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XXII. *On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids.* By G. G. Stokes, M.A., *Fellow of Pembroke College.*

[Read *April* 14, 1845.]

The equations of Fluid Motion commonly employed depend upon the fundamental hypothesis that the mutual action of two adjacent elements of the fluid is normal to the surface which separates them. From this assumption the equality of pressure in all directions is easily deduced, and then the equations of motion are formed according to D'Alembert's principle. This appears to me the most natural light in which to view the subject; for the two principles of the absence of tangential action, and of the equality of pressure in all directions ought not to be assumed as independent hypotheses, as is sometimes done, inasmuch as the latter is a necessary consequence of the former*. The equations of motion so formed are very complicated, but yet they admit of solution in some instances, especially in the case of small oscillations. The results of the theory agree on the whole with observation, so far as the time of oscillation is concerned. But there is a whole class of motions of which the common theory takes no cognizance whatever, namely, those which depend on the tangential action called into play by the sliding of one portion of a fluid along another, or of a fluid along the surface of a solid, or of a different fluid, that action in fact which performs the same part with fluids that friction does with solids.

Thus, when a ball pendulum oscillates in an indefinitely extended fluid, the common theory gives the arc of oscillation constant. Observation however shows that it diminishes very rapidly in the case of a liquid, and diminishes, but less rapidly, in the case of an elastic fluid. It has indeed been attempted to explain this diminution by supposing a friction to act on the ball, and this hypothesis may be approximately true, but the imperfection of the theory is shown from the circumstance that no account is taken of the equal and opposite friction of the ball on the fluid.

Again, suppose that water is flowing down a straight aqueduct of uniform slope, what will be the discharge corresponding to a given slope, and a given form of the bed? Of what magnitude must an aqueduct be, in order to supply a given place with a given quantity of water? Of what form must it be, in order to ensure a given supply of water with the least expense of materials in the construction? These, and similar questions are wholly out of the reach of the common theory of Fluid Motion, since they entirely depend on the laws of the transmission of that tangential action which in it is wholly neglected. In fact, according to the common theory the water ought to flow on with uniformly accelerated velocity; for even the supposition of a certain friction against the bed would be of no avail, for such friction could not be transmitted through the mass. The practical importance of such questions as those above mentioned has made them the object of numerous experiments, from which empirical formulæ have been constructed. But such formulæ, although fulfilling well enough the purposes for which they were

* This may be easily shown by the consideration of a tetrahedron of the fluid, as in Art. 4.

constructed, can hardly be considered as affording us any material insight into the laws of nature; nor will they enable us to pass from the consideration of the phenomena from which they were derived to that of others of a different class, although depending on the same causes.

In reflecting on the principles according to which the motion of a fluid ought to be calculated when account is taken of the tangential force, and consequently the pressure not supposed the same in all directions, I was led to construct the theory explained in the first section of this paper, or at least the main part of it, which consists of equations (13), and of the principles on which they are formed. I afterwards found that Poisson had written a memoir on the same subject, and on referring to it I found that he had arrived at the same equations. The method which he employed was however so different from mine that I feel justified in laying the latter before this Society*. The leading principles of my theory will be found in the hypotheses of Art. 1, and in Art. 3.

The second section forms a digression from the main object of this paper, and at first sight may appear to have little connexion with it. In this section I have, I think, succeeded in shewing that Lagrange's proof of an important theorem in the ordinary theory of Hydrodynamics is untenable. The theorem to which I refer is the one of which the object is to show that $udx + vdy + wdz$, (using the common notation,) is always an exact differential when it is so at one instant. I have mentioned the principles of M. Cauchy's proof, a proof, I think, liable to no sort of objection. I have also given a new proof of the theorem, which would have served to establish it had M. Cauchy not been so fortunate as to obtain three first integrals of the general equations of motion. As it is, this proof may possibly be not altogether useless.

Poisson, in the memoir to which I have referred, begins with establishing, according to his theory, the equations of equilibrium and motion of elastic solids, and makes the equations of motion of fluids depend on this theory. On reading his memoir, I was led to apply to the theory of elastic solids principles precisely analogous to those which I have employed in the case of fluids. The formation of the equations, according to these principles, forms the subject of Sect. III.

The equations at which I have thus arrived contain two arbitrary constants, whereas Poisson's equations contain but one. In Sect. IV. I have explained the principles of Poisson's theories of elastic solids, and of the motion of fluids, and pointed out what appear to me serious objections against the truth of one of the hypotheses which he employs in the former. This theory seems to be very generally received, and in consequence it is usual to deduce the measure of the cubical compressibility of elastic solids from that of their extensibility, when formed into rods or wires, or from some quantity of the same nature. If the views which I have explained in this section be correct, the cubical compressibility deduced in this manner is too great, much too great in the case of the softer substances, and even the softer metals. The equations of Sect. III. have, I find, been already obtained by M. Cauchy in his *Exercises Mathématiques*, except that he has not considered the effect of the heat developed by sudden compression. The method which I have employed is different from his, although in some respects it much resembles it.

The equations of motion of elastic solids given in Sect. III. are the same as those to which different authors have been led, as being the equations of motion of the luminiferous ether in vacuum. It may seem strange that the same equations should have been arrived at for cases so different; and I believe this has appeared to some a serious objection to the employment of those equations in the case of light. I think the reflections which I have made at the end of Sect. IV., where I have examined the consequences of the law of continuity, a law which seems to pervade nature, may tend to remove the difficulty.

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* The same equations have also been obtained by Navier in the case of an incompressible fluid, (*Mém. de l'Institut*, T. VI.) but his principles differ from mine still more than do Poisson's.

SECTION I.

Explanation of the Theory of Fluid Motion proposed. Formation of the Differential Equations. Application of these Equations to a few simple cases.

1. Before entering on the explanation of this theory, it will be necessary to define, or fix the precise meaning of a few terms which I shall have occasion to employ.

In the first place, the expression "the velocity of a fluid at any particular point" will require some notice. If we suppose a fluid to be made up of ultimate molecules, it is easy to see that these molecules must, in general, move among one another in an irregular manner, through spaces comparable with the distances between them, when the fluid is in motion. But since there is no doubt that the distance between two adjacent molecules is quite insensible, we may neglect the irregular part of the velocity, compared with the common velocity with which all the molecules in the neighbourhood of the one considered are moving. Or, we may consider the mean velocity of the molecules in the neighbourhood of the one considered, apart from the velocity due to the irregular motion. It is this regular velocity which I shall understand by the *velocity of a fluid at any point*, and I shall accordingly regard it as varying continuously with the co-ordinates of the point.

Let P be any material point in the fluid, and consider the instantaneous motion of a very small element E of the fluid about P. This motion is compounded of a motion of translation, the same as that of P, and of the motion of the several points of E relatively to P. If we conceive a velocity equal and opposite to that of P impressed on the whole element, the remaining velocities form what I shall call the *relative velocities* of the points of the fluid about P; and the motion expressed by these velocities is what I shall call the *relative motion* in the neighbourhood of P.

It is an undoubted result of observation that the molecular forces, whether in solids, liquids, or gases, are forces of enormous intensity, but which are sensible at only insensible distances. Let E' be a very small element of the fluid circumscribing E, and of a thickness greater than the distance to which the molecular forces are sensible. The forces acting on the element E are the external forces, and the pressures arising from the molecular action of E'. If the molecules of E were in positions in which they could remain at rest if E were acted on by no external force and the molecules of E' were held in their actual positions, they would be in what I shall call a state of *relative equilibrium.* Of course they may be far from being in a state of actual equilibrium. Thus, an element of fluid at the top of a wave may be sensibly in a state of relative equilibrium, although far removed from its position of equilibrium. Now, in consequence of the intensity of the molecular forces, the pressures arising from the molecular action on E will be very great compared with the external moving forces acting on E. Consequently the state of relative equilibrium, or of relative motion, of the molecules of E will not be sensibly affected by the external forces acting on E. But the pressures in different directions about the point P depend on that state of relative equilibrium or motion, and consequently will not be sensibly affected by the external moving forces acting on E. For the same reason they will not be sensibly affected by any motion of rotation common to all the points of E; and it is a direct consequence of the second law of motion, that they will not be affected by any motion of translation common to the whole element. If the molecules of E were in a state of relative equilibrium, the pressure would be equal in all directions about P, as in the case of fluids at rest. Hence I shall assume the following principle:—

That the difference between the pressure on a plane in a given direction passing through any point P *of a fluid in motion and the pressure which would exist in all directions about* P *if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about* P; *and that the relative motion*

due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned.

Let us see how far this principle will lead us when it is carried out.

2. It will be necessary now to examine the nature of the most general instantaneous motion of an element of a fluid. The proposition in this article is however purely geometrical, and may be thus enunciated:—"Supposing space, or any portion of space, to be filled with an infinite number of points which move in any continuous manner, retaining their identity, to examine the nature of the instantaneous motion of any elementary portion of these points."

Let u, v, w be the resolved parts, parallel to the rectangular axes Ox, Oy, Oz, of the velocity of the point P, whose co-ordinates at the instant considered are x, y, z. Then the relative velocities at the point P', whose co-ordinates are $x + x'$, $y + y'$, $z + z'$, will be

$$\frac{du}{dx}x' + \frac{du}{dy}y' + \frac{du}{dz}z' \text{ parallel to } x,$$

$$\frac{dv}{dx}x' + \frac{dv}{dy}y' + \frac{dv}{dz}z' \ldots\ldots\ldots\ldots y,$$

$$\frac{dw}{dx}x' + \frac{dw}{dy}y' + \frac{dw}{dz}z' \ldots\ldots\ldots\ldots z,$$

neglecting squares and products of x', y', z'. Let these velocities be compounded of those due to the angular velocities ω', ω'', ω''' about the axes of x, y, z, and of the velocities U, V, W parallel to x, y, z. The linear velocities due to the angular velocities being $\omega''z' - \omega'''y$, $\omega'''x' - \omega'z'$, $\omega'y' - \omega''x'$ parallel to the axes of x, y, z, we shall therefore have

$$U = \frac{du}{dx}x' + \left(\frac{du}{dy} + \omega'''\right)y' + \left(\frac{du}{dz} - \omega''\right)z',$$

$$V = \left(\frac{dv}{dx} - \omega'''\right)x' + \frac{dv}{dy}y' + \left(\frac{dv}{dz} + \omega'\right)z',$$

$$W = \left(\frac{dw}{dx} + \omega''\right)x' + \left(\frac{dw}{dy} - \omega'\right)y' + \frac{dw}{dz}z'.$$

Since ω', ω'', ω''' are arbitrary, let them be so assumed that

$$\frac{dU}{dy'} = \frac{dV}{dx'},\ \frac{dV}{dz'} = \frac{dW}{dy'},\ \frac{dW}{dx'} = \frac{dU}{dz'},$$

which gives

$$\omega' = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right),\ \omega'' = \tfrac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right),\ \omega''' = \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right), \ldots\ldots\ldots(1)$$

$$\left.\begin{aligned} U &= \frac{du}{dx}x' + \tfrac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right)y' + \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right)z', \\ V &= \tfrac{1}{2}\left(\frac{dv}{dx} + \frac{du}{dy}\right)x' + \frac{dv}{dy}y' + \tfrac{1}{2}\left(\frac{dv}{dz} + \frac{dw}{dy}\right)z', \\ W &= \tfrac{1}{2}\left(\frac{dw}{dx} + \frac{du}{dz}\right)x' + \tfrac{1}{2}\left(\frac{dw}{dy} + \frac{dv}{dz}\right)y' + \frac{dw}{dz}z'. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The quantities ω', ω'', ω''' are what I shall call the *angular velocities of the fluid* at the point considered. This is evidently an allowable definition, since, in the particular case in which

the element considered moves as a solid might do, these quantities coincide with the angular velocities considered in rigid dynamics. A further reason for this definition will appear in Sect. III.

Let us now investigate whether it is possible to determine x', y', z' so that, considering only the relative velocities U, V, W, the line joining the points P, P' shall have no angular motion. The conditions to be satisfied, in order that this may be the case, are evidently that the increments of the relative co-ordinates x', y', z' of the second point shall be ultimately proportional to those co-ordinates. If e be the rate of extension of the line joining the two points considered, we shall therefore have

$$\left.\begin{aligned} Fx' + hy' + gz' &= ex', \\ hx' + Gy' + fz' &= ey', \\ gx' + fy' + Hz' &= ez'; \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where $F = \dfrac{du}{dx}$, $G = \dfrac{dv}{dy}$, $H = \dfrac{dw}{dz}$, $2f = \dfrac{dv}{dz} + \dfrac{dw}{dy}$, $2g = \dfrac{dw}{dx} + \dfrac{du}{dz}$, $2h = \dfrac{du}{dy} + \dfrac{dv}{dx}$.

If we eliminate from equations (3) the two ratios which exist between the three quantities x', y', z', we get the well known cubic equation

$$(e - F)(e - G)(e - H) - f^2(e - F) - g^2(e - G) - h^2(e - H) - 2fgh = 0, \ldots\ldots (4)$$

which occurs in the investigation of the principal axes of a rigid body, and in various others. As in these investigations, it may be shewn that there are in general three directions, at right angles to each other, in which the point P' may be situated so as to satisfy the required conditions. If two of the roots of (4) are equal, there is one such direction corresponding to the third root, and an infinite number of others situated in a plane perpendicular to the former; and if the three roots of (4) are equal, a line drawn in any direction will satisfy the required conditions.

The three directions which have just been determined I shall call *axes of extension.* They will in general vary from one point to another, and from one instant of time to another. If we denote the three roots of (4) by e', e'', e''', and if we take new rectangular axes $Ox_{,}$, $Oy_{,}$, $Oz_{,}$, parallel to the axes of extension, and denote by $u_{,}$, $U_{,}$ &c. the quantities referred to these axes corresponding to u, U, &c., equations (3) must be satisfied by $y_{,}' = 0$, $z_{,}' = 0$, $e = e'$, by $x_{,}' = 0$, $z_{,}' = 0$, $e = e''$, and by $x_{,}' = 0$, $y_{,}' = 0$, $e = e'''$, which requires that $f_{,} = 0$, $g_{,} = 0$, $h_{,} = 0$, and we have

$$e' = F_{,} = \frac{du_{,}}{dx_{,}}, \quad e'' = G_{,} = \frac{dv_{,}}{dy_{,}}, \quad e'' = H_{,} = \frac{dw_{,}}{dz_{,}}.$$

The values of $U_{,}$, $V_{,}$, $W_{,}$, which correspond to the residual motion after the elimination of the motion of rotation corresponding to ω', ω'' and ω''', are

$$U_{,} = e'x_{,}', \quad V_{,} = e''y_{,}', \quad W_{,} = e'''z_{,}'.$$

The angular velocity of which ω', ω'', ω''' are the components is independent of the arbitrary directions of the co-ordinate axes: the same is true of the directions of the axes of extension, and of the values of the roots of equation (4). This might be proved in various ways; perhaps the following is the simplest. The conditions by which ω', ω'', ω''' are determined are those which express that the relative velocities U, V, W, which remain after eliminating a certain angular velocity, are such that $Udx' + Vdy' + Wdz'$ is ultimately an exact differential, that is to say when squares and products of x', y' and z' are neglected. It appears moreover from the solution that there is only one way in which these conditions can be satisfied for a given system of co-ordinate axes. Let us take new rectangular axes Ox, Oy, Oz, and let U, V, W be the resolved parts along these axes of the velocities U, V, W, and x′, y′, z′, the relative co-ordinates of P'; then

$$U = \mathrm{U} \cos x\mathrm{x} + \mathrm{V} \cos x\mathrm{y} + \mathrm{W} \cos x\mathrm{z},$$
$$dx' = \cos x\mathrm{x}\, d\mathrm{x}' + \cos x\mathrm{y}\, d\mathrm{y}' + \cos x\mathrm{z}\, d\mathrm{z}', \text{ \&c.};$$

whence, taking account of the well known relations between the cosines involved in these equations, we easily find

$$U dx' + V dy' + W dz' = \mathrm{U}\, d\mathrm{x}' + \mathrm{V}\, d\mathrm{y}' + \mathrm{W}\, d\mathrm{z}'.$$

It appears therefore that the relative velocities U, V, W, which remain after eliminating a certain angular velocity, are such that $\mathrm{U}\, d\mathrm{x}' + \mathrm{V}\, d\mathrm{y}' + \mathrm{W}\, d\mathrm{z}'$ is ultimately an exact differential. Hence the values of U, V, W are the same as would have been obtained from equations (2) applied directly to the new axes, whence the truth of the proposition enunciated at the head of this paragraph is manifest.

The motion corresponding to the velocities $U_{,}$, $V_{,}$, $W_{,}$ may be further decomposed into a motion of dilatation, positive or negative, which is alike in all directions, and two motions which I shall call *motions of shifting*, each of the latter being in two dimensions, and not affecting the density. For let δ be the rate of linear extension corresponding to a uniform dilatation; let $\sigma x_{,}'$, $-\sigma y_{,}'$ be the velocities parallel to $x_{,}$, $y_{,}$, corresponding to a motion of shifting parallel to the the plane $x_{,}y_{,}$, and let $\sigma' x_{,}'$, $-\sigma' z_{,}'$ be the velocities parallel to $x_{,}$, $z_{,}$, corresponding to a similar motion of shifting parallel to the plane $x_{,}z_{,}$. The velocities parallel to $x_{,}$, $y_{,}$, $z_{,}$ respectively corresponding to the quantities δ, σ and σ' will be $(\delta + \sigma + \sigma') x_{,}'$, $(\delta - \sigma) y_{,}'$, $(\delta - \sigma') z_{,}'$, and equating these to $U_{,}$, $V_{,}$, $W_{,}$ we shall get

$$\delta = \tfrac{1}{3}(e' + e'' + e'''), \quad \sigma = \tfrac{1}{3}(e' + e''' - 2e''), \quad \sigma' = \tfrac{1}{3}(e' + e'' - 2e''').$$

Hence the most general instantaneous motion of an elementary portion of a fluid is compounded of a motion of translation, a motion of rotation, a motion of uniform dilatation, and two motions of shifting of the kind just mentioned.

3. Having determined the nature of the most general instantaneous motion of an element of a fluid, we are now prepared to consider the normal pressures and tangential forces called into play by the relative displacements of the particles. Let p be the pressure which would exist about the point P if the neighbouring molecules were in a state of relative equilibrium: let $p + p_{,}$ be the normal pressure, and $t_{,}$ the tangential action, both referred to a unit of surface, on a plane passing through P and having a given direction. By the hypotheses of Art. 1. the quantities $p_{,}$, $t_{,}$ will be independent of the angular velocities ω', ω'', ω''', depending only on the residual relative velocities U, V, W, or, which comes to the same, on e', e'' and e''', or on σ, σ' and δ. Since this residual motion is symmetrical with respect to the axes of extension, it follows that if the plane considered is perpendicular to any one of these axes the tangential action on it is zero, since there is no more reason why it should act in one direction rather than in the opposite; for by the hypotheses of Art. 1. the change of density and temperature about the point P is to be neglected, the constitution of the fluid being ultimately uniform about that point. Denoting then by $p + p'$, $p + p''$, $p + p'''$ the pressures on planes perpendicular to the axes of $x_{,}$, $y_{,}$, $z_{,}$, we must have

$$p' = \phi(e', e'', e'''), \quad p'' = \phi(e'', e''', e'), \quad p''' = \phi(e''', e', e''),$$

$\phi(e', e'', e''')$ denoting a function of e', e'' and e''' which is symmetrical with respect to the two latter quantities. The question is now to determine, on whatever may seem the most probable hypothesis, the form of the function ϕ.

Let us first take the simpler case in which there is no dilatation, and only one motion of shifting, or in which $e'' = -e'$, $e''' = 0$, and let us consider what would take place if the fluid consisted of smooth molecules acting on each other by actual contact. On this supposition, it is clear, considering the magnitude of the pressures acting on the molecules compared with their masses, that they would be sensibly in a position of relative equilibrium, except when the equilibrium of any one of them became impossible from the displacement of the adjoining

ones, in which case the molecule in question would start into a new position of equilibrium. This start would cause a corresponding displacement in the molecules immediately about the one which started, and this disturbance would be propagated immediately in all directions, the nature of the displacement however being different in different directions, and would soon become insensible. During the continuance of this disturbance, the pressure on a small plane drawn through the element considered would not be the same in all directions, nor normal to the plane: or, which comes to the same, we may suppose a uniform normal pressure p to act, together with a normal pressure $p_{,,}$ and a tangential force $t_{,,}$, $p_{,,}$ and $t_{,,}$ being forces of great intensity and short duration, that is being of the nature of impulsive forces. As the number of molecules comprised in the element considered has been supposed extremely great, we may take a time τ so short that all summations with respect to such intervals of time may be replaced without sensible error by integrations, and yet so long that a very great number of starts shall take place in it. Consequently we have only to consider the average effect of such starts, and moreover we may without sensible error replace the impulsive forces such as $p_{,,}$ and $t_{,,}$, which succeed one another with great rapidity, by continuous forces. For planes perpendicular to the axes of extension these continuous forces will be the normal pressures p', p'', p'''.

Let us now consider a motion of shifting differing from the former only in having e' increased in the ratio of m to 1. Then, if we suppose each start completed before the starts which would be sensibly affected by it are begun, it is clear that the same series of starts will take place in the second case as in the first, but at intervals of time which are less in the ratio of 1 to m. Consequently the continuous pressures by which the impulsive actions due to these starts may be replaced must be increased in the ratio of m to 1. Hence the pressures p', p'', p''' must be proportional to e', or we must have

$$p' = Ce', \quad p'' = C'e', \quad p''' = C''e'.$$

It is natural to suppose that these formulæ held good for negative as well as positive values of e'. Assuming this to be true, let the sign of e' be changed. This comes to interchanging x and y, and consequently p''' must remain the same, and p' and p'' must be interchanged. We must therefore have $C'' = 0$, $C' = -C$. Putting then $C = -2\mu$ we have

$$p' = -2\mu e', \quad p'' = 2\mu e', \quad p''' = 0.$$

It has hitherto been supposed that the molecules of a fluid are in actual contact. We have every reason to suppose that this is not the case. But precisely the same reasoning will apply if they are separated by intervals as great as we please compared with their magnitudes, provided only we suppose the force of restitution called into play by a small displacement of *any one* molecule to be very great.

Let us now take the case of two motions of shifting which coexist, and let us suppose $e' = \sigma + \sigma'$, $e'' = -\sigma$, $e''' = -\sigma'$. Let the small time τ be divided into $2n$ equal portions, and let us suppose that in the first interval a shifting motion corresponding to $e' = 2\sigma$, $e'' = -2\sigma$ takes place parallel to the plane $x_{,}\, y_{,}$, and that in the second interval a shifting motion corresponding to $e' = 2\sigma'$, $e''' = -2\sigma'$ takes place parallel to the plane $x_{,}\, z_{,}$, and so on alternately. On this supposition it is clear that if we suppose the time $\frac{\tau}{2n}$ to be extremely small, the continuous forces by which the effect of the starts may be replaced will be $p' = -2\mu(\sigma + \sigma')$, $p'' = 2\mu\sigma$, $p''' = 2\mu\sigma'$. By supposing n indefinitely increased, we may make the motion considered approach as near as we please to that in which the two motions of shifting coexist; but we are not at liberty to do so, for in order to apply the above reasoning we must suppose the time $\frac{\tau}{2n}$ to be so large that the average effect of the starts which occur in it may be taken. Consequently it must be taken as an additional assumption, and not a matter of absolute demonstration, that the effects of the two motions of shifting are superimposed.

Hence if $\delta = 0$, *i.e.* if $e' + e'' + e''' = 0$, we shall have in general

$$p' = -2\mu e', \quad p'' = -2\mu e'', \quad p''' = -2\mu e''' \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

It was by this hypothesis of starts that I first arrived at these equations, and the differential equations of motion which result from them. On reading Poisson's memoir however, to which I shall have occasion to refer in Section IV., I was led to reflect that however intense we may suppose the molecular forces to be, and however near we may suppose the molecules to be to their positions of relative equilibrium, we are not therefore at liberty to suppose them *in* those positions, and consequently not at liberty to suppose the pressure equal in all directions in the intervals of time between the starts. In fact, by supposing the molecular forces indefinitely increased, retaining the same ratios to each other, we may suppose the displacements of the molecules from their positions of relative equilibrium indefinitely diminished, but on the other hand the force of restitution called into action by a given displacement is indefinitely increased in the same proportion. But be these displacements what they may, we know that the forces of restitution make equilibrium with forces equal and opposite to the effective forces; and in calculating the effective forces we may neglect the above displacements, or suppose the molecules to move in the paths in which they would move if the shifting motion took place with indefinite slowness. Let us first consider a single motion of shifting, or one for which $e'' = -e'$, $e''' = 0$, and let $p_{,}$ and $t_{,}$ denote the same quantities as before. If we now suppose e' increased in the ratio of m to 1, all the effective forces will be increased in that ratio, and consequently $p_{,}$ and $t_{,}$ will be increased in the same ratio. We may deduce the values of p', p'' and p''' just as before, and then pass by the same reasoning to the case of two motions of shifting which coexist, only that in this case the reasoning will be demonstrative, since we *may* suppose the time $\frac{\tau}{2n}$ indefinitely diminished. If we suppose the state of things considered in this paragraph to exist along with the motions of starting already considered, it is easy to see that the expressions for p', p'' and p''' will still retain the same form.

There remains yet to be considered the effect of the dilatation. Let us first suppose the dilatation to exist without any shifting: then it is easily seen that the relative motion of the fluid at the point considered is the same in all directions. Consequently the only effect which such a dilatation could have would be to introduce a normal pressure $p_{,}$, alike in all directions, in addition to that due to the action of the molecules supposed to be in a state of relative equilibrium. Now the pressure $p_{,}$ could only arise from the aggregate of the molecular actions called into play by the displacements of the molecules from their positions of relative equilibrium; but since these displacements take place, on an average, indifferently in all directions, it follows that the actions of which $p_{,}$ is composed neutralize each other, so that $p_{,} = 0$. The same conclusion might be drawn from the hypothesis of starts, supposing, as it is natural to do, that each start calls into action as much increase of pressure in some directions as diminution of pressure in others.

If the motion of uniform dilatation coexists with two motions of shifting, I shall suppose, for the same reason as before, that the effects of these different motions are superimposed. Hence subtracting δ from each of the three quantities e', e'' and e''', and putting the remainders in the place of e', e'' and e''' in equations (5), we have

$$p' = \tfrac{2}{3}\mu(e'' + e''' - 2e'), \quad p'' = \tfrac{2}{3}\mu(e''' + e' - 2e''), \quad p''' = \tfrac{2}{3}\mu(e' + e'' - 2e''') \ldots\ldots\ldots (6)$$

If we had started with assuming $\phi(e', e'', e''')$ to be a linear function of e', e'' and e''', avoiding all speculation as to the molecular constitution of a fluid, we should have had at once $p' = Ce' + C'(e'' + e''')$, since p' is symmetrical with respect to e'' and e'''; or, changing the constants, $p' = \frac{2}{3}\mu(e'' + e''' - 2e') + \kappa(e' + e'' + e''')$. The expressions for p'' and p''' would be obtained by interchanging the requisite quantities. Of course we may at once put $\kappa = 0$ if we assume that in the case of a uniform motion of dilatation the pressure at any instant depends only on the actual density and temperature at that instant, and not on the rate at which the

former changes with the time. In most cases to which it would be interesting to apply the theory of the friction of fluids the density of the fluid is either constant, or may without sensible error be regarded as constant, or else changes slowly with the time. In the first two cases the results would be the same, and in the third case nearly the same, whether κ were equal to zero or not. Consequently, if theory and experiment should in such cases agree, the experiments must not be regarded as confirming that part of the theory which relates to supposing κ to be equal to zero.

4. It will be easy now to determine the oblique pressure, or resultant of the normal pressure and tangential action, on any plane. Let us first consider a plane drawn through the point P parallel to the plane yz. Let $Ox_{,}$ make with the axes of x, y, z angles whose cosines are l', m', n'; let l'', m'', n'' be the same for $Oy_{,}$, and l''', m''', n''' the same for $Oz_{,}$. Let P_1 be the pressure, and (xty), (xtz) the resolved parts, parallel to y, z respectively, of the tangential force on the plane considered, all referred to a unit of surface, (xty) being reckoned positive when the part of the fluid towards $-x$ urges that towards $+x$ in the positive direction of y, and similarly for (xtz). Consider the portion of the fluid comprised within a tetrahedron having its vertex in the point P, its base parallel to the plane yz, and its three sides parallel to the planes $x_{,}y_{,}$, $y_{,}z_{,}$, $z_{,}x_{,}$ respectively. Let A be the area of the base, and therefore $l'A$, $l''A$, $l'''A$ the areas of the faces perpendicular to the axes of $x_{,}$, $y_{,}$, $z_{,}$. By D'Alembert's principle, the pressures and tangential actions on the faces of this tetrahedron, the moving forces arising from the external attractions, not including the molecular forces, and forces equal and opposite to the effective moving forces will be in equilibrium, and therefore the sums of the resolved parts of these forces in the directions of x, y and z will each be zero. Suppose now the dimensions of the tetrahedron indefinitely diminished, then the resolved parts of the external, and of the effective moving forces will vary ultimately as the cubes, and those of the pressures and tangential forces on the sides as the squares of homologous lines. Dividing therefore the three equations arising from equating to zero the resolved parts of the above forces by A, and taking the limit, we have

$$P_1 = \Sigma l'^2 (p + p'),\ (xty) = \Sigma l'm' (p + p'),\ (xtz) = \Sigma l'n' (p + p'),$$

the sign Σ denoting the sum obtained by taking the quantities corresponding to the three axes of extension in succession. Putting for p', p'' and p''' their values given by (6), putting $e' + e'' + e''' = 3\delta$, and observing that $\Sigma l'^2 = 1$, $\Sigma l'm' = 0$, $\Sigma l'n' = 0$, the above equations become

$$P_1 = p - 2\mu \Sigma l'^2 e' + 2\mu\delta,\quad (xty) = -2\mu \Sigma l'm'e',\quad (xtz) = -2\mu \Sigma l'n'e'.$$

The method of determining the pressure on any plane from the pressures on three planes at right angles to each other, which has just been given, has already been employed by MM. Cauchy and Poisson.

The most direct way of obtaining the values of $\Sigma l'^2 e'$ &c. would be to express l', m' and n' in terms of e' by any two of equations (3), in which x', y', z' are proportional to l', m', n', together with the equation $l'^2 + m'^2 + n'^2 = 1$, and then to express the resulting symmetrical function of the roots of the cubic equation (4) in terms of the coefficients. But this method would be excessively laborious, and need not be resorted to. For after eliminating the angular motion of the element of fluid considered the remaining velocities are $e'x_{,}'$, $e''y_{,}'$, $e'''z_{,}'$, parallel to the axes of $x_{,}$, $y_{,}$, $z_{,}$. The sum of the resolved parts of these parallel to the axis of x is $l'e'x_{,}' + l''e''y_{,}' + l'''e'''z_{,}'$. Putting for $x_{,}'$, $y_{,}'$, $z_{,}'$ their values $l'x' + m'y' + n'z'$ &c., the above sum becomes

$$x'\Sigma l'^2 e' + y'\Sigma l'm'e' + z'\Sigma l'n'e';$$

but this sum is the same thing as the velocity U in equation (2), and therefore we have

$$\Sigma l'^2 e' = \frac{du}{dx},\quad \Sigma l'm'e' = \tfrac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right),\quad \Sigma l'n'e' = \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dw}{px}\right).$$

It may also be very easily proved directly that the value of 3δ, the rate of cubical dilatation, satisfies the equation

$$3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \quad \text{.....} \quad (7)$$

Let P_2, (ytz), (ytx) be the quantities referring to the axis of y, and P_3, (ztx), (zty) those referring to the axis of z, which correspond to P_1 &c. referring to the axis of x. Then we see that $(ytz) = (zty)$, $(ztx) = (xtz)$, $(xty) = (ytx)$. Denoting these three quantities by T_1, T_2, T_3, and making the requisite substitutions and interchanges, we have

$$\left.\begin{aligned} P_1 &= p - 2\mu\left(\frac{du}{dx} - \delta\right), \\ P_2 &= p - 2\mu\left(\frac{dv}{dy} - \delta\right), \\ P_3 &= p - 2\mu\left(\frac{dw}{dz} - \delta\right), \\ T_1 &= -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right), \quad T_2 = -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right), \quad T_3 = -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right). \end{aligned}\right\} \text{.....} (8)$$

It may also be useful to know the components, parallel to x, y, z, of the oblique pressure on a plane passing through the point P, and having a given direction. Let l, m, n be the cosines of the angles which a normal to the given plane makes with the axes of x, y, z; let P, Q, R be the components, referred to a unit of surface, of the oblique pressure on this plane, P, Q, R being reckoned positive when the part of the fluid in which is situated the normal to which l, m and n refer is urged by the other part in the positive directions of x, y, z, when l, m and n are positive. Then considering as before a tetrahedron of which the base is parallel to the given plane, the vertex in the point P, and the sides parallel to the co-ordinate planes, we shall have

$$\left.\begin{aligned} P &= l P_1 + m T_3 + n T_2, \\ Q &= l T_3 + m P_2 + n T_1, \\ R &= l T_2 + m T_1 + n P_3. \end{aligned}\right\} \text{.....} (9)$$

In the simple case of a sliding motion for which $u = 0$, $v = f(x)$, $w = 0$, the only forces, besides the pressure p, which act on planes parallel to the co-ordinate planes are the two tangential forces T_3, the value of which in this case is $-\mu\dfrac{dv}{dx}$. In this case it is easy to show that the axes of extension are, one of them parallel to Oz, and the two others in a plane parallel to xy, and inclined at angles of 45^0 to Ox. We see also that it is necessary to suppose μ to be positive, since otherwise the tendency of the forces would be to increase the relative motion of the parts of the fluid, and the equilibrium of the fluid would be unstable.

5. Having found the pressures about the point P on planes parallel to the co-ordinate planes, it will be easy to form the equations of motion. Let X, Y, Z be the resolved parts, parallel to the axes, of the external force, not including the molecular force; let ρ be the density, t the time. Consider an elementary parallelepiped of the fluid, formed by planes parallel to the co-ordinate planes, and drawn through the point (x, y, z) and the point $(x + \Delta x, y + \Delta y, z + \Delta z)$. The mass of this element will be ultimately $\rho\,\Delta x\,\Delta y\,\Delta z$, and the moving force parallel to x arising from the external forces will be ultimately $\rho X\,\Delta x\,\Delta y\,\Delta z$; the effective moving force parallel to x will be ultimately $\rho\dfrac{Du}{Dt}\Delta x\,\Delta y\,\Delta z$, where D is used, as it will be in the rest of this paper,

to denote differentiation in which the independent variables are t and three parameters of the particle considered, (such for instance as its initial co-ordinates,) and not t, x, y, z. It is easy also to show that the moving force acting on the element considered arising from the oblique pressures on the faces is ultimately $\left(\frac{dP}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz}\right) \Delta x \, \Delta y \, \Delta z$, acting in the negative direction. Hence we have by D'Alembert's principle

$$\rho \left(\frac{Du}{Dt} - X\right) + \frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} = 0, \text{ \&c.,} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which equations we must put for $\frac{Du}{Dt}$ its value $\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}$, and similarly for $\frac{Dv}{dt}$ and $\frac{Dw}{Dt}$. In considering the general equations of motion it will be needless to write down more than one, since the other two may be at once derived from it by interchanging the requisite quantities. The equations (10), the ordinary equation of continuity, as it is called,

$$\frac{d\rho}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} + \frac{d\rho w}{dz} = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

which expresses the condition that there is no generation or destruction of mass in the interior of a fluid, the equation connecting p and ρ, or in the case of an incompressible fluid the equivalent equation $\frac{D\rho}{Dt} = 0$, and the equation for the propagation of heat, if we choose to take account of that propagation, are the only equations to be satisfied at every point of the interior of the fluid mass.

As it is quite useless to consider cases of the utmost degree of generality, I shall suppose the fluid to be homogeneous, and of a uniform temperature throughout, except in so far as the temperature may be raised by sudden compression in the case of small vibrations. Hence in equations (10) μ may be supposed to be constant as far as regards the temperature; for, in the case of small vibrations, the terms introduced by supposing it to vary with the temperature would involve the square of the velocity, which is supposed to be neglected. If we suppose μ to be independent of the pressure also, and substitute in (10) the values of P_1 &c. given by (8), the former equations become

$$\rho \left(\frac{Du}{Dt} - X\right) + \frac{dp}{dx} - \mu \left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \text{ \&c.} \ldots\ldots (12)$$

Let us now consider in what cases it is allowable to suppose μ to be independent of the pressure. It has been concluded by Dubuat, from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure. The total retardation depends, partly on the friction of the water against the sides of the pipe or canal, and partly on the mutual friction, or tangential action, of the different portions of the water. Now if these two parts of the whole retardation were separately variable with p, it is very unlikely that they should when combined give a result independent of p. The amount of the internal friction of the water depends on the value of μ. I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of the pressure. On this supposition, we have from equations (11) and (12)

$$\rho \left(\frac{Du}{Dt} - X\right) + \frac{dp}{dx} - \mu \left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) = 0, \text{ \&c.} \ldots\ldots\ldots\ldots (13)$$

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

These equations are applicable to the determination of the motion of water in pipes and canals, to the calculation of the effect of friction on the motions of tides and waves, and such questions.

If the motion is very small, so that we may neglect the square of the velocity, we may put $\frac{Du}{Dt} = \frac{du}{dt}$, &c. in equations (13). The equations thus simplified are applicable to the determination of the motion of a pendulum oscillating in water, or of that of a vessel filled with water and made to oscillate. They are also applicable to the determination of the motion of a pendulum oscillating in air, for in this case we may, with hardly any error, neglect the compressibility of the air.

The case of the small vibrations by which sound is propagated in a fluid, whether a liquid or a gas, is another in which $\frac{d\mu}{dp}$ may be neglected. For in the case of a liquid reasons have been shown for supposing μ to be independent of p, and in the case of a gas we may neglect $\frac{d\mu}{dp}$, if we neglect the small change in the value of μ, arising from the small variation of pressure due to the forces X, Y, Z.

6. Besides the equations which must hold good at any point in the interior of the mass, it will be necessary to form also the equations which must be satisfied at its boundaries. Let M be a point in the boundary of the fluid. Let a normal to the surface at M, drawn on the outside of the fluid, make with the axes angles whose cosines are l, m, n. Let P', Q', R' be the components of the pressure of the fluid about M on the solid or fluid with which it is in contact, these quantities being reckoned positive when the fluid considered presses the solid or fluid outside it in the positive directions of x, y, z, supposing l, m and n positive. Let S be a very small element of the surface about M, which will be ultimately plane, S' a plane parallel and equal to S, and directly opposite to it, taken within the fluid. Let the distance between S and S' be supposed to vanish in the limit compared with the breadth of S, a supposition which may be made if we neglect the effect of the curvature of the surface at M; and let us consider the forces acting on the element of fluid comprised between S and S', and the motion of this element. If we suppose equations (8) to hold good to within an insensible distance from the surface of the fluid, we shall evidently have forces ultimately equal to PS, QS, RS, (P, Q and R being given by equations (9),) acting on the inner side of the element in the positive directions of the axes, and forces ultimately equal to $P'S$, $Q'S$, $R'S$ acting on the outer side in the negative directions. The moving forces arising from the external forces acting on the element, and the effective moving forces will vanish in the limit compared with the forces PS, &c.: the same will be true of the pressures acting about the edge of the element, if we neglect capillary attraction, and all forces of the same nature. Hence, taking the limit, we shall have

$$P' = P, \quad Q' = Q, \quad R' = R.$$

The method of proceeding will be different according as the bounding surface considered is a free surface, the surface of a solid, on the surface of separation of two fluids, and it will be necessary to consider these cases separately. Of course the surface of a liquid exposed to the air is really the surface of separation of two fluids, but it may in many cases be regarded as a free surface if we neglect the inertia of the air: it may always be so regarded if we neglect the friction of the air as well as its inertia.

Let us first take the case of a free surface exposed to a pressure Π, which is supposed to be the same at all points, but may vary with the time; and let $L = 0$ be the equation to the surface. In this case we shall have $P' = l\Pi$, $Q' = m\Pi$, $R' = n\Pi$; and putting for P, Q, R their values given by (9), and for P_1 &c. their values given by (8), and observing that in this case $\delta = 0$, we shall have

$$l(\Pi - p) + \mu\left\{2l\frac{du}{dx} + m\left(\frac{du}{dy} + \frac{dv}{dx}\right) + n\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right\} = 0, \text{ \&c.,} \ldots\ldots\ldots (14)$$

in which equations l, m, n will have to be replaced by $\frac{dL}{dx}$, $\frac{dL}{dy}$, $\frac{dL}{dz}$, to which they are proportional.

If we choose to take account of capillary attraction, we have only to diminish the pressure Π by the quantity $H\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$, where H is a positive constant depending on the nature of the fluid, and r_1, r_2 are the principal radii of curvature at the point considered, reckoned positive when the fluid is concave outwards. Equations (14) with the ordinary equation

$$\frac{dL}{dt} + u\frac{dL}{dx} + v\frac{dL}{dy} + w\frac{dL}{dz} = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

are the conditions to be satisfied for points at the free surface. Equations (14) are for such points what the three equations of motion are for internal points, and (15) is for the former what (11) is for the latter, expressing in fact that there is no generation or destruction of fluid at the free surface.

The equations (14) admit of being differently expressed, in a way which may sometimes be useful. If we suppose the origin to be in the point considered, and the axis of z to be the external normal to the surface, we have $l = m = 0$, $n = 1$, and the equations become

$$\frac{dw}{dx} + \frac{du}{dz} = 0, \quad \frac{dw}{dy} + \frac{dv}{dz} = 0, \quad \Pi - p + 2\mu\frac{dw}{dz} = 0. \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

The relative velocity parallel to z of a point $(x', y', 0)$ in the free surface, indefinitely near the origin, is $\frac{dw}{dx}x' + \frac{dw}{dy}y'$: hence we see that $\frac{dw}{dx}$, $\frac{dw}{dy}$ are the angular velocities, reckoned from x to z and from y to z respectively, of an element of the free surfaee. Subtracting the linear velocities due to these angular velocities from the relative velocities of the point (x', y', z'), and calling the remaining relative velocities U, V, W, we shall have

$$U = \frac{du}{dx}x' + \frac{du}{dy}y' + \left(\frac{du}{dz} + \frac{dw}{dx}\right)z',$$

$$V = \frac{dv}{dx}x' + \frac{dv}{dy}y' + \left(\frac{dv}{dz} + \frac{dw}{dy}\right)z',$$

$$W = \frac{dw}{dz}z'.$$

Hence we see that the first two of equations (16) express the conditions that $\frac{dU}{dz'} = 0$ and $\frac{dV}{dz'} = 0$, which are evidently the conditions to be satisfied in order that there may be no sliding motion in a direction parallel to the free surface. It would be easy to prove that these are the conditions to be satisfied in order that the axis of z may be an axis of extension.

The next case to consider is that of a fluid in contact with a solid. The condition which first occurred to me to assume for this case was, that the film of fluid immediately in contact with the solid did not move relatively to the surface of the solid. I was led to try this condition from the following considerations. According to the hypotheses adopted, if there was a very large relative

motion of the fluid particles immediately about any imaginary surface dividing the fluid, the tangential forces called into action would be very large, so that the amount of relative motion would be rapidly diminished. Passing to the limit, we might suppose that if at any instant the velocities altered discontinuously in passing across any imaginary surface, the tangential force called into action would immediately destroy the finite relative motion of particles indefinitely close to each other, so as to render the motion continuous; and from analogy the same might be supposed to be true for the surface of junction of a fluid and solid. But having calculated, according to the conditions which I have mentioned, the discharge of long straight circular pipes and rectangular canals, and compared the resulting formulæ with some of the experiments of Bossut and Dubuat, I found that the formulæ did not at all agree with experiment. I then tried Poisson's conditions in the case of a circular pipe, but with no better success. In fact, it appears from experiment that the tangential force varies nearly as the square of the velocity with which the fluid flows past the surface of a solid, at least when the velocity is not very small. It appears however from experiments on pendulums that the total friction varies as the first power of the velocity, and consequently we may suppose that Poisson's conditions, which include as a particular case those which I first tried, hold good for very small velocities. I proceed therefore to deduce these conditions in a manner conformable with the views explained in this paper.

First, suppose the solid at rest, and let $L = 0$ be the equation to its surface. Let M' be a point within the fluid, at an insensible distance h from M. Let ϖ be the pressure which would exist about M if there were no motion of the particles in its neighbourhood, and let $p_{,}$ be the additional normal pressure, and $t_{,}$ the tangential force, due to the relative velocities of the particles, both with respect to one another and with respect to the surface of the solid. If the motion is so slow that the starts take place independently of each other, on the hypothesis of starts, or that the molecules are very nearly in their positions of relative equilibrium, and if we suppose as before that the effects of different relative velocities are superimposed, it is easy to show that $p_{,}$ and $t_{,}$ are linear functions of u, v, w and their differential coefficients with respect to x, y, and z; u, v, &c. denoting here the velocities of the fluid about the point M', in the expressions for which however the co-ordinates of M may be used for those of M', since h is neglected. Now the relative velocities about the points M and M' depending on $\frac{du}{dx}$ &c. are comparable with $\frac{du}{dx}h$, while those depending on u, v and w are comparable with these quantities, and therefore in considering the action of the fluid on the solid it is only necessary to consider the quantities u, v and w. Now since, neglecting h, the velocity at M' is tangential to the surface at M, u, v, and w are the components of a certain velocity V tangential to the surface. The pressure $p_{,}$ must be zero; for changing the signs of u, v, and w the circumstances concerned in its production remain the same, whereas its analytical expression changes sign. The tangential force at M will be in the direction of V, and proportional to it, and consequently its components along the axes of x, y, z will be proportional to u, v, w. Reckoning the tangential force positive when, l, m, and n being positive, the solid is urged in the positive directions of x, y, z, the resolved parts of the tangential force will therefore be νu, νv, νw, where ν must evidently be positive, since the effect of the forces must be to check the relative motion of the fluid and solid. The normal pressure of the fluid on the solid being equal to ϖ, its components will be evidently $l\varpi$, $m\varpi$, $n\varpi$.

Suppose now the solid to be in motion, and let u', v', w' be the resolved parts of the velocity of the point M of the solid, and ω', ω'', ω''' the angular velocities of the solid. By hypothesis, the forces by which the pressure at any point differs from the normal pressure due to the action of the molecules supposed to be in a state of relative equilibrium about that point are independent of any velocity of translation or rotation. Supposing then linear and angular velocities equal and opposite to those of the solid impressed both on the solid and on the fluid, the former will be for an instant at rest, and we have only to treat the resulting velocities of the fluid as in the first case.

Hence $P' = l\varpi + \nu(u - u')$, &c.; and in the equations (8) we may employ the actual velocities u, v, w, since the pressures P, Q, R are independent of any motion of translation and rotation common to the whole fluid. Hence the equations $P' = P$, &c. give us

$$l(\varpi - p) + \nu(u - u') + \mu\left\{2l\left(\frac{du}{dx} - \delta\right) + m\left(\frac{du}{dy} + \frac{dv}{dx}\right) + n\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right\} = 0, \text{ \&c.,} \ldots\ldots(17)$$

which three equations with (15) are those which must be satisfied at the surface of a solid, together with the equation $L = 0$. It will be observed that in the case of a free surface the pressures P', Q', R' are given, whereas in the case of the surface of a solid they are known only by the solution of the problem. But on the other hand the form of the surface of the solid is given, whereas the form of the free surface is known only by the solution of the problem.

Dubuat found by experiment that when the mean velocity of water flowing through a pipe is less than about one inch in a second, the water near the inner surface of the pipe is at rest. If these experiments may be trusted, the conditions to be satisfied in the case of small velocities are those which first occurred to me, and which are included in those just given by supposing $\nu = \infty$.

I have said that when the velocity is not very small the tangential force called into action by the sliding of water over the inner surface of a pipe varies nearly as the square of the velocity. This fact appears to admit of a natural explanation. When a current of water flows past an obstacle, it produces a resistance varying nearly as the square of the velocity. Now even if the inner surface of a pipe is polished we may suppose that little irregularities exist, forming so many obstacles to the current. Each little protuberance will experience a resistance varying nearly as the square of the velocity, from whence there will result a tangential action of the fluid on the surface of the pipe, which will vary nearly as the square of the velocity; and the same will be true of the equal and opposite reaction of the pipe on the fluid. The tangential force due to this cause will be combined with that by which the fluid close to the pipe is kept at rest when the velocity is sufficiently small.

There remains to be considered the case of two fluids having a common surface. Let u', v', w', μ', δ' denote the quantities belonging to the second fluid corresponding to u, &c. belonging to the first. Together with the two equations $L = 0$ and (15) we shall have in this case the equation derived from (15) by putting u', v', w' for u, v, w; or, which comes to the same, we shall have the two former equations with

$$l(u - u') + m(v - v') + n(w - w') = 0 \ldots\ldots\ldots\ldots\ldots(18)$$

If we consider the principles on which equations (17) were formed to be applicable to the present case, we shall have six more equations to be satisfied, namely (17), and the three equations derived from (17) by interchanging the quantities referring to the two fluids, and changing the signs of l, m, n. These equations give the value of ϖ, and leave five equations of condition. If we must suppose $\nu = \infty$, as appears most probable, the six equations above mentioned must be replaced by the six $u' = u$, $v' = v$, $w' = w$, and

$$lp - \mu f(u, v, w) = lp' - \mu' f(u', v', w'), \text{ \&c.,}$$

$f(u, v, w)$ denoting the coefficient of μ in the first of equations (17). We have here six equations of condition instead of five, but then the equation (18) becomes identical.

7. The most interesting questions connected with this subject require for their solution a knowledge of the conditions which must be satisfied at the surface of a solid in contact with the fluid, which, except perhaps in case of very small motions, are unknown. It may be well however to give some applications of the preceding equations which are independent of these conditions. Let us then in the first place consider in what manner the transmission of sound in a fluid is affected by the tangential action. To take the simplest case, suppose that no forces act on the fluid, so that the pressure and density are constant in the state of

equilibrium, and conceive a series of plane waves to be propagated in the direction of the axis of x, so that $u = f(x, t)$, $v = 0$, $w = 0$. Let $p_{,}$ be the pressure, and $\rho_{,}$ the density of the fluid when it is in equilibrium, and put $p = p_{,} + p'$. Then we have from equations (11) and (12), omitting the square of the disturbance,

$$\frac{1}{\rho_{,}}\frac{d\rho}{dt} + \frac{du}{dx} = 0, \quad \rho_{,}\frac{du}{dt} + \frac{dp'}{dx} - \frac{4}{3}\mu\frac{d^2u}{dx^2} = 0 \ldots\ldots\ldots\ldots(19)$$

Let $A\Delta\rho$ be the increment of pressure due to a very small increment $\Delta\rho$ of density, the temperature being unaltered, and let m be the ratio of the specific heat of the fluid when the pressure is constant to its specific heat when the volume is constant; then the relation between p' and ρ will be

$$p' = mA(\rho - \rho_{,}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

Eliminating p' and ρ from (19) and (20) we get

$$\frac{d^2u}{dt^2} - mA\frac{d^2u}{dx^2} - \frac{4\mu}{3\rho_{,}}\frac{d^3u}{dt\,dx^2} = 0.$$

To obtain a particular solution of this equation, let $u = \phi(t)\cos\frac{2\pi x}{\lambda} + \psi(t)\sin\frac{2\pi x}{\lambda}$. Substituting in the above equation, we see that $\phi(t)$ and $\psi(t)$ must satisfy the same equation, namely,

$$\phi''(t) + \frac{4\pi^2}{\lambda^2}mA\phi(t) + \frac{16\pi^2\mu}{3\lambda^2\rho_{,}}\phi'(t) = 0,$$

the integral of which is

$$\phi(t) = \epsilon^{-ct}\left(C\cos\frac{2\pi bt}{\lambda} + C'\sin\frac{2\pi bt}{\lambda}\right),$$

where $c = \frac{8\pi^2\mu}{3\lambda^2\rho_{,}}$, $b^2 = mA - \frac{16\pi^2\mu^2}{9\lambda^2\rho_{,}^2}$, C and C' being arbitrary constants. Taking the same expression with different arbitrary constants for $\psi(t)$, replacing products of sines and cosines by sums and differences, and combining the resulting sines and cosines two and two, we see that the resulting value of u represents two series of waves propagated in opposite directions. Considering only those waves which are propagated in the positive direction of x, we have

$$u = C_1\epsilon^{-ct}\cos\left\{\frac{2\pi}{\lambda}(bt - x) + C_2\right\} \ldots\ldots\ldots\ldots\ldots(21)$$

We see then that the effect of the tangential force is to make the intensity of the sound diminish as the time increases, and to render the velocity of propagation less than what it would otherwise be. Both effects are greater for high, than for low notes; but the former depends on the first power of μ, while the latter depends only on μ^2. It appears from the experiments of M. Biot, made on empty water pipes in Paris, that the velocity of propagation of sound is sensibly the same whatever be its pitch. Hence it is necessary to suppose that for air $\frac{\mu^2}{\lambda^2\rho_{,}^2}$ is insensible compared with A or $\frac{p_{,}}{\rho_{,}}$. I am not aware of any similar experiments made on water, but the ratio of $\left(\frac{\mu}{\lambda\rho_{,}}\right)^2$ to A would probably be insensible for water also. The diminution of intensity as the time increases is, in the case of plane waves, due *entirely* to friction; but as we do not possess any means of measuring the intensity of sound the theory cannot be tested, nor the numerical value of μ determined, in this way.

The velocity of sound in air, deduced from the note given by a known tube, is sensibly less than that determined by direct observation. Poisson thought that this might be due to the retardation of the air by friction against the sides of the tube. But from the above investigation it seems unlikely that the effect produced by that cause would be sensible.

The equation (21) may be considered as expressing in all cases the effect of friction; for we may represent an arbitrary disturbance of the medium as the aggregate of series of plane waves propagated in all directions.

8. Let us now consider the motion of a mass of uniform inelastic fluid comprised between two cylinders having a common axis, the cylinders revolving uniformly about their axis, and the fluid being supposed to have attained its permanent state of motion. Let the axis of the cylinders be taken for that of z, and let q be the actual velocity of any particle, so that $u = -q \sin\theta$, $v = q\cos\theta$, $w = 0$, r and θ being polar co-ordinates in a plane parallel to xy.

Observing that $\dfrac{d^2f}{dx^2} + \dfrac{d^2f}{dy^2} = \dfrac{d^2f}{dr^2} + \dfrac{1}{r}\dfrac{df}{dr} + \dfrac{1}{r^2}\dfrac{d^2f}{d\theta^2}$, where f is any function of x and y, and that $\dfrac{dp}{d\theta} = 0$, we have from equations (13), supposing after differentiation that the axis of x coincides with the radius vector of the point considered, and omitting the forces, and the part of the pressure due to them,

$$\frac{dp}{dr} - \rho\frac{q^2}{r} = 0,$$

$$\frac{d^2q}{dr^2} + \frac{1}{r}\frac{dq}{dr} - \frac{q}{r^2} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

and the equation of continuity is satisfied identically.

The integral of (22) is $$q = \frac{C}{r} + C'r.$$

If a is the radius of the inner, and b that of the outer cylinder, and if q_1, q_2 are the velocities of points close to these cylinders respectively, we must have $q = q_1$ when $r = a$, and $q = q_2$ when $r = b$, whence

$$q = \frac{1}{b^2 - a^2}\left\{(bq_1 - aq_2)\frac{ab}{r} + (bq_2 - aq_1)r\right\} \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

If the fluid is infinitely extended, $b = \infty$, and

$$\frac{q}{q_1} = \frac{a}{r}.$$

These cases of motion were considered by Newton, (*Principia*, Lib. II. Prop. 51.) The hypothesis which I have made agrees in this case with his, but he arrives at the result that the velocity is constant, not, that it varies inversely as the distance. This arises from his having taken, as the condition of there being no acceleration or retardation of the motion of an annulus, that the force tending to turn it in one direction must be equal to that tending to turn it in the opposite, whereas the true condition is that the moment of the force tending to turn it one way must be equal to the moment of the force tending to turn it the other. Of course, making this alteration, it is easy to arrive at the above result by Newton's reasoning. The error just mentioned vitiates the result of Prop. 52. It may be shown from the general equations

that in this case a permanent motion in annuli is impossible, and that, whatever may be the law of friction between the solid sphere and the fluid. Hence it appears that it is necessary to suppose that the particles move in planes passing through the axis of rotation, while they at the same time move round it. In fact, it is easy to see that from the excess of centrifugal force in the neighbourhood of the equator of the revolving sphere the particles in that part will recede from the sphere, and approach it again in the neighbourhood of the poles, and this circulating motion will be combined with a motion about the axis. If however we leave the centrifugal force out of consideration, as Newton has done, the motion in annuli becomes possible, but the solution is different from Newton's, as might have been expected.

The case of motion considered in this article may perhaps admit of being compared with experiment, without knowing the conditions which must be satisfied at the surface of a solid. A hollow, and a solid cylinder might be so mounted as to admit of being turned with different uniform angular velocities round their common axis, which is supposed to be vertical. If both cylinders are turned, they ought to be turned in opposite directions, if only one, it ought to be the outer one; for if the inner were made to revolve too fast, the fluid near it would have a tendency to fly outwards in consequence of the centrifugal force, and eddies would be produced. As long as the angular velocities are not great, so that the surface of the liquid is very nearly plane, it is not of much importance that the fluid is there terminated; for the conditions which must be satisfied at a free surface are satisfied for any section of the fluid made by a horizontal plane, so long as the motion about that section is supposed to be the same as it would be if the cylinders were infinite. The principal difficulty would probably be to measure accurately the time of revolution, and distance from the axis, of the different annuli. This would probably be best done by observing motes in the fluid. It might be possible also to discover in this way the conditions to be satisfied at the surface of the cylinders; or at least a law might be suggested, which could be afterwards compared more accurately with experiment by means of the discharge of pipes and canals.

If the rotations of the cylinders are in opposite directions, there will be a certain distance from the axis at which the fluid will not revolve at all. Writing $-q_1$ for q_1 in equation (23), we have for this distance $\sqrt{\dfrac{ab\,(bq_1 + aq_2)}{bq_2 + aq_1}}$.

9. Although the discharge of a liquid through a long straight pipe or canal, under given circumstances, cannot be calculated without knowing the conditions to be satisfied at the surface of contact of the fluid and solid, it may be well to go a certain way towards the solution.

Let the axis of z be parallel to the generating lines of the pipe or canal, and inclined at an angle α to the horizon; let the plane yz be vertical, and let y and z be measured downwards. The motion being uniform, we shall have $u = 0$, $v = 0$, $w = f(x, y)$, and we have from equations (13)

$$\frac{dp}{dx} = 0, \quad \frac{dp}{dy} = g\rho \cos \alpha, \quad \frac{dp}{dz} = g\rho \sin \alpha + \mu \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2}\right).$$

In the case of a canal $\dfrac{dp}{dz} = 0$; and the calculation of the motion in a pipe may always be reduced to that of the motion in the same pipe when $\dfrac{dp}{dz}$ is supposed to be zero, as may be shown by reasoning similar to Dubuat's. Moreover the motion in a canal is a particular case of the motion in a pipe. For consider a pipe for which $\dfrac{dp}{dz} = 0$, and which is divided symmetrically by the plane xz. From the symmetry of the motion, it is clear that we must have $\dfrac{dw}{dy} = 0$ when $z = 0$;

but this is precisely the condition which would have to be satisfied if the fluid had a free surface coinciding with the plane xz; hence we may suppose the upper half of the fluid removed, without affecting the motion of the rest, and thus we pass to the case of a canal. Hence it is the same thing to determine the motion in a canal, as to determine that in the pipe formed by completing the canal symmetrically with respect to the surface of the fluid.

We have then, to determine the motion, the equation

$$\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + \frac{g\rho \sin \alpha}{\mu} = 0.$$

In the case of a rectangular pipe, it would not be difficult to express the value of w at any point in terms of its values at the several points of the perimeter of a section of the pipe. In the case of a cylindrical pipe the solution is extremely easy: for if we take the axis of the pipe for that of z, and take polar co-ordinates r, θ in a plane parallel to xy, and observe that $\frac{dw}{d\theta} = 0$, since the motion is supposed to be symmetrical with respect to the axis, the above equation becomes

$$\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{g\rho \sin \alpha}{\mu} = 0.$$

Let a be the radius of the pipe, and U the velocity of the fluid close to the surface; then, integrating the above equation, and determining the abitrary constants by the conditions that w shall be finite when $r = 0$, and $w = U$ when $r = a$, we have

$$w = \frac{g\rho \sin \alpha}{4\mu}(a^2 - r^2) + U.$$

SECTION II.

Objections to Lagrange's proof of the theorem that if udx + vdy + wdz *is an exact differential at any one instant it is always so, the pressure being supposed equal in all directions. Principles of M. Cauchy's proof. A new proof of the theorem. A physical interpretation of the circumstance of the above expression being an exact differential.*

10. The proof of this theorem given by Lagrange depends on the legitimacy of supposing u, v and w capable of expansion according to positive integral powers of t, for a sufficiently small finite value of t. It is clear that the expansion cannot contain negative powers of t, since u, v and w are supposed to be finite when $t = 0$; but it may be objected to Lagrange's proof that there are functions of t of which the expansion contains fractional powers of t, and that we do not know but that u, v and w may be such functions. This objection has been considered by Mr. Power*, who has shown that the theorem is true if we suppose u, v and w capable of expansion according to any powers of t. Still the proof remains unsatisfactory, in fact inconclusive, for these are functions of t, (for instance $e^{-\frac{1}{t^2}}$, $t \log t$,) which do not admit of expansion according

* *Cambridge Philosophical Transactions*, Vol. VII. Part 3.

to powers of t, integral or fractional, and we do not know but that u, v and w may be functions of this nature. I do not here mention the proof which Poisson has given of the theorem in his *Traité de Méchanique*, because it appears to me liable to an objection to which I shall presently have occasion to refer: in fact, Poisson himself did not think the theorem generally true.

It is remarkable that Mr. Power's proof, if it were legitimate, would establish the theorem even when account is taken of the variation of pressure in different directions, according to the theory explained in Section 1, if we suppose that $\frac{d\mu}{dp} = 0$. To show this we have only got to treat equations (12) as Mr. Power has treated the three equations of fluid motion formed on the ordinary hypothesis. Yet in this case the theorem is evidently untrue. Thus, conceive a mass of fluid which is bounded by a solid plane coinciding with the plane yz, and which extends infinitely in every direction on the positive side of the axis of x, and suppose the fluid at first to be at rest. Suppose now the solid plane to be moved in any manner parallel to the axis of y; then, unless the solid plane exerts no tangential force on the fluid, (and we may suppose that it does exert some,) it is clear that at a given time we shall have $u = 0$, $v = f(x)$, $w = 0$, and therefore $udx + vdy + wdz$ will not be an exact differential. It will be interesting then to examine in this case the nature of the function of t which expresses the value of v.

Supposing X, Y, Z to be zero in equations (12), and observing that in the case considered we have $\frac{dp}{dy} = 0$, we get

$$\frac{dv}{dt} = \frac{\mu}{\rho}\frac{d^2v}{dx^2} \quad \text{.....} \quad (24)$$

Differentiating this equation $n - 1$ times with respect to t, we easily get

$$\frac{d^nv}{dt^n} = \left(\frac{\mu}{\rho}\right)^n \frac{d^{2n}v}{dx^{2n}}:$$

but when $t = 0$, $v = 0$ when $x > 0$, and therefore for a given value of x all the differential coefficients of v with respect to t are zero. Hence for indefinitely small values of t the value of v at a given point increases more slowly than if it varied ultimately as any power of t, however great; hence v cannot be expanded in a series according to powers of t. This result is independent of the condition to be satisfied at the surface of the solid plane.

I think what has just been proved shows clearly that Lagrange's proof of the theorem considered, even with Mr. Power's improvement of it, is inadmissible.

11. The theorem is however true, and a proof of it has been given by M. Cauchy*, which appears to me perfectly free from objection, and which is very simple in principle, although it depends on rather long equations. M. Cauchy first eliminates p from the three equations of motion by means of the conditions that $\frac{d^2p}{dxdy} = \frac{d^2p}{dydx}$ &c., he then changes the independent variables from x, y, z, t to a, b, c, t, where a, b, c are the initial co-ordinates of the particles. The three transformed equations admit each of being once integrated with respect to t; and determining the arbitrary functions of a, b, c by the initial values of u, v and w, the three integrals have the form

$$\omega_0' = F\omega' + G\omega'' + H\omega''', \text{ \&c.,}$$

* *Mémoire sur la Théorie des Ondes*, in the first volume of the *Mémoires présentés à l'Institut.* M. Cauchy has not had occasion to enunciate the theorem, but it is contained in his equations (16). This equation may be obtained in the same manner in the more general case in which p is supposed to be a function of ρ.

ω', ω'' and ω''' denoting here the same as in Art. 2, and ω_0' &c. denoting the initial values of ω', &c. for the same particle. Solving the above equations with respect to ω', ω'' and ω''', the resulting equations are

$$\omega' = \frac{1}{S}\left(\frac{dx}{da}\omega_0' + \frac{dx}{db}\omega_0'' + \frac{dx}{dc}\omega_0'''\right), \text{ \&c.,}$$

where S is a function of the differential coefficients of x, y and z with respect to a, b and c, which by the condition of continuity is shown to be equal to $\frac{\rho}{\rho_0}$, ρ_0 being the initial density about the particle whose density at the time considered is ρ. Since $\frac{dx}{da}$ &c. are finite, (for to suppose them infinite would be equivalent to supposing a discontinuity to exist in the fluid,) it follows at once from the preceding equations that if $\omega_0' = 0$, $\omega_0'' = 0$, $\omega_0''' = 0$, that is if $u_0 da + v_0 db + w_0 dc$ be an exact differential, either for the whole fluid or for any portion of it, then shall $\omega' = 0$, $\omega'' = 0$, $\omega''' = 0$, i. e. $udx + vdy + wdz$ will be an exact differential, at any subsequent time, either for the whole mass or for the above portion of it.

12. It is not from seeing the smallest flaw in M. Cauchy's proof that I propose a new one, but because it is well to view the subject in different lights, and because the proof which I am about to give does not require such long equations. It will be necessary in the first place to prove the following lemma.

LEMMA. If $\omega_1, \omega_2 \ldots \omega_n$ are n functions of t, which satisfy the n differential equations

$$\left.\begin{aligned} \frac{d\omega_1}{dt} &= P_1\omega_1 + Q_1\omega_2 \ldots + V_1\omega_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d\omega_n}{dt} &= P_n\omega_1 + Q_n\omega_2 \ldots + V_n\omega_n, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(25)$$

where $P_1, Q_1 \ldots V_n$ may be functions of t, $\omega_1 \ldots \omega_n$, and if when $\omega_1 = 0$, $\omega_2 = 0 \ldots \omega_n = 0$, none of the quantities $P_1, \ldots V_n$ is infinite for any value of t from 0 to T, and if $\omega_1 \ldots \omega_n$ are each zero when $t = 0$, then shall each of these quantities remain zero for all values of t from 0 to T.

DEMONSTRATION. Let τ be a finite value of t, then by hypothesis τ may be taken so small that the values of $\omega_1 \ldots \omega_n$ are sufficiently small to exclude all values which might render any one of the quantities $P_1 \ldots V_n$ infinite. Let L be a superior limit to the numerical values of the several quantities $P_1 \ldots V_n$ for all values of t from 0 to τ; then it is evident that $\omega_1 \ldots \omega_n$ cannot increase faster than if they satisfied the equations

$$\left.\begin{aligned} \frac{d\omega_1}{dt} &= L(\omega_1 + \omega_2 \ldots + \omega_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d\omega_n}{dt} &= L(\omega_1 + \omega_2 \ldots + \omega_n), \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

vanishing in this case also when $t = 0$. But if $\omega_1 + \omega_2 \ldots + \omega_n = \Omega$, we have by adding together the above equations

$$\frac{d\Omega}{dt} = nL\Omega:$$

if now Ω be not equal to zero, dividing this equation by Ω and integrating, we have

$$\Omega = Ce^{nLt};$$

but no value of C different from zero will allow Ω to vanish when $t = 0$, whereas by hypothesis it does vanish; hence $\Omega = 0$; but Ω is the sum of n quantities which evidently cannot be negative, and therefore each of these must be zero. Since then $\omega_1 \ldots \omega_n$ would have to be equal to zero for all values of t from 0 to τ even if they satisfied equations (26), they must *à fortiori* be equal to zero in the actual case, since they satisfy equations (25). Hence there is no value of t from 0 to T at which any one of the quantities $\omega_1 \ldots \omega_n$ can begin to differ from zero, and therefore these quantities must remain equal to zero for all values of t from 0 to T.

This lemma might be extended to the case in which $n = \infty$, with certain restrictions as to the convergency of the series. We may also, instead of the integers $1, 2 \ldots n$, have a continuous variable a which varies from 0 to a, so that ω is a function of the independent variables a and t, satisfying the differential equation

$$\frac{d\omega}{dt} = \int_0^a \psi(a, \omega, t)\, \omega\, da,$$

where $\psi(a, 0, t)$ does not become infinite for any value of a from 0 to a combined with any value of t from 0 to T. It may be shown, just as before, that if $\omega = 0$ when $t = 0$ for all values of a from 0 to a, then must $\omega = 0$ for all values of t from 0 to T. The proposition might be further extended to the case in which $a = \infty$, with a certain restriction as to the convergency of the integral, but equations (25) are already more general than I shall have occasion to employ.

It appears to me to be sometimes assumed as a principle that two variables, functions of another, t, are proved to be equal for all values of t when it is shown that they are equal for a certain value of t, and that whenever they are equal for the same value of t their increments for the same increment of t are *ultimately* equal. But according to this principle, if two curves could be shown always to touch when they meet they must always coincide, a conclusion manifestly false. I confess I cannot see that Newton in his *Principia*, Lib. I. Prop. 40 has proved more than that if the velocities of the two bodies are equal at equal distances, the increments of those velocities for equal increments of the distances are ultimately equal: at least something additional seems required to put the proof quite out of the reach of objection. Again it is usual to speak of the condition, that the motion of a particle of fluid in contact with the surface of a solid at rest is tangential to the surface, as the same thing as the condition that the particle shall always remain in contact with the surface. That it is the same thing might be shown by means of the lemma in this article, supposing the motion continuous; but independently of proof I do not see why a particle should not move in a curve not coinciding with the surface, but touching it where it meets it. The same remark will apply to the condition that a particle which at one instant lies in a free surface, or is in contact with a solid, shall ultimately lie in the free surface, or be in contact with the solid, at the consecutive instant. I refer here to the more general case in which the solid is at rest or in motion. For similar reasons Poisson's proof of the Hydrodynamical theorem which forms the principal subject of this section has always appeared to me unsatisfactory, in fact far less satisfactory than Lagrange's. I may add that Poisson's proof, as well as Lagrange's, would apply to the case in which friction is taken into account, in which case the theorem is not true.

13. Supposing ρ to be a function of p, $\frac{1}{f'(p)}$, the ordinary equations of Hydrodynamics are

$$\frac{df(p)}{dx} = X - \frac{Du}{Dt}, \quad \frac{df(p)}{dy} = Y - \frac{Dv}{Dt}, \quad \frac{df(p)}{dz} = Z - \frac{Dw}{Dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

The forces X, Y, Z will here be supposed to be such that $Xdx + Ydy + Zdz$ is an exact differential, this being the case for any forces emanating from centres, and varying as any functions

of the distances. Differentiating the first of equations (27) with respect to y, and the second with respect to x, subtracting, putting for $\frac{Du}{Dt}$ and $\frac{Dv}{Dt}$ their values, adding and subtracting $\frac{du}{dz}\frac{dv}{dz}$, and employing the notation of Art. 2, we obtain

$$\frac{D\omega'''}{Dt} = \frac{du}{dz}\omega' + \frac{dv}{dz}\omega'' - \left(\frac{du}{dx} + \frac{dv}{dy}\right)\omega''' \text{.....}(28)$$

By treating the first and third, and then the second and third of equations (27) in the same manner, we should obtain two more equations, which may be got at once from that which has just been found by interchanging the requisite quantities. Now for points in the interior of the mass the differential coefficients $\frac{du}{dz}$, &c. will not be infinite, on account of the continuity of the motion, and therefore the three equations just obtained are a particular case of equations (25). If then $udx + vdy + wdz$ is an exact differential for any portion of the fluid when $t = 0$, that is, if ω', ω'' and ω''' are each zero when $t = 0$, it follows from the lemma of the last article that ω', ω'' and ω''' will be zero for any value of t, and therefore $udx + vdy + wdz$ will always remain an exact differential. It will be observed that it is for the same portion of fluid, not for the fluid occupying the same portion of space, that this is true, since equations (28), &c. contain the differential coefficients $\frac{D\omega'}{Dt}$ &c., and not $\frac{d\omega'}{dt}$, &c.

14. The circumstance of $udx + vdy + wdz$ being an exact differential admits of a physical interpretation which may be noticed, as it is well to view a subject of this nature in different lights.

Conceive an indefinitely small element of a fluid in motion to become suddenly solidified, and the fluid about it to be suddenly destroyed; let the form of the element be so taken that the resulting solid shall be that which is the simplest with respect to rotatory motion, namely, that which has its three principal moments about axes passing through the centre of gravity equal to each other, and therefore every axis passing through that point a principal axis, and let us enquire what will be the linear and angular motion of this element just after solidification.

By the instantaneous solidification, velocities will be suddenly generated or destroyed in the different portions of the element, and a set of mutual impulsive forces will be called into action. Let x, y, z be the co-ordinates of the centre of gravity G of the element at the instant of solidification, $x + x'$, $y + y'$, $z + z'$ those of any other point in it. Let u, v, w be the velocities of G along the three axes just before solidification, u', v', w' the relative velocities of the point whose relative co-ordinates are x', y', z'. Let $\bar{u}$, $\bar{v}$, $\bar{w}$ be the velocities of G, $u_{,}$, $v_{,}$, $w_{,}$ the relative velocities of the point above mentioned, and ω', ω'', ω''' the angular velocities just after solidification. Since all the impulsive forces are internal, we have

$$\bar{u} = u, \ \bar{v} = v, \ \bar{w} = w.$$

We have also, by the principle of the conservation of areas,

$$\Sigma m \{y'(w_{,} - w') - z'(v_{,} - v')\} = 0, \text{ \&c.},$$

m denoting an element of the mass of the element considered. But $u_{,} = \omega''z' - \omega'''y'$, u' is ultimately equal to $\frac{du}{dx}x' + \frac{du}{dy}y' + \frac{du}{dz}z'$, and similar expressions hold good for the other

quantities. Substituting in the above equations, and observing that $\Sigma m y' z' = \Sigma m' z' x' = \Sigma m x' y' = 0$, and $\Sigma m x'^2 = \Sigma m y'^2 = \Sigma m z'^2$, we have

$$\omega' = \frac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \text{ \&c.}$$

We see then that an indefinitely small element of the fluid, of which the three principal moments about the centre of gravity are equal, if suddenly solidified and detached from the rest of the fluid will begin to move with a motion simply of translation, which may however vanish, or a motion of translation combined with one of rotation, according as $u\,dx + v\,dy + w\,dz$ is, or is not an exact differential, and in the latter case the angular velocities will be the same as in Art. 2.

The principle which forms the subject of this section might be proved, at least in the case of a homogeneous incompressible fluid, by considering the change in the motion of a spherical element of the fluid in the indefinitely small time dt. This method of proving the principle would show distinctly its intimate connexion with the hypothesis of normal pressure, or the equivalent hypothesis of the equality of pressure in all directions, since the proof depends on the impossibility of an angular velocity being generated in the element in the indefinitely small time dt by the pressure of the surrounding fluid, inasmuch as the direction of the pressure at any point of the surface ultimately passes through the centre of the sphere. The proof I speak of is however less simple than the one already given, and would lead me too far from my subject.

SECTION III.

Application of a method analogous to that of Sect. 1. *to the determination of the equations of equilibrium and motion of elastic solids.*

15. ALL solid bodies are more or less elastic, as is shown by the capability they possess of transmitting sound, and vibratory motions in general. The solids considered in this section are supposed to be homogeneous and uncrystallized, so that when in their natural state the average arrangement of their particles is the same at one point as at another, and the same in one direction as in another. The natural state will be taken to be that in which no forces act on them, from which it may be shown that the pressure in the interior is zero at all points and in all directions, neglecting the small pressure depending on attractions of the nature of capillary attraction.

Let x, y, z be the co-ordinates of any point P in the solid considered when in its natural state, α, β, γ the increments of those co-ordinates at the time considered, whether the body be in a state of constrained equilibrium or of motion. It will be supposed that α, β and γ are so small that their squares and products may be neglected. All the theorems proved in Art. 2. with reference to linear and angular velocities will be true here with reference to linear and angular displacements, since these two sets of quantities are resolved according to the same laws, as long as the angular displacements are supposed to be very small. Thus, the most general displacement of a very small element of the solid consists of a displacement of translation, an angular displacement, and three displacements of extension in the direction of three rectangular axes, which may be called in this case, with more propriety than in the former, *axes of extension.* The three displacements of extension may be resolved into two displacements of shifting, each in two dimensions, and a displacement of uniform dilatation, positive or negative. The pressures about the element considered will depend on the displacements of extension only;

there may also, in the case of motion, be a small part depending on the relative velocities, but this part may be neglected, unless we have occasion to consider the effect of the internal friction in causing the vibrations of solid bodies to subside. It has been shown (Art. 7.) that the effect of this cause is insensible in the case of sound propagated through air; and there is no reason to suppose it greater in the case of solids than in the case of fluids, but rather the contrary. The capability which solids possess of being put into a state of isochronous vibration shows that the pressures called into action by small displacements depend on homogeneous functions of those displacements of one dimension. I shall suppose moreover, according to the general principle of the superposition of small quantities, that the pressures due to different displacements are superimposed, and consequently that the pressures are linear functions of the displacements. Since squares of α, β and γ are neglected, these pressures may be referred to a unit of surface in the natural state or after displacement indifferently, and a pressure which is normal to any surface after displacement may be regarded as normal to the original position of that surface. Let $-A\delta$ be the pressure corresponding to a uniform linear dilatation δ when the solid is in equilibrium, and suppose that it becomes $-mA\delta$, in consequence of the heat developed, when the solid is in a state of rapid vibration. Suppose also that a displacement of shifting parallel to the plane xy, for which $\alpha = kx$, $\beta = -ky$, $\gamma = 0$, calls into action a pressure $-Bk$ on a plane perpendicular to the axis of x, and a pressure Bk on a plane perpendicular to that of y; the pressures on these planes being equal and of opposite signs, that on a plane perpendicular to the axis of z being zero, and the tangential forces on those planes being zero, for the same reasons as in Sect. 1. It may also be shown as before that it is necessary to suppose B positive, in order that the equilibrium of the solid medium may be stable, and it is easy to see that the same must be the case with A for the same reason.

It is clear that we shall obtain the expressions for the pressures from those already found for the case of a fluid by merely putting α, β, γ, B for u, v, w, μ and $-A\delta$ or $-mA\delta$ for p, according as we are considering the case of equilibrium or of vibratory motion, the body being in the latter case supposed to be constrained only in so far as depends on the motion.

For the case of equilibrium then we have from equations (8)

$$P_1 = -A\delta - 2B\left(\frac{d\alpha}{dx} - \delta\right), \quad T_1 = -B\left(\frac{d\beta}{dz} + \frac{d\gamma}{dy}\right), \text{ \&c.,} \ldots\ldots\ldots\ldots (29)$$

δ being here $= \frac{1}{3}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right)$; and the equations of equilibrium will be obtained from (12) by putting $\frac{Du}{Dt} = 0$, $p = -A\delta$, making the same substitution as before for u, v, w and μ. We have therefore, for the equations of equilibrium,

$$\rho X + \tfrac{1}{3}(A + B)\frac{d}{dx}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) + B\left(\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right) = 0, \text{ \&c.} \ldots\ldots (30)$$

In the case of a vibratory motion, when the body is in its natural state except so far as depends on the motion, we have from equations (8)

$$P_1 = -mA\delta - 2B\left(\frac{d\alpha}{dx} - \delta\right), \quad T_1 = -B\left(\frac{d\beta}{dz} + \frac{d\gamma}{dy}\right), \text{ \&c.,} \ldots\ldots\ldots\ldots (31)$$

and the equations of motion will be derived from (12) as before, only $\frac{Du}{Dt}$ &c. must be replaced by $\frac{d^2\alpha}{dt^2}$ &c., and X, Y, Z put equal to zero. The equations of motion, then, are

$$\rho\frac{d^2\alpha}{dt^2} = \tfrac{1}{3}(mA + B)\frac{d}{dx}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) + B\left(\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right), \text{ \&c.} \ldots\ldots (32)$$

16. The conditions to be satisfied at the surface of the solid may be easily deduced from the analogous conditions in the case of a fluid with a free surface, only it will be necessary to replace the normal pressure Π by an oblique pressure, of which the components will be denoted by X_1, Y_1, Z_1. We have then, making the necessary changes in the quantities involved in (14),

$$X_1 + lA\delta + B\left\{2l\frac{d\alpha}{dx} + m\left(\frac{d\alpha}{dy} + \frac{d\beta}{dx}\right) + n\left(\frac{d\alpha}{dz} + \frac{d\gamma}{dx}\right)\right\} = 0, \text{ \&c.}$$

for the case of equilibrium, and for the case of motion such as that just considered it will only be necessary to replace A by mA in these equations. If we measure the angles of which l, m, n are the cosines from the external normal, the forces X_1, Y_1, Z_1 must be reckoned positive when l, m and n being positive, the surface of the solid is urged in the negative directions of x, y, z, and in other cases the signs must be taken conformably.

If the solid considered is in a state of constraint when at rest, and is moreover put into a state of vibration, the pressures and displacements due to these two causes must be calculated separately and added together. If m were equal to 1, they could be calculated together from the same equations.

SECTION IV.

Principles of Poisson's theory of elastic solids, and of the oblique pressures existing in fluids in motion. Objections to one of his hypotheses. Reflections on the constitution, and equations of motion of the luminiferous ether in vacuum.

17. IN the twentieth *Cahier* of the *Journal de l'Ecole Polytechnique* may be found a memoir by Poisson, entitled *Mémoire sur les Equations générales de l'Equilibre et du Mouvement des Corps solides élastiques et des Fluides*, which contains the substance of two memoirs presented by him to the Academy, brought together with some additions. In this memoir the author treats principally of the equations of equilibrium and motion of elastic solids, of the equations of equilibrium of fluids, with reference especially to capillary attraction, and of the equations of motion of fluids supposing the pressure not to be equal in all directions.

It is supposed by Poisson that all bodies, whether solid or fluid, are composed of ultimate molecules, separated from each other by vacant spaces. In the cases of an uncrystallized solid in its natural state, and of a fluid in equilibrium, he supposes that the molecules are arranged irregularly, and that the average arrangement is the same in all directions. These molecules he supposes to act on each other with forces, of which the main part is a force in the direction of the line joining the centres of gravity, and varying as some function of the distance of these points, and the remainder a secondary force, or it may be two secondary forces, depending on the molecules not being mathematical points. He supposes that it is on these secondary forces that the solidity of solid bodies depends. He supposes however that in calculating the pressures these secondary forces may be neglected, partly because they become insensible at much smaller distances than the main part of the forces, and partly because they act, on the average, alike in all directions. He supposes that the molecular force decreases very rapidly as the distance increases, yet not so rapidly but that the sphere in which the molecular action is sensible contains an immense number of molecules. He supposes consequently that in estimating the resultant force of a hemisphere of the medium on a molecule in the centre of its base the action of the neighbouring molecules, which are situated irregularly, may be neglected compared with the action of those

more remote, of which the average may be taken. The consequence of this supposition of course is that the total action is normal to the base of the hemisphere, and sensibly the same for one molecule as for an adjacent one.

The rest of the reasoning by which Poisson establishes the equations of motion and equilibrium of elastic solids is purely mathematical, sufficient data having been already assumed. It might appear that the reasoning in Art. 16 of his memoir, by which the expression for N is simplified, required the fresh hypothesis of a symmetrical arrangement of the molecules; but it really does not, being admissible according to the principle of averages. Taking for the natural state of the body that in which the pressure is zero, the equations at which Poisson arrives contain only *one* unknown constant k, whereas the equations of Sect. III. of this paper contain *two*, A or mA and B. This difference depends on the assumption made by Poisson that the irregular part of the force exerted by a hemisphere of the medium on a molecule in the centre of its base may be neglected in comparison with the whole force. As a result of this hypothesis, Poisson finds that the change in direction, and the proportionate change in length, of a line joining two molecules are continuous functions of the co-ordinates of one of the molecules and the angles which determine the direction of the line; whereas in Sect. III., if we adopt the hypothesis of ultimate molecules at all, it is allowable to suppose that these quantities vary irregularly in passing from one pair of molecules to an adjacent pair. Of course the equations of Sect. III. ought to reduce themselves to Poisson's equations for a particular relation between A and B. Neglecting the heat developed by compression, as Poisson has done, and therefore putting $m = 1$, this relation is $A = 5B$.

18. Poisson's theory of fluid motion is as follows. The time t is supposed to be divided into a number n of equal parts, each equal to τ. In the first of these the fluid is supposed to be displaced as an elastic solid would be, according to Poisson's previous theory, and therefore the pressures are given by the same equations. If the causes producing the displacement were now to cease, the fluid would re-arrange itself, so that the average arrangement about each point should be the same in all directions after a very short time. During this time, the pressures would have altered, in an unknown manner, from those corresponding to a displaced solid to a normal pressure equal to $p + \frac{Dp}{Dt}\tau$, the pressures during the alteration involving an unknown function of the time elapsed since the end of the interval τ. Another displacement and another re-arrangement may now be supposed to take place, and so on. But since these very small relative motions will take place independently of each other, we may suppose each displacement to begin at the expiration of the time during which the preceding one is supposed to remain, and we may suppose each re-arrangement to be going on during the succeeding displacements. Supposing now n to become infinite, we pass to the case in which the fluid is supposed to be continually beginning to be displaced as a solid would, and continually re-arranging itself so as to make the average arrangement about each point the same in all directions.

Poisson's equations (9), page 152, which are applicable to the motion of a liquid, or of an elastic fluid in which the change of density is small, agree with equations (12) of this paper. For the quantity ψt is the pressure p which would exist at any instant if the motion were then to cease, and the increment, $\frac{d\psi t}{dt}\tau$ or $\frac{Dp}{Dt}\tau$, of this quantity in the very small time τ will depend only on the increment, $\frac{d\chi t}{dt}\tau$ or $\frac{D\rho}{Dt}\tau$, of the density χt or ρ. Consequently the value of $\frac{d\psi t}{dt}\tau$ will be the same as if the density of the particle considered passed from χt to $\chi t + \frac{d\chi t}{dt}\tau$ in the time τ by a uniform motion of dilatation. I suppose that according to Poisson's views such a motion would not require a re-arrangement of the molecules, since the pressure remains equal

in all directions. On this supposition we shall get the value of $\frac{d\psi t}{dt}$ from that of $R_{,}' - K$ in the equations of page 140 by putting $\frac{du}{dx} = \frac{dv}{dy} = \frac{dw}{dz} = -\frac{1}{3\chi t}\frac{d\chi t}{dt}$. We have therefore

$$a\frac{d\psi t}{dt} = \frac{a}{3}(K - 5k)\frac{d\chi t}{\chi t dt}.$$

Putting now for $\beta + \beta'$ its value $2ak$, and for $\frac{1}{\chi t}\frac{d\chi t}{dt}$ its value given by equation (2), the expression for ϖ, page 152, becomes

$$\varpi = p + \frac{a}{3}(K + k)\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right).$$

Observing that $a(K + k) = \beta$, this value of ϖ reduces Poisson's equations (9) to the equations (12) of this paper.

Poisson himself has not made this reduction of his equations, nor any equivalent one, so that his equations, as he has left them, involve two arbitrary constants. The reduction of these two to one depends on the assumption that a uniform expansion of any particle does not require a re-arrangement of the molecules, as it leaves the pressure still equal in all directions. If we do not make this assumption, but retain the two arbitrary constants, the equations will be the same as those which would be obtained by the method of this paper, supposing the quantity κ of Art. 3 not to be zero.

19. There is one hypothesis made in the common theory of elastic solids, the truth of which appears to me very questionable. That hypothesis is the one to which I have already alluded in Art. 17, respecting the legitimacy of neglecting the irregular part of the action of the molecules in the immediate neighbourhood of the one considered, in comparison with the total action of those more remote, which is regular. It is from this hypothesis that it follows as a result that the molecules are not displaced among one another in an irregular manner, in consequence of the directive action of neighbouring molecules. Now it is obvious that the molecules of a fluid admit of being displaced among one another with great readiness. The molecules of solids, or of most solids at any rate, must admit of new arrangements, for most solids admit of being bent, permanently, without being broken. Are we then to suppose that when a solid is constrained it has no *tendency* to relieve itself from the state of constraint, in consequence of its molecules *tending towards* new relative positions, provided the amount of constraint be very small? It appears to me to be much more natural to suppose *à priori* that there should be some such tendency.

In the case of a uniform dilatation or contraction of a particle, a re-arrangement of its molecules would be of little or no avail towards relieving it from constraint, and therefore it is natural to suppose that in this case there is little or no tendency towards such a re-arrangement. It is quite otherwise, however, in the case of what I have called a displacement of shifting. Consequently B will be less than if there were no tendency to a re-arrangement. On the hypothesis mentioned in this article, of which the absence of such tendency is a consequence, I have said that a relation has been found between A and B, namely $A = 5B$. It is natural then to expect to find the ratio of A to B greater than 5, approaching more nearly to 5 as the solid considered is more hard and brittle, but differing materially from 5 for the softer solids, especially such as Indian rubber, or, to take an extreme case, jelly. According to this view the relation $A = 5B$ belongs only to an ideal elastic solid, of which the solidity, or whatever we please to call the property considered, is absolutely perfect.

To show how implicitly the common theory of elasticity seems to be received by some, I may mention that MM. Lamé and Clapeyron mention Indian rubber among the substances to which it would seem they consider their theory applicable*. I do not know whether the coefficient of elasticity, according to that theory, has been determined experimentally for Indian rubber, but one would fancy that the cubical compressibility thence deduced, by a method which will be seen in the next article, would turn out comparable with that of a gas.

20. I am not going to enter into the solution of equations (30), but I wish to make a few remarks on the results in some simple cases.

If k be the cubical contraction due to a uniform pressure P, then will

$$k = \frac{3P}{A}.$$

If a wire or rod, of which the boundary is any cylindrical surface, be pulled in the direction of its length by a force of which the value, referred to a unit of surface of a section of the rod, in P, the rod will extend itself uniformly in the direction of its length, and contract uniformly in the perpendicular direction; and if e be the extension in the direction of the length, and c the contraction in any perpendicular direction, both referred to a unit of length, we shall have

$$e = \frac{A+B}{3AB}P, \quad c = \frac{A-2B}{6AB}P:$$

also, the cubical dilatation $= e - 2c = \dfrac{P}{A}$.

If a cylindrical wire of radius r be twisted by a couple of which the moment is M, and if θ be the angle of torsion for a length z of the wire, we shall have

$$\theta = \frac{2Mz}{\pi B r^4}.$$

The expressions for k, c, e and θ, and of course all expressions of the same nature, depend on the reciprocals of A and B. Suppose now the value of e, or θ, or any similar quantity not depending on A alone, be given as the result of observation. It will easily be conceived that we might find very nearly the same value for B whether we supposed $A = 5B$ or $A = nB$, where n may be considerably greater than 5, or even infinite. Consequently the observation of two such quantities, giving very nearly the same value of B, might be regarded as confirming the common equations.

If we denote by E the coefficient of elasticity when A is supposed to be equal to $5B$ we have, neglecting the atmospheric pressure†,

$$e = \frac{2P}{5E}, \quad \theta = \frac{2Mz}{\pi E r^4}.$$

If now we denote by E_1 the value of E deduced from observation of the value of e, and by E_2 the value of E obtained by observing the value of θ, or else, which comes to the same, by observing the time of oscillation of a known body oscillating by torsion, we shall have

$$\frac{2}{5E_1} = \tfrac{1}{3}\left(\frac{1}{A} + \frac{1}{B}\right), \quad E_2 = B, \quad \text{whence } \frac{1}{A} = \frac{6}{5E_1} - \frac{1}{E_2}.$$

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* *Mémoires présentés à l'Institut*, Tom. IV. p. 469.

† Lamé, *Cours de Physique*, Tom. I.

If A be greater than $5B$, E_1 ought to be a little greater than E_2. This appears to agree with observation. Thus the following numbers are given by M. Lamé* $E_1 = 8000$, $E_2 = 7500$ for iron; $E_1 = 2510$, $E_2 = 2250$ for brass†. The difference between the values of E_1 and E_2 is attributed by M. Lamé to the errors to which the observation of the small quantity e is liable. If the above numbers may be trusted, we shall have

$$A = 60000,\quad B = 7500,\quad \frac{A}{B} = 8 \text{ for iron};$$

$$A = 29724,\quad B = 2250,\quad \frac{A}{B} = 13{\cdot}21 \text{ for brass.}$$

The cubical contraction k is almost too small to be made the subject of direct observation‡, it is therefore usually deduced from the value of e, or from the coefficient of elasticity E found in some other way. On the supposition of a single coefficient E, we have $\frac{k}{e} = \frac{3}{2}$, but retaining the two, A and B, we have $\frac{k}{e} = \frac{9B}{A+B} = 9\left(1 + \frac{B}{A}\right)^{-1}\frac{B}{A}$, which will differ greatly from $\frac{3}{2}$ if $\frac{A}{B}$ be much greater than 5. The whole subject therefore requires, I think, a careful examination, before we can set down the values of the coefficients of cubical contraction of different substances in the list of well ascertained physical data. The result, which is generally admitted, that the ratio of the velocity of propagation of normal, to that of tangential vibrations in a solid is equal to $\sqrt{3}$, is another which depends entirely on the supposition that $A = 5B$. The value of m, again, as deduced from observation, will depend upon the ratio of A to B; and it would be highly desirable to have an accurate list of the values of m for different substances, in hopes of thereby discovering in what manner the action of heat on those substances is related to the physical constants belonging to them, such as their densities, atomic weights, &c.

The observations usually made on elastic solids are made on slender pieces, such as wires, rods, and thin plates. In such pieces, all the particles being at no great distance from the surface, it is easy to see that when any small portion is squeezed in one direction it has considerable liberty of expanding itself in a direction perpendicular to this, and consequently the results must depend mainly on the value of B, being not very different from what they would be if A were infinite. This is not so much the case with thick, stout pieces. If therefore such pieces could be put into a state of isochronous vibration, so that the musical notes and nodal lines could be observed, they would probably be better adapted than slender pieces for determining the value of mA. The value of m might be determined by comparing the value of mA, deduced from the observation of vibrations, with the value of A, deduced from observations made in cases of equilibrium, or, perhaps, of very slow motion.

21. The equations (32) are the same as those which have been obtained by different authors as the equations of motion of the luminiferous ether in vacuum. Assuming for the present that the equations of motion of this medium ought to be determined on the same principles as the equations of motion of an elastic solid, it will be necessary to consider whether the equations (32) are altered by introducing the consideration of a uniform pressure Π existing in the medium

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* Lamé, *Cours de Physique*, Tom. I.

† These numbers refer to the French units of length and weight.

‡ I find however that direct experiments have been made by Prof. Oersted. According to these experiments the cubical compressibility of solids which would be obtained from Poisson's theory is in some cases as much as 20 or 30 times too great. See the *Report of the British Association* for 1833, p. 353, or *Archives des découvertes, &c.* for 1834, p. 94.

when in equilibrium; for we have evidently no right to assume, either that no such pressure exists, or, supposing it to exist, that the medium would expand itself but very slightly if it were removed. It will now no longer be allowable to confound the pressure referred to a unit of surface as it was, in the position of equilibrium of the medium, with the pressure referred to a unit of surface as it actually is. The latter mode of referring the pressure is more natural, and will be more convenient. Let the pressure, referred to a unit of surface as it is, be resolved into a normal pressure $\Pi + p_1$ and a tangential pressure t_1. All the reasoning of Sect. III. will apply to the small forces p_1 and t_1; only it must be remembered that in estimating the whole oblique pressure a normal pressure Π must be compounded with the pressures given by equations (31). In forming the equations of motion, the pressure Π will not appear, because the resultant force due to it acting on the element of the medium which is considered is zero. The equations (32) will therefore be the equations of motion required.

If we had chosen to refer the pressure to a unit of surface in the original state of the surface, and had resolved the whole pressure into a pressure $\Pi + p_1$ normal to the original position of the surface, and a pressure t_1 tangential to that position, the reasoning of Sect. III. would still have applied, and we should have obtained the same expressions as in (31) for the pressures P_1, T_1, &c., but the numerical value of A would have been different. According to this method, the pressure Π would have appeared in the equations of motion. It is when the pressures are measured according to the method which I have adopted that it is true that the equilibrium of the medium would be unstable if either A or B were negative. I must here mention that from some oversight the right-hand sides of Poisson's equations, at page 68 of the memoir to which I have referred, are wrong. The first of these equations ought to contain $\frac{\Pi}{\rho}\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)$, instead of $\frac{\Pi}{\rho}\frac{d^2u}{dx^2}$, and similar changes must be made in the other two equations.

It is sometimes brought as an objection to the equations of motion of the luminiferous ether, that they are the same as those employed for the motion of solid bodies, and that it seems unnatural to employ the same equations for substances which must be so differently constituted. It was, perhaps, in consequence of this objection that Poisson proposes, at page 147 of the memoir which I have cited, to apply to the calculation of the motion of the luminiferous ether the same principles, with a certain modification, as those which he employed in arriving at his equations (9) page 152, *i. e.* the equations (12) of this paper. That modification consists in supposing that a certain function of the time $\phi(t)$ does not vary very rapidly compared with the variation of the pressure. Now the law of the transmission of a motion transversal to the direction of propagation depending on equations (12) of this paper is expressed, in the simplest case, by the equation (24); and we see that this law is the same as that of the transmission of heat, a law extremely different from that of the transmission of vibratory motions. It seems therefore unlikely that these principles are applicable to the calculation of the motion of light, unless the modification which I have mentioned be so great as wholly to alter the character of the motion, that is, unless we suppose the pressure to vary extremely fast compared with the function $\phi(t)$, whereas in ordinary cases of the motion of fluids the function $\phi(t)$ is supposed to vary extremely fast compared with the pressure.

Another view of the subject may be taken which I think deserves notice. Before explaining this view however it will be necessary to define what I mean in this paragraph by the word *elasticity*. There are two distinct kinds of elasticity; one, that by which a body which is uniformly compressed tends to regain its original volume, the other, that by which a body which is constrained in a manner independent of compression tends to assume its original form. The constants A and B of Sect. III. may be taken as measures of these two kinds of elasticity. In the present paragraph, the word will be used to denote the second kind. Now many highly

elastic substances, as iron, copper, &c., are yet to a very sensible degree plastic. The plasticity of lead is greater than that of iron or copper, and, as appears from experiment, its elasticity less. On the whole it is probable that the greater the plasticity of a substance the less its elasticity, and *vice versâ*, although this rule is probably far from being without exception. When the plasticity of the substance is still further increased, and its elasticity diminished, it passes into a viscous fluid. There seems no line of demarcation between a solid and a viscous fluid. In fact, the practical distinction between these two classes of bodies seems to depend on the intensity of the *extraneous* force of gravity, compared with the intensity of the forces by which the parts of the substance are held together. Thus, what on the Earth is a soft solid might, if carried to the Sun, and retained at the same temperature, be a viscous fluid, the force of gravity at the surface of the Sun being sufficient to make the substance spread out and become level at the top: while what on the Earth is a viscous fluid might on the surface of Pallas be a soft solid. The gradation of viscous, into what are called perfect fluids seems to present as little abruptness as that of solids into viscous fluids; and some experiments which have been made on the sudden conversion of water and ether into vapour, when enclosed in strong vessels and exposed to high temperatures, go towards breaking down the distinction between liquids and gases.

According to the law of continuity, then, we should expect the property of elasticity to run through the whole series, only, it may become insensible, or else may be masked by some other more conspicuous property. It must be remembered that the elasticity here spoken of is that which consists in the tangential force called into action by a displacement of continuous sliding: the displacements also which will be spoken of in this paragraph must be understood of such displacements as are independent of condensation or rarefaction. Now the distinguishing property of fluids is the extreme mobility of their parts. According to the views explained in this article, this mobility is merely an extremely great plasticity, so that a fluid admits of a *finite*, but exceedingly small amount of constraint before it will be relieved from its state of tension by its molecules assuming new positions of equilibrium. Consequently the same oblique pressures can be called into action in a fluid as in a solid, provided the amount of relative displacement of the parts be exceedingly small. All we know for certain is that the effect of elasticity in fluids, (elasticity of the second kind be it remembered,) is quite insensible in cases of equilibrium, and it is probably insensible in all ordinary cases of fluid motion. Should it be otherwise, equations (8) and (12) will not be true, or only approximately true. But a little consideration will show that the property of elasticity may be quite insensible in ordinary cases of fluid motion, and may yet be that on which the phenomena of light entirely depend. When we find a vibrating string, the small extent of vibration of which can be actually seen, filling a whole room with sound, and remember how rapidly the intensity of the vibrations of the air must diminish as the distance from the string increases, we may easily conceive how small in general must be the amount of the relative motion of adjacent particles of air in the case of sound. Now the extent of the vibration of the ether, in the case of light, may be as small compared with the length of a wave of light as that of the air is compared with the length of a wave of sound: we have no reason to suppose it otherwise. When we remember then that the length of a wave of sound in air varies from some inches to several feet, while the greatest length of a wave of light is about .00003 of an inch, it is easy to imagine that the *relative* displacement of the particles of ether may be so small as not to reach, nor even come near to the greatest relative displacement which could exist without the molecules of the medium assuming new positions of equilibrium, or, to keep clear of the idea of molecules, without the medium assuming a new arrangement which might be permanent.

It has been supposed by some that air, like the luminiferous ether, ought to admit of transversal vibrations. According to the views of this article such would, mathematically speaking, be the case; but the extent of such vibrations would be necessarily so very small as to render them utterly insensible, unless we had organs with a delicacy equal to that of the retina adapted to receive them.

It has been shown to be highly probable that the ratio of A to B increases rapidly according as the medium considered is softer and more plastic. For fluids therefore, and among them for the luminiferous ether, we should expect the ratio of A to B to be extremely great. Now if N be the velocity of propagation of normal vibrations in the medium considered in Sect. III., and T that of transversal vibrations, it may be shown from equations (32) that

$$N^2 = \frac{mA + 4B}{3\rho}, \quad T^2 = \frac{B}{\rho}.$$

This is very easily shown in the simplest case of plane waves: for if $\beta = \gamma = 0$, $\alpha = f(x)$, the equations (32) give $\rho \frac{d^2\alpha}{dt^2} = \frac{1}{3}(mA + 4B)\frac{d^2\alpha}{dx^2}$, whence $\alpha = \phi(Nt - x) + \psi(Nt + x)$; and if $\alpha = \gamma = 0$, $\beta = f(x)$, the same equations give $\rho \frac{d^2\beta}{dt^2} = B\frac{d^2\beta}{dx^2}$, whence $\beta = \zeta(Tt - x) + \xi(Tt + x)$. Consequently we should expect to find the ratio of N to T extremely great. This agrees with a conclusion of the late Mr. Green's*. Since the equilibrium of any medium would be unstable if either A or B were negative, the least possible value of the ratio of N^2 to T^2 is $\frac{4}{3}$, a result at which Mr. Green also arrived. As however it has been shown to be highly probable that $A > 5B$ even for the hardest solids, while for the softer ones $\frac{A}{B}$ is much greater than 5, it is probable that $\frac{N}{T}$ is greater than $\sqrt{3}$ for the hardest solids, and much greater for the softer ones.

If we suppose that in the luminiferous ether $\frac{A}{B}$ may be considered infinite, the equations of motion admit of a simplification. For if we put $mA\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) = -p$ in equations (32), and suppose mA to become infinite while p remains finite, the equations become

$$\left.\begin{aligned} &\rho\frac{d^2\alpha}{dt^2} = -\frac{dp}{dx} + B\left(\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2}\right), \text{ \&c.} \\ &\text{and} \quad \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots (33)$$

When a vibratory motion is propagated in a medium of which (33) are the equations of motion, it may be shown that $p = \psi(t)$ if the medium be indefinitely extended, or else if there be no motion at its boundaries. In considering therefore the transmission of light in an uninterrupted vacuum the terms involving p will disappear from equations (33); but these terms are, I believe, important in explaining Diffraction, which is the principal phenomenon the laws of which depend only on the equations of motion of the lumniferous ether in vacuum. It will be observed that putting $A = \infty$ comes to the same thing as regarding the ether as incompressible with respect to those motions which constitute Light.

G. G. STOKES.

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* *Cambridge Philosophical Transactions*, Vol. VII. Part I. p. 2.