Nine Chapters on the Mathematical Art

Chinese Mathematics History Topics Index

The *Jiuzhang suanshu* or *Nine Chapters on the Mathematical Art* is a practical handbook of mathematics consisting of 246 problems intended to provide methods to be used to solve everyday problems of engineering, surveying, trade, and taxation. It has played a fundamental role in the development of mathematics in China, not dissimilar to the role of Euclid's *Elements* in the mathematics which developed from the foundations set up by the ancient Greeks. There is one major difference which we must examine right at the start of this article and this is the concept of proof.

It is well known what that Euclid, for example, gives rigorous proofs of his results. Failure to see similar rigorous proofs in Chinese works such as the *Nine Chapters on the Mathematical Art* led to historians believing that the Chinese gave formulas without justification. This however is simply an example of historians well versed in mathematics which is essentially derived from Greek mathematics, thinking that Chinese mathematics was inferior since it was different. Recent work has begun to correct this false impression and understand that there are different understandings of "proof". For example in [8] Chemla shows that Chinese mathematicians certainly understood how to give convincing arguments that their methodology for solving particular problems was correct.

Let us now give a short description of each of the nine chapters of the book.

Chapter 1: Land Surveying.
This consists of 38 problems on land surveying. It looks first at area problems, then looks at rules for the addition, subtraction, multiplication and division of fractions. The Euclidean algorithm method for finding the greatest common divisor of two numbers is given. It then proceeds to further area problems which do not use the material on fractions which appears somewhat misplaced. The types of shapes for which the area is calculated include triangles, rectangles, circles, trapeziums. In Problem 32 an accurate approximation is given for π. This is discussed in detail in Liu Hui's biography.

Chapter 2: Millet and Rice.
This chapter contains 46 problems concerning the exchange of goods, particularly the exchange rates among twenty different types of grains, beans, and seeds. The mathematics involves a study of proportion and percentages and introduces the rule of three for solving proportion problems. Many of the problems seem simple an excuse to give the reader practice at handling difficult calculations with fractions.

Chapter 3: Distribution by Proportion.
Here there are 20 problems which again involve proportion, many involving different sums given to or owed by officials of various different ranks. Direct proportion, inverse proportion and compound proportion are all studied. In particular arithmetic and geometric progressions are used in some of the problems.

Chapter 4: Short Width.
This chapter contains 24 problems and takes its name from the first eleven problems which ask what the length of a field will be if the width is increased but the area kept constant. These first eleven problems involve unit fractions are all of the following type, where $n = 2, 3, 4, ..., 12$:

> *Suppose a field has width* $1 + {}^1/_2 + {}^1/_3 + ... + {}^1/_n$. *What must its length be if its area is* 1?

Problems 12 to 18 involve the extraction of square roots, and the remaining problems involve the extraction of cube roots. Notions of limits and infinitesimals appear in this chapter. Liu Hui whose commentary of 263 AD has become part of the text attempts to find the volume of a sphere, gives an approximate formula which he shows to be incorrect, then charmingly writes:-

Let us leave the problem to whoever can tell the truth.

Chapter 5: Civil Engineering.
Here there are 28 problems on the construction of canals, ditches, dykes, etc. Volumes of solids such as prisms, pyramids, tetrahedrons, wedges, cylinders and truncated cones are calculated. Liu Hui, in his commentary, discusses a "method of exhaustion" he has invented to find the correct formula for the volume of a pyramid.

Chapter 6: Fair Distribution of Goods.
This chapter contains 28 problems involving ratio and proportion. The problems are varied and concern problems about travelling, taxation, sharing etc. Problem 12 is a pursuit problem:-

> *A good runner can go* 100 *paces while a poor runner covers* 60 *paces. The poor runner has covered a distance of* 100 *paces before the good runner sets off in pursuit. How many paces does it take the good runner before he catches up the poor runner.*
> [*Answer:* 250 *paces*]

Problem 26 has become a classic type still used today:-

> *A cistern is filled through five canals. Open the first canal and the cistern fills in* 1/3 *day; with the second, it fills in* 1 *day; with the third, in* 21/2 *days; with the fourth, in* 3 *days, and with the fifth in* 5 *days. If all the canals are opened, how long will it take to fill the cistern?*
> [*Answer:* 15/74 *of a day*]

Chapter 7: Excess and Deficit.
The 20 problems give a rule of double false position. Essentially linear equations are solved by making two guesses at the solution, then computing the correct answer from the two errors. For example to solve

$$ax + b = c$$

we try $x = i$, and instead of c we get $c + d$. Then we try $x = j$, and instead of c we obtain $c + e$. Then the correct solution is

$$x = (jd - ie)/(d - e).$$

The first problem essentially contains the "guesses" in its formulation:-

> *Certain items are purchased jointly. If each person pays* 8 *coins, the surplus is* 3 *coins, and if each person gives* 7 *coins, the deficiency is* 4 *coins. Find the number of people and the total cost of the items.*
> [*Answer: There are* 7 *people and the total cost of the items is* 53 *coins.*]

Problem 18, although not formulated as a "guessing problem" is solved in that manner:-

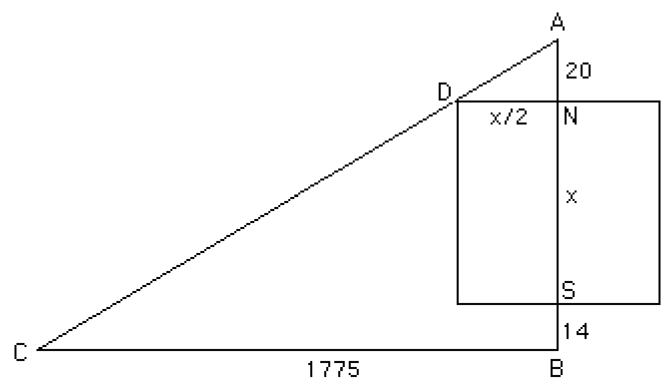
> *There are two piles, one containing* 9 *gold coins and the other* 11 *silver coins. The two piles of coins weigh the same. One coin is taken from each pile and put into the other. It is now found that the pile of mainly gold coins weighs* 13 *units less than the pile of mainly silver coins. Find the weight of a silver coin and of a gold coin.*

Chapter 8: Calculation by Square Tables.
Here 18 problems which reduce to solving systems of simultaneous linear equations are given. However the method given is basically that of solving the system using the augmented matrix of coefficients. The problems involve up to six equations in six unknowns and the only difference with the modern method is that the coefficients are placed in columns rather than rows. The matrix is then reduced to triangular form, using elementary column operations as is done today in the method of Gaussian elimination, and the answer

interpreted for the original problem. Negative numbers are used in the matrix and the chapter includes rules to compute with them.

Chapter 9: Right angled triangles.
In this final chapter there are 24 problems which are all based on right angled triangles. The first 13 problems are solved using an application of Pythagoras's theorem, which the Chinese knew as the Gougu rule. Two problems study what are now called Pythagorean triples, while the remainder use the theory of similar triangles. Here is an example of one using similar triangles; it is Problem 20:-

> *There is a square town of unknown dimensions. There is a gate in the middle of each side. Twenty paces outside the North Gate is a tree. If one leaves the town by the South Gate, walks* 14 *paces due south, then walks due west for* 1775 *paces, the tree will just come into view. What are the dimensions of the town.*



In the diagram the North Gate is N, the South Gate is S, and the tree is A. Walking south from S 14 paces reaches B, turn west and walk 1775 paces to C. From C the tree at A is just visible so the line CA passes through the corner D of the square.

Now triangles AND and ABC are similar so

$AN/ND = AB/BC$

giving

$20/(x/2) = (20 + x + 14)/1775.$

Then $x^2 + x(20 + 14) = 2\,(20\times1775)$, or

$x^2 + 34x = 71000.$
[Answer: The side of the town is 250 paces]

Quadratic equations are considered for the first time in Chapter 9, are solved by an analogue of division using ideas from geometry, in fact from the Chinese square-root algorithm, rather than from algebra.

Having looked at the content of the work, let us think next about its date. Liu Hui wrote a commentary on the *Nine Chapters on the Mathematical Art* in 263 AD. He believed that the text which he was commentating on was originally written around 1000 BC but incorporated much material from later eras. He wrote in the Preface:-

> *In the past, the tyrant Qin burnt written documents, which led to the destruction of classical knowledge. Later, Zhang Cang, Marquis of Peiping and Geng Shouchang, Vice-President of the Ministry of Agriculture, both became famous through their talent for calculation. Because of the ancient texts had deteriorated, Zhang Cang and his team produced a new version removing the poor parts and filling in the missing parts. Thus, they revised some parts with the result that these were different from the old parts ...*

Let us give some dates for the events Liu Hui describes. The Qin dynasty preceded the Han dynasty and it was the Qin ruler Shih Huang Ti who tried to reform education by destroying all earlier learning. He ordered all books to be burnt in 213 BC and Zhang Cang, who Liu Hui refers to, did his reconstruction around 170 BC.

Most historians, however, would not believe that the original text of the *Nine Chapters on the Mathematical Art* was nearly as old as Liu Hui believed. In fact most historians think that the text originated around 200 BC after the burning of the books by Shih Huang Ti. Others give dates between 100 BC and 50 AD.

What methods are used to try to date the material? Perhaps the most important is to examine the units of length, volume and weight which appear in the various problems. Standard decimal units of length were established in China around 200 BC and later further subdivisions occurred. That the basic units are used, but not the later subdivisions, leads to a date of shortly after 200 BC. In Liu Hui's commentary subdivisions introduced around 250 AD are used, which is in line with this commentary being written in 263 AD.

Of course, the dating using units of length is not conclusive. Consider the fact that Britain changed to a decimal currency in 1970. If you pick up a book with mathematics problems given in decimal currency then we could argue as above and say that the book was written after 1970. However new editions of popular textbooks were brought out when the currency changed, so many older books appeared in decimal editions. The *Nine Chapters on the Mathematical Art* was certainly an important text, so may have had its units of length brought up to date as it evolved.

Is there other evidence for dating parts of the *Nine Chapters on the Mathematical Art* other than units of measurement? Yes, there are. Problems contain references to taxes, methods of distributing goods, towns, and parks which all point to slightly different dates for different parts of the text but 206 BC to 50 AD covering these different dates.

In addition to Liu Hui's commentary of 263, there was another important later commentary, namely that of Li Chunfeng whose commentary was written around 640 when he headed a team asked to annotate The Ten Classics. Li Chunfeng corrected and clarified some of Liu Hui's comments, expanding on much of what had been pretty concisely written.

The *Nine Chapters on the Mathematical Art* [4]:-

> *... has dominated the history of Chinese mathematics. It served as a textbook not only in China but also in neighbouring countries and regions until western science was introduced from the Far East around* 1600 *AD.*

Now although European science does not appear to have reached China in sixteenth century, it has been pointed out that a number of mathematical formulas and rules which were widely used in Europe during that century are essentially identical to formulas written down in the *Nine Chapters on the Mathematical Art.* This leads to an interesting question which historians have as yet no convincing answer, namely were the European formulas taken directly from those of China.

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List of References (32 books/articles)

Other Web sites

1. Simon Fraser University
