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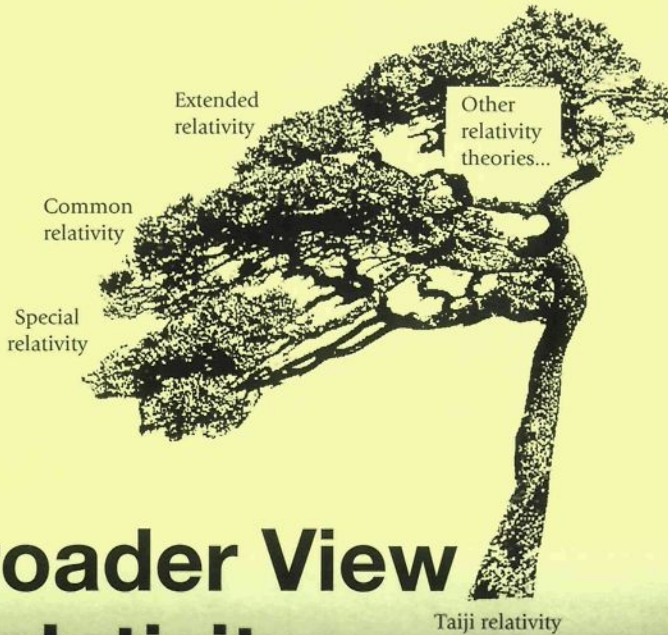

A Broader View of Relativity

General Implications of Lorentz and Poincaré Invariance

2nd Edition

Jong-Ping Hsu
Leonardo Hsu

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(J.-P. Hsu & L. Hsu)

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Jong-Ping Hsu
University of Massachusetts Dartmouth, USA

Leonardo Hsu
University of Minnesota, USA

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A Broader View

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A Broader View of Relativity

To

Bonnie Mei-Chu and Leslie

A Broader View of Relativity

A Broader View of Relativity

A Broader View of Relativity

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A Broader View

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"Those who from time to time send to *Nature*, as manuscripts intended for publication, proofs that the special theory of relativity must be wrong will in future, at least in the first instance, be referred to the issue of *Il Nuovo Cimento* (**74B**, 67; 1983) in which Dr. J. P. Hsu from the National Aeronautics and Space Administration's Goddard Space Flight Center* works out some of the consequences of rejecting the notion that the velocity of light is the same in all frames of reference moving relative to each other (or, more strictly, in all inertial frames). For, although Dr. Hsu's professed object is to demonstrate that some "fundamental" constants are more fundamental than others, he embarks on his argument by throwing away the assumption that the velocity of light is constant (and isotropic) for the sake of a system for measuring time which satisfies one of the rudimentary goals of the anti-relativists---doing away with the problem of simultaneity. .....

Hsu's starting point, explicitly and emphatically not that of an anti-relativist, is that it should be philosophically permissible to require a system for the measurement of what he calls "common time" that should be valid in all relatively moving frames of reference. The procedure is straightforward, and follows Einstein's original discussion of the problem. ..."

NEWS AND VIEWS
Nature Editorial
303, 129 (1983).

*It should read "Marshall Space Flight Center"

"We are not saying that special relativity is wrong in some way. Instead, taiji relativity shows that relativistic time, or any particular time system for that matter, is not a necessary ingredient of a theory for it to correctly reproduce all known experimental results. The four-dimensional symmetry (i.e., Lorentz and Poincaré invariance) of the physical framework is all that matter."

J. P. Hsu and Leonardo Hsu
Physics Letters A, **196**, 1 (1994), p. 3

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Preface

In this book, we present a broader view of relativity in order to understand the fundamental concepts of space, time, and truly universal constants based on the smallest number of postulates. Certain new viewpoints, such as the use of a common time for all observers within a four-dimensional symmetry framework, are non-trivial and need time to digest. The impetus for many of the ideas in this book arose from a term paper (see Appendix B) written by LH for a college physics seminar course in the Fall of 1990. The purpose of that paper was to explore the long-standing question of whether the Lorentz transformation could be unambiguously determined from the results of various experiments. The answer turned out to be negative and this result stimulated our collaboration to step back and look at the big picture of relativity and to explore the physical implications of the principle of relativity — a basic theme of this book.

The distinct features of this book are to stress

(A) a broader view of the relationship between the principle of relativity and our concept of time,

(B) a unification of the spacetime transformations for inertial and non-inertial frames based on limiting Lorentz and Poincaré invariance, and

(C) the truly universal and fundamental constants in both inertial and non-inertial frames.

The title of this edition has been changed to "A Broader View of Relativity — General Implications of Lorentz and Poincaré Invariance" to reflect the focus and the main ideas expounded in the book. The revision for this second edition summarizes our collaborative work performed over the past 15 years. In addition to the new chapters 1, 7, 9, 10, 20, 21, 24, 25 and 27, and new appendices A, B and D, many updates and corrections were made. Moreover, the book has been largely rewritten to reflect updates in our understanding of the conceptual basis of taiji and common relativity including the role that human defined systems of units play in those theories.

Much of the new material is related to the physics of accelerated frames. The physics and spacetime properties of accelerated frames are smoothly connected to those of inertial frames through the limiting Lorentz and Poincaré invariance because all accelerated frames become inertial frames in the limit of zero accelerations. Quantum electrodynamics and gravity as a (generalized) Yang-Mills theory are discussed in both inertial and non-inertial frames.

Special acknowledgment is made to Andreas Ernst who helped to correct many typos and translated the original book into German and to Mei-Chu for her tireless support throughout our lives.

This work was supported in part by the Potz Science Fund and the Jing Shin Research Fund of the UMassD Foundation.

Jong-Ping Hsu
(Univ. of Massachusetts Dartmouth)
Leonardo Hsu
(Univ. of Minnesota, Twin Cities)

Preface to the First Edition

The purpose of this book is primarily to expound the idea that the first principle of relativity (i.e., the form of a physical law is the same in any inertial frame) taken by itself, implies a new theoretical framework which is consistent with the Lorentz and Poincaré group properties and with all previous experiments. The motivation of this work is *not* to show that special relativity is wrong in some way, but instead to show that special relativity is in some sense over specified and that removal of the over specification leads to a fresh view of the physical world with new concepts and results which are unobtainable through special relativity.

This new framework uses a four-dimensional symmetry formalism in which the three spatial variables as well as the temporal variable are all expressed in units of length. Furthermore, because it is based on only one postulate, it is the logically simplest theory which has Lorentz and Poincaré invariance. This is important because it is absolutely essential to insist that a fundamental physical theory should be derivable from the smallest possible set of basic principles. The 4-dimensional symmetry by itself dictates the kinematics of particles and fields in inertial frames in a manner which can also be extended to non-inertial frames through a limiting procedure.

Three aspects related to this simplest 4-dimensional symmetry framework are discussed:

(A) The first principle of relativity is shown to be the essence of relativity theory. All previous experimental results related to Einstein's theory of special relativity can be derived and understood based solely on this symmetry principle, without invoking an additional postulate regarding the universal constancy of the speed of light. We call such a theory "taiji relativity." As a result, we also find that the number of truly universal and fundamental constants in inertial frames is reduced to two, the atomic fine structure constant $\alpha_e=1/137.03604$ and a second constant $J=3.51773\times10^{-38}$ gram·cm. The speed of light (measured in cm/sec) and the Planck constant are found not to be truly fundamental.

(B) The renaissance of the common-sense concept of time $t=t'$ is shown to be possible. By embedding this "common time" in a 4-dimensional symmetry framework of the Lorentz and Poincaré groups, rather than the 3-dimensional framework of the Galilean group, we can construct a viable theory ("common relativity") which is consistent with all known experiments. Such a theory has certain advantages over special relativity in that we can introduce the notion of the canonical evolution for a system of N particles and derive the invariant Liouville equation, a basic equation of the statistical physics. This cannot be done, in principle, in special relativity because each particle has its own relativistic time, so that one cannot have a canonical evolution and can only derive N *one-particle* Liouville equations rather than one single invariant Liouville equation for N particles.

(C) The first principle of relativity can also lead to an 'absolute' theory of spacetime physics in non-inertial frames undergoing a constant linear acceleration or a uniform rotation. This is accomplished on the basis of limiting 4-dimensional symmetry, the idea that transformations for non-inertial frames must smoothly reduce to the familiar 4-dimensional transformation of relativity in the limit of zero acceleration. The relativity theory of spacetime for inertial frames is simply the limiting case of such a theory of spacetime ("taiji spacetime") for non-inertial frames. Particle dynamics and quantization of fields in linearly accelerated frames are discussed. The universal constants in non-inertial frames are found to be the same as the ones in inertial frames mentioned above. Thus, α_e and J are the truly universal and fundamental constants in the physical world since almost all reference frames in the universe are, strictly speaking, non-inertial.

I want to express my gratitude to Bonnie Hsu for her patience and assistance in the preparation of this book. The writing of the book was supported in part by the Potz Science Fund and the Jing Shin Research Fund of the University of Massachusetts Dartmouth. I would like to thank the Academia Sinica and the Beijing Normal University in Beijing for their hospitality. I am indebted to Leslie Hsu, George Leung, Wolfhard Kern, John Dowd, Ed King and Kevin Smith for reading many chapters and improving the text. I would also like to thank Leonardo Hsu for his many valuable suggestions made during our discussions of the ideas presented here.

As the author, I must take sole responsibility for any mistakes in the book and would be grateful to have my attention called to them by readers.

Institute of Theoretical Physics
Academia Sinica, Beijing
November, 1999

J. P. Hsu
(UMassD)

Overview

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A.

The Historical and Physical Context of Relativity Theory

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1.

Introduction and Overview

"Truth loves its limits, for there it meets the beautiful."
R. Tagore, *Fireflies*

1a. Special relativity is NOT incorrect!

When one encounters a book about relativity theory whose title is not some straightforward variation of the words "special relativity" or "general relativity" and whose purpose is not simply to describe those theories and their applications, there is always the sneaking suspicion that the authors have some kind of vendetta against Einstein's theory and that somewhere along the line, the reader will be treated to some elaborately constructed, but ultimately incorrect, explanation of why Einstein's relativity theories must be wrong. It is the purpose of this first paragraph to assure the reader that this is not the case with the book presently sitting in the reader's hands. For over a century, experiments have been performed to test Einstein's most famous work and none has ever been found to be inconsistent with the predictions of the theory of special relativity. The rigorous testing to which special relativity has been subjected has cemented its place in physics as solidly as Newton's theory of gravitation. The existence of any major flaws in the theory has virtually been ruled out and if corrections to the theory need to be made in the future, the regimes in which those modifications might need to be made (e.g., at very short distances) are well-recognized.[1]

Instead, the purpose of this book is, as its title suggests, to step back and to take a broader view of relativity and in particular, the principle of relativity upon which it is based. In the one hundred years since Einstein formulated the theory of special relativity, our way of thinking about the nature of space and time, especially the units of measurement we use in quantifying them, has undergone a significant conceptual shift. If we were to look back at the principle of relativity now, would we see anything that might have been missed by the great physicists of the early 1900's who, though visionary for their time, were still constrained to some extent by the prevailing notions of space and time? Our answer is yes. In this initial chapter, we give an overview of the content of this book, laying out our ideas and the motivation behind them. Some of the ideas will undoubtedly seem strange, or even absurd,[2] as special relativity must have seemed to many physicists of the early 20th century. While we make no pretentions to our intellectual capacity or insight *vis à vis* that of Einstein or to the effects that our ideas might have compared to the revolution in scientific thought sparked by special relativity, we do ask that the reader keep in mind the ultimate criteria by which the usefulness of theories and ideas are judged in science, namely whether they are

(1) consistent with experimental results, and
(2) useful for predicting and explaining the phenomena we observe in the universe around us.

1b. Idea #1: Einstein's first postulate of relativity (the principle of relativity) is the only necessary ingredient of a viable theory

Part A of this book, chapters 2-6, are devoted to a short history of the theory of special relativity, with particular emphasis on the contributions of those other than Einstein. This is not meant to diminish Einstein's role in the creation of the theory by any means, but instead an attempt to give credit where it is due to some of the other physicists who, though unable to make the great leap that Einstein did, still came close.

In part B of the book, comprised of chapters 7-17, we explore the relationship between the principle of relativity and the units that we use to quantify our measurements of space and time. In particular, we argue that our present definitions of those units, the meter and second in the SI system of units, has restricted the way we think about spacetime[3] and that removing some of those restrictions opens up new ways of thinking about relativity, in addition to putting additional tools in the physics toolbox from which we draw when trying to solve physical problems. This exploration was motivated originally by a careful analysis[4] of some of the precision experimental tests of special relativity and by the realization that "relativistic time, or any particular time system for that matter, is not a necessary ingredient of a theory for it to correctly reproduce all known experimental results. The four-dimensional symmetry (i.e., the Lorentz and Poincare invariance) of the physical framework is all that matters."[5]

Our argument is in some sense an extension of that made by Taylor and Wheeler[6] in their excellent Spacetime Physics text. In that book, Taylor and Wheeler argue that since special relativity has shown that space and time can be put on an equal footing (though they are not exactly the same thing), it is more logical to express both spatial and temporal intervals using the same units.[7] Though numerous other authors have made such a statement, Taylor and Wheeler actually follow through on this line of thought by quantifying all space and time measurements in their text using the unit meter. They further argue that expressing both the spatial and temporal components of the spacetime interval using the same unit better emphasizes the unified nature of spacetime and that doing so reveals that the value one typically associates with the speed of light c, namely 299792458 m/s, is not a fundamental constant of our universe at all, but merely a conversion constant between the independently developed units of meter and second that humans created before being aware of the four-dimensional nature of spacetime.

Taylor and Wheeler[6] also state (and we show explicitly) that if one expresses both spatial and temporal intervals using the unit meter, the principle of relativity by itself implies that the speed of light is both isotropic and has the same value of one meter per meter in all inertial frames, without any need for a second postulate. Einstein's second postulate is thus effectively a definition for the unit second, making it proportional to the unit meter and insuring that the speed of light expressed in units of meter per second is a universal constant. In Einstein's time, when space and time were thought to be completely independent entities, it was a necessary postulate in the formulation of special relativity. Given that we now view space and time as parts of a unified spacetime however, it is superfluous. Furthermore, this view leads us to see that the unit second is completely superfluous to physics.

We take Taylor and Wheeler's line of thought one step further. If the unit second is superfluous to physics and is a human construct, this implies that we may define it in any way we like. In particular, we should feel free to define it in a way that makes the problems we want to solve more convenient to attack. The second postulate of special relativity in 1905 and the 1983 General

Conference on Weights and Measures established a particular definition for the second.[8] While this definition is convenient in that the meter and second are directly proportional, making the speed of light a universal constant and simplifying calculations for problems involving the propagation of light signals, it does introduce other inconveniences when trying to solve certain other kinds of problems. For example, when studying many particle systems, the use of relativistic time means that one must use the proper time for each particle to express the covariant equations of motion and thus, one cannot derive an invariant Liousville equation in special relativity.[9] In addition, relativistic time presents some difficulties in defining an invariant temperature, leading to problems in developing an invariant Planck law for black-body radiation, and in providing a theoretical framework for calculating the radiative reaction force for accelerating charges.[9]

In chapters 7 through 17, we propose a new definition of the unit second (which we call the common-second to distinguish it from the traditionally defined unit of time) that can overcome these difficulties and in addition, carries other benefits such as the ability to define an invariant quantity we call the "generg y" that allows for a fuzziness of the position operator of a quantum particle at short distances, and for the construction of an invariant Planck's law for black-body radiation.[10] Of course, this new definition carries some inconveniences of its own. The speed of light for example, measured in meters per common-second, is no longer a universal constant. However, as we show, this effect is merely an artifact of a particular human definition of the unit of time and does not affect any of the underlying physics (the speed of light measured in natural units of meter per meter is still a universal constant). Because the present definition of our unit of length, the meter, is dependent on the definition of the second, we redefine that unit as the length of some standard object. Although such a definition is not as convenient for experimental studies requiring high precision, it has no drawbacks in terms of the theoretical discussion.

A good analogy to use in thinking about the relationship between the traditional definition of the second and the common-second is the relationship between the different coordinate systems, Cartesian, spherical, parabolic, etc., we use in physics. All of the systems are "right" in the sense that they can all be used to solve any problem. However, some coordinate systems are clearly more convenient for solving certain problems than others. For example, the use of parabolic coordinates when solving a short-range kinematics problem in a flat-Earth approximation is no more desirable than the use of Cartesian coordinates to solve the Schrödinger equation for a hydrogen atom. In special relativity, there is a simple relationship between the speed of light in different inertial frames $c = c'$ which leads to a complicated relationship between time coordinates (measured in the unit second) $t' = \gamma(t - \beta x/c)$. In the relativity theory using the common-second (which we call common relativity),[11] there is a simple relationship between the time coordinates $t'=t$ which leads to a complicated relationship between the speed of light in different inertial frames. Both theories are consistent with all known experiments and so are equally "correct." However, one might choose to use special relativity when solving problems involving the propagation of light and common relativity when solving problems involving the evolution of a many particle system, in which each particle occupies a different inertial frame and has a different proper time.

In summary, our goal in part B is to show that the principle of relativity, by implying that space and time are on equal footings and thus that only one unit is necessary to quantify both, allows us flexibility in the definition of an additional unit, should we choose to use it, in expressing spatial and temporal quantities. The careful construction of such a definition can lead to an alternative unit system, analogous to an alternative coordinate system, that can be of help in solving certain kinds of physics problems.

1c. Idea #2: The principle of relativity is useful as a limiting principle in the discussion of the physics of accelerated frames

Despite the advances in our knowledge of spacetime brought about by the theory of special relativity, our understanding of spacetime is by no means complete. The principle of relativity applies only to physics in inertial frames, which are only approximations and idealizations of the physically realized reference frames within which the vast majority of phenomena in our universe take place. Because of the infinite range of the gravitational force, virtually all physical frames of reference in our universe are non-inertial.

In inertial frames, the Lorentz and Poincaré transformations, which embody the properties of the Lorentz and Poincaré groups, enable physical theories to be formulated covariantly and tested experimentally. In non-inertial frames, the tensor calculus of general relativity leads to an analogous group consisting of all point transformations of spacetime which leave the differential form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ invariant. However, this group is too general for quantum field theory and results in few testable predictions.

It would be reasonable to assume that that the future of relativity theories for the spacetime of inertial frames lie in their unification with the absolute theory for spacetime of accelerated frames. Although there is no relativity in accelerated motion, we believe that the concept of "limiting four-dimensional symmetry" or "limiting Lorentz and Poincaré invariance," which states that spacetime coordinate transformations between accelerated frames must reduce to the Lorentz transformations in the limit of zero acceleration, can pave the way to this unification. One generalization of the Lorentz transformation leads to the Wu transformation, which relates the spacetime coordinates of an inertial frame to those in a frame undergoing a constant-linear-acceleration.[12] The Wu transformation includes the better known Møller transformation as a special case and provides a general framework within which to understand the physics of both inertial and non-inertial frames. In this framework of limiting Lorentz and Poincaré invariance, the spacetime of non-inertial frames is characterized by a vanishing Riemann-Christoeffel curvature tensor, implying a flat spacetime that avoids the complications of the curved spacetime of general relativity.

In part C of this book we use the principle of limiting four-dimensional symmetry to explore the physics of non-inertial frames, first developing a spacetime transformation between an inertial frame and one undergoing a constant linear acceleration, then between and inertial frame and one with an arbitrary acceleration along a straight line, and finally between an inertial frame and one undergoing a uniform rotation. In addition to the kinematical transformations among such frames, we discuss the dynamics of classical and quantum particles as well as possible experimental tests of the theories.

Finally, we use our examination of non-inertial frames and the flexibility in the definition of our units of space and time to investigate the status of what are traditionally known as fundamental physical constants. At present, the majority of our physical theories are developed for inertial frames. However, true fundamental physical constants of our universe would be those whose values are the same not only in all inertial frames, but all frames in general, both inertial and non-inertial. For example, the speed of light is of course not a constant in non-inertial frames nor is it even necessarily a constant in all inertial frames when measured in a unit system other than natural units if one chooses a different definition than the conventional one for the unit second. On the other hand, the dimensionless electromagnetic coupling strength $\alpha_e = 1/137.036$ not only is invariant under a change of unit systems, but also appears to be a universal constant in both inertial and non-inertial frames.

References

1. Physics at short distances is in many ways equivalent to physics at large momenta or high energies, according to quantum mechanics. It is safe to say that all known laws of physics can be applied to phenomena occurring at length scales larger than 10^{-17}cm, as indicated by high-energy experiments. In other words, Lorentz and Poincaré invariance (or the 4-dimensional symmetry of the Lorentz and Poincaré groups) has been confirmed experimentally down to distances as small as 10^{-17}cm. It is possible that physical laws such as the Coulomb force or physical principles such as Lorentz and Poincaré invariance may become only approximately true at length scales smaller than 10^{-17}cm. See J. Schwinger, *Quantum Electrodynamics* (Dover Publications, New York, 1958), p. xvi. R. P. Feynman, *Theory of Fundamental Processes* (Benjamin, New York, 1962), p. 145; *Quantum Electrodynamics* (Benjamin, New York, 1962), pp. 138-139. For a related discussion, see S. Weinberg, *The Quantum Theory of Fields*, vol. 1, Foundations (Cambridge Univ. Press, Cambridge, 1995), pp. 31-38. For possible modifications at short distances, see discussions in chapters 14 and 15.
2. Editorial, Nature, **303**, 129 (1983). Discussions and comments were made by the editor on the idea of common time embedded in a 4-dimensional symmetry framework. We elaborate this idea in chapter 8.
3. See Appendix A for a discussion of units and the development of relativity theories.
4. This analysis of the ability of precision experiments to determine all of the parameters of the Lorentz transformation was submitted as a term paper for an undergraduate seminar course in Fall 1990. This term paper is reproduced in Appendix B. The main result was published in Leonardo Hsu and Jong-Ping Hsu, Nuovo Cimento **112B**, 1147 (1997).
5. Jong-Ping Hsu and Leonardo Hsu, Phys. Letters A **196**, 1 (1994).
6. E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (Second Ed., W. H. Freeman and Company, 1992), p. 60.
7. Mathematicians and physicists in the early 20th century wrote the Maxwell equations setting $c=1$ for simplicity. See J. P. Hsu and T. Kleinschmidt, in *Lorentz and Poincare Invariance* (J. P. Hsu and Y. Z. Zhang, World Scientific 2001) pp. 43-53. In the context of special relativity, Poincare

stated this explicitly in his relativity paper, finished in 1905. (See the discussion in section 5c.)

8. Officially, the second is defined in terms of the oscillations based on the frequency of radiation from cesium atoms, while the meter is defined in terms of the second. However, this is merely for reasons of better measurement precision. Logically, the definitions of meter and second are directly linked and it makes no difference which one is dependent on the other.
9. These ideas are discussed in chapters 13 and 14.
10. J. P. Hsu, Nuovo Cimento **74B**, 67 (1983); **93B**, 178 (1986).
11. J. P. Hsu, Found. Phys. **8**, 371 (1978); **6**, 317 (1976); J. P. Hsu and T. N. Sherry, *ibid* **10**, 57 (1980).
12. Jong-Ping Hsu and Leonardo Hsu, Nuovo Cimento **112**, 575 (1997) and Chin. J. Phys. **35**, 407 (1997). This is called the Wu transformation to honor Ta-You Wu's idea of a kinematic approach to finding an accelerated transformation in a spacetime with vanishing Riemann curvature tensor.

2.

Space, Time and Inertial Frames

2a. Space

Human beings have an intuitive sense of space and time and its existence is taken for granted by all but the most hard-boiled of philosophers. In basic terms, space is the grand stage for all physical phenomena, while time is intimately related to our perception of changes in our universe. As human beings, our perceptions of space and time are inherently different. We can move an object in space left or right, forwards or backwards, up or down, or some combination of the three. This implies that the space we live in is three-dimensional. At present, however we cannot move objects in time in the same way we can in space, except in stories such as H. G. Wells' *The Time Machine*.[1] Although we can easily return to a previous location in space as many times as we wish, we appear to be able to visit a point in time only once. Time seems to flow uniformly without being affected by anything.

Space itself is an "empty stage,"[2] without reference marks, so the only way to denote the position of an object (which for now we idealize as a point particle) on a line is to specify its distance relative to another object with the help of some standardized measuring device, such as a meter stick. Such a method of measuring space is possible only because of the invariance in the *form or size of solid bodies under certain transformations, namely spatial* translations and rotations. As will become clear later, the invariance of solid bodies under spatial motion is closely related to the invariant forms of physical laws.

A convenient way to specify the spatial positions of objects is to set up a coordinate system relative to which the position of any object can be measured. For example, in Cartesian coordinates, one sets up three mutually perpendicular lines and the position of an object at any instant is given by three numbers which are the shortest distances of the object from each of the axes.

Ignoring gravitational effects, the physical space we inhabit appears to be described by Euclidean geometry. The concepts of Euclidean geometry probably originated from the intuitions and world experiences of the ancients. As an independent branch of mathematics, however, Euclidean geometry now has a life of its own and extends beyond any physical structure we might encounter. If gravity is not ignored, then according to general relativity, our space is curved and described by non-Euclidean geometry, which was discovered independently by G. F. Gauss (1777-1855), J. Bolyai (1802-1860), and N. I. Lobachevsky (1793-1856). A more general non-Euclidean geometry (now called Riemannian geometry) was later constructed by G. F. B. Riemann (1826-1866).

As science has progressed through the ages, the ability to make precise and reproducible measurements of space and time has become more and more important. The standards of measurement have been modified and improved in these respects. For example, the definition of the meter has changed several times. The original standard meter was a bar made of platinum in 1793. Its length at 0°C was supposedly one ten-millionth of the length of the Earth's meridian at sea level. As the techniques of spectroscopy became available, the definition of the meter was changed to 1,553,164.13 times the wavelength of the red line of cadmium in air at standard temperature and pressure (1 atm and 0°C). In 1960, the meter was redefined as 1,650,763.73 wavelengths of the orange-red line of Krypton-86. Using interferometers, one could now make measurements of extremely high accuracy and reproducibility. Most recently, in 1983, the definition of the meter was once again altered, this time being identified with the distance traveled by light through vacuum in 1/299792458 of a second, where the speed of light is defined to be 299792458 meters per second. This latest definition is not as hard to realize as one may think because there are atomic clocks which are capable of marking such precise time intervals. For example, using light signals, the distance between the Earth and the moon can be measured to within a few meters.

2b. Time

If the state of the universe never changed, then it would be impossible to define time. Newtonian mechanics gives us an intuitive concept of space and time: Space is absolute and is defined in terms of a reference coordinate system; time, on the other hand, is treated as a separate entity consistent with human perceptions.[3]

As with the measurement of space, the standard by which we measure time has also undergone many changes and improvements throughout the ages, but the one thing that all time pieces have had in common is that they all have used some regularly recurring event to mark the passage of time. In the early days, water clocks, hourglasses and sundials were used to measure time. In the early 1600's, Galileo discovered the constancy of the period of a swinging pendulum and for hundreds of years afterwards, that property was used to construct many different clocks. The Dutch scientist C. Huygens was probably the first to invent a pendulum clock around 1656 using Galileo's results. Electric clocks were invented in the second half of the 19th century, but were not used extensively until after 1930. Their hands are driven by a synchronous electric motor controlled by an alternating current with a stable frequency. The year 1929 saw the invention of the quartz clock, which used the vibration of a quartz crystal to drive a synchronous motor at a very precise rate. The turning of a balance wheel and the oscillation of a quartz crystal have enabled the reduction in size of timepieces. For the most precise time measurements in modern research labs, atomic clocks, in which oscillations based on the frequency of radiation from atomic and molecular transitions, are used to mark time. Cesium atomic clocks are currently used as the standard of measurement. One second is defined as 9,192,631,770 periods of the radiation associated with the transition between the two ground state hyperfine levels of a cesium-133 atom, with a precision of a few parts per billion.

2c. Inertial frames of reference

For a description of physical phenomena, we must have a frame of reference, which includes both a system of coordinates to indicate the spatial position of a particle and clocks that have fixed coordinates in the frame to indicate the time. With the help of meter sticks and clocks fixed in a frame, one can give operational definitions of space and time in that frame, provided that the clocks at different coordinates are synchronized by some procedure. For simplicity, we assume the use of identical meter sticks and identical clocks for all inertial frames. Roughly speaking, this means that the length of a meter stick on the ground (F frame) relative to observers that remain at a fixed set of coordinates in the ground frame is the same as that of *another* meter stick in a train (F' frame) relative to observers that remain at a fixed set of coordinates in the train's frame, as long as the train moves with constant velocity relative to the ground frame. Furthermore, the rate of ticking of a clock at rest in the ground frame with respect to ground-observers is the same as that of *another* clock at rest relative to the train with respect to the train-observers. These assumed properties are consistent with the postulate that all inertial frames are equivalent.

There exist frames of reference in which a particle has a constant velocity if the particle is not acted upon by external forces. These frames are called ***inertial frames.*** Clearly, all inertial frames move with constant velocities relative to one another. We may remark that why a particle has this inertial property, i.e., it has uniform motion under zero net force, is not known. This inertial property of all matter was discovered by Galileo and is called the **principle of inertia.** His discovery marked a great advance in our understanding of motion.

2d. Coordinate transformations

We stress that simply having a grid of identical clocks (synchronized according to a certain procedure) and meter sticks *in one inertial frame* is insufficient for a complete description of events. One may ask an elementary question:

If a particle is at position (x,y,z) at time t as measured by an observer in a particular inertial frame, what is the corresponding position (x',y',z') at time t' for *the same particle* as measured by a second observer in a different inertial frame?

In order to answer this question, *there must be a specific relation, or transformation, between the space and time coordinates of any two inertial frames.* Only after we have such a space and time transformation is the mathematical framework for describing physical phenomena complete.

The space and time transformations for inertial frames are of fundamental importance because they determine the basic space-time symmetry of the frame and restrict possible forms of the dynamics of interactions, even though they do not dictate the actual dynamics themselves. It is fair to say that the basic dynamics of particles and fields that is known so far is determined by several symmetry principles including local gauge symmetry, and by simplicity. For example, the interaction between a charged particle and the electromagnetic fields cannot be uniquely determined by the gauge symmetry principle; simplicity (e.g., an uncomplicated algebraic form for the electromagnetic coupling) also must be invoked.

In this connection, one may say that space and time in physics as a whole are not completely known because their properties in non-inertial frames have not yet been satisfactorily defined and tested experimentally. In chapters 18-26, we discuss space, time and physics in simple non-inertial frames with a constant linear acceleration based on limiting Lorentz and Poincaré invariance.

2e. Units of space and time

In a given frame of reference, one defines three spatial coordinates and an evolution variable to describe the motion of particles. Prior to 1983, the units of the spatial coordinates (meter) and the evolution variable (second) were independently defined, in accordance with human perceptions of space and time as two very different entities. In the present day, however, the definitions are linked, reflecting our understanding that both are parts of a four-dimensional spacetime. We can thus express both spatial coordinates and the evolution variable in units of meter (or second).

Although this unification of space and time is now a foundation of our understanding of the universe, this was not true at the time when our theories of space and time, i.e., the relativity theories, were developed. As a result, even though the theories themselves led to and were based on the unification of space and time, the mathematics were formulated in such a way that space and time were still differentiated, i.e., the two quantities measured using two independently defined units. As we shall see in later chapters, this particular formulation has limited the generalizations that could be made with the new theories.[4] Even today, though our understanding of the four-dimensional symmetry has progressed over the past century, we still have been limited by the blinders imposed by the initial mathematical formulation.

References

1. H. G. Wells, *The Time Machine* (1895). There are a few interesting passages in the book. The Time Traveler ardently explains to his guests: "The geometry, for instance, they taught you at school is founded on a misconception.... There are really four dimensions, three which we call the three planes of Space, and a fourth, Time. There is, however, a tendency to draw an unreal distinction between the former three dimensions and the latter, because it happens that our consciousness moves intermittently in one direction along the latter from the beginning to the end of our lives.... There is no difference between Time and any of the three dimensions of space, except that our consciousness moves along it." (pp. 1-2) Thus, it appears that long before Poincaré (1906) and Minkowski (1909) published their views that the Lorentz transformations could be interpreted as a 'rotation' of the coordinate system in a 4-dimensional spacetime, etc., Wells was the first to argue for a 4-dimensional vision of the universe. Of course, the four dimensions of spacetime are, strictly speaking, not *completely* equivalent. Sometime, people call it (3+1)-dimensional spacetime.
2. Space is usually considered to be a set of points satisfying specified geometric postulates. Physically, this is true if and only if one ignores the effects of quantum fields. See T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood Academic Publishers, New York; and Science Press, Beijing, 1981), pp. 378-405. This book gives a comprehensive discussion on fundamental aspects of particle physics and quantum fields related to the vacuum (i.e., the "ether," in old language). For a brief discussion of changing views of geometry and space, see O. Veblen and J. H. C. Whitehead, *The Foundations of Differential Geometry* (Cambridge Univ. Press, London, 1954), pp. 31-33.
3. Newton stated: "Absolute, true, and mathematical time, of itself, and from its own nature, flows equally without relation to anything external..." "Absolute space, in its own nature, without relation to anything external, remains always similar and immovable..." See I. Newton, *The Principia*, (translated by A. Motte, Prometheus Books, Amherst, N.Y., 1995) p. 13.

4. J. P. Hsu and L. Hsu, Phys. Letters **A 196**, 1 (1994); (Erratum) ibid **217**, 359 (1996); Leonardo Hsu and Jong-Ping Hsu, Nuovo Cimento **111B**, 1283 (1996); Jong-Ping Hsu and Leonardo Hsu, in *JingShin Physics Symposium in Memory of Professor Wolfgang Kroll* (World Scientific, Singapore. New Jersey, 1997), pp. 176-193.

3.

The Nontrivial Pursuit of Earth's Absolute Motion

3a. Newton's frame of absolute rest

Once an inertial frame is set up with operational definitions of space, time, and observers, one can proceed to discuss mechanical phenomena. Mechanics is the study of the motions and interactions of objects and is probably the oldest branch of physics. The first person to undertake a systematic study of mechanics was Isaac Newton (1642-1727).[1] In his *Principia* (1687), Newton established his three now-famous laws of motion and derived a large number of interesting and useful results. This work forms the foundation of what we now refer to as classical mechanics. Based on the method used to obtain the vast majority of his results, it is fair to say that Newtonian mechanics is as complete and rigorous as Euclidean geometry.[2]

Though not all of Newton's work has remained valid, Newtonian mechanics has two very important and fundamental properties which have had great significance in the later development of all branches of physics.

The first property is that Newton's laws of motion are invariant under the Galilean transformations. What do we mean by invariant? Simply that the mathematical expressions of the laws of mechanics have the same form for all observers whose relative motion is described by a constant velocity. This is equivalent to making the statement that, as far as the laws of mechanics are concerned, all inertial frames are completely equivalent. In classical mechanics, this invariance of laws is a basic symmetry property,[3] which is a simple generalization of the concept of symmetry as applied to geometry. The **Symmetry** of an object is, in the words of Herman Weyl, that "*one does something to it, and after one finishes doing it, the object looks the same.*" For example, a sphere rotated by any angle or displaced by any amount in space is still a sphere. This concept can be generalized in several ways--for example, the shortest distance between two points in space remains the same under any translation or rotation of the coordinate axes describing that space.

It is extremely fortunate that Newton's laws of motion are the same in every inertial frame. If the law took a different form in different inertial frames, it would be much more complicated to understand the physical world. In this sense, the *invariance of physical laws is interlocked with the simplicity of physics.*

As an example, consider a set of Cartesian axes (x', y', z') for an inertial frame F', say, a train that moves with a constant velocity $\mathbf{V}=(V_x, V_y, V_z)$ relative to the inertial frame F (the ground) with Cartesian axes (x, y, z). Then

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t\,, \qquad t' = t\,. \tag{3.1}$$

This gives the coordinate transformation between the two frames, with the stipulation that the origins of F and F' coincide at time t=0 and that the two sets of axes are parallel to each other. This is the Galilean transformation. The spatial interval between two points is independent of velocity, and is absolute. It is also explicitly assumed that there exists a unique time which is independent of any reference to special frames of coordinates. This is another important point that we will return to later. The surface of Earth or a laboratory on the rotating Earth may be considered to be an inertial frame to a very good approximation during short time intervals, if gravity is neglected.

Intuitively, one might expect that the laws of physics should remain the same when going from a particular coordinate system to one which is rotated or translated by a fixed amount from the original system. This is indeed true. This stands to reason as moving to a different place or tilting one's head does not change the outcome or characteristics of physical phenomena. This is a fundamental property of our universe, as it cannot be explained in terms of something more basic. Let us see how one of Newton's laws holds up under a Galilean transformation.

Newton's law of motion can be expressed as

$$m \frac{d^2\mathbf{r}_b}{dt^2} = \mathbf{F}(\mathbf{r}_b - \mathbf{r}_a), \tag{3.2}$$

where m is the mass of an object and $\mathbf{r}_b$ is its position. The vector $\mathbf{r}_a$ can be interpreted as the position of the source of the force on our object or as an equilibrium point for our system. Using equation (3.1) above, one can verify that Newton's motion law (3.2) is invariant under a Galilean transformation, i.e., that the equation looks exactly the same, except that all quantities are now measured relative to the coordinate axes of the primed frame:

$$m \frac{d^2\mathbf{r}'_b}{dt^2} = \mathbf{F}(\mathbf{r}'_b - \mathbf{r}'_a) \;, \tag{3.3}$$

and the mass m is a scalar, m=m'. This simple property, which holds for all of Newton's laws, implies that the F frame (typically associated with the ground) and the F' frame (typically associated with a moving train) are physically equivalent. Since the mechanical laws of motion are the same in both inertial frames, one cannot use observations of mechanical phenomena to determine which frame one is in or equivalently, to say which frame is "at rest" and which one is "moving." This is the *primitive principle of relativity for mechanics*, sometimes called the Galilean principle of relativity.

This particular symmetry property of mechanics had been known qualitatively to both Eastern and Western scientists for quite a while before Newton's time. In the western literature, Galileo (1564-1642) appears to be the first one to have discussed it in a famous passage from his book "Dialogue Concerning the Two Chief World Systems" (1632). He observed that if one was below decks on a large ship moving with constant velocity in smooth waters, one could not tell whether or not the ship was moving. In Chinese literature, there is a very similar observation in a book called "*Shang-Shu Weei*" (*An Appendix to the Book of History*), dating about 1400 years before Galileo's, which reads: "The earth (ground) always moves and never stops, and yet one does not know it; just like if one sits in a boat with windows closed, the boat moves and one does not feel it."[4]

Unfortunately, this observation was not taken seriously by Chinese thinkers and did not have any further impact on the development of science in China.

Returning to our discussion of Newtonian mechanics, the second important property of Newton's laws is that he postulated an absolute time as well as an absolute space, corresponding to the frame of absolute rest. As far as his concept of time was concerned, it appeared so obvious and intuitive that it was taken for granted as correct for more than two hundred years until the time of Poincaré and Einstein. Newton called it "absolute, true and mathematical time," which "of itself and from its own nature, flows equally without relation to anything external." However, his idea of absolute space was criticized by Leibniz, Huygens and others. Leibniz argued that there is no philosophical need for any conception of space apart from the relations of material objects. Given that mechanical phenomena were not sufficient to determine a frame of absolute rest, Newton was probably aware of the logical weakness in his postulated absolute space, but he employed theological arguments to strengthen his idea of the absolute space, declaring that the absolute space was simply the space relative to God. About 150 years later, when the idea of the stationary ether permeating all space was popular, it was only natural to associate this ether with the frame of absolute rest.

3b. Measuring Earth's velocity

In the nineteenth century, the study of electric and magnetic phenomena was still a relatively new subject. When electromagnetic waves were found to be associated with the oscillations of fields, the similarity to mechanical oscillations was too great for people to believe that the waves could be transmitted through empty space. The theory of the ether that permeated all space and matter and provided a medium for the oscillations of the fields made perfect sense at that time and provided a convenient and pleasing analogy to the mechanical case. It may be somewhat puzzling for modern physicists to see why the idea of the ether was so popular among physicists in the nineteenth century. However, the very existence of the electromagnetic field was considered to be unequivocal and firm evidence for the existence of the ether.

Given the ether and a frame of reference that could be associated with absolute rest, it was only natural to try to discover the velocity of the Earth through this medium. As mentioned before, mechanical phenomena do not provide meaningful tests because of Galilean invariance. However, when light was discovered to be a form of electromagnetic wave, optics seemed to present a possible solution. Initially, such experiments appeared to be impossible, considering the precision that would be required. Maxwell (1831-1879) wrote that "all methods . . . by which it is practicable to determine the velocity of light from terrestrial experiments depend on the measurement of the time required for the double journey from one station to the other and back again, and the increase of this time on account of a relative velocity of the ether equal to that of Earth would be only about one hundred millionth part of the whole time of transmission, and would therefore be quite insensible."

This apparently insurmountable difficulty was overcome by the genius of A. A. Michelson (1852-1931) in 1881 with his invention of the interferometer. The Michelson experiment split a monochromatic beam of light into two beams propagating in perpendicular directions for equal distances. When the two were rejoined, an interference pattern of light and dark rings could be observed due to the phase difference in the beams. According to the Galilean transformation, the velocity of light should not be isotropic in a terrestrial laboratory that was moving with respect to the ether. Therefore, when the apparatus was rotated through 90°, changing the directions of the two arms, the interference bands should have been observed to shift by a measurable amount dependent on the earth's velocity. As is well known, however, no shift was ever observed in this or any of the numerous similar experiments that followed.[5]

The same experiment was repeated by Michelson together with Morley in 1887 resulting in the same null outcome with a much smaller upper bound on the magnitude of any possible fringe shift and the results were published in the *American Journal of Science*.[6] The null result was a disappointment to Lorentz, Michelson and others who believed in the existence of the ether. However, a positive reaction to this null result was received by the editor of *Science* in 1889, in a very short paper with the title "The Ether and the Earth's Atmosphere" by G. F.

FitzGerald:[7]

"I have read with much interest Messrs. Michelson and Morley's wonderfully delicate experiment attempting to decide the important question as to how far the ether is carried along by the earth. Their result seems opposed to other experiments showing that the ether in the air can be carried along only to an inappreciable extent. I would suggest that almost the only hypothesis that can reconcile this opposition is that the length of material bodies changes, according as they are moving through the ether or across it, by an amount depending on the square of the ratio of their velocities to that of light. We know that electric forces are affected by the motion of the electrified bodies relative to the ether, and it seems a not improbable supposition that the molecular forces are affected by the motion, and that the size of a body alters consequently. It would be very important if secular experiments on electric attractions between permanently electrified bodies, such as in a very delicate quadrant electrometer, were instituted in some of the equatorial parts of the earth to observe whether there is any diurnal and annual variation of attraction —diurnal due to the rotation of the earth being added and subtracted from its orbital velocity, and annual similarly for its orbital velocity and the motion of the solar system."

This suggestion, the FitzGerald contraction, was the first spark of theoretical thought that eventually ignited a prairie fire in theoretical physics in the 20th century.

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1. Newton is arguably the greatest scientist and genius. However, as a man in the eye of his own friends and contemporaries, "Newton was profoundly neurotic of a not unfamiliar type." "His deepest instincts were occult, esoteric, semantic -- with profound shrinking from the world, a paralyzing fear of exposing his thoughts, his beliefs and his discoveries, in all nakedness to the inspection and criticism of the world. 'Of the most fearful, cautious and suspicious temper that I ever knew' said Whiston, his successor in the Lucasian Chair." "He parted with and published nothing except under the extreme pressure of friends. ... His peculiar gift was the power of holding continuously in his mind a purely mental problem until he had seen straight through it." "He regarded the Universe as a cryptogram set by the Almighty — just as he himself wraps the discovery of the calculus in a cryptogram when he communicated with Leibnitz. By pure thought, by concentration of mind, the riddle, he believed, would be revealed to the initiate." "He believed that the clues to the riddle of the universe were to be found partly in the evidence of the heavens and in the constitution of elements,... but also partly in certain papers and traditions handed down by the brethren in an unbroken chain back to the original cryptic revelation in Babylonia." "Very early in life Newton abandoned orthodox belief in the Trinity. ... He was persuaded that the revealed documents give no support to the Trinitarian doctrines which were due to late falsification. The revealed God was one God." This belief is presumably related to his idea of Absolute Space, being the one and only one that is relative to God. See "Newton, the Man" by J. M. Keynes, in *The World of Physics*, (Ed. J. H. Weaver, Simon and Schuster, New York, 1987), Vol. 1, pp. 537-547.
2. Euclid's ELEMENTS started with definitions for point, line, etc., followed by 5 postulates; while Newton's PRINCIPIA started with definitions for mass, kinetic energy, etc., followed by the 3 laws of motion.
3. A. Zee, *Fearful Symmetry, The Search for Beauty in Modern Physics*

(Macmillan, New York, 1986). J. P. Hsu and Y. Z. Zhang, *Lorentz and Poincaré Invariance—100 Years of Relativity* (World Scientific, 2001), pp. xxi-xxxi and p. 4.

4. "*Shang-Shu Weei*" (*An Appendix to the Book of History*), published in the East Han Dynasty (AD 23-AD 221), the author was unknown.

5. A. A. Michelson, Am. J. Sci. **22**, 120 (1881). See also A. P. French, *Special Relativity* (W. W. Norton & Company, New York, 1968), pp. 51-58; A. Pais, *Subtle is the Lord..., The Science and the Life of Albert Einstein* (Oxford Univ. Press, Oxford, 1982), pp. 111-122.

6. A. A. Michelson and E. W. Morley, Am. J. Sci. **34**, 333 (1887).

7. G. F. FitzGerald, Science **13**, 349 (1889). FitzGerald was a tutor at Trinity College in Dublin in 1879 and became a professor in 1881. He was the first physicist to suggest a method of producing radio waves by an oscillating electric current, which was verified experimentally by H. R. Hertz between 1885 and 1889. FitzGerald told his friends frankly that he was not in the least sensitive to making mistakes. His habit was to rush out with all sorts of crude notions in hope that they might set others to thinking and lead to some advance. He did precisely this in the short paper he submitted to *Science*.

4.

On the Right Track: Voigt, Lorentz and Larmor

4a. Lorentz's heuristic local time

The Michelson-Morley experiment carried out in 1887 confirmed Michelson's original experimental null result of 1881 with a 20-fold increase in accuracy. In 1889, G. F. FitzGerald (1851-1901) proposed that the null result obtained could be explained if bodies in motion through the ether underwent a length contraction in the direction of motion "by an amount depending on the square of the ratio of their velocities to that of light." Of course, there was no other experimental evidence for this conclusion, so this was merely an ad hoc explanation. However, since the ether was generally believed to exist at that time, FitzGerald's idea was "natural" and attractive. This idea of FitzGerald influenced Larmor and Lorentz in obtaining the relativistic transformation of space and time.

The next development was carried out by H. A. Lorentz (1853-1928). He finished his doctoral dissertation in 1875, in which he refined Maxwell's electromagnetic theory so that it explained more satisfactorily the reflection and the refraction of light. He became a professor of mathematical physics at Leiden University in 1878 at the age of 25. His main effort was to complete Maxwell's theory to explain the relationship among electricity, magnetism and light by introducing the Lorentz force and by suggesting that the oscillations of charged particles inside the atom was the source of light. To test his idea on the source of light, he reasoned that a strong magnetic field should have an effect on the oscillations of these charged particles. P. Zeeman, a student of Lorentz, carried out experiments to confirm this idea in 1896. This phenomenon is now known as the Zeeman effect. For these experiments, both Zeeman and Lorentz were awarded the Nobel Prize in 1902.

In 1895, Lorentz was investigating the physical effects of Earth's motion through the ether on electric and optical phenomena, taking Maxwell's equations to be a description of such phenomena in the ether. This amounted to finding the proper transformation for Maxwell's equation from the ether frame F_a (the frame of absolute rest) to the (moving) Earth's frame, F'. At this time, it was known that while Newton's mechanical equations were mathematically invariant under the Galilean transformation, Maxwell's equations were not. In order to make them invariant from frame to frame to first order in V/c, Lorentz discovered that one had to introduce a new time t' in the moving frame F':

$$t' = t - \frac{Vx}{c^2}, \quad x' = x - Vt, \quad y' = y, \quad z' = z, \tag{4.1}$$

where t is the time measured in the ether frame F_a and the relative motion of the two frames is taken to be solely along the parallel x and x' axes. Lorentz was not the first to write down such a transformation, however. In 1887, Woldemar Voigt (1850-1919) had introduced a non-absolute time $t' = t - Vx/c^2$ while studying Doppler effects.[1]

Lorentz considered this time t' in (4.1) to be a local time which had no physical meaning since it contradicted the absolute time t' = t in the Galilean transformation (3.1), which was too intuitively obvious to be incorrect. He later remarked that

> "a transformation of the time was necessary, so I introduced the conception of local time which is different for different frames of reference which are in motion relative to each other. But I never thought that this had anything to do with real time. This real time for me was still represented by the older classical notion of an absolute time, which is independent of any reference to special frames of coordinates. There existed for me only one true time. I considered my time transformation only as a heuristic working hypothesis, so the theory of relativity is really solely Einstein's work."[2]

On the other hand, Poincaré regarded Lorentz as the one who had conceived of the principle of relativity for electromagnetic phenomena and was the first physicist to call the formulae obtained by Lorentz the "Lorentz transformation".[3] In one of his essays, "Space and Time," Poincaré wrote, prophetically: "Will not the principle of relativity, as conceived by Lorentz, impose upon us an entirely new conception of space and time and thus force us to abandon some conclusions which might have seemed established?"[4]

4b. Development of the Lorentz transformations

Sir Joseph Larmor (1857-1942) was educated at Belfast and Cambridge. He taught at Cambridge from 1885 to 1932 and was the Lucasian Professor of Mathematics in the University of Cambridge. Larmor was knighted in 1909. He did pioneering work in determining the rate of energy radiation from an accelerated electron and in explaining the splitting of spectral lines by a magnetic field. Like many physicists at that time, he believed that matter consisted entirely of electric particles moving in the ether. Today, he is mainly known for his work on the wobbling motion of an atomic orbit when an atom is subjected to an external magnetic field, called the "Larmor precession." The rate of the precession is known as the "Larmor frequency." Larmor also derived the "Larmor formula" for the total power radiated by an accelerated charge. In retrospect, however, Larmor's most significant contribution is his discovery of the exact relativistic spacetime transformation which is now called the "Lorentz transformations." In the fierce competition of scientific research, to be the first to make a discovery is everything. To be second is nothing. In light of this, one may be surprised that, before he died in 1942, Larmor never made any claim of being the first to conceive of this important exact spacetime transformation, which specifies the transformation of space and time between inertial frames and forms the basis of Einstein's special theory of relativity.

Larmor studied Lorentz's paper of 1895 which contained the first order transformation (4.1). He was able to improve it and obtained an exact transformation which can be written in the familiar form:

$$x' = \gamma(x - Vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{Vx}{c^2}\right); \quad \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}, \tag{4.2}$$

after a change of variables. This gives the coordinate transformation between the two frames F and F', with the stipulation that the origins of F and F' coincide at time t=0 and that the two sets of axes are parallel to each other. This important result was presented and discussed in his book *Aether and Matter*[5] which was completed in 1898 and published in 1900. Apparently, since this set of equation is known as the Lorentz transformation, this work went largely unnoticed. Larmor also derived what we now know as the relativistic length contraction from transformation (4.2). At the time, it was thought that this was the contraction that FitzGerald had proposed. However, this is not the case since the FitzGerald contraction is an actual physical contraction of the length of an object caused by its motion in absolute space, while the contraction implied by (4.2) is a relativistic effect resulting from the alteration of the notion of simultaneity due to the relative motion of the observer. Larmor did not discuss the physical meaning of t'. Presumably, his viewpoint on that account was the same as that of Lorentz.

In 1899, Lorentz wrote down the exact transformation with an additional factor K:

$$x' = K\gamma(x - Vt), \quad y' = Ky, \quad z' = Kz, \quad t' = K\gamma\left(t - \frac{Vx}{c^2}\right). \tag{4.3}$$

It is unknown whether or not he was aware of Larmor's work at the time. He noted that the scale factor K could not be determined by the Michelson-Morley experiment and that it required "a deeper knowledge of the phenomena." Nevertheless, he stressed that the length contraction implied by the spatial components in (4.3) were precisely those which one had to assume in order to explain the Michelson-Morley experiment.

In 1904, Lorentz wrote down the transformation

$$x' = \gamma(x - Vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma\left(t - \frac{Vx}{c^2}\right). \tag{4.4}$$

He attempted to fix the value of K in (4.3) to be 1 by considering the transformation properties of the equation of motion of an electron in an external field. However, because he made a mistake in the transformation equations for velocities, his proof was valid only to first order in V/c.[6] This mistake was corrected by Poincaré in 1905.

As mentioned earlier, Voigt[7] (1850-1919) had obtained a similar transformation as early as 1887 while studying Doppler shifts in his paper 'Über das Dopplersche Prinzip.'[1] His transformation was the same as (4.3), but with the scale factor K equal to $1/\gamma$:

$$x' = (x - Vt), \qquad y' = y/\gamma, \qquad z' = z/\gamma, \qquad t' = \left(t - \frac{Vx}{c^2}\right); \qquad \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}. \tag{4.5}$$

Reflecting for a moment, one sees that Voigt had actually introduced two revolutionary ideas into physics,

(a) the concept of non-absolute time t', which is, to first order, the correct relativistic time, and

(b) the universal and constant speed c for the propagation of light in all inertial frames.

Furthermore, Voigt actually derived his transformation (4.5) based on the postulate of the invariance of the wave equation,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi = 0, \tag{4.6}$$

for the propagation of light in the ether. If the physicists of the time had been imaginative enough, they might have recognized the potential of these ideas to open up a whole new view of physics. As we know, this never happened. In the time when the ideas of Newtonian absolute time and space dominated physics, presumably people simply dismissed Voigt's ideas as nonsense. Later, in 1906, Lorentz said that regrettably, Voigt's transformations had escaped his notice all those years.[8]

It would be fair to say that Lorentz first conceived of the *invariance* of the laws governing electromagnetic fields and that his transformations embody the mathematical essence of the new concepts of space and time in special relativity. However, Lorentz believed in the ether.[9] Consequently, the relativity of space and time and the principle of relativity were not apparent to him before 1905. He did not believe in a physical interpretation of t' even though he discovered it himself and moreover, he did not attempt to formulate a theory of relativity[10] for mechanics and electrodynamics as Poincaré and Einstein did. Lorentz arrived at the new concept of time by struggling with the old idea of absolute time. Unfortunately, the preconceptions of absolute time and ether remained the language of his thinking for the rest of his life.

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3. H. Poincaré, Compt. Rend. Acad. Sci. Paris 140, 1504 (1905) and Rend. Circ. Mat. Palermo **21**, 129 (1906).
4. H. Poincaré, *Mathematics and Science: Last Essays* (Dover, N.Y. 1963; originally published in 1913).
5. J. Larmor, *Aether and Matter* (Cambridge University Press, Cambridge, 1900.), pp. 167-170 and pp. 173-175. This book has a long subtitle: "A development of the dynamical relations of the ether to material systems on the basis of the ATOMIC CONSTITUTION OF MATTER including a discussion of the influence of the earth's motion on optical phenomena." The notations of Larmor for his transformations are messy. For example, he wrote down the expression: $\varepsilon^{1/2}x'$, y', z', $\varepsilon^{-1/2}t'-(v/c^2)\varepsilon^{1/2}x'$, where $\varepsilon = (1-v^2/c^2)^{-1}$. One has to follow the notation used in the first order approximation to find the relations $t' = t$, $z' = z$, $y' = y$ and $x' = (x-vt)$ and to obtain the familiar expression: $\varepsilon^{1/2}(x-vt)$, y, z, $\varepsilon^{1/2}(t-vx/c^2)$. The title of chapter XI is "MOVING MATERIAL SYSTEM: APPROXIMATION CARRIED TO THE SECOND ORDER." This seems to suggest that, at that time, he was not aware of or did not regard his transformation to be correct and exact to all orders. Like most physicists at that time, Larmor apparently believed that the ether played a fundamental role in the physical universe. In section 123, he speculated on a hydrodynamic theory "which would construct an atom out of vortex rings" and "the circulation of the vortex is however in the dynamical theory an unalterable constant, so that the one system cannot be changed by natural processes into the other."

6. A. Pais, *Subtle is the Lord..., The Science and the Life of Albert Einstein* (Oxford Univ. Press, Oxford, 1982), p. 126.

7. W. Voigt was a colleague of Klein, Hilbert, Minkowski, Runge, Wiechert, Prandtl and Schwarzschild at Göttingen, which had been developed by Gauss into a world famous center of learning in the sciences. Voigt published a book: *Magneto- und Elektrooptik*, Leipzig, 1908. He also gave lectures on optics and taught an advanced course in optical experiments. He was Max Born's teacher and used to investigate crystal properties with the help of symmetry considerations. Voigt appreciated his talented former student Born who later became Voigt's friend Minkowski's assistant. In 1909, after the death of Minkowski, the young Born gave his first report on his work concerning the constant acceleration of an electron and its electromagnetic mass to the Mathematical Society. There were continual interruptions and attacks, so that Born was confused and got entangled in his formulae. Klein stopped the talk and declared that he had never listened to such a bad lecture in all his life. Born was completely broken and seriously considered giving up physics and entering one of the technical colleges in order to become an engineer. However, with the help of Runge and Hilbert, Born got a second chance and convinced Klein. After the second report, Voigt encouraged Born to stay and become a lecturer at Göttingen by saying that Born's paper seemed to him a very suitable thesis, and Born could count on his support in the faculty, offering Born a lectureship (Privatdozent). It appears that Voigt's encouragement to the young Born eventually made a bigger contribution to physics than his own research on crystals and magneto-optical phenomena.

8. Lorentz and Voigt corresponded regarding the Michelson-Morley experiment around 1888, and Lorentz knew some of Voigt's work, though not the Voigt transformation (4.5). In 1906, Lorentz said in his Columbia University lectures that Voigt had applied the transformation (4.5) to show the invariance of the wave equation $[\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - (1/c)^2\partial^2/\partial t^2]\Phi = 0$. He also commented: "The idea of the transformations....might therefore have been borrowed from Voigt and the proof that it does not alter the form

of the equations for the free ether is contained in his paper.[1]" See H. A. Lorentz, *The Theory of Electrons* (ser. 169. Teubner, Leipzig, 1909; Dover, New York, 1952), p. 198, and, also, A. Pais, ref. 6, pp. 121-122.

9. Lorentz said in 1909 that he could not but regard the ether as endowed with a certain degree of substantiality, however different it may be from all ordinary matter. See H. A. Lorentz, ref. 8, p. 230.

10. The main difference between Lorentz's and Einstein's attitude toward relativity can be seen clearly from Lorentz's statement: "....the chief difference being that *Einstein simply postulates what we have deduced, with some difficulty and not altogether satisfactorily, from the fundamental equations of the electromagnetic field.* By doing so, he may certainly take credit for making us see in the negative result of experiments like those of Michelson, Rayleigh and Brace, not a fortuitous compensation of opposing effects, but *the manifestation of a general and fundamental principle.*" See H. A. Lorentz, ref. 8, p. 230. Nevertheless, Lorentz believed in the ether to the end of his life.

5.

The Contributions of Poincaré

5a. Poincaré's insight into physical time

If one were to examine the writings of the great French mathematician and physicist Jules Henri Poincaré (1854-1912) at the turn of the 20th century, one would see a foreshadowing of most of the notions which would later become part of the special theory of relativity.[1] In 1895, he noted the impossibility of detecting Earth's absolute motion and in July of 1905, completed his theory of relativity based on *the principle of relativity* and a definition—choosing the units of length and of time so that the speed of light was equal to unity. Many physicists consider Poincaré's main contributions to relativity to be of a mathematical nature. However, this does not adequately represent his work. In the literature, historical accounts of Poincaré's contribution to relativity theory have been controversial. At one extreme, Whittaker[2] has claimed that Poincaré discovered the principle of relativity for physical laws in 1904 and that the originators of the relativity theory are Lorentz and Poincaré rather than Einstein. At the opposite end of the spectrum, Holton[3] asserts that Poincaré's principle of relativity is equivalent to the Galilean-Newtonian principle of relativity, ignoring his comprehensive papers on relativity and electrodynamics finished in June and July 1905. It seems fair to say that the truth is somewhere in between these two views.[4]

In 1895, Poincaré had already noted the impossibility of detecting Earth's absolute motion. "Experiment has revealed a multitude of facts which can be summed up in the following statement: It is impossible to detect the absolute motion of matter, or rather the relative motion of ponderable matter with respect to the ether; all that one can exhibit is the motion of ponderable matter with respect to ponderable matter."[5]

Around 1898, Poincaré examined the concepts of absolute time and absolute simultaneity in response to the then oft-debated question of the measurement of time intervals. He appears to have been the first physicist to discuss and analyze the concept of time from what people later called the "operational viewpoint." In an article entitled "La Mesure du Temps",[6] Poincaré stressed that "*we have no direct intuition about the equality of two time intervals.* [emphasis Poincaré's] People who believe they have this intuition are the dupes of an illusion." This was a remarkable insight. Noting earlier definitions of simultaneity which he found unsatisfactory, he wrote that "it is difficult to separate the qualitative problems of simultaneity from the quantitative problem of the measurement of time; either one uses a chronometer, or one takes into account a transmission velocity such as the one of light, since one cannot measure such a velocity without measuring a time." He finally concluded that "the simultaneity of two events or the order of their succession, as well as the equality of two time intervals, must be defined in such a way that the statement of the natural laws be as simple as possible. In other words, all rules and definitions are but the result of an unconscious opportunism."

With this penetrating understanding of physical time, it was natural that Poincaré[7] in 1900 showed a strong interest in and gave the first correct physical interpretation to the "local time" t' which was introduced by Lorentz in 1895. Lorentz investigated the influence of the Earth's motion on electric and optical phenomena. He realized that if the experiments carried out in a terrestrial laboratory could not reveal any effect on the motion, then the equations of the theory must have the same form when going from the absolute rest frame $F_{ar}(t,x)$ of the ether to the terrestrial frame $F'(t',x')$. To obtain this result to the first order in V/c, Lorentz introduced approximate transformations of time $t' = t - Vx/c^2$ and space $x' = x - Vt$, where y, z and the second order terms in V/c were neglected.

Local time t' in a moving frame F' differs from the true time t in the frame of absolute rest F_{ar} by the amount Vx/c^2 at each point on the x axis in F. Lorentz considered the local time t' to be nothing but a convenient mathematical quantity to simplify Maxwell's equations in a moving frame F' and did not believe that it had any physical meaning at all. However, in a paper published in 1900,[7] Poincaré showed that the local time t' could be given a simple physical interpretation: Suppose observers at various points along the x' axis of the moving frame F' synchronize their respective clocks by exchanging light signals with an observer at the origin of F' and that the speed of light is independent of the motion of its source. The difference between true time and local time would be accounted for by an observer at rest in F_{ar} by the amount each clock in F' had been thrown out of "true synchrony" by its translation during the exchange of signals. Because the exact Lorentz transformations had not yet been derived, these discussions hold only to the first order approximation in (V/c). Historically, this was the first operational definition of time. An exact operational definition of time was discussed by Poincaré later in 1904, as we shall see below.

5b. Poincaré and the principle of relativity

The idea of relativity had already engrossed Poincaré's mind for about ten years when he proposed it as the "*principle of relativity*" in 1904 and reformulated physics in accordance with it in 1905. It was a long evolutionary process for him finally to grasp the principle based on an inductive approach. As mentioned before, in 1895, Poincaré had already noted the impossibility of detecting Earth's absolute motion.[5] Five years later in 1900, he called it the "principle of relative motion."[7] In his book *Science and Hypothesis* published in 1902, Poincaré's principle of relative motion is stated in chapter VII, "RELATIVE AND ABSOLUTE MOTION," as follows: "The movement of any system whatever ought to obey the same laws, whether it is referred to fixed axes or to the movable axes which are implied in uniform motion in a straight line."[7] Although Lorentz had initiated research in "relativity" as early as 1895 and attempted to show that Maxwell's equations are invariant under a new space and time transformation, he did not conceive the principle of relativity to be generally and rigorously valid and never believed in the relativistic time, even though he himself discovered it.

In his address to the Paris Congress of 1900, Poincaré discussed hypotheses in physics and the theories of modern physics.[8] It was a comprehensive and impressive survey of all branches of physics up to that time, the very beginning of the 20th century. In his address, he asked a burning question, "Does the ether really exist?" which was one of the central questions in physics at that time. The continuing importance of this question can be seen by the fact that Einstein came back to this question in 1920 from the viewpoint of gravity, that Dirac wrote a paper with essentially the same title in 1951 based on relativistic quantum electrodynamics, and that particle physicists such as T. D. Lee, Weisskopf, Bjorken and others continued to discuss it in the 1980's and afterward based on modern gauge field theory, as we shall see in section 5f.

The essence of Poincaré's argument was that the ether exists but that there is a conspiracy of dynamical effects so that the velocity of an object moving through the ether cannot be detected by optical phenomena. He believed that the conspiracy causing the cancellations of the velocity-dependent terms should be rigorous and absolute, holding to all orders, rather than just to the first order.

In 1904, Poincaré delivered an important address to the International Congress of Arts and Sciences in St. Louis with the title "THE PRINCIPLES OF MATHEMATICAL PHYSICS," in which he listed and discussed six general principles of physics.[9] One of them is what he called the "principle of relativity, according to which the laws of physical phenomena should be the same, whether for an observer fixed, or for an observer carried along in a uniform movement of translation; so that we have not, and could not have any means of discerning whether or not we are carried along in such a motion." He explained it as follows: "Indeed, experience has taken on itself to ruin this interpretation of the principle of relativity; all attempts to measure the velocity of the earth in relation to the ether have led to negative results. This time experimental physics has been more faithful to the principle than mathematical physics; ... but experiment has been stubborn in confirming it. .....And finally Michelson has pushed precision to its limit: nothing has come of it." In this connection, it is worthwhile to note that Poincaré's principle of relativity is clearly a new one and is *not* the Galilean-Newtonian principle of relativity in classical mechanics. In fact, this principle is exactly the same as his "principle of relative motion" proposed in 1900,[7] as mentioned previously. We stress that *this is a pure symmetry principle because it asserts that the form of physical laws should be the same in any inertial frame.* In modern language, it asserts that physical laws must display the four-dimensional symmetry of the Lorentz and the Poincaré groups.

Taken alone, Poincaré's principle of relativity does not provide sufficient foundation for special relativity. *However, with this principle of relativity, Poincaré has laid the foundation for a broad four-dimensional symmetry framework, of which special relativity is, in the restricted sense of adding an extra postulate, a particular case.* We elaborate this statement in chapter 7.

Poincaré even went one step further by treating Lorentz's ingenious idea of a local time t' as a physical time and gave t' a rigorous operational definition based on his principle of relativity. Consider two observers (or the station A and the station B) in uniform relative motion who wish to synchronize their clocks by means of light signals. According to Poincaré, clocks synchronized in this manner do not mark 'true' time if the frame of reference has an absolute motion, *since the velocity of light is not isotropic in that frame.* However, this leads to no contradiction because all physical laws in the moving frame are the same as those in a rest frame. Thus, an observer in the moving frame has no means of detecting the anisotropy and thereby ascertaining a difference between his local time and "true" time. The difference "matters little since we have no means of perceiving it." "All phenomena which happen at A, for example, will be late, but all will be equally so, and the observer who ascertains them will not perceive it since his watch is slow; so as the principle of relativity would have it, he will have no means of knowing whether he is at rest or in absolute motion."

Here, in 1904, appeared for the first time the procedure for what is now called the operational definition of physical time. Poincaré continued: "Unhappily, that does not suffice, and complementary hypotheses are necessary; it is necessary to admit that bodies in motion undergo a uniform contraction in the sense of the motion.... Thus, the last little difference finds themselves compensated. And then there is still the hypothesis about force. Force, whatever be their origin, gravity as well as elasticity, would be reduced in a certain proportion in a world animated by a uniform translation; or, rather, this would happen for the components perpendicular to the translation; the components parallel would not change."

Poincaré concluded his talk with a marvelous vision: "Perhaps we should construct a whole new mechanics, that we only succeed in catching a glimpse of, where inertia increasing with the velocity, the velocity of light would become an impassable limit."[9]

5c. Poincaré's theory of relativity

As mentioned previously, by 1904, Poincaré had discussed both a postulate related to relativity, i.e., the principle of relativity, and a closely related property, i.e., that "no velocity could surpass that of light" (which was stated as a characteristic of the mechanics of the future). These correspond closely, though they are not identical to, the two postulates Einstein made in discussing special relativity one year later. Poincaré also discussed two hypotheses of mechanics: (a) that bodies in motion suffer a uniform contraction in their direction of motion and (b) that the components of forces perpendicular to the translation would be reduced. Because his idea of a new mechanics based on these two hypotheses and the principle of relativity was not stated until his lectures[10] in Göttingen in 1909 and the lack of explicit physical interpretations of space, time and relativity,[4] many physicists tend not to consider Poincaré as an originator of special relativity.

In June and July 1905, Poincaré completed two papers, both entitled "On the Dynamics of the Electron,"[1] regarding his principle of relativity and the Lorentz transformations. The first was a summary of the second, much longer paper, which was Poincaré's last major work on relativity and was written at about the same time that Einstein finished his first paper on special relativity. It is interesting to note that, roughly speaking, the mathematician Poincaré took a more physical approach to discussing his relativity theory, while the physicist Einstein took a more mathematical (axiomatic) approach in his formulation of the theory. In his work, Poincaré stayed close to experimental evidence, noting that neither the aberration of starlight and related phenomena nor the work of Michelson revealed any evidence for an absolute motion of the earth.

He said, "It seems that this impossibility to disclose experimentally the absolute motion of the earth is a general law of nature; we are led naturally to admit this law, which we shall call the *Postulate of Relativity*, and to admit it unrestrictedly. Although this postulate, which up till now agrees with experiment, must be confirmed or disproved by later more precise experiments, it is in any case of interest to see what consequences can flow from it." Furthermore, before Poincaré wrote down Maxwell's equations and the Lorentz force, he chose the units of length and of time so as to make the speed of light unity. In other words, he defined c=1 by a suitable choice of units rather than by postulating the speed of light to be a universal constant, in sharp contrast with Einstein. Poincaré never mentioned an experimental test of the universality of the speed of light, although he did mention that the postulate of relativity must be tested experimentally. This indicates that he did not perceive the invariance of physical laws and the constancy of the speed of light to be equally fundamental. Although this is usually considered to be a weakness in Poincaré's understanding of physics, we will argue later that this is actually Poincaré's insight.

In developing his theory of relativity, Poincaré wrote down the Maxwell equations, the continuity equation for the conservation of charge, the wave equations for the scalar and vector potentials, and the Lorentz force. He stressed that "these equations admit a remarkable transformation discovered by Lorentz, which is of interest because it explains why no experiment is capable of making known to us the absolute motion in the universe." The transformations discovered by Lorentz in 1899 are given by (4.3), i.e.,

$$x' = K\gamma(x - \beta t), \quad y' = Ky, \quad z' = Kz, \quad t' = K\gamma(t - \beta x); \tag{5.1}$$

$$\gamma = 1/\sqrt{1-\beta^2}, \qquad \beta = V/c, \qquad c = 1.$$

Poincaré called (5.1) the Lorentz transformations and rigorously proved the invariance of Maxwell's equations under these transformations. Poincaré derived the transformation equations for the force per unit charge and pointed out that they differ significantly from those found by Lorentz. He further discussed in detail the *complete* group properties of the transformations (5.1).[1] When we talk about an F and an F' frame in (5.1), we imply two inertial frames with origins that coincide at $t=t'=0$ and where the relative motion is only along the x axis and the velocity of F' relative to F is V.

When Poincaré discussed the Lorentz group, it was clear that his derivation of the Lorentz transformation (5.1) was no different from that used in many of today's textbooks (with $K=1$): He explained that every transformation of this group can be decomposed into a transformation of the form $x_i'=Kx_i$, $t'=Kt$ and a linear transformation that leaves invariant the quadratic form $\mathbf{x}^2 - t^2$, where $\mathbf{x}=(x_1,x_2,x_3)$. Perhaps because his paper was not widely read and because this simple derivation was not written down explicitly in terms of the invariance of physical laws, many people are still not aware of Poincaré's derivation of the Lorentz transformations. Most physicists believe that he had simply assumed the transformation, as Lorentz and Larmor did. As a result, it is usually believed (incorrectly) that Einstein was the first to derive the Lorentz transformations from first principles.

In his Rendiconti paper, Poincaré did not discuss time and clock synchronizations because they were discussed in his previous papers. He did however discuss a procedure for making spatial measurements, writing "How do we make our measurements? By transporting to mutual juxtaposition objects considered as invariable solids, one would reply at first; but this is no longer true in the present theory if one admits Lorentzian contraction. In this theory two equal lengths are by definition two lengths which are traversed by light in equal times."

The elegance of Poincaré's style is most evident in his use of the principle of least action (or the Lagrangian formulation) for Maxwell's equations with sources in vacuum and the equations of motion for charged particles which were shown to be invariant under the Lorentz transformations. He was the first physicist to employ the Lorentz-covariant Lagrangian formalism for discussions of physical theories. Today, this formalism has become a standard method for treating quantum field theories and relativistic particle dynamics. Poincaré also used the Lorentz transformation to discuss the dynamics and the contraction of an extended electron in detail. The electromagnetic field due to the presence of the electron can affect the electron itself. The interaction energy between the electron and its own electromagnetic field is called the self-energy and is an inherent part of the electron. He discussed the self-energy of the electron and noted that an extended electron with negative charge cannot be stable.

In these discussions, Poincaré wrote down the complete Lorentz invariant action integrals for the electron, the electromagnetic fields and their interaction. For example, he discussed quasi-stationary and arbitrary motions of an electron and obtained the exact Lagrangian for the electron, in which he defined a quantity that he called the experimental mass (now known as the rest mass) of the electron.[1] Once the Lagrangian $L=m(1-u^2)^{1/2}$ for the electron is given, its kinetic properties and equations of motion are completely determined. The correct expressions for the relativistic energy $\left(L-\mathbf{u}\bullet(\partial L/\partial\mathbf{u})\right)$, momentum ($\partial L/\partial\mathbf{u}=-D\mathbf{u}/u$, $D=-dL/du$), and the transverse and the longitudinal mass (D/u and dD/du) were also derived by Poincaré in addition to the exact covariant equation of motion. This covariant equation of motion for a charged particle is important because it shows for the first time that Newton's second law, **F**=m**a**=d(m**u**)/dt, in classical mechanics can be generalized to be consistent with the principle of relativity. This task is difficult to carry out if one does not use the Lorentz-invariant Lagrangian and the variational calculus. (In contrast, Einstein failed in deriving the exact covariant equation of motion for a charged particle in 1905, as we shall see in chapter 6 below.) All these show Poincaré's mastery of electrodynamics and mathematics and his difference from Einstein in style and taste.

Poincaré concluded that "if one admits this impossibility (of manifesting absolute motion) one must admit that electrons when in motion contract so as to become ellipsoid of revolution whose two axes remain constant; one must then admit ... the existence of a supplementary potential proportional to the volume of the electron." Thus, his principle of relativity led him to develop the "Poincaré stresses" to maintain the stability of an extended electron, a model discussed by Lorentz and others. This was the climax of his paper. However, the electron is now assumed to be a point-like object in quantum mechanics and quantum electrodynamics, so that Poincaré stresses do not play a significant role in modern physics, although they are frequently mentioned. We note that since the assumption of point-like particles is related to the divergence difficulty in quantum field theory, it is possible that a model for non-point-like particles might emerge in the physics of the future and that Poincaré stresses may again become important, unless the quantization of the electric charge and its stability can be understood on the basis of some as yet undiscovered new principles.

In this connection, it is interesting to note Poincaré's outlook of the physical world: When an extended electron is at rest, the equilibrium of the Coulomb force and the Poincaré stresses result in a spherical electron. However, when in motion, an electron suffers a contraction such that the extended electron undergoes a deformation precisely according to the Lorentz transformation. In this sense, *he appeared to believe that his non-electromagnetic stresses gave a dynamical explanation to the contraction of the extended electron.*[1]

In the last part of his regular paper, he discussed a "relativistic theory of gravitation" which was a generalization of Newtonian gravity to be consistent with the principle of relativity. His generalized gravitational theory involved a retarded action-at-a-distance interaction and the concept of gravitational waves which were "assumed to propagate with the velocity of light." It was the first such attempt and not satisfactory. However, the significant result is the 4-dimensional symmetry framework (with 4-vectors and Lorentz invariants, etc.) that he developed, anticipating Minkowski, as a powerful tool to study physics. After discussing general properties of the basic scalar equation of propagation and transformation properties of the gravitational force, Poincaré said: "In order to make further progress, it is necessary to look for the *invariants of the Lorentz group* (the italics are Poincaré's.) We know that the transformations of this group (taking K=1) are the linear transformations which do not change the quadratic form $x^2 + y^2 + z^2 - t^2$. .... Let us consider x, y, z, $t\sqrt{-1}$; dx, dy, dz, $dt\sqrt{-1}$;.... as coordinates of ... points...in a space of four-dimensions. We see that the Lorentz transformation is just a rotation of this space about the origin, regarded as fixed. We shall have as invariants only the...distances between the points among themselves and the origin,... $x^2 + y^2 + z^2 - t^2$, $x\delta x + y\delta y + z\delta z - t\delta t$, etc...." (c=1).

Indeed, Poincaré's statement concerning the "the *invariants of the Lorentz group*" is precisely a modern view of the theory: "Mathematically speaking, therefore, the special theory of relativity is the theory of invariants of the Lorentz group," as Pauli said in 1921 (when he was 21 years old) in the article on the theory of relativity written for the *Mathematical Encyclopedia*, which later became his book "*Theory of Relativity.*"[2]

The essence of Poincaré's argument for K=1 in (5.1) is the principle of relativity which implies that both frames, F and F', are physically completely equivalent. Otherwise, if one sets $K=\sqrt{1-V^2/c^2}=1/\gamma$ then the transformation (5.1) becomes

$$x' = x - Vt, \quad y' = y/\gamma, \quad z' = z/\gamma, \quad t' = t - Vx/c^2. \tag{5.2}$$

This particular transformation was first obtained by Voigt in 1887 in a work in which he discussed Doppler shifts and invariance of wave equations. Clearly, Voigt's transformation preserves the "four-dimensional intervals": $s(0)^2 = s'(V)^2$, where

$$\begin{aligned} s(0)^2 &= c^2t^2 - \mathbf{r}^2, && \text{for frame F (at rest)}, \\ s'(V)^2 &= (c^2t'^2 - \mathbf{r}'^2)\left(1-\frac{V^2}{c^2}\right)^{-1}, && \text{for frame F' (moving)}, \end{aligned} \tag{5.3}$$

or $s(V)^2 = s'(0)^2$, where

$$\begin{aligned} s(V)^2 &= (c^2t^2 - \mathbf{r}^2)\left(1-\frac{V^2}{c^2}\right), && \text{in F (moving)}, \\ s'(0)^2 &= c^2t'^2 - \mathbf{r}'^2, && \text{in F' (at rest)}. \end{aligned} \tag{5.4}$$

Note that the four-dimensional intervals are not those from special relativity but are multiplied by a velocity-dependent factor. This transformation implies that one of the frames has an "absolute velocity" V. Whether such an "absolute velocity" V, if it exists, can be detected turns out to be non-trivial and is still an open question,[11] as we shall see in the next section.

5d. Conformal transformations and a frame of "absolute rest"

Theoretical physicists and mathematicians like to discuss general cases. In fact, Poincaré derived most of the equations in his relativity paper[1] based on the transformation (5.1) with the arbitrary constant $K(\beta)\neq 1$. He stated that every transformation in this group, which he called the Lorentz group, can be decomposed into a transformation of the form, $\mathbf{r}'=K\mathbf{r}$, $t'=Kt$, and a linear transformation that leaves invariant the quadratic form $\mathbf{r}^2- c^2t^2$. Thus, in modern language, Poincaré actually discussed the invariance of Maxwell's equations under conformal transformations. The conformal transformation is a larger class of coordinate transformations for which $c^2t'^2-\mathbf{r}'^2$ is proportional, though not necessarily equal, to $c^2t^2-\mathbf{r}^2$, and which therefore also leaves the speed of light universal and invariant, as shown in equations (5.2)-(5.4). Concerning the special case K=1, Poincaré said: "...we have to suppose that K is a function of β and it is a question of choosing this function in such a way that this group portion, which I shall call P, forms also a group." In this connection, it is worthwhile to note that the concept of conformal invariance was explicitly introduced into physics in 1909 by Cunningham who showed that Maxwell's equations are invariant not only under the Lorentz group, but also under the conformal group C_o.[12]

In connection with the existence of an ether, let us briefly examine further possible physical implications of an absolute velocity (i.e., a velocity relative to a frame of absolute rest) based on the 4-dimensional framework with the conformal symmetry. Suppose we have three inertial frames F_a = F(0), F' = F'(V) and F" = F"(U), where V and U are the absolute velocities of F' and F" relative to the absolute rest frame F_a=F(0) along the +x direction. For a general K(V), we assume the four-dimensional intervals to be[9]

$$s^2 = c^2t^2 - \mathbf{r}^2, \qquad \text{for } F(0) \,,$$

$$s'^2 = K^{-2}(V)(c^2 t'^2 - \mathbf{r}'^2), \qquad \text{for } F'(V), \tag{5.5}$$

$$s''^2 = K^{-2}(U)(c^2 t''^2 - \mathbf{r}''^2), \qquad \text{for } F''(U).$$

If we insist that the four-dimensional interval be invariant, we obtain the following "absolute" transformations:

$$x' = K(V)\left(\frac{x - Vt}{\sqrt{1-\beta_V^2}}\right), \quad y' = K(V)y, \quad z' = K(V)z, \quad t' = K(V)\left(\frac{t - Vx/c^2}{\sqrt{1-\beta_V^2}}\right), \tag{5.6}$$

and

$$x'' = K(U)\left(\frac{x - Ut}{\sqrt{1-\beta_U^2}}\right), \quad y'' = K(U)y, \quad z'' = K(U)z, \quad t'' = K(U)\left(\frac{t - Ux/c^2}{\sqrt{1-\beta_U^2}}\right), \tag{5.7}$$

where $\beta_V = V/c$ and $\beta_U = U/c$.

It can be shown that the transformations (5.6) and (5.7) form a conformal group because, for example, the transformation between F'(V) and F''(U) has the same form

$$x'' = \rho\left(\frac{x' - V't'}{\sqrt{1-\beta'^2}}\right), \quad y'' = \rho y', \quad z'' = \rho z', \quad t'' = \rho\left(\frac{t' - V'x'/c^2}{\sqrt{1-\beta'^2}}\right), \tag{5.8}$$

where

$$\rho = \frac{K(U)}{K(V)}, \qquad V' = \frac{U - V}{1 - UV/c^2}, \qquad \beta' = \frac{V'}{c}. \tag{5.9}$$

Mathematically, the Voigt transformation (5.2) is clearly a special case of a conformal transformation (5.6). The physical relevance of the conformal transformations is, in general, not yet clear.[13]

One may interpret absolute transformations such as those shown above as a theory of "conformal four-dimensional symmetry" with an absolute motion and with a constant speed of light c. Thus, it will be interesting to see whether these transformations can be excluded by modern high precision experiments of special relativity. Surprisingly, it is nontrivial to exclude experimentally this type of "absolute" transformation.[11] For example, let us consider experiments designed to measure the Doppler shift to high precision. The transformations of the contravariant wave 4-vector $k^{\mu}=(k^0, k^1, k^2, k^3)=(\omega/c, k_x, k_y, k_z)$ between two moving frames, $F'=F'(V)$ and $F''=F''(U)$, are given by

$$k''_x = \rho\frac{k'_x - \beta' k'^0}{\sqrt{1-\beta'^2}}, \quad k''_y = \rho k'_y, \quad k''_z = \rho k'_z, \quad k''^0 = \rho\frac{k'^0 - \beta' k'_x}{\sqrt{1-\beta'^2}}, \tag{5.10}$$

where $(k'_x, k'_y, k'_z) = (k'^1, k'^2, k'^3)$ and

$$\rho = \frac{K(U)}{K(V)}, \quad V' = \frac{U-V}{1-UV/c^2}, \quad \beta' = \frac{V'}{c}. \tag{5.11}$$

Suppose one performs the experiment in the $F'(V)$ frame and that the atoms are at rest in $F''(U)$ so that $k''_x = 2\pi/\lambda''$ and $k''^0 = 2\pi\nu''/c$. In practice, one cannot know the wavelength and frequency λ'' and ν'' of the light emitted by the atoms as measured by observers in the $F''(U)$ frame. One can only compare the shifted λ' and ν' with the unshifted quantities λ'_o and ν'_o associated with the same kind of atoms at rest in the laboratory frame $F'(V)$. Since $F'(V)$ and $F''(U)$ are not completely equivalent, as shown in (5.5), the wavelength and the frequency of light emitted by atoms at rest in F' and measured by observers in F' are not the same as those emitted by the same kind of atoms at rest in F'' and measured in F''. Thus, one does not have the usual relations $\lambda'' = \lambda'_o$ and $\nu'' = \nu'_o$ of special relativity. Rather, one has in general

$$\frac{k''_x}{K(U)} = \frac{k'_{xo}}{K(V)} \quad \text{and} \quad \frac{(\omega''/c)}{K(U)} = \frac{(\omega'_o/c)}{K(V)} \tag{5.12}$$

for the contravariant wave 4-vector $k'^{\mu} = (k'^0, k'^1, k'^2, k'^3)$ because of the metric tensor in (5.5) for the two frames F'(V) and F"(U). Thus, one obtains

$$\frac{1}{\lambda'' K(U)} = \frac{1}{\lambda'_o K(V)} \quad \text{and} \quad \frac{\nu''}{K(U)} = \frac{\nu'_o}{K(V)}. \tag{5.13}$$

It follows from (5.10) and (5.13) that the observable Doppler shifts in the moving frame F'(V) are given by:[11]

$$\frac{1}{\lambda'_o} = \frac{1}{\lambda'} \frac{(1-\beta')}{\sqrt{1-\beta'^2}}, \qquad \nu'_o = \nu' \frac{(1-\beta')}{\sqrt{1-\beta'^2}}, \tag{5.14}$$

which are exactly the same as those in special relativity. Therefore, we see that the absolute velocities V and U cannot be determined individually by the Doppler shift experiment. The best one can do is to determine the "relative velocity" β' between F'(V) and F"(U), as shown by (5.14). Note that if one uses the covariant wave 4-vectors, $k'_{\mu} = K^{-2}(V) k'^{\mu}$ and $k''_{\mu} = K^{-2}(U) k''^{\mu}$, the relations corresponding to (5.10) and (5.12) will be different. Nevertheless, the final result (5.14) for Doppler shift remains the same. Furthermore, the Michelson-Morley experiment also cannot rule out the validity of the "absolute transformation" (5.6) and (5.7) because the speed of light is still the universal constant c in such a theory with "conformal 4-dimensional symmetry."

Incidentally, these results are consistent with Poincaré's view that the ether exists but that the absolute velocity of a moving frame cannot be detected. Nevertheless, there is a difference: Namely, there is *no conspiracy* of dynamical effects due to the interaction of ether and matter so that the velocity of an object moving through the ether cannot be detected by optical phenomena. Rather, the effects of the absolute velocities V and U are suppressed and disappear from the Doppler shift experiment and the Michelson-Morley experiment because of the inherent properties of the conformal 4-dimensional symmetry.

5e. Poincaré's impact on relativity and symmetry principles

Poincaré's paper[1] gave a complete logical foundation for relativity theory, including the mathematical framework and the basic invariant equations for both electromagnetic fields and charged particles. There is little doubt that Einstein formulated the theory of special relativity independently, even though conceptually he may have benefited from Poincaré's non-technical articles and Lorentz's early papers. As we shall discuss in the next chapter, there are no differences in experimental implications between Poincaré's and Einstein's relativity theories because their equations are identical, provided Einstein's approximate and non-Lorentz invariant dynamical equations for charged particles are properly corrected. (See chapter 6.)

Both Poincaré and Einstein should be credited for the discovery and the establishment of the fundamental principle of relativity for physical laws. Thus we shall call this fundamental symmetry principle in physics the "Poincaré-Einstein Principle." In our discussions, the following statements may be taken to be equivalent:

(a) the Poincaré-Einstein principle,

(b) the principle of relativity,

(c) the 4-dimensional symmetry,

(d) the equivalence of all inertial frames,

(e) the Lorentz and Poincaré invariance.

Henri Poincaré was generally acknowledged to be the most outstanding mathematician alive at the turn of the 20th century. He was a man with enormous interest and rare insight in all branches of mathematics, astronomy and theoretical physics. After Gauss, he was the last mathematical universalist—doing creative work of high quality in all four main divisions of mathematics: arithmetic, algebra, geometry and analysis.[14] In addition, Poincaré made significant contributions towards the theory of special relativity in the years from 1895 to 1905.

The main reason that Poincaré did not achieve a complete grasp of the very essence of relativity appears to be because of his viewpoint that the ether did exist and that there was a conspiracy of *dynamical effects* due to the interaction of ether and matter such that the velocity of an object moving through the ether could not be detected. This outlook of the physical world urged him to search for a dynamical explanation for the length contraction of the electron, as he did in his Rendiconti paper.[1] He did not have a satisfactory answer to explain the conspiracy of dynamical effects. Apparently, a dynamical explanation for the length contraction was very natural and popular among physicists at that time. FitzGerald also speculated that "the molecular forces are affected by the motion, and that the size of a body alters consequently."[15] Even as late as 1912, three months before his death, Poincaré delivered a lecture (to the French Society of Physics, April 11, 1912) entitled "The Relations Between Matter and Ether."[16]

In the modern theory of gauge fields, an ether may indeed exist, but it is quite different from the type of ether that Lorentz and Poincaré had in mind. Furthermore, one probably cannot prove experimentally that Poincaré's belief in a conspiracy of *dynamical effects* due to interaction of ether and matter is wrong. However, as far as relativity theory is concerned, it is not necessary and simply makes the theory more complicated conceptually. In this aspect, Einstein had a more profound understanding: Namely, that such an effect does not need to be explained!

Poincaré was not alone in having had an incorrect conceptual interpretation of his equations. Maxwell also had wrong ideas of molecular vortices and idle wheels for the electromagnetic field in his equations, but physicists still call them Maxwell's equations.

Why did this thing happen?

In an interesting discussion of innovation in physics, Dyson made the piercing observation[17] "When the great innovation appears, it will almost certainly be in a muddled, incomplete and confusing form. To the discoverer himself it will be only half-understood; to everybody else it will be a mystery. For any speculation which does not at first glance look crazy, there is no hope." He explained that "the reason why new concepts in any branch of science are hard to grasp is always the same; contemporary scientists try to picture the new concept in terms of ideas which existed before. The discoverer himself suffers especially from this difficulty; he arrived at the new concept by struggling with the old ideas, and the old ideas remain the language of his thinking for a long time afterward." This makes it clear why a young scientist has an edge over a mature scientist in grasping a new concept. For example, one can see a difference between the young Einstein and the mature Poincaré regarding having a sound grip of the "crazy" properties of space, time and relativity related to the Lorentz transformations around 1900; or of the young Heisenberg versus the mature Einstein in understanding the "crazy" uncertainty principle in quantum mechanics around 1927.

It seems fair to say that Poincaré ushered symmetry principles onto the stage of twentieth century physics through:

(i) his 1904 proposal[9] of the symmetry principle of relativity for all laws of physics,

(ii) his recognition[1] that the Lorentz transformations of space and time preserve the invariant $x^2 + y^2 + z^2 - c^2t^2$ and form a symmetry group, and that the Lorentz group has 6 generators (three for spatial rotations and three for boosts),

(iii) his insight into the relation between the Lorentz transformations and a 'rotational symmetry' in a 4-dimensional space,[1]

(iv) his use of the *invariants of the Lorentz group* and the Lorentz covariant Lagrangian formalism for treating fields and particle dynamics, which is particularly powerful for treating a physical system with symmetry properties.

These insights show that his understanding of the symmetry properties of the physical world was deeper than anybody else's at the beginning of the twentieth century. With the help of the enormous impact of Einstein's special relativity and general relativity, symmetry principles are now generally believed to play a universal and fundamental role in revealing the simplicity of the physical world. In particular, the view that *symmetry dictates interactions* took root mainly through the works of H. Weyl and of C. N. Yang and R. L. Mills in modern quantum field theory and was stressed in particular by Utiyama and by Yang.[18] The most spectacular results of the power of symmetry principles can be seen in Dirac's prediction of the existence of anti-particles and in the establishment of the unified electroweak theory and quantum chromodynamics based on non-Abelian gauge fields discovered by Yang and Mills. Today, one hundred years after Poincaré proposed the symmetry principle of relative motion for all physical laws, symmetry principles in physics have transcended both kinetic and dynamic properties and are at the very heart of our understanding of the universe.

In this connection, we may remark that Poincaré did not discuss the most general symmetry group in 4-dimensional spacetime (i.e., the inhomogeneous Lorentz group generated by four translations and six rotations). However, this group is nowadays called the "Poincaré group"[19] by Wigner and many physicists and mathematicians. Furthermore, "Lorentz and Poincaré invariance" is now a standard term in the 1997 ICSU/AB International Classification System for Physics, and in Physics and Astronomy Classification Scheme. These attributions appear to be in memory of his insight of the 4-dimensional symmetry of spacetime. The Poincaré group also has important applications in particle physics, as shown by Wigner and others.[20]

5f. Retro physics: Past and present views of the ether

The concept of ether is an old and cherished one which is intertwined with the physical properties of the vacuum as a whole. In the age of Newton and Maxwell, ether meant, among other things, the medium for the propagation of light and *was identified with the classical electromagnetic field.* Its existence was taken for granted, just like the existence of the electromagnetic field. Nowadays, it is simply called the "vacuum" in modern quantum field theory. The properties of the vacuum are so complicated and non-classical from the viewpoint of quantum fields that the modern concept is completely beyond anything Faraday, Maxwell, Lorentz, Poincaré and Einstein could have dreamed.

In the 19th century, the universe was believed to be built of objects with mechanical properties. In particular, light was believed to be supported by a material substance called ether when it propagated in vacuum. Maxwell and other physicists tried to understand and visualize the electromagnetic field as a mechanical stress in a material substance. Some physicists believed ordinary matter to be condensed ether. Others such as Lord Kelvin held that matter is only the locus of those points at which the ether is animated by vortex motions. Riemann believed it to be the locus of those points at which ether is constantly destroyed. The properties and the names associated with the ether may change, but the essential idea has never faded away even after Einstein declared it to be superfluous based on his theory of relativity. In the early 20th century, the concept of a state of motion relative to the ether had to be given up because (a) it was unobservable and (b) it became superfluous to a physical formalism since the electromagnetic field became regarded as an independent physical reality.

Nevertheless, many well-known physicists continued to reexamine and discuss the ether from time to time. For example, in 1920 Einstein himself gave a talk on ether and general relativity. He identified the ether with the gravitational field and said that "the ether of the general theory of relativity is a medium without mechanical and kinematic properties, but which codetermines mechanical and electromagnetic events."[21] However, there was no further investigation along this line of thought.

In 1951, Dirac published a paper in the journal Nature with the title "Is there an Ether?"—which is the same question that Poincaré asked in 1900. Dirac discussed the existence of an ether on the basis of modern physics:

"If one reexamines the question in the light of present-day knowledge, one finds that the ether is no longer ruled out by relativity, and good reasons can now be advanced for postulating an ether."[22]

After the significance of gauge theories[21,23] based on Yang-Mills fields was recognized, many theorists discussed the physical properties of the vacuum in the 1970s. In his 1981 book *Particle Theory and Introduction to Field Theory*, T. D. Lee discussed the vacuum as the source of asymmetry in quantum field theory (related to CP non-conservation and spontaneous symmetry breaking in the unified electroweak theory, etc.) and as a color dielectric medium for permanent quark confinement.[24] Lee said that

"...since at that [Faraday's] time the nonrelativistic Newtonian mechanics was the only one available, the vacuum was thought to provide an absolute frame which could be distinguished from other moving frames by measuring the velocity of light.... We know now that vacuum is Lorentz-invariant, which means that just by our running around and changing the reference system we are not going to alter the vacuum."

Here, the vacuum is the ether in old language. Lee pointed out that from Dirac's hole theory we know that the vacuum is actually quite complicated, even though it is Lorentz-invariant and is the lowest energy state of the system. Then Lee asked an important and long-standing question:

"*Could the vacuum be regarded as a physical medium?*"

This is a question that has engrossed the minds of many physicists such as Huygens, Faraday, Maxwell, Michelson, Lorentz, Poincaré, Einstein, Dirac and others throughout the ages.[14] Lee's answer was as follows:

"*If under suitable conditions the properties of the vacuum, like those of any medium, can be altered physically, then the answer would be affirmative. Otherwise, it might degenerate into semantics.*"

He then proceeded to analyze and discuss the physical properties of the vacuum (or the ether) on the basis of two remarkable phenomena in modern physics, missing symmetry and permanent quark confinement.

Who says the concept of the ether is dead ?

Some may wonder whether Poincaré's acceptance of the principle of relativity is only provisional because he did not give up the concept of the ether. In contrast, Einstein put his faith in the two postulates of relativity and abandoned the notion of an ether completely. In this connection, it should be stressed that from the viewpoint of modern quantum field theory, the existence of an 'ether' is not incompatible with the principle of relativity or Lorentz and Poincaré invariance. In this sense, Poincaré's belief turns out to be not completely wrong from a modern field-theoretic viewpoint. Since the basic laws for the electromagnetic fields and for the motion of a charged particle in his 1905 papers are Lorentz invariant and identical to those in special relativity, his idea of "absolute rest" appears to be only a question of semantics in the sense of T. D. Lee.

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2. Sir E. Whittaker, *A History of the Theories of Ether and Electricity: The Modern Theories 1900-1926* (Philosophical Library, New York, 1954), p. 40 and in *Biographical Memoirs of Fellows of the Royal Society* (London, 1955), p. 42.
3. G. Holton, Am. J. Phys. **28**, 627 (1960).
4. B. Hoffmann, Relativity, in *Dictionary of the History of Ideas*, vol. IV (Ed. P. P. Wiener, Charles Scribner's Sons, New York, 1973), pp. 74-92. Hoffmann observed that "along with the later fame of Einstein there grew a popular mythology correctly attributing the theory of relativity to him, but seriously slighting the work of Poincaré." In ref. 1, C. Cuvaj also said: "Frequent misunderstanding of the work of Poincaré and Einstein has resulted in controversy tending to obscure the main achievements of Poincaré." Nevertheless, physicists such as Pauli, Feynman, J. J. Sakurai, Pais and many others gave Poincaré credit for his endeavors towards relativity theory. For example, Feyman said: "Einstein realized, and Poincaré too, that the only possible way in which a person moving and a person standing still could measure the speed (of light) to be the same was that their sense of time and their sense of space are not the same, ..." "It was Poincaré's suggestion to make this analysis of what you can do to the equations and leave them alone. It was Poincaré's attitude to pay attention to the symmetries of

physical laws." See R. P. Feynman, *The Character of Physical Law* (MIT Press, 1965), pp. 91-92 and p. 94; W. Pauli, *Theory of Relativity* (Pergamon Press, London, 1958), p. 1 and p. 21 ; J. J. Sakurai, *Advanced Quantum Mechanics* (Benjamin/Cummings, 1984), p. 12 and Pais, in ref. 1. For different views of Poincaré's contributions, see also C. Scribner, Jr., Am. J. Phys. **32**, 672 (1964); C. Lanczos (one of Einstein's closest collaborators), *The Variational Principles of Mechanics* (4th ed. Univ. of Toronto Press, 1977), pp. 291-293; C. Cuvaj, ref. 1, and references therein.

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7. H. Poincaré, Arch. Néerl. [2], **5**, 271 (1900). See also *Science and Hypothesis* (first published in 1902, Dover Publications, Inc. 1952), p.111. Some physicists interpreted Poincaré's principle of relative motion to be simply the principle of relativity for mechanical experiments. The interpretation is untenable: In his address to the Paris Congress in 1900, he discussed whether optical and electrical phenomena are influenced by the motion of the earth and said that "many experiments have been made on the influence of the motion of the earth. The results have always been negative."
8. The English translation of this address can be found in H. Poincaré, *Science and Hypothesis* (first appeared in 1902, Dover Publications, Inc. 1952), pp. 140-182.
9. H. Poincaré, Bull. Sci. Math. **28**, 302 (1904). This address was translated into English by G. B. Halsted and published in The Monist, **15**, 1 (1905) and also in chapters 7 to 9 of *The Value of Science* by Poincaré, (in *The Foundations of Science*, G. R. Halsted, Tran. Science Press, New York, 1913; or Dover, New York, 1958). To appreciate the vision of Poincaré in this talk, it is interesting to note that he even speculated about the future of physical law: "Physical law will then take an entirely new aspect; it will no longer be solely a differential equation, it will take the character of a statistical law." This prophecy was finally realized by the discoveries of Schrödinger's wave

equation and Born's statistical interpretation of the wave function more than 20 years later.

10. A. Pais, in ref. 1, p. 150 and pp. 167-168. Pais discussed Poincaré's third postulate for a new mechanics in 1909 and reached a conclusion: "It is evident that as late as 1909 Poincaré did not know that the contraction of rods is a consequence of the two Einstein's postulates. Poincaré therefore did not understand one of the most basic traits of special relativity." However, this third hypothesis may also be viewed from a different angle: We note that Poincaré made a great effort (in his Rendiconti paper in ref. 1) to give a consistent dynamical theory of an extended electron: "It is necessary to assume a special force which explains simultaneously both the contraction and the constancy of two of the axes. I have sought to determine this force, and I have found that it can be represented by a constant external pressure acting on the deformable and compressible electron, whose work is proportional to the change in volume of this electron." Furthermore, he also stated in his Rendiconti paper: "How do we make our measurements? By transporting to mutual juxtaposition objects considered as invariable solids, one would reply at first; but this is no longer true in the present theory if one admits Lorentzian contraction. In this theory two equal lengths are by definition two lengths which are traversed by light in equal times." Thus, he knew that the length contraction follows from the Lorentz transformations which can be obtained on the basis of the principle of relativity and the definition $c=1$ for the speed of light. In the last part of his Rendiconti paper, Poincaré explained: "We know that the transformations of this group (taking $K=1$) are the linear transformations which do not change the quadratic form $x^2 + y^2 + z^2 - t^2$." Therefore, it appears to be more reasonable to interpret his third hypothesis as a manifestation of his views that the contraction of length is a dynamical result rather than simply a kinematic property of the Lorentz transformations. Since he considered the electron has a finite size and he did not have a satisfactory explanation for the conspiracy of dynamical effects in general, he cast it as a hypothesis. It is worthy to note that the formulation of electrodynamics with an extended electron is

extremely difficult. Poincaré continued to work on it even after he correctly wrote down the relativistic equation for the motion of a charged particle in 1905, as shown in his 1909 Göttingen lecture. Einstein did not discuss this problem of instability of an extended electron. Even in 2005, one hundred years later, we still do not have a satisfactory theory for a non-point-like electron, although such a theory is probably crucial for a logically consistent quantum electrodynamics free from divergence difficulties. Of course, Poincaré's view differs from our present understanding of length contraction in special relativity as a kinematic property of the Lorentz transformations. However, the difference is perhaps only a semantic one in the sense that there is no experimental difference between his "dynamical contraction" and Einstein's kinematic contraction of lengths because the spacetime transformations, and the basic dynamical equations for particles and fields in Poincaré's relativity are identical to those in Einstein's relativity.

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Symposium on Particle Physics, Univ. of Madras, Madras, India, January, 1971).

23. For example, theoretical physicists J. Bjorken gave a talk on "The New Ether" in the Niels Bohr Centennial Symposium (Nov. 1985, Boston), and V. Weisskopf on "The Vacuum in Quantum Field Theories" at Harvard University in Nov., 1985. More recently, in explaining his theory of QCD Lite, F. Wilczek stated: "Most of the mass of ordinary matter, for sure, is the pure energy of moving quarks and gluons. The remainder, a quantitatively small but qualitatively crucial remainder – it includes the mass of electrons – is all ascribed to the confounding influence of a pervasive medium, the Higgs field condensate." In other words, "what we call empty space, or vacuum, is filled with a condensate spawned by that field [the Higgs field]." (See Physics Today, January 2000, page 13.) In this connection, it is interesting to note that if the Higgs field does not exist in nature, one could simply assume constant isospinors in vacuum (i.e., a "vacuum field" or in a certain sense, an "ether") to generate the lepton and gauge boson masses. The reason is that the gauge-covariant property of gauge fields naturally provides the gauge-covariant derivative to act on these constant isospinors. This suffices to generate the masses for the $W^{\pm}$ and Z^{0} gauge bosons and the electron mass in unified electroweak theory. See J. P. Hsu, Nuovo Cimento **89B**, 30 (1985) (p.43).

24. T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood Academic Publishers, New York; and Science Press, Beijing, 1981), pp. 378-405. This book gives comprehensive discussions on fundamental aspects of particle physics related to the vacuum (i.e., the "ether," in old language).

6.

The Novel Creation of the Young Einstein

"*Concepts which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labeled as 'conceptual necessities,' 'a priori situations,' etc. The road of scientific progress is frequently blocked for long periods by such errors.*"

A. Einstein, Phys. Zeitschr. 17,101 (1916)

ALBERT EINSTEIN: Person of the Century

"*He was the pre-eminent scientist in a century dominated by science. The touchstones of the era---the Bomb, the Big Bang, quantum physics and electronic---all bear his imprint.*"

Frederic Golden, TIME, Dec. 31, 1999, page 62

6a. Fresh thoughts from a young mind

Unlike Lorentz and Poincaré, Einstein was a young physicist at the turn of the century and had more fresh ideas and flexibility in his thinking. Rather than being rooted in the same ideas as the traditional physicists of his time, he was able to discard the idea of an ether without any reservations. His approach to the problem was quite novel in that it was mostly aesthetic and logically deductive, rather than mathematical. Furthermore, the young Einstein showed a more profound understanding of the heart of the matter, namely, the physical properties of space and time.

Although Einstein did not refer to other physicists' earlier works in his first 1905 paper on relativity, it is certain that he had known of the work of Poincaré and Lorentz.[1] In modern times, if an unknown physicist were to submit a research paper without any references to previous work by other scientists, such a paper would be rejected by any of the well-known journals such as *Physical Review* or *Nuovo Cimento.* Perhaps this absence of references was the sign of a young rebel outside the academic circle. It certainly made it more difficult for later historians to trace the origin of special relativity and to assess the originality of his paper.

Around the time Einstein was formulating his ideas about relativity, his personal life was in turmoil. He was a young theoretical physicist who had tried and failed for years to find an academic position after graduating. He was passionately in love with his classmate, Mileva Maric, and they had a baby girl, Lieserl, before their marriage but were forced to give the baby away. Moreover, his father had passed away around that time and his mother was vehemently opposed to Einstein's marriage. It was the worst time in Einstein's life, yet the best time for Einstein's physics. This curious circumstance is not unlike that of great artists, whether in music, painting, or poetry, and some of their greatest works.

In his first 1905 paper on relativity,[2] Einstein derived all of the essential results contained in Poincaré's papers of 1905 except for the exact covariant equations of motion for a charged particle. These results are now standard textbook material and hence, will not be discussed in detail here. To read his paper is to see the exuberance of new ideas and the beauty of nature. Although most people tend to think that Einstein created special relativity all by himself, it is more reasonable to consider special relativity as both an effect and a cause of scientific progress.

6b. The theory of special relativity

Einstein's approach to special relativity was as follows: He simply observed that "the phenomena of electrodynamics and mechanics possess no properties corresponding to the idea of absolute rest," and from this idea, proceeded to formulate special relativity entirely on the basis of two postulates or principles:

(PI) The form of a physical law is the same in any inertial frame.

(PII) In all inertial frames, the speed of light c is the same whether the light is emitted from a source at rest or in motion.

The first postulate was originally stated by Einstein as follows: "The laws in accordance with which the states of physical systems vary are not dependent on whether these changes of state are referred to one or to the other of two coordinate systems that are in uniform translational relative motion."[2] It is the same as Poincaré's 'principle of relative motion'[1] proposed in 1900 or his 'principle of relativity' as discussed in the 1904 address in St. Louis: ".... the laws of physical phenomena must be the same for a 'fixed' observer or for an observer who has a uniform motion of translation relative to him ..." This first postulate implies that all inertial frames are equivalent.

The second postulate is also known as the principle of the constancy of the speed of light. This particular property of the speed of light in vacuum (or of electromagnetic waves in general) was actually built into the Voigt transformation (4.5) in 1887 and the exact Lorentz transformations (4.4) obtained by Larmor (1898) and Lorentz (1899). From this historical viewpoint, neither of the two statements above postulated by Einstein was anything original. However, it took Einstein's extraordinary insight to understand the role they play, to combine these two postulates of physics together consistently, and to extract their revolutionary implications regarding space and time, which were so different from the widely accepted Newtonian ideas of space and time.

6c. Derivation of the Lorentz transformation

Einstein showed that a coordinate transformation between F and F' can be derived from the two postulates. He started from the invariance of the form of the law for the propagation of light with $c'=c$ and then obtained the transformation equations in (4.3) where K is an arbitrary scale factor depending on V only. Based on the arguments that

(a) the product of transformation (4.3) and its inverse should yield the identity, (i.e., K(V)K(-V)=1), and

(b) the transformations of y and z should not change if $V \to -V$ (i.e., K(V)=K(-V)), it follows that K(V)=1 for any V, because K(0)=+1 for V=0. The crucial point in this argument is that the inverse transformation must have the same form as the original transformation, according to the principle of relativity. (Note that eq. (4.5) corresponds to (4.3) with $K(v) = (1 - V^2/c^2)^{1/2} = 1/\gamma$. In this case, the transformation (4.5) and its inverse do not have the same form.)

In modern textbooks, the Lorentz transformation is usually derived from the invariance of the four-dimensional spacetime interval,

$$s^2 = c^2t^2 - r^2 = c^2t'^2 - r'^2 = s'^2. \tag{6.1}$$

One immediately obtains the Lorentz transformation without having to find K(V). Since mathematically, (6.1) is a definition of the four-dimensional spacetime interval, but the Lorentz transformation is supposed to be derived from the form invariance of a physical law, one may ask:

To what physical law does the equation in (6.1) correspond?

The answer depends on the values of s^2. When $s^2>0$, (6.1) is equivalent to the equation of motion of a non-interacting particle with a mass $m>0$ moving with a constant velocity $v=r/t$ in an inertial frame:

$$(E/c)^2 - p^2 = m^2 c^2 , \tag{6.2}$$

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \qquad \mathbf{v} = \frac{\mathbf{r}}{t}, \qquad m > 0 . \tag{6.3}$$

When $s^2=0$, (6.1) is the equation of motion of a massless non-interacting particle such as a photon. In this case, we still have the relation, $(E/c)^2-p^2=0$. However, (6.3) is no longer applicable. When $s^2<0$, (6.1) has no physical meaning, i.e. it is not a law of motion for any known physical object. (One might say that (6.1) with

$s^2<0$ describes the motion of a "tachyon" with $v>c$ and $m^2<0$, but tachyons have never been detected experimentally.)

A rigorous proof of the equivalence of the spacetime interval (6.1) and the physical laws (6.2) can be done with the help of a covariant variational calculus.[3]

In his 1905 paper, after Einstein derived the Lorentz transformations, noted that they formed a group, and proved the invariance of Maxwell's equations under the Lorentz transformations, he proceeded to derive and discuss many interesting physical results. Apart from discussions of physical properties of space and time, he obtained the law of velocity addition, the law of aberration, the Doppler shift for any angle between a monochromatic light ray and the x-axis, and the relativistic kinetic energy $mc^2(\gamma - 1)$. All these are exact results.

However, when he discussed the equation of motion of a charged particle in an external electromagnetic field, he obtained only approximate results that hold for terms that are first order in v/c:

$$m\gamma^3 \frac{d^2x}{dt^2} = eE_x, \qquad m\gamma \frac{d^2y}{dt^2} = e\left(E_y - \frac{vH_z}{c}\right), \qquad m\gamma \frac{d^2z}{dt^2} = e\left(E_z - \frac{vH_y}{c}\right), \tag{6.4}$$

where the velocity v is in the x direction. This unsatisfactory situation was noted and resolved by Max Planck in 1906 by using the correct relativistic momentum $\mathbf{p} = m\mathbf{v}/\sqrt{1 - v^2/c^2}$ and its transformation laws, apparently without being aware of Poincaré's previous work published in 1906.[4]

We shall not go into details of Einstein's 1905 paper since all physical results in special relativity are well-known and described in many widely available references.

6d. Relativity of space and time

The new properties of space and time introduced by Einstein are the most difficult concepts to understand in special relativity. Let us consider the novel properties of space and time in the Lorentz transformations (4.2), i.e.,

$$x' = \gamma(x - Vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma\left(t - \frac{Vx}{c^2}\right), \qquad \gamma = \frac{1}{\sqrt{1-(V/c)^2}}. \tag{6.5}$$

When we talk about an F and an F' frame, we imply two inertial frames with origins that coincide at $t=t'=0$ and where the relative motion is only along the x axis and the velocity of F' relative to F is V.

(A) Relativity of length.

The length of a rod does not have an absolute meaning because its length depends on who does the measuring and the conditions under which the measurement is performed. Suppose there is a rod at rest relative to the F' frame, parallel to the x'-axis. Its length measured by observers in F' is called the *proper length* of the rod:

$$L'_o = x'_2 - x'_1, \tag{6.6}$$

where x'_2 and x'_1 are the coordinates of the two ends of the rod in this frame F'. What is its length as measured by an observer at rest relative to another frame F? To determine its length L in F, one must find the coordinates x_2 and x_1 for the two ends of the rod at the same time $t_2=t_1=t$ in F. According to the Lorentz transformation (6.5), one finds

$$x'_2 = (x_2 - Vt)/\sqrt{1 - V^2/c^2}, \qquad x'_1 = (x_1 - Vt)/\sqrt{1 - V^2/c^2}, \tag{6.7}$$

which are the two ends of the rod as measured at the same time t in F. Thus, one obtains the relation

$$L = L'_o\sqrt{1 - V^2/c^2}, \quad L = x_2 - x_1, \quad t = t_2 = t_1, \tag{6.8}$$

which shows that the length of a moving rod is contracted by a factor $\sqrt{1 - V^2/c^2}$ relative to its proper length in its direction of motion. Furthermore, the rod's length is greatest when measured from the frame in which the rod is at rest. If the rod is parallel to the y- or z-axis, its length will not be changed due to motion along the x-axis, but its thickness will be.

The contraction of length in (6.8) is called the Lorentz contraction in special relativity. We note that the original FitzGerald-Lorentz contraction is absolute rather than relative, because it is assumed to be a real physical contraction when the rod has absolute motion, i.e. motion relative to the ether. However, in special relativity, there is no absolute motion and hence, no absolute contraction. If one puts a rod L_o at rest in F and parallel to the x-axis, the length L' of this rod as measured by observers in F' at a certain time t' will be

$$L' = L_o\sqrt{1 - V^2/c^2}, \qquad (t' = t'_2 = t'_1). \tag{6.9}$$

Comparing (6.8) and (6.9), we see the relativity of length contraction. (This is intimately related to the relativity of simultaneity, i.e. $t'_2 - t'_1 = 0$ and $t_2 - t_1 = 0$ cannot be both true when the relative velocity V between F and F' frames is not zero.)

Now suppose there is a meter stick at rest in F and another meter stick at rest in F'. One may wonder: if one compares them, which meter stick is really contracted? This question cannot be answered until the conditions for their comparison are defined in terms of measurement of length. Different results for length contraction such as (6.8) and (6.9) are obtained under different conditions of measurement (i.e., whether the positions of the ends of the rods are measured at the same time t or the same time t'). Nevertheless, if one does not use the Lorentz transformations with a certain simultaneity condition to compare them, can one know whether the length of a meter stick at rest in F has the same length as another meter stick at rest in F'? The answer is affirmative, according to the Poincaré-Einstein principle of relativity, which implies the equivalence of the two inertial frames F and F'.

Why? If one is not satisfied with this answer, there is really no deeper reason than the postulated Poincaré-Einstein principle of relativity. As mentioned before in chapter 5, sec. 5e, the symmetry principles, as we understand them at the present state of knowledge of physics, have transcended both kinetic and dynamic properties and gone right into the very heart of our understanding of the universe. Without the Poincaré-Einstein principle of relativity, it appears to be impossible to construct a viable theory that explains and predicts the phenomena we observe in our universe. However, one could also explain the equivalence in the following way: Suppose two identical meter sticks are made by the same factory in F, and one moves a meter stick to another inertial frame F'. The distance between the atoms that make up the meter stick depends on the interaction strengths of those atoms which, in turn, depends on the Dirac equation and the values of certain fundamental constants. If the values of the fundamental constants and the Dirac equation are the same in F and F', then the Bohr radius of a hydrogen atom at rest in F must be the same as that of another hydrogen atom at rest in F'. Therefore, the length of a meter stick at rest in F has the same length as a meter stick at rest in F':

$$\begin{aligned} &L'_o(\text{at rest in F', measured by F' observers}) \\ &\quad = L_o(\text{at rest in F, measured by F observers}). \end{aligned} \tag{6.10}$$

(B) Relativity of time.

In special relativity, the time interval between two events does not have an absolute meaning because it, too, depends on the conditions of measurement. To discuss this, we must first set up a clock system in each frame. Suppose one has clocks with identical mechanisms of ticking, some are at rest in F and others are at rest in F'. Einstein's procedure of synchronizing them is as follows:

A light signal is sent out from a point a at time t_{1a} (according to the clock at a) to another point b, arriving at time t_b (according to the clock at b), and reflected back by a mirror to point a at time t_{2a} (again, according to the clock at a). We adjust the reading of the clock at b to satisfy the conditions

$$t_b = \frac{1}{2}(t_{2a} - t_{1a}) + t_{1a} \qquad \text{and} \qquad t_{2a} - t_{1a} = \frac{2L}{c}, \tag{6.11}$$

where L is the distance between points a and b as measured by an observer in that frame. This is called *Einstein's synchronization of clocks in an inertial reference frame.* By this procedure, one can synchronize all clocks in any given frame. As a result, the times t' and t in F' and F respectively will be related by the last relation in the Lorentz transformation (6.5). Einstein's synchronization procedure is the operational definition of relativistic time. It gives the precise physical meaning of time in the Lorentz transformation. Furthermore, it helps us to see that Newtonian absolute time cannot be realized by an operational procedure to be consistent with experiments and hence, that absolute and unique time for all observers is unphysical.

Suppose a clock is at rest in the F frame at the point $\mathbf{r}=(x,y,z)$. If a ball strikes the clock at time t_1 and another ball strikes the clock at time t_2, the time interval between these two events is

$$\Delta t = t_2 - t_1, \tag{6.12}$$

according to the clock at the point $\mathbf{r}$. This is known as the proper time between the two events. Now, what is the time which elapses between the two balls striking the clock as viewed by an observer in another frame F'? From transformation (6.5), we find

$$t'_1 = \frac{t_1 - Vx/c^2}{\sqrt{1 - V^2/c^2}}, \qquad t'_2 = \frac{t_2 - Vx/c^2}{\sqrt{1 - V^2/c^2}}. \tag{6.13}$$

It follows that

$$\Delta t' = t'_2 - t'_1 = \frac{\Delta t}{\sqrt{1 - V^2/c^2}}, \qquad (\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}). \tag{6.14}$$

To observers in F', the clock at **r** in F is moving and the result (6.16) shows the relativistic time dilation, i.e., a moving clock appears to run slowly: $\Delta t' > \Delta t$.

We stress again, the time dilation is always relative in special relativity. To compare the rate of ticking of clocks in two frames, we require two or more clocks in one frame and one clock in the other frame. The clock that slows down is always the one which is being compared with different clocks in the other system.

6e. The completion of special relativity by Minkowski's idea of 4-dimensional spacetime

The theory of special relativity is usually considered to have been developed in its entirety in 1905, with the later 4-dimensional symmetry framework of Minkowski and Poincaré viewed as a purely mathematical development. However, in light of the later development of general relativity, covariant quantum electrodynamics, unified electroweak theory and quantum chromodynamics, both the ideas introduced by special relativity and the explicit 4-dimensional symmetry framework are necessary for the covariant formulation of theories and for calculating physical results (e.g., using Feynman rules.) Thus, it would be more reasonable to consider the idea of the 4-dimensional spacetime as an integral and necessary part of the physics of special relativity. Just as quantum mechanics is incomplete without the probabilistic interpretation of the wave functions, special relativity is not complete without the 4-dimensional symmetry interpretation of the Lorentz transformation and of the form of physical laws.

The radical idea of a 4-dimensional spacetime for physical laws was first conceived by Poincaré in 1905, as discussed in section 5c. In his Rendiconti paper, Poincaré considered (x, y, z, it), (with $c = 1$), as coordinates of a point in a space of 4-dimensions and stated that the Lorentz transformation is just a rotation of this space about the origin, regarded as fixed. In this way, one has an invariant distance between the point and the origin,

$$x^2 + y^2 + z^2 - t^2 = x^2 + y^2 + z^2 + (it)^2 , \qquad i = \sqrt{-1}. \qquad (6.15)$$

Thus, the geometry of such a 4-dimensional space (x, y, z, it) is closely related to Euclidean geometry. Poincaré further explained: "We know that the transformations of this group (with K=1) are the linear transformations which do not change the quadratic form $x^2 + y^2 + z^2 - t^2$."

Unfortunately, Poincaré's idea remained almost completely unnoticed until 1908. The same idea, its elegance and important applications were again suggested and expounded by the Russian mathematician Hermann Minkowski (1864-1909) at Göttingen.[5] The impact of Minkowski's work on 4-dimensional spacetime has been enormous. His essential idea was that the invariant theory of the Lorentz group can be represented geometrically, so that it is a natural generalization of the tensor calculus for a 4-dimensional space. The presence of tensor calculus forever changed the landscape of theoretical physics. It greatly simplified calculations and proofs in theoretical physics just as the earlier introduction of the vector notation did. Einstein used nearly six pages to prove the invariance of Maxwell's equations with sources in his original paper.[6] The same proof can be done in about six lines using tensor notation. Furthermore, a few years later the ideas of 4-dimensional spacetime and tensor calculus paved the way for Einstein's creation of general relativity — a theory of gravity based on a 4-dimensional curved spacetime with Riemann geometry.

We know that in principle, the electrodynamics of moving bodies was solved in 1905 by Einstein and Poincaré. Nevertheless, Minkowski further discussed phenomenological electrodynamics of moving bodies in 1908. When the structure of matter is not completely known, the prediction of macroscopic phenomena of moving bodies is not trivial even if phenomena associated with bodies at rest are known experimentally. Minkowski showed that within the framework of special relativity, the equations and the boundary conditions for a phenomenological electrodynamics of moving bodies can be derived from Maxwell's equations and the boundary conditions for bodies at rest.[7]

6f. Einstein and Poincaré

Let us briefly compare Einstein's and Poincaré's work up to 1905. Poincaré finished his last major paper on relativity at the peak of his fame and intellectual power. The young Einstein wrote his first paper on special relativity and started vigorously to pursue relativity and its generalization with all his ingenuity and with many more fruitful results later. Both were influenced by the pioneering work of Lorentz. These three physicists laid the foundations of the theory of special relativity.

It is fair to say that both Einstein and Poincaré realized that "the only possible way in which a person moving and a person standing still could measure the speed (of light) to be the same was that their sense of time and their sense of space are not the same, that the clocks inside the space ship are ticking at a different speed from those on the ground."[8]

Einstein constructed the logical foundation of special relativity based on two postulates and gave a clear and complete physical interpretation of space and time in his theory, including the definition of simultaneity, the relativity of length and time, and the physical meaning of the equations obtained with respect to moving bodies and moving clocks. The development of this physical interpretation was the most difficult part of the theory and turned out to be the greatest strength of Einstein's paper. In contrast, Poincaré's formulation was based on one postulate, the principle of relativity, and a definition $c=1$ by choosing suitable units of length and time. He did not give a clear and complete physical interpretation of space, time and relativity. Although there are differences in their concepts and interpretations, the two formulations lead to exactly the same set of basic equations for field and particle dynamics, provided Einstein's approximate dynamical equations for charged particles are corrected. Thus, there is no experimentally detectable difference between them.

Einstein used about six pages to derive the Lorentz transformation based on the invariance of the law of the propagation of light $c^2t^2 - x^2 - y^2 - z^2 = 0$, his operational definitions of space and time, and the universal constancy of the speed of light. In contrast, Poincaré used just one sentence to explain that the Lorentz transformation can be derived as the result of "a linear transformation that leaves invariant the quadratic form $c^2t^2 - x^2 - y^2 - z^2$."[9]

As a mathematician, Poincaré understood the Lorentz group completely as a group with six parameters, three for spatial rotations and three for constant linear motions, while Einstein viewed it as a group with one parameter for the constant motion in a given direction. The understanding of the group property is crucial because mathematically, the theory of special relativity is the theory of invariants of the Lorentz group and physically, the symmetry of the theory depends completely on the group properties.

Poincaré believed in the existence of an as yet undetected ether, while Einstein did not believe in the ether. It was widely believed by most people that Einstein was right and Poincaré was wrong. However, this belief is no longer tenable from the viewpoint of modern gauge field theory and particle physics. Based on the unified electroweak theory and quantum chromodynamics, the physical vacuum is quite complicated, contrary to Einstein's belief. (See chapter 5, sec. 5f.) Nevertheless, it was very important to separate Maxwell's equations from a mechanical model of the ether, and this was accomplished by Einstein.

The basic formulation of relativistic electrodynamics consists of two parts: the invariant equations of the electromagnetic fields and the invariant equations of motion for charged particles. Being a grandmaster of mathematics, Poincaré gave a complete formulation of electrodynamics, which included the mathematical framework and the Lorentz invariant fundamental equations of motion for charged particles and electromagnetic fields in his theory of relativity. He did all this elegantly by using the principle of least action (i.e., the Lagrangian formulation) and by choosing units so that $c=1$.[10] In contrast, Einstein's 1905 formulation of the covariant electrodynamics is mathematically less elegant and not completely satisfactory: His result for the transverse mass $m/(1 - v^2/c^2)$ differs from the correct expression $m/\sqrt{1 - v^2/c^2}$ obtained by Lorentz and Poincaré.[4] Moreover, his resultant equations of motion for charged particles are not Lorentz invariant because they are only approximations which hold for small velocities and accelerations.

It appears that the physicist Einstein and the mathematician Poincaré had quite different styles and tastes, as shown by the notations and emphases in their papers. In particular, Einstein believed in the speed of light as a truly universal constant. His paper was more appealing to physicists and had a great impact on the development of physics. He derived results which could be directly tested experimentally. In contrast, Poincaré's Rendiconti paper published in a mathematical journal was not widely read by physicists and hence, did not have much influence in the early development of relativity.[11] He treated the constant speed of light as nothing more than a definition, which was consistent with his conventionalism.[12] However, his paper showed elegance and generality based on the Lorentz covariant Lagrangian formulation. The difference in their views concerning the constancy of the speed of light is conceptually significant (though not experimentally differentiable), and we shall return to this point in chapters 7 and 16.

Having said all this, Einstein is generally considered to have had a more profound understanding of physical space, time and relativity. In the early years, Lorentz was probably in the best position to appreciate and assess the work of Poincaré and Einstein. He was particularly impressed by "a remarkable reciprocity that has been pointed out by Einstein" and credited Einstein for "making us see in the negative result of experiments like those of Michelson, Rayleigh and Brace, not a fortuitous compensation of opposing effects, but the manifestation of a general and fundamental principle."[13]

When Pauli discussed Einstein and the development of physics, he said: 'Nowadays we speak with some justification of the "Lorentz Group"; but as a matter of history it was precisely the group property of his transformations that *Lorentz* failed to recognize; this was reserved for Poincaré and Einstein independently. It is regrettable that a certain amount of dispute about priority has arisen over this.'[14]

References

1. See, for example, A. Pais, *Subtle is the Lord..., The Science and the Life of Albert Einstein* (Oxford Univ. Press, Oxford, 1982), pp. 133-134 and pp. 163-166. Pais stresses that in 1902, "Einstein and his friends did much more than just browse through Poincaré's writings. Solovine has left us a detailed list of books which the Akademic members read together. Of these, he singles out one and only one, *La Science et l'Hypothese*, for the following comment: '[This] book profoundly impressed us and kept us breathless for weeks on end'! " Poincaré's book *La Science et l'Hypothese* was published in 1902. Pais therefore believes that, "prior to his own first paper on relativity, Einstein knew the Paris address in which Poincaré suggested that the lack of any evidence for motion relative to the ether should hold generally to all orders in v/c and that 'the cancellation of the [velocity-dependent] terms will be rigorous and absolute.' But there is more. In *La Science et l'Hypothese*, there is a chapter on classical mechanics in which Poincaré writes, 'There is no absolute time; to say that two durations are equal is an assertion which has by itself no meaning and which can acquire one only by convention.... Not only have we no direct intuition of the equality of two durations, but we have not even direct intuition of the simultaneity of two events occurring in different places; this I have explained in an article entitled "La Mesure du Temps".' " Furthermore, in chapter VII, RELATIVE AND ABSOLUTE MOTION, of this book, Poincaré discussed the 'principle of relative motion': "The movement of any system whatsoever ought to obey the same laws, whether it is referred to fixed axes or to the movable axes which are implied in uniform motion in a straight line." This is precisely the principle of relativity that Poincaré proposed two years later in his 1904 address in St. Louis. On page 166, Pais states "It cannot be said, however, that the content of Einstein's June 1905 paper depends in any technical sense on these important remarks by Poincaré. Others in Einstein's position might perhaps have chosen to mention Poincaré at the earliest opportunity. However, it does

not seem to me that Einstein had compelling reasons to do so in 1905." Nevertheless, based on today's guidance for referees to accept papers submitted to say, Physical Review or Nuovo Cimento, it is very unlikely that such an omission of any reference to relevant works would be allowed in a published paper.

2. A. Einstein, Ann. Phys. (Leipzig) **17**, 891 (1905); translated by H. M. Schwartz, in Am. J. Phys. **45**, 18 (1977).

3. This equivalence can be shown by using a covariant variational calculus in which one treats the second end point as variable and admits only actual paths. The usual variational calculus fixed both end points. See for example L. Landau and E. Lifshitz, "*The Classical Theory of Fields*" (Addison-Wesley press, Cambridge, 1951), pp. 28-29. If $ds^2 = c^2dt^2 - dr^2$, we have the usual action S for a free particle with a mass m,

$$S = -mc\int_a ds, \tag{R6.1}$$

where the first end point a is fixed. The variational principle $\delta S = 0$ under the stated conditions will lead to

$$p^\mu = -\frac{\partial S}{\partial x_\mu} = mc\frac{dx^\mu}{ds}, \qquad \mu = 0, 1, 2, 3. \tag{R6.2}$$

Therefore, one has the energy-momentum relation for a free particle

$$p^\mu p_\mu = m^2c^2, \qquad p_\mu = \eta_{\mu\nu}p^\nu, \qquad \eta_{\mu\nu} = (1, -1, -1, -1). \tag{R6.3}$$

If (R6.3) holds, then using (R6.2), one can obtain $ds^2 = c^2dt^2 - dr^2$. For a photon with mass m=0, the above calculations cannot be applied. Rather, one can still show that $c^2dt^2 - dr^2 = 0$ is not an independent assumption. It can be derived from Maxwell's equations. The solution of a free plane wave in the vacuum is given by

$$E(ct,r) = E_0 \sin(k_0 ct - \mathbf{k}\cdot\mathbf{r}) \ , \qquad k_0 = \omega/c, \tag{R6.4}$$

where $k_0^2 - \mathbf{k}^2 = 0$. To calculate the velocity of the propagation of this wave, one simply looks at the motion of a node $E(ct,r)=0$, or $k_0 ct - \mathbf{k}\cdot\mathbf{r} = k_0 ct - kr\cos\theta = 0$. Along the direction of the wave propagation $\mathbf{k}$ (i.e., $\theta=0$), one has $dr/dt = ck_0/k = c$, i.e.,

$$c^2dt^2 - dr^2 = 0 \, . \tag{R6.5}$$

4. M. Planck, Verh. Deutsch. Phys. Ges. 4, 136 (1906). Planck also used the covariant Lagrangian formalism to obtain the correct equations of motion for a charged particle. It is worthwhile to note that not all of the kinematic properties inherent in the Lorentz transformation were immediately made clear by Einstein's 1905 paper. For example, the increase in the mass of a particle as a function of its velocity is one such kinematical property that was not discovered until later. As late as 1909, Max Born, though familiar with special relativity, still took the time to calculate the electromagnetic mass of the electron to find its velocity dependence, similar to what Lorentz had done previously. In light of this, one may wonder: Who was the referee of Einstein's original paper on relativity? It is not known with certainty who the referee was. Max Planck and Paul Drude were two possible candidates. Planck was not only immediately interested in Einstein's work on relativity, he also asked von Laue to discuss it in a colloquium and sent him to Bern to meet with Einstein soon after the colloquium. These events are well documented. (Helmut Rechenberg, private e-mail correspondence, Sept. 1, 1999.) Paul Drude was the editor of Ann. Phys. (Leipzig) when Einstein submitted his paper in 1905. He was knowledgeable in both theoretical and experimental physics. His view of the ether in 1900 was as follows: "The conception of an ether absolutely at rest is the most simple and the most natural—at least if the ether is conceived to be not a substance but merely

space endowed with certain physical properties." [P. Drude, *The Theory of Optics* (Dover, New York, 1959) p. 457.] Unfortunately, he committed suicide in 1906. Planck became the new editor of Ann. Phys. shortly afterward. It seems likely that Paul Drude was the referee of Einstein's 1905 paper on relativity.

5. H. Minkowski (1864-1909), F. Klein and D. Hilbert were considered three "prophets" in the mecca of German mathematics at Göttingen in the early 20th century. Minkowski and Hilbert were friends since their schooldays in Königsberg. Around 1882, the French Academy proposed the problem of the representation of a number as the sum of five squares. Minkowski's investigation led him far beyond the stated problem. In the spring of 1883, the 18 year old Minkowski and a well-known English mathematician H. Smith were awarded jointly the French Academy Grand Prize. The young Hilbert was attracted to become friends with the shy, kind-hearted and gifted Minkowski. They often took long walks in the woods. They engrossed themselves in the problems of mathematics and exchanged their newly acquired understandings and thoughts on every subject. In this way, they formed a friendship for life. In 1902, when Hilbert was offered a very honored position in Berlin, he grabbed this opportunity and proposed that a new professorship be created expressly for Minkowski at Göttingen. As a result, these two great mathematicians were able to continue their walks in the woods at Göttingen. The joy of Klein, the head of the group, and mathematics students knew no limit. "To me, he was a gift from heaven" wrote Hilbert. Unfortunately, Minkowski died suddenly after an operation for appendicitis in 1909. Apart from his well-known 4-dimensional spacetime as the mathematical framework for special relativity, he also made important contributions on "Minkowskian geometry" by modifying Euclid's congruence axioms (i.e., the axiom of the free mobility of rigid point systems) and on the "geometry of numbers." In the latter, his theorem on lattice points in convex regions provides insight into apparently unrelated subjects such as algebraic number fields, finite groups, and the arithmetic theory of quadratic forms.

6. H. A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, *The Principle of Relativity* (translated by W. Perrett and G. B. Jeffery, Methuen and Company, 1923) pp. 51-55 and pp. 59-61.
7. W. Pauli, *Theory of Relativity* (Pergamon Press, London, 1958) pp. 99-111.
8. R. Feynman, *The Character of Physical Law* (MIT Press, Cambridge, MA. 1965), p. 92.
9. Some people believe that Einstein derived the Lorentz transformations, while Poincaré assumed them in order to get the invariance of the electromagnetic equations. This belief appears to be untenable. In the last part of his Rendiconti paper, Poincaré considered (x, y, z, $t\sqrt{-1}$) to be the coordinates of a point in a space of four dimensions and stated that the Lorentz transformation is just a rotation of this space about the origin, regarded as fixed. Poincaré explained: "We know that the transformations of this group (taking K=1) are the linear transformations which do not change the quadratic form $x^2 + y^2 + z^2 - t^2$." As a high-caliber mathematician, this explanation clearly showed that he knew that the Lorentz transformation could be derived based on an invariant $x^2 + y^2 + z^2 - t^2$. (See also ref. 7.) It was understood that the calculations were simple and need not have been written explicitly in the paper. Poincaré's method of derivation has been used by many authors of textbooks to obtain the Lorentz transformations, see for example, V. Barger and M. Olsson, *Classical Mechanics: A Modern Perspective* (2nd ed., McGraw-Hill, New York, 1973), pp. 347-348.
10. Today, quantum field theorists all use the Lorentz covariant Lagrangian formulation and choose units in which $c = \hbar = 1$ in their formulations and investigations, just as Poincaré did in 1905.
11. A very important and useful reference book of relativity for decades, DAS RELATIVITÄTSPRINZIP, which collected the important papers of Lorentz, Einstein, Minkowski and Weyl, appeared in a series on monographs edited by Otto Blementhal (based on the suggestion of Sommerfeld) and was first published in German by Teubner, Leipzig, 1913. In this book, Sommerfeld stated in his NOTES that "Whereas Minkowski's ideas on the vector of the

first kind, or four-vector, were in part anticipated by Poincaré (Rend. Circ. Mat. Palermo, 21, 1906), the introduction of the six-vector is new" and that "Minkowski's relativistic form of Newton's law for the special case of zero acceleration mentioned in the text included in the more general form proposed by Poincaré (loc. cit)." Its English translation, THE PRINCIPLE OF RELATIVITY, was first published in 1923. It is regrettable that Poincaré's original and comprehensive paper was included in neither of them. It was translated into English around 1970 by C. W. Kilmister (*Special Theory of Relativity* [Pergamon, New York, 1970]) and by H. M. Schwartz, Am. J. Phys. **39**, 1287 (1971), **40**, 862 (1972), **40**, 1282 (1972).

12. This appears to be consistent with conventionalism (H. Poincaré and Pierre Duhem) which attempted to do justice to the arbitrary elements in theory construction and held that all coherent scientific theories have equal validity and that therefore, one theory cannot be more true than another. For example, according to Poincaré, the geometrical axioms are "*neither synthetic a priori intuitions nor experimental facts.*" "They are conventions. Our choice among all possible conventions is *guided* by experimental facts; but it remains *free*, and is only limited by the necessity of avoiding every contradiction, and thus it is that postulates may remain rigorously true even when the experimental laws which have determined their adoption are only approximate. In other words, *the axioms of geometry* (I do not speak of those of arithmetic) *are only definitions in disguise.* What, then, are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true and if the old weights and measures are false; if Cartesian coordinates are true and polar coordinates false. One geometry cannot be more true than another; it can only be more convenient." (The italics are Poincaré's.) See H. Poincaré, *Science and Hypothesis* (first appeared in 1902, Dover Publications, Inc. 1952), p. 50.

13. Lorentz and Pauli were familiar with and understood the works of both Poincaré and Einstein. It is somewhat odd that Pauli credited Poincaré for the principle of relativity, while Lorentz credited Einstein for it. Both Pauli and Lorentz discussed relativity theory and regarded it as Einstein's work.

See, for example, H. A. Lorentz, *The Theory of Electrons* (first appeared in 1909, Dover, New York, 1952), pp. 223-224, 226-228 and 321-325. In his book, Lorentz discussed only the "Poincaré stress" for the equilibrium of an extended electron and not the formulation of relativity theory for the electromagnetic fields and the dynamics of charged particles based on the principle of relativity (and the definition c=1.)

14. W. Pauli, *Writings on Physics and Philosophy* (Edited by C. P. Enz and Karl von Meyenn, translated by R. Schlapp, Springer-Verlag, Berlin, 1994), p. 118; *Neue Zürcher Zeitung*, 12. Januar 1958.

B.

A Broader View of Relativity: The Central Role of the Principle of Relativity

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7.

Relativity Based Solely on the Principle of Relativity

7a. Motivation

Because one of the primary goals of physics is to explain the wide range of phenomena we observe in our universe using the smallest possible number of laws, a fundamental physical theory should be based on the smallest and simplest set of postulates. Special relativity, as formulated by Einstein in 1905, is based on two: (1) the principle of relativity, which states that the laws of physics have the same form in every inertial frame, and (2) the universality of the speed of light, which states that the speed of light is independent of the motion of the source and of the observer.

The aims of the chapters in this part are to show that:

(i) a relativity theory consistent with the results of all known experiments can be formulated solely on the basis of the principle of relativity,[1] and

(ii) the resulting relativity theory can lead to a fresh view of old physics, including making more tractable the solution of the relativistic many-body problem and, in a related vein, helping to develop more convenient definitions for thermodynamics quantities in a relativistic framework.[1]

The first aim may be confusing to some readers in the sense that throughout the literature, one can find statements to the effect that the universality of the speed of light is a direct consequence of the principle of relativity and that the second postulate is not needed.[2] However, based on his formulation of the theory of special relativity, Einstein evidently thought that a second postulate was needed to establish the universality of the speed of light. By the end of this chapter, we hope to make clear under what conditions a second postulate is or is not required, to show that the postulate of the universality of the speed of light is tied to our concepts of space and time, and that historically, the construction of a relativity theory based solely on the principle of relativity would not have been possible without Einstein's work on special relativity based on the two postulates named above. As a point of clarification, we will define a "relativity theory," to refer to a theory that has among its postulates the principle of relativity.

7b. A brief digression: natural units and their physical basis

In our discussion, we will adopt a modern viewpoint of our concepts of space and time in regards to the units used to quantify them. As we will see, such a viewpoint is essential to understanding the role of the postulate on the universality of the speed of light and why though necessary at first, it can now (under the right circumstances) be considered a consequence of the principle of relativity. By a modern viewpoint, we refer to the use of natural units, in which all quantities are expressed in terms of a single unit, traditionally taken to be a unit of length. Although natural units may seem to be purely a calculational convenience, it has in fact a physical basis. We merely outline the argument here and refer the reader to Appendix A for a more detailed discussion.

For a combination of reasons, both technological and historical, the modern international system of units (SI) is founded on seven base units corresponding to the dimensions of length (meter), mass (kilogram), time (second), electric current (ampere), thermodynamic temperature (kelvin), amount of substance (mole), and luminous intensity (candela).[3] Although these are considered "base" units, they are not mutually independent. For example, the definitions of the meter, ampere, and candela are given in terms of other SI units and so of the seven base units, only four, the second, kilogram, kelvin, and mole have definitions that are, at present, independent. We say "at present" because the independence of the definitions of those four units is not based on any physical reasons, but only on reasons of practicality and precision.

As has been discussed in the literature,[4] continuing improvements in technology and the precision with which we can measure certain types of quantities will lead to a future reduction in the number of independently defined units and there is no reason not to expect that one day in the future, all seven base units will be defined in terms of one single unit. Such a change will not matter much to physicists, who are already used to working natural units. However, defining all units in terms of a single unit implies that the system of natural units is not an artificial construct developed solely for calculational convenience, but is grounded in the physical properties of our universe. Physics *requires* only a single unit. The other units have been developed as a result of human conventions and a perceptual system rooted in the particular anatomy of *Homo sapiens.* Although their use makes the investigation of our universe much more convenient for us, we must keep in mind that their definitions are historical artifacts of human activity and as such can be changed to suit our convenience without affecting physical laws.

Although as a practical matter, the base unit in terms of which all other units will eventually be defined seems likely to be the second for reasons of precision and reproducibility of measurements, we will use length as our base dimension in the following discussion (with the unit of meter) because it is the one traditionally used in the natural unit system.

7c. Taiji relativity: A relativity theory based solely on the principle of relativity

We now derive the coordinate transformations for a relativity theory based solely on the principle of relativity and with all quantities expressed in terms of a single unit. We are not the first to do this. Both Taylor and Wheeler, in their excellent *Spacetime Physics*, and Thomas Moore, in volume R of his text *Six Ideas that Shaped Physics*,[5] develop special relativity using only a single unit to express both lengths and time intervals. However, we will go much further. Rather than limiting ourselves to a discussion of special relativity under natural units, we will also explore implications of such a relativity theory not discussed by Einstein. We call this theory that we propose "taiji relativity"[1] because it is based on what we believe to be the smallest and simplest set of postulates and system of units, and to distinguish it from special relativity. In ancient Chinese thought, the word "taiji" denotes the ultimate principles or the conditions that existed before the creation of the world.[6] Although the equations of taiji relativity will look the same as those in special relativity, there is an important conceptual difference between the two, which we will discuss at the end of this chapter. This difference is similar to the difference between Einstein's and Poincaré's use of the Lorentz transformations. Though mathematically identical, the two men had very different ideas in mind when writing them down.

As usual, we describe an "event" using the four coordinates

$$(w, x, y, z) = x^{\mu}, \quad \text{and} \quad (w', x', y', z') = x'^{\mu}, \quad \mu = 0,1,2,3, \tag{7.1}$$

in inertial frames F(w,x,y,z) and F'(w',x',y',z') respectively. The variables w and w' are the evolution variables of a physical system (time) measured in units of length. We use the letter w to distinguish it from t, which we will take to be the evolution variable expressed using the conventional unit second and also refer to w and w' as the "taiji time." In special relativity, the quantities w and t are related by w = (299792458 m/s) t in every inertial frame. However, in taiji relativity, since the unit of second does not exist, the variable t is not defined.

Using the usual simplifying conventions that the relative motion between F and F' frames is along the parallel x and x' axes, that the origins of F and F' coincide at w = w' = 0, and that the transformation between the two sets of coordinates is linear,[7] the transformation between the two sets of coordinates can be written as

$$w'= a_1 w+a_2 x, \qquad x'= b_1 w+ b_2 x, \qquad y' = y, \qquad z' = z \tag{7.2}$$

where a_1, a_2, b_1, and b_2 are constants to be determined.

The principle of relativity implies that physical laws must have the same form in any frame. This statement is equivalent to demanding that the four-dimensional interval $s^2 = \eta_{\mu\nu}x^{\mu}x^{\nu} = x_{\mu}x^{\mu} = w^2 - \mathbf{r}^2$, $\eta_{\mu\nu} = (1,-1,-1,-1)$, be invariant so that[8]

$$w'^2 - \mathbf{r}'^2 = w^2 - \mathbf{r}^2$$

Substituting the expressions for w' and x' from (7.2) into (7.3) and equating the coefficients of w^2, wx, and x^2 in addition to using the condition that when $dx'/dw'=0$, $dx/dw=\beta$, we can then determine all four unknown constants in terms of β, where β is a dimensionless constant that characterizes the magnitude of the relative motion between F and F'. The four-dimensional coordinate transformation can then be written as

$$w' = \gamma(w - \beta x), \quad x' = \gamma(x - \beta w), \quad y' = y, \quad z' = z; \tag{7.4}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{7.5}$$

with a corresponding inverse transformation, derived from (7.4) and (7.5)

$$w = \gamma(w' + \beta x'), \quad x = \gamma(x' + \beta w'), \quad y = y', \quad z = z'. \tag{7.6}$$

From the solution of the Maxwell equations for a plane wave, it follows that the propagation of a light signal emitted from any source is described by $s^2 = 0$ or equivalently $ds^2 = dw^2 - d\mathbf{r}^2 = 0$.[9] In that case, the speed of light β_L implied by (7.3) is thus

$$\beta_L = |dr/dw| = 1. \tag{7.7}$$

Since ds^2 is invariant, this result, which is a consequence of the principle of relativity, holds in all inertial frames: the speed of a light signal is isotropic and independent of the motion of the source and of the observer.

From equations (7.4)-(7.5), we can also derive the taiji velocity transformation,

$$u'_x = \frac{u_x - \beta}{1 - u_x\beta}, \qquad u'_y = \frac{u_y}{\gamma(1 - u_x\beta)}, \qquad u'_z = \frac{u_z}{\gamma(1 - u_x\beta)}; \tag{7.8}$$

$$\mathbf{u}' = (dx'/dw', dy'/dw', dz'/dw'), \qquad \mathbf{u} = (dx/dw, dy/dw, dz/dw),$$

where **u** is the velocity of a particle as measured by an F frame observer and **u'** is the velocity of the same particle as measured by an F' observer. Since we use natural units, these velocities are unitless quantities, the components of which range in value from -1 to 1.

7d. Realization of taiji time

If taiji relativity is to be a physical theory, then the taiji time w must have an operational definition. As we are accustomed to the usual kinds of clocks that measure time in seconds, one might ask how the measurement of time as expressed by w can be physically realized. The answer is straightforward. Because taiji relativity and special relativity are mathematically identical if one sets c=1 in special relativity, one can use the usual procedure for synchronizing clocks in special relativity to set up a clock system in taiji relativity. In order to obtain taiji time measurements in units of meters, we merely re-label the clocks from seconds to meters after synchronization and scale the readings so that a time interval Δt of 1 second corresponds to a time interval Δw of 299792458 m.

A taiji-time Δw can also be understood in terms of an *optical path length.* For a finite interval, the propagation of a light signal starting from the origin $\mathbf{r} = 0$ at $w = 0$ is described by

$$w^2 - \mathbf{r}^2 = 0, \tag{7.9}$$

where r is the distance traveled by the light signal during the taiji-time interval $\Delta w = w$. Since equation (7.9) implies that $r = w$ and the speed of light $\beta_L \equiv r/w = 1$ is invariant, w is also the distance traveled by the light signal. Thus, the time interval Δw between two events can be interpreted as the distance traveled by a light signal between the occurrence of the two events. To signify this, we will refer to the quantity w as the "lightime."

This interpretation of w suggests another way of operationalizing the taiji time, which is to use clocks whose rates of ticking are based on the motion of a light signal. Such clocks, which are discussed in many books on special relativity, consist of a light bulb and light detector mounted next to each other facing in the same direction, and a mirror placed half a meter away facing the two devices. The light bulb flashes, sending out a light signal that reflects off the mirror and returns to the detector. When the detector receives the signal, it triggers the light bulb to flash again. Because a light signal travels a total distance of 1 meter between flashes, the time interval Δw marked by the clock between flashes is $\Delta w = 1$ meter.

7e. The conceptual difference between taiji relativity and special relativity

As noted previously, the taiji coordinate transformation equations (7.4) and (7.5) are identical to the conventional Lorentz transformation equations if one sets $c = 1$ and $t = w$. Therefore, taiji relativity shares all of the mathematical and operational characteristics of special relativity. For example, a clock system for measuring the taiji-time w can be defined operationally in the same way as clock systems are synchronized for relativistic time, the taiji coordinate transformations share all of the Lorentz and Poincaré group properties of special relativity (see chapter 11 for details), and all of the experimental predictions of special relativity are duplicated by taiji relativity. However, there is an important conceptual difference between the two theories.

We propose a new way of thinking about the relationship between the taiji time w measured in units of meter and time t measured in units of second. In special relativity (and as defined by the 1983 meeting of the General Conference on Weights and Measures), the two units are directly proportional and related by $w = (299792458\ m/s)\ t$. However, as Taylor and Wheeler have pointed out, the number 299792458 is simply a conversion factor, "a factor that arose out of historical accident in humankind's choice of units for space and time, with no deeper physical significance."[10] If the use of two different units for describing space and time is a product of the human perceptual system and the conversion factor between the two units a human convention, why not explore the implications of a new relationship between w and t? Just because $w = ct$ leads to a relativity theory that is consistent with all experiments does not imply that it is the ONLY relationship that is consistent with all experiments. In fact, because doing so is merely equivalent to changing the official definition of the second in terms of the meter (and vice-versa), a human convention, none of the experimental predictions of the theory verified over the last century would change. As we stated at the beginning of this chapter, our motivation is to explore a new relationship that makes solving certain kinds of problems more convenient, either through simplifying the calculations involved or through opening the possibility of defining new useful quantities based new symmetries exposed.

Some types of new relationships, such as $w = (9.83571... \times 10^8 \text{ ft/s})t$ are trivial. However, what if we were to adopt a relationship which is more radically different, such as $w = bt$ where b is not a constant but a function? [11]

At first glance, such a relationship may seem impractical. Not only would the speed of light measured in units of meter per second no longer be a universal constant (although the unitless speed as measured in natural units would still be a universal constant as demonstrated in (7.7)), clocks set up to display the time t in seconds would appear to run in some crazy non-uniform way. However, it is precisely the goal of the following chapters to show that one particular such relationship with its corresponding "crazy" definition of the second nevertheless results in a relativity theory

(A) in which all inertial frames are equivalent,

(B) that still correctly predicts experimental results,

(C) that still has all of the Lorentz and Poincaré group properties of special relativity, and most importantly

(D) can be more convenient for certain calculations than special relativity (though consequently, it will be less convenient than special relativity for others).

A useful analogy to use in thinking about such alternative relationships between w and t is the different coordinate systems: Cartesian, spherical, cylindrical, hyperbolic, etc. One can think of special relativity, with its particularly simple relationship between w and t, as analogous to the Cartesian coordinate system. Changing the scaling factor between w and t while keeping it constant is analogous to adopting a different Cartesian coordinate system with a different grid spacing. Moving to a relationship w=bt where b is a function is analogous to changing to a coordinate system such as spherical coordinates, with its complex relationships between (x, y, z) and (r, θ, φ). Neither system is more "correct" than any other and all predict the same physics. However, depending on the problem to be analyzed, certain coordinate systems are more convenient and are better suited to revealing symmetries and invariants of the physical systems under consideration.

7f. The role of a second postulate

We end this chapter with a short discussion on the necessity of a second postulate for the formulation of an experimentally consistent relativity theory. As mentioned previously, although Einstein thought it necessary to postulate the universality of the speed of light separately from the principle of relativity, some authors claim that this universality is a direct consequence of the principle of relativity and thus that a second postulate is unnecessary. In light of equation (7.7) and our discussion on the relationship between w and t, we can now resolve these two points of view.

As we saw in the derivation of equation (7.7), the principle of relativity by itself implies that the speed of light, when measured in terms of a unitless velocity within the framework of natural units, is a universal constant. However, the principle of relativity does not, by itself, specify any particular relationship between the time w measured in units of meter and the time t measured in units of second. Because the use of two separate units to measure space and time is purely a human convention (as evidenced by the fact that the official definition of the unit meter is cast in terms of the unit second), there is no relationship between the meter and the second that is determined by any property of our universe. Thus, we are free to make that choice ourselves, according to whatever is most convenient for us.

Choosing w=ct in every inertial frame, as Einstein did, corresponds to making a second postulate that "The speed of light, *measured in units of meter per second*, is independent of the motion of the source and of the observer" and is equivalent to defining the unit second as "The interval of time that passes when a light signal travels a distance 299792458 m in a vacuum." Choosing w=bt where b is a function results in a different second postulate and a different definition of the second. Although the speed of light measured in units of meter per second is no longer a universal constant, this is purely because we have chosen a particular type of "second" and the unitless speed of light expressed in natural units would still be a universal constant.

Thus, whether or not a second postulate is necessary for the formulation of special relativity can be seen as a difference in the viewpoint one has of space and time. In Einstein's era, space and time were thought to be two very different entities as evidenced by the use of two independently defined units to quantify them.[12] Einstein was able to free himself sufficiently from that mode of thought to develop special relativity, in which space and time are unified in a four-dimensional spacetime. However, he did not have the decades of experience working with special relativity as a natural part of physics that modern physicists have, and was not able to see all of the implications of his theory, such as that the unit of length need not be defined independently of the unit of time and that it is in fact more natural to express both quantities in terms of the same unit. For him, it was necessary to introduce a second postulate in order to obtain the universality of the speed of light in the system of units in use at the time. From our modern point of view in which both spatial and temporal intervals can be expressed using the same units, it is revealed that the universality of the speed of light is indeed a direct consequence of the principle of relativity. From a historical perspective, we also can see that in order to arrive at our modern point of view from which a relativity theory can be derived from a single postulate, it was necessary first to derive it from two, to set us on the right track of regarding space and time as a unified spacetime.

Finally, in closing, we note that the preceding discussion allows us to give a definite answer to a question that vexed many physicists in the early part of the 20^{th} century:[13] Is it possible to formulate an experimentally-consistent relativity theory in which the one-way speed of light is not a universal constant? Answer: If by speed one means the speed expressed in terms of the most fundamental set of units (natural units), the answer is no. A universal one-way speed of light is a direct consequence of the principle of relativity, without which it seems to be impossible to formulate an experimentally viable theory. On the other hand, if one means the speed expressed in terms of units that are the result of human invention, the answer is yes.

References

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2. L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Transl. M. Hamermesh, Addison-Wesley, 1957) p. 2; E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (Second Ed., W. H. Freeman and Company, 1992) p. 60. See also chapter 5, section 5c; and A. A. Tyapkin, Sov. Phys. Usp. **15**, 205 (1972).
3. A good review of the SI units system and its historical context can be found at http://physics.nist.gov/cuu/Units/index.html.
4. B. N. Taylor and P. J. Mohr, "On the redefinition of the kilogram," Metrologia **36**, 63-64 (1999); Edwin R. Williams, Richard, L. Steiner, David B. Newell, and Paul T. Olson, "Accurate Measurement of the Planck Constant," Phys. Rev. Lett. **81**, 2404-2407 (1998).
5. E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (Second Ed., W. H. Freeman and Company, 1992); T. Moore, *Six Ideas that Shaped Physics* (McGraw-Hill Science, 2002).
6. See, for example, *The New Lin Yutang Chinese-English Dictionary* (Ed. Lai Ming and Lin Tai-Yi, Panorama Press, Hong Kong, 1987) p. 90.
7. We can prove that the principle of relativity demands that the transformation be linear in the following way: Let us consider the relations between (w,x) and (w',x'). Suppose they are related by non-linear relations: $w'=aw+bx+cwx+ew^2+fx^2+.....$, and $x'=Ax+Bw+Cwx+....$, where a, b, c,....A, B, C,... are constant. After taking differentiations, one has $dw'=a^*dw+b^*dx$ and $dx'=A^*dw+B^*dx$, where a^*, b^*, A^* and B^* are, in general, functions of w and x. For an object at rest in the inertial frame F'(w',x'), i.e., dx'=0, its velocity as measured in the inertial frame F(w,x) is $dx/dw= \beta$ = constant. This relation, together with the infinitesimal 4-dimensional interval $ds^2 = dw'^2 - d\mathbf{r}'^2 = dw^2 - d\mathbf{r}^2$, leads to the result $a^* = A^* = \gamma$ and $b^* = B^*= -\beta\gamma$, where γ is given by (7.5). This implies that

a*, b*, A* and B* are constants. It follows that the relations between (w',x') and (w,x) are linear.

8. For a free particle, $\mathbf{r}^2/w^2 = \boldsymbol{\beta}^2 =$ constant. Eq. (7.3) can be written as $m^2w^2/s^2 - m^2r^2/s^2 = m^2$, which is exactly the same as the 'energy-momentum" relation $(p^0)^2 - \mathbf{p}^2 = m^2$ because $p^0 = m/(1-\beta^2)^{1/2}$ and $\mathbf{p} = m\boldsymbol{\beta}/(1-\beta^2)^{1/2}$. For a rigorous proof based on covariant variational calculus, see ref. 3 in ch. 6.
9. The finite transformation (7.4) is equivalent to the transformation of the differentials, $dw' = \gamma(dw - \beta dx)$, $dx' = \gamma(dx - \beta dw)$, $dy' = dy$, $dz' = dz$. In the same sense, finite 4-dimensaional interval (7.3) is equivalent to the differential interval $ds^2 = dw'^2 - d\mathbf{r}'^2 = dw^2 - d\mathbf{r}^2$.
10. E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (Second Ed., W. H. Freeman and Company, 1992) p. 5. See also J.-M. Lévy-Leblond, Rivista del Nuovo Cimento **7**, 187-214 (1977).
11. J. P. Hsu, Nuovo Cimento B**74**, 67 (1983); Phys. Letters A**97**, 137 (1983); Editorial, Nature **303**, 129 (1983); J. P. Hsu, Found. Phys. **8**, 371 (1978); **6**, 317 (1976); J. P. Hsu and T. N. Sherry, ibid **10**, 57 (1980);); J. P. Hsu and C. Whan, Phys. Rev. A**38**, 2248 (1988), Appendix. These papers discuss a four-dimensional symmetry framework based on the usual first postulate and a different second postulate, namely, a common time t' = t (or w=bt and w'=b't) for all observers in different inertial frames.
12. In 1905, the meter was defined as the distance between two marks on a bar made of a platinum-iridium alloy, measured at the melting point of ice. The second was defined as the fraction 1/86400 of the mean solar day.
13. This question was discussed by Ritz (1908), Tolman (1910), Kunz (1910), Comstock (1910) and Pauli (1921, 1958), among others. See W. Pauli, *Theory of Relativity* (Pergamon Press, London, 1958) pp. 5-9 and references therein.

8.

Common Relativity

8a. A new unit for time

We have seen in the previous chapter that the principle of relativity, applied in the context of natural units, results in a theory that is mathematically equivalent to Einstein's special relativity with the constant c set to unity and the evolution variable expressed with the dimension of length. If we wish to express the coordinate variables using two different units as is more commonly done, for example where the spatial coordinates x, y, and z have the unit meter and the evolution variable t has the unit second, it is necessary to define the unit second. Adopting the SI definitions of the meter and second, in which the two are directly proportional leads to special relativity as discussed in the vast majority of texts, in which the speed of light is a universal constant not only when expressed in natural units, but also in SI units of meter per second. However, as we argued in the previous chapter, the use of two separate units to specify the coordinate variables and the relationship between the units meter and second are human conventions, and as such, it is possible to define a different relationship between the two without altering any physics. At this point the question, "For what reason would one want to choose a different definition?" comes to mind.

Although the particularly simple relationship between the units meter and second is convenient for many purposes such as when discussing the propagation light signals, one inconvenient side effect is evidenced when tackling problems involving many bodies, such as the problem of describing the canonical evolution of a system of many particles in statistical mechanics.[1] Because the particles in such a system travel with a range of different velocities (up to N different velocities for a system of N particles), we are forced to describe such a situation using a system of up to N coupled differential equations, each with a different t.[2] Although a sufficiently powerful computer can help to alleviate this calculational nightmare, this inconvenience also has implications in developing a convenient definition for the temperature of a system in a relativistic framework. The most obvious way around these difficulties is to define a new unit of time in which all inertial frames use the same time t. However, this definition must differ from that of Galilean relativity however, which we know to be inconsistent with experiments.

We will call this new unit of time the "common-second" because it is shared by all inertial frames and to distinguish it from the usual unit of time "second" in the SI unit system. As we have discussed in the previous chapter, making such a new definition is analogous to developing a new coordinate system within which to express physical laws and solve physics problems. In this chapter, we develop the coordinate transformation equations based on the principle of relativity and the unit of common-second. In analogy with special relativity, we may say that this new relativity theory, which we call "common relativity"[3,4] is based on the following two postulates:

(I). The form of a physical law is the same in any inertial frame (the principle of relativity).

(II). The physical time t_c (expressed in units of common-second) is the same in any inertial frame ($t_c = t_c'$ for all pairs of inertial frames).

In the following discussion, we will use t_c to denote the evolution variable in common relativity. This will help to distinguish it from the conventional time t specified by the unit second as defined by special relativity and at the 1983 Meeting of General Conference on Weights and Measures.

We note that the postulate of common time (II) is not really a postulate in the same sense as the principle of relativity. The principle of relativity is a postulate in the sense that it leads to specific experimental predictions that can and have been tested. The second postulate in common relativity and in special relativity as developed by Einstein is really a definition because it defines a relationship between the unit meter and the unit second. Because only one unit is physically necessary and the additional unit merely serves to make the theory more convenient by putting its experimental predictions in terms of quantities that are more familiar to our human senses, it is a human convention and as such, can never be proven wrong. This is similar to the situation with the length units meter and nanometer. Although it is useful to have both, only one of the two is necessary and the relationship that 1 meter = 10^9 nanometer is not something that can be proven incorrect experimentally.

The claim that the relationship $t_c = t_c'$ can never be proven to be inconsistent with experiments may seem strange as the same "definition" in the context of the Galilean transformations leads to a physical theory with demonstrably incorrect predictions. In that case however, the blame lies not with the definition $t = t'$, but with the three-dimensional symmetry framework of Galilean relativity, under which physical laws such as Maxwell's equations were NOT postulated to be invariant in inertial frames. Implementing $t_c = t_c'$ within a four-dimensional symmetry framework, in which Maxwell's equations ARE invariant (see chapter 12), leads to a physical theory with correct experimental predictions, as we shall see.

8b. Operationalizing the common-second and the equivalence of inertial frames

In order for common relativity to be a "physical" theory, one must be able to define the common time t_c operationally. In the subsequent paragraphs we discuss how the relationship $t_c = t_c'$ can be realized.

In many texts on special relativity, the synchronization of clocks (in the gedanken experiment sense) in any pair of inertial frames F and F' is achieved by placing identical clocks on the vertices of a grid in each inertial frame. Because we have postulated the equivalence of all inertial frames, the rate of ticking of each of these clocks is supposed to be identical. One then chooses an origin for the spatial coordinates in each frame and by appropriate adjustment of the readings on the clocks, insures that at the instant when the origins of the two different inertial frames occupy the same point in space, the clocks in each frame located at the origin read $t = t' = 0$. To synchronize the readings on the rest of the clocks in each inertial frame, one can stand at the halfway point between the clock to be synchronized and the clock at the origin of that frame and use light signals to adjust the clock to be synchronized to have the same reading as the clock at the origin.

In common relativity, one can use a similar but slightly simpler procedure. One first chooses any inertial frame (we will call it F), places clocks at the vertices of a grid in that frame and then synchronizes the clocks in the same way as in special relativity. This has the effect of defining the speed of light (measured in meter per common-second) to be isotropic in the F frame. All observers in all inertial frames then use that one set of clocks. Whenever a time reading is desired, an observer in any inertial frame simply records the reading on the clock nearest to him or her.

Using the unit common-second, many of the effects and properties ordinarily associated with relativity, such as the universality of the speed of light (measured in units of meter per second), the lack of a common concept of simultaneity (of time measured in second), and the relativistic length contraction and time dilation are lost. However, it is important to keep in mind that these properties are lost only when one uses the human-conceived unit "common-second" to measure time and that the loss in that case is due to the particular relationship between the meter and the common-second. Because common relativity retains the principle of relativity, its consistency with experimental results is retained (as we shall see explicitly in the following two chapters). A mere human definition cannot alter the physical equivalence of inertial frames and thus as long as a physical theory is based on the principle of relativity, as is common relativity, it will be consistent with experimental results.

Some readers might still object that common time, even as a human convention, is untenable because the speed of light is isotropic in one frame only and that such a definition violates the principle of relativity by singling out that frame in particular. However, we note that although our scheme for clock synchronization defines the speed of light (measured in meter per common-second) to be isotropic in only the F frame,[5] the frame chosen to be F could be any inertial frame. Common time differs from the absolute time of Galilean relativity in that Galilean relativity assumes that there exists somewhere in the universe a frame of absolute rest (the frame of rest relative to the ether or to God, as Newton might have put it), that this would be the only frame in which the speed of light would be isotropic, and that one could measure the velocity of Earth relative to that frame. In common time, on the other hand, *any* frame could be arbitrarily chosen to be the one in which clocks are synchronized such that the speed of light in that frame (measured in meter per common-second) is isotropic. Thus, all inertial frames are equivalent in the sense that any of them could be chosen (by human definition) to be this frame and thus the question of measuring the absolute velocity of Earth relative to that frame makes no sense.

One might also claim that the requirement that the speed of light be isotropic in one frame constitutes a third postulate or assumption. However, the existence of such a frame is not necessary to common relativity. One could imagine synchronizing clocks in F as specified above, placing identical clocks at the vertices of a grid in F', synchronizing the F' clocks by adjusting their readings to match that of the nearest F clocks at a particular time, and then destroying all the F clocks along with any objects in the universe that were at rest with respect to those F clocks. In effect, all physical traces of the F frame would have been eliminated from the universe. However, no physics would change. The existence of an F frame in which the speed of light (measured in meter per common-second) is merely a technical assumption for simplifying our calculations, analogous to the assumption made when synchronizing clocks in special relativity that the clocks at the origins of the F and F' frames read $t=t'=0$ when the origins coincide.

In closing this section, we appeal to the ultimate criteria by which a theory is judged to be useful in physics: consistency with experimental results and ability to predict and explain the phenomena we see around us. In the following pages and chapters, we will show that common relativity satisfies both these criteria.

8c. Coordinate transformations in common relativity

In order to derive the coordinate transformations in common relativity, let us consider two inertial frames F and F', whose relative motion is along parallel x and x' axes, with the usual simplifying assumptions that the origins of the frames coincide at $t_c = 0$. Suppose that we have synchronized the clocks in F according to the procedure described previously and that F' moves with a constant velocity $\mathbf{V}=(V,0,0)$ (measured in meter per common-second) as measured by observers in F.

The coordinate 4-vectors of an event as recorded by observers in F and F' can be denoted by

$$x^{\mu} = (bt_c, x, y, z) \qquad \text{and} \qquad x'^{\mu} = (b't_c, x', y', z'), \tag{8.1}$$

in F and F' respectively. Because the common time t_c is a scalar, it does not transform as a component of a four-vector. In order to obtain a quantity with the correct transformation properties, we must pair the common time t_c with a function b such that $bt_c = w$ and $b't_c = w'$, since we know from taiji relativity that w does have the correct transformation properties. In the F frame, where the speed of light (in meter per common-second) is isotropic and a constant, b=c. In F', where the speed of light (in meter per common-second) is not isotropic, b' is a function. Because we have named w the "lightime," we will refer to the function b as the "ligh."

As in the derivation of the taiji coordinate transformations, we know that the transformation equations must be linear in the coordinates

$$b't_c = a_1 ct_c + a_2 x, \qquad x' = b_1 ct_c + b_2 x, \qquad y' = y, \qquad z' = z, \tag{8.2}$$

where a_1, a_2, b_1, and b_2 are constants to be determined. The condition that an object at rest in F' ($d\mathbf{r}'/dt_c=0$) must have the velocity $d\mathbf{r}/dt_c=\mathbf{V}=(V,0,0)$ as measured by F observers leads to $-b_1 c/b_2 = V$ and thus that $-b_1/b_2 = V/c = \beta$, where β is again a dimensionless constant characterizing the magnitude of the relative motion between F and F'. From the invariant law for the motion of particles in F and F', $s^2 = x_\mu x^\mu = x'_\mu x'^\mu = s'^2$. Combining these conditions, the 4-dimensional space-lightime transformation in common relativity is found to be

$$b't_c = \gamma(ct_c - \beta x), \quad x' = \gamma(x - \beta ct_c), \quad y' = y, \quad z' = z; \tag{8.3}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = V/c,$$

with a corresponding inverse transformation

$$ct_c = \gamma(b't_c + \beta x'), \quad x = \gamma(x' + \beta b't_c), \quad y = y', \quad z = z'. \tag{8.4}$$

As one expects, this is precisely the result obtained by replacing w with ct_c and w' with $b't_c$ in the taiji transformations (7.4).

The ligh function b is peculiar to common relativity and does not always have a simple physical interpretation. We examine this function more closely in the following section.

8d. Physical interpretation of the ligh function b

Taking the derivatives of the first equation in both (8.3) and (8.4) with respect to the common time t_c gives

$$\frac{d(b't_c)}{dt_c} = \gamma\left(c - \beta\frac{dx}{dt_c}\right), \tag{8.5}$$

$$c = \gamma\left(\frac{d(b't_c)}{dt_c} + \beta\frac{dx'}{dt_c}\right). \tag{8.6}$$

Because (8.5) and (8.6) are inverses of each other, it is natural to associate b' with the speed of light as measured by F' observers. However, the presence of the dx/dt_c term can be confusing. How is it possible that the speed of light depends on the velocity of some object that has not even been specified?

To answer this question, first let us consider the case where the equations describe the propagation of a light signal in a vacuum, so that $\Delta s^2 = 0$. In this situation, since $\Delta(b't_c) = \Delta r$ in every frame, it is clear that the lightime interval $\Delta(b't_c)$ in the F' frame can be interpreted as the vacuum optical path of the light, i.e., the distance traveled by the light signal. For a light signal propagating along the +x/x' axes, the ligh function b is given by

$$b = c \quad \text{in } F(ct_c,x,y,z)\ ,$$

$$\frac{d(b't_c)}{dt_c} = b' = \gamma\left(c - \beta\frac{dx}{dt_c}\right) = \gamma(c - \beta c) \quad \text{in } F'(b't_c,x',y',z'). \tag{8.7}$$

We know that $d(b't_c)/dt_c = b'$ because $\gamma(c-\beta c)$ is independent of t_c. In this case, b' can be interpreted as the one-way speed of light in F'. As a check, one can verify that $d(b't_c)/dt_c = dx'/dt_c$. Furthermore, in the more general case of a light signal propagating in an arbitrary direction, $b'^2 = v'^2_x + v'^2_y + v'^2_z$.

In a situation when the equations are used to describe the motion of a particle with non-zero mass so that $\Delta s^2 \neq 0$ and $dx/dt_c \neq c$, the lightime interval $\Delta(b't_c)$ does not correspond to the distance traveled by any actual light signal. In this case, *the ligh function b' is the average speed, as measured by F' observers, of a light signal that travels a closed path in an inertial frame moving with a velocity dx/dt_c relative to F.* Thus, the ligh function does, in a way, refer to the speed of light as measured by F' observers. However, the interpretation is somewhat complex, because of the way the unit common-second has been defined.

To see that the given interpretation is correct, consider the simple case $dx'/dt_c=0$, so that $dx/dt_c=\beta c$. We can think of this as corresponding to the motion of an object at rest relative to F'. In this situation,

$$\frac{d\left(b' t_c\right)}{dt_c} = b' = \gamma\left(c - \beta\frac{dx}{dt_c}\right) = \gamma\left(c - \beta^2 c\right) = \frac{c}{\gamma}. \tag{8.8}$$

Comparing this result with equation (9.6) in chapter 9, we see that b' in this case is the average speed, measured by F' observers, of a light signal that travels a closed path in the F' frame. Thus, the interpretation that b' is the average speed of a light signal as measured by F' observers that travels a closed path in a frame that moves at a velocity dx/dt_c relative to F holds.

In the more complex case $dx'/dt_c \neq 0$, consider a set of three inertial frames, F, F', and F", all with relative motion along the x/x'/x" axes along with the condition that $dx/dt_c=\beta' c$ where β' characterizes the motion of F" relative to F (thus, $dx''/dt_c=0$). The relevant equations of the coordinate transformation between the F' and F" frames is

$$b'' t_c = \gamma_1(b' t_c - \beta_1 x'), \qquad x'' = \gamma_1(x' - \beta_1 b' t_c), \tag{8.9}$$

where β_1 and γ_1 characterize the velocity of the F" frame relative to the F' frame. Taking derivatives of these equations yields

$$\frac{d(b''t_c)}{dt_c} = \gamma_1\left(\frac{d(b't_c)}{dt_c}t_c - \beta_1\frac{dx'}{dt_c}\right), \qquad \frac{dx''}{dt_c} = \gamma_1\left(\frac{dx'}{dt_c} - \beta_1\frac{d(b't_c)}{dt_c}\right). \tag{8.10}$$

If we consider a light signal that makes a closed path in the F" frame, then the interval $\Delta x''$ between the events corresponding to the beginning and end of the light signal's trip is equal to zero, and thus that $dx''/dt_c=0$. The second equation of (8.10) then implies that

$$\frac{dx'}{dt_c} = \beta_1\frac{d(b't_c)}{dt_c} \tag{8.11}$$

and substituting this quantity into the first equation of (8.10) yields

$$\frac{d(b''t_c)}{dt_c} = \gamma'\left(\frac{d(b't_c)}{dt_c} - \beta'^2\frac{d(b't_c)}{dt_c}\right) = \frac{1}{\gamma'}\frac{d(b't_c)}{dt_c}. \tag{8.12}$$

This equation is analogous to (8.8). Since (8.12) was derived under the condition that $dx''/dt_c=0$ (or equivalently, $dx/dt_c=\beta'c$), the quantities $d(b''t_c)/dt_c$ and $d(b't_c)/dt_c$ refer to a light signal that makes a closed round trip in the F" frame. Thus $d(b't_c)/dt_c$ is the average speed, measured by F' observers, of a light signal that travels a closed path in F", just as in (8.8), c was the average speed, measured by F observers, of a light signal that travels a closed path in F'. This then, is the role of the mysterious dx/dt_c that appears in the equation for b'. If we interpret b' as the average speed of a light signal measured by F' observers, then dx/dt_c specifies the frame in which that light signal travels a closed path.

As is evident from this discussion, for problems involving calculations of the propagation of light signals, it will usually be easier to use the clock synchronization procedure specified by special relativity, in which the quantity c is an invariant and has a simpler physical interpretation. The purpose of the above discussion is to serve as a caution to readers that the function b', although related to the speed of light in F', is not simply the speed of a light signal as measured by F' observers.

8e. Implications of common time

Let us now consider some of the implications of the coordinate transformations of common relativity (8.3) for the reciprocity of velocities in two inertial frames, the speed of light, and the relativistic length contraction and time dilation effects. Although these all differ from the results in special relativity, we shall see in the following chapters that they differ in such a way that allows the experimental predictions of common relativity to be identical to those of special relativity.

8e.1 Reciprocity of velocities

From the coordinate transformations (8.3) and (8.4), the velocity (measured in meter per common-second) of the F' frame relative to the F frame is βc while the velocity (measured in meter per common-second) of the F frame relative to the F' frame is $-\gamma\beta c$. However, the fact that these reciprocal velocities (measured in units of meter per common-second) are asymmetric is purely an artifact of using the common-second as a unit of time. The relative velocities of the two inertial frames in natural units (obtained by finding $dx'/d(b't_c)$ when $dx/d(bt_c) = 0$ and by finding $dx/d(bt_c)$ when $dx'/d(b't_c)=0$) are β and $-\beta$, as we expect.

More generally, in order to construct a theory with the correct Lorentz and Poincaré group properties, one must define the relative velocity between frames as the quantity obtained by taking derivatives of the coordinate transformation equations with respect to the quantity $w=bt_c$ rather than simply t_c (in special relativity, where $b=c$ is a constant in all frames, it does not make a difference whether one takes derivatives with respect to w or t). These properties can be seen explicitly in the steps from equations (11.2) to (11.4), which also hold for common relativity with the replacements: $w \to bt_c$, $w' \to b't_c$ and $w'' \to b''t_c$. Thus the ratio $d\mathbf{r}/d(bt_c)$ more properly characterizes the rate of change of an object's spatial location and demonstrates the relativity of motion and the equivalence of inertial frames. Because t_c is a scalar and is not simply proportional to w, the components of $d\mathbf{r}/dt_c$ do not directly reveal the relativity of motion nor the equivalence of inertial frames. This stipulation applies only to objects that do not move at the vacuum speed of light. For light signals in vacuum, both $\mathbf{dr}/dt_c$ and $\mathbf{dr}/d(bt_c)$ give the same result.

This differences between the two types of ratios is not a problem for common relativity as long as one keeps in mind which quantities are tied to physical phenomena (such as w) and which are free inventions of the human mind for convenience's sake (such as t_c).

8e.2 One-way speed of light

It is evident that since clocks in common relativity are synchronized differently from those in special relativity, the speed of light (measured in meter per common-second) will no longer isotropic nor will it be the same in all inertial frames. To see how it differs, we first take derivatives of the coordinate transformations (8.3) with respect to t_c (since we discuss the propagation of light signals, the results will be the same as if we took derivatives with respect to bt_c)

$$\frac{d(b' t_c)}{dt_c} = \gamma(c - \beta v_x), \qquad v'_x = \gamma(v_x - \beta c), \qquad v'_y = v_y, \qquad v'_z = v_z . \tag{8.13}$$

Setting $dx/dt = c\cos\theta$ and $dy/dt = c\sin\theta$ for a light signal traveling at an arbitrary angle relative to the x-axis, we obtain

$$v'_x = \gamma(c\cos\theta - \beta c), \qquad v'_y = c\sin\theta, \qquad v'_z = v_z = 0, \tag{8.14}$$

for the velocity components of the light signal measured by an F' observer or

$$c' = \sqrt{v'^2_x + v'^2_y} = \gamma c(1 - \beta\cos\theta) \tag{8.15}$$

for the speed of the light signal. Note that in this case, c' is equal to $d(b't_c)/dt_c$. Thus we see that the one-way speed of light in F' (measured in meter per common second) is non-isotropic.

As noted previously, the non-universality of the speed of light in common relativity makes special relativity a better choice for solving problems involving the propagation of light. On the other hand, the trade-off is that the transformation of the evolution variable using the unit second is more complicated in special relativity than in common relativity, making common relativity a better choice for solving problems involving systems of many particles.

8e.3 Time dilation

In common relativity, because all observers may use the same set of clocks, there is no relativistic time dilation (with time measured in the unit common-second). A time interval Δt_c between two events is the same in any inertial frame $\Delta t_c' = \Delta t_c$. This is a direct consequence of the common time transformation $t_c = t_c'$. However, time intervals Δw as measured in units of meter by a clock whose rate of ticking depends on the movement of a light signal, as described in section 7d, are dilated in the same way as found in special relativity.

8e.4 Length contraction

If we have a rod of length L at rest relative to the F frame lying parallel to the x and x' axes ($\Delta x = L$), its length $\Delta x'$ as measured by observers in F' will be γL as obtained from the coordinate transformations (8.3) with $\Delta x = L$ and $\Delta t_c = 0$. On the other hand, if a rod of length L is at rest relative to the F' frame, lying parallel to the x and x' axes ($\Delta x' = L$), its length Δx as measured by observers in F will be L/γ.[6] In both cases, the observer in F' measures a longer length for the rod and the relativistic length contraction does not exist, again because of the common time. Note however that the principle of relativity still demands that an F-frame observer measuring the length of a rod at rest relative to the F frame obtains the same result as an F' observer measuring the length of an identical rod at rest relative to the F' frame.

Like the absence of a relativistic time dilation (in common-seconds), the absence of a relativistic length contraction in common relativity does not present any obstacle to the correct prediction of experimental results because the length contraction and time dilation effects are an artifact of how the unit second or common-second is defined relative to the unit meter. The units of common-second and second specify different conditions for measuring the locations of the two ends of a rod in order to find its length. As long as these measurement conditions are taken into account, the results cannot be in contradiction with experiment.

Although it may seem counter-intuitive that common relativity can correctly predict experimental results, we will see in the next two chapters that the results of experiments always depend on a combination of the above effects in such a way that the experimental predictions of common relativity remain identical to those of special relativity.

References

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2. R. Hakim, J. Math. Phys. (N.Y.) **8**, 1315 (1967).
3. J. P. Hsu, Nuovo Cimento B, **74**, 67 (1983).
4. J. P. Hsu, Found. Phys. **8**, 371 (1978); **6**, 317 (1976); J. P. Hsu and T. N. Sherry, *ibid* **10**, 57 (1980).
5. Editorial, Nature **303**, 129 (1983).
6. If one uses the inverse transformation (8.4) in common relativity, one has $c\Delta t_c = \gamma[\Delta(b' t_c) + \beta\Delta x']$. This relation implies that when $\Delta t_c = 0$, the quantity $\Delta(b' t_c) = -\beta\Delta x'$ is, in general, not equal to zero. This property is related to the fact that as $t_c \to 0$, the ligh function approaches infinity, $b' \to \infty$, such that their product $b' t_c = -\beta x' = -\gamma\beta x$ in (8.3) and (8.4) remains finite. Such a singular property for the ligh function b' is necessary in order to preserve the 4-dimensional symmetry of common relativity and does not lead to any physical impossibilities.

9.

Experimental Tests I

9a. Time intervals versus optical path length

Because the coordinate transformations in taiji relativity and special relativity are mathematically identical, it is straightforward to show that the experimental predictions of special relativity and of taiji relativity are also identical. Nevertheless in this chapter, we revisit several of the foundational experiments of special relativity related to the propagation of light, re-interpreting them in terms of the optical path length of light w, rather than in terms of the time t (as measured in the conventional unit of second) or in terms of velocities (measured in the conventional units of meter per second). Such an interpretation is not only more consistent with the spirit of taiji relativity, but also makes clear that the experimental results are consistent with other relativity theories such as common relativity. We will also compare the experimental results to the predictions of common relativity to show that the definition of the common second $t_c = t_c'$ in no way contradicts experimental results.

9b. The Michelson-Morley experiment

The Michelson-Morley experiment was first carried out by A. A. Michelson in 1881, more than twenty years before the birth of special relativity. It had been suggested by Maxwell in 1878 that such an experiment could reveal the absolute velocity of the Earth as it moved through the ether. However, the experiment produced a null result, indicating that the absolute motion of the Earth could not be detected by optical phenomena.[1,2] This "alarming result" was the first experiment that stimulated the search for the theory of relativity and it played a central role in the earlier work (1886-1905) of Lorentz and Poincaré, although not in Einstein's work. Lorentz was deeply concerned about the null result of the Michelson-Morley experiment for a long time and corresponded with W. Voigt around 1888.

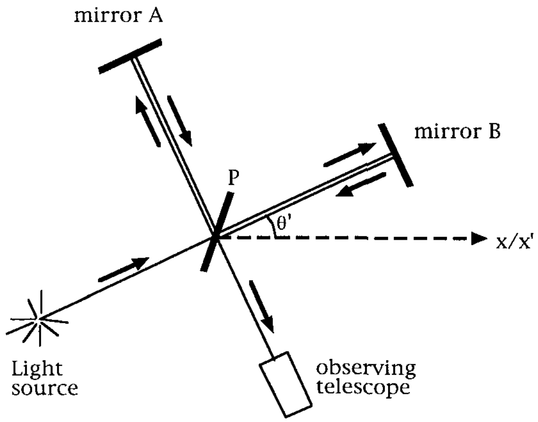

Fig. 9.1 A Michelson-Morley interferometer at an arbitrary orientation.

For purposes of the following discussion, let us consider the simplified Michelson-Morley apparatus shown in figure 9.1. Light from a source is split into two beams by a half-silvered mirror (beamsplitter) P. One beam travels to mirror A and is reflected back to P. On its return, a fraction of this beam passes through the beamsplitter into a telescope. The second beam travels to a mirror B and on its return, part of this beam is reflected by the beamsplitter into the telescope, interfering with the part of beam A that travels to the telescope. If the light source is monochromatic, an interference pattern of light and dark bands or rings will be seen in the telescope.

After observing the interference pattern produced with their apparatus is a particular orientation, Michelson and Morley rotated the apparatus about an axis perpendicular to the plane of the interferometer, changing the angle θ' indicated in the diagram. They expected to see a shift in the interference fringes as the apparatus was rotated enabling them to determine the speed of the Earth relative to ether. However, no such shift was ever detected.

The null result of the Michelson-Morley experiment has been interpreted in different ways at different times: Originally, it was interpreted by Fitzgerald and Lorentz as the contraction of the absolute length of a rod along the direction of its motion through the ether. Nowadays the Michelson-Morley experiment is interpreted as evidence that the two-way speed of light is isotropic in any inertial frame.[3]

9b.1 The Michelson-Morley experiment and taiji relativity

Because the unit second is not defined in taiji relativity, we will analyze the Michelson-Morley experiment strictly in terms of the optical path lengths traveled by the two beams along perpendicular arms of the interferometer. From this point of view, the Michelson-Morley experiment can be said to demonstrate that the difference between the optical path lengths of the two arms of the interferometer is independent of the orientation of the interferometer in any inertial frame.

Let us first consider the direct interpretation of the Michelson-Morley experiment, in which the experiment has a null result in any frame when carried out by an observer at rest with respect to the apparatus. In this case, the rotational symmetry of the four-dimensional spacetime guarantees that no fringe shift will be observed as the interferometer is rotated. Because both the arm lengths of the interferometer and the wavelength of the light emitted by the source do not depend on the orientation of the apparatus or the direction in which the light is emitted, the number of wavelengths traveled by the light signals along each arm of the interferometer remains the same. Thus, the phase difference between the two light signals when they combine and interfere at the observing telescope also remains the same while the apparatus is rotated.

9b.2 The Michelson-Morley experiment and common relativity

Because common relativity is based on taiji relativity and simply defining a second unit for the evolution variable (the common-second) cannot change any physical properties implied by a theory, one could use the same argument with optical path lengths to demonstrate that common relativity also is consistent with the Michelson-Morley experiment. However, it is useful to analyze it from the point of view of the common-time t_c to show explicitly that the two-way speed of light (measured in meter per common-second) is isotropic in any inertial frame even though the one-way speed no longer is.

Let us again consider the case where both the apparatus and observer are at rest relative to the F' frame and that one arm of the interferometer makes an angle θ' with the x' axis. If the arm of the interferometer with mirror B has length L', then the amount of time (measured in common-second) it takes a light signal to traverse the arm in both directions (going out and coming back) is

$$t'_{c,tot} = \frac{L'}{c'_{out}} + \frac{L'}{c'_{back}}. \tag{9.1}$$

From equation (8.14), the x'-components of the velocity of the light signal traveling out and along the interferometer arm are

$$\begin{aligned} v'_{x\text{-}out} &= \gamma(c\cos\theta_1 - \beta c), \quad \text{and} \\ v'_{x\text{-}back} &= \gamma(c\cos\theta_2 - \beta c), \end{aligned} \tag{9.2}$$

where θ_1 and θ_2 are the angles between the x-axis and the velocity vector of the outbound and returning light signals as measured from the F frame. Since $v'_{x\text{-}out} = c'_{out}\cos\theta'_1$ and $v'_{x\text{-}back} = c'_{back}\cos\theta'_2$, equations (9.2) can be re-written as

$$\begin{aligned} c'_{out} &= \frac{\gamma(c\cos\theta_1 - \beta c)}{\cos\theta'_1}, \quad \text{and} \\ c'_{back} &= \frac{\gamma(c\cos\theta_2 - \beta c)}{\cos\theta'_2}, \qquad \theta'_2 = \theta'_1 + \pi. \end{aligned} \tag{9.3}$$

The inverse transformation for the x-component of the velocity in common relativity is $v_x = \gamma(v_x' + \beta c')$, from which we obtain

$$\begin{aligned} c\cos\theta_1 &= \gamma(c'_{out}\cos\theta'_1 + \beta c'_{out}), \quad \text{and} \\ c\cos\theta_2 &= \gamma(c'_{back}\cos\theta'_2 + \beta c'_{back}). \end{aligned} \tag{9.4}$$

Finally, substituting these expressions for $c\cos\theta_1$ and $c\cos\theta_2$ into equations (9.3) yields

$$c'_{out} = \frac{c\sqrt{1-\beta^2}}{1+\beta\cos\theta'_1}, \qquad c'_{back} = \frac{c\sqrt{1-\beta^2}}{1-\beta\cos\theta'_1}. \tag{9.5}$$

Thus, the amount of time it takes a light signal to make a round trip along one arm of the interferometer and the average speed of the round-trip c'_{rt} are respectively

$$t'_{c,tot} = \frac{2L'\gamma}{c} \qquad \text{and} \qquad c'_{rt} = \frac{2L'}{t'_{c,tot}} = \frac{c}{\gamma}, \tag{9.6}$$

independent of the orientation of the arm.

9c. The Kennedy-Thorndike experiment

The Kennedy-Thorndike (KT) experiment[4] is similar to the Michelson-Morley experiment, except that it used an interferometer with two arms of unequal lengths as shown schematically in figure 9.2. Rather than searching for a shift in the interference fringes as the orientation of the apparatus is changed, Kennedy and Thorndike looked for fringe shifts over time, as Earth (and the apparatus) instantaneously occupied a series of different inertial frames as it traveled in its orbit around the sun. As with the Michelson-Morley experiment, no fringe shift was ever observed, indicating that the relative phase of the two beams when they combine at the beamsplitter after making a round trip along each of the interferometer arms is the same in all inertial frames. Equivalently, one can say that the number of extra wavelengths traveled by the light beam in making a round trip along the longer of the two arms of the interferometer has the same value in all inertial frames. Since the wavelength of a light signal is related to its frequency and speed, the null result has been interpreted as evidence that the two-way speed of light has the same numerical value (expressed in meter per second) in all inertial frames.

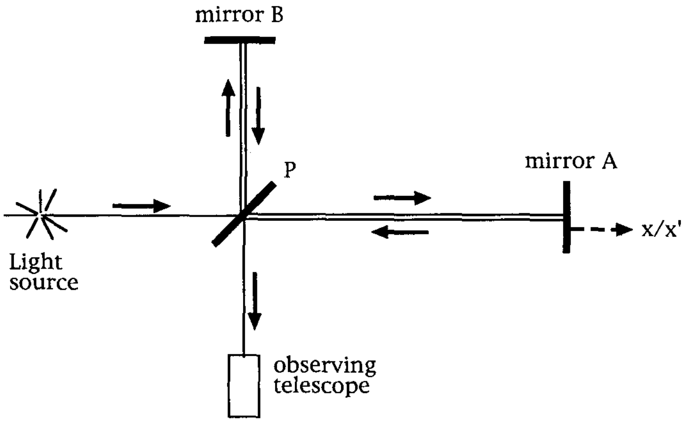

Fig. 9.2 The apparatus used in the Kennedy-Thorndike experiment was an interferometer with two arms of unequal lengths.

To analyze the null result of the Kennedy-Thorndike experiment from the point of view of taiji relativity, we return to the most basic interpretation of the null result, i.e., that the phase difference between the two light beams after making round trips along their respective arms of the interferometer has the same value in all inertial frames. Since the phase difference is a result of the number of extra wavelengths traveled by one light beam along the longer of the two arms, we may also conclude that the difference in length of the arms (measured in the number of wavelengths of the type of light used) is the same in all inertial frames.

Verification that taiji relativity correctly predicts the outcome of the Kennedy-Thorndike experiment is thus a direct consequence of the principle of relativity. Since all inertial frames are postulated to be equivalent, then the wavelength of the light emitted from the source used in the experiment and the lengths of the interferometer arms are also postulated to be equal in all inertial frames.[5] Thus, the number of extra wavelengths traveled by the light beam traversing the longer of the two arms must be equal in all inertial frames.

9c.2 The Kennedy-Thorndike experiment and common relativity

We now examine the prediction of common relativity as regards the Kennedy-Thorndike experiment. One could, of course, use the same reasoning as in the analysis based on taiji relativity, where the equivalence of all inertial frames led to the prediction of a null result. However, let us see more explicitly how the non-universal speed of light (measured in meter per common-second) can lead to a correct prediction in this case. To simplify the mathematics, we assume that the arms of the Kennedy-Thorndike interferometer are oriented such that one is parallel to the x/x' axes and other is parallel to the y/y' axes as shown in figure 9.2.

The phase difference δ between the light beams (in radians) can be written as

$$\delta = \frac{1}{2\pi}\left[\frac{c'_{A\text{-out}}\Delta t_{A\text{-out}}}{\lambda'} + \frac{c'_{A\text{-back}}\Delta t_{A\text{-back}}}{\lambda'} - \frac{c'_{B\text{-out}}\Delta t_{B\text{-out}}}{\lambda'} - \frac{c'_{B\text{-back}}\Delta t_{B\text{-back}}}{\lambda'}\right] \tag{9.7}$$

where the c's are the speeds of the light signals traveling out and back along interferometer arms A and B and the Δt's are the time intervals (measured in common-seconds) for the one-way trips along those arms. However, since the time intervals are simply the arm lengths of the interferometer divided by the relevant speed of light,

$$\Delta t_{A\text{-out}} = \frac{L'_A}{c'_{A\text{-out}}}, \quad \Delta t_{A\text{-back}} = \frac{L'_A}{c'_{A\text{-back}}}, \quad \Delta t_{B\text{-out}} = \frac{L'_B}{c'_{B\text{-out}}}, \quad \Delta t_{B\text{-back}} = \frac{L'_B}{c'_{B\text{-back}}}, \tag{9.8}$$

then the phase difference is simply

$$\delta = \frac{1}{2\pi}\frac{2(L'_A - L'_B)}{\lambda'} \tag{9.9}$$

in any inertial frame since the principle of relativity implies that the interferometer arm lengths and wavelength of the light must be the same in all inertial frames when measured by an observer at rest with respect to the apparatus.

Relative to special relativity, we can see that introducing the common-second has two effects that cancel each other in this case. On the one hand, the speed of light (measured in meters per common-second) differs from the speed of light in special relativity. However, the time interval required for light to make one-way trips along the interferometer arms is also different in common relativity and is different in such a way that the experimentally measured quantity, the difference in phase between the two beams, is exactly the same in both special and common relativity.

9d. The Fizeau experiment

The Fizeau experiment[1,6] was first performed in 1851, over fifty years before Einstein published his work on the theory of special relativity. In it, an interferometer similar to the one shown schematically in figure 9.3 was constructed. Light from a source S traveled to a beamsplitter P. Part of this light was then reflected so as to travel to mirrors M1, M2, and then M3 before returning to the beamsplitter. The other part of the light was transmitted by the beamsplitter and traveled to mirrors M3, M2, and then M1 before returning

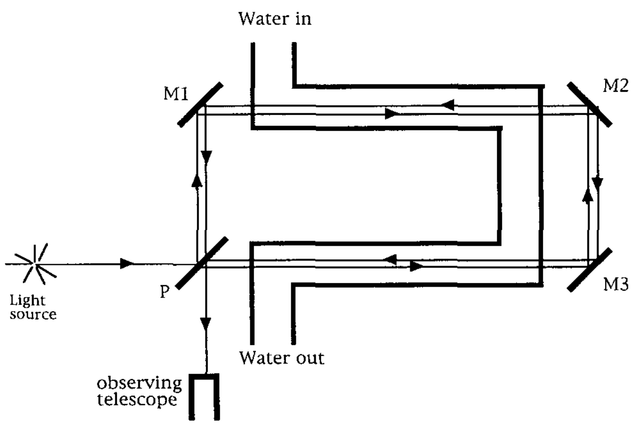

Fig. 9.3 Apparatus used in the Fizeau experiment.

to the beamsplitter. Part of each of the returning beams was reflected or transmitted into the observing telescope, producing an interference pattern of light and dark bands. Water was flowed through the tubes between the mirrors in the direction shown, so that one of the two light beams produced by the beamsplitter always traveled in the same direction as the velocity of the water and the other beam always traveled in the opposite direction to the velocity of the water. The fringe shift observed when the water was made to flow was found to be dependent on the speed of the water flow and corresponded to introducing a phase difference between the two beams of

$$\delta = \frac{1}{2\pi}\frac{2Ln^2}{\lambda c}\left(1 - \frac{1}{n^2}\right)v_w, \tag{9.10}$$

where n is the index of refraction of water, L is the total distance traveled by the light beams through the water, and v_w is the speed of the water flow. Since this phase difference is related to the optical path difference $c\Delta t$ of the beams traveling in the two directions by $2\pi\delta\lambda = c\Delta t$, this experiment is often interpreted as showing that the speed of light through a moving medium is

$$c_m = \frac{c}{n} + \left(1 - \frac{1}{n^2}\right)v_m, \tag{9.11}$$

where v_m is the velocity of the medium and n is the index of refraction of the medium at rest. Later, it was shown that this result was a natural consequence of the relativistic addition of velocities.

9d.1 The Fizeau experiment and taiji relativity

Because of the mathematical equivalence of the coordinate transformation equations in taiji and special relativity, this result can easily be calculated in the framework of taiji relativity using the velocity transformation equations (7.8). Suppose that the apparatus is constructed so as to be at rest in the F frame. The taiji speed of light in the rest frame of the water F' is $\beta'_L=1/n$, where $n>1$. If the velocity of the water is parallel to the velocity of the light signal β_L, then the taiji-speed of light observed by an observer stationary with respect to F can be calculated from the inverse of transformation (7.8) with $u_x'=\beta'_L$ and $u_x=\beta_L$ as

$$\beta_L = \frac{\beta'_L + \beta}{1+\beta\beta'_L} = \frac{1}{n} + \beta\left(1 - \frac{1}{n^2}\right), \tag{9.12}$$

where β is the velocity of the water relative to the F frame. Thus the Fresnel drag coefficient $(1 - 1/n^2)$ in taiji relativity is exactly the same as that obtained in special relativity.

In terms of the difference in optical path lengths w, suppose that the total distance the light beams travel through water is L and that the taiji-speed of the water in Fizeau's experiment is β. The difference in the optical path length is

$$\Delta w = \frac{L}{\frac{1}{n} - \beta\left(1 - \frac{1}{n^2}\right)} - \frac{L}{\frac{1}{n} + \beta\left(1 - \frac{1}{n^2}\right)} = 2L\beta n^2\left(1 - \frac{1}{n^2}\right), \tag{9.13}$$

leading to the experimentally observed phase difference.

9d.2 The Fizeau experiment and common relativity

From the point of view of common relativity, we can also derive the phase difference between the two beams by calculating the effective index of refraction of the moving water. To simplify the calculations, let us consider a stripped-down version of the apparatus, with a single straight tube of length L oriented parallel to the x/x' axes and with water inside the tube flowing in the +x/+x' direction. The tube is at rest in the F frame and the water is at rest with respect to the F' frame, which moves with a velocity β relative to the F frame. The phase difference developed between two beams that travel through the water, one in the same direction as the water flow and the other in the opposite direction, is

$$\delta = \frac{1}{2\pi}\left[\frac{L}{\lambda_+} - \frac{L}{\lambda_-}\right] = \frac{1}{2\pi}\left[\frac{Ln_{eff+}}{\lambda} - \frac{Ln_{eff-}}{\lambda}\right], \qquad \frac{1}{n_{eff\pm}} = \frac{c_{water\pm}}{c}, \tag{9.14}$$

where λ_+ and λ_- are the wavelengths of the light signal traveling in the same and opposite directions to the water flow, respectively. The quantities n_{eff+} and n_{eff-} are the effective indices of refraction, i.e., the ratios of the speed of light in the moving water to the speed of light in vacuum for light traveling in the same and opposite directions to the water flow, respectively. These differ from the ordinary index of refraction of water because the water is moving. Since we analyze the experiment from the point of view of the F frame, the vacuum speed of light is isotropic, although we would obtain the same result in any inertial frame.

As touched upon at the end of chapter 8, when discussing motion that does not occur at the vacuum speed of light, one must use the ratio $\mathbf{dr}/d(bt_c)$ to characterize the velocity, rather than the ratio $\mathbf{dr}/dt_c$, in order for the theory to have the correct group properties. Because $\mathbf{dr}/d(bt_c)$ is equal to the ratio $\mathbf{dr}/dt_c$ divided by the vacuum speed of light, $\mathbf{dr}/d(bt_c)$ turns out to be exactly the $c_{water\pm}/c = 1/n_{eff\pm}$ ratio that we are interested in calculating for a light signal traveling through a medium.

Let us consider a light signal traveling in the +x/+x' direction through the water, in the same direction as the water. In the F' frame, where the medium is at rest, the speed of this light signal is $c'_{water}=c'/n'$ where n' is the usual index of refraction for water with the value 1.33. Applying the inverse velocity transformations (8.6) to obtain the desired ratio, we find that

$$\frac{c_{water+}}{c} = \frac{dx/dt_c}{d(bt_c)/dt_c} = \frac{\gamma\left(v'_x + \beta\frac{d(b't_c)}{dt_c}\right)}{\gamma\left(\frac{d(b't_c)}{dt_c} + \beta v'_x\right)}. \tag{9.15}$$

Since we are discussing the propagation of a light signal in water, $v'_x = c'/n'$. Also, because $b't_c$ is the optical path traveled by a light signal in vacuum, $d(b't_c/dt_c) = c'$. Equation (9.15) then becomes

$$\frac{c_{water+}}{c} = \frac{\gamma\left(\frac{c'}{n'} + \beta c'\right)}{\gamma\left(c' + \beta\frac{c'}{n'}\right)} = \frac{\left(\frac{1}{n'} + \beta\right)}{\left(1 + \beta\frac{1}{n'}\right)} \tag{9.16}$$

and expanding this expression in powers of β yields a velocity ratio

$$\frac{c_{water+}}{c} = \left(\frac{1}{n'} + \beta\right)\left(1 + \beta\frac{1}{n'}\right)^{-1} = \frac{1}{n'} + \beta - \frac{\beta}{n'^2} + \ldots \tag{9.17}$$

which is the same result as obtained in special relativity. Inserting (9.17) into equation (9.14) for two light signals, one traveling in the same direction as the water and one traveling in the opposite direction, gives the experimentally determined expression for the phase difference between the two beams.

References

1. A. A. Michelson, Am. J. Sci. **22**, 120 (1881); A. A. Michelson and E. W. Morley, Am, J. Sci. **34**, 333 (1887). For discussions of the Michelson-Morley experiment and the Fizeau experiment in special relativity see, for example, A. P. French, *Special Relativity* (Norton & Company, New York, 1968) pp. 51-57, pp. 46-48 and pp. 131-132.
2. For a discussion of the Michelson-Morley experiment in "extended relativity" based on Reichenbach's concept of time (cf. chapter 17) and Edwards' universal 2-way speed of light, see Leonardo Hsu, Jong-Ping Hsu and Dominik A. Schneble, Nuovo Cimento **111B**, 1305 and 1310 (1996). See also Y. Z. Zhang, *Special Relativity and Its Experimental Foundations.* (World Scientific, Singapore 1997), chapter 11.
3. The 2-way speed of light was introduced and discussed in physics long time ago. For example, Maxwell noted in 1878 that "all methods....by which it is practicable to determine the velocity of light from terrestrial experiments depends on the measurement of the time required for the double journey from one station to the other and back again." J. C. Maxwell, *The Scientific Papers of James Clerk Maxwell,* (Dover, New York) Vol. 2, p.763. Let us consider two inertial frames F and F', where F' is moving along the x-axis. Suppose there is a clock 1 located at the origin of the F frame and a mirror at point x=L on the x-axis. A light signal starts from the origin at time t_1. it reaches the mirror and returns to the origin at time t_3. The two-way speed c_{2w} of light moving along the x-axis and returning to the starting point in F is defined by $c_{2w} = 2L/(t_3 - t_1)$. The round-trip trajectory of this light signal forms a "closed loop" in F. However, from the viewpoint of observers in F', the trajectory of this light signal forms an "open-loop" because the returning point differs from the starting point in the F' frame.
4. R. J. Kennedy and E. M. Thorndike, Phys. Rev. **42**, 400 (1932).
5. The physical properties that the wavelength of the light emitted from the source and the lengths of the interferometer arms are equal in all inertial frames follow from the principle of relativity. (To be more specific, the arm length of an interferometer at rest in F and measured by an F-observer is the same as the arm length of the same interferometer at rest in F' and measured by an F'-observer). These properties are determined by

the atomic structure which is determined by the laws of physics and the fundamental constants such as the fine structure constant, the electron mass and Planck's constant, etc. The principle of relativity implies that the laws of physics (e.g., the Dirac wave equation for the electron) are the same in all inertial frames. Therefore, the Bohr radius of the hydrogen atom, the interferometer arm length and the wavelength of light emitted from atomic transitions will be the same in all inertial frames.

6. A. Fizeau, C. R. Ac. Sci. Paris **33**, 349 (1851).

10.

Experimental Tests II

In this section, we apply taiji relativity and common relativity to two more of the foundational experiments of relativity: the Ives-Stilwell experiment investigating the second order Doppler shift of light from a moving source and the observation of the lifetime dilation of unstable particles moving at high speeds. Because the analysis of these experiments requires more than just the kinematical properties of a relativity theory given by the coordinate transformations, we also develop some additional physics by examining how the use of natural units and common time affect the interpretation and equations of the physics of electromagnetic interactions and some elementary aspects of quantum field theory.

10a. The Ives-Stilwell experiment

In 1938, H. E. Ives and G. R. Stilwell[1] published the results of an experiment in which they measured the second order Doppler shift of light emitted by excited hydrogen ions (H_2^+ and H_3^+) that had been accelerated to speeds of about 10^6 m/s. Rather than measuring this shift directly by trying to observe the radiation emitted from the atoms perpendicular to the direction of motion of the atoms, which would have been very difficult to carry out experimentally with the required precision, Ives and Stilwell collected the light emitted by the atoms in the same and opposite directions as their velocities and calculated the deviation of the average of the two wavelengths from the wavelength of light emitted by stationary excited hydrogen atoms. This deviation is exactly the second order Doppler shift $\Delta\lambda = (1/2)\ (V/c)^2 \lambda$.

10b. Atomic energy levels and Doppler shifts in taiji relativity

The covariant Dirac Hamiltonian for the electron of a hydrogen atom is

$$H_D = -\boldsymbol{\alpha}_D \cdot \mathbf{p} - \beta_D m_e - \frac{e^2}{4\pi r}, \qquad \mathbf{p} = -i\nabla, \tag{10.1}$$

where $\boldsymbol{\alpha}_D$ and β_D are the usual 4 x 4 Dirac matrices. Because natural units are used in taiji relativity, the electron charge e is expressed as a unitless number:

$$e = \sqrt{4\pi} / \sqrt{137.036}.$$

The covariant Dirac equation in taiji relativity is given by

$$i \frac{\partial}{\partial w} \Psi = H_D \Psi, \tag{10.2}$$

where $\partial/\partial w$ and H_D both transform as the zeroth component of p^μ. Since the taiji-time w has the dimension of length, H_D has dimensions of inverse length and the Dirac equation leads to electronic energy levels also with dimensions of inverse length. Solving (10.2) gives the electronic energy levels for an hydrogen atom[2]

$$E_n = \frac{m_e}{\sqrt{1 + \dfrac{\alpha_e^2}{\left\{ n - (j + \frac{1}{2}) + \sqrt{(j + \frac{1}{2})^2 - \alpha_e{}^2} \right\}^2}}}, \tag{10.3}$$

where α_e is the fine structure constant $\alpha_e = e^2/4\pi = 1/137.036$. In natural units, the electron mass m_e is $2.59..x10^{12}$ cm^{-1}. Thus in taiji relativity, an atomic system can only emit or absorb energy quanta characterized by a 'taiji frequency' k^0 (with dimensions of inverse length) that is equal to the difference between two energy levels $k^0 = |E_n - E_k|$.

It is important to note that in taiji relativity, where we have only a unit for length, the traditional concept of frequency (ω or ν) is not defined. Instead, an electron making a transition from a higher to a lower energy level emits radiation that travels with a speed β_L=1 and that has a wavelength λ=$2\pi/k$ where $k = |\mathbf{k}| = k^0$ is the difference in energy between the two levels. This wavelength λ

can be measured experimentally by passing the radiation through a diffraction grating and measuring the positions of the light and dark bands that result from the interference. Although we will call k^0 the taiji frequency because it is analogous to the frequency in special relativity, it is more closely related to an energy in taiji relativity.

Because the invariance of the energy-momentum law implies that the quantity $k_\mu k^\mu = (k_0)^2 - \mathbf{k}^2 = 0$ is invariant, the wave 4-vector k^μ must transform in the same way as the coordinate 4-vector x^μ

$$k'^0 = \gamma(k^0 - \beta k_x), \qquad k'_x = \gamma(k_x - \beta k^0), \qquad k'_y = k_y, \qquad k'_z = k_z. \tag{10.4}$$

Since

$$k_x = |\mathbf{k}|\cos\theta = k^0 \cos\theta, \tag{10.5}$$

the Doppler effect in taiji relativity is

$$k'^0 = k^0 \gamma\left(1 - \beta\cos\theta\right). \tag{10.6}$$

The second order term of the transverse Doppler shift (when $\theta = 90°$) is thus $\Delta\lambda = (1/2)\,\beta^2\lambda$, the same as in special relativity.[3]

10c. Atomic energy levels and Doppler shifts in common relativity

In common relativity, where we use the more familiar set of units (with the exception of the replacement of the second by the common-second), the covariant Dirac Hamiltonian for a hydrogen atom in the F frame is

$$H_D = -\,\boldsymbol{\alpha}_D\cdot\mathbf{p} - \beta_D m - \frac{\bar{e}^2}{4\pi r}, \qquad \mathbf{p} = -iJ\nabla, \tag{10.7}$$

$$\bar{e} = -1.602\times10^{-20}\sqrt{4\pi}\,(g\cdot cm)^{1/2}, \qquad J = 3.5177293\times10^{-38}\,g\cdot cm,$$

where $\bar{e}$ has the numerical value e/c and J has the numerical value $\hbar$/c. The covariant Dirac equation in a general inertial frame is

$$iJ\frac{\partial}{\partial(bt_c)}\Psi = H_D\Psi, \tag{10.8}$$

where both $\partial/\partial(bt_c)$ and H_D transform as the zeroth component of a 4-vector. Since the quantities H_D and $J(\partial/\partial(bt_c))$ both have the dimension of mass, the Dirac equation leads to electronic energy levels with the dimension of mass. Solving the Dirac equation (10.8) leads to[2]

$$M_n = \frac{m_e}{\sqrt{1+\dfrac{\alpha_e^2}{\left\{n-(j+\frac{1}{2})+\sqrt{(j+\frac{1}{2})^2-\alpha_e^2}\right\}^2}}} \tag{10.9}$$

again, where $\alpha_e = \bar{e}^2/(4\pi J) = 1/137.036$ is the fine structure constant. Thus in common relativity, we can think of an atomic system as having "mass levels" M_n and such systems can only emit or absorb "mass quanta" that are equal to the difference between two mass levels. These quanta, with a value

$$Jk^0 = |M_n - M_k|, \tag{10.10}$$

are directly proportional to the zeroth component of the wave 4-vector k^0.

Since common relativity has a separate unit of time, we can define a frequency ω so that the wave 4-vector can be written as

$$k^\mu = \left(\frac{\omega}{c},\mathbf{k}\right) \text{ in F} \qquad \text{and} \qquad k'^\mu = \left(\frac{\omega'}{c'},\mathbf{k}'\right) \text{ in F'}. \tag{10.11}$$

To see an important difference between common relativity and special relativity regarding frequency and the speed of light, consider first an atom at rest in the F frame in which the speed of light is isotropic. As measured by an F observer, an electron making a transition between two mass levels M_n and M_k will radiate a photon with a frequency ω and speed c

$$|M_n - M_k| = \hbar\omega/c^2 = (\hbar/c)k^0. \qquad (10.12)$$

In contrast, consider the same kind of atom at rest in the F' frame, where the speed of light is not isotropic. For measurements made by an F' observer, we have

$$|M^\dagger_n - M^\dagger_k| = \hbar^\dagger\omega^\dagger/c^{\dagger 2} = (\hbar^\dagger/c^\dagger)\,k^{\dagger 0}, \qquad (10.13)$$

where we have used the $\dagger$ symbol to indicate that these are quantities for a different physical system (one at rest relative to F', rather than relative to F).[4]

Since the atoms are at rest in F and F' respectively, the principle of relativity requires that $(M_n - M_k)$ and $(M^\dagger_n - M^\dagger_k)$ both be equal and isotropic as measured by observers at rest relative to those respective systems. Thus, we get

$$(\hbar/c)k^0 = (\hbar^\dagger/c^\dagger)\,k^{\dagger 0}, \qquad (10.14)$$

because all inertial frames are equivalent. Since the principle of relativity also demands that $\omega^\dagger/c^\dagger = \omega/c$, or equivalently that $k^{\dagger 0} = k^0$, then[4]

$$\hbar/c = \hbar^\dagger/c^\dagger = J. \qquad (10.15)$$

Thus, the ratio $\hbar/c$ is a universal constant in common relativity even though the values of $\hbar$ and c, separately, are not. Because in the equations of common relativity, this ratio appears everywhere in place of $\hbar$, we will denote it by the letter J. This concept of the universal constant J in common relativity and the relations (10.9)-(10.15) are very important to understanding the experimental results of Doppler shifts within the framework of common relativity.

Within the framework of common relativity, it is not the shift in the frequency ω that is measured in experiments, but the quantities $\omega'/c' = k'^0$ and

$\omega/c=k^0$. Therefore, the Doppler shift is given by the four-dimensional transformation of the wave vectors k^μ and k'^μ (in F and F' frames):

$$k'^0 = \gamma(k^0 - \beta k_x), \qquad k'_x = \gamma(k_x - \beta k^0), \qquad k'_y = k_y, \qquad k'_z = k_z. \tag{10.16}$$

This satisfies the invariance relation,

$$(k'^0)^2 - \mathbf{k}'^2 = (k^0)^2 - \mathbf{k}^2 = 0, \tag{10.17}$$

for electromagnetic waves. Since $k_x = |\mathbf{k}|\cos\theta = k^0\cos\theta$, we obtain the same Doppler shift as in special relativity[3]

$$k'^0 = k^0\left[\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\right], \tag{10.18}$$

consistent with the measurements obtained by Ives and Stilwell.[1]

10d. Lifetime dilation of cosmic-ray muons

The final experiment that we consider from the point of view of taiji and common relativity is the dilation of the lifetime of unstable muons produced in the upper atmosphere by cosmic rays. In 1941, B. Rossi and D. B. Hall published the results of a study in which they investigated the flux of muons with energies within a certain range at various altitudes.[5] The flux of muons with a particular range of energies is measured on a mountain top, a few kilometers above sea level. Since these muons travel at nearly the speed of light, the amount of time required for the muons to reach sea level can be computed and that number compared with the decay rate of muons at rest to obtain the expected muon flux at sea level. However, measurements made at sea level show a much larger flux of muons than expected from the calculations. It is as if, to the muons, the trip took much less time than was calculated. These measurements have been interpreted as confirmation of the relativistic time dilation effects resulting from the particular clock synchronization procedure of special relativity. Can the results of this experiment possibly be predicted by common relativity, in which a different clock synchronization procedure is used? As we shall see, the answer is yes.

10e. The cosmic-ray muon experiment and taiji relativity

As with the other experiments, because the spacetime transformations in taiji relativity are mathematically identical to special relativity, it is clear that taiji relativity must predict the correct outcome for this experiment. Again, however, we analyze this experiment in terms of natural units to expose what is truly revealed by the experimental results and to set the stage for how common relativity with its clock synchronization procedure that differs from the one in special relativity, can produce the same predictions as special relativity.

The cosmic ray muon experiment can be analyzed in two ways, one in terms of a simple dilation of the "decay length" of the muons and the second in terms of the transformation of transition probabilities as calculated by quantum field theory in general. We begin with the former analysis.

We consider an unstable particle such as a muon at rest in the F' frame ($\Delta x' = x'_2 - x'_1 = 0$) and moving with a velocity β as measured in the F frame. If the time interval between the creation and decay of the particle is $\Delta w'_o$ as measured by an F' observer, then the corresponding interval Δw as measured by an observer in F is

$$\Delta w = \gamma \Delta w'_o, \tag{10.19}$$

the same result as in special relativity. Because the unit second is not defined in taiji relativity, we can think of this result as a dilation of the "decay length" of the particle, where "decay length" means "the vacuum optical path length traversed by a light signal that travels a closed path in F'($\Delta x'=0$) between the time of creation and the time of decay of the particle." Such an optical length helps us understand the concept of "rest decay length" $\Delta w'_o$ for a particle decay at rest in taiji relativity.

10f. Decay-length dilation in quantum field theory and taiji relativity

We now turn our attention to an analysis of the muon decay from the point of view of transition probabilities in quantum field theory (see Appendix C). In the covariant formalism of quantum field theory based on taiji relativity, one starts with an invariant action and uses the taiji time w in a general inertial frame as the evolution variable for a state in the Schrödinger representation. Because the evolution of a physical system is assumed to be described by a Hamiltonian operator $H^{(S)}(w)$ with the same transformation property as the taiji-time w or the operator $\partial/\partial w$, the equation of a physical state $\Phi^{(S)}(w)$ in the Schrödinger representation is given by

$$i\,\frac{\partial\Phi^{(S)}(w)}{\partial w} = H^{(S)}(w)\Phi^{(S)}(w)\,, \qquad H^{(S)} = H_o^{(S)} + H_I^{(S)}. \tag{10.20}$$

The usual covariant formalism of perturbation theory can also be applied to quantum field theory by considering the interaction representation and the S-matrix based on taiji relativity.

To see the dilation of the mean decay length of an unstable particle, let us examine the decay rate $\Gamma(1\to 2+3+...+N)$ for a physical process $1\to 2+3+...+N$. This rate is given by

$$\Gamma(1\to 2+3+...+N) = \lim_{W\to\infty}\int \frac{1}{w}\left|\langle f\,|S|\,i\rangle\right|^2 \frac{d^3x_2 d^3p_2}{(2\pi)^3}\cdots\frac{d^3x_N d^3p_N}{(2\pi)^3}\,, \tag{10.21}$$

which has the dimensions of inverse length in taiji relativity. Since a particle's lifetime is measured in terms of taiji-time, which has the units of length, this decay rate can be interpreted as the inverse of the particle's "decay-length" or the mean distance which a group of such particles would travel before their number is decreased by a factor of 1/e=1/2.71828 due to decay. The decay length D is given by

$$D = 1/\Gamma(1\to 2+3+...+N)\,. \tag{10.22}$$

Thus, in taiji relativity, one has the "rest decay length" D_0 for a particle decay at rest, corresponding to the "rest lifetime" in the conventional theory. Also, instead of the dilation of the lifetime of a particle in flight, we have dilation of the distance it travels before decaying. Such a dilation is physically correct because it is equal to the experimentally determined distance traveled by the particles at high energies and at a speed very close to the speed of light.[5]

Let us consider a simple specific example, i.e., the muon decay $\mu^-(p_1) \to e^-(p_2)+\nu_\mu(p_3)+\bar{\nu}_e(p_4)$ with the usual V-A coupling. The decay S-matrix in the momentum space is given by

$$\langle f|S|i\rangle \propto (p_{01}p_{02}p_{03}p_{04})^{-1}\delta^4(p_1-p_2-p_3-p_4)M_{sc}, \tag{10.23}$$

$$M_{sc} = (G/\sqrt{2}\,)[\bar{\nu}_\mu(p_3)\gamma^\lambda(1-\gamma_5)\mu(p_1)][\bar{e}(p_2)\gamma_\lambda(1-\gamma_5)\nu_e(p_4)]\,.$$

The decay length, defined by $D=1/\Gamma(1\to 2+3+...+N)$, is found to be

$$\frac{1}{D} \propto \frac{1}{p_{01}}\int \frac{d^3p_2}{p_{02}}\frac{d^3p_3}{p_{03}}\frac{d^3p_4}{p_{04}}\delta^4(p_1-p_2-p_3-p_4)\sum_{spin}|M_{sc}|^2. \tag{10.24}$$

Everything to the right of $1/p_{01}$ in (10.24) is invariant under a taiji transformation so that the decay length D is indeed proportional to $p_{01} = m_1\gamma_1 = m_1/\sqrt{1-\beta_1^2}$.[5] We note that, when a particle at rest decays, its decay-length is not zero and can be expressed in terms of taiji-time w. It should be stressed that result (10.24) is sufficient to understand all previous experiments of the "lifetime dilation" of unstable particles because it is the decay-length of an unstable particle in flight which is the quantity that is directly measured in these experiments. For example, in the experiment described previously involving the counting of the muon flux at various altitudes above sea level, one is actually measuring the distance a group of muons travels in order for their number to be reduced by a certain factor and one then infers information about decay times from combining the distance information with calculated velocities of the muons.[6]

Similarly, the experimental setup in laboratories for measuring "lifetime dilation" is roughly as follows:

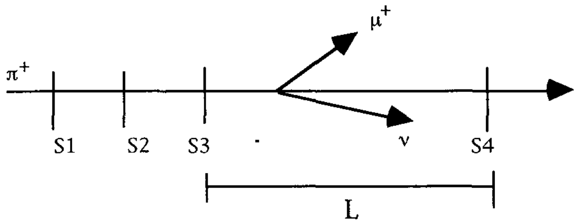

Fig. 10.1 Experimental arrangement for measuring the mean lifetime of particle decays in flight.

A narrow beam of unstable particles, for example, pions, passes through three detectors, S_1, S_2 and S_3. Some of the pions decay between detectors S_3 and S_4, according to the reaction $\pi^+ \to \mu^+ + \nu_\mu$. If the distance L between S_3 and S_4 is sufficiently large, the probability of a μ^+ entering the detector S_4 is negligible because the μ^+'s produced in the decay move out in all directions. The coincidence of signals in S_1, S_2 and S_3 indicates a high-speed charged particle passing all three detectors, i.e., moving in the horizontal beam direction. By counting these coincident signals, one has the total number of charged pions N_0 in the beam before decay. Similarly, the coincidence of signals in S_1, S_2, S_3 and S_4 gives the number of pions, N' remaining after traveling a length L in the horizontal direction of the beam. In taiji relativity, the relationship between N_0 and N' can be given in terms of taiji-time as,

$$N' = N_0 \exp[-\Gamma w'], \qquad \Gamma = \Gamma(\pi^+ \to \mu^+ + \nu_\mu), \tag{10.25}$$

where w' is the proper taiji-time of the pion (at rest in the F' frame, i.e., $\Delta x'=0$) moving from S_3 to S_4. This relationship can also be expressed in terms of the taiji-time w of the laboratory frame F,

$$w' = w\sqrt{1-\beta^2}, \tag{10.26}$$

where $\beta=dx/dw$ is the known taiji-velocity of the pion. Thus, using $w=L/\beta$, equation (10.25) can be written as

$$N' = N_0 \exp[-\Gamma w\sqrt{1-\beta^2}\ ] = N_0 \exp[-\Gamma\sqrt{1-\beta^2}\ L/\beta]\ . \tag{10.27}$$

Experimentally, one changes L and measures N' and N_0 for each L and then plots a graph of $\ln(N'/N_0)$ versus L to get a straight line with a slope of $-\Gamma\sqrt{1-\beta^2}\ /\beta$. From the slope and the known taiji-velocity β, one can then obtain the mean life of a positive pion,

$$w(\pi^+) = D(\pi^+) = 1/\Gamma = 780.45\ \text{cm}\ . \tag{10.28}$$

Since the time interval $\Delta t=1$ second corresponds to $\Delta w=299792458$ meters, the result (10.28) corresponds to the conventional pion mean life $\tau(\pi^+)=2.603\times10^{-8}$ seconds in special relativity.

Within the framework of special relativity with a universal speed of light c, the decay-length D can be converted to the decay lifetime τ by

$$\tau = \frac{D}{c} = \frac{1}{c\Gamma} \tag{10.29}$$

in any frame. In special relativity, both the decay-length and the lifetime are dilated. However, within the framework of taiji relativity with its use of natural units, one can only talk about the dilation of the decaylength. The dilation of the decay length is completely relative within the framework of taiji relativity and special relativity.

10g. Cosmic-ray muons and common relativity

In common relativity, because all observers use the same set of clocks, the time interval between the creation of a muon and its subsequent decay has the same value (measured in common-seconds) in all inertial frames. Can common relativity also predict correctly the result of the cosmic-ray muon experiment?

Conceptually, it is important to understand the role of common time in this situation. Suppose that the F frame is at rest relative to Earth and that there is a muon μ' at rest in the F' frame. In common time, the lifetime $\Delta t_c'[\mu']$ of this muon measured by an F'-observer is the same as the lifetime $\Delta t_c[\mu']$ of the same muon as measured by an F-observer on Earth

$$\Delta t'_c[\mu'] = \Delta t_c[\mu']. \qquad (10.30)$$

Now, suppose there is a second muon μ which is at rest relative to Earth and the F frame. In common time, the lifetime $\Delta t'_c[\mu]$ of this muon as measured by an F'-observer is the same as the lifetime $\Delta t_c[\mu]$ of the same muon as measured by an F-observer

$$\Delta t'_c[\mu] = \Delta t_c[\mu]. \qquad (10.31)$$

Experimentally, we know that $\Delta t_c[\mu']$ must be different from $\Delta t_c[\mu]$. In fact, those two time intervals must differ by a factor dependent on the speed of the muon in flight and this inequality implies that the time interval $\Delta t'_c[\mu']$ which represents the lifetime (in common seconds) of a muon at rest in F' measured by an F' observer, must differ from $\Delta t_c[\mu]$, which represents the lifetime (in common seconds) of a muon at rest in F measured by an F observer. Although this may appear at first to violate the principle of relativity, it actually does not because the principle of relativity implies only that the laws of physics be invariant in all inertial frames, and not the human convention used to define units.

To clarify this point, consider the intervals $\Delta w = \Delta(bt_c)$ associated with the situation. Because this interval is associated with an actual physical quantity (i.e., the distance traveled by a light signal in a vacuum during a particular time interval) the principle of relativity implies that $\Delta w'[\mu'] = \Delta w[\mu]$. The relationship between $\Delta w[\mu]$ and $\Delta t_c[\mu]$ is determined by how human beings have defined the unit of time (second or common-second) relative to the unit of length. In common relativity, $\Delta w[\mu]$ and $\Delta t_c[\mu]$ are related by the factor c, $\Delta w[\mu] = c\Delta t_c[\mu]$. However, $\Delta w'[\mu']$ and $\Delta t_c'[\mu']$ are not so related because the speed of light is not isotropic in F'. Because the value of any intervals Δt_c are purely the result of how humans have decided to mark off the dials on a clock and to set its rate of ticking, the relationship between different Δt_C's is not governed by the principle of relativity. Instead, as we shall see in chapters 13-16, it can be set to whatever makes the calculations for the problem at hand the most convenient to perform.

To see that common relativity makes the correct experimental prediction of the dilation of the decay lifetime of muons in motion, we must consider the interval $\Delta w = \Delta(bt_C)$, rather than Δt_C. The analysis then is identical to that under taiji relativity. If a muon μ' at rest relative to the F' frame decays with a 'lifetime' $\Delta w'[\mu']$ as measured by an F' observer, then the decay 'lifetime' of that same muon $\Delta w[\mu']$ as measured by an observer in F is related to $\Delta w'[\mu']$ by

$$\Delta w[\mu'] = \gamma \Delta w'[\mu']. \tag{10.32}$$

Since the principle of relativity implies that $\Delta w'[\mu'] = \Delta w[\mu]$, i.e., the inertial frames F' and F are equivalent, then we obtain the familiar result from special relativity

$$\Delta w[\mu'] = \gamma \Delta w[\mu]. \tag{10.33}$$

10h. Quantum field theory and the decay length in common relativity

Within the 4-dimensional symmetry framework of common relativity, the usual covariant formalism of perturbation theory can also be applied to quantum field theory by considering the interaction representation and the S-matrix. If one calculates scattering cross sections and decay rates (with respect to the lightime $w=bt_C$) of a physical process, one will get formally the same result as that in conventional quantum field theory. For example, one has the equations (10.20)-(10.29), in which the taiji time w, if any, is replaced by the lightime bt_C in a general inertial frame.

To be specific, the mean decay length of an unstable particle in the decay process $1\rightarrow 2+3+...+N$ is given by the decay rate $\Gamma(1\rightarrow 2+3+...+N)$

$$\Gamma(1\to 2+3+...+N) = \lim_{w\to\infty} \int \frac{|\langle f|S|i\rangle|^2}{w} \frac{d^3x_2 d^3p_2}{(2\pi J)^3} \cdots \frac{d^3x_N d^3p_N}{(2\pi J)^3}, \quad w=bt_c. \quad (10.34)$$

The quantity $\Gamma(1\to 2+3+...+N)$ has the dimension of inverse length. Its inverse is particle's "lifetime" measured in terms of the lightime bt_c which has the dimension of length and hence, may be called "decay length." The decay length D is given by

$$D = 1/\Gamma(1\to 2+3+...+N). \quad (10.35)$$

Thus, in common relativity, one has the "rest decay length" D_0 for the decay of a particle at rest, corresponding to the "rest lifetime" multiplied by the speed of light in the conventional theory. The results in equations (10.34) and (10.35) imply that common relativity also predicts the dilation of the decay-length, consistent with experimental results.

References

1. H. E. Ives and G. R. Stilwell, J. Opt. Soc. Am. **28**, 215 (1938); **31**, 369 (1941). A hydrogen discharge tube is the source of the ions H_2^+ and H_3^+. After acceleration, these ions could produce neutral but still excited hydrogen atoms, which emit the Balmer lines.
2. The integer n is identical with the principal quantum number in non-relativistic quantum mechanics. For detailed calculations for the hydrogen atom, see J. J. Sakurai, Advanced Quantum Mechanics (Addison-Wesley, Reading, MA, 1967) pp. 122-127.
3. The numerical value of β in taiji relativity (and common relativity) is the same as that of V/c in special relativity.
4. We use the † symbol to avoid possible confusion with the ' symbol which is usually employed in the Lorentz transformations of two wave four-vectors such as those in (10.11). The symbols ω and ω' denote quantities of same atomic system as measured by different observers, i.e., F and F' observers. For a discussion of universal and fundamental constants in taiji and common relativities, see Appendix A.
5. The dimensionless velocity β_1 in taiji relativity has the same numerical value as (v_1 / c) for particle 1 in special relativity. See chapter 12 for a detailed discussion of the relativistic energy and momentum for a particle. Particle Data Group, *Particle Physics Booklet* (July, 1996, American Institute of Physics) p. 11. A muon's mean life τ is 2.197×10^{-6} seconds and its decay-length is given by $c\tau$=658.654 meters. Recently, the lifetime (or the decay-length) dilatation predicted by the 4-dimensional symmetry of the Lorentz group has been tested and confirmed by measuring the decay lifetime of k_s^0 in flight at several hundreds of GeV (i.e.,$\gamma \sim 10^3$.) See N. Grossman, K. Heller, C. James, et al, Phys. Rev. Lett. **59**, 18 (1987).
6. See for example, A. P. French, *Special Relativity* (W. W. Norton & Company, New York, 1968) pp. 98-103.

11.

Group Properties of Taiji Relativity and Common Relativity

11a. General group properties

Lorentz and Poincaré invariance is closely intertwined with the group properties of the 4-dimensional coordinate transformations. Because there is an obvious one-to-one correspondence between the 4-dimensional transformations of taiji relativity (7.4) or common relativity (8.3) and the Lorentz transformation (6.5), it seems mathematically trivial that the transformations (7.4) and (8.3) form a Lorentz group, just as the Lorentz transformations do. However, the new concepts behind the taiji transformations (e.g., that there are infinitely many possibilities for specifying the time t using a unit other than the meter through the relations w=bt and w'=b't') are highly nontrivial.

In this chapter, we show that the taiji transformations, which are derived solely from the first postulate of special relativity, (i.e., the Poincaré-Einstein principle of relativity) have precisely the properties of the Lorentz group and the Poincaré group. This is crucial[1] for the formulation of quantum field theories and particle physics on the basis of taiji relativity.

Once the 4-dimensional group properties of the taiji transformations are established it follows that, regardless of one's definition of the ligh function b one always has a valid 4-dimensional symmetry framework with which to understand and investigate phenomena in the physical world.

Mathematically, a group consists of a set of elements (or operators) $G^o = \{g_0, g_1, ...g_k, ..\}$ and an operation (also known as a "multiplication rule" and denoted by a •) such that:

(a) Combining any two elements g_i and g_k using the operation leads to another elements g_n in the set G^o,

$$g_i \bullet g_k = g_n.$$

(b) There exists a unit element g_0 in G^o such that for any element g_i, one has the relationship

$$g_0 \bullet g_i = g_i \bullet g_0 = g_i.$$

(c) For any element g_k, there exists an inverse element $g_f = g_k^{-1}$ such that

$$g_k \bullet g_k^{-1} = g_k^{-1} \bullet g_k = g_0.$$

(d) The operation obeys the associative rule,

$$(g_i \bullet g_k) \bullet g_n = g_i \bullet (g_k \bullet g_n).$$

The most important groups in fundamental physics are those related to geometrical or physical transformations.[2] For the Lorentz group, these elements are the transformation matrices (11.15) below. The Lorentz group can be defined as the set of all 4×4 real matrices that leave the 4-dimensional interval $w^2-x^2-y^2-z^2$ invariant. These matrices are singled-valued and continuous functions of six parameters; three angles for rotations and three velocities for motion in 3-dimensional space. The values of these parameters are continuous, so that the Lorentz group is a continuous or Lie group.

Let us consider the group properties of the taiji transformations which relate two inertial frames, F and F', which have mutually parallel axes. First, we consider the special case where the relative motion between the F and F' frames is characterized by a dimensionless constant taiji-velocity β, measured in terms of taiji-time w, along parallel x and x' axes. The origins of F(w,x,y,z) and F'(w',x',y',z') coincide at the taiji-time w=w'=0. In addition to these two standard reference frames, let us also consider a third frame F"(w",x",y",z"), with axes similarly oriented, which moves with a constant taiji-velocity $(\beta^1,0,0)$ as measured in terms of the taiji-time w by observers in the F frame. For simplicity, we shall call β or β^1 the velocity. The constant velocity of F" measured by observers in F' is then denoted by $\boldsymbol{\beta}'=(\beta'^1,0,0)$ and is related to β and β^1 by (7.8):

$$\beta'^1 = \frac{\beta^1 - \beta}{1-\beta\beta^1}, \qquad \beta'^1 = u'_x, \qquad \beta^1 = u_x. \tag{11.1}$$

In analogy with equation (7.4), the 4-dimensional transformations between F and F" are given by

$$w'' = \gamma_1(w - \beta^1 x), \qquad x'' = \gamma_1(x - \beta^1 w), \qquad y'' = y, \qquad z'' = z; \tag{11.2}$$

$$\gamma_1 = \frac{1}{\sqrt{1-(\beta^1)^2}} .$$

The inverse transformations of (7.4) can be easily obtained and are:

$$w = \gamma(w'+\beta x'), \qquad x = \gamma(x'-\beta w'), \qquad y = y', \qquad z = z'; \tag{11.3}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

Based on (11.2), (11.3) and (7.8), one can show that the taiji transformations between F' and F" are

$$w'' = \gamma'_1(w'-\beta'^1 x'), \qquad x'' = \gamma'_1(x'-\beta'^1 w'), \qquad y'' = y', \qquad z'' = z'; \tag{11.4}$$

$$\beta'^1 = \frac{\beta^1 - \beta}{1-\beta^1\beta}, \qquad \gamma'_1 = \gamma\gamma_1(1-\beta^1\beta) = \frac{1}{\sqrt{1-(\beta'^1)^2}} ,$$

which have the same form as the taiji transformations between F and F', or F and F".

Other group properties, such as the existence of an identity transformation [e.g., $\beta=0$ in (11.3)] , the existence of an inverse transformation [e.g., (11.3) is the inverse transformation of (7.4)] and the fact that the transformations obey the associative rule, can be verified. These properties and the result (11.4) show that the set of 4-dimensional taiji transformations forms precisely the Lorentz group.[3] These 4-dimensional group properties are critical to the ability of taiji relativity to predict correctly the results of all known experiments.

This demonstrates that the Lorentz group can actually accommodate a wide class of different concepts of the time t (measured in a unit other than meter) and that postulating the speed of light (in meters per second) to be a universal constant is not a requirement for transformations to form such a group.

11b. Lorentz group properties

In the previous section, we discussed group properties of the taiji transformations for inertial frames with relative motion solely along the x and x' axes. In general, the taiji transformations relate quantities in inertial frames with relative motion along x, y, and z directions that leave the 4-dimensional interval $w^2-x^2-y^2-z^2$ invariant. Consequently, the contents of the Lorentz group are much richer than what we have previously discussed. There are many different transformations which leave $w^2-x^2-y^2-z^2$ invariant. For example, there is a subset involving three discrete transformations: a spatial inversion ($w \to w$, $\mathbf{r} \to -\mathbf{r}$), a time inversion ($w \to -w$, $\mathbf{r} \to \mathbf{r}$) and a spacetime inversion ($w \to -w$, $\mathbf{r} \to -\mathbf{r}$). Another subset is a six-parameter (three angles for rotations and three constant velocities for boosts) continuous group, which was first noted and discussed by Poincaré in 1905.

As stated before, the Lorentz group is defined as the set of all 4×4 real matrices that leave the interval s^2 invariant, where

$$s^2 = w^2 - x^2 - y^2 - z^2 = \eta_{\mu\nu} x^\mu x^\nu = x_\mu x^\mu; \qquad (11.5)$$

$$\eta_{00} = 1, \qquad \eta_{11} = \eta_{22} = \eta_{33} = -1, \qquad \eta_{\mu\nu} = 0 \qquad \text{for} \qquad \mu \neq \nu,$$

$$x^\mu = (w, \mathbf{r}), \qquad x_\mu = \eta_{\mu\nu} x^\nu = (w, -\mathbf{r}), \qquad \mu, \nu = 0,1,2,3. \qquad (11.6)$$

The metric tensors $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ satisfy the relation $\eta_{\mu\nu}\eta^{\mu\rho} = \delta_\nu{}^\rho$.

If x_1^μ and x_2^μ denote two coordinate 4-vectors (related by the taiji transformations), then we have the following three types of invariant intervals in the 4-dimensional spacetime:

$$\Delta s^2 = (x_{2\mu} - x_{1\mu})(x_2{}^\mu - x_1{}^\mu) > 0: \quad \text{a "time-like" interval,} \tag{11.7}$$

$$\Delta s^2 = (x_{2\mu} - x_{1\mu})(x_2{}^\mu - x_1{}^\mu) = 0: \quad \text{a "light-like" interval,} \tag{11.8}$$

$$\Delta s^2 = (x_{2\mu} - x_{1\mu})(x_2{}^\mu - x_1{}^\mu) < 0: \quad \text{a "space-like" interval.} \tag{11.9}$$

Physically, these invariant expressions describe the invariant laws of motion for three types of particles:

(11.7) for massive particles, $m^2 > 0$,

(11.8) for massless particles, $m^2 = 0$,

(11.9) for tachyons (moving faster than the speed of light $\beta_L=1$), $m^2 < 0$.

The first two laws have been experimentally demonstrated. However, it is not known why the third law (11.9) for "tachyons" is not physically realized in nature[4] or why tachyons, if they exist at all, have no interaction with ordinary particles.

Let us consider the continuous Lorentz group with six parameters. With the help of tensor notation,[3] the general 4-dimensional taiji transformations can be parameterized as follows,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{11.10}$$

where we have employed the summation convention for repeated indices. Any quantity Q^μ with four components that satisfy the transformation (11.10) (i.e., $Q^\mu=\Lambda^\mu{}_\nu Q^\nu$) is called a contravariant 4-vector. The transformation tensor $\Lambda^\mu{}_\nu$ and the metric tensor $\eta_{\alpha\beta}$ satisfy the relation

$$\eta_{\mu\nu} = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu}\eta_{\alpha\beta}. \tag{11.11}$$

The metric tensors $\eta_{\alpha\beta}$ and $\eta^{\alpha\beta}$ can be used to raise and lower the indices of any tensor. (For example, a covariant 4-vector Q_μ is, by definition, related to the corresponding contravariant vector Q^ν by $Q_\mu = \eta_{\mu\nu}Q^\nu$ and in general, one has $A^{\cdots\mu\rho\cdots\alpha} = \eta^{\mu\nu}A^{\cdots}{}_{\nu}{}^{\rho\cdots\alpha}$.) The metric of the Lorentz group, given in (11.6), is $\eta_{\mu\nu}$= (1,-1,-1,-1) which has three negative signs and one positive sign. Thus, the Lorentz group is denoted by O(3,1).

The 4-dimensional taiji transformation (11.10) can be written in matrix form and $(\Lambda^{\mu}{}_{\nu})$ can be considered a 4×4 matrix. (See (11.15) below.) The matrix form of (11.11) is $\eta = \Lambda^T \eta \Lambda$, where the superscript T denotes the transpose of a matrix. The set of matrices $\{\Lambda\} = \{\Lambda, \Lambda', \Lambda'', ...\}$ leaves the interval s^2 in (11.5) invariant and can be considered elements of the Lorentz group. Indeed, one can verify that $\{\Lambda\}$ satisfies the group properties mentioned previously in section 11a.

The anti-symmetric operator $L^{\mu\nu}$ of the Lorentz group was first discussed by Poincaré in his Rendiconti paper. It is defined as

$$L^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}), \qquad \partial^{\mu} = \partial/\partial x_{\mu}. \tag{11.12}$$

It can be verified that $L^{\mu\nu}$ generates the algebra of the Lorentz group,

$$[L^{\mu\nu}, L^{\alpha\beta}] = i(\eta^{\mu\beta}L^{\nu\alpha} + \eta^{\nu\alpha}L^{\mu\beta} - \eta^{\mu\alpha}L^{\nu\beta} - \eta^{\nu\beta}L^{\mu\alpha}). \tag{11.13}$$

We can parameterize the matrix Λ of the taiji transformations (11.10) in terms of ϕ by using equation (7.4):

$$\begin{aligned} w' &= \gamma(w - \beta x) = w\cosh\phi - x\sinh\phi, \\ x' &= \gamma(x - \beta w) = x\cosh\phi - w\sinh\phi, \end{aligned} \tag{11.14}$$

$$y' = y, \qquad z' = z;$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\phi, \qquad \gamma\beta = \sinh\phi.$$

The Lorentz group is non-compact because the range of the parameter β does not include the endpoint 1. If the metric tensor is $\overline{g}_{\mu\nu}=(1,1,1,1)$ [i.e., $\overline{s}^2=\overline{g}_{\mu\nu}x^\mu x^\nu=w^2+x^2+y^2+z^2$] instead of that in (11.5)), then the group would be the 4-dimensional orthogonal group O(4) rather than O(3,1). Since the coordinates (w,x,y,z) are real numbers, the group O(4) is actually the 4-dimensional rotational group. The parameters of the group O(4) have finite ranges that include the endpoints and hence the group O(4) is compact. Mathematically, the 4-dimensional space associated with O(4) is the perfect generalization of the ordinary 3-dimensional space with the distance $x^2+y^2+z^2$. However, such a perfect generalization is not perfect from the physical viewpoint. The reason is that the invariant interval $\overline{s}^2=\overline{g}_{\mu\nu}x^\mu x^\nu=w^2+x^2+y^2+z^2$ has only a geometric meaning and has nothing to do with laws of physics. This is in sharp contrast with the invariant interval (11.5) or (11.7)-(11.9), which is also applicable to the laws of physics.

The taiji transformation (11.14) can also be written in matrix form

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \equiv \Lambda(10,\phi) \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}, \tag{11.15}$$

where the transformation matrix $\Lambda(10,\phi)$ denotes a rotation in the wx (or x^0x^1) plane in the 4-dimensional spacetime. For an infinitesimal transformation, Λ, in (11.15) takes the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon\, \lambda^\mu{}_\nu, \qquad \lambda^{\mu\nu} = -\lambda^{\nu\mu}. \tag{11.16}$$

The infinitesimal generator m^{10} for the rotation in (11.15) is defined as

$$m^{10} = -i\frac{d}{d\phi}\Lambda(10,\phi)\Big|_{\phi=0} = -i\begin{pmatrix}0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{11.17}$$

Similarly, for rotations in the x^0x^2 and x^0x^3 planes, the infinitesimal generators are given by

$$m^{20} = -i\begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \qquad m^{30} = -i\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}. \tag{11.18}$$

In equations (11.1)-(11.4), we see that two boosts in the x direction generate another boost. However, a boost in the x direction followed by a boost in the y direction does not generate another boost. Instead, it generates a spatial rotation. In fact, this is the physical origin of the Thomas precession.[5] Therefore, we must also introduce three generators for spatial rotations in order to have a closed algebra. This important property was recognized by Poincaré, but not by Einstein, in their original work on relativity in 1905. The infinitesimal generators for spatial rotations are

$$m^{12} = -i\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \qquad m^{23} = -i\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix},$$

$$m^{31} = -i\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}. \tag{11.19}$$

In general, the infinitesimal generator $m^{\mu\nu}$ is defined to satisfy the relation $m^{\mu\nu}=-m^{\nu\mu}$. An arbitrary infinitesimal taiji transformation can be written as

$$\Lambda(\omega) = 1 + \frac{i}{2}\omega_{\mu\nu}m^{\mu\nu}, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{11.20}$$

A finite rotation in the $\mu\nu$ plane (in the sense μ to ν) is given by exponentiation

$$\Lambda(\mu\nu, \phi) = \exp(i\phi m^{\mu\nu}). \tag{11.21}$$

It can be shown that the generators $m^{\mu\nu}$ satisfy the commutation relation

$$[m^{\mu\nu}, m^{\alpha\beta}] = i(\eta^{\mu\beta}m^{\nu\alpha} + \eta^{\nu\alpha}m^{\mu\beta} - \eta^{\mu\alpha}m^{\nu\beta} - \eta^{\nu\beta}m^{\mu\alpha}). \tag{11.22}$$

Let us define the spatial rotational generators J^i and boost-generators K^i as follows:

$$J^i = \frac{1}{2}\varepsilon^{ijk}L^{jk}, \qquad k^i = L^{0i}, \tag{11.23}$$

where $L^{\mu\nu}$ given in (11.12) are the generators of the Lie algebra of the Lorentz group. They satisfy the commutation relation (11.13) which is the same as that of $m^{\mu\nu}$ in (11.22). One can verify the following commutation relations:

$$[J^i, J^j] = i\varepsilon^{ijk}J^k, \tag{11.24}$$

$$[K^i, K^j] = -i\varepsilon^{ijk}J^k, \tag{11.25}$$

$$[J^i, K^j] = i\varepsilon^{ijk}K^k. \tag{11.26}$$

If one makes the following linear combinations of these rotational and boost generators,

$$M^k = \frac{1}{2}(J^k + iK^k), \tag{11.27}$$

$$N^k = \frac{1}{2}(J^k - iK^k), \tag{11.28}$$

one can show that M^i and N^k commute,

$$[M^i, N^k] = 0, \tag{11.29}$$

so that the algebra of the Lorentz group has been split into two pieces. Each piece generates a separate group called SU(2). One can use the irreducible representations of SU(2) to construct representations of the Lorentz group.[6]

11c. Poincaré group properties

In order to have the most general transformations in flat 4-dimensional spacetime, one can generalize (11.10) to include translations such as

$$x'^\mu = \Lambda^\mu_\nu x^\nu + b^\nu, \tag{11.30}$$

where b^ν is a constant and real 4-vector. These are known as the Poincaré transformations. Note that (11.30) does not leave the 4-dimensional interval (11.5) invariant. Nevertheless, the interval Δs^2 in (11.7)-(11.9) is still invariant. The set of transformations (11.30) forms a Poincaré group with 10 generators (3 angles for rotations, 3 constant velocities for boosts and 4 constants in b^ν for spacetime translations.) Thus, in addition to the six generators $L_{\mu\nu}$ given by (11.12), there are four translation generators $P_\mu = i\partial_\mu$ (with a suitable choice of units, J=1). These generators satisfy the following commutation relations:

$$[L^{\mu\nu}, L^{\alpha\beta}] = i(\eta^{\mu\beta}L^{\nu\alpha} + \eta^{\nu\alpha}L^{\mu\beta} - \eta^{\mu\alpha}L^{\nu\beta} - \eta^{\nu\beta}L^{\mu\alpha}), \tag{11.31}$$

$$[L_{\mu\nu}, P_\alpha] = i(\eta_{\nu\alpha}P_\mu - \eta_{\mu\alpha}P_\nu), \qquad (11.32)$$

$$[P_\mu, P_\nu] = 0. \qquad (11.33)$$

This is the Lie algebra of the Poincaré group (or the Poincaré algebra).

Based on these generators, one can construct two invariant (or Casimir) operators:

$$P_\mu P^\mu \qquad \text{and} \qquad W_\mu W^\mu, \qquad (11.34)$$

where

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}P_\nu L_{\alpha\beta}. \qquad (11.35)$$

The tensor $\varepsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric unit 4-tensor of the fourth rank and satisfies $\varepsilon^{0123}=1$. Its components change sign upon interchange of any two indices, so that the components different from zero are equal to ± 1.

The two invariant operators in (11.34), $P_\mu P^\mu$ and $W_\mu W^\mu$, commute with all generators of the Poincaré group. The operator $P_\mu P^\mu$ has a clear physical interpretation. Namely, it is the square of the mass of a particle. It is easier to see the meaning of $W_\mu W^\mu$, if one makes a taiji transformation to the rest frame of the particle. In this case, the eigenvalues of P_μ are given by (m,0,0,0) and we have

$$W^0 = 0, \qquad W^i = \frac{1}{2}m\ \varepsilon^{ijko}L_{jk}, \qquad (11.36)$$

where W^i/m, i=1,2,3 are simply the usual rotational matrices in the three spatial dimensions which obey the angular momentum commutation relations. Since the particle is at rest, W^i is just the spin operator. The eigenvalues of the square of $\mathbf{W}=(W^i)$ are therefore given by

$$W^2 = m^2 s(s+1), \qquad m > 0, \tag{11.37}$$

where s is the spin eigenvalue of a particle, s = 0, 1/2, 1, 3/2, 2,... . Thus, if $m > 0$, a particle with spin s has 2s+1 components (i.e., 2s+1 independent states for a given momentum 4-vector). However, if m = 0, a particle with a spin s can have only 2 components. For example, the photon has spin s=1, but only two independent polarization states. This property of the photon is a purely kinematical property dictated by the 4-dimensional symmetry of Lorentz and Poincaré groups.

It is interesting that all particles in the physical world can be classified according to the eigenvalues of the two invariant operators in (11.34):

$$P_\mu P^\mu > 0, \qquad s = 0, 1, 2, 3, \ldots \text{(bosons)}, \qquad s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \text{(fermions)}, \tag{11.38}$$

$$P_\mu P^\mu = 0, \qquad \text{spin} = \pm s, \tag{11.39}$$

$$P_\mu P^\mu = 0, \qquad \text{continuous } s, \tag{11.40}$$

$$P_\mu P^\mu < 0, \qquad \text{(tachyons)}. \tag{11.41}$$

Particles with the properties in (11.40) and (11.41) have not been detected experimentally. Nevertheless, it is of great significance to the Poincaré group that all particles (or fields) can be classified using the eigenvalues of the two invariant operators, one related to the mass of the particle and the other to its spin.[7] In contrast, the Lorentz group does not have the translation operator P_μ (or $P_\mu P^\mu$) and so is inadequate for the classification of elementary particles. Mass and spin are two fundamental properties of elementary particles. The fact that particles have a discrete spin can be understood on the basis of spacetime symmetry.[8] Also, the CPT invariance for all interactions[9] of particles is closely related to Lorentz invariance. However, the fact that elementary particles (such as quarks and leptons) also have discrete masses is far from understood.[10]

These discussions for Lorentz and Poincaré groups hold for both the 4-dimensional transformations in taiji relativity and common relativity.

References

1. Although the form of transformations with 4-dimensional symmetry is simple, its significance to the formulation of particle physics and quantum field theory has not been fully appreciated by some authors who have attempted to modify the synchronization procedure for clocks and/or to test a different ideas of time or the speed of light. For example, Winnie and Edwards attempted to obtain new transformations of space and time. However, the resultant transformations they obtained are untenable because the 4-dimensional symmetry of the Lorentz and Poincaré groups was lost. See discussions in chapter 17.
2. See, for example, P. Roman, *Theory of Elementary Particles* (2nd. ed., North-Holland, Amsterdam, 1961), pp. 8-51; Wu-Ki Tung, *Group Theory in Physics* (World Scientific, Singapore, 1985) pp. 173-211.
3. See, for example, J. D. Jackson, *Classical Electrodynamics* (3rd ed., John Wiley & Sons, New York, 1999) pp. 539-548.
4. For a discussion of tachyons, see G. Feinberg, Phys. Rev. 159, 1089 (1967); E. C. G. Sudarshan, in *The Encyclopedia of Physics* (edited by R.M. Besancon, Van Nostrand Reinhold, New York, 1974) p. 309 and references therein.
5. J. D. Jackson, ref. 3, pp. 548-553.
6. S. Weinberg, *The Quantum Theory of Fields, I. Foundations* (Cambridge Univ. Press, New York, 1995) pp. 49-74; M. Kaku, *Quantum Field Theory, A Modern Introduction* (Oxford Univ. Press, New York, 1993) pp. 50-56.
7. The physical content of the Poincaré group is richer than that of the Lorentz group and has not been fully explored. See E. P. Wigner, Ann. Math. **40**, 149 (1939); E. Inönü and E. P. Wigner, Nuovo Cimento IX, 705 (1952); V. Bargmann, Ann. Math. **59**, 1 (1954); S. Weinberg, ref. 5, pp. 55-81
8. In the world of elementary particles and quantum fields, there is a necessary and fundamental connection between spin and statistics. This can be understood with the help of the 4-dimensional symmetry of the Lorentz group together with other requirements such as local fields, unique ground states, and microscopic causality. See J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965) pp. 170-172.

9. For discussions of CPT invariance and weak interactions, see J. P. Hsu, Phys. Rev. D**5**, 981 (1972); J. P. Hsu and M. Hongh, Phys. Rev. D**6**, 256 (1972).
10. This suggests that symmetry principles alone are inadequate for understanding the phenomenon of mass and that perhaps, interactions of particles should be taken into consideration. However, it is puzzling that the electron and the muon appear to have exactly the same interactions (except for the "extremely" weak gravitational interaction), yet the muon mass is about two hundred times larger than the electron mass.

12.

Invariant Actions in Relativity Theories and Truly Fundamental Constants

12a. Invariant actions for classical electrodynamics in relativity theories

Thus far, we have discussed different relativity theories such as taiji relativity, common relativity and special relativity. Later, in chapter 17, we will see how special relativity is a special case of extended relativity, which is based on the principle of relativity with a universal two-way speed of light. All these relativity theories possess the four-dimensional spacetime symmetry that is equivalent to the Lorentz and Poincaré invariance. Therefore, we can develop Lorentz invariant actions for classical and quantum electrodynamics in these theories.

In taiji relativity and common relativity, the speed of light (measured in units of meter per second) is either undefined or not an invariant quantity, so the invariant action for a free particle in, for example, the F' frame *cannot* be written in the usual form $-\int mc'ds'$, where $ds'^2 = dx'_\mu dx'^\mu$. Nevertheless, $mds' = mds$ *is* invariant because the interval ds (=ds') and the (rest) mass m are scalars. In all relativity theories, regardless of whether or not the speed of light (in meter per second) is universal constant, the invariant action S_{CED} for classical electrodynamics (CED) can be assumed to take the following form,[1]

$$S_{CED} = \int \{-mds - \bar{e}a_\mu dx^\mu\} - \frac{1}{4}\int f^{\mu\nu}f_{\mu\nu}d^3r dw, \tag{12.1}$$

$$x^\mu = (w,\mathbf{r}), \qquad x_\mu = \eta_{\mu\nu}x^\nu = (w,-\mathbf{r}), \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu,$$

$$ds^2 = dx^\mu dx_\mu = [dw]^2 - [d\mathbf{r}]^2, \qquad \bar{e} = -1.6021891\times 10^{-20}\sqrt{4\pi}\ (g\cdot cm)^{1/2},$$

where x^μ is the coordinate 4-vector, a_ν is the electromagnetic 4-potential, and ds^2 is the differential form of the 4-dimensional interval. The electric charge $\bar{e}$ is a universal constant, which happens to be the same thing as the charge expressed in Heaviside-Lorentz (rationalized) units, If the time variable t (measured in seconds) is introduced in a relativity theory, then one has w=bt and w'=b't', where b is the ligh function (e.g., in special relativity, b=b'=c). The action (12.1) has different dimensions from the usual action in special relativity. Comparing (12.1) with the corresponding action S_{sr} in special relativity for the F frame, we see that $S=S_{sr}/c, \bar{e}=e/c$ (e in esu) and $a_\mu=a_\mu(w,\mathbf{r})=A_\mu(w,\mathbf{r})/c$, where A_μ is the usual electromagnetic vector potential in special relativity. Note however that (12.1) is more general in that it can be used as the action in any relativity theory, including special relativity.

In preparation for the discussions in chapters 13-16 related to common relativity, let us first consider the action (12.1) within the framework of common relativity. As mentioned before, common relativity has two universal constants in classical and quantum electrodynamics, J and $\bar{e}$, given in (10.7).

In a general reference frame, the 4-coordinates are denoted by (w,x,y,z), where $w=bt_c$. The invariant action for a charged particle interacting with the electromagnetic field is given by the first two terms in (12.1):[2]

$$S_c=\int \{-m ds - \bar{e}a_\mu dx^\mu\} = \int L_c dt_c , \qquad a_\mu = (a_0, -\mathbf{a}), \tag{12.2}$$

$$L_c = -m\sqrt{C^2-\mathbf{v}^2} - \bar{e}a_0 C + \bar{e}\mathbf{a}\cdot\mathbf{v}, \qquad \frac{ds}{dt_c}=\sqrt{C^2-\mathbf{v}^2}, \quad C \equiv \frac{d(bt_c)}{dt_c}.$$

In the following discussion, we use C, which is in general not a constant, as a shorthand for $d(bt_c)/dt_c$. The canonical momentum $\mathbf{P}$ and the Hamiltonian H_c for a charged particle are defined, as usual, by

$$\begin{aligned} \mathbf{P} &= \frac{\partial L_c}{\partial \mathbf{v}} = \mathbf{p} + \bar{e}\mathbf{a}, \\ H_c &= [(\partial L_c/\partial \mathbf{v})\cdot\mathbf{v} - L_c] = Cp_0 + \bar{e}a_0 C, \end{aligned} \tag{12.3}$$

$$p_0 = \frac{m}{\sqrt{1-v^2/C^2}}, \qquad \mathbf{p} = \frac{mv/C}{\sqrt{1-v^2/C^2}}, \qquad p^\mu p_\mu = m^2,$$

where $p^\mu=(p^0,\mathbf{p})$ and $p_\mu=(p_0,p_1,p_2,p_3)=(p^0,-\mathbf{p})$ are the 4-momenta of the particle. Note that the canonical momentum $(P^0,\mathbf{P})$, where $P^0=H_c/C$, also forms a 4-vector. The Lagrange equation of motion for a charged particle in the electromagnetic field can be derived from (12.2) by the variational calculus. One has

$$\frac{dp^\mu}{ds} = \bar{e} f^{\mu\nu} \frac{dx_\nu}{ds}, \tag{12.4}$$

$$p^\mu = m\frac{dx^\mu}{ds} = \left(\frac{m}{\sqrt{1-v^2/C^2}}, \frac{mv/C}{\sqrt{1-v^2/C^2}}\right),$$

where $p^\mu=(p^0,\mathbf{p})$ is the usual momentum for a particle. Multiplying both sides of (12.4) by ds/dt_c, one obtains the equation of motion expressed in terms of the common time t_c,

$$\frac{dp^\mu}{dt_c} = \bar{e} f^{\mu\nu} \frac{dx_\nu}{dt_c}, \tag{12.5}$$

$$\text{i.e.,} \quad \frac{dp^0}{dt_c} = \bar{e}\mathbf{E}\cdot\mathbf{v}, \qquad \frac{d\mathbf{p}}{dt_c} = \bar{e}(\mathbf{E}C + \mathbf{v}\times\mathbf{B}),$$

where p^0 and $\mathbf{p}$ are given in (12.4) and $\bar{e}(\mathbf{E}C+\mathbf{v}\times\mathbf{B})$ is the Lorentz force in common relativity.

For a continuous charge distribution in space, the second term in (12.1) becomes $-\int a_\mu J^\mu d^3r dw$, where $w=bt_c$. The invariant Maxwell equations in a general inertial frame in common relativity are thus

$$\partial_\mu f^{\mu\nu} = J^\nu, \qquad \partial_\lambda f^{\mu\nu} + \partial_\mu f^{\nu\lambda} + \partial_\mu f^{\lambda\mu} = 0; \tag{12.6}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \qquad x^\lambda = (bt_c, \mathbf{r}), \qquad f^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu.$$

One can write the field-strength tensor $f^{\mu\nu}$ in matrix form

$$f^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{12.7}$$

$$\mathbf{E} = \frac{\partial \mathbf{a}}{\partial(bt_c)} - \nabla a_0, \qquad \mathbf{B} = \nabla \times \mathbf{a}. \tag{12.8}$$

Since $a^\mu(w,\mathbf{r}) \Leftrightarrow A^\mu(ct,\mathbf{r})/c$, the fields $\mathbf{E}(w,\mathbf{r})$ and $\mathbf{B}(w,\mathbf{r})$ are related to the usual $\mathbf{E}_{SR}(ct,\mathbf{r})$ and $\mathbf{B}_{SR}(ct,\mathbf{r})$ in special relativity by the following correspondences:

$$\mathbf{E}(w,\mathbf{r}) \Leftrightarrow \mathbf{E}_{SR}(ct,\mathbf{r})/c, \qquad \mathbf{B}(w,\mathbf{r}) \Leftrightarrow \mathbf{B}_{SR}(ct,\mathbf{r})/c. \tag{12.9}$$

The transformations of the field tensor $f^{\mu\nu}$ are given by

$$f'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} f^{\alpha\beta}, \tag{12.10}$$

where $\partial x'^\mu/\partial x^\alpha$ can be calculated from the coordinate transformation (8.3). Since the fields $\mathbf{E}$ and $\mathbf{B}$ are elements of the field tensor $f^{\alpha\beta}$, the transformation equations for the electromagnetic fields are

$$\begin{aligned} E'_x &= E_x, \qquad E'_y = \gamma(E_y - \beta B_z), \qquad E'_z = \gamma(E_z + \beta B_y), \\ B'_x &= B_x, \qquad B'_y = \gamma(B_y + \beta E_z), \qquad B'_z = \gamma(B_z - \beta E_y). \end{aligned} \tag{12.11}$$

Note that one can use transformations (8.13) and (12.11) for velocities and electromagnetic fields to show that $\bar{e}\mathbf{E}\cdot\mathbf{v}$ and the Lorentz force $\bar{e}(\mathbf{E}C+\mathbf{v}\times\mathbf{B})$ in

(12.5) form a 4-vector ($\bar{e}\mathbf{E}\cdot\mathbf{v}$, $\bar{e}[\mathbf{E}C+\mathbf{v}\times\mathbf{B}]$), i.e., they transform like a 4-vector.

In terms of **E**, **B**, and the 4-current $J^{\mu}=(\rho,\mathbf{J})$, the first equation in (12.6) can be written as

$$\nabla\cdot\mathbf{E}=\rho, \qquad \nabla\times\mathbf{B}-\frac{\partial\mathbf{E}}{\partial(bt_c)}=\mathbf{J}. \tag{12.12}$$

The second equation in (12.6) can be written as

$$\nabla\cdot\mathbf{B}=0, \qquad \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial(bt_c)}=0. \tag{12.13}$$

In common relativity, the speed of light does not explicitly appear in the new formulation of the Maxwell equations in a general inertial frame. The Maxwell equations can be written in the familiar form

$$\nabla\cdot\mathbf{B}=0, \qquad \nabla\times\mathbf{E}-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0, \qquad \text{etc.,} \tag{12.14}$$

if and only if the speed of light is assumed to be a constant c. Nevertheless, when $\rho=0$ and $\mathbf{J}=0$, the wave equations (12.12) and (12.13) can be written in the 4-dimensional form,

$$\frac{\partial^2 g}{\partial w^2}-\nabla^2 g=0, \qquad w=bt_c, \tag{12.15}$$

where the function $g=g(bt_c,\mathbf{r})$ stands for any component of **E** or **B**. A plane wave solution g_p of (12.15) takes the form

$$g_p=A_o\sin(k_o w-\mathbf{k}\cdot\mathbf{r}), \qquad k_0^2-\mathbf{k}^2=0. \tag{12.16}$$

When y=z=0, the wave equation (12.13) with $g=g(bt_c,x)$ leads to the general solution in any inertial frame, $g(bt_c,x)=g_1(x+bt_c)+g_2(x-bt_c)$, where g_1 and g_2 are two arbitrary functions. Since b is in general, a function, it is not the speed of the wave. To obtain the speed of the wave, we may consider the amplitude g_2 which satisfies the equation $x - bt_c$ = constant. Differentiating with respect to the common time t_c, we find the speed of this wave to be

$$\frac{dx}{dt_c} = \frac{d(bt_c)}{dt_c} = C, \tag{12.17}$$

where the speed of light C as measured by the common time t_c in a general inertial frame is not isotropic, as shown in equation (8.5). Only in the inertial frame $F(ct_c,x,y,z)$ does one have a constant speed of light $b=C=c$.

12b. Universal constants and invariant actions

In taiji relativity, where we use natural units, the only constants that appear in the equations are dimensionless constants such as α_e. All other constants such as c, $\hbar$, etc. turn out to have the value 1 (see Appendix A for a more detailed discussion). In common relativity however, in which we define the common-second as our unit of time t_c, universal constants with units exist. In this section, we examine what they are and compare them with the usual ones found in special relativity.

According to quantum mechanics, the wave 4-vector k^μ of a quantum particle must be proportional to the 4-momentum p^μ. Thus, in general we have

$$p'^\mu = J' k'^\mu, \qquad \text{in F' frame;} \tag{12.18}$$

$$p^\mu = J k^\mu, \qquad \text{in F frame.} \tag{12.19}$$

Based on symmetry considerations, the same proportionality relation should hold in any inertial frame so that $J'=J$. Therefore, the constant J is a universal constant in common relativity.[1] Its value can be obtained by comparing (12.19) in F with the usual relation in special relativity (SR) $p^\mu_{SR}=\hbar k^\mu$, where $p^\mu_{SR}/c \Leftrightarrow p^\mu$, $k^0_{SR}=\omega/c \Leftrightarrow k^0$ and $\mathbf{k}_{SR}=\mathbf{k}$ in F. Note that the last relation must be true because the spatial components $\mathbf{r}$ in common relativity are the same as the usual $\mathbf{r}_{SR}$. As a result, we can deduce that the value of J must be $\hbar/c = h/(2\pi c)$:[1]

$$J = 3.5177293 \times 10^{-38}\ \mathrm{g{\cdot}cm}. \tag{12.20}$$

This result is consistent with (10.7) and (10.15) in chapter 10.

The quantity J is unusual because it is a universal constant regardless of what one uses for a second postulate in the 4-dimensional symmetry framework. Equivalently, it is independent of how one defines an additional unit (other than meter) for the evolution variable. As long as one makes any kind of definition at all, it is a constant. This is in sharp contrast to the conventional universal constants c and $\hbar$, whose universality depends on a particular choice for a second postulate, i.e., the one chosen by Einstein for special relativity, which dictates a certain relationship between w and t. The universal constant J in common relativity plays the role of the Planck constant in the conventional theory. For example, we have expressions such as $\mathbf{p} = -iJ\nabla$, $\exp(ip_\mu x^\mu/J)$, $d^3r d^3p/(2\pi J)^3$ and $[J^2(\partial^2/\partial w^2) - J^2\nabla^2 - m^2]\Phi(x^\mu) = 0$.

If one compares equation (12.3) with the corresponding special relativistic relation in the F frame, one has $\bar{e} = e/c$ (where e is measured in electrostatic units, esu,) and $a^\mu(w,\mathbf{r}) \Leftrightarrow A^\mu(ct,\mathbf{r})/c$; the latter is necessary, so that the action for the free electromagnetic field

$$-(1/4)\int f_{\mu\nu} f^{\mu\nu} d^3r dw\ , \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \tag{12.21}$$

has the same dimension as S_c in (12.2), as is required for consistency. Thus, the universal constant $\bar{e}$ in common relativity is

$$\bar{e} = -1.6021891\times 10^{-20}\ (4\pi)^{1/2}\ (\mathrm{g{\cdot}cm})^{1/2}. \tag{12.22}$$

Like J, $\bar{e} = e/c$ is a universal constant in all relativity theories, regardless of the choice of second postulate. It can be verified that the dimensionless electromagnetic coupling strength α_e has the usual dimensionless value of

$$\alpha_e = \bar{e}^2/(4\pi J) = 1/137.0359895.$$

Note that although from the conventional point of view of special relativity, there are three universal constants $\hbar$, e and c in quantum electrodynamics, we see that in relativity theories in general, there are only two such constants, J and $\bar{e}$, the universality of which *are independent of the specific transformation property of the evolution variable.* In other words, they are independent of how the unit of time t is defined.

12c. Dirac's conjecture regarding the fundamental constants

In his 1963 article "The Evolution of the Physicist's Picture of Nature", Dirac gave an interesting account of how physical theory had developed in the past and how it could be expected to develop in the future. Dirac noted that in the conventional framework of physics based on special relativity and quantum mechanics, three fundamental constants in nature are the charge e, the Planck constant $\hbar$, and the speed of light c and that using these constants, one can construct a dimensionless number $\alpha_e = e^2/(4\pi\hbar c)$, which has a value very close to 1/137. Dirac wrote that "there will be a physics in the future that works when $\hbar c/e^2$ has the value 137 and that will not work when it has any other value." (The factor of 4π in α_e is not important since one can choose suitable units such as Gaussian units for the charge e such that α_e does not contain the factor of 4π.)

Dirac then gave the following outlook on physics:
"The physics of the future, of course, cannot have the three quantities $\hbar$, e, and c all as fundamental quantities. Only two of them can be fundamental, and the third must be derived from those two. It is almost certain that c will be one of the two fundamental ones. The velocity of light, c, is so important in the four-dimensional picture, and it plays such a fundamental role in the special theory of relativity, correlating our units of space and time, that it has to be fundamental. Then we are faced with the fact that of the two quantities $\hbar$ and e, one will be fundamental and one will be derived. If $\hbar$ is fundamental, e will have to be explained in some way in terms of the square root of $\hbar$, and it seems most unlikely that any fundamental theory can give e in terms of a square root, since square roots do not occur in basic equations. It is much more likely that e will be the fundamental quantity and that $\hbar$ will be explained in terms of e^2.

Then there will be no square root in the basic equations. I think one is on safe ground if one makes the guess that in the physical picture we shall have at some future stage e and c will be fundamental quantities and $\hbar$ will be derived."[3]

We can compare Dirac's outlook in 1963 with our present understanding of relativity and of the 4-dimensional symmetry of physical laws. Because one can formulate an experimentally consistent theory based solely on the first postulate of relativity (using the same unit to quantify both space and time), and because the constant c (with units of meter/second) does not appear in such a theory, one sees that the constant, c=299792458 m/s, is not necessary to the formulation and understanding of physics. *Dirac's conjecture was based on the implicit assumption that the evolution (or temporal) variable must be measured with a unit different from that used for spatial variables.* As our understanding of physics has progressed, we have seen that this turns out not to be necessary. It is precisely because c merely correlates our units of space and time that its value is not a universal and fundamental constant. As Taylor and Wheeler state in *Spacetime Physics*, the value of c is simply "the work of two centuries of committees."[4] More fundamentally, the quantity that correlates the evolution and spatial variables is simply 1, which is the dimensionless and universal taiji-speed of light derived from Maxwell's equations in taiji relativity.

12d. Truly fundamental constants

At this point, one might ask: If the speed of light c is neither a necessary nor a truly universal constant, why do all experiments (before 1983) that measure the speed of light in different inertial frames give the same value c=299792458 m/s? To answer this question, let us analyze the origin of the value of c from the viewpoint of the Poincaré-Einstein principle alone, i.e., in the framework of taiji relativity. Suppose that a physicist introduces the time t' as the evolution variable in an F' frame. This time t' must have some relation to w', say w'=b't', as required by 4-dimensional symmetry. In taiji relativity, the invariant phase of an electromagnetic wave can be written as

$$k'^{\mu} x'_{\mu} = \frac{\omega'}{C'} b't' - \mathbf{k'} \cdot \mathbf{r'}, \text{ where } C' \equiv d(b't')/dt', \tag{12.23}$$

where C' and b' are functions which are separately undefined. These two functions are different but related, because the speed of light C' is given by the invariant law of propagation (7.9) in the frame F' with $w'=b't'$, $[d(b't')]^2-dr'^2 \equiv (C'dt')^2-r'^2=0$, i.e., $d(b't') \equiv C'dt'$. Now, suppose our physicist chooses to measure time t' using the electromagnetic frequency ω'. At first glance one may think that this can be done without making any assumptions about the speed of light, but this is not true. In order to measure the time t' using the frequency ω', one must adopt a specific relationship between the two quantities. Typically, the relationship that is adopted is the one that assumes that the invariant phase has the more restricted form, $\omega't'-\mathbf{k'}\cdot\mathbf{r'}$ and thus that the electromagnetic oscillation has the form $\sin(\omega't')$, as it does in special relativity. This amounts to making an additional assumption that $C'=b'=c=$constant in the invariant phase of electromagnetic waves. In taiji relativity, ω' and t' have no specified relationship (indeed, t' does not even exist), and thus one cannot be used to measure the other. With the pre-1983 definitions of meter and second, the speed of light that results experimentally from the aforementioned hidden assumption turns out to have the value 299792458 m/s. Therefore, the fact that previous experiments seems to imply a particular universal value is merely the result of a subtle assumption that fixes a relationship where no a priori relationship existed, combined with the historical definitions of our units of length and time.

If one looks at the physical world from the simplest viewpoint, i.e., based solely on the 4-dimensional symmetry of the Poincaré-Einstein principle alone, electromagnetic waves have two types of periodic behavior, one related to our perception of spatial length determined by $|\mathbf{k}|$ (or the wave length $\lambda=2\pi/|\mathbf{k}|$) and the other related to our perception of "duration" determined by k_0, as shown in (12.16). Since $|\mathbf{k}|=k_0$, then it is clear that both spatial and temporal intervals can be measured using the same unit and these waves propagate with a dimensionless speed of 1. In other words, the relationship between units for the spatial interval $\Delta x=1$ m and the "time" interval $\Delta w=1$ m are completely dictated by the Maxwell equations and the Poincaré-Einstein principle. Since our ancestors were not aware of the Poincaré-Einstein principle and their intuitive sense of time was different from that of space, they invented a quantity with a different unit to measure time intervals. As a result, we have a quantity called frequency ω measured in the unit of 1/second and it is purely a matter of history and human convention that the duration $\Delta t=1$ s is equivalent to the taiji-time interval $\Delta w=299792458$ meter.

If the numerical value 299792458 m/s is not an inherent property of our universe it is not a fundamental physical constant. Thus, we conclude that there is only one truly universal and fundamental constants in classical electrodynamics, the dimensionless fine structure constant $\alpha_e = 1/137.0359895$. Based on a similar line of reasoning, we further conclude (see chapter 23 and Appendices A and C) that α_e is the only truly fundamental physical constant in quantum electrodynamics in both inertial and accelerated frames, in contrast to Dirac's conjecture.

References

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2. Jong-Ping Hsu, Phys. Lett. A **97**, 137 (1983); Nuovo Cimento B **74**, 67 (1983); J. P. Hsu and C. Whan, Phys. Rev. A **38**, 2248 (1988), Appendix. These papers discuss a four-dimensional symmetry framework based on the usual first postulate and a different second postulate, namely, a common time $t'=t$ for all observers in different inertial frames.
3. P. A. M. Dirac, Sci. Am. **208**, 48 (1963).
4. E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (Second Ed., W. H. Freeman and Company, 1992), p. 58.
5. For detailed discussions, see chapter 23, and Appendices A and C.

13.

Common Relativity and Many-body Systems

13a. Advantages of common time

In classical mechanics, the motion of a many-body system can be described in terms of the position of the center of mass $\mathbf{r}_C$, which has a well-defined properties under the Galilean transformation. However, in the mechanics under special relativity, the concept of the center of mass for a system of particles is not well-defined because the moving mass is not a constant. The center of energy $\mathbf{R}_{CE}$ defined by

$$\mathbf{R}_{CE} = \sum_i \frac{E_i \mathbf{r}_i}{E_{tot}}, \qquad E_{tot} = \sum_k E_k,$$

in special relativity corresponds to the center of mass $\mathbf{r}_C$ in classical physics, where E_i and $\mathbf{r}_i$ are the relativistic energy and position of the ith particle, respectively. Clearly, $\mathbf{R}_{CE}$ reduces to $\mathbf{r}_C$ in the nonrelativistic limit because E_i becomes the mass m_i. However, the components of $\mathbf{R}_{CE}$ do not transform as components of a 4-vector under the Lorentz transformations. A similar situation occurs with the total momentum

$$P_\mu = \sum_i p_{i\mu},$$

which, in general, is not a 4-vector. The reason is that the sums P_μ are, in fact, sums of the values of $p_{i\mu}$ at a given instant in the given frame. However, in a different frame these values are no longer the momenta of each of the individual particles at the same instant. Therefore, when one transforms the total momentum P_μ to a new frame F', one has not only to form

$$\sum_i p'_{i\mu} ,$$

according to the 4-vector rule, but also must recalculate these quantities appropriately according to the simultaneity of the F' frame.[1] Only for noninteracting particles is the sum P_μ a 4-vector because each $p_{i\mu}$ is constant.

Apart from these problems, the task of characterizing many-body systems has other difficulties within the framework of special relativity:[2]

(i) The concept of a Hamiltonian system with many degrees of freedom breaks down because of the lack of a single time for a system of N particles. Relativistic time appears to be incompatible with the notion of the canonical evolution of a many-body system and with the evolution of a stochastic process.

(ii) Since the notion of a 3-dimensional spatial volume is not a covariant concept, the uniform density in configuration space is not normalizable.

These properties lead to several difficulties in treating many-body systems within the framework of special relativity because, in contrast to Newtonian mechanics, it does not have a single time which is applicable to all particles or to all inertial frames of references.[2,3]

Common relativity, on the other hand, does not have these difficulties. *It is the natural and unique four-dimensional generalization of the classical three-dimensional framework with one single time.* As demonstrated in chapter 12, common time is universal but differs from the Newtonian absolute time in that it also is consistent with the principle of relativity. It provides a universal simultaneity, just like the absolute time in classical mechanics. As a result, common time embedded in a 4-dimensional symmetry framework enables us to formulate the Hamiltonian dynamics of a many-body system and also to define an effective and invariant 6-dimensional μ-space and 6N-dimensional phase space for a covariant statistical mechanics, which special relativity does not allow.[3]

The assumption of common time in space-lightime necessitates the existence of the ligh function b in a general inertial frame, $b = x_0/t_c = w/t_c$, which transforms as the zeroth component of a 4-vector because the common time t_c is a scalar. This is important because the ligh function b enables us to define an invariant volume V_I in a general inertial frame:

$$V_I = \int \frac{1}{c_o} d^4x\delta(t_c - \frac{x^0}{b}) = \int \frac{|b|}{c_o} dxdydz, \qquad c_o \equiv c, \tag{13.1}$$

where $d^4x=dx^0dx^1dx^2dx^3=dx^0dxdydz$ and the *scaling constant* $c_o \equiv c$ is introduced to preserve the usual dimension of the volume V_I. Note that there is no restriction on the zeroth component x^0 in $d^4x=dx^0dxdydz$, just as $x^0 \equiv w$ in equations (7.1) and (7.4) within taiji relativity. The δ-function in (13.1) imposes on x^0 the restriction that one uses common time in all inertial frames. Note that the condition imposed by the δ-function in (13.1), i.e., $t_c - x^0/b=0$, is invariant. This invariance can be seen as follows:

$$t_c - \frac{x^0}{b} = t_c - \frac{x'^0 M}{b'N} = 0, \qquad M = \left(1 + \frac{\beta x'}{x'^0}\right), \qquad N = \left(1 + \frac{\beta x'}{b' t_c}\right). \tag{13.2}$$

From the expression for N and the constraint $[t_c - x'^0M/(b'N)] = 0$, one can obtain the relation between M and N, $N=1/[1-\beta x'/(x'^0M)]$. Thus, one has

$$t_c - \frac{x'^0 M}{b'N} = t_c - \frac{x'^0 M[1-\beta x'/(x'^0 M)]}{b'}$$

$$= t_c - \frac{[x'^0 (1-\beta x'/x'^0) - \beta x')]}{b'} = t_c - \frac{x'^0}{b'}. \tag{13.3}$$

It follows from (13.2) and (13.3), that the volume element $dV_I= c_0^{-1}d^4x\delta(t_c - x^0/b) =(|b|/c_0)dxdydz$ is invariant under the 4-dimensional coordinate transformation in common relativity. The numerical value of c_o in a general inertial frame is defined to be identical to the speed of light in F, $c_0=c=b$, so that V_I is the same as the usual volume in the frame F. Even though the speed of light is not isotropic in a general inertial frame F', one can still make this definition without violating the 4-dimensional symmetry of common relativity because c_0 is simply a scaling constant. In the F'-frame, an invariant integral involving $dV_I=(|b'|/c)dx'dy'dz'$ is, in general, not convenient for carrying out calculations because of the presence of the ligh function b'. However, we can always transform quantities to the F-frame, in which b=c and dV_I=dxdydz, for ease of calculating invariant integrals.

One can also understand the new invariant volume dV_I in (13.3) as the volume of the parallelepiped in the space-lightime defined by four vectors a_μ, b_ν, c_α, and d_β:

$$dV_I = \varepsilon^{\mu\nu\alpha\beta} a_\mu b_\nu c_\alpha d_\beta, \tag{13.4}$$

where $a_\mu=(b/c_0,0,0,0)$, $b_\nu=(0,dx,0,0)$, $c_\alpha=(0,0,dy,0)$ and $d_\beta=(0,0,0,dz)$. It is the same as the invariant expression (13.4) provided a_μ, b_ν, c_α, d_β are vectors lying in the direction of the coordinate axes and of length dw, dx, dy, dz, respectively. The hypersurface $d\Sigma^\mu$ in the 4-dimensional space-lightime,

$$d\Sigma^\mu = (d\Sigma^0, d\Sigma^1, d\Sigma^2, d\Sigma^3), \tag{13.5}$$

transforms like a 4-vector, where $d\Sigma^0=dxdydz$ is equal to the three-dimensional volume element. Geometrically, the 4-vector $d\Sigma^\mu$ is equal in absolute magnitude to the "area" of the element of hypersurface and points in the direction perpendicular to all lines lying in this element.

The existence of the invariant volume V_I is important to four-dimensional thermodynamics and statistical mechanics. It enables us to define the notion of a box (or a 3-dimensional spatial volume) and to normalize the uniform density in configuration space.

13b. Hamiltonian dynamics in common relativity

In order to see what new quantities are covariant under the common relativity transformation and properties of the invariant 6N-dimensional phase space, let us briefly consider kinematics and the invariant Hamiltonian dynamics. The invariant "action function" for a free particle is assumed to be

$$S = -\int m ds = -\int m\sqrt{v^\mu v_\mu}\; dt_c = \int L dt_c,$$

$$L = -m\sqrt{v^\mu v_\mu} = -m\sqrt{C^2 - v^2}, \tag{13.6}$$

where $ds^2=dw^2-dr^2=(C^2-v^2)dt_c^2$ and $C=dw/dt_c=d(bt_c)/dt_c$ in a general inertial frame. The 4-momentum p^μ of a free particle is defined as

$$p^\mu = -\frac{\partial L}{\partial v_\mu} = \frac{mv^\mu / C}{\sqrt{1 - v^2/C^2}} = Gv^\mu, \qquad v_\mu = (C,-\mathbf{v}), \quad v^\mu = (C,\mathbf{v}), \tag{13.7}$$

where p^μ has the dimension of mass. This definition is consistent with the usual definition of the momentum $\mathbf{p}=\partial L/\partial \mathbf{v}$. Although mv^μ is a 4-vector, it is not the physical momentum. The quantity G,

$$G = \frac{m}{\sqrt{C^2 - v^2}} = \frac{\sqrt{\mathbf{p}^2 + m^2}}{C} = \frac{p^0}{C}, \tag{13.8}$$

is an invariant, being the ratio of corresponding components of two 4-vectors, p^μ and v^μ, with $p_x/v_x = p^0/C$. Its invariance can also be seen from (13.7) because p^μ and v^μ are 4-vectors with the same transformation properties in common relativity. We call G the "genergy" because it is equivalent (except for a multiplicative constant) to the conserved energy p^0 in the F-frame, in which C=c so that the conservation of energy implies the conservation of the invariant genergy. Since special relativity does not have a four-velocity measured in terms of a scalar common time, it does not have an invariant quantity analogous to the genergy (13.8).

With the help of the invariant genergy G, we can define the four-coordinate R^μ_C of the "center of mass" for an N-particle system in common relativity,

$$R^\mu_C = \frac{Q^{\mu *}}{\sum_k G_k}, \qquad Q^{\mu *} = \sum_{i=1}^{N} G_i x^\mu_i. \tag{13.9}$$

The spatial components $\mathbf{R}_C$ can be satisfactorily identified with the relativistic center of mass for a system of particles in common relativity. It can be verified that $\mathbf{R}_C$ reduces to the classical center of mass when the velocities of the particles are small. Similarly, the total momentum

$$P_\mu = \sum_i p_{i\mu}$$

in common relativity is a 4-vector because the common time gives us a universal definition of simultaneity which is applicable to all observers in all inertial frames of reference.

Let us consider N charged particles with masses m_a, a=1,2,..., N, within the framework of the common-relativistic Hamiltonian dynamics.[3] We begin with an 8N-dimensional "extended phase space" with the basic variables x_v^a, p_v^a, to define the Poisson brackets and the generators of the Poincaré group.

$$\{x_a^\mu, x_b^\nu\} = 0, \qquad \{x_a^\mu, p_b^\nu\} = \eta^{\mu\nu}\delta_{ab}, \qquad \{p_a^\mu, p_b^\nu\} = 0, \tag{13.10}$$

where

$$\{A, B\} \equiv \frac{\partial A}{\partial x_a^\mu}\frac{\partial B}{\partial p_{a\mu}} - \frac{\partial A}{\partial p_{a\mu}}\frac{\partial B}{\partial x_a^\mu}, \qquad x_a^\mu = (w_a, x_a, y_a, z_a).$$

We impose 2N constraints of the form

$$K_a \equiv (P_a^\mu - \bar{e}\, a_a^\mu)^2 - m_a^2 = 0, \qquad Z_a(x_a^0, t_c) = x_a^0 - b_a t_c = 0, \tag{13.11}$$

where P_a^μ is the canonical momentum 4-vector, $P_a^\mu = p_a^\mu - \bar{e} a_a^\mu$, and a_a^μ is the electromagnetic potential acting on the particle a. Thus, we end up with 6N basic independent variables. The Dirac Hamiltonian is

$$H_D = W_a K_a, \tag{13.12}$$

where W_a is determined by the constraints Z_a=0 which must hold for all times t_c, i.e. dZ/dt_c=0.

The Hamiltonian equations of motion with the common time t_c are then[3]

$$\frac{dx_a^\mu}{dt_c} = \{x_a^\mu, H_D\} = \frac{\partial H_D}{\partial p_{a\mu}}, \qquad \frac{dp_a^\mu}{dt_c} = \{p_a^\mu, H_D\} = -\frac{\partial H_D}{\partial x_{a\mu}}, \tag{13.13}$$

where

$$W_a = -A_{ab}\frac{\partial Z_b}{\partial t_c}, \qquad A_{ab}\{Z_b, K_c\} = \delta_{ac}.$$

One can verify that Hamilton's equations of motion (13.13) with the Dirac Hamiltonian H_D in (13.12) and the constraints (13.11) lead to the equations of motion for a charged particle,

$$\frac{dp_a^\mu}{dt_c} = \bar{e}\, f_a^{\mu\nu} v_{a\nu}, \qquad a = 1, 2, ...N, \tag{13.14}$$

$$v_{a\nu} = \frac{C_a(P_{a\nu} - \bar{e}a_{a\nu})}{(P_a^0 - \bar{e}a_a^0)},$$

$$dZ_a(x_a^0, t_c) = dx_a^0 - \frac{d(b_a t_c)}{dt_c}dt_c = dx_a^0 - C_a dt_c, \qquad \frac{\partial Z_a}{\partial t_c} = -C_a,$$

$$A_{ac} = \frac{\delta_{ac}}{2(P_a^0 - \bar{e}a_a^0)},$$

$$W_a = \frac{C_a}{2(P_a^0 - \bar{e}a_a^0)},$$

where a is not summed. The resultant equation of motion (13.14) is the same as that in (12.5), as it should be. Thus, the evolution of the N-particle system in terms of the common time t_c is completely determined by Hamilton's equations (13.13) and the initial data, i.e., the initial momenta and positions of the particles.

A basic unsolved problem in the statistical mechanics based on special relativity is related to the phase space in different frames of reference. In a general inertial frame, it appears to be impossible to have a meaningful Lorentz-invariant phase space related to the initial data at a given time for all observers in different frames, in contrast with the classical case based on Newtonian laws. In Newtonian physics, observers can, in principle, perform measurements at a given absolute time. Thus one can specify a system of N particles by the vectors $(\mathbf{q}_1,...,\mathbf{q}_N,\mathbf{p}_1,...,\mathbf{p}_N)$ which span the phase space of the system.

In common relativity, suppose that there is a clock and an observer at each point in space in every inertial frame. The F-frame observers can perform instantaneous measurements of $\mathbf{r}$ and $\mathbf{v}$ for each particle at a given common time t_c using the nearest clock so as to remove the need to account for the finiteness of the speed of light. Thus, we have the coordinate and momentum data of particles (r_a, p_a), a=1,2,..., N, at the common time t_c. These data, along with the dynamical equations, completely determine the evolution of the system. Of course, the data can also be transformed to another frame F' using the space-lightime transformations which relate

$$(ct_c, \mathbf{r}_a) \quad \text{and} \quad (c, \mathbf{v}_a) \qquad \text{in F,} \qquad a=1,2,\ldots, N,$$

to

$$(b'_a t_c, \mathbf{r}'_a) \quad \text{and} \quad (c'_a, \mathbf{v}'_a) \qquad \text{in F',} \qquad a=1,2,\ldots, N.$$

The initial positions and velocities of the particles in F can also be directly obtained by the F' observers by measuring $\mathbf{r}'_a, \mathbf{v}'_a$ and computing b'_a and c'_a $=d(b'_a t_c))/dt_c$ with the help of eq. (8.3). In this way, we have global knowledge of the whole system at a given instant of common time in the statistical mechanics based on common relativity.

With the help of (13.3) and (13.11), we can define in μ-space an effective six-dimensional volume element (for particles with $m \geq 0$)

$$d\mu = d^4x \ \delta(\frac{x^0}{b} - t_c) d^4p \ \delta(p_\lambda^2 - m^2) \ \theta(p^0) 2G. \tag{13.15}$$

This volume element is an invariant because the common time t_c, the rest mass m, the genergy G of a particle, and the ratio x_0/b are all scalars in common relativity. In the particular frame F, one can see that eq. (13.15) reduces to $d^3x d^3p$ by carrying out the integrations over x^0 and p^0. The 6N-dimensional phase space in common relativity has the invariant volume element

$$d\Gamma = \prod_{a=1}^{N} d\mu_a . \tag{13.16}$$

The independent physical variables in phase space are the only really important ones. We can eliminate the nonphysical variables from the theory using the constraint equations. *However, this elimination may be awkward and may spoil the four-dimensional properties of the equations; therefore, we shall retain the constraint equations in the general theory.* That is, we will treat the

N-particle phase space as an invariant 8N-dimensional extended phase space with 2N invariant constraints (13.11). In this way we have, effectively, a physical 6N-dimensional phase space that is invariant under the space-lightime transformations.

13c. Invariant kinetic theory of gases

Let us consider kinetic theory based on a one-particle distribution function $D_1(x^A,t_c)$ and a kinetic equation for $D_1(x^A,t_c)$ within the framework of common relativity. In order to show explicitly the 4-dimensional symmetry of the theory of gases we use $x^A=(x^\mu,p^\mu)$, where p^μ is the 4-momentum of the particle in (13.7), as the coordinates of the extended μ-space, i.e., the one-particle phase space. The reason for this is that the 4-dimensional transformations of coordinates can only be expressed in terms of the Cartesian coordinates x^μ. The invariant time-dependent distribution function $D_1(x^A,t_c)$ in the μ-space is normalized by the expression

$$\int D_1(x^A,t_c)d\mu = \int D^* d^4x d^4p = N,$$

$$D^* \equiv D_1(x^A,t)\ \delta(t_c - \frac{x^0}{b})\ \delta(p_\lambda{}^2 - m^2)\ \theta(p^0)2G, \tag{13.17}$$

where $d\mu$ is given by (13.15) and $N(t_c)$ is the number of particles in the gas at time t_c. The function $D^*d^4xd^4p/N$ can be formally interpreted as the probability of finding the particle at common time t_c in the volume element d^4xd^4p around the point x^A in the 8-dimensional space. If one carries out the integration over x^0 and p^0 from $-\infty$ to $+\infty$, then $[(1/N)\int D^*dx^0dp^0]d^3xd^3p$ is the probability of finding the particle at time t_c in the spatial volume element d^3x around the point **r** and with a momentum in the region d^3p around the value **p**.

This is in sharp contrast with the usual relativistic kinetic theory in which the distribution D' is normalized on a six-dimensional manifold through a current[4]

$$\int D' 2p_\mu \theta(p^0)\delta(p_\lambda^2 - m^2)d^4p d\Sigma_\mu = N, \tag{13.18}$$

because of the lack of a single evolution variable for a system of particles. In other words, the usual relativistic kinetic theory is a statistics of curves rather than a statistics of points. In the conventional theory, such a six-dimensional manifold is, in general, not meaningful because of the arbitrariness of Σ (except the case when the current is conserved) and the fact that the distribution D' is not actually the probability density.[2]

The invariant distribution $D_1(x^A,t_c)$ is a function in the eight-dimensional space with properties as defined by (13.17). The common-relativistic one-particle Liouville equation for the invariant distribution $D_1(x^A,t_c)$ can be derived by considering the eight-vectors $x^A(t_c)$ of the particle coordinates, the corresponding velocities

$$\dot{x}^A(t_c) = \frac{dx^A(t_c)}{dt_c} = (\dot{x}^\mu(t_c), \dot{p}^\mu(t_c)) = (v^\mu, F^\mu) , \tag{13.19}$$

and the current density $j^A(x^A,t_c)$,

$$j^A(x^A,t_c) = D_1(x^A,t_c)\, \dot{x}^A(t_c). \tag{13.20}$$

The current density $j^A(x^A,t_c)$ satisfies the continuity equation in the eight-dimensional space

$$\frac{\partial}{\partial t_c} D_1(x^A,t_c) + \frac{\partial}{\partial x^A}[D_1(x^A,t_c)\dot{x}^A(t_c)] = 0 . \tag{13.21}$$

Using Hamilton's equations of motion (13.13), we have

$$\frac{\partial}{\partial x^A}\dot{x}^A(t_c) = \frac{\partial}{\partial x^\mu}\frac{\partial H_D}{\partial p_\mu} + \frac{\partial}{\partial p_\mu}\left(-\frac{\partial H_D}{\partial x^\mu}\right) = 0 , \tag{13.22}$$

and therefore, we can write (13.21) as

$$\frac{\partial}{\partial t_c} D_1(x^A,t_c) + \left[v^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu}\right] D_1(x^A,t_c) = \frac{d}{dt_c} D_1(x^A,t_c) = 0 . \tag{13.23}$$

This is a simple example of the Liouville theorem, to be discussed in section 13d below. Clearly, if the one-particle distribution does not depend on t_c explicitly, i.e., $D_1 = D_1(x^A)$, then we have an invariant distribution $D_1(x^A)$ which obeys equation (13.23) where the first term is zero.

In order to see the connection to the nonrelativistic case, we write the invariant distribution $D_1(x^A,t_c)$ in the following form

$$D_1(x^A,t_c) \equiv \overline{D}(x^0(t_c),\mathbf{r},p^0(t_c),\mathbf{p},t_c)\delta\left(x^0 - x^0(t_c)\right)\delta\left(p^0 - p^0(t_c)\right), \tag{13.24}$$

$$x^0(t_c) = bt_c\,, \quad p^0(t_c) = \sqrt{\mathbf{p}^2(t_c)+m^2}.$$

It follows from (13.23) and (13.24) that

$$\frac{\partial\overline{D}}{\partial t_c} + \mathbf{v}\cdot\frac{\partial\overline{D}}{\partial\mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial\overline{D}}{\partial\mathbf{p}} = 0, \tag{13.25}$$

because $(\partial/\partial t_c)\ \delta(x^0-x^0(t_c))\ \delta(p^0-p^0(t_c))=0$ and

$$\overline{D}\left[v^\mu\frac{\partial}{\partial x^\mu} + F^\mu\frac{\partial}{\partial p^\mu}\right]\delta(x^0-x^0(t_c))\delta(p^0-p^0(t_c))$$

$$=\overline{D}\left[\dot{x}^0\frac{\partial}{\partial x^0} + \dot{p}^0\frac{\partial}{\partial p^0}\right]\delta(x^0-x^0(t_c))\delta(p^0-p^0(t_c))$$

$$=-\left\{\left[\dot{x}^0\frac{\partial}{\partial x^0} + \dot{p}^0\frac{\partial}{\partial p^0}\right]\overline{D}\right\}\delta(x^0-x^0(t_c))\delta(p^0-p^0(t_c)). \tag{13.26}$$

In the nonrelativistic case, we have $|\mathbf{p}(t_c)| \approx |m\mathbf{v}| << p^0(t_c) \approx m$ and $C \approx b \approx c$ in the frame F, so that

$$\int_{[x^0p^0]} d\mu = d^3x d^3p\,,$$

$$\int D_1(x^A,t_c)\delta\left(x^0 - x^0(t_c)\right)\delta\left(p^0 - p^0(t_c)\right)dx^0dp^0 \tag{13.27}$$

$$=D_1(x^A,t_c)\big|_{x^0=bt_c;p^0=p^0(t_c)} = \overline{D}(\mathbf{r},\mathbf{p},t_c).$$

The last expression in (13.27) is the one-particle distribution in the classical six-dimensional μ-space $(\mathbf{r},\mathbf{p})$, which satisfies eq. (13.25). The non-relativistic distribution $\overline{D}(\mathbf{r},\mathbf{p},t_c)$ is the probability density of finding the particle at common time t_c near the point $(\mathbf{r},\mathbf{p})$ in the F frame in which C=b=c.

A general 4-dimensional kinetic equation for a gas not in equilibrium is given by

$$\left(\frac{\partial}{\partial t_c}+v^\mu\frac{\partial}{\partial x^\mu}+F^\mu\frac{\partial}{\partial p^\mu}\right)D_1=P(D_1), \qquad D_1=D_1(x^A,t_c), \qquad F^\mu=\frac{dp^\mu}{dt_c}, \tag{13.28}$$

where $P(D_1)$ is given by Boltzmann's collision postulate (i.e. the variation of D_1 per unit time due to collisions, see eq. (13.78) below) and it should fulfill certain requirements.[5]

Once the solution of the kinetic equation (13.28) is obtained, one can compute the energy-momentum tensor of the relativistic fluid

$$T^{\mu\nu}(x)=\int d^4p\,2\theta(p^0)\delta(p_\lambda{}^2-m^2)p^\mu p^\nu D_1(x^A), \qquad p^\mu=\frac{mdx^\mu}{ds}=Gv^\mu . \tag{13.29}$$

The average value of a measurable quantity $Q(x,p)$ is defined as

$$\langle Q(x,p)\rangle=\int Q(x,p)D_1(x,p)d\mu, \qquad x^A=(x,p) . \tag{13.30}$$

The particle current density $j^\mu(x)$ in the 4-dimensional spacetime can be defined in terms of $j^\mu(x,p)=D_1(x,p)v^\mu$ in (13.20) with A=0,1,2,3, as follows,

$$j^\mu(x)=\int d^4p\,2G\theta(p^0)\delta(p_\lambda{}^2-m^2)D_1(x,p)v^\mu, \qquad Gv^\mu=p^\mu . \tag{13.31}$$

Also, the invariant density of a fluid can be defined as

$$\int D_1(x,p)\delta(p_\lambda{}^2-m^2)2G\theta(p^0)d^4p \ . \tag{13.32}$$

This can be verified in the simple case where $D_1(x,p)=A\rho(x)\exp(G/\tau)$.

In the presence of the external electromagnetic field tensor $f^{\mu\nu}$, when

$P(D_1)=0$ and $\partial D_1/\partial t_c=0$, (13.29), (13.28), (13.29) and $Gv^\mu = p^\mu$ lead to

$$\partial_\mu T^{\mu\nu} = j_\mu(x) f^{\mu\nu}, \tag{13.33}$$

where $j_\mu(x)$ is given by (13.31). Clearly, the result (13.33) reduces to the conservation law $\partial_\mu T^{\mu\nu}=0$ when $f^{\mu\nu}=0$.

13d. Invariant Liouville equation

We consider the Gibbs ensemble of similar systems of N particles, i.e. identical in composition and macroscopic conditions, but in different states. (cf., section (13.e) below) Such an ensemble is represented by an invariant N-particle density function $D_N(x^A_{a,c}) \equiv \rho_I(a, p_a, t_c)$, a=1,2,...N, which is normalized by the relation

$$\int \rho_I(x_a, p_a, t_c) d\Gamma = \int \rho^* \prod_{b=1}^{N} d^4x_b d^4p_b = N, \qquad (x_a, p_a) \equiv (x^\mu_a, p^\mu_a), \tag{13.34}$$

$$\rho^* \equiv \rho_I(x_a, p_a, t_c) \prod_{a=1}^{N} \delta(t_c - x_{0a}/b_a)\delta(p^2_a - m^2_a)\theta(p^0_a)(2G_a), \tag{13.35}$$

where $d\Gamma$ is given by (13.16). The density ρ_I satisfies the continuity equation

$$\frac{\partial}{\partial t_c}\rho_I + \frac{\partial}{\partial x^\mu}(v^\mu_a \rho_I) + \frac{\partial}{\partial p^\mu_a}(\dot{p}^\mu_a \rho_I) = 0, \tag{13.36}$$

where one sums over both a=1,2,...,N and μ=0,1,2,3. With the help of the Dirac Hamiltonian and eq. (13.13), we are able to obtain one single 4-dimensional invariant Liouville equation in common relativity,

$$\left(\frac{\partial}{\partial t_c} + v^\mu_a \frac{\partial}{\partial x^\mu_a} + F^\mu_a \frac{\partial}{\partial p^\mu_a}\right)\rho_I = \frac{d}{dt_c}\rho_I = 0, \tag{13.37}$$

for a system of N particles. We note that, in the framework of special relativity, one can only obtain N *one-particle* Liouville equations[2] rather than one single invariant Liouville equation:

$$\left(\frac{\partial}{\partial \tau_b}+\frac{dx_b^{\mu}}{d\tau_b}\frac{\partial}{\partial x_b^{\mu}}+\frac{dp_b^{\mu}}{d\tau_b}\frac{\partial}{\partial p_b^{\mu}}\right)\rho_1(x_b,p_b,\tau_b)=\frac{d}{d\tau_b}\rho_1(x_b,p_b,\tau_b)=0, \tag{13.38}$$

b is not summed, $\quad b=1,2,...N$,

where τ_b is the proper time of the particle b. The basic reason for this property is the absence of a universal (or common) time for all observers in special relativity. As a result, within the framework of special relativity, there is no notion of the canonical evolution of an N-particle system and the entire history of the system of N particles is represented by an N-dimensional manifold rather than a one-dimensional manifold.

With the help of (13.35) and the relation (13.26) for each particle, the Liouville equation (13.37) takes the more familiar form

$$\left(\frac{\partial}{\partial t_c}+\sum_a \mathbf{v}_a\cdot\frac{\partial}{\partial \mathbf{r}_a}+\sum_a \mathbf{F}_a\cdot\frac{\partial}{\partial \mathbf{p}_a}\right)\rho=0, \tag{13.39}$$

$$\rho_I(x_a,p_a,t_c)\equiv\rho\left(x_a^0(t_c),\mathbf{r}_a,p_a^0(t_c),\mathbf{p}_a,t_c\right)\prod_{b=1}^{N}\delta\left(x_b^0-x_b^0(t_c)\right)\delta\left(p_b^0-p_b^0(t_c)\right),$$

$$x_a^0(t_c)=b_at_c\,,\quad p_a^0(t_c)=\sqrt{\mathbf{p}_a^2(t_c)+m^2},\quad G_a=\frac{p_a^0(t_c)}{C_a},\qquad \text{(a is not summed)}.$$

Thus, we see that the explicit four-dimensional feature of (13.37) has been lost in this case, and the 3-dimensional form in (13.39) is the same as the usual Liouville equation in the nonrelativistic limit.

The ensemble average of a measurable property $f(x_a,p_a)$ of a system is defined by

$$\langle f(x_a,p_a)\rangle=\frac{\int f(x_a,p_a)\rho_I(x_a,p_a)d\Gamma}{\int\rho_I(x_a,p_a)d\Gamma}. \tag{11.40}$$

Relations $p_{0a}=\sqrt{p_a^2+m_a^2}$ and $x_{0a}=b_a t_c$ are understood in the functions $\rho(x_a,p_a,t_c)$ and $f(x_a,p_a)$ in (13.40).

13e. Invariant entropy, temperature and the Maxwell-Boltzmann distribution

Let us consider the distribution of momenta of a relativistic gas at equilibrium in a box having the invariant volume (13.2). Suppose one divides all the quantum states of an individual particle of the gas into groups denoted by i=1,2,3,... Each group contains neighboring states (with approximately the same genergies). Both the number of states w_i in group i and the number of particles n_i in these states are very large. Then the set of numbers n_i completely describes the macroscopic state of the gas. Each particle in group i has an invariant genergy G_i. These quantities satisfy the conditions

$$\sum_i n_i = N, \qquad \sum_i n_i G_i = G_{tot}. \tag{13.41}$$

Putting each of the n_i particles into one of the w_i states, one obtains $(w_i)^{n_i}$ possible distributions. Since all particles are identical, these possible distributions contain some identical ones which differ only by a permutation of the particles. The number of permutations of n_i particles is $n_i!$. Thus the statistical weight of the distribution of n_i particles over w_i states is $(w_i)^{n_i}/n_i!$. We define the entropy S_c of the gas as a whole to be

$$S_c = \sum_i \ln \Delta\Gamma_i, \qquad \Delta\Gamma_i = (w_i)^{n_i}/(n_i!). \tag{13.42}$$

When the gas is in the equilibrium state, the entropy of the system must be a maximum. Suppose the average number of particles in each of the quantum states of the group i is denoted by $<n_i>$, i.e., $<n_i>=n_i/w_i$. One can find those $<n_i>$ which give S_c in (13.42) its maximum possible value subject to the conditions in (13.41). By the usual methods of maximization of S_c and the Lagrange multipliers, we obtain the most probable distribution

$$< n > = A \quad \exp[-G_i / \tau], \tag{13.43}$$

where A is a normalization constant and τ is a "scalar temperature." We stress that $<n_i>$ is an invariant function and can be used to define invariant thermodynamic quantities.[3] (For a derivation of (13.43) using the genergy, see (13.95).)

Similarly, we can obtain the invariant Maxwell-Boltzmann equilibrium distribution (normalized by (13.17))

$$D_I^{MB}(p) = BN \quad \exp[-G / \tau], \tag{13.44}$$

with

$$G = \frac{\sqrt{p^2 + m^2}}{C}, \qquad B = \frac{1}{4} pm^2 V_I \tau c K_2\left(\frac{m}{c\tau}\right),$$

where K_v (z) is the Bessel function with an imaginary argument and can be expressed in terms of the Hankel function $H_v^{(1)}(iz)$: $K_v(z)=(\pi i/2)\exp[-i\pi v/2] \times H^{(1)}(iz)$. In the nonrelativistic limit, we choose the frame F in which C=c and make the approximation of large mass m or $\sqrt{p^2+m^2} \approx m+mv^2/2c^2$. We have

$$D_I^{MB} \approx \frac{N \ \exp[-m / c\tau - p^2 / 2mc\tau]}{V_I 4m^2 c\tau\sqrt{\pi c\tau / 2m} \ \exp[-m / c\tau]}$$

$$= \frac{N / V_I}{(2\pi mc)^{3/2}} \exp[-mv^2 / 2c^3\tau]. \tag{13.45}$$

Comparing this with the Maxwell-Boltzmann distribution $\propto \exp[-mv^2/2k_BT]$, we see that the invariant temperature τ is related to the usual temperature T and the Boltzmann constant k_B by the relation (in the F frame):

$$\tau = k_B T / c^3. \tag{13.46}$$

Note that the concept of an invariant temperature τ is closely related to the fact that the genergy G of a particle is invariant and proportional to the energy in the F frame. In special relativity, the usual procedure is to introduce an inverse temperature 4-vector for the invariant form of the Maxwell-Boltzmann distribution. However, it is not at all clear that such a notion of a temperature 4-vector has any physical meaning.[2]

13f. Invariant Boltzmann-Vlasov equation

For a non-equilibrium gas, one may describe the system in terms of the coordinates x^μ and velocities v^μ of the particles which comprise it. Suppose the system is described by the distribution $f(x^\mu,v^\mu)$. The rate of change of the distribution $f(x^\mu,v^\mu)$ in a general inertial frame is given by the invariant equation

$$\frac{df}{dt_c} = \frac{dx^\mu}{dt_c}\frac{\partial f}{\partial x^\mu} + \frac{dv^\mu}{dt_c}\frac{\partial f}{\partial v^\mu} = v^\mu \frac{\partial f}{\partial x^\mu} + a^\mu \frac{\partial f}{\partial v^\mu}, \tag{13.47}$$

$$x^\mu = (bt_c,x,y,z)\ , \qquad \frac{dx^\mu}{dt_c} = v^\mu = (C,\mathbf{v}), \qquad \frac{dv^\mu}{dt_c} = a^\mu = (a^0,\mathbf{a}), \qquad C = \frac{d(bt_c)}{dt_c},$$

where the velocities v^μ and accelerations a^μ are 4-vectors in common relativity because the common time t_c is a scalar. The physical meaning of the distribution $f(x^\mu,v^\mu)$ can be seen from the following invariant relations

$$\int f(x^\mu,v^\mu)d^4v\delta(\dot{s}^2-C^2+\mathbf{v}^2) = n(x^\mu) = n(bt_c,x,y,z), \tag{13.48}$$

$$\dot{s}^2 \equiv \left(\frac{ds}{dt_c}\right)^2 = \left(\frac{dw}{dt_c}\right)^2 - \left(\frac{d\mathbf{r}}{dt_c}\right)^2 = C^2 - \mathbf{v}^2, \qquad \text{and} \qquad \frac{dw}{dt_c} = \frac{d(bt_c)}{dt_c} = C,$$

where $n(bt_c,x,y,z)$ is the number of molecules per unit volume at the time t_c and position $\mathbf{r}=(x,y,z)$ in a general inertial frame. The function $\delta(\dot{s}^2-C^2+\mathbf{v}^2)$ is an invariant constraint of the 4-dimensional velocity integration $\int d^4v$. This is the same as a mass-shell constraint, $\delta(p_\lambda{}^2-m^2)$, in the momentum integration $\int d^4p$. In the framework of common relativity, we have, by definition, the simple familiar form of the density function $n(t,x,y,z)$ of molecules because b=C=c=constant in the frame F. In a general inertial frame, integrating over the invariant volume given by (13.1) yields the total number N of molecules,

$$\int n(x^{\mu})dV_I = N, \tag{13.49}$$

which is a constant, independent of the common time t_C and the reference frame.

If the molecules do not collide, then $df/dt_C = 0$, i.e.,

$$v^{\mu} \frac{\partial f}{\partial x^{\mu}} + a^{\mu} \frac{\partial f}{\partial v^{\mu}} = 0, \qquad f = f(x^{\mu}, v^{\mu}). \tag{13.50}$$

This is a special case of the Liouville theorem. If the molecules collide with each other, then df/dt_C is not equal to zero and the function f can be understood based on Boltzmann's collision postulate.[5]

For a high temperature plasma, let us consider the generalized 4-dimensional Boltzmann-Vlasov equation based on common relativity. Vlasov first applied the equation to plasmas. The equation in a general inertial frame is assumed to have the form of (13.50) with the acceleration $a^{\mu}=(a^0,\mathbf{a})$ of charged particles expressed in terms of the electromagnetic fields:

$$C\frac{\partial f}{\partial(bt_C)} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + a^0(\mathbf{E},\mathbf{B},\mathbf{v})\frac{\partial f}{\partial C} + \mathbf{a}(\mathbf{E},\mathbf{B},\mathbf{v})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{13.51}$$

where $f=f(bt_C,\mathbf{r},C,\mathbf{v})\equiv f_1$ is the one-particle distribution function. The relation between the 4-acceleration $a^{\mu}=(a^0,\mathbf{a})$ and the electromagnetic fields can be obtained from the equation of motion (13.14) with the 4-coordinate $x^{\mu}=(w,\mathbf{r})\equiv(bt_C,\mathbf{r})$ and the 4-dimensional transformations in common relativity. The equation of motion of a charged particle can be written as

$$\frac{dp^{\mu}}{ds} = \bar{e}\, f^{\mu\nu}\frac{dx_{\nu}}{ds}, \qquad \text{or} \qquad \frac{dp^{\mu}}{dt_C} = \bar{e}\, f^{\mu\nu}\frac{dx_{\nu}}{dt_C}, \tag{13.52}$$

where $x_{\nu}=(bt_C,-\mathbf{r})$ and

$$p^{\mu} = m\frac{dx^{\mu}}{ds} = (p^0, \mathbf{p}) = \left(\frac{m}{\sqrt{1-\beta^2}}, \frac{m\boldsymbol{\beta}}{\sqrt{1-\beta^2}}\right), \qquad \boldsymbol{\beta} = \frac{\mathbf{v}}{C}. \tag{13.53}$$

The fourth component of the equation of motion in (13.52) is not independent of the three spatial components. The three independent equations of motion can be written as

$$\frac{d\mathbf{p}}{dt_c} = \bar{e}(C\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{13.54}$$

Using the expression for $\mathbf{p}$ in (13.53) and $dv^{\mu}/dt_c = (dC/dt_c, d\mathbf{v}/dt_c) = (a^0, \mathbf{a})$, eq. (13.54) becomes

$$\frac{m\gamma}{C}\left[\mathbf{a} + \gamma^2 \frac{\mathbf{v}}{C^2}(\mathbf{v}\cdot\mathbf{a})\right] - m\gamma^3 \frac{\mathbf{v}a^0}{C^2} = \bar{e}(C\mathbf{E} + \mathbf{v} \times \mathbf{B}). \qquad \gamma = \frac{1}{\sqrt{1 - v^2/C^2}}. \tag{13.55}$$

Now let us consider the Boltzmann-Vlasov equation in the F frame, in which the speed of light is constant, i.e., $a^0 = \dot{C} = 0$ or $C = c =$ constant. One can solve for the acceleration $\mathbf{a}$ in terms of $\mathbf{E}$ and $\mathbf{B}$ by setting $\mathbf{a} = Z_1(C\mathbf{E} + \mathbf{v} \times \mathbf{B}) + Z_2\mathbf{v}(\mathbf{v}\cdot\mathbf{E})$ and using (13.55) with $\dot{C} = 0$ to determine Z_1 and Z_2. We obtain

$$\mathbf{a} = \frac{c\bar{e}}{m\gamma}\left(c\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) - \frac{\bar{e}}{m\gamma}\mathbf{v}(\mathbf{v}\cdot\mathbf{E}), \qquad \dot{C} = 0. \tag{13.56}$$

Thus, the generalized 4-dimensional Boltzmann-Vlasov equation based on common relativity can be written as

$$\frac{\partial f}{\partial t_c} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \left\{\frac{c\bar{e}}{m\gamma}\left(c\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) - \frac{\bar{e}}{m\gamma}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\right\}\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{13.57}$$

in the F frame. If the second order terms in (v/c) are neglected, equation (13.57) reduces to the usual Boltzmann-Vlasov equation,

$$\frac{\partial f}{\partial t_c}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{c^2\bar{e}}{m\gamma}\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f}{\partial \mathbf{v}}=0. \qquad (13.58)$$

Let us consider another inertial frame F' in which the speed of light c' is not a constant, $\dot{C}=\dot{c}'\neq 0$. In this case, (13.55) is written in terms of primed quantities measured by observers in the F' frame,

$$\frac{m\gamma'}{c'}\left[\boldsymbol{a}'+\gamma'^2\frac{\mathbf{v}'}{c'^2}(\mathbf{v}'\cdot\boldsymbol{a}')\right]-m\gamma'^3\frac{\mathbf{v}'a'^0}{c'^2}=\bar{e}(c'\mathbf{E}'+\mathbf{v}'\times\mathbf{B}'). \qquad (13.59)$$

The quantity $a'_0=\dot{c}'$ in F' can be solved by multiplying (13.59) with $\mathbf{v}'$. We obtain

$$a'_0=\frac{c'}{\beta'^2}\left[\frac{\mathbf{v}'\cdot\boldsymbol{a}'}{c'^2}-\frac{\bar{e}}{m\gamma'^3}(\mathbf{v}'\cdot\mathbf{E}')\right], \qquad \boldsymbol{\beta}'=\frac{\mathbf{v}'}{c'}, \qquad \gamma'=\frac{1}{\sqrt{1-\beta'^2}}. \qquad (13.60)$$

Substituting (13.60) into (13.59), one has

$$\frac{m\gamma'}{c'}\left[\boldsymbol{a}'+\frac{\mathbf{v}'}{\beta'^2c'^2}(\mathbf{v}'\cdot\boldsymbol{a}')\right]-\frac{\bar{e}\mathbf{v}'}{\beta'^2c'}(\mathbf{E}'\cdot\mathbf{v}')=\bar{e}(c'\mathbf{E}'+\mathbf{v}'\times\mathbf{B}'). \qquad (13.61)$$

One can solve for the acceleration $\boldsymbol{a}'$ in terms of $\mathbf{E}'$ and $\mathbf{B}'$ by letting $\boldsymbol{a}' = Y_1(c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}') + Y_2\mathbf{v}'(\mathbf{v}'\cdot\mathbf{E}')$ and using (13.61) to determine Y_1 and Y_2. One finds that

$$a'_0=\left\{\frac{c'\bar{e}}{m\gamma'}+Y_2\right\}\mathbf{v}'\cdot\mathbf{E}', \quad \text{and}$$

$$\boldsymbol{a}'=\frac{\bar{e}c'}{m\gamma'}\left\{c'\mathbf{E}'+\mathbf{v}'\times\mathbf{B}'+Y_2\frac{\mathbf{v}'}{c'}(\mathbf{E}'\cdot\mathbf{v}')\right\}, \qquad (13.62)$$

where the function Y_2 cannot be determined because there are four unknown variables, but only three independent equations of motion. However, the 4-acceleration a^μ in F and a'^μ in F' must be related by the following 4-dimensional transformation:

$$a_0=\gamma(a'_0+\beta a'_x), \quad a_x=\gamma(a'_x+\beta a'_0), \quad a_y=a'_y, \quad a_z=a'_z. \qquad (13.63)$$

For simplicity, let us consider the y-component in (13.63). From (13.56) and (13.62), we have

$$\frac{c\bar{e}}{m\gamma}\left(CE + \mathbf{v}\times\mathbf{B}\right)_y + \frac{\bar{e}}{m\gamma}v_y(\mathbf{v}\cdot\mathbf{E})$$

$$= \frac{\bar{e}c'}{m\gamma'}\left\{(c'\mathbf{E}'+\mathbf{v}'\times\mathbf{B}')_y + Y_2\frac{v'_y}{c'}(\mathbf{E}'\cdot\mathbf{v}')\right\}. \qquad (13.64)$$

Based on the transformations for **E**, **B** and v^μ in equations (13.33) and (13.9), one can show that

$$\frac{c\bar{e}}{m\gamma} = \frac{\bar{e}}{m}\sqrt{c^2 - v^2} = \frac{\bar{e}}{m}\sqrt{c'^2 - v'^2} = \frac{c'\bar{e}}{m\gamma'},$$

$$(c\mathbf{E} + \mathbf{v}\times\mathbf{B})_y = (c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}')_y, \qquad \text{and} \qquad v_y = v'_y . \qquad (13.65)$$

It follows from (13.64) and (13.65) that

$$Y_2 = -\frac{c\bar{e}}{m\gamma}\frac{(\mathbf{E}\cdot\mathbf{v})/c}{(\mathbf{E}'\cdot\mathbf{v}')/c'}. \qquad (13.66)$$

If one wishes, one may express c/γ and $(\mathbf{E}\cdot\mathbf{v})/c$ in (13.66) in terms of quantities measured by observers in the F' frame with the help of the transformation properties of $(\mathbf{E}\cdot\mathbf{v})/c$ (or the 4-vector in (13.73) below) and the invariant $\sqrt{c^2-v^2} = \sqrt{c'^2-v'^2}$. Note that this function Y_2 can also be obtained from other relations in the transformation (13.63) for the 4-acceleration a^μ in common relativity.

Thus, the 4-acceleration a'^μ is completely determined by (13.62) and (13.66)

$$a'_0 = \frac{\bar{e}c'}{m\gamma'}\left[\mathbf{v}'\cdot\mathbf{E}' - c'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right],$$

$$\mathbf{a}' = \frac{\bar{e}c'}{m\gamma'}\left[c'\mathbf{E}'+\mathbf{v}'\times\mathbf{B}' - \mathbf{v}'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right]. \qquad (13.67)$$

Thus, the generalized Boltzmann-Vlasov equation (13.51) in the F' frame can be written as

$$\frac{\partial f'}{\partial t_c}+\mathbf{v}'\cdot\frac{\partial f'}{\partial \mathbf{r}'}+\frac{\bar{e}c'}{m\gamma'}\left[\mathbf{v}'\cdot\mathbf{E}'-c'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right]\frac{\partial f'}{\partial c'}$$

$$+\frac{\bar{e}c'}{m\gamma'}\left[c'\mathbf{E}'+\mathbf{v}'\times\mathbf{B}'-\mathbf{v}'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right]\cdot\frac{\partial f'}{\partial \mathbf{v}'}=0, \qquad (13.68)$$

where $f'=f(b't_c,\mathbf{r}',c',\mathbf{v}')$. Since $(\mathbf{v}\cdot\mathbf{E},c\mathbf{E}+\mathbf{v}\times\mathbf{B})$ and $(c,\mathbf{v})$ are 4-vectors, the quantity $(\mathbf{v}\cdot\mathbf{E})/c$ in (13.68) can be expressed in terms of the primed quantities $\mathbf{v}'$, $\mathbf{E}'$, $\mathbf{B}'$ and c' measured by observers in F' by using transformations (13.9) and (13.33). Nevertheless, it is convenient to leave the unprimed quantity in (13.67) and (13.68). Using the velocity transformation (13.9), one can also show that the accelerations $a^\mu=(0,\mathbf{a})$ in (13.56) and a'^μ in (13.67) are related by a 4-dimensional transformation.

Combining (13.56) and (13.67) gives

$$a_0^2-\mathbf{a}^2=-\left\{\frac{c\bar{e}}{m\gamma}\left(c\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)-\frac{\bar{e}}{m\gamma}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\right\}^2$$

$$=-\left(\frac{c\bar{e}}{m\gamma}\right)^2\left[\left(c\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)^2-(\mathbf{v}\cdot\mathbf{E})^2-\frac{(\mathbf{v}\cdot\mathbf{E})^2}{\gamma^2}\right], \qquad (13.69)$$

where $a_0=0$, $\gamma=1/\sqrt{1-v^2/c^2}$, and

$$a_0'^2-\mathbf{a}'^2=\left(\frac{c'\bar{e}}{m\gamma'}\right)^2\left\{\left[\mathbf{v}'\cdot\mathbf{E}'-c'\frac{\mathbf{v}\cdot\mathbf{E}}{c}\right]^2-\left[\left(c'\mathbf{E}'+\mathbf{v}'\times\mathbf{B}'\right)-\mathbf{v}'\frac{(\mathbf{v}\cdot\mathbf{E})}{c}\right]^2\right\}$$

$$=-\left(\frac{c\bar{e}}{m\gamma}\right)^2\left[\left(c'\mathbf{E}'+\mathbf{v}'\times\mathbf{B}'\right)^2-(\mathbf{v}'\cdot\mathbf{E}')^2-\frac{(\mathbf{v}\cdot\mathbf{E})^2}{\gamma^2}\right], \qquad (13.70)$$

where we have used the invariant relation $c/\gamma = c'/\gamma'$. Note that the unprimed terms $(\mathbf{v}\cdot\mathbf{E})^2/\gamma^2$ in (13.69) and (13.70) are the same. Thus, the invariant relation

$$a_0^2 - a^2 = -a^2 = a_0'^2 - a'^2 \tag{13.71}$$

is equivalent to the invariant relation

$$(c\mathbf{E} + \mathbf{v}\times\mathbf{B})^2 - (\mathbf{v}\cdot\mathbf{E})^2 = (c'\mathbf{E}' + \mathbf{v}'\times\mathbf{B}')^2 - (\mathbf{v}'\cdot\mathbf{E}')^2. \tag{13.72}$$

This is not surprising because

$$\frac{dp^\mu}{dt_c} = \bar{e}f^{\mu\nu}\frac{dx_\nu}{dt_c} = \bar{e}\Big((\mathbf{v}\cdot\mathbf{E}), (c\mathbf{E} + \mathbf{v}\times\mathbf{B})\Big) \tag{13.73}$$

is a 4-vector in common relativity. It is stressed that the acceleration is the quantity with the simplest physical properties in common relativity because of the presence of the common time t_c and the 4-dimensional symmetry.

13g. Boltzmann's transport equation with 4-dimensional symmetry

For the classical kinetic theory of dilute gases for a system of N molecules enclosed in a box of volume V_I, we are interested in the invariant distribution function f under a 4-dimensional transformation. Let us define the distribution f as a function of the coordinates and momenta, $(x^\mu,p^\nu)\equiv(x,p)$, instead of a function of the coordinates and velocities (x^μ,v^ν). Such a definition of f(x,p) will make the 4-dimensional symmetry more explicit, as we shall see below. In a general inertial frame, the invariant volume element in the μ-space given by (13.15) can be rewritten as

$$d\mu = d^4x\delta\left(\frac{x^0}{b} - t_c\right)d^4p\delta(p_\lambda^2 - m^2)\theta(p^0)(2G)$$

$$= d^3xdx^0\delta\left(\frac{x^0}{b} - t_c\right)d^3pdp^0\delta(p_\lambda^2 - m^2)\theta(p^0)\left(\frac{2p^0}{C}\right).$$

The distribution function f(x,p) is defined so that classically, the number of molecules dN at common time t_c in a volume element d^3x about **r** and within a momentum space volume element d^3p around $\mathbf{p}$ is given by

$$dN = dN = \int_{[x^0p^0]} f(x,p)d\mu = \int f(x,p)dV_I d^3p \frac{c_o}{C}, \tag{13.74}$$

$$dV_I = \int_{[x^0]} \frac{d^4x}{c_o}\delta\left(t_c - \frac{x^0}{b}\right), \tag{13.75}$$

$$d^3p\frac{c_o}{C} = \frac{d^3p}{2p^0}(2Gc_o), \qquad p^0 = \sqrt{\mathbf{p}^2 + m^2}, \tag{13.76}$$

where integrations over x^0 and p^0 have been carried out in (13.74) and c_o is a scaling constant defined in (13.1) to give the correct dimensions for dV_I.

Note that both dV_I and $d^3p(c_o/C)$ are invariant in common relativity and that they are large enough so that dN is a very large number ($\sim 10^{10}$) of molecules. Nevertheless, the volume elements dV_I and $d^3p(c_o/C)$ are microscopic in size. Also, if the distribution f(x,p) is independent of the position $\mathbf{r}$, we have the usual relation, $\int f(x,p)d^3p = N/V$, in the inertial frame $F(ct_c,\mathbf{r})$ in which $c_o=c=C$ and $\int dV_I = V_I$, where V_I is the volume of a box. The distribution f(x,p) can be identified with D_1 in (13.28) when D_1 does not depend on t_c explicitly, $f(x,p)=D_1(x,p)$.

Let us consider equation (13.28) with the collision term P(D), where we use D to denote the one-particle distribution function for the following discussions,

$$\left(\frac{\partial}{\partial t_c} + v^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu}\right)D = P(D), \qquad D = D(x,p,t_c), \tag{13.77}$$

$$v^\mu = \frac{dx^\mu}{dt_c} = (C,\mathbf{v}), \qquad F^\mu = \frac{dp^\mu}{dt_c} = \left(\frac{dp^0}{dt_c}, \frac{d\mathbf{p}}{dt_c}\right),$$

where the collision term $P(D) = (\partial D/\partial t_c)_{coll}$ must be specified for (13.77) to be meaningful. In order to have an invariant expression for the collision term P(D), we use quantum mechanics to treat the scattering process. Although the molecules are regarded as classical objects, "they see each other as plane waves of definite momenta rather than wave packets of well-defined positions"[6] in scattering processes. By considering the collision process, $p_1 + p_2 \rightarrow p'_1 + p'_2$, one can express $(\partial D/\partial t_c)_{coll} = (\partial D_1/\partial t_c)_{coll}$ in the invariant form,

$$P(D_1) = \left(\frac{\partial D_1}{\partial t_c}\right)_{coll} = \int \frac{d^3p_2}{2p_{20}} \frac{d^3p'_1}{2p'_{10}} \frac{d^3p'_2}{2p'_{20}} \delta^4\left(p_1 + p_2 - p'_1 - p'_2\right)$$

$$\times \left|M_{fi}\right|^2 \left(D'_1 D'_2 - D_1 D_2\right), \tag{13.78}$$

where $D_i=D(x,p_i,t_c)$ and $D'_i=D(x,p'_i,t_c)$, i=1,2. From equations (13.77) and (13.78), we have the 4-dimensional Boltzmann transport equation based on common relativity:

$$\left(\frac{\partial}{\partial t_c} + C\frac{\partial}{\partial(bt_c)} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{dp_1^0}{dt_c}\frac{\partial}{\partial p_1^0} + \frac{d\mathbf{p}_1}{dt_c} \cdot \frac{\partial}{\partial \mathbf{p}_1}\right) D_1$$

$$= \int \frac{d^3p_2}{2p_{20}} \frac{d^3p'_1}{2p'_{10}} \frac{d^3p'_2}{2p'_{20}} \delta^4(p_1 + p_1 - p'_1 - p'_2)\left|M_{fi}\right|^2 (D'_1 D'_2 - D_1 D_2). \tag{13.79}$$

In the inertial frame $F(ct_c,\mathbf{r})$ in which b=c=C, if the distribution function D_1 is a function of x and p only,[7] then the left side of (13.79) reduces to the more familiar form

$$\frac{\partial D_1}{\partial t_c} + \mathbf{v}_1 \cdot \frac{\partial D_1}{\partial \mathbf{r}} + \frac{dp_1^0}{dt_c}\frac{\partial D_1}{\partial p_1^0} + \frac{d\mathbf{p}_1}{dt_c} \cdot \frac{\partial D_1}{\partial \mathbf{p}_1}, \qquad D_1 = D_1(x,p). \tag{13.80}$$

Note that the third and fourth terms are not independent because of the relation $p_1^0=\sqrt{\mathbf{p}_1^2 + m^2}$. However, it is more convenient to leave Boltzmann's transport equation in the form (13.80) [or the left side of (13.79)] to have an explicit 4-dimensional form. We also note that $(dp_1{}^0/dt_c)$ is smaller than $|d\mathbf{p}_1/dt_c|$ by a factor v/c, as one can see from (13.73). Thus, if the velocity $v=|\mathbf{v}|$ is much smaller than c, (13.80) reduces to the usual form:

$$\frac{\partial D_1}{\partial t_c} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial D_1}{\partial \mathbf{r}} + \frac{d\mathbf{p}_1}{\partial t_c} \cdot \frac{\partial D_1}{\partial \mathbf{p}_1}. \tag{13.81}$$

In this connection, it is worthwhile to note that if the Boltzmann-Vlasov equation for high temperature plasma is expressed in terms of the distribution function $D_1(x,p)$, we have

$$\frac{\partial D_1}{\partial(bt_c)} + \frac{\mathbf{p}}{p_0}\cdot\frac{\partial D_1}{\partial \mathbf{r}} + \bar{e}\frac{\mathbf{E}\cdot\mathbf{p}}{p^0}\frac{\partial D_1}{\partial p^0} + \bar{e}\left(\mathbf{E} + \frac{\mathbf{p}\times\mathbf{B}}{p^0}\right)\frac{\partial D_1}{\partial \mathbf{p}} = 0\ , \tag{13.82}$$

in a general inertial frame, where we have used the following relations

$$p_0 = \sqrt{\mathbf{p}^2 + m^2} = \frac{m}{\sqrt{1 - v^2/C^2}}, \qquad \mathbf{p} = \frac{m\mathbf{v}/C}{\sqrt{1 - v^2/C^2}}, \qquad \frac{\mathbf{v}}{C} = \frac{\mathbf{p}}{p_0}, \tag{13.83}$$

$$\frac{1}{C}\frac{dp_0}{dt_c} = \frac{\bar{e}\mathbf{E}\cdot\mathbf{v}}{C} = \bar{e}\frac{\mathbf{E}\cdot\mathbf{p}}{p^0}, \qquad \frac{1}{C}\frac{d\mathbf{p}}{dt_c} = \bar{e}\left(\mathbf{E} + \frac{\mathbf{p}\times\mathbf{B}}{p^0}\right). \tag{13.84}$$

13h. Boltzmann's H-theorem with 4-dimensional symmetry

In common relativity, the invariant Boltzmann functional $H(t_c)$ can be defined in terms of the invariant distribution $D(p^\mu,t_c) \equiv D(p,t_c)$,

$$H(t_c) = \int D(p_1,t_c) \ln D(p_1,t_c) \frac{d^3p_1}{2p_{10}}, \qquad p_{10} = \sqrt{\mathbf{p}_1^2 + m^2}, \tag{13.85}$$

where the invariant distribution D is assumed to be a function of the 4-momentum p_1^μ and common time t_c and which satisfies[8]

$$\frac{\partial D_1}{\partial t_c} = \int \frac{d^3p_2}{2p_{20}}\frac{d^3p'_1}{2p'_{10}}\frac{d^3p'_2}{2p'_{20}}\delta^4(p_1 + p_2 - p'_1 - p'_2)$$

$$\times\left|M_{fi}\right|^2 (D'_1D'_2 - D_1D_2), \qquad D_1 = D(p_1,t_c). \tag{13.86}$$

Note that both the distribution D_1 and $d^3p_1/(2p_{10})$ are invariant. It follows from (13.85) and (13.86) that

$$\frac{dH(t_c)}{dt_c} = \int \frac{d^3p_1}{2p_{10}} \frac{d^3p_2}{2p_{20}} \frac{d^3p'_1}{2p'_{10}} \frac{d^3p'_2}{2p'_{20}} \delta^4(p_1 + p_2 - p'_1 - p'_2)$$

$$\times |M_{fi}|^2 (D'_1 D'_2 - D_1 D_2)[1 + \ln D_1]. \tag{13.87}$$

Note that the transition matrix M_{fi} is invariant under the interchange of $\mathbf{p}_1$ and $\mathbf{p}_2$, so that the integrand in (13.87) is unchanged under $\mathbf{p}_1 \leftrightarrow \mathbf{p}_2$. From (13.87) and the new expression obtained by $\mathbf{p}_1 \leftrightarrow \mathbf{p}_2$, we have

$$\frac{dH(t_c)}{dt_c} = \int \frac{d^3p_1}{2p_{10}} \frac{d^3p_2}{2p_{20}} \frac{d^3p'_1}{2p'_{10}} \frac{d^3p'_2}{2p'_{20}} \delta^4(p_1 + p_2 - p'_1 - p'_2)$$

$$\times |M_{fi}|^2 (D'_1 D'_2 - D_1 D_2)\left[1 + \frac{1}{2}\ln(D_1 D_2)\right]. \tag{13.88}$$

For every scattering there is an inverse scattering with the same transition matrix M_{fi}. Thus, the integral (13.88) is invariant under the interchange $\{\mathbf{p}_1, \mathbf{p}_2\} \leftrightarrow \{\mathbf{p}'_1, \mathbf{p}'_2\}$,

$$\frac{dH(t_c)}{dt_c} = -\int \frac{d^3p_1}{2p_{10}} \frac{d^3p_2}{2p_{20}} \frac{d^3p'_1}{2p'_{10}} \frac{d^3p'_2}{2p'_{20}} \delta^4(p_1 + p_2 - p'_1 - p'_2)$$

$$\times |M_{fi}|^2 (D'_1 D'_2 - D_1 D_2)\left[1 + \frac{1}{2}\ln(D'_1 D'_2)\right]. \tag{13.89}$$

It follows from (13.88) and (13.89) that

$$\frac{dH(t_c)}{dt_c} = \frac{1}{4}\int \frac{d^3p_1}{2p_{10}} \frac{d^3p_2}{2p_{20}} \frac{d^3p'_1}{2p'_{10}} \frac{d^3p'_2}{2p'_{20}} \delta^4(p_1 + p_2 - p'_1 - p'_2)$$

$$\times |M_{fi}|^2 (D'_1 D'_2 - D_1 D_2)[\ln(D_1 D_2) - \ln(D'_1 D'_2)]. \tag{13.90}$$

Since the integrand of this equation is non-negative, we have

$$\frac{dH(t_c)}{dt_c} \leq 0 \tag{13.91}$$

in all inertial frames.

We observe that equation (13.91) vanishes, i.e., $dH(t_C)/dt_C=0$, if and only if the integrand of (13.90) vanishes identically. In other words, the condition $dH(t_C)/dt_C=0$ is the same as

$$D'_o(\mathbf{p}'_1)D'_o(\mathbf{p}'_2) = D_o(\mathbf{p}_1)D_o(\mathbf{p}_2), \tag{13.92}$$

where $D_o(\mathbf{p})$ denotes the equilibrium distribution. This has an interesting implication concerning the explicit form of the invariant equilibrium (or Maxwell-Boltzmann) distribution $D_o(\mathbf{p})$. Equation (13.92) leads to

$$\ln D'_o(\mathbf{p}'_1) + \ln D'_o(\mathbf{p}'_2) = \ln D_o(\mathbf{p}_1) + \ln D_o(\mathbf{p}_2), \tag{13.93}$$

for any possible collision process, $\mathbf{p}_1 + \mathbf{p}_2 \rightarrow \mathbf{p}'_1 + \mathbf{p}'_2$. Thus, the relation (13.93) can be interpreted as a conservation law. Since the function $\ln D_o(\mathbf{p})$ is a scalar in common relativity, the only quantity which is both a scalar and conserved is the genergy $G(\mathbf{p})$. In the F frame, in which $C=c$, we have $G(\mathbf{p})=p_0/c=\sqrt{\mathbf{p}^2+m^2}/c$. Clearly, the conservation of the genergy,

$$G(\mathbf{p}_1) + G(\mathbf{p}_2) = G(\mathbf{p}'_1) + G(\mathbf{p}'_2), \tag{13.94}$$

is the same as the conservation of the energy p_0 in the F frame. Since the genergy $G(\mathbf{p}_1)=p_{10}/C_1$ is an invariant, the relation (13.94) holds for all inertial frames. Thus, the only solution of (13.93) is

$$\ln D_o(\mathbf{p}) = -AG(\mathbf{p}) + \ln B, \qquad \text{or} \qquad \ln D_o(\mathbf{p}) = B\exp[-AG(\mathbf{p})], \tag{13.95}$$

where A and B are arbitrary constants that can be determined by measurements of the physical properties of a system.

In the literature, most of the formalisms on relativistic statistical mechanics are not manifestly covariant. Some of them are, strictly speaking, not covariant because of the approximations involved. Some formalisms lack proof of their effective covariance.[2]

We have demonstrated some novel and interesting features of the 4-dimensional symmetry framework of common relativity. This framework has a considerable advantage over the usual spacetime framework with regard to the formulation of statistical mechanics within a 4-dimensional symmetry framework. The existence of a scalar common time t_C, the 4-coordinate (bt_C,x,y,z), and the genergy G in common relativity is the key to preserving the invariant phase space of initial particle positions and velocities, the concept of a Hamiltonian system with many degrees of freedom, the Liouville equation and the invariant Maxwell-Boltzmann distribution with a scalar temperature.

References

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7. In the case where $D_1 = D_1(x,p)$, the 4-dimensional transport equation (13.79) or (13.80) can be used to derive the conservation theorem associated with the distribution function (see ref. 5). For example, one can show that

$$\int \frac{d^3p}{2p_0}\, Z(x,p)\left(\frac{\partial D}{\partial t_c}\right)_{coll} = 0,$$

and

$$\int \frac{d^3p}{2p_0}\, Z(x,p)\left(v^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu}\right) D(x,p) = 0,$$

where $(\partial D/\partial t_c)_{coll}$ is given by (13.78), and $Z(x,p)$ denotes the conserved quantity associated with the collision process $p_1 + p_2 \rightarrow p'_1 + p'_2$, i.e., $Z(x,p_1) + Z(x,p_2) = Z(x,p'_1) + Z(x,p'_2)$. These results are useful for deriving conservation laws of mass, momentum, etc. in hydrodynamics in a 4-dimensional symmetry framework.
8. This is equivalent to assuming that the distribution function $D(p_1,t_c)$ in a general inertial frame (1) is independent of the 4-coordinate x^μ and (2) satisfies Boltzmann's transport equation (13.28) without an external force, $F^\mu=0$, and with the collision term $P(D_1)$ given by (13.78).

14.

Common Relativity and the 3K Cosmic Microwave Background

14a. Implications of an invariant and non-invariant Planck's law for blackbody radiation

Since the anisotropy of the 3K cosmic background radiation was experimentally established,[1] many physicists have concluded that it is now possible to talk about "absolute motion".[2] This conclusion was reached within the framework of special relativity, in which the Planck law for blackbody radiation is not invariant under the Lorentz transformation.[3] If one is not aware that the 4-dimensional symmetry can be understood from a broader viewpoint than that of just special relativity, this conclusion appears to be unavoidable. However, for Planck's law for blackbody radiation not to be invariant is unnatural and inconsistent with the fundamental Poincaré-Einstein principle of relativity for physical laws.

As we have shown in previous chapters, nature can be viewed from the standpoint of a 4-dimensional theory of relativity with a common (but not absolute) time for all observers. Existing experiments cannot distinguish between common relativity and special relativity because they both have the 4-dimensional symmetry of the Lorentz and the Poincaré groups. Nevertheless, the conceptual framework of common relativity is not exactly the same as that of special relativity. For example, we can introduce the concepts of an invariant genergy $G=p_0/C$ and the invariant volume of a box for a many-particle system. With the help of these concepts, we can formulate a new covariant thermodynamics which leads to an invariant Planck law.[4]

14b. Invariant partition function

Let us consider systems with discrete states labelled by an index i, since the generalization to systems with continuous states is straightforward. Each state of the system corresponds to an invariant genergy $G = p^0/C > 0$. Because of its invariance, the genergy G corresponds more closely to the scalar energy in classical physics than the non-scalar energy p^0. We assume, as usual, that the interaction between molecules (or particles) is sufficiently weak so that there is an exchange of genergy between molecules but no change in the structure and properties of a molecule.

Consider a gas with N particles, where $N \gg 1$. Suppose the μ space volume is divided into K cells with $K \gg 1$. Each particle in the cell ω_i is in the state i and has genergy G_i. Also suppose there are n_i particles in the cell ω_i. Assuming that the *a priori* probability of the cell ω_i to be occupied is 1, we have the probability

$$W(n_1, n_2, ...) = N!/\left(\prod_i n_i!\right) \tag{14.1}$$

for n_1 particles in ω_1, n_2 particles in ω_2, etc. The equilibrium state of the gas corresponds to a state of maximum $W(n_1, n_2, ...)$ consistent with the constraints

$$N = \sum_i n_i, \tag{14.2}$$

$$G_t = \sum_i n_i G_i = N\overline{G}, \tag{14.3}$$

where $\overline{G}$ is the average genergy of the constituent particles. (For simplicity, we do not consider modifications due to the radical length.) In equilibrium, we have $n_i = \overline{n}_i$, where

$$\overline{n}_i = \frac{N \exp[-\overline{\beta} G_i]}{Z}, \qquad \overline{\beta} = \text{parameter}, \tag{14.4}$$

which is obtained by varying $W(n_1, n_2, ...)$ with constraints (14.2) and (14.3). The invariant partition function Z is defined by

$$Z = \sum_i \exp[-\bar{\beta} G_i], \qquad \bar{\beta} > 0, \tag{14.5}$$

provided the summation over the states i does not change the invariance of (14.5). From (14.3) and (14.5), one has $\overline{G} = -\partial \ln Z / \partial \bar{\beta}$.

Once the partition function Z is known, all other thermodynamic quantities can now be derived. We can define an invariant "free genergy" F_c so that

$$Z = \exp[-\bar{\beta} F_c]. \tag{14.6}$$

We note that (14.5) is the invariant partition function of the Gibbs ensemble, in which there is no exchange of momentum, within the framework of common relativity. This can be compared with the conventional Lorentz-invariant partition function, in which one must introduce a 4-vector of the inverse temperature whose operational meaning is, in general, not at all clear.

In a particular frame F, the partition function in (14.5) can be identified with the usual partition function of the Gibbs ensemble $\sum_i \exp\left(-E_i / k_B T\right)$, and the invariant parameter $\bar{\beta}$ in (14.5) can be related to the Boltzmann constant $k_B = 1.38 \times 10^{-16}$ erg/deg K and the usual absolute temperature T. Since $G_i \approx m/c + p_i{}^2/2mc$, $p_i \approx m v_i/c$, and $E_i \approx m v_i{}^2/2$ in the F frame, we have

$$1/\bar{\beta} = k_B T / c^3 \equiv \tau. \tag{14.7}$$

We will call the invariant quantity τ the "common temperature". The dimension of τ is the same as that of genergy, rather than that of energy. Note that the usual temperature T is closely related to the energy E_i and that neither T nor E_i are invariants in common relativity.

14c. Covariant thermodynamics

In conventional thermodynamics, the relation between the entropy S, the heat Q, and the absolute temperature T is given by

$$dS = \frac{(dQ)_{rev}}{T}. \qquad (14.8)$$

This suggests that we relate the "common entropy" S_C and "common heat" Q_c by the relation[4]

$$dS_C = \frac{(dQ_c)_{rev}}{\tau} \qquad (14.9)$$

in the covariant thermodynamics formulated on the basis of common relativity. Relations (14.8) and (14.9) are the same in the frame F in which the speed of light has been defined to be isotropic, provided that $S_C=S/k_B$ and $Q_c= Q/c^3$. In general, the definitions of S_C and Q_c must be consistent with (14.9) and the Boltzmann equation

$$S_c = \ln W(n_1, n_2, ...) = \frac{S}{k_B}. \qquad (14.10)$$

Let us consider a change of the genergy of a closed system. It is given by the invariant equation

$$N\delta\overline{G} = \sum_i \left(\overline{n}_i \delta G^i + G^i \delta\overline{n}_i\right), \qquad (14.11)$$

where the first term is always considered to be the "work" and the second term to be the "heat." Thus we can define the common heat Q_C as

$$N(\delta Q_c)_{rev} = \sum_i G^i \delta\overline{n}_i, \qquad (14.12)$$

where the reversibility of the change is achieved by letting $\delta\overline{n}_i$ correspond to a change in which the probability W defined in (14.1) remains a maximum. We see that the common heat Q_c is the change of genergy of the whole system resulting from a change in the statistical arrangement of its components. On the other hand, "common work" is due to the change of genergy of every component of a system without changing their statistical arrangement.[5]

The reversible change of $\overline{n}_i$ in (14.12) can be defined as

$$\delta\overline{n}_i = -N\frac{\partial^2 \ln Z}{\partial z_i \partial z_j}\delta z_j, \delta z_j, \tag{14.13}$$

because Z in (14.5) can be considered to be a function of $z_i = \overline{\beta}G_i$ and (14.4) can also be written as

$$\delta\overline{n}_i = -N\frac{\partial \ln Z}{\partial z_i}. \tag{14.14}$$

It follows from (14.13) and (14.12) that

$$\frac{1}{\tau}\left(\delta Q_c\right)_{rev} = -\sum_{i,j} z_i \delta z_j \frac{\partial^2 \ln Z}{\partial z_i \partial z_j}, \tag{14.15}$$

where $G_i = z_i / \overline{\beta} = \tau z_i$. Thus $(\delta Q_c)_{rev} / \tau$ can be represented by a Pfaffian in the variable z_i and it is equal to dS_c according to (14.9). After integration, we find the average common entropy $\overline{S}_c$ of a particle in the ensemble:

$$\overline{S}_c = \frac{\overline{G}}{\tau} + \ln Z, \tag{14.16}$$

where the constant of integration is neglected. Using (14.2)-(14.5) and (14.16), one can verify that

$$S_c = N\overline{S}_c = \ln W = -\sum_i \frac{\exp(-\overline{\beta}G_i)}{Z}\ln\left(\frac{\exp(-\overline{\beta}G_i)}{Z}\right), \tag{14.17}$$

where we have used the Stirling approximation: $\ln(N!) \approx N(\ln N - 1)$.

For a thermodynamic system, the invariant entropy S_c can be expressed as a function of N, $G_t = N\bar{G}$ and V_I. The equations for thermodynamic equilibrium are invariant:

$$\frac{1}{\tau} = \left(\frac{\partial S_c}{\partial G}\right)_{V,N}, \tag{14.18}$$

$$\frac{P_c}{\tau} = \left(\frac{\partial S_c}{\partial V}\right)_{G,N}, \tag{14.19}$$

$$\frac{\mu_c}{\tau} = \left(\frac{\partial S_c}{\partial N}\right)_{G,V}, \tag{14.20}$$

where $G \equiv G_t$ and $V \equiv V_I$. The invariant pressure P_c and chemical potential μ_c are related to the usual pressure P and chemical potential μ by the relations

$$p_c = \frac{P}{c^3}, \qquad \mu_c = \frac{\mu}{c^3}. \tag{14.21}$$

in the F frame. One can verify that in the F frame, the present formalism reduces to the usual thermodynamics at low energies.

14d. The canonical distribution and blackbody radiation

Because of the existence of the new invariant genergy G in (13.8) and the invariant volume V_I in (13.1), we are able to define a meaningful invariant density for particles. One cannot do this within the framework of special relativity because of the lack of an invariant genergy. The number of states available to a particle in dV_I can be written as

$$d\Gamma = 2Gc_0 dV_I \delta(p_\lambda^2 - m^2)\theta(p^0)d^4p \frac{1}{(2\pi J)^3}, \tag{14.22}$$

$$G = \frac{\sqrt{p^2 + m^2}}{C}, \qquad J = 3.5177 \times 10^{-33} g \cdot cm,$$

in an arbitrary frame, where p_λ and $m \geq 0$ are the 4-momentum and the mass of the particle, respectively. The invariance of $d\Gamma$ in (14.22) ensures that the summation over the states of a system (see eq. (14.5) does not change the invariance of the expression. In the particular frame F, we have $C=c=c_0$ and hence

$$d\Gamma = \frac{d^3x d^3p}{(2\pi J)^3}, \qquad \text{(in the F frame)}, \tag{14.23}$$

as expected.

For an ideal gas containing N particles with the same mass $m \geq 0$, we have the invariant canonical (or Gibbs) distribution

$$d\rho = \frac{N(2\pi J)^3}{4\pi m^2 \tau V_I c_0 K_2(m/\tau c_0)} \exp\left(-\frac{G}{\tau}\right) d\Gamma, \tag{14.24}$$

$$m^2 K_2\left(\frac{m}{\tau c_0}\right) = \frac{1}{\tau c_0}\int_0^\infty dp p^2 \exp\left[-\frac{\sqrt{p^2+m^2}}{\tau c}\right] \tag{14.25}$$

$$= \frac{1}{\tau c_0}\int_0^\infty \sqrt{p_0^2 - m^2}\, p_0 dp_0 \exp\left[\frac{-p^0}{\tau c_0}\right]$$

$$\to \quad 2(c\tau)^2 \qquad \text{as} \qquad m \to 0.$$

For blackbody radiation, we have the partition function Z

$$Z = \sum_n \exp[-(G_n/\tau)], \qquad G_n = nG = \frac{nJk_0}{C}, \tag{14.26}$$

where the wave 4-vector $k^\mu=(k_0, \mathbf{k})$ is related to the momentum p^μ by

$$p^\mu = Jk^\mu, \qquad k^\mu = \left(\frac{\omega}{C}, \mathbf{k}\right). \tag{14.27}$$

Note that for a massless particle, G is still invariant under the space-lightime transformation:

$$\frac{G}{J} = \frac{k'_0}{c'} = \frac{\gamma(k_0 - \beta k_x)}{\gamma(c - \beta v_x)} = \frac{k_0}{c}, \tag{14.28}$$

because $k_x=k_0\cos\theta$ and $v_x=c\cos\theta$. It follows from (14.26) that

$$< n >= \frac{1}{Z}\sum_n n \exp(-nG/\tau) = \frac{1}{\exp(G/\tau) - 1}, \tag{14.29}$$

which is the *invariant* Planck distribution for blackbody radiation in the framework of common relativity. Comparing equation (14.29) with the usual expression $<n>=1/(\exp[\hbar k_0 c/k_B T]-1)$ in the F-frame, we obtain

$$\frac{Jk_0}{c\tau} = \frac{\hbar k_0 c}{k_B T}, \qquad \frac{\hbar}{c} = J, \tag{14.30}$$

which is consistent with (14.7).

In the present formalism, we also have an invariant grand-canonical distribution, the invariant "thermodynamic potential" Ω_c, and the invariant Fermi-Dirac distribution, etc.:

$$\Omega_c = -\tau \ln\left[\sum_n \exp\left(\frac{\mu_c}{\tau} - \frac{G_n}{\tau}\right)\right],$$

$$< n >_{FD} = \frac{1}{\exp[(G - \mu_c)/\tau] + 1}, \qquad \text{etc.} \tag{14.31}$$

14e. The question of Earth's "absolute" motion relative to the 3K cosmic microwave background

The 3k cosmic microwave background was detected by Penzias and Wilson in 1965. It is an important discovery for understanding the universe and is believed to be radiation left over from the early epoch of the Universe in which matter and radiation were in thermal equilibrium. The 3k background radiation involves about 10^3 photons per cm^3 and this temperature is assumed to be the present temperature of the universe because the effect of hot stars is negligible when one averages their temperature over all space. The 3k cosmic radiation appears to be isotropic, having the same radiation intensity in all directions, as observed on the Earth. However, a very small anisotropy in the cosmic blackbody radiation has been detected and this anisotropy is usually attributed to Earth's motion relative to the background radiation. This interpretation is made within the framework of special relativity with certain implicit assumptions. Consider two frames F and F' (which is moving with the speed V along the x-axis of F) immersed in the blackbody radiation. The usual Planck distribution in the F frame is

$$\langle n\rangle = \frac{1}{\exp[h\nu/k_B T]-1}. \tag{14.32}$$

Observers in the F' frame, however, will detect a modified distribution,

$$\langle n\rangle' = \frac{1}{\exp\left[\dfrac{h\nu'(1+V\cos\theta'/c)}{k_B T\sqrt{1-V^2/c^2}}\right]-1}, \tag{14.33}$$

which is derived from (14.32) using the special relativistic transformations

$$\nu' = \frac{\nu(1-V\cos\theta/c)}{\sqrt{1-V^2/c^2}}, \qquad \cos\theta' = \frac{\cos\theta - V/c}{1-V\cos\theta/c}, \tag{14.34}$$

and the assumptions that the Boltzmann constant k_B and the temperature T are Lorentz scalars. It follows that the spectral distribution of the energy of the blackbody radiation is given by

$$dE(k_0) = \frac{2d^3xd^3k}{(2\pi)^3}\frac{h\nu}{\exp[h\nu/k_BT]-1}, \qquad \text{in F}$$

$$dE'(k'_0) = \frac{2d^3x'd^3k'}{(2\pi)^3}\frac{h\nu'}{\exp[h\nu'(1+V\cos\theta'/c)\gamma/k_BT]-1}, \qquad \text{in F'}. \tag{14.35}$$

The photon energies $h\nu$ and $h\nu'$ in (14.35) are related by the relativistic Doppler effect (14.34). Thus, the cosmic microwave background plays the role of a new "ether" and the frame F is the privileged and absolute frame in which the law of the Planck distribution takes a particular simple form, as one can see from (14.35). According to this view, nature provides a natural reference frame, i.e., the cosmic 3k radiation represents a fixed system of coordinates in the Universe. Many physicists thus concluded that we now are justified in talking about "absolute" motion.

Nevertheless, this interpretation is not in harmony with the Poincaré-Einstein principle of relativity for physical laws. Furthermore, this conclusion is untenable from the viewpoint of the covariant thermodynamics based on common relativity. According to the common relativistic framework, the Planck formula corresponding to (14.35) in F and F' can be obtained by combining the invariant density of states (14.22) with two polarizations (i.e. $2d\Gamma$), the invariant Planck distribution (14.29) and the covariant photon "energy" Jk_0 (which has the dimension of mass):

$$dM(k_0) = \frac{2d\Gamma Jk_0}{\exp[G/\tau]-1}, \qquad \text{in F,}$$

$$dM'(k'_0) = \frac{2d\Gamma Jk'_0}{\exp[G/\tau]-1}, \qquad \text{in F',} \tag{14.36}$$

where we have used the invariant genergy in (14.28). The photon "energy" Jk_0 and Jk_0' in (14.36) are related by the Doppler effect in common relativity:

$$k'_0 = \gamma(k_0 - \beta k_x), \qquad k'_x = \gamma(k_x - \beta k_0), \qquad k'_y = k_y, \qquad k'_z = k_z,$$
$$k^{\mu} = \left(\frac{\omega}{c}, \mathbf{k}\right), \qquad k'^{\mu} = \left(\frac{\omega'}{c'}, \mathbf{k}'\right). \tag{14.37}$$

Since the density of states (14.22) and the Planck distribution $<n>=\{\exp[G/\tau]-1\}^{-1}$ are invariant, the quantity $dM'(k_0')$ is covariant and transforms like k_0' in common relativity. Therefore, from this viewpoint, the observed anisotropy in the cosmic 3k radiation should not be interpreted as being caused by the motion of Earth relative to the cosmic microwave background. This is in contrast with the corresponding result of special relativity, in which $dE'(k'_0)$ in (14.35) does not have a covariant form and does not transform like k_0' because the Planck distribution is not invariant and takes the simplest form only in the frame F. Note that the Doppler shifts of $h\nu'$ in (14.35) and Jk'_0 in (14.36) are both consistent with previous experimental results. The reason for this can be seen clearly by considering the quantities involved in the Doppler shift experiment.

The relation between the frequency $\nu=ck_0/2\pi$ and $\nu'=c'k'_0/2\pi$ in (14.37) is quite different from that given by (14.34) in special relativity. This is because we have used common time to measure the speed of light c and c' and to define the frequency of a wave in our formalism. However, this does not contradict laboratory experiments testing the Doppler effect involving shifts of atomic-level structure, lasers, etc. The reason has been discussed in chapter 10 [eqs. (10.7)-(10.18)].

Experimentally, the anisotropy of the cosmic 3k radiation was established at the level of one part in 10^3 to 10^4. According to common relativity, this anisotropy should not be interpreted as being related to the angular dependence of the Planck distribution due to the motion of the Earth relative to the 3K cosmic radiation. Instead, the anisotropy of the 3k radiation indicates a new phenomenon on a cosmological scale. For example, it may be related to a small nonsymmetric expansion of the Universe or large-scale irregularities in the distribution of energy in the Universe. Whether or not these interpretations are viable could be clarified in the future when one is able to measure the dipole anisotropy ($\sim 10^{-5}$) due to the Earth's orbital motion and the quadruple anisotropy.

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15.

Common Relativity and Quantum Mechanics

15a. Fuzziness at short distances and the invariant genergy

One of the basic problems in quantum mechanics is the fact that the non-square-integrable coordinate representation is incompatible with the usual probabilistic interpretation. This situation suggests the need to introduce *fuzzy coordinates* to modify physics at short distances.[1] The introduction of fuzzy coordinates is also motivated by the problem of locality and the related ultraviolet divergence in quantum field theories. Fuzzy coordinates may be significant because an inherent fuzziness at short distances is characterized by a radical length R. This radical length R is related to the fundamental length R_D in quantum field theory, a concept that has was introduced many years ago by Heisenberg and Dirac.[2] Such an inherent fuzziness is also related to Feynman's idea[3] of a basic width for a modified δ-function for interactions. In 1949, Dirac made the following comments:

> "Presentday atomic theories involve the assumption of localizability, which is sufficient but is very likely too stringent ··· A less drastic assumption may be adequate, e.g., that there is a fundamental length λ such that the Poisson bracket of two dynamical variables must vanish if they are localized at two points whose separation is space-like and greater than λ, but need not vanish if it is less than λ."

Schwinger,[4] Feynman[5] and Wigner[6] also had similar views.

As we have seen, common relativity has a new invariant $G(\mathbf{p})=p^0/C$ $=\sqrt{\mathbf{p}^2+m^2}/C$ which is called the "genergy." In the F frame in which the speed of light is, by definition, a constant, i.e., C=c, the genergy $G(\mathbf{p})$ is essentially the same as the conserved quantity, energy, $p^0=\sqrt{\mathbf{p}^2+m^2}$. The invariant genergy $G(\mathbf{p})$ can help us to operationalize an inherent fuzziness at short distances without upsetting the 4-dimensional symmetry. This is not possible to do in the framework of special relativity because the genergy $G(\mathbf{p})$ is not an invariant quantity in special relativity. The concept of the genergy $G(\mathbf{p})$ makes possible a departure from the 4-dimensional symmetry of special relativity at short distances. This is but another example in which common relativity, by allowing the definition of new invariant quantities not possible in special relativity, may be more convenient than special relativity for certain types of calculations.

First, let us show the invariance of the genergy G by calculations based on the 4-dimensional space-lightime transformation:[1] The transformations of the 4-momentum $p^\mu = m dx^\mu/ds$ are given by :

$$p'^0 = \gamma\left(p^0 - \beta p_x\right), \quad p'_x = \gamma\left(p_x - \beta p^0\right), \quad p'_y = p_y, \quad p'_z = p_z, \tag{15.1}$$

$$\gamma = 1/\sqrt{1 - \beta^2} ,$$

where the spatial components of the momentum are $\mathbf{p}=(p_x,p_y,p_z)=(p^1,p^2,p^3)$. It follows from (15.1) and the velocity transformations (8.5) that the ratio p^0/c is invariant under the 4-dimensional transformations:

$$\frac{p'^0}{c'} = \frac{p^0}{c} = G(\mathbf{p}), \quad p'^0 = \sqrt{\mathbf{p}'^2 + m^2}, \qquad p^0 = \sqrt{\mathbf{p}^2 + m^2}. \tag{15.2}$$

$$c' \equiv \frac{d(b't_c)}{dt_c}, \qquad \frac{p_x}{p_0} = \frac{1}{c}\frac{dx}{dt_c}.$$

Note that the genergy $G(\mathbf{p})$ involves the spatial momentum vector $\mathbf{p}$. This invariance is quite interesting because it suggests the existence of an invariant function involving $\mathbf{p}$ and a new universal constant I which can be related to a radical (or fundamental) length. The constant I would allow us to define a dimensionless function $D(\mathbf{p})$ in terms of the scalar genergy $G(\mathbf{p})$ associated with a physical particle of mass m,

$$D(\mathbf{p}) = \frac{1}{1 + I^2 G^2(\mathbf{p})}. \tag{15.3}$$

The constant I has the dimension of length/(mass·time). Thus, D(**p**) is a pure number and may be related to a probability. The constant I can be related to a radical length R defined by

$$R = \frac{IJ}{c_o}, \tag{15.4}$$

where c_o is the scaling constant defined in (13.1). The radical length R in nature represents a basic physical quantity which is independent of the artificial choices of units for mass and time. Naturally, the numerical value of R still depends on our choice of a unit for length. However, the radical length R, if it exists, should be regarded as the unit of length *chosen by nature rather than by humans.* This has physical significances, as we shall see later. Note that the invariant relations (15.2) and (15.3) do not exist in special relativity.

15b. Fuzzy quantum mechanics with an inherent fuzziness in the position of a point particle

In quantum mechanics and quantum field theories, *basic physical objects are assumed to be point particles which may have a definite and precise position in space (or a definite momentum, but not both at the same time because of the uncertainty principle).* As a result, the Coulomb potential produced by a point electron takes the form $-e/(4\pi r)$, which diverges at $r=0$. This divergence is intimately related to the fundamental divergence difficulty in quantum electrodynamics and other field theories. In addition, the point particle picture implies that a quantum particle can have a position eigenstate $|q\rangle$ of a position operator Q:

$$Q\,|q\rangle = q\,|q\rangle. \tag{15.5}$$

Therefore, in principle, the position q of a quantum particle can be measured precisely, i.e.,

$$\Delta q_{min} = 0. \tag{15.6}$$

However, the properties (15.5) and (15.6) do not have operational meanings because they cannot be realized. Furthermore, the position eigenstate (15.5) is, strictly speaking, not physically meaningful because its wave function is not square-integrable and hence, cannot be normalized to have an elementary probabilistic interpretation. Thus, in quantum mechanics, one gives up the fundamental idea of a probabilistic interpretation of a state for the sake of introducing certain mathematical idealizations. This is inconsistent with the notion of operational definitions in physics, which states that one should formulate a physical theory using observable quantities and realizable states.

On the other hand, if one attempts to avoid the above difficulties by assuming the particle to have a finite size, then one immediately encounters other difficult problems related to non-local interactions, non-local wave functions and so on.

One way to depart from the conventional point particle picture without these difficulties is to assume that the particle itself has no size and no structure, but that its position cannot be measured with unlimited accuracy:[1]

$$\Delta q_{min} = R > 0, \tag{15.7}$$

where R is a very small radical length. The postulate (15.7) enables us to avoid unknown tacit assumptions such as indefinitely short distances in space.[5] This suggests a picture of a fuzzy-point particle, which closely resembles a fuzzy point in the fuzzy set theory of Zadeh.[7]

According to postulate (15.7), we replace the improperly idealized state $\langle q|$ with zero width by a new base state $(q|$, which has a width R. To accomplish this, it is convenient to use Klauder's continuous representation[8] of Hilbert space to express the new base $(q|$:

$$(q| = \int_{-\infty}^{\infty} dp \, \langle p| D(p)\, e^{ipq/J} \frac{1}{\sqrt{2\pi J}}, \tag{15.8}$$

where $\langle p|$ is the usual momentum eigenstate. In order to preserve known properties of a physical state at low momenta, the function D(p) must satisfy

$$D(p) = \begin{cases} 1 & |p| << J/R; \\ 0 & |p| >> J/R. \end{cases} \tag{15.9}$$

Similarly, we can replace the usual momentum eigenstate $\langle p|$ with a momentum state $(p|$ which has an inherent fuzziness

$$\Delta p_{min} = \frac{J}{S} > 0, \tag{15.10}$$

where p and S have the dimensions of mass and length, respectively. To be consistent with known physical states at low momenta, the length scale S should be extremely large. Presumably, it is related to the size of the observable physical universe $S \geq 10^{10}$ light years. For simplicity, we postulate that the physical effects of a finite S are too small to be detected experimentally and work in the limit in which S approaches infinity so that

$$(p| = \langle p|. \tag{15.11}$$

Instead, we will concentrate on the new physical properties implied by the radical length R>0.

Consider the fuzzy base states. In the limit $S \to \infty$, the base momentum states $(p|$ and $|p)$ have the usual orthogonal and completeness relationships

$$(p|p') = \delta(p-p'), \qquad \int_{-\infty}^{\infty} dp\, |p)(p| = 1,$$

$$(p|P = (p|p, \qquad (p|Q = iJ\frac{\partial}{\partial p}(p|, \tag{15.12}$$

where Q and P are Hermitian operators for positions and momenta. Because R>0, the position vector $(q|$, its dual vector $|q)$ and the Hermitian operators Q and P satisfy

$$(q|q') = (q'|q)^* = \int \frac{dp}{2\pi J} D^2(p) e^{i(q-q')p/J} = D^2(\partial)\delta(q-q'), \tag{15.13}$$

$$\partial \equiv -iJ\frac{\partial}{\partial q} \qquad \int \equiv \int_{-\infty}^{+\infty},$$

$$(q \mid P = -iJ\frac{\partial}{\partial q}(q \mid, \qquad P \mid q) = iJ\frac{\partial}{\partial q}\mid q), \qquad \int dq \mid q)(q \mid = D^2(p), \tag{15.14}$$

$$\int dq[D^{-1}(\partial) \mid q)][D^{-1}(\partial)(q \mid] = \int dq \mid q)[D^{-2}(\partial)(q \mid] = 1, \tag{15.15}$$

$$(q \mid Q = (q \mid [q - iJ\frac{\partial \ln D(P)}{\partial P}], \qquad Q \mid q) = [q + iJ\frac{\partial \ln D(P)}{\partial P}] \mid q), \tag{15.16}$$

where $D^2(\partial)$ and $D^{-1}(\partial)$ are integral operators defined by

$$f(\partial)\phi(q) = \frac{1}{2\pi J}\int dp e^{ipq/J} f(p) \int dq' e^{-ipq'/J}\phi(q'). \tag{15.17}$$

Thus, the continuous base states $(q|$ defined in (15.8) satisfy the minimum requirement for a coherent state, i.e., that $(q|$ is a strongly continuous function of q that satisfies the relation (15.15). Nevertheless, the most important physical property of $(q|$ is given by (15.16), (15.13), (15.7), namely that the position operator Q has neither eigenstates nor eigenvalues. Instead, it has only a fuzzy value with an uncertainty $\Delta q \geq R$. In other words, a "point particle" (i.e., one with no size and no structure), if measured, will never be found at one and only one point q at a time. Thus, it is appropriate to term $(q|$ and Q *fuzzy states* and *fuzzy dynamical variables,* respectively. (In principle, both P and Q should be fuzzy dynamical variables if the length scale S is treated as a finite quantity.) Formally, $(q|$ is an eigenstate of the operator

$$Q_0 = Q + iJ\frac{\partial \ln D(P)}{\partial P},$$

with the eigenvalue q. However, such an operator Q_0 has no operational meaning in the present formalism of fuzzy quantum mechanics.

The fuzziness of the base state $(q|$ is completely determined by the function D(p). Let us consider a plausibility argument for the function D(p) to have the following form:

$$D(p) = \frac{1}{a^2p^2 + 1}, \qquad a^2 = \frac{2R^2}{J^2}. \tag{15.18}$$

Classically, one can determine a particle's position by confining the particle in a certain range Δx by an attractive square-well potential. If the potential becomes narrower and deeper, we can determine the position more accurately. In the limiting case, we have a δ-function potential, $-V_0\delta(x-x_0)$, and the classical particle is precisely located at the point x_0. However, a quantum particle is described by the Schrödinger (or Klein-Gordon) equation

$$\left[-\frac{J^2}{2m}\frac{d^2}{dx^2} - V_0\delta(x - x_0)\right]\phi(x) = E\phi(x), \tag{15.19}$$

rather than the classical Newtonian equation. The solution to (15.19) has the form

$$\phi(x) = A\exp\left[-\frac{|x - x_0|}{\sqrt{2}R}\right], \qquad R = \frac{J^2}{\sqrt{2mV_0}}, \tag{15.20}$$

which shows that the position of the particle is fuzzy (with the uncertainty $\Delta x \sim R$). The Fourier transform of (15.20) leads to

$$\int_{-\infty}^{\infty} A\exp\left[-\frac{|x - x_0|}{\sqrt{2}R}\right]\exp\left[-\frac{i(x - x_0)p}{J}\right]dx = \frac{2\sqrt{2}AR}{p^2R^2/J^2 + 1}, \tag{15.21}$$

which has the same form as that in (15.18). Thus, it is reasonable to use $D(p)$ in (15.18) to describe the fuzzy base state $\langle q|$ in (15.8) for a quantum particle. It appears that the value R can only be determined by future experiments. There is, as yet, no compelling reason to identify R with the Planck length $\sim 10^{-33}$ cm or any other known length in physics.

According to the "probability axiom" of quantum mechanics, the physical base states associated with the dynamical variables Q and P must have a probabilistic meaning. Thus, these states should satisfy Klauder's postulates of continuous representation rather than the usual eigenbras and eigenkets. One might think that this is nothing new because one can always transform the fuzzy state $(q|$ in (15.8) to the eigenstate $\langle q|$ by a nonunitary transformation. However, this is not true: Under a nonunitary transformation that preserves $PQ-QP=-iJ$, the position operator Q and the states $(q|$ and $|q)$ become

$$Q \to Q' = D(p)QD^{-1}(p) = Q - iJ\frac{\partial \ln D(P)}{\partial P}, \tag{15.22}$$

$$(q| \mapsto \{q| = (q|D^{-1}(P) = \langle q|, \quad |q) \to |q\} = D(P)|q) \neq |q\rangle. \tag{15.23}$$

Thus we see that the expectation value of the coordinate Q remains the same

$$(q_1|Q|q_2) = \{q_1|Q'|q_2\} \neq \langle q_1|Q|q_2\rangle, \tag{15.24}$$

i.e., the fuzzy feature of coordinates is unchanged by a nonunitary transformation.

We postulate fuzziness as a fundamental and inherent property of a particle's position. Regardless of how one improves the experimental techniques and the apparatus, there will always be an uncertainty $\Delta q \geq R$ associated with each measurement, even if the particle itself has no size and structure. From this emerges a strange new picture of the fuzzy point particle: Namely, *a particle is, at a given instant of time, partially located at one point and partially elsewhere, and can never be completely at one point.* This is analogous to Zadeh's original idea of fuzzy set theory[7] and in sharp contrast to both the classical and the conventional quantum-mechanical concept of a point particle.

15c. Fuzzy point and modified Coulomb potential at short distances

The Klauder representation in Hilbert space naturally allows a basic length scale R to characterize continuous states. As long as R>0, our results indicate that space is continuous but fuzzy at short distances. The length scale R characterizes the smallest width of a wave packet that can be physically realized and the fuzzy base vectors $(q|$ form a submanifold in Hilbert space.

Since the radical length R seems to be so small that its effects are yet undetected, one might ask what types of physical effects we might expect from this fuzziness and how they might be detected experimentally. The fuzziness of an electron's coordinates implies that the electron must have an "R-inherent charge distribution" given by

$$\rho_R(r) = -\frac{\bar{e}}{(2\pi J)^3}\int_{-\infty}^{\infty} D^2(p)\exp[-i\mathbf{p}\cdot\mathbf{r}/J]d^3p$$

$$= -\frac{\bar{e}}{10\sqrt{2}R^3}\exp\left[-\frac{r}{\sqrt{2}R}\right], \qquad (15.25)$$

$$D(\mathbf{p}) = \frac{1}{a^2\mathbf{p}^2+1}, \qquad a^2 = \frac{2R^2}{J^2},$$

where the form of D(**p**) is assumed to be given by (15.18). Note that

$$\rho_R(r) \to -\bar{e}\delta^3(r),$$

as R approaches 0, as expected. This R-inherent charge distribution is consistent with the fuzzy point picture of a particle. The modified Coulomb potential $V_R(r)$ produced by such a fuzzy point electron is given by

$$\int d^3r' \frac{\rho_R(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|},$$

i.e.,

$$V_R(r) = -\bar{e}\int \frac{d^3k}{(2\pi)^3}\frac{D^2(k)}{\mathbf{k}^2}\exp(-i\mathbf{k}\cdot\mathbf{r}), \qquad \mathbf{k} = \mathbf{p}/J,$$

$$= \frac{-\bar{e}}{4\pi r}\left[1-\left(1+\frac{r}{2\sqrt{2}R}\right)\exp[-r/(\sqrt{2}R)]\right] \qquad (15.26)$$

$$= \begin{cases} -\dfrac{\bar{e}}{4\pi r} & r \gg R, \\[2ex] -\dfrac{\bar{e}}{8\sqrt{2}\pi R} & r \ll R, \end{cases}$$

which is finite at r=0. This suggests that the electromagnetic interaction will be asymptotically free at very high energies (or $p \gg \hbar/R$). The results (15.26) and

(15.25) indicate that R should make its presence known in say, e^+e^- scattering at very high energies

$$e^-e^+ \rightarrow e^-e^+ \qquad \text{or} \qquad e^-e^+ \rightarrow \mu^-\mu^+ . \tag{15.27}$$

If one measures the differential cross sections carefully, a deviation from the usual quantum electrodynamics results could be attributed to the radical length R, provided that the electron is not a composite particle. The details of the deviation are related to the specific form of the function $D(\mathbf{p})$.

Of course, there is also the possibility that the constant R in Klauder's continuous representation is so small that it will forever elude detection. In this case, the present formalism is still advantageous to ordinary quantum mechanics because this formalism enables us to apply the probabilistic interpretation to all states of all observables regardless of whether they are discrete or continuous.

It is likely that Klauder's continuous representations for coordinates and momenta are not merely a matter of mathematical purism, but rather an inherent property of the physical world at very small and very large distances. Such representations are particularly interesting because they suggest a connection between the microscopic physical world and the new mathematics related to fuzzy set theory. Also, introducing fuzziness at short distances might lead to a solution to the divergence problem in field theories.

Note that it is highly nontrivial to accommodate the fuzziness of the coordinates and the usual 4-dimensional symmetry of spacetime in special relativity. If the fuzziness of Q is really fundamental, then it would imply that the usual 4-dimensional symmetry of special relativity becomes exact only at low energies or in the limit $R \rightarrow 0$.

15d. Suppression of the contribution of large momentum states to physical processes

Let us assume that the inherent fuzziness of the coordinate variables (or of a quantum particle's position) is a fundamental property of physics and is independent of the 4-dimensional symmetry of spacetime. We would expect that, apart from the usual relativistic time dilation (or the dilation of the decay length), the lifetime of an unstable particle decay in flight with a momentum p will be further dilated due to the existence of the radical length R when $p \geq J/R$. (This new effect may be called the 'radical dilation.')

The new 'radical dilation' effect for the lifetime of a moving particle is related to the radical length R. The fuzzy coordinate base state (15.8) leads to

$$(\mathbf{q}'|\mathbf{q}) = (2\pi J)^{-3}\int_{-\infty}^{\infty} D^2(\mathbf{p})e^{i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{p}/J}d^3p = D^2(\partial')\delta^3(\mathbf{q}'-\mathbf{q}), \tag{15.28}$$

where

$$D^2(\partial')f(q') = (2\pi J)^{-3}\int_{-\infty}^{\infty} d^3p e^{i\mathbf{p}\cdot\mathbf{q}'/J}D^2(\mathbf{p})\int_{-\infty}^{\infty} d^3q e^{-i\mathbf{p}\cdot\mathbf{q}/J}f(\mathbf{q}), \tag{15.29}$$

$$D^2(\mathbf{p}) = \frac{1}{[2R^2\mathbf{p}^2/J^2+1]^2}.$$

This property, which will be discussed in section 16c, can be properly implemented only in quantum field theory. However, we can still see its partial effect on a physical process through an effective density of states, without invoking quantized fields. We first note that mathematically, the momentum p can take any value between $-\infty$ and $+\infty$. However, the region in which the momentum of a particle can be physically realized is effectively finite. Thus, the three-dimensional momentum space appears to be non-Euclidean because one may picture the momentum space as having a volume element $D^2(\mathbf{p})d^3p$, as shown in (15.28). As a result, the number of one-particle states with a momentum between $\mathbf{p}$ and $\mathbf{p}+d\mathbf{p}$ is given by the following "effective density of states"

$$\left[\int d^3q\right]\frac{D^2(\mathbf{p})d^3p}{(2\pi J)^3} = [V]\frac{D^2(\mathbf{p})d^3p}{(2\pi J)^3}, \tag{15.30}$$

for a spin-zero particle. Thus the contribution of large momentum states to scattering cross sections or decay rates of physical processes is suppressed due to the new restriction (15.7) on the Heisenberg uncertainty relation.

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16.

Common Relativity and Fuzzy Quantum Field Theory

16a. Fuzzy quantum field theories

It has been observed that finite quantum electrodynamics cannot be formulated consistently within the framework of special relativity.[1] The reason for this is that limiting the magnitude of interactions while retaining the customary coordinate description is contradictory, since no mechanism is provided for precisely localized measurements. This observation suggests that the physical properties of space at very short distances may be very different from those at macroscopic or even atomic distances. The fact that no mechanism is provided for precisely localized measurements indicates that there might be an inherent fuzziness at short distances and that the concept of a point-like particle, which works very well in the atomic and nuclear domains, must be modified so that it resembles some sort of non-point-like object. Spatially extended particles, fuzzy particles and string-like objects have all been discussed in the literature.[2] However, it is very difficult to make extended particles and string-like objects consistent with the 4-dimensional symmetry of the Lorentz and the Poincaré groups. Thus far, people have found strings that fit a symmetry framework of a 26-dimensional or 10-dimensional spacetime, but this has as yet little relevance to present experiments and observations. Rather than trying to explain why a 10- or 26-dimensional spacetime has only 4 observable dimensions, a different approach would be to find a string-like object (no matter how strange) that can be used to formulate a consistent theory in the 4-dimensional spacetime. If this can be accomplished, it would be an important discovery.

On the other hand, one may ask: Is the 4-dimensional symmetry of Lorentz and Poincaré invariance so sacred that it must be absolutely preserved at all costs? A physical principle is sacred only as long as it is supported by experiments. However, experiments have limitations: Newton's law of motion is known to be good for the macroscopic world and small speeds, and the Dirac equation is good for spatial regions where $\Delta r > 10^{-17}$ cm and for energies smaller than roughly 10^3 GeV. Nevertheless, whether or not these physical principles are valid for describing the early universe or for particles when their energy reaches 10^{30} GeV is far from certain. Based on the evolution of physical theories and discoveries of physical laws in the past 100 years, it seems reasonable to conjecture that most currently established physical laws will not last another 100 years without modification.

Dirac commented on the divergence difficulties in the quantum theory of fields: "The difficulties, being of a profound character, can be removed only by some drastic change in the foundations of the theory, probably a change as drastic as the passage from Bohr's orbit theory to the present quantum mechanics."[3]

It is reasonable to assume that the 4-dimensional symmetry of Lorentz and Poincaré groups is a good starting principle, which can be realized in different conceptual frameworks, as discussed in previous chapters. We now explore the physical origin and implications of the concept of fuzzy particles, which can be accommodated in the 4-dimensional symmetry of common relativity (but not that of special relativity).

It appears quite possible that the divergence difficulties originate from the naive analogy between field quantities ($\varphi(t,\mathbf{r})$, $\partial\varphi(t,\mathbf{r})/\partial t$) and the generalized coordinates and velocities ($q_i(t)$, $dq_i(t)/dt$). For decades, it seemed quite certain that one had to follow this "exact analogy" in order to apply the canonical procedure for quantizing the fields. However, if one insists on preserving this "exact analogy," one is led to the customary coordinate description at short distances and thus encounters profound divergence difficulties. This leads to some doubts about to what lengths this analogy can be extended. According to this analogy, one simply replaces the Fourier coefficients c_k and c_k^* of a free classical field φ by the corresponding annihilation and creation operators:

$$c_k \to \frac{a_k}{\sqrt{2\omega_k}}, \qquad c^*_k \to \frac{a_k^\dagger}{\sqrt{2\omega_k}}. \tag{16.1}$$

In this way, the usual Hamiltonian H_{usu} of the field φ is

$$H_{usu} = \sum_{k} \omega_k \left(a_k a_k^\dagger + \frac{1}{2} \right), \tag{16.2}$$

where the field φ is enclosed in a cubic box of side $L=V^{1/3}$ and satisfies the periodic boundary conditions. Result (16.2) shows the analogy with the harmonic oscillator.

In this section, we attempt to modify the conventional quantum field which allows, in principle, the existence of a physical wave with an infinitesimal wavelength, so that one can measure spatial positions to an arbitrary accuracy. The basic new idea is that such a wave does not exist even in principle. Thus, spatial measurements cannot be made as accurate as one wants, even in principle. In this sense, space is "fuzzy at short distances." Such a fuzziness suggests that there exists a fundamental probability distribution $P(k,m)$ that approaches zero as the wavelength or $2\pi/k$ approaches zero. We assume that this is inherent in each harmonic oscillator with the momentum $\mathbf{k}$ of a field with a mass m.[4] Operationally, this means that instead of (16.1), one should replace the Fourier coefficients of a free classical field by

$$\begin{aligned} c_k &\to \frac{a_k D(k,m)}{\sqrt{2\omega_k}}, \\ c^*_k &\to \frac{a_k^\dagger D(k,m)}{\sqrt{2\omega_k}}; \qquad D(k,m) = \sqrt{P(k,m)}, \quad k = |\mathbf{k}|. \end{aligned} \tag{16.3}$$

Thus, the Hamiltonian of the field φ becomes

$$H = \sum_{k} \omega_k P(k,m) \left(a_k a_k^\dagger + \frac{1}{2} \right). \tag{16.4}$$

The specific form of the probability distribution $P(k,m)$ can only be determined by future experiments. Nevertheless, based on known experiments at low energies, the probability distribution $P(k,m)$ must satisfy the following limits:

$$P(k,m) \to \begin{cases} 1, & k \to 0, \\ 0, & k \to \infty. \end{cases} \tag{16.5}$$

As we saw earlier in chapter 15, the concept of an inherent probability distribution has the effect of suppressing the large momentum contributions, this time to the harmonic oscillator Hamiltonian (16.4). In addition, field oscillators would have a finite zero-point energy, in contrast to the usual expression (16.2) which has an infinite zero-point energy. To make our discussion concrete, we will assume the invariant function D(k,m) in common relativity to be given by (15.3) with **p**=J**k**,

$$P(k,m) = \left[\frac{1}{1+I^2G^2(\mathbf{k})}\right]^2, \qquad G(\mathbf{k}) = \frac{\sqrt{k^2+m^2}}{C}, \tag{16.6}$$

where the quantity I is a new fundamental constant. Presumably, the value of I is very small so that $P(k,m) \approx 1$ for presently attainable energies in the laboratory.

We may picture a field as a set of harmonic oscillators where each harmonic oscillator is associated with an intrinsic probability determined by its momentum. In the limit $I \to 0$, the probability distribution $P(k,m) \to 1$ and we have the usual field in which all states of the harmonic oscillator of the field are equally probable.

The probability distribution P(k,m) in (16.5) must have the following two important properties:

(i) P(k,m) depends only on the spatial components of the four-momentum vector of an oscillator. This dependence is necessary for the unitarity of quantum field theories, i.e., the probability for any physical process to occur must be non-negative.

(ii) P(k,m) must be an invariant function, as required by the four-dimensional symmetry.

These two requirements can only be satisfied by field theories formulated within the framework of common relativity because special relativity does not have the *invariant* genergy (15.2), which is related only to the spatial components of the momentum four-vector. Thus, quantum field theories *based* *on special relativity* cannot satisfy (16.5) or possess the two properties described above. Furthermore, the experimental evidence for the presence of $P(k,m)\neq 1$ in the future would imply that the spacetime symmetry of special relativity is only approximate at "low energies" and is not exact at very high energies or short distances.

According to Maxwell's equations and the postulate of the universal probability distribution for field oscillators, we expand the photon field $\mathbf{A}(\omega,\mathbf{r})$ as follows,

$$\mathbf{A}(\omega,\mathbf{r}) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3k}{\sqrt{2k_0}} \sum_{\lambda=1}^{2} \boldsymbol{\varepsilon}(k,\lambda)\sqrt{P(k,0)}\left[a(k,\lambda)e^{-ik\cdot x} + a^{\dagger}(k,\lambda)e^{ik\cdot x}\right], \quad (16.7)$$

$$k_0 = |\mathbf{k}|; \quad \mathbf{k}\cdot\boldsymbol{\varepsilon}(k,\lambda)=0; \quad \boldsymbol{\varepsilon}(k,\lambda)\cdot\boldsymbol{\varepsilon}(k,\lambda') = \delta_{\lambda\lambda'}, \quad k^2 = k_\mu k^\mu = 0,$$

in the radiation gauge. Since photons are massless, they are associated with the invariant probability distribution $P(k,0)$ in (16.7). The operators $a(k,\lambda)$ and $a^{\dagger}(k,\lambda)$ are assumed to satisfy the commutation relations,

$$\left[a(k,\lambda), a^{\dagger}(k',\lambda')\right] = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'},$$

$$\left[a(k,\lambda), a(k',\lambda')\right] = \left[a^{\dagger}(k,\lambda), a^{\dagger}(k',\lambda')\right] = 0. \quad (16.8)$$

As usual, we assume the gauge invariant Lagrangian L_a for the photon field a_μ:

$$L_a = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu}; \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \quad (16.9)$$

Note that $a^\mu = (a^0,\mathbf{a})$ in common relativity has the same dimensions as A^μ/c in special relativity because $L_a d^4x$ and J have the same dimensions. The Hamiltonian H_a and the momentum $\mathbf{Q}_a$ of the photon field can be written as

$$H_a = \int d^3k \mid \mathbf{k} \mid P(k,0) \sum_{\lambda=1}^{2} a(k,\lambda)a^\dagger(k,\lambda), \tag{16.10}$$

$$Q_a = \int d^3k\mathbf{k}P(k,0) \sum_{\lambda=1}^{2} a(k,\lambda)a^\dagger(k,\lambda).$$

The modified photon propagator is now given by

$$iD^{tr}_{\mu\nu}(x,I) = \langle 0|a_\mu(x)a_\nu(0)\theta(w)|0\rangle + \langle 0|a_\nu(0)a_\mu(x)\theta(-w)|0\rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{P(k,0)e^{-ik\cdot x}}{k^2+i\varepsilon} \sum_{\lambda=1}^{2} \varepsilon_\mu(k,\lambda)\varepsilon_\nu(k,\lambda), \tag{16.11}$$

$$\varepsilon^\mu(k,\lambda) = \left(0, \varepsilon(k,1), \varepsilon(k,2), \mathbf{k}/|\mathbf{k}|\right) = (0, \boldsymbol{\varepsilon}(k,\lambda)).$$

As expected, this propagator reduces to the Feynman propagator $iD^{tr}_{F\mu\nu}(x)$ in the limit $I \to 0$,

$$iD^{tr}_{\mu\nu}(x,0) = iD^{tr}_{F\mu\nu}(x). \tag{16.12}$$

The Coulomb potential of a charged particle is now produced by a fuzzy source rather than a point source. The fuzzy source is determined by the probability function P(k,0) in (16.11).[5] The modified Coulomb potential $V(r) = a_0(r)$ is

$$V(r) = -\bar{e} \int \frac{d^3k}{(2\pi)^3} \frac{P(k,0)}{k^2} e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{16.13}$$

which is consistent with (14.26). It is important to remember that P(k,0) should not be understood or interpreted as the form factor due to charge distribution in the classical sense. Rather, it is a quantum mechanical property related to the inherent probability distribution of photon field oscillators. In other words, photons should be pictured as a "fuzzy particle" rather than a point particle. A "fuzzy photon" can produce an effect similar to that produced by a non-point charge distribution. In general, any departure from the point picture of particles in electrodynamics implies a modification of the Coulomb potential at short distances which can be tested by very high-energy experiments in the future.

16b. Fuzzy quantum electrodynamics based on common relativity

In fuzzy quantum electrodynamics (fuzzy QED), the invariant action Sq, involving a fuzzy electron field ψ and a photon field a_μ, is assumed to have the usual form

$$S_q = \int L d^4x, \qquad L = \overline{\psi}\left[\gamma^\mu\left(iJ\partial_\mu - \overline{e}a_\mu\right) - m\right]\psi - \frac{1}{4}f_{\mu\nu}f^{\mu\nu}, \tag{16.14}$$

$$J = 3.5177\times10^{-38}\, g\cdot cm, \qquad \overline{e} = -1.602\times10^{-20}\sqrt{4\pi}\ (g\cdot cm)^{1/2}. \tag{16.15}$$

The wave equation of a free Dirac field can be derived from (16.14). It has the usual form

$$\left[\gamma^\mu iJ\partial_\mu - m\right]\psi = 0. \tag{16.16}$$

Fuzzy quantum electrodynamics is based on the general postulate that there is an inherent probability distribution P(p,m) associated with a field oscillator with momentum **p** and mass m. This is a natural generalization of the idea of the "fuzzy quantum field" discussed in section 15a.

Based on the 4-dimensional formalism with a covariant gauge condition, we have the following free photon and electron fields:

$$a_\mu(w,\mathbf{r}) = \sum_{\mathbf{p};\alpha}\sqrt{\frac{JP(\mathbf{p},0)}{2Vp_0}}\left[a(\mathbf{p},\alpha)\varepsilon_\mu(\alpha)e^{-ip\cdot x/J} + a^\dagger(\mathbf{p},\alpha)\varepsilon_\mu(\alpha)e^{ip\cdot x/J}\right], \tag{16.17}$$

$$\psi(w,\mathbf{r}) = \sum_{\mathbf{p};s}\sqrt{\frac{mP(\mathbf{p},0)}{Vp_0}}\left[b(\mathbf{p},s)u(\mathbf{p},s)e^{-ip\cdot x/J} + d^{\dagger}(\mathbf{p},s)v(\mathbf{p},s)e^{ip\cdot x/J}\right], \tag{16.18}$$

where $p \cdot x = p_\mu x^\mu$,

$$\left[a(\mathbf{p},\alpha), a^{\dagger}(\mathbf{p}',\alpha')\right] = \delta_{\mathbf{pp}'}\delta_{\alpha\alpha'},$$

$$\left[b(\mathbf{p},s), b^{\dagger}(\mathbf{p}',s')\right] = \delta_{\mathbf{pp}'}\delta_{ss'}, \qquad \left[d(\mathbf{p},s), d^{\dagger}(\mathbf{p}',s')\right] = \delta_{\mathbf{pp}'}\delta_{ss'}, \tag{16.19}$$

and all other commutators vanish. Of course, commutators for quantized fields $\psi(w,\mathbf{r})$ and $a_\mu(w,\mathbf{r})$ can be derived from (16.17)-(16.19). In a general inertial frame, the Dirac equation in (16.16) can be written as

$$iJ\frac{\partial\psi}{\partial w} = [-iJ\boldsymbol{\alpha}_D \cdot \boldsymbol{\nabla} + \beta_D m]\,\psi. \tag{16.20}$$

As before, $w = bt_c$ denotes the lightime, t_c is common time, and $\boldsymbol{\alpha}_D$, β_D are the usual constant Dirac matrices.

Since the lightime w plays the role of evolution variable in the invariant equations of motion (16.20), it is also the evolution variable for a state $\Phi^{(S)}(w)$ in the Schrödinger representation

$$iJ\,\frac{\partial\Phi^{(S)}(w)}{\partial w} = H^{(S)}(w)\Phi^{(S)}(w), \qquad H^{(S)} = H_o^{(S)} + H_I^{(S)}, \tag{16.21}$$

because the evolution of a physical system is assumed to be described by a Hamiltonian operator which has the same transformation property as the lightime w or $\partial/\partial w$.

The conventional covariant formalism of perturbation theory can also be applied to fuzzy QED. For simplicity, we set J=1, so that the dimensions of the Lagrangian density L, the photon fields a_μ and the electron field ψ are

$$\left[L^{1/4}\right]=\left[a_\mu\right]=\left[\psi^{2/3}\right]=\left[\text{mass}\right]=\left[1/\text{length}\right], \qquad J=1. \tag{16.22}$$

To obtain the modified rules for Feynman diagrams in fuzzy QED, we follow the usual quantization procedure and define the fuzzy QED Lagrangian L_{FQED} by adding a gauge fixing term in the Lagrangian (16.14),

$$L_{FQED}=L-\frac{1}{2\rho}\left(\partial^\mu a_\mu\right)^2, \qquad J=1, \tag{16.23}$$

where ρ is a gauge parameter. We define the M-matrix as follows:

$$S_{if}=\delta_{if}-i(2\pi)^4\delta^4\left(p_i^{(tot)}-p_f^{(tot)}\right)\sqrt{\Pi_{ext..par}(n_j/V)}\ M_{if}, \tag{16.24}$$

where "ext..par" stands for "external particles." The quantity n_j denotes m_j/p_{0j} for spin 1/2 fermions and $1/(2p_{0j})$ for bosons. Because of the 4-dimensional symmetry in (16.23) and (16.24), the rules for writing M_{if} are formally the same as those in the usual QED:

(a) The fuzzy photon propagator is now given by

$$\frac{-iP(k,0)[\eta_{\mu\nu}-(1-\rho)k_\mu k_\nu/k^2]}{(k^2+i\varepsilon)}, \qquad k^2=k_\mu k^\mu. \tag{16.25}$$

(b) The fuzzy electron propagator is

$$\frac{iP(p,m)}{\gamma^\mu p_\mu-m+i\varepsilon}. \tag{16.26}$$

(c) The electron-photon vertex is

$$-(i\bar{e})\gamma^\mu. \tag{16.27}$$

(d) Each external photon line has an additional factor $\sqrt{P(k,0)}\ \varepsilon_\mu$. Each external electron line has $\sqrt{P(p,m)}u(s,p)$ for the absorption of an electron and $\sqrt{P(p,m)}\ \bar{u}(s,p)$ for the emission of an electron, etc.

Other rules such as taking the trace with a factor (-1) for each closed electron loop, integration with $d^4k/(2\pi)^4$ over a momentum k_μ not fixed by the conservation of four-momentum at each vertex, etc., are the same as the usual Feynman rules.

Thus, calculating scattering cross sections and decay rates (with respect to the lightime w) of a physical process, one obtains formally the same result as in conventional QED. For example, let us consider again the decay rate $\Gamma(1\to 2+3+...+N)$ for a physical process $1\to 2+3+...+N$ given by

$$\Gamma(1\to 2+3+...+N)=\lim_{w\to\infty}\int\frac{|\langle f|S|i\rangle|^2}{w}\frac{d^3x_2d^3p_2}{(2\pi J)^3}...\frac{d^3x_Nd^3p_N}{(2\pi J)^3}, \tag{16.28}$$

where $w=bt_c$ and $\Gamma(1\to 2+3+...+N)$ has the dimensions of inverse length. Its inverse is a particle's lifetime measured in terms of the lightime $w=bt_c$ which has the dimension of length and hence, $1/\Gamma$ may be called the "decay length." The decay length D is given by

$$D=\frac{1}{\Gamma(1\to 2+3+...+N)}. \tag{16.29}$$

Thus in common relativity, one has the "rest decay length" D_0 for the decay of a particle at rest, corresponding to the "rest lifetime" in the conventional theory. Also, instead of the dilation of the lifetime of a particle in flight, we have a dilation of the decay length it travels before decaying. This result is consistent with experiments because it is equal to the experimentally measured distance traveled by a decaying particle. (See chapter 10, sections 10g and 10h.)

For a scattering process $1+2\to 3+4+....+N$, the differential cross section $d\sigma$, which has the dimension of area, is given by

$$d\sigma = \frac{1}{4\left[(p_1 \cdot p_2)^2 - (m_1 m_2)^2\right]^{1/2}} |M_{if}|^2 \left[\prod_{ext..fer}(2m_{fer})\right] \frac{d^3\mathbf{p}_3}{(2\pi)^3 2p_{03}} \cdots\cdots$$

$$\cdots\cdots \frac{d^3\mathbf{p}_N}{(2\pi)^3 2p_{0N}} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - \cdots - p_N) S_o, \qquad (16.30)$$

where $p_0 = (\mathbf{p}^2 + m^2)^{1/2}$ and S_o denotes a factor $1/n!$ for each kind of (n) identical particles in the final state. If the initial particles are unpolarized, one must take the average over initial spin states. When there is no external fermion in a process, then $[\Pi_{ext\ fer}(2m_{fer})]$ in (16.30) is replaced by 1.

The formal expressions (16.28) and (16.30) are the same as the conventional ones. The new and different effects in fuzzy QED come from the *M*-matrix, which involves modified propagators related to an inherent fuzziness at short distances as shown in eqs. (16.25) through (16.27).

16c. Experimental tests of the 4-dimensional symmetry of special relativity at very high energies

Let us elaborate what new physical effects may be obtained due to an inherent probability distribution P(k,m) for field oscillators. Suppose one calculates the differential cross-section of unpolarized electrons scattered by an external potential (or a point-like nucleus). One obtains the result

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}(Z\alpha)^2 P(p_i - p_f, 0) P(p_i, m) P(p_f, m) \left(\frac{\omega_p}{|\mathbf{p}_i|^4 \sin^4(\theta/2)}\right)$$

$$\times\left(1 - \beta^2 \sin^2(\theta/2)\right), \qquad (16.31)$$

$$P(p_i - p_f, 0) = \frac{\sqrt{(\mathbf{p}_i - \mathbf{p}_f)^2}}{C_d}, \quad P(p_i, m) = \frac{\sqrt{\mathbf{p}_i^2 + m^2}}{C_i} \equiv \frac{\omega_p}{C_i}, \qquad (16.32)$$

$$P(p_f, m) = \frac{\sqrt{p_f^2 + m^2}}{C_f}, \qquad |\mathbf{p}_i - \mathbf{p}_f|^2 = 4|\mathbf{p}_i|^2 \sin^2\left(\frac{\theta}{2}\right), \qquad \beta = \frac{|\mathbf{p}_i|}{\omega_p}. \tag{16.33}$$

If the laboratory is chosen to be the F frame in which the speed of light is defined to be isotropic, then $C_d = C_i = C_f = c = 29979245800 cm/s$ (see section 12c). The differential cross section for the scattering of unpolarized electrons (16.31) is suppressed at large momenta by the inherent probability distribution associated with the photon in the intermediate state and the electrons in external states. In this calculation, the point-like nucleus is assumed to be at rest and its inherent probability distribution is negligible.

Next, let us consider the differential cross section for Møller scattering, (after C. Møller who first discussed the process in 1931). Møller scattering is the scattering of two electrons, $e^-(p_1)+e^-(p_2) \rightarrow e^-(p_1')+e^-(p_2')$. Since we are interested in the effects of the inherent probability distribution, we calculate the differential cross section at "high energies," $p_{10} >> 0$. The result is relatively simple to obtain in the center-of-mass frame,

$$\frac{d\sigma}{d\Omega} = \frac{r_o^2}{8}\left(\frac{m}{p_{10}}\right)^2 P^4(p,m)\left[\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}P_-^2 + \frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}P_+^2 \right.$$
$$\left. + \frac{2P_-P_+}{\cos^2(\theta/2)\sin^2(\theta/2)}\right], \tag{16.34}$$

$$\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2, \qquad \mathbf{p}' = \mathbf{p}_1' = -\mathbf{p}_2', \qquad p = |\mathbf{p}| = |\mathbf{p}'|,$$

$$P_\pm = P(k_\pm, 0), \qquad k_\pm = |\mathbf{p} \pm \mathbf{p}'|, \qquad r_o = \frac{\alpha_e}{m} = 2.82 \times 10^{-13} \text{cm},$$

where one may choose the center-of-mass frame to be the frame F in which the speed of light is isotropic.

One may also consider the pair annihilation process, i.e., $e^-(p_-) + e^+(p_+) \rightarrow \gamma(k_1) + \gamma(k_2)$. In the laboratory frame, in which $p_-=(m,0,0,0)$, the differential cross section is

$$\frac{d\sigma}{d\Omega}=\frac{r_o^2 k_{10}^2}{8}\left(\frac{1}{|\mathbf{p}_+|(m+p_{+0})}\right)P(p_-,m)P(p_+,m)P(k_1,0)P(k_2,0)$$

$$\times\left\{\left[\frac{k_{10}}{k_{20}}-\frac{2(\varepsilon_1 . k_2)^2}{k_{20}m}\right]P_a^2+\left[\frac{k_{20}}{k_{10}}-\frac{2(\varepsilon_2 . k_1)^2}{k_{10}m}\right]P_b^2\right.$$

$$\left.+\left[2-4(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2)^2+\frac{2(\varepsilon_2\cdot k_1)^2}{k_{10}m}+\frac{2(\varepsilon_1\cdot k_2)^2}{k_{20}m}\right]P_aP_b\right\}, \qquad (16.35)$$

$$P_a = P(p_- - k_1, m), \qquad P_b = P(p_- - k_2, m), \qquad k_i^\mu = (k_{i0},\mathbf{k}_i), \qquad i = 1, 2.$$

In the limit of zero radical length, $R\to 0$, (i.e., $I\to 0$ in (16.6)), all inherent probability distributions reduce to 1, and the terms involving $(\varepsilon_2{\cdot}k_1)$ and $(\varepsilon_1{\cdot}k_2)$ cancel, so that (16.35) reduces to the usual result.

The new properties of the scattering cross sections (16.31) to (16.35) due to fuzziness at short distances can all be tested by future high-energy experiments. These are also experimental tests of the possible inexactness of the 4-dimensional symmetry as instantiated by special relativity at very high energies and short distances.

At present, there exists experimental data for the decay of unstable particles in flight at high energies (several hundreds of GeV), $K_S^o(p) \to \pi^+(p_+) + \pi^-(p_-)$. This can be used to estimate the upper limit of the radical length R in this theory. The decay rate of K_S^o in flight is roughly modified by the inherent probability distribution as follows

$$\Gamma(K_S^o \to \pi^+ + \pi^-) = \frac{1}{p_0}P(p, m_K)P(p_+, m_{\pi^+})P(p_-, m_{\pi^-}) \times \text{const.} \qquad (16.36)$$

At several hundred GeV, the masses of the mesons can be neglected in the estimate. The departure of $P(p, m_K)P(p_+, m_{\pi+})P(p_-, m_{\pi-})$ from 1 must be within the experimental error. Based on the result of the decay rate of K_S^o in flight,[6] we estimate that

$$I < 2\times 10^{-4}\,\frac{c}{m_p} \qquad \text{or} \qquad R < 4\times 10^{-13}\text{cm}, \tag{16.37}$$

where we have ignored the meson masses. Since $m_p \approx 1\text{GeV}/c^2$, (16.37) and (16.6) indicate that the inherent probability distribution $P(k,m)$ becomes important only at extremely large momenta ($p >> 10^4$ GeV/c^2).

In summary, if we treat the concept of probability in quantum mechanics as fundamental so that the realizable position states of a particle must have a probabilistic interpretation, then the contributions of the corresponding states with large momenta will be suppressed by an inherent probability distribution $P(p,m)$. This implies that dilation of decay lengths should come from both the usual relativistic effect and the 'radical dilation' due to $R \neq 0$ at very high energies. Note that the inherent probability $P(p,m)$ is not a scalar under the Lorentz transformations of special relativity. Therefore, the spacetime 4-dimensional symmetry as expressed by the equations of special relativity is only approximate at extremely large momentum if $R \neq 0$ or $P(p,m) \neq 1$. Only in the limit of low energies or $R \to 0$, does the spacetime symmetry of special relativity become exact. However, because the inherent probability $P(p,m)$ can be expressed in terms of the genergy $G(\mathbf{p})$, which is an invariant in common relativity, common relativity provides a framework in which the 4-dimensional symmetry remains exact even in the presence of the inherent probability distribution $P(p,m) \neq 1$.

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17.

Extended Relativity: A Weaker Postulate for the Speed of Light

17a. Four-dimensional symmetry as a guiding principle

Only at the beginning of the twentieth century after the creation of special relativity was it recognized that the concept of symmetry played a fundamental role in physics.[1] The 4-dimensional spacetime symmetry has been one of the most thoroughly tested symmetry principles in physics. In his Nobel Lecture, C. N. Yang made the following piercing observation:

> "*Nature seems to take advantage of the simple mathematical representations of the symmetry laws. When one pauses to consider the elegance and the beautiful perfection of the mathematical reasoning involved and contrast it with the complex and far-reaching physical consequences, a deep sense of respect for the power of the symmetry laws never fails to develop.*"[2]

This quotation summarizes the essence of symmetry in physics, which will be illustrated below by an analysis of different viewpoints of the physical world to show how 4-dimensional symmetry is critical to any theory.

Let us consider the viewpoint of Reichenbach's more general concept of time and Edwards' universal two-way speed of light, which includes relativistic time and special relativity as a special case. Unfortunately, Reichenbach's and Edwards' treatments were, in general, not consistent with the Lorentz and Poincaré symmetry groups and so their efforts never led to an experimentally consistent theory. The lack of the 4-dimensional symmetry of the Lorentz group will cause many problems in the formulation of quantum field theories, especially the Feynman rules for calculations in quantum electrodynamics or chromodynamics.[3] For the purpose of comparison, we shall explain in some detail the application of 4-dimensional symmetry to the ideas of Reichenbach and Edwards. In this way, we will also show how their work can be made consistent with experiments.

We term such a 4-dimensional theory "extended relativity." In the context of the family of relativity theories that we have been discussing, including taiji relativity, common relativity, and special relativity, extended relativity is simply taiji relativity with a second postulate stating that the 2-way speed of light is a universal constant.

In the following discussions of “extended relativity”, involving the universal two-way speed of light c, we shall use the usual units for space, time and mass, and retain the usual constants such as e, c and $\hbar$.

Using 4-dimensional symmetry as a guiding principle for discussing physical laws, we first analyze Edwards' original attempt in 1963 to formulate a relativity theory based on a weaker postulate for the speed of light. Edwards postulated that "the two-way speed of light in a vacuum as measured in two coordinate systems moving with constant relative velocity is the same constant regardless of any assumptions concerning the one-way speed."[4] He derived space and time transformations involving infinitely many possible physical times which could be physically realized through Reichenbach's convention of clock synchronization (which includes relativistic time as a special case). Edwards' transformations were shown to be consistent with many experiments related to the propagation of light. Nevertheless, we show that they do not form the Lorentz group *so that, in general, physical laws are not invariant under such transformations.* As a result, Edwards' work leads to an incorrect expression for the relativistic energy-momentum of a particle in the Lagrangian formalism.[3]

In view of these results, one may conclude that assuming the universality of the 2-way speed of light and Reichenbach's concept of time is experimentally untenable. However, this is not the case. In the following paragraphs we show that Reichenbach's general convention of time and Edwards' universal 2-way speed of light can be accommodated in a 4-dimensional formalism which is consistent with the relativistic energy-momentum of a particle and the Lorentz and Poincaré groups. Such a theory, extended relativity, includes special relativity as a special case.

These results are physically and pedagogically interesting for the following reason: Physics students are usually puzzled by discussions of

Reichenbach's convention of time and the impossibility of the unambiguous measurement of the one-way speed of light in the literature.[4] They ask: Can Reichenbach's convention of time be consistent with the Lorentz group properties of transformation between two inertial frames? Are the ideas of Reichenbach and Edwards viable? Our results suggest that the key to answering these questions is the 4-dimensional symmetry of the Lorentz group. The physical theory of taiji relativity based solely on the first postulate of relativity, i.e., the principle of relativity for physical laws, has been discussed in chapter 7. We show that the correct formalism of Edwards' theory based on a weaker postulate for the speed of light can be obtained by imposing an additional postulate of the universal 2-way speed of light onto taiji relativity. Thus, extended relativity gives a more restricted view of the 4-dimensional physical world than that of taiji relativity. It also shows the power of the first postulate of relativity from the vantage point of 4-dimensional symmetry.

17b. Edwards' transformation with Reichenbach's time

Let us consider two inertial frames F and F', with relative motion along the x/x'-axes. Suppose there are two identical clocks, clock 1 located at the origin of the F frame and clock 2 at point x on the x-axis. A light signal starts from the origin (event 1) at time t_1, it reaches clock 2 (event 2) at time t_2 and returns to the origin (event 3) at time t_3. Reichenbach's concept of time is realized by synchronizing clock 2 to read t_2 by the relation[4]

$$t_2 = t_1 + \varepsilon(t_3 - t_1), \tag{17.1}$$

where ε is restricted by $0< \varepsilon <1$, so that causality is preserved, i.e., t_2 cannot be earlier than t_1. The same synchronization procedure can be performed on clocks in the F' frame:

$$t'_2 = t'_1 + \varepsilon'(t'_3 - t'_1). \tag{17.2}$$

By definition, the two-way speed c_{2w} of light along the x-axis in F is given by

$$c_{2w} = \frac{2L}{t_3 - t_1}, \tag{17.3}$$

which is independent of t_2 and Reichenbach's parameter ε. Similarly, in the F' frame, we also have a constant two-way speed of light,

$$c'_{2w} = \frac{2L'}{t'_3 - t'_1}. \tag{17.4}$$

Following Edwards, we shall assume these two-way speeds of light to have the same value,

$$c'_{2w} = c_{2w} = c. \tag{17.5}$$

The synchronization of clocks in F and F' according to (17.1)-(17.5) defines the Reichenbach time which clearly includes relativistic time as a special case ($\varepsilon = \varepsilon' = 1/2$).

In special relativity, an Einstein clock at x is synchronized according to $t_E = t_o + x/c$ (where t_o is the time of the clock at the origin of F). The corresponding Reichenbach clock at x reads $t_R = t_o + 2\varepsilon x/c_{2w} \equiv t$, which follows from (17.1), (17.4) and (17.5). Thus, t_E and t are related by

$$t_E = t - (2\varepsilon - 1)\frac{x}{c} = t - \frac{qx}{c}, \qquad q = 2\varepsilon - 1. \tag{17.6}$$

Similarly, in the F' frame, we have the relation

$$t'_E = t' - \frac{q'x'}{c}, \qquad q' = 2\varepsilon' - 1. \tag{17.7}$$

For simplicity and without loss of generality, we set $q = 0$ in (17.6), so that physics in the F frame is identical to that in special relativity. We now concentrate on physical implications of (17.7) in the F' frame to see whether Edwards' transformation is inconsistent with experimental results. Using (17.6) with $q = 0$, (17.7) and the Lorentz transformation involving Einstein time t_E and t'_E, one obtains the Edwards transformation between inertial frames F and F',

$$t' = \gamma\left((1-\beta q')t - (\beta - q')\frac{x}{c}\right), \qquad x' = \gamma(x - \beta ct), \qquad y' = y \qquad z' = z; \tag{17.8}$$

$$\beta = \frac{V}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}},$$

where V is the speed of F' as seen from F. This type of transformation with Reichenbach's time was first derived and discussed by Edward.[4]

It is important to note the following properties:

(A) The coordinate vector (ct', x', y', z') is no longer a 4-vector under the transformation (17.8) because (17.8) does not have the explicit 4-dimensional symmetry of the Lorentz group in general.

(B) In the limit $V \to 0$, i.e., F and F' become the same inertial frame, but (17.8) does not reduce to the identity transformation:

$$t' = \left(t - q'\frac{x}{c}\right), \qquad x' = x, \qquad y' = y \qquad z' = z. \tag{17.9}$$

This shows that Edwards' transformations do not form the Lorentz group, except in the special case $q' = q = 0$.

(C) Under Edwards' transformations, one has

$$c^2 t'^2 + 2q' x' ct' - x'^2 (1 - q'^2) - y'^2 - z'^2 = c^2 t^2 - x^2 - y^2 - z^2, \qquad q = 0. \tag{17.10}$$

Note that if $q \neq 0$, then $c^2t^2 - x^2 - y^2 - z^2$ will be replaced by $c^2t^2 + 2qxct - x^2(1-q^2) - y^2 - z^2$. It can be shown that this quadratic form holds also for infinitesimal intervals.

17c. Difficulties of Edwards' transformation

The Edwards transformation (17.8) has been shown to be consistent with many experiments related to the propagation of light.[4] Moreover, since Edwards' transformation appears to be obtainable by a "change of time variables," (17.6) and (17.7), in the Lorentz transformation, one might think that it is equivalent to the Lorentz transformation. However, this is not the case because Edwards' transformation (17.8) does not have the Lorentz group properties (except when q=q'=0) due to the transformation of t.[3] Thus, Edwards' transformation (17.8), as it stands, violates the 4-dimensional symmetry of the Lorentz group. As a result, one can show explicitly (see (17.14) below) that the time t' leads to an incorrect expression for the relativistic energy-momentum of a particle in the Lagrange formalism. This contradicts experiments in general, except when q'=q=0.

To wit, let us consider Edwards' transformation (17.8), where we have chosen a frame F in which the Reichenbach time equals the Einstein time by setting q=0. We stress that this frame can be chosen arbitrarily. (If one wishes, one can set q'=0 instead of q=0. The following arguments still hold with F and F' interchanged.) For a free particle, we have the actions S and S' in F and F' respectively:

$$S = -\int mc\ ds = \int L\ dt, \qquad ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2,$$

$$L = -mc^2\sqrt{1 - v^2/c^2}, \qquad \text{in F (with } q = 0), \qquad (17.11)$$

$$S' = -\int mc\ ds' = \int L'dt', \quad ds'^2 = c^2dt'^2 + 2q'dx'cdt' - dx'^2(1 - q'^2) - dy'^2 - dz'^2,$$

$$L'(E) = -mc^2\sqrt{(1 - q'v'_x/c)^2 - v'^2/c^2}, \qquad \text{in F'(with } 1>q'>-1). \qquad (17.12)$$

Note that S = S' because ds = ds' which can be verified by taking differentials of (17.8). The constant c in (17.12) is the universal 2-way speed of light measured in F'. The Lagrangians L and L' lead to the following momenta in F and F' respectively,

$$p_x(E) = \frac{\partial L}{\partial v_x} = \frac{mv_x}{\sqrt{1-(v/c)^2}} \equiv p_x, \quad \text{etc.} \quad \text{in } F, \tag{17.13}$$

$$p'_x(E) = \frac{\partial L'}{\partial v'_x} = \frac{mcq' - mq'v'_x + mv'_x}{\sqrt{(1-q'v'_x/c)^2 - (v'/c)^2}}, \quad \text{etc.} \quad \text{in } F'. \tag{17.14}$$

We see clearly that the momentum $p'_x(E)$ in F' contradicts experimental results, except when $q'=q=0$. Similarly, one can show that the result for the kinetic energy, defined by $(\mathbf{p}'(E)\cdot\mathbf{v}' - L')$, in F' is also incorrect.

Before dismissing the expression (17.14), one should check whether (17.14) is the same as the result of changing the time variable in the normal special relativistic momentum $p'_x(SR)$,

$$p'_x(SR) = \frac{mu'_x}{\sqrt{1-u'^2/c^2}}, \qquad u'_x = \frac{dx'}{dt'_E}, \quad \text{etc.} \quad \text{in } F'. \tag{17.15}$$

Under a change of time variable given in (17.7), we have the relation $dx'/dt'=(dx'/dt'_E)(dt'_E/dt')$, etc., i.e.,

$$\mathbf{u}' = \frac{\mathbf{v}'}{1-q'v'_x/c}. \tag{17.16}$$

From equations (17.15) and (17.16), we obtain

$$p'_x(SR) = \frac{mv'_x}{\sqrt{(1-q'v'_x/c)^2 - v'^2/c^2}} \neq p'_x(E), \tag{17.17}$$

which differs from the momentum $p'_x(E)$ in (17.14). *This example shows that the lack of 4-dimensional symmetry in Edwards' formalism based on transformation (17.8) with $1 > q' > -1$ make it untenable.* Of course, there are many other difficulties related to the lack of symmetry such as the non-invariance of the Maxwell equations, the Dirac equation and laws in quantum electrodynamics under the transformations (17.8).

In the next section, however, we show that if one uses 4-dimensional symmetry of taiji relativity as a guiding principle, all these difficulties can be resolved.

17d. Extended relativity: A 4-dimensional theory with Reichenbach's time (a universal 2-way speed of light)

The method for the construction of a 4-dimensional symmetry framework without the usual relativistic time has been discussed before.[3,5] The logically simplest case is the 4-dimensional symmetry of taiji relativity which is based solely on the first postulate of relativity. It can be applied to guide the construction of a 4-dimensional framework for the present case with Reichenbach's time. For simplicity and without loss of generality, we choose $q=0$ in synchronizing clocks in the F frame, so that we have the usual relativistic time $t = t_E$. In the F' frame, clocks are synchronized to read the Reichenbach time t'. An event is, as usual, denoted by (w,x,y,z), where w=ct, in F. However, following taiji relativity discussed in chapter 7, the same event must be denoted by (w',x',y',z'), with w'=b't', in F' within the 4-dimensional symmetry framework. We stress that *it is necessary to introduce the function b' so that (w',x',y',z')= (w',**r**') transforms like a 4-vector and the laws of physics can display 4-dimensional symmetry.* Otherwise, the laws of physics cannot be invariant under the transformation from F to F'. With clocks in F and F' synchronized as discussed previously, the times t and t' are related as in (17.8):

$$t'=\gamma\left[(1-\beta q')t-(\beta-q')\frac{x}{c}\right], \qquad q'=2\varepsilon'-1. \tag{17.18}$$

This may be considered an assumption. Indeed, from the viewpoint of taiji relativity, this is a second postulate which is analogous to assuming the universality of the two-way speed of light over a closed path in any inertial frame, as discussed in section 16b. As usual, we start with the invariance of the 4-dimensional interval s^2

$$s^2=(b't')^2-x'^2-y'^2-z'^2=c^2t^2-x^2-y^2-z^2, \tag{17.19}$$

to derive the 4-dimensional transformation. Note that relation (17.19) follows from the first postulate of relativity, i.e., the Poincaré-Einstein principle, because it is equivalent to the law of motion, $p_0{}^2-\mathbf{p}^2=m^2$, for a free particle with mass m>0. (See ref. 3 in chapter 6.) By the usual method for deriving the Lorentz transformation or the taiji transformation (7.4), we derive the extended 4-dimensional transformation involving Reichenbach's time,

$$w'=b't'=\gamma\left(ct-\beta x\right), \qquad x'=\gamma(x-\beta ct), \qquad y'=y \qquad z'=z; \tag{17.20}$$

$$\beta=\frac{V}{c}, \qquad \gamma=\frac{1}{\sqrt{1-\beta^2}},$$

where t' is given in (17.18). This extended transformation completely determines the function b'

$$b'=\frac{ct-\beta x}{t(1-\beta q')-(\beta-q')x/c}, \tag{17.21}$$

i.e., if x and t are known, we can always calculate the value of b' in (17.20) and (17.21). Note that the *fourth dimension in F' is now b't'≡w' rather than t' or ct',* where w'=b't' is called the "lightime." Although the function b' in (17.21) and the time t' in (17.18) separately have complicated transformation properties, b' and t' are separately well-defined and the product b't'≡w' transforms as the fourth component of the coordinate 4-vector. The properties of b' are completely determined by (17.21) or the transformation (17.20). From here on, we will use w'≡b't' to display explicitly the 4-dimensional symmetry of physical laws.

The extended transformation of velocities can be derived from (17.20):

$$c'\equiv\frac{dw'}{dt'}=\frac{dt}{dt'}\gamma(c-\beta v_x),$$

$$v'_x=\frac{dt}{dt'}\gamma(v_x-\beta c), \tag{17.22}$$

$$v'_y = \frac{dt}{dt'} v_y, \qquad v'_z = \frac{dt}{dt'} v_z,$$

where dt/dt' may be obtained from differentiation of (17.18),

$$\frac{1}{(dt/dt')} = \frac{dt'}{dt} = \gamma\left((1-\beta q') - (\beta - q')\frac{dx}{cdt}\right) . \qquad (17.23)$$

This definition $c' \equiv d(w')/dt'$ in (17.22) is quite natural because $dw'^2 - d\mathbf{r}'^2 = 0$ can be interpreted as the law of light propagation, $c'^2 dt'^2 - d\mathbf{r}'^2 = 0$. Note that the transformation of the ratios $\mathbf{v}'/c'$ are precisely the same as those in special relativity

$$\frac{v'_x}{c'} = \frac{(v_x/c) - \beta}{1 - v_x\beta/c} \; ; \qquad \frac{v'_y}{c'} = \frac{v_y/c}{\gamma(1 - v_x\beta/c)} \; , \qquad \text{etc.} \qquad (17.24)$$

Let us consider the property of c' in F'. Its value depends on its direction of propagation. Suppose a light signal propagates along an angle θ as measured in F and θ' as measured in F'. We have

$$v_x = c \cos\theta, \qquad v'_x = c' \cos\theta'. \qquad (17.25)$$

From the inverse transformation of (17.22), we obtain $c = (dt'/dt)[\gamma(c' - \beta v'_x)]$, i.e.,

$$c' = \frac{c}{\{(dt'/dt)[\gamma(1 + \beta\cos\theta')]\}} = c'(\theta'). \qquad (17.26)$$

Evidently, the average speed of light over any closed path in F is a constant which is equal to c. Now we shall show that the average speed of a light signal over any closed path is also c, even though the speed of light is no longer isotropic in F'.

Suppose a light signal travels along the vectors $\mathbf{r}'_i$, where $i = 1,2,...N$, which form a closed path on the x'-y' plane in F'. The total distance L'_{tot} and the total time T'_{tot} are given by

$$L'_{tot} = \sum_{i=1}^{N} r'_i , \quad \text{and} \quad T'_{tot} = \sum_{i=1}^{N} \frac{r'_i}{c'(\theta'_i)} \; ; \; N > 1. \tag{17.27}$$

The average speed c'_{av} of the light signal over this closed path is

$$c'_{av} = \frac{L'_{tot}}{T'_{tot}} = \frac{L'_{tot}}{L'_{tot}/c + (q'/c) \sum_{i=1}^{N} r'_i \cos\theta'_i} = c, \tag{17.28}$$

where we have used

$$v'_{xi} = c'(\theta'_i)\cos\theta'_i, \tag{17.29}$$

$$c'(\theta'_i) = \frac{c}{(dt'/dt)_i \left[\gamma(1+\beta\cos\theta'_i)\right]}, \tag{17.30}$$

$$\left(\frac{dt'}{dt}\right)_i = \frac{1+q'\cos\theta'_i}{\gamma\left(1+\beta\cos\theta'_i\right)}, \qquad i = 1,2,....,N, \tag{17.31}$$

$$\sum_{i=1}^{N} r'_i \cos\theta'_i = 0. \tag{17.32}$$

Equation (17.32) is a property of a closed path in F'. Thus, the average speed of light over an arbitrary closed path is a universal constant c in extended relativity.

We note that a closed path for a light signal as observed by F observers is, in general, not a closed path as observed by F' observers. Suppose a signal starting from O(or O'), i.e., the origin of F (F'), travels to a point A (A') on the x-axis and is reflected back to B = O (B'$\neq$ O) as observed in F (F'). The two-way speed of this signal in F is c. However, one may ask, what is the average speed of this light signal as measured in F'? From Reichenbach's time (17.18) and extended transformation (17.20), we derive the space and time coordinates of these events in F',

$$t'(O') = x'(O') = 0, \tag{17.33}$$

$$t'(A') = \gamma\left([1-\beta q')t(A) - (\beta + q')\frac{x(A)}{c}\right), \qquad x'(A') = \gamma\big((x(A) - \beta ct(A)\big), \tag{17.34}$$

$$t'(B') = \gamma\left([1-\beta q')t(B) - (\beta + q')\frac{x(B)}{c}\right), \qquad x'(B') = \gamma\big((x(B) - \beta ct(B)\big). \tag{17.35}$$

Since t(O)=x(O)=x(B)=0, x(A)=ct(A)=L and t(B)=2L/c, the equations in (17.33) -(17.35) lead to the "average speed" $c'_{av}(nc)$ for such a non-closed path in F':

$$c'_{av}(nc) = \frac{x'(A')+|x'(B')-x'(A')|}{t'(A')+|t'(B')-t'(A')|} = \frac{c}{1-\beta q'}. \tag{17.36}$$

Result (17.36) can also be derived from (17.22) directly because this average speed of light can be considered as two events which satisfy $\Delta x = x(B) - x(O) = 0$. As far as constant velocities are concerned, $\Delta x = 0$ is equivalent to $dx = 0$ or $dx/dt = v_x = 0$. Using relation (17.23) with the condition $v_x = 0$, we obtain

$$\frac{dt'}{dt} = \gamma(1-\beta q'), \qquad v_x = 0. \tag{17.37}$$

Thus using the expression for c' in (17.22) for a non-closed path in F', we also have the relation

$$c'(v_x = 0) = \frac{c}{(1-\beta q')}, \tag{17.38}$$

which is precisely the result (17.36).

17e. The two basic postulates of extended relativity

The physical foundation of the extended transformation (17.20) is based on two postulates: (1) The invariance of the form of physical laws and (2)

Reichenbach's time (17.18). The latter may also be regarded as a definition of time much like the definition of common time introduced in chapter 8.

As we have discussed previously, relation (17.18) is directly related to clock synchronization. To be specific, we have used the relation between Reichenbach's and Einstein's times, together with the Lorentz transformation, to obtain (17.18).

One may ask: Is it possible to derive the extended transformations (17.20) and (17.18) without using Einstein's time and the Lorentz transformation as a crutch? The answer is yes. Instead of postulating (17.18), one can use its equivalent postulate: Namely, that the 2-way speed of light over a closed path in any frame is a universal constant and independent of the motion of the light sources. This (second) postulate was first made by Edwards for his formalism.[4]

Let us now derive the relation (17.18) based on the two basic postulates of extended relativity:

[1]. The principle of relativity for physical laws: The form of a physical law must be invariant under coordinate transformations. (In other words, physical laws must display 4-dimensional symmetry.)

[2]. The 2-way speed of light over a closed path is a universal constant and is independent of the motion of its source.

To derive (17.18) from these two postulates, we first observe that, starting from (17.19) and following the steps from (7.2) to (7.4), we obtain (17.20) but not (17.18). Since the F' frame moves along the x-axis, its time t' can only be a linear function of t and x,

$$t' = Pt + Qx, \tag{17.39}$$

where unknown coefficients P and Q are to be determined. From (17.22) and (17.39), we have

$$c'(+) = \left(\frac{dt}{dt'}\right)_1 c\gamma(1-\beta), \qquad \frac{dx}{dt} = +c, \tag{17.40}$$

$$c'(-) = \left(\frac{dt}{dt'}\right)_2 c\gamma(1+\beta), \qquad \frac{dx}{dt} = -c, \tag{17.41}$$

where

$$\left(\frac{dt}{dt'}\right)_1 = P + Qc, \tag{17.42}$$

$$\left(\frac{dt}{dt'}\right)_2 = P - Qc. \tag{17.43}$$

Postulate [2] implies that

$$\frac{L'}{c'(+)} + \frac{L'}{c'(-)} = \frac{2L'}{c'_{2w}} = \frac{2L'}{c}, \tag{17.44}$$

and it follows from equations (17.40)-(17.44) that

$$P + \beta c Q = \frac{1}{\gamma}. \tag{17.45}$$

Without loss of generality, we may express Q in terms of another parameter q' such that (17.39) is more closely related to the form (17.18) in Edwards' transformation,

$$Q = -\frac{\gamma(\beta + q')}{c}. \tag{17.46}$$

Relations (17.39), (17.45) and (17.46) lead to

$$t' = \gamma\left\{(1 - \beta q')t - (\beta + q')\frac{x}{c}\right\}. \tag{17.47}$$

This result is precisely the same as the basic transformation of Reichenbach's time (17.18).

17f. Invariant action for a free particle in extended relativity

Although extended relativity involves a class of different concepts of time, realized by Reichenbach's procedures for clock synchronization based on a universal 2-way speed of light, all physical laws have the 4-dimensional form that are identical to those in special relativity.

Let us first demonstrate that extended relativity leads to a correct expression for momentum, in contrast to Edwards' formalism. The invariant action for a free particle in F' is

$$S' = -\int mc\, ds' = \int L' dt', \qquad c = c'_{2w} = c_{2w}, \tag{17.48}$$

$$ds'^2 = c'^2 dt'^2 - dx'^2 - dy'^2 - dz'^2, \qquad c' = \frac{dw'}{dt'}.$$

Note that c in (17.48) is the two-way speed of light, c_{2w}, which is a universal constant. The Lagrangian L' in the F' frame takes the form

$$L' = -mcc'\sqrt{1 - v'^2/c'^2}, \tag{17.49}$$

which leads to the momentum **p**',

$$\mathbf{p}' = \frac{mc(\mathbf{v}'/c')}{\sqrt{1 - v'^2/c'^2}}. \tag{17.50}$$

The "energy" p'_0 is defined as the zeroth component of the momentum by

$$p'_0 = \frac{(\mathbf{p}' \cdot \mathbf{v}' - L')}{c'} = \frac{mc}{\sqrt{1 - v'^2/c'^2}}. \tag{17.51}$$

These form the momentum 4-vector $p'_\mu = (p'_0, \mathbf{p}')$ which satisfies the 4-dimensional invariant relation

$$p'^2_0 - \mathbf{p}'^2 = m^2 c^2. \tag{17.52}$$

By Noether's theorem, this is the conserved energy-momentum in extended relativity.

We see clearly that the momentum $\mathbf{p}'$ and the energy p'_0 in F' are consistent with those measured in high energy experiments. As a matter of fact, the momentum p'_x given by (17.50) is precisely the same as the momentum $p'_x(SR)$ in (17.15) for the F' frame in special relativity. The reason is that the invariant ds^2 in F' for special relativity is the same as ds'^2 given by (17.48), [i.e. $ds^2 = c^2 dt'^2_E - dx'^2 - dy'^2 - dz'^2 = c'^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 = ds'^2$], so that

$$\mathbf{p}'(SR) = mc\frac{d\mathbf{r}'}{ds} = mc\frac{d\mathbf{r}'}{ds'} = \mathbf{p}', \qquad \mathbf{u}' = \frac{d\mathbf{r}'}{dt'_E}, \qquad \mathbf{v}' = \frac{d\mathbf{r}'}{dt'}. \tag{17.53}$$

This example shows that in order for Reichenbach's time to be viable it must be embedded within a 4-dimensional framework.

The Klein-Gordon equation takes the form

$$\left(\hbar^2 \partial_\mu^2 + m^2 c^2\right)\varphi = 0. \tag{17.54}$$

It can be shown to be invariant under the extended transformation (17.20) as well because the differential operators

$$\partial_\mu = \partial / \partial x^\mu, \qquad \text{and} \qquad \partial'_\mu = \partial / \partial x'^\mu, \tag{17.55}$$

are 4-vectors, where $x^\mu = (ct, \mathbf{r})$, $x'^\mu = (w', \mathbf{r}') = (b't', \mathbf{r}')$. It is interesting to see from (17.54) that, although one has times t and t' in extended relativity, the evolution variable in basic 4-dimensional equation is the lightime $w' = b't'$ rather than Reichenbach's time t' in the F' frame. This is dictated by the 4-dimensional symmetry of the Lorentz and Poincaré groups and is compatible with the evolution variable of taiji relativity.

17g. Comparison of extended relativity and special relativity

Although Edwards' original transformations (17.8) can be obtained by a "change" of time variables (17.7) in the Lorentz transformations, this does not imply that the two transformations are completely equivalent *experimentally and theoretically.* The reason is that it is not simply a change of variable within the framework of special relativity. To be specific, such a change of variable violates the second postulate (i.e., the universal constancy of the speed of light) of special relativity. As a result, after the "change" of time variable in (17.7), one is no longer within the same conceptual framework because, strictly speaking, special relativity is a theory which possess the 4-dimensional symmetry with the Einstein time t'_E which has the usual time dilation property. However, after the change of time variable in (17.7), one has extended relativity which possess the 4-dimensional symmetry with the Reichenbach time t' given by (17.47) which does not have the usual time dilatation property in general. Nevertheless, we show in section 17j that the experimental result of the lifetime dilation of unstable particles decay in flight can be understood within extended relativity in terms of the dilation of the decay length, just as we did for common relativity.

We have shown in (17.12) that Edwards' original transformations lead to an incorrect momentum in the Lagrangian formalism because of its lack of 4-dimensional symmetry, as shown in equations (17.13)-(17.17). *This shows that the first principle of relativity for physical laws has not been incorporated in Edwards' formalism of relativity.* In particular, (ct',x',y',z') is not a four-vector. In sharp contrast, the present formalism of extended relativity based on the new transformations (17.20) between the coordinate four-vectors (w,x,y,z) and (w',x',y',z'), where w=ct and w'=b't', is explicitly consistent with the 4-dimensional symmetry of the Lorentz and the Poincaré groups or the Poincaré-Einstein principle of relativity.

If one compares the space-lightime transformations (17.20) with the Lorentz transformations, one sees that

$$ct'_E = b't', \tag{17.56}$$

where t'_E and t' are respectively Einstein's time and Reichenbach's time. We note that relation (17.56) does not imply that extended relativity is *an ordinary change of time variables within the framework of special relativity.* All that it implies is that if the fourth component ct'_E of a coordinate 4-vector in special relativity is replaced by b't', where b' is a function, one obtains another 4-dimensional symmetry framework without the universal constant of the speed of light c. Therefore, (17.56) *is simply the direct consequence of replacing Einstein's second postulate by a universal two-way speed of light (Edwards' weaker second postulate).*

If one takes the viewpoint that extended relativity is just a change of time variables given by (17.56), one should be able to obtain all results in extended relativity from the corresponding result in special relativity by the relation (17.56). To wit, the momentum (17.50) indeed can be obtained from usual relativistic momentum (17.15) by a simple change of variable (17.56). However, this is not always the case. For example, a plane wave in F' is described by an invariant function

$$\exp[i(\omega' t'_E - \mathbf{k}' \cdot \mathbf{r}')] \tag{17.57}$$

in special relativity. By a simple change of time variable (17.56), one obtains

$$\exp\left[i\left(b' t' \frac{\omega'}{c} - \mathbf{k}' \cdot \mathbf{r}'\right)\right], \tag{17.58}$$

which leads to an *incorrect* wave 4-vector (ω'/c, $\mathbf{k}'$). The correct wave 4-vector in F' frame should be

$$(k'^0, \mathbf{k}') = \left(\frac{\omega'}{c'}, \mathbf{k}'\right). \tag{17.59}$$

This is the only wave 4-vector which is consistent with the defined relation

$$\nu' \lambda' = c', \tag{17.60}$$

in extended relativity. We note that if quantum mechanics is formulated on the basis of extended relativity, the 4-momentum of a photon should be proportional to the extended wave 4-vector (17.59) rather than (ω'/c, $\mathbf{k}'$).

As a matter of fact, based on equation (17.57) one can obtain the correct invariant phase, involving (17.60) and the 4-coordinate (w',x',y',z') by changing both time and quantities related to time (e.g., the frequency). *To do this, one must first know the correct wave 4-vector in extended relativity, i.e., one must be able to formulate 4-dimensional extended relativity independent of special relativity.* In this sense, it is gratifying that we are able to formulate a 4-dimensional theory of extended relativity, as shown in section 17e.

From previous discussions, we conclude that *extended relativity and special relativity are two logically different theories,* one that has a universal one-way speed of light and the other that has a universal two-way speed of light (which includes the former as a special case.) Nevertheless, both have the 4-dimensional symmetry of the Lorentz group, so that physical laws have the same forms in all inertial frames. (See section 18c.)

As with common relativity, extended relativity is consistent with all known experiment. For any invariant law in special relativity, there is a corresponding law of the same form in extended relativity. *This suggests that extended relativity is experimentally equivalent to special relativity.* We believe that this is yet another piece of evidence that concepts of time (e.g., Einstein's time or Reichenbach's time) and the resulting speed of light (measured in meter per second) in the 4-dimensional symmetry framework are human conventions rather than an inherent property of the physical world. This is the same conclusion that was already indicated by the results of taiji relativity.

17h. An unpassable limit and a non-constant speed of light

From the expressions for the momentum and energy in (17.50) and (17.51) or the transformations of velocities (17.22) under extended relativity, one can see an interesting property of the non-constant speed of light c'. Namely, although c' differs in different directions in F', if one compares speeds of various physical objects in a given direction in F', the speed of light is *the maximum speed in the universe.* This holds for all values of the parameter q' in (17.22). We stress that this "maximum speed" is a more general property than the speed of light being a universal constant because the latter corresponds to the special case q'=0 in (17.22). It is interesting to note that this property was discussed as a "character" of an entirely new mechanics by Poincaré in his vision of a relativity theory in 1904:

"The velocity of light is an unpassable limit." (See sec. 5c in chapter 5.)

Furthermore, even though the velocities c' and **v'** measured using

Reichenbach's time depend on the parameter q', as shown in (17.22) and (17.23), experimental results turn out to be independent of q'. For example, the momentum **p'** in (17.50) depends on the ratio **v'**/c'=(d**r'**/dt')/(dw')/dt'=d**r'**/dw' which is independent of time t' or q'. Thus, the restriction on q' in (17.12) is not necessary from experimental viewpoint. The basic reason for these properties is that, within the 4-dimensional symmetry framework, the inherent evolution variable in the laws of physics is the lightime w' rather than Reichenbach's time t', as already indicated by taiji relativity. (See chapter 7.)

17i. Lorentz group and the space-lightime transformations

If an object is at rest in the F frame, i.e., **v** = (0,0,0), the extended velocity transformations (17.22) lead to

$$c' = \frac{c}{1-\beta q'} \equiv c'(0),$$

$$v'_x = \frac{-\beta c}{1-\beta q'} \equiv v'_x(0), \qquad v'_y = v'_z = 0. \tag{17.61}$$

This shows that the speed of the F frame, measured from F', is different from the speed of F' measured from F and, moreover, the difference depends on the parameter q'. This might be puzzling because common sense tells us that the speed of F measured from F' should be -V. However, as can be seen from special relativity and quantum mechanics, common sense is often a poor guide in modern physics. Let us consider the ratio $v'_x(0)/c'(0)$ in (17.61). This ratio is a constant, independent of dt/dt', and satisfies

$$\beta' = -\frac{v'_x(0)}{c'(0)} = +\beta. \tag{17.62}$$

Thus, using the ratios of velocities, β' and β, one has the symmetry of velocities between F and F'. This is the same situation as in common relativity. The reason for the simplicity shown in (17.62) is simply that the real evolution variable in extended relativity is the lightime w'=b't' rather than the Reichenbach time t' in F'. The inverse transformation of (17.20) can be obtained. It takes the form:

$$ct = \gamma(b't'+\beta'x') = \gamma'(b't'+\beta'x'),$$

$$x = \gamma'(x'+\beta'b't'), \qquad y = y', \qquad z = z'; \qquad \gamma' = \gamma. \tag{17.63}$$

Let us consider also another frame F" moving with a constant velocity $\mathbf{V}_1 = (V_1,0,0)$, as measured from F and $\mathbf{V'}_1 = (V'_1,0,0)$ as measured in F'. Note that the ratio V'_1/c' *is* related to V_1 by (17.24):

$$\frac{V'_1}{c'} = \frac{(V_1/c-\beta)}{(1-\beta V_1/c)}. \tag{17.64}$$

From (17.61), (17.62) and (17.64), we see that instead of V or V', one should use the ratio V/c or V'/c' to characterize the relative motion between inertial frames in extended relativity, as *this ratio will always be constant and more importantly, is independent of the parameter q'.*

The 4-dimensional transformations between F and F" are given by

$$b''t'' = \gamma_1(ct - \beta_1 x), \quad x'' = \gamma_1(x - \beta_1 ct), \quad y'' = y, \quad z'' = z, \tag{17.65}$$

where

$$t'' = \gamma_1\left[(1-\beta_1 q'')ct - (\beta_1 - q'')x\right]\frac{1}{c}. \tag{17.66}$$

From (17.20) and (17.65), we can obtain the transformations between F' and F",

$$b''t'' = \gamma'_1(b't' - \beta'_1 x'), \quad x'' = \gamma'_1(x' - \beta'_1 b't'), \quad y'' = y', \quad z'' = z', \tag{17.67}$$

where

$$\beta'_1 = \frac{(\beta_1 - \beta)}{(1-\beta_1\beta)} = \frac{V'_1}{c'}, \qquad \beta_1 = \frac{V_1}{c}, \tag{17.68}$$

$$\gamma'_1 = \gamma_1\gamma\left[1-\beta_1\beta\right] = \frac{1}{\sqrt{1-\beta'^2_1}}, \qquad \gamma_1 = \frac{1}{\sqrt{1-\beta_1^2}}, \tag{17.69}$$

$$t'' = \gamma'_1\left[(1-\beta'_1 q'')b't' - (\beta'_1 - q'')x'\right]\frac{1}{c}. \tag{17.70}$$

This result (17.67), together with other properties such as the existence of an inverse transformation and associativity, demonstrates that *the set of extended 4-dimensional transformations forms precisely the Lorentz group.* This shows explicitly that the Lorentz group can accommodate a weaker postulate for the speed of light. The Lorentz group properties are the core of 4-dimensional symmetry and are *crucial for extended relativity to be consistent with experiments and to be applicable to quantum field theories.*

17j. Decay rate and "lifetime dilation" of unstable particles

Fundamental wave equations such as (17.53) with (17.54) show that 4-dimensional symmetry dictates the evolution parameter to be the lightime w rather than t in a general inertial frame. If we consider the decay rate $\Gamma(1\to 2+3+...+N)$ for a physical process $1\to 2+3+...+N$, we can follow the steps in Appendix C because we have 4-dimensional symmetry in extended relativity. This cannot be done in Edwards' original formulation because of the lack of explicit 4-dimensional symmetry in his transformations. In a general inertial frame with (w,x,y,z), we obtain precisely the same form as (C.43) in Appendix C for the decay rate:

$$\Gamma(1\to 2+3+...+N) = \lim_{w\to\infty}\int \frac{c\left|\langle f\,|S|\,i\rangle\right|^2}{w}\frac{d^3x_2 d^3p_2}{(2\pi\hbar)^3}\cdots\frac{d^3x_N d^3p_N}{(2\pi\hbar)^3}, \tag{17.71}$$

where w=bt in a general inertial frame. This definition of decay rate has the dimension of inverse time because of the presence of the constant c in (17.71). The decay lifetime τ is given by $\tau=1/\Gamma(1\to 2+3+...+N)$. To illustrate the calculation of a decay rate, let us consider a simple example, i.e., the muon decay

$\mu^-(p_1) \rightarrow e^-(p_2) + \nu_\mu(p_3) + \overline{\nu}_e(p_4)$ with the usual V-A coupling. Following the steps in Appendix C, the muon lifetime τ can be calculated and the result is[6]

$$\frac{1}{\tau} = \Gamma(1 \rightarrow 2 + 3 + \ldots + N)$$

$$\propto \frac{1}{p_{01}} \int \frac{d^3p_2}{p_{02}} \frac{d^3p_3}{p_{03}} \frac{d^3p_4}{p_{04}} \, \delta^4(p_1 - p_2 - p_3 - p_4) \sum_{\text{spin}} |M_{sc}|^2 . \tag{17.72}$$

Everything to the right of $1/p_{01}$ in (17.72) is invariant under the extended transformation so that the "decay lifetime" τ is indeed dilated by the usual γ factor:

$$\tau \propto \sqrt{\mathbf{p}_1^2 + m_1^2} = p_{01} \propto \gamma . \tag{17.73}$$

One should keep in mind, however, that it is really the decay length τc which is measured in the laboratory. Such a "decay time" is directly related to the lightime, rather than Reichenbach's time, as one can see from (17.71).

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C.

The Role of the Principle of Relativity in the Physics of Accelerated Frames

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18.

The Principle of Limiting Lorentz and Poincaré Invariance

> Every generalization is a hypothesis.... It is clear that any fact can be generalized in an infinite number of ways, and it is a question of choice. The choice can only be guided by considerations of simplicity.
>
> H. Poincaré, *Science and Hypothesis*

18a. An answer to the young Einstein's question and its implications

Ever since the precise formalisms for mechanics and electrodynamics in inertial frames were discovered and understood, physicists have attempted to discover the exact laws of physics in non-inertial frames.

"*Is it conceivable that the principle of relativity also holds for systems which are accelerated relative to each other?*"

This was the question the young Einstein asked in his 1907 paper, two years after he created special relativity for inertial frames.[1] He said that this question must occur to everyone who has followed the applications of the relativity principle. He considered two reference frames F_1 and F_2. The frame F_1 is accelerated in the +x direction with a constant acceleration g, and the frame F_2 is at rest in a homogeneous gravitational field which gives all objects an acceleration of -g in the x-direction. *Einstein extended the principle of relativity to the case of the "uniformly accelerated frame" by assuming the complete physical equivalence of a homogenenous gravitational field and the corresponding constant acceleration of a reference frame.*[1]

However, such an idea of extension does not seem to enable physicists to derive an exact generalization of the Lorentz transformation for frames with uniform accelerations or rotations. One might think that perhaps this is partially due to the fact that the essential 4-dimensional symmetry of the

Lorentz transformation was not understood at that time. But even today, although special relativity can be used to analyze situations in which particles are accelerated, the exact operational meanings of constant linear accelerations and uniform rotations of non-inertial frames are still not completely clear.[2]

There is a difficulty in the conventional notion of constant acceleration. A constant acceleration g implies a velocity $v_I = gt_I$ as measured in an inertial frame, in which the time t_I is not limited in any way. Therefore, this velocity v_I will eventually exceed the speed of light after a sufficiently long time t_I, however small g may be. This violates special relativity and experimental results of particle kinematics. Therefore, the exact operational meaning of "constant linear acceleration" (CLA) cannot be satisfactorily based on the conventional concept of acceleration dv_I/dt_I = constant *in an inertial frame.* Nevertheless, by considering small gt_I, Einstein in 1907 was able to obtain two results related to time in accelerated frames:

(A)
$$\tau(x) = \tau(0)\left(1 + \frac{gx}{c^2}\right). \tag{18.1}$$

(B) The Maxwell equations in a uniformly accelerated frame F_1 have the same form as in an inertial frame, but with the velocity of light c replaced by

$$c\left(1 + \frac{gx}{c^2}\right). \tag{18.2}$$

Based on the equivalence principle, result (A) is interpreted to be equivalent to the metric tensor component $g_{00} \sim (1+2gx/c^2)$ for the spacetime associated with an accelerated frame, and is consistent with the gravitational red shift. On the other hand, result (B) was not completely satisfactory.

The *4-dimensional* spacetime symmetry of relativity was introduced by Poincaré and Minkowski,[3] and is one of the most thoroughly tested symmetry principles of the 20th century. It is the mathematical manifestation of the first postulate of special relativity, i.e., the symmetry or invariance of physical laws. The Lorentz and Poincaré invariance (or the 4-dimensional symmetry of the

Lorentz and Poincaré groups) is fundamental and is extremely powerful in helping us to understand physics.[4]

Although there is a definite non-equivalence between an inertial and a non-inertial frame, we shall demonstrate in this chapter that the question raised by Einstein in 1907, namely whether "the principle of relativity also holds for systems which are accelerated relative to each other" can be answered based on the following two reasons:

(A) *The properties and characteristics of any accelerated frame $F(w,x,y,z)$ must smoothly reduce to those of an inertial frame $F_I(w_I,x_I,y_I,z_I)$ in the limit of zero acceleration.*

(B) *The principle of relativity states that the laws of physics are invariant under a coordinate transformation between inertial frames. Logically, this is equivalent to the statement that the laws of physics in inertial frames must display the four-dimensional symmetry of the Lorentz and the Poincaré groups.*

Thus, it is natural and worthwhile to investigate spacetime transformations for accelerated frames on the basis of the principle of "limiting Lorentz and Poincaré invariance" or "limiting 4-dimensional symmetry". *This principle states that the laws of physics in non-inertial frames must display the 4-dimensional symmetry of the Lorentz and Poincaré groups in the limit of zero acceleration.* As we shall later demonstrate in section 19d, the only relativistically consistent definition for "constant-linear-acceleration (CLA)" is one in which a particle undergoing such an acceleration experiences a constant change of its relativistic momentum per unit time. Equivalently, if a particle moves with a constant linear acceleration, then the change per unit length of its energy is constant as measured by observers in any inertial frame. Thus, accelerations of reference frames can be investigated on the basis of a purely kinematical approach, independent of the gravitational field. Physical results can be obtained without using gravity as a crutch, in contrast to Møller's approach to deriving accelerated transformations of spacetime based on general relativity.[5]

The "4-dimensional" spacetime (w,x,y,z) of non-inertial frames is, of course, no longer the same as the relative spacetime of inertial frames in taiji relativity. This is obvious because there is no relativity or reciprocity between an inertial frame and a non-inertial frame. In other words, such a spacetime (w,x,y,z) cannot have the 4-dimensional symmetry of the Lorentz and the Poincaré groups. It is a more general 4-dimensional spacetime which includes the 4-dimensional spacetime of inertial frames as a special case when the acceleration approaches zero. To avoid confusion, we will call the more general spacetime *"taiji spacetime."* The evolution variable w of non-inertial frames will remain the *"taiji time."* *Thus, taiji spacetime contains both non-relative spacetime and the relative spacetime of taiji relativity.* As usual, we use *non-relativistic* to describe properties or quantities in Newtonian physics; but we shall use *non-relative* to denote properties or quantities associated with non-inertial frames. We will use the subscript I to denote quantities that are measured by observers in an inertial frame. Finally, taiji spacetime for non-inertial frames is not a general covariant theory (such as general relativity) because only a particular type of coordinates can be used in an accelerated transformation, just as only Cartesian coordinates can be used in the Lorentz transformation.

Based on this limiting 4-dimensional symmetry principle, the answer to Einstein's question is *affirmative* in the sense that we can obtain a "minimal generalization" of Lorentz transformations for non-inertial frames with a constant-linear-acceleration (CLA). "Minimal generalization" means that the resultant equations involve a minimal departure from the simple case with zero acceleration. (See (18.8) below.)

The 4-dimensional symmetry of the Lorentz and the Poincaré groups is the essence of the Lorentz and Poincaré invariance or the Poincaré-Einstein principle of relativity. It must be stressed that *"limiting Lorentz and Poincaré invariance" is simply Lorentz and Poincaré invariance applied to non-inertial frames in the limit of zero acceleration.* We note that it is only the first postulate of relativity theory that is applied to non-inertial frames. The reason is because the speed of light (measured in any units) cannot be a universal constant in non-inertial frames.

It is interesting and gratifying that the principle of limiting Lorentz and Poincaré invariance has further physical implications. As we shall see later, *it can also lead to transformations of spacetime for frames with general-linear-accelerations and rotations, and reveal which constants are fundamental in both inertial and non-inertial frames.*

18b. Generalizing Lorentz transformations from inertial frames to accelerated frames

Since 1909, accelerated motion and transformations for accelerated motion have been discussed by many.[5-8] The results not been satisfactory because they lacked the limiting Lorentz and Poincaré invariance. In section 18d and chapter 19 below, we show that the idea of limiting Lorentz and Poincaré invariance leads to a coordinate transformation between an inertial frame and a frame with a constant acceleration with two parameters, the constant acceleration and the initial velocity, that reduces to the coordinate transformations of taiji or special relativity theory in the limit of zero acceleration.

Let us first briefly consider the history of attempts to generalize the Lorentz transformations to accelerated frames. Soon after the introduction of special relativity, there were a number of attempts to define a transformation between the evolution variable in an inertial frame and the evolution variable in a frame with a constant linear acceleration. In 1907, through an ingenious trick of clock synchronization with the help of three reference frames, Einstein was able to obtain an important result (18.1) for clocks at different positions in a CLA frame F with small accelerations in the x direction:[1]

$$t_x = t_o\left(1 + \frac{gx}{c^2}\right), \qquad F=F(ct,x,y,z). \tag{18.3}$$

However, this approximate result is inadequate to reveal the relationship between time in an inertial frame and that in a CLA frame.

In 1909, Born solved the relativistic equation of motion, $dp_I/dt_I = F$, using a constant force F=mg in an inertial frame. Using the relativistic momentum for p_I, he obtained[6]

$$\left(x_I + \frac{c^2}{g}\right)^2 - \left(ct_I\right)^2 = \left(\frac{c^2}{g}\right)^2, \tag{18.4}$$

which is a hyperbola in the $x_I t_I$ plane. The above results still do not lead to an exact coordinate transformation between an inertial frame and a CLA frame.

More ambitious attempts to develop full coordinate transformations between inertial and non-inertia frames came later. In 1943, Møller[5] obtained a spacetime transformation between an inertial frame $F_I(ct_I,x_I,y_I,z_I)$ and a CLA frame $F(ct,x,y,z)$ moving with a constant acceleration "a" in the x_I (or x) direction:

$$ct_I = \left(x + \frac{c^2}{a}\right)\sinh\frac{at}{c}, \quad x_I = \left(x + \frac{c^2}{a}\right)\cosh\frac{at}{c} - \frac{c^2}{a}, \; y_I = y, ... \tag{18.5}$$

based on Einstein's vacuum equation $R_{\mu\nu}=0$ and a postulated time-independent metric tensor of the form, $g_{\mu\nu}=(g_{oo}(x),-1,-1,-1)$. However, his transformation involved only one parameter, i.e., the constant acceleration, and hence it is of limited applicability because it does not reduce to the Lorentz transformation with a non-zero constant velocity in the limit of zero acceleration.

In 1972, Wu and Lee[7] employed a kinematical approach to derive a uniformly accelerated transformation in the x-direction by assuming a time-independent metric tensor $g_{\mu\nu}(x) = (g_{oo}(x),-1,-1,-1)$ and a local Lorentz contraction of length. Their results turned out identical to those of Møller.

Later, Hsu and Kleff[10] used Wu-Lee's kinematical approach to obtain a generalized transformation of time from an inertial frame to a CLA frame $F(ct,x,y,z)$ with an arbitrary initial velocity v_0 and a constant acceleration "a" in the x direction. We define w=ct in the CLA frame to facilitate comparison with Møller's and Wu-Lee's work. This may be called the generalized Møller-Wu-Lee (MWL) transformation. One has the following transformation of the evolution variable between an inertial frame F_I and a CLA frame F,

$$ct_I = \left(x + \frac{c^2}{a\gamma_o}\right)\sinh\left(\frac{a\gamma_o t}{c} + \tanh^{-1}\beta_o\right) - \frac{\beta_o c^2}{a}, \tag{18.6}$$

$$x_I = \left(x + \frac{c^2}{\gamma_o a}\right)\cosh\left(\frac{a\gamma_o t}{c} + \tanh^{-1}\beta_o\right) - \frac{c^2}{a}, \quad y_I=y, \quad z_I=z \; ;$$

$$\gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}, \qquad \beta_o = \frac{v_o}{c}, \qquad \beta = \frac{v}{c} = \tanh\left(\frac{a\gamma_o t}{c} + \tanh^{-1}\beta_o\right). \tag{18.7}$$

One can verify that eq. (18.6) reduces to (18.5) in the limit of zero initial velocity, $\beta_0 \to 0$. Moreover, (18.6) reduces to the usual Lorentz transformation, $ct_I = \gamma_0[ct+\beta_0 x]$, in the limit of zero acceleration, $a \to 0$. The generalized MWL transformations (18.6) will be derived and discussed in the next chapter.

However, based on the limiting 4-dimensional symmetry (see chapter 19), one can obtain yet another transformation between an inertial frame $F_I(w_I,x_I,y_I,z_I)$ and a CLA frame $F(w,x,y,z)$:[11]

$$\begin{aligned} w_I &= \gamma\beta\left(x + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}, \\ x_I &= \gamma\left(x + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}, \qquad y_I = y, \qquad z_I = z; \end{aligned} \tag{18.8}$$

where one may write, if one wishes, $w_I = ct_I$, and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}, \qquad \beta = \alpha_o w + \beta_o .$$

We call this the Wu transformations for CLA frames and it will also be discussed in the next chapter. Note that β in (18.8) has the usual form of a linear function of the evolution parameter w, $\beta = \alpha_o w+\beta_o$, in contrast to the more complicated function of $\beta = v/c = \tanh(gt/c + \tanh^{-1}\beta_o)$ in (18.5) and (18.6). In this sense, (18.8) is formally a minimal generalization of the classical transformations for accelerated frames.

18c. Physical time and 'spacetime clocks' in linearly accelerated frames

The limiting 4-dimensional symmetry naturally dictates the smooth connection between the physical time of inertial frames and that of accelerated frames in the limit of zero acceleration. Since lifetime dilations of particles with constant velocities can be described by the physical time in inertial frames, the corresponding "physical time" in accelerated frames must be able to describe, say, the accelerated "lifetime dilation" or "decay-length dilation" of an accelerated unstable particle in flight. In this sense, the "time" or "evolution variable" in both inertial and non-inertial frames should have equal *physical significance.*

Thus, it is important first to define properly what the "physical time" is in accelerated frames and how one can realize it operationally through the setup of a clock system. This is a very difficult problem in a general non-inertial frame, especially in rotating frames, because the speed of light is no longer constant so that the usual synchronization procedure based on light signals cannot be applied.

Since there is no constant speed of light c in an accelerated frame F from the operational viewpoint, one can only denote the 4-coordinate of an event by (w,x,y,z) in general, according to the principle of limiting 4-dimensional symmetry. Again, w has the dimension of length and plays the role of the evolution variable. We avoid introducing the constant c = 299792458 cm/s into the formalism of physics in non-inertial frames. Although it is not physically or logically wrong to do so, defining w=ct in non-inertial frames is unnecessary.[9]

As discussed in chapters 2 (sec. 2c.) and 7 (sec. 7d.), the spacetime of an inertial frame can be pictured as a rigid grid of meter sticks and identical clocks that have been synchronized assuming the universality of the speed of light. For simplicity, let us use the term "*spacetime clock*" to denote a clock (on the grid) shows both the time w_I and the position $\mathbf{r}_I$ of the clock, i.e., (w_I,x_I,y_I,z_I). One can always use the spacetime clock an event to record the spacetime coordinates of that event. Since the spacetime transformation between an inertial and an accelerated frame is non-linear, one can no longer picture the spacetime coordinates of a CLA frame as a uniform grid with spacetime clocks. To be more specific, let us consider the inverse of the Wu transformation (18.8):

$$w = \frac{w_I + \beta_o/\alpha_o\gamma_o}{\alpha_o x_I + 1/\gamma_o} - \frac{\beta_o}{\alpha_o}, \qquad w_I = ct_I,$$

$$x = \sqrt{(x_I + 1/\gamma_o\alpha_o)^2 - (w_I + \beta_o/\alpha_o\gamma_o)^2} - \frac{1}{\alpha_o\gamma_o^2}, \qquad y = y_I, \qquad z = z_I. \tag{18.9}$$

With this transformation in hand, the spacetime clocks in the CLA frame can be synchronized without any further information. This is possible because any "clock" (in the sense of a device which shows the time and position) can be adjusted to run at an arbitrary rate and show an arbitrary time and position.[9] For example, our F "clocks" could be some kind of computerized devices which have the capability to measure their position $\mathbf{r}_I$ in the F_I frame, to obtain $w_I = ct_I$ from the nearest F_I clock, and then to compute and display w and $\mathbf{r}$=(x,y,z) using (18.9) [with known velocity β_o and acceleration α_o] on some readout. As a result, if the clocks in the CLA frame are uniformly spaced on a regular grid, the differences in their time readings will not be uniform. However, this grid of computerized spacetime clocks will automatically become the more familiar Einstein clocks of inertial frames, provided that $w \equiv ct$ and $w' \equiv ct'$, when the acceleration α_o in (18.9) approaches zero. This indicates that *the coordinates (w,x,y,z) in (18.8) for a CLA frame play the same role and have a similar physical meaning to the spacetime coordinates in the transformations for inertial frames.*

In the following discussions, we shall use

$$(w,x,y,z) \qquad \text{And} \qquad (w_I,x_I,y_I,z_I) \tag{18.10}$$

as coordinates of non-inertial and inertial frames respectively. For simplicity, w and w_I are called the "time."

18d. Møller's gravitational approach to accelerated transformations

It appears that physical phenomena in a constant-linear-acceleration (CLA) frame have not been thoroughly investigated. Although there have been lifetime dilation experiments for particles in uniform circular motion, these results cannot be applied to the case of CLA motion in which a particle's speed changes. Also, we still do not have a satisfactory transformation between an inertial frame and a CLA frame which can be smoothly and naturally connected to the Lorentz transformation when the accelerated frame becomes an inertial frame. Møller used Einstein's vacuum equations $R_{ik} = 0$ and postulated a time-independent g_{00} and $g_{11} = g_{22} = g_{33} = -1$, $ds^2 = g_{00}(x)dw^2 - dr^2$, to obtain $g_{00}(x) = (1+gx/c^2)^2$ and a transformation between an accelerated frame and an inertial frame.[6] But his accelerated transformation cannot be smoothly connected to the Lorentz transformation in the limit of zero acceleration.

In this section, Einstein's equation $R_{ik} = 0$ is used as a guiding principle to obtain a transformation between an inertial frame and a CLA frame. Although Einstein's covariant equation holds for any coordinates, the Lorentz transformation prefers the Cartesian coordinates. Thus, the natural assumption of smooth connection between a CLA frame and an inertial frame dictates that the CLA transformation be expressed in quasi-Cartesian coordinates which become Cartesian in the limit of zero acceleration. The reason for using $R_{ik} = 0$ is suggested by a heuristic view that the 'inertial force' of accelerated frames and the 'gravitational force' may be considered as having been "unified" by Einstein's equation. Nevertheless, these two forces satisfy different "boundary conditions:" Namely, in contrast to the gravitational force, the inertial force does not vanish at spatial infinity, and the transformations for accelerated frames should reduce to the Lorentz transformation when inertial forces vanish.

Let us first consider an inertial frame $F_I(w_I, x_I, y_I, z_I)$, and a CLA frame $F(w,x,y,z)$ moving with a constant acceleration along the x-axis. Based on the preceding discussions, it is natural to assume that ds^2 takes the form

$$\begin{aligned} ds^2 &= c^2 dt_I^2 - dx_I^2 - dy_I^2 - dz_I^2 \\ &= g_{00}(x,w)dw^2 + g_{11}(x)dx^2 + g_{22}(x)dy^2 + g_{33}(x)dz^2, \end{aligned} \tag{18.11}$$

where g_{00} is assumed, in general, to be a function of w and x. The time t_I is shown by the Einstein clocks in the inertial frame F_I. Although $R_{ik} = 0$ holds for arbitrary coordinates, we postulate the metric (18.11) so that ds^2 and the resultant transformations are consistent with both Einstein's vacuum equation

$R_{ik}=0$ and the new "boundary condition" of limiting four-dimensional symmetry when the acceleration vanishes. Since the CLA frame F moves along the x-axis, we look for axially symmetric solutions with

$$g_{22} = g_{33} = -Y^2(x) \tag{18.12}$$

and where all metric tensors components are functions of x, except that g_{00} may be a function of x and w.

These physical properties of $g_{00}(x,w)$ are crucial to the new CLA transformation; and will be determined later. Now we can calculate Christoffel symbols $G^i{}_{jk} = g^{im}(\partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk})$ and the Ricci tensor $R_{ik} = \partial_m G^m{}_{ik} + \partial_k G^m{}_{im} + G^n{}_{ik}G^m{}_{nm} - G^m{}_{in}G^n{}_{km}$, where $\partial_k = \partial/\partial x^k$, k=0,1,2,3. The equations $R_{ii} = 0$, i=0,1,2, lead to

$$\partial_x^2 W - \partial_x W\left(\frac{\partial_x X}{X} - 2\frac{\partial_x Y}{Y}\right) = 0, \qquad \partial_x = \frac{\partial}{\partial x}, \tag{18.13}$$

$$\frac{\partial_x^2 W}{W} + 2\frac{\partial_x^2 Y}{Y} - \frac{\partial_x X}{X}\left(2\frac{\partial_x Y}{Y} + \frac{\partial_x W}{W}\right) = 0, \tag{18.14}$$

$$\frac{\partial_x^2 Y}{Y} - \frac{\partial_x Y \partial_x X}{YX} + \left(\frac{\partial_x Y}{Y}\right)^2 + \frac{\partial_x Y \partial_x W}{YW} = 0, \tag{18.15}$$

respectively, where Y is given by (18.12) and

$$W^2 = g_{00}(x, w), \qquad X^2 = -g_{11}(x); \qquad W = W_1(x)W_2(w). \tag{18.16}$$

Here, $R_{33} = 0$ gives the same equation as (18.15) and other components of R_{ik} vanish identically.

If $\partial^2{}_x Y = 0$, equations (18.13)-(18.15) lead to an exact solution

$$W_1 = \frac{f_3}{\sqrt{f_1 x + f_0}}, \qquad X = f_2\sqrt{f_1 x + f_0}, \qquad Y = f_1 x + f_0\ , \tag{18.17}$$

where f_1, f_2 and f_3 are constants. The quantity $W_2(w)$ in (18.16) is arbitrary because it cannot be determined by Einstein's equation. Physically, one expects that the metric tensor g_{22} should satisfy $-g_{22} = Y^2 = 1$ rather than $Y^2 = (f_1 x + f_0)^2$, since there is no motion along the y-axis. Furthermore, the accelerated transformation based on this solution cannot be smoothly connected to the Lorentz transformation (i.e., it does not satisfy the "limiting four-dimensional symmetry" or integrability conditions in (18.23) below.) Therefore, the solution (18.17) with $f_1 \neq 0$ is not physically meaningful. The case $f_1 = 0$ in (18.17) is trivial and uninteresting because it is equivalent to that of zero acceleration.

Let us concentrate on the nontrivial case $\partial_x Y = 0$ and $\partial_x W = 0$. We have the solution

$$Y = 1, \tag{18.18}$$

which satisfies the boundary conditions $g_{22}(0) = g_{33}(0) = -1$ at the origin. From (18.13) and (18.18), we deduce a general relation between $W_1(x)$ and $X(x)$:

$$\frac{dW_1(x)}{dx} = fX(x), \tag{18.19}$$

where f is a constant of integration. The function $W_1(x)$ can be determined if $X(x)$ is given or postulated. However, the time-dependent part of g_{00}, i.e., $W_2(w)$, still cannot be determined by Einstein's equation, just as in the previous case because $\partial_x Y = 0$. Thus we have seen that Einstein's covariant equation by itself does not lead to a specific form for $X(x)$, $W_1(x)$ and $W_2(w)$. One must have specific functional forms for $X(x)$ and W in order to have finite transformations between a linearly accelerated frame and an inertial frame. Møller postulated $W=W_1(x)$ and $X(x)=1$ in (18.16) (i.e., time-independent g_{00} and $g_{11} = g_{22} = g_{33} = -1$) to obtain $g_{00}(x) = (1+gx/c^2)^2$ and the transformation between an accelerated frame and an inertial frame given in (18.5).[6]

18e. Accelerated transformations with the limiting Lorentz and Poincaré invariance

Let us now consider the implications of the limiting Lorentz and Poincaré invariance for the transformation equations between a CLA frame F and an inertial frame F_I. Suppose that a CLA frame F moves along parallel x and x_I axes and that the origins of F and F_I coincide at the taiji-time $w = w_I = 0$. We postulate that the invariant infinitesimal intervals for a CLA frame F and an inertial frame F_I are given by

$$ds^2 = W^2(w,x)dw^2 - dx^2 - dy^2 - dz^2 = dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2 . \qquad (18.20)$$

The principle of relativity dictates that W(w,x) must approach 1 in the limit of zero acceleration. This is the requirement of the limiting Lorentz and Poincaré invariance (4-dimensional symmetry). The presence of the function $W(w,x) \neq 1$ implies the physical non-equivalence of F and F_I. The local relation between $F(x^\mu)$ and $F_I(x_I^\mu)$ may be written in the form,

$$\begin{aligned} dw_I &= \gamma(Wdw + \beta dx), \\ dx_I &= \gamma(dx + \beta W dw), \qquad dy_I = dy, \qquad dz_I = dz , \end{aligned} \qquad (18.21)$$

where γ and β are functions of the time w in general, and

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} . \qquad (18.22)$$

The local transformation (18.21) preserves the invariance of ds^2 in (18.20). In order for F(w,x,y,z) to reduce smoothly to an inertial frame in the limit of zero acceleration, there must exist a *global* transformation related to (18.21). In other words, the limiting 4-dimensional symmetry also dictates that the two unknown function W(w,x) and $\beta(w)$ must satisfy the following two integrability conditions for the differential relations in (18.21):

$$\frac{\partial(\gamma W)}{\partial x} = \frac{\partial(\gamma\beta)}{\partial w}, \qquad \text{and} \qquad \frac{\partial\gamma}{\partial w} = \frac{\partial(\gamma\beta W)}{\partial x}. \tag{18.23}$$

By separation of variables, $W(w,x)=W_w W_x$, the two equations in (18.23) lead to the same relation

$$\frac{dW_x}{dx} = \frac{\gamma^2}{W_w}\frac{d\beta}{dw} = k_1 , \tag{18.24}$$

where k_1 is a constant. Thus we have two solutions

$$W_x = k_1 x + k_2, \qquad \text{and} \tag{18.25}$$

$$W_w = \frac{d\beta/dw}{k_1(1-\beta^2)}. \tag{18.26}$$

The implication of the principle of limiting 4-dimensional symmetry is as follows: the solution of the differential equation (18.26) depends on the physical properties of either β or W_w. If the "velocity function" β is known, then W_w can be determined, and vice versa. The principle of limiting 4-dimensional symmetry by itself cannot uniquely determine the function W(w,x). In general, there are infinitely many solutions for (18.24). We observe that the present case resembles the situation in which gauge symmetry cannot determine uniquely the electromagnetic action,[12] and one must further postulate a minimal electromagnetic coupling. In the present case, we postulate a minimal generalization of the Lorentz transformation.

The minimal generalization of the Lorentz transformation is to give up the untenable relation $v_I = gt_I + v_{Io}$ in an inertial frame F_I (as discussed in section 18a,) and to retain the velocity function β as the usual linear function of time w in the CLA frame,

$$\beta = \alpha_o w + \beta_o . \tag{18.27}$$

It follows from (18.25) and (18.26) that we have a time-dependent metric tensor $g_{oo}(w,x)=W^2$, where

$$W = W_w W_x = \gamma^2(\gamma_o^{-2} + \alpha_o x) = \frac{(\gamma_o^{-2} + \alpha_o x)}{(1-\beta^2)}, \tag{18.28}$$

and in which the constants k_2 and k_1 in the product $W_w W_x$ are determined by the limiting condition $W(w,x) \to 1$ as $\alpha_o \to 0$.

However, there is another relatively simple generalization of the Lorentz transformation that can be obtained by postulating a time-independent metric tensor g_{oo} , i.e.,

$$g_{00} = W_x \qquad \text{or} \qquad W_w = 1. \tag{18.29}$$

It then follows from (18.25) and (18.26) that,

$$W_x = k_1 x + k_2, \tag{18.30}$$

$$\beta = \tanh(k_1 w + k_3), \tag{18.31}$$

with parameters k_1, k_2, and k_3 determined as follows. The constant k_3 in (18.31) can be determined by the initial condition that $\beta = \beta_o$ when time $w = 0$, i.e., $k_3 = \tanh^{-1}\beta_o$. Furthermore, since β is equal to β_o when the acceleration approaches zero, k_1 in (18.31) should be proportional to the constant acceleration, although k_1 may also depend on β_o, as shown in eq. (18.35) below. Finally, since W_x must be 1 if the acceleration (and thus k_1) vanishes, we see that $k_2=1$. Relations (18.30) and (18.31) are now determined as follows:

$$W_x = 1 + k_1 x, \qquad \text{and} \qquad \beta = \tanh(k_1 w^* + \tanh^{-1}\beta_o), \tag{18.32}$$

where we have denoted the time as w^* in order to distinguish it from w in (18.27). In this case, the simple relationship between the velocity and time variables in (18.27) has been replaced by a more complicated relationship.

Thus, we do not consider (18.32) to be a minimal generalization. Nevertheless, we will also discuss the solution (18.32) because the assumption of a time-independent metric tensor was made by Møller and also by Wu and Lee,[6,7]. Equation (18.32) leads to a generalization of the transformation obtained by them to the case of non-zero initial velocity, although it is based on entirely different considerations.[10]

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19.

Extended Lorentz Transformations for Frames with Constant-Linear-Accelerations

19a. Generalized Møller-Wu-Lee transformations

We are now in a position to generalize the Lorentz transformations to reference frames with constant linear accelerations. Substituting (18.32) in (18.21), one can carry out the integration and obtain the relations

$$w_I = \left(x + \frac{1}{k_1}\right)\sinh(k_1 w^* + \tanh^{-1}\beta_o) + k_6,$$

$$x_I = \left(x + \frac{1}{k_1}\right)\cosh(k_1 w^* + \tanh^{-1}\beta_o) + k_7, \tag{19.1}$$

$$y_I = y, \qquad z_I = z.$$

Using the usual initial condition that $x_I = x = 0$, when $w_I = w^* = 0$, the constants of integration k_6 and k_7 in (19.1) can be determined in terms of k_1

$$k_6 = \frac{-\gamma_o\beta_o}{k_1}, \qquad k_7 = \frac{-\gamma_o}{k_1}. \tag{19.2}$$

In order to determine the role of a constant acceleration in (19.1) and to compare it with the transformation obtained by Møller and by Wu and Lee,[1,2] we follow the approach of Wu and Lee and impose the boundary condition that the velocity β is related to the constant acceleration α^* by $\beta = \alpha^* w_I + \beta_o$ at $x_I = 0$.[2] From (18.32), (19.1) and (19.2), k_1 is found to be

$$k_1 = \gamma_o\alpha^*, \tag{19.3}$$

because

$$\beta = \frac{\sinh(k_1 w^* + \tanh^{-1}\beta_o)}{\cosh(k_1 w^* + \tanh^{-1}\beta_o)} = \frac{w_I + \gamma_o\beta_o / k_1}{x_I + \gamma_o / k_1} = \alpha^* w_I + \beta_o, \qquad \text{for} \qquad x_I = 0.$$

Thus, the generalized Møller-Wu-Lee (MWL) transformation for the frames F_I and F with non-zero initial velocity β_o is given by

$$w_I = \left(x + \frac{1}{\gamma_o\alpha^*}\right)\sinh(\gamma_o\alpha^* w^* + \tanh^{-1}\beta_o) - \frac{\beta_o}{\alpha^*},$$

$$x_I = \left(x + \frac{1}{\gamma_o\alpha^*}\right)\cosh(\gamma_o\alpha^* w^* + \tanh^{-1}\beta_o) - \frac{1}{\alpha^*}, \tag{19.4}$$

$$y_I = y, \qquad z_I = z.$$

In the limit of zero acceleration $\alpha^* \to 0$, transformation (19.4) indeed reduces to a form with the four-dimensional symmetry of the Lorentz group

$$w_I = \gamma_o(w^* + \beta_o x), \quad x_I = \gamma_o(x + \beta_o w^*), \quad y_I = y, \quad z_I = z, \tag{19.5}$$

where

$$\sinh(\tanh^{-1}\beta_o) = \gamma_o\beta_o, \qquad \text{and} \qquad \cosh(\tanh^{-1}\beta_o) = \gamma_o.$$

Furthermore, when β_o approaches zero the accelerated transformations (19.4) reduce to the Møller-Wu-Lee transformations (18.5) with $a/c^2 = \alpha^*$, $w = w^*$ and $ct_I = w_I$. The result (19.4) is the same as (18.6) with $a/c^2 = \alpha^*$, $w = w^*$ and $ct_I = w_I$.[3]

The inverse transformations of (19.4) are found to be

$$w^* = \frac{1}{\gamma_o\alpha^*}\left\{\tanh^{-1}\left[\frac{w_I + \beta_o/\alpha^*}{x_I + 1/\alpha^*}\right] - \tanh^{-1}\beta_o\right\},$$

$$x = \sqrt{(x_I + 1/\alpha^*)^2 - (w_I + \beta_o/\alpha^*)^2} - \frac{1}{\gamma_o\alpha^*}, \tag{19.6}$$

$$y_I = y, \qquad z_I = z.$$

Differentiation of the generalized MWL transformation (19.4) gives

$$dw_I = \gamma(W_x dw^* + \beta dx), \quad dx_I = \gamma(dx + \beta W_x dw^*), \quad dy_I = dy, \quad dz_I = dz. \qquad (19.7)$$

$$W_x = 1 + \gamma_o \alpha^* x, \qquad \beta = \tanh(\gamma_o \alpha^* w^* + \tanh^{-1}\beta_o), \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

It follows from (19.7) that

$$ds^2 = dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2 = W_x^2 dw^{*2} - dx^2 - dy^2 - dz^2 = g^*_{\mu\nu} dx^\mu dx^\nu, \quad (19.8)$$

$$g^*_{\mu\nu} = (W_x{}^2, -1, -1, -1), \qquad\qquad W_x = 1 + \gamma_o \alpha^* x,$$

where $\mu, \nu = 0,1,2,3$.

As is the case in which Cartesian coordinates are the preferred coordinate system for the Lorentz transformations, the coordinates $x^\mu = (w^*, x, y, z)$ with metric tensor $g^*_{\mu\nu}$ given by (19.8) are the preferred coordinates for the generalized MWL transformation. The coordinates x^μ cannot be arbitrary because the principle of limiting 4-dimensional symmetry dictates that the transformation (19.6) or (19.4) must reduce smoothly to the Lorentz transformation in the limit of zero acceleration. In a gedanken sense, both the position x and the time w can be operationalized using a grid of computers that determine their own position x and the time w= w* they should display using equation (19.6) and the values of x_I and w_I obtained from the nearest F_I clock.

The propagation of a light signal in the CLA frame is described by ds=0, i.e.,

$$\left|\frac{dr}{dw^*}\right| = W_x = 1 + \alpha^* \gamma_o x \qquad (19.9)$$

and is consistent with Einstein's approximate result (18.2) when one sets w*=ct, β_o=0 and $\alpha^*=g/c^2$.

For a particle at rest in F_I, we obtain the expression for its proper lifetime (or decay-length) $w_{Ip} = \int dw_I = \int ds$ using (19.7) and (19.8) with $dx_I = 0$ and $y_I = z_I = 0$, i.e.,

$$w_{Ip} = \int_0^{w^*} dw^* \sqrt{W_x^2 - \left(\frac{dx}{dw^*}\right)^2} = \alpha^* \gamma_o \int_0^{w^*} dw^* \frac{1}{\cosh^2(\gamma_o \alpha^* w^* + \tanh^{-1}\beta_o)}$$

$$= \left(\frac{1}{\alpha^*} + x_I\right)\left[\tanh(\gamma_o \alpha^* w^* + \tanh^{-1}\beta_o) - \beta_o\right], \qquad (19.10)$$

where x_I is the fixed position of the particle. On the other hand, if a particle is at rest at x in F, its proper lifetime (or decay-length) is $w^*_p = \int dw^* = (1/W_x)\int ds = (1/W_x)\int dw_I[1 - (dx_I/dw_I)^2]^{1/2}$, where x is fixed, so that we have

$$w^*_p = \frac{1}{(1 + \alpha^* \gamma_o x)} \int_0^{w^*} dw_I \sqrt{1 - \beta^2}$$

$$= \frac{1}{\gamma_o \alpha^*}\left[\tanh^{-1} Z(x, w_I) - \tanh^{-1} Z(x, 0)\right], \qquad (19.11)$$

where

$$Z(x, w_I) = \frac{(w_I + \beta_o/\alpha^*)}{\sqrt{(x + 1/\gamma_o\alpha^*)^2 + (w_I + \beta_o/\alpha^*)^2}},$$

$$\left.\frac{dx_I}{dw_I}\right|_{dx=0} = \beta = \tanh(\gamma_o \alpha^* w^* + \tanh^{-1}\beta_o) = \frac{w_I + \beta_o/\alpha^*}{x_I + 1/\alpha^*}.$$

The usual acceleration, $d^2x_I/dw_I^2 = d\beta/dw_I$ of a fixed point **r** in F, as measured in F_I, is

$$\frac{d^2x_I}{dw_I^2} = \frac{(x + 1/\gamma_o\alpha^*)^2}{[(x + 1/\gamma_o\alpha^*)^2 + (w_I + \beta_o/\alpha^*)^2]^{3/2}}, \qquad (19.12)$$

which is clearly not a constant. However,

$$\frac{d}{dw_I}\frac{\beta}{\sqrt{1-\beta^2}} = \gamma^3\frac{d\beta}{dw_I} = \frac{\alpha^*\gamma_o}{1+x\gamma_o\alpha^*}, \tag{19.13}$$

is a constant because x is fixed. This is consistent with Born's result in (18.4) where dp_I/dt_I is a constant.

19b. Minimal generalization of Lorentz transformations: The Wu transformations

For the minimal generalization of the Lorentz transformations (described in section 18e), we retain the usual relation $\beta=\alpha_o w+\beta_o$. The postulate of limiting 4-dimensional symmetry and the minimal generalization lead to the interval (18.20) with W given by (18.28). Substituting (18.28) into (18.21) and integrating, one obtains the CLA transformation:

$$w_I = \gamma\beta\left(x+\frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o},$$

$$x_I = \gamma\left(x+\frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}, \quad y_I = y, \quad z_I = z; \quad w_I = ct_I, \tag{19.14}$$

where the constants of integration, $\beta_o/\alpha_o\gamma_o$ and $1/\alpha_o\gamma_o$, can be determined by the initial conditions $x_I=x=0$ when $w_I=w=0$. We call (19.14) the Wu transformation.[4] One can verify that (19.14) indeed reduces to the Lorentz transformation with $w^*\to w$ in the limit of zero acceleration.

The inverse Wu transformation derived from (19.14) is

$$w = \frac{(w_I+\beta_o/\alpha_o\gamma_o)}{(\alpha_o x_I+1/\gamma_o)} - \frac{\beta_o}{\alpha_o},$$

$$x = \sqrt{(x_I+1/\gamma_o\alpha_o)^2-(w_I+\beta_o/\alpha_o\gamma_o)^2} - \frac{1}{\alpha_o\gamma_o^2}, \quad y = y_I, \quad z = z_I. \tag{19.15}$$

Differentiation of (19.14) gives

$$dw_I = \gamma(Wdw+\beta dx), \quad dx_I = \gamma(dx+\beta Wdw), \quad dy_I = dy, \quad dz_I = dz; \tag{19.16}$$

$$\beta = \alpha_o w + \beta_o, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad W = \gamma^2(\gamma_o^{-2} + \alpha_o x). \tag{19.17}$$

The invariant infinitesimal interval is thus

$$ds^2 = dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{19.18}$$

$$g_{\mu\nu} = (W^2, -1, -1, -1) = (\gamma^4[\gamma_0^{-2} + \alpha x]^2, -1, -1, -1).$$

The factor W in the metric tensor (19.18) and the differential form of the Wu transformation (19.16) may be called the Wu factor. As with the generalized MWL transformation, the coordinates (w,x,y,z) with the metric tensor $g_{\mu\nu}$ given by (19.18) are the preferred coordinates for the accelerated Wu transformation. The time w in (19.14) and (19.15) differs from that in (19.4) and (19.6), of course. We shall discuss physical implications of this important difference in section 20e.

The propagation of light is described by setting ds=0 in (19.18), so that the speed of light measured in terms of w in a CLA frame is now given by the Wu factor W:

$$\left|\frac{dr}{dw}\right| = W = \gamma^2(\gamma_o^{-2} + \alpha_o x), \tag{19.19}$$

which is consistent with the result (19.9) and Einstein's result (18.2) for small velocities, $\beta_o \sim 0$ and $|\beta^2| << 1$. At large velocities however, they have significant differences. Such differences are related to the differences between the two functions $\beta(w^*)$ in (19.7) and $\beta(w)$ in (19.17).

If a particle is at rest in F_I at the location $(x_I,0,0)$, (19.18) implies that its proper lifetime w_{Ip} is $w_{Ip} = \int dw_I = \int ds$, i.e.,

$$w_{Ip} = \int_0^w dw\sqrt{W^2 - (dx/dw)^2} = \left(\frac{1}{\gamma_o} + \alpha_o x_I\right)w, \qquad dr_I = 0. \tag{19.20}$$

On the other hand, if a particle is at rest in F (r=constant), (19.18) gives the proper lifetime w_p as

$$w_p = \int_0^w dw = \int \frac{ds}{W} = \frac{1}{\gamma_o^{-2} + \alpha_o x} \int_o^{w_I} dw_I \left[1 - \left(\frac{dx_I}{dw_I}\right)^2\right]^{3/2}$$

$$= \frac{1}{\alpha} \left\{ \frac{w_I + \beta_o / \alpha_o \gamma_o}{\sqrt{(x + 1/\gamma_o^2 \alpha_o)^2 + (w_I + \beta_o / \alpha_o \gamma_o)^2}} \right.$$

$$\left. - \frac{\beta_o / \alpha_o \gamma_o}{\sqrt{(x + 1/\gamma_o^2 \alpha_o)^2 + (\beta_o / \alpha_o \gamma_o)^2}} \right\}. \tag{19.21}$$

This can also be written as

$$w_I = \left(x + \frac{1}{\gamma_o^2 \alpha_o}\right) \left[\gamma\beta - \gamma_o \beta_o\right], \qquad \beta = \alpha_o w + \beta_o. \tag{19.22}$$

These relations for the dilation of the lifetimes of unstable particles could be tested experimentally by measurements of the decay-lengths (or "lifetimes") of a particles undergoing a constant linear acceleration.

19c. A comparison of the generalized MWL and Wu transformations

In the preceding discussion, we obtained two simple transformations between an inertial frame and a CLA frame on the basis of the principle of limiting 4-dimensional symmetry. One was the Wu transformation (19.14) which was obtained by assuming a minimal generalization of the Lorentz transformations preserving the usual linear relationship between acceleration and velocity, $\beta = \alpha_o w + \beta_o$, in accelerated frames. The other was the generalized Møller-Wu-Lee (MWL) transformations (19.4), which were obtained by assuming the metric tensor $(g_{oo},-1,-1,-1)$ to be time-independent. This transformation has a more complicated and unusual relationship between the acceleration and velocity, $\beta = \tanh(\gamma_o \alpha^* w^* + \tanh^{-1} \beta_o)$. Both of the evolution variables w and w* in these two transformations can be considered as a generalization of the 'time' in inertial frames. We have seen that the physical 'time' in CLA frames cannot be uniquely determined by the limiting 4-dimensional symmetry principle. Let us summarize the properties of the generalized MWL transformations and the Wu transformations, both of which are based on the limiting 4-dimensional symmetry principle and an additional assumption:

(A) Generalized MWL transformations:

Additional assumption: $g_{00}(x^{\mu}) = g_{00}(x) \quad \Rightarrow$

$$g_{00} = \left[1 + \alpha^* \gamma_o x\right]^2, \qquad \left(\frac{dx_I}{dw_I}\right)_x = \tanh(\alpha^* \gamma_o w^* + \tanh^{-1}\beta_o). \tag{19.23}$$

(B) Wu transformations:

Additional assumption: $\beta = \alpha_o w + \beta_o = \left(\frac{dx_I}{dw_I}\right)_x \quad \Rightarrow$

$$g_{00}(w, x) = \left[\gamma^2(\gamma_o^{-2} + \alpha_o x)\right]^2. \tag{19.24}$$

It is interesting to observe that time w^* in the generalized MWL transformation (19.4) and time w in the Wu transformation (19.14) are related by the following relation:[5]

$$w = \frac{1}{\alpha_o}\left[\tanh(\alpha^* \gamma_o w^* + \tanh^{-1}\beta_o) - \beta_o\right], \qquad \alpha^* = \alpha_o \gamma_o. \tag{19.25}$$

This relation between w and w^* can be seen by comparing $(dx_I/dw_I)_x$ in (19.23) and (19.24). It follows that

$$W_x dw^* = (1 + \alpha^* \gamma_o x)\left(\frac{1}{\alpha^* \gamma_o}\right) d\left[\tanh^{-1}(\alpha_o w + \beta_o)\right] = W(w, x) dw, \tag{19.26}$$

and

$$ds^2 = dw_I^2 - dr_I^2 = W_x^2 dw^{*2} - dr^2 = W(w,x)^2 dw^2 - dr^2. \tag{19.27}$$

Although both times w* in (19.4) and w in (19.14) reduce to the same "taiji time" (measured in the unit meter rather than second) for inertial frames in the limit of zero acceleration, a relevant question is:

"*Which physical time in constant-linear-acceleration frames is correct?*" Since the physical time in inertial frames is directly related to the dilation of the "lifetime" or to the decay-length of an unstable particle decaying in flight, the same must hold also in CLA frames because of the limiting 4-dimensional symmetry principle. Therefore, the question of which physical time is correct for accelerated frames can only be settled by experiments. One way to answer this question is to measure the decay-length dilation to test the predictions of the Wu transformations and the generalized MWL transformations. We will discuss such a test in more detail in chapter 20.

19d. Four-momentum and constant-linear-acceleration of an accelerated particle

Now let us demonstrate that, in the framework limiting 4-dimensional symmetry, a "constant linear acceleration" means a constant change in the kinetic energy p_{I0} of an object per unit length, as measured in an inertial frame. Consider the invariant action S_f for a "free particle" in a CLA frame (which may be associated either with the generalized MWL transformations with time w* or the Wu transformations with time w):

$$S_f = -\int_a^b m ds = \int_{w_a}^{w_b} L_w dw, \tag{19.28}$$

$$ds^2 = W^2 dw^2 - dx^2 - dy^2 - dz^2, \qquad g_{\mu\nu} = (W^2, -1, -1, -1),$$

$$L_w = -m\sqrt{g_{\mu\nu}u^\mu u^\nu} = -m\sqrt{W^2 - (\beta^i)^2}, \qquad u^\mu = \frac{dx^\mu}{dw} = (1, \beta^i).$$

The function W in (19.28) may be either $W_x = (1+\alpha^*\gamma_o x)$ with w replaced by w* or $W(w,x) = \gamma^2(\gamma_o^{-2} + \alpha_o x)$.

As usual, the covariant momentum p_i and the corresponding "energy" p_0 (or the Hamiltonian H) with the dimension of mass are given by

$$p_i = -\frac{\partial L_w}{\partial \beta^i} = \left(\frac{-m\Gamma\beta_x}{W}, \frac{-m\Gamma\beta_y}{W}, \frac{-m\Gamma\beta_z}{W}\right), \quad p_i = g_{ik}p^k = -p^i, \quad i = 1,2,3;$$

$$p_0 = \left(\frac{\partial L_w}{\partial \beta^i}\beta^i - L_w\right) = m\Gamma W = g_{00}p^0, \qquad p^0 = m\frac{dx^0}{ds} = m\frac{dw}{ds}; \tag{19.29}$$

$$\Gamma = \frac{1}{\sqrt{1-\beta^2/W^2}}; \qquad \beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 = -\beta_i\beta^i, \quad \beta^i = \frac{dx^i}{dw} = u^i.$$

Thus, the covariant momentum 4-vector $p_\mu = (p_0, p_1, p_2, p_3) = (p_0, -\mathbf{p})$ transforms like the covariant coordinate $dx_\mu = g_{\mu\nu}dx^\nu$. We have

$$p_{I0} = \gamma(\frac{p_0}{W} - \beta p_1), \quad p_{I1} = \gamma(p_1 - \frac{\beta p_0}{W}), \qquad p_{I2} = p_2, \qquad p_{I3} = p_3;$$
$$p_{I0}^2 - \mathbf{p}_I^2 = (p_0/W)^2 - \mathbf{p}^2 = m^2. \tag{19.30}$$

Now suppose a particle is at rest in the CLA frame ($\mathbf{p} = 0$). From (19.30), $p_0 = mW$ and $p_{I0} = m\gamma$. Thus, when $\mathbf{r}$ is fixed, we obtain

$$\left(\frac{dp_{I0}}{dx_I}\right)_x = m\gamma^3\frac{d\beta}{dx_I}, \tag{19.31}$$

$$= \begin{cases} m\alpha/(\gamma_o^{-2} + \alpha_o x); & \beta = \alpha_o w + \beta_o; \\ \\ m\gamma_o\alpha^*/(1+\alpha^*\gamma_o x); & \beta = \tanh[\alpha^*\gamma_o w^* + \tanh^{-1}\beta_o]. \end{cases}$$

The last relation involving α^* is consistent with the constant acceleration given in (19.13).

The result (19.31) is intimately related to the "limiting four-dimensional symmetry." The concept of linear and uniform acceleration of a particle can be defined in the sense of (19.31), i.e., a constant change of a particle's "energy" p_{I0} per unit length, as measured in an inertial frame F_I. It is gratifying to see that this is precisely the type of acceleration that is realized in high energy laboratories. Therefore, the theories with limiting 4-dimensional symmetry already have partial experimental support.

19e. Experiments on Wu-Doppler effects of waves emitted from accelerated atoms

The new accelerated transformations (19.4) and (19.14) can also be experimentally tested by measuring a Doppler-type shift of wavelength of light emitted from an accelerated source.[6] From eq. (19.30) one obtains the Wu transformation of the covariant wave 4-vector $k_\mu = p_\mu/J$ between an inertial frame F_I and a CLA frame F:

$$k_{I0} = \gamma(k_0 / W - \beta k_1), \quad k_{I1} = \gamma(k_1 - \beta k_0 / W), \quad k_2 = k_3 = 0, \tag{19.32}$$

where $k_{I\mu} = (k_{I0}, -\mathbf{k}_I)$ and $k_\mu = (k_0, -\mathbf{k})$. As discussed earlier in section 10b, Jk_{I0} and Jk_0 are moving masses of the same photon measured by observers in F_I and F respectively. Suppose the radiation source is located at the origin of the F frame and is co-moving with that frame, $\mathbf{r} = 0$. Experimentally, it is difficult for observers in the accelerated frame F to measure k_0(rest) and k_1(rest) for such a radiation source. Thus we express the shifts in terms of quantities measured by observers in the inertial frame (or laboratory) F_I. Using (10.14) as the approximate relation,, i.e. $k_0(\text{rest}) \cong k_{I0}(\text{rest})$, and $W(w,0) = \gamma^2\gamma_o^{-2}$, we obtain a Doppler-type shift of k_{I0} (related to the photon's moving mass or the difference between the atomic mass levels of the radiation source) and wavelength λ_I

$$\begin{aligned} k_{I0} &= k_{I0}(\text{rest}) \, \frac{(1+\beta)}{\gamma\gamma_o^{-2}}, \\ \frac{1}{\lambda_I} &= \frac{1}{\lambda_I(\text{rest})} \, \frac{(1+\beta)}{\gamma\gamma_o^{-2}}, \qquad \beta = \alpha_o W + \beta_o, \end{aligned} \tag{19.33}$$

for waves emitted from a radiation source undergoing a CLA. These new results predicted by the Wu transformation are termed the Wu-Doppler shift.[5]

For an experimental test, it is more convenient to express 'velocities' in (19.33) in terms of distances. Suppose the radiation source (at $\mathbf{r} = 0$) enters an accelerated potential at $x_I = 0$ in F_I with an initial velocity β_o and emits radiation when it has been accelerated to the point $x_I = L$ with a velocity β_I. We have

$$\beta_I = \left(\frac{dx_I}{dw_I}\right)_{x=0} = \beta = \sqrt{1 - \gamma_o^{-2}(1 + L\alpha_o\gamma_o)^{-2}} \,, \tag{19.34}$$

$$\gamma_I = \frac{1}{\sqrt{1-\beta_I^2}} = \gamma_o(1 + \alpha_o L \gamma_o) \,. \tag{19.35}$$

It follows from (19.33)-(19.35) that the wavelength shift is given by

$$\delta\lambda_I = \lambda_I - \lambda_I(\text{rest}) \approx \lambda_I(\text{rest})\left[\alpha_o L - \sqrt{\beta_o^2 + 2\alpha_o L} \; - \frac{1}{2}\beta_o^2\right]. \tag{19.36}$$

Also, according to (19.33), we expect the shift of mass levels of an atom at rest in a CLA frame to be

$$\delta M_I = Jk_{I0} - Jk_{I0}(\text{rest}) \approx Jk_{I0}(\text{rest})\left[\sqrt{\beta_o^2 + 2\alpha_o L} \; + \frac{1}{2}\beta_o^2 - \alpha_o L\right]. \tag{19.37}$$

The predictions (19.36) and (19.37) can be measured in the laboratory frame F_I using a method similar to that in the Ives-Stilwell experiment.[7]

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20.

Physical Properties of Spacetime in Accelerated Frames

20a. A general transformation for a CLA frame with an arbitrary $\beta(w)$

Our discussions so far have been based on the principle of limiting 4-dimensional symmetry which requires that all accelerated transformations reduce to the Lorentz and Poincaré transformations in the limit of zero acceleration and hence, that physical laws in an accelerated frame F can be obtained by transforming the laws of physics from their commonly known forms in inertial frames to an accelerated frame F.[1] This suggests that the spacetime coordinates of accelerated frames are as meaningful as those of inertial frames. In this chapter, we illustrate these properties and the distortions in the spacetime of accelerated frames graphically, including singularities and horizons. Physical implications of various accelerated transformations for the lifetime and decay-length dilations of unstable particles undergoing accelerated motion are calculated in section 20e.[2]

A generalized Lorentz transformation must be consistent with the fact that any accelerated frame reduces to an inertial frame in the limit of zero acceleration. This is a natural, basic and necessary requirement from a physical viewpoint. But such a generalization turns out not to be unique from a theoretical viewpoint and there are infinitely many generalizations. So far, no established theoretical principle leads to a simple and unique generalization. Indeed, the original transformations derived by Wu and Møller are just two of the many possible generalizations.[1] In all of these cases, the spacetime of CLA frames is characterized a metric tensor of the form $(W^2, -1, -1, -1) \equiv P_{\mu\nu}$, which may be called the Poincaré metric tensor. Furthermore, because there finite transformations between inertial and CLA frames exist, the spacetime of these CLA frames is flat, i.e., they have a vanishing Riemann curvature tensor. This property is useful for formulations of field theory, including gauge theory of gravity in non-inertial frames.

Let us consider the transformations between an inertial frame $F_I(w_I, x_I, y_I, z_I)$ and a constant-linear-acceleration (CLA) frame F(w,x,y,z) which moves with a time-dependent velocity $\beta(w)$ along the x-axis. Suppose the metric tensors in the inertial and the CLA frames are:

$$\eta_{\mu\nu} = (1,-1,-1,-1), \tag{20.1}$$

$$P_{\mu\nu} = (W^2,-1,-1,-1), \tag{20.2}$$

where $W = W(w,\mathbf{r})$ is any real-valued function of spacetime in F(w,x,y,z) with the property $W \to 1$ for vanishing acceleration. Thus, the Poincaré metric tensor $P_{\mu\nu}$ reduces to the Minkowski metric tensor $\eta_{\mu\nu}$ in the limit of zero acceleration.

Let us consider a class of transformations for an inertial frame $F_I(w_I, x_I, y_I, z_I)$ and a constant-linear-acceleration (CLA) frame F(w,x,y,z),[2] including constant spacetime translations:

$$\begin{aligned}
w_I &= \gamma\beta\left(x + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o} + w_o, \\
x_I &= \gamma\left(x + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o} + x_o, \qquad y_I = y + y_o, \qquad z_I = z + z_o;
\end{aligned} \tag{20.3}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}, \qquad \beta = \beta(w) = \text{arbitrary function},$$

where $x_o^\mu = (w_o, x_o, y_o, z_o)$ are constants and the arbitrary velocity function $\beta(w)$ will be specified and discussed below. The Poincaré metric tensor for the accelerated frame F(w,x,y,z) is given by (20.2) with

$$W = \gamma^2 \left(x + \frac{1}{\alpha_o \gamma_o^2} \right) \frac{d\beta}{dw}.$$

This general transformation of spacetime is interesting because it includes the Wu transformation (19.14) and the Møller transformation for accelerated frames as special cases. Even though $\beta(w)$ is an arbitrary function of time w in the F frame, it actually has a specific and unique expression when we express it in terms of the spacetime coordinates of the inertial frame F_I

$$\beta(w) = \frac{w_I + \beta_o / \alpha_o \gamma_o - w_o}{x_I + 1/\alpha_o \gamma_o - x_o}. \tag{20.4}$$

The fact that the right-hand-side of (20.4) does not involve any arbitrary functions implies that from the viewpoint of observers in the inertial frame F_I, the motion of the F frame is independent of the function $\beta(w)$. (See also equation (20.17) below.) This may be interpreted as a flexibility or a 'gauge symmetry' of the time w of the accelerated frame F.

When the acceleration in (20.3) approaches zero, i.e., $\beta \to \beta_o$, the linear-acceleration transformations (20.3) must reduce to the Poincaré transformations (given w=ct and $w_I = ct_I$),

$$w_I = \gamma_o(w + \beta_o x) + w_o, \qquad x_I = \gamma_o(x + \beta_o w) + w_o,$$
$$y_I = y + y_o, \qquad z_I = z + z_o, \tag{20.5}$$

where

$$\beta(w) \to \beta_o + \alpha_o w, \text{ for small } \alpha_o. \tag{20.6}$$

As long as the arbitrary function $\beta(w)$ satisfies this limiting property as $\alpha_o \to 0$, equation (20.3) will reduce correctly to the Poincaré transformation (for w=ct and

$w_I = ct_I$) or to the Lorentz transformation (when $x_o^\mu = 0$) in the limit of zero acceleration. If a spacetime transformation for non-inertial frames does not have this limiting 4-dimensional symmetry, it is incomplete and probably not viable.

20b. The singular wall and horizons in the Wu transformation

Making the minimal departure from the Lorentz transformation and easiest non-constant choice for $\beta(w)$ in (20.3), we assume $\beta(w) = \beta_o + \alpha_o w$. The spacetime transformations (20.3) take the following form:

$$w_I = \gamma(\beta_o + \alpha_o w)\left(x + \frac{1}{\alpha_o \gamma_o^2}\right) - \frac{\beta_o}{\alpha_o \gamma_o},$$

$$x_I = \gamma\left(x + \frac{1}{\alpha_o \gamma_o^2}\right) - \frac{1}{\alpha_o \gamma_o}, \qquad y_I = y, \qquad z_I = z; \tag{20.7}$$

where we have set $w_o = x_o = y_o = z_o = 0$ for simplicity, and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}, \qquad \beta = \beta(w) = \beta_o + \alpha_o w. \tag{20.8}$$

This Wu transformation is a minimal generalization of the Lorentz transformation obtained on the basis on limiting 4-dimensional symmetry.[1] The transformation for a differential 4-vector dx^μ=(dw,dx,dy,dz) and the explicit expression for the invariant interval ds^2 are given by (19.16)-(19.18). Note that there is a singular wall at $x = -1/(\alpha_o \gamma_o^2)$ in (20.7), where W or the metric tensor P_{00} vanish, which is unphysical. One can also verify that, on the other side of the singular wall, the differentials dw_I and Wdw turn out to have different signs.

The inverse transformation of (20.7) is given by

$$w = \frac{1}{\alpha_o}\left(\frac{w_I + \beta_o / \alpha_o\gamma_o}{x_I + 1/\alpha_o\gamma_o} - \beta_o\right)$$

$$x = \sqrt{(x_I + 1/\alpha_o\gamma_o)^2 - (w_I + \beta_o/\alpha_o\gamma_o)^2} - \frac{1}{\alpha_o\gamma_o^2}\,, \tag{20.9}$$

$$y = y_I, \qquad z = z_I\ .$$

As $\alpha_o \to 0$, the last terms in (20.7) and (20.9) are singular. However, they are cancelled by other singular terms in the Wu transformations. Thus, the inertial limit $\alpha_o \to 0$ is well-defined for the Wu transformations.

We are used to picturing the physical world from the viewpoint of observers in an inertial frame. Let us use the inverse Wu transformations (20.9) to demonstrate the physical properties of the space and time axes (w,x). For constant w and $x_I \neq -1/(\alpha_o\gamma_o)$, the time transformation in (20.9) leads to straight lines. All lines corresponding to constant w pass through the same point $(w_I, x_I) = (-\beta_o/(\alpha_o\gamma_o), -1/(\alpha_o\gamma_o))$. In contrast, for constant x, the Wu transformation (20.9) leads to 2 sets of hyperbolic lines that satisfy the condition $\left| x_I + 1/(\alpha_o\gamma_o) \right| > \left| w_I + \beta_o/(\alpha_o\gamma_o) \right|$.

The graphs in Fig. 20.1 show lines of constant x and of constant w (straight lines) in the (w_I, x_I) plane for the Wu transformations with zero inertial velocity, $\beta_o = 0$, and different values for constant acceleration, $\alpha_o = -0.001, +0.001$.

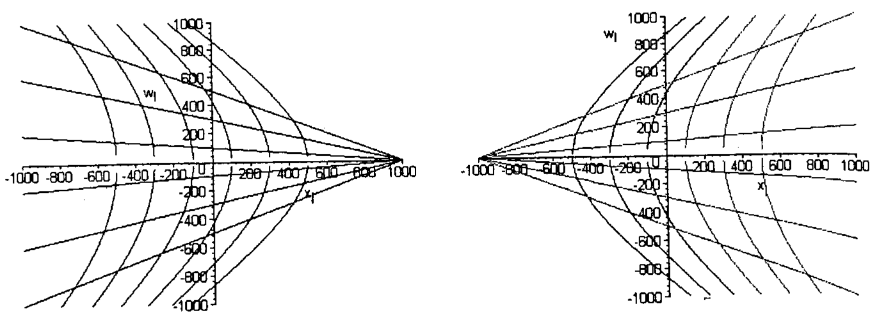


Fig. 20.1 Lines of constant w and x for accelerated frames with velocity $\beta_o = 0$ and accelerations $\alpha_o = \pm 0.001$.

Fig. 20.2 shows the same spacetime picture with an initial velocity, $\beta_0 = 0.5$, and two values of the acceleration $\alpha_0 = -0.001, +0.001$.

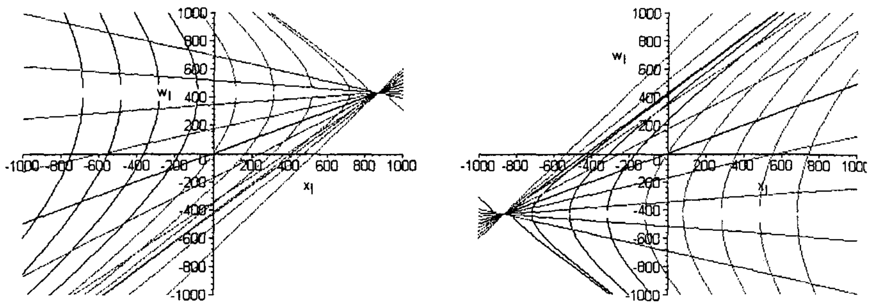


Fig. 20.2 Lines of constant w and x for accelerated frames with velocity $\beta_0 = 0.5$ and accelerations $\alpha_0 = \pm 0.001$.

Fig. 20.3 shows the lines with an acceleration of zero so that all lines are straight, corresponding to an inertial frame with a velocity $\beta_0 = 0.5$, as we expect from the Lorentz transformation.

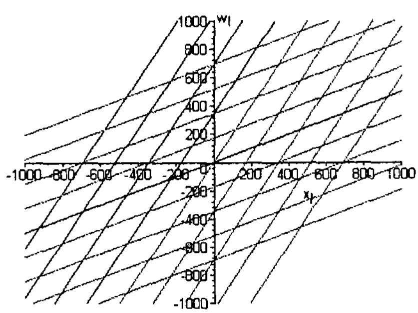
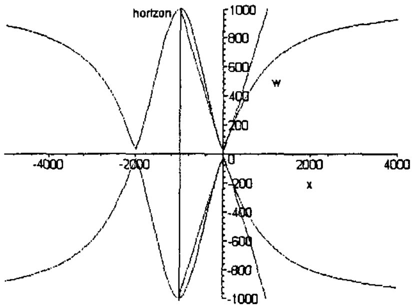


Fig. 20.3 Lines of constant w and x for an inertial frame with $\beta_0 = 0.5$.

The graphs in Fig. 20.4 show the change of a lightcone and of the relation $s^2 = w_I^2 - x_I^2$, where w_I^2 and x_I^2 can be expressed in terms of w and x by using the Wu transformation (20.7).



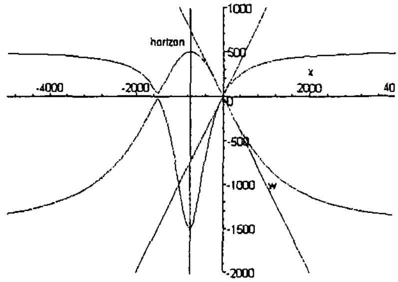


Fig. 20.4 Graphs of lightcones in a frame with velocity $\beta_o = 0$ and accelerations $\alpha_o = +0.001$ (left figure) and velocity $\beta_o = 0.5$ and accelerations $\alpha_o = +0.001$ (right figure).

It is interesting to observe that the region of spacetime consisting of all points $[x + 1/(\alpha_o\gamma_o^2)] > 0$ and all w from $(-1 - \beta_o)/\alpha_o$ to $(1 - \beta_o)/\alpha_o$ corresponds to the sector of the (w_I, x_I) plane between $(w_I + \beta_o/(\alpha_o\gamma_o)) = (x_I + 1/(\alpha_o\gamma_o))$ and $(w_I + \beta_o/(\alpha_o\gamma_o)) = -(x_I + 1/(\alpha_o\gamma_o))$. As $\alpha_o \to 0$, this region becomes larger and larger, eventually covering the whole (w_I, x_I) plane. Thus, this region of spacetime is identified with the physical world of the CLA frame F. Similarly, the region of spacetime consisting of all $[x + 1/(\alpha_o\gamma_o^2)] < 0$ is represented by the opposite sector with negative $(x_I + 1/(\alpha_o\gamma_o)$. This opposite region of spacetime may be called the 'mirror spacetime' of the CLA frame. The rest of the (w_I, x_I) plane cannot be reached for any real w and x for $\alpha_o \neq 0$. The time w is physical only within the range, $(-1 - \beta_o)/\alpha_o < w < (1 - \beta_o)/\alpha_o$. Outside this range, the CLA frame ceases to exist because the velocity $\beta(w)$ is greater than 1, which is unphysical. The limits of time, $w \to (\pm 1 - \beta_o)/\alpha_o$, in the CLA frame implies $w_I \to \pm\infty$ in inertial frame, (everywhere in physical space except at the singular wall). This property is interlocked with the assumption $\beta(w) = \beta_o + \alpha_o w$. If one makes a different assumption for the arbitrary function $\beta(w)$ in (20.3), one has a different time for a CLA frame.

The Wu transformations suggest that acceleration distorts spacetime and forms a horizon that corresponds to a 'wall singularity' $x = -1/(\alpha_o\gamma_o^2)$ (with arbitrary y and z coordinates) of the coordinates for the CLA frame F or $(w_I + \beta_o / \alpha_o\gamma_o)^2 = (x_I + 1/\alpha_o\gamma_o)^2$. This 'singular wall' separates physical spacetime from the 'mirror spacetime.' The location of the wall singularity depends on the sign of the acceleration α_o and $|\alpha_o\gamma_o^2|$. The 'mirror spacetime' emerges from the quadratic equation associated with the second equation in (20.9):

$$\left(x + \frac{1}{\alpha_o\gamma_o^2}\right)^2 = \left(x_I + \frac{1}{\alpha_o\gamma_o}\right)^2 - \left(w_I + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2.$$

Note that we also have a quadratic equation for the relativistic energy-momentum relation that leads to solutions with 'negative energy.' The mirror spacetime resembles the negative energy solution for a particle in some ways, so one may wonder whether the mirror spacetime exists in some physical sense. Such a question probably cannot be answered because even if it exists there is no possibility of communication between these two sectors of spacetime separated by the singular wall or the horizon.

The rate of ticking of a clock and the observable speed of light in the CLA frame F near this singular wall have very peculiar properties. Namely, as the coordinate x approaches $-1/(\alpha_o\gamma_o^2)$ in (20.9) (i.e., $(x_I + 1/\alpha_o\gamma_o) \to (w_I + \beta_o / \alpha_o\gamma_o)$), the Wu transformation leads to the following results:

(a) The clock stops ticking

$$\left(\frac{dw}{dw_I}\right)_{dx=0} = \frac{1}{\gamma W} = \frac{(x_I + 1/\alpha_o\gamma_o)^2 - (w_I + \beta_o / \alpha_o\gamma_o)^2}{\alpha_o(x_I + 1/\alpha_o\gamma_o)^3} \to 0. \qquad (20.10)$$

(b) The speed of light in the CLA frame F increases without bound

$$\left(\frac{dx}{dw}\right)_{ds=0} = \pm|W| = \pm\frac{\alpha_o(x_I + 1/\alpha_o\gamma_o)^2}{\sqrt{(x_I + 1/\alpha_o\gamma_o)^2 - (w_I + \beta_o/\alpha_o\gamma_o)^2}} \to \pm\infty. \tag{20.11}$$

Result (a) shows that the rate of ticking of a clock at rest in F at (x,0,0), i.e., dx=0, slows down in comparison with a clock at $(x_I, 0, 0)$ in the inertial frame. Result (b) is due to the fact that the law for the propagation of light is given by $ds^2 = 0$ and is naturally related to (a). The ratio dw/dw_I in (20.10) is positive, consistent with $(x_I + 1/\alpha_o\gamma_o) > 0$ and $(x_I + 1/\alpha_o\gamma_o)^2 > (w_I + \beta_o/\alpha_o\gamma_o)^2$, where the last relation is consistent with $\beta^2 < 1$.

20c. Generalized MØller-Wu-Lee transformation for an accelerated frame

Suppose one assumes the velocity function β in (20.3) for the CLA frame F(w*,x,y,z) to be

$$\beta(w^*) = \tanh(\gamma_o\alpha^* w^* + \tanh^{-1}\beta_o), \tag{20.12}$$

where w* denotes time in the accelerated frame F. The transformations in (20.3) are then the generalized MØller-Wu-Lee transformations[3]

$$w_I = \left(x + \frac{1}{\alpha^*\gamma_o}\right)\sinh\left(\alpha^*\gamma_o w^* + \tanh^{-1}\beta_o\right) - \frac{\beta_o}{\alpha^*},$$

$$x_I = \left(x + \frac{1}{\alpha^*\gamma_o}\right)\cosh\left(\alpha^*\gamma_o w^* + \tanh^{-1}\beta_o\right) - \frac{1}{\alpha^*}, \tag{20.13}$$

$$y_I = y, \qquad z_I = z.$$

When $\beta_o = 0$, the generalized Møller-Wu-Lee (MWL) transformation (20.13) reduces to the original Møller transformation,[4]

$$w_I = \sinh(\alpha^* w^*)\left(x + \frac{1}{\alpha^*}\right),$$

$$x_I = \cosh(\alpha^* w^*)\left(x + \frac{1}{\alpha^*}\right) - \frac{1}{\alpha^*}, \tag{20.14}$$

$$y_I = y, \qquad z_I = z.$$

This has been discussed by many authors.[4,5] However, the original Møller transformation did not involve a constant velocity β_o and hence, did not satisfy the principle of limiting 4-dimensional symmetry. We have generalized it so that it involves two constants β_o and α_o and thus is smoothly connected to the Lorentz transformation in the limit of zero acceleration.[3]

The inverse generalized MWL transformation can be derived from (20.13):

$$w^* = \frac{1}{\gamma_o \alpha^*}\left(\tanh^{-1}\left[\frac{w_I + \beta_o/\alpha^*}{x_I + 1/\alpha^*}\right] - \tanh^{-1}\beta_o\right),$$

$$x = \sqrt{(x_I + 1/\alpha^*)^2 - (w_I + \beta_o/\alpha^*)^2} - \frac{1}{\alpha^* \gamma_o}, \tag{20.15}$$

$$y = y_I, \qquad z = z_I.$$

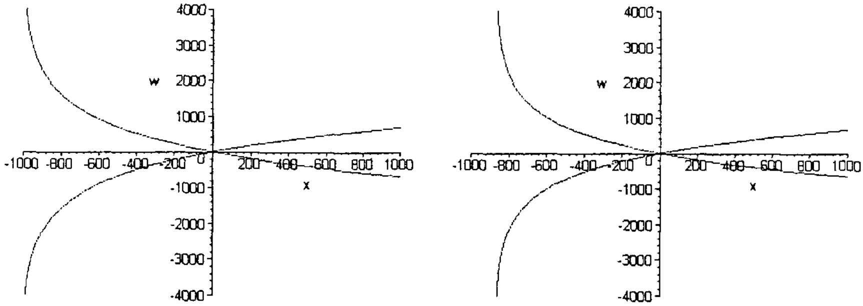
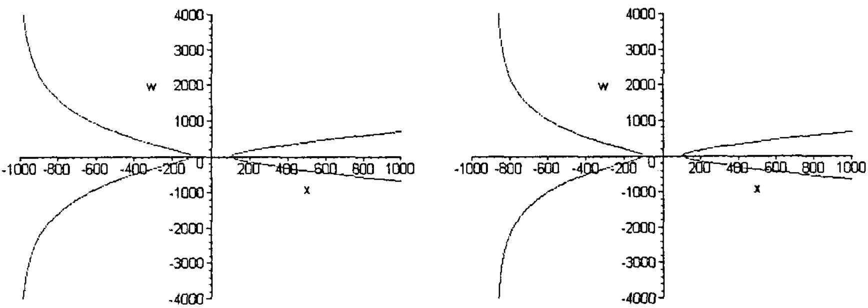
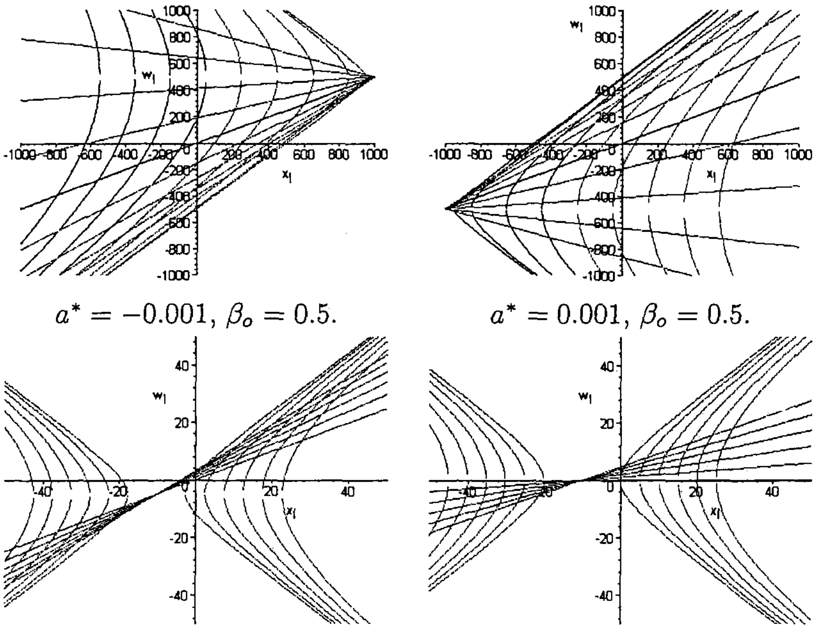


Fig 20.5 Graphs of the MWL lightcone (s=0) with acceleration $\alpha^* = 0.001$ and velocities β_o=0 (left diagram) and β_o=0.5 (right diagram).

The graphs in Figs. 20.5 and 20.6 shows the changes of the relation $s^2 = w_I^2 - x_I^2$ for different velocities β_o, where w_I^2 and x_I^2 can be expressed in terms of w=w* and x by using the MWL transformation (20.13).



Fig, 20.6 MWL graph for $s^2 = w_I^2 - x_I^2$ with velocity (left) β_o=0, (right) β_o=0.5, acceleration $\alpha^* = 0.001$, s=10000.



$a^* = -0.001,\ \beta_o = 0.5.$ $a^* = 0.001,\ \beta_o = 0.5.$

$a^* = -0.1,\ \beta_o = 0.5.$ $a^* = -0.1,\ \beta_o = 0.005.$

Fig. 20.7 Diagrams showing lines of constant w and x in accelerated frames with various acceleration and velocities under the MWL transformation.

The graphs in Fig. 20.7 show lines of constant w (straight lines) and of constant x in the (w_I, x_I) plane with various velocities β_o and accelerations α^*. These geometric properties of the accelerated spacetime are similar to those of the corresponding figures for the Wu transformations. This is not surprising because the main difference between (20.9) and (20.15) is the physical property of time:

$$\frac{(-1-\beta_o)}{\alpha_o} < w < \frac{(1-\beta_o)}{\alpha_o}, \qquad -\infty < w^* < +\infty. \tag{20.16}$$

In fact, based on (20.4), (20.7) and (20.15), one can show that the times w and w* are related by

$$\tanh(\gamma_o \alpha^* w^* + \tanh^{-1}\beta_o) = \beta_o + \alpha_o w = \frac{w_I + \beta_o/(\alpha_o\gamma_o)}{x_I + 1/(\alpha_o\gamma_o)}, \tag{20.17}$$

provided $\alpha^* = \alpha_o\gamma_o$.

The region of physical spacetime in F(w*,x,y,z) consisting of all points $[x+1/(\alpha^*\gamma_o)] > 0$ with w* between $-\infty$ and $+\infty$ corresponds to the sector of the (w_I, x_I) plane between $(w_I + \beta_o/\alpha^*) = +(x_I + 1/\alpha^*)$ and $(w_I + \beta_o/\alpha^*) = -(x_I + 1/\alpha^*)$. Similarly, the region of spacetime consisting of all $[x+1/(\alpha^*\gamma_o)] < 0$ is represented by the opposite sector with negative $(x_I + 1/\alpha^*)$. This region may be called the mirror spacetime. The rest of the (w_I, x_I) plane cannot be reached for any real w and x.

Based on the MWL transformation (20.13), as one approaches the wall singularity at $x = -1/(\alpha^*\gamma_o)$, the rate of ticking of clocks and the observable speed of light in the F(w*,x,y,z) frame show the following unusual properties:

(A) The rate of ticking of a clock at rest in the accelerated frame F(w*,x,y,z) at (x,0,0), i.e., dx=0, varies as a function of x_I (or x and w) in comparison with a clock at $(x_I,0,0)$ in the inertial frame

$$\left(\frac{dw^*}{dw_I}\right)_{dx=0} = \frac{\sqrt{1-(w_I+\beta_o/\alpha^*)^2/(x_I+1/\alpha^*)^2}}{\alpha^*\gamma_o(x+1/\alpha^*\gamma_o)} = \frac{1}{\gamma_o(1+\alpha^* x_I)}. \tag{20.18}$$

(B) Because the propagation of light is given by $ds^2 = W_x^2 dw^{*2} - dr^2 = 0$, where $W_x = (1+\alpha^*\gamma_o x) > 0$, the speed of light in CLA frame F(w,x) slows down to zero,

$$\left(\frac{dr}{dw^*}\right)_{ds=0} = \pm\left|1+\alpha^*\gamma_o x\right| \to 0. \tag{20.19}$$

Thus, there is no possibility of communication between the physical spacetime and the mirror spacetime. These two properties in (20.18) and (20.19) are again related. They differ from (20.10) and (20.11) because of different properties of times w and w*, as shown in (20.16) and (20.17).

The contravariant 4-momentum p^μ of a particle has the same transformation properties as dx^μ, which can be derived by differentiation of (20.3). Thus, we have

$$p_I^0 = \gamma(Wp^0 + \beta p^1), \qquad p_I^1 = \gamma(p^1 + \beta W p^0), \qquad W = \gamma^2\left(x + \frac{1}{\alpha_o \gamma_o^2}\right)\frac{d\beta}{dw}, \tag{20.20}$$

and $p_I^k = p^k$, k=1,2. Following the steps from (19.30) to (19.31), one can show that the general transformation (20.3) with an arbitrary $\beta(w)$, the Wu transformation (20.7), and the generalized MWL transformation (20.13) all have a common property, namely that a particle at rest at (x,0,0) in the F frame has the same constant-linear-acceleration as measured by observers at rest relative to the inertial frame F_I because

$$\left(\frac{dp_I^0}{dx_I}\right)_X = \left(\frac{dp_I^1}{dw_I}\right)_X = \frac{md\gamma}{\gamma\beta W dw} = \frac{\alpha_o m}{\gamma_o^{-2} + \alpha_o x} = \text{const.}, \tag{20.21}$$

where X stands for the general transformation (20.3) with an arbitrary velocity $\beta(w)$, or the Wu transformations (20.7). We also have

$$\left(\frac{dp_I^0}{dx_I}\right)_{MWL} = \left(\frac{dp_I^1}{dw_I}\right)_{MWL} = \frac{\alpha^* m}{\gamma_o^{-1} + \alpha^* x} = \text{const.}, \tag{20.22}$$

for a particle at rest in the accelerated frame at, say, a certain point (x,0,0), where x is a fixed constant. Note that (20.21) and (20.22) are the same when $\alpha^* = \alpha_o \gamma_o$.

20d. Decay-length dilations due to particle acceleration

Although the concept of time in the accelerated frame F differs in all of these transformations, all are smoothly connected to the Lorentz transformation in the limit of zero acceleration. One might thus wonder whether any of them can be considered physical in the sense that it can be related to some physical phenomena in an accelerated frame. The frequency of atomic radiation, the Doppler shift of the wavelength of electromagnetic radiation and the lifetime for the decay of an unstable particle are some of these phenomena.

Let us concentrate on the possible effect on a particle's decay-length due to acceleration.[8] A simple setup to perform such experiment is to create a potential drop over a distance L, say from x_{I1} to $x_{I2} = x_{I1} + L$. One could place detectors at x_{I1} and x_{I2} to count the number of particles (charged pions for example) in the beam before and after traveling a length L. There is an additional complication however, i.e., the spacetime transformations we have been discussing have a metric tensor $P_{00} = W^2$ with a non-trivial space-dependence. One can usually neglect this nuisance property by setting x=0, i.e., assuming that the particle is located at the origin of the accelerated coordinate system. However here, we want to examine this x-dependence explicitly and graphically for a particle located at any point in the CLA frame.

Because quantum field theory has not yet been formulated for accelerated frames, we shall make some reasonable assumptions. For example, we will assume that the usual exponential decay-law holds in accelerated frames,

$$N = N_o \exp[-\Delta w / w_o], \qquad \Delta w = w_2 - w_1, \tag{20.23}$$

where w_o is the decay lifetime of a particle at rest in an accelerated frame. We shall assume that w_o is approximately the same as the lifetime that has been measured in (inertial) laboratories. This approximation should be good when the acceleration is small. (For further discussions, see section 20e below.) Based on this law, we shall now deduce and examine the predictions for the lifetime dilation of an unstable particle in flight that is being accelerated by a potential field.

We use the inverse Wu transformation (20.9) to express $w_2 - w_1$ in terms of spatial coordinates in the inertial laboratory frame. We have

$$w_2 - w_1 = \frac{1}{\alpha_o}\left[\frac{w_{I2} + \beta_o/(\alpha_o\gamma_o)}{x_{I2} + 1/(\alpha_o\gamma_o)} - \frac{w_{I1} + \beta_o/(\alpha_o\gamma_o)}{x_{I1} + 1/(\alpha_o\gamma_o)}\right]$$

$$= \frac{1}{\alpha_o}\left[\sqrt{1 - \left(\frac{x_2 + 1/(\alpha_o\gamma_o^2)}{x_{I2} + 1/(\alpha_o\gamma_o)}\right)^2} - \sqrt{1 - \left(\frac{x_1 + 1/(\alpha_o\gamma_o^2)}{x_{I1} + 1/(\alpha_o\gamma_o)}\right)^2}\right], \tag{20.24}$$

where the particles are at rest at $x = -x_2 = x_1$ in the CLA frame. Therefore, if there are N_o particles at x_{I1}, then the number of particles remaining at $x_{I2} = x_{I1} + L$ is given by (20.23) and (20.24)

$$N(x_{I1} + L) = N_o(x_{I1})\exp\left(\frac{-1}{\alpha_o w_o}\left[\sqrt{1 - \left(\frac{x_2 + 1/(\alpha_o\gamma_o^2)}{x_{I2} + 1/(\alpha_o\gamma_o)}\right)^2} - \sqrt{1 - \left(\frac{x_1 + 1/(\alpha_o\gamma_o^2)}{x_{I1} + 1/(\alpha_o\gamma_o)}\right)^2}\right]\right), \tag{20.25}$$

where $x = x_2 = x_1$ is fixed. This is the formula for the number of unstable particles remaining after starting from an arbitrary point x_I and traveling a distance L with a constant-linear-acceleration. Note that the exponential part in (20.25) is dependent on the coordinate variable x or x_I, where these two variables are related through the transformation (20.9). In the limit of zero acceleration, $\alpha_o \to 0$, the formula reduces to the usual formula in special relativity

$$N(x_{I1} + L) = N_o(x_{I1})\exp\left(\frac{-L\sqrt{1 - \beta_o^2}}{\beta_o w_o}\right), \tag{20.26}$$

for particle decay in flight with the velocity β_o. Note that the exponential part in (20.26) is now independent of the coordinate variables x and x_I, as it should be for inertial frames.

On the other hand, if one uses the generalized MWL transformation (20.15) to express the law (20.24) with $w \to w^*$, one has the result:

$$N(x_{I1} + L) = N_o(x_{I1}) \exp\left(\frac{-T}{\alpha^* \gamma_o w_o}\right),$$

$$T = \tanh^{-1}\sqrt{1 - \left(\frac{x + 1/(\alpha^* \gamma_o)}{x_{I2} + 1/\alpha^*}\right)^2} - \tanh^{-1}\sqrt{1 - \left(\frac{x + 1/(\alpha^* \gamma_o)}{x_{I1} + 1/\alpha^*}\right)^2}\ . \tag{20.27}$$

Figs. 20.8 and 20.9 show graphically the different predictions of the two formulas. Both of them are insensitive to the coordinate x in (20.25) and (20.27).

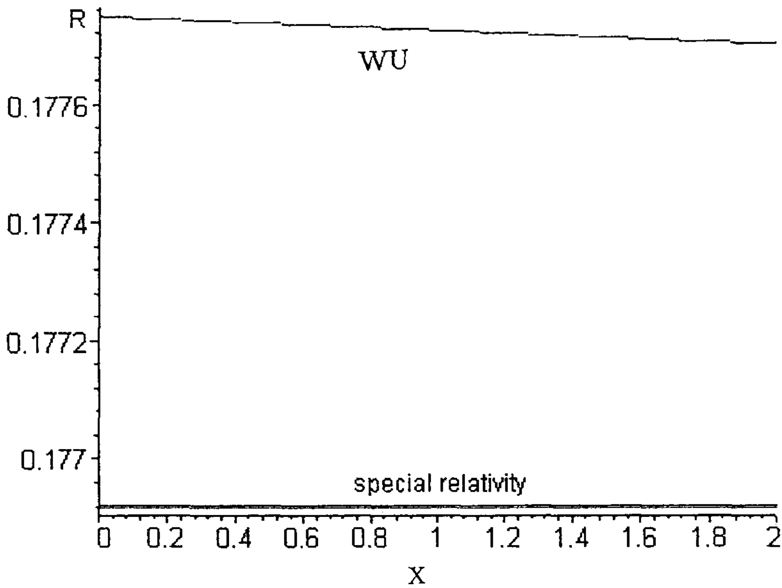


Fig. 20.8 Coordinate dependence of the concentration $R = N/N_o$. Comparing Wu transformation and special relativity $\beta_o = 0.5, \alpha_o = 0.001, L = 7.8, w_o = 7.8$.

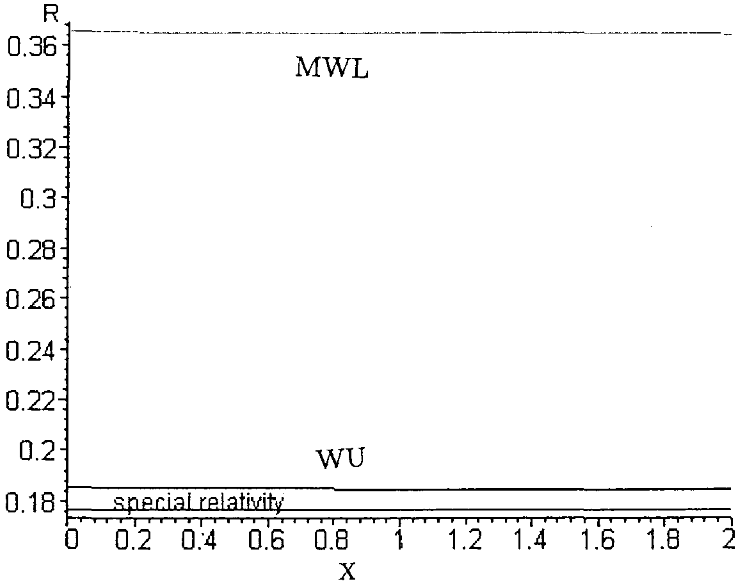


Fig. 20.9 Coordinate dependence of the concentration $R = N/N_0$. Comparing MWL and Wu transformations $\beta_o = 0.5$, $\alpha_o = 0.001$, $L = 7.8$, $w_o = 7.8$.

20e. Discussion

Although the velocity function $\beta(w)$ in (20.3) is arbitrary, the transformations for the differentials dx^μ are as simple as that in (19.16) (where $\beta(w) = \beta_o + \alpha_o w$). From (20.3) with arbitrary velocity $\beta(w)$, we obtain

$$dw_I = \gamma(\overline{W}dw + \beta dx), \quad dx_I = \gamma(dx + \beta\overline{W}dw), \quad dy_I = dy, \quad dz_I = dz, \qquad (20.28)$$

where $\overline{W} = \gamma^2(x + 1/\alpha_o\gamma_o^2)d\beta/dw$. As usual, the set of the generalized Lorentz transformations (20.7) form the Wu group that includes the Lorentz group as a limiting case of zero acceleration.[1] However, in order to see the group properties one must consider, for example, the Wu transformation between two CLA frames F(w,x,y,z) and F'(w',x',y',z'),

$$\gamma(\beta_o + \alpha_o w)\left(x + \frac{1}{\alpha_o \gamma_o^2}\right) - \frac{\beta_o}{\alpha_o \gamma_o} = \gamma'(\beta'_o + \alpha'_o w')\left(x' + \frac{1}{\alpha'_o \gamma'^2_o}\right) - \frac{\beta'_o}{\alpha'_o \gamma'_o},$$

$$\gamma\left(x + \frac{1}{\alpha_o \gamma_o^2}\right) - \frac{1}{\alpha_o \gamma_o} = \gamma'\left(x' + \frac{1}{\alpha'_o \gamma'^2_o}\right) - \frac{1}{\alpha'_o \gamma'_o}, \qquad y = y', \qquad z = z'; \tag{20.29}$$

rather than just (20.7) for an inertial and a CLA frame. After some calculations, one can express the unprimed variables in (20.29) in terms of primed variables and constants. Similarly, the set of transformations (20.13) also form a group. These symmetry group properties are important for the formulation of field theories in accelerated frames.

To test the two decay laws (20.25) and (20.27) involving the two constant accelerations α_0 and α^*, it is reasonable to compare them with the relation $\alpha^* = \alpha_o \gamma_o$ because their corresponding times w* and w are related by equation (20.17). If there is enough data, we expect that the formula (20.25), if it is correct, will enable us to determine experimentally the 'rest lifetime' w_0 of a particle in an accelerated frame.

Experimental tests of the sole α_0-dependent effect in the decay formulas is particularly interesting because it is dependent purely on the acceleration rather than on the velocity $\beta(w) = \beta_o + \alpha_o w$. In discussions of accelerations, physicists usually make the simplifying assumption that any effects of acceleration appear only through the velocity $\beta(w) = \beta_o + \alpha_o w$). However, strictly speaking, this assumption appears to be inconsistent with the principle of limiting 4-dimensional symmetry, which lead to a distinct α_0-dependence in the decay formulas (20.25) and (20.27). To see this α_0-dependence in a simple form, we set $x = x_I = 0$ in (20.27) and obtain

$$R = \frac{N(L)}{N_o(0)} = \exp\left(\frac{-1}{\alpha_o w_o}\left[\sqrt{1 - \frac{1-\beta_o^2}{(1+\alpha_o \gamma_o L)^2}} - \beta_o\right]\right). \tag{20.30}$$

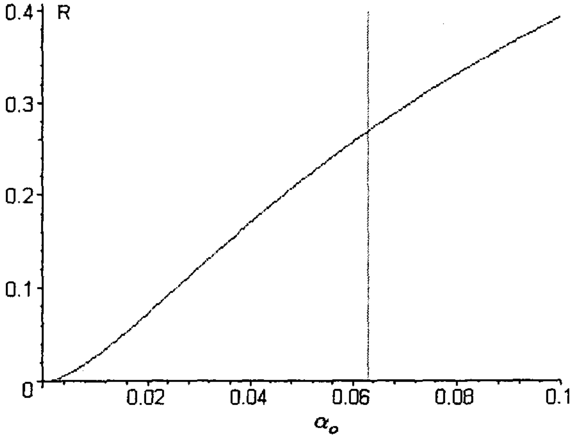


Fig. 20.10 R=N(L)/N_0(0) in respect to α_0 (see eq. (20.30)) (L=7.8, $w_0 = 7.8$, $\beta_0 = 0.1$, vertical line at α_0=0.063).

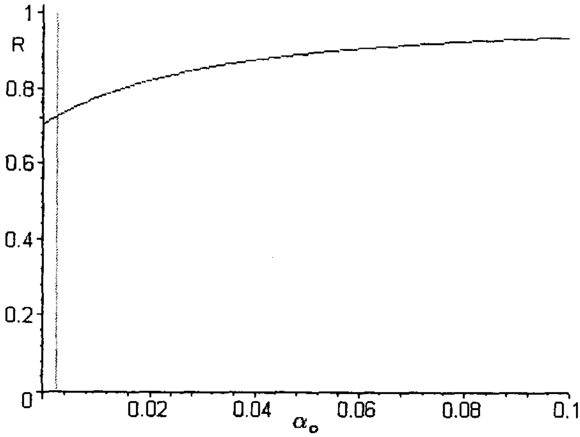


Fig. 20.11 R=N(L)/N_0(0) in respect to α_0 (see eq. (20.30)) (L=7.8 $w_0 = 7.8$, $\beta_0 = 0.943$, vertical line at α_0=0.0025).

Suppose a charged pion enters a potential gradient of 10 MeV per meter, $\Delta V = 10$ MeV/m, with an initial velocity β_o. Its acceleration can be calculated using (20.21). One obtains α_o=0.063/m for β_o=0.1, and α_o=0.0025/m for β_o= 0.943. The α_o-dependent behavior in (20.30) is shown in Figures 20.10 and 20.11. When β_o=0.1, we use the values α_o=0.030/m, 0.063/m, 0.09/m. And when β_o=0.943, we use the values α_o= 0.0012/m, 0.0025/m, 0.005/m.[6]

In quantum field theory, the decay rate for accelerated unstable particles is formally the same as in (17.72) with a time-dependent factor $\Gamma(\beta)=\Gamma_o(1-\beta^2)^{1/2}$. Making a natural generalization of the usual exponential decay law by assuming that its differential form, $dN/dw=-\Gamma N$ is also valid in an inertial frame with a time-dependent decay rate Γ, the decay law for particles at rest in a CLA frame is then

$$N/N_o=\exp[-\Gamma_o\int_1^2\sqrt{1-\beta^2}dw_I]=\exp\left[-\Gamma_o A\left\{\sinh^{-1}\left(\frac{B_2}{A}\right)-\sinh^{-1}\left(\frac{B_1}{A}\right)\right\}\right], \tag{20.31}$$

where we have used (18.9), $\beta=(w_I+\beta_o/\alpha_o\gamma_o)/(x_I+1/\alpha_o\gamma_o)$, $A=x+1/\alpha_o\gamma_o^2$ and $B_k = w_{Ik}+\beta_o/\alpha_o\gamma_o$, k=1,2.

On the other hand, from the point of view of limiting Lorentz and Poincare invariance, the differential form of the invariant decay law $dN/ds=-\Gamma_o N$ should hold not only in all inertial frames, but also in non-inertial frames in the limit where the acceleration goes to zero. If we make the additional assumption that this differential form is valid in non-inertial frames not only in the limit of zero acceleration but also for non-zero accelerations, then we obtain the same result for N/N_o as the one based on the field theory approach. As expected, this expression for N/N_o reduces to the usual decay law for inertial frames in the limit of zero acceleration. Thus, the results in equations (20.23) and (20.24) are valid only for CLA frames in the limit of small accelerations.[7]

It is important to determine experimentally the correct physical time in accelerated frames before attempting to formulate physical theories in accelerated frames. Thus, experiments that can determine the correct physical time in accelerated frames are crucial to our understanding of the physics of non-inertial frames.

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6. The constant acceleration α_o (in unit 1/m) is related to the usual acceleration a_o (in unit m/s^2) by $\alpha_o=a_o/(\gamma_o c^2)$, as one can see by comparing equations (18.6) and (18.8). We can estimate that α_o=0.005/m corresponds to an extremely large acceleration $a_o \sim 10^{13}$ m/s^2 for $\gamma_o \sim 1$.
7. The decay law in a CLA frame differs from (20.31). Suppose particles are at rest in a CLA frame. Based on the Wu transformation (19.14), we obtain $(N/N_o)_{CLA}=\exp(Z_{Wu})$, $\quad Z_{Wu}=-\Gamma_o\int_1^2 W dw = -\Gamma_o(x+1/\alpha_o\gamma_o^2)[\tanh^{-1}\beta_2 - \tanh^{-1}\beta_1]$, where $\beta_k = \beta_o + \alpha_o w_k$, k=1,2. We have used the decay law $dN/ds=-\Gamma_o N$. On the other hand, if one uses the generalized MWL transformation (19.7), one has $(N/N_o)_{CLA}=\exp(Z_{MWL})$, where $Z_{MWL}=-\Gamma_o\int_1^2 W_x dw^* = -\Gamma_o(1+\alpha^*\gamma_o x)[w_2^* - w_1^*]$.

21.

Extended Lorentz Transformations for Accelerated Frames and a Resolution to the "Two-Spaceship Paradox"

21a. The two-spaceship paradox

For years, the "two-spaceship paradox," including the effect of a constant linear acceleration has been discussed in the literature[1] based on special relativity (with $d^2s = c^2dt_I^2 - dr_I^2$ for an inertial frame) and $d^2s = (1+\alpha_o x/c^2)^2c^2dt^2 - dr^2 \equiv ds_\alpha^2$ for a non-inertial frame with a constant linear acceleration α_o. However, the discussions have been based on intuitive arguments rather than an explicit spacetime transformation between the two frames, one of which is accelerated. In chapters 16 and 17, we obtained a spacetime transformation between an inertial frame $F_I(w_I, x_I, y_I, z_I)$ and a frame undergoing constant linear acceleration (CLA) frame $F(w,x,y,z)$ as an extension of the Lorentz transformation, which is also characterized by a metric with the form $d^2s = W^2dw^2 - dx^2 - dy^2 - dz^2$. In this chapter, we use the transformation to resolve the apparent paradox.

The essential parts of the 'two-spaceship paradox' are as follows:[1]

(a) Consider two identical spaceships, A and B, which at first are at rest in an inertial frame F_I, a distance L apart. At time $w_I = ct_I = 0$ the spaceships accelerate in the same direction parallel to the line that joins them. As viewed from F_I, they undergo the same acceleration for the same duration, then stop accelerating at the same time after reaching a steady speed u.

(b) The apparent paradox arises when asking the question "What is the distance between the two spaceships after they reach the steady speed u, observed from F_I?" Arguments can be advanced either for the distance to remain unchanged (because both spaceships have undergone the same motion) or for the distance between the ships to be Lorentz contracted L' $(=L\sqrt{1-(u/c)^2})$ (treating the spaceships as the two ends of a long rod in motion).

Although the dynamics of a point particle undergoing a constant acceleration can be treated in the framework of special relativity, the constant-linear-acceleration of a reference frame is, strictly speaking, beyond special relativity because of the extended nature of the reference frame. In order to discuss the two-spaceship problem, we must have transformations that include two parameters, the relative instantaneous velocity and the acceleration of the non-inertial frame. As we shall see, such transformations for accelerated frames (with a metric of the form, $d^2s = (1+\alpha_o x/c^2)^2c^2dt^2 - dr^2 \equiv ds_\alpha^2$ imply that if the engines of spaceships A and B are turned off at the same time $w_{I1} = w_{I2}$ as measured in the inertial frame F_I, then their final velocities relative to F_I cannot be the same, as was assumed in the statement of the paradox. Surprisingly, this peculiar property (see eq. (21.14) below) is not in conflict with experimental results and is consistent with accelerated Galilean transformations at small accelerations and velocities. It is a natural result of the limiting four-dimensional symmetry of accelerated frames.

As discussed in chapters 16 and 17, one can find two transformations relating the coordinates of an inertial and a non-inertial frame with constant linear acceleration, the Møller and Wu transformations.[2,3] However, in discussing the two-spaceships paradox, neither is ideal. The Møller transformation involves only the acceleration of the non-inertial frame as a parameter and reduces to the identity transformation rather than the Lorentz transformation, when the acceleration is zero.[2] The Wu transformation involves both the acceleration of the non-inertial frame α_o and the relative instantaneous velocity of the two frames β_o at time w=0. However, when the acceleration is zero, it reduces to the Lorentz transformation with a relative velocity β_o between the two inertial frames whose origins coincide at $w = 0$. Thus, it is inconvenient to use the Wu transformation when discussing a situation such as the "two-spaceship paradox" because the acceleration of a frame (the frame of the two spaceships) becomes zero at a time other than $w = 0$.

To work around this problem, we first use generalized forms of both the Møller and Wu transformations, both of which smoothly reduce to the Lorentz transformation when the acceleration α_o is zero. The generalized forms of these transformations can be used to obtain the Lorentz transformation between the inertial frame F_I and the inertial frame occupied by the two spaceships after they stop accelerating without any additional adjustments. Such transformations for accelerated frames are necessary in order to discuss operationally any changes of lengths that are related to constant linear accelerations. The generalized Møller and Wu transformations are particularly suited for the clarification of the "two-spaceship paradox" because they enable us to discuss length contractions from the viewpoint of both inertial observers and the spaceship pilots before, during and after the acceleration.

21b. Generalized Møller and Wu transformations

Let us consider first the Wu transformations. In the following discussion, we assume that both the velocity and the acceleration are along the x-axis, ignoring the unimportant y and z axes. The general Wu spacetime transformation between an inertial frame $F_I(w_I, x_I)$ and a CLA frame F(w,x) can be written as[3]

$$ct_I = \gamma\beta\left(x + \frac{1}{\alpha\gamma_a^2}\right) + w_A,$$

$$x_I = \gamma\left(x + \frac{1}{\alpha\gamma_a^2}\right) + x_A, \tag{21.1}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \gamma_a = \frac{1}{\sqrt{1-\beta_a^2}}, \qquad \beta = \alpha_o(w - w_a) + \beta_a, \tag{21.2}$$

$$w_A = \gamma_a w_a - \frac{\beta_a}{\alpha_o\gamma_a}, \qquad x_A = \gamma_a\beta_a w_a - \frac{1}{\alpha_o\gamma_a}. \tag{21.3}$$

The quantity β_a is the velocity at a given time w_a, $\beta_a = \beta_o + \alpha_o w_a$. If one chooses $w_a = 0$, then β_a is the initial velocity β_o. In this special case, (21.1) becomes the Wu transformations discussed in section 19b. From the relation for β in (21.2), we see that the limit $\alpha_o \to 0$ is the same as $w \to w_a$. Therefore, the limit $\alpha_o \to 0$ is always understood as taking the time w to be $w = w_a$, so that we have $\beta = \beta_a$ in this limit.

The general Wu transformation (1) can be derived on the basis of limiting 4-dimensional symmetry together with the linear function $\beta = \beta(w)$ in (21.2).[3] It preserves the invariant interval

$$d^2s = dw_I^2 - dx_I^2 = W^2dw^2 - dx^2, \tag{21.4}$$

$$W^2 = \gamma^4(\gamma_a^{-2} + \alpha_o x)^2. \tag{21.5}$$

If one introduces another time variable w* given by

$$w^* = \frac{1}{\alpha_o\gamma_a^2}\left(\tanh^{-1}[\beta_a + \alpha_o(w - w_a)] - \tanh^{-1}[\beta_a - \alpha_o w_a]\right), \tag{21.6}$$

the metric in (4) can be written as

$$d^2s = W_M^2 dw^{*2} - dx^2, \qquad W_M = (1 + \gamma_a^2\alpha_o x), \tag{21.7}$$

where the metric tensor $g_{00} = W_M^2$ is time-independent, and

$$dw^* = \gamma_a^{-2}\gamma^2 dw, \tag{21.8}$$

so that the metric ds^2 in the CLA frame F(w*,x) can be written as

$$ds^2 = (1 + \gamma_a^2\alpha_o x)^2 dw^{*2} - dx^2. \tag{21.9}$$

The spacetime transformation (21.1) can then be expressed in terms of (w*,x,):

$$w_I = \left(x + \frac{1}{\alpha_o \gamma_a^2}\right)\sinh\left(\gamma_a^2 \alpha_o w^* + \tanh^{-1}\beta_o\right) + w_A,$$

$$x_I = \left(x + \frac{1}{\alpha_o \gamma_a^2}\right)\cosh\left(\gamma_a^2 \alpha_o w^* + \tanh^{-1}\beta_o\right) + x_A, \tag{21.10}$$

where $\beta_o = \beta_a - \alpha_o w_a$. This is the generalized MØller transformations. One can verify that both the generalized MØller transformation (21.10) and the Wu transformation (21.1) include the Lorentz transformations as the limiting case when $\alpha_o \to 0$ at any time w_a of the CLA frame F: $w_I = \gamma_a(w + \beta_a x)$, $x_I = \gamma_a(x + \beta_a w)$. The times in these transformations (21.1) and (21.10) can be operationalized by computerized clocks as described previously in chapter 18.[4]

Our results for the two different times w and w* can be summarized as follows:

(A') When $w_a = 0$ (i.e., $\beta_a = \beta_o$), the transformation (21.1) corresponds to the usual Wu transformations (19.14), characterized by a linear relation

$$\beta = \beta_o + \alpha_o w \tag{21.11}$$

and a finite range for the time variable w in the CLA frame F(w,x), $(-1-\beta_o)/\alpha_o < w < (1-\beta_o)/\alpha_o$. Note that the maximum value $w_{max} = (1-\beta_o)/\alpha_o$ for the time w in the CLA frame F corresponds to $\beta = 1$ and, hence, $w_I = +\infty$ for the time in the inertial frame $F_I(w_I, x_I)$. The inverse transformation to (21.1) is:

$$w = \frac{w_I - w_A}{\alpha_o(x_I - x_A)} - \frac{\beta_a}{\alpha_o} + w_a, \tag{21.12}$$

$$x = \sqrt{(x_I - x_A)^2 - (w_I - w_A)^2} - \frac{1}{\alpha_o \gamma_a^2}.$$

(B') When $\beta_a = 0$ and $w_a = 0$ (or $\beta_o = 0$), the transformation (21.10) reduces to the usual Møller transformation, i.e.,

$$w_I = \left(x + \frac{1}{\alpha^*}\right)\sinh(\alpha^* w^*), \qquad x_I = \left(x + \frac{1}{\alpha^*}\right)\cosh(\alpha^* w^*) - \frac{1}{\alpha^*},$$

and $y_I = y$, $z_I = z$. The Møller transformation is characterized by

(i) a non-linear relation between velocity β and time w^*,

$$\beta = \tanh(\alpha_o \gamma_a^2 w^* + \tanh^{-1}\beta_o) = \tanh(\alpha_o \gamma_a^2 w^*), \text{ and} \tag{21.13}$$

(ii) an infinite range for the time variable w^* of the CLA frame $F(w^*,x)$, $-\infty < w^* < +\infty$.

The inverse of the generalized Møller transformations can be derived from equation (21.10).

21c. Motion and length contraction involving accelerations

Let us now consider the linearly accelerated motion of an arbitrary point x, i.e., x=k=constant or dx=0, as measured in an inertial frame F_I,

$$\left(\frac{dx_I}{dw_I}\right)_{x=k} = \left[\frac{w_I - w_A}{x_I - x_A}\right]_{x=k} = \left[\frac{w_I - w_A}{\sqrt{(x + 1/\alpha_o \gamma_a^2)^2 + (w_I - w_A)^2}}\right]_{x=k}, \tag{21.14}$$

where we have used (21.1) and (21.12). This result holds for both the generalized Wu and Møller transformations, implying that at a given time w_I, the velocity dx_I / dw_I of a particle at $(k_1, 0, 0)$ is different from the velocity of a particle at point $(k_2, 0, 0)$, as measured by an observer in an inertial frame F_I. This result substantiates our previous observation concerning the final velocities of the two spaceships in section 1. Note that when one compares the velocities of two objects at the same time w_I in the inertial frame, one should not set $\alpha_o \to 0$ because this limit corresponds, by definition, to events that occur at the same time $w = w_a$ in the CLA frame F.

Suppose the two spaceships are at rest in the CLA frame F along the x-axis. What is the distance the two points occupied by the spaceships x_1 and x_2? Since there is a wall singularity at $x_s = -1/(\alpha_o \gamma_a^2)$ in the extended Lorentz transformations (21.1), (21.10) and (21.12),[2,3] the values of x_1 and x_2 must both be larger (more positive) than x_s in order for them to be on the physically occupiable side of the wall singularity. The left-hand-side of the singular wall is unphysical because it cannot be reached by any physical signal from the origin of a CLA frame. However, there is no restriction on the distance between these two points. If the distance between these two points is measured by two inertial observers at the same time, $w_{I2} = w_{I1} = w_{Io}$, the inverse transformations (21.12) lead to

$$\left(x_2 - x_1\right)_{QI} = \left(x_{I2} - x_A\right)\sqrt{1-\beta_2^2} - \left(x_{I1} - x_A\right)\sqrt{1-\beta_1^2} \ , \tag{21.15}$$

$$\beta_i = \frac{w_{Ii} - w_A}{x_{Ii} - x_A}, \qquad i = 1,2 \ ;$$

where the subscript QI denotes the condition $w_{I2} = w_{I1} = w_{Io}$. To the first order in α_o, we have the following approximation:

$$\left(x_2 - x_1\right)_{QI} \approx \gamma_a\left(x_{I2} - x_{I1}\right) + \frac{\alpha_o\gamma_a^2}{2}\left[-2\gamma_a w_a \beta_a\left(x_{I2} - x_{I1}\right) + x_{I2}^2 - x_{I1}^2\right]$$

$$-\frac{\alpha_o\gamma_a^4}{2}\left[-2\beta_a w_{Io}\left(x_{I2} - x_{I1}\right) + x_{I2}^2 - x_{I1}^2\right] \ . \tag{21.16}$$

On the other hand, suppose the two corresponding distances $x_{I2} - x_{I1}$ and $x_2 - x_1$ are measured at the same time $w_2 = w_1 = w_a$ of the CLA frame F. The generalized Wu transformations (21.1) lead the following exact result:

$$\left[x_{I2} - x_{I1}\right]_Q = \gamma_a\left(x_2 - x_1\right), \qquad \gamma_a = \frac{1}{\sqrt{1 - (\beta_o + \alpha_o w_a)^2}}, \tag{21.17}$$

where the subscript Q denotes the condition $w_2 = w_1 = w_a$. The generalized Møller transformation (21.10) then yields

$$\left[x_{I2} - x_{I1}\right]_{Q^*} = \left(x_2 - x_1\right)\cosh\left[\gamma_a^2\alpha_o w^* + \tanh^{-1}\beta_o\right] = \gamma_a\left(x_2 - x_1\right), \tag{21.18}$$

where the subscript Q* denotes the condition $w^*_2 = w^*_1 = w^*_a$, and we have used the relation (21.6) for times w* and w. Thus both the generalized Møller and Wu transformations of spacetime lead to exactly the same result (21.17) or (21.18) which depends only on the velocity β_a at the time w_a of measurement. This result is similar to that in special relativity, i.e., roughly speaking, a "moving" meter stick (at rest in F_I) appears to be shorter, as measured by observers in the CLA frame F, as shown in (21.17) and (21.18).

21d. Discussion

In the previous argument, it does not matter whether the distance $\left(x_2 - x_1\right)$ in (21.15) is the length of a spaceship or the distance between two spaceships A and B that are at rest in the CLA frame F. During acceleration, both the distances between the two spaceships and the lengths of the spaceships undergo the usual Lorentz contraction in addition to a change due to the constant acceleration. If both the initial velocity and the acceleration are zero at time $w_a = 0$, i.e. $\beta_a = \beta_o = 0$, and $\alpha_o = 0$, then equation (16) reduces to $x_2 - x_1 = x_{I2} - x_{I1}$ for $w_{I2} - w_{I1} = 0$. Thus, there is no length contraction before acceleration. After accelerating to reach a constant velocity β_a, i.e. setting $\alpha_o = 0$ at time w_a in (16), both the distances between two spaceships and the length of a spaceship are contracted by just the usual Lorentz contraction, $[x_2 - x_1]_{QI} = \gamma_a(x_{I2} - x_{I1})$.

A qualitative picture of a reference frame F with a constant-linear-acceleration is as follows: The spacetime of accelerated frame is non-uniform, in sharp contrast to the uniform spacetime of inertial frame. An inertial frame can be pictured as an (idealized) rigid scaffolding, while a CLA frame F corresponds to scaffolding with a non-uniform deformation. Along the x-axis, the Møller and Wu transformations map only the section of spacetime between the singular wall and x $=+\infty$ in an accelerated frame F to the entire spacetime extending from x = $-\infty$ to x = $+\infty$ in an inertial frame. The limits on the time variable $w \to [(\pm 1 - \beta_a)/\alpha_o + w_a]$ in the CLA frame implies $w_I \to \pm\infty$ in the inertial frame everywhere in physical space except at the `singular wall,' $x = -1/(\alpha_o \gamma_a^2)$. It is interesting that as one approaches this 'singular wall' in the accelerated frame the clocks slow down relative to the nearby clocks in an inertial frame F_I, and the speed of light increases without limit. At the singular wall, the clocks stop completely and the speed of light becomes infinity.[6] These properties are interlocked with the assumed relation $\beta(w) = \beta_a + \alpha_o(w - w_a)$ for the Wu transformation (21.1).

In general, intuitive arguments related to acceleration are not reliable because of the non-trivial physical properties of accelerated frames, as shown in (21.1) and (21.10). Once we postulate the metric $d^2s = (1 + \alpha_o x/c^2)^2 c^2 dt^2 - dr^2$ for a constant-linear-acceleration (CLA) frame F(w,x), then a particle at rest in this CLA frame does not satisfy d^2x_I / dw_I^2 = constant, as measured by observers in an inertial frame F_I. This is because the velocity of a particle moving with d^2x_I / dw_I^2 = constant will become larger than the speed of light for sufficiently large w_I, leading to unphysical results. As argued previously, the only relativistically consistent meaning for "constant-linear-acceleration" is one in which the relativistic momentum p_{Ix} (or energy p_{I0}) of a particle undergoing CLA satisfies dp_{Ix}/dw_I= constant (or dp_{I0}/dx_I=constant), as measured in an inertial frame.[3] For example, particles in a high energy linear accelerator that has a constant potential drop per unit length along the particle beam have a constant linear acceleration.

Thus, the generalized MØller and Wu transformations can be used to discuss the two-spaceship problem involving a constant-linear-acceleration. Using these specific transfomations gives the measurements of length and time well-defined operational meanings and hence, there is no "two-spaceship paradox."

References

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2. C. MØller, Danske Vid. Sel. Mat.-Fys. **20**, No. 19 (1943); see also *The Theory of Relativity*, (Oxford University Press, 1952), Chapter VII. Ta-You Wu and Y. C. Lee, Intern. J. Theoretical Phys. **5**, 307 (1972). Ta-You Wu, *Theory of Relativity* (Lian Jing Publishing Co., Taipei, 1978) pp. 172-175. J. P. Hsu, Nuovo Cimento B **108**, 949 (1993), and B **109**, 645 (1994).
3. Limiting 4-dimensional symmetry means that any accelerated transformation of spacetime must reduce to the Lorentz transformation (that displays the 4-dimensional symmetry) in the limit of zero acceleration. J. P. Hsu, *Einstein's Relativity and Beyond-New Symmetry Approaches* (World Scientific, 2000) Ch. 21, and J. P. Hsu and Silvia M. Cleff, Chin. J. Phys. **36**, 768 (1998).
4. One can set up a grid of clocks in an inertial frame by the usual procedure. The clock system in CLA frame F can be realized as follows: Suppose a computer clock can accept information concerning its position in the F_I frame, obtain w_I from the nearest F_I clock, and then compute and display w [or w*] using the inverse transformations (21.11) [or (21.12)]. See J. P. Hsu in ref. 2. The problem of physical time for the CLA frame is still open. It is reasonable to assume that the physical time for the CLA frame should be able to describe the lifetime dilatation of particles decay in flight (and the Doppler effect of a moving source) with constant-linear-acceleration. This can be determined by experiments in the future.
5. D. T. Schmitt and J. P. Hsu, Intl. J. Mod. Phys. A **20**, 5989 (2005).
6. If one uses another time variable w* in (6) for the accelerated frame F, then the speed of light slows down to zero at the singular wall.

22.

Dynamics of Classical and Quantum Particles in Constant-Linear-Acceleration Frames

22a. Classical electrodynamics in constant-linear-acceleration frames

Within the four-dimensional symmetry framework of taiji relativity with the usual basic units (length, time, mass), quantum electrodynamics has only two universal and fundamental constants, $J = 3.5177293\times10^{-38}$g·cm and $\bar{e} = -1.6021891\times10^{-20}(4\pi)^{1/2}(\text{g}\cdot\text{cm})^{1/2}$, instead of the usual three, c, $\hbar$, e (in the electrostatic unit, esu).[1] However, based on physical considerations, we have also argued in chapter 12 that from the most fundamental point of view, in which there is only one unit of measurement, quantum electrodynamics has only one truly universal and fundamental constant $\bar{e} = e = \sqrt{4\pi}/\sqrt{137.036}$. In this chapter, we shall show that these results also hold in the present formalism of physics in accelerated frames, in which the speed of light is not a constant.

Let us assume that accelerated frames satisfy the Wu transformations in (19.14). Since the speed of light in an accelerated frame F is no longer a universal constant, we assume that the invariant action for a charged particle moving in the electromagnetic potential $a_\mu(x)$ can be written as[2]

$$S = \int (-mds - \bar{e}a_\mu dx^\mu) - \frac{1}{4}\int f_{\mu\nu}f^{\mu\nu}Wd^4x$$

$$= \int Ldw - \frac{1}{4}\int f_{\mu\nu}f^{\mu\nu}Wd^4x, \tag{22.1}$$

$$ds^2 = W^2dw^2 - dx^2 - dy^2 - dz^2, \qquad \sqrt{-\det g_{\mu\nu}} = W = \gamma^2(\gamma_o^{-2} + \alpha_o x), \tag{22.2}$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \quad dx^\mu = (dw, dr), \qquad g_{\mu\nu} \equiv P_{\mu\nu}, \tag{22.3}$$

$$dx_\mu = g_{\mu\nu}dx^\nu = (W^2 dw, -d\mathbf{r}), \qquad g_{\mu\nu} = (W^2, -1, -1, -1),$$

in a reference frame F that moves with a constant-linear-acceleration (CLA). The differential of the coordinate $dx^\mu = (dw, d\mathbf{r})$ is, by definition, a contravariant vector. The covariant coordinate dx_μ is given by (22.3). Note that the invariant action S for a CLA frame does not involve the constant c and that $Wd^4x = \sqrt{-\det g_{\mu\nu}}$ $\times dwdxdydz$ is the invariant volume element in taiji spacetime. We use taiji-time w as the evolution variable, so that the Lagrangian L, defined in (22.1), takes the form,

$$L = -m\sqrt{W^2 - \beta_x^2 - \beta_y^2 - \beta_z^2} - \bar{e}(a_0 + a_i\beta^i), \quad i = 1,2,3, \tag{22.4}$$

$$\boldsymbol{\beta} = d\mathbf{r}/dw = (\beta^1, \beta^2, \beta^3) \equiv (\beta_x, \beta_y, \beta_z).$$

Note that $\boldsymbol{\beta}$ in (22.4) is the 'velocity' of the particle measured in terms of taiji time w. In the limit $\alpha_o \to 0$, F is an inertial frame and, hence $\bar{e}$ and a_μ correspond to the charge e(in esu) and the usual electromagnetic potential $A_\mu(ct, \mathbf{r})$ by $\bar{e} = e/c$ and $a_\mu(w, \mathbf{r}) \Leftrightarrow A_\mu(ct, \mathbf{r})/c$ respectively. The canonical momentum P_i of a particle in the CLA frame F is defined by

$$P_i = -\frac{\partial L}{\partial \beta^i} = p_i + \bar{e}a_i, \quad P_i = -P^i, \quad i = 1,2,3; \tag{22.5}$$

$$p_i = \left(\frac{-m\Gamma\beta_x}{W}, \frac{-m\Gamma\beta_y}{W}, \frac{-m\Gamma\beta_z}{W}\right) = g_{ik}p^k, \qquad \beta_x \equiv \beta^1 = -\beta_1, \text{ etc.,} \tag{22.6}$$

$$\Gamma = \frac{1}{\sqrt{1 - \beta^2/W^2}}, \qquad \beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 = -\beta_i\beta^i, \quad \beta^i = \frac{dx^i}{dw}. \tag{22.7}$$

The "Hamiltonian" $H = P_0$, which has the same dimension as that of P_i, is defined by

$$P_0 = \left(\frac{\partial L}{\partial \beta^i}\beta^i - L\right) = p_0 + \bar{e}a_0 = \sqrt{(P_i - \bar{e}a_i)^2 + m^2} + \bar{e}a_0 \equiv H\,; \tag{22.8}$$

$$p_0 = m\Gamma W = g_{00}p^0, \qquad p^0 = m\frac{dx^0}{ds} = m\frac{dw}{ds}. \tag{22.9}$$

Note that the contravariant momentum p^μ and the covariant momentum p_μ are related by $p_\mu = g_{\mu\nu}p^\mu$, i.e., $p_\mu = (p_0, p_1, p_2, p_3) = (W^2p^0, -p^1, -p^2, -p^3) \equiv (W^2p^0, -\mathbf{p})$ where the function W is given in (22.2).

We observe that the covariant momentum $p_\mu = (p_0, p_1, p_2, p_3)$ in (22.6) and (22.9) can be written as $p_\mu = m dx_\mu/ds = m g_{\mu\nu}dx^\nu/ds$. Since m and ds are invariant, p_μ should transform as $g_{\mu\nu}dx^\nu$. From the transformations of dx^ν in (19.16) and $g_{\mu\nu}$ given in (19.18), we obtain the transformation of the covariant momentum

$$p_{I0} = \gamma\left(\frac{p_0}{W} - \beta p_1\right), \qquad p_{I1} = \gamma\left(p_1 - \beta\frac{p_0}{W}\right), \qquad p_{I2} = p_2, \quad p_{I3} = p_3; \tag{22.10}$$

$$\beta = \alpha_o W + \beta_o = \frac{(w_I + \beta_o/\alpha_o\gamma_o)}{(x_I + 1/\alpha_o\gamma_o)} = \frac{\sqrt{(x_I + 1/\gamma_o\alpha_o)^2 - (x + 1/\alpha_o\gamma_o^2)^2}}{(x_I + 1/\alpha_o\gamma_o)},$$

where we have used $dx_\mu = g_{\mu\nu}dx^\nu$, $dx_{I\mu} = \eta_{\mu\nu}dx_I^\nu$, $\eta_{\mu\nu} = (1,-1,-1,-1)$ and (19.15). Note that β in (22.10) is the velocity of the CLA frame F and differs from $\boldsymbol{\beta}$ in (22.4)-(22.9), where $\boldsymbol{\beta} = (\beta^1, \beta^2, \beta^3) \equiv (\beta_x, \beta_y, \beta_z)$ is the velocity of a particle.

This transformation (22.10) allows us to see how the particle's energy p_{I0} increases as a function of distance it travels in an inertial laboratory. Suppose the particle is in the CLA frame at $x^i = (x_o, 0, 0) = \text{constant}$, so that $dx^i/dw = 0$, $\Gamma = 1$ and $p_0 = mW$ in (22.7) and (22.9). We have $p_{I0} = m\gamma = m/\sqrt{1-\beta^2}$, which leads to

$$\frac{dp_{I0}}{dx_I} = \frac{m\alpha_o}{(\gamma_o^2 + \alpha_o x_o)} = \text{constant}, \tag{22.11}$$

where we have used β in (22.10) with $x=x_o$=constant. This result gives the operational meaning to the concept of constant acceleration in an inertial laboratory such as the Stanford Linear Accelerator Center. To be more specific, if $x_O=0$ and the initial velocity $\beta_0=0$, the constant acceleration α_o of a charged particle with mass m can be obtained by measuring $(1/m)(dp_{I0}/dx_I)$, where (dp_{I0}/dx_I) is related to the potential gradient of the accelerator.

The Lagrange equation of motion of a charged particle can be derived from the invariant action (22.1). We obtain

$$m\frac{Du_\mu}{ds} = \bar{e}f_{\mu\nu}u^\nu, \qquad \bar{e}=e=\sqrt{4\pi}/\sqrt{137.036}. \tag{22.12}$$

$$u^\nu = \frac{dx^\nu}{ds}, \qquad Du_\mu = u_{\mu;\nu}dx^\nu \equiv D_\nu u_\mu dx^\nu,$$

$$D_\nu u_\mu = \frac{\partial u_\mu}{\partial x^\nu} - \Gamma^\rho{}_{\mu\nu}u_\rho \equiv \partial_\nu u_\mu - \Gamma^\rho{}_{\mu\nu}u_\rho, \qquad D_\nu u^\mu = \partial_\nu u^\mu + \Gamma^\mu{}_{\nu\rho}u^\rho,$$

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) = \Gamma^\rho{}_{\nu\mu},$$

where ;ν or D_ν denotes the partial covariant differentiation[3] with respect to x^ν. The Christoffel symbols $\Gamma^\rho{}_{\mu\nu}$ can be calculated with the metric tensor $g_{\mu\nu}$ given by (22.3) for the space of a CLA frame with the Wu transformation. On the other hand, if one works in the space of a frame with the generalized MWL transformation, one uses the metric tensor in (19.8), i.e., $g^*{}_{\mu\nu}=(W_x{}^2,-1,-1,-1)$ where $W_x=1+\gamma_o\alpha^*x$, to calculate the Christoffel symbols and so on.

For a continuous charge distribution in space, the invariant action for the electromagnetic fields and their interaction is assumed to be

$$S_{em} = -\int a_\mu j^\mu W d^4x - \frac{1}{4}\int f_{\mu\nu}f^{\mu\nu}W d^4x, \tag{22.13}$$

which leads to the following Maxwell's equations

$$f_{\mu\nu} = D_\mu a_\nu - D_\nu a_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu ,$$

$$D_\alpha f_{\mu\nu} = \partial_\alpha f_{\mu\nu} - \Gamma^\rho{}_{\mu\alpha} f_{\rho\nu} - \Gamma^\rho{}_{\nu\alpha} f_{\mu\rho}. \qquad (22.14)$$

These Maxwell's equations are invariant under the Wu transformations for CLA frames. Note that the coordinates in the Wu transformations (19.14) for a CLA frame F(w,x,y,z) is not arbitrary, in contrast to the coordinate transformations in general relativity. Rather, they are a particular set of coordinates with a metric tensor given by (19.18) and with a vanishing Riemann-Christoffel curvature tensor. Since the Wu transformations is smoothly connected to the Lorentz transformations in the limit of zero acceleration, the coordinates of a CLA frame have the same physical significance as that of an inertial frame.

22b. Quantum particles and Dirac's equation in a CLA frame

Equation (22.10) implies the invariant relation $g^{\mu\nu}p_\mu p_\nu = m^2$, which can be written in the form $g^{\mu\nu}(P_\mu - \bar{e}a_\mu)(P_\nu - \bar{e}a_\nu) = m^2$ by using equations (22.5) through (22.9) and the relation $P^\mu = g^{\mu\nu}P_\nu$ for CLA frames. This equation for the canonical momentum P_μ of a classical charged particle suggests that the generalized Klein-Gordon equation for a quantum charged particle in a CLA frame should have the form

$$\left[g^{\mu\nu}(iD_\mu - \bar{e}a_\mu)(iD_\nu - \bar{e}a_\nu) - m^2\right]\Phi = 0, \qquad g^{\mu\nu} = (W^{-2}, -1, -1, -1),$$

i.e., $$\Big[W^{-2}(iD_0 - \bar{e}a_0)^2 - (iD_1 - \bar{e}a_1)^2 - (iD_2 - \bar{e}a_2)^2$$

$$-(iD_3 - \bar{e}a_3)^2 - m^2\Big]\Phi = 0. \qquad (J \equiv 1) \qquad (22.15)$$

Note that the canonical momentum P_μ/i must be replaced by covariant derivatives D_μ as defined in the Riemannian geometry with a metric tensor $g_{\mu\nu}$

given in (22.3). Similarly, the generalized Dirac equation for a CLA frame F should have the form

$$\left[\Gamma^{\mu}(x)(P_{\mu} - \bar{e}a_{\mu}) - m\right]\psi = 0, \tag{22.16}$$

or $\quad [W^{-1}\gamma^0(P_0 - \bar{e}a_0) + \gamma^1(P_1 - \bar{e}a_1) + \gamma^2(P_2 - \bar{e}a_2) + \gamma^3(P_3 - \bar{e}a_3) - m]\psi = 0,$

where

$$P_{\mu} \equiv (P_0, -\mathbf{P}), \quad \psi = \psi(w,x,y,z); \qquad \Gamma^{\mu}(x) = (W^{-1}\gamma^0, \gamma^1, \gamma^2, \gamma^3), \tag{22.17}$$

$$\{\gamma_D^a, \gamma_D^b\} = \eta^{ab}, \qquad \eta^{ab} = (1,-1,-1,-1), \qquad a,b = 0,1,2,3.$$

The explicit form of the operator P_{μ} for the Dirac spinor turns out to be slightly different from that for a scalar function. (See equations (22.19) below.) Evidently, the generalized Dirac equation (22.16) reduces to the usual Dirac equation in the limit of zero acceleration, $\alpha_o \to 0$ or $W \to 1$. If one wishes, one can relate $\Gamma^{\mu}(x)$ in (22.16) to the constant Dirac matrices γ^a, $a = 0,1,2,3$, using the relationship $\Gamma^{\mu}(x) = e_a^{\mu}(x)\gamma^a$, where $e_a^{\mu}(x)$ is a tetrad (i.e., a unit tangent vector) and satisfies the relations

$$\sum_{a=0}^{3} e_a^{\mu}(x)e_a^{\nu}(x) = g^{\mu\nu} \qquad \text{and} \qquad \sum_{a=0}^{3} e_{a\mu}(x)e_{a\nu}(x) = g_{\mu\nu}. \tag{22.18}$$

Note that the subscript a is of no significance to the covariance.

When $\bar{e} = 0$, we have the "free" Dirac equation in a CLA frame involving a "gauge covariant differentiation":

$$\left(i\Gamma^{\mu}\nabla_{\mu} - m\right)\psi = 0, \qquad \nabla_{\mu} = \left([\partial_0 + \frac{1}{2}(\partial_k W)\gamma_D^0\gamma_D^k], \ \partial_1, \ \partial_2, \ \partial_3\right), \tag{22.19}$$

which will be discussed in section 23c. In the absence of the electromagnetic potential $a_\mu(x)$, or when the potentials do not involve time explicitly, one can use the separation of variables in (22.19) to find the w-dependent part of ψ :

$$\psi = F_W(w)\psi_r(\mathbf{r}), \tag{22.20}$$

$$F_W = F_{WO} \exp\left\{ \frac{-i}{\gamma_o^2} \int_0^W \gamma^2(w')Mdw' \right\}$$

$$= F_{WO} \exp\left\{ \frac{-iM}{2\alpha_o\gamma_o^2} \ln\left|\frac{(1+\beta)(1-\beta_o)}{(1-\beta)(1+\beta_o)}\right| \right\}, \tag{22.21}$$

where M is a constant, $\beta = \alpha_o w + \beta_o$, and we have combined the space-dependent terms involving the same Dirac matrices together (e.g., $\gamma^k W^{-1}\partial_k W$ and $\gamma^k \partial_k$, k=1,2,3). Note that (22.21) reduces to the usual form $F_W = F_{WO}\exp(-iMw)$ in the limit of zero acceleration, $\alpha_o \to 0$.

22c. Stability of atomic levels against constant accelerations

A physical system is described by a Hamiltonian (e.g., P_0 in (22.16)) and the Hamiltonian in our formalism for accelerated frames has the dimension of mass. Thus, let us now consider the "mass levels" of a hydrogen atom at rest in a CLA frame F. First, we must determine the generalized Coulomb potential produced by a charged particle at rest in a CLA frame. Such a Coulomb potential can be obtained by solving the generalized Maxwell's equation $g^{\mu\nu}D_\mu f_{\nu\alpha} = j_\alpha$, (or $g^{\mu\nu}[D_\mu D_\nu a_\alpha - D_\mu D_\alpha a_\nu] = j_\alpha$) given by (22.14) for a CLA frame with the 4-potential $a_\mu = (a_0,0,0,0)$ and the current density $j_\mu = (\bar{e}\,\delta(\mathbf{r}),0,0,0)$ in a CLA frame.

Based on the covariant differentiations in (22.12) and (22.14), and the metric tensor $g_{\mu\nu}$ in (19.18) and that in (19.8), one can verify that $D_\mu D_\alpha A_\nu = D_\alpha D_\mu A_\nu$ for an arbitrary vector A_ν. (This is related to the fact that the Riemann-Christoffel curvature tensor $R^\nu{}_{\sigma,\mu\alpha}$ vanishes, $(D_\mu D_\alpha - D_\alpha D_\mu)A^\nu = -R^\nu{}_{\sigma,\mu\alpha}A^\sigma = 0$, for the spaces of CLA frames with both the metric tensors given in (19.18) and in (19.8).) As usual, we can choose a gauge condition $D^\nu a_\nu = g^{\nu\mu}D_\mu a_\nu = 0$ to simplify Maxwell's equations. Thus, we have $g^{\mu\nu}D_\mu D_\nu a_\mu = j_\alpha$. When $a_\mu(w,\mathbf{r})=(a_0,0,0,0)$, this equation leads to the following generalized Coulomb equations for CLA frames:

$$g^{\mu\nu}D_\mu D_\nu a_0 = g^{\mu\nu}\partial_\mu\partial_\nu a_0 - \frac{2\alpha_o\beta}{1-\beta^2}g^{00}\partial_0 a_0 + \frac{\alpha_o}{\gamma_o^2+\alpha_o x}\partial_1 a_0 = j_0$$

$$\text{for} \quad g^{\mu\nu}=(\gamma^4(\gamma_0^{-2}+\alpha_o x)^2,-1,-1,-1), \tag{22.22}$$

$$g^{\mu\nu}D_\mu D_\nu a_0 = g^{\mu\nu}\partial_\mu\partial_\nu a_0 + \frac{\alpha^*}{1+\gamma_o\alpha^* x}\partial_1 a_0 = j_0 \,,$$

$$\text{for} \quad g^{\mu\nu}=((1+\gamma_0\alpha^* x)^2,-1,-1,-1).$$

In general, these equations give complicated potentials produced by a simple point charge. Let us consider the static case, $a_0 = a_0(\mathbf{r})$ and $j_0 = \bar{e}\,\delta(\mathbf{r})$, with the first order approximation, $\alpha_o/(\gamma_0^{-2}+\alpha_o x)\approx\alpha_o$. Under these conditions, both equations in (22.22) lead to the differential equation and the solution for the generalized Coulomb potential in CLA frame:

$$\left(\nabla^2 - \alpha_o\frac{\partial}{\partial x}\right)a_0 = -\bar{e}\,\delta(\mathbf{r}),$$

$$a_0 = \frac{\bar{e}}{4\pi r}\left(1+\frac{\alpha_o x}{2}\right). \tag{22.23}$$

As expected, they become the usual equation and Coulomb potential in the limit of zero acceleration, $\alpha_o \to 0$.

The generalized Dirac equation for the electron in an accelerated hydrogen atom (i.e., at rest in a CLA frame) is given by (22.19) with Γ^μ given by (22.17) with the usual replacement, $i\nabla_\mu \to (i\nabla_\mu - \bar{e}a_\mu)$ [for $\bar{e}<0$, J=1]. We have

$$\left\{ \frac{i}{W}\frac{\partial}{\partial w} + \frac{i}{2W}\frac{\partial W}{\partial w}\alpha_{Dx} - \frac{1}{W}\frac{\bar{e}^2}{4\pi r}\left(1+\frac{\alpha_o x}{2}\right)\right.$$

$$\left. +\alpha_{Dx} i\frac{\partial}{\partial x} + \alpha_{Dy} i\frac{\partial}{\partial y} + \alpha_{Dz} i\frac{\partial}{\partial z} - \beta m \right\}\psi = 0, \qquad (22.24)$$

where $\boldsymbol{\alpha}_D = (\alpha_{Dx}, \alpha_{Dy}, \alpha_{Dz}) = (\beta_D\gamma^1, \beta_D\gamma^2, \beta_D\gamma^3)$ and $\beta_D \equiv \gamma^0$ are the Dirac matrices.

If one uses the W given by (22.2), i.e., $W = \gamma^2(\gamma_o^{-2} + \alpha_o x)$ associated with the Wu transformation, the effective potential $\bar{e}a_0/W = -(\bar{e}^2/4\pi r)(1+\alpha_o x/2)/W$ becomes time-dependent due to acceleration of the whole atom because γ involves w. In this case, the solution is more complicated due to the w-dependent W. On the other hand, suppose one uses the time-independent W given by (19.8), i.e., $W = W_x = (1+\gamma_o\alpha^* x)$ which is related to the generalized MWL transformation. The effective Coulomb potential $\bar{e}a_0/W_x$ is then also time-independent and one has a stable atom, just as in an inertial frame.

In most cases, the generalized Dirac equation in (22.24) is complicated and does not have the usual spherical symmetry because of the presence of the metric tensor $g_{\mu\nu}$ and the generalized Coulomb potential given by (22.23). However, the violation of spherical symmetry turns out to be extremely small and the Dirac equation for the hydrogen atom can be well approximated by the usual form in an inertial frame. The reason is that we are interested only in the atomic domain, $r{\sim}10^{-8}$cm. We have at present linear accelerators in an inertial frame F_I (laboratory) with a maximum voltage gradient of about 70 MeV per meter. We estimate the acceleration of a hydrogen atom to be

$$\alpha^* \sim 0.05/m \quad \text{for} \quad \beta_o \sim 0.1\,. \qquad (22.25)$$

Thus, the extra x-dependent part in $(1+\gamma_o\alpha^* x)$ and the generalized Coulomb potential given by (22.23) is much smaller than 1,

$$\alpha^* x \sim 10^{-11}, \tag{22.26}$$

where x is roughly the size of the atom, $x \sim 10^{-8}$cm. Therefore, the violation of spherical symmetry in the generalized Dirac equation is negligible for all practical purposes. As a result, the generalized Dirac equation in (22.24) can be well approximated by the usual Dirac equation, so that the mass levels of an accelerated hydrogen atom are given by[1]

$$M(n) = \frac{m}{\sqrt{1+\dfrac{\alpha_e^2}{\left(n-h_d+\sqrt{h_d^2-\alpha_e^2}\right)^2}}}, \qquad h_d = j+\frac{1}{2}, \tag{22.27}$$

where $\alpha_e = \bar{e}^{\,2}/(4\pi)$ ($\sim 1/137$) in natural units. Thus, the atomic mass level structure is relatively insensitive to linear accelerations. When an electron jumps from a state n_1 to another state n_2, it will emit or absorb a 'mass quantum' k_o (with J=1):

$$M_I(n_2) - M_I(n_1) = k_{Io}, \qquad \text{in } F_I, \tag{22.28}$$

$$M(n_2) - M(n_1) = k_o, \qquad \text{in } F. \tag{22.29}$$

If two photons with "moving masses," k_{Io} and k_o, are emitted from two hydrogen atoms at rest in F_I and F respectively and measured immediately, then the results (22.27) through (22.29) imply

$$k_o(\text{rest}) = k_{Io}(\text{rest}), \tag{22.30}$$

where the "(rest)" associated with k_o and k_{Io} refers to the state of motion of the source of photons. The result in (22.30) is isotropic when the photon is emitted and immediately measured in F. On the other hand, if the measurement is delayed, then the acceleration of the F frame will cause a Doppler-type shift.

22d. Electromagnetic fields produced by a charge with a constant-linear-acceleration

In 1909, Max Born first discussed the motion of a charge with a constant linear acceleration based on special relativity.[4] In the case where the motion of the charge is along the x_I-axis of an inertial frame $F_I(w_I,x_I,y_I,z_I)$, Born obtained the hyperbolic motion (18.4), i.e.,

$$x_I = \sqrt{(c^2/g)^2 + (ct_I)^2} - \frac{c^2}{g}, \qquad v_I = \frac{c^2 t_I}{\sqrt{(c^2/g)^2 + (ct_I)^2}}. \tag{22.31}$$

This motion can be viewed as the motion of a particle under a constant external force, $\mathbf{F}_I^{ext}$=const., along the x-axis,

$$\frac{d}{dt_I}\mathbf{p}_I = \mathbf{F}_I^{ext}, \qquad \mathbf{p}_I = \frac{m\mathbf{v}_I}{\sqrt{1 - v_I^2/c^2}}, \qquad \mathbf{v}_I = (v_I, 0, 0)\,, \tag{22.32}$$

in special relativity. Born also obtained the electromagnetic fields associated with this type of motion of a charged particle.

Let us compare the motion described by (22.31) with the motion implied by the Wu transformation (19.14) for a CLA frame F(w,x,y,z). Suppose a particle is at rest in the CLA frame F at $x=x_o$ and $y=z=0$. Its position $(x_I,0,0)$ and velocity dx_I/dw_I can be expressed in terms of the time w_I in the inertial laboratory frame $F_I(w_I,x_I,y_I,z_I)$,

$$x_I = \sqrt{\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2 + \left(w_I + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2} - \frac{1}{\alpha_o\gamma_o}, \tag{22.33}$$

$$\frac{dx_I}{dw_I} = \alpha_o w + \beta_o = \frac{w_I + \dfrac{\beta_o}{\alpha_o\gamma_o}}{\sqrt{\left(x_o + \dfrac{1}{\alpha_o\gamma_o^2}\right)^2 + \left(w_I + \dfrac{\beta_o}{\alpha_o\gamma_o}\right)^2}}, \qquad \text{for} \quad x = x_o, \tag{22.34}$$

where we have used the Wu transformations (19.14) and (19.15). We observe that there are non-trivial dependencies on the initial velocity β_o and position x_o in the CLA motion in (22.33) and (22.34). The initial position $x_I(0)$ at $w_I = 0$ can be obtained from (22.33),

$$x_I(0) = \sqrt{\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2 + \left(\frac{\beta_o}{\alpha_o\gamma_o}\right)^2} - \frac{1}{\alpha_o\gamma_o}, \qquad w_I = 0, \tag{22.35}$$

$$= x_o \qquad \text{for} \quad \beta_o = 0, \qquad \gamma_o = 1.$$

From (22.31), (22.33) and (22.34), we see that (22.31) corresponds to the special case of (22.33) and (22.34) when both the initial position and the initial velocity of the F frame are zero, $x_o = 0$ and $\beta_o = 0$. However, an important difference is that the acceleration corresponding to the hyperbolic motion described by the equations in (22.31) is $d^2x_I/dt_I^2 = (c^6/g^2)/[(c^2/g)^2+(ct_I)^2]^{3/2}$, so that in the limit of zero acceleration $g \to 0$ one has the result $x_I \to 0$. It does not reduce to a constant motion with non-zero velocity. Note that the constant acceleration of a particle in special relativity is defined in its instantaneous rest frame $v_I = 0$, i.e., $t_I = 0$ from (22.31) and hence, $d^2x_I/dt_I^2 = g =$ constant. On the contrary, the limit of zero acceleration of the motion implied by the Wu transformation has a well-defined constant motion: $x_I = (x_o+\gamma_o\beta_o w_I)/\gamma_o$ as $\alpha_o \to 0$. In other words, in the limit of zero acceleration, we obtain exactly the Lorentz transformation with the constant velocity β_o. The reason for this difference is that the CLA motion in (22.33) and (22.34) satisfies the limiting 4-dimensional symmetry, while the hyperbolic motion (22.31) in general does not.

In the CLA motion, the acceleration of this particle can be derived from (22.34)

$$\frac{d^2x_I}{dw_I^2} = \frac{\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2}{\left[\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2 + \left(w_I + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2\right]^{3/2}}, \quad \text{for} \quad x = x_o. \tag{22.36}$$

Thus, the Wu transformation implies that the acceleration of this particle depends on its position x_o and the initial velocity β_o of the CLA frame. The relativistic equation of motion (22.32) cannot give this detailed information. As a result, if one uses (22.31) or (22.32) to derive transformations for reference frames with constant linear acceleration, one obtains the 4-dimensional group of conformal transformations in spacetime,[5] which turns out not to be physically meaningful and does not satisfy the limiting 4-dimensional symmetry.

Now, however, we have the much more satisfactory Wu transformation which does satisfy the limiting 4-dimensional symmetry and also provides more detailed information concerning accelerations. We now apply it to the investigation of the electromagnetic fields produced by an accelerated charge, classical radiation and energy conservation.

First, let us consider the electromagnetic fields produced by a charge $\bar{e}$, whose motion is described by (22.33) and (22.34). According to Maxwell's equations, the retarded 4-potentials $a_{I\mu}(f)$ at the field point f are the covariant Liénard-Wiechert potentials[6] as observed in an inertial frame F_I

$$a_{I\mu}(f) = \frac{\bar{e}u_\mu(s)}{4\pi(x_I^f - x_I^s)^\lambda u_\lambda(s)}, \tag{22.37}$$

$$f \equiv (w_I^f, x_I^f, y_I^f, z_I^f), \qquad s \equiv (w_I^s, x_I^s, y_I^s, z_I^s),$$

$$u_\mu(s) = \eta_{\mu\nu}u^\nu(s), \qquad \eta_{\mu\nu} = (1,-1,-1,-1),$$

where s denotes the source point and $u^\nu(s)$ the 4-velocity of the charged particle (the source)

$$u^\nu(s) = \left(\frac{dw_I^s}{ds}, \frac{dx_I^s}{ds}, 0, 0\right),$$

$$\frac{dw_I^s}{ds} = \frac{1}{\sqrt{1-\beta^2}} = \frac{\sqrt{\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2 + \left(w_I^s + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2}}{x_o + \frac{1}{\alpha_o\gamma_o^2}}, \tag{22.38}$$

$$\frac{dx_I^s}{ds} = \frac{dx_I^s/dw_I^s}{\sqrt{1-\beta^2}} = \frac{w_I^s + \frac{\beta_o}{\alpha_o\gamma_o}}{x_o + \frac{1}{\alpha_o\gamma_o^2}},$$

where we have used (22.33) and (22.34). Note that the charged particle is assumed to be at rest at the position $(x_o,0,0)$ in the CLA frame F(w,x,y,z), i.e., $\mathbf{r}^s = (x_I^s,0,0)$ in the inertial frame F_I.

It is important to note that the field time w_I^f is related to the source (or emission) time w_I^s by the relation

$$w_I^f - w_I^s = |\mathbf{r}_I^f - \mathbf{r}_I^s| > 0. \tag{22.39}$$

This is the causality condition that specifies the amount by which the Liénard-Wiechert potential (22.37) must be retarded in order to be consistent with experiments and observations.

To find the potential $a_{I\mu}(f)$ at the field point $f \equiv (w_I^f, x_I^f, y_I^f, z_I^f)$, we must express the source coordinate $(w_I^s, x_I^s, y_I^s, z_I^s)$ in terms of the field point f. This can be done because the square of the causality condition (22.39) enables us to solve for w_I^s in terms of $(w_I^f, x_I^f, y_I^f, z_I^f)$,

$$(w_I^f - w_I^s)^2 = (\mathbf{r}_I^f - \mathbf{r}_I^s)^2$$

$$= \left[x_I^f - \sqrt{\left(x_o + \frac{1}{\alpha_o \gamma_o^2}\right)^2 + \left(w_I^s + \frac{\beta_o}{\alpha_o \gamma_o}\right)^2} + \frac{1}{\alpha_o \gamma_o} \right]^2 + \rho^2, \qquad (22.40)$$

$$\rho^2 = (x_I^f - y_I^s)^2 + (z_I^f - z_I^s)^2 = (y_I^f)^2 + (z_I^f)^2.$$

For calculations, it is convenient to set

$$w_f = w_I^f + \frac{\beta_o}{\alpha_o \gamma_o}, \qquad w_s = w_I^s + \frac{\beta_o}{\alpha_o \gamma_o}, \qquad x_f = x_I^f + \frac{1}{\alpha_o \gamma_o}, \qquad k = x_o + \frac{1}{\alpha_o \gamma_o^2}.$$

It is then straightforward to obtain

$$w_I^s + \frac{\beta_o}{\alpha_o \gamma_o} = \frac{\left(w_I^f + \frac{\beta_o}{\alpha_o \gamma_o}\right) A^* - \left(x_I^f + \frac{1}{\alpha_o \gamma_o}\right) B^*}{2\left[\left(x_I^f + \frac{1}{\alpha_o \gamma_o}\right)^2 - \left(w_I^f + \frac{\beta_o}{\alpha_o \gamma_o}\right)^2\right]}, \qquad (22.41)$$

$$A^* = \left(x_I^f + \frac{1}{\alpha_o \gamma_o}\right)^2 + \rho^2 + k^2 - \left(w_I^f + \frac{\beta_o}{\alpha_o \gamma_o}\right)^2,$$

$$B^* = \sqrt{\left[\left(w_I^f + \frac{\beta_o}{\alpha_o \gamma_o}\right)^2 - \left(x_I^f + \frac{1}{\alpha_o \gamma_o}\right)^2 - \rho^2 + k^2\right]^2 + 4k^2\rho^2}.$$

Similarly, we can also express the coordinate x_I^s of the charged particle in terms of the field point f,

$$x_I^s + \frac{1}{\alpha_o\gamma_o} = \sqrt{k^2 + \left(w_I^s + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2} = \frac{\left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)A^* - \left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)B^*}{2\left[\left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)^2 - \left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2\right]}, \tag{22.42}$$

where we have used (22.33), (22.34) and (22.41). It follows from (22.38), (22.41) and (22.42) that

$$(x_I^f - x_I^s)^\mu u_\mu(s) = \frac{B^*}{2\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2}. \tag{22.43}$$

Thus, the retarded potentials $a_{I\mu}(f)$ in (22.37) produced by a charged particle at rest at a point $(x_o,0,0)$ in the CLA frame F are given by

$$a_{I0}(f) = \frac{\bar{e}}{4\pi} \frac{\left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)A^* - \left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)B^*}{B^*\left[\left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)^2 - \left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2\right]}, \tag{22.44}$$

$$a_{I1}(f) = -\,a_I{}^1(f) = -\frac{\bar{e}}{4\pi} \frac{\left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)A^* - \left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)B^*}{B^*\left[\left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)^2 - \left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2\right]}, \tag{22.45}$$

$$a_{I2}(f) = a_{I3}(f) = 0\,. \tag{22.46}$$

The electromagnetic field $\mathbf{E}(f)$ can be calculated from equations (22.44) through (22.46). We have

$$E_{I1}(f) = \frac{-\bar{e}}{\pi(B^*)^3}\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2\left[\left(x_o + \frac{1}{\alpha_o\gamma_o}\right)^2 - \left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)^2 + \left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2 + \rho^2\right],$$

$$E_{I2}(f) = \frac{2\bar{e}}{\pi(B^*)^3}\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2 y_I^f\left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right), \tag{22.47}$$

$$E3(f) = \frac{2\bar{e}}{\pi(B^*)^3}\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2 z_I^f\left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right).$$

Similarly, the magnetic induction $\mathbf{B}_I(f)$ is found to be

$$B_{I1}(f) = 0,$$

$$B_{I2}(f) = \frac{-\bar{e}}{\pi(B^*)^3}\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2 z_I^f\left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right), \tag{22.48}$$

$$B_{I3}(f) = \frac{\bar{e}}{\pi(B^*)^3}\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2 y_I^f\left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right).$$

In these calculations, we have used

$$\mathbf{E}_I(f) = \left[-\frac{\partial \mathbf{a}_I}{\partial w_I} - \frac{\partial a_{I0}}{\partial \mathbf{r}_I}\right]_f, \qquad \mathbf{B}_I(f) = \left[\nabla \times \mathbf{a}_I\right]_f, \tag{22.49}$$

where $[\ ]_f$ denotes that the quantities within the brackets [ ] refer to the field point f. The function B^* is given by (22.41) and the coordinates of the charged particle in F_I are $(x_I^s, y_I^s, z_I^s) = (x_I^s, 0, 0)$. One can verify that Born's results correspond to the results (22.44)-(22.48) in the special case $x_o = 0$ and $\beta_o = 0$. Although these expressions involve $1/\alpha_o$, they do not diverge in the limit of zero acceleration $\alpha_o \to 0$. As a matter of fact, in the limit $\alpha_o \to 0$, the retarded potentials in (22.44)-(22.46) reduce to the usual ones found in standard textbooks.[6] Moreover, the electromagnetic fields in equations (22.47)-(22.48) vanish at the "black wall" $x_o = -1/\alpha_o\gamma_o^2$, the singularity wall of the CLA frame.

The causality condition (22.39) can be expressed in terms of the field point $f \equiv (w_I^f, x_I^f, y_I^f, z_I^f)$,

$$w_I^f - w_I^s = \frac{1}{2\left[\left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)^2 - \left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2\right]}$$

$$\times\left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)B^*\left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)\left[\left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2 - \left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)^2 + \rho^2 + k^2\right] > 0.$$

$$= \frac{x_f B^* w_f}{2\left[x_f^2 - w_f^2\right]}\left[\left(w_I^f + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2 - \left(x_I^f + \frac{1}{\alpha_o\gamma_o}\right)^2 + \rho^2 + k^2\right] > 0. \tag{22.50}$$

When $\alpha_o \to 0$, it reduces to

$$w_I^f - w_I^s = \gamma_o^2\left[\left(x_I^f - \frac{x_o}{\gamma_o}\right)\beta_o - \beta_o^2 w_I^f\right] + \gamma_o^2\sqrt{\left(x_I^f - \frac{x_o}{\gamma_o} - \beta_o w_I^f\right)^2 + \rho^2} > 0. \tag{22.51}$$

Thus, the restriction on $(w_I^f, x_I^f, y_I^f, z_I^f)$ to satisfy the causality condition is not simple even in the limit $\alpha_o \to 0$.

According to classical electrodynamics, an accelerated charge emits electromagnetic radiation. The radiation rate for a charge with arbitrary motion can be calculated using the Liénard-Wiechert potentials (22.37):

$$R_{rad} = \frac{2}{3}\bar{e}^2 \frac{du_\mu(s)}{ds}\frac{du^\mu(s)}{ds}. \tag{22.52}$$

This holds generally for any source point s with the retarded velocity 4-vector $u^\mu(s) = (dw_I^s/ds, dx_I^s/ds, 0, 0)$ of any source point s. For the CLA motion of a charge, the acceleration 4-vector $du^\mu(s)/ds$ can be obtained from (22.38). We have

$$\frac{du^0(s)}{ds} = \frac{w_I^s + \dfrac{\beta_o}{\alpha_o\gamma_o}}{\left(x_o + \dfrac{1}{\alpha_o\gamma_o^2}\right)^2}, \tag{22.53}$$

$$\frac{du^1(s)}{ds} = \frac{\sqrt{\left(x_o + \dfrac{1}{\alpha_o\gamma_o^2}\right)^2 + \left(w_I^s + \dfrac{\beta_o}{\alpha_o\gamma_o}\right)^2}}{\left(x_o + \dfrac{1}{\alpha_o\gamma_o^2}\right)^2}, \tag{22.54}$$

$$\frac{du^2(s)}{ds} = \frac{du^3(s)}{ds} = 0.$$

These lead to the relation

$$\frac{du_\mu(s)}{ds}\frac{du^\mu(s)}{ds} = -\frac{1}{\left[x_o + \dfrac{1}{\alpha_o\gamma_o}\right]^2}. \tag{22.55}$$

Thus, the radiation rate for a charge with a CLA motion is a constant

$$R_{rad} = -\frac{2\bar{e}^2}{3}\frac{1}{\left[x_o + \dfrac{1}{\alpha_o\gamma_o^2}\right]^2}. \tag{22.56}$$

The radiation rate R_{rad} has the dimension of $1/\text{length}^2$ because $\bar{e}$ is dimensionless in natural units, as given by (22.12), and x_o has the dimension of length.

We can thus estimate the radiation rate for an electron at a high energy laboratory such as the Stanford Linear Accelerator with w_I=ct and a potential gradient of 60 MeV per meter. Setting x_o=0, β_o=0 and using $\bar{e}$ in (22.12), the result (22.11) with (dp_{I0}/dx_I)=60MeV/c^2 per meter, and the electron mass $m \approx 0.5\text{MeV}/c^2 \approx 10^{-30}$ Kg, we obtain

$$R_{rad} \approx -\bar{e}^2 \alpha_o^2 \approx -10^{-40}\ \text{Kg/m} \approx -0.3\times 10^3/\text{m}^2, \tag{22.57}$$

which corresponds to about 10^{-23} watts in conventional units. This is extremely small compared to loss of energy by an electron moving in a circular path. For example, at the 10 GeV Cornell electron synchrotron with an orbital radius of 100 meters, the loss of electron energy per orbit is about 9 MeV,[6] which corresponds to about 10^{-2} MeV/m or 10^{-6} Watts. This is a significant loss of the electron energy. Thus, r-f power must be supplied to maintain electrons at a constant energy as they circulate. For an electron with CLA motion, the loss of energy is completely negligible at the presently available accelerations.

22e. Covariant radiative reaction force in special relativity and common relativity

A complete satisfactory treatment of the radiative reaction forces and their effects exists neither in classical electrodynamics nor in quantum electrodynamics. This is a profound difficulty which is intimately related to the conceptual framework of spacetime and elementary particles. Whenever a charged particle is accelerated by an external force, it emits radiation which carries away linear momentum and angular momentum. Thus, the emitted radiation influences the particle's subsequent motion.

However, the situation in special relativity is not satisfactory because the covariant radiative reaction force vanishes for a charged particle which moves with a constant linear acceleration and radiates. In contrast, common relativity does not have this problem and is more satisfactory.

In the following discussions in this section, all coordinates and momenta are those of the charged particle (i.e., the source of electromagnetic emission) in a general inertial frame. The superscript s for the source and the subscript I for quantities measured in an inertial frame will be suppressed in this section. Only the constant $(x_o,0,0)$ refers to the position of the particle in a CLA frame.

First, let us consider the covariant radiative reaction force in special relativity. The usual form of the equations of motion such as (12.4) and (22.12) is not completely satisfactory because they do not take the radiative reaction force into account. Once the radiative reaction force appears, the system becomes non-holonomic because such a force cannot be derived from a potential in the Lagrangian formalism. The Abraham-Lorentz model of an electron with a finite size has been studied in detail and leads to the Abraham radiative reaction 4-vector.[7] Dirac's formalism of Lorentz covariant classical electrodynamics[8] (c=1) also gives the same result

$$\frac{dp^\mu}{ds} = F^\mu_{ext} + F^\mu_{rad}, \tag{22.58}$$

$$F^\mu_{ext} = \bar{e}F^{\mu\nu}u_\nu \,, \tag{22.59}$$

$$F^\mu_{rad} = \frac{2\bar{e}^2}{3m}\left(\frac{d^2p^\mu}{ds^2} + \frac{p^\mu}{m^2}\frac{dp_\alpha}{ds}\frac{dp^\alpha}{ds}\right). \tag{22.60}$$

Let us consider the CLA motion described by (22.33) and (22.34), which includes hyperbolic motion as a special case. Since $p^\mu = mu^\mu$, differentiating eq. (22.38) leads to

$$\frac{1}{m}\frac{d^2p^\mu}{ds^2} = \left(\frac{\sqrt{\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2 + \left(w + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2}}{\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^3},\ \frac{w + \frac{\beta_o}{\alpha_o\gamma_o}}{\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^3},\ 0,\ 0\right)$$

$$= \frac{p^\mu}{m\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2} . \tag{22.61}$$

This second order derivative of momentum may be called the "jerk" 4-vector, since the time derivative of the acceleration is known as jerk in mechanics. Although this quantity "jerk" never appears in the fundamental laws of nature (i.e., elementary particle physics or quantum field theories), it is relevant to robotics design and in tracking systems for fast moving objects. From (22.54), (22.60), (22.61) and p^μ=mu^μ we have the following result based on special relativity (c=1):

$$F^\mu_{rad} = \frac{2\bar{e}^2}{3m}\left\{\frac{p^\mu}{\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2} + \frac{(-p^\mu)}{\left(x_o + \frac{1}{\alpha_o\gamma_o^2}\right)^2}\right\} = 0. \tag{22.62}$$

This implies that for CLA motions, the radiative reaction force vanishes, so that equation (22.58) turns out to be the same as the conventional equation $dp^\mu/ds = \bar{e} F^{\mu\nu}u_\nu$, even though the charge emits electromagnetic radiation. This is indeed a very strange result. In the literature, Pauli and von Laue said that charges with CLA motion (i.e., the hyperbolic motion) do not radiate.[9] If they were right, the result (22.62) would be satisfactory. However, their statement is incorrect.[5] Other physicists such as Langevin, Poincaré and Heitler have concluded that there is indeed radiation.[4] The result (22.60) for the radiative reaction force of Abraham and Dirac appears to be unique because any force must satisfy the relation $p^\mu F_\mu$=0 which follows logically from the basic 4-dimensional law $ds^2 = dx_\mu dx^\mu$, x^μ = (ct,x,y,z), and the definitions $p^\mu = mu^\mu$ and c = 1 in special relativity. Furthermore, the 4-dimensional symmetry of the constant linear acceleration requires that the jerk 4-vector be proportional to the velocity 4-vector, as shown in (22.61). So the situation appears to be quite hopeless because 4-dimensional symmetry of special relativity seems to be exceedingly restrictive.

Fortunately, common relativity offers a way around this difficulty. Within the framework of common relativity, one has the basic law $ds^2 = dx_\mu dx^\mu$ with $x^\mu = (bt_c,x,y,z)$, but the common time t_c is an invariant quantity, just like ds. Consequently, one also has $p^\mu p_\mu = m^2$, where $p^\mu = mdx^\mu/ds$. By differentiating $p^\mu p_\mu = m^2$ with respect to the common time t_c twice, one obtains $(dp^\mu/dt_c)p_\mu = 0$ and $(d^2p^\mu/dt_c^2)\, p_\mu+(dp^\mu/dt_c)(dp_\mu/dt_c) = 0$. These relations imply that the covariant radiative reaction force has the form

$$\frac{d^2p^\mu}{dt_c^2} + \frac{p^\mu}{m^2}\frac{dp_\alpha}{dt_c}\frac{dp^\alpha}{dt_c}. \tag{22.63}$$

Note that in the absence of the radiative reaction force F^μ_{rad} in (22.58), the equation of motion, $mdu^\mu/ds = \bar{e}\, F^{\mu\nu}dx_\nu/ds$, can be written in terms of invariant common time t_c, $dp^\mu/dt_c = \bar{e}\, F^{\mu\nu}dx_\nu/dt_c$. Thus, in the framework of common relativity, we can postulate the basic equation of motion for a charged particle emitting radiation to be

$$\frac{dp^\mu}{dt_c} = F^\mu_{ext} + F^\mu_{rad}, \tag{22.64}$$

$$F^\mu_{ext} = \bar{e} f^{\mu\nu}\frac{dx_\nu}{dt_c}, \tag{22.65}$$

$$F^\mu_{rad} = \frac{2\bar{e}^2}{3m}\frac{dt_c}{ds}\left(\frac{d^2p^\mu}{dt_c^2} + \frac{p^\mu}{m^2}\frac{dp_\alpha}{dt_c}\frac{dp^\alpha}{dt_c}\right). \tag{22.66}$$

The invariant factor dt_c/ds in (22.66) is necessary for the dimensions to be correct.

Let us now consider the conservation of "energy" p^0 in the general equation of motion (22.64). The zeroth components F^0_{ext} and F^0_{rad} in (22.64) are respectively the rate of work done by the external force (i.e., dW_{ext}/dt_c) and by the radiative reaction force, as measured using common time t_c. Since $d/dt_c = (ds/dt_c)(d/ds) = (C^2-v^2)^{1/2}(d/ds)$, where $C = d(bt_c)/dt_c$, we have

$$\frac{d^2p^\mu}{dt_c^2} + \frac{p^\mu}{m^2}\frac{dp_\alpha}{dt_c}\frac{dp^\alpha}{dt_c}$$

$$= \sqrt{C^2 - v^2}\left(\frac{d}{ds}\sqrt{C^2 - v^2}\right)\frac{dp^\mu}{ds} + \left(C^2 - v^2\right)\left(\frac{d^2p^\mu}{ds^2} + \frac{p^\mu}{m^2}\frac{dp_\alpha}{ds}\frac{dp^\alpha}{ds}\right)$$

$$= v^\alpha \frac{dv_\alpha}{ds}\frac{dp^\mu}{ds}, \qquad v^\mu = \frac{dx^\mu}{dt_c} = (C,\mathbf{v}), \tag{22.67}$$

where we have used (22.61). The result in (22.38) gives

$$v^\alpha = \frac{dx^\alpha}{dt_c} = \left(\frac{dw}{dt_c}, \frac{dx}{dt_c}, 0, 0\right)$$

$$= \left(C, \frac{\left(w + \dfrac{\beta_o}{\alpha_o\gamma_o}\right)C}{\sqrt{\left(x_o + \dfrac{1}{\alpha_o\gamma_o^2}\right)^2 + \left(w + \dfrac{\beta_o}{\alpha_o\gamma_o}\right)^2}}, 0, 0\right). \tag{22.68}$$

From equations (22.66) through (22.68), we obtain a non-zero radiative reaction force in common relativity

$$F^0_{rad} = \frac{2\bar{e}^2}{3m}\frac{dt_c}{ds} v^\alpha \frac{dv_\alpha}{ds}\frac{dp^\mu}{ds}$$

$$= -\frac{2\bar{e}^2}{3m} \frac{C\left(w + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2}{\left(x_0 + \frac{1}{\alpha_o\gamma_o^2}\right)^2 \sqrt{\left(x_0 + \frac{1}{\alpha_o\gamma_o^2}\right)^2 + \left(w + \frac{\beta_o}{\alpha_o\gamma_o}\right)^2}}, \tag{22.69}$$

where we have used $dC/dt_c = 0$ by choosing the inertial frame to be the one in which the speed of light is constant. This can be done because $v^\alpha(dv_\alpha/ds)$ is an invariant in common relativity. Note that F^0_{rad} never positive, as expected.

In general, the zeroth component of (22.64) can be written as the law of conservation of "energy" (with the dimension of mass) for an arbitrary motion,

$$\frac{dp^0}{dt_c} = \frac{dW_{ext}}{dt_c} - \frac{dW_{rad}}{dt_c}, \tag{22.70}$$

$$\frac{dW_{rad}}{dt_c} = -\frac{\gamma}{C}\frac{d}{dt_c}\left(\frac{2\bar{e}^2}{3}\frac{du^0}{dt_c}\right) - CR_{rad}, \qquad u^0 = \frac{p^0}{m}, \tag{22.71}$$

where R_{rad} is a scalar quantity given by (22.52). Its physical meaning is that the rate of change in the kinetic energy of a particle is equal to the rate of work done by the external force minus the rate of work done by the radiative reaction force. The work done by the radiative reaction force is always negative and involves two parts:

(i) One part is related to the rate of change in $(2\bar{e}^2/3)(du^0/dt_c)$ which may be called the "acceleration charge energy." The idea of "acceleration energy" has discussed by Schott.[4] This "acceleration charge energy" is independent of the sign of the charge and depends only on the rate of change of $u^0 = \gamma$, just like the energy p^0 of a particle.

(ii) The second part is related to the radiation rate R_{rad} and is never positive.

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9. W. Pauli, *Theory of Relativity* (Pergamon Press, New York, 1958) p.93. "For the electromagnetic fields produced in the frame in which the field point and the center of the hyperbola are simultaneous," Pauli obtained the result that "there is no formation of a wave zone nor any corresponding radiation." But he also said parenthetically that radiation does occur when two uniform, rectilinear motions are connected by a "portion" of hyperbolic motion.

23.

Quantizations of Scalar, Spinor and Electromagnetic Fields in Constant-Linear-Acceleration Frames

23a. Scalar fields in constant-linear-acceleration frames

In constant-linear-acceleration (CLA) frames, physical equations and calculations are generally more mathematically complicated than those in inertial frames. The reason for this is that the differential time-axis dw in CLA frames is altered by the Wu factor $W(w,x)=\gamma^2(\gamma_o^{-2}+\alpha_o x) > 0$, as shown in (22.2), so that the physical space is bounded by a "black wall" at $x=-1/\alpha_o\gamma_o^2$. This "black wall" is a wall-singularity where the speed of any physical signal or particle becomes infinite. Nevertheless, the CLA frame with a constant-linear-acceleration is the simplest non-inertial frame, and offers the best opportunity to explore physical properties of quantum fields beyond inertial frames or the framework of relativity theory.

The investigation of quantum field theory in CLA frames can shed light on more complicated non-inertial frames and perhaps on quantum gravity. Furthermore, when considering the S-matrix and the Feynman rules in quantum field theory, we must use a plane wave (for a free particle) which is very simple in inertial frames, but complicated in CLA frames. As a result, many calculations in CLA frames are most easily carried out by a change of variables. Such a change of variables effectively transforms calculations from a CLA frame to an inertial frame. Of course, there are non-trivial problems which must be solved in CLA frames, such as finding the Coulomb potential generated by a charged particle at rest in a CLA frame and solving for energy states in such a potential, as discussed in chapter 22.

Because we have specific coordinate transformations between inertial frames and CLA frames, we can explore their implications for possible new physics of particles and fields in non-inertial frames. The equation of motion of a classical particle in CLA frames has been discussed in chapter 22. Even the simplest motion, e.g., constant velocity in an inertial frame, will have more complicated properties from the viewpoint of a CLA frame, as shown in the momentum transformations in (22.10).

A similar complication occurs in field theory: The Klein-Gordon equation for a "free" (i.e., non-interacting) particle in a CLA frame can be obtained from (22.15) by setting $\bar{e}=0$:

$$\left[g^{\mu\nu}D_\mu D_\nu + m^2\right]\Phi(w,\mathbf{x}) = 0, \tag{23.1}$$

where $\mathbf{x}$ denotes the position vector and D_μ is the partial covariant derivative as defined in (22.12) with the fundamental metric tensor given by (19.8) (associated with the generalized MWL transformations),

$$g^*_{\mu\nu} = (W_x^2, -1, -1, -1) = ([1+\gamma_o\alpha^* x]^2, -1, -1, -1), \tag{23.2}$$

or given by (19.18) (associated with the Wu transformations)

$$g_{\mu\nu} = (W^2, -1, -1, -1) = (\gamma^4[1+\alpha_o x]^2, -1, -1, -1). \tag{23.3}$$

In the following discussion, the form of physical equations (or laws) involving W hold both for metric tensors in (23.2) and in (23.3). In other words, the function W stands for either $[1 + \gamma_o\alpha^* x] > 0$ or $\gamma^2[\gamma_o^{-2} + \alpha_o x] > 0$.

Using the covariant expression for d'Alembert's operator $g^{\mu\nu}D_\mu D_\nu$ we find that

$$g^{\mu\nu}D_\mu D_\nu\Phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\ g^{\mu\nu}\partial_\nu\Phi\right)$$

$$= \left\{W^{-2}\partial_0^2 - \nabla^2 - W^{-3}(\partial_0 W)\partial_0 - W^{-1}(\partial_1 W)\partial_1\right\}\Phi, \tag{23.4}$$

$$\sqrt{-g} = \sqrt{-\det g_{\sigma\rho}} = W > 0,$$

where

$$\partial_\mu = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial/\partial w, \partial/\partial \mathbf{x}), \qquad \mathbf{x} = (x^1, x^2, x^3). \tag{23.5}$$

The partial differential operators (23.5) are related to those in an inertial frame $\partial_{I\mu} = (\partial_{I0}, \partial_{I1}, \partial_{I2}, \partial_{I3}) = (\partial/\partial w_I,\ \partial/\partial x_I^1,\ \partial/\partial x_I^2,\ \partial/\partial x_I^3)$ by the relations,

$$\partial_{I0} = \gamma(W^{-1}\partial_0 - \beta\partial_1), \quad \partial_{I1} = \gamma(\partial_1 - \beta W^{-1}\partial_0), \quad \partial_{I2} = \partial_2, \quad \partial_{I3} = \partial_3, \tag{23.6}$$

which can be derived from the transformations in (19.14) (or (19.4) with W in (23.6) replaced by W_x). The covariant derivatives ∂_μ transform in the same way as the covariant momentum 4-vector p_μ in (22.10), as expected. It follows from (23.6) that

$$\partial_{I0}^2 - \nabla_I^2 = W^{-2}\partial_0^2 - \nabla^2 - W^{-3}(\partial_0 W)\partial_0 - W^{-1}(\partial_1 W)\partial_1, \tag{23.7}$$

which is consistent with the invariant expression for d'Alembert's operator $g^{\mu\nu}D_\mu D_\nu$ in (23.4).

For CLA frames under the Wu transformations, W is given by (23.3), and the free Klein-Gordon equation (23.1) takes the form,

$$\left\{\frac{1}{W^2}\partial_0^2 - \nabla^2 - \frac{2\alpha\beta\gamma^2}{W^2}\partial_0 - \frac{\alpha_o}{(\gamma_0^{-2} + \alpha_o x)}\partial_1 + m^2\right\}\Phi = 0. \tag{23.8}$$

The solution of this "free" Klein-Gordon equation in a CLA frame takes the form

$$\Phi(x) = \Phi_o \exp\{-iP(x)\}, \qquad x \equiv x^\mu, \tag{23.9}$$

$$P(x) = p_{I0} w_I(w, x^1) + p_{I1} x_I^1(w, x^1) + p_{I2} x_I^2 + p_{I3} x_I^3$$

$$= p_{I0}\left[\gamma\beta\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}\right] + p_{I1}\left[\gamma\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}\right] + p_{I2}x^2 + p_{I3}x^3$$

$$= \frac{\gamma p_o}{W}\left[\gamma\beta\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}\right] - \gamma\beta p_1\left[\gamma\beta\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}\right]$$

$$+\gamma p_1\left[\gamma\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}\right] - \frac{\gamma\beta p_o}{W}\left[\gamma\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}\right] + p_{I2}x^2 + p_{I3}x^3,$$

where $W = W(w,x^1)$ and we have used (22.10). The phase $P(x)$ reduces to the usual invariant phase $p_o w + p_1 x^1 + p_2 x^2 + p_3 x^3$ in the limit of zero acceleration $\alpha \to 0$. Note that p_{I0} and p_{I1} are constant momenta measured in F_I and that they are related to the non-constant momenta p_μ in the CLA frame F by the transformation given in (22.10). The "free" wave function (23.9) with the complicated phase $P(x)$ in a CLA frame is actually a simple plane wave in an inertial frame F_I. If one substitutes the solution (23.9) with the phase $P(x)$ into (23.8), one obtains

$$g^{\mu\nu}p_\mu p_\nu - m^2 = (W^2 p_0{}^2 - \mathbf{p}^2 - m^2) = 0. \tag{23.10}$$

This is the invariant relation for the momentum 4-vector, which takes the Lorentz invariant form $p_I{}^\nu p_{I\nu} - m^2 = 0$ in an inertial frame.

23b. Quantization of scalar fields in CLA frames

The canonical quantization of fields mimics the dynamics in quantum mechanics. In fact, it is a generalization of the quantum mechanics of a finite system to an infinite system. In the following discussion, we follow closely the canonical quantization of fields in inertial frames. The invariant action S for a neutral scalar field in CLA frames is assumed to take the form:

$$S = \int L_s d^4x, \quad L_S = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - m^2\Phi^2\right), \tag{23.11}$$

$$g = \det g_{\sigma\rho}\,, \quad J \equiv 1,$$

where the metric tensors are given in (23.2) or (23.3). The extra function of spacetime $\sqrt{-g}$ in the scalar Lagrangian density L_S comes from the invariant volume element $\sqrt{-g}\, d^4x = W d^4x$. In the conventional field theory formulated in inertial frames, the Lagrangian density is required to be a functional only of the fields and their first derivatives. In addition, the Lagrangian density must not have an explicit dependence on the spacetime coordinates.[1] However, when one generalizes field theory from inertial frames to non-inertial frames, these requirements become too stringent and must be relaxed. This is because the fundamental metric tensor $g_{\mu\nu}(x)$ is present in the invariant action. As a result, the Lagrangian density will have an explicit spacetime dependence through $g_{\mu\nu}$ which stems from the linear acceleration and is not a physical field in our formalism. The geometrical property of spacetime in a CLA frame is completely described by the metric tensor $g_{\mu\nu}$ and the Riemann curvature tensor can be shown to be zero, implying a flat spacetime without gravity.[2]

By varying the scalar field Φ one obtains the free Klein-Gordon equation for CLA frames

$$\frac{1}{\sqrt{-g}}\, \partial_\mu(\sqrt{-g}\;\; g^{\mu\nu}\partial_\nu\Phi) + m^2\Phi = 0, \tag{23.12}$$

which is the same as (23.1). The energy-momentum tensor $T_{\mu\nu}$ for the scalar field is defined by

$$T^\mu_\nu = \frac{\partial L_S}{\partial(\partial_\mu\Phi)}\,\partial_\nu\Phi - \delta^\nu_\mu L_S$$

$$= \sqrt{-g}\;\; \partial^\mu\Phi\partial_\nu\Phi - \frac{1}{2}\sqrt{-g}\;\; \delta^\mu_\nu(g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi - m^2\Phi^2). \tag{23.13}$$

The momentum $\pi(w,\mathbf{x})$ conjugate to $\Phi(w,\mathbf{x})$ is defined by

$$\pi(w,\mathbf{x}) = \frac{\partial L_s}{\partial(\partial_0\Phi)} = \sqrt{-g}\ \ g^{00}\partial_0\Phi = \frac{1}{W}\ \partial_0\Phi, \qquad \mathbf{x} = (x^1, x^2, x^3), \tag{23.14}$$

where the presence of the function W in (23.14) is due to the acceleration of the reference frame. The Hamiltonian density H_s and the momentum density P_i of the scalar field are defined by

$$H_s = \pi(w,\mathbf{x})\partial_0\Phi - L_s = \frac{1}{2}\sqrt{-g}\,(g^{00}\pi^2 + |\nabla\,\Phi|^2 + m^2\Phi^2) = T^0_0\,, \tag{23.15}$$

$$P_i = T^0_i = \pi(w,\mathbf{x})\partial_i\Phi.$$

To quantize the neutral scalar field, we postulate the equal-time commutation relation

$$[\Phi(w,\mathbf{x}), \pi(w,\mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}'), \tag{23.16}$$

$$[\Phi(w,\mathbf{x}),\Phi(w,\mathbf{x}')] = [\pi(w,\mathbf{x}),\pi(w,\mathbf{x}')] = 0.$$

As usual, we may express the scalar field operator $\Phi(w,\mathbf{x})$ in terms of an operator-valued amplitude A(k), $k_\mu = p_\mu/J \equiv p_\mu$, and its Hermitian conjugate $A^\dagger(k)$:

$$\Phi(w,x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^4k}{W}\delta(k_\mu k^\mu - m^2)\theta(k_0)[A(k)e^{-iP} + A^\dagger(k)e^{iP}], \tag{23.17}$$

$$\delta(k_\mu k^\mu - m^2) = \delta(k_0k_0/W^2 - \mathbf{k}^2 - m^2)$$

$$= \frac{\delta(k_0 - W\omega_k)}{2k_0/W^2} + \frac{\delta(k_0 + W\omega_k)}{2|k_0|/W^2}, \qquad \omega_k = \sqrt{\mathbf{k}^2 + m^2}\,, \tag{23.18}$$

$$P = P(x) = \left(\gamma\frac{k_0}{W} - \gamma\beta k_1\right)\left[\gamma\beta\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}\right]$$

$$+\left(\gamma k_1 - \gamma\beta\frac{k_0}{W}\right)\left[\gamma\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}\right] + k_2x^2 + k_3x^3, \tag{23.19}$$

where $W=W(w,x^1)$ and $P(x)=P(x^\mu)$. The validity of the commutation relations (23.16) for canonical quantization and the Fourier transform (23.17) for the field $\Phi(w,\mathbf{x})$ are independent of the detailed form of the Lagrangian density. This is the reason why it is possible to carry out the canonical quantization of fields in non-inertial frames. Before quantization, $A^\dagger(k)$ in (23.17) is the complex conjugate amplitude of $A(k)$. After the quantization postulate (23.16) is made, one has quantum fields and the amplitude becomes an operator with $A^\dagger(k)$ the Hermitian conjugate to $A(k)$.

Relation (23.18) follows from the formula[3]

$$\delta^{(\kappa)}(f(x)) = \left(\frac{d}{dx}\right)^\kappa \delta(f(x)) = \sum_n \frac{1}{|f'(x_n)|}\left(\frac{1}{f'(x)}\frac{d}{dx}\right)^\kappa \delta(x - x_n), \tag{23.20}$$

$$f(x_n) = 0, \qquad n=1,2,3... \qquad f'(x) = df(x)/dx,$$

with $\kappa=0$ and $n=2$. It is important to note that the quantities $(\gamma k_0/W-\gamma\beta k_1)$ and $(\gamma k_1-\gamma\beta k_0/W)$ in (23.19) are both constants associated with "plane waves,"

$$(\gamma k_0/W - \gamma\beta k_1) = \text{constant} \quad \text{and} \quad (\gamma k_1 - \gamma\beta k_0/W) = \text{constant}, \tag{23.21}$$

even though β, γ, k_0, k_1 and W are separately non-constants in a CLA frame, as shown in (22.10). The invariant volume element $d^4k/\sqrt{g_{00}} = d^4k/W$ in the momentum space of a CLA frame can be obtained from d^4k_I in an inertial frame through the momentum transformation (22.10) with the covariant momentum 4-vector p_μ replaced by k_μ:

$$d^4k_I = J(k_{I\lambda}/k_\lambda)d^4k = d^4k/W, \tag{23.22}$$

where $J(k_{I\lambda}/k_\lambda)=1/W$ is the Jacobian.

With the help of (23.18), we can write the Fourier decomposition (23.17) for $\Phi(w,\mathbf{x})$ and (23.14) for $\pi(w,\mathbf{x})$ in the form

$$\Phi(w,\mathbf{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}[a(\mathbf{k})e^{-iP} + a^\dagger(\mathbf{k})e^{iP}], \quad a(\mathbf{k}) = \frac{A(k)}{\sqrt{2\omega_k}}, \tag{23.23}$$

$$\pi(w,\mathbf{x}) = \int \frac{d^3k \ i\omega_k}{\sqrt{(2\pi)^3 2\omega_k}}[-a(\mathbf{k})e^{-iP} + a^\dagger(\mathbf{k})e^{iP}], \quad \omega_k = \sqrt{\mathbf{k}^2 + m^2}, \tag{23.24}$$

where k_0 in the phase P is equal to $W\omega_k$, i.e., $k_0 = W\sqrt{\mathbf{k}^2+m^2}$. We can also express the operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$, which are functions of the spatial momentum vector $\mathbf{k}$, in terms of the scalar field operator Φ:

$$a(\mathbf{k}) = i\int \frac{d^3x}{W\sqrt{(2\pi)^3 2\omega_k}}[e^{iP}\partial_0\Phi - (\partial_0 e^{iP})\Phi], \tag{23.25}$$

$$a^\dagger(\mathbf{k}) = -i\int \frac{d^3x}{W\sqrt{(2\pi)^3 2\omega_k}}[e^{-iP}\partial_0\Phi - (\partial_0 e^{-iP})\Phi]. \tag{23.26}$$

One can verify that the commutation relations

$$\begin{aligned} &[a(\mathbf{k}), a^\dagger(\mathbf{k'})] = \delta^3(\mathbf{k} - \mathbf{k'}), \\ &[a(\mathbf{k}), a(\mathbf{k'})] = [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k'})] = 0, \end{aligned} \tag{23.27}$$

are consistent with the equal-time commutators in (23.16). To see this, let us substitute (23.23) and (23.24) into the equal-time commutator (23.16). We have

$$[\Phi(w,\mathbf{x}),\pi(w,\mathbf{x'})] = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \int \frac{d^3k' \ i\omega_{k'}}{\sqrt{(2\pi)^3 2\omega_{k'}}}$$

$$\times \left[a(\mathbf{k})e^{-iP(x)} + a^\dagger(\mathbf{k})e^{iP(x)}, -a(\mathbf{k})e^{-iP'(x')} + a^\dagger(\mathbf{k})e^{iP'(x')}\right]\Big|_{W=W'}$$

$$= \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \int \frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_{k'}}} \, i\omega_{k'} \, \delta^3(\mathbf{k} - \mathbf{k}')\left[e^{-iP+iP'} + e^{iP-iP'}\right]\Big|_{W=W'}$$

$$= \int \frac{i d^3k}{2(2\pi)^3}\left[e^{-iP+iP'} + e^{iP-iP'}\right]\Big|_{W=W',\, \mathbf{k}=\mathbf{k}'}, \tag{23.28}$$

where the phase functions P and P' are given by (23.19),

$$P = P(x) = P(x,k), \qquad P' = P(x \to x', k \to k'), \quad x \equiv x^\mu, \qquad k \equiv k^\mu, \text{etc.} \tag{23.29}$$

Let us calculate the term $e^{-iP+iP'}$ in (23.28). We obtain

$$e^{-iP+iP'}\Big|_{W=W',\,\mathbf{k}=\mathbf{k}'} = \exp\Bigg[-i\Bigg\{\left(\gamma\frac{k_0}{W} - \gamma\beta k_1\right)\left[\gamma\beta\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}\right]$$

$$+\left(\gamma k_1 - \gamma\beta\frac{k_0}{W}\right)\left[\gamma\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}\right] + k_2x^2 + k_3x^3\Bigg\}$$

$$-i\Bigg\{\left(\gamma\frac{k_0}{W} - \gamma\beta k_1\right)\left[\gamma\beta\left(x'^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}\right]$$

$$+\left(\gamma k_1 - \gamma\beta\frac{k_0}{W}\right)\left[\gamma\left(x'^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}\right] + k_2x'^2 + k_3x'^3\Bigg\}\Bigg]$$

$$= \exp\Big[-i\frac{\gamma^2\beta k_0}{W}(x^1 - x'^1) + i\gamma^2\beta^2k_1(x^1 - x'^1) - i\gamma^2k_1(x^1 - x'^1)$$

$$+ i\frac{\gamma^2\beta k_0}{W}(x^1 - x'^1) - ik_2(x^2 - x'^2) - ik_3(x^3 - x'^3)\Big]$$

$$= \exp[-i\mathbf{k}\cdot(\mathbf{x} - \mathbf{x}')]. \tag{23.30}$$

The other term $e^{iP-iP'}$ is just the complex conjugate of (23.30). Thus, equation (23.28) is the same as the equal-time commutator (23.16).

In general, the Hamiltonian H of the scalar field,

$$H = \int H_s d^3x = \int d^3x \sqrt{-g}\, \frac{1}{2}[g^{00}\pi^2 + |\nabla\Phi|^2 + m^2\Phi^2], \tag{23.31}$$

cannot be expressed in terms of the operators a(k) and $a^\dagger$(k) because of the presence of the spacetime-dependent function $\sqrt{-g} = W$ in the integrand. Nevertheless, one still can derive physical results from the theory. The Hamiltonian H in (23.31) can be expressed in terms of the operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ only in the limit of zero acceleration, i.e., $W \to 1$. This suggests that one can define H_ω by the relation

$$H_\omega \equiv \int d^3x \frac{H_s}{\sqrt{-g}} = \int \frac{1}{2} d^3k\ \omega_k[a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})]. \tag{23.32}$$

This H_ω is not exactly the "Hamiltonian" of the scalar field defined in (23.31). It corresponds to p_0/W and is related to an "instantaneous conservation of energy" in a collision process. Nevertheless, the momentum P_i of the scalar field can be expressed in terms of the operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$:

$$P_i = \int \mathcal{P}_i d^3x = \int \frac{1}{2} d^3k\, k_i[a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})], \tag{23.33}$$

where $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are respectively the annihilation and the creation operators. We observe that the momentum P_i is not constant:

$$[H, P_i] \neq 0, \tag{23.34}$$

in contrast to the corresponding P_i in quantum field theory in inertial frames.

The reason is obvious: a constant momentum in an inertial frame can no longer be a constant when it is measured in a CLA frame. The Hamiltonian H in (23.31) contains $\sqrt{-g} = W$ which is an explicit function of space and time. Only in the limit of zero acceleration does (23.34) vanish as it must.

One can verify that the operators $P_\mu=(P_0,P_i)=(H,P_i)$ and $M_{\mu\nu}$ generate translations and rotations in CLA frames:

$$i[P_\mu,\Phi] = \partial_\mu\Phi, \tag{23.35}$$

$$i[M_{\mu\nu},\Phi] = (x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi, \tag{23.36}$$

where $M^{\mu\nu}$ is given by

$$M^{\mu\nu} = \int d^3x\sqrt{-g}\,[T^{0\nu}x^\mu - T^{0\mu}x^\nu]. \tag{23.37}$$

The Heisenberg equations of motion are given by

$$\frac{\partial\Phi}{\partial w} = \frac{1}{i}(\Phi H - H\Phi) = \frac{1}{i}[\Phi,H], \tag{23.38}$$

$$\frac{\partial\pi}{\partial w} = \frac{1}{i}[\pi,H], \qquad \pi = \pi(w,\mathbf{x}). \tag{23.39}$$

As expected to insure consistency of quantization, equations (23.14), (23.16) and (23.39) reproduce the field equation (23.12).

23c. Quantization of spinor fields in CLA frames

The previous discussion of canonical quantization of scalar fields in non-inertial frames with constant linear acceleration can be applied to other fields. In this section, we will give a brief derivation of the main results for spinor fields.

The invariant action S_ψ for a free electron field ψ in CLA frames is assumed to be given by

$$S_\psi = \int L_\psi d^4x, \quad L_\psi = \frac{1}{2}\sqrt{-g}\,\bar{\psi}\Gamma^\mu i\partial_\mu\psi - \frac{1}{2}\sqrt{-g}\,(\partial_\mu\bar{\psi})\Gamma^\mu i\psi - \sqrt{-g}\,m\bar{\psi}\psi, \tag{23.40}$$

$$\{\Gamma^\mu,\Gamma^\nu\} = 2g^{\mu\nu}(x), \quad \sqrt{-g} = W > 0, \qquad \Gamma^\mu = (W^{-1}\gamma^0, \gamma^1, \gamma^2, \gamma^3),$$

where $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ are the constant Dirac matrices. The presence of the metric tensor $g^{\mu\nu}(x)$ is a new feature in the formalism of quantum field theory in CLA frames. The fermion Lagrangian must be symmetrized because Γ_μ is now a function of spacetime rather than a constant. Also, the non-universal speed of light does not appear in the Lagrangian density L_ψ. Thus, with the help of integration by parts, the Lagrangian in (23.40) can be written in the usual form in terms of a "gauge covariant derivative" ∇_μ,

$$L_\psi = \sqrt{-g}\,\bar{\psi}\Gamma^\mu i\partial_\mu\psi + \frac{1}{2}\bar{\psi}[i(\partial_\mu\sqrt{-g}\,\Gamma^\mu)]\psi - \sqrt{-g}\,m\bar{\psi}\psi$$

$$= W\bar{\psi}\Gamma^\mu i\nabla_\mu\psi - Wm\bar{\psi}\psi, \tag{23.41}$$

$$\nabla_\mu = (\partial_0 + \frac{1}{2}(\partial_k W)\gamma^0\gamma^k, \partial_1, \partial_2, \partial_3).$$

The presence of the gauge covariant derivative can also be seen explicitly if one introduces vierbeins (or tetrads) to express the matrices Γ^μ. Four mutually orthogonal unit vectors, denoted by $e^a{}_\nu$, a=0,1,2,3 form an orthonormal tetrad, where ν is the covariant tensor index and the Latin suffix "a" is a label distinguishing the particular vector. The contravariant components of the same tetrad can be obtained by using $g^{\mu\nu}$ to raise the index ν, $e^{a\mu} = g^{\mu\nu}e^a{}_\nu$. The labels on the vectors have no tensorial meaning, but for convenience we shall raise and lower them by using $\eta_{ab} = (1, -1, -1, -1)$. We have

$$\Gamma^\mu = \gamma^a e_a{}^\mu, \qquad e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}, \qquad e^a{}_\mu e^b{}_\nu g^{\mu\nu} = \eta^{ab}, \qquad e^a{}_\mu = g_{\mu\nu} e^{a\nu}, \tag{23.42}$$

where γ^a are the usual constant Dirac matrices, and the Latin suffixes a, b, c,... may be called Lorentz indices because they are related to the Lorentz group. We have the following expressions for the non-zero diagonal elements:

$$g_{\mu\nu} = (W^2, -1, -1, -1) \qquad \eta^{ab} = (1, -1, -1, -1), \qquad e^a{}_\mu = (W, 1, 1, 1), \tag{23.43}$$

$$e_{a\mu} = (W, -1, -1, -1), \qquad e_a{}^\mu = (W^{-1}, 1, 1, 1), \qquad e^{a\mu} = (W^{-1}, -1, -1, -1).$$

These vierbeins in (23.41) correspond to "scale gauge fields" (which differ from Yang-Mills fields or gauge fields) in the Poincaré gauge-invariant Lagrangian involving fermions.[4]

One can introduce the local Lorentz transformation of the spinor

$$\psi \to \exp[i\varepsilon^{ab}(x)\sigma_{ab}]\psi, \qquad \sigma_{ab} = (i/2)[\gamma_a\gamma_b - \gamma_b\gamma_a]. \tag{23.44}$$

Corresponding to this local transformation, one has the gauge covariant partial derivative

$$\nabla_\mu\psi = [\partial_\mu - \frac{i}{4}\,\omega_\mu{}^{ab}\sigma_{ab}]\,\psi. \tag{23.45}$$

The connection $\omega_\mu{}^{ab}(x)$ is introduced so that one can *gauge the Lorentz group.* It is given by[5]

$$\omega_\mu{}^{ab}(x) = \frac{1}{2}\, e^{\lambda a}(\partial_\mu e_\lambda{}^b - \partial_\lambda e_\mu{}^b) + \frac{1}{4}\, e^{\rho a} e^{\lambda b}(\partial_\lambda e_\rho{}^c - \partial_\rho e_\lambda{}^c)\, e_{\mu c} - (a \leftrightarrow b)$$

$$= - e^{\lambda a}\partial_\lambda W \delta^{bo}\delta_{\mu 0} + e^{\lambda b}\partial_\lambda W \delta^{ao}\delta_{\mu 0}\,. \tag{23.46}$$

There are only two non-vanishing components for $\omega_\mu{}^{ab}(x)$:

$$\omega_0{}^{01} = -\,\omega_0{}^{10} = -\,\partial_1 W. \tag{23.47}$$

It follows from (23.45) and (23.47) that

$$\nabla_0\psi = [\partial_0 + \frac{1}{2}\,\gamma^0\gamma^1\partial_1 W]\psi, \tag{23.48}$$

$$\nabla_k\psi = \partial_k\psi, \qquad \text{for} \qquad k = 1,2,3.$$

One can verify that the results in (23.48) are precisely the same as $\nabla_\mu\psi$, with $\nabla_\mu = (\partial_0 + \frac{1}{2}(\partial_k W)\gamma^0\gamma^k, \partial_1, \partial_2, \partial_3)$, in (23.41) because the metric tensor $g_{\mu\nu}$ or the Wu factor W does not depend on y and z, i.e. $\partial_2 W = \partial_3 W = 0$.

From the Lagrangian (23.41), we obtain the generalized Dirac equation for CLA frames:

$$(i\Gamma^\mu\nabla_\mu - m)\psi = 0, \qquad \bar{\psi}(-\overleftarrow{\nabla}_\mu i\Gamma^\mu - m) = 0. \tag{23.49}$$

For covariant quantization of the spinor fields in such CLA frames, we use the gauge covariant derivative (23.45) or (23.41) consistently and follow the canonical quantization as closely as possible. The "canonical momentum" π_b conjugate to ψ_b is defined as

$$\pi_b = \partial L_\psi/\partial(\nabla_0\psi_b) = i\psi^\dagger/W, \qquad L_\psi = W\bar{\psi}\Gamma^\mu i\,\nabla_\mu\psi - Wm\bar{\psi}\psi\,, \tag{23.50}$$

where the presence of W(w,x) in the Lagrangian, the canonical momentum (23.50) and the field equations (23.49) are the only source of complication of field quantization in CLA frames. The Hamiltonian density H_ψ for a free electron is defined as

$$H_\psi = \pi\nabla_0\psi - L_\psi = -\sqrt{-g}\;\bar{\psi}\gamma^k i\partial_k\psi + \sqrt{-g}\;m\bar{\psi}\psi\,, \tag{23.51}$$

where we have used Γ^k, k=1,2,3, in (23.40) and ∇_k in (23.41). The energy-momentum tensor $T_{\mu\nu}$ for the spinor field is defined by

$$T^{\mu}{}_{\nu} = \frac{\partial L_\psi}{\partial(\nabla_\mu \psi)}\nabla_\nu\psi - \delta^{\mu}{}_{\nu}L_\psi \,. \tag{23.52}$$

Thus, we have the energy H and momentum P_k of the spinor field:

$$H = \int d^3x T_0^0 = \int d^3x\sqrt{-g}(-i\overline{\psi}\gamma^k\partial_k\psi + m\overline{\psi}\psi), \quad T_0^0 = H_\psi, \tag{23.53}$$

$$P_k = \int d^3x T_k^0 = \int d^3x\sqrt{-g}\, i\,\overline{\psi}\gamma^0\nabla_k\psi = \int d^3x\frac{i}{W}\psi^\dagger\partial_k\psi. \tag{23.54}$$

The fundamental equal-time anticommutation relations for spinor fields are postulated to be

$$\{\psi_\alpha(w, \mathbf{x}), \pi_\beta(w, \mathbf{y})\} = i\delta^3(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta}, \tag{23.55}$$

$$\{\psi_\alpha(w, \mathbf{x}), \psi_\beta(w, \mathbf{y})\} = 0, \qquad \{\pi_\alpha(w, \mathbf{x}), \pi_\beta(w, \mathbf{y})\} = 0,$$

where $\pi_\alpha(w,\mathbf{y})$ is given by (23.50) and W must be treated as an ordinary function rather than an operator. From equations (23.53) through (23.55), one can verify that

$$i[P_\mu, \psi(x)] = \nabla_\mu\psi(x), \qquad P_\mu = (H, P_k). \tag{23.56}$$

In the calculation of $[P_k, \psi(x)]$, the extra function 1/W in (23.54) disappears after the anticommutator (23.55) is used. Similar cancellations of W's occur in $[H, \psi(x)]$ when (23.55) and the free equation $i\Gamma^k\nabla_k\psi - m\psi = -\, i\Gamma^0\nabla_0\psi$ in (23.49) are used.

In order to carry out quantization in momentum space in terms of the creation and annihilation operators, we must first find the solution to the generalized free Dirac equation (23.49) in linearly accelerated frames. Because of the presence of the gauge covariant derivative ∇_μ, the time and space dependence of the wave function ψ is not just the phase $iP(x)$ for a free fermion. There is an additional gauge factor $G(w)$, which changes the magnitude of the "plane wave function," due to the acceleration of reference frames:

$$\psi = u(k)\exp[-iP(x) - G(w)], \qquad P(x) = P(x^\mu) = P(w,x,y,z), \tag{23.57}$$

$$(\Gamma^\mu k_\mu - m)u(k) = 0, \tag{23.58}$$

where $u(k)$ is a 4-component spinor. The phase factor $P(x)$ is given by (23.19) and the time-dependent gauge factor $G(w)$ is found to be

$$G(w) = \int^{w} dw \frac{1}{2}\gamma^0\gamma^1(\partial_1 W) = \frac{1}{2}\gamma^0\gamma^1 \tanh^{-1}(\alpha_o w + \beta_o), \quad \text{for} \quad W = \gamma^2[\gamma_o{}^{-2} + \alpha_o x],$$

$$(23.59)$$

$$G(w) = \frac{1}{2}\gamma^0\gamma^1\alpha^* w, \qquad \text{for} \quad W = [1+\gamma_o\alpha^* x]\,,$$

where $\gamma = 1/\sqrt{1-(\alpha_o w+\beta_o)^2}$, $\gamma_o = 1/\sqrt{1-\beta_o{}^2}$, and $\gamma^0\gamma^1$ satisfies the relation $(\gamma^0\gamma^1)^2 = 1$ because of the anti-commutation in (23.40).

For a free spinor field ψ, the Fourier expansion of a solution (23.57) for a "free" electron takes the form

$$\psi(w,\mathbf{x}) = \int \sqrt{\frac{m}{\omega_k}} \frac{d^3k}{\sqrt{(2\pi)^3}} \sum_{s=1}^{2} [b(\mathbf{k},s)u(\mathbf{k},s)\exp(-iP - G)$$

$$+ d^\dagger(\mathbf{k},s)v(\mathbf{k},s)\exp(iP - G)], \tag{23.60}$$

$$k_0 = W\omega_k, \qquad \omega_k = \sqrt{\mathbf{k}^2 + m^2}\ ,$$

where the phase $P=P(x)$ is given by (23.19) and the gauge factor $G=G(w)$ is given by (23.58) and (23.59). According to (23.58), the spinors $u(\mathbf{k},s)$ and $v(\mathbf{k},s)$

satisfy the equations

$$(\Gamma^\mu k_\mu - m)u(\mathbf{k},s) = 0, \qquad \bar{u}(\mathbf{k},s)(\Gamma^\mu k_\mu - m) = 0,$$

$$(\Gamma^\mu k_\mu + m)v(\mathbf{k},s) = 0, \qquad \bar{v}(\mathbf{k},s)(\Gamma^\mu k_\mu + m) = 0, \tag{23.61}$$

and the orthonormality relations

$$\bar{u}(\mathbf{k},s)u(\mathbf{k},s') = -\bar{v}(\mathbf{k},s)v(\mathbf{k},s') = \delta_{ss'},$$

$$u^\dagger(\mathbf{k},s)u(\mathbf{k},s') = v^\dagger(\mathbf{k},s)v(\mathbf{k},s') = \frac{k_0}{m}\delta_{ss'}, \qquad k_0 = W\sqrt{\mathbf{k}^2+m^2}. \tag{23.62}$$

$$\bar{v}(\mathbf{k},s)u(\mathbf{k},s') = \bar{u}(\mathbf{k},s)v(\mathbf{k},s') = 0.$$

We normalize the spinors so that $u^\dagger(\mathbf{k},s)u(\mathbf{k},s') = (k_0/m)\delta_{ss'}$. This is a covariant relation since both sides transform like the zeroth component of a 4-vector under accelerated Wu (or MWL) transformations. We also have the following completeness relations

$$\sum_{s=1}^{2} [u_\alpha(\mathbf{k},s)\bar{u}_\beta(\mathbf{k},s) - v_\alpha(\mathbf{k},s)\bar{v}_\beta(\mathbf{k},s)] = \delta_{\alpha\beta},$$

$$\sum_{s=1}^{2} u_\alpha(\mathbf{k},s)\bar{u}_\beta(\mathbf{k},s) = \left(\frac{m + \Gamma^\mu k_\mu}{2m}\right)_{\alpha\beta}, \tag{23.63}$$

$$-\sum_{s=1}^{2} v_\alpha(\mathbf{k},s)\bar{v}_\beta(\mathbf{k},s) = \left(\frac{m - \Gamma^\mu k_\mu}{2m}\right)_{\alpha\beta}.$$

Upon second quantization of the spinor field, the Fourier expansion coefficients $b(\mathbf{k},s)$, $b^\dagger(\mathbf{k}',s')$, $d(\mathbf{k},s)$ and $d^\dagger(\mathbf{k}',s')$ become operators that annihilate and create particles. They are assumed to satisfy the anticommutation relations,

$$\{b(\mathbf{k},s),\ b^\dagger(\mathbf{k}',s')\} = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'},$$
$$\{d(\mathbf{k},s),\ d^\dagger(\mathbf{k}',s')\} = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \tag{23.64}$$

where all other anticommutators vanish. The anticommutators for the quantized fields $\psi(w,\mathbf{x})$ and $\psi^\dagger(w,\mathbf{x})$ in (23.55) can be shown to be consistent with (23.60) and (23.64). In the calculation of these anticommutators, the gauge factor $G(w)$ in ψ and $\psi^\dagger$ cancels because of the relation $\gamma^0\gamma^1 = -\gamma^1\gamma^0$, or

$$e^{-G(w)}\gamma^0 = \gamma^0 e^{G(w)}, \qquad G(w) = \int dw \frac{1}{2}\gamma^0\gamma^1(\partial_1 W). \tag{23.65}$$

Finally, the Hamiltonian operator $H = \int d^3x T^0{}_0$ cannot be expressed in terms of the operators b, $b^\dagger$, d and $d^\dagger$ because of the presence of W in the integrand. This restriction occurs because the Hamiltonian H transforms like k_0 and $k_0 = W\omega_k = W\sqrt{\mathbf{k}^2+m^2}$. However, we have

$$H_\omega = \int d^3x T_0^0 / W = \int d^3k\, \omega_k [b^\dagger(\mathbf{k})b(\mathbf{k}) + d(\mathbf{k})d^\dagger(\mathbf{k})], \tag{23.66}$$

$$P_i = \int d^3x T_i^0 = \int d^3k\, k_i\, [b^\dagger(\mathbf{k})b(\mathbf{k}) + d(\mathbf{k})d^\dagger(\mathbf{k})]. \tag{23.67}$$

The quantity H_ω in (23.66) is not exactly the Hamiltonian, in contrast to H in (23.53). Instead, it corresponds to p_0/W which is useful for treating scattering processes in field theory.

23d. Quantization of the electromagnetic field in CLA frames

The action of the electromagnetic field is assumed to be that of the usual invariant form with a gauge fixing term involving a gauge parameter ρ,

$$S = \int d^4x L_{em}, \qquad L_{em} = -\frac{1}{4}\sqrt{-g}\; f_{\mu\nu}f^{\mu\nu} - \frac{1}{2\rho}\sqrt{-g}\,(D^{\mu}a_{\mu})^2, \tag{23.68}$$

$$f_{\mu\nu} = D_{\mu}a_{\nu} - D_{\nu}a_{\mu} = \partial_{\mu}a_{\nu} - \partial_{\nu}a_{\mu}, \tag{23.69}$$

$$D^{\mu}a_{\mu} = D_{\mu}a^{\mu} = D_{\mu}(a_{\alpha}g^{\mu\alpha}) = \partial_{\mu}(a_{\alpha}g^{\mu\alpha}) - a_{\alpha}g^{\mu\alpha}\frac{1}{\sqrt{-g}}\partial_{\mu}\sqrt{-g}$$

$$= g^{\mu\alpha}\partial_{\mu}a_{\alpha} - a_{\alpha}\frac{1}{\sqrt{-g}}\partial_{\mu}(g^{\mu\alpha}\sqrt{-g}) \equiv \chi. \tag{23.70}$$

Varying the action S, we obtain the free electromagnetic equation

$$\partial_{\mu}(\sqrt{-g}\; f^{\mu\nu}) - \frac{1}{\rho}g^{\mu\nu}\sqrt{-g}\;\partial_{\mu}\chi = 0. \tag{23.71}$$

Since $f^{\mu\nu}$ is antisymmetric, the scalar field $\chi \equiv D^{\mu}a_{\mu}$ satisfies the free equation in CLA frames

$$\frac{1}{\sqrt{-g}}\partial_{\nu}(g^{\mu\nu}\sqrt{-g}\;\partial_{\mu}\chi) = D^{\mu}D_{\mu}\chi = 0, \tag{23.72}$$

which corresponds to the free equation $\partial^{\mu}\partial_{\mu}\chi=0$ in inertial frames.

The electromagnetic 4-potential can be written in the form

$$a_{\mu}(w,\mathbf{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{\alpha=0}^{3}\Big[a(\mathbf{k},\alpha)\varepsilon_{\mu}(\alpha)\exp(-iP(x))$$

$$+ a^{\dagger}(\mathbf{k},\alpha)\varepsilon_{\mu}(\alpha)\exp(iP(x))\Big], \qquad k_0 = W\omega_k = W|\mathbf{k}|. \tag{23.73}$$

$$\sum_{\alpha=0}^{3}\varepsilon_{\mu}(\alpha)\varepsilon_{\nu}(\alpha) = -g_{\mu\nu} + (1-\rho)\frac{k_{\mu}k_{\nu}}{k^{\lambda}k_{\lambda}}, \tag{23.74}$$

$$[a(\mathbf{k},\alpha),\ a^\dagger(\mathbf{k}',\alpha')] = -\,\delta^3(\mathbf{k}-\mathbf{k}')\,g_{\alpha\alpha'}, \tag{23.75}$$

$$[a(\mathbf{k},\alpha),\ a(\mathbf{k}',\alpha')] = 0, \qquad [a^\dagger(\mathbf{k},\alpha),\ a^\dagger(\mathbf{k}',\alpha')] = 0.$$

The sum over α in (23.73) for four linearly independent unit polarization vectors $\varepsilon_\mu(\alpha)$, α=0,1,2,3 is an ordinary sum and not a scalar product in 4-dimensional space. It can be quantized covariantly using the Gupta-Bleuler method generalized for non-inertial frames.[6]

Equation (23.71) is important for covariant quantization of the electromagnetic field by the Gupta-Bleuler method which involves both physical photons (i.e., the two transverse components) and unphysical photons (i.e., longitudinal and time-like photons). In the Gupta-Bleuler method, equation (23.71) guarantees that the unphysical photons do not interact and hence, do not contribute to physical amplitudes and can be excluded from physical states.[7] In other words, it assures that when the condition for physical states $|\Psi\rangle$,

$$k^\mu a_\mu(\mathbf{k})|\Psi\rangle = 0, \quad \text{for all } \mathbf{k},\ \ k_0 = |\mathbf{k}|, \quad a_\mu(\mathbf{k}) = \sum_{\alpha=0}^{3} a(\mathbf{k},\alpha)\varepsilon_\mu(\alpha), \tag{23.76}$$

is imposed at say, time w=0, it will hold for all times.

References

1. J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields.* Vol. 2 (McGraw-Hill Book Company, New York, 1965) p. 14. C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954), see note added in proof on p. 193. The standard canonical formalism in quantum mechanics is not applicable to the quantization of a Lagrangian with an explicit time dependence.
2. The Wu transformation does not change the curvature of spacetime, so it implies that the spacetime of CLA frames is flat (i.e., the Riemann curvature tensor vanishes), just like the spacetime of inertial frames. C. Møller, *The Theory of Relativity* (Oxford Univ. Press, London, 1952), p. 285.
3. I. M. Gel'fand and G.E. Shilov, *Generalized Functions* Vol. 1 (transl. by E. Saletan, Academic Press, New York, 1964) p. 185.
4. J. P. Hsu, Phys. Letters. **119B**, 328 (1982). (See appendix C.) It is shown that a gauge-invariant fermion Lagrangian with external spacetime symmetry groups (e.g., Lorentz and Poincaré groups) dictates the presence of two distinct gauge fields: the conventional Yang-Mills fields $b_\mu^A = (b_\mu^i, b_\mu^{jk})$ and the new scale gauge fields e_A^μ .
5. The commutator of two gauge covariant derivatives ∇_μ gives a gauge curvature tensor $R_{\mu\nu}{}^{ab}$: $[\nabla_\mu, \nabla_\nu]\psi = (-i/4)R_{\mu\nu}{}^{ab}\sigma_{ab}\psi$, where $R_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_\nu{}^{db}\eta_{cd} - \omega_\nu{}^{ac}\omega_\mu{}^{db}\eta_{cd}$. (This gauge curvature tensor $R_{\mu\nu}{}^{ab}$ resembles the Riemann curvature tensor $R^\alpha{}_{\beta\mu\nu}$ for curved spaces. The spacetime of CLA frames with the metric tensor $g_{\mu\nu}$ given by (23.2) and (23.3) is flat because the Riemann curvature tensor vanishes.) The connection $\omega_\mu{}^{ab}$ can be expressed in terms of the vierbein by imposing a constraint $\nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \Gamma^\lambda{}_{\mu\nu} e_\lambda{}^a + \omega_\mu{}^{ac} e_\nu{}^b \eta_{cb} = 0$, where $\Gamma^\lambda{}_{\mu\nu}$ is the Christoffel symbol given in (22.12).
6. For the Gupta-Bleuler method, see M. Kaku, *Quantum Field Theory, A Modern Introduction* (Oxford Univ. Press, New York, 1993) pp. 112-114.
7. In contrast, unphysical bosons (ghost particles) in Yang-Mills fields do not satisfy a free equation and can interact with other particles to give unphysical contributions which upset the unitarity and gauge invariance

of scattering amplitudes. It is much more involved to remove these unphysical contributions in gauge theory. See, for example, J. P. Hsu and J. A. Underwood, Phys. Rev. D **12**, 620 (1975); J. P. Hsu and E.C.G. Sudarshan, Phys. Rev. D **9**, 1678 (1974); M. Kaku, *Quantum Field Theory, A Modern Introduction* (Oxford Univ. Press, New York, 1993) pp. 298-306.

24.

Group and Lie Algebra Properties of Accelerated Spacetime Transformations

24a. The Wu transformation with acceleration in an arbitrary direction

The symmetry principles are at the very heart of our understanding of the physical world. Symmetry in physics means that observable results are not changed under a transformation. Transformations of spacetime can be understood as a change of observer's viewpoint. To understand symmetry physically, it is necessary to know mathematically the group properties of the transformation and the associated Lie algebra.[1]

The Wu transformation is one of the simplest spacetime transformations beyond the Lorentz and Poincaré transformations in which one can hope to discuss field theory and particle physics. Even with such a minimal generalization of the Lorentz group, the group properties and the associated algebra of such spacetime transformations for accelerated frames are much more involved. Specifically, the Lie algebra of the generators of the Wu transformation fails to close.[1]

Let x_I^μ denote inertial coordinates in Minkowski spacetime and x^μ denote coordinates for the non-inertial frame.[2] Take the (constant) direction of acceleration to be the x^1-axis in both frames. Then the Wu transformation is[3]

$$x_I^0 = \gamma u\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}, \tag{24.1}$$

$$x_I^1 = \gamma\left(x^1 + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}, \qquad x_I^2 = x^2, \qquad x_I^3 = x^3. \tag{24.2}$$

Here $u = \alpha_o x^0 + \beta_o$, for some constants α_o and $-1 < \beta_o < 1$, $\gamma = (1-u^2)^{-1/2}$, and $\gamma_o = (1-\beta_o^2)^{-1/2}$. Because $\gamma = (1-u^2)^{-1/2}$ is a function of x^2 these transformations are non-linear in general. Despite the $1/\alpha_o$ terms, these expressions are continuous in α_o.

When the velocity $\mathbf{u}$ of the accelerating frame is in an arbitrary spatial direction $\mathbf{u}/|\mathbf{u}|$ in the inertial frame, the finite Wu transformation takes the following general form:

$$x_I^0 = u\gamma\left[-\frac{\mathbf{x}\cdot\mathbf{u}}{u} + \frac{1}{\alpha_o\gamma_o^2}\right] - \frac{\beta_o}{\alpha_o\gamma_o}, \tag{24.3}$$

$$x_I^k = x^k + \left[-(\gamma-1)\frac{\mathbf{x}\cdot\mathbf{u}}{u} + \left(\frac{\gamma}{\gamma_o} - 1\right)\frac{1}{\gamma_o\alpha_o}\right]\frac{u^k}{u}, \tag{24.4}$$

where we have used the metric tensor $\eta_{\mu\nu} = (1,-1,-1,-1)$. In other words, we have the relation, $u_i = -u^i = (-u_x, -u_y, -u_z)$ for i=1,2,3.

To see the generators of the Wu transformation, we first expand the finite transformation in α_o for given β_o to obtain

$$x_I^0 = w\gamma_o + \beta_o\gamma_o\left(-\frac{\mathbf{x}\cdot\mathbf{u}}{u}\right) + \alpha_o\gamma_o w\left[\frac{1}{2}\beta_o w(3\gamma_o^2) + \gamma_o^2\left(-\frac{\mathbf{x}\cdot\mathbf{u}}{u}\right)\right] + O(\alpha_o^2), \tag{24.5}$$

$$x_I^k = x^k + \left[(\gamma_o - 1)\left(-\frac{\mathbf{x}\cdot\mathbf{u}}{u}\right) + w\beta_o\gamma_o\right]\frac{u^k}{u}$$

$$+ w\alpha_o\gamma_o\left[\beta_o\gamma_o^2\left(-\frac{\mathbf{x}\cdot\mathbf{u}}{u}\right) + \frac{1}{2}w(3\gamma_o^2 - 2)\right]\frac{u^k}{u} + O(\alpha_o^2). \tag{24.6}$$

Next we assume $\beta_o = O(\alpha_o)$ and expand the transformation in powers of β_o to obtain[1]

$$x_I^\mu = x^\mu + \beta_o \frac{u_i}{u}\left(-\eta^{\mu 0}\delta_\nu^i - \eta^{\mu i}\delta_\nu^0\right)x^\nu + \alpha_o \frac{u_i}{u}\left(-\eta^{\mu 0}\delta_\nu^i - \frac{1}{2}\eta^{\mu i}\delta_\nu^0\right)x^0 x^\nu, \qquad (24.7)$$

where equations (24.5) and (24.6) have been combined together. The coefficients of β_o and α_o give the generators of the Wu transformations. For example, the coefficients of α_o involve an additional factor x^0. The source of this additional factor is the fact that the constant acceleration has the dimension of inverse length, in contrast to the dimensionless constant velocity. This additional factor implies that these are new Lie algebra elements. To continue to compute the Lie algebra would require bracketing these new elements with the others. Clearly this would introduce further new generators with factors. Therefore, there are infinitely-many generators in the Lie algebra. This conclusion holds in general for finite Wu transformations between non-inertial frames.[1]

24b. Generators of the Wu transformation in cotangent spacetime

We have seen that there are complications associated with the generators in equations (24.7) for the finite Wu transformation. It is more convenient instead to fix a spacetime point and consider the effects of an infinitesimal CLA transformation on the cotangent space at this point. Here the transformations of the differential 4-vectors dx_I^μ and dx^μ can be obtained from (24.3)-(24.4), and can be approximated by

$$dw_I = \gamma(Wdw + \mathbf{u}\cdot d\mathbf{r}) = Wdw + (\beta_o + \alpha_o w)\frac{\mathbf{u}\cdot d\mathbf{r}}{u} + O(2), \qquad (24.8)$$

$$d\mathbf{r}_I = d\mathbf{r} + (\gamma - 1)\frac{\mathbf{u}\cdot d\mathbf{r}}{u^2}\mathbf{u} + \gamma\mathbf{u}Wdw = d\mathbf{r} + (\beta_o + \alpha_o w)\frac{\mathbf{u}Wdw}{u} + O(2), \qquad (24.9)$$

for small α_o and β_o, where we have kept W fixed and used $u(w) \to \beta_o + \alpha_o w$ for small acceleration α_o.

To transform from F to F_I, i.e., (dw,dx,dy,dz) to (dw_I,dx_I, dy_I,dz_I) while preserving the invariance of ds^2 in (35), one can first re-scale dw,

$$(dw,dx,dy,dz) \to (Wdw,dx,dy,dz), \qquad (24.10)$$

and then perform a rotation in the 4-dimensional cotangent spacetime $dx^\mu=(Wdw,dx,dy,dz)$. However, if one first performs a rotation,

$$(dw,dx,dy,dz) \to (dw',dx',dy',dz')=(\gamma[dw+\beta dx],\ \gamma[dx+\beta dw],\ dy,\ dz), \qquad (24.11)$$

and then re-scales the time differential dw',

$$(dw',dx',dy',dz') \to (Wdw',dx',dy',dz'), \qquad (24.12)$$

one does not get the transformation from F to F_I because the invariance of ds^2 has been upset. In particular, $(Wdw')^2-dx'^2-dy'^2-dz'^2 \neq (Wdw)^2-dx^2-dy^2-dz^2$. Therefore, the group operations of a scale-change and that of a 4-dimensional rotation cannot be combined. The separation of these two group operations is necessary in order to see the fact that the Lorentz group generators are actually hidden in the accelerated Wu transformations.

Using matrix notation, the CLA transformations of the differentials dx^μ can be expanded as follows

$$\begin{pmatrix} dw_I \\ dx_I \\ dy_I \\ dz_I \end{pmatrix} = \left[1 - \beta_x i m^{10} - \beta_y i m^{20} - \beta_z i m^{30}\right] \begin{pmatrix} dw^* \\ dx \\ dy \\ dz \end{pmatrix} \equiv \Lambda \begin{pmatrix} dw^* \\ dx \\ dy \\ dz \end{pmatrix}, \qquad (24.13)$$

$$im^{10} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad im^{20} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad im^{30} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\beta_x = \beta_{ox} + \alpha_{ox} w, \quad \text{etc.} \qquad dw^* = W dw,$$

for small velocities and accelerations. This expansion and the following treatment also hold for the small arbitrary velocity function u(w) discussed previously.

With a locally re-scaled time differential $dw^*=Wdw$, transformation (24.13) is formally the same as the Lorentz transformation, except that the velocity $\bar{\beta} = (\beta_x, \beta_y, \beta_z)$ is now time-dependent. The dimensionless "boost-generators" $m^{0k} = K^k$ are defined by

$$m^{0k} = i\left(\frac{\partial \Lambda}{\partial \beta_k}\right)_{\beta_{0i}=0, \alpha_{0i}=0}$$

$$= \left[\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \right] \tag{24.14}$$

$$\beta_k = (\beta_1, \beta_2, \beta_3) = (-\beta_x, -\beta_y, -\beta_z) = -\beta^k.$$

By explicit calculation, the Lie algebra of the Wu transformation closes on the set $\left[m^{\mu\nu}\right]$, where $m^{\mu\nu} = -m^{\nu\mu}$ are given in (24.49) for m^{ko}, and

$$-im^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad -im^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad -im^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{24.15}$$

The dimensionless generators $m^{\mu\nu}$ satisfy the well-known commutation relations

$$\left[m^{\mu\nu}, m^{\alpha\beta}\right] = i\left(\eta^{\nu\alpha}m^{\mu\beta} - \eta^{\nu\beta}m^{\mu\alpha} - \eta^{\mu\alpha}m^{\nu\beta} + \eta^{\mu\beta}m^{\nu\alpha}\right). \tag{24.16}$$

It is interesting to observe that the group of the scale-change of dw is quite different from the 4-dimensional rotational group because the scale change W cannot be obtained through a succession of small changes in the acceleration α_o. Rather, this particular scale-change W of the time differential dw can only be obtained by exponentiation of a 4x4 matrix C with a spacetime-dependent function ln W

$$\exp[-i(\ln W)\, C] \begin{pmatrix} dw \\ dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} Wdw \\ dx \\ dy \\ dz \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{24.17}$$

where C and Λ_c are respectively defined by $C = i\partial\Lambda_c / \partial W$ and

$$\begin{pmatrix} dw^* \\ dx \\ dy \\ dz \end{pmatrix} = \left[1 + (W-1)C\right] \begin{pmatrix} dw \\ dx \\ dy \\ dz \end{pmatrix} \equiv \Lambda_c \begin{pmatrix} dw \\ dx \\ dy \\ dz \end{pmatrix}. \qquad 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{24.18}$$

We note that the momentum 4-vector p^μ has the same transformation property as dx^μ because of the relation $p^\mu = m\,dx^\mu / ds$. Consider the effect of the infinitesimal form of

$$\exp\left(-\frac{1}{2}\varepsilon^{\mu\nu}J_{\mu\nu}\right) \text{ and } \exp\left(-ibD_{00}\right) \tag{24.19}$$

on the momentum 4-vector p^μ, where $J_{\mu\nu}$ is the 'angular-momentum' operator in the 4-momentum space and D_{00} may be termed a 'scale-change' operator:

$$J_{\mu\nu} = i(p_\mu \partial^*_\nu - p_\nu \partial^*_\mu) = -J_{\nu\mu}, \qquad D_{00} = i(p_0 \partial^*_0). \quad \partial^*_\nu = \frac{\partial}{\partial p^\nu}. \tag{24.20}$$

We have

$$\left(1 - \frac{1}{2} i\varepsilon^{\mu\nu} J_{\mu\nu}\right)p^\lambda = p^\lambda + \varepsilon^{\alpha\lambda} p_\alpha, \qquad (-ibD_{00})p^\lambda = +bp_0 \delta^\lambda_0, \tag{24.21}$$

$$\varepsilon^{i0} = -\beta^i = (-\beta^i_o - \alpha^i_o w), \qquad \varepsilon^{ik} = 0, \qquad dx_i = (-dx, -dy, -dz), \tag{24.22}$$

where i, k =1,2,3. The transformation relations in (24.21) for the momentum 4-vector are equivalent to the CLA transformation for the differentials dx^μ. One can verify that the operators $J^{\mu\nu}$ and D^{00} separately form closed Lie algebras. These operators satisfy the commutation relations

$$\left[J^{\mu\nu}, J^{\alpha\beta}\right] = i\left(\eta^{\nu\alpha} J^{\mu\beta} - \eta^{\nu\beta} J^{\mu\alpha} - \eta^{\mu\alpha} J^{\nu\beta} + \eta^{\mu\beta} J^{\nu\alpha}\right), \tag{24.23}$$

$$\left[D^{00}, D^{00}\right] = 0. \tag{24.24}$$

24c. The Wu algebra in a modified momentum space and the classification of particles

Since x^μ is not a 4-vector, one cannot have the conventional covariant generators in the underlying spacetime. In order to construct generators of the Wu algebra, it is essential to use the coordinate 4-vector dx^μ of the cotangent spacetime. Physically, the cotangent spacetime can be interpreted as the 4-dimensional momentum space. A particle's contravariant 4-momentum p^μ is given in (19.29).

In the momentum space, the p^μ are variables while W is fixed, just as in the cotangent spacetime, the dx^μ are variables while W is fixed.

In analogy to (24.10), we may introduce a rescaled momentum vector

$$\overline{p}^\mu = (Wp^0, \mathbf{p}), \tag{24.25}$$

so that the invariant energy-momentum relation in (19.30) can be expressed in terms of $\overline{p}^\mu$ and the metric (4.10):

$$\eta_{\mu\nu}\overline{p}^\mu \overline{p}^\nu = m^2. \tag{24.26}$$

Following Wigner, this invariant relation can be employed to define little groups for both inertial and CLA frames. The differential Wu transformation in the momentum space (associated with both CLA frames and inertial frames) can be written as

$$\overline{p}'^\mu = \Lambda^\mu_\nu \overline{p}^\nu, \tag{24.27}$$

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \varepsilon^\mu_\nu, \qquad \varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}. \tag{24.28}$$

Based on the infinitesimal form of Λ^μ_ν, one can obtain infinitesimal generators $m^{\mu\nu}$, which are 4x4 matrices and satisfy the commutation relation (24.16).

Note that Λ^μ_ν has precisely the same form as that in the Lorentz transformations except that the time-independent velocity β_o is now replaced by a time-dependent velocity β. Thus, it involves three parameters for spatial rotations and three for pure boosts, where these parameters involve both constant acceleration and velocity.

It can be verified that the generators of the differential Wu group in general are given by the operators

$$W^{\mu\nu} = i(\overline{p}^{\mu}\delta^{\nu} - \overline{p}^{\nu}\delta^{\mu}), \qquad \delta_{\nu} = \frac{\partial}{\partial\overline{p}^{\nu}} = \eta_{\nu\lambda}\delta^{\lambda}, \tag{24.29}$$

which also satisfies the commutation relation,

$$\left[W^{\mu\nu}, W^{\alpha\beta}\right] = i\left(\eta^{\nu\alpha}W^{\mu\beta} - \eta^{\nu\beta}W^{\mu\alpha} - \eta^{\mu\alpha}W^{\nu\beta} + \eta^{\mu\beta}W^{\nu\alpha}\right). \tag{24.30}$$

In the 4-momentum space, apart from the invariant (24.26), one can construct another invariant

$$W_{\mu}W^{\mu}, \qquad \text{where} \qquad W^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\overline{p}_{\nu}W_{\alpha\beta}. \tag{24.31}$$

The invariant $\eta_{\mu\nu}\overline{p}^{\mu}\overline{p}^{\nu}$ in (24.26) has a clear physical interpretation as the square of the mass of a particle. To see the physical meaning of $W_{\mu}W^{\mu}$, we make a transformation to the rest frame of the particle, $\overline{p}^{\mu} = (Wp^0, 0, 0, 0) = (m, 0, 0, 0)$. We have

$$W^0 = 0, \qquad W^i = \frac{1}{2}\varepsilon^{i0jk}mW_{jk}\ , \tag{24.32}$$

where the commutation relation of the operators, W^i/m, $i=1,2,3$, behaves like the usual rotational matrices in the 3-dimensional space. Thus, W^i/m can be identified as the spin operator in linearly accelerated frames. This identification is consistent with the limit of zero acceleration in which the Wu transformation reduces to the Lorentz transformations.

In light of previous discussions, the differential Wu spacetime transformation enables us to generalize Wigner's results for the little group for inertial frames to non-inertial frames. In particular, we can classify elementary particles according to the two invariants, $\eta_{\mu\nu}\overline{p}^{\mu}\overline{p}^{\nu}$ and $W_{\mu}W^{\mu}$, in the momentum space as follows

$$\eta_{\mu\nu}\overline{p}^{\mu}\overline{p}^{\nu} > 0, \qquad s=0,1,2,3,.... \text{ (bosons)} \tag{24.33}$$

$$s=1/2,\ 3/2,\ 5/2,.... \text{ (fermions)} \tag{24.34}$$

$$\eta_{\mu\nu}\overline{p}^{\mu}\overline{p}^{\nu} = 0, \qquad s=+s,\ -s, \tag{24.35}$$

$$\eta_{\mu\nu}\overline{p}^{\mu}\overline{p}^{\nu} = 0, \qquad \text{continuous } s \tag{24.36}$$

$$\eta_{\mu\nu}\overline{p}^{\mu}\overline{p}^{\nu} < 0, \qquad \text{(tachyons).} \tag{24.37}$$

These results hold for both inertial frames and non-inertial frames with constant-linear-accelerations. They show a deeper and broader physical meaning for the fundamental concepts of mass and spin.

References

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2. Lorentz and Poincaré invariance alone implies that the spacetime coordinates of an inertial frames must be denoted by the 4-variables $x^\mu = (w,x,y,z)$, so that physical laws can be written in the 4-dimensional symmetry form. This symmetry principle dictates that we treat all 4 variables (w,x,y,z) on an equal footing. It is natural for them all to have the same dimension (e.g., length) and to be measured using the same unit (e.g., centimeter or meter).
3. Jong-Ping Hsu and Leonardo Hsu, Nuovo Cimento B **112**, 575 (1997) and Chin. J. Phys. **35**, 407 (1997). Jong-Ping Hsu, *Einstein's Relativity and Beyond-New Symmetry Approaches,* (World Scientific, Singapore, 2000), Chapters 22-23.

25.

Coordinate Transformations for Frames with a General-Linear-Acceleration

25a. Spacetime transformations based on limiting Lorentz and Poincaré invariance

Inertial frames are idealizations or approximations of non-inertial frames with small accelerations or during short time intervals. Experiments have established that physical laws in inertial frames display the 4-dimensional symmetry of the Lorentz and Poincaré groups. However, almost all physical frames of reference in the universe are, strictly speaking, non-inertial because of the long-range action of the 'gravitational force.' Thus, it is not completely satisfactory that the basic laws of physics and the universal constants are understood or known only in inertial frames but not in non-inertial frames.[1,2] In particular, we already know that the speed of light is not a universal constant in non-inertial frames.

The principle of limiting Lorentz and Poincaré invariance (4-dimensional symmetry) has been used to derive a simple generalization of the Lorentz transformation, i.e., a spacetime transformation involving a constant velocity and a constant linear-acceleration.[2] This set of transformations forms the Wu group, which includes the Møller group and the Lorentz group as limiting cases.[3,4] In this chapter, we apply the same limiting Lorentz and Poincaré invariance to generalize spacetime transformations for frames with arbitrary accelerations along a straight line. These non-inertial frames are called general-linear-acceleration (GLA) frames.

Physically, non-inertial frames are not equivalent to inertial frames. The 4-dimensional spacetime (w,x,y,z) of non-inertial frames can be considered a generalization of the Minkowski spacetime for inertial frames. Let us call such a spacetime "general taiji (GT) spacetime." Since the constant c of the speed of light has no operational meaning in GLA frames, we use w (with the dimension of length) rather than t (or ct) for the generalized evolution variable and call it the "GT time" (or simply time) for GLA frames.[5]

With the help of such GLA transformations, the basic wave equations, truly universal constants and physical properties of spacetime for both inertial and non-inertial frames can be discussed. In addition, the quantization of gauge fields in GT spacetime with non-constant metric tensors could shed light on the quantum theory of gravity.[6]

Let us consider first the implications of limiting 4-dimensonal symmetry for spacetime transformations between a simple non-inertial frame $F_c(w,x,y,z)$ (with constant linear acceleration α_o) and an inertial frame $F_I(w_I, x_I, y_I, z_I)$. Suppose that the frame F_c has both its velocity and acceleration directed along parallel x and x_I axes and that the origins of F_c and F_I coincide at the time $w = w_I = 0$. We have transformations for the constant-linear-acceleration frame $F_c(x)$ and an inertial frame $F_I(x_I)$[4]

$$w_I = \gamma\beta x + \frac{\gamma\beta}{\alpha_o\gamma_o^2} + a_o + w_o, \quad x_I = \gamma x + \frac{\gamma}{\alpha_o\gamma_o^2} + b_o + x_o, \quad y_I = y + y_o, \quad z_I = z + z_o,$$

$$a_o = -\frac{\beta_o}{\alpha_o\gamma_o}, \quad b_o = -\frac{1}{\alpha_o\gamma_o}, \quad \beta = \beta_o + \alpha_o w, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}, \tag{25.1}$$

where the velocity $\beta = \beta(w)$ is a linear function of the time w. The transformation (25.1) with the condition $x_o^\mu = (w_o, x_o, y_o, z_o) = 0$ is called the Wu transformation. It reduces to the Møller transformation when $\beta_o = 0$ and $x_o^\mu = 0$, provided that we make a change of the time variable, $w = (1/\alpha_o)\tanh(\alpha_o w^*)$.[2] One can verify that in the limit of zero acceleration, $\alpha_o \to 0$, the transformation (25.1) reduces to the 4-dimensional transformation

$$w_I = \gamma_o(w + \beta_o x) + w_o, \quad x_I = \gamma_o(x + \beta_o w) + x_o, \quad y_I = y + y_o, \quad z_I = z + z_o. \qquad (25.2)$$

Thus, the limiting 4-dimensional symmetry of the Lorentz and Poincaré invariance is satisfied. The differential form of the Wu transformation (25.1) for constant-linear-acceleration is given by

$$dw_I = \gamma(W_c dw + \beta dx), \qquad dx_I = \gamma(dx + \beta W_c dw), \quad dy_I = dy, \quad dz_I = dz; \qquad (25.3)$$

$$W_c = \gamma^2(\gamma_o^{-2} + \alpha_o x).$$

Thus we have

$$ds_c^2 = g_{c\mu\nu} dx^\mu dx^\nu = dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2, \qquad (25.4)$$

$$g_{c\mu\nu} = (W_c^2(w, x), -1, -1, -1),$$

where $g_{c\mu\nu}$ is the metric tensor for the spacetime of constant-linear-acceleration frames.

Now let us consider a generalization of the inhomogeneous Wu transformation (25.1) to the most general non-inertial frame F(w,x,y,z) moving with an arbitrary velocity $\beta(w)$ or arbitrary acceleration $\alpha(w)$ along the x-axis,

$$\beta(w) = \beta_1(w) + \beta_o, \qquad \alpha(w) = \frac{d\beta(w)}{dw} = \frac{d\beta_1}{dw}, \qquad \beta(0) = \beta_o, \qquad \alpha(0) = \alpha_o. \qquad (25.5)$$

The last two initial conditions are related to the fact that the origins of F(w,x,y,z) and $F_I(w_I, x_I, y_I, z_I)$ coincide at the GT time $w = w_I = 0$. The velocity $\beta(w)$ is an arbitrary function of the GT time w and characterizes the general-linear-acceleration in the x-direction of a non-inertial frame F. This function must be given in order to specify the acceleration of a non-inertial frame. If

$\beta = J_{eo}w^2/2 + \alpha_o w + \beta_o$, then the non-inertial frame has a constant jerk J_{eo} and a variable acceleration $d\beta/dw = J_{eo}w + \alpha_o$, which is linear in the time w. When the jerk J_{eo} vanishes, the non-inertial frame has a constant acceleration α_o.

One of the simplest generalizations of the constant-acceleration case (25.3) is to write the local relation for F(x) and $F_I(x_I)$ in the following form,

$$dw_I = \gamma(W_a dw + \beta dx), \quad dx_I = \gamma(dx + \beta W_b dw), \quad dy_I = dy, \quad dz_I = dz; \tag{25.6}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \beta_1(w) + \beta_o, \qquad \beta^2 < 1,$$

where in general, the two functions $W_a = W_a(w,x)$ and $W_b = W_b(w,x)$ may be different, in contrast to the constant-linear-acceleration case in (25.1). The differential form (25.6) is a minimal departure from the Wu transformations (25.3) for constant accelerations.

The limiting 4-dimensional symmetry dictates that the two unknown functions $W_a(w,x)$ and $W_b(w,x)$ must satisfy the following two integrability conditions for the differential equations in (25.6)

$$\frac{\partial}{\partial x}(\gamma W_a) = \frac{\partial}{\partial w}(\gamma\beta), \qquad \frac{\partial}{\partial x}(\gamma\beta W_b) = \frac{\partial}{\partial w}(\gamma). \tag{25.7}$$

Since γ and β are functions of w only, (25.6) and (25.7) lead to the results

$$w_I = \gamma\beta x + \int \gamma A(w) dw, \qquad x_I = \gamma x + \int \gamma\beta B(w) dw, \tag{25.8}$$

where we obtain W_a and W_b, $W_a = \gamma^2\alpha(w)x + A(w)$ ($W_b = \gamma^2\alpha(w)x + B(w)$), from (25.7) and substitute them in (25.6) to carry out the integration. The guiding principles of limiting 4-dimensional symmetry and minimal departure from (25.1) suggest that the two integrals in (25.8) have the following forms involving an arbitrary acceleration function $\alpha(w)$ [7]

$$\int \gamma A(w)dw = \gamma\beta\frac{1}{\alpha(w)\gamma_o^2} + a_o, \tag{25.9}$$

$$\int \gamma\beta B(w)dw = \gamma\frac{1}{\alpha(w)\gamma_o^2} + b_o. \tag{25.10}$$

Such a generalization for spacetime transformations is essentially an assumption guided by the 4-dimensional symmetry of the Lorentz and Poincaré groups and by minimal departure from the Wu and Møller transformations for constant-linear-acceleration frames. By differentiations of (25.9) and (25.10), we can determine the two functions A(w) and B(w),

$$A(w) = \frac{\gamma^2}{\gamma_o^2} - \frac{\beta J_e}{\alpha^2(w)\gamma_o^2}, \qquad B(w) = \frac{\gamma^2}{\gamma_o^2} - \frac{J_e}{\beta\alpha^2(w)\gamma_o^2}, \qquad J_e(w) = \frac{d\alpha}{dw} = \frac{d^2\beta}{dw^2}, \tag{25.11}$$

where $J_e(w)$ is the 'jerk,' which is the third-order time derivative of the coordinates.

From equations (25.8) through (25.10), we obtain a simple and general spacetime transformations for non-inertial frames with the most general linear-acceleration[8]

$$w_I = \gamma\beta\left(x + \frac{1}{\alpha(w)\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o} + w_o, \tag{25.12}$$

$$x_I = \gamma\left(x + \frac{1}{\alpha(w)\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o} + x_o, \quad y_I = y + y_o, \quad z_I = z + z_o,$$

where the two constants of integration $a_o = -\beta_o/(\alpha_o\gamma_o)$ and $b_o = -1/(\alpha_o\gamma_o)$ are determined by the limiting 4-dimensional symmetry as $\alpha(w) = \alpha_o \to 0$. The relations in (25.12) may be termed "general taiji (GT) transformations" for a frame F(x) with a general linear acceleration $\alpha(w)$, which is an arbitrary function of w. In the following discussions, we shall set $x_o^\mu = 0$.

The GT transformations for the differentials dx^μ and dx_I^μ can be obtained from (25.12):

$$dw_I = \gamma(W_a dw + \beta dx), \quad dx_I = \gamma(dx + \beta W_b dw), \quad dy_I = dy, \quad dz_I = dz; \tag{25.13}$$

$$W_a = \gamma^2\left(\alpha x + \frac{1}{\gamma_o^2}\right) - \frac{\beta J_e(w)}{\alpha^2\gamma_o^2} > 0, \qquad W_b = \gamma^2\left(\alpha x + \frac{1}{\gamma_o^2}\right) - \frac{J_e(w)}{\beta\alpha^2\gamma_o^2} > 0 .$$

The invariant interval ds^2 in GLA frames can be obtained from (25.13),

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{25.14}$$

$$g_{00} = W^2, \qquad g_{01} = g_{10} = U, \qquad g_{11} = g_{22} = g_{33} = -1,$$

$$W^2 = \left[\gamma^2\left(\alpha x + \frac{1}{\gamma_o^2}\right)\right]^2 - \left[\frac{J_e(w)}{\alpha^2\gamma_o^2}\right]^2 > 0, \qquad U = \frac{J_e(w)}{\alpha^2\gamma_o^2}.$$

Mathematically, transformation (25.13) can be obtained by making the replacement $(dw,\ dx) \to (dw^*,\ dx^*) = (W^*dw,\ dx - Udw)$, where $W^* = \gamma^2(\alpha x + 1/\gamma_o^2)$, and then performing a 4-dimensional rotation in the w-x plane with y and z axes fixed:

$$dw_I = \gamma(dw^* + \beta dx^*), \qquad dx_I = \gamma(dx^* + \beta dw^*), \qquad dy_I = dy, \qquad dz_I = dz;$$

$$dw^* \equiv W^* dw, \qquad dx^* \equiv dx - Udw .$$

When the jerk $J_e(w) = d\alpha(w)/dw$ vanishes, we have $\alpha(w) = \alpha_o$ and one can see that the GT transformation (25.12) reduces to the Wu transformation (25.1) for a constant-linear-acceleration frame $F_c(x)$ in which the time axis is everywhere orthogonal to the spatial coordinate curves. The contravariant metric tensors for a general non-inertial frame can be obtained from (25.14):

$$g^{00} = \frac{1}{W^2 + U^2}, \qquad g^{01} = g^{10} = \frac{U}{W^2 + U^2},$$

$$g^{11} = \frac{-W^2}{W^2 + U^2}, \qquad g^{22} = g^{33} = -1. \tag{25.15}$$

Using (25.14) and (25.15), one can verify that $g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$ and that all other components vanish.

If $x^\mu_o = 0$, the inverse of the GT transformation (12) is found to be

$$\beta(w) = \frac{w_I + \beta_o/(\alpha_o \gamma_o)}{x_I + 1/(\alpha_o \gamma_o)}, \tag{25.16}$$

$$x = \sqrt{\left(x_I + \frac{1}{\alpha_o \gamma_o}\right)^2 - \left(w_I + \frac{\beta_o}{\alpha_o \gamma_o}\right)^2} - \frac{1}{\alpha(w)\gamma_o^2}, \qquad y = y_I, \qquad z = z_I.$$

If a specific function for $\beta(w)$ is given and one can solve for the time w in terms of $\beta(w)$, then the first equation in (25.16) can be written in the form $w=w(\beta)$. For example, the simplest generalization of constant-linear acceleration is the case with a constant jerk, J_{eo}=constant or $\beta = J_{eo} w^2/2 + \alpha_o w + \beta_o$. When w is positive, we have

$$w = \frac{1}{J_{eo}} \left[-\alpha_o + \sqrt{\alpha_o^2 + 2J_{eo}(\beta - \beta_o)} \right], \tag{25.17}$$

where $\beta = \beta(w)$ is given by (25.16), so that we have the transformation for time, $w=w(w_I, x_I)$.

25b. Physical implications and discussion

The coordinates x^μ specified by the metric tensor in (25.15) for a general-linear-acceleration frame are the general taiji (GT) spacetime. They are the preferred coordinates for the general taiji transformation with limiting 4-dimensional symmetry.[9] Other choices of coordinates will not satisfy the limiting 4-dimensional symmetry. Thus, the present theory of spacetime for general-linear-acceleration frames is not a general covariant theory, in contrast to the general theory of relativity. Furthermore, the Riemann curvature tensor of the GT spacetime vanishes, as implied by the GT transformation (25.12). Other physical implications are as follows:

(A) Spacetime-Dependent Speed of Light in Non-Inertial Frames

It is known that the law for the propagation of light is ds=0 in an inertial frame. Thus, the propagation of light in a non-inertial frame is described by the same invariant law (25.14) with ds=0. In order to see the property of light in a GLA frame, let us consider some specific and simple cases. Suppose a light signal moves along the x-axis, i.e., dy=dz=0, the speed of light β_{Lx} is found to be

$$\beta_{Lx} = \frac{dx}{dw} = \gamma^2(\alpha x + \gamma_o^{-2}) + \frac{J_e(w)}{\alpha^2\gamma_o^2}, \qquad \frac{dy}{dw} = \frac{dz}{dw} = 0, \tag{25.18}$$

which is certainly different from the speed of light $\beta_L = 1$ (derived from $ds^2 = dw_I^2 - dx_I^2 = 0$) in an inertial frame. If the signal moves in the y-direction, i.e., dx=dz=0, eq. (25.14) with ds=0 leads to the speed of light β_{Ly},

$$\beta_{Ly} = \frac{dy}{dw} = \sqrt{\left[\gamma^2\left(\alpha x + \frac{1}{\gamma_o^2}\right)\right]^2 - \left[\frac{J_e(w)}{\alpha^2\gamma_o^2}\right]^2}, \qquad \frac{dx}{dw} = \frac{dz}{dw} = 0. \tag{25.19}$$

(B) The Velocity-Addition Laws in Non-Inertial Frames

In general, the law for velocity addition can be obtained from (25.13),

$$\frac{dx_I}{dw_I} = \frac{dx/dw + \beta W_b}{W_a + \beta dx/dw}, \tag{25.20}$$

$$\frac{dy_I}{dw_I} = \frac{dy/dw}{\gamma(W_a + \beta dx/dw)}, \qquad \frac{dz_I}{dw_I} = \frac{dz/dw}{\gamma(W_a + \beta dx/dw)}.$$

In particular, if $\beta_L = dx_I/dw_I = 1$, then the velocity-addition law (25.20) leads to the same spacetime-dependent speed of light (25.18) in GLA frames.

(C) The General Taiji Group

Let us consider two other GLA frames F' and F'', which are respectively characterized by arbitrary velocities $\beta'(w')$, $\beta''(w'')$, initial accelerations α'_o, α''_o, and initial velocities β'_o, β''_o. We have GT transformations among F_I, F, F' and F''. One can show that the GT transformation between F'(x') and F''(x'') has the same form as that for F and F' frames. Other group properties can also be verified.

Thus, the general taiji transformations form a group, which may be called 'general taiji (GT) group.'[10] This GT group involves one arbitrary acceleration function $\alpha(w)$ and two parameters, i.e., the initial acceleration α_o and the initial velocity β_o. The GT group includes the Wu and Møller groups (for constant linear accelerations) and the Lorentz and Poincaré groups as special limiting cases.

(D) Physical Time in Inertial and Non-Inertial Frames

The usual time t (measured in seconds, for example) and the universal constant c (measured in m/s) do not exist within this theory in general. The physical time w, called taiji time, has the dimension of length and can be realized physically by 'computerized clocks'[5] (see chapter 18). Since the variable speed of light is given by the function (25.18), it is very complicated to use such a non-constant speed of light to synchronize clocks in a GLA frame F(w,x,y,z). However, it is not necessary in general to use light signals to synchronize clocks.[2] One can use a grid of 'computer clocks' in F to realize the taiji-time w. A computer clock can accept information concerning its position in the F_I frame, obtain w_I from the nearest F_I clock, and then compute and display w using the inverse transformation of (25.12). In the case of constant jerk, we have the transformation of time w (25.17), which includes the transformation of time, $w = \gamma_o(w_I - \beta_o x_I)$ in relativity theory as a special limiting case when $J_{eo} \to 0$, and $\alpha_o \to 0$.

In a GLA frame, time w is restricted by $\beta^2(w) < 1$ and space is limited by $x > -1/(\alpha(w)\gamma_o^2$, as shown in (25.12) and (25.16).

(E) Classical Electrodynamics in Inertial and Non-Inertial Frames

Let us consider classical electrodynamics in GLA Frames. For a continuous charge distribution in space, the invariant action for the electromagnetic fields and their interaction is assumed to be

$$S_{em} = -\int\left[a_\mu j^\mu + \frac{1}{4} f_{\mu\nu} f^{\mu\nu}\right]\sqrt{-g}\, d^4x, \tag{25.21}$$

$$\sqrt{-g} = \sqrt{-\det g_{\alpha\beta}} = \sqrt{W^2 + U^2} = \gamma^2\left(\alpha x + \frac{1}{\gamma_o^2}\right) > 0,$$

$$f_{\mu\nu} = D_\mu a_\nu - D_\nu a_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu,$$

where D_ν denotes the partial covariant derivative. The most general Maxwell equations for both inertial and non-inertial frames can be derived from the action S_{em}.

(F) Truly Universal Constants in Both Inertial and Non-Inertial Frames

Since the speed of light c in an accelerated frame F is no longer a universal constant, the invariant action for a charged particle moving in the electromagnetic 4-potential $a_\mu(x)$ in a GLA frame is assumed to be

$$S=\int \left(-mds-\bar{e}a_\mu dx^\mu\right), \qquad \bar{e}=-1.6021891\times10^{-20}\sqrt{4\pi}\sqrt{g\cdot cm}, \tag{25.22}$$

where $\bar{e}$ is the electric charge. The action S is formally the same as that in a CLA frame.[2]

Following the same reasoning as that for CLA frames,[2] the truly universal and fundamental constants in both inertial and non-inertial frames are the quantum constant, $J=3.5177293\times10^{-38}\,g\cdot cm$, and the electric charge given by (25.22) [or $\alpha_e=\bar{e}^2/(4\pi J)=1/137.036$]. It is interesting that these universal constants for non-inertial frames with general-linear-accelerations turn out to be precisely the same as those in taiji relativity, which was formulated solely on the basis of the first principle of relativity.[5]

(G) Generalized Klein-Gordon and Dirac Equations for Non-Inertial Frames

The general Klein-Gordon and Dirac equations for both inertial and non-inertial frames are

$$\left[g^{\mu\nu}(iJD_\mu-\bar{e}a_\mu)(iJD_\nu-\bar{e}a_\nu)-m^2\right]\phi=0, \tag{25.23}$$

$$J\Gamma^\mu(i\partial_\mu-\bar{e}a_\mu)\psi+\frac{1}{2}iJ(\partial_\mu\Gamma^\mu)\psi+\frac{1}{2}(\partial_\mu \ln\sqrt{-g})iJ\Gamma^\mu\psi-m\psi=0, \tag{25.24}$$

where $\{\Gamma^\mu,\Gamma^\nu\}=2g^{\mu\nu}(x)$. The Dirac equation (25.24) can be derived from a symmetrized Lagrangian.[11]

References

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5. Such a taiji time w appears naturally in the theory of spacetime for inertial frames based solely on the first postulate of relativity, without making any postulate regarding the speed of light. See Jong-Ping Hsu and Leonardo Hsu, Phys. Lett., A**196**, 1 (1994) and ref. 2
6. See, for example, Ed. J. P. Hsu and D. Fine, *100 Years of Gravity and Accelerated Frames---The Deepest Insights of Einstein and Yang-Mills* (World Scientific, 2005) pp. xix-xxxvii and Chapters A, C, and D.
7. The generalization with this replacement $\alpha_o \to \alpha(w)$ (for the accelerations in the denominators of (25.9) and (25.10)) is crucial. Mathematically, this replacement implies that the two variables w and x in W(w,x) cannot be separated. In contrast, a general transformation for GLA frames was discussed in a previous paper based on the separation of w and x in W(w,x). See J. P. Hsu, in *FRONTIERS OF PHYSICS AT THE MILLENNIUM, SYMP* (Ed. Y. L.

Wu and J. P. Hsu, World Scientific, 2001). The generality in this conference paper turned out to be restricted and hence, not completely satisfactory because the additional assumption of separation of variables, $W(w,x)=W_1(w)W_2(x)$, prevents the general-linear-acceleration transformation from being full realized.

8. Note that although both transformations in (25.12) and (20.3) involve an arbitrary function $\beta(w)$, the factor $[x+1/(\alpha(w)\gamma_o^2)]$ in (25.12) differs from that $[x+1/(\alpha_o\gamma_o^2)]$ in (25.3). Following the steps in (19.30)-(19.31) and using (25.13)-(25.14), one can show that $(dp_I^0/dx_I)_x=(m/\gamma\beta W_b)d(\gamma W_a/W)/dw \neq$ constant, for a particle at rest in the accelerated frame F. This result is in sharp contrast to constant $(dp_I^0/dx_I)_x$ in (20.21).

9. This is analogous to the fact that Cartesian coordinates are the preferred coordinates for the Lorentz transformations.

10. Jong-Ping Hsu, Chin. J. Phys. **40**, 265 (2002).

11. Such a Lagrangian involves $(iJ/2)[\overline{\psi}\Gamma^\mu\partial_\mu\psi-(\partial_\mu\overline{\psi})\Gamma^\mu\psi]$. Γ^μ can be related to the constant Dirac matrix γ^a by $\Gamma^\mu=\gamma^a e_a^\mu$, where the tetrad, e_a^μ, a=0,1,2,3, is a set of four mutually orthogonal unit vectors. In field theory with the Poincaré or the de Sitter group as the gauge group, the gauge invariant Lagrangian must have the tetrad as a 'scale gauge field,' in addition to the usual gauge fields. See J. P. Hsu, Phys. Lett. **119B**, 328 (1982).

26.

A Taiji Rotational Transformation with Limiting 4-Dimensional Symmetry

26a. A smooth connection between rotational and inertial frames

The 4-dimensional symmetry framework is the foundation of relativity theory and is perhaps the most thoroughly tested symmetry principle in the 20th century. It is the mathematical manifestation of the Poincaré-Einstein principle of relativity, i.e., the 4-dimensional symmetry (or invariance) of physical laws, which is extremely powerful in helping us to understand physics.[1] The power of 4-dimensional symmetry will be demonstrated by an analysis from the novel viewpoint of rotating non-inertial frames.

"Is it conceivable that the principle of relativity also holds for systems which are accelerated relative to each other?" This was the question the young Einstein asked in 1907.[2] The answer to Einstein's question is *affirmative* only in a limiting sense because any non-inertial frame must reduce to an inertial frame in the limit of zero acceleration, as discussed in section 18a. This indicates that the 4-dimensional symmetry of physical laws must hold in the limit of zero acceleration for all non-inertial frames. We call this the principle of limiting 4-dimensional symmetry. As we have seen in previous chapters, we are able to obtain a "minimal generalization" of the Lorentz transformation for non-inertial frames with constant-linear-accelerations. A "minimal generalization" means that the resultant equations involve a minimal departure from the limiting case of zero acceleration. The resultant spacetime transformations for frames with constant-linear-acceleration have shown that a "constant acceleration" is better defined relativistically as a constant change in a particle's energy per unit distance traveled rather than the traditional definition of the rate of change of velocity with time in an inertial frame or laboratory.

In this chapter, we show that the affirmative answer to Einstein's question also holds in a limiting sense for rotational frames.[3] The principle of "limiting 4-dimensional symmetry" enables us to obtain a satisfactory minimal extension of the conventional classical rotational transformation to a frame whose origin moves in a circle of non-zero radius around a fixed point in an inertial frame. Our results are consistent with high energy experiments involving unstable particles in a circular storage ring and are smoothly connected to the 4-dimensional transformations for inertial frames in the limit of zero acceleration. Furthermore, they turn out to support Pellegrini-Swift's analysis of the Wilson experiment,[4] in which they pointed out that rotational transformations cannot be locally replaced by Lorentz transformations.

26b. A taiji rotational transformation with limiting 4-dimensional symmetry

Suppose $F_I(w_I,x_I,y_I,z_I)$ is an inertial laboratory frame and $F(w,x,y,z)$ is a non-inertial frame whose origin moves in a circle of radius R around the origin of F_I with an 'angular velocity' Ω in such a way that the y-axis of the F frame always points to the origin of $F_I(x_I^\mu)$. Physical quantities without the subscript "I" are those measured by observers in the orbiting frame $F(w,x,y,z)$.

In our discussions, we use w_I and w, respectively, as the evolution variables in F_I and F. As previously described, this 'time' w can be physically realized in an orbiting frame without relying on light signals. To avoid confusion, let us call w the "taiji time" which will reduce to Einstein time in the limit of zero acceleration. An operational meaning for w will be given later.

Based on limiting 4-dimensional symmetry considerations and the classical rotational transformation

$$w_I = w, \qquad x_I = x\cos(\Omega w) - y\sin(\Omega w),$$
$$y_I = x\sin(\Omega w) + y\cos(\Omega w), \qquad z_I = z; \tag{26.1}$$

we write the general rotational transformation for $F_I(x_I^\mu)$ and $F(x^\mu)$ in the form:

$$w_I = Aw + B\boldsymbol{\rho}\cdot\boldsymbol{\beta}, \qquad x_I = Gx\cos(F\Omega w) - E(y-R)\sin(F\Omega w),$$

$$y_I = Ix\sin(F\Omega w) + H(y-R)\cos(F\Omega w), \qquad z_I = z; \tag{26.2}$$

$$\boldsymbol{\rho} = (x,y), \qquad \mathbf{S} = (x, y-R), \quad \boldsymbol{\beta} = \boldsymbol{\Omega} \times \mathbf{S}, \qquad \boldsymbol{\Omega} = (0,0,\Omega).$$

Note that the constant 'angular velocity' Ω is measured in terms of w by observers in F:

$$\Omega = \frac{d\phi}{dw}. \tag{26.3}$$

Thus, the angular velocity Ω has the dimension of inverse length and hence, the 'velocity' $\boldsymbol{\beta} = \boldsymbol{\Omega}\times\mathbf{S}$ and Ωw are dimensionless. The functions A, B, E, F, G, H and I may, in general, depend on the coordinates x^μ and will be determined by the limiting 4-dimensional symmetry principle.

In the limit $R\rightarrow 0$ and small Ω (or $|\boldsymbol{\Omega}\times\mathbf{S}|=\beta<<1$), transformation (26.1) should reduce to the classical rotational transformation (26.1). Thus, we have

$$-E\approx G\approx H\approx I\approx 1, \qquad F\approx 1, \qquad \text{for small } \Omega. \tag{26.4}$$

Furthermore, when R=0, the rotational transformation (26.1) must exhibit an x-y symmetry. This implies that

$$-E = G = H = I; \qquad \text{for } R = 0. \tag{26.5}$$

On the other hand, in the limit where $R\rightarrow\infty$ and $\Omega\rightarrow 0$ such that their product $R\Omega = \beta_o$ is a non-zero constant velocity along the x_I-axis, (26.1) must reduce to the 4-dimensional form

$$w_I = \gamma_o(w + \beta_o x), \qquad x_I = \gamma_o(x + \beta_o w), \qquad y_I = y, \qquad z_I = z; \tag{26.6}$$

$$\gamma_o = \frac{1}{\sqrt{1-\beta_o^2}}, \qquad \beta_o = R\Omega.$$

Note that we have $y_I = -\infty$ instead of $y_I = y$ because $R\rightarrow\infty$ in this limit.

Finally, the existence of the limiting spacetime transformation should hold for *both finite and differential forms* of the transformation. Thus, we also have the well-defined differential form,

$$dw_I = \gamma_o(dw + \beta_o dx), \quad dx_I = \gamma_o(dx + \beta_o dw), \quad dy_I = dy, \quad dz_I = dz. \tag{26.7}$$

The limiting requirements in (26.6) and (26.7) lead respectively to

$$A = B = G = -EF = \gamma_o, \tag{26.8}$$

and $\quad H = 1. \qquad (26.9)$

These limiting 4-dimensional symmetry relations in (26.5), (26.8) and (26.9) do not lead to a unique solution for the unknown functions. This situation is analogous to the case in which gauge symmetry does not uniquely determine the electromagnetic action[5] and one must in addition postulate a minimal electromagnetic coupling. Here, we postulate the minimal generalization of the classical rotational transformation (26.1). Based on the relations in (26.5), (26.8) and (26.9), it is natural for our transformation to have the following two properties: (i) The functions A, B, G, I and $-$ E are simply extended from $\gamma_o=(1-\beta_o^2)^{-1/2}$ to $\gamma=(1-\beta^2)^{-1/2}$, where $\beta=S\Omega$, for non-vanishing 'angular velocity' Ω and finite R. (ii) Only the function H depends on R, i.e., $H = \gamma$ for $R = 0$ and $H = 1$ for $R\to\infty$. Thus, the limiting relations (26.5), (26.8), (26.9) together with the requirement of a minimal generalization of (26.1) lead to the simplest and the most natural solution for the unknown functions in (26.1):

$$\begin{gathered} A = B = G = I = \gamma = (1 - \beta^2)^{-1/2}, \qquad E = -\gamma, \\ F = 1, \quad H = \frac{(\gamma + R/R_o)}{(1 + R/R_o)}, \end{gathered} \tag{26.10}$$

where $\beta = |\boldsymbol{\Omega}\times\mathbf{S}| < 1$ and H is also obtained by requiring it to be the simplest function involving only the first power of γ. As we shall see below, this simplest solution (26.10) turns out to be consistent with all existing experiments of particles' energy-momentum in high energy accelerators involving rotational motion. The exact general rotational transformation of spacetime is thus given by (26.1) and (26.10)

$$w_I = \gamma(w + \boldsymbol{\rho}\cdot\boldsymbol{\beta}) = \gamma(w + xR\Omega), \quad x_I = \gamma[x\cos(\Omega w) - (y - R)\sin(\Omega w)], \tag{26.11}$$

$$y_I = \gamma[x\sin(\Omega w) + (H/\gamma)(y-R)\cos(\Omega w)], \qquad z_I = z.$$

It is necessary for A, B, G, I and -E in (26.2) to approach γ in the limit $R \to \infty$ so that the rotational spacetime transformation satisfies the limiting 4-dimensional symmetry. To avoid confusion, let us call (26.11) the "taiji rotational transformation" in which w_I and w are the evolution variables. Let us call $(w,\mathbf{r})$ the '*limiting Cartesian coordinates*' to distinguish them from the usual Cartesian coordinates $(w_I,\mathbf{r}_I)$. The metric tensor $g_{\mu\nu}$ for the space of a rotating frame is given in (26.19) below. The presence of the γ-factor and the 'scaling factor' H in (26.11) are new properties implied by the limiting 4-dimensional symmetry.

It is tedious but straightforward to show that the set of taiji rotational transformations (26.11) form a "taiji rotational group" which includes the Lorentz group as a special case.

26c. Physical properties of the taiji rotational transformation

Presumably, the physical effects related to the terms R/R_o in (26.10) can be observed only in rotational motion with a large R. When the term R/R_o is small and negligible, $H/\gamma \approx 1$. In this case, the taiji rotational transformation (26.11) become very simple and closely resembles the classical form (26.1). In fact, all previous rotational experiments have been performed under the condition R=0. It appears that this approximation is sufficient for experiments in Earth laboratories. From now on, we shall ignore R/R_o so that (26.11) becomes

$$w_I = \gamma(w + \boldsymbol{\rho}\cdot\boldsymbol{\beta}) = \gamma(w + xR\Omega), \qquad x_I = \gamma[x\cos(\Omega w) - (y-R)\sin(\Omega w)],$$
$$y_I = \gamma[x\sin(\Omega w) + (y-R)\cos(\Omega w)], \qquad z_I = z. \tag{26.12}$$

The inverse transformation of (26.12) is then

$$w = (w_I - \gamma xR\Omega)/\gamma, \qquad x = [x_I\cos(\Omega w) + y_I\sin(\Omega w)]/\gamma,$$
$$y - R = [-x_I\sin(\Omega w) + y_I\cos(\Omega w)]/\gamma, \qquad z = z_I. \tag{26.13}$$

We stress that the exact taiji rotational transformation (26.11) can only be expressed by using the *limiting Cartesian coordinates* for the rotating frame F and cannot be written in terms of say, cylindrical coordinates, in sharp contrast to the conventional classical rotational transformation. This property is dictated by the limiting 4-dimensional symmetry and is analogous to the fact that Cartesian coordinates are preferred in the Lorentz transformation.

In order to discuss experimental results, it suffices to concentrate on the special case R=0 of (26.11):

$$w_I = \gamma w, \quad x_I = \gamma[x\cos(\Omega w) - y\sin(\Omega w)], \quad y_I = \gamma[x\sin(\Omega w) + y\cos(\Omega w)],$$

$$z_I = z; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \rho\Omega. \tag{26.14}$$

Let us compare the taiji rotational transformation (26.14) and the classical rotational transformation (26.1):

(A) The taiji rotational transformation and the Lorentz transformation are both exact and consistent with four-dimensional symmetry. In contrast, the classical rotational transformation (26.1) and the Galilean transformation are only approximately true to the first order in the 'velocity' β.

(B) The taiji rotational transformation (26.14) predicts that *the length of a rotating radius* $(x^2+y^2)^{1/2}$ *is contracted by a* γ *factor*:

$$x_I^2 + y_I^2 = \gamma^2[x^2 + y^2], \tag{26.15}$$

independent of the variables w and w_I. If one replaces a rotational transformation during a very short time interval by a Lorentz transformation, one would be led to a completely different conclusion, namely, that a rotating radius does not contract because it is always perpendicular to the direction of motion. The property (26.15) is interlocked with the flat spacetime of the rotating frame, as shown in (26.14).

(C) For a fixed γ, (26.14) gives $\Delta t_I = \gamma\Delta t$ which is independent of the spatial distance between two events, if we set $w = ct$ and $w_I = ct_I$. That is, clocks at rest relative to a rotating frame and located at a distance $\rho = (x^2 + y^2)^{1/2}$ from the center of rotation slow down by a factor of $\gamma = (1 - \beta^2)^{-1/2}$ in comparison with clocks in the inertial frame F_I. Because there is no relativity for accelerated frames, this is an absolute effect in that observers in both F_I and F agree that it is the accelerated clocks that are slowed. This effect is implied by the limiting four-dimensional symmetry.

Certain properties of the taiji rotational transformation (26.14) have been confirmed by high-energy particle kinematics experiments and "lifetime dilatation",[6] as we shall see later. In a sense, the unexpected relation (26.15) has been indirectly verified because the confirmed results (see equations in (26.31) and (26.32) below) are intimately related to the presence of the γ factors in (26.14) and, hence, (26.15).

26d. The metric tensors for the spacetime of rotating frames

The transformation of the contravariant 4-vectors $dx_I^\mu=(dw_I,dx_I,dy_I,dz_I)$ and $dx^\mu=(dw,dx,dy,dz)$ can be derived from (26.11)

$$dw_I = \gamma[dw + (\gamma^2\Omega^2 wx)dx + (\gamma^2\Omega^2 wy)dy],$$

$$dx_I = \gamma\{[\cos(\Omega w) + \gamma^2\Omega^2 x^2 \cos(\Omega w) - \gamma^2\Omega^2 xy\sin(\Omega w)]dx$$

$$-[\sin(\Omega w) + \gamma^2\Omega^2 y^2 - \gamma^2\Omega^2 xy]dy - [\Omega x\sin(\Omega w) + \Omega y\cos(\Omega w)]dw\},$$

$$dy_I = \gamma\{[\sin(\Omega w) + \gamma^2\Omega^2 x^2 \sin(\Omega w) + \gamma^2\Omega^2 xy\cos(\Omega w)]dx \tag{26.16}$$

$$+[\cos(\Omega w) + \gamma^2\Omega^2 y^2 \cos(\Omega w) + \gamma^2\Omega^2 xy\sin(\Omega w)]dy$$

$$+[\Omega x\cos(\Omega w) - \Omega y\sin(\Omega w)]dw\},$$

$$dz_I = dz.$$

To find the metric tensor $g_{\mu\nu}$ in the rotating frame F with $R = 0$, it is convenient to use (26.11) to write $ds^2 = dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2$ in the following form first:

$$ds^2 = d(\gamma w)^2 - (x^2 + y^2)\gamma^2\Omega^2 dw^2 - d(\gamma x)^2 - d(\gamma y)^2 - dz^2$$

$$+2\gamma\Omega y d(\gamma x)dw - 2\Omega\gamma x d(\gamma y)dw. \tag{26.17}$$

Then with the help of the relation $d\gamma = \gamma^3\Omega^2(xdx+ydy)$, (26.17) can be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \qquad \mu, \nu = 0, 1, 2, 3; \tag{26.18}$$

where the non-vanishing components of $g_{\mu\nu}$ are given by

$$g_{00} = 1, \qquad g_{33} = -1, \qquad g_{11} = -\gamma^2\left[1 + 2\gamma^2\Omega^2 x^2 - \gamma^4\Omega^4 x^2(w^2 - x^2 - y^2)\right],$$

$$g_{22} = -\gamma^2\left[1 + 2\gamma^2\Omega^2 y^2 - \gamma^4\Omega^4 y^2(w^2 - x^2 - y^2)\right],$$

$$g_{01} = \gamma^2\left[\Omega y + \gamma^2\Omega^2 wx\right], \qquad g_{02} = \gamma^2\left[-\Omega x + \gamma^2\Omega^2 wy\right], \tag{26.19}$$

$$g_{12} = -\gamma^4\Omega^2 xy\left[2 - \gamma^2\Omega^2(w^2 - x^2 - y^2)\right].$$

The contravariant metric tensor $g^{\mu\nu}$ is found to be

$$g^{00} = \gamma^{-2}\left[1 - \Omega^4 w^2(x^2 + y^2)\right], \qquad g^{33} = -1,$$

$$g^{11} = -\gamma^{-2}\left[\gamma^{-2}(1 - \Omega^2 x^2) - 2\gamma^{-2}\Omega^3 wxy + \Omega^6 w^2 y^2(x^2 + y^2)\right],$$

$$g^{22} = -\gamma^{-2}\left[\gamma^{-2}(1 - \Omega^2 y^2) + 2\gamma^{-2}\Omega^3 wxy + \Omega^6 w^2 x^2(x^2 + y^2)\right], \tag{26.20}$$

$$g^{01} = -\gamma^{-2}\left[-\Omega y - \gamma^{-2}\Omega^2 wx + \Omega^5 w^2 y(x^2 + y^2)\right],$$

$$g^{02} = -\gamma^{-2}\left[\Omega x - \gamma^{-2}\Omega^2 wy - \Omega^5 w^2 x(x^2 + y^2)\right],$$

$$g^{12} = \gamma^{-2}\left[\gamma^{-2}\Omega^2 xy - \gamma^{-2}\Omega^3 w(x^2 - y^2) + \Omega^6 w^2 xy(x^2 + y^2)\right].$$

The components of this tensor can be obtained using the momentum transformation (26.29) below and the invariant relation, $p_{I\nu}p_I^{\nu} = g^{\mu\nu}p_\mu p_\nu$. All other components in (26.19) and (26.20) are zero. Indeed, we have also verified $g_{\mu\lambda}g^{\lambda\nu} = \delta^\nu{}_\mu$ based on (26.19) and (26.20).

26e. The invariant action for electromagnetic fields and charged particles in rotating frames and truly fundamental constants

We are now able to write the invariant action S_{em} in a rotating frame for a charged particle with mass m and charge $\bar{e}$ moving in the 4-potential a_μ:

$$S_{em} = \int [-mds - \bar{e}a_\mu dx^\mu - \frac{1}{4}f_{\mu\nu}f^{\mu\nu}(-\det g_{\alpha\beta})^{1/2}d^4x], \tag{26.21}$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \qquad \partial_\lambda = \frac{\partial}{\partial x^\lambda}, \tag{26.22}$$

where ds is given by (26.18) and (26.19). Each term in (26.21) must have the same dimension of mass·length. In other words, $\bar{e}a_\mu$ must have the dimension of mass. This can be satisfied if $\bar{e}$ and a_μ are related to the usual charge e measured in electrostatic units (esu) and the usual 4-potential A_μ in special relativity (SR) by the relation

$$\bar{e} = \frac{e}{c} = -1.6021891 \times 10^{-20}\sqrt{4\pi} \ \ (\text{g cm})^{1/2}, \tag{26.23}$$

$$a_\mu(w,\mathbf{r}) \leftrightarrow \left[\frac{A_\mu(t,\mathbf{r})}{c}\right]_{SR},$$

where e/c is in Heaviside-Lorentz units and the symbol $\leftrightarrow$ denotes correspondence. If one directly uses w as the evolution variable and does not set w=ct in (26.21), then the *truly universal* electric charge[7] for rotating frames is $\bar{e}$ which happens to be the same as the charge measured in electromagnetic unit (emu). Also, the potential a_μ has the dimension $(g/cm)^{1/2}$, so that $\bar{e}\, a_\mu$ has the correct dimension of mass.

The Lagrange equation of motion of a charged particle can be derived from (26.21). We obtain

$$m\frac{Du_\mu}{ds} = \bar{e} f_{\mu\nu}u^\nu, \qquad u^\nu = \frac{dx^\nu}{ds}, \qquad u_\mu = g_{\mu\nu}u^\nu, \tag{26.24}$$

where $Du_\mu = u_{\mu;\nu}dx^\nu$, and $;\nu$ denotes covariant differentiation with respect to x^ν.

Starting with the invariant action (26.21) and replacing the second term with

$$-\int a_\mu j^\mu \sqrt{-\det g_{\lambda\rho}}\ d^4x \tag{26.25}$$

for a continuous charge distribution in space, we obtain the invariant Maxwell equations in a rotating frame

$$f^{\mu\nu}{}_{;\nu} = j^\mu, \qquad \partial_\lambda f_{\mu\nu} + \partial_\mu f_{\nu\lambda} + \partial_\nu f_{\lambda\mu} = 0. \tag{26.26}$$

Based on gauge invariance and the taiji rotational invariance of the action (26.21), the electromagnetic potential must be a covariant vector a_μ in non-inertial frames. Since the force **F** and the fields **E** and **B** are naturally related to a change of the potential a_μ with respect to a change of coordinates (i.e., x^μ, by definition), the fields **E** and **B** are naturally identified with components of the covariant tensor $f_{\mu\nu}$ as given by (26.22) in non-inertial frames. Equation (26.22) can also be written in terms of partial covariant derivatives, $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu = a_{\nu;\mu} - a_{\mu;\nu}$, and the metric tensor $g_{\mu\nu}$ behaves like a constant under covariant differentiation, $g_{\mu\nu;\mu} = 0$.

26f. The 4-momentum and the 'lifetime dilation' of a particle at rest in a rotating frame

Although the speed of light is not a universal constant in a rotating frame F, we still can formulate a covariant four-momentum $p_\mu = (p_o, p_i)$ by using w, with the dimension of length, rather than the more traditional t, with the dimension of time, as the evolution variable in the Lagrangian formalism. From the invariant action $S_f = -\int mds = \int Ldw$ for a 'non-interacting particle' with mass m in the rotating frame F, we have the spatial components of the covariant four-momentum,

$$p_i = -\frac{\partial L}{\partial v^i} = mg_{\nu i}\frac{dx^\nu}{ds} = g_{\nu i}p^\nu, \qquad i = 1,2,3; \tag{26.27}$$

$$L = -m\sqrt{g_{\mu\nu}v^\mu v^\nu},$$

where L and p_i both have the dimension of mass and $v^\mu \equiv dx^\mu/dw = (1, v^i)$. The zeroth component p_0 (or the Hamiltonian) with the dimension of mass is defined as usual

$$p_0 = v^i\frac{\partial L}{\partial v^i} - L = mg_{\nu 0}\frac{dx^\nu}{ds} = g_{\nu 0}p^\nu. \tag{26.28}$$

The taiji rotational transformation of the differential operators $\partial/\partial x_I{}^\mu$ and $\partial/\partial x^\mu$ can be calculated from the inverse of (26.16). The covariant momentum p_μ has the same transformation properties as the covariant differential operator $\partial/\partial x^\mu$. This can also be seen from the quantum-mechanical relation $p_\mu \propto \partial/\partial x^\mu$. Thus, we have

$$p_{I0} = \gamma^{-1}\left(p_0 + \Omega y p_1 - \Omega x p_2\right),$$

$$p_{I1} = \left[-\gamma^{-2}\Omega^2 w x_I\right]p_0 + \gamma^{-2}\left[\gamma\cos(\Omega w) - \Omega^2 x_I x - \Omega^3 w x_I y\right]p_1$$

$$+\gamma^{-2}\left[-\gamma\sin(\Omega w) - \Omega^2 x_I y + \Omega^3 w x_I x\right]p_2,$$

$$p_{I2} = \left[-\gamma^{-2}\Omega^2 w y_I\right]p_0 + \gamma^{-2}\left[\gamma\sin(\Omega w) - \Omega^2 y_I x - \Omega^3 w y_I y\right]p_1$$

$$+\gamma^{-2}\left[\gamma\cos(\Omega w) - \Omega^2 y_I y + \Omega^3 w y_I x\right]p_2, \qquad (26.29)$$

$$p_{I3} = p_3.$$

Let us consider the kinematical properties of a particle. Suppose a particle is at rest in the rotating frame F, so that $dx^i = 0$ and hence, $ds = dw$. Based on $p^\nu = m dx^\nu/ds$ in (26.27), the contravariant momenta are $p^i = 0, i = 1,2,3$, and $p^0 = m$. In this case the covariant momenta of this particle in F are

$$p_0 = m, \qquad p_1 = m\gamma^2(\Omega y + \gamma^2\Omega^2 wx),$$
$$p_2 = m\gamma^2(-\Omega x + \gamma^2\Omega^2 wy), \qquad p_3 = 0. \qquad (26.30)$$

This difference between p_μ and p^μ is due to the presence of the metric tensor components g_{01} and g_{02}. Its covariant momenta, as measured in an inertial frame F_I, are given by (26.29)-(26.30):

$$p_{I0} = \gamma m, \qquad p_{I1} = m\gamma\left[\Omega x\sin(\Omega w) + \Omega y\cos(\Omega w)\right],$$
$$p_{I2} = -m\gamma\left[\Omega x\cos(\Omega w) - \Omega y\sin(\Omega w)\right], \qquad p_{I3} = 0. \qquad (26.31)$$

It is gratifying that the expression for the energy of a rotating particle p_{I0} in (26.31) agrees with the results of high energy experiments performed in an inertial laboratory frame F_I. If one uses the classical rotational transformation (26.1), one would obtain $p_{I0} = m(1+ \rho^2\Omega^2)$ for the energy of such a particle at rest in F, which is clearly inconsistent with high energy experiments.

For the decay of a particle orbiting in a storage ring, the particle can be considered to be at rest in the frame F. The taiji rotational transformation (26.14) gives

$$\Delta w_I = \gamma \Delta w, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \rho\Omega. \tag{26.32}$$

Since w and w_I both have the dimension of length, (26.32) can be understood as the dilation of the decay length which is the quantity that is directly measured in experiments measuring the dilation of the decay lifetime of muons in flight in a storage ring.[6] Again, this is an absolute effect in that observers in both F_I and F agree that it is the accelerated muons whose mean decay length is dilated. If one replaces a rotational transformation during a very short time interval by a Lorentz transformation, one obtains the same expression for the time interval $\Delta t_I = \gamma \Delta t'_I$ or, $\Delta w_I = \gamma \Delta w'_I$ with $w_I = ct_I$ and $w'_I = ct'_I$, by imposing the condition $\Delta x'_I = 0$ for the two events. There is a conceptual difference however, as the time dilatation effect implied by the Lorentz transformations is relative rather than absolute. In this connection, we note that the relation $w_I = w$ for any $\beta = \rho\Omega \neq 0$ in the classical rotational transformation (26.1) is inconsistent with the muon experiment in a storage ring.

The new transformation (26.11) has several advantages for purposes of analyzing physical phenomena in rotating reference frames. One advantage relative to the Lorentz transformation is that since we derive our transformation from first principles, there are no conceptual difficulties arising from the improper use of a transformation designed to treat only inertial reference frames. Secondly, the coordinates involved in our transformation are physically meaningful and operationally defined. In contrast, the arbitrary coordinate systems devised for non-inertial frames based on general relativity have, in general, no correspondence with physically measurable distances and times.[8]

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27.

Epilogue

We end this account of a broader view of relativity by summarizing the general physical implications of Lorentz and Poincaré invariance, including an overview of different relativity theories and their relationships (as shown in Figure 27.1) and several generalizations of the spacetime transformation equations to include frames with a general linear acceleration (as shown in Figure 27.2).

In his well-known book, The Theory of Relativity,[1] Pauli discussed the following interesting question

> '*Could one not avoid such radical deductions and yet retain agreement with experiment, by rejecting the constancy of the velocity of light and retaining only the first postulate?* '

This question has been raised independently by W. Ritz,[2] Tolman,[3] Kunz[4] and Comstock.[5] Our answer to this question is unambiguously affirmative, in contrast with the previous answers given by Pauli and others. The Poincaré-Einstein principle of relativity (i.e., the form of a physical law is the same in any inertial frame) is indeed a fundamental and necessary component of any theory if it is to correctly explain and predict phenomena in the physical world. In particular, we have shown that the theory of taiji relativity can be formulated solely on the basis of the first principle of relativity, i.e., the Poincaré-Einstein principle, without assuming the constancy of the speed of light.

Einstein's second postulate of special relativity, namely that the speed of light c is a universal constant, is not a necessary component of an experimentally consistent theory, according to our analysis of taiji relativity. Although the constant c has been described as "the foundation stone of Einstein's special theory of relativity"[6] and "so important in the four-dimensional picture, [playing] such a fundamental role in the special theory of relativity...that it has to be fundamental,"[7] *the particular definitions of meter and second that lead to c being a universal constant are not crucial or necessary parts of a relativistic theory.* Alternative definitions for the unit of time, such as that for the common-second under which c is not a universal constant, also lead to experimentally consistent theories. The understanding of this property is crucial for understanding the primacy of the Poincaré-Einstein principle of relativity or the Lorentz and Poincaré invariance. Furthermore, such an understanding enables one to see the big picture of relativity theories, as shown in Figure 27.1, which was hidden beneath the relationship between the principle of relativity and our concept of time.

Once the spacetime properties of inertial frames and the nature of physical laws in those frames are understood, the next question we can ask is

Can the principle of relativity be used as a limiting principle for understanding the physics of accelerated frames? In other words, can one generalize the Lorentz transformations for inertial frames to non-inertial frames with linear accelerations or rotations? Which physical constants of inertial frames remain constants in non-inertial frames?

We have seen that we can answer these questions as well. Based on limiting Poincaré and Lorentz invariance, we have obtained and discussed some of the simplest generalizations of the Lorentz transformation to non-inertial frames with constant-linear-accelerations and general-linear-accelerations. (See Figure 27.2.) It can also be used to derive a relativistic transformation for rotating coordinate systems. These results suggest that the physical laws and taiji spacetime of non-inertial frames are more closely related to the limiting Poincaré and Lorentz invariance than to general relativity with the gravitational field equation and the equivalence principle. It is natural to hope that basic physical theories of the future can be formulated and understood on the basis of a unified spacetime transformation for both inertial and non-inertial frames.

From this viewpoint, the physics of Yang-Mills fields, matter fields and gravitational field in non-inertial frames, especially the quantum aspects of fields, is far from being understood. This presents an ongoing challenge to physicists and mathematicians.

Nevertheless, we can already see certain basic aspect of physics in the future. Truly universal constants should be inherent characteristics of the physical world, independent of human convention. Therefore, they should be the same in all relativity theories and should be universal in both inertial and non-inertial frames of reference. According to this criterion, the constants J and $\bar{e}$ (or $\alpha_e = \bar{e}^2/J \approx 1/137$) in quantum electrodynamics are truly universal, while $\hbar$ and c are not. This result has been substantiated by the generalization of quantum electrodynamics for inertial frames to non-inertial frames with a linear acceleration or rotation. Furthermore, we must also take the human convention of physical units into consideration. Inherent constants of nature should be independent of the human conventions of units. It follows that the constant J is not inherent in nature, as shown in Appendix A. The results of these analyses appear to indicate that the only truly universal (and fundamental) constants in physics are the dimensionless coupling constants of Yang-Mills gauge fields and matter fields.[8]

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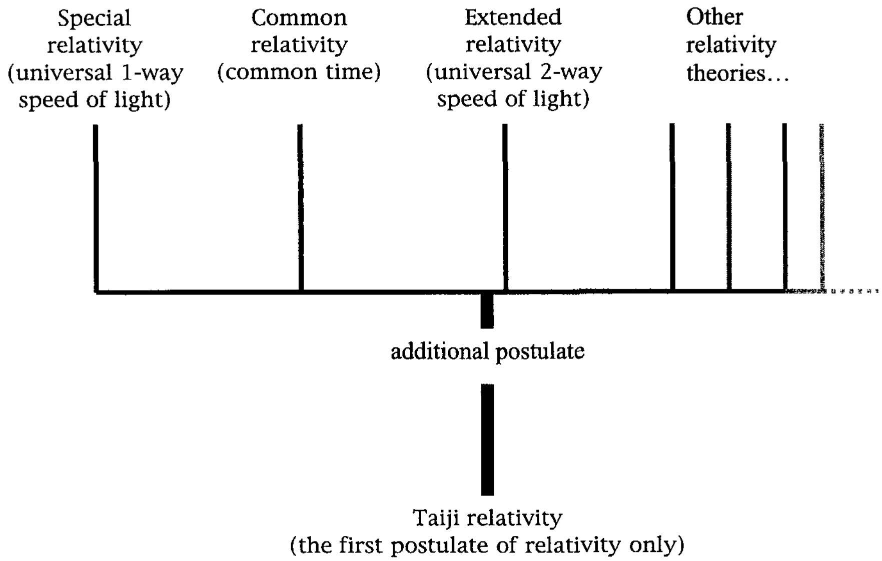

Figure 27.1. "Tree" of relativity theories showing the logical connection between taiji relativity, based solely on the Principle of Relativity, and other relativity theories for which an additional postulate has been made.

Spacetime transformation between two inertial frames
($\beta=\beta_o=$ constant; $\gamma_o = (1-\beta_o{}^2)^{-1/2}$)

$$w_I = \gamma_o(w - \beta_o x), \quad x_I = \gamma_o(x - \beta_o w), \quad y' = y, \quad z' = z\,;$$

$$\Big\downarrow \quad \beta=\beta_o \rightarrow \beta=\beta_o+\alpha_o w$$

Spacetime transformation between an inertial frame and a frame with a Constant-Linear-Acceleration
[$\beta=\beta_o+\alpha_o w$, $\gamma = (1-\beta^2)^{-1/2}$]

$$w_I = \gamma\beta\left(x + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}, \qquad x_I = \gamma\left(x + \frac{1}{\alpha_o\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}, \qquad y_I = y, \qquad z_I = z;,$$

$$\Big\downarrow \quad \beta=\beta_o+\alpha_o w \rightarrow \beta=\beta(w) = \text{arbitrary}$$

Spacetime transformation between an inertial frame and a frame with a General-Linear-Acceleration
[$\beta=\beta(w)$ = arbitrary, $\gamma = (1-\beta^2)^{-1/2}$]

$$w_I = \gamma\beta\left(x + \frac{1}{\alpha(w)\gamma_o^2}\right) - \frac{\beta_o}{\alpha_o\gamma_o}, \qquad x_I = \gamma\left(x + \frac{1}{\alpha(w)\gamma_o^2}\right) - \frac{1}{\alpha_o\gamma_o}, \qquad y_I = y, \qquad z_I = z;,$$

Figure 27.2. The hierarchy of spacetime transformations between inertial and accelerated frames from least to most general.

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D.

Appendices

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Appendix A.

Systems of Units and the Development of Relativity Theories

Aa. Units, convenience and physical necessity

"Natural units" is a system of units used widely by theoretical and particle physicists in which the vacuum speed of light c and Planck's constant $\hbar$ are defined to be dimensionless with unit magnitude. All physical quantities are expressed in terms of a power of the same single unit[1]. A desirable feature of natural units is that the equations expressing physical laws are simpler to write down and have fewer constants obscuring the essential physics they embody.

One may ask however, if the natural unit system is simply a calculational convenience or whether there is a physical justification for setting c and $\hbar$ to one. In order to connect natural units to the internationally accepted SI system of units, one must be able to define operationally each of the base SI units in terms of one single unit in a physically meaningful way. For example, one might arbitrarily decide that 1 meter is equivalent to 5 kelvin, but such an equivalence would not be physically meaningful. On the other hand, saying that 1 second is equivalent to 299 792 458 meter **is** physically meaningful because the success of the special theory of relativity has established that space and time are on an equal footing in a four-dimensional spacetime and because the ultimate speed possible in our universe is 299 792 458 meter in one second as measured by our independently defined units of meter and second. This issue of a physical justification for natural units is intimately related to a question which students sometimes ask: "What is the minimum number of units and fundamental constants which are necessary for physics?" In this appendix, we argue that there is indeed a physical basis for the natural unit system. We discuss possible definitions for all the base SI units in terms of a single unit and the implications of the natural unit system for the status of constants such as the speed of light c and Planck's constant $\hbar$.

In the following section, we discuss how the dimensions of mass, length, and time can be expressed in terms of the same single unit using the results of special relativity and quantum mechanics. In sections Ac and Ad, we derive definitions for the remaining four "base" SI units in terms of our single unit and discuss units for other common physical quantities. Finally, in section Ae, we discuss the status of some common physical constants in light of the new definitions.

Ab. Time, length, and mass

Ab.1 Time and length

Prior to the twentieth century, space and time seemed to be completely different entities and thus different units, the meter and second, were invented to measure the two. From a modern viewpoint however, the theory of special relativity implies that space and time are not independent and separate, but parts of a four-dimensional spacetime. This view is reflected in the fact that in the SI system, the unit of length, meter, is defined in terms of the unit of time, second[2].

Such a definition is possible because in our physical laws, four coordinates–three spatial and one temporal–are necessary to specify the spacetime location of an event. Although there is nothing wrong with using different units to express these four quantities it is logically simpler to use the same single unit, either meter or second. As an analogy, there is nothing wrong with expressing north-south distances in meters and east-west distances in miles. Doing so, however, introduces an extra conversion constant that is clearly artificial and unnecessary. The mathematical form of physical laws is much simpler when distances in both directions are expressed using the same unit. The same is true for intervals of both space and time.

In order to create a physically meaningful equivalence between the meter and the second, there must be a physical phenomenon that involves both dimensions of length and time. The maximum speed c of a particle with a non-imaginary mass provides a convenient conversion factor between the two units. For practical reasons of precision and reproducibility, the unit of time second is chosen to be the base unit in the SI system. The modern (1983) definition of the meter is then "The meter is the length of the path traveled by light in vacuum during a time interval of 1/299792458 of a second."[2] From a purely theoretical point of view, either the meter or the second could be used as the fundamental unit, with all other units defined in terms of it. In the natural unit system, a unit of length such as the meter is customarily chosen to be the base unit.

One could imagine that had the four-dimensional nature of spacetime been understood from the very beginning, we might now only talk about the dimension of length, instead of the dimensions of length and time and only a single unit for both spatial and temporal intervals would have arisen, rather than the two separate units of second and meter. In fact, some textbooks on special relativity[3,4] purposely write the Lorentz transformations using the same unit for both spatial and temporal coordinates to emphasize this point. Thus, we see that the constant $c = 299792458$ m/s originates from an incomplete understanding of physics and is merely a conversion factor between the human-defined dimensions of length and time[5].

Although expressing length and time intervals in terms of a single unit may seem more complex because of our familiarity with using both meter and second, a single unit system is simpler from a logical standpoint, since it requires only a single definition for a single unit. Furthermore, since one of the primary goals of physics is to explain the diverse phenomena around us using the smallest possible set of laws, it is also desirable to use the simplest system of units, i.e., one with the smallest number of independent units.

Ab.2 Time and mass

At present, the definition of the kilogram, the SI unit of mass, is based on the international prototype of the kilogram, a cylindrically-shaped object made of a platinum-iridium alloy stored at the International Bureau of Weights and Measures in Sèvres, France. This definition is independent of the definitions of the second and meter. However, just as the four-dimensional symmetry of our universe indicates that a single unit is sufficient to express both spatial and time intervals, the dual wave-particle nature of matter allows the unification of the unit of mass with that of time and length. The strength of a particle's gravitational interactions, its resistance to acceleration as a result of an applied force, and the wavelength of a photon with an energy equal to the particle's rest mass are all physically related quantities and thus can be expressed in terms of the same unit.

As before, in order to develop a physically meaningful equivalence between the unit of mass and the unit of time (or length), there must be a physical phenomenon involving both types of quantities. The equivalence of mass and energy provides just such a phenomenon. When both lengths and times are expressed using the same unit using the conversion 1 m = 1/299792458 s (i.e., c=1), the relation $E = mc^2 = m$ implies that both energy and mass can be expressed in terms of the same unit. The equation $E = m = h\nu$ then gives a quantitative relationship between a particle's mass and the frequency of a photon with an energy equal to the particle's rest mass. Thus we see that, Planck's constant h ($=2\pi\hbar$) can be used as a conversion factor between our units of mass and time. Using the fact that 1 meter is equivalent to 1/299792458 of a second, we have

$$h = 6.626... \text{ x } 10^{-34} \text{ J·s} = 6.626... \text{ x } 10^{-34} \frac{\text{kg} \cdot \text{m}^2}{\text{s}}$$

$$= 6.626...\text{x}10^{-34} \left(\frac{1}{299792458}\right)^2 \text{kg·s} = 1$$

so that 1 kg is equivalent to 135639274 x 10^{42} s^{-1} or similarly, that 1 kg is equivalent to 4.524... x 10^{41} m^{-1}.

Using h as the conversion factor, the definition "The kilogram is the mass of a body at rest whose equivalent energy equals the energy of a collection of photons whose frequencies sum to 135 639 274 x 10^{42} hertz"[6] has been proposed for the kilogram.

We now see that expressing distances, time intervals, and masses in terms of the same unit is not merely an artificial choice made for purposes of simplifying some mathematical calculations. Instead, it is a reflection of our understanding that in our universe, these three quantities are all related in a fundamental way through the four-dimensional symmetry of spacetime and the wave-particle duality of matter. Expressing all three using the same unit leads to the simplest mathematical form for physical theories.

Furthermore, we see that rather than being fundamental constants which are inherent properties or "characteristic numbers" of our universe, Planck's constant h and the speed of light c are merely conversion factors between the different units that humans have invented as a result of an incomplete understanding of physics.

Ab.3 Practical considerations

At present, the SI unit kilogram is not defined in terms of atomic quantities because the stability and reproducibility of the mass of the international prototype of the kilogram is better than that of any atomic standard which we can yet achieve.[6] However, with continuing advances in technology and increasing precision in measurements[7], it is only a matter of time before a redefinition in terms of atomic quantities, takes place. From a theoretical viewpoint however, there is no need to wait for practical measurements to reach a particular threshold of precision.

If the kilogram were to be redefined in terms of the frequency of a collection of photons, one side effect would be the fixing of the value of Planck's constant h. This is analogous to the case where the redefinition of the meter in 1983 had the effect of fixing the speed of light to an exact value, with no uncertainty. This further emphasizes the fact that h and c are human inventions, which can be given an exact value, rather than an inherent characteristic of our universe, which one could never measure with infinite precision.

While the particular redefinition of the kilogram proposed above has the effect of setting the value of Planck's constant h to one, an equivalent and possibly more desirable definition would be "The kilogram is the mass of a body at rest whose equivalent energy equals the energy of a collection of photons whose <u>angular</u> frequencies sum to 852 246 694 x 10^{42} s^{-1}." This definition would result in a corresponding conversion factor of 1 kg = 8.522... x 10^{50} s^{-1} = 2.843... x 10^{42} m^{-1} since $\hbar$, rather than h, would then be the conversion factor. For the remainder of this paper, we will use this alternate definition (i.e., $\hbar = c = 1$) because it is more consistent with the natural units in use today.

Ac. Other SI base units

Although the ability to express lengths, masses, and time intervals using the same single unit implies that all physical quantities can be expressed in terms of powers of that one unit (see section Ad for a discussion of electric charge), one can also develop corresponding redefinitions for the other base SI units. The classification of the ampere, kelvin, candela, and mole as "base" units is a historical one and does not imply that those units are truly independent of the meter, second, and kilogram[8]. For example, the definitions of the ampere, mole, and candela are already dependent on the definitions of the meter, second, and kilogram in the SI unit system.

Ac.1 Current

As stated above, the SI unit of current ampere is already defined in terms of other base SI units. Thus, its classification as a "base" SI unit is only a historical artifact. The present definition is "The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2 x 10^{-7} newton per meter of length."[9]

This definition can be rewritten in terms of the unit second as follows: "The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 second apart in vacuum, would produce between these conductors a force equal to 5.683... x 10^{35} second^{-2} per second of length."
The distance of 1 second used in this definition is of course, much larger than the 1 meter used in the present definition, but a force of 5.683... x 10^{35} second^{-2} is equal to 2 x 10^{-7} newton. In natural units, a force of 1 s^{-2} can be interpreted as the applied force which causes a mass of 1 s^{-1} (about 1.173 x 10^{-51} kg) to accelerate at 1 s^{-1} (about 2.998 x 10^{8} $m{\cdot}s^{-2}$).

We derive the conversion factor between ampere and second using the relation

$$\frac{F}{\ell} = \frac{2I_1I_2}{c^2d} \tag{1}$$

written in Gaussian units where F/ℓ is the force per unit length between two infinite current-carrying wires, I_1 and I_2 are the currents in the two wires, c is the speed of light, and d is the perpendicular distance between the wires. The relation written in SI units cannot be used because that it is used to fix the value of μ_0 and thus leads to an identity.

In natural units, $c = 1$. Inserting 2×10^{-7} newton per meter for F/ℓ, 1 ampere for both I_1 and I_2, and 1 meter for d, we obtain the conversion 1 ampere = $5.331... \times 10^{17}$ second^{-1} = $1.778... \times 10^{9}$ meter^{-1}.

Ac.2 Luminous intensity

The candela, like the ampere, is already defined in terms of other SI units. The definition of the SI unit of luminous intensity is:

"The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian."[10]

One candela is thus equivalent to 1/683 watt per steradian for radiation with a frequency of 540 THz (approximately green light with a wavelength of 555 nm). Converting the unit watt to our single base unit, using the conversion factors obtained in section II, we obtain

1 watt = $9.479... \times 10^{33}$ s^{-2} = $1.055... \times 10^{17}$ m^{-2}. A new definition expressed entirely in terms of the base unit second would then read "The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of $1.388... \times 10^{31}$ s^{-2} per steradian." Thus, 1 candela = $1.388... \times 10^{31}$ s^{-2} = $1.544... \times 10^{14}$ m^{-2} for 540 THz radiation.

Ac.3 Mole

As with the ampere and candela, the mole is already defined in terms of other units and is not an independent unit. As a unit, the mole is similar to "dozen" in that it represents a specific number of individual objects. The present definition is:

"1. The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12; its symbol is 'mol.'

2. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles."[11]

This definition can be re-written in terms of the unit second as:

"1. The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in an amount of carbon 12 with an equivalent energy equal to a collection of photons whose angular frequencies sum to 1.023... x 10^{49} s^{-1}; its symbol is 'mol.'

2. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles."

Ac.4 Temperature

The present definition of the SI unit of temperature is "The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water."[12] This is somewhat similar to the definition of the mole, which is also based on a particular substance.

However, just as special relativity shows that space and time are both parts of the four-coordinate and quantum mechanics demonstrates that mass and frequency (or wavelength) are simply different aspects of the same quantity, the kinetic theory of gases demonstrates that temperature is not an independent characteristic of a system, but simply one type of energy scale. For example, the temperature of a monatomic ideal gas is related to the average kinetic energy of its individual gas molecules through $\overline{E} = \frac{3}{2} k_B T$. Since energy has the dimension of mass in natural units, the unit kelvin will be dimensionally equivalent to the kilogram, the inverse meter, and the inverse second.

To establish a physically meaningful conversion factor between the kelvin and these other units, we note that Boltzmann's constant k_B already provides a conversion between the energy and the temperature of a system. In analogy with using c and $\hbar$ as conversion factors among the meter, second, and kilogram, a conversion which fixes the value of Boltzmann's constant k_B to one ($k_B = 1$) is desirable and results in the equivalencies 1 kelvin = 1.309... x 10^{11} second^{-1} = 436.7... meter^{-1}. A new definition of the unit kelvin might then be "The kelvin, unit of thermodynamic temperature, is the thermodynamic temperature of a system whose energy is equal to the energy of a collection of photons whose angular frequencies sum to 1.309... x 10^{11} s^{-1}."

As was the case for the speed of light c and Planck's constant $\hbar$, we see from kinetic theory that the Boltzmann constant k_B is not a fundamental constant of nature either, but instead the result of the different units humans have devised as a result of an incomplete understanding of the atomic basis of temperature and energy. In this sense, the Boltzmann constant is similar to Joule's constant, which was used to relate mechanical energy (in joules) and heat (in calories). When it was realized that both were the same type of quantity, the unit joule was used to express both and Joule's constant of 4.184 J/cal became merely another conversion constant.

Ac.5 Practical considerations

As with the current definition of the kilogram, the current definition of the kelvin is based on a macroscopic physical property of a substance because such a definition offers greater stability and reproducibility than allowed by our knowledge of the value of k_B.[13] The precision with which k_B is known is limited by the precision with which we can measure the values of Avogadro's number N_A and the gas constant R. In the future, if the precision with which these two constants are known can be improved to the point where the uncertainty in k_B is roughly one part in 10^7, then serious consideration would be given to defining the unit kelvin in terms of atomic quantities.

Ad. Other units

Table I lists some common quantities and their units in both the SI and the natural unit system. In it, we have taken the base unit to be the meter in order to be consistent with the commonly used system of natural units. Note that electrical charge is dimensionless in the natural unit system. The electron charge in natural units can be obtained from the definition of the fine structure constant $\alpha = e^2/(\hbar c) = e^2 = 1/137.036$, since $\hbar$ and c are both equal to 1 in natural units. The electron charge is then $e = \alpha^{1/2} = 0.08542$ and a charge of 1 coulomb is equal to 5.332×10^{17}.

It is also interesting to note that angular momentum is dimensionless in the natural unit system. Thus in quantum mechanics, angular momentum is quantized in integers and half integers.

Ae. Status of the fundamental constants

From the previous discussion, it is clear that not all of the quantities we call "fundamental physical constants" are alike. Levy-Leblond[14] grouped the fundamental physical constants into two categories, (1) Constants characterizing whole classes of physical phenomena (such as the electric charge e and the universal gravitational constant G), and (2) Universal constants (such as c and h) which act as concept or theory synthesizers (for example, Planck's constant h synthesizes the concepts of momentum and wavelength through the relation $p = h/\lambda$).

From the point of view of natural units, the dimensionless constants in the former category are true fundamental constants in the sense that they are inherent properties of our universe and have values which are not determined by human convention.[15] Constants in the latter category are not fundamental constants in the sense that they are historical products of an incomplete physical understanding of our universe and their particular values are determined by human convention. If we were to rebuild physics from the ground up knowing what we do today and using the logically simplest set of units (natural units, which requires only one definition) the constants in the former category would still appear as parameters in our theories, whose values could only be determined by experiment, while constants in the latter category would not appear at all.[16] Thus, many of the papers in metrology which discuss the measurement of the fundamental physical constants[13,17,18] are discussing two different types of experiments. One is an experiment which seeks to determine more precisely one of the fundamental numbers which characterize our universe. Another is an experiment which seeks to determine more precisely the conversion factor between the historically independent definitions of two units.

One criterion which can be used to determine the category to which a particular constant belongs is if the value of a constant can be made unity by a suitable definition of units, then that constant is merely a conversion factor between different human-invented units. This is analogous to the criterion that if the value of a constant can be made exact by a suitable definition of units, then that constant is not a true fundamental constant. For example, quantities such as c, $\hbar$, and k_B are not fundamental constants because they can be made to have the value unity by suitable definitions of second, meter, kilogram, and kelvin. The vacuum permittivity μ_0 and permeability ε_0 are likewise merely conversion constants. Although they are not equal to one under present definitions of the units, they could be made unity by a suitable redefinition of the ampere.

In contrast, if the value of a constant cannot be made equal to one by a suitable choice of units, then that quantity is a true fundamental constant. For example, the electron charge e or the weak mixing angle θ_w (found in the unified electroweak theory) are fundamental constants since they are dimensionless and cannot, under any circumstances, be made to have the value one. The Josephson constant J_K and the von Klitzing constant R_K are $J_K = 4\pi e$ and $R_K = (2\pi e^2)^{-1}$ in natural units and are also fundamental constants, although we would not want to treat e, α, J_K, and R_K as four *independent* fundamental constants, since they are all powers or multiples of each other. At present, the only quantities which qualify as independent fundamental constants under this criterion are the coupling constants for each of the three fundamental forces - G (gravitational), α and θ_w (electroweak) and α_s (strong).[15]

Table 2 gives values of some common constants in natural units.

Af. Discussion and conclusion

Although the determination of conversion constants such as c or $\hbar$ is of no interest to the theoretician, it is of great importance to experimental physics. Despite the fact that all units can be operationally defined in terms of a single unit, the present state of technology is such that better accuracy and precision is obtained by using a system (the SI) in which at least some of the units are defined independently. Furthermore, even after some future time at which technology progresses to the point where all of the SI units can be defined in terms of a single unit, the awkwardness of some of the new definitions, the need to communicate with other disciplines, and plain old inertia will likely conspire to preserve the use of multiple units and unit systems (c.f., the continuing widespread use of British units in the United States).

Natural units are not merely a calculational convenience, but have a conceptual basis rooted in the nature of our physical universe. For example, because physical laws have the same form in all inertial frames and the three spatial coordinates and time appear in coordinate transformations in a very symmetric way, both distances and time intervals should logically be expressed in the same units. Without this four-dimensional symmetry, there would be no physical basis for defining the meter in terms of the second or vice versa. Similarly, the dual wave-particle nature of matter in our universe provides a physical basis for defining mass in terms of length or time.

The use of natural units also provides a criterion for deciding which of the "fundamental constants" are truly fundamental and inherent characteristics of our universe and which are merely conversion constants resulting from an imperfect understanding of the connections between physical quantities. At this point, it appears that the only true fundamental constants are the coupling constants associated with the gravitational, electroweak, and strong.

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distinct from all other coupling constants in gauge field theories such as the electroweak theory or quantum chromodynamics.

16. In a recent column in Physics Today (F. Wilczek, "Analysis and Synthesis I: What matters for matter," Physics Today **56(5)**, 10-11 (2003)), Frank Wilczek states that parameters such as e, $\hbar$, and the mass of the electron m_e can be eliminated by a suitable choice of a system of units for length ($\hbar^2/m_e e^2$), time ($\hbar^3/m_e e^4$), and mass (m_e). This is much the same as choosing a system of natural units, in which parameters such as c, $\hbar$ and k_B are eliminated. As Wilczek mentions later in the article, for a more complete theory, parameters such as the fine structure constant a inevitably come in.
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Quantity	SI	natural
time	s	m
length	m	m
mass	kg	m^{-1}
electric current	A	m^{-1}
temperature	K	m^{-1}
amount of substance	mol	m^{-1}
luminous intensity	cd	m^{-2}
velocity	$m\ s^{-1}$	dimensionless
acceleration	$m\ s^{-2}$	m^{-1}
force	$N = kg\ m\ s^{-2}$	m^{-2}
energy	$J = kg\ m^2\ s^{-2}$	m^{-1}
momentum	$m\ kg\ s^{-1}$	m^{-1}
angular momentum	$m^2\ kg\ s^{-1}$	dimensionless
electric charge	$C = A\ s$	dimensionless
electric potential	$V = m^2\ kg\ s^{-3}\ A^{-1}$	m^{-1}
electric field	$V\ m^{-1} = m\ kg\ s^{-3}\ A^{-1}$	m^{-2}
magnetic field	$A\ m^{-1}$	m^{-2}
resistance	$W = m^2\ kg\ s^{-3}\ A^{-2}$	dimensionless
capacitance	$F = m^{-2}\ kg^{-1}\ s^4\ A^2$	m
inductance	$H = m^2\ kg\ s^{-2}\ A^{-2}$	m
entropy	$J\ K^{-1}$	dimensionless

Table I. Units of common quantities in the SI and natural unit system. The meter is used as the single unit in the natural unit system, as is customary.

Constant	Value in SI units	Value in natural units
speed of light (c)	299 792 458 $\frac{m}{s}$	1
Planck's constant ($\hbar$)	1.055... x 10-34 J • s	1
fine structure constant (α)	1/137.0...	1/137.0
weak mixing angle (θ_w)	0.2224...	0.2224
Boltzmann constant (k_B)	1.381... x 10^{-23} $\frac{J}{K}$	1
Gravitational constant (G)	6.673... x 10^{-11} $\frac{m^3}{kg \cdot s^2}$	2.611... x 10^{-70} m^2
Vacuum permittivity (ε_0)	8.854... x 10^{-12} $\frac{F}{m}$	$1/4\pi$
Vacuum permeability (μ_0)	4π x 10^{-7} $\frac{N}{A^2}$	4π

Table 2. Values of common physical constants in SI and natural unit systems.

Appendix B.

Can one Derive the Lorentz Transformation from Precision Experiments?

The following is the text of a term paper written by one of the authors (LH) for an undergraduate research seminar in the fall semester of 1990. It provided some of the impetus for the work described in this book. For consistency, the section numbering has been changed to Ba, Bb,...Be. Because the rest of the paper has been taken verbatim, some of the notation and formatting is different from that in the other parts of this book.

Abstract: Investigation of two derivations of the Lorentz transformation from the results of three fundamental experiments reveals that an additional theoretical assumption has been made, making the special relativity transformations the inevitable result. A more careful derivation shows that the three fundamental experiments of special relativity are not enough to determine a space-time transformation uniquely.

"This is a nice presentation and idea. Clearly written." I. F. S.

Ba. Introduction

Ever since its introduction in 1905 by Einstein's paper, "On the Electrodynamics of Moving Bodies," the theory of special relativity has generated much discussion and has had a profound impact on physics. By putting space and time on an equal footing and postulating the equivalency of all inertial frames, the theory changed the whole framework within which physics is done and the way in which physical laws are formulated. As J. D. Jackson points out, relativistic effects play a significant role in physics, "from the lowest energies in atomic systems (where the precision is so high that the tiny relativistic effects must be included) to the highest laboratory energies in the giant particle accelerators (where relativistic effects are gross and must enter even the crudest considerations)." [0]

Because so much of the theory is based on two postulates which, though familiar now, seemed to fly in the face of common sense at the time, a fair amount of effort has been devoted to trying to replace the foundational postulates with laboratory observations, i.e., to derive the Lorentz transformations from experimental results such as the Michelson-Morley experiment, measurements of the Doppler shift, and so on. In this paper, I will discuss two such attempts, one made by H. P. Robertson[1] in 1949, and a much more recent one made by Dieter Hils and John L. Hall[2] (following a parameterization of the space-time transformation by R. Mansouri and R. Sexl[3-5]) just this past year. Both start with a general linear transformation containing a number of unknown parameters and, using previous experimental results, obtain values for the parameters which uniquely fix the transformation to be the one obtained by Lorentz. However, I believe that in the two derivations, the authors have, perhaps unconsciously, made certain assumptions which inevitably led them to the Lorentz transformation. In this paper, I will discuss what these assumptions are and then, using a transformation even more general than the ones used by Robertson or Hils and the same experimental data, see to what extent the transformation parameters can really be fixed.

The derivations of the Lorentz transformation by Robertson and Hils used the results of only three experiments to completely specify the transformation parameters. These are the Michelson-Morley, Kennedy Thorndike, and Ives-Stilwell experiments. Before going on, here is a quick summary of these three fundamental experiments of special relativity and their results.

Bb. Three classical tests of special relativity

The earliest experiment which suggested that the then modern view of physics needed revising was the Michelson-Morley experiment, first performed in 1881 by A. A. Michelson alone, and again with greater precision in 1887 in a collaboration with Edward Morley[6]. This experiment was originally designed to determine the velocity of the earth with respect to the ether, a hypothetical medium that permeated the universe and was responsible for the transmission of electromagnetic waves. The setup consisted of a number of full and half-silvered mirrors, a light source, and an observing telescope. Light from the source was split into two equal beams by a half-silvered mirror. The two beams would travel mutually perpendicular paths, one aligned parallel with the Earth's motion through the ether, of equal length before being recombined by the half-silvered mirror to form an interference pattern. The light in the first arm would travel at the velocity of light plus or minus the speed of the Earth through the ether while the second beam would travel only at the speed of light. The beams would thus take different amounts of time to travel equal path lengths and would be out of phase when recombined to produce a pattern of light and dark bands. By observing the change in the interference pattern as the setup was rotated by 90 degrees, one could calculate the velocity of the Earth relative to the ether.

As we know, no change was ever observed in the interference pattern. The motion of the Earth through the ether, if indeed there was an ether, was undetectable. Since this experiment has been performed at many orientations and times of the year, the null result is usually interpreted as meaning: The time required for light to travel a distance L and back is independent of its direction.

Since Lorentz and Fitzgerald showed that a particular length contraction along the direction of motion could provide the same null result to the Michelson-Morley experiment as an isotropic speed of light, modern versions of the experiment have sought to test the isotropy of an "etalon of length" in space. One such test was carried out by A. Brillet and J. L. Hall[7]. A He-Ne laser, a Fabry-Perot cavity and a servo mechanism forming a feedback loop are mounted on a rotating optical bench. The laser beam passes through the cavity with the servo continuously adjusting the frequency of the laser so that its beam satisfies standing wave boundary conditions inside the cavity. Thus, changes in the length of the cavity are seen as changes in the frequency of the laser light. Part of the beam is diverted from the loop, directed upwards along the table's axis of rotation, and combined with the beam of a non-rotating, highly stable laser. The combined beam is then fed into a beat detector, the beats being produced because of the differing frequency of the two beams, and the output of the beat detector is finally sent to a computer which stores the information over the course of the table's rotation. If space is not isotropic and lengths are contracted along a particular direction, then the frequency of the He-Ne laser will change periodically as the length of the cavity changes in response to its orientation, and the beat frequency will change periodically. Taking into account many sources of errors, Brillet and Hall found that one could rule out the possibility of a length contraction to a very high degree of accuracy ($\Delta L/L$ would have to be $< 1 \times 10^{-15}$).

The second of the three experiments, chronologically, is the Kennedy-Thorndike test, the results of which were first published in 1932.[8] The original purpose was to see whether absolute time could be ruled out experimentally, since the results of all other experiments to date could be explained by means other than modifying the concept of absolute time as found in the Galilean transformation. The setup for the experiment resembles that of the Michelson-Morley experiment except that the two arm lengths are different from each other. Light from a source is split into two beams by a half silvered mirror, the beams travel in two different directions and are reflected back upon themselves. The half-silvered mirror then recombines the two beams and reflects them into a telescope for observation of their interference pattern.

If there really was absolute time, it would manifest itself as a change in the interference pattern over time as the velocity of the apparatus through space changed due to the rotational motion of the Earth. Over a period of several days' observation however, no change in the interference pattern could be found that might correspond to the changing velocity of the earth. Kennedy and Thorndike noted that "there is no effect corresponding to absolute time unless the velocity of the solar system in space is no more than about half that of the earth in its orbit," a possibility that has been ruled out by modern astronomical measurements. The usual interpretation of this null result is: The time for light to travel out to a point and back in an inertial frame is independent of the velocity of the inertial frame.

As with the Michelson-Morley experiment, modern Kennedy-Thorndike experiments have utilized lasers to make very precise measurements. In a very recent experiment by D. Hils and J. L. Hall[2], a setup similar to the one described above (different lasers were used and the apparatus could not rotate) for a modern Michelson-Morley experiment was used to try to detect a 24 hour sidereal variation in the beat frequency corresponding to the earth's rotation. This measurement is the physical equivalent to that made by the original KT experiment, comparing the transformation of time and length in a moving frame. Again, the looked for variation could not be found with any certainty and so the original null result is reconfirmed, but to a level of accuracy 300 times better than the first attempt.

The third and last experiment to be described here is the Ives-Stilwell experiment[9], which measured the second-order Doppler shift of a moving light source by comparing the spectra produced by stationary and moving hydrogen canal rays (an old term for positively charged ions produced in a gas by electrical discharge). The ions were produced in an arc between some filaments and a grounded aluminum electrode and then accelerated between that and a second electrode held at some high voltage.

In order to eliminate the effects of the first order shift (which would have masked any second order effects), one must measure the spectrum of the moving particles at right angles to the particles' direction of motion. Since this is extremely difficult to accomplish experimentally, Ives and Stilwell instead took two simultaneous measurements of the rays, one with and one opposite to their direction of motion, resulting in one red and one blue shifted line, where the second order displacement can be calculated by comparing the center of gravity of the two lines with the line produced by a stationary source. They discovered that the wavelength of the light emitted from the moving ions was shifted by a factor of $(1-v^2/c^2)^{-1/2}$, where v is the velocity of the emitting ions with respect to the observer.

As with the previous two experiments, modern reproductions of the Ives-Stilwell experiment have used lasers and beat frequency detectors. In one version[10], the frequency of the beam from a laser was tuned to a particular transition of Neon atoms at rest while another was tuned to the same transition of moving Neon atoms. The two beams were combined and the beat frequency of the combined beam was measured for several velocities of the moving Neon atoms to measure the Doppler shift of the frequency of the transition.

Now let us see how the results of these experiments are used to determine a unique space-time transformation.

Bc. Deriving the Lorentz transformation?

The first of the two attempts to find a purely experimental basis for the Lorentz transformation was made by H. P. Robertson[1] in 1949. Following is a short summary of his work and my criticism of his derivation.

To start, Robertson defines two inertial frames; Σ, a "rest" frame with coordinates τ, ξ, η and ζ (or ξ^0, ξ^1; ξ^2, and ξ^3) where τ is the time coordinate, and S, a moving frame with coordinates t, x, y, and z (or x^0, x^1, x^2, and x^3), with t the time coordinate. He also defines the metric

$$(0) \qquad d\sigma^2 = \sum_{\mu,\nu=0}^{3} \gamma_{\mu\nu} d\xi^\mu d\xi^\nu = d\tau^2 - \frac{1}{c^2}(d\xi^2 + d\eta^2 + d\zeta^2),$$

assuming that the speed of light is isotropic in the Σ frame. The problem is to find the transformation T which expresses the greek letters in terms of the roman ones. In the most general case, we have

$$(1) \qquad \xi^\mu = \sum_{i=0}^{3} a_i^\mu x^i, \qquad \mu = 0,1,2,3.$$

Using appropriate symmetry arguments, the 16 coefficients can be reduced to four parameters as follows. Take the components of the velocity vector of the S frame measured from the Σ frame as v^α (α = 1, 2, 3) so the equation of motion for the spatial origin of the S frame is

$$\tau = a_0^0 t, \qquad \xi^\alpha = a_0^\alpha t, \qquad \text{for} \quad x^1 = x^2 = x^3 = 0, \tag{2}$$

$$\frac{d\xi^\alpha}{d\tau} = v^\alpha, \qquad \therefore\ a_0^\alpha = a_0^0 v^\alpha, \qquad (a_0^0 \neq 0).$$

If time is to work normally on both frames, a_0^0 cannot be zero.

Now consider a light signal propagated from the origin at t = 0 which is reflected at coordinates p^a (a = 1, 2, 3) and returns to the origin at t_0. The outgoing signal is on the light cone defined by the equation

$$\Delta\sigma^2 = \gamma_{\mu\nu}(a_0^\mu \Delta t + a_a^\mu \Delta x^a)(a_0^\nu \Delta t + a_b^\nu \Delta x^b) = 0, \tag{3}$$

$$\Delta x^\mu = x_B^\mu - x_A^\mu, \qquad t_A = 0, \qquad t_B = t, \qquad x_A^a = 0, \qquad x_B^a = x^a,$$

while the reflected light is described by the equation

$$\gamma_{\mu\nu}(a_0^\mu \Delta t + a_a^\mu \Delta x^a)(a_0^\nu \Delta t + a_b^\nu \Delta x^b) = 0, \tag{4}$$

$$t_A = t_0, \qquad t_B = t, \qquad x_A^a = 0, \qquad x_B^a = x^a.$$

Assuming that the speed of light is the same in both directions along a line in the S frame, the point $(t_0/2, p^a)$ must satisfy both equations. Setting the equations equal to each other, we find that all the cross terms must vanish, or

$$(5) \qquad \gamma_{\mu\nu} a_0^\mu a_a^\nu t p^a = g_{0a} t p^a = 0, \qquad g_{ij} = \gamma_{\mu\nu} a_i^\mu a_j^\nu .$$

Since this must hold for all p^a, equations (2) and (5) give

$$(6) \qquad g_{0a} = a_0^0 a_a^0 - \frac{1}{c^2} a_0^\alpha a_a^\alpha = a_0^0 (a_a^0 - v^\alpha a_a^\alpha / c^2) = 0.$$

To simplify the transformation without loss of generality, one can align the axes of both frames such that the relative motion is purely along the ξ^1-axis. Then $v^2 = v^3 = 0$. By choosing the rest of the axes in the two frames to be parallel, requiring azimuthal symmetry about the x-axis, and constraining all motion to be in the x or ξ direction, the transformation equations simplify to

$$(7) \qquad \begin{aligned} \tau &= a_0^0 t + \frac{v a_1^1}{c^2} x, \qquad & \eta &= a_2^2 y, \\ \xi &= v a_0^0 t + a_1^1 x, \qquad & \zeta &= a_2^2 z. \end{aligned}$$

The metric (0) can now be rewritten in terms of t, x , y, and z as

$$(8) \qquad \begin{aligned} d\sigma^2 &= (g_0 dt)^2 - \frac{1}{c^2}\left[(g_1 dx)^2 + (g_2)^2 (dy^2 + dz^2)\right], \\ g_0(v) &= a_0^0 \sqrt{1 - v^2/c^2}, \qquad g_1(v) = a_1^1 \sqrt{1 - v^2/c^2}, \qquad g_2(v) = a_2^2 . \end{aligned}$$

Of course, as v goes to 0, all g's and a's should become 1.

To determine the remaining parameters, Robertson now turns to the three aforementioned experiments. As stated above, the results of the Michelson-Morley experiment tell us that the time required for light to travel out to a point and back is independent of the direction. Using (8), Robertson finds that the time it takes for light to travel out a distance L in a direction making an angle h with the x-axis and back is

$$(9) \quad t = \frac{L}{cg_0}\left[(g_1(v)\cos h)^2 + (g_2(v)\sin h)^2\right] + \frac{L}{cg_0}\left[[g_1(v)\cos(h+\pi)]^2 + [g_2(v)\sin(h+\pi)]^2\right].$$

For this expression to be independent of h, a necessary condition is

$$(10) \qquad g_1(v) = g_2(v),$$

where both g's are positive, since the a's are positive.

Next, Robertson uses the Kennedy-Thorndike experiment to further fix the values of the g's. From (9) and (10), the time it takes for light to travel a distance L in any direction is

$$(11) \qquad t = \frac{L\,g_1(v)}{c\,g_0(v)}.$$

Therefore, the difference in time Δt for the two light beams to return to the mirror which initially split them is related to the difference in the lengths of the arms of the interferometer ΔL by the equation:

$$(12) \qquad \Delta t = \frac{g_1(v)\,\Delta L}{c\,g_0(v)}.$$

The parameter v in this equation is the velocity of the apparatus, which changes periodically as a result of the earth's orbit about the sun and its rotation about its own axis. A change in Δt would appear as a shift in the interference pattern. Because the pattern did not change over a long period, Robertson deduces that the ratio g_l/g_0 is independent of v and since $g_l/g_0 = 1$ when v=0 (both are identically 1), this ratio must hold for all v.

To summarize to this point, we have determined that $g_1 = g_2 = g_3$. Setting all three equal to g(v), the transformation T then looks like

$$\tau = g(v)\frac{t + vx/c^2}{\sqrt{1-v^2/c^2}}, \qquad \xi = g(v)\frac{x+vt}{\sqrt{1-v^2/c^2}},$$

(13)

$$\eta = g(v)y, \qquad \zeta = g(v)z.$$

To find definite values for g, Robertson turns to the Ives-Stilwell experiment. He introduces a third reference frame S', moving with velocity v' with respect to Σ where v' > v, with clocks set such that at t = t' = 0, the origins of S and S' coincide. By setting $d\xi/d\tau = v'$ and $dx/dt = u$ in (13), the velocity u of S' with respect to S is found to be

(14) $$U = \frac{v - v'}{1 - vv'/c^2}.$$

Suppose that there is a light source at the origin of S' which is co-moving with that frame. Its equations of motion relative to the S frame are obtained by setting x', y', and z' to 0

$$\tau = g(v')\frac{t'}{\sqrt{1-v'^2/c^2}} = g(v)\frac{x+vt}{\sqrt{1-v^2/c^2}},$$

(15)

$$\xi = g(v')\frac{v't'}{\sqrt{1-v'^2/c^2}} = g(v)\frac{vt+x}{\sqrt{1-v^2/c^2}},$$

and solving for x and t

(16) $$t = p\,t', \qquad x = upt', \qquad p = \frac{g(v')}{g(v)}\frac{1}{\sqrt{1-u^2/c^2}}.$$

If the light source sends out a signal at some $t' < 0$, while it's approaching an observer R located at the origin of S, the signal will be received by R at a time

$$(17) \qquad t_+ = t + \frac{x}{c} = pt' - \frac{upt'}{c} = (1 - \frac{u}{c})pt'.$$

Two signals sent by the moving source separated by a time $\Delta t'$ would be seen by observer R separated by a time

$$(18) \qquad \Delta t_+ = (1 - \frac{u}{c})p\,\Delta t'.$$

For a periodically emitting source, we can think of the wavelength of the radiation emitted as $\lambda' = c\,\Delta t'$ in the S' frame and

$$(19) \qquad \lambda_+ = (1 - \frac{u}{c})p\,\lambda'$$

in the S frame. By a similar argument, radiation emitted as the source is moving away from the observer has its wavelength altered like

$$(20) \qquad \lambda_- = (1 + \frac{u}{c})p\,\lambda'.$$

The same radiation emitted from a source stationary to the observer has wavelength $\lambda' = \lambda$, so the center of gravity of the two shifted lines is displaced from the non-moving source's line by an amount

$$(21) \qquad \Delta\lambda = \frac{1}{2}(\lambda_+ + \lambda_-) - \lambda = (p-1)\lambda \approx \lambda\left[\frac{g(v')}{g(v)}\left(1 + \frac{u^2}{2c^2}\right) - 1\right].$$

In their experiment, Ives and Stilwell determined that the second order Doppler shift $\Delta\lambda/\lambda$ could be given by the expression $u^2/2c^2$. Since this result holds for motion of the source at an arbitrary angle from the x axis and there is no correlation between v and the x-component of u, Robertson determines g(v) and g(v') to both be unity (to second order). Thus he claims to have arrived at the familiar Lorentz transformation and four-dimensional metric, purely on the basis of experimental results.

While each step in this derivation seems reasonable, I believe that Robertson has made an unwarranted assumption near the beginning of his calculation which has made the Lorentz transformation the inevitable result. It comes while he is reducing the number of undetermined coefficients in the transformation T. When we consider a light signal being sent out from the origin and reflected back from a point, Robertson writes: "We *agree* to set the auxiliary clock situated at $x^a=p^a$ [in the S frame] in such a way that it records the time $t_0/2$ for the...reflection" (emphasis his). In other words, he has constrained the velocity of light along a line in S to be equal in both directions along that line. Now, even before this, Robertson had assumed that the speed of light is isotropic in his "rest" frame Σ. This is a perfectly reasonable assumption and is useful for simplifying the discussion, but, in order to truly derive a transformation using only experimental results, one should not put any restrictions on the speed of light in any other frames. Putting his assumption about the speed of light in S together with the results of the Michelson-Morley experiment, we soon arrive at the conclusion that the speed of light must be isotropic in all inertial frames. This constraint results in a great loss of generality and contradicts an earlier statement made by Robertson, "No assumption is here made concerning the velocity of light...in S." Later on, we shall see how the absence of this constraint affects the derivation.

The second derivation of the Lorentz transformation is more direct. The parameterization of the general transformation and calculations were first done by Reza Mansouri and Roman Sexl in a 1977 series of papers[3-5] using the best data available at the time. In 1990, Dieter Hils and John Hall used modern laser techniques to greatly improve the resolution of the Kennedy-Thorndike experiment and using Mansouri and Sexl's framework, confirmed their Lorentz derivation to a much higher level of confidence[2].

In this derivation, one first writes down a general linear transformation between two frames, S the preferred frame, and S', the moving frame:

$$t=a(v)T + ex, \qquad y=d(v)Y, \tag{22}$$
$$x=b(v)(X-vT), \qquad z=d(v)Z,$$

where

$$a(v) = 1+ \alpha\left(\frac{v}{c}\right)^2 +\ldots \qquad b(v) = 1+ \beta\left(\frac{v}{c}\right)^2 +\ldots \tag{23}$$
$$d(v) = 1+ \delta\left(\frac{v}{c}\right)^2 +\ldots$$

As before, the three spatial axes in both frames have been aligned and the relative motion of S and S' is restricted to the x-direction. Hils and Hall set e by the convention for clock synchronization and determine a(v), b(v), and d(v) from experimental data. Using Einstein's method of clock synchronization, they fix

$$e = -\frac{v}{c^2}\,. \tag{24}$$

The speed of a light signal traveling at an angle θ with respect to the x-axis is then

$$c(\theta) = \sqrt{(c\cos\theta)^2+(c\sin\theta)^2} = \sqrt{(x/t)^2+(y/t)^2} \tag{25}$$
$$= c\left[1+(\frac{1}{2}-\beta+\delta)\frac{v^2}{c^2}\sin^2\theta+(\beta-\alpha-1)\frac{v^2}{c^2}\right]$$

to second order. This result holds quite generally since we have already fixed c to be azimuthally symmetric around the x-axis.

With e determined, α is the only unknown parameter left in the transformation between t and T. From the results of modern measurements of the Doppler shift it is determined to be

$$(26) \qquad \alpha = -\frac{1}{2} \pm 10^{-7}.$$

The most accurate Michelson-Morley experiments, showing that the speed of light is independent of θ, fixes

$$(27) \qquad \frac{1}{2} - \beta + \delta = 0 \pm 5 \times 10^{-9}.$$

And in light of this value, the Kennedy-Thorndike experiment performed by Hils and Hall, showing that the two way speed of light does not depend on the velocity of the reference frame, gives us

$$(28) \qquad \beta - \alpha - 1 = 0 \pm 7 \times 10^{-5}.$$

Therefore, we can deduce that

$$(29) \qquad \beta = \frac{1}{2} \pm 7 \times 10^{-5}, \qquad \delta = 0 \pm 7 \times 10^{-5}.$$

Putting these figures into (25), we see that c is now a universal constant to an accuracy of 7×10^{-5}, the biggest uncertainty in any of the three measurements, and equations (22) now have the form of the Lorentz transformation to second order.

However, this derivation has the same problem as Robertson's. By using Einstein's method of synchronization to determine the parameter e, Hils and Hall have, just as Robertson did, constrained the speed of light along a line to be the same in both directions along that line. This is evident from (25), where $\sin\theta$ is squared so that $c(\theta) = c(\theta + \pi)$.

An additional effect of choosing e to be $-v/c^2$ that Robertson's assumption does not have is that in order for this set of transformation equations to have four dimensional symmetry, the parameters α, β, and δ MUST assume the values -1/2, 1/2, and 0 respectively. This fact is noted indirectly when Hils and Hall mention that "the kinematical parameters a(v), b(v), and d(v) might be determined by theory." Then the remainder of the derivation is merely a check to see whether or not the results of the three experiments are consistent with the four-dimensional symmetry of physical laws. Since this framework has been verified again and again by countless experiments, most notably high energy particle experiments, the verification of those values for the parameters is rather trivial. Once e has been set in that manner, the Lorentz transformation is the only possibility with four-dimensional symmetry permitted by (22). For any other values of α, β, and δ, we get only three-dimensional symmetry (Galilean invariance) at best, which has been conclusively ruled out by experiment.

In sum, my criticism concerning the previous derivations is that by using the Einstein clock synchronization convention, both have actually assumed that the one-way speed of light is the same in both directions along a line in the moving frame, instead of deducing it from any experimental results. Of course, for purposes of setting up any kind of clock system at all, one may assume an isotropic and constant speed of light in a "rest" or "preferred" frame. However, in order to truly derive a transformation from experimental results, one should not make the least assumption regarding the speed of light in any other frame. With this in mind, let us now see how precisely a space-time transformation can be specified based on the results of the three experiments.

Bd. A more general form

To start with, I choose

$$x' = A(v)\,x + B(v)\,ct = D(v)[A'(v)\,x + B'(v)\,ct],$$

(30) $$y' = D(v)\, y, \qquad z' = D(v)\, z,$$

$$b't' = E(v)\, ct + F(v)\, x = D(v)[E'(v)\, ct + F'(v)\, x]$$

as a parameterization of a more general form for a linear transformation between reference frames S (a "rest" frame) and S' (a moving frame) than either of those proposed by Robertson or Hils and Hall. The coordinates in S and S' are (ct, x, y, z) and (b't', x', y', z') respectively and v is the velocity of S' with respect to S. As in the previous derivations, I have aligned the coordinate axes such that the motion of S' is along the x-axis of the S frame and have introduced azimuthal symmetry about the axis of motion (y and z transform in the same way). Also, I have assumed that the speed of light is isotropic in S only, so we can use Einstein's procedure to synchronize the clocks in S. In the moving frame, the speed of light is not, in general, isotropic, so we can only write b't' in the transformation equations, instead of ct'. Although we don't yet know of a relation between t and t' and so cannot separate b' from t', I make the further assumption that the time coordinates are linearly related and write

(31) $$t' = M(v)\, t + N(v)\, x.$$

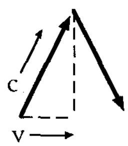
Physically, this relation can be realized for any choice of M and N since the rate of ticking and reading of a clock can be arbitrarily adjusted and all (31) means is that an S' clock located at x shows the time Mt + Nx while an S clock at the same position shows time t. In using the above equations, I have defined my metric in S to be

(32) $$(ct)^2 - x^2 - y^2 - z^2 = \sigma^2.$$

To begin the determination of the seven parameters (A, B, D, E, F, M, and N), we start with the requirement that an object located at the origin of S' and co-moving with that frame must have a velocity v in the rest frame. Introducing the condition x'= 0 in (30), we get

$$(33) \qquad 0 = A'(v)\,x + B'(v)\,ct, \qquad \frac{x}{t} = v = -\frac{B'(v)\,c}{A'(v)}\,.$$

Next, we use the results of the Michelson-Morley experiment, which tell us that: For a given v, the time required for light to make a trip out to a point and back is independent of the direction. This is equivalent to saying that the speed of light, averaged over a round trip, is the same in any direction. Suppose that we are in the rest frame S watching the (moving) apparatus in S' from the side (see Fig. 1). We see the light signal traversing the vertical arm of the experiment make a triangular path as shown below. The velocity of this light beam measured from the moving frame is (using (30))



$$\frac{dx}{dt} = v, \qquad \frac{dy}{dt} = \sqrt{c^2 - v^2}, \qquad \frac{dx'}{dt'} = 0\,;$$

$$(34) \qquad c'_\uparrow = \sqrt{\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2} = \frac{D\sqrt{c^2 - v^2}}{M + Nv} = \frac{Dc\sqrt{1-\beta^2}}{M + Nv}\,, \qquad c'_\uparrow = c'_\downarrow = c'_{\perp avg}\,,$$

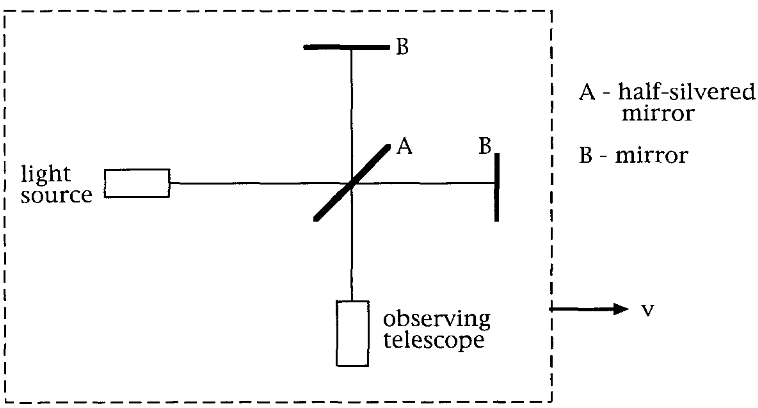
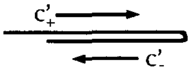


Fig. 1. Our view of simplified MM experiment

where $\beta = v/c$. From the same vantage point, we see the horizontal beam describe a path as shown in (35). The average speed of this light beam is (taking the arm lengths to be L' measured from S')



$$\frac{dx}{dt} = c, \qquad \frac{dy}{dt} = 0, \qquad \frac{dy'}{dt'} = 0,$$

$$c'_+ = \sqrt{\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2} = D\frac{A'(v) + B'(v)c}{M + Nv},$$

(35)

$$c'_-\left(\frac{dx}{dt} = -c,\ \frac{dy}{dt} = 0,\ \frac{dy'}{dt'} = 0\right) = D\frac{-A'(v)c + B'(v)c}{M - Nv},$$

$$\frac{L'}{c'_+} - \frac{L'}{c'_-} = \frac{2L'}{c'_{\|avg}}, \qquad c'_{\|avg} = 2\left(\frac{1}{c'_+} - \frac{1}{c'_-}\right)^{-1} = \frac{Dc}{A'}\frac{A'^2 - B'^2}{M + Nv}.$$

Setting the horizontal and vertical averages equal to each other and using (33), we determine that

$$Dc\frac{\sqrt{1-\beta^2}}{M + Nv} = \frac{Dc}{A'}\frac{A'^2 - B'^2}{M + Nv}, \qquad A' = -\frac{B'c}{v},$$

(36)

$$\therefore\ A' = \gamma, \qquad B' = -\beta\gamma,$$

where $\gamma = (1 - \beta^2)^{-1/2}$. I have taken the positive root of A to make the x and x' axes parallel rather than antiparallel.

We now apply the Kennedy-Thorndike results, first writing the metric given by (32) in terms of primed variables

(37) $$(BF - EA)^{-2}[(b't')^2(A^2 - B^2) - x'^2(E^2 - F^2) + 2b't'x'(EB - AF)] - D^{-2}[y'^2 + z'^2] = d\sigma^2.$$

At this point, Robertson used the metric to find an expression for the time it takes for a light signal to make a trip out to a point and back in the moving frame and made this expression independent of the direction of the trip and v. Since we do not have expressions for M and N, it is impossible for us to do the analogous step without creating an expression loaded with more unknown parameters than we will eventually be able to solve for. To circumvent this problem, we opt for a different, but equivalent interpretation of the Kennedy-Thorndike experiment.

The unchanging interference pattern seen in the original experiment and the constant frequency of the cavity laser in modern experiments indicate that the recombining light beams in the interferometer maintain the same absolute and relative phases as the velocity of the apparatus changes. We might say that this test shows that the final phase of a light signal making a round trip between two points is independent of the velocity the observer. Therefore, instead of making time be the round-trip invariant, I will use a quantity I call the "light path," equal to c't' (b' = c' where the motion of light is concerned). The light path is the distance traveled by a light beam in a time t' and since the wavelength of the light does not depend on the type of clock synchronization we use (relation between t and t'), requiring the round trip light path to be invariant is the same as requiring the final phase of the light signal to be independent of the direction of its trip. Solving for c't' from the metric (c't' = b't' when $d\sigma^2 = 0$), we obtain

(38) $$[c't'](\theta) = \frac{\ell'\cos\theta(A'F' - E'B') + \ell'(E'A' - B'F')}{A'^2 - B'^2} \,,$$

where

(39) $$x' = \ell'\cos\theta, \qquad y'^2 + z'^2 = \ell'^2\sin^2\theta \,.$$

So the round trip light path is

(40) $$[c't'](\theta) + [c't'](\theta + \pi) = \frac{2\ell'(E'A' - B'F')}{A'^2 - B'^2} \,.$$

The Kennedy-Thorndike experiment requires that this expression be independent of velocity. As we can see from (30), the parameters A', E', and D must reduce to 1 and B' and F' must become 0 when v=0, so

$$(41)\qquad \left.\frac{(E'A'-B'F')}{A'^2-B'^2}\right|_{v=0}=1, \qquad \therefore\ \frac{(E'A'-B'F')}{A'^2-B'^2}=1$$

for all v. Using (36), the relation between E' and F' is then

$$(42)\qquad \gamma E'=1-\beta\gamma F'.$$

Finally, we turn to the Ives-Stilwell experiment to try to determine the remaining parameters D, E', F', M and N. In order to proceed, we must make one last, small assumption regarding the invariant form of a plane wave. I postulate that, as in special relativity, the form for a plane wave which is invariant under (30) is

$$(43)\qquad \exp[i(x^0k_0-\mathbf{r}\bullet\mathbf{k})]=\exp[i(x'^0k'_0-\mathbf{r}'\bullet\mathbf{k}')].$$

Under this very reasonable assumption, the transformation equations of the wave four-vector are

$$k_0=D(E'k'_0+B'k'_x)=D(E'k'_0-\gamma\beta k'_x),$$

$$(44)\qquad k_x=D(A'k'_x+F'k'_0)=D(\gamma k'_x+F'k'_0),$$

$$k_y=Dk'_y, \qquad k_z=Dk'_z,$$

where

$$k_x=k_0\cos\theta, \qquad k_y=k_0\sin\theta,$$

$$(45)$$

$$k'_x=k'_0\cos\theta', \qquad k'_y=k'_0\sin\theta'.$$

Note that for this part, I have rotated the coordinate axes so that the wave has no z dependence to simplify the ensuing math.

In the Ives-Stilwell experiment, only the second order Doppler shift was measured. In order to eliminate the first order shift, the light source must be viewed perpendicularly to its direction of motion. This is equivalent to setting θ = 90 degrees. We first use the k_x and k_y equations to solve for k_0,

(46)
$$k_x = 0 = D(\gamma k'_0 \cos\theta' + F' k'_0), \qquad \cos\theta = -F'/\gamma,$$
$$k_y = D k'_0 \sqrt{1-\cos^2\theta'} = D k'_0 \sqrt{1 - F'^2/\gamma^2} = k_0 .$$

Using the relation between E' and F obtained above and the transformation equation for k_0, we find

(47)
$$k_0 = D\left[\frac{1-\beta\gamma F'}{\gamma} k'_0 - \gamma\beta k'_0\left(\frac{-F'}{\gamma}\right)\right] = D\frac{k'_0}{\gamma}.$$

Setting the expressions for k_0 in (46) and (47) equal to each other gives

(48)
$$D k'_0 \sqrt{1-F'^2/\gamma^2} = D\frac{k'_0}{\gamma}, \qquad F' = -\beta\gamma,$$
$$E' = \frac{1-\beta\gamma(-\beta\gamma)}{\gamma} = \gamma .$$

I have chosen F' to be negative so that t' and t will have the same sign.

The only remaining task is to determine D, M and N. Since $k=2\pi/\lambda$, we can rewrite (47) as

(49)
$$\lambda = \gamma\lambda'/D,$$

so to second order accuracy

$$\frac{\Delta\lambda}{\lambda} = D\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) - 1 = (D-1) + \frac{D}{2}\frac{v^2}{c^2}. \tag{50}$$

From the Ives-Stilwell experiment, we know that the second order wavelength shift $\Delta\lambda/\lambda$ of a moving light source is related to its velocity by $V^2/2c^2$. For our proposed transformation to correctly predict the outcome of this experiment, we see that we must have $D = 1$.

Be. Discussions and conclusions

Having culled as much information as possible out of the three fundamental experiments, we can now write the transformation equations (30) as

$$\begin{aligned} x' &= \gamma(x - \beta ct), \qquad & y' &= y, \\ b't' &= \gamma(ct - \beta x), \qquad & z' &= z, \end{aligned} \tag{51}$$

with the invariant metric

$$(b't')^2 - x'^2 - y'^2 - z'^2 = s^2 = (ct)^2 - x^2 - y^2 - z^2. \tag{52}$$

Notice that at no point did we ever postulate a method of clock synchronization, as Robertson or Hils and Hall did, which would have been equivalent to assuming a certain relation between t and t' (or equivalently, c and b'). As a result, it is impossible, on the basis of these three experiments alone, to separate the variables b' and t' in the transformation or to determine the parameters M and N. By choosing a method of clock synchronization equivalent to that of special relativity and thus assuming that the speed of light was the same in both directions along a line joining the two clocks, Robertson and Hils and Hall have actually fixed the speed of light to be a universal constant. Without this stipulation, we see that an infinite number of transformations, all consistent with the results of the three experiments, are allowed, corresponding to arbitrary choices of M and N. Each of these transformations has the Lorentz group properties (this is evident because v'/c' is independent of M and N) and four-dimensional symmetry, reducing the "tests of the isotropy of light" to mere tests of the four dimensional symmetry of natural laws. In the future, consideration of these many possibilities may lead to new ways of looking at physics.

It is my belief that no experiment to date contains enough information to exactly specify a transformation. To do so, one would have to devise a method to measure the one way velocity of light without introducing a method of clock synchronization, a task whose feasibility is a much debated subject. In the meantime, more and more accurate versions of these experiments will challenge the limits of our measurement ability and provide more and more precise verification of the four-dimensional symmetry of our world.

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Appendix C.

Quantum Electrodynamics in Both Linearly Accelerated and Inertial Frames

Ca. Quantum electrodynamics based on taiji relativity

In an inertial frame, the 4-dimensional symmetry of the Lorentz and Poincaré groups is shared by both special relativity and taiji relativity. Thus, the usual formalism of quantum field can be applied to field theory based on taiji relativity. In fact, the following discussion on quantum electrodynamics holds for all relativity theories, including special relativity, common relativity, and extended relativity, provided one consistently uses the 4-coordinate $x^{\mu}=(w,x,y,z)$.

For quantum electrodynamics (QED) in an inertial frame, the invariant action S_q, involving Dirac's electron field ψ, the photon field a_{μ}, is assumed to take the usual form,[1]

$$S_q = \int L\, d^4x, \qquad L = \overline{\psi}[\gamma^{\mu}(i\partial_{\mu} - \bar{e}a_{\mu}) - m]\psi - \frac{1}{4} f_{\mu\nu}f^{\mu\nu}, \tag{C.1}$$

$$\alpha_e = \frac{\bar{e}^2}{4\pi} \approx \frac{1}{137}, \qquad \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}, \tag{C.2}$$

where $d^4x=dwd^3r$, and we have used the natural units for simplicity. Each term in the Lagrangian density L has the dimension of $1/(\text{length})^4$.

For quantization of Dirac fields, the "canonical momentum" π_b conjugate to ψ_b, b=1,2,3,4, and the Hamiltonian density for a free electron are defined as

$$\pi_b = \frac{\partial L_{\psi}}{\partial(\partial_0\psi_b)}, \qquad L_{\psi} = \overline{\psi}[\gamma^{\mu}i\partial_{\mu} - m]\psi, \tag{C.3}$$

$$H = \pi\partial_0\psi - L_\psi .$$

For free photon (a_μ) and electron (ψ) fields, we have

$$a_\mu(w,\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p};\alpha}\sqrt{1/(2p_0)}\Big[a(\mathbf{p},\alpha)\varepsilon_\mu(\alpha)\exp(-ip\cdot x)$$

$$+ a^\dagger(\mathbf{p},\alpha)\varepsilon_\mu(\alpha)\exp(ip\cdot x)\Big], \tag{C.4}$$

$$\psi(w,\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p};s}\sqrt{m/p_0}[b(\mathbf{p},s)u(\mathbf{p},s)\exp(-ip\cdot x)$$

$$+ d^\dagger(\mathbf{p},s)v(\mathbf{p},s)\exp(ip\cdot x)], \qquad p\cdot x = p_\mu x^\mu , \tag{C.5}$$

where

$$[a(\mathbf{p},a),\ a^\dagger(\mathbf{p}',a')] = \delta_{\mathbf{pp}'}\delta_{aa'},$$

$$[b(\mathbf{p},s),\ b^\dagger(\mathbf{p}',s')] = \delta_{\mathbf{pp}'}\delta_{ss'} , \qquad [d(\mathbf{p},s),\ d^\dagger(\mathbf{p}',s')] = \delta_{\mathbf{pp}'}\delta_{ss'} , \tag{C.6}$$

and all other commutators such as $[a(\mathbf{p},\alpha),\ a(\mathbf{p}',\alpha')]$ and $[a^\dagger(\mathbf{p},\alpha),\ a^\dagger(\mathbf{p}',\alpha')]$ vanish. Commutators for quantized fields $\psi(w,\mathbf{r})$ and $a_\mu(w,\mathbf{r})$ can be derived from (C.4)-(C.6). The Dirac equation based on taiji relativity can be derived from (C.1),

$$i\,\frac{\partial\psi}{\partial w} = [\boldsymbol{\alpha}_D\cdot\ (-i\nabla - \bar{e}\mathbf{a}) + \beta_D m + \bar{e}a_0]\psi , \tag{C.7}$$

where $\boldsymbol{\alpha}_D=(\gamma^0\gamma^1, \gamma^0\gamma^2, \gamma^0\gamma^3)$ and $\beta_D=\gamma^0$ are the usual constant Dirac matrices.

In view of the equations of motion (C.7), we must use the taiji-time w in a general frame as the evolution variable for a state $\Phi^{(S)}(w)$ in the Schrödinger representation:

$$i\,\frac{\partial\Phi^{(S)}(w)}{\partial w}=H^{(S)}(w)\Phi^{(S)}(w), \qquad H^{(S)}=H_o^{(S)}+H_I^{(S)}, \tag{C.8}$$

because the evolution of a physical system is assumed to be described by a Hamiltonian operator which has the same transformation property as the taiji-time w or $\partial/\partial w$.

The usual covariant formalism of perturbation theory can now be applied.[2] To illustrate this, let us briefly consider the interaction representation and the S-matrix based on taiji relativity. The transformations of the state vector $\Phi(w)$ and operator O from the Schrödinger representation to the interaction representation are

$$\Phi(w) \equiv \Phi^{(I)}(w) = \exp[iH_o^{(S)}w]\Phi^{(S)}(w), \tag{C.9}$$

$$O(w) \equiv O^{(I)}(w) = \exp[iH_o^{(S)}w]O^{(S)}\exp[-iH_o^{(S)}w]. \tag{C.10}$$

Because $O^{(S)}$ and $O(w)$ are the same for $w=0$, we have,

$$i\,\frac{\partial\Phi(w)}{\partial w}=H_I(w)\Phi(w), \qquad H_I=\exp[iH_o^{(S)}w]H_I^{(S)}\exp[-iH_o^{(S)}w], \tag{C.11}$$

$$O(w) = \exp[iH_o^{(S)}w]O(0)\exp[-iH_o^{(S)}w]. \tag{C.12}$$

The U-matrix can be defined in terms of the taiji-time w: $\Phi(w)=U(w,w_0)\Phi(w_0)$, $U(w_0,w_0)=1$. It follows from (C.11) and (C.12), that

$$i\,\frac{\partial U(w,w_0)}{\partial w}=H_I(w)U(w,w_0). \tag{C.13}$$

If a physical system is in the initial state Φ_i at taiji-time w_0, the probability of finding it in the final state Φ_f at a later taiji-time w is given by

$$|\langle\Phi_f|U(w,w_0)\Phi_i\rangle|^2 = |U_{fi}(w,w_0)|^2. \tag{C.14}$$

The average transition probability per unit taiji-time for $\Phi_i \to \Phi_f$ is

$$\frac{|U_{fi}(w,w_0) - \delta_{fi}|^2}{(w - w_0)}. \tag{C.15}$$

As usual, we can express the S-matrix in terms of the U-matrix, i.e. $S=U(\infty,-\infty)$ and obtain the following form

$$S = 1 - i\int_{-\infty}^{\infty} H_I(w)dw + (-i)^2\int_{-\infty}^{\infty} dw\int_{-\infty}^{w} H_I(w)H_I(w')dw' + \ldots\ldots \tag{C.16}$$

For w-dependent operators, one can introduce a w-product W* (corresponding to the usual chronological product), so that (C.16) can be written in an exponential form:

$$S = W^*\{\exp[-i\int_{-\infty}^{\infty} H_I(x^\mu)dwd^3r]\}, \qquad \int_{-\infty}^{\infty} H_I(x^\mu)d^3r = H_I(w). \tag{C.17}$$

In natural units, $L^{1/4}$, a_μ, and $\psi^{2/3}$ all have the dimension of inverse length and consequently, the classical electron radius r_e and electron Compton wavelength λ_e, for example, are given by

$$r_e=\alpha_e/m_e \qquad \text{and} \qquad \lambda_e=1/m_e\,, \tag{C.18}$$

respectively.

To obtain the rules for Feynman diagrams in taiji QED, we follow the usual quantization procedure and define L_{TQED} by adding a gauge fixing term to the Lagrangian (C.1),

$$L_{TQED} = L - \frac{1}{2\rho}(\partial^\mu a_\mu)^2 \tag{C.19}$$

where ρ is a gauge parameter. As usual, we define the *M*-matrix as follows:

$$S_{if} = \delta_{if} - i(2\pi)^4\delta^4(p_f^{(tot)} - p_i^{(tot)})\Big[\prod_{\text{ext par}} (n_j/V)\Big]^{1/2} M_{if}, \qquad (C.20)$$

where "ext par" denotes external particles and $n_j = m_j/p_{0j}$ for spin 1/2 fermions and $1/2p_{0j}$ for bosons. Because of the 4-dimensional symmetry in (C.19) and (C.20), the rules for writing M_{if} are formally the same as those in the usual QED, except that certain quantities (e.g., w, p_μ and $\bar{e}$) have different dimensions from the corresponding quantities in conventional QED. To wit,

(a) the covariant photon propagator is now given by

$$\frac{-i[\eta_{\mu\nu} - (1-\rho)k_\mu k_\nu/(k^2+i\epsilon)]}{(k^2+i\epsilon)}, \qquad k^2 = k_\mu k^\mu, \qquad (C.21)$$

(b) the electron propagator is

$$\frac{i}{(\gamma^\mu p_\mu - m + i\epsilon)}, \qquad (C.22)$$

(c) the electron-photon vertex is

$$-i\,\bar{e}\gamma^\mu \qquad (C.23)$$

and (d) each external photon line has an additional polarization 4-vector $\varepsilon_\mu(\alpha)$. Furthermore, each external electron line has u(s,p) for the absorption of an electron and $\bar{u}$(s,p) for the emission of an electron, etc.

Other rules such as taking the trace with a factor of –1 for each closed electron loop, integration with $d^4k/(2\pi)^4$ over a momentum k_μ not fixed by the conservation of four-momentum at each vertex, etc. are the same as usual.

Thus, if one calculates scattering cross sections and decay rates (with respect to the taiji-time w) of a physical process, one will get formally the same result as that in conventional QED.[1] Let us consider an example. For a scattering process $1+2 \to 3+4+....+N$, the differential cross section $d\sigma$, which has the dimension of area, is given by

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} |M_{if}|^2 \left[\prod_{ext\ fer} (2m_{fer}) \right] \frac{d^3p_3}{(2\pi)^3 2p_{03}} .....$$

$$...... \frac{d^3p_N}{(2\pi)^3 2p_{0N}} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 .... - p_N) S_o, \qquad \text{(C.24)}$$

where $p_0 = \sqrt{\mathbf{p}^2 + m^2}$ and S_o denotes a factor $1/(n!)$ for each kind of (n) identical particles in the final state. If the initial particles are unpolarized, one takes the average over initial spin states. When there is no external fermion in a process, then $[\prod_{ext\ fer} (2m_{fer})]$ in (C.24) is replaced by 1.

Cb. Experimental measurements of dilations of decay-lengths and decay-lifetimes in inertial frames

Since the time t is undefined in taiji relativity, one may wonder:
How can taiji relativity explain the well-established experimental results of the "lifetime dilatation" of unstable particles?
The answer is that experiments that purport to measure the mean lifetime of unstable particles in flight actually measure the mean decay-lengths rather than the lifetimes, where the decay length is the distance traveled by an unstable particle before decaying. Furthermore, the decay-length dilatation can be calculated and it can be shown that in the quantum field theory based on taiji relativity, this length is dilated by a factor of γ. For a specific example, see section 10f.

Cc. Quantum electrodynamics of bosons in accelerated and inertial frames

Quantum scalar field operators obey equation (23.35) in CLA frames. This suggests that we use the taiji-time w in a general frame as the evolution variable for a state $\Phi^{(S)}(w)$ in the Schrödinger representation:

$$i\frac{\partial\Phi^{(S)}(w)}{\partial w}=H^{(S)}(w)\Phi^{(S)}(w)\ , \qquad H^{(S)}=H_o^{(S)}+H_I^{(S)}, \tag{C.25}$$

which is the same as (C.8). The reason is that the evolution of a physical system is assumed to be described by a Hamiltonian operator which has the same transformation property as $\partial/\partial w$. A covariant partial derivative is the same as an ordinary partial derivative, $D_\mu = \partial_\mu$, when they operate on scalar functions. We note that the form (C25) is no longer true if the Hamiltonian involves spinor fields; in this case, the time derivative ∂_0 has to be replaced by the 'gauge covariant derivative' ∇_0, according to equation (23.48).

There is a class of phenomena in particle physics and in the conventional quantum field theory in which the particles move with constant velocities as measured in an inertial frame.[2] For these physical phenomena, it is natural to assume that the usual covariant formalism of perturbation theory can also be applied to QED of scalar bosons in CLA frames, which are smoothly connected to inertial frames in the limit of zero acceleration. Thus, we assume that equations (C.9) to (C.23) hold also for CLA frames. In this case, one can define the S-matrix and obtain the generalized Feynman rules in both linearly accelerated and inertial frames.

To obtain the rules for Feynman diagrams in CLA frames, we follow the usual procedure[1] and assume L_{SQED} to be

$$L_{SQED} = L_{sp} - \sqrt{-g}(\partial^\mu a_\mu)^2/(2\rho), \tag{C.26}$$

$$L_{sp} = \sqrt{-g}\,[g^{\mu\nu}(i\partial_\mu - \bar{e}\,a_\mu\}\Phi^*(i\partial_\nu - \bar{e}a_\nu\}\Phi - m^2\Phi^*\Phi] - \frac{1}{4}\sqrt{-g}\,f_{\mu\nu}f^{\mu\nu}\ ,$$

$$f_{\mu\nu} = D_\mu a_\nu - D_\nu a_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu, \qquad \bar{e} = -\sqrt{4\pi\alpha_e}\,,$$

where ρ is a gauge parameter. The theory is gauge invariant, namely, physical results are independent of the gauge parameter ρ.

To see that there is a "conservation" of 4-momentum at each vertex of the Feynman diagram in CLA frames, let us consider the wave function $\Phi(w,\mathbf{x})$ =

$\Phi(x)$ for a "free particle" given by (23.23) with the phase P given by (23.19) and the condition (23.21) for a plane wave. In CLA frames, one can verify that

$$\frac{\partial}{\partial w}P = k_{I0}(x^1 + 1/\alpha_o\gamma_o^2)\gamma^3\alpha_o + k_{I1}(x^1 + 1/\alpha_o\gamma_o^2)\gamma^3\alpha_o\beta$$

$$= (\gamma k_{I0} + \gamma\beta k_{I1})W = k_0, \tag{C.27}$$

$$\frac{\partial}{\partial x^1}P = k_{I0}\gamma\beta + k_{I1}\gamma = k_1, \qquad \frac{\partial}{\partial x^2}P = k_2, \qquad \frac{\partial}{\partial x^3}P = k_3,$$

where we have used $(\partial/\partial w)\gamma\beta = \gamma^3\alpha_o$ and $(\partial/\partial w)\gamma = \gamma^3\alpha_o\beta$. Thus, the relation

$$i\,\frac{\partial}{\partial x^\mu}\,e^{-iP} = k_\mu\,e^{-iP} \tag{C.28}$$

holds for a "free wave" in CLA frames.

However, the zeroth component p_0 (or k_0) of the covariant 4-momentum depends on the Wu factor W(w,x), as shown in (22.9), and is not conserved in a particle collision process. Fortunately, the conservation of "momentum" in a collision process, e.g., a+b→c+d, as observed in CLA frames can be expressed in terms of

$$(p_0/W, p_1, p_2, p_3) \equiv \bar{p}_\mu \equiv (\bar{p}_0, \bar{p}_1, \bar{p}_2, \bar{p}_3), \tag{C.29}$$

$$g^{\mu\nu}p_\mu p_\nu = (p_0/W)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = \eta^{\mu\nu}\bar{p}_\mu\bar{p}_\nu, \quad \eta^{\mu\nu} = (1,-1,-1,-1).$$

The rescaled momentum 4-vector $\bar{p}_\mu$ is not a 4-vector under the Wu or the MWL transformation. However, the momentum space of $\bar{p}_\mu$ is formally more similar to the space of the 4-momentum $p_{I\mu}$ in inertial frames than that of the true 4-momentum p_μ as far as the S-matrix and Feynman rules are concerned. For the scattering process a+b→c+d, we have the following relations for momenta:

$$\bar{p}_{0a}+\bar{p}_{0b}=\gamma(p_{I0a}+p_{I0b}+\beta[p_{I1a}+p_{I1b}]),$$

$$\bar{p}_{1a}+\bar{p}_{1b}=\gamma(p_{I1a}+p_{I1b}+\beta[p_{I0a}+p_{I0b}]),$$

$$p_{2a}+p_{2b}=p_{I2a}+p_{I2b}, \qquad p_{3a}+p_{3b}=p_{I3a}+p_{I3b}; \tag{C.30}$$

$$\bar{p}_{0c}+\bar{p}_{0d}=\gamma(p_{I0c}+p_{I0d}+\beta[p_{I1c}+p_{I1d}]),$$

$$\bar{p}_{1c}+\bar{p}_{1d}=\gamma(p_{I1c}+p_{I1d}+\beta[p_{I0c}+p_{I0d}]),$$

$$p_{2c}+p_{2d}=p_{I2c}+p_{I2d}, \qquad p_{3c}+p_{3d}=p_{I3c}+p_{I3d}; \qquad \bar{\mathbf{p}}=\mathbf{p},$$

which can be derived from the inverse transformation of (22.10) and (C.29). Since the 4-momentum is conserved in the inertial frame F_I, i.e., $p_{I0c}+p_{I0d}=p_{I0a}+p_{I0b}$ = constant and $\mathbf{p}_{Ic}+\mathbf{p}_{Id}=\mathbf{p}_{Ia}+\mathbf{p}_{Ib}$ = constant, we have the conservation of the rescaled momentum in CLA frames

$$\bar{p}_{0c}+\bar{p}_{0d}=\bar{p}_{0a}+\bar{p}_{0b} \quad \text{and} \quad \bar{\mathbf{p}}_c+\bar{\mathbf{p}}_d=\bar{\mathbf{p}}_a+\bar{\mathbf{p}}_b, \tag{C.31}$$

at the "instant of collision," so to speak. Although both sides of the equations in (C.31) are, in general, not constant as shown in (C.30), they must be the same at the instant of collision. In this sense, we have conservation of energy-momentum for both inertial and non-inertial frames. Based on the Wu transformation for coordinates and 4-momenta, we have

$$\int d^4x_I \exp(-ip_{I\mu}x_I^{\mu}) = (2\pi)^4\delta^4(p_I) = \int\sqrt{-g}\,d^4x e^{-iP(x)}$$

$$= (2\pi)^4\delta(\gamma\bar{p}_0-\gamma\beta\bar{p}_1)\delta(\gamma\bar{p}_1-\gamma\beta\bar{p}_0)\delta(\bar{p}_2)\delta(\bar{p}_3), \tag{C.32}$$

$$= (2\pi)^4\delta(\bar{p}_0)\delta(\bar{p}_1)\delta(\bar{p}_2)\delta(\bar{p}_3),$$

$$\delta(p_{I0})\delta(p_{I1}) = \delta(\gamma\bar{p}_0 - \gamma\beta\bar{p}_1)\delta(\gamma\bar{p}_1 - \gamma\beta\bar{p}_0) = \frac{\delta(\bar{p}_0)\delta(\bar{p}_1)}{J(p_{I\lambda}/\bar{p}_\lambda)} = \delta(\bar{p}_0)\delta(\bar{p}_1),$$

where we have used (23.8), (22.10) and (19.14). In the last equation, $J(k_{I\lambda}/k_\lambda)$ is the Jacobian of the $p_{I\lambda}$ with respect to the $\bar{p}_\lambda$ which can be calculated using (22.10). This result is the 2-dimensional generalization of the 1-dimensional case given by (23.20) with $\kappa = 0$.[3] In a CLA frame, the integral of a "plane wave" over the "whole spacetime" is limited and complicated by the presence of a "black wall" (i.e., a wall singularity) at $x = -1/(\alpha_o\gamma_o^2)$. The integration can be carried out by a change of variables and this amounts to using variables in an inertial frame as a crutch to obtain the result. The relation (C.31) or (C.32) implies that momentum is conserved at a vertex in the generalized Feynman rules in CLA frames. These properties of equations (C.27) through (C.32) are convenient for writing down the generalized Feynman rules for quantum electrodynamics in CLA frames.

As usual, if there are no identical particles in the final state, we define the relationship between the M and S matrices for initial (i) and final (f) states as follows:

$$S_{fi} = \delta_{fi} - i(2\pi)^4\delta^4(\bar{p}_f^{(tot)} - \bar{p}_i^{(tot)})\Big[\prod_{\text{ext par}} (n_j/V)\Big]^{1/2} M_{fi}, \tag{C.33}$$

where "ext par" denotes external particles, $n_j = m_j/\omega_{kj}$ for spin 1/2 fermions and $n_j = 1/2\omega_{kj}$ for bosons. Note that the S-matrix elements for physical processes which are observed and measured in CLA frames are defined only for those cases in which the momenta of the initial and final states are constant in an inertial frame.

Because of the 4-dimensional symmetry in (C26) and (C.33), the generalized Feynman rules for writing the amplitude M_{fi} are formally the same as those in the usual QED in the natural units. The generalized Feynman rules for the amplitude M_{fi} in both constant-linear-acceleration frames and inertial frames are as follows:

(a) The covariant photon propagator is given by

$$\frac{-i[\eta_{\mu\nu} - (1-\rho)\bar{k}_\mu \bar{k}_\nu / (\bar{k}^2 + i\varepsilon)]}{(\bar{k}^2 + i\varepsilon)} \tag{C.34}$$

where $\rho = 1$ is the Feynman gauge, and $\rho = 0$ the Landau gauge.

(b) The scalar boson propagator is

$$\frac{i}{(\bar{p}^\mu \bar{p}_\mu - m^2 + i\varepsilon)} . \tag{C.35}$$

(c) The vertex $\Phi(\bar{p}\ )+\gamma(\bar{k},\mu) \to \Phi(\bar{p}')$ is

$$- i\, \bar{e}(\bar{p}_\mu + \bar{p}'_\mu), \tag{C.36}$$

where $\gamma(\bar{k},\mu)$ is an incoming photon line toward the vertex with the alteration momentum $\bar{k}_\lambda$ and a polarization index μ.

(d) The vertex $\Phi+\gamma(\mu) \to \Phi+\gamma(\nu)$ has the factor

$$2i\, \bar{e}^2 \eta_{\mu\nu} . \tag{C.37}$$

(e) Each external photon line with an index μ has a polarization vector ε_μ.

(f) A factor 1/2 for each closed loop containing only two photon lines, e.g., $\Phi+\Phi \to \gamma(\mu)+\gamma(\nu) \to \Phi+\Phi$.

Other rules such as integration with $d^4\bar{k}/(2\pi)^4$ $(=W^{-1}d^4k/(2\pi)^4)$ over a momentum $\bar{k}_\mu$ not fixed by the "conservation" of momentum at each vertex etc. are the same as usual.

Cd. Feynman rules for QED with fermions in both CLA and inertial frames

To obtain the rules for Feynman diagrams of spinor QED in CLA frames, we have to replace the time derivative $\partial_0 = \partial/\partial w$ by the gauge covariant time derivative ∇_0, according to equation (23.48). We follow the usual quantization procedure and define L_{TQED} by adding a gauge fixing term in the Lagrangian density,

$$L_{TQED} = L - \sqrt{-g}(D^\mu a_\mu)^2/2\rho, \tag{C.38}$$

$$L = \sqrt{-g}\,\bar{\psi}\Gamma^\mu(i\nabla_\mu - \bar{e}a_\mu)\psi - \sqrt{-g}\,m\bar{\psi}\psi, \tag{C.39}$$

$$\nabla_\mu = (\partial_0 + \frac{1}{2}(\partial_k W)\gamma^0\gamma^k, \partial_1, \partial_2, \partial_3),$$

where ρ is a gauge parameter. As usual, the M-matrix is defined in (C.33). One can verify that

$$\nabla_\mu e^{(-iP(x) - G(w))} = k_\mu e^{(-iP(x) - G(w))}, \tag{C.40}$$

for a "plane wave" (23.57) of a free fermion in CLA frames.

The generalized Feynman rules for the amplitude M_{fi} of QED in both CLA frames and inertial frames are as follows:

(a) The covariant photon propagator is given by (C.34).

(b) The electron propagator is

$$\frac{i}{(\gamma^\mu\bar{p}_\mu - m + i\varepsilon)}, \qquad \gamma^\mu\bar{p}_\mu = \Gamma^\mu p_\mu. \tag{C.41}$$

(c) The electron-photon vertex is

$$-i\bar{e}\gamma^\mu. \tag{C.42}$$

(d) Each external photon line has an additional factor $\varepsilon_\mu(\alpha)$ Each external electron line has $u(s,\mathbf{p})$ for the annihilation of an electron and $\bar{u}(s,\mathbf{p})$ for the creation of an electron. Each external positron line has $v(s,\mathbf{p})$ for the annihilation of a positron and $\bar{v}(s,\mathbf{p})$ for the creation of a positron.

Other rules such as taking the trace with a factor (-1) for each closed electron loop, integration with $d^4\bar{k}/(2\pi)^4$ over a momentum $\bar{k}_\mu$ not fixed by the conservation of alteration momentum at each vertex are the same as usual.

Thus, if one calculates scattering cross sections and decay rates (with respect to the taiji-time w) of a physical process, one will get formally the same result as in conventional QED. For example, let us consider the decay rate $\Gamma(1\to 2+3+...+N)$ for a physical process $1\to 2+3+...+N$. It is given by the expression

$$\Gamma(1\to 2+3+...+N) = \lim_{w\to\infty} \int \frac{|S_{fi}|^2}{w} \frac{d^3x_2 d^3p_2}{(2\pi)^3} \frac{d^3x_3 d^3p_3}{(2\pi)^3} \cdots \frac{d^3x_N d^3p_N}{(2\pi)^3}$$

$$= \int \frac{1}{2\omega_{p_1}} |M_{fi}|^2 \left[\prod_{ext\ fer} (2m_{fer})\right] \frac{d^3p_2}{(2\pi)^3 2\omega_{p_2}} \cdots\cdots\cdots$$

$$\times \frac{d^3p_N}{(2\pi)^3 2\omega_{p_N}} (2\pi)^4 \delta^4(\bar{p}_1 + \bar{p}_2 - \bar{p}_3 - \bar{p}_4 \cdots - \bar{p}_N) S_o, \qquad \text{(C.43)}$$

where S_{fi} is the S-matrix element between the initial state i and the final state f given by (C.33). The decay rate $\Gamma(1\to 2+3+...+N)$ has the dimension of inverse length and S_o denotes a factor 1/n! for each kind of (n) identical particles in the final state. When there is no external fermion in a process, then $[\Pi_{ext\ fer}(2m_{fer})]$ in (C.43) is replaced by 1. The decay length D is given by $D = 1/\Gamma(1\to 2+3+...+N)$. For a scattering process $1+2\to 3+4+....+N$, the differential cross section $d\sigma$, which has the dimension of area, is given by

$$d\sigma = \frac{1}{4[(\bar{p}_1\cdot\bar{p}_2)^2 - (m_1 m_2)^2]^{1/2}} |M_{fi}|^2 \left[\prod_{ext\ fer} (2m_{fer})\right] \frac{d^3p_3}{(2\pi)^3 2\omega_{p_3}} \cdots$$

$$\cdots\cdots \frac{d^3p_N}{(2\pi)^3 2\omega_{p_N}}(2\pi)^4\delta^4(\bar{p}_1 + \bar{p}_2 - \bar{p}_3 - \bar{p}_4 \cdots - \bar{p}_N) S_O, \tag{C.44}$$

where $p_i = \bar{p}_i$, $i = 1, 2, 3$ and $\bar{p}_0 = \omega_p = (\mathbf{p}^2 + m^2)^{1/2}$. If the initial particles are unpolarized, one takes the average over initial spin states. When there is no external fermion in a process, then $[\Pi_{ext\ fer}(2m_{fer})]$ in (C.44) is replaced by 1.

Ce. Some QED results in both CLA and inertial frames

Let us consider some well-known physical processes in QED[1] to illustrate the generalized Feynman rules in both CLA frames and inertial frames, and see how the conventional results in inertial frames are modified if they are measured in a laboratory with constant-linear-acceleration.

A. Electron Scattering from a Point-Like Proton

According to the generalized Feynman rules, the amplitude M_{fi} for such an electron scattering from a point-like proton, $e(\bar{p}_i + p(\bar{P}_i) \rightarrow e(\bar{p}_f) + p(\bar{P}_f)$ with the exchange of a photon $\gamma(\bar{q})$, where $\bar{q}_\mu = \bar{p}_{f\mu} - \bar{p}_{i\mu}$, is given by

$$M_{fi} = \bar{u}(s_f, \mathbf{p}_f)\,[-\,i\,\bar{e}\gamma^\mu]u(s_i, \mathbf{p}_i)\left[\frac{-i\eta_{\mu\nu}}{(\bar{q}_\sigma\bar{q}^\sigma + i\varepsilon)}\right]\bar{u}(S_f, \mathbf{P}_f)[-\,i\bar{e}\gamma^\mu]u(S_i, \mathbf{P}_i), \tag{C.45}$$

where we have used the Feynman gauge for the photon propagator ($\rho = 1$). The S-matrix element S_{fi} in (C.33) takes the form

$$S_{fi} = -\,i(2\pi)^4\delta^4(\bar{P}_f + \bar{p}_f - \bar{P}_i - \bar{p}_i)\left[\frac{m}{\omega_{pf}}\frac{m}{\omega_{pi}}\frac{M}{\omega_{Pf}}\frac{M}{\omega_{Pi}}\frac{1}{V^4}\right]^{1/2} M_{fi}\,, \tag{C.46}$$

where m and M are, respectively, masses for the electron and the proton. The differential cross section is given by (C.44),

$$d\sigma = \frac{mM}{[(\bar{p}_i\cdot\bar{P}_i)^2 - (mM)^2]^{1/2}}|M_{fi}|^2\frac{m d^3\bar{p}_f}{(2\pi)^3\omega_{pf}}\frac{M d^3\bar{P}_f}{(2\pi)^3\omega_{Pf}}$$

$$\times(2\pi)^4\delta^4(\bar{p}_f + \bar{P}_f - \bar{p}_i - \bar{P}_i), \tag{C.47}$$

where $\bar{p}_0 = \omega_{pf} = (\mathbf{p}_f^2+m^2)^{1/2}$, $\omega_{Pf} = (\mathbf{P}_f^2+M^2)^{1/2}$ and $|M_{fi}|^2$ is given by

$$|M_{fi}|^2 = \frac{\bar{e}^4}{2m^2M^2\bar{q}^4}[\bar{P}_f\cdot\bar{p}_f\bar{P}_i\cdot\bar{p}_i + \bar{P}_f\cdot\bar{p}_i\bar{P}_i\cdot\bar{p}_f - m^2\bar{P}_f\cdot\bar{P}_i$$

$$- M^2\bar{p}_f\cdot\bar{p}_i + 2M^2m^2]. \tag{C.48}$$

Since $\bar{P}_f\cdot\bar{p}_f = \bar{P}_{f\mu}\bar{p}_f^{\mu}$, etc. we see that the differential cross section is formally the same as the one given by the conventional theory in an inertial frame, except that each individual momentum is not constant in CLA frames,

$$d\sigma(\text{CLA frame}) = d\sigma(\text{inertial frame}). \tag{C.49}$$

The origin of this identical result is the limiting 4-dimensional symmetry which dictates the invariance of the action or the S-matrix.

After integration, (C.49) gives the total cross section which can be pictured as the effective size of the target particle, in this case, a proton. This effective size of the proton depends of the strength of the interaction. For the electromagnetic interaction, the coupling strength is $\alpha_e \sim 1/137 \sim 10^{-2}$, the weak interaction coupling strength is about 10^{-12}. The size (or cross section) of a proton is about 10^{-24} cm^2 from the viewpoint of the electron. But from the viewpoint of a neutrino, which has only weak interactions with the proton, the size of a proton is extremely small, about 10^{-44} cm^2.

B. Compton Scattering

The S-matrix element for the Compton scattering process, $\gamma(\bar{k}) + e(\bar{p}_i) \rightarrow \gamma(\bar{k}') + e(\bar{p}_f)$, is given by

$$S_{fi} = -i(2\pi)^4\delta^4(\bar{k}' + \bar{p}_f - \bar{k} - \bar{p}_i)\left[\frac{m}{\omega_{pf}}\frac{m}{\omega_{pi}}\frac{1}{2\omega_k}\frac{1}{2\omega_{k'}}\frac{1}{V^4}\right]^{1/2} M_{fi}\ , \tag{C.50}$$

where $\omega_k = |\bar{\mathbf{k}}| = |\mathbf{k}|$, $\omega_{pf} = \sqrt{\bar{\mathbf{p}}_f^2 + m^2}$ and the M-matrix element is given by

$$M_{fi} = \bar{u}(s_f, \mathbf{p}_f) \Big\{ [-i\bar{e}\, \gamma^\alpha \varepsilon'_\alpha] \Big[\frac{-i}{\gamma^\mu(\bar{p}_{i\mu} + \bar{k}_\mu) - m + i\varepsilon} \Big] [-i\bar{e}\gamma^\nu \varepsilon_\nu]$$

$$+ [-i\bar{e}\, \gamma^\alpha \varepsilon_\alpha] \Big[\frac{-i}{\gamma^\mu(\bar{p}_{i\mu} - \bar{k}'_\mu) - m + i\varepsilon} \Big] [-i\bar{e}\gamma^\nu \varepsilon'_\nu] \Big\} u(s_i, \mathbf{p}_i), \tag{C.51}$$

according to the generalized Feynman rules.

We obtain the result that the differential cross section for the Compton scattering is also the same as that in an inertial frame,

$$d\sigma_{Compton}(\text{CLA frame}) = d\sigma_{Compton}(\text{inertial frame}). \tag{C.52}$$

C. Self-Mass of the Electron

The self-mass of the electron is given by the expression

$$\delta m = \int \frac{d^4\bar{k}}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{[\bar{k}_\sigma \bar{k}^\sigma + i\varepsilon]} [-\, i\bar{e}\, \gamma^\mu] \frac{-i}{[\gamma^\rho(\bar{p}_{i\rho} - \bar{k}_\rho) - m + i\varepsilon]} [-i\, \bar{e}\, \gamma^\nu]$$

$$= \delta m(\text{inertial frame}). \tag{C.53}$$

This is consistent with the fact that the (rest) mass of the electron in an inertial frame is the same as that of the electron in CLA frames, as shown in (23.8).

D. Anomalous Magnetic Moment of the Electron

The anomalous magnetic moment of the electron can be found from the calculation of the vertex correction,

$$-\, i\, \bar{e}\, \gamma^\mu \;\to\; -\, i\, \bar{e}\, \gamma^\mu - i\, \bar{e}\, \Lambda^\mu \tag{C.54}$$

due to higher order interactions. Up to the third order, we have

$$\Lambda^{\mu} = \int \frac{d^4\bar{k}}{(2\pi)^4}\left[\frac{-i\eta_{\nu\lambda}}{(\bar{k}_{\sigma}\bar{k}^{\sigma}+i\varepsilon)}\right][\gamma^{\nu}]\left[\frac{-i}{\gamma^{\rho}(\bar{p}'_{\rho}-\bar{k}_{\rho})-m+i\varepsilon}\right][-i\bar{e}\gamma^{\mu}]$$

$$\times\left[\frac{-i}{\gamma^{\sigma}(\bar{p}_{\sigma}-\bar{k}_{\sigma})-m+i\varepsilon}\right][-i\bar{e}\gamma^{\lambda}]. \tag{C.55}$$

The result can be written in the form

$$\Lambda^{\mu} = \Gamma^{\nu}G_1(\bar{q}^{\,2}) + \frac{1}{2m}\bar{q}_{\,\nu}\sigma^{\nu\mu}G_2(\bar{q}^2), \qquad \bar{q}_{\mu} = \bar{p}'_{\mu} - \bar{p}_{\mu}. \tag{C.56}$$

The anomalous magnetic moment of the electron measured in a CLA frame is the same as that measured in an inertial frame,

$$G_2(0)_{\text{CLA frame}} = \left(\frac{\alpha_e}{2\pi}\right) = G_2(0)_{\text{inertial frame}}. \tag{C.57}$$

Finally, we note that these discussons for QED can also be applied to non-inertial frames with a constant rotational motion, since we have the taiji rotational transformations (26.11) with limiting 4-dimensional symmetry.

References

1. See, for example, J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, MA, 1967) pp.171-172 and pp. 181-188; S. Weinberg, *The Quantum Theory of Fields,* vol. I, Foundations (Cambridge University Press, New York, N.Y., 1995) pp.134-147. J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964) chapter 7.
2. There is another class of phenomena in which particles move with a constant acceleration relative to an inertial frame. In this case, such particles are at rest relative to a particular CLA frame. The physical laws governing this class of phenomena appear to be more involved. For example, consider the mathematical law governing the decay of unstable particles. In quantum field theory, the decay rate for accelerated unstable particles is formally the same as in (17.72) with a time-dependent factor $\Gamma(\beta)=\Gamma_o(1-\beta^2)^{1/2}$.
3. I. M. Gel'fand and G.E. Shilov, *Generalized Functions* Vol. 1 (transl. by E. Saletan, Academic Press, New York, 1964) p. 185.

Appendix D

Yang-Mills Gravity with Translation Gauge Symmetry in Inertial and Non-inertial Frames

Da. Translation gauge transformations and an 'effective metric tensor' in flat spacetime

The limiting 4-dimensional symmetry of the Lorentz and Poincaré groups leads to generalized spacetime transformations for non-inertial frames, as we have discussed in chapter 19. For non-inertial frames with constant-linear-accelerations, we have discussed the Wu transformation and the generalized MWL transformation. The existence of these spacetime transformations implies that the spacetimes of both inertial and non-inertial frames have zero Riemann-Christoffel curvature tensor. This provides a simple mathematical framework for discussions of Yang-Mills theory of gauge fields in both inertial and non-inertial frames.[1]

It is well-known that the form of basic interactions for strong, weak and electromagnetic forces in the physical world can be determined by gauge symmetry of actions involving quadratic gauge curvatures.[2] Yang-Mills theory is characterized by two basic features; (A) a gauge invariant action involving quadratic gauge curvature, and (B) an underlying flat spacetime, in harmony with the grand tradition of all classical and quantum fields. However, Einstein's theory of gravity based on general coordinate invariance is a departure from this tradition. Dyson stressed that "the most glaring incompatibility of concepts in contemporary physics is that between Einstein's principle of general coordinate invariance and all the modern schemes for a quantum-mechanical description of nature."[3] It would be very interesting if gravity could be understood within the framework of Yang-Mills theory with the spacetime translation gauge symmetry. The spacetime translation symmetry is particularly interesting because it implies the conserved energy-momentum tensor which is the source of the gravitational field.

The relation between such a translation gauge theory and fiber bundles is not completely clear. Usually, a fiber bundle over the 4-dimensional spacetime manifold attaches to each point of spacetime some internal space. In Yang-Mills theory, the internal gauge group is associated with the internal space. However, the spacetime translation group is associated with the external spacetime rather than an internal space. In this connection, we note that Einstein's gravitational field is a metric which determines the Levi-Civita connection on the tangent vector bundle, while Yang-Mills fields are connections on principle fiber bundles. Besides, the Hilbert-Einstein action involves a linear spacetime curvature, while the Yang-Mills action involves a quadratic gauge curvature. Thus, both Einstein's gravitational field and the translation gauge field are not exactly the same as the Yang-Mills field.

Nevertheless, we shall assume that Yang-Mills' approach to an internal gauge group can be generalized to the external spacetime translation group. Such a framework facilitates the quantization of gauge fields by the conventional method and hopefully, may shed light on a theory of quantum gravity based on Yang-Mills-type action involving quadratic gauge curvature.

In Einstein's theory of gravity, the structure of couplings for $g_{\mu\nu}$ is very complicated and also has non-trivial difficulties in both technical and conceptual aspects from the viewpoint of quantum field theory. However, as a classical field theory, one of the strengths of general relativity lies in its successful equation of motions for objects and light rays, based on the metric $g_{\mu\nu}dx^{\mu}dx^{\nu}$, which was first discussed by Einstein and Grossmann.[4] As for renormalizable quantum field theory, Yang-Mills fields have the best track record in theory and experiment, providing that the underlying spacetime is flat. Yang-Mills fields with spacetime translation gauge symmetry have special features which provide a natural union of the Einstein-Grossmann metric and the gravitational Yang-Mills field. The union of the two implies that (i) The framework is applicable to all general frames of reference (both inertial and non-inertial) in which the spacetime is characterized by the vanishing Riemann-Christoffel curvature tensor. (ii) The 'effective Einstein-Grossmann metric' originates physically from a spin-2 Yang-Mills field in flat spacetime.

The formulations for electromagnetic and Yang-Mills fields associated with internal gauge groups have been developed extensively. They are based on the replacement

$$\partial_\mu \to \partial_\mu + igB_\mu^a \tau^a,$$

where τ^a is the constant matrix representations of the gauge groups which have little to do with external spacetime. For external gauge groups related to spacetime, e.g., the de Sitter group or the Poincaré group, the gauge invariant Lagrangian involving fermions turns out to be richer in content.[5]

Let us concentrate on a specific simple external gauge group of translations T(4) in flat spacetime. The translation group T(4) is the Abelian subgroup of the Poincaré group. This group is particularly interesting because it is the minimal gauge group related to the conserved energy-momentum tensor that couples to a spin-2 field $\phi_{\mu\nu}$. However, the generators of the translational group are the displacement operators, $p_\mu = i\partial_\mu$ (J=1) in inertial frames. In a general frame (inertial or non-inertial) with a metric tensor $P_{\mu\nu}$, we replace ∂_μ by D_μ, i.e., the partial covariant derivative with respect to the Levi-Civita connection (or the metric tensor $P_{\mu\nu}$ in flat spacetime. Thus, the replacement in Yang-Mills gravity takes on the different form

$$D_\mu \to D_\mu - ig\phi_{\mu\nu}D^\nu \equiv J_{\mu\nu}D^\nu,$$

where $D_\mu = P_{\mu\nu}D^\nu$ in such a gauge theory. The generators of this non-compact translation group do not have a constant matrix representation. It is precisely this unique property that leads naturally to an 'effective Einstein-Grossmann metric' in flat spacetime. Such an effective metric emerges for matter fields (as shown in equations (D.5) and (D.6) below). Furthermore, the displacement operator of the translation gauge group dictates that the coupling constant g in $J_{\mu\nu}$ must have the dimension of length and that the interaction cannot have both attractive and repulsive forces, in sharp contrast to the dimensionless real coupling constants in electrodynamics and other Yang-Mills theories associated with internal gauge groups.

The new formalism of external gauge symmetry for translations in flat spacetime leads to a gauge-invariant action involving fermions. It suggests that (a) the massless Yang-Mills spin-2 field in flat spacetime can be identified as the gravitational gauge field,[6] (b) a new gravitational gauge equation can be formulated in both inertial and non-inertial frames and (c) an 'effective metric' $G_{\mu\nu}dx^\mu dx^\nu$ can be used to describe the motion of classical objects in flat spacetime. In the post-Newtonian approximation, the present gauge field equation is consistent with classical tests such as the perihelion shift of the Mercury and the time delay of radar echoes.

Let us consider translation gauge symmetry, i.e., the local spacetime translation with an arbitrary infinitesimal vector gauge-function $\Lambda^\mu(x)$,

$$x^\mu \to x'^\mu = x^\mu + \Lambda^\mu(x), \qquad x \equiv x^\mu = (w, x, y, z). \tag{D.1}$$

The basic point is that this transformation has a dual interpretation:

(i) a shift of the spacetime coordinates by an infinitesimal vector gauge-function $\Lambda^\mu(x)$, and

(ii) an arbitrary infinitesimal transformation.

These two mathematical implications of the transformation (D.1) dictate the following gauge transformation of spacetime translations for physical quantities in the Lagrangian of fields

$$Q^{\mu_1..\mu_m}_{\alpha_1...\alpha_n}(x) \to Q^{\$\mu_1..\mu_m}_{\alpha_1...\alpha_n}(x) = \left(Q^{\nu_1...\nu_m}_{\beta_1..\beta_n}(x) - \Lambda^\lambda(x)\partial_\lambda Q^{\nu_1...\nu_m}_{\beta_1...\beta_n}(x)\right)$$

$$\times \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\mu_m}}{\partial x^{\nu_m}} \frac{\partial x^{\beta_1}}{\partial x'^{\alpha_1}} \cdots \frac{\partial x^{\beta_n}}{\partial x'^{\alpha_n}}, \tag{D.2}$$

where $\mu_m, \nu_m, \alpha_m, \beta_m$ are spacetime indices.

The spinors $\psi(x)$ and $D_\mu\psi$ transform, by definition, as a scalar function $Q(x)$ and a covariant vector $Q_\mu(x)$ respectively under the translation gauge transformation

$$Q(x) \to Q^\$(x) = \left(Q(x) - \Lambda^\lambda(x)\partial_\lambda Q(x)\right),$$

$$D_\mu\psi(x) \to \left(D_\mu\psi(x)\right)^\$ = \left(D_\nu\psi(x) - \Lambda^\lambda(x)\partial_\lambda D_\nu\psi(x)\right)\frac{\partial x^\nu}{\partial x'^\mu},$$

$$A^\mu \to (A^\mu)^\$ = A^\mu - \Lambda^\lambda\partial_\lambda A^\mu + A^\lambda\partial_\lambda\Lambda^\mu, \tag{D.3}$$

$$Q^{\mu\nu} \to (Q^{\mu\nu})^\$ = Q^{\mu\nu} - \Lambda^\lambda\partial_\lambda Q^{\mu\nu} + Q^{\lambda\nu}\partial_\lambda\Lambda^\mu + Q^{\mu\lambda}\partial_\lambda\Lambda^\nu,$$

$$T_{\mu\nu} \to (T_{\mu\nu})^\$ = T_{\mu\nu} - \Lambda^\lambda\partial_\lambda T_{\mu\nu} - T_{\mu\alpha}\partial_\nu\Lambda^\alpha - T_{\alpha\nu}\partial_\mu\Lambda^\alpha.$$

We use D_μ to denote the partial covariant derivative with respect to the metric tensor $P_{\mu\nu}(x)$ in a general reference frame. The translation gauge transformation (D.2) is formally similar to the Lie variations in the coordinate transformation.

For an example of $P_{\mu\nu}(x)$ in the flat spacetime of a general reference frame, let us consider the accelerated Wu transformation between an inertial frame $F_I(w_I, x_I, y_I, z_I)$ and a non-inertial frame $F(w,x,y,z)$. The Wu transformations preserve the invariant infinitesimal interval[7]

$$ds^2 = dw_I^2 - dx_I^2 - dy_I^2 - dz_I^2 = P_{\mu\nu}dx^\mu dx^\nu, \tag{D.4}$$

$$P_{\mu\nu} = (W^2, -1, -1, -1) = \left(\gamma^4[\gamma_0^{-2} + \alpha x]^2, -1, -1, -1\right).$$

All constant-linear-acceleration frames of reference have the metric tensor of the form $P_{\mu\nu} = (W^2, -1, -1, -1)$.[8,9] The metric tensor $P_{\mu\nu}(x)$ for a general frame of reference with zero Riemann-Christoffel curvature tensor may be called the Poincaré metric tensor. In the limit of zero acceleration, $\alpha_o \to 0$, $P_{\mu\nu}(x)$ in (D.4) reduces to the Minkowski metric tensor $\eta_{\mu\nu} = (1,-1,-1,-1)$ of inertial frames.

In order to see the relation of such a spin-2 field $\phi_{\mu\nu}(x)$ to the gravitational field, let us consider the kinetic term in the Lagrangian of a scalar field Φ: (1/2) $P^{\mu\nu}(x)\, D_\mu\Phi D_\nu\Phi$. In the presence of the spin-2 field $\phi_{\mu\nu}(x)$, the translation gauge symmetry dictates the replacement, $D_\mu \to D_\mu - ig\phi_{\mu\nu}D^\nu \equiv J_{\mu\nu}D^\nu$. Thus, we have

$$\frac{1}{2}P^{\mu\nu}D_\mu\Phi D_\nu\Phi \to \frac{1}{2}P^{\mu\nu}J_{\mu\alpha}(D^\alpha\Phi)J_{\nu\beta}(D^\beta\Phi) = \frac{1}{2}G_{\alpha\beta}(D^\alpha\Phi)(D^\beta\Phi), \tag{D.5}$$

$$G_{\alpha\beta} = P^{\mu\nu}J_{\mu\alpha}J_{\nu\beta} = P_{\alpha\beta} + 2g\phi_{\alpha\beta} + g^2P^{\lambda\sigma}\phi_{\alpha\lambda}\phi_{\beta\sigma}, \qquad D_\mu\Phi = \partial_\mu\Phi,$$

where the function $G_{\alpha\beta}$ formally resembles a metric tensor. One can see from (D.5) that it appears as if the geometry of the spacetime is changed from a Euclidean spacetime to a non-Euclidean spacetime due to the presence of the spin-2 field $\phi_{\mu\nu}(x)$. As suggested by the action for a quantum field involving the kinetic term in (D.5), the action S_p for the motion of classical objects effectively is assumed to take the form,

$$S_p = -\int_a^b m\, ds_{ei}, \qquad ds_{ei}^2 = G_{\mu\nu}dx^\mu dx^\nu, \tag{D.6}$$

where $G_{\mu\nu}dx^\mu dx^\nu$ denotes the effective Einstein-Grossmann metric for motions of classical objects.

This action (D.6) for particles suggests a simple and natural union of the Einstein-Grossmann metric for motions of classical objects with the Yang-Mills field for gravity with a flat-spacetime translation gauge group: namely, that the spin-2 gauge field and its interaction with fermion matter actually take place in a flat spacetime and that only the equation of motion of classical objects is derived from the classical action in (D.6) which happens to have a form similar to the Einstein-Grossmann metric $g_{\mu\nu}dx^\mu dx^\nu$. From the viewpoint of Yang-mills theory, the presence of the effective metric tensor $G_{\alpha\beta}$ in (D.5) and (D.6) indicates that the underlying symmetry is the translation gauge symmetry in flat spacetime rather than the general covariance in curved spacetime.

Db. Yang-Mills theory with translation gauge symmetry

The present theory of Yang-Mills gravity is formulated on the basis of the translation gauge symmetry and the postulate of the effective metric tensor in (D.6) for the motion of a classical particle in such a spin-2 fields. However, from the field-theoretic viewpoint, the real physical spacetime is still flat and the fundamental metric tensor is still $P_{\mu\nu}(x)$ in general frames of reference. Thus, $G_{\mu\nu}(x)$ in (D.6) is treated merely as an "effective metric tensor" for the motion of a classical object in the presence of the spin-2 gauge field, in the sense that the 4-dimensional effective interval is $ds_{ei}^2 = G_{\mu\nu}dx^\mu dx^\nu$ in the action (D.6) for classical objects such as planets, stars or light rays. We note that the Poincaré metric tensor $P_{\mu\nu}(x)$ is a purely geometrical property of spacetime and does not contain the physical field $\phi_{\mu\nu}$. This property is important because it enables the Yang-Mills gravity to have a very simple coupling, namely, the maximum coupling is a 4-vertex (in Feynman rules).

The translational gauge invariance requires that a symmetric spin-2 field $\phi_{\mu\nu} = \phi_{\nu\mu}$ couple to the fermion field $\psi(x)$ via the energy-momentum tensor. We postulate the following gauge-invariant fermion action S_ψ in a general frame:

$$S_\psi = \int L_\psi \sqrt{-P}\ d^4x, \qquad L_\psi = \frac{i}{2}\left(\overline{\psi}\Gamma^\mu \Delta_\mu \psi - (\Delta_\mu \overline{\psi})\Gamma^\mu \psi\right) - m\overline{\psi}\psi, \tag{D.7}$$

$$\{\Gamma^\mu, \Gamma^\nu\} = 2P^{\mu\nu}(x), \qquad \Gamma^\mu = \gamma^a e_a^\mu, \qquad P = \det(P_{\mu\nu}), \tag{D.8}$$

$$\Delta_\mu\psi = J_{\mu\nu}D^\nu\psi, \qquad D^\nu\psi = \partial^\nu\psi, \qquad J_{\mu\nu} = P_{\mu\nu} + g\phi_{\mu\nu}, \tag{D.9}$$

where γ^a and e_a^μ are respectively the constant Dirac matrices and the tetrads, while Δ_μ is the translational gauge-covariant derivative. If one considers $e_b^\mu J_{\mu\nu}$ in the action (D.7) as an 'effective tetrad,' $E_{a\nu} = e_a^\mu J_{\mu\nu}$, then the corresponding 'effective metric tensor' is $E_{a\alpha}E_{b\beta}\eta^{ab} = e_a^\mu J_{\mu\alpha}e_b^\nu J_{\nu\beta}\eta^{ab} = P^{\mu\nu}J_{\mu\alpha}J_{\nu\beta} = G_{\alpha\beta}$, which is the same as that in (D.5).

The translational gauge transformation (D.2) leads to

$$L_\psi \rightarrow L_\psi - \Lambda^\lambda\partial_\lambda L_\psi.$$

Therefore, the fermion Lagrangian $\sqrt{-P}\, L_\psi$ changes only by a divergence,

$$\sqrt{-P}\; L_\psi \rightarrow \sqrt{-P}\; L_\psi - \partial_\mu\left(\Lambda^\mu\sqrt{-P}\; L_\psi\right), \tag{D.10}$$

where we have used the gauge transformation,

$$\sqrt{-P} \rightarrow \sqrt{-P^\$} = \left[(1 - \Lambda^\sigma\partial_\sigma)\sqrt{-P}\right]\left(1 - \partial_\lambda\Lambda^\lambda\right).$$

This gauge transformation can be obtained using

$$P_{\mu\nu} - \Lambda^\lambda\partial_\lambda P_{\mu\nu} - P_{\mu\beta}\partial_\nu\Lambda^\beta - P_{\alpha\nu}\partial_\mu\Lambda^\alpha = \left[(1 - \Lambda^\sigma\partial_\sigma)P_{\alpha\beta}\right]\left(\delta_\mu^\alpha - \partial_\mu\Lambda^\alpha\right)\left(\delta_\nu^\beta - \partial_\nu\Lambda^\beta\right)$$

for an infinitesimal vector function $\Lambda^\mu(x)$. The last term in (D.10) does not contribute to the action (D.7) because of Gauss' theorem. Thus, the action S_ψ is translational gauge invariant.

Dc. Gravitational action with quadratic gauge-curvature

As usual, the gauge-curvature (or the gauge-field strength) is given by the commutator of two gauge covariant derivatives $\Delta_\mu = J_{\mu\nu}D^\nu$:

$$\left[J_{\mu\nu}D^\nu, J_{\alpha\beta}D^\beta\right] = \left(J_{\mu\nu}(D^\nu J_{\alpha\beta}) - J_{\alpha\nu}(D^\nu J_{\mu\beta})\right)D^\beta \equiv C_{\mu\alpha\beta}D^\beta. \tag{D.11}$$

The gauge-curvature $C_{\mu\alpha\beta}$ satisfies the algebraic identities,

(i) cyclicity, $C_{\mu\alpha\beta} + C_{\alpha\beta\mu} + C_{\beta\mu\alpha} = 0$, and

(ii) antisymmetry, $C_{\mu\alpha\beta} = - C_{\alpha\mu\beta}$.

These identities can be directly verified.

By examining all possible terms of the scalar quadratic gauge-curvature such as

$$C_{\mu\beta\alpha}C^{\beta\alpha\mu}, \qquad C_{\alpha\mu}{}^{\alpha}C^{\beta\mu}{}_{\beta}, \text{ etc.}$$

using the anti-symmetric property (ii) and by interchanging dummy indices, we find that there are only two independent quadratic terms,

$$C_{\mu\alpha\beta}C^{\mu\beta\alpha} \text{ and } C_{\mu\alpha}{}^{\alpha}C^{\mu\beta}{}_{\beta}$$

for the T(4) gauge curvature. All other terms can be expressed in terms of these two quadratic terms. For example, we have

$$C_{\mu\alpha\beta}C^{\mu\alpha\beta} = 2C_{\mu\alpha\beta}C^{\mu\beta\alpha}.$$

Thus, the T(4) gauge invariant Lagrangian density is naturally assumed to be a linear combination of these two independent terms, $C_{\mu\alpha\beta}C^{\mu\beta\alpha} + fC_{\mu\alpha}{}^{\alpha}C^{\mu\beta}{}_{\beta}$. For simplicity, the constant f is chosen to be $f = -1$, i.e., one has the simplest linearized equation when a gauge condition is imposed. In this case, the linearized gauge field equation turns out to be the same as the linearized Einstein's equation in general relativity. Thus, we postulate the action for the spinor-tensor fields to be

$$S_{\psi\phi} = \int L_{\psi\phi}\sqrt{-P}\ d^4x, \qquad L_{\psi\phi} = -\frac{1}{2g^2}\left(C_{\mu\alpha\beta}C^{\mu\beta\alpha} - C_{\mu\alpha}{}^{\alpha}C^{\mu\beta}{}_{\beta}\right) + L_{\psi}, \tag{D.12}$$

where $L_{\psi\phi}\sqrt{-P}$ changes only by a divergence under the gauge transformation (D.2). Note that the quadratic gauge-curvature in (D.12) can be written as

$$-\frac{1}{2g^2}\left(\frac{1}{2}C_{\mu\alpha\beta}C^{\mu\alpha\beta} - C_{\mu\alpha}{}^{\alpha}C^{\mu\beta}{}_{\beta}\right).$$

Dd. Linearlized equations of the tensor field and the Hamilton-Jacobi equation for particles

For simplicity, let us choose inertial frames rather than general non-inertial frames in the following discussions of experimental implications. One of the ways to do this is to consider the case,

$$P_{\mu\nu} = \eta_{\mu\nu}, \qquad J_{\mu\nu} = \eta_{\mu\nu} + g\phi_{\mu\nu}, \tag{D.13}$$

where all tensors indices are raised and lowered by $\eta_{\mu\nu}$. For weak fields, the linearized gauge equation for spin-2 field can be derived from (D.12)

$$\partial_\lambda\partial^\lambda\phi_{\mu\nu} - \partial^\lambda\partial_\mu\phi_{\lambda\nu} - \eta_{\mu\nu}\partial_\lambda(\partial^\lambda\phi^\alpha_\alpha - \partial^\alpha\phi^\lambda_\alpha) + \partial_\mu\partial_\nu\phi^\alpha_\alpha - \partial_\mu\partial^\beta\phi_{\nu\beta} = -gT_{\mu\nu}, \tag{D.14}$$

where μ and ν are understood to be symmetrized, and

$$T_{\mu\nu} = \frac{1}{2}\left(\overline{\psi}i\gamma_\mu\partial_\nu\psi - (\partial_\nu\overline{\psi})i\gamma_\mu\psi\right)$$

is the energy-momentum tensor of a free fermion. In the calculations of the effective metric tensor and the gravitational quadrupole radiation, it is not necessary to symmetrize μ and ν explicitly in the gauge-field equation. The gauge symmetry allows us to impose the usual gauge condition for the massless spin-2 field in (D.14),

$$\partial^{\mu}\phi_{\mu\nu} = \frac{1}{2}\partial_{\nu}\phi^{\lambda}_{\lambda}. \tag{D.15}$$

The gauge equation (D.14) takes a simple form:

$$\partial_{\lambda}\partial^{\lambda}\phi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial_{\lambda}\partial^{\lambda}\phi^{\alpha}_{\alpha} = -gT_{\mu\nu}, \tag{D.16}$$

which turns out to be formally the same as the linearized Einstein's equation for a weak gravitational field.

The classical particle action (D.6), with the effective interval d^2s_{ei} leads to the variation

$$\delta S_p = -mG_{\mu\nu}\frac{dx^{\mu}}{ds_{ei}}\delta x^{\nu}, \tag{D.17}$$

if one considers only the actual path with one of its end points variable.[10] Thus we have the Hamilton-Jacobi equation for a particle with mass m,

$$I^{\mu\nu}(\partial_{\mu}S_p)(\partial_{\nu}S_p) - m^2 = 0, \qquad I^{\mu\nu}G_{\nu\lambda} = \delta^{\mu}_{\lambda}. \tag{D.18}$$

In the Newtonian limit, (D.16) with the only non-vanishing component, $T_{00} = m\delta^3(\mathbf{r})$, leads to

$$g\phi_{00} = -\frac{g^2 m}{(8\pi r)}.$$

Also, I^{00} in the Hamilton-Jacobi equation (D.18) has the usual result

$$I^{00} = 1 + \frac{2Gm}{r} ,$$

in this limit, where G is the gravitational constant. Based on these results, together with $I^{\mu\nu}$ in (D.18) and $G_{\mu\nu}$ in (D.6), we obtain the relation

$$g = \sqrt{8\pi G} , \qquad \left(g\phi_{00} = g\phi_{11} = -\frac{Gm}{r}, \ \text{etc.} \right), \tag{D.19}$$

by solving (D.16) with the spherical coordinate, $x^{\mu} = (w, r, \theta, \varphi)$ to the first order approximation.

De. The gauge field equation in inertial and non-inertial frames

In order to show that the theory is viable beyond the first-order approximation (D.19), let us compare the result (D.18) with the perihelion shift of the Mercury,[10,11] which is sensitive to the coefficient appearing in the second-order term in I^{00} or G_{00}. We solve the non-linear gauge field equations by the method of successive approximation and carry out the related post-Newtonian approximation to the second order. To accomplish this calculation and to have a well-defined gauge fields, we include the usual gauge condition (D.15), so that the total Lagrangian in a general frame takes the form

$$L_t \sqrt{-P} = \left\{ L_{\psi\phi} - \frac{\xi}{2g^2} \left(D_{\mu} J^{\mu\alpha} - \frac{1}{2} D^{\alpha} J^{\mu}_{\mu} \right) \left(D^{\nu} J_{\nu\alpha} - \frac{1}{2} D_{\alpha} J^{\nu}_{\nu} \right) \right\} \sqrt{-P} . \tag{D.20}$$

We derive the following exact gauge field equations in a general frame

$$H^{\mu\nu} + \xi A^{\mu\nu} = -g^2 T^{\mu\nu}, \tag{D.21}$$

$$H^{\mu\nu} \equiv D_\lambda\left(J^\lambda_\rho C^{\rho\mu\nu} - J^\lambda_\alpha C^{\alpha\beta}{}_\beta P^{\mu\nu} + C^{\mu\beta}{}_\beta J^{\nu\lambda}\right) - C^{\mu\alpha\beta} D^\nu J_{\alpha\beta} + C^{\mu\beta}{}_\beta D^\nu J^\alpha_\alpha - C^{\lambda\beta}{}_\beta D^\nu J^\mu_\lambda,$$

$$A^{\mu\nu} \equiv D^\mu\left(D_\lambda J^{\lambda\nu} - \frac{1}{2} D^\nu J^\lambda_\lambda\right) - \frac{1}{2} P^{\mu\nu}\left(D_\alpha D_\lambda J^{\lambda\alpha} - \frac{1}{2} D_\alpha D^\alpha J^\lambda_\lambda\right),$$

where μ and ν are understood to be symmetrized.

It suffices to consider an inertial frame and a static and spherically symmetric system, in which gauge fields are produced by an object at rest with mass m. Based on symmetry considerations,[12] the non-vanishing components of the exterior solutions $\phi_{\mu\nu}(r)$ are $\phi_{00}(r)$, $\phi_{11}(r)$, $\phi_{22}(r)$ and $\phi_{33}(r) = \phi_{22}(r)\sin^2\theta$, where $x^\mu = (w, r, \theta, \varphi)$.

To solve the static gauge field, let us write

$$J_{00} = J^0_0 = S, \qquad -J_{11} = J^1_1 = R, \qquad -\frac{J_{22}}{r^2} = J^2_2 = -\frac{J_{33}}{r^2\sin^2\theta} = J^3_3 = T. \qquad \text{(D.22)}$$

For $(\mu,\nu) = (0,0),(1,1),(2,2),(3,3)$, the gauge-field equation (D.21) can be written respectively as

$$\frac{d}{dr}\left(R^2\frac{dS}{dr}\right) + \frac{2R^2}{r}\frac{dS}{dr} + \left(R\frac{d}{dr} + \frac{dR}{dr} + \frac{2}{r}R\right)\left(-R(\frac{dS}{dr} + 2\frac{dT}{dr}) - \frac{2}{r}T^2 + \frac{2}{r}TR\right) \qquad \text{(D.23)}$$

$$+\xi\left[\frac{1}{4}\frac{d^2}{dr^2}(S - R + 2T) + \frac{1}{2r}\frac{d}{dr}(S - 3R + 4T) - \frac{1}{r^2}(R - T)\right] = 0,$$

$$R\left(\frac{dS}{dr}\right)^2 + 2r^3\left(\frac{d}{dr}\frac{T}{r}\right)\left[\frac{R}{r}\frac{d}{dr}\frac{T}{r} + \frac{T^2}{r^3}\right] + \left(\frac{dS}{dr} + 2\frac{dT}{dr} - \frac{2T}{r}\right)\times \qquad \text{(D.24)}$$

$$\left(-R(\frac{dS}{dr} + 2\frac{dT}{dr}) - \frac{2}{r}T^2 + \frac{2}{r}TR\right)$$

$$+\xi\left[\frac{1}{4}\frac{d^2}{dr^2}(S-R+2T)-\frac{1}{2r}\frac{d}{dr}(S+R)+\frac{3}{r^2}(R-T)\right]=0,$$

$$\left(R\frac{d}{dr}+\frac{dR}{dr}+\frac{5R}{r}-\frac{2T}{r}\right)\left(\frac{R}{r}(\frac{d}{dr}\frac{T}{r})+\frac{T^2}{r^3}\right)+ \tag{D.25}$$

$$+\left(\frac{1}{r^2}(R\frac{d}{dr}+\frac{dR}{dr})+\frac{3R}{r^3}-\frac{2T}{r^3}\right)\left(-R(\frac{dS}{dr}+2\frac{dT}{dr})-\frac{2}{r}T^2+\frac{2}{r}TR\right)$$

$$+\xi\left[\frac{1}{4r^2}\frac{d^2}{dr^2}(S-R+2T)-\frac{1}{r^3}\frac{d}{dr}(R-T)+\frac{1}{r^4}(R-T)\right]=0$$

after tedious but straightforward calculations. The equation for the component $(\mu,\nu)=(3,3)$ is the same as equation (D.25).

The second-order approximation of the fields in (D.23) through (D.25) are solved and hence, (D.22) leads to the following results

$$S=1+\frac{a_o}{r}+\frac{a_o^2}{2r^2},\qquad R=1-\frac{a_o}{r}-\frac{a_o^2}{r^2}\left(\frac{1}{2}+\frac{2}{\xi}\right), \tag{D.26}$$

$$T=1-\frac{a_o}{r}+2\frac{a_o^2}{r^2}\left(\frac{1}{\xi}-1\right).$$

The parameter a_o can be determined by the Newtonian approximation of $g\phi_{00}$

$$S=1+g\phi_{00}=1-\frac{Gm}{r},\qquad a_o=-Gm. \tag{D.27}$$

Equations (D.22) and (D.26) lead to the second order approximation of the gauge field $g\phi_{00}$:

$$g\phi_{00} = -\frac{Gm}{r} + \frac{G^2m^2}{2r^2}, \qquad g\phi_{11} = -\frac{Gm}{r} + \frac{G^2m^2}{r^2}\left(\frac{1}{2} + \frac{2}{\xi}\right), \tag{D.28}$$

$$g\phi_{22} = -r^2\left[\frac{Gm}{r} + \frac{2G^2m^2}{r^2}\left(\frac{1}{\xi} - 1\right)\right], \qquad g\phi_{33} = g\phi_{22}\sin^2\theta;$$

which is consistent with the ξ-independent first order solutions in (D.19).

Based on (D.21) and (D.6) with $P_{\mu\nu} = (1, -1, -r^2, -r^2\sin^2\theta)$, we obtain the effective metric tensor,

$$G_{00}(r) = 1 - \frac{2Gm}{r} + \frac{2G^2m^2}{r^2}, \qquad G_{11}(r) = -\left[1 + \frac{2Gm}{r} - \frac{4G^2m^2}{r^2\xi}\right], \tag{D.29}$$

$$G_{22}(r) = -r^2\left[1 + \frac{2Gm}{r} + \frac{G^2m^2}{r^2}\left(\frac{4}{\xi} - 3\right)\right], \qquad G_{33}(r) = G_{22}(r)\sin^2\theta,$$

which are ξ-dependent, just like the gauge field $\phi_{\mu\nu}$. In order to see the ξ-independent physical results and their agreement with gravitational experiments, let us carry out the expansion to the second order in all components G_{00}, G_{11}, G_{22} and G_{33} in the usual spherical coordinates. For any given value of ξ, we can make a change of variable

$$\rho^2 = r^2\left[1 + \frac{2Gm}{r} + \frac{G^2m^2}{r^2}\left(\frac{4}{\xi} - 3\right)\right],$$

where the zeroth and the first orders terms are independent of ξ. We obtain the following effective metric tensors $G_{\mu\nu}(\rho)$:

$$G_{00}(\rho) = 1 - \frac{2Gm}{\rho} + \frac{A_o}{\rho^2}, \qquad G_{11}(\rho) = -\left[1 + \frac{2Gm}{\rho} - \frac{2G^2m^2}{\rho^2}\right], \tag{D.30}$$

$$G_{22}(\rho) = -\rho^2 \, , \qquad G_{33}(\rho) = -\rho^2 \sin^2\theta, \qquad A_o = 0,$$

which are ξ-independent. This is interesting because, although Yang-Mills gravity is formulated in flat spacetime, the spacetime translation gauge symmetry leads to an effective metric tensor $G_{\mu\nu}(\rho)$ in (D.5), (D.6) and (D.30) to the second order. Note that only the second order term in $G_{11}(\rho)$, i.e., $2G^2m^2/\rho^2$, differs from the corresponding term in Einstein's theory. As we shall see below, this difference is too small to be detected with the instruments currently available.[9] We have shown that the effective metric tensor $G_{\mu\nu}(\rho)$ in spherical coordinates is independent of the gauge parameter ξ in the post-Newtonian approximation.

The inverse of the non-vanishing components of $G_{\mu\nu}(\rho)$ are given by

$$I^{00}(\rho) = \frac{1}{G_{00}(\rho)} = 1 + \frac{2Gm}{\rho} + \frac{4G^2m^2}{\rho^2},$$

$$I^{11}(\rho) = \frac{1}{G_{11}(\rho)} = -\left[1 - \frac{2Gm}{\rho} + \frac{6G^2m^2}{\rho^2}\right], \tag{D.31}$$

$$I^{22}(\rho) = \frac{1}{G_{22}(\rho)} = \frac{-1}{\rho^2}, \qquad I^{33}(\rho) = \frac{1}{G_{33}(\rho)} = \frac{-1}{\rho^2 \sin^2\theta}.$$

Df. Perihelion shifts and bending of light

Let us choose $\theta = \pi/2$, the Hamilton-Jacobi equation for a planet with mass m_p has the following form

$$I^{00}(\rho)\left(\frac{\partial S}{\partial w}\right)^2 + I^{11}(\rho)\left(\frac{\partial S}{\partial \rho}\right)^2 + I^{22}(\rho)\left(\frac{\partial S}{\partial \varphi}\right)^2 - m_p^2 = 0. \tag{D.32}$$

By the general procedure for solving this equation, we look for an action S in the form

$$S = -E_o w + M\varphi + f(\rho).$$

The action S is found to be

$$S = -E_o w + M\varphi + \int \frac{1}{\sqrt{|I^{11}(\rho)|}} \sqrt{E_o^2 I^{00}(\rho) - m_p^2 - \frac{M^2}{\rho^2}}\, d\rho, \qquad \text{(D.33)}$$

where E_o and M are respectively constant energy and angular momentum. As usual, the trajectory is determined by $\partial S / \partial M = \text{constant}$. We find

$$\varphi = \int \frac{(M/\rho^2) d\rho}{\sqrt{E_o^2 I^{00}(\rho)|I^{11}(\rho)| - m_p^2 |I^{11}(\rho)| - M^2 |I^{11}(\rho)| / \rho^2}}, \qquad \text{(D.34)}$$

$$I^{00}(\rho)|I^{11}(\rho)| = 1 + \frac{6G^2 m^2}{\rho^2},$$

where $I^{00}(\rho)$ and $I^{11}(\rho)$ are given by (D.31). This term $I^{00}(\rho) \mid I^{11}(\rho) \mid$ differs from the corresponding term in Einstein's theory, in which $g^{00}(\rho) \mid g^{11}(\rho) \mid = 1$. For the approximate trajectory, we write (D.34) as a differential equation with $\sigma = 1/\rho$. We have

$$M^2 \left(\frac{d\sigma}{d\varphi}\right)^2 = E_o^2 I^{00}(\rho)|I^{11}(\rho)| - m_p^2 |I^{11}(\rho)| - M^2 \sigma^2 |I^{11}(\rho)|. \qquad \text{(D.35)}$$

We differentiate equation (D.35) with respect to φ and obtain

$$\frac{d^2\sigma}{d\varphi^2} = \frac{1}{P} - \sigma(1-Q) + 3Gm\sigma^2, \tag{D.36}$$

$$P = \frac{M^2}{m_p^2 Gm}, \qquad Q = \frac{6Gm}{P}\left(\frac{E_o^2 - m_p^2}{m_p^2}\right).$$

Einstein's theory does not have this type of correction term Q because it has the relation $g^{00}(\rho)\,|\,g^{11}(\rho)\,| = 1$. The new correction term Q is of the order of $(Gm/P)\beta^2$ which is the result of the relation $I^{00}(\rho)\,|\,I^{11}(\rho)\,| > 1$ in (D.34). This correction is extremely small and at present, undetectable because of velocity β of planets is very small compared to the speed of light.

To see the effect of this correction Q to the perihelion shift, we solve (D.36) by a change of variable

$$\overline{\sigma} = \sigma(1-Q).$$

We can write equation (D.36) as

$$\frac{d^2\overline{\sigma}}{d\overline{\varphi}^2} = \frac{1}{P} - \overline{\sigma} + \frac{3Gm}{(1-Q)^2}\overline{\sigma}^2, \qquad \overline{\varphi} = \varphi\sqrt{1-Q}. \tag{D.37}$$

By the usual successive approximations,[7] we obtain the solution

$$\overline{\sigma} = \frac{1}{P}\left[1 + e\cos\left(\overline{\varphi}\left(1 - \frac{3Gm}{P(1-Q)^2}\right)\right)\right], \tag{D.38}$$

which can be written as

$$\sigma = \frac{1}{P(1-Q)}\left[1 + e\cos\left(\varphi\left(1 - \frac{\alpha}{P}\right)\right)\right], \qquad \alpha = 3Gm\left(1 + \frac{E_o^2 - m_p^2}{m_p^2}\right), \tag{D.39}$$

where e is the eccentricity. The semi-major axis a can be expressed in terms of e and P, $a = P/(1-e^2)$. The perihelion shift for one revolution of the planet is given by

$$\delta\varphi = \frac{2\pi\alpha}{P} = \frac{6\pi Gm}{P}\left(1 + \frac{E_o^2 - m_p^2}{m_p^2}\right), \tag{D.40}$$

where the second term in the parentheses is the difference between the Yang-Mills gravity and Einstein's theory. This result shows that the observable perihelion shift is independent of the gauge parameter ξ, which appears in the second order approximation of the solution of the spin-2 gauge field $g\phi$. Since the observational accuracy of the perihelion shift of the Mercury is about 1%, the prediction (D.40) of Yang-Mills gravity can be tested only when

$$\frac{E_o^2 - m_p^2}{m_p^2} \approx \beta^2 \approx 0.01. \tag{D.41}$$

Thus, it is not possible at present to test the small correction in (D.40) of Yang-Mills gravity in the solar system.

For the bending of light, the wave 4-vector satisfies

$$I^{\mu\nu}k_\mu k_\nu = 0, \qquad I^{\mu\nu}G_{\nu\lambda} = \delta^\mu_\lambda. \tag{D.42}$$

Substituting $\partial\psi/\partial x^\mu$ for k_μ, where ψ is the eikonal, we have the eikonal equation in the spin-2 field $\phi_{\mu\nu}$,

$$I^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu} = 0, \tag{D.43}$$

where $I^{\mu\nu}$ is given by (D.31). The trajectory of a light ray is determined by the eikonal equation (D.43), which is the same as (D.36) with $m_p \to 0$ and E_o replaced with $\omega_o = -\partial\psi/\partial w$, $(c = 1)$. The trajectory of the light ray is

$$\varphi = \int \frac{(M/\rho^2)d\rho}{\sqrt{\omega_o^2 I^{00}(\rho)\left|I^{11}(\rho)\right| - M^2\left|I^{11}(\rho)\right|/\rho^2}}, \tag{D.44}$$

$$= \int \frac{d\rho}{\rho^2\sqrt{\frac{\omega_o^2}{M^2}\left(1+\frac{6G^2m^2}{\rho^2}\right) - \frac{1}{\rho^2}\left(1-\frac{2Gm}{\rho}+\frac{6G^2m^2}{\rho^2}\right)}}.$$

Following the usual procedure,[7] we can write (D.34) in the form of a differential equation

$$\left(\frac{d\sigma}{d\varphi}\right)^2 = \frac{\omega_o^2}{M^2}\left(1+6G^2m^2\sigma^2\right) - \sigma^2 + 2Gm\sigma^3 - 6G^2m^2\sigma^4. \tag{D.45}$$

Differentiating with respect to φ, we obtain

$$\frac{d^2\sigma}{d\varphi^2} = -\sigma\left(1-\frac{6G^2m^2\omega_o^2}{M^2}\right) + 3Gm\sigma^2 - 12G^2m^2\sigma^3, \tag{D.46}$$

where the second term in the parentheses and the last term in (D.46) are small corrections to the present gauge theory of gravity. If one keeps just the term $-\sigma$ on the right-hand-side of (D.46), one has the usual (zeroth order approximation) solution

$$\sigma = \sigma_o = \frac{\cos\varphi}{R_o}, \qquad R_o = \frac{M}{\omega_o}. \tag{D.47}$$

The trajectory is a straight line $r = R_o / \cos\varphi$ passing at a distance R_o from the origin. For the next higher approximation, we set $\sigma = \sigma_0 + \sigma_1$ and obtain

$$\frac{d^2\sigma_1}{d\varphi^2} + \sigma_1 = 3Gm\sigma_0^2 = \frac{3Gm\omega_o^2}{M^2}\cos^2\varphi, \qquad \sigma = \sigma_0 + \sigma_1. \tag{D.48}$$

A particular solution for σ_1 is

$$\sigma_1 = \frac{Gm\omega_o^2}{M^2}\left(1 + \sin^2\varphi\right). \tag{D.49}$$

Therefore, there is no observable difference between this Yang-Mills gravity and Einstein's theory because both give the same result for $\sigma_0 + \sigma_1$:

$$\sigma = \sigma_0 + \sigma_1 = \frac{\cos\varphi}{R_o} + \frac{Gm\omega_o^2}{M^2}\left(1 + \sin^2\varphi\right). \tag{D.50}$$

This trajectory is a curve. To estimate the order of magnitude of the new correction to this curve in Yang-Mills gravity, we ignore the last term $-12G^2m^2\sigma^3$ in (D.46), and calculate the correction to (D.50). We follow a similar method to that used in (D.36) and (D.37), and set $\sigma = 0$ to determine the direction of the light ray at large distances from the origin. As usual, we look for a solution φ of the form

$$\varphi = \pm\frac{\pi}{2} + \frac{1}{2}\Delta\varphi, \tag{D.51}$$

where $\Delta\varphi$ is small. We find that the angle between the two asymptotes is given by the usual result $(4Gm\omega_o / M)$ and a correction

$$\Delta\varphi = \frac{4Gm\omega_o}{M}\left(1 + \frac{3\pi Gm\omega_o}{4M}\right), \tag{D.52}$$

where the second term in the bracket is a very small correction to the Yang-Mills gravity. Because these correction terms are too small to be observed in the solar system, there is no observable difference at present between Yang-Mills gravity and Einstein's theory.

Dg. The Yang-Mills gravitational force

If one compares the fermion equation, $[i\Gamma^{\mu}(P_{\mu\nu} + g\phi_{\mu\nu})D^{\nu} - m]\psi = 0$, derived from (D.12) with Dirac's equation for charged fermions in quantum electrodynamics [i.e., $(i\gamma^{\mu}\partial_{\mu} - e\gamma^{\mu}A_{\mu} - m)\psi = 0$], one can see a distinct difference: the kinematic term $i\gamma^{\mu}\partial_{\mu}$ and the electromagnetic coupling term $e\gamma^{\mu}A_{\mu}$ have a different relative sign, if one takes the complex conjugate of the Dirac equations . This implies the presence of both repulsive and attractive forces between two charges. However, there is no change in the relative sign of the kinematical term and the spin-2 coupling term in our fermion equation $[i\Gamma^{\mu}(P_{\mu\nu} + g\phi_{\mu\nu})D^{\nu} - m]\psi = 0$ when one takes the complex conjugate. This provides a natural explanation for the solely attractive nature of the gravitational force between two fermions, in contrast to that in electrodynamics. Furthermore, the electromagnetic U(1) gauge group and the gravitational T(4) gauge group have different group generators. The electromagnetic coupling constant e is dimensionless while the gravitational coupling constant G has the dimension of length in natural units ($c=\hbar=1$).

Yang-Mills gravity with the translational symmetry[13] has a well-defined conservation law for the energy-momentum tensor in flat spacetime, similar to the conservation of charge in electrodynamics. It is believed that such a Yang-Mills gravity can shed light on quantum gravity because

(i) the maximum interaction vertex is 4-vertex, just as in the usual Yang-Mills theory with internal Lie groups, and

(ii) it is based on gauge symmetry, which could minimize the ultraviolet divergences.

In light of these discussions, it appears to be possible to understand gravity based on the spacetime gauge theory with the translational symmetry[14] in general frames of reference.

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13. Within the framework of curved spacetime, gravity with translation gauge symmetry was discussed by Y. M. Cho, [Phys. Rev. **14**, 2515 (1976)]. His formulation and results are very different from ours. For example, Cho made an additional assumption that $\xi_\mu\phi = 0$ for the gauge covariant derivative, so that one has $\Delta_i\phi = (\partial_i + B_i^\mu\xi_\mu)\phi = \partial_i\phi$, where ϕ is a scalar field and ξ_μ are generators of the translation group. Furthermore, he assumed $\partial_i\phi = h_i^\mu\partial_\mu\phi$, where $h_i^\mu = \delta_i^\mu + fB_i^\mu$. This assumption is equivalent to adopting a curved spacetime. The Yang-Mills' formulation of gauge symmetry and gauge fields does not make these additional assumptions.

14. The energy-momentum tensor $t_{\mu\nu}$ of the gravitational field can be defined by (D.21) with ξ=0: $D^\lambda D_\lambda\phi^{\mu\nu} = -g(T^{\mu\nu} + t^{\mu\nu})$. In an inertial frame, using the usual approximation and gauge condition (D.15), we can calculate the average energy-momentum of a gravitational plane wave and the radiation power. The approximate result turns out to be the same as that in general relativity. For more complete discussions on gauge invariance based on explicit calculations, the classical Hamilton-Jacobi equation derived from wave equations and gravitational radiations, see Jong-Ping Hsu, 'Yang-Mills Gravity in Flat Space-time, I. Classical Gravity with Translation Gauge Symmetry' (Int. J. Mod. Phys. A, to be published in 2006).

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