A_i \quad , \quad B_i) \qquad \rightarrow \qquad (\mathbf{A} \wedge \mathbf{B})_{ij} = \frac{1}{2} \, (A_i B_j - A_j B_i) \, . \tag{4.49}$$

The exterior product of a vector by a second rank (antisymmetric) tensor gives

$$(A_i \quad , \quad B_{ij}) \qquad \rightarrow \qquad (\mathbf{A} \wedge \mathbf{B})_{ijk} = \frac{1}{3} \, (A_i B_{jk} + A_j B_{ki} + A_k B_{ij}) \, . \tag{4.50}$$

Finally, it can be seen that the exterior product of three vectors gives

$$\begin{aligned}
(A_i \quad , \quad B_i \quad , \quad C_i) \qquad & \rightarrow \qquad \qquad \qquad (4.51) \\
(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C})_{ijk} \quad &= \quad \frac{1}{6} \, (A_i(B_j C_k - B_k C_j) + A_j(B_k C_i - B_i C_k) + A_k(B_i C_j - B_j C_i)) \\
&= \quad \frac{1}{6} \, (B_i(C_j A_k - C_k A_j) + B_j(C_k A_i - C_i A_k) + B_k(C_i A_j - C_j A_i)) \\
&= \quad \frac{1}{6} \, (C_i(A_j B_k - A_k B_j) + C_j(A_k B_i - A_i B_k) + C_k(A_i B_j - A_j B_i)) \, .
\end{aligned}$$

Let us examine with more detail the formulas above in the special case of a 3-D space.

The dual of the exterior product of two vectors (equation 4.49) gives

$$^*\overline{(\mathbf{a} \wedge \mathbf{b})}^i = \frac{1}{2} \, \overline{\varepsilon}^{ijk} a_j b_k \, , \tag{4.52}$$

i.e., one half the usual vector product of the two vectors:

$$^*\overline{(\mathbf{a} \wedge \mathbf{b})} = \frac{1}{2} \overline{(\mathbf{a} \times \mathbf{b})} \, . \tag{4.53}$$

The dual of the exterior product of a vector by a second rank (antisymmetric) tensor (equation 4.50) is

$$^*\overline{(\mathbf{a} \wedge \mathbf{b})} = \frac{1}{3} \, a_i \left(\frac{1}{2!} \, \overline{\varepsilon}^{ijk} b_{jk} \right) \, , \tag{4.54}$$

or, introducing the vector ${}^*\overline{b}^i$, dual of the tensor b_{ij} ,

$$ {}^*\overline{(\mathbf{a} \wedge \mathbf{b})} = \frac{1}{3}\, a_i\ {}^*\overline{b}^i \, . \tag{4.55} $$

This shows that the exterior product contains, via the duals, the contraction of a form and a vector.

Finally, the dual of the exterior product of three vectors (equation 4.51) is

$$ {}^*\overline{(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})} = \frac{1}{3!}\, \overline{\varepsilon}^{ijk}\, a_i\, b_j\, c_k \, , \tag{4.56} $$

i.e., one sixth of the triple product of the three vectors.

Comment: explain that the triple product of three vectors is $\mathbf{a} \cdot \overline{(\mathbf{b} \times \mathbf{c})} = \mathbf{b} \cdot \overline{(\mathbf{c} \times \mathbf{a})} = \mathbf{c} \cdot \overline{(\mathbf{a} \times \mathbf{b})}$.

4.4 Exterior Derivative of tensors

Let $\mathbf{T}$ be a totally antisymmetric tensor with components $T_{i_1 i_2 \ldots i_p}$. The exterior product of "nabla" with $\mathbf{T}$ is called the *exterior derivative* of $\mathbf{T}$, and is denoted $\nabla \wedge \mathbf{T}$:

$$ (\nabla \wedge \mathbf{T})_{i j_1 j_2 \ldots j_p} = \delta^{k \ell_1 \ell_2 \ldots \ell_p}_{i j_1 j_2 \ldots j_p}\, \nabla_k\, T_{\ell_1 \ell_2 \ldots \ell_p} \, . \tag{4.57} $$

Here, $\nabla_i T_{jk\ldots}$ denotes the covariant derivative defined in section XXX.

The "nabla" notation allows to use direclty the formulas developed for the exterior product of a vector by a tensor to obtain formulas for exterior derivatives. For instance, from equation 4.49 it follows the definition of the exterior derivative of a vector

$$ (\nabla \wedge \mathbf{b})_{ij} = \frac{1}{2}\, (\nabla_i b_j - \nabla_j b_i) \, , \tag{4.58} $$

or, if we use the dual (equations 4.52–4.53),

$$ {}^*\overline{(\nabla \wedge \mathbf{b})}^i = \frac{1}{2}\, \overline{\varepsilon}^{ijk} \nabla_j b_k \, , \tag{4.59} $$

i.e.,

$$ {}^*\overline{(\nabla \wedge \mathbf{b})} = \frac{1}{2} \overline{(\nabla \times \mathbf{b})} \, . \tag{4.60} $$

The exterior derivative of a vector equals one-half the rotational (curl) of the vector.

The exterior derivative of a second rank (antisymmetric) tensor is directly obtained from equation 4.50:

$$ (\nabla \wedge \mathbf{b})_{ijk} = \frac{1}{3}\, (\nabla_i b_{jk} + \nabla_j b_{ki} + \nabla_k b_{ij}) \, . \tag{4.61} $$

Taking the dual of the expression and introducing the vector ${}^*\overline{b}^i$, dual of the tensor b_{ij} , gives (see equation 4.55)

$$ {}^*\overline{(\nabla \wedge \mathbf{b})} = \frac{1}{3}\, \nabla_i\ {}^*\overline{b}^i \, , \tag{4.62} $$

which shows that the dual of the exterior derivative of a second rank (antisymmetric) tensor equals one-third of the divergence of the dual of the tensor. The exterior derivative contains, via the duals, the divergence of a vector.

Chapter 5

Integration theory, in short

5.1 The volume element

Consider, in a space with n dimensions, p linearly independent vectors $\{d\mathbf{r}_1, d\mathbf{r}_2, \ldots, d\mathbf{r}_p\}$. As they are linear independent, $p \leq n$.

We define the "differential element"

$$d^{(p)}\sigma = p!\,(d\mathbf{r}_1 \wedge d\mathbf{r}_2 \wedge \ldots \wedge d\mathbf{r}_p)\,. \qquad \text{Note : } \sigma \text{ should be bold} \tag{5.1}$$

Using equation 4.47 (Note: in fact this equation with indices changed of place) gives the components

$$d^{(p)}\sigma^{i_1\ldots i_p} = \delta^{i_1\ldots i_p}_{j_1\ldots j_p}\, dr_1^{i_1}\, dr_2^{i_2} \ldots dr_p^{i_p}\,. \tag{5.2}$$

In a space with n dimensions, the dual of the differential element of dimension p will have q indices, with $p+q=n$. The general definition of dual (equation 4.12) gives

$$^*d^{(p)}\underline{\sigma}_{i_1\ldots i_q} = \frac{1}{p!}\,\underline{\varepsilon}_{i_1\ldots i_q j_1\ldots j_p}\, d^{(p)}\sigma^{j_1\ldots j_p} \tag{5.3}$$

The definition 5.2 and the property 1.152 give

$$^*d^{(p)}\underline{\sigma}_{i_1\ldots i_q} = \underline{\varepsilon}_{i_1\ldots i_q j_1\ldots j_p}\, dr_1^{j_1}\, dr_2^{j_2} \ldots dr_p^{j_p}\,. \tag{5.4}$$

In order to simplify subsequent notations, it is better not to keep the $*$ notation. Instead, we will write

$$^*d^{(p)}\underline{\sigma}_{i_1\ldots i_q} = d^{(p)}\underline{\Sigma}_{i_1\ldots i_q} \tag{5.5}$$

For reasons to be developed below, $d^{(p)}\underline{\Sigma}_{i_1\ldots i_q}$ will be called the *capacity element*.

We can easily see, for instance, that the differential elements of dimensions 0, 1, 2 and 3 have components

$$\begin{aligned}
d^0\sigma &= 1 && (5.6)\\
d^1\sigma^i &= dr_1^i && (5.7)\\
d^2\sigma^{ij} &= dr_1^i dr_2^j - dr_1^j dr_2^i && (5.8)\\
d^3\sigma^{ijk} &= dr_1^i(dr_2^j dr_3^k - dr_2^k dr_3^j) + dr_1^j(dr_2^k dr_3^i - dr_2^i dr_3^k) + dr_1^k(dr_2^i dr_3^j - dr_2^j dr_3^i) \\
&= dr_2^i(dr_3^j dr_1^k - dr_3^k dr_1^j) + dr_2^j(dr_3^k dr_1^i - dr_3^i dr_1^k) + dr_2^k(dr_3^i dr_1^j - dr_3^j dr_1^i) \\
&= dr_3^i(dr_1^j dr_2^k - dr_1^k dr_2^j) + dr_3^j(dr_1^k dr_2^i - dr_1^i dr_2^k) + dr_3^k(dr_1^i dr_2^j - dr_1^j dr_2^i)\,. && (5.9)
\end{aligned}$$

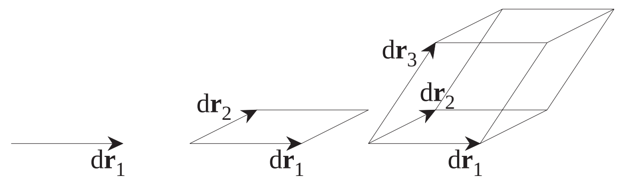

Figure 5.1: From vectors in a three-dimensional space we define the one-dimensional capacity element $d^1\underline{\Sigma}_{ij} = \underline{\varepsilon}_{ijk}\, dr_1^k$, the two-dimensional capacity element $d^2\underline{\Sigma}_i = \underline{\varepsilon}_{ijk}\, dr_1^j dr_2^k$ and the three-dimensional capacity element $d^3\underline{\Sigma} = \underline{\varepsilon}_{ijk}\, dr_1^i dr_2^j dr_3^k$. In a metric space, the rank-two form $d^1\underline{\Sigma}_{ij}$ defines a surface perpendicular to $d\mathbf{r}_1$ and with a surface magnitude equal to the length of $d\mathbf{r}_1$. The rank-one form $d^2\underline{\Sigma}_i$ defines a vector perpendicular to the surface defined by $d\mathbf{r}_1$ and $d\mathbf{r}_2$ and with length representing the surface magnitude (the vector product of the two vectors). The rank-zero form $d^3\underline{\Sigma}$ is a scalar representing the volume defined by the three vectors $d\mathbf{r}_1$, $d\mathbf{r}_2$ and $d\mathbf{r}_3$ (the triple product of the vectors). Note: clarify all this.

For a given dimemsion of the differential element, the number of indices of the capacity elements depends on the dimension of the space. **In a three-dimensional space**, for instance, we have

$$
\begin{aligned}
d^0\underline{\Sigma}_{ijk} &= \underline{\varepsilon}_{ijk} && (5.10)\\
d^1\underline{\Sigma}_{ij} &= \underline{\varepsilon}_{ijk}\, dr_1^k && (5.11)\\
d^2\underline{\Sigma}_{i} &= \underline{\varepsilon}_{ijk}\, dr_1^j dr_2^k && (5.12)\\
d^3\underline{\Sigma} &= \underline{\varepsilon}_{ijk}\, dr_1^i dr_2^j dr_3^k \,. && (5.13)
\end{aligned}
$$

Note: explain that I use the notation $d^{(p)}$ but $d^1, d^2, \dots$ in order not to suggest that p is a tensor index and, at the same time, for not using too heavy notations..

Note: refer here to figure 5.1, and explain that we have, in fact, vector products of vectors and triple products of vectors.

5.2 The Stokes' theorem

Comment: I must explain here first what integration means.

Let, in a space with n dimensions, $(\mathbf{T})$ be a totally antisymmetric tensor of rank p , with $(p < n)$. The Stokes' theorem

$$
\boxed{\int_{(p+1)D} d^{(p+1)}\sigma^{i_1\dots i_{p+1}}\, (\nabla \wedge \mathbf{T})_{i_1\dots i_{p+1}} = \int_{pD} d^{(p)}\sigma^{i_1\dots i_p}\, T_{i_1\dots i_p}} \qquad (5.14)
$$

holds. Here, the symbol $\int_{(p+1)D} d^{(p+1)}$ stands for an integral over a p+1)-dimensional "volume", (embedded in an space of dimension n), and $\int_{pD} d^{(p)}$ for the integral over the p-dimensional boundary of the "volume".

This fundamental theorem contains, as special cases, the divergence theorem of Gauss-Ostrogradsky, and the rotational theorem of Stokes (stricto sensu). Rather than deriving it here, we will explore its consequences. For a demonstration, see, for instance, Von Westenholz (1981).

In a three-dimensional space $(n = 3)$, we may have p respectively equal to 2 , 1 and 0 . This gives the three theorems

$$
\begin{aligned}
\int_{3D} d^3\sigma^{ijk}\, (\nabla \wedge \mathbf{T})_{ijk} &= \int_{2D} d^2\sigma^{ij}\, T_{ij} && (5.15)\\
\int_{2D} d^2\sigma^{ij}\, (\nabla \wedge \mathbf{T})_{ij} &= \int_{1D} d^1\sigma^{i}\, T_{i} && (5.16)
\end{aligned}
$$

$$\int_{1D} d^1\sigma^i \, (\nabla \wedge \mathbf{T})_i \quad = \quad \int_{0D} d^0\sigma \, T \, . \tag{5.17}$$

It is easy to see (appendix XXX) that these equation can be written

$$\frac{1}{0!}\int_{3D} d^3\underline{\Sigma} \left(\frac{1}{2!}\overline{\varepsilon}^{ijk}\nabla_i T_{jk} \right) \quad = \quad \frac{1}{1!}\int_{2D} d^2\underline{\Sigma}_i \left(\frac{1}{2!}\overline{\varepsilon}^{ijk} T_{jk} \right) \tag{5.18}$$

$$\frac{1}{1!}\int_{2D} d^2\underline{\Sigma}_i \left(\frac{1}{1!}\overline{\varepsilon}^{ijk}\nabla_j T_{k} \right) \quad = \quad \frac{1}{2!}\int_{1D} d^1\underline{\Sigma}_{ij} \left(\frac{1}{1!}\overline{\varepsilon}^{ijk} T_{k} \right) \tag{5.19}$$

$$\frac{1}{2!}\int_{1D} d^1\underline{\Sigma}_{ij} \left(\frac{1}{0!}\overline{\varepsilon}^{ijk}\partial_k \, T \right) \quad = \quad \frac{1}{3!}\int_{0D} d^0\underline{\Sigma}_{ijk} \left(\frac{1}{0!}\overline{\varepsilon}^{ijk} T \right) . \tag{5.20}$$

Simplifying equation 5.18 and introducing the vector density $\overline{t}^i$, dual to the tensor T_{ij}, (i.e., $\overline{t}^i = \frac{1}{2!}\overline{\varepsilon}^{ijk} T_{jk}$), gives

$$\int_{3D} d^3\underline{\Sigma} \; \nabla_i \overline{t}^i = \int_{2D} d^2\underline{\Sigma}_i \; \overline{t}^i \, . \tag{5.21}$$

This corresponds to the divergence theorem of Gauss-Ostrogradsky: The integral over a (3-D) volume of the divergence of a vector equals the flux of the vector across the surface bounding the volume.

It is worth to mention here that expression 5.21 has been derived without any mention to a metric in the space. We have sen elsewhere that densities and capacities can be defined even if there is no notion of distance. If there is a metric, then from the capacity element $d^3\underline{\Sigma}$ we can introduce the *volume element* $d^3\Sigma$ using the standard rule for putting on and taking off bars

$$d^3\Sigma = \overline{g} \, d^3\underline{\Sigma} \, , \tag{5.22}$$

as well as the *surface element*

$$d^2\Sigma_i = \overline{g} \, d^2\underline{\Sigma}_i \, . \tag{5.23}$$

$d^3\Sigma$ is now the familiar volume inside a prism, and $d^2\Sigma_i$ the vector (if we raise the index with the metric) representing the surface inside a lozenge.

Equation 5.21 then gives

$$\int_{3D} d^3\Sigma \; \nabla_i t^i = \int_{2D} d^2\Sigma_i \; t^i \, , \tag{5.24}$$

which is the familiar form for the divergence theorem.

Keeping the compact expression for the capacity element in the lefthand side of equation 5.19, but introducing its explicit expression in the right hand side gives, after simplification,

$$\int_{2D} d^2\underline{\Sigma}_i \, (\overline{\varepsilon}^{ijk}\nabla_j T_k) = \int_{1D} dr_1^i \, T_i \, , \tag{5.25}$$

which corresponds to the rotational theorem (theorem of Stokes stricto sensu): the integral of the rotational (curl) of a vector on a surface equals the circulation of the vector along the line bounding the surface.

Finally, introducing explicit expressions for the capacity elements at both sides of equation 5.20 gives

$$\int_{1D} dr_1^i \, \partial_i \, T = \int_{0D} T \, . \tag{5.26}$$

Writing this in the more familiar form gives

$$\int_a^b dr^i \partial_i \, T = T(b) - T(a) \, , \tag{5.27}$$

which corresponds the fundamental theorem of integral calculus: the integral over a line of the gradient of a scalar equals the difference of the values of the scalar at the two end-points.

Note: the demonstration that follows is to be put in an appendix:

In a three-dimensional space $(n = 3)$, we may have p respectively equal to 2 , 1 and 0 . This gives the three theorems

$$\int_{3D} d^3\sigma^{ijk} \, (\nabla \wedge \mathbf{T})_{ijk} = \int_{2D} d^2\sigma^{ij} \, T_{ij} \tag{5.28}$$

$$\int_{2D} d^2\sigma^{ij} \, (\nabla \wedge \mathbf{T})_{ij} = \int_{1D} d^1\sigma^{i} \, T_{i} \tag{5.29}$$

$$\int_{1D} d^1\sigma^{i} \, (\nabla \wedge \mathbf{T})_{i} = \int_{0D} d^0\sigma \, T \, . \tag{5.30}$$

Explicitly, using the results of sections 4.3 and 4.4, this gives

$$\int_{3D} d^3\sigma^{ijk} \, \frac{1}{3} \left(\nabla_i T_{jk} + \nabla_j T_{ki} + \nabla_k T_{ij}\right) = \int_{2D} d^2\sigma^{ij} \, T_{ij} \tag{5.31}$$

$$\int_{2D} d^2\sigma^{ij} \, \frac{1}{2} \left(\nabla_i T_j - \nabla_j T_i\right) = \int_{1D} d^1\sigma^{i} \, T_{i} \tag{5.32}$$

$$\int_{1D} d^1\sigma^{i} \, \nabla_i T = \int_{0D} d^0\sigma \, T \, , \tag{5.33}$$

or, we use the antisymmetry of the tensors,

$$\int_{3D} d^3\sigma^{ijk} \, \nabla_i T_{jk} = \int_{2D} d^2\sigma^{ij} \, T_{ij} \tag{5.34}$$

$$\int_{2D} d^2\sigma^{ij} \, \nabla_i T_j = \int_{1D} d^1\sigma^{i} \, T_{i} \tag{5.35}$$

$$\int_{1D} d^1\sigma^{i} \, \partial_i T = \int_{0D} d^0\sigma \, T \, . \tag{5.36}$$

We can now introduce the capacity elements instead of the differential elements:

$$\frac{1}{0!} \int_{3D} d^3\underline{\Sigma} \left(\frac{1}{2!} \overline{\varepsilon}^{ijk} \nabla_i T_{jk}\right) = \frac{1}{1!} \int_{2D} d^2\underline{\Sigma}_i \left(\frac{1}{2!} \overline{\varepsilon}^{ijk} T_{jk}\right) \tag{5.37}$$

$$\frac{1}{1!} \int_{2D} d^2\underline{\Sigma}_i \left(\frac{1}{1!} \overline{\varepsilon}^{ijk} \nabla_j T_{k}\right) = \frac{1}{2!} \int_{1D} d^1\underline{\Sigma}_{ij} \left(\frac{1}{1!} \overline{\varepsilon}^{ijk} T_{k}\right) \tag{5.38}$$

$$\frac{1}{2!} \int_{1D} d^1\underline{\Sigma}_{ij} \left(\frac{1}{0!} \overline{\varepsilon}^{ijk} \partial_k \, T\right) = \frac{1}{3!} \int_{0D} d^0\underline{\Sigma}_{ijk} \left(\frac{1}{0!} \overline{\varepsilon}^{ijk} T\right) \, . \tag{5.39}$$

Introducing explicit expressions for the capacity elements gives

$$\int_{3D} (\underline{\varepsilon}_{jk\ell} dr_1^j dr_2^k dr_3^\ell) \, \nabla_i \overline{t}^i = \int_{2D} (\underline{\varepsilon}_{ijk} dr_1^j dr_2^k) \, \overline{t}^i \tag{5.40}$$

$$\int_{2D} (\underline{\varepsilon}_{i\ell m} dr_1^\ell dr_2^m) \, (\overline{\varepsilon}^{ijk} \nabla_j T_k) = \int_{1D} dr_1^i \, T_i \tag{5.41}$$

$$\int_{1D} dr_1^i \, \partial_i \, T = \int_{0D} T \, , \tag{5.42}$$

where, in equation 5.40, $\overline{t}^i$ stands for the vector dual to the tensor T_{ij} , i.e., $\overline{t}^i = \frac{1}{2!} \overline{\varepsilon}^{ijk} T_{jk}$.

Equations 5.37 and 5.40 correspond to the divergence theorem of Gauss-Ostrogradsky, equations 5.38 and 5.39 correspond to the rotational theorem of Stokes (stricto sensu), and equation 5.42, when written in its more familiar form

$$\int_a^b dr^i \partial_i \, T = T(b) - T(a) \tag{5.43}$$

corresponds the fundamental theorem of integral calculus.

Part II

Second part: Gravitation and Dynamics

Chapter 6

The 4-D space-time

Summary

The tensors describing curvature and torsion in an arbitray manifold satisfy two identities — the *contracted Bianchi identities* —. When considering the four-dimensional space-time as a manifold with curvature and torsion, the fundamental field equations — the Einstein-Cartan equations — are obtained by identifying the "conserved" quantities in the Bianchi identities with the mass tensor and the spin tensor. Without any extra assumption, this gives a theory describing the dynamics of continuous media (including general gravitational interactions). Simplifying hypothesis give then classical limits (e.g., non relativistic, elastic).

6.1 Introduction

Any mathematical structure may be used to represent different physical objects. For instance, a 3-D linear vector space may represent the space of *forces* acting on a particle, as well as the space of *colors*: each vector of the basis represents then one fundamental color (e.g., yellow, cyan, magenta) with some standard intensity.

The major Einstein's discovery was that the formalism of differentiable manifolds (i.e., the geometry of curved surfaces and spaces) may also be used to describe the 4-D space-time, and that, then, gravitational forces can be interpreted as space-time curvature. One corollary of the theory was that the space itself has curvature (i.e., is not euclidean).

The fact that two things as different as the geometry of the space, and the dynamics of matter can be described in a single formalism was surprising at the time. Of course, it is futile to ask if this is well "understood" or not. Any moderately intelligent human being can understand enough of the formalism as to be able to make predictions on the outcome of experiments.

The claim that in the universe there are no forces, but only space-time curvature, gives a biased description of physics. The 4-D space-time of general relativity has two very different sort of dimensions: three of the dimensions are spatial dimensions, and one dimension is a time dimension. Vey precisely, this means that if a suitable choice of space-time coordinates makes the space-time metric diagonal, we will necessarily have something like $ds^2 = -(d\widehat{x}^0)^2 + (d\widehat{x}^1)^2 + (d\widehat{x}^2)^2 + (d\widehat{x}^3)^2$: we can say that the coordinate $\widehat{x}^0$ is the time coordinate.

At any moment we can switch between a representation of the universe as a four-dimensional space-time continuum, with space-time curvature and torsion, where all instants of time are simultaneously considered, and where the physical laws are 4-D geometrical laws, and a representation of the universe as a three-dimensional space continuum (with curvature and torsion), evolving in time, and where the physical laws invoke forces.

Those two representations are both perfectly valid. The advantage of the first is its great compacity of formulation. The second has the advantage of corresponding better to our perception of beings evolving with time, and perceiving forces. We will see that, when considering all the gravitational interactions in a continuous medium, it is the second representation that will give us the best insight into the phenomenology involved.

The original Einstein's theory of gravitation related gravitational forces to the *curvature* of the space-time, while space-time *torsion* was assumed to vanish. This approximation is not valid if in addition to mass, we consider that matter possesses spin. The coupling of spin to space-time torsion leads to the so-called Einstein-Cartan theory of gravitation, which predicts gravitational forces related to the distribution of spin. Inside a continuous medium, the predicted spin-spin gravitational force density is proportional to the gradient of the squared spin density, thus tending to concentrate spin. It is then for instance possible to imagine a spherical distribution of matter where repulsive electrostatical forces are compensated by the attractive spin-spin gravitational forces.

In the previous chapter, we have used an underbar to represent a tensor density. In this chapter we only use true tensors (i.e., tensor densities of weight 0). The underbar will then be reserved here to 4-D tensors, like in $\underline{T}_\alpha{}^\beta$. The space components of the tensor, $\underline{T}_i{}^j$, when considered as the components of a 3-D tensor will be denoted without the underbar: $T_i{}^j$.

6.2 Chapters' overview

Our physical 3-D space is not euclidean: it may have (time-dependent) curvature and torsion. If we denote by g_{ij} the space metric and by $S_{ij}{}^k$ the space torsion, then all other space properties can be computed, as, for instance, the space connection $\Gamma_{ij}{}^k$ or the space Riemann tensor $R_{ijk}{}^\ell$.

If instead of being interested in the geometry of space, we are interested in the geometry of the space-time, in addition to g_{ij} and $S_{ij}{}^k$ we need a scalar field φ , a vector field I_i , an antisymmetric tensor field $\widehat{J}_{ij}$, and a tensor field $K_i{}^j$ (all these being time-dependent) in order to build the space-time metric $\underline{g}_{\alpha\beta}$:

$$\begin{pmatrix} \underline{g}_{00} & \underline{g}_{0j} \\ \underline{g}_{i0} & \underline{g}_{ij} \end{pmatrix} = \begin{pmatrix} -\varphi^2 & 0 \\ 0 & g_{ij} \end{pmatrix} \tag{6.1}$$

and the space-time torsion $\underline{S}_{\alpha\beta}{}^\gamma$:

$$\begin{pmatrix} \underline{S}_{00}{}^0 & \underline{S}_{0j}{}^0 \\ \underline{S}_{i0}{}^0 & \underline{S}_{ij}{}^0 \end{pmatrix} = \begin{pmatrix} 0 & c^2 I_j \\ -c^2 I_i & c\widehat{J}_{ij}/\varphi \end{pmatrix} \tag{6.2}$$

$$\begin{pmatrix} \underline{S}_{00}{}^k & \underline{S}_{0j}{}^k \\ \underline{S}_{i0}{}^k & \underline{S}_{ij}{}^k \end{pmatrix} = \begin{pmatrix} 0 & cK_j{}^k\varphi \\ -cK_i{}^k\varphi & S_{ij}{}^k \end{pmatrix} . \tag{6.3}$$

We can then compute all the other properties of the space-time, as, for instance, the space-time connection $\underline{\Gamma}_{\alpha\beta}{}^\gamma$ or the space-time Riemann tensor $\underline{R}_{\alpha\beta\gamma}{}^\delta$: we have a "model of the Universe" (if we are interested in cosmology) or, more simply, a model of a continuous medium. Different choices for φ , g_{ij} , I_i , $\widehat{J}_{ij}$, $K_i{}^j$, and $S_{ij}{}^k$, give different models of continuous media.

The material properties of the universe are introduced through the mass (stress-energy) tensor

$$\begin{pmatrix} \underline{T}_0{}^0 & \underline{T}_0{}^j \\ \underline{T}_i{}^0 & \underline{T}_i{}^j \end{pmatrix} = \begin{pmatrix} -\rho c^2 & -q^j c\varphi \\ cp_i/\varphi & T_i{}^j \end{pmatrix} \tag{6.4}$$

and the spin tensor

$$\begin{pmatrix} \underline{M}_{00}{}^0 & \underline{M}_{0j}{}^0 \\ \underline{M}_{i0}{}^0 & \underline{M}_{ij}{}^0 \end{pmatrix} = \begin{pmatrix} 0 & c^2 t_j \\ -c^2 t_i & c\sigma_{ij}/\varphi \end{pmatrix} \tag{6.5}$$

$$\begin{pmatrix} \underline{M}_{00}{}^k & \underline{M}_{0j}{}^k \\ \underline{M}_{i0}{}^k & \underline{M}_{ij}{}^k \end{pmatrix} = \begin{pmatrix} 0 & c\pi_j{}^k\varphi \\ -c\pi_i{}^k\varphi & M_{ij}{}^k \end{pmatrix} . \tag{6.6}$$

Here ρ , p_i , q^i , and $T_i{}^j$ are respectively the mass density, the linear momentum density, the mass flux, and the stress, while σ_{ij} , $M_{ij}{}^k$, t^i , and $\pi_i{}^j$ are respectively the spin density, the moment stress, the unbalance, and the unbalance flux.

The Einstein-Cartan equations

$$\underline{T}_\alpha{}^\beta = \frac{1}{\chi}\left(\underline{R}_\alpha{}^\beta - \frac{1}{2}\,\underline{\delta}_\alpha{}^\beta \underline{R}\right) , \tag{6.7}$$

and

$$\underline{M}_{\alpha\beta}{}^\gamma = \frac{1}{\chi}\left(\underline{S}_{\alpha\beta}{}^\gamma + \underline{S}_\alpha \underline{\delta}_\beta{}^\gamma - \underline{S}_\beta \underline{\delta}_\alpha{}^\gamma\right) , \tag{6.8}$$

link then these material properties to the geometrical quantities φ , g_{ij} , I_i , $\widehat{J}_{ij}$, $K_i{}^j$, and $S_{ij}{}^k$, introduced above.

In particular, this gives the "conservation equations" for the material properties:

$$\frac{\partial\rho}{\partial\tau} + \nabla_i q^i = \kappa\,, \tag{6.9}$$

$$\frac{\partial p_i}{\partial\tau} + \nabla_j T_i{}^j = f_i\,, \tag{6.10}$$

$$\frac{\partial t_i}{\partial\tau} + \nabla_j \pi_i{}^j = \omega_i\,, \tag{6.11}$$

$$\frac{\partial\sigma_{ij}}{\partial\tau} + \nabla_k M_{ij}{}^k = \psi_{ij}\,, \tag{6.12}$$

with the corresponding expressions for κ , f_i , ω_i , and ψ_{ij} .

The term f_i is important, as it represents the force density acting on the medium. In addition to the Newtonian gravitational force density ρg_i , responsible of apples' fall, it contains, a spin-spin gravitational force density $(\pi G/c^2)\nabla_i(\sigma_{jk}\sigma^{jk})$ that may have important implications for understanding the microscopical structure of matter.

The gravitational field g_i will appear as follows: the field φ will be the (squared root of the) ratio between proper time and coordinate time. The scalar field U defined (up to an additive constant) by

$$\varphi = \exp(U - U_0)\,, \tag{6.13}$$

is the *gravitational potential*, and exactly corresponds to the classical Newtonian potential (times c^{-2}). Its (negative) gradient

$$g_i = -\frac{1}{c^2}\nabla_i U \tag{6.14}$$

is the Newtonian *gravitational field* (times c^{-4}).

From the conservation equations for the material properties it is easy to obtain the nonrelativistic limit. The elastic hypothesis then leads to the equations governing the propagation of elastic waves in solids, but for a general medium, where the stress tensor is not necessarily symmetric (called Cosserat's or "micro polar" medium).

6.3 Measuring proper time

For long, humans thought that there was an absolute way of defining the length of a line joining two points in space.

Today, that hypothesis has been dropped, and replaced by the hypothesis that the only "absolute" possible definition is the "proper time" associated to every "particle" of the universe.

That our space-time has some property that is close to what we call the proper time of particles is suggested by the consistency we get in the description of physical processes with “regular” evolution, like radioactive decay of particles, pulsar rotation, assumed periodicity of electromagnetic radiation, etc.

The only way for an observer to measure “his” proper time is by observing near him one of those processes with regular evolution, and use it as a clock. The more regular the process, the more accurate will be his estimation of proper time.

Discovering more and more regular clocks (pulsars, cesium clocks, ...) is an important task of experimental physics.

Of course, we cannot exclude that a future, more precise, vision of the universe will force us to admit that the proper time of a particle depends on some extra property of the universe in a way not yet understood. But, for the time being, as we do not know of any experimental facts contradicting the hypothesis, we assume the the concept of proper time of a particle has an absolute sense, and that the proper time of the particle can be measured as precisely as we wish using clocks whose “regularity” is checked by the internal consistency of the physics so obtained.

When this book is printed, physicists choose as unit of proper time the *second*, defined as the duration of 9 192 631 770 periods of the electromagnetic radiation corresponding to the transition between the two hyperfine levels of the fundamental state of an atom of cesium (Comment: check that definition when the book goes to print).

6.4 Measuring (improper) distances

For long, the definition of the unit of distance has been independent of that of time. It was made by selecting a given object, accessible to all, and by deciding what was, by definition, its length (a given rod in Paris measured, by definition, one *meter*).

Since then, a major discovery was made: that if two observers, moving one relatively to the other, measure the speed of electromagnetic waves in the vacuum, they obtain the same value, denoted by c .

Should we be able to define the “absolute motion” of an observer, physics as seen by an abserver “absolutely” at rest should be different to physics as seen by any other observer. Our experiments show that this is not the case (in particular, we know that the equations describing the electromagnetic and similar fields are the same for all observers, irrespectively of their relative velocity, as they have to predict the same velocity of waves, c).

In fact, there is a way of defining an absolute motion in the universe, as the “ 3 K ” background radiation observed is interpreted as a fossil radiation from the early universe. We can say that an observer is “absolutely at rest” if that radiation appears isotropic (i.e., if there is no Doppler shift effect on the radiation).

Thus, we postulate that no signal can propagate faster than the electromagnetic waves, and current theories predict that other sort of waves (gravitational, ...) and particles (with zero mass) also propagate at that speed.

Again, we cannot exclude that, some day, we will discover finer facts (for instance, c at a given point could weakly depend on the direction of propagation), but, for the time being, the postulate of the existence of that limiting velocity gives a consistent physical vision of the universe.

This being so, we have two possibilities: we can continue to define the unit of distance (say, the *meter*) independently of the velocity of light, and try to made ever-more-precise measurements of c , or we can fix c , at an arbitrary value, and try to made ever-more-precise realizations of the meter.

It is the second way that has been chosen by physicists now, by fixing

$$c = 299\ 792\ 458 \text{ meter/second} . \tag{6.15}$$

Then, a meter is the distance traveled in the vacuum by an electromagnetic wave in 1/299 792 458 seconds.

An experimental realization of this definition shows that the line here below measures approximately one tenth of a meter (Comment: in the final print, some reduction may be applied; check and correct the length of the line).

This is 1/10 of a meter

If instead of keeping the book at rest in front of you, you look at it while walking (assuredly, you should have to walk quite rapidly. . .), you may find that the above rod measures less than announced. This is the so-called “Lorentz contraction of length”, not discussed here.

Notice that, instead of the meter, we could have chosen the *second-light* as unit of distance, or, if the qualificative “light” is dropped (as the “radian” when measuring angles), we could have chosen a time (the *second*) to measure distances. In the first case, the value of c would be $c = 1$ second-light/second , while with the second choice it would be $c = 1$, without any unit.

6.5 Defining space-time coordinates

Let us call a *clock* an instrument that emits signals at some predetermined time intervals.

To define a system of space-time coordinates in some region of the space-time, we drop in the space many clocks, in a configuration that is sort of “continuous — and arbitrary — deformation of a 3-D regular grid”. Each clock will then carry three integers $\{i, j, k\}$ identifying it, and which, by definition, will denote its *position*. If we label the signals each clock emits by the integer n , then, the collection of all the *events* $\{n, i, j, k\}$ define a space-time grid of points. By interpolation between the points of the grid we define then the space-time coordinates $\{\underline{x}^0, \underline{x}^1, \underline{x}^2, \underline{x}^3\}$ of any event in the space-time continuum.

By no means we need to assume that the clocks are “fixed” or follow a particular movement, or that the clocks emit their “proper time”.

In that way, we have perfectly defined a system of coordinates in a region of the space-time. Obviously, that system of coordinates is (quite) arbitrary. Nevertheless, one coordinate $(\underline{x}^0)$ is very different from the others: the coordinate line $(\underline{x}^0)$ can be the space-time trajectory of a particle, while the other coordinate lines cannot (this would imply velocities higher than c .)

Later on, we will encounter an expression like $\underline{g}_{\alpha\beta}\ d\underline{x}^\alpha\ d\underline{x}^\beta$, where every term in the sum has a dimension of a squared *length*. As we will choose $\underline{g}_{00}$ adimensional, then $\underline{x}^0$ will have the dimension of a length, and will define the *coordinate time*, t , by

$$t = \frac{\underline{x}^0}{c} \ . \tag{6.16}$$

Obviously, t has a dimension of time. The dimensions of the $\underline{x}^i$ are variable (think, for instance, that a choice of spherical coordinates gives different dimensions for $\underline{x}^1 = r$ and for $\{\underline{x}^2, \underline{x}^3\} = \{\theta, \varphi\}$).

We have defined our coordinate system $\{\underline{x}^0, \underline{x}^1, \underline{x}^2, \underline{x}^3\}$ by sprinkling the space with clocks. Another observer may decide to use his own clocks, in his own way, to define another coordinate system $\{\underline{x}'^0, \underline{x}'^1, \underline{x}'^2, \underline{x}'^3\}$.

Pamameterizing the trajectory of a particle by an arbitrary parameter λ will give, in our coordinate system, $\underline{x}^\alpha(\lambda)$, and, in the coordinate system of the other observer, $\underline{x}'^\alpha(\lambda)$.

The proper time of the particle in the trajectory from, say, λ_A to λ_B , is an invariant (evaluating it when the trajectory is followed from the first or the second coordinate system has to give identical results). This is very similar to the property of a *distance*, and means that we can introducing a *metric* *tensor* in the space-time, with components $\underline{g}_{\alpha\beta}(\underline{x}^0, \underline{x}^1, \underline{x}^2, \underline{x}^3)$, in such a way that the proper time can be computed, in each coordinate system, by the expresions

$$\begin{aligned} \tau_B - \tau_A &= \int_{\lambda_A}^{\lambda_B} \sqrt{-\frac{1}{c^2}\ \underline{g}_{\alpha\beta}\ d\underline{x}^\alpha\ d\underline{x}^\beta} \\ &= \int_{\lambda_A}^{\lambda_B} \sqrt{-\frac{1}{c^2}\ \underline{g}'_{\alpha\beta}\ d\underline{x}'^\alpha\ d\underline{x}'^\beta}\ , \end{aligned} \tag{6.17}$$

or, equivalently, by

$$\tau_B - \tau_A = \int_{\lambda_A}^{\lambda_B} d\tau\ , \tag{6.18}$$

where

$$\begin{aligned} -c^2\ d\tau^2 &= \underline{g}_{\alpha\beta}\ d\underline{x}^\alpha\ d\underline{x}^\beta \\ &= \underline{g}'_{\alpha\beta}\ d\underline{x}'^\alpha\ d\underline{x}'^\beta\ . \end{aligned} \tag{6.19}$$

6.6 The space-time metric

The component $\underline{g}_{00}$ of the metric can esaily be interpreted as the (square root of the) ratio between proper time and coordinate time (see below). The components $\underline{g}_{ij}$ correspond to the metric of the 3-D space. What about the components $\underline{g}_{0i}$? A suitable (and very reasonable) choice of space-time coordinates just makes them to vanish: we show in appendix that if our choice of space-time coordinates is such that a light signal leaving a clock at (coordinate) time t_0 and reaching a neighbouring clock at $t_0 + \Delta t$, comes reflected back at $t_0 + 2\Delta t$, then $\underline{g}_{0i} = 0$.

As there is no physical restriction in such a choice, we always will assume it. Then, the following notations can be used:

$$\begin{pmatrix} \underline{g}_{00} & \underline{g}_{0j} \\ \underline{g}_{i0} & \underline{g}_{ij} \end{pmatrix} = \begin{pmatrix} -\varphi^2 & 0 \\ 0 & g_{ij} \end{pmatrix}\ . \tag{6.20}$$

Obviously, g_{ij} is the metric of the 3-D subspace of the 4-D space-time defined by a section $\underline{x}^0 = ct =$ const. , i.e., the metric of the 3-D *space* in the classical sense. As we will later see, that space — our space — may have curvature and torsion.

Using the general arguments of previous chapter, it is possible to see that the space components of space-time tensors are 3-D tensors, and that we can “raise and lower indices” of those tensors with the space metric g_{ij} .

For a particle following the time coordinate line, $dx^i = 0$. Using the expression 6.19 for the time element and the decomposition 6.20 of the 4-D metric in 3-D fields, we found that the proper time of the particle satisfies then

$$\begin{aligned} -c^2\ d\tau^2 &= \underline{g}_{\alpha\beta}\ d\underline{x}^\alpha\ d\underline{x}^\beta \\ &= \underline{g}_{00}\ d\underline{x}^0\ d\underline{x}^0 \\ &= -\varphi^2\ d\underline{x}^0\ d\underline{x}^0 \\ &= -\varphi^2\ c^2 dt^2\ , \end{aligned} \tag{6.21}$$

i.e.,

$$d\tau = \varphi\ dt\ . \tag{6.22}$$

This gives a physical interpretation for φ as the ratio between the proper time and the coordinate time for a particle describing a time coordinate line, i.e., for a particle “at rest” in the coordinate system.

6.7 The space-time torsion

Let us denote by $\underline{S}_{\alpha\beta}{}^{\gamma}$ the space-time torsion. From this 4-D tensor field we can introduce the space torsion $S_{ij}{}^{k}$ and three 3-D fields I_i , $\widehat{J}_{ij}$, and $K_i{}^j$:

$$
\begin{aligned}
\begin{pmatrix} \underline{S}_{00}{}^{0} & \underline{S}_{0j}{}^{0} \\ \underline{S}_{i0}{}^{0} & \underline{S}_{ij}{}^{0} \end{pmatrix} &= \begin{pmatrix} 0 & I_j c^2 \\ -I_i c^2 & \widehat{J}_{ij} c/\varphi \end{pmatrix} \\
\begin{pmatrix} \underline{S}_{00}{}^{k} & \underline{S}_{0j}{}^{k} \\ \underline{S}_{i0}{}^{k} & \underline{S}_{ij}{}^{k} \end{pmatrix} &= \begin{pmatrix} 0 & K_j{}^k c\varphi \\ -K_i{}^k c\varphi & S_{ij}{}^k \end{pmatrix} .
\end{aligned}
\tag{6.23}
$$

The space torsion $S_{ij}{}^{k}$ has the standard interpretation of the 3-D torsion of a constant-time section of the 4-D space-time. We will later see that this torsion of the physical 3-D space is essentially proportional to the moment stress tensor of the matter filling the space.

The 3-D field $\widehat{J}_{ij}$ will be proportional to the matter spin density, and the 3-D fields I_i and $K_i{}^j$ will essentially be proportional to two other properties of matter: the unbalance, and the unbalance flux.

In that sense, the introduction of space-time torsion (i.e., of spin) in the theory, does not complicate the equations. The new quantities introduced are simply algeabrically related to the quantites describing the matter properties: we do not have new “field (differential) equations”.

Chapter 7

Matter

Mass and spin are obvious properties of matter that are easily accommodated within the framework of the general relativistic theory developed here. Other fundamental properties (like electric charge) can only be introduced in an ad-hoc way.

Precisely, this means that, while mass and spin curve and twist the space-time (and the space itself), electric charge does not. It is not forbidden by the theory that such properties may exist, but they are not "geometrized". In that sense, they are extraneous to the theory.

The theory will predict that mass and spin are "conserved", or, more precisely, that they satisfyme equations traditionally called "conservation equations".

The theory will also predict that two other quantities are conserved. One will easily be identified with the linear momentum. The other (the unbalance) will not have (to our knowledge) a classical equivalent.

Of special interest for us will be the equation of conservation of linear momentum, as the "source term" will describe the whole set of *forces* acting in a medium submitted to gravitational interaction. When discussing the properties of space-time (Section 6.4), we saw that we could either choose to arbitrarily fix the unit of distance (the meter) and made ever-more-precise measurements of c, or arbitrarily fixe the value of c and made ever-more-precise realizations of the meter. Equivalently, we can either choose to arbitrarily fix the unit of mass (the kilogram) and made ever-more-precise measurements of the gravitational constant G, or arbitrarily fixe the value of G and made ever-more-precise realizations of the kilogram. For the time being, it is the first of these options that has been selected, and the kilogram is the mass of a given iridioplatinum cylinder deposited at the *Pavillon de Breteuil*, in Sèvres, near Paris.

Choosing to fix the value of G, would lead to two options: fixing it to a value close to its actual estimate, and with its present units, or fixing it at $G = 1$, without any unit, in which case, the kilogram would not be an independent unit, but derived from the second and the meter. If, as discussed before, also the value of c was fixed to $c = 1$, then we would measure time intervals, distances, and masses, all in *seconds*.

To be complete, we should made here a complete operational definition of all matter properties (linear momentum, spin, ...), but will not attempt such a task.

7.1 Recall: classical dynamics

The fundamental equations of the classical dynamics of continuous media are the equations describing conservation of mass, linear momentum and angular momentum:

$$\frac{\partial \rho}{\partial t} + \nabla_j(\rho v^j) = 0\,, \tag{7.1}$$

$$\frac{\partial p_i}{\partial t} + \nabla_j(p_i v^j - \tau_i{}^j) = f_i\,, \tag{7.2}$$

$$\frac{\partial \sigma_{ij}}{\partial t} + \nabla_k(\sigma_{ij} v^k - m_{ij}{}^k) = \psi_{ij} + (\tau_{ji} - \tau_{ij})\,. \tag{7.3}$$

Here, ρ is the *mass density*, v_i the *velocity* of the medium, $p_i = \rho v_i$ the *linear momentum density*, and τ_{ij} the *stress*. When considering an arbitrary interface inside the medium, with unit normal n_i , then $\phi_i = \tau_i{}^j\, n_j$ represents the *traction*, i.e., the force per unit area that one side of the surface exerts on the other side. The antisymmetric tensor σ_{ij} represents the *spin density* (or proper angular moment density). The tensor $m_{ij}{}^k$ is antisymmetric in its two lower indices, and when considering an arbitrary interface inside the medium, with unit normal n_i , then $\varphi_{ij} = m_{ij}{}^k\, n_k$ represents the surface density of torque that one side of the surface exerts on the other side. We call $m^k{}_{ij}$ the *moment stress* or *couple stress*.

The term f_i represents all the external densities of force acting on the medium. These, for instance, can be electromagnetic forces (see section XXX). We are specially interested in the gravitational forces, as they are going to be naturally predicted by the theory to be developed below. The Newton's force density is

$$f_i = \rho g_i\,, \tag{7.4}$$

where the *gravitational field* $\mathbf{g}$ relates to the mass density distribution by

$$\nabla_i\, g^i = -4\pi G \rho \tag{7.5}$$

and is curl-free, i.e., derives from a potential U , called the *gravity potential*:

$$g_i = -\nabla_i U\,. \tag{7.6}$$

In equation 7.5, G is the *Newtonian constant of gravitation* with the experimental value

$$G = 6.672\ 59 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{sec}^{-2}\,. \tag{7.7}$$

The term ψ_{ij} represents all the external torque densities acting on the medium, for instance, when an electromagnetic field acts on a medium with magnetic properties.

To better understand equations 7.1–7.3, let us consider a volume inside the medium that follows the matter on his movement and call it $V(t)$. The surface of the volume will be denoted $S(t)$. We define, for that volume, the following quantities:

- **Mass**

$$M(t) = \int_{V(t)} dV(\mathbf{x})\ \rho(\mathbf{x},t) \tag{7.8}$$

- **Linear momentum**

$$P_i(t) = \int_{V(t)} dV(\mathbf{x})\ p_i(\mathbf{x},t) \tag{7.9}$$

- **Force**

$$F_i(t) = \int_{V(t)} dV(\mathbf{x})\ f_i(\mathbf{x},t) + \int_{S(t)} dS(\mathbf{x})\ \phi_i(\mathbf{x},t) \tag{7.10}$$

(as stated above, $\phi_i = \tau_i{}^j\, n_j$ are the surface tractions)

- **Angular momentum with respect to the origin of coordinates**

$$\Sigma_{ij}(t) = \int_{V(t)} dV(\mathbf{x})\ \{\sigma_{ij}(\mathbf{x},t) + (r_i(\mathbf{x})\, p_j(\mathbf{x},t) - r_j(\mathbf{x})\, p_i(\mathbf{x},t))\} \tag{7.11}$$

($\mathbf{r}(\mathbf{x})$ is the position vector: as we are only interested here in flat (Euclidean) spaces [with possibly non Cartesian coordinates] the introduction of this vector is easy, which is not the case in curved spaces).

- **Torque with respect to the origin of coordinates**

$$\begin{aligned}\Gamma_{ij}(t) &= \int_{V(t)} dV(\mathbf{x})\ \{\psi_{ij}(\mathbf{x},t) + (r_i(\mathbf{x})\, f_j(\mathbf{x},t) - r_j(\mathbf{x})\, f_i(\mathbf{x},t))\} \\ &+ \int_{S(t)} dS(\mathbf{x})\ \{\varphi_{ij}(\mathbf{x},t) + (r_i(\mathbf{x})\, \phi_j(\mathbf{x},t) - r_j(\mathbf{x})\, \phi_i(\mathbf{x},t))\} \end{aligned} \tag{7.12}$$

(as stated above, $\varphi_{ij} = m_{ij}{}^k\ n_k$ represents the surface density of torque))

It is easy to show (see appendix XXX) that the set of local conservation equations 7.1–7.3 are equivalent to the set of (global) conservation equations

$$\boxed{\begin{aligned} \frac{dM}{dt}(t) &= 0 && (7.13)\\ \frac{dP_i}{dt}(t) &= F_i(t) && (7.14)\\ \frac{d\Sigma_{ij}}{dt}(t) &= \Gamma_{ij}(t)\,. && (7.15)\end{aligned}}$$

Comment: explain somewhere how important are these equations.

Comment: the term $p_i v_j - p_j v_i$ accounts for the effect discussed by Sedov (transfer from intrinsic angular momentum into extrinsic angular momentum).

Comment: explain that equation 7.3, setting the conservation of angular momentum is not a consequence of the other two. We have seen that we have intrinsic as well as extrinsic angular momentum density (σ_{ij} versus $r_i p_j - r_j p_i$), and only the conservation of the extrinsic (orbital) angular momentum density is a consequence of the conservation of the linear momentum.

Comment: explain that the purpose of this chapter is to *demonstrate* the equations 7.1–7.3 from very general principles.

7.2 Mass and spin

In our general relativistic theory, matter possesses two fundamental properties: mass and spin, described respectively by the 4-D tensors $\underline{T}_\alpha{}^\beta$ and $\underline{M}_{\alpha\beta}{}^\gamma$, the last being antisymmetric in its two lower indices:

$$\underline{M}_{\alpha\beta}{}^\gamma = -\underline{M}_{\beta\alpha}{}^\gamma\,. \tag{7.16}$$

Introducing right now a general system of space-time coordinates will complicate our goal (to obtain the 3-D equations governing the dynamics of continuous media, valid for arbitrary reference systems). For didactical purposes, it is better to consider first a very special reference system, where the space coordinate lines are attached to the medium, and move with it. In that reference system, the medium is "at rest".

Once the corresponding 3-D dynamical equations will be found — and they already are going to contain many terms —, it is possible to change the reference system to a general one, to get the general equations.

So, from now, and up to the section on relative velocities (section 8.2), we consider that we, local observer, see a continuous medium around us that evolves, but has zero local velocity.

From the four-dimensional mass-tensor $\underline{T}_\alpha{}^\beta$ we can introduce the following three-dimensional fields:

$$\begin{pmatrix} \underline{T}_0{}^0 & \underline{T}_0{}^j \\ \underline{T}_i{}^0 & \underline{T}_i{}^j \end{pmatrix} = \begin{pmatrix} -\rho c^2 & -q^j c\ \varphi \\ p_i c/\varphi & T_i{}^j \end{pmatrix} , \tag{7.17}$$

where ρ is the mass density and $T_i{}^j$ the stress (what we call the "stress" here will be related with the classical stress $\tau^j{}_i$ through $T_i{}^j = -\tau^j{}_i$). Both ρ and $\tau^j{}_i$ have been mentioned in section 7.1. We will see below that p_i represents that part of linear momentum density that is not of convective origin — i.e., apart from ρv_i —. Also, q^i is the mass flux that is not of convective origin — like the Poynting vector if the medium is electromagnetically charged —. Finally, from the spin tensor $\underline{M}_{\alpha\beta}{}^\gamma$ we can introduce the three-dimensional spin density σ_{ij} and the moment stress $M_{ij}{}^k$, (what we call the "moment stress" here will be related with the classical moment stress $m^k{}_{ij}$ through $M_{ij}{}^k = -m^k{}_{ij}$). Both, σ_{ij} and $m^k{}_{ij}$ have been already seen in section 7.1. We can also introduce two new tensors, t_i and $\pi_\alpha{}^\beta$, to be analised below:

$$\begin{pmatrix} \underline{M}_{00}{}^0 & \underline{M}_{0j}{}^0 \\ \underline{M}_{i0}{}^0 & \underline{M}_{ij}{}^0 \end{pmatrix} = \begin{pmatrix} 0 & t_j c^2 \\ -t_i c^2 & \sigma_{ij} c/\varphi \end{pmatrix} \tag{7.18}$$

$$\begin{pmatrix} \underline{M}_{00}{}^k & \underline{M}_{0j}{}^k \\ \underline{M}_{i0}{}^k & \underline{M}_{ij}{}^k \end{pmatrix} = \begin{pmatrix} 0 & \pi_j{}^k c\ \varphi \\ -\pi_i{}^k c\ \varphi & M_{ij}{}^k \end{pmatrix} . \tag{7.19}$$

Chapter 8

From geometry to physics

8.1 Media at rest

8.1.1 Fundamental constraints

Assume that, for all times, a space metric g_{ij} and a space torsion $S_{ij}{}^k$ are given, satisfying

$$g_{ij} = g_{ji} \, , \tag{8.1}$$

and

$$S_{ij}{}^k = -S_{ji}{}^k \, . \tag{8.2}$$

If the constraint

$$\nabla_i \; g_{jk} = 0 \tag{8.3}$$

is postulated, then the connection can be computed through

$$\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) + \frac{1}{2}(S_{ijk} + S_{kij} + S_{kji}) \, , \tag{8.4}$$

and the curvature (Riemann) tensor

$$R_{ijk}{}^\ell = \partial_i \Gamma_{jk}{}^\ell - \partial_j \Gamma_{ik}{}^\ell + \Gamma_{is}{}^\ell \Gamma_{jk}{}^s - \Gamma_{js}{}^\ell \Gamma_{ik}{}^s \tag{8.5}$$

is antisymmetric in its two last indices:

$$R_{ij}{}^{k\ell} = -R_{ij}{}^{\ell k} \tag{8.6}$$

(the antisymmety in the first two indices is obvious).

We are going to show that if an arbitrary scalar field φ , an arbitrary vector field I_i , an arbitrary antisymmetric tensor field $\widehat{J}_{ij}$, and an arbitrary tensor field $K_i{}^j$ are given, then the space-time metric

$$\begin{pmatrix} \underline{g}_{00} & \underline{g}_{0j} \\ \underline{g}_{i0} & \underline{g}_{ij} \end{pmatrix} = \begin{pmatrix} -\varphi^2 & 0 \\ 0 & g_{ij} \end{pmatrix} \, , \tag{8.7}$$

and the space-time torsion

$$\begin{pmatrix} \underline{S}_{00}{}^0 & \underline{S}_{0j}{}^0 \\ \underline{S}_{i0}{}^0 & \underline{S}_{ij}{}^0 \end{pmatrix} = \begin{pmatrix} 0 & c^2 I_j \\ -c^2 I_i & c\widehat{J}_{ij}/\varphi \end{pmatrix} \tag{8.8}$$

$$\begin{pmatrix} \underline{S}_{00}{}^k & \underline{S}_{0j}{}^k \\ \underline{S}_{i0}{}^k & \underline{S}_{ij}{}^k \end{pmatrix} = \begin{pmatrix} 0 & cK_j{}^k\varphi \\ -cK_i{}^k\varphi & S_{ij}{}^k \end{pmatrix} \tag{8.9}$$

are consistent, i.e., if from these, we compute the space-time connection through the usual expression

$$\underline{\Gamma}_{\alpha\beta\gamma} = \frac{1}{2}(\partial_\alpha \underline{g}_{\beta\gamma} + \partial_\beta \underline{g}_{\alpha\gamma} - \partial_\gamma \underline{g}_{\alpha\beta}) + \frac{1}{2}(\underline{S}_{\alpha\beta\gamma} + \underline{S}_{\gamma\alpha\beta} + \underline{S}_{\gamma\beta\alpha}) \tag{8.10}$$

and when defining covariant space-time derivatives through usual expressions, like

$$\underline{\nabla}_\alpha \underline{g}_{\beta\gamma} = \partial_\alpha \underline{g}_{\beta\gamma} - \underline{\Gamma}_{\alpha\beta}{}^\sigma \, \underline{g}_{\sigma\gamma} - \underline{\Gamma}_{\alpha\gamma}{}^\sigma \, \underline{g}_{\beta\sigma} \, , \tag{8.11}$$

we satisfy

$$\underline{\nabla}_\alpha \underline{g}_{\beta\gamma} = 0 \, . \tag{8.12}$$

Then, the space-time Riemann tensor can be computed through the usual expression

$$\underline{R}_{\alpha\beta\gamma}{}^\delta = \partial_\alpha \underline{\Gamma}_{\beta\gamma}{}^\delta - \partial_\beta \underline{\Gamma}_{\alpha\gamma}{}^\delta + \underline{\Gamma}_{\alpha\sigma}{}^\delta \underline{\Gamma}_{\beta\gamma}{}^\sigma - \underline{\Gamma}_{\beta\sigma}{}^\delta \underline{\Gamma}_{\alpha\gamma}{}^\sigma \, , \tag{8.13}$$

and we have the antisymmetry property

$$\underline{R}_{\alpha\beta}{}^{\gamma\delta} = -\underline{R}_{\alpha\beta}{}^{\delta\gamma} \, . \tag{8.14}$$

We have then a “model of the Universe”. Different choices for φ , g_{ij} , I_i , $\widehat{J}_{ij}$, $K_i{}^j$, and $S_{ij}{}^k$, give different universe models. We will express the material properties of the universe (mass density, spin density,. . .) as a function of these fields, and will obtain the corresponding (3-D) evolution equations.

The “hat” over $\widehat{J}_i{}^j$ will later mean that this tensor is the antisymmetric part of a tensor $J_i{}^j$ to be defined.

8.1.2 All definitions together

From now on, given an arbitrary tensor A_{ij} , we will define its *symmetric part* by

$$\widetilde{A}_{ij} = A_{ij} + A_{ji} \, , \tag{8.15}$$

and its *antisymmetric part* by

$$\widehat{A}_{ij} = A_{ij} - A_{ji} \, . \tag{8.16}$$

Then, we have

$$A_{ij} = \frac{1}{2} \, (\widetilde{A}_{ij} + \widehat{A}_{ij}) \, . \tag{8.17}$$

Instead of considering partial derivatives with respect to the time coordinate, much simpler expressions will be obtained is we consider partial derivatives with respect to the proper time of the observer (defined below). If dt represents a time interval, then, as shown in Section 6.6, the the proper time of the observer is $d\tau = \varphi dt$, where φ is the field introduced in the space-time metric (equation 8.7). Then, we write

$$\begin{pmatrix} d\underline{x}^0 \\ d\underline{x}^i \end{pmatrix} = \begin{pmatrix} c \; dt \\ dx^i \end{pmatrix} = \begin{pmatrix} c \; d\tau/\varphi \\ dx^i \end{pmatrix} \, , \tag{8.18}$$

and use the notation

$$\partial_0 \equiv \frac{\partial}{\partial \underline{x}^0} \equiv \frac{\varphi}{c} \frac{\partial}{\partial \tau} \, . \tag{8.19}$$

The field φ will not have any simple physical interpretation. Instead, the scalar field U defined (up to an additive constant) by

$$\varphi = \exp(U - U_0) \, , \tag{8.20}$$

i.e.,

$$U = U_0 + \operatorname{Log} \varphi \, , \tag{8.21}$$

will be the *gravitational potential*, and will exactly correspond to the classical Newtonian potential (times c^{-2}). In particular, its (negative) gradient

$$g_i = -\frac{1}{c^2}\nabla_i U \tag{8.22}$$

will be the *gravitational field*, and, as shown later, the product ρg_i will give the gravitational force density (times c^{-4}).

In the equations below, we will also use the time derivative of the potential:

$$\alpha = \frac{1}{c^4}\frac{\partial U}{\partial \tau} \,. \tag{8.23}$$

Defining

$$G_{ij} = \frac{1}{2}\left(\frac{1}{c^2}\frac{\partial g_{ij}}{\partial \tau} + \widehat{J}_{ij} + \widehat{K}_{ij}\right) , \tag{8.24}$$

gives

$$\widetilde{G}_{ij} = \frac{1}{c^2}\frac{\partial g_{ij}}{\partial \tau} , \tag{8.25}$$

and

$$\widehat{G}_{ij} = \widehat{J}_{ij} + \widehat{K}_{ij} \,. \tag{8.26}$$

Defining

$$J_{ij} = G_{ij} - K_{ij} \,, \tag{8.27}$$

i.e.,

$$J_{ij} = \frac{1}{2}\left(\frac{1}{c^2}\frac{\partial g_{ij}}{\partial \tau} + \widehat{J}_{ij} - \widetilde{K}_{ij}\right) , \tag{8.28}$$

gives

$$\widetilde{J}_{ij} = \widetilde{G}_{ij} - \widetilde{K}_{ij} \,, \tag{8.29}$$

and the antisymmetric part of J_{ij} corresponds, as it should, to the antisymmetric tensor $\widehat{J}_{ij}$ introduced in equation 8.8.

The following definition is going to be useful for expressing the space-time connection:

$$h_i = \; g_i - I_i \,, \tag{8.30}$$

while the following ones are going to be useful to express the curvature:

$$\gamma_i{}^j = \frac{1}{c^2}\frac{\partial J_i{}^j}{\partial \tau} + \nabla_i h^j - c^2 g_i h^j + J_i{}^k G_k{}^j \,, \tag{8.31}$$

$$\delta_{ij}{}^k = \nabla_i J_j{}^k - \nabla_j J_i{}^k + S_{ij}{}^l J_l{}^k \,, \tag{8.32}$$

$$\varepsilon_{ij}{}^k = \frac{1}{c^2}\frac{\partial \Gamma_{ij}{}^k}{\partial \tau} - \nabla_i G_j{}^k + c^2(g_i G_j{}^k + J_i{}^k h_j - J_{ij} h^k) \,, \tag{8.33}$$

and

$$r_{ij}{}^{k\ell} = R_{ij}{}^{k\ell} + c^2(J_i{}^l J_j{}^k - J_j{}^l J_i{}^k) \,, \tag{8.34}$$

where $R_{ij}{}^{k\ell}$ has been defined in equation 8.5.

It is not obvious, but it is possible to show that

$$\varepsilon_i{}^{jk} = -\varepsilon_i{}^{kj} \,. \tag{8.35}$$

8.1.3 Space-time connection and curvature

When computing the space-time connection using equation 8.10, we obtain

$$\begin{pmatrix} \underline{\Gamma}_{00}{}^{0} & \underline{\Gamma}_{0j}{}^{0} \\ \underline{\Gamma}_{i0}{}^{0} & \underline{\Gamma}_{ij}{}^{0} \end{pmatrix} = \begin{pmatrix} c^3\alpha\varphi & -c^2 h_j \\ -c^2 g_i & cJ_{ij}/\varphi \end{pmatrix} \tag{8.36}$$

$$\begin{pmatrix} \underline{\Gamma}_{00}{}^{k} & \underline{\Gamma}_{0j}{}^{k} \\ \underline{\Gamma}_{i0}{}^{k} & \underline{\Gamma}_{ij}{}^{k} \end{pmatrix} = \begin{pmatrix} -c^2 h^k\varphi^2 & cG_j{}^k\varphi \\ cJ_i{}^k\varphi & \Gamma_{ij}{}^k \end{pmatrix} , \tag{8.37}$$

where $\Gamma_{ij}{}^k$ was defined in equation 8.4.

When computing the components of the Riemann tensor using equation 8.13, we obtain

$$\begin{pmatrix} \underline{R}_{00}{}^{00} & \underline{R}_{0j}{}^{00} \\ \underline{R}_{i0}{}^{00} & \underline{R}_{ij}{}^{00} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \tag{8.38}$$

$$\begin{pmatrix} \underline{R}_{00}{}^{0\ell} & \underline{R}_{0j}{}^{0\ell} \\ \underline{R}_{i0}{}^{0\ell} & \underline{R}_{ij}{}^{0\ell} \end{pmatrix} = \begin{pmatrix} 0 & -c^2\gamma_j{}^l \\ c^2\gamma_i{}^l & -c\delta_{ij}{}^l/\varphi \end{pmatrix} , \tag{8.39}$$

$$\begin{pmatrix} \underline{R}_{00}{}^{k0} & \underline{R}_{0j}{}^{k0} \\ \underline{R}_{i0}{}^{k0} & \underline{R}_{ij}{}^{k0} \end{pmatrix} = \begin{pmatrix} 0 & c^2\gamma_j{}^k \\ -c^2\gamma_i{}^k & c\delta_{ij}{}^k/\varphi \end{pmatrix} , \tag{8.40}$$

$$\begin{pmatrix} \underline{R}_{00}{}^{k\ell} & \underline{R}_{0j}{}^{k\ell} \\ \underline{R}_{i0}{}^{k\ell} & \underline{R}_{ij}{}^{k\ell} \end{pmatrix} = \begin{pmatrix} 0 & c\varepsilon_j{}^{k\ell}\varphi \\ -c\varepsilon_i{}^{k\ell}\varphi & r_{ij}{}^{k\ell} \end{pmatrix} . \tag{8.41}$$

The Ricci tensor $\underline{R}_\alpha{}^\beta = \underline{R}_{\gamma\alpha}{}^{\beta\gamma}$ is given by

$$\begin{pmatrix} \underline{R}_{0}{}^{0} & \underline{R}_{0}{}^{j} \\ \underline{R}_{i}{}^{0} & \underline{R}_{i}{}^{j} \end{pmatrix} = \begin{pmatrix} c^2\gamma_k{}^k & -c\varepsilon_k{}^{jk}\varphi \\ -c\delta_{ki}{}^k/\varphi & c^2\gamma_i{}^j + r_{ki}{}^{jk} \end{pmatrix} , \tag{8.42}$$

and the scalar curvature $\underline{R} = \underline{R}_\sigma{}^\sigma$ by

$$\underline{R} = 2c^2\gamma_i{}^i + r_{ji}{}^{ij} . \tag{8.43}$$

The contracted torsion

$$\underline{S}_\alpha = \underline{S}_{\sigma\alpha}{}^\sigma \tag{8.44}$$

is given by

$$\begin{pmatrix} \underline{S}_0 \\ \underline{S}_i \end{pmatrix} = \begin{pmatrix} -cK_s{}^s\varphi \\ c^2 I_i + S_{si}{}^s \end{pmatrix} . \tag{8.45}$$

8.1.4 Recall: Bianchi identities

We have seen in the previous chapter how to define, for an arbitrary differentiable manifold (i.e., for any *geometrical* space), the *curvature* and the *torsion*. In general relativity, as we consider the "geometry" of the space-time, we also have to introduce its curvature and torsion. Let us denote them by $\underline{R}_{\alpha\beta\gamma}{}^\delta$ and $\underline{S}_{\alpha\beta}{}^\gamma$ respectively.

The contracted curvature is defined by

$$\underline{R}_{\alpha\beta} = \underline{R}_{\sigma\alpha\beta}{}^\sigma , \tag{8.46}$$

and is called the *Ricci tensor.* The scalar

$$\underline{R} = \underline{g}^{\rho\sigma}\underline{R}_{\rho\sigma} = \underline{R}_\sigma{}^\sigma , \tag{8.47}$$

is called the *scalar curvature*. The torsion tensor is antisymmetric,

$$\underline{S}_{\alpha\beta}{}^{\gamma} = -\underline{S}_{\beta\alpha}{}^{\gamma} \,, \tag{8.48}$$

the *contracted torsion* is defined by

$$\underline{S}_{\alpha} = \underline{S}_{\sigma\alpha}{}^{\sigma} \,, \tag{8.49}$$

and does not receive any particular name.

We have seen in the previous chapter that curvature and torsion satisfy, by their very definition, the *contracted Bianchi identities*

$$\underline{\nabla}_{\alpha}\Big(\underline{R}_{\beta}{}^{\alpha} - \frac{1}{2}\underline{\delta}_{\beta}{}^{\alpha}\underline{R}\Big) = \underline{S}_{\lambda\rho}{}^{\sigma}\left(\frac{1}{2}\underline{R}_{\beta\sigma}{}^{\lambda\rho} + \underline{\delta}_{\beta}{}^{\lambda}\underline{R}_{\sigma}{}^{\rho}\right) \,, \tag{8.50}$$

and

$$\underline{\nabla}_{\alpha}\Big(\underline{S}_{\beta\gamma}{}^{\alpha} + \underline{S}_{\beta}\delta_{\gamma}{}^{\alpha} - \underline{S}_{\gamma}\delta_{\beta}{}^{\alpha}\Big) = \Big(\underline{R}_{\beta\gamma} - \underline{R}_{\gamma\beta}\Big) + \underline{S}_{\sigma}\underline{S}_{\beta\gamma}{}^{\sigma} \,. \tag{8.51}$$

8.1.5 4-D field equations (Einstein-Cartan Equations)

The form of the contracted Bianchi identities 8.50–8.51 suggests to introduce the *Einstein tensor*

$$\underline{E}_{\beta}{}^{\alpha} = \underline{R}_{\beta}{}^{\alpha} - \frac{1}{2}\underline{\delta}_{\beta}{}^{\alpha}\underline{R} \tag{8.52}$$

and the *Cartan tensor*

$$\underline{C}_{\beta\gamma}{}^{\alpha} = \underline{S}_{\beta\gamma}{}^{\alpha} + \underline{S}_{\beta}\underline{\delta}_{\gamma}{}^{\alpha} - \underline{S}_{\gamma}\underline{\delta}_{\beta}{}^{\alpha} \,. \tag{8.53}$$

Then, the contracted Bianchi identities can be written

$$\underline{\nabla}_{\alpha}\underline{E}_{\beta}{}^{\alpha} = \underline{S}_{\lambda\rho}{}^{\sigma}\left(\frac{1}{2}\underline{R}_{\beta\sigma}{}^{\lambda\rho} + \underline{\delta}_{\beta}{}^{\lambda}\underline{R}_{\sigma}{}^{\rho}\right) \,, \tag{8.54}$$

and

$$\underline{\nabla}_{\alpha}\underline{C}_{\beta\gamma}{}^{\alpha} = \Big(\underline{R}_{\beta\gamma} - \underline{R}_{\gamma\beta}\Big) + \underline{S}_{\sigma}\ \underline{S}_{\beta\gamma}{}^{\sigma} \,. \tag{8.55}$$

Einstein demonstrated that a consistent gravitation theory can be obtained by identifying the tensor $\underline{E}_{\alpha}{}^{\beta}$ with the mass (i.e., stress-energy) tensor:

$$\boxed{\underline{E}_{\alpha}{}^{\beta} = \chi\ \underline{T}_{\alpha}{}^{\beta} \,,} \tag{8.56}$$

where the proportionality constant is

$$\chi = \frac{8\pi G}{c^4} \,. \tag{8.57}$$

As when Einstein developed his gravitation theory, the spin was not yet discovered, the theory did not account for spin. The Einstein equation 8.56 couples mass with space-time curvature. Today we understand that spin couples with space-time torsion:

$$\boxed{\underline{C}_{\alpha\beta}{}^{\gamma} = \chi\ \underline{M}_{\alpha\beta}{}^{\gamma} \,.} \tag{8.58}$$

We will later see (Section 8.3) that in order to get, in the classical limit, the right equations of conservation of angular momentum, the proportionality constant in the Cartan equation 8.58 *must* be the same than in the Einstein equation 8.56, i.e., the χ defined by equation 8.57. So, even if this Einstein-Cartan theory has two fundamental equations, there is only one fundamental constant.

8.1.6 Expressing the matter properties

Recall that from the space-time metric $\widehat{g}_{\alpha\beta}$ we have introduced the space metric g_{ij} and the scalar field φ , while from the space-time torsion $\underline{S}_{\alpha\beta}{}^{\gamma}$ we have introduced the space torsion $S_{ij}{}^{k}$ and the 3-D tensor fields I_i , $\widehat{J}_{ij}$, and $K_i{}^j$.

From those, we can express all the components of the space-time curvature $\underline{R}_{\alpha\beta\gamma}{}^{\delta}$ and of the space-time Einstein tensor $\underline{E}_{\alpha}{}^{\beta}$, and all the components of the space-time torsion $\underline{S}_{\alpha\beta}{}^{\gamma}$ and of the space-time Cartan tensor $\underline{C}_{\alpha\beta}{}^{\gamma}$. We have then expressions for the left-hand side of the Einstein-Cartan equations 8.56 and 8.58.

The Einstein equation

$$\underline{T}_{\alpha}{}^{\beta} = \frac{1}{\chi}\left(\underline{R}_{\alpha}{}^{\beta} - \frac{1}{2}\,\underline{\delta}_{\alpha}{}^{\beta}\underline{R}\right) , \tag{8.59}$$

allows to compute the material properties ρ , p_i , q^i , and $T_i{}^j$ (respectively mass density, linear momentum density, mass flux, and stress). This gives

$$\begin{pmatrix} \underline{T}_0{}^0 & \underline{T}_0{}^j \\ \underline{T}_i{}^0 & \underline{T}_i{}^j \end{pmatrix} = \begin{pmatrix} -\rho c^2 & -q^j c\varphi \\ c p_i/\varphi & T_i{}^j \end{pmatrix} , \tag{8.60}$$

where

$$\rho = \frac{1}{2\chi c^2} r_{k\ell}{}^{\ell k} , \tag{8.61}$$

$$p_i = -\frac{1}{\chi}\delta_{ki}{}^{k} , \tag{8.62}$$

$$q^j = +\frac{1}{\chi}\varepsilon_k{}^{jk} , \tag{8.63}$$

and

$$T_i{}^j = \frac{1}{\chi}\left(c^2(\gamma_i{}^j - \delta_i{}^j\gamma_k{}^k) + (r_{ki}{}^{jk} - \frac{1}{2}\delta_i{}^j r_{k\ell}{}^{\ell k})\right) . \tag{8.64}$$

The Cartan equation

$$\underline{M}_{\alpha\beta}{}^{\gamma} = \frac{1}{\chi}\left(\underline{S}_{\alpha\beta}{}^{\gamma} + \underline{S}_{\alpha}\underline{\delta}_{\beta}{}^{\gamma} - \underline{S}_{\beta}\underline{\delta}_{\alpha}{}^{\gamma}\right) , \tag{8.65}$$

allows to compute the material properties σ_{ij} , $M_{ij}{}^k$, t^i , and $\pi_i{}^j$ (respectively spin density, moment stress, unbalance, unbalance flux). This gives

$$\begin{pmatrix} \underline{M}_{00}{}^0 & \underline{M}_{0j}{}^0 \\ \underline{M}_{i0}{}^0 & \underline{M}_{ij}{}^0 \end{pmatrix} = \begin{pmatrix} 0 & c^2 t_j \\ -c^2 t_i & c\sigma_{ij}/\varphi \end{pmatrix} , \tag{8.66}$$

$$\begin{pmatrix} \underline{M}_{00}{}^k & \underline{M}_{0j}{}^k \\ \underline{M}_{i0}{}^k & \underline{M}_{ij}{}^k \end{pmatrix} = \begin{pmatrix} 0 & c\pi_j{}^k\varphi \\ -c\pi_i{}^k\varphi & M_{ij}{}^k \end{pmatrix} , \tag{8.67}$$

where

$$t_i = -\frac{1}{\chi c^2} S_{ji}{}^{j} , \tag{8.68}$$

$$\sigma_{ij} = \frac{1}{\chi}\widehat{J}_{ij} , \tag{8.69}$$

$$\pi_i{}^j = \frac{1}{\chi}\left(K_i{}^j - K_k{}^k\delta_i{}^j\right) , \tag{8.70}$$

and

$$M_{ij}{}^k = \frac{1}{\chi}\left(S_{ij}{}^k + S_{\ell i}{}^l\delta_j{}^k - S_{\ell j}{}^l\delta_i{}^k + c^2(I_i\delta_j{}^k - I_j\delta_i{}^k)\right) . \tag{8.71}$$

8.1.7 The "conservation equations"

From the previous expressions for the material properties of the continuous medium, ρ , p_i , t^i , and σ_{ij} , we can try to derive the corresponding "conservation equations". It is much simpler to directly use the contracted Bianchi identities

$$\underline{\nabla}_\alpha \left(\underline{R}_\beta{}^\alpha - \frac{1}{2} \underline{\delta}_\beta{}^\alpha \underline{R} \right) = \underline{S}_{\lambda\rho}{}^\sigma \left(\frac{1}{2} \underline{R}_{\beta\sigma}{}^{\lambda\rho} + \underline{\delta}_\beta{}^\lambda \underline{R}_\sigma{}^\rho \right) , \tag{8.72}$$

and

$$\nabla_\alpha \left(\underline{S}_{\beta\gamma}{}^\alpha + \underline{S}_\beta \underline{\delta}_\gamma{}^\alpha - \underline{S}_\gamma \underline{\delta}_\beta{}^\alpha \right) = \left(\underline{R}_{\beta\gamma} - \underline{R}_{\gamma\beta} \right) + \underline{S}_\sigma \ \underline{S}_{\beta\gamma}{}^\sigma , \tag{8.73}$$

which directly give

$$\frac{\partial \rho}{\partial \tau} + \nabla_i q^i = \kappa , \tag{8.74}$$

$$\frac{\partial p_i}{\partial \tau} + \nabla_j T_i{}^j = f_i , \tag{8.75}$$

$$\frac{\partial t_i}{\partial \tau} + \nabla_j \pi_i{}^j = p_i - q_i + \omega_i , \tag{8.76}$$

$$\frac{\partial \sigma_{ij}}{\partial \tau} + \nabla_k M_{ij}{}^k = T_{ij} - T_{ji} + \psi_{ij} , \tag{8.77}$$

where

$$\begin{aligned} \kappa = & -c^2 \Big(\rho J_i{}^i - h_i (p^i + q^i) \Big) - T_i{}^j J_j{}^i \\ & + \frac{1}{\chi} \left(-K_i{}^j r_{kj}{}^{ik} - \frac{1}{2} S_{ij}{}^k \varepsilon_k{}^{ij} + c^2 I_i \varepsilon_j{}^{ij} \right) , \end{aligned} \tag{8.78}$$

$$\begin{aligned} f_i = & +c^4 \rho h_i - c^2 \Big(-h_j T_i{}^j + J_{ji} q^j - G_i{}^j p_j + J_j{}^j p_i \Big) + \frac{1}{\chi} \Bigg(c^4 \Big(\gamma_i{}^j I_j - \gamma_j{}^j I_i \Big) \\ & + c^2 \left(-\frac{1}{2} \varepsilon_i{}^{jk} \widehat{J}_{jk} - \widehat{J}_{ij} \varepsilon_k{}^{jk} - \delta_{ij}{}^k K_k{}^j + K_i{}^j \delta_{kj}{}^k \right) + \frac{1}{2} r_{ij}{}^{k\ell} S_{k\ell}{}^j + S_{ij}{}^k (c^2 \gamma_k{}^j + r_{\ell k}{}^{j\ell}) \Bigg) , \end{aligned} \tag{8.79}$$

$$\begin{aligned} \omega_i = & -c^2 \Big(J_j{}^j t_i - G_i{}^j t_j - \sigma_{ij} h^j - \pi_i{}^j h_j \Big) - M_{ij}{}^k J_k{}^j \\ & + \frac{1}{\chi} \Big(-c^2 K_j{}^j I_i + K_i{}^j (c^2 I_j + S_{kj}{}^k) \Big) , \end{aligned} \tag{8.80}$$

and

$$\begin{aligned} \psi_{ij} = & +c^4 (t_i h_j - t_j h_i) - c^2 \Big(-(J_{ki} \pi_j{}^k - J_{kj} \pi_i{}^k) - (\sigma_{ik} G_j{}^k - \sigma_{jk} G_i{}^k) \Big) \\ & - c^2 \Big(J_k{}^k \sigma_{ij} - M_{ij}{}^k h_k \Big) + \frac{1}{\chi} \Big(-c^2 K_k{}^k \widehat{J}_{ij} + S_{ij}{}^k (c^2 I_k + S_{\ell k}{}^l) \Big) . \end{aligned} \tag{8.81}$$

The term κ corresponds to the rate of creation of mass density. The terms f_i and ψ_{ij} are respectively the force density and the torque density acting on the medium (they can also be named the rate of creation of linear momentum and of angular momentum respectively). See below for an interpretation of ω_i (and of t_i).

8.1.8 The simplest model with spin

We have seen that, to define a model of the universe, we can freely choose the fields φ , g_{ij} , I_i , $\widehat{J}_{ij}$, ${K_i}^j$, and ${S_{ij}}^k$. In standard gravitation theory (i.e., in the theory without spin), the fundamental fields are φ and g_{ij} , while the fields I_i , $\widehat{J}_{ij}$, ${K_i}^j$, and ${S_{ij}}^k$ identically vanish. This corresponds to a universe with mass and without spin. Let us be interested in the opposite case, where φ is constant (so the gravitational potential U is constant), and the space metric g_{ij} is that of an euclidean space, independent on time. The simplest model is then obtained when only the field $\widehat{J}_{ij}$ is non vanishing, while the three fields I_i , ${K_i}^j$, and ${S_{ij}}^k$ vanish.

Formula 8.69 shows that the field $\widehat{J}_{ij}$ is then just proportional to the spin density of the universe:

$$\widehat{J}_{ij} = \chi \sigma_{ij} . \tag{8.82}$$

From now on, let us use, instead of χ , its expression as a function of the Newtonian Gravitational constant G (as given by equation 8.57). Then, the previous equation writes

$$\widehat{J}_{ij} = \frac{8\pi G}{c^4} \sigma_{ij} . \tag{8.83}$$

An easy use of the formulas of section 8.1.6 allows then to express all the material properties of the universe as a function of the spin density:

$$\rho = \frac{\pi G}{c^4} \sigma_{ij} \sigma^{ij} , \tag{8.84}$$

$$p_i = q_i = -\frac{1}{2} \nabla_j {\sigma_i}^j , \tag{8.85}$$

and

$${T_i}^j = \frac{1}{2} \partial_\tau {\sigma_i}^j + \frac{\pi G}{c^2} {\delta_i}^j \sigma_{k\ell} \sigma^{k\ell} , \tag{8.86}$$

while the material fields t_i , ${\pi_i}^j$, and ${M_{ij}}^k$, all vanish.

Of particular interest are the right-hand side of the evolution equations 8.74 – 8.77:

$$\kappa = \frac{\pi G}{c^4} \partial_\tau (\sigma_{ij} \sigma^{ij}) , \tag{8.87}$$

$$f_i = \frac{\pi G}{c^2} \nabla_i (\sigma_{jk} \sigma^{jk}) , \tag{8.88}$$

$$\omega_i = 0 , \tag{8.89}$$

and

$$\psi_{ij} = \partial_\tau \sigma_{ij} . \tag{8.90}$$

In particular, we see that within a fluid with spin density σ_{ij} there is a *force density* f_i proportional to the squared spin density (equation 8.88). Like the Newtonian gravitational force, the gravitational force due to the spin density tends to concentrate matter, as is directed towards regions of high spin density.

If, like the mass-mass force, the spin-spin gravitational force is an attractive force (it tends to concentrate spin), unlike the mass-mass force it has vanishing range: it only depends on the local value of the spin density.

At the molecular (or larger) scale, where matter is concentrated in atoms surrounded by vacuo, the spin-spin gravitational interaction should not play any significant role.

On the contrary, such a force could play a major role in accounting for the stability of elementary particles. For instance, it is well known that simple models of an electron, can not explain its stability, as the electrostatic repulsive forces dominate, by orders of magnitude, the attractive gravitational forces. This is only true if we consider only the mass-mass gravitational forces. It is possible to imagine matter distributions where the spin gravitational force equilibrates other repulsive forces, like Coulomb's electrostatic repulsion in elementary models of an electron as a continuous medium. For instance, appendix 8.5.1 shows a simple "model of an electron" where the electrostatic repulsive forces are exactly compensated by the attractive spin-spin gravitational force just described.

Let us mention that Kerlick [1975], and O'Connell [1977] give a discussion on the spin-spin gravitational interaction discussed here, but based on Lagrangian arguments.

8.1.9 The gravitational forces

The gravitational force density acting in a fluid has been given by equation 8.79, with reference to equation 8.75.

We see that 8.79 has many terms: there are many diferent gravitational forces acting on a continuous medium.

Let us focus in the first term,

$$f_i = c^4 \rho h_i \,. \tag{8.91}$$

Equation 8.30 gives the definition of h_i :

$$h_i = g_i - I_i \,, \tag{8.92}$$

equation 8.31 gives the definition of $\gamma_i{}^j$:

$$\gamma_i{}^j = \frac{1}{c^2}\frac{\partial J_i{}^j}{\partial \tau} + \nabla_i h^j - c^2 g_i h^j + J_i{}^k G_k{}^j \,, \tag{8.93}$$

and one of the reversed form of the Einstein equations (equation 8.248) gives

$$\gamma_s{}^s = -\frac{\chi}{2}\left(\rho + \frac{1}{c^2} T_s{}^s\right) \,, \tag{8.94}$$

i.e.,

$$\gamma_s{}^s = -\frac{4\pi G}{c^4}\left(\rho + \frac{1}{c^2} T_s{}^s\right) \,. \tag{8.95}$$

For ordinary matter, the term $T_s{}^s/c^2$ can be neglected in front of ρ , the field I_i can be neglected in front of g_i , and the time derivative of $J_i{}^j$ and the quadratic terms can be neglected. This gives for the gravitational force density

$$f_i = c^4 \rho g_i \,, \tag{8.96}$$

where

$$\nabla_i g^i = -\frac{4\pi G}{c^4}\rho \,, \tag{8.97}$$

and we should remember that g_i was defined by (equation 6.14)

$$g_i = -\frac{1}{c^2}\nabla_i U \,. \tag{8.98}$$

The force density 8.96 is the standard Newtonian gravitational force density. The field g_i is the Newtonian gravitational field (times c^{-4}), satisfying the Poisson's equation 8.97, and U is the classical Newtonian potential (times c^{-2}).

So far, for the gravitational force density in "ordinary matter", where only the "massive" properties of the medium are taken into account.

The previous section has shown that when taking into account only the spin properties of the medium, we have the gravitational force density

$$f_i = \frac{\pi G}{c^2} \nabla_i (\sigma_{jk} \sigma^{jk}) \, . \tag{8.99}$$

This means that, among the many forces acting in a fluid in gravitational interaction, two are particularly simple, depending respectively on the mass and spin only:

$$f_i = c^4 \rho g_i + \frac{\pi G}{c^2} \nabla_i (\sigma_{jk} \sigma^{jk}) \, . \tag{8.100}$$

8.1.10 3-D evolution equations for a medium locally at rest

What we would have liked to solve is the following problem of initial conditions: given ρ , p_i , t_i , σ_{ij} , g_{ij} (and some other fields) at $t = t_0$, and assuming given some "constitutive equations" that allow us to compute q^i , ${T_i}^j$, ${\pi_i}^j$, and ${M_{ij}}^k$ at any time as a function of the "history" of the medium, then, it is possible to extrapolate the values of ρ , p_i , t_i , σ_{ij} , g_{ij} (and some other fields) at all times $t > t_0$.

Instead, what we have solved is: given g_{ij} , $\widetilde{G}_{ij}$, σ_{ij} , and t_i , at $t = t_0$, and assuming given some "constitutive equations" that allow us to compute $\Delta p^i \equiv p^i - q^i$, ${T_i}^j$, ${\pi_i}^j$, and ${M_{ij}}^k$ at any time as a function of the "history" of the medium, as well as assuming given U at all times, then, it is possible to extrapolate the values of g_{ij} , $\widetilde{G}_{ij}$, σ_{ij} , and t_i , at all times $t > t_0$.

This is shown in Appendix 8.5.3.

8.2 Moving media

The equations seen so far are valid in a coordinate system attached to the continuous medium, with respect to which the medium is "at rest". Usually we want to measure forces in the "laboratory" reference system, with respect to which the medium is moving. This introduces many more terms in the equations ("convective terms") and many more forces similar to the magnetic force in a moving electric medium.

Let us start by introducing the velocity.

8.2.1 Measuring (improper) velocities

Assume we have, in the laboratory, a rod of length $\Delta\ell$ and a clock, both at rest.

If a particle passes by, and it takes a time $\Delta\tau$ (in the laboratory clock) to the particle to travel between the two extremities of the rod, then, by definition, the velocity of the particle is

$$v = \lim_{\Delta\ell \to 0} \frac{\Delta\ell}{\Delta\tau} \, . \tag{8.101}$$

How small has to be in practice $\Delta\ell$ for an accurate enough estimation of the velocity will depend of the regularity of the displacement of the particle.

Should the particle travel at the speed of light, then, by the operational definition of distance, $\Delta\ell = c\Delta\tau$, and $v = c$. For any other particle, $v < c$.

8.2.2 Defining the 4-velocity

Consider, again, an arbitrary particle. In section 6.5, we have parameterized the space-time trajectory of a particle by an arbitrary parameter λ . Instead, we can now choose the particle's proper time:

$\underline{x}^\alpha(\tau)$. The tangent to the trajectory is called the 4-velocity:

$$\underline{U}^\alpha = \frac{d\underline{x}^\alpha}{d\tau} \, . \tag{8.102}$$

It satisfies

$$\begin{aligned} \underline{g}_{\alpha\beta}\underline{U}^\alpha\underline{U}^\beta &= \underline{g}_{\alpha\beta}\frac{d\underline{x}^\alpha}{d\tau}\frac{d\underline{x}^\beta}{d\tau} \\ &= \frac{\underline{g}_{\alpha\beta}\; d\underline{x}^\alpha\; d\underline{x}^\beta}{d\tau^2} \\ &= \frac{-c^2 d\tau^2}{d\tau^2} \, , \end{aligned} \tag{8.103}$$

i.e.,

$$\underline{g}_{\alpha\beta}\; \underline{U}^\alpha\; \underline{U}^\beta = -c^2 \, . \tag{8.104}$$

It is important to realize that the definition of four velocity uses the proper time *of the particle*, while the 3-D definition of (improper) velocity in section 8.2.1 used the proper time *of the observer*.

Let us consider a particle that in the time interval dt has made the space displacement dx^i , and let us denote by $d\tau_{obs}$ and $d\tau_{part}$ the proper time elapsed as measured by the particle itself or by the observer respectively.

The proper time for the observer is, from equation 6.22,

$$d\tau_{obs} = \varphi dt \, , \tag{8.105}$$

while the proper time for the particle is, from equation 6.19,

$$\begin{aligned} d\tau_{part} &= \sqrt{\varphi^2 dt^2 - \frac{1}{c^2} dx_i dx^i} = \varphi\; dt\; \sqrt{1 - \frac{1}{c^2\varphi^2}\frac{dx_i}{dt}\frac{dx^i}{dt}} \\ &= \varphi\; dt\; \sqrt{1 - \frac{1}{c^2}\frac{dx_i}{d\tau_{obs}}\frac{dx^i}{d\tau_{obs}}} \, . \end{aligned}$$

The 3-velocity was defined in section 8.2.1 as

$$v^i = \frac{dx^i}{d\tau_{obs}} \, . \tag{8.106}$$

Then,

$$d\tau_{part} = \varphi\; dt\; \sqrt{1 - \frac{v_i\; v^i}{c^2}} \, , \tag{8.107}$$

or, if we introduce the standard notations

$$\beta^i = \frac{v^i}{c} \qquad\qquad \beta^2 = v_i v^i \, , \tag{8.108}$$

we obtain the equation giving the relation between the proper time of a particle, and the coordinate and proper times of the observer:

$$d\tau_{part} = \varphi\; dt\sqrt{1-\beta^2} = d\tau_{obs}\sqrt{1-\beta^2} \, , \tag{8.109}$$

where we have used 6.22.

If, as it is customary, we introduce the "gamma factor"

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \, , \tag{8.110}$$

then the previous equation gives

$$\gamma \; d\tau_{part} = \varphi \; dt = d\tau_{obs} \, . \tag{8.111}$$

As, obviously, $\gamma > 1$, equations 8.109 and 8.111 show that the proper time for the particle is always smaller that the proper time of the observer: this is the relativistic "contraction of time", seen here in the context of general relativity.

Using the previous equations we easily obtain the components of the particle's 4-velocity in terms of the 3-velocity measured by the observer:

$$\begin{pmatrix} \underline{U}^0 \\ \underline{U}^i \end{pmatrix} = \begin{pmatrix} \gamma c/\varphi \\ \gamma v^i \end{pmatrix} \, . \tag{8.112}$$

8.2.3 Lorentz transformation

When two observers cross each other, they may want to compare the components of the tensors in their respective reference systems. In particular, the material properties (mass,...)are usually measured in a reference system where the matter is at rest, and from those measured values we may need to predict the values that should be measured in reference systems where the matter is in movement.

It is the Lorentz transformation that performs such a task. If the components of a space-time vector are $\underline{V}^\alpha$ in some reference system, they will be

$$\underline{V}'^{\alpha'} = \underline{\Lambda}^{\alpha'}{}_{\alpha} \underline{V}^\alpha \, , \tag{8.113}$$

in another reference system whose relative velocity with respect to the first reference system is $v^i = c\beta^i$. Here, the $\underline{\Lambda}^{\alpha'}{}_{\alpha}$ are the components of the "Lorentz transformation". It is given by

$$\begin{pmatrix} \underline{\Lambda}^0{}_0 & \underline{\Lambda}^0{}_j \\ \underline{\Lambda}^i{}_0 & \underline{\Lambda}^i{}_j \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta_j/\varphi \\ \gamma\beta^i\varphi & \delta^i{}_j + \frac{(\gamma-1)}{\beta^2}\beta^i\beta_j \end{pmatrix} \, . \tag{8.114}$$

It can be introduced in many ways. For instance, it may be said that it is a "space-time *rotation*": the length of any vector has to remain unchanged. Equivalently me may say that it has to leave the components of the metric tensor invariant.

Notice that, although the Lorentz transformation is used in the theory of special relativity, the expression given above is fully valid within the framework of the general relativistic theory developed here.

It is easily seen that the inverse transformation is obtained by simply reversing the velocity ($\beta_i \to -\beta_i$).

Note: Check if I say somewhere that the product of the Lorentz transformation timest its transpose gives the Kronecker tensor.

Note: Check if I say somewhere that the determinant of the Lorentz transformation equals -1.

For the sake of completeness, let us mention explicitly that the components of second and third order tensors change respectively according to

$$\underline{T}'^{\alpha'\beta'} = \underline{\Lambda}^{\alpha'}{}_{\alpha} \underline{\Lambda}^{\beta'}{}_{\beta} \underline{T}^{\alpha\beta} \tag{8.115}$$

and

$$\underline{M}'^{\alpha'\beta'\gamma'} = \underline{\Lambda}^{\alpha'}{}_{\alpha} \underline{\Lambda}^{\beta'}{}_{\beta} \underline{\Lambda}^{\gamma'}{}_{\gamma} \underline{M}^{\alpha\beta\gamma} \, . \tag{8.116}$$

For instance, if in the proper reference system, the mass tensor (stress-energy tensor) is given by

$$\begin{pmatrix} \underline{T}_0{}^0 & \underline{T}_0{}^j \\ \underline{T}_i{}^0 & \underline{T}_i{}^j \end{pmatrix} = \begin{pmatrix} -\rho c^2 & -q^j c\varphi \\ c p_i/\varphi & T_i{}^j \end{pmatrix} , \tag{8.117}$$

then, in a system moving with velocity v^i with respect to it, it is given by

$$\begin{pmatrix} \underline{T}_0{}^0 & \underline{T}_0{}^j \\ \underline{T}_i{}^0 & \underline{T}_i{}^j \end{pmatrix} = \begin{pmatrix} -\widetilde{\rho} c^2 & -\widetilde{q}^j c\varphi \\ c \widetilde{p}_i/\varphi & \widetilde{T}_i{}^j \end{pmatrix} , \tag{8.118}$$

where

$$\begin{aligned}
\widetilde{\rho} &= \gamma^2 \left(\rho + \frac{1}{c^2}(v_k p^k + v_l q^l) + \frac{1}{c^4} v_k v_l T^{k\ell} \right) , \\
\widetilde{p}^i &= \gamma \left(p^i + \frac{1}{c^2} v_l T^{i\ell} + \left(\gamma\rho + \frac{\gamma}{c^2} v_l q^l + \frac{\gamma-1}{v^2} v_k p^k + \frac{1}{c^2}\frac{\gamma-1}{v^2} v_k v_l T^{k\ell} \right) v^i \right) , \\
\widetilde{q}^j &= \gamma \left(q^j + \frac{1}{c^2} v_k T^{kj} + \left(\gamma\rho + \frac{\gamma}{c^2} v_k p^k + \frac{\gamma-1}{v^2} v_l q^l + \frac{1}{c^2}\frac{\gamma-1}{v^2} v_k v_l T^{k\ell} \right) v^j \right) , \\
\widetilde{T}^{ij} &= T^{ij} + \left(\frac{\gamma-1}{v^2} v_l T^{i\ell} + \gamma p^i \right) v^j + \left(\frac{\gamma-1}{v^2} v_k T^{kj} + \gamma q^j \right) v^i \\
&+ \left(\gamma\rho + \frac{\gamma-1}{v^2}(v_k p^k + v_l q^l) + \frac{(\gamma-1)^2}{v^4} v_k v_l T^{k\ell} \right) v^i v^j .
\end{aligned}$$

Analogous (but more lengthy) expressions are found for the spintensor.

Later on, we are going to be interested in the limit $v^i \ll c$. This obviously gives

$$\begin{aligned}
\widetilde{\rho} &= \rho , \\
\widetilde{p}^i &= p^i + \rho v^i , \\
\widetilde{q}^j &= q^j + \rho v^j , \\
\widetilde{T}^{ij} &= T^{ij} + p^i v^j + q^j v^i + \rho v^i v^j .
\end{aligned}$$

Similarly, in the limit $v^i \ll c$ we obtain for the spin tensor

$$\begin{aligned}
\widetilde{t}^i &= t^i , \\
\widetilde{\pi}^{ij} &= \pi^{ij} + t^i v^j , \\
\widetilde{\sigma}^{ij} &= \sigma^{ij} + t^i v^j - t^j v^i , \\
\widetilde{M}^{ijk} &= M^{ijk} + (\pi^{ik} v^j - \pi^{jk} v^i) + (\sigma^{ij} + t^i v^j - t^j v^i) v^k ,
\end{aligned}$$

and similar expressions for all other tensors involved.

8.2.4 3-D evolution equations for moving media (low velocities)

After a Lorentz transformation of all the fields appearing in the evolution equations of section 8.1.10, we obtain exact 3-D evolution equations, valid for any value of the velocity of the medium.

The equations are easy to obtain, but so cumbersome, that we prefer not to explicitly list them here. That's where the advantage of 4-D notations becomes obvious. But the advantage of the the 3-D notations is that while the 3-D space has curvature and torsion, the geometry of the 4-D space-time is no more into consideration: space-time curvature and torsion have been replaced by ordinary 3-D forces. We emphasize that we are talking about *exact* expressions.

There is a second reason why we have chosen not to explicitly list the exact 3-D equations here.

At the right-hand side of the "conservation equations" 8.74–8.77, there are the "source terms" κ , f_i , ψ_{ij} , and ω_i , given in equations 8.78–8.81. It is very interesting to look at the expressions of these terms after a Loretz transformation has added all the terms depending on the velocity of the medium. In particular, the force density term f_i shows all the gravitational forces acting in a continuous medium under gravitational interaction — and there are many —. For instance, besides the classical Newtonian force density, there is the spin-spin gravitational force density already discussed in section 8.1.9. Adding the terms depending on the velocity of the medium will add a lot of terms, like in electromagnetism, where if the medium is at rest, there is only the electric force, while for a moving medium the magnetic force appears.

The second reason why we have chosen not to explicitly list the exact 3-D equations is that we would have liked to give the expression of these forces (in particular) as a function of some "fundamental" quantities, instead as giving them as a function of the quatities appearing in equations 8.78–8.81. Our hope was to discover what those fundamental quantities could be when solving the problem of initial conditions, where, in addition to the material quantities ρ , p_i , t_i , and σ_{ij} , some other quatities should have been given at $t = t_0$ to have uniqueness of the evolution of the system.

The evolution theorem in Appendix 8.5.3 is not as fundamental as we have been looking for. We have the inpression of having missed to recognize those fundamental quantities, and, therefore, we have not made the effort of writing down explicitly all the source terms in the 3-D conservation equations and interpreting them physically.

Let us now turn to the simplest problem where we are only interested in the limit $v^i \ll c$. The conservation equations 8.78–8.81 become (replacing the proper time τ by the Newtonian time t),

$$\frac{\partial \rho}{\partial t} + \nabla_i (q^i + \rho v^i) = \kappa \,, \tag{8.119}$$

$$\frac{\partial}{\partial t}(p_i + \rho v_i) + \nabla_j \left((\rho v_i + p_i)\, v^j + v_i q^j + T_i{}^j \right) = f_i \,, \tag{8.120}$$

$$\frac{\partial t_i}{\partial t} + \nabla_j (t_i v^j + \pi_i{}^j) = (p_i - q_i) + \omega_i \,, \tag{8.121}$$

$$\frac{\partial}{\partial t}(\sigma_{ij} + t_i v_j - t_j v_i) + \nabla_k \left((\sigma_{ij} + t_i v_j - t_j v_i)\, v^k + \pi_i{}^k v_j - \pi_j{}^k v_i + M_{ij}{}^k \right) = (T_{ij} - T_{ji}) + \psi_{ij} \,, \tag{8.122}$$

where the source terms are, again, long to express, but easy to obtain (Comment: say here that the source terms are, in classical physics given in an ad-hoc way).

8.2.5 Global conservation equations

Comment: say here that the local conservation equations 8.122 can, equivalently, be written as global conservation equations. Considering a volume $\mathcal{V}(t)$ attached to the medium (comment: explain), with surface $\mathcal{S}(t)$ these equations can be written

$$\begin{aligned} \frac{\partial M}{\partial t} + Q(t) &= K(t) \\ \frac{\partial \mathbf{P}}{\partial t} + \mathbf{S}(t) &= \mathbf{F}(t) \\ \frac{\partial \mathbf{T}}{\partial t} + \mathbf{A}(t) &= \mathbf{Z}(t) \\ \frac{\partial \mathcal{D}}{\partial t} + \mathcal{G}(t) &= \mathcal{C}(t) \,, \end{aligned} \tag{8.123}$$

where I have defined the four fundamental conserved quantities, the *mass*

$$M(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\,\rho(\mathbf{x},t)\,, \tag{8.124}$$

the *linear momentum*

$$\mathbf{P}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\left(p_i(\mathbf{x},t) + \rho(\mathbf{x},t)\,v_i(\mathbf{x},t)\right)\mathbf{e}^i(\mathbf{x})\,, \tag{8.125}$$

the *unbalance*

$$\mathbf{T}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\,t_i(\mathbf{x},t)\,\mathbf{e}^i(\mathbf{x})\,, \tag{8.126}$$

and the *angular momentum*

$$\mathcal{D}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\left[\sigma_{ij}(\mathbf{x},t) + \left(t_i(\mathbf{x},t)v_j(\mathbf{x},t) - t_j(\mathbf{x},t)v_i(\mathbf{x},t)\right)\right]\mathbf{e}^i(\mathbf{x})\otimes\mathbf{e}^j(\mathbf{x})\,, \tag{8.127}$$

where I have also defined four "external sources," the *rate of creation of mass*

$$K(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\,\kappa(\mathbf{x},t)\,, \tag{8.128}$$

the *force* (or *rate of creation of linear momentum*)

$$\mathbf{F}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\,f_i(\mathbf{x},t)\,\mathbf{e}^i(\mathbf{x})\,, \tag{8.129}$$

the *rate of creation of unbalance*

$$\mathbf{Z}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\left[\left(p_i(\mathbf{x},t) - q_i(\mathbf{x},t)\right) + \omega_i(\mathbf{x},t)\right]\mathbf{e}^i(\mathbf{x})\,, \tag{8.130}$$

and the *torque* (or *rate of creation of angular momentum*)

$$\mathcal{C}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\left[\left(T_{ij}(\mathbf{x},t) - T_{ji}(\mathbf{x},t)\right) + \psi_{ij}(\mathbf{x},t)\right]\mathbf{e}^i(\mathbf{x})\otimes\mathbf{e}^j(\mathbf{x})\,, \tag{8.131}$$

and where, finally, I have defined four "internal sources" (note that these are surface integrals), the *energy(?) heat(?) mass(?) flux*

$$Q(t) = \int_{\mathcal{S}(t)} dS(\mathbf{x})\,\varphi(\mathbf{x},t)\,, \tag{8.132}$$

the *traction (stress flux(?))*

$$\mathbf{S}(t) = \int_{\mathcal{S}(t)} dS(\mathbf{x})\left(\varphi(\mathbf{x},t)\,v_i(\mathbf{x},t) + s_i(\mathbf{x},t)\right)\mathbf{e}^i(\mathbf{x})\,, \tag{8.133}$$

the *unbalance flux (?)*

$$\mathbf{A}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\,a_i(\mathbf{x},t)\,\mathbf{e}^i(\mathbf{x})\,, \tag{8.134}$$

and the *angular momentum flux (?)*

$$\mathcal{G}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\left[\left(a_i(\mathbf{x},t)v_j(\mathbf{x},t) - a_j(\mathbf{x},t)v_i(\mathbf{x},t)\right) + d_{ij}(\mathbf{x},t)\right]\mathbf{e}^i(\mathbf{x})\otimes\mathbf{e}^j(\mathbf{x})\,, \tag{8.135}$$

where I have introduced four fluxes (?), the *xxx*

$$\varphi(\mathbf{x},t) = q^i(\mathbf{x},t)\, n_i(\mathbf{x},t)\,, \tag{8.136}$$

the *traction*

$$s_i(\mathbf{x},t) = T_i{}^j(\mathbf{x},t)\, n_j(\mathbf{x},t)\,, \tag{8.137}$$

the *xxx*

$$a_i(\mathbf{x},t) = \pi_i{}^j(\mathbf{x},t)\, n_j(\mathbf{x},t)\,, \tag{8.138}$$

and the *xxx*

$$d_{ij}(\mathbf{x},t) = M_{ij}{}^k(\mathbf{x},t)\, n_k(\mathbf{x},t)\,. \tag{8.139}$$

8.2.6 Global conservation equations in Euclideanspaces with Cartesian coordinates

Comment: This is certainly not the good place for this section, but I have to give somewhere the equivalent of the equations of the previous section for Euclidean spaces with Cartesian coordinates:

$$\begin{aligned} \frac{\partial M}{\partial t} + Q(t) &= K(t) \\ \frac{\partial P_i}{\partial t} + S_i(t) &= F_i(t) \\ \frac{\partial T_i}{\partial t} + A_i(t) &= Z_i(t) \\ \frac{\partial D_{ij}}{\partial t} + G_{ij}(t) &= C_{ij}(t)\,, \end{aligned} \tag{8.140}$$

$$M(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\, \rho(\mathbf{x},t)\,, \tag{8.141}$$

$$P_i(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x}) \Big(p_i(\mathbf{x},t) + \rho(\mathbf{x},t)\, v_i(\mathbf{x},t) \Big)\,, \tag{8.142}$$

$$T_i(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\, t_i(\mathbf{x},t)\,, \tag{8.143}$$

$$D_{ij}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x}) \Big[\sigma_{ij}(\mathbf{x},t) + \Big(t_i(\mathbf{x},t) v_j(\mathbf{x},t) - t_j(\mathbf{x},t) v_i(\mathbf{x},t) \Big) \Big]\,, \tag{8.144}$$

$$K(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\, \kappa(\mathbf{x},t)\,, \tag{8.145}$$

$$F_i(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\, f_i(\mathbf{x},t)\,, \tag{8.146}$$

$$Z_i(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x}) \Big[\Big(p_i(\mathbf{x},t) - q_i(\mathbf{x},t) \Big) + \omega_i(\mathbf{x},t) \Big]\,, \tag{8.147}$$

$$C_{ij}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x}) \Big[\Big(T_{ij}(\mathbf{x},t) - T_{ji}(\mathbf{x},t) \Big) + \psi_{ij}(\mathbf{x},t) \Big]\,, \tag{8.148}$$

$$Q(t) = \int_{\mathcal{S}(t)} dS(\mathbf{x})\, \varphi(\mathbf{x},t)\,, \tag{8.149}$$

$$S_i(t) = \int_{\mathcal{S}(t)} dS(\mathbf{x}) \Big(\varphi(\mathbf{x},t)\, v_i(\mathbf{x},t) + s_i(\mathbf{x},t) \Big)\,, \tag{8.150}$$

$$A_i(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x})\, a_i(\mathbf{x},t)\,, \tag{8.151}$$

$$G_{ij}(t) = \int_{\mathcal{V}(t)} dV(\mathbf{x}) \left[\Big(a_i(\mathbf{x},t) v_j(\mathbf{x},t) - a_j(\mathbf{x},t) v_i(\mathbf{x},t) \Big) + d_{ij}(\mathbf{x},t) \right] , \tag{8.152}$$

$$\varphi(\mathbf{x},t) = q^i(\mathbf{x},t)\, n_i(\mathbf{x},t)\,, \tag{8.153}$$

$$s_i(\mathbf{x},t) = T_i{}^j(\mathbf{x},t)\, n_j(\mathbf{x},t)\,, \tag{8.154}$$

$$a_i(\mathbf{x},t) = \pi_i{}^j(\mathbf{x},t)\, n_j(\mathbf{x},t)\,, \tag{8.155}$$

$$d_{ij}(\mathbf{x},t) = M_{ij}{}^k(\mathbf{x},t)\, n_k(\mathbf{x},t)\,. \tag{8.156}$$

8.2.7 Ordinary matter

A simple, but quite general, model of matter is obtained when $p^i = q^i = 0$, $t^i = 0$, and $\pi_i{}^j = 0$. We talk about "ordinary matter". For instance, $p^i = 0$ means that the only linear momentum density in the medium is that of convective origin, ρv^i , and that there are no "currents" (like in an electromagnetic medium).

Then, the equations above simplify to

$$\frac{\partial \rho}{\partial t} + \nabla_i(\rho v^i) = \kappa\,, \tag{8.157}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \nabla_j(\rho v_i v^j + T_i{}^j) = f_i\,, \tag{8.158}$$

$$\frac{\partial \sigma_{ij}}{\partial t} + \nabla_k(\sigma_{ij} v^k + M_{ij}{}^k) = (T_{ij} - T_{ji}) + \psi_{ij}\,. \tag{8.159}$$

Here, to match classical notations, we have to define

$$\tau^j{}_i = -T_i{}^j \tag{8.160}$$

and

$$m^k{}_{ij} = -M_{ij}{}^k\,. \tag{8.161}$$

The equations above then give

$$\frac{\partial \rho}{\partial t} + \nabla_i(\rho v^i) = \kappa\,, \tag{8.162}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \nabla_j(\rho v_i v^j - \tau^j{}_i) = f_i\,, \tag{8.163}$$

$$\frac{\partial \sigma_{ij}}{\partial t} + \nabla_k(\sigma_{ij} v^k - m^k{}_{ij}) = (\tau_{ji} - \tau_{ij}) + \psi_{ij}\,. \tag{8.164}$$

8.2.8 Interpretation of the conserved quantities

The first conservation equation (8.119) corresponds to the conservation of mass (or, equivalently, of energy). ρ being the mass density, the vector ρv^i gives, when integrated on a surface, the mass crossing the surface per unit of time due to the convective motion of the medium. Traditionally, this vector ρv^i is called the (convective) *mass (*or energy*) flux*. The vector q^i is then the *intrinsic mass flux*, i.e, the mass flux not due to convection. For instance, if there is an electromagnetic field, then, independently of the motion of the medium, there will be an electromagnetic mass (or energy) flux (the Poynting vector), and the total mass flux will be the sum $\rho v^i + q^i$.

The second conservation equation (8.120) corresponds to the conservation of linear momentum. While ρv^i is the linear momentum density due to convection (i.e, due to the motion of the medium),

p^i is the *intrinsic linear momentum density*, an example being obtained when considering a medium with electric currents. The term $T_i{}^j$ is the stress tensor, and is sometimes called the (intrinsic) linear momentum flux.

Let's skip for a moment equation 8.121 and give the interpretation of equation 8.122. It obviously corresponds to the conservation of angular momentum. While the antisymmetric tensor σ_{ij} represents the *spin density*, the tensor $M_{ij}{}^k$ is the moment stress (comment: explain). The source of spin is (among others) the antisymmetric part of the stress. Comment: say that this is classical for micropolar media. We see that the total spin density at a point of a continuous medium equals the intrinsic spin density, σ_{ij} , and a convective part, $t_i v_j - t_j v_i$. As this is non classical, let us examine it with some care.

If instead of using the antisymmetric tensor σ_{ij} to represent the spin density, we use its dual vector

$$s^i = \frac{1}{2}\varepsilon^{ijk}\sigma_{jk} \, , \tag{8.165}$$

the total spin density is given by the vector

$$\mathbf{S} = \mathbf{s} + \mathbf{t} \times \mathbf{v} \, , \tag{8.166}$$

i.e, the total spin vector equals the intrinsic spin density plus a convective part. This convective part equals the vector product of the material property described by the vector $\mathbf{t}$ by the velocity $\mathbf{v}$.

A very imperfect analogy of this material property corresponds to a wheel attached to an axle but able to run and glide on a plane. If we associate to the wheel a vector perpendicular to the plane (and to the axle), and with magnitude equal to the ratio (moment of inertia/radius of the wheel), then, the kinetic moment of the wheel, when running at velocity $\mathbf{v}$, equals the vector product of the two vectors.

Comment: make a drawing here.

Equation 8.121 says that this property, described by the vector $\mathbf{t}$, is conserved. Thus, we have four conserved quantities: the mass, the linear and angular momentum, and the vector $\mathbf{t}$. Although it is not common that a theory predicts the conservation of a quantity not yet interpreted, this conservation equation has not received much attention so far. Comment: mention here Halbwachhhhsss (balourd, unbalance) and the personnal communication with Vigier.

Equation 8.121 also says that the source of $\mathbf{t}$ is the difference between the linear momentum density $\mathbf{p}$ anf the energy flux $\mathbf{q}$. For instance, as shown in appendix xxx, an electromagnetic field has a linear momentum density

$$\mathbf{p} = \mathbf{D} \times \mathbf{B} \tag{8.167}$$

and an energy flux

$$\mathbf{q} = \frac{1}{c^2}\mathbf{E} \times \mathbf{H} \, . \tag{8.168}$$

In vacuo,

$$\varepsilon = \varepsilon_0 \tag{8.169}$$

and

$$\mu = \mu_0 \, , \tag{8.170}$$

and, as

$$\varepsilon_0\mu_0 = \frac{1}{c^2} \, , \tag{8.171}$$

the linear momentum density equals the energy flux:

$$\mathbf{p} = \mathbf{q} \, . \tag{8.172}$$

In a general medium, the linear momentum density is not equal to the energy flux, and there is a source for the quantity $\mathbf{t}$.

8.3 Classical limit (low velocities and weak fields)

In the equations of the previous section, the fields were not assumed to be weak. If they are, then we obtain the classical limit, where the physical 3-D space is Euclidean, the time is Newtonian, and from the many gravitational forces acting in a continuous medium, only a few are assumed to be significant.

We say that the gravitational fields are *weak* when the equations 8.162–8.164 of previous section can be approximated by

$$\frac{\partial \rho}{\partial t} + \nabla_j(\rho v^j) = 0\,, \tag{8.173}$$

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_j(\rho v_i v^j - \tau^j{}_i) = c^4 \rho g_i + \frac{\pi G}{c^2} \nabla_i(\sigma_{jk}\sigma^{jk})\,, \tag{8.174}$$

and

$$\frac{\partial \sigma_{ij}}{\partial t} + \nabla_k(\sigma_{ij} v^k - m^k{}_{ij}) = \tau_{ij} - \tau_{ji}\,, \tag{8.175}$$

where we have kept in the right-hand terms only the classical Newtonian gravitational force density, end the spin-spin force density analysed above.

Note that, should we have taken, instead of equations 8.56 and 8.58:

$$\underline{E}_\alpha{}^\beta = \chi\ \underline{T}_\alpha{}^\beta\,, \tag{8.56 again}$$

and

$$\underline{C}_{\alpha\beta}{}^\gamma = \chi\ \underline{M}_{\alpha\beta}{}^\gamma\,, \tag{8.58 again}$$

a different coupling constant:

$$\underline{E}_\alpha{}^\beta = \chi\ \underline{T}_\alpha{}^\beta\,, \tag{8.176}$$

and

$$\underline{C}_{\alpha\beta}{}^\gamma = \chi'\ \underline{M}_{\alpha\beta}{}^\gamma\,, \tag{8.177}$$

then, equation 8.175 would have been, instead,

$$\frac{\partial \sigma_{ij}}{\partial t} + \nabla_k(\sigma_{ij} v^k - m^k{}_{ij}) = \frac{\chi}{\chi'}\,(\tau_{ij} - \tau_{ji})\ . \tag{8.178}$$

If we want to obtain, in this classical limit, the classical equation 7.3, we are forced to take

$$\chi = \chi'\,, \tag{8.179}$$

i.e., to take the same coupling constant between space-time curvature and mass (Einstein equation) that between space-time torsion and spin (Cartan equation).

8.4 Classical elasticity

All classical (i.e., non relativistic) media are assumed to satisfy the equations of the dynamics of continous media 8.173–8.175.

The stress $\tau^j{}_i$ and the moment stress $m^k{}_{ij}$ are defined ad-hoc, as functions of all the past history of the medium. These are the “constitutive equations” that define the “rheology” of the medium. Different choices of rheology allow the equations describing the dynamics of continous media to describe, for instance, fluids, plastics, or solids.

The elastic approximation consists in assuming that the stress $T_i{}^j$ and the moment stress $M_{ij}{}^k$ depend, at any time t, only on the state of *deformation* of the “solid” at time t.

To define deformation, we have to refer to an "undeformed state" of the solid, where, by definition, stress and moment stress vanish. Usually, the undeformed state is assumed to be the state of the medium at some initial time t_0 .

So far, we have used, as fundamental variable for describing the movement of the particles of the continuous medium, the velocity $v^i(\mathbf{x}, t)$. Now, instead, we consider the *displacement* field $u^i(\mathbf{x}, t)$ as giving the total displacement of the "particle" that is at point $(\mathbf{x})$ at time t with respect to the position the particle had at time t_0 .

The relation

$$v^i(\mathbf{x}, t) = \frac{\partial u^i}{\partial t}(\mathbf{x}, t) \tag{8.180}$$

is not valid in general: While $v^i(\mathbf{x}, t)$ represents the velocity at time t of the particle that *is* at point $\{\mathbf{x}\}$, $(\partial u^i/\partial t)(\mathbf{x}, t)$ represents the velocity at time t of the particle that *was* at point $\{\mathbf{x}\}$ at time t_0 (and that, at time t is not more there, having been displaced by an amount $u^i(\mathbf{x}, t)$). Usually, one is interested (as in here) in *small deformations*. Then equation 8.180 is considered a valid approximation.

We are interested in describing elastic waves. Usually, the change in density at a point when a wave passes through is small and can be neglected. Then, the equation describing the conservation of mass 8.173 can be dropped, and the density term in the equation describing the conservation of linear momentum 8.174 can be taken out from the time derivative term, thus leading to

$$\rho \frac{\partial v_i}{\partial t} - \nabla_j \tau^j{}_i = 0 \, , \tag{8.181}$$

or, if we use the approximation 8.180,

$$\rho \frac{\partial^2 u^i}{\partial t^2} - \nabla_j \tau^{ji} = 0 \, . \tag{8.182}$$

Let us now turn to the equation describing the conservation of angular momentum 8.175.

The spin density σ_{ij} is antisymmetric. It can be represented by a vector s^i (the "dual" tensor of the previous chapter):

$$s^i = \frac{1}{2} \epsilon^{ijk} \sigma_{jk} \; ; \qquad \sigma_{ij} = \epsilon_{ijk} s^k \, . \tag{8.183}$$

Similarly, the moment stress $m^k{}_{ij}$ can also be represented by the tensor μ^{ij} :

$$\mu^{\ell i} = \frac{1}{2} \epsilon^{ijk} m^\ell{}_{jk} \; ; \qquad m^\ell{}_{ij} = \epsilon_{ijk} \mu^{\ell k} \, . \tag{8.184}$$

Then, equation 8.175 writes

$$\frac{\partial s^i}{\partial t} - \nabla_j \mu^{ji} = \epsilon^{ijk} \tau_{jk} \, . \tag{8.185}$$

The spin density s^i may have different interpretations in different models of continuous media. In an elastic medium, it can be directly linked with the rotations of the particles of the medium. If φ_i is the local rotation vector, then

$$s^i = J^{ij} \frac{\partial \varphi_j}{\partial t} \, , \tag{8.186}$$

where J^{ij} the local moment of inertia. As an example, consider a body with large molecules, like a polymer, or a granular body, like an ordinary rock. In addition to the ordinary displacements, "points" may obviously undergo rotations.

Equation 8.185 becomes then

$$J^{ij} \frac{\partial^2 \varphi_j}{\partial t^2} - \nabla_j \mu^{ji} = \epsilon^{ijk} \tau_{jk} \, . \tag{8.187}$$

Equations 8.182 and 8.187 are the fundamental equations for elastic media. The stress τ_{ij} and the moment stress μ^{ij} are functions of the deformation, which is, in turn, function of the displacements u^i and rotations φ_i .

A final modification will lead to the classical form of the equations. The theory developed here ignores electromagnetic, chemical, and other phenomena. It is usually assumed that those only appear as new right-hand (source) terms in the dynamical equations. Calling those respectively f^i and ψ^i gives

$$\rho \frac{\partial^2 u^i}{\partial t^2} - \nabla_j \tau^{ji} = f^i \tag{8.188}$$

and

$$J^{ij} \frac{\partial^2 \varphi_j}{\partial t^2} - \nabla_j \mu^{ji} = \epsilon^{ijk} \tau_{jk} + \psi^i \,. \tag{8.189}$$

Given the sources f^i and ψ^i of the field, the parameters describing the medium, ρ and J^{ij} , and the particular relation between stresses and deformations (see for instance the linearized theory of the next section), the set of equations 8.188– 8.189 have an unique solution if we prescribe initial conditions,

$$u^i(\mathbf{x}, t_0) \qquad \frac{\partial u^i}{\partial t}(\mathbf{x}, t_0) \qquad \varphi^i(\mathbf{x}, t_0) \qquad \frac{\partial \varphi^i}{\partial t}(\mathbf{x}, t_0) \,, \tag{8.190}$$

and conditions at the space boundaries. For instance, we can impose surface tractions and moment tractions:

$$\tau^{ji}(\mathbf{x}, t) n_j(\mathbf{x}) \qquad \mu^{ji}(\mathbf{x}, t) n_j(\mathbf{x}) \qquad\qquad \text{for } \mathbf{x} \text{ at the surface} \,. \tag{8.191}$$

8.4.1 Elasticity with nonsymmetric stresses

A medium with non symmetric stresses is called a “Cosserat’s micropolar medium”. Early references to media with non symmetric stress can be found in Cosserat and Cosserat (1896, 1907, 1909). Truesdell and Toupin (1960) also mention in their general exposition, the basic equations of Cosserat’s media. Nowacki (1986) gives a modern vision of the theory, and discusses the physical properties of micropolar media.

The requirement that the expressions describing the conservation of mechanical energy should be invariant by space translation and rotation imply (Nowacki, 1986; see also the Appendix)

$$\tau^{ij} \frac{\partial \varepsilon_{ij}}{\partial t} + \mu^{ij} \frac{\partial \gamma_{ij}}{\partial t} = 0 \,, \tag{8.192}$$

where

$$\varepsilon_{ij} = \nabla_i u_j - \epsilon_{ijk} \varphi^k \tag{8.193}$$

and

$$\gamma_{ij} = \nabla_i \varphi_j \,. \tag{8.194}$$

Note: say somewhere that the physical dimensions of the tensors above are as follows:

$$[\gamma] = L^{-1} \tag{8.195}$$

$$[\varphi] = 1 \tag{8.196}$$

$$[u] = L \tag{8.197}$$

$$[\varepsilon] = 1 \,. \tag{8.198}$$

That the conservation of mechanical energy has a simple expression as a function of ε_{ij} and γ_{ij} suggests that those are good measures of *deformation*. The (non symmetric) tensor ε_{ij} is called the *translational deformation* or, simply, the *deformation*. The antisymmetric tensor γ_{ij} is called the *rotational deformation* or, also, the *torsion*.

In an elastic medium, we can express stress and moment stress as functions of the deformation. If deformations are small enough, a first order development can be used:

$$\tau_{ij} = c_{ijk\ell}\, \varepsilon^{k\ell} + b_{ijk\ell}\, \gamma^{k\ell} \tag{8.199}$$

and

$$\mu_{ij} = b_{ijk\ell}\, \varepsilon^{k\ell} + a_{ijk\ell}\, \gamma^{k\ell} \,. \tag{8.200}$$

Note: say somewhere that the physical dimensions of the tensors above are as follows:

$$[c] = [\tau] \tag{8.201}$$

$$[b] = [\tau] L \tag{8.202}$$

$$[a] = [\tau] L^2 \,. \tag{8.203}$$

The form of equations 8.199–8.200, and, in particular, the fact that the same tensor $b_{ijk\ell}$ appears in both equations comes from a thermodynamic simplifying hypothesis, that the internal energy density is quadratic on the deformations:

$$U = \frac{1}{2} c_{ijk\ell} \varepsilon^{ij} \varepsilon^{k\ell} + \frac{1}{2} a_{ijk\ell} \gamma^{ij} \gamma^{k\ell} + b_{ijk\ell} \varepsilon^{ij} \gamma^{k\ell} \,. \tag{8.204}$$

Then,

$$c_{ijk\ell} = \left(\frac{\partial^2 U}{\partial \varepsilon^{ij} \partial \varepsilon^{k\ell}} \right) \,, \qquad a_{ijk\ell} = \left(\frac{\partial^2 U}{\partial \gamma^{ij} \partial \gamma^{k\ell}} \right) \,, \qquad b_{ijk\ell} = \left(\frac{\partial^2 U}{\partial \varepsilon^{ij} \partial \gamma^{k\ell}} \right) \,. \tag{8.205}$$

The expression of the internal energy density (equation 8.204) implies the symmetries

$$c_{ijk\ell} = c_{k\ell ij} \tag{8.206}$$

and

$$a_{ijk\ell} = a_{k\ell ij} \,, \tag{8.207}$$

while the tensor $b_{ijk\ell}$ does not have any particular symmetry. This makes a total of $45+45+81 = 171$ independent elastic coefficients.

The asymmetric stresses play a role in situation of strong stress *gradients* occurring for instance in the vicinity of cracks, or, more generally, in highly heterogeneous media. It could, for instance, well happen that the understanding of the process of seismic rupture in elastic solids requires to drop the usual hypothesis that the stress tensor is symmetric.

For isotropic media (Nowacki, 1986),

$$c_{ijk\ell} = \lambda \delta_{ij} \delta_{k\ell} + \mu (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) + \nu (\delta_{ik} \delta_{j\ell} - \delta_{i\ell} \delta_{jk}) \,, \tag{8.208}$$

$$a_{ijk\ell} = \eta \delta_{ij} \delta_{k\ell} + \xi (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) + \zeta (\delta_{ik} \delta_{j\ell} - \delta_{i\ell} \delta_{jk}) \,, \tag{8.209}$$

and

$$b_{ijkl} = 0 \,. \tag{8.210}$$

This gives

$$\tau_{ij} = \lambda {\varepsilon_k}^k \delta_{ij} + \mu (\varepsilon_{ij} + \varepsilon_{ji}) + \nu (\varepsilon_{ij} - \varepsilon_{ji}) \tag{8.211}$$

and

$$\mu_{ij} = \eta {\gamma_k}^k \delta_{ij} + \xi (\gamma_{ij} + \gamma_{ji}) + \zeta (\gamma_{ij} - \gamma_{ji}) \,. \tag{8.212}$$

Note: say somewhere that the physical dimensions of the tensors above are as follows:

$$[\lambda] = [\mu] = [\nu] = [\tau] \, . \tag{8.213}$$

In addition to the Lamé's parameters λ and μ there are four other elastic parameters: ν , η , ξ , and ζ . While λ , μ, and ν have the physical dimension of a pressure, η , ξ , and ζ have the physical dimension of a force.

As the tensors $c_{ijk\ell}$ and $a_{ijk\ell}$ have to be positive definite, the elastic parameters satisfy some inequalities:

$$\mu > 0 \quad \xi > 0 \quad \nu > 0 \quad \zeta > 0 \quad 3\lambda + 2\mu > 0 \quad 3\eta + 2\xi > 0 \, . \tag{8.214}$$

Example (torsional waves): Replacing 8.211 and 8.193 in the dynamical equation 8.188 gives the differential equation for the displacement field u^i . Applying the operator ∇_i to the equation so obtained, and defining

$$\theta = \nabla_i u^i \, , \tag{8.215}$$

leads to

$$\rho \frac{\partial^2 \theta}{\partial t^2} - (\lambda + 2\mu) \nabla^2 \theta = \nabla_i f^i \, , \tag{8.216}$$

which demonstrates that the dilatation θ obeys the wave equation, with the (compressional wave) velocity

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}} \, . \tag{8.217}$$

This is a clasical wave. Let us, instead, be interested in purely torsional waves. The variable φ^i represents the angle of rotation of every point of the medium. We define its divergence,

$$\vartheta = \nabla_i \varphi^i \, , \tag{8.218}$$

and assume isotropy of local moment of inertia:

$$J_i{}^j = J \ \delta_i{}^j \, . \tag{8.219}$$

Equation 8.189 then leads to

$$\frac{1}{c^2} \ \frac{\partial^2 \vartheta}{\partial t^2} - \nabla^2 \vartheta + \frac{1}{L^2} \ \vartheta = \nabla_i \psi^i \, , \tag{8.220}$$

where

$$c = \sqrt{\frac{\eta + 2\xi}{J}} \tag{8.221}$$

has the dimension of a speed, and where

$$L = \sqrt{\frac{\eta + 2\xi}{4\nu}} \tag{8.222}$$

has the dimension of a length. This is the well known Klein-Gordon equation, whose solution is analyzed in Appendix **??**.

This "wave" is missing in the theory where the stress is assumed symmetric. When this wave arrives to a region of the space, all points experience a rotation, without any displacement.

Few experiments have been performed to test the theory of elasticity with non symmetric stresses. On the other side, it is well known that "beam models", where a regular collection of beams is assembled to form a 3-D structure (like in concrete buildings), have as limit such a Cosserat medium (Herrmann, 1989; Herrmann et al., 1989; see also Kaliski, 1963, and Askar and Cakmak, 1968).

8.4.2 Elasticity with symmetric stresses

If local rotations and moment stresses may be neglected, we have the standard theory of elastic media. Then equation 8.187 simply tells us that

$$\tau_{ij} = \tau_{ji} \, , \tag{8.223}$$

i.e., the stress tensor is symmetric.

We are then left with

$$\rho \frac{\partial^2 u^i}{\partial t^2} - \nabla_j \tau^{ji} = f^i \, , \tag{8.224}$$

$$\tau_{ij} = c_{ijk\ell} \, \epsilon^{k\ell} \tag{8.225}$$

and

$$\epsilon_{ij} = \frac{1}{2} (\nabla_j u_i + \nabla_i u_j) \, . \tag{8.226}$$

This makes a total of 21 independent elastic parameters.

For isotropic media,

$$c_{ijk\ell} = \lambda \delta_{ij} \delta_{k\ell} + \mu (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) \, . \tag{8.227}$$

This gives

$$\tau_{ij} = \lambda {\varepsilon_k}^k \delta_{ij} + 2\mu \varepsilon_{ij} \, . \tag{8.228}$$

Note: I have to explain that the tensor δ_{ij} in equation 8.227 is the metric, g_{ij} . Thus, I should rather write

$$c_{ijk\ell} = \lambda g_{ij} g_{k\ell} + \mu (g_{ik} g_{j\ell} + g_{i\ell} g_{jk}) \tag{8.229}$$

and

$$\tau_{ij} = \lambda g_{ij} {\varepsilon_k}^k + 2\mu \varepsilon_{ij} \, . \tag{8.230}$$

I should also correct equations 8.208–8.209.

8.5 Appendices

8.5.1 Appendix: Model of an electron

Let us first define the constants

$$A_s = \frac{e^2 c^2}{8\pi^3 G \varepsilon_0 \hbar} \ , \tag{8.231}$$

$$A_e = \frac{e^3 c^2}{4\pi^3 G \varepsilon_0 \hbar^2} \ , \tag{8.232}$$

and

$$L = \frac{\hbar}{ec}\sqrt{\pi G \varepsilon_0} \ , \tag{8.233}$$

where $-e$ is the electric charge of an electron and ε_0 is the dielectric constant of vacuo.

Numerically, we have

$$A_s = 1.5 \ 10^{32} \ \mathrm{kg \ s^{-1}} \ , \tag{8.234}$$

$$A_e = 4.5 \ 10^{47} \ \mathrm{Cm^{-2}} \ , \tag{8.235}$$

and

$$L = 9.5 \ 10^{-35} \ \mathrm{m} \ . \tag{8.236}$$

Let k_i be an unit (constant) vector "pointing upwards", and r a radial variable.

Then, the radial distribution of spin density is (we use here definition 8.183)

$$s_i = k_i \ A_s \frac{1}{r} sin \frac{r}{L} \tag{8.237}$$

and the radial distribution of electric charge

$$\rho_e = -A_e \frac{1}{r} sin \frac{r}{L} \tag{8.238}$$

have the following properties (note that πL is the first zero of the sinc function):

- The total spin inside $r < \pi L$ is $\hbar/2$,
- The total electric charge inside $r < \pi L$ is $-e$,
- *At every point, the electrostatic (repulsive) force density exactly equilibrates the (attractive) gravitational spin-spin force density*,
- The (attractive) gravitational mass-mass force density is orders of magnitude smaller.

I.e., we have a "model of an electron".

The spin-spin force density is computed from

$$f_i = \frac{\pi G}{c^2} \nabla_i (\sigma_{jk} \sigma^{jk}) = \frac{2\pi G}{c^2} \nabla_i (s_j s^j) \ , \tag{8.239}$$

while the Coulomb's force density is computed from Coulomb's law:

$$f_i = \frac{1}{4\pi\varepsilon_0} \frac{\rho_e(r) Q(r)}{r^2} \ , \tag{8.240}$$

where $Q(r)$ is the total electric charge inside the sphere of radius r .

We obtain for both force densities the expression

$$|f_i| = \frac{e^5 c^3}{16\pi^{1/2} G^{3/2} \varepsilon^{3/2} \hbar^3} \frac{1}{r^2} \sin\frac{r}{L} \cos\frac{r}{L} \left(1 - \frac{L}{r} tg \frac{r}{L}\right) \ , \tag{8.241}$$

the gravitational spin-spin force density pointing inward and the electrostatic one pointing outward.

The choice $r < \pi L$ for the "radius of the electron" is quite arbitrary: Any finite value of r would have done as well.

This shows that simple models of an electron exist where the Coulomb's electrostatic force is exactly compensated by the gravitational (spin-spin) force.

8.5.2 Appendix: The Einstein-Cartan equations in 3-D form

The Einstein equations

The Einstein equation give the stress-energy tensor as a function of the Ricci tensor:

$$\underline{T}_\alpha{}^\beta = \frac{1}{\chi}\left(\underline{R}_\alpha{}^\beta - \frac{1}{2}\,\underline{\delta}_\alpha{}^\beta\underline{R}\right) \ . \tag{8.242}$$

It is easy to obtain the inverse relation:

$$\underline{R}_\alpha{}^\beta = \chi\left(\underline{T}_\alpha{}^\beta - \frac{1}{2}\,\underline{\delta}_\alpha{}^\beta\underline{T}\right) \ . \tag{8.243}$$

In 3-D notations, using the definitions introduced in sections 8.1.2, 8.1.3, and 8.1.6 gives, for the direct Einstein equation,

$$\rho = \frac{1}{2\chi c^2} r_s{}^s \ , \tag{8.244}$$

$$p_i = -\frac{1}{\chi}\delta_{si}{}^s \ , \tag{8.245}$$

$$q^j = +\frac{1}{\chi}\varepsilon_s{}^{js} \ , \tag{8.246}$$

and

$$T_i{}^j = \frac{1}{\chi}\left(c^2\left(\gamma_i{}^j - \delta_i{}^j\gamma_k{}^k\right) + \left(r_i{}^j - \frac{1}{2}\delta_i{}^j r_k{}^k\right)\right) \ , \tag{8.247}$$

and, for the inverse Einstein equation, the (equivalent) expressions

$$\gamma_s{}^s = -\frac{\chi}{2}\left(\rho + \frac{1}{c^2}T_s{}^s\right) \ , \tag{8.248}$$

$$\delta_{si}{}^s = -\chi p_i \ , \tag{8.249}$$

$$\varepsilon_s{}^{js} = +\chi q^j \ , \tag{8.250}$$

and

$$r_i{}^j + \gamma_i{}^j c^2 = \chi\left(T_i{}^j + \frac{1}{2}\delta_i{}^j\left(\rho c^2 - T_k{}^k\right)\right) \ . \tag{8.251}$$

The Cartan equations

The Cartan equation gives the spin tensor as a function of the torsion tensor:

$$\underline{M}_{\alpha\beta}{}^\gamma = \frac{1}{\chi}\left(\underline{S}_{\alpha\beta}{}^\gamma + \underline{S}_\alpha\underline{\delta}_\beta{}^\gamma - \underline{S}_\beta\underline{\delta}_\alpha{}^\gamma\right) \ . \tag{8.252}$$

The inverse relation is

$$\underline{S}_{\alpha\beta}{}^\gamma = \chi\left(\underline{M}_{\alpha\beta}{}^\gamma + \frac{1}{2}\left(\underline{M}_\alpha\underline{\delta}_\beta{}^\gamma - \underline{M}_\beta\underline{\delta}_\alpha{}^\gamma\right)\right) \ . \tag{8.253}$$

In 3-D notations, using the definitions introduced in sections 8.1.1, 8.1.2, 8.1.3, and 8.1.6 gives, for the direct Cartan equation,

$$\sigma_{ij} = \frac{1}{\chi}\widehat{J}_{ij} \ , \tag{8.254}$$

$$\pi_i{}^j = \frac{1}{\chi}\left(K_i{}^j - K_k{}^k\delta_i{}^j\right) \ , \tag{8.255}$$

$$t_i = -\frac{1}{\chi c^2} S_{ji}{}^j \ , \tag{8.256}$$

and

$$M_{ij}{}^k = \frac{1}{\chi} \left(S_{ij}{}^k + S_{\ell i}{}^\ell \delta_j{}^k - S_{\ell j}{}^\ell \delta_i{}^k + c^2 (I_i \delta_j{}^k - I_j \delta_i{}^k) \right) \tag{8.257}$$

and, for the inverse Cartan equation, the (equivalent) expressions

$$\widehat{J}_{ij} = \chi \sigma_{ij} \ , \tag{8.258}$$

$$K_i{}^j = \chi \left(\pi_i{}^j - \frac{1}{2} \pi_k{}^k \delta_i{}^j \right) \ , \tag{8.259}$$

$$S_{ji}{}^j = -\chi c^2 t_i \ , \tag{8.260}$$

$$S_{ij}{}^k = \chi \left(M_{ij}{}^k + \frac{1}{2} \left(M_{\ell i}{}^\ell \delta_j{}^k - M_{\ell j}{}^\ell \delta_i{}^j \right) + \frac{c^2}{2} \left(t_i \delta_j{}^k - t_j \delta_i{}^k \right) \right) \ . \tag{8.261}$$

The Cartan equations are somewhat simplified if, from the spin tensor $M_{ij}{}^k$ and the torsion tensor $S_{ij}{}^k$, we introduce their traces

$$M_i = M_{ji}{}^j \tag{8.262}$$

$$S_i = S_{ji}{}^j \tag{8.263}$$

and the traceless tensors

$$\breve{M}_{ij}{}^k = M_{ij}{}^k + \frac{1}{2} \left(M_i \delta_j{}^k - M_j \delta_i{}^k \right) \tag{8.264}$$

and

$$\breve{S}_{ij}{}^k = S_{ij}{}^k + \frac{1}{2} \left(S_i \delta_j{}^k - S_j \delta_i{}^k \right) \ . \tag{8.265}$$

Then, obviously,

$$M_{ij}{}^k = \breve{M}_{ij}{}^k - \frac{1}{2} \left(M_i \delta_j{}^k - M_j \delta_i{}^k \right) \tag{8.266}$$

and

$$S_{ij}{}^k = \breve{S}_{ij}{}^k - \frac{1}{2} \left(S_i \delta_j{}^k - S_j \delta_i{}^k \right) \ . \tag{8.267}$$

With those definitions, the whole set of direct Cartan equations are now written

$$\sigma_{ij} = \frac{1}{\chi} \widehat{J}_{ij} \ , \tag{8.268}$$

$$\pi_i{}^j = \frac{1}{\chi} \left(K_i{}^j - K_k{}^k \delta_i{}^j \right) \ , \tag{8.269}$$

$$t_i = -\frac{1}{\chi c^2} S_i \ , \tag{8.270}$$

$$M_i = -\frac{1}{\chi} \left(S_i + 2c^2 I_i \right) \ , \tag{8.271}$$

and

$$\breve{M}_{ij}{}^k = \frac{1}{\chi} \breve{S}_{ij}{}^k \ , \tag{8.272}$$

while the inverse Cartan set of equations is written

$$\widehat{J}_{ij} = \chi \sigma_{ij} \ , \tag{8.273}$$

$$K_i{}^j = \chi \left(\pi_i{}^j - \frac{1}{2} \pi_k{}^k \delta_i{}^j \right) \ , \tag{8.274}$$

$$S_i = -\chi c^2 t_i \ , \tag{8.275}$$

$$I_i = \frac{\chi}{2} \left(t_i - \frac{1}{c^2} M_i \right) \ , \tag{8.276}$$

and

$$\breve{S}_{ij}{}^k = \chi \breve{M}_{ij}{}^k \ . \tag{8.277}$$

8.5.3 Appendix: 3-D evolution equations for a medium locally at rest

We show here that, given g_{ij} , $\widetilde{G}_{ij}$, σ_{ij} , and t_i , at $t = t_0$, and assuming given some "constitutive equations" that allow us to compute $\Delta p^i \equiv p^i - q^i$, T_i^j , $\pi_i{}^j$, and $M_{ij}{}^k$ at any time as a function of the "history" of the medium, as well as assuming given U at all times, then, it is possible to extrapolate the values of g_{ij} , $\widetilde{G}_{ij}$, σ_{ij} , and t_i , at all times $t > t_0$.

In section 8.1.6, we have seen the Einstein-Cartan equations. Rather, we will use here the more convenient form of these equations derived in appendix 8.5.2, equations 8.248 and 8.251:

$$\gamma_s{}^s = -\frac{\chi}{2} \left(\rho + \frac{1}{c^2} T_s{}^s \right) \ , \tag{8.278}$$

$$r_i{}^j + \gamma_i{}^j c^2 = \chi \left(T_i^j + \frac{1}{2} \delta_i{}^j \left(\rho c^2 - T_k{}^k \right) \right) \ , \tag{8.279}$$

and equations 8.275–8.277:

$$S_i = -\chi c^2 t_i \ , \tag{8.280}$$

$$I_i = \frac{\chi}{2} \left(t_i - \frac{1}{c^2} M_i \right) \ , \tag{8.281}$$

and

$$\breve{S}_{ij}{}^k = \chi \breve{M}_{ij}{}^k \ . \tag{8.282}$$

Let us first see what we can compute at any t .

As U is given for all times, using 6.13 we compute φ :

$$\varphi = \exp(U - U_0) \tag{8.283}$$

(U_0 being an arbitrary constant). From U we can get the gravitational field g_i (equation 6.14):

$$g_i = -\frac{1}{c^2} \nabla_i U \ . \tag{8.284}$$

As we assume tha the constitutive equations will give us at any time the moment stresses, we will be able to compute, using 8.264,

$$\breve{M}_{ij}{}^k = M_{ij}{}^k + \frac{1}{2} \left(M_i \delta_j{}^k - M_j \delta_i{}^k \right) \tag{8.285}$$

and, using 8.277,

$$\breve{S}_{ij}{}^k = \chi \breve{M}_{ij}{}^k \ . \tag{8.286}$$

The constitutive equations giving also $\pi_i{}^j$, using 8.274we can obtain

$$K_i{}^j = \chi \left(\pi_i{}^j - \frac{1}{2} \delta_i{}^j \pi_k{}^k \right) \ . \tag{8.287}$$

The definitions of symmetric and antisymmetric part of a tensor (equations 8.15 and 8.16) give then

$$\widetilde{K}_{ij} = K_{ij} + K_{ji} \tag{8.288}$$

and

$$\widehat{K}_{ij} = K_{ij} - K_{ji} \ . \tag{8.289}$$

Now, at $t = t_0$ we can successively computeusing equations all written in the text

$$S_i = -\chi c^2 t_i \tag{8.290}$$

$$S_{ij}{}^k = \breve{S}_{ij}{}^k - \frac{1}{2}\left(S_i \delta_j{}^k - S_j \delta_i{}^k\right) \tag{8.291}$$

$$I_i = \frac{\chi}{2}\left(t_i - \frac{1}{c^2} M_i\right) \tag{8.292}$$

$$h_i = g_i - I_i \tag{8.293}$$

$$\widehat{J}_{ij} = \chi \sigma_{ij} \tag{8.294}$$

$$\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) + \frac{1}{2}(S_{ijk} + S_{kij} + S_{kji}) \tag{8.295}$$

$$R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l - \partial_j \Gamma_{ik}{}^l + \Gamma_{is}{}^l \Gamma_{jk}{}^s - \Gamma_{js}{}^l \Gamma_{ik}{}^s \tag{8.296}$$

$$\widehat{G}_{ij} = \widehat{J}_{ij} + \widehat{K}_{ij} \tag{8.297}$$

$$G_{ij} = \frac{1}{2}\left(\widetilde{G}_{ij} + \widehat{G}_{ij}\right) \tag{8.298}$$

$$\widetilde{J}_{ij} = \widetilde{G}_{ij} - \widetilde{K}_{ij} \tag{8.299}$$

$$J_{ij} = \frac{1}{2}\left(\widetilde{J}_{ij} + \widehat{J}_{ij}\right) \tag{8.300}$$

$$\widetilde{r}_{ij} = \widetilde{R}_{ij} + c^2 J_k{}^k \widetilde{J}_{ij} - c^2 (J_i{}^k J_{kj} + J_j{}^k J_{ki}) \tag{8.301}$$

$$\widehat{R}_{ij} = \widehat{R}_{ij} + c^2 J_k{}^k \widehat{J}_{ij} - c^2 (J_i{}^k J_{kj} - J_j{}^k J_{ki}) \tag{8.302}$$

$$\rho = \frac{1}{2\chi c^2} r_k{}^k \tag{8.303}$$

$$\delta_i = \nabla_k J_i{}^k + \nabla_i J_k{}^k + S_{ki}{}^l J_l{}^k \tag{8.304}$$

$$p_i = -\frac{1}{\chi}\delta_i \tag{8.305}$$

$$\varepsilon^i = \chi q^i = \chi (p^i + \Delta p^i) \tag{8.306}$$

$$c^2 \widetilde{\gamma}_i{}^j = -\widetilde{r}_i{}^j + \chi(\widetilde{T}_i{}^j + \delta_i{}^j (\rho c^2 - T_k{}^k)) \tag{8.307}$$

and

$$c^2 \widehat{\gamma}_i{}^j = -\widehat{r}_i{}^j + \chi \widehat{T}_i{}^j \ . \tag{8.308}$$

Then, at $t = t_0$ we can deduce

$$\partial_\tau g_{ij} = c^2 \widetilde{G}_{ij} \tag{8.309}$$

$$\partial_\tau \widetilde{G}_{ij} = \widetilde{\gamma}_{ij} + \partial \widetilde{K}_{ij} - (\nabla_i h_j + \nabla_j h_I) + (g_i h_j + g_j h_i) + (J_i{}^s G_{js} + J_j{}^s G_{is}) \tag{8.310}$$

$$\chi \partial_\tau \sigma_{ij} = \widehat{\gamma}_{ij} - (\nabla_i h_j - \nabla_j h_I) + (g_i h_j - g_j h_i) + (J_i{}^s G_{js} - J_j{}^s G_{is}) \tag{8.311}$$

and

$$-\chi \partial_\tau t_i = \varepsilon_i - \frac{1}{2}\nabla_i G_k{}^k + \nabla_k G_i{}^k - g_k G_i{}^k - J_k{}^k h_i + J_{ik} h^k \ . \tag{8.312}$$

As these are the quantities given at $t = t_0$, we can extrapolate them for $t > t_0$, and the problem of extrapolation of initial conditions is solved.

8.5.4 Appendix: Strain tensor

Homogeneous 1-D strain

Assume, in a 1-D world, an object of length ℓ which, in an homogeneous deformation, elongates by $d\ell$. The "incremental strain" is defined as the relative elongation:

$$d\varepsilon = \frac{d\ell}{\ell} \, . \tag{8.313}$$

The integration $\varepsilon = \int_{\ell_0}^{\ell} \frac{d\ell}{\ell}$ defines then the *strain*:

$$\varepsilon = \log \frac{\ell}{\ell_0} \tag{8.314}$$

Equivalently, we can write

$$\ell = \ell_0 \, e^{\varepsilon} \, . \tag{8.315}$$

General 1-D strain

A general (i.e., inhomogeneous) deformation of a 1-D medium can be described, for instance, by giving the function $s(s_0)$ defining the distance s to some reference point, after deformation, of the material point whose distance to the same reference point was s_0 before deformation[1]. When applied to this situation, formula 8.314 gives (see figure 8.1)

$$\varepsilon = \log \left[\lim_{\Delta s_0 \to 0} \frac{\Delta s}{\Delta s_0} \right] \tag{8.316}$$

i.e.,

$$\boxed{\varepsilon(s_0) = \log \left[\frac{ds}{ds_0}(s_0) \right] \, ,} \tag{8.317}$$

which is the main definition of strain.

An integration gives the position as a function of the strain:

$$s(s_0) = \int_0^{s_0} ds' \; e^{\varepsilon(s')} \, . \tag{8.318}$$

We have

$$\begin{aligned} s(s_0 + \Delta s_0) &= \int_0^{s_0+\Delta s_0} ds' \; e^{\varepsilon(s')} \\ &= \int_0^{s_0} ds' \; e^{\varepsilon(s')} + \int_{s_0}^{s_0+\Delta s_0} ds' \; e^{\varepsilon(s')} \\ &= s(s_0) + \int_{s_0}^{s_0+\Delta s_0} ds' \; e^{\varepsilon(s')} \, . \end{aligned} \tag{8.319}$$

Assume that Δs_0 is small enough, and use the notation δs_0. We can then use the first order approximation

$$s(s_0 + \delta s_0) = s(s_0) + e^{\varepsilon(s_0)} \, \delta s_0 \, , \tag{8.320}$$

i.e,

$$s(s_0 + \delta s_0) - s(s_0) = e^{\varepsilon(s_0)} \, \delta s_0 \, , \tag{8.321}$$

[1]These points are "material points," i.e., points attached to the medium undergoing deformation.

which can be written

$$\delta s = e^{\varepsilon}\, \delta s_0 \tag{8.322}$$

or

$$\delta s^2 = e^{2\varepsilon}\, \delta s_0^2 \,, \tag{8.323}$$

i.e.,

$$\boxed{\delta s^2 - \delta s_0^2 = \left(e^{2\varepsilon} - 1 \right) \delta s_0^2 \,.} \tag{8.324}$$

This is a very important equation. It relates the change in length of a small segment at a given point to the strain at that point. Notice that although the segment is assumed small, the deformation may arbitrarily large: this equation may be used to define a *finite strain*.

For a small deformation (i.e., for a small value of ε),

$$\boxed{\delta s^2 - \delta s_0^2 \;\approx\; 2\,\varepsilon\, \delta s_0^2 \;\approx\; 2\,\varepsilon\, \delta s^2 \,.} \tag{8.325}$$

3-D deformation

Consider, in an arbitrary space furnished with an arbitrary coordinate system, a deformable medium and, on it, two neighbouring points $\mathbf{x}_0$ and $\mathbf{x}_0 + \delta\mathbf{x}_0$ with respective coordinates x_0^i and $x_0^i + \delta x_0^i$. They are separated by a distance δs_0 that can be computed using the metric tensor:

$$\delta s_0^2 = g_{ij}(\mathbf{x}_0)\, \delta x_0^i\, \delta x_0^i \,. \tag{8.326}$$

If the medium undergoes (inhomogeneous) deformation, the two points will have new coordinates x^i and $x^i + \delta x^i$, and the new distance between the two points, δs can be computed from the expression

$$\delta s^2 = g_{ij}(\mathbf{x})\, \delta x^i\, \delta x^i \,. \tag{8.327}$$

Equation 8.324 suggests to define the *strain field* by the expression[2].

$$\boxed{\delta s^2 - \delta s_0^2 = \left(e^{2\varepsilon_{ij}} - g_{ij} \right) \delta x_0^i\, \delta x_0^j \,.} \tag{8.328}$$

(Note: I should explain here why the coordinate increments δx^i have replaced the length increment δs of equation 8.324.)

For a small deformation (i.e., for a small value of ε),

$$\boxed{\delta s^2 - \delta s_0^2 = 2\, \varepsilon_{ij}(\mathbf{x}_0)\, \delta x_0^i\, \delta x_0^j} \tag{8.329}$$

or, equivalently,

$$\boxed{\delta s^2 - \delta s_0^2 = 2\, \varepsilon_{ij}(\mathbf{x})\, \delta x^i\, \delta x^j \,,} \tag{8.330}$$

the two expressions corresponding respectively to a Lagrangian or Eulerian description of the deformation.

In a space of dimension N , to compute the strain tensor at a given point we may, for instance, consider at the given point point $N(N+1)/2$ small noncolinear vectors. The comparison of their lengths before and after deformation through equations 8.329 or 8.330 provides the necessary conditions to obtain the $N(N+1)/2$ independent components of the strain tensor. Practically, choosing $N(N+1)/2$ small noncolinear vectors may mean, for instance, considering the three sides of a small triangle in 2-D, of the six edges of a small tetrahedron in 3-D.

[2]The exponential of a tensor $\psi_i{}^j$ should, in all rigor, be written $(e^{\psi})_i{}^j$, and is defined by $e^{\psi_i{}^j} = \delta_i{}^j + \psi_i{}^j + \frac{1}{2!}\psi_i{}^k\psi_k{}^j + \frac{1}{3!}\psi_i{}^k\psi_k{}^\ell\psi_\ell{}^j + \dots$. Remember that the metric tensor g_{ij} appearing in the equation is identical to the Kronecker's tensor: $g_i{}^j = \delta_i{}^j$ (see chapter 1).

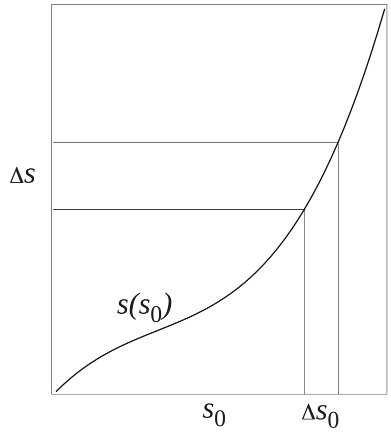

Figure 8.1: Deformation af a 1-D medium.

Example: Homogeneous deformation in polar coordinates

Consider the Euclidean plane. Using first Cartesian coodinates, it is well known that a homogeneous deformation is described by a constant deformation field, with components

$$\begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} , \tag{8.331}$$

where α is a dimensionless constant taking any positive or negative small value. Two points separated by a distance δs_0 before deformation, will be separated, after deformation, by a distance $\delta s = (1+\alpha)\,\delta s_0$.

The components of this deformation tensor when using polar coordinates may be obtained from equation 8.331 using, for instance, the general rules for transformation of the components of a tensor under a coordinate change (see section XXX). This gives

$$\begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\varphi} \\ \varepsilon_{\varphi r} & \varepsilon_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha r^2 \end{pmatrix} . \tag{8.332}$$

Let us obtain this result from a direct use of equations 8.329 and 8.330.

In the left (resp. right) column, we will make use of equation 8.329 (resp. 8.330).

double colonne

The radial segment shown in figure 8.3 has a length, before deformation,

$$\delta s_0 = \delta r_0 \,, \tag{8.333}$$

and, after deformation,

$$\delta s = \delta r = (1+\alpha)\, \delta r_0 \,. \tag{8.334}$$

This gives

$$\begin{aligned} \delta s^2 - \delta s_0^2 &= \Big((1+\alpha)^2 - 1 \Big)\, \delta r_0^2 \\ &= \Big(2\alpha + \alpha^2 \Big)\, \delta r_0^2 \\ &\approx 2\,\alpha\, \delta r_0^2 \,. \end{aligned} \tag{8.335}$$

The length of the tangential segment is, before deformation,

$$\delta s_0 = r_0\, \delta\varphi_0 \tag{8.336}$$

and, after deformation,

$$\delta s = r\, \delta\varphi = (1+\alpha)\, r_0\, \delta\varphi_0 \,. \tag{8.337}$$

This gives

$$\begin{aligned} \delta s^2 - \delta s_0^2 &= \Big((1+\alpha)^2 - 1 \Big)\, r_0^2\, \delta\varphi_0^2 \\ &= \Big(2\alpha + \alpha^2 \Big)\, r_0^2\, \delta\varphi_0^2 \\ &\approx 2\,\alpha\, r_0^2\, \delta\varphi_0^2 \,. \end{aligned} \tag{8.338}$$

The strain tensor was defined by equation 8.329:

$$\delta s^2 - \delta s_0^2 = 2\, \varepsilon_{ij}(\mathbf{x}_0)\, \delta x_0^i\, \delta x_0^j \,.$$

Using the results 8.335 and 8.338 gives then

$$\varepsilon_{rr}(r_0, \varphi_0) = \alpha \tag{8.339}$$

and

$$\varepsilon_{\varphi\varphi}(r_0, \varphi_0) = \alpha r_0^2 \tag{8.340}$$

(demonstrating that $\varepsilon_{r\varphi} = 0$ would require the consideration of a third vector).

The radial segment shown in figure 8.3 has a length, before deformation,

$$\delta s_0 = \delta r_0 = \frac{1}{1+\alpha}\, \delta r \,, \tag{8.341}$$

and, after deformation,

$$\delta s = \delta r \,. \tag{8.342}$$

This gives

$$\begin{aligned} \delta s^2 - \delta s_0^2 &= \left(1 - \frac{1}{(1+\alpha)^2} \right) \delta r^2 \\ &= \frac{2\alpha + \alpha^2}{(1+\alpha)^2}\, \delta r^2 \\ &\approx 2\,\alpha\, \delta r^2 \,. \end{aligned} \tag{8.343}$$

The length of the tangential segment is, before deformation,

$$\delta s_0 = r_0\, \delta\varphi_0 = \frac{1}{1+\alpha}\, r\, \delta\varphi \tag{8.344}$$

and, after deformation,

$$\delta s = r\, \delta\varphi \,. \tag{8.345}$$

This gives

$$\begin{aligned} \delta s^2 - \delta s_0^2 &= \left(1 - \frac{1}{(1+\alpha)^2} \right) r^2\, \delta\varphi^2 \\ &= \frac{2\alpha + \alpha^2}{(1+\alpha)^2}\, r^2\, \delta\varphi^2 \\ &\approx 2\,\alpha\, r^2\, \delta\varphi^2 \,. \end{aligned} \tag{8.346}$$

The strain tensor was defined by equation 8.330:

$$\delta s^2 - \delta s_0^2 = 2\, \varepsilon_{ij}(\mathbf{x})\, \delta x^i\, \delta x^j \,.$$

Using the results 8.343 and 8.346 gives then

$$\varepsilon_{rr}(r, \varphi) = \alpha \tag{8.347}$$

and

$$\varepsilon_{\varphi\varphi}(r, \varphi) = \alpha r^2 \tag{8.348}$$

(demonstrating that $\varepsilon_{r\varphi} = 0$ would require the consideration of a third vector).

8.5.5 Appendix: The deformation as a function of the displacement vector

It should first be understood that small deformations is not equivalent to small displacements. For consider that a solid translation gives no deformation but arbitrarily large displacements.

Second, it is important to realize that the notion of “displacement vector” makes only general sense in Euclidean spaces, or in general spaces only if displacements are so small that we can consider that they belong to the tangent space.

In the sections before we have seen that the direct definition of the strain tensor has operational meaning, and that no other expression is needed. But as it is common usage to introduce a displacement vector and to use it for defining the strain, we are going to obtain the corresponding formulas here.

We face here one notational problem. The equations below, when developed in a general space, need an extensive use of the parallel transport of vectors and tensors, and become quite intricate. To avoid this complication, we are going to work here only with Euclidean spaces, and using Cartesian coordinates. The final formulas will then be generalized to arbitrary coordinate systems (but still for Euclidean spaces).

In what follows we will define the deformation of a medium at some time t with respect to an “undeformed state” at time t_0 .

As we did before, we will develop at left (resp. at right) the formulas correspondig to the definition of the strain tensor given by equation 8.329 (resp. 8.330), corresponding respectively to a Lagrangian or Eulerian description.

Let us define the deformation of the medium by the functions

$$x^i = x^i(\mathbf{x}_0, t) \tag{8.349}$$

giving the coordinates at time t of the material point whose coordinates at time t_0 were x_0^i . Two points that, before deformation, were separated by a (squared) distance

$$\delta s^2(t_0) = g_{ij}\, \delta x_0^i\, \delta x_0^j \tag{8.350}$$

will be separated, after deformation, by the (squared) distance

$$\begin{aligned} \delta s^2(t) &= g_{k\ell}\, \delta x^k\, \delta x^\ell \qquad (8.351) \\ &= g_{k\ell}\, \frac{\partial x^k}{\partial x_0^i}(\mathbf{x}_0, t) \delta x_0^i\, \frac{\partial x^\ell}{\partial x_0^j}(\mathbf{x}_0, t) \delta x_0^j\,. \end{aligned}$$

This easily gives

$$\delta s^2(t) - \delta s^2(t_0) = \tag{8.352}$$

$$= \left(g_{k\ell}\, \frac{\partial x^k}{\partial x_0^i}(\mathbf{x}_0, t)\, \frac{\partial x^\ell}{\partial x_0^j}(\mathbf{x}_0, t) - g_{ij} \right) \delta x_0^i\, \delta x_0^j\,,$$

and, using the definition 8.329 of the strain tensor,

$$\varepsilon_{ij}(\mathbf{x}_0, t) = \frac{1}{2} \left(g_{k\ell}\, \frac{\partial x^k}{\partial x_0^i}(\mathbf{x}_0, t)\, \frac{\partial x^\ell}{\partial x_0^j}(\mathbf{x}_0, t) - g_{ij} \right) . \tag{8.353}$$

If a *displacement vector* $u^i(\mathbf{x}_0, t)$ is introduced through

$$x^i(\mathbf{x}_0, t) = x_0^i + u^i(\mathbf{x}_0, t) \tag{8.354}$$

then, developing the expression 8.353 we obtain (dropping the variables $(\mathbf{x}_0, t)$ everywhere)

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_0^j} + \frac{\partial u_j}{\partial x_0^i} + \frac{\partial u_k}{\partial x_0^i} \frac{\partial u^k}{\partial x_0^j} \right) . \tag{8.355}$$

In a general coordinate system, this equation generalizes into

$$\boxed{\varepsilon_{ij} = \frac{1}{2} \left(\nabla_i u_j + \nabla_j u_i + \nabla_i u_k\, \nabla_j u^k \right) . \qquad (8.356)}$$

Let us define the deformation of the medium by the functions

$$x_0^i = x_0^i(\mathbf{x}, t) \tag{8.357}$$

giving the coordinates at time t_0 of the material point whose coordinates at time t are x^i . Two points that, after deformation, are separated by a (squared) distance

$$\delta s^2(t) = g_{ij}\, \delta x^i\, \delta x^j \tag{8.358}$$

were separated, before deformation, by the (squared) distance

$$\begin{aligned} \delta s^2(t_0) &= g_{k\ell}\, \delta x_0^k\, \delta x_0^\ell \qquad (8.359) \\ &= g_{k\ell}\, \frac{\partial x_0^k}{\partial x^i}(\mathbf{x}, t) \delta x^i\, \frac{\partial x_0^\ell}{\partial x^j}(\mathbf{x}, t) \delta x^j\,. \end{aligned}$$

This easily gives

$$\delta s^2(t) - \delta s^2(t_0) = \tag{8.360}$$

$$= \left(g_{ij} - g_{k\ell}\, \frac{\partial x_0^k}{\partial x^i}(\mathbf{x}, t)\, \frac{\partial x_0^\ell}{\partial x^j}(\mathbf{x}, t) \right) \delta x^i\, \delta x^j\,,$$

and, using the definition 8.330 of the strain tensor,

$$\varepsilon_{ij}(\mathbf{x}_0, t) = \frac{1}{2} \left(g_{ij} - g_{k\ell}\, \frac{\partial x_0^k}{\partial x^i}(\mathbf{x}, t)\, \frac{\partial x_0^\ell}{\partial x^j}(\mathbf{x}, t) \right) . \tag{8.361}$$

If a *displacement vector* $v^i(\mathbf{x}, t)$ is introduced through

$$x_0^i(\mathbf{x}, t) = x^i - v^i(\mathbf{x}, t)\,, \tag{8.362}$$

then, developing the expression 8.361 we obtain (dropping the variables $(\mathbf{x}, t)$ everywhere)

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - \frac{\partial v_k}{\partial x^i} \frac{\partial v^k}{\partial x^j} \right) . \tag{8.363}$$

In a general coordinate system, this equation generalizes into

$$\boxed{\varepsilon_{ij} = \frac{1}{2} \left(\nabla_i v_j + \nabla_j v_i - \nabla_i v_k\, \nabla_j v^k \right) . \qquad (8.364)}$$

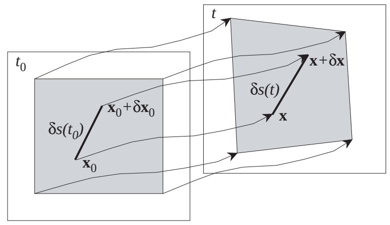

Figure 8.2: Two neighbouring points $\mathbf{x}_0$ and $\mathbf{x}_0 + \delta\mathbf{x}_0$ are separated by a distance δs_0 that can be computed using the metric tensor: $\delta s_0^2 = g_{ij}(\mathbf{x}_0)\,\delta x_0^i\,\delta x_0^i$. After a deformation, the two points will have new coordinates x^i and $x^i + \delta x^i$, and the new distance between the two points, δs can be computed from the expression $\delta s^2 = g_{ij}(\mathbf{x})\,\delta x^i\,\delta x^i$. The *strain field* can be defined by the expression $\delta s^2 - \delta s_0^2 = 2\,\varepsilon_{ij}(\mathbf{x}_0)\,\delta x_0^i\,\delta x_0^j$ or, equivalently, by $\delta s^2 - \delta s_0^2 = 2\,\varepsilon_{ij}(\mathbf{x})\,\delta x^i\,\delta x^j$ . .

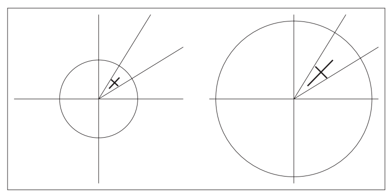

Figure 8.3: Under an homogeneous deformation, the two small vectors defined, in polar coordinates, by the components $(\delta r_0, 0)$ and $(0, \delta\varphi_0)$, transform into vectors whose components are $((1+\alpha)\,\delta r_0, 0)$ and $(0, \delta\varphi_0)$. The definition $\delta s^2 - \delta s_0^2 = 2\,\varepsilon_{ij}(\mathbf{x}_0)\,\delta x_0^i\,\delta x_0^j$ and/or the definition $\delta s^2 - \delta s_0^2 = 2\,\varepsilon_{ij}(\mathbf{x})\,\delta x^i\,\delta x^j$ allow a direct calculation of the strain tensor (see text).

8.5.6 Appendix: Physical dimensions of tensors

Let us be specific about the physical dimension of all fields considered. Mass, lenght and time dimensions will be respectively denoted by M , L , and T .

By a language abuse, we will write something like $[g_i] = \mathrm{L}^{-3}\mathrm{T}^2$, which, in all rigor, would mean that the components of the vector $\mathbf{g}$ have as dimension $\mathrm{L}^{-3}\mathrm{T}^2$. This, of course would not be correct, as the physical dimension of the components of any tensor depend on the coordinates chosen (think, for instance, that the physical dimension of the components of a tensor change if we replace, say, cartesian by spherical coordinates). So when we write $[g_i] = \mathrm{L}^{-3}\mathrm{T}^2$, we mean that it is the *norm* of the vector $\mathbf{g}$ that has such a dimension.

The space-time vector $\{d\underline{x}^0, d\underline{x}^i\} = \{c\ dt, d\underline{x}^i\}$ is assumed to have the physical dimension of a length.

Then, the main tensors considered in this chapter have the following physical dimensions:

$$[U] = [\varphi] = [g_{ij}] = 1$$

$$[S_{ij}{}^k] = [\Gamma_{ij}{}^k] = \mathrm{L}^{-1}$$

$$[J_{ij}] = [K_i{}^j] = [A_{ij}] = [B_{ij}] = \mathrm{L}^{-2}\mathrm{T}$$

$$[I_i] = [h_i] = [g_i] = \mathrm{L}^{-3}\mathrm{T}^2$$

$$[\alpha] = \mathrm{L}^{-4}\mathrm{T}^3$$

$$[r_{ij}{}^{k\ell}] = \mathrm{L}^{-2}$$

$$[\varepsilon_i{}^{jk}] = [\delta_{ij}{}^k] = \mathrm{L}^{-3}\mathrm{T}$$

$$[\gamma_i{}^j] = \mathrm{L}^{-4}\mathrm{T}^2$$

$$[T_i{}^j] = \mathrm{ML^{-1}T^{-2}}$$

$$[p_i] = [q^i] = \mathrm{ML^{-2}T^{-1}}$$

$$[\rho] = \mathrm{ML^{-3}}$$

$$[M_{ij}{}^k] = \mathrm{MT^{-2}}$$

$$[\pi_i{}^j] = [\sigma_{ij}] = \mathrm{ML^{-1}T^{-1}}$$

and

$$[t_i] = \mathrm{ML^{-2}} \ .$$

Note: Say somewhere that, in the section on torsional waves, while λ , μ, and ν have the physical dimension of a pressure, η , ξ , and ζ have the physical dimension of a force.

8.5.7 Appendix: Shearing, stretching, bending and twisting

Consider a point on a surface inside a continuous medium, and let n_i be the normal to the surface at the considered point. Let $T_i{}^j$ be the stress at that point, and $M_{ij}{}^k$ the moment stress.

The vector

$$\tau_i = T_i{}^j n_j \tag{8.365}$$

is called the *traction*, and the antisymmetric tensor

$$\mu_{ij} = M_{ij}{}^k n_k \tag{8.366}$$

is called the *moment traction*. Instead of the antisymmetric tensor μ_{ij} , we can use its dual vector

$$\breve{\mu}^i = \frac{1}{2}\varepsilon^{ijk}\mu_{jk} \ . \tag{8.367}$$

The decompositions

$$\underbrace{\underbrace{\boldsymbol{\tau}}}_{\text{traction}} = \underbrace{\underbrace{(\mathbf{n}\cdot\boldsymbol{\tau})\,\mathbf{n}}}_{\text{normal traction}} + \underbrace{\underbrace{(\mathbf{n}\times\boldsymbol{\tau})\times\mathbf{n}}}_{\text{shearing traction}} \tag{8.368}$$

and

$$\underbrace{\underbrace{\breve{\boldsymbol{\mu}}}}_{\text{couple (s. d.)}} = \underbrace{\underbrace{(\mathbf{n}\cdot\breve{\boldsymbol{\mu}})\,\mathbf{n}}}_{\text{twisting couple (s. d.)}} + \underbrace{\underbrace{(\mathbf{n}\times\breve{\boldsymbol{\mu}})\times\mathbf{n}}}_{\text{bending couple (s. d.)}} \tag{8.369}$$

project $\boldsymbol{\tau}$ and $\breve{\boldsymbol{\mu}}$ respectively onto the normal and onto the considered plane.

The letters (s. d.) stand there because we do not have couples, but surface densities of couples. A surface density of a force is called a traction, but there is no common name for the surface density of a couple.

To understand how these different efforts act, we may first imagine a crystal, and a plane separating two atomic planes (see figure). It is easy to imagine which sort of forces between atoms correspond to normal tension and shearing stress. An homogeneous twisting couple in the plane should correspond to a twisting couple of each atom facing the plane on the atom at the other side of the plane. The reader should realize that a global twist of the crystal (obtained by some external forces) does not generate this sort of (microscopical) twisting couples: the microscopical effects of a global twist in an ordinary elastic homogeneous medium generate only tractions, and not moment tractions.

8.5.8 Appendix: Tolo's letter

Primer punto de vista (que no me gusta)

Usando un sistema de coordenadas arrastradas por el cuerpo en su deformación, todo punto material tiene coordenadas constantes. El tensor métrico depende del tiempo. En un punto de coordenadas x^i , un pequeño vector de componentes dx^i tiene, at instante t , una longitud $ds(t)$ definida por la expresión

$$ds^2(t) = g_{ij}(\mathbf{x},t)\ dx^i\ dx^j \ . \tag{8.370}$$

Tenemos

$$ds^2(t) - ds^2(t_0) = (g_{ij}(\mathbf{x},t) - g_{ij}(\mathbf{x},t_0))\ dx^i\ dx^j \ , \tag{8.371}$$

y se define el *tensor de deformación*

$$2\ e_{ij}(\mathbf{x};t,t_0) = g_{ij}(\mathbf{x},t) - g_{ij}(\mathbf{x},t_0) \ . \tag{8.372}$$

Entonces

$$ds^2(t) - ds^2(t_0) = 2\ e_{ij}(\mathbf{x};t,t_0)\ dx^i\ dx^j \ . \tag{8.373}$$

Segundo punto de vista (que me gusta más)

El sistema de coordenadas está definido independientemente del movimiento del cuerpo. El tensor métrico no depende del tiempo. Todo punto material se representa por las coordenadas $\{x^i\}$ que tenia en cierto instante t_0 , antes de la deformación. La deformación del cuerpo estará perfectamente definida si se conocen las coordenadas $\{y^i\}$ de todo punto en todo tiempo t . Estas coordenadas pueden estar dadas, por ejemplo, por las funciones

$$y^i = y^i(\mathbf{x},t) \ . \tag{8.374}$$

Por hypótesis,

$$y^i(\mathbf{x},t_0) = x^i \ . \tag{8.375}$$

Sea, en el instante t_0 y en el punto de coordenadas $\{x^i\}$, un pequeño vector de componentes dx^i , y sea $ds(t_0)$ su longitud. En el instante t el vector estará en el punto de coordenadas $\{y^i(\mathbf{x},t)\}$, tendrá componentes $dy^i(t)$ y longitud $ds(t)$.

Tenemos

$$ds^2(t_0) = g_{ij}(\mathbf{y}(\mathbf{x},t_0))\ dy^i(t_0)\ dy^j(t_0) = g_{ij}(\mathbf{x})\ dx^i\ dx^j \tag{8.376}$$

y

$$ds^2(t) = g_{ij}(\mathbf{y}(\mathbf{x},t))\ dy^i(t)\ dy^j(t) = g_{ij}(\mathbf{y}(\mathbf{x},t))\ \frac{\partial y^i}{\partial x^k}(\mathbf{x},t)\ dx^k\ \frac{\partial y^j}{\partial x^\ell}(\mathbf{x},t)\ dx^\ell \ . \tag{8.377}$$

Definiendo

$$\begin{aligned} 2\ e_{ij}(\mathbf{x};t,t_0) &= g_{k\ell}(\mathbf{y}(\mathbf{x},t))\ \frac{\partial y^k}{\partial x^i}(\mathbf{x},t)\ \frac{\partial y^\ell}{\partial x^j}(\mathbf{x},t) - g_{ij}(\mathbf{y}(\mathbf{x},t_0)) \\ &= g_{k\ell}(\mathbf{y}(\mathbf{x},t))\ \frac{\partial y^k}{\partial x^i}(\mathbf{x},t)\ \frac{\partial y^\ell}{\partial x^j}(\mathbf{x},t) - g_{ij}(\mathbf{x}) \ , \end{aligned} \tag{8.378}$$

se tiene

$$ds^2(t) - ds^2(t_0) = 2\ e_{ij}(\mathbf{x};t,t_0)\ dx^i\ dx^j \ . \tag{8.379}$$

La deformación así definida no ha sido linearizada. Por consiguente, es válida incluso para deformaciones finitas.

A fin de cuentas la definición 8.378 es idéntica a 8.372, si interpretamos los $\{y^i\}$ como un nuevo sistema de coordenadas.

Medios elásticos (por ejemplo)

La deformación introducida arriba es la que entra en la definición de un cuerpo elástico, donde el tensor de los esfuerzos, σ_{ij} en el punto $\mathbf{x}$ y en el instante t , depende sólo de la deformación en éste punto y en éste instante:

$$\sigma_{ij}(\mathbf{x},t) = f_{ij}\left(\mathbf{e}(\mathbf{x},t)\right) \ . \tag{8.380}$$

Para un cuerpo elástico lineal ("ley" de Hooke),

$$\sigma_{ij}(\mathbf{x},t) = \sigma_{ij}^{(0)}(\mathbf{x}) + c_{ij}{}^{k\ell}(\mathbf{x})\ e_{k\ell}(\mathbf{x},t) \ . \tag{8.381}$$

Medio micropolar

Nota: aquí, de momento, se olvida todo rigor, y se hace todo a la americana, "quick and dirty".

Consider a body with large molecules, like a polymer, or a granular body, like an ordinary rock. In addition to the ordinary displacements u^i , "points" may undergo rotations φ^i .

We have the equations of conservation of linear and angular momentum:

$$\rho\frac{\partial^2 u^i}{\partial t^2} - \nabla_j\sigma^{ji} = 0 \tag{8.382}$$

and

$$J^{ij}\frac{\partial^2 \varphi_j}{\partial t^2} - \nabla_j\mu^{ji} = \epsilon^{ijk}\sigma_{jk} \ . \tag{8.383}$$

Aquí, J^{ij} representa la densidad de momento de inercia, y μ^{ij} el tensor "momento de esfuerzos". Un tal medio, con un tensor de esfuerzos σ^{ij} no simétrico, y un tensor μ^{ij} no nulo es un "medio micropolar de Cosserat".

Las "deformación ordinaria" está definida por

$$\varepsilon_{ij} = \nabla_i u_j - \epsilon_{ijk}\varphi^k \tag{8.384}$$

y la "deformación angular" por

$$\gamma_{ij} = \nabla_i\varphi_j \ . \tag{8.385}$$

If deformations are small enough, a first order development can be used:

$$\sigma_{ij} = c_{ijk\ell}\ \varepsilon^{k\ell} + b_{ijk\ell}\ \gamma^{k\ell} \tag{8.386}$$

and

$$\mu_{ij} = b_{ijk\ell}\ \varepsilon^{k\ell} + a_{ijk\ell}\ \gamma^{k\ell} \ . \tag{8.387}$$

Arguments based on the internal energy density imply the symmetries

$$c_{ijk\ell} = c_{k\ell ij} \tag{8.388}$$

and

$$a_{ijk\ell} = a_{k\ell ij} \ , \tag{8.389}$$

while the tensor $b_{ijk\ell}$ does not have any particular symmetry. This makes a total of $45+45+81 = 171$ independent elastic coefficients.

For isotropic media (Nowacki, 1986),

$$c_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \nu(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) \ , \tag{8.390}$$

$$a_{ijk\ell} = \eta\delta_{ij}\delta_{k\ell} + \xi(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \zeta(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) \ , \tag{8.391}$$

and

$$b_{ijkl} = 0 \ . \tag{8.392}$$

This gives

$$\sigma_{ij} = \lambda \varepsilon_k{}^k \delta_{ij} + \mu(\varepsilon_{ij} + \varepsilon_{ji}) + \nu(\varepsilon_{ij} - \varepsilon_{ji}) \tag{8.393}$$

and

$$\mu_{ij} = \eta \gamma_k{}^k \delta_{ij} + \xi(\gamma_{ij} + \gamma_{ji}) + \zeta(\gamma_{ij} - \gamma_{ji}) \ . \tag{8.394}$$

La pregunta

La deformación ε_{ij} utilizada para el medio micropolar se define correctamente como en las secciones 8.5.8–8.5.8 arriba. Cómo se define correctamente la deformación angular γ_{ij} ?

Mi impresión es que si la deformación ordinaria es simplemente la diferencia entre el tensor métrico final e inicial, la deformación angular deber ser la diferencia entre el tensor de torsión final e inicial.

Si la torsión se escribe, en función de la conexión como

$$S_{ij}{}^k = \Gamma_{ij}{}^k - \Gamma_{ji}{}^k \ , \tag{8.395}$$

se puede introducir el tensor

$$s^{i\ell} = \varepsilon^{ijk} \ S_{jk}{}^\ell \ . \tag{8.396}$$

Entonces, la deformación angular seria

$$\gamma^{ij}(\mathbf{x}; t, t_0) = s^{ij}(\mathbf{x}, t) - s^{ij}(\mathbf{x}, t_0) \ . \tag{8.397}$$

Pero no veo cómo justificar esto. Crees tu que es cierto? Estando tú convencido como estás de que la torsión del espacio es digna de estudio, no seria interesante tener para la torsión la misma analogia que para la métrica con la deformación de un cuerpo elástico?

8.5.9 Appendix: Maxwell's equations

Comment: for the time being, this section is written for special relativity (generalize?), and for media at rest (generalize?).

Note: I can not underline here the ε , that may represent the Levi-Civita capacity.

The first electromagnetic tensor: The first electromagnetic tensor will be here denoted $\underline{F}^{\alpha\beta}$. We will see that its 3-D components correspond to the fields $\mathbf{E}$ and $\mathbf{B}$. The dual of this tensor will be denoted $\underline{f}_{\alpha\beta}$. We have then the equivalent equations

$$\underline{F}^{\alpha\beta} = \frac{1}{2!}\,\underline{\varepsilon}^{\alpha\beta\gamma\delta}\,\underline{f}_{\gamma\delta} \qquad \Longleftrightarrow \qquad \underline{f}_{\alpha\beta} = \frac{1}{2!}\,\underline{\varepsilon}_{\alpha\beta\gamma\delta}\,\underline{F}^{\gamma\delta} \ . \tag{8.398}$$

In terms of 3-D fields we have (note: given the 4-velocity, the introduction of the 3-D can be done rigorously [to be done?]), equivalently,

$$\begin{pmatrix} \underline{F}^{00} & \underline{F}^{0j} \\ \underline{F}^{i0} & \underline{F}^{ij} \end{pmatrix} = \begin{pmatrix} 0 & B^j c \\ -B^i c & -e^{ij} \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{pmatrix} \underline{f}_{00} & \underline{f}_{0j} \\ \underline{f}_{i0} & \underline{f}_{ij} \end{pmatrix} = \begin{pmatrix} 0 & -E_j \\ E_i & b_{ij}\ c \end{pmatrix} \ , \tag{8.399}$$

where we have the 3-D duality relations

$$e_{ij} = \frac{1}{1!}\,\varepsilon_{ijk}\,E^k \qquad \Longleftrightarrow \qquad E^i = \frac{1}{2!}\,\varepsilon^{ijk}\,e_{jk} \tag{8.400}$$

$$B^i = \frac{1}{2!}\,\varepsilon^{ijk}\,b_{jk} \qquad \Longleftrightarrow \qquad b_{ij} = \frac{1}{1!}\,\varepsilon_{ijk}\,B^k \ . \tag{8.401}$$

The first Maxwell equation: The first Maxwell equation states that the tensor $\underline{F}^{\alpha\beta}$ is divergence free. This gives (writing also the equivalent dual equation)

$$\nabla_\beta \underline{F}^{\alpha\beta} = 0 \qquad \Longleftrightarrow \qquad \nabla_\alpha \underline{f}_{\beta\gamma} + \nabla_\beta \underline{f}_{\gamma\alpha} + \nabla_\gamma \underline{f}_{\alpha\beta} = 0 \ . \tag{8.402}$$

The second electromagnetic tensor: The second electromagnetic tensor will be here denoted $\underline{K}^{\alpha\beta}$. We will see that its 3-D components correspond to the fields $\mathbf{H}$ and $\mathbf{D}$. The dual of this tensor will be denoted $\underline{k}_{\alpha\beta}$. We have then the equivalent equations

$$\underline{K}^{\alpha\beta} = \frac{1}{2!}\,\underline{\varepsilon}^{\alpha\beta\gamma\delta}\,\underline{k}_{\gamma\delta} \qquad \Longleftrightarrow \qquad \underline{k}_{\alpha\beta} = \frac{1}{2!}\,\underline{\varepsilon}_{\alpha\beta\gamma\delta}\,\underline{K}^{\gamma\delta} \ . \tag{8.403}$$

In terms of 3-D fields we have, equivalently,

$$\begin{pmatrix} \underline{K}^{00} & \underline{K}^{0j} \\ \underline{K}^{i0} & \underline{K}^{ij} \end{pmatrix} = \begin{pmatrix} 0 & D^j c \\ -D^i c & h^{ij} \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{pmatrix} \underline{k}_{00} & \underline{k}_{0j} \\ \underline{k}_{i0} & \underline{k}_{ij} \end{pmatrix} = \begin{pmatrix} 0 & H_j \\ -H_i & d_{ij}\ c \end{pmatrix} \ . \tag{8.404}$$

where we have the 3-D duality relations

$$h_{ij} = \frac{1}{1!}\,\varepsilon_{ijk}\,H^k \qquad \Longleftrightarrow \qquad H^i = \frac{1}{2!}\,\varepsilon^{ijk}\,h_{jk} \tag{8.405}$$

$$D^i = \frac{1}{2!}\,\varepsilon^{ijk}\,d_{jk} \qquad \Longleftrightarrow \qquad d_{ij} = \frac{1}{1!}\,\varepsilon_{ijk}\,D^k \ . \tag{8.406}$$

The electric current: The electric current is described using the vector $\underline{J}^\alpha$, or the dual tensor $\underline{j}_{\alpha\beta\gamma}$. We have the equivalent equations

$$\underline{J}^\alpha = \frac{1}{3!}\,\underline{\varepsilon}^{\alpha\beta\gamma\delta}\,\underline{j}_{\beta\gamma\delta} \qquad \Longleftrightarrow \qquad \underline{j}_{\alpha\beta\gamma} = \frac{1}{1!}\,\underline{\varepsilon}^{\alpha\beta\gamma\delta}\,\underline{J}^\delta \ . \tag{8.407}$$

In terms of 3-D fields we have, equivalently (CHECK SIGNS HERE!),

$$\begin{pmatrix} \underline{J}^0 \\ \underline{J}^i \end{pmatrix} = \begin{pmatrix} \rho c \\ j^i \end{pmatrix} \qquad \Longleftrightarrow \qquad \left.\begin{matrix} \underline{j}_{123} \\ \underline{j}_{230} \\ \underline{j}_{301} \\ \underline{j}_{012} \end{matrix}\right\} = \left\{\begin{matrix} \rho c \\ j^1 \\ j^2 \\ j^3 \end{matrix}\right. \ , \tag{8.408}$$

where, the other components not shown at the right hand side are zero or are obtained by simple permutaion of indices.

The second Maxwell equation: The second Maxwell equation states that the divergence of the tensor $\underline{K}^{\alpha\beta}$ equals the electric current vector, i.,e., (writing also the equivalent dual equation)

$$\nabla_\beta \underline{K}^{\alpha\beta} = \underline{J}^\alpha \qquad \Longleftrightarrow \qquad \nabla_\alpha \underline{k}_{\beta\gamma} + \nabla_\beta \underline{k}_{\gamma\alpha} + \nabla_\gamma \underline{k}_{\alpha\beta} + \underline{j}_{\alpha\beta\gamma} = 0 \tag{8.409}$$

3-D form of Maxwell equations: The first 4-D Maxwell equation can be written, in 3-D, using any of the two equivalent set of expressions

$$\left.\begin{matrix} \nabla_i B^i = 0 \\ \nabla_j e^{ij} + \frac{\partial B^i}{\partial t} = 0 \end{matrix}\right\} \qquad \Longleftrightarrow \qquad \left\{\begin{matrix} \nabla_i b_{jk} + \nabla_j b_{ki} + \nabla_k b_{ij} = 0 \\ (\nabla_i E_j - \nabla_j E_i) + \frac{\partial b_{ij}}{\partial t} = 0 \end{matrix}\right. \ , \tag{8.410}$$

although the standard (pseudo-vectorial) form of the equations is

$$\text{div } \mathbf{B} = 0 \tag{8.411}$$

$$\mathbf{rot}\ \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \ . \tag{8.412}$$

The second 4-D Maxwell equation can be written, in 3-D, using any of the two equivalent set of expressions (note: sign of the last equation not checked)

$$\left.\begin{matrix} \nabla_i D^i = \rho \\ \nabla_j h^{ij} - \frac{\partial D^i}{\partial t} = j^i \end{matrix}\right\} \qquad \Longleftrightarrow \qquad \left\{\begin{matrix} \nabla_i d_{jk} + \nabla_j d_{ki} + \nabla_k d_{ij} = \rho \\ (\nabla_i H_j - \nabla_j H_i) - \frac{\partial d_{ij}}{\partial t} = \varepsilon_{ijk}\, j^k \end{matrix}\right. \ , \tag{8.413}$$

although the standard (pseudo-vectorial) form of the equations is

$$\text{div } \mathbf{D} = \rho \tag{8.414}$$

$$\mathbf{rot}\ \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j} \ . \tag{8.415}$$

The polarization tensor: The polarization tensor $\underline{\mathcal{P}}_{\alpha\beta}$ is defined by

$$\underline{\mathcal{P}}_{\alpha\beta} = \underline{f}_{\alpha\beta} - \underline{K}_{\alpha\beta} \ . \tag{8.416}$$

Writing

$$\begin{pmatrix} \underline{\mathcal{P}}_{00} & \underline{\mathcal{P}}_{0j} \\ \underline{\mathcal{P}}_{i0} & \underline{\mathcal{P}}_{ij} \end{pmatrix} = \begin{pmatrix} 0 & P_j c \\ -P_i c & m_{ij} \end{pmatrix} \tag{8.417}$$

gives

$$P_i + E_i = c\, D_i \qquad ; \qquad m_{ij} + h_{ij} = c\, b_{ij} \ . \tag{8.418}$$

Minkowski's energy-momentum tensor: It is defined by

$$\underline{T}_\alpha^\beta = \frac{1}{2c}\left(\underline{f}_{\alpha\sigma}\underline{K}^{\beta\sigma} - \underline{k}_{\alpha\sigma}\underline{F}^{\beta\sigma}\right) \ . \tag{8.419}$$

Using the definition 8.60

$$\begin{pmatrix} \underline{T}_0{}^0 & \underline{T}_0{}^j \\ \underline{T}_i{}^0 & \underline{T}_i{}^j \end{pmatrix} = \begin{pmatrix} -\rho c^2 & -q^j c \\ c p_i & T_i{}^j \end{pmatrix} \tag{8.420}$$

gives

$$\rho = \frac{1}{2c^2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right) \tag{8.421}$$

$$\mathbf{p} = \mathbf{D}\times\mathbf{B} \tag{8.422}$$

$$\mathbf{q} = \frac{1}{c^2}\,\mathbf{E}\times\mathbf{H} \tag{8.423}$$

$$T_i{}^j = \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)\delta_i{}^j - \left(E_i D^j + H_i B^j\right) \ . \tag{8.424}$$

Here, ρ represents the energy density of the electromagnetic field. $T_i{}^j$ is the Maxwell's stress tensor. $\mathbf{p}$ is the linear momentum density and $\mathbf{q}$ is the energy flux (the *Poynting* vector). (Comment: names from Møller, definitions from Minkowski). We see that, in general,

$$\mathbf{p} \neq \mathbf{q}\,, \tag{8.425}$$

i.e., the linear momentum density is not identical to the energy flux.

Antisymmetric part of the energy-momentum tensor: We have seen that the antisymmetric part of the energy-momentum tensor

$$\underline{\psi}_{\alpha\beta} = \underline{T}_{\alpha\beta} - \underline{T}_{\alpha\beta} \tag{8.426}$$

is the "source term" for the continuity equations concerning the spin. The antisymmetric tensor $\underline{\psi}_{ij}$ is equivalent to a 3-D vector and to an antisymmetric 3-D tensor.

The 3-D vector is $\mathbf{p}-\mathbf{q}$. The antisymmetric 3-D tensor is

$$\psi_{ij} = T_{ij} - T_{ji} = (B_i H_j - B_j H_i) + (D_i E_j - D_j E_i) \ . \tag{8.427}$$

Associating to ψ_{ij} the vector

$$\Psi^i = \frac{1}{2!}\,\varepsilon^{ijk}\,\psi_{jk} \tag{8.428}$$

we can write

$$\Psi = \mathbf{B}\times\mathbf{H} + \mathbf{D}\times\mathbf{E} \ . \tag{8.429}$$

Using equations 8.418, i.,e.,

$$\mathbf{D} = \frac{1}{c}\left(\mathbf{P}+\mathbf{E}\right) \qquad ; \qquad \mathbf{B} = \frac{1}{c}\left(\mathbf{M}+\mathbf{H}\right) . \tag{8.430}$$

gives

$$\Psi = \frac{1}{c}\left(\mathbf{M}\times\mathbf{H} + \mathbf{P}\times\mathbf{E}\right) \ , \tag{8.431}$$

which is the standard expression for the couple acting on a medium magnetized and electrically polarized.

Note: emphasize here that the nonsymmetry of the Minkowsky tensor gives this couple. It is too bad to see most books on electromagnetism overlook this important property of the Minkowsky tensor. There has been considerable debate in the literature, and some have proposed symmetrized versions of it (see for instance Abraham, 1909; and Abraham and Becker, 1933). Brevik (1970) showed that the Minkowski's tensor more conveniently describes some experimental results. The reader may refer to the short discussion in Møller (1972).

Note: say that for a linear and isotropic medium,

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{8.432}$$

and

$$\mathbf{B} = \mu \mathbf{H} \,, \tag{8.433}$$

and the Maxwell's stress is symmetric:

$$T_{ij} = T_{ji} \,. \tag{8.434}$$

In vacuo,

$$\varepsilon = \varepsilon_0 \tag{8.435}$$

and

$$\mu = \mu_0 \,, \tag{8.436}$$

and, as

$$\varepsilon_0 \mu_0 = \frac{1}{c^2} \,, \tag{8.437}$$

the linear momentum density equals the energy flux:

$$\mathbf{p} = \mathbf{q} \,. \tag{8.438}$$

The Lorentz density of force:

$$\underline{\Phi}_\alpha = -\frac{1}{c} \nabla_\beta \underline{T}_\alpha{}^\beta \tag{8.439}$$

Comment: try to justify why the force density plus the divergence of the stress should be zero.

This gives:

$$\underline{\Phi}_\alpha = \frac{1}{c} \left(\underline{f}_{\alpha\beta} \underline{J}^\beta + \frac{1}{4} \left(\underline{f}_{\beta\gamma} \nabla_\alpha \underline{K}^{\beta\gamma} - \underline{k}_{\beta\gamma} \nabla_\alpha \underline{F}^{\beta\gamma} \right) \right) \tag{8.440}$$

The space components of the force density are

$$\varphi_i = \rho E_i + \varepsilon_{ik\ell} j^k B^\ell + \frac{1}{2} \left(B^k \nabla_i H_k - H_k \nabla_i B^k + D^k \nabla_i E_k - E_k \nabla_i D^k \,. \right) \tag{8.441}$$

If the medium is linear and isotropic ($\mathbf{D} = \varepsilon \mathbf{E}$; $\mathbf{B} = \mu \mathbf{H}$) and homogeneous ($\nabla_i \varepsilon = 0$; $\nabla_i \mu = 0$), then the Lorentz Force density reduces to the usual expression

$$\boldsymbol{\varphi} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} \,. \tag{8.442}$$

Comment: Say somewhere that Sédov mentions the moment stress tensor $M_{ij}{}^k$ for the electromagnetic field, but that it does not give any expression for it.

Physical dimensions:

$$\begin{gathered} [q] = Q \\ [E] = Q^{-1} M L T^{-2} \\ [D] = Q L^{-2} \\ [H] = Q L^{-1} T^{-1} \\ [B] = Q^{-1} M T^{-1} \\ [\varepsilon] = Q^2 M^{-1} L^{-3} T^2 \\ [\mu] = Q^{-2} M L \\ [c] = L T^{-1} \end{gathered} \tag{8.443}$$

Values of physical constants:

$$c = 299\ 792\ 458 \text{ m s}^{-1} \tag{8.444}$$

Comment: this is an exact value, the unit of length being a derived quantity.

$$\mu_0 = 4\pi 10^{-7} \text{ C}^{-2} \text{ kg m} \tag{8.445}$$

Comment: exact value.

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.854\ 187\ 817\ldots \text{ C}^2 \text{ kg}^{-1} \text{ m}^{-3} \text{ s}^2 \tag{8.446}$$

Comment: exact value.

Comment: C stands for the Coulomb, the unit of charge. The Ampère is $\text{A} = \text{C s}^{-1}$.

Note: for the values of the physical constants, see for instance Cohen and Taylor (1991).

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8.5.10 Appendix: Demonstration: Time derivative and space integration (Euclidean spaces)

This appendix demonstrates that if we define the vector

$$\begin{aligned} \mathbf{P}(t) &= \int_{V(t)} dV(\mathbf{x})\, \mathbf{p}(\mathbf{x},t) \\ &= \int_{V(t)} dV(\mathbf{x})\, p^i(\mathbf{x},t)\, \mathbf{e}_i(\mathbf{x})\,, \end{aligned} \tag{8.447}$$

then,

$$\frac{d\mathbf{P}}{dt}(t) = \int_{V(t)} dV(\mathbf{x}) \left\{ \frac{\partial p^i}{\partial t}(\mathbf{x},t) + \nabla_j \left(v^j(\mathbf{x},t)\, p^i(\mathbf{x},t) \right) \right\} \mathbf{e}_i(\mathbf{x})\,. \tag{8.448}$$

Here, the notation $V(t)$ indicates that the considered volume follows the matter in his movement (i.e., the relative velocity of the matter with respect to the surface of the volume is zero).

As a corollary, if the condition

$$\frac{d\mathbf{P}}{dt}(t) = 0 \tag{8.449}$$

holds for any volume inside the medium, then, at any point,

$$\frac{\partial p^i}{\partial t}(\mathbf{x},t) + \nabla_j \left(v^j(\mathbf{x},t)\, p^i(\mathbf{x},t) \right) = 0\,. \tag{8.450}$$

The demonstration for a general tensor is procedes along the same lines.

Warning: the integrals here are not using the concept of parallel transport. Therefore they are only defined in Euclidean spaces.

Demonstration:

$$\frac{d\mathbf{P}}{dt}(t) = \frac{d}{dt} \int_{V(t)} dV(\mathbf{x})\, p^i(\mathbf{x},t)\, \mathbf{e}_i(\mathbf{x}) \tag{8.451}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_{V(t+\Delta t)} dV(\mathbf{x})\, p^i(\mathbf{x}, t+\Delta t)\, \mathbf{e}_i(\mathbf{x}) - \int_{V(t)} dV(\mathbf{x})\, p^i(\mathbf{x},t)\, \mathbf{e}_i(\mathbf{x}) \right\} \tag{8.452}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_{V(t+\Delta t)} dV(\mathbf{x}) \left(p^i(\mathbf{x},t) + \frac{\partial p^i}{\partial t}(\mathbf{x},t)\Delta t \right) \mathbf{e}_i(\mathbf{x}) - \int_{V(t)} dV(\mathbf{x})\, p^i(\mathbf{x},t)\, \mathbf{e}_i(\mathbf{x}) \right\} \tag{8.453}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_{V(t)} dV \left(p^i + \frac{\partial p^i}{\partial t}\Delta t \right) \mathbf{e}_i + \int_{\Delta V} dV \left(p^i + \frac{\partial p^i}{\partial t}\Delta t \right) \mathbf{e}_i - \int_{V(t)} dV\, p^i\, \mathbf{e}_i \right\} \tag{8.454}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \Delta t \int_{V(t)} dV\, \frac{\partial p^i}{\partial t}\, \mathbf{e}_i + \int_{\Delta V} dV \left(p^i + \frac{\partial p^i}{\partial t}\Delta t \right) \mathbf{e}_i \right\} \tag{8.455}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \Delta t \int_{V(t)} dV\, \frac{\partial p^i}{\partial t}\, \mathbf{e}_i + \int_{S(t)} dS\, n_j\, v^j\, \Delta t \left(p^i + \frac{\partial p^i}{\partial t}\Delta t \right) \mathbf{e}_i \right\} \tag{8.456}$$

$$= \int_{V(t)} dV\, \frac{\partial p^i}{\partial t}\, \mathbf{e}_i + \int_{S(t)} dS\, n_j\, v^j p^i\, \mathbf{e}_i \tag{8.457}$$

$$= \int_{V(t)} dV\, \frac{\partial p^i}{\partial t}\, \mathbf{e}_i + \int_{V(t)} dV\ \nabla_j \left(v^j p^i\, \mathbf{e}_i \right) \tag{8.458}$$

$$= \int_{V(t)} dV\, \frac{\partial p^i}{\partial t}\, \mathbf{e}_i + \int_{V(t)} dV\ \nabla_j \left(v^j p^i \right) \mathbf{e}_i \tag{8.459}$$

$$= \int_{V(t)} dV \left(\frac{\partial p^i}{\partial t} + \nabla_j \left(v^j p^i \right) \right) \mathbf{e}_i\,. \tag{8.460}$$

Equation 8.451 is just the original definition. Equation 8.452 explicits the time derivative. Note that, as the considered volume follows the matter movement, at time $t + \Delta t$ we have to integrate over a volume that, in general, will be different from $V(t)$. Equation 8.453 takes the first order of a Taylor's development (higher orders are not needed). Equation 8.454 says that the volume at time $t + \Delta t$ will equal the volume at time t plus a certain volume change ΔV to be evaluated below. Equation 8.455 is just a simplification of the previous one. Equation 8.456 says that the volume change ΔV can be obtained by integrating on the surface $S(t)$ the normal component of the velocity (times Δt). Equation 8.457 takes the limit $\Delta t \to 0$, which amounts to dropping a second order term and simplifying. Equation 8.458 uses the Green's theorem (the surface integral of a flux equals the volume integral of the divergence). Equation 8.459 uses the formal property $\nabla_i \mathbf{e}_j = 0$ (see section XXX). This directly leads to equation 8.460.

8.5.11 Appendix: Demonstration: Time derivative and space integration (general case)

This appendix demonstrates the formula

$$\frac{d}{dt} \int_{\mathcal{V}(t)} dV(\mathbf{x}) \varphi^i(\mathbf{x})\, p_i(\mathbf{x}, t) = \tag{8.461}$$

$$= \int_{\mathcal{V}(t)} dV(\mathbf{x})\, \varphi^i(\mathbf{x}) \left\{ \frac{\partial p_i}{\partial t} + \nabla_j \left(p_i\, v^j \right) \right\} (\mathbf{x}, t) + \int_{\mathcal{V}(t)} dV(\mathbf{x})\, p_i(\mathbf{x}, t)\, v^j(\mathbf{x}, t) \left(\nabla_j \varphi^i \right) (\mathbf{x}, t) \, .$$

Here, the notation $\mathcal{V}(t)$ indicates that the considered volume follows the matter in his movement (i.e., the relative velocity of the matter with respect to the surface of the volume is zero).

Demonstration:

$$\frac{d}{dt} \int_{\mathcal{V}(t)} dV(\mathbf{x}) \varphi^i(\mathbf{x})\, p_i(\mathbf{x}, t) =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_{\mathcal{V}(t+\Delta t)} dV(\mathbf{x})\, \varphi^i(\mathbf{x})\, p_i(\mathbf{x}, t + \Delta t) - \int_{\mathcal{V}(t)} dV(\mathbf{x})\, \varphi^i(\mathbf{x})\, p_i(\mathbf{x}, t) \right\} \tag{8.462}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_{\mathcal{V}(t+\Delta t)} dV(\mathbf{x})\, \varphi^i(\mathbf{x}) \left(p_i(\mathbf{x}, t) + \frac{\partial p_i}{\partial t}(\mathbf{x}, t) \Delta t \right) - \int_{\mathcal{V}(t)} dV(\mathbf{x})\, \varphi^i(\mathbf{x})\, p_i(\mathbf{x}, t) \right\} \tag{8.463}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_{\mathcal{V}(t)} dV\, \varphi^i \left(p_i + \frac{\partial p_i}{\partial t} \Delta t \right) + \int_{\Delta \mathcal{V}} dV\, \varphi^i \left(p_i + \frac{\partial p_i}{\partial t} \Delta t \right) - \int_{\mathcal{V}(t)} dV\, \varphi^i\, p^i \right\} \tag{8.464}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \Delta t \int_{\mathcal{V}(t)} dV\, \varphi^i \frac{\partial p_i}{\partial t} + \int_{\Delta \mathcal{V}} dV\, \varphi^i \left(p_i + \frac{\partial p_i}{\partial t} \Delta t \right) \right\} \tag{8.465}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \Delta t \int_{\mathcal{V}(t)} dV\, \varphi^i \frac{\partial p_i}{\partial t} + \int_{\mathcal{S}(t)} dS\, n_j\, v^j\, \Delta t\, \varphi^i \left(p_i + \frac{\partial p_i}{\partial t} \Delta t \right) \right\} \tag{8.466}$$

$$= \int_{\mathcal{V}(t)} dV\, \varphi^i \frac{\partial p_i}{\partial t} + \int_{\mathcal{S}(t)} dS\, n_j\, v^j\, \varphi^i\, p^i \tag{8.467}$$

$$= \int_{\mathcal{V}(t)} dV\, \varphi^i \frac{\partial p_i}{\partial t} + \int_{\mathcal{V}(t)} dV\, \nabla_j \left(v^j\, \varphi^i\, p_i \right) \tag{8.468}$$

$$= \int_{\mathcal{V}(t)} dV\, \varphi^i \frac{\partial p_i}{\partial t} + \int_{\mathcal{V}(t)} dV\, \varphi^i\, \nabla_j \left(v^j\, p_i \right) + \int_{\mathcal{V}(t)} dV\, p_i\, v^j\, \nabla_j \varphi^i \tag{8.469}$$

$$= \int_{\mathcal{V}(t)} dV\, \varphi^i \left(\frac{\partial p^i}{\partial t} + \nabla_j \left(v^j\, p_i \right) \right) + \int_{\mathcal{V}(t)} dV\, p_i\, v^j\, \nabla_j \varphi^i \, . \tag{8.470}$$

Equation 8.462 explicits the time derivative. Note that, as the considered volume follows the matter movement, at time $t+\Delta t$ we have to integrate over a volume that, in general, will be different from $V(t)$. Equation 8.463 takes the first order of a Taylor's development (higher orders are not needed). Equation 8.464 says that the volume at time $t+\Delta t$ will equal the volume at time t plus a certain volume change ΔV to be evaluated below. Equation 8.465 is just a simplification of the previous one. Equation 8.466 says that the volume change ΔV can be obtained by integrating on the surface $S(t)$ the normal component of the velocity (times Δt). Equation 8.467 takes the limit $\Delta t \to 0$, which amounts to dropping a second order term and simplifying. Equation 8.468 uses the Green's theorem (the surface integral of a flux equals the volume integral of the divergence). Equation 8.469 uses the formal property $\nabla_i \mathbf{e}_j = 0$ (see section XXX). This directly leads to equation 8.470.

8.5.12 Appendix: Demonstration: Classical conservation equations

Consider a continuous medium from the viewpoint of nonrelativistic mechanics, and let, at a point, $\mathbf{x}$ and time t , $\rho(\mathbf{x},t)$ denote the density of mass, $\sigma_{ij}(\mathbf{x},t)$ the density of intrinsic angular momentum (spin), and $v^i(\mathbf{x},t)$ the velocity with respect a Galilean (i.e., nonaccelerated) frame. The space is here assumed Euclidean, but the coordinates are not necessarily chosen Cartesian.

We consider a volume inside the medium that follows the matter on his movement and call it $V(t)$. The surface of the volume will be denoted $S(t)$.

The *mass* inside the volume is:

$$M(t) = \int_{V(t)} dV(\mathbf{x})\ \rho(\mathbf{x},t)\,. \tag{8.471}$$

Denoting $p_i = \rho v_i$, the *linear momentum* is:

$$P_i(t) = \int_{V(t)} dV(\mathbf{x})\ p_i(\mathbf{x},t)\,. \tag{8.472}$$

The total *force* acting on that portion of matter equals the sum of the forces acting inside the volume plus the sum of the forces acting on the surface of the volume:

$$F_i(t) = \int_{V(t)} dV(\mathbf{x})\ f_i(\mathbf{x},t) + \int_{S(t)} dS(\mathbf{x})\ \phi_i(\mathbf{x},t)\,. \tag{8.473}$$

Here, f_i is the *force density* of external origin, ϕ_i denotes the *surface traction* that the exterior medium exerts on the surface, related to the *stress* $\tau_i{}^j$ through $\phi_i = \tau_i{}^j\, n_j$. (Comment explain the normal).

The total *angular momentum* (with respect to the origin of coordinates) is obtained as the sum of the intrinsic angular momentum density (or *spin*) plus the extrinsic (or orbital) angular momentum density:

$$\Sigma_{ij}(t) = \int_{V(t)} dV(\mathbf{x})\ \{\sigma_{ij}(\mathbf{x},t) + (r_i(\mathbf{x})\, p_j(\mathbf{x},t) - r_j(\mathbf{x})\, p_i(\mathbf{x},t))\}\ , \tag{8.474}$$

where $\mathbf{r}(\mathbf{x})$ is the position vector of the point $\mathbf{x}$. As we are only interested here in flat (Euclidean) spaces the introduction of this vector poses no problem.

The total *torque* (with respect to the origin of coordinates) is obtained as the sum of four terms: The result of the external torque density, ψ_{ij} , the couple resulting from the external force density f_i , and the two surface contributions: the result of the *surface torque density* that the exterior medium exerts on the surface, φ_{ij} , and the couple resulting from the surface traction ϕ_i :

$$\begin{aligned}\Gamma_{ij}(t) \ &= \int_{V(t)} dV(\mathbf{x})\ \{\psi_{ij}(\mathbf{x},t) + (r_i(\mathbf{x})\, f_j(\mathbf{x},t) - r_j(\mathbf{x})\, f_i(\mathbf{x},t))\} \\ &+ \int_{S(t)} dS(\mathbf{x})\ \{\varphi_{ij}(\mathbf{x},t) + (r_i(\mathbf{x})\, \phi_j(\mathbf{x},t) - r_j(\mathbf{x})\, \phi_i(\mathbf{x},t))\}\ .\end{aligned} \tag{8.475}$$

The surface torque density φ_{ij} is related to the couple stress ${m_{ij}}^k$ through $\varphi_{ij} = {m_{ij}}^k\, n_k$.

We are going to show that if the (global) conservation equations

$$\frac{dM}{dt}(t) = 0 \tag{8.476}$$

$$\frac{dP_i}{dt}(t) = F_i(t) \tag{8.477}$$

$$\frac{d\Sigma_{ij}}{dt}(t) = \Gamma_{ij}(t) \tag{8.478}$$

hold for any volume inside the medium, then, they are equivalent to the set of local conservation equations

$$\frac{\partial \rho}{\partial t} + \nabla_j(\rho v^j) = 0 \tag{8.479}$$

$$\frac{\partial p_i}{\partial t} + \nabla_j(p_i v^j - {\tau_i}^j) = f_i \tag{8.480}$$

$$\frac{\partial \sigma_{ij}}{\partial t} + \nabla_k(\sigma_{ij} v^k - {m_{ij}}^k) = \psi_{ij} + (\tau_{ji} - \tau_{ij}) + (p_i v_j - p_j v_i)\ . \tag{8.481}$$

Demonstrations

The mass: Equation 8.479 follows directly from equation 8.476 using the results of the previous annex.

The linear momentum: We remark that from $\phi_i = {\tau_i}^j\, n_j$ it follows, using Green's theorem,

$$\begin{aligned} F_i(t) &= \int_{V(t)} dV(\mathbf{x})\ f_i(\mathbf{x},t) + \int_{S(t)} dS(\mathbf{x})\ \phi_i(\mathbf{x},t) \\ &= \int_{V(t)} dV(\mathbf{x})\ f_i(\mathbf{x},t) + \int_{S(t)} dS(\mathbf{x})\ n_j(\mathbf{x})\, {\tau_i}^j(\mathbf{x},t) \\ &= \int_{V(t)} dV(\mathbf{x})\ f_i(\mathbf{x},t) + \int_{V(t)} dV(\mathbf{x})\ (\nabla_j\, {\tau_i}^j)(\mathbf{x},t) \\ &= \int_{V(t)} dV(\mathbf{x})\ \Big(f_i(\mathbf{x},t) + (\nabla_j\, {\tau_i}^j)(\mathbf{x},t)\Big) \end{aligned} \tag{8.482}$$

Then, equation 8.480 follows directly from the previous appendix.

The angular momentum: We have defined (equation 8.475):

$$\Gamma_{ij}(t) = \Gamma^{(\mathrm{V})}_{ij}(t) + \Gamma^{(\mathrm{S})}_{ij}(t)\,, \tag{8.483}$$

where

$$\Gamma^{(\mathrm{V})}_{ij}(t) = \int_{V(t)} dV(\mathbf{x})\ \{\psi_{ij}(\mathbf{x},t) + (r_i(\mathbf{x})\, f_j(\mathbf{x},t) - r_j(\mathbf{x})\, f_i(\mathbf{x},t))\} \tag{8.484}$$

and

$$\Gamma^{(\mathrm{S})}_{ij}(t) = \int_{S(t)} dS(\mathbf{x})\ \{\varphi_{ij}(\mathbf{x},t) + (r_i(\mathbf{x})\, \phi_j(\mathbf{x},t) - r_j(\mathbf{x})\, \phi_i(\mathbf{x},t))\}\ . \tag{8.485}$$

Introducing $\varphi_{ij} = m_{ij}{}^k\, n_k$ and $\phi_i = \tau_i{}^j\, n_j$, using the Green's theorem, and using $\nabla_i\, r^j = \delta_i{}^j$, gives

$$
\begin{aligned}
\Gamma^{(\mathrm{S})}_{ij}(t) &= \int_{S(t)} dS\ n_k \left\{ m_{ij}{}^k + \left(r_i\, \tau_j{}^k - r_j\, \tau_i{}^k \right) \right\} \\
&= \int_{V(t)} dV\ \nabla_k \left\{ m_{ij}{}^k + \left(r_i\, \tau_j{}^k - r_j\, \tau_i{}^k \right) \right\} \\
&= \int_{V(t)} dV\ \left(\nabla_k m_{ij}{}^k + g_{ki}\, \tau_j{}^k + r_i \nabla_k \tau_j{}^k - g_{kj}\, \tau_i{}^k - r_j \nabla_k \tau_i{}^k \right) \\
&= \int_{V(t)} dV\ \left\{ \nabla_k m_{ij}{}^k + (\tau_{ji} - \tau_{ij}) + \left(r_i \nabla_k \tau_j{}^k - r_j \nabla_k \tau_i{}^k \right) \right\} . \qquad (8.486)
\end{aligned}
$$

This gives, for the total torque,

$$
\Gamma_{ij}(t) = \int_{V(t)} dV\ \left\{ \nabla_k m_{ij}{}^k + (\tau_{ji} - \tau_{ij}) + \psi_{ij} + \left(r_i \left(\nabla_k \tau_j{}^k + f_j \right) - r_j \left(\nabla_k \tau_i{}^k + f_i \right) \right) \right\} . \qquad (8.487)
$$

Using equation 8.474 and the result of the previous appendix, we have

$$
\begin{aligned}
\frac{d\Sigma_{ij}}{dt}(t) &= \int_{V(t)} dV(\mathbf{x})\ \left\{ \frac{\partial}{\partial t} (\sigma_{ij} + r_i\, p_j - r_j\, p_i) + \nabla_k \left(v^k\, (\sigma_{ij} + r_i\, p_j - r_j\, p_i) \right) \right\} \\
&= \int_{V(t)} dV(\mathbf{x})\ \left\{ \frac{\partial \sigma_{ij}}{\partial t} + r_i \frac{\partial p_j}{\partial t} - r_j \frac{\partial p_i}{\partial t} + \nabla_k \left(v^k \sigma_{ij} \right) \right. \\
&\qquad\qquad \left. + r_i \nabla_k \left(v^k p_j \right) - r_j \nabla_k \left(v^k p_i \right) + g_{ki} v^k p_j - g_{kj} v^k p_i \right\} \\
&= \int_{V(t)} dV(\mathbf{x})\ \left\{ \frac{\partial \sigma_{ij}}{\partial t} + \nabla_k \left(v^k \sigma_{ij} \right) + r_i \left(\frac{\partial p_j}{\partial t} + \nabla_k \left(v^k p_j \right) \right) \right. \\
&\qquad\qquad \left. - r_j \left(\frac{\partial p_i}{\partial t} + \nabla_k \left(v^k p_i \right) \right) + (v_i p_j - v_j p_i) \right\} . \qquad (8.488)
\end{aligned}
$$

Then, the condition that equation 8.478 must hold for any volume is equivalent to the condition that, at any point,

$$
\begin{aligned}
&\frac{\partial \sigma_{ij}}{\partial t} + \nabla_k \left(v^k \sigma_{ij} - m_{ij}{}^k \right) + r_i \left(\frac{\partial p_j}{\partial t} + \nabla_k \left(v^k p_j - \tau_j{}^k - f_j \right) \right) - r_j \left(\frac{\partial p_i}{\partial t} + \nabla_k \left(v^k p_i - \tau_i{}^k - f_i \right) \right) = \\
&\qquad = \psi_{ij} + (\tau_{ji} - \tau_{ij}) + (p_i v_j - p_j v_i) \ . \qquad (8.489)
\end{aligned}
$$

If the conservation of linear momentum holds (equation 8.480), then this equation simplifies to equation 8.481.

Chapter 9

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