

November 2016

Prospects for Infrared Quantum Gravity: From Cosmology to Black Holes

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PROSPECTS FOR INFRARED QUANTUM GRAVITY: FROM COSMOLOGY TO BLACK HOLES

A Dissertation Presented

by

BASEM MAHMOUD EL-MENOUFI

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2016

Physics Department

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I have not failed. I've just found 10,000 ways that won't work.

ACKNOWLEDGMENTS

First and for most, I thank my Lord Allah whose grace and mercy were the only reason I was able to complete this dissertation.

I would also like to express my gratitude towards everyone who helped me in writing this dissertation.

On the faculty side, I would like to thank Professor John F. Donoghue for his supervision during my Ph.D. Also, I am grateful to Professors Jennie Traschen, David Kastor, Lorenzo Sorbo, Barry Holstein, Eugene Golowich, Michael Ramsey-Musolf for our collaboration and their classroom instruction during various stages in my Ph.D. Last but not least, many thanks to Professor Houjun Mo for being a member of my committee.

I am quite indebted to Gregory Ovanesyan, Ben Ett and Graham White for their collaboration and emotional support during a tough stage in my Ph.D. I would also like to thank my friends Dominique Cambou, Mohamed Anber, Selcuk Yassar, Sheema Rahmanseresht, Alphan Aksoyoglu, Peker Milas, Amir Azadi.

Many thanks to the physics department staff Jane Knapp, Mary Ann Ryan, Kristine Reopell and Ingrid Pollard.

ABSTRACT

PROSPECTS FOR INFRARED QUANTUM GRAVITY: FROM COSMOLOGY TO BLACK HOLES

SEPTEMBER 2016

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Although perturbatively non-renormalizable, general relativity is a perfectly valid quantum theory at low energies. Treated as an effective field theory one is able to make genuine quantum predictions by applying the conventional rules of quantum field theory. The low energy degrees of freedom and couplings of quantum gravity are fully dictated by the symmetries of general relativity. To realize the full EFT treatment one has to supplement the theory with experimental input necessary to fix the Wilson coefficients of the most general Lagrangian. In spite of the fact that this is not feasible, one can still extract the leading quantum corrections which are precisely induced by the low-energy fluctuations of the massless graviton. The long-distance portion of loops is non-analytic in momentum space or equivalently non-local in position space. In this thesis we are going to study the construction, properties and phenomenology of these non-local effects.

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CHAPTER 1

INTRODUCTION

What is quantum gravity? Do we have any hints about what it might turn out to be? Can we learn something about quantum gravity from our current knowledge? These questions are arguably the most important in theoretical physics. Besides quantum gravity there are a few other pressing questions awaiting to be resolved. The nature of dark matter/energy, the matter anti-matter asymmetry and the origin of neutrino masses reside on top of the list. The answer to these questions will inevitably lead us to new physics beyond the Standard Model (SM) but most likely without any drastic modifications to the underlying principles of quantum field theory. On the contrary, many believe that the precise formulation of quantum gravity will lead to a totally new insight into both sets of principles upon which quantum mechanics and general relativity are based.

One inquires at this stage about the intrinsic complexity in quantum gravity. Conventionally the problem is stated as follows: Take general relativity and run it through the machinery of quantum field theory and you get non-sense. The technical phrase used here is that gravity is a non-renormalizable field theory. When early workers embarked on constructing the SM, they (successfully) invoked renormalizability as a condition a candidate field theory must satisfy. This criterion is indeed central to the construction of the SM but with a bit of skepticism one should ask: is it really that fundamental? With the advent of the Wilsonian approach to quantum field theory, it became clear that renormalizability should not be viewed as a principle of nature. In other words, all our theories are effective. They are only capable of describing the

physics at certain length/energy scales. Once we go beyond the regime of validity of the effective theory, new degrees of freedom and interactions start to appear. This is the modern point of view held in the physics community.

In [1] , John F. Donoghue raised the following question: Are there any conflicts between gravity and quantum mechanics at the energy scales that are presently accessible? Given that we have tested both theories over a wide range of length scales with extreme success, the answer to the last question must be negative. This point of view opened the door to treat quantum gravity as an effective field theory (GEFT) with the alternative definition of renormalizability in an EFT: The theory is renormalizable order by order in the power counting. In particular, GEFT is a fully predictive theory. One of the major powers of EFT techniques is the ability to separate the effects of the unknown high energy physics from the known low energy physics. When viewed at low energies, high energy physics decouples and merely introduces local interactions involving the light degrees of freedom. The last trace of the unknown physics is encoded in the Wilson coefficients of the local effective Lagrangian, which could be measured experimentally where all low energy observables are expressed in terms of the measured parameters. On the other hand, the low energy fluctuations of massless (light) fields are non-analytic in momentum space or non-local in position space. Being non-analytic and finite, they clearly do not renormalize any of the Wilson coefficients in the effective Lagrangian. In other words, they represent reliable predictions associated with low energy dynamics.

The only obstacle to fully realizing the EFT program is the lack of precise measurements of the Wilson coefficients. Nevertheless, the local operators in the effective theory only affect the short-distance physics. At long-distances however, the non-analytic portion of loops dominate any observable. In particular, at the one-loop order only the knowledge of Newton's constant G_N is required to extract the leading modifications to the Newtonian potential. One elegant prediction of quantum gravity

is the leading correction to the Newtonian potential computed in [1, 2]

$$V(r) = -\frac{GMm}{r} \left[1 + \frac{3G(M+m)}{rc^2} + \frac{41}{10\pi^2} \frac{l_p^2}{r^2} \right] . \quad (1.1)$$

Unfortunately, the quantum correction shown above is extremely small to be detected. The deviation from the Newtonian inverse square law is only appreciable at the Planck length. Should we stop considering these non-local effects? Absolutely not. This thesis is primarily concerned with showing the wealth of physical phenomena where non-local effects play a central role. In particular, the following questions constitutes the backbone of the dissertation:

- How to incorporate non-local effects in the effective action of quantum gravity?
- What are the properties of non-local field theories?
- What are the cosmological implications of non-locality?
- A statistical description of black hole thermodynamics can tell us a lot about quantum gravity. What can we learn from the infrared about the entropy of black holes?

Here, I will describe some of the work that has been done to answer the above questions.

Part I: Construction and properties of non-local Lagrangians

The building block of non-local field theories is *form factors* which encode the low-energy dynamics of massless particles. What are the properties of these form factors in curved space? This is mandatory to enable us to construct non-local covariant actions. With this goal in mind, I revisited the conformal anomaly of quantum electrodynamics (QED). QED with massless fermions is conformally invariant on the classical level but the classical symmetry is known to be broken by quantum effects.

The conventional derivation of anomalies highlights the UV behavior of Feynman graphs as the origin of the anomaly. This picture is inaccurate, since the symmetry is truly broken due to the running of the electric charge which is an infrared effect.

In chapter 2, I compute the correction to the Maxwell action due to the one-loop effects of massless fermions and charged scalars. If the action is expressed in quasi-local form, it is easy to see how the anomalous contribution to the trace of the energy-momentum tensor arises. Although the anomalous trace is a local operator, the full energy-momentum tensor is a non-local object. To construct the latter, the charged particles are integrated out from the path integral. Focusing on the flat-space limit, I was able to define the full non-local energy-momentum tensor which exhibits some nice properties. One important aspect in this work was the infrared safety of the action if the gauge field was assumed to satisfy the classical equations of motion. This non-local coupling of photons to gravity violates some aspects of the Equivalence Principle since it modifies the classical prediction of light bending to exhibit dependence on the photon energy.

Next in chapter 3, I moved on to construct the non-local action on a curved spacetime background. In particular, I aimed to study the covariant nature of the non-local form factors which turned out to be highly non-trivial. One puzzle that revealed itself in the perturbative calculation is the apparent absence of some terms needed to build the covariant non-local form factors. The existing literature on non-local field theories, as little as it is, do not provide an accurate description of this particular issue. I initiated a rigorous systematic way of applying a *non-linear completion* procedure, which enabled me to express the effective action as an expansion in curvatures. I also introduced the technique of *counter-terms* crucial to realize the non-linear completion procedure. Moreover, a convenient choice of the curvature basis proved indispensable. Employing the Weyl tensor revealed the relation between the coefficients of different operators and the electric charge beta function. Finally, I sorted out a tension that

has been accumulating in the literature regarding the non-local action that yields the trace anomaly.

Part II: Cosmological singularities and cosmic magnetic fields

Cosmology is perhaps the most interesting and phenomenologically relevant arena to study modifications to Einstein gravity. This is precisely the content of my work in Part II. The initial focus in chapter 4 is mainly on the singularity problem omnipresent in classical general relativity. As proven by the singularity theorems [3], they are unavoidable in both cosmological and black hole spacetimes. The most important physical assumption of the theorems is that gravity be described by Einstein equations and that the matter sources satisfy the strong energy condition. Although singularities are conventionally thought to be a pathology that will only get resolved with the ultimate formulation of quantum gravity, I argue that this might not be the case. In fact, I showed that it is possible that the quantum-induced modifications to Einstein gravity lead to singularity avoidance. This is brought about by the leading logarithmic non-locality induced in the effective action by massless vacuum fluctuations.

Precisely I calculated the one-loop gravitational effective action due to a massless minimally coupled free scalar field. To pursue my investigation, I had to introduce a *non-linear completion* technique by which the perturbative calculation of the effective action is expressed as an expansion in the curvatures. This method is described in detail in chapter 3. This step is crucial to fully study the physics in the non-linear regime of gravity. The universality of the non-analytic pieces allowed me to provide the appropriate results for other spin fields including gravitons. The effective action I studied are due to massless particles, and hence are genuinely non-local in position space. Another field theoretic subtlety is the issue of *causality*. In this set-up, the resulting equations of motion are integro-differential and to obtain reliable results I

had to ensure the modified Friedmann equation is causal and real. This is done by implementing the in-in formalism to find the correct causal prescription for the non-local kernel. Moreover, I explored writing the effective action in a different curvature basis which better encodes the properties of the action under Weyl transformations. This helped me explain why the modified cosmological evolution due to conformally invariant fields (e.g. fermions) is insensitive to the renormalization scale. This is interesting enough in its own right since above the electroweak scale the SM could be made conformal by simply adding the conformal coupling term to the Higgs potential. Being free from the arbitrary renormalization scale makes the phenomenology fully trustworthy.

In chapter 5, I study the possibility for achieving magneto-genesis during inflation by employing the one-loop effective action of massless QED obtained in chapter 3. Cosmic magnetic fields have been detected at various length scales. In particular, the primordial origin of the extra-galactic medium field is very attractive. The non-local action found in chapter 3 encodes the conformal anomaly of QED which is crucial to avoid the vacuum preservation in classical electromagnetism. If electroweak symmetry is not broken during inflation, massless fermions are ubiquitous at the inflationary scale. It is then rather important to systematically study the possibility of magneto-genesis through the SM conformal anomaly. This does not require any physics beyond the SM. In particular, I found a blue spectrum for the magnetic field with spectral index $n_B = 2 - \alpha$ where α is a small number that depends on both the number of e-folds during inflation as well as the coefficient of the one-loop beta function. The sign of the beta function in particular has important implications on the final result. Carefully following the evolution of the coherence scale shows that a reheating temperature lower than 100GeV is required to obtain present day magnetic fields consistent with the lower bound on the fields in the intergalactic medium.

Part III: Quantum-induced correction to the Bekenstein-Hawking entropy

An understanding of the statistical description of the entropy associated to black holes is one of the holy grails of modern physics. It is widely believed that unraveling this bit of physics would give strong hints on the formulation of quantum gravity. It has been known for a very long time that corrections to Einstein gravity modifies the entropy of black holes by modifying the Bekenstein-Hawking area law. The effect of local corrections can easily be studied, for example, quadratic gravity corrections contributes a trivial constant to the entropy. One can essentially argue based on simple dimensional analysis that local curvature corrections would only yield corrections which are *analytic* in the area of the event horizon. Nevertheless, a full knowledge of the UV completion of gravity is required to quantify the entropy corrections arising from local interactions. One then should ask: What about quantum gravity in the infrared? This is the question I answered in chapter 6.

For the special class of Kerr-Schild (KS) spacetimes, I performed an exact computation of the non-local gravitational effective action which results from the long distance propagation of massless matter fields and gravitons. I changed my technique from previous chapters and rather employed the non-local heat kernel expansion. The reason is simple: I wanted to compare the result to that obtained using Feynman graphs. I found that this method captures some portions of the action which slip detection using Feynman graphs. Most notably, KS spacetimes encompass both the Schwarzschild as well as the Kerr solutions. In fact, the Kerr-Schild ansatz was the main reason Roy Kerr was able to obtain his famous solution in the first place. The central point in my work is the ability to *exactly* resolve the heat kernel on the KS background spacetime. The effective action is then obtained at face value which is a non-trivial result.

Having an exact result for the action enabled me to study the general covariance of the form factors. For KS spacetimes, the form factors are uniquely defined by

flat-space operators. This in particular showed that computing the effective action over a fixed spacetime leads to tractable and simpler results. My second and most important study is the resulting correction to the entropy of a Schwarzschild black hole. In the Euclidean approach one determines the partition function from the Euclidean path integral. The *scaling* properties of the partition function follows that of the effective action. From the latter, I was able to deduce how the entropy would scale under an arbitrary scale transformation. I found that the correction to the area law exhibits a logarithmic dependence on the horizon area. The latter arises precisely from the non-local portion of the action. My study opens the door to better understand the quantum corrections to the thermo-dynamic behavior of black holes. In particular, the quantum-induced correction to the Bekenstein-Hawking entropy hints at the possibility of rendering a black hole thermodynamically stable.

CHAPTER 2

QED TRACE ANOMALY, NON-LOCAL LAGRANGIANS AND QUANTUM EQUIVALENCE PRINCIPLE VIOLATIONS

2.1 Introduction

We are used to dealing with local effective Lagrangians. However, one can also use non-local effective actions to summarize the one-loop predictions of a theory containing light or massless particles (see e.g. [4]). The non-locality occurs because light particles propagate a long distance within loop processes. In this chapter, we explore some of the properties of such non-local effective actions in a simple context - that of the energy momentum tensor in gauge theories with massless particles.

One of the simplest and most instructive derivations of the QED trace anomaly is also one of the least known. Let us present a quick treatment of this derivation, which we will then explore in more detail in the body of this chapter. In the massless limit, the classical electromagnetic action with charged matter is invariant under the continuous rescaling

$$\begin{aligned} A_\mu(x) &\rightarrow A'_\mu(x') = \lambda^{-1} A_\mu(x), & \psi(x) &\rightarrow \psi'(x') = \lambda^{-3/2} \psi(x) \\ \phi(x) &\rightarrow \phi'(x') = \lambda^{-1} \phi(x) \quad . \end{aligned} \tag{2.1}$$

with $x' = \lambda x$. Associated with this symmetry is a scale or dilatation current¹

¹There are subtleties associated with the exact relation between the dilatation current and the energy-momentum tensor [5, 6] which we briefly discuss in an appendix

$$J_D^\mu = x_\nu T^{\mu\nu} \quad (2.2)$$

and the invariance of the action then leads to the tracelessness of the energy momentum tensor

$$\partial_\mu J_D^\mu = T_\mu{}^\mu = \left. \frac{\partial \hat{\mathcal{L}}_\lambda}{\partial \lambda} \right|_{\lambda=1} = 0 \quad (2.3)$$

where $\hat{\mathcal{L}}_\lambda = \lambda^4 \mathcal{L}(A', \psi', \phi')$ is independent of λ when the action is scale invariant.

With the symmetric energy momentum tensor for the photon,

$$T_{\mu\nu} = -F_{\mu\sigma} F_\nu{}^\sigma + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \quad (2.4)$$

this property is readily apparent.

If we consider loops of the massless charged fields², the vacuum polarization diagram will contain a divergent piece which goes into the renormalization of the electric charge. It also contains a $\ln q^2$ in momentum space, where q_μ refers to the momentum of the photon. Rescaling the gauge field by the bare electric charge $A_\mu \rightarrow A_\mu/e_0$, we can write a one-loop effective action describing both of these effects

$$S = \int d^4x \left[-\frac{1}{4} F_{\rho\sigma} \left[\frac{1}{e^2(\mu)} - b_i \ln(\square/\mu^2) \right] F^{\rho\sigma} \right] \quad (2.5)$$

where b_i is the leading coefficient of the beta function, $b_s = 1/(48\pi^2)$ for a charged scalar and $b_f = 1/(12\pi^2)$ for a charged fermion, and $\square = \partial^2$

Under a scale transformation, we see that the $\ln \square$ term violates the scaling invariance since $\ln \square \rightarrow \ln \square - \ln \lambda^2$. From eq. (2.3), we now infer that

$$\partial_\mu J_D^\mu = \frac{b_i}{2} F_{\rho\sigma} F^{\rho\sigma} \quad (2.6)$$

²All fields will be treated as massless in this chapter. While there are no strictly massless charged particles, the results will apply at momentum transfer well above the particle mass. Moreover, these massless calculations are illustrative of other interesting situations, such as QCD or gravity, where strictly massless particles do appear.

After reverting to the usual definition of the field this yields the usual form of the trace anomaly

$$T_\mu{}^\mu = \frac{b_i e^2}{2} F_{\rho\sigma} F^{\rho\sigma} \quad . \quad (2.7)$$

This derivation is instructive because it highlights the key physics - that the anomaly is related to the scale dependence of the running coupling, which breaks the classical scale invariance. However, the procedure is also unusual in that the anomaly is associated with an *infrared* effect, the $\ln q^2$ or $\ln \square$ behavior. Most derivations and discussions of anomalies emphasize the ultraviolet origin of the effect, either through regularization of the path integral or through the UV properties of Feynman diagrams. Of course, the UV (the renormalization of the charge) and the IR (the $\ln q^2$) are tied together when using dimensional regularization with massless fields, so there is not a contradiction. However, it is satisfying to our effective field theory sensibilities to see a derivation that is insensitive to the UV regularization. No matter how one regulates or modifies the high energy end of the theory (consistent with gauge invariance of course) the infrared behavior and the trace anomaly will remain unaffected³.

The Lagrangian of eq. (2.5) is written in *quasi-local* form, which we will explain in more detail below. The $\ln \square$ term is a shorthand for a non-local object

$$\langle x | \ln \left(\frac{\square}{\mu^2} \right) | y \rangle \equiv L(x - y) = \int \frac{d^4 q}{(2\pi)^4} e^{-iq \cdot (x-y)} \ln \left(\frac{-q^2}{\mu^2} \right) \quad . \quad (2.8)$$

However, under rescaling, this behaves in the same way as described above with a local term

$$L(x - y) \rightarrow \lambda^{-4} \left(L(x - y) - \ln \lambda^2 \delta^4(x - y) \right) \quad (2.9)$$

³There are also infrared derivations of the chiral anomaly [7] and the trace anomaly [8, 9] which make use of dispersion relations, with the integrand in the dispersive integral being dominated by low energy contributions.

yielding the same trace anomaly equation. It is well known that the anomaly does not follow from any local Lagrangian. Here, we have seen that it does follow from the variation of a non-local Lagrangian.

As far as we know, this derivation was first sketched by Deser, Duff and Isham in a paper on gravitational conformal anomalies [10]. One can find echoes of it throughout the gravitational literature, for example in [11, 12, 13, 14, 15, 16, 17], which is surely an incomplete list. The local anomaly itself has been thoroughly discussed in the literature and we have little new to add. However, our objective in this chapter is two-fold. The first concerns the connection of anomalies to non-local effective actions which is not regularly discussed in the gauge theory literature. Our purpose here will be to give a thorough discussion of this non-local effect for QED and to use this simple example to make a concrete exploration of non-local effective actions. A second goal is to discuss the extra novel features when we include the gravitational coupling in the non-local actions. This provides a simple example of non-local gravitational actions, which is an interesting but more complicated subject.

After finding a local trace anomaly from a non-local action, it is natural to consider the full energy-momentum tensor which yields the appropriate trace. Due to the propagation of massless particles in the loop, it will also be a non-local object. To our knowledge, this object has not been constructed before in the literature. This step is indeed important if one wants to fully understand the phenomenology of the trace anomaly. We will construct this object for a charged scalar field in the loop and later display the result for fermions by consulting the matrix element calculation of [18, 19]. An extra motivation for using a charged scalar is that, unlike fermions, the scalar's minimally coupled action is *not* conformally invariant. This provides an interesting insight into the connection between conformal/scale invariance and the anomaly. Our non-local form also has several interesting properties, which we discuss.

In regard to gravity, we also provide a *partial* non-linear completion of the perturbative result using the gravitational curvatures, although we reserve a detailed discussion of this aspect to the next chapter. Our result for the traceful part of the energy-momentum tensor can be obtained by varying a covariant action

$$T_{\mu\nu}^{anom.} = \left(\frac{2}{\sqrt{g}} \frac{\delta\Gamma[g, A]}{\delta g^{\mu\nu}} \right)_{g=\eta} \quad (2.10)$$

where

$$\Gamma[g, A] = \int d^4x \sqrt{g} \left(n_R F_{\rho\sigma} F^{\rho\sigma} \frac{1}{\square} R + n_C F^{\rho\sigma} F_{\lambda}^{\gamma} \frac{1}{\square} C_{\rho\sigma\gamma}{}^{\lambda} \right) . \quad (2.11)$$

Here, $C_{\rho\sigma\gamma}{}^{\lambda}$ is the Weyl tensor and \square is the covariant d' Alembertian. We will find that the first coefficient is determined by the beta functions of fermions or bosons

$$n_R^{(s,f)} = -\frac{\beta^{(s,f)}}{12e} \quad (2.12)$$

while the last coefficient is not related to the beta functions and does not contribute to the trace. Note the $1/\square$ pole which appears in the action which is required by direct calculation of the effective action.

Since the energy momentum-tensor describes the coupling of photons to gravity, we also look at the scattering of a photon by the gravitational field of a massive object. The quantum corrections carry an extra energy dependence that leads to violations of some of the predictions of classical general relativity. For example, the equivalence principle requires that the bending of light is the same for photons of all energies. We show that this is no longer the case when non-local loop effects are present. We should expect that this quantum violation of the equivalence principle should be a general phenomenon, as noted in [20]. Within our calculation it could be described as a “tidal” effect since the photon’s coupling is no longer a local object but samples the

gravitational field over a long distance through quantum loops of massless particles. Quantum mechanics does this in general by producing spatial non-localization and our example provides a non-trivial demonstration of this property⁴.

2.2 The background field method and the non-local effective action

Here we give a brief derivation of the non-local effective action using the background field method. The classical action for QED coupled to a charged field reads

$$S = S_{EM} + \int d^4x (D_\mu \phi)^\star D^\mu \phi \quad (2.13)$$

where

$$D_\mu \phi = (\partial_\mu + ie_0 A_\mu) \phi, \quad S_{EM} = \int d^4x -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (2.14)$$

and e_0 is the bare electric charge.

The one loop effective action is obtained by integrating out the charged scalar field

$$\begin{aligned} \Gamma[A] &= \frac{1}{e_0^2} S_{EM} - i \ln \left(\int \mathcal{D}\phi^\star \mathcal{D}\phi e^{iS} \right) \\ &= \frac{1}{e_0^2} S_{EM} + i \ln (\text{Det } D^2) \end{aligned} \quad (2.15)$$

where we rescaled the gauge field. The operator reads

$$D^2 = \square + i(\partial \cdot A) + 2iA^\mu \partial_\mu - A^2 \quad . \quad (2.16)$$

⁴Of course, since all charged particles in Nature have mass, the results will only be applicable in the real world for photons with energies well above the electron mass.

In perturbation theory we can expand the logarithm in powers of the interaction

$$\ln (\text{Det } D^2) = \text{Tr} \left(\frac{1}{\square} v - \frac{1}{2} \frac{1}{\square} v \frac{1}{\square} v + \dots \right) + \text{const.} \quad (2.17)$$

where

$$v = i(\partial \cdot A) + 2iA^\mu \partial_\mu - A^2 \quad . \quad (2.18)$$

Introducing position-space eigenstates such that

$$\langle x | \frac{1}{\square} | y \rangle = i\Delta_F(x - y) \quad (2.19)$$

and using dimensional regularization, we have that $\Delta_F(0) = 0$, and hence the first term in the expansion vanishes. Integrating by parts to place the derivatives on the propagators and noting that the latter is a function of the geodesic distance $|x - y|$, we find the order- A^2 contribution

$$\Gamma[A] = \frac{1}{e_0^2} S_{EM} + i \int d^D x d^D y A^\mu(x) M_{\mu\nu}(x - y) A^\nu(y) \quad (2.20)$$

and

$$M_{\mu\nu}(x - y) = \partial_\mu \Delta_F(x - y) \partial_\nu \Delta_F(x - y) - \Delta_F(x - y) \partial_\nu \partial_\mu \Delta_F(x - y) \quad (2.21)$$

and all derivatives act on x . By Fourier transforming and using standard manipulations in momentum space, one obtains the following relations

$$\begin{aligned} \Delta_F(x) \partial_\mu \Delta_F(x) &= \frac{1}{2} \partial_\mu \Delta_F^2(x) \\ \Delta_F(x) \partial_\mu \partial_\nu \Delta_F(x) &= [d \partial_\mu \partial_\nu - g_{\mu\nu} \square] \frac{\Delta_F^2(x)}{4(d-1)} \\ \partial_\mu \Delta_F(x) \partial_\nu \Delta_F(x) &= [(d-2) \partial_\mu \partial_\nu + g_{\mu\nu} \square] \frac{\Delta_F^2(x)}{4(d-1)} \quad . \end{aligned} \quad (2.22)$$

These combine to produce a tensor

$$M_{\mu\nu}(x-y) = [g_{\mu\nu}\square - \partial_\mu\partial_\nu] \frac{\Delta_F^2(x-y)}{2(d-1)} \quad (2.23)$$

which is conserved in any dimension. Converting one x -derivative back to one with respect to y and integrating by parts we convert the result to a manifestly gauge invariant form

$$\Gamma[A] = \frac{1}{e_0^2} S_{EM} - i \int d^D x d^D y F_{\mu\nu}(x) \left[\frac{\Delta_F^2(x-y)}{4(d-1)} \right] F^{\mu\nu}(y) \quad . \quad (2.24)$$

We can represent the squared propagator by a Fourier transformation

$$\Delta_F^2(x-y) = - \int \frac{d^D q}{(2\pi)^D} e^{-iq(x-y)} I_2(q) \quad (2.25)$$

where $I_2(q)$ is the scalar bubble function which reads

$$I_2(q) = \frac{i}{16\pi^2} \left[\frac{1}{\bar{\epsilon}} - \ln \left(\frac{-q^2}{\mu^2} \right) \right], \quad \frac{1}{\bar{\epsilon}} = \frac{1}{\epsilon} - \gamma + \ln 4\pi \quad . \quad (2.26)$$

with $\epsilon = (4-D)/2$. Now it is easy to renormalize the electric charge⁵ and hence express the 4D effective action in a quasi-local form

$$\Gamma[A] = \int d^4 x \quad - \frac{1}{4} F_{\mu\nu} \left[\frac{1}{e^2(\mu)} - b_i \ln \left(\frac{\square}{\mu^2} \right) \right] F^{\mu\nu} \quad (2.27)$$

where we find for the scalar loop (and by analogy for the fermion loop)

$$b_s = \frac{1}{48\pi^2}, \quad b_f = \frac{1}{12\pi^2} \quad . \quad (2.28)$$

⁵Note that since $[1/(D-1)]1/\epsilon = 1/(3\epsilon) + 2/3$, there is an extra constant factor of 2/3 when using modified Minimal Subtraction renormalization. This constant is irrelevant for our purposes and we do not display it.

2.3 Including the energy momentum tensor in the effective action

The trace of the energy momentum tensor is a local object. What about the full energy-momentum tensor $T_{\mu\nu}$ itself? One might try following the conventional procedure by employing the translation invariance of the quasi-local action in eq. (2.5) to find $T_{\mu\nu}$, but the non-local term renders this task impossible. One elegant pathway is to compute the effective action in curved space from which we can identify the energy momentum tensor through the relation

$$\delta\Gamma[g, A] = \frac{1}{2} \int d^4x \sqrt{g} \delta g^{\mu\nu} T_{\mu\nu} \quad . \quad (2.29)$$

Hence we are interested in the non-local effective action including gravity. Of course we cannot complete this program for an arbitrary gravitational field. However it is sufficient to use perturbation theory if our aim is just the flat space result. Moreover, as we show in section 6, perturbation theory can be used to propose a non-linear completion of the effective action apart from subtleties that we address in the next chapter. We perform the computation for bosons and consult [18, 19] to read off the result for fermions. The starting point is the action

$$S = S_{EM} + \int d^Dx \sqrt{g} [g^{\mu\nu} (D_\mu \phi)^* (D_\nu \phi) - \xi \phi^* \phi R] \quad (2.30)$$

where all derivative operators are covariant.

We have included the $\xi \phi^* \phi R$ coupling, with $\xi = 0$ being minimally coupled and $\xi = 1/6$ being conformally coupled, in order to separately follow scale and conformal symmetry. For $\xi = 1/6$ the above action is invariant under *local* Weyl transformations, i.e. conformal transformations. Namely,

$$g_{\mu\nu} \rightarrow e^{2\sigma(x)} g_{\mu\nu}, \quad \phi \rightarrow e^{-\sigma(x)} \phi, \quad A_\mu \rightarrow A_\mu \quad . \quad (2.31)$$

On the other hand, the minimally coupled action is invariant only under scale transformations. The scalar field energy-momentum tensor

$$T_{\mu\nu} = (\partial_\mu\phi)^*(\partial_\nu\phi) + (\partial_\nu\phi)^*(\partial_\mu\phi) - g_{\mu\nu}(\partial_\lambda\phi)^*(\partial^\lambda\phi) + 2\xi(g_{\mu\nu}\square - \partial_\mu\partial_\nu)\phi^*\phi - 2\xi(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\phi^*\phi \quad (2.32)$$

is traceless only for $\xi = 1/6$. For future reference, we point out that the trace of the energy-momentum tensor could be directly determined by performing a conformal transformation and then varying the action with respect to σ , namely

$$\delta_\sigma S = - \int d^4x \sigma T^\mu_\mu . \quad (2.33)$$

Turning to our calculation, we start by performing the path-integral which yields eq. (2.15) but with the curved space operator

$$D^2 = \sqrt{g} (\nabla^\mu \nabla_\mu + 2iA^\mu \partial_\mu + i\nabla_\mu A^\mu - A_\mu A^\mu + \xi R) . \quad (2.34)$$

The perturbative calculation is set up by expanding the metric around flat space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (2.35)$$

and all other geometric quantities accordingly. From eq. (2.29), it suffices to compute the effective action linear in the perturbation $h_{\mu\nu}$ up to terms quadratic in the gauge field. There exist three diagrams which contribute at this order, a triangle figure [2.1] and two bubble-like diagrams figure [2.2]. We evaluate the effective action *on-shell*, and thus impose both on-shellness of external photons $p^2 = p'^2 = 0$ and transversality $p \cdot A(p) = p' \cdot A(p') = 0$.

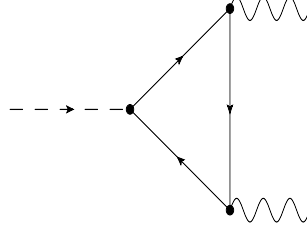


Figure 2.1. Triangle diagram.

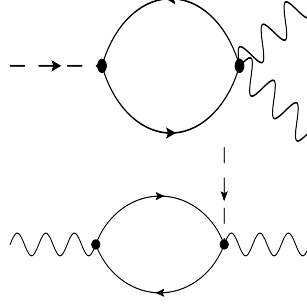


Figure 2.2. Bubble diagrams.

The calculation is performed using the Passarino-Veltman (P-V) reduction technique [21], the details of which are included in an appendix. The result of the triangle diagram is

$$\mathcal{T} = \int_p \int_{p'} \tilde{h}^{\mu\nu}(-q) \tilde{A}^\alpha(p) \tilde{A}^\beta(-p') \mathcal{P}_{\mu\nu,\alpha\beta}^T \quad (2.36)$$

where

$$\begin{aligned} \mathcal{P}_{\mu\nu,\alpha\beta}^T = & [4H + Bq^2]\eta_{\mu\nu}\eta_{\alpha\beta} + 4H(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}) + [4I - 4J + Cq^2 - Dq^2]\eta_{\mu\nu}p'_\alpha p_\beta \\ & + [4I + 4E + B]Q_\mu Q_\nu \eta_{\alpha\beta} + [4J - B]q_\mu q_\nu \eta_{\alpha\beta} \\ & + [4K + 4F + C - 4M - 4G - D]Q_\mu Q_\nu p'_\alpha p_\beta + [4M - C - 4L + D]q_\mu q_\nu p'_\alpha p_\beta \\ & + [4I + 2E - 4J](p'_\alpha p_\mu \eta_{\nu\beta} + p'_\mu p_\beta \eta_{\nu\alpha} + p'_\alpha p_\nu \eta_{\mu\beta} + p'_\nu p_\beta \eta_{\mu\alpha}) \\ & - 4\xi(q_\mu q_\nu - q^2 \eta_{\mu\nu})(B\eta_{\alpha\beta} + (C - D)p'_\alpha p_\beta) \quad . \end{aligned} \quad (2.37)$$

Here the various coefficients are the result of performing the momentum integration - these are given in the appendix. The first of the bubble diagrams reads

$$\mathcal{B}_1 = \int_p \int_{p'} \tilde{h}^{\mu\nu}(-q) \tilde{A}^\alpha(p) \tilde{A}^\beta(-p') \mathcal{P}_{\mu\nu,\alpha\beta}^{B_1} \quad (2.38)$$

where

$$\mathcal{P}_{\mu\nu,\alpha\beta}^{B_1} = \left[\frac{D-2}{4(D-1)} - \xi \right] (q^2 \eta_{\mu\nu} - q_\mu q_\nu) \eta_{\alpha\beta} I_2(q) \quad . \quad (2.39)$$

The last diagram reads

$$\mathcal{B}_2 = 2 \int_p \int_{p'} \tilde{h}^{\mu\nu}(-q) \tilde{A}^\alpha(p) \tilde{A}^\beta(-p') \mathcal{P}_{\mu\nu,\alpha\beta}^{B_2} \quad (2.40)$$

where

$$\begin{aligned} \mathcal{P}_{\mu\nu,\alpha\beta}^{B_2} = & \frac{1}{2} \left(\eta_{\beta\mu} p_\nu p_\alpha + \eta_{\beta\nu} p_\mu p_\alpha - \frac{1}{2} \eta_{\mu\nu} p_\beta p_\alpha \right) I_2(p) \\ & - \frac{D}{4(D-1)} \left(\eta_{\beta\mu} p_\nu p_\alpha - \eta_{\beta\nu} p_\mu p_\alpha + \frac{1}{2} \eta_{\mu\nu} p_\alpha p_\beta \right) I_2(q) \quad . \end{aligned} \quad (2.41)$$

This last diagram vanishes simply due to the condition $p \cdot \tilde{A}(p) = 0$.

Combining the three diagrams we find that to this order in perturbation theory the effective action reads

$$\Gamma[g, A] = S_{EM} - i \int_p \int_{p'} \tilde{h}^{\mu\nu}(-q) \tilde{A}^\alpha(p) \tilde{A}^\beta(-p') \mathcal{M}_{\mu\nu,\alpha\beta} \quad (2.42)$$

where

$$\begin{aligned} \mathcal{M}_{\mu\nu,\alpha\beta} = & \mathcal{P}_{\mu\nu,\alpha\beta}^T - \mathcal{P}_{\mu\nu,\alpha\beta}^{B_1} \\ = & \left(\frac{1}{12} \mathcal{M}_{\mu\nu,\alpha\beta}^0 + \frac{1}{q^2} [a Q_\mu Q_\nu (p'_\alpha p_\beta - p \cdot p' \eta_{\alpha\beta}) \right. \\ & \left. + b (q_\mu q_\nu - q^2 \eta_{\mu\nu}) (p'_\alpha p_\beta - p \cdot p' \eta_{\alpha\beta})] \right) I_2(q) \end{aligned} \quad (2.43)$$

and

$$a = -\frac{1}{24}(D-4), \quad b = \left[\frac{5}{24} - \xi \right] (D-4) \quad (2.44)$$

and $\mathcal{M}_{\mu\nu,\alpha\beta}^0$ is the lowest order photon energy momentum matrix element

$$\begin{aligned} \mathcal{M}_{\mu\nu,\alpha\beta}^0 = & p'_\mu p_\nu \eta_{\alpha\beta} + p_\mu p'_\nu \eta_{\alpha\beta} + \eta_{\mu\nu} p'_\alpha p_\beta - p_\mu p'_\alpha \eta_{\nu\beta} - p'_\mu p_\beta \eta_{\alpha\nu} - p_\nu p'_\alpha \eta_{\mu\beta} \\ & - p'_\nu p_\beta \eta_{\alpha\mu} + p \cdot p' (\eta_{\mu\alpha} \eta_{\beta\nu} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}) \quad . \end{aligned} \quad (2.45)$$

We have the limit $D = 4$ in all terms except for those of eq. 2.44.

There are a couple of interesting calculational features in this computation. One is that although we are calculating a triangle diagram, the scalar triangle integral

$$I_3(p, p') = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(l^2 + i0)((k+p)^2 + i0)((k+p')^2 + i0)} \quad (2.46)$$

does not appear in the result. The above integral is infrared divergent, and thus despite the massless loops the on-shell conditions yielded an infrared finite effective action up to this order in perturbation theory. The P-V reduction has expressed all of the integrals in terms of the bubble integral and the answer only contains

$$I_2(q) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + i0)((k+q)^2 + i0)} = \frac{i}{16\pi^2} \left[\frac{1}{\bar{\epsilon}} - \ln \left(\frac{-q^2}{\mu^2} \right) \right] \quad (2.47)$$

with $\frac{1}{\bar{\epsilon}} = \frac{1}{\epsilon} - \gamma + \ln 4\pi$. Also interesting is that the bubble integral as a function of an external momenta

$$I_2(p^2 = \lambda^2) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + i0)((k+p)^2 + i0)} = \frac{i}{16\pi^2} \left[\frac{1}{\bar{\epsilon}} - \ln \left(\frac{-\lambda^2}{\mu^2} \right) \right] \quad (2.48)$$

does not appear in the answer. In doing the P-V reduction shown in the appendix, we kept the off-shell condition $p^2 = p'^2 = \lambda^2$ in potentially divergent contributions in

order to regulate the infrared aspects of the integrals, and inspection of these integrals shows $I_2(\lambda^2)$ occurring frequently. However, all such terms drop out of the final result.

2.3.1 Renormalization

It is expected that the divergent part of the effective action is proportional to S_{EM} which reads in momentum space

$$S_{EM} = \frac{1}{4e_0^2} \int_p \int_{p'} \tilde{h}^{\mu\nu}(-q) \tilde{A}^\alpha(p) \tilde{A}^\beta(-p') \mathcal{M}_{\mu\nu,\alpha\beta}^0 \quad (2.49)$$

and e_0 is the bare electric charge. As usual, the bare electric charge is replaced by its renormalized counterpart via

$$e_0 = \mu^\epsilon Z_3^{-1/2} e \quad . \quad (2.50)$$

Working in the modified MS -scheme the renormalization constant is easily determined to be

$$Z_3 = 1 - \frac{e^2}{48\pi^2 \bar{\epsilon}} \quad . \quad (2.51)$$

It is now easy to determine the beta function from the RGE

$$\beta^s(e) = \frac{e^3}{48\pi^2} \quad . \quad (2.52)$$

After renormalization, we pass to the limit $D = 4$ and write down the renormalized effective action

$$\begin{aligned} \Gamma^{\text{ren}}[g, A] = & \frac{1}{4} \int_p \int_{p'} \tilde{h}^{\mu\nu}(-q) \tilde{A}^\alpha(p) \tilde{A}^\beta(-p') \\ & \left[\left(\frac{1}{e^2(\mu)} - \frac{1}{48\pi^2} \ln \left(\frac{-q^2}{\mu^2} \right) \right) \mathcal{M}_{\mu\nu,\alpha\beta}^0 + \mathcal{M}_{\mu\nu,\alpha\beta}^s \right] \end{aligned} \quad (2.53)$$

where we identified the finite tensor for the charged scalar leaving the value of the conformal coupling arbitrary

$$\mathcal{M}_{\mu\nu,\alpha\beta}^s(\xi) = \frac{1}{48\pi^2 q^2} (Q_\mu Q_\nu - (5 - 24\xi)(q_\mu q_\nu - q^2 \eta_{\mu\nu})) (p'_\alpha p_\beta - p \cdot p' \eta_{\alpha\beta}) \quad . \quad (2.54)$$

We see that only for $\xi = 1/6$ does the photon's energy momentum tensor have the expected trace relation. The lack of Weyl invariance in the scalar sector when $\xi \neq 1/6$ carries over to the photon interaction and modifies the trace. As we show below, this feature is not present for fermions since the classical theory is Weyl invariant. On the other hand, it is satisfying to observe that, using the beta function, the renormalized effective action is indeed scale-independent.

2.3.2 Fermions and non-universality

At this stage, it is quite straightforward to read off the result for fermions from the matrix-element computation of [18]

$$\begin{aligned} \Gamma^{\text{ren}}[g, A] = & \frac{1}{4} \int_p \int_{p'} \tilde{h}^{\mu\nu}(-q) \tilde{A}^\alpha(p) \tilde{A}^\beta(-p') \\ & \left[\left(\frac{1}{e^2} - \frac{1}{12\pi^2} \ln \left(\frac{-q^2}{\mu^2} \right) \right) \mathcal{M}_{\mu\nu,\alpha\beta}^0 + \mathcal{M}_{\mu\nu,\alpha\beta}^f \right] \end{aligned} \quad (2.55)$$

where the finite tensor now becomes

$$\mathcal{M}_{\mu\nu,\alpha\beta}^f = \frac{1}{24\pi^2 q^2} (-Q_\mu Q_\nu - q_\mu q_\nu + q^2 \eta_{\mu\nu}) (p'_\alpha p_\beta - p \cdot p' \eta_{\alpha\beta}) \quad . \quad (2.56)$$

We also find the fermionic beta function

$$\beta^f(e) = \frac{e^3}{12\pi^2} \quad . \quad (2.57)$$

An interesting aspect of this result is the *non-universality* of the structure of the finite tensor which is responsible for the anomalous trace. However, we will show below

that the trace of this tensor reproduces the correct anomaly for both bosons and fermions.

2.3.3 Position space effective action

Let us collect these calculations into a position space effective action. After integrating out the massless charged particle, it has the general structure

$$\Gamma[A, h] = \frac{1}{e^2(\mu)} S_{EM}[A, h] + \Gamma^{(0)}[A] + \Gamma^{(1)}[A, h] \quad (2.58)$$

where

$$S_{EM}[A, h] = -\frac{1}{4} \int d^4x \left(F_{\mu\nu} F^{\mu\nu} + 2 h^{\mu\nu} T_{\mu\nu}^{cl} \right) \quad (2.59)$$

with $T_{\mu\nu}^{cl}(x)$ given by eq. (2.4) and $\Gamma^{(0)}[A]$ being the non-local piece in eq. (3.1). The loop corrections linear in $h_{\mu\nu}$ are contained in $\Gamma^{(1)}[A, h]$. Written in quasi-local form, it has the structure ⁶

$$\Gamma^{(1)}[A, h] = -\frac{1}{2} \int d^4x h^{\mu\nu} \left[b_s \log \left(\frac{\square}{\mu^2} \right) T_{\mu\nu}^{cl} - \frac{b_s}{2} \frac{1}{\square} \tilde{T}_{\mu\nu}^s \right] \quad (2.60)$$

for conformally coupled scalars, where b_s is the beta function coefficient and $\tilde{T}_{\mu\nu}^s$ is the operator

$$\tilde{T}_{\mu\nu}^s = 2\partial_\mu F_{\alpha\beta} \partial_\nu F^{\alpha\beta} - \eta_{\mu\nu} \partial_\lambda F_{\alpha\beta} \partial^\lambda F^{\alpha\beta} \quad (2.61)$$

For fermions, the structure is similar

$$\Gamma^{(1)}[A, h] = -\frac{1}{2} \int d^4x h^{\mu\nu} \left[b_f \log \left(\frac{\square}{\mu^2} \right) T_{\mu\nu}^{cl} - \frac{b_f}{2} \frac{1}{\square} \tilde{T}_{\mu\nu}^f \right] \quad (2.62)$$

⁶From now onwards, we use $\xi = 1/6$.

except now $\tilde{T}_{\mu\nu}^f$ is a different operator

$$\tilde{T}_{\mu\nu}^f = -F_{\alpha\beta}\partial_\mu\partial_\nu F^{\alpha\beta} - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda F_{\alpha\beta}\partial^\lambda F^{\alpha\beta} . \quad (2.63)$$

Both of these are genuine non-local actions. To display the non-locality we recall that the $\log \square$ factor is to be interpreted as in eq. (2.8), and equivalently the $1/\square$ term is the representation of the Feynman propagator as in eq. (2.19). Then the explicitly non-local form reads⁷

$$\begin{aligned} \Gamma^{(1)}[A, h] = & -\frac{1}{2} \int d^4x h^{\mu\nu}(x) \int d^4y \left[b_i L(x-y) T_{\mu\nu}^{cl}(y) \right. \\ & \left. - i \frac{b_i}{2} \Delta_F(x-y) \tilde{T}_{\mu\nu}^i(y) \right], \quad i = s, f . \end{aligned} \quad (2.64)$$

We see both a logarithmic non-locality and a mass-less pole non-locality.

From eq. (2.29), one can readily obtain the energy momentum tensor itself from these expressions. In doing so, we rescale the photon field by a factor of $e(\mu)$ in order to obtain the conventional normalization. The result is given by the non-local object

$$\begin{aligned} T_{\mu\nu}^i(x) = & T_{\mu\nu}^{cl}(x) - e^2 b_i \int d^4y \left[L(x-y) T_{\mu\nu}^{cl}(y) \right. \\ & \left. + \frac{i}{2} \Delta_F(x-y) \tilde{T}_{\mu\nu}^i(y) \right], \quad i = s, f . \end{aligned} \quad (2.65)$$

Note that this form does contain a dependence on the scale μ within the logarithm.

Using the on-shell condition $\square A_\mu = 0$ we have that

$$\partial_\lambda F_{\alpha\beta} \partial^\lambda F^{\alpha\beta} = \frac{1}{2} \square (F_{\mu\nu} F^{\mu\nu}) \quad (2.66)$$

⁷When using the in-in formalism, the causal prescription for the $\ln \square$ piece was computed in chapter 3 and evidently the $1/\square$ is the retarded propagator.

and thus one can easily verify that the above tensor reproduces the correct trace anomaly. Moreover, to show that it is conserved one merely notices that both non-local functions are functions of the geodesic distance and hence convert derivatives to be with respect to the y variable and then uses integration by parts. Eq. (2.65) is one of the main results of this chapter.

One can gain some insight into this structure if one decomposes the boson and fermion tensors into a universal term which yields the proper trace and a non-universal term that is traceless. Here we find

$$\tilde{T}_{\mu\nu}^i = a_1^i A_{\mu\nu} + a_2^i S_{\mu\nu}, \quad i = s, f \quad (2.67)$$

where

$$A_{\mu\nu} = \partial_\mu F_{\alpha\beta} \partial_\nu F^{\alpha\beta} + F_{\alpha\beta} \partial_\mu \partial_\nu F^{\alpha\beta} - \eta_{\mu\nu} \partial_\lambda F_{\alpha\beta} \partial^\lambda F^{\alpha\beta} \quad (2.68)$$

$$S_{\mu\nu} = 4\partial_\mu F_{\alpha\beta} \partial_\nu F^{\alpha\beta} - 2F_{\alpha\beta} \partial_\mu \partial_\nu F^{\alpha\beta} - \eta_{\mu\nu} \partial_\lambda F_{\alpha\beta} \partial^\lambda F^{\alpha\beta} \quad (2.69)$$

and

$$a_1^s = a_1^f = \frac{2}{3}, \quad a_2^s = \frac{1}{3}, \quad a_2^f = -\frac{1}{6} . \quad (2.70)$$

The trace of $A_{\mu\nu}$ gives the anomaly, while $S_{\mu\nu}$ is traceless. There is of course an ambiguity in any such decomposition - one can add any traceless tensor to $A_{\mu\nu}$ while subtracting it from $S_{\mu\nu}$. We have chosen the linear combinations to match the non-linear completion that we will display in section 6, such that $A_{\mu\nu}$ corresponds to the $F^2(1/\square)R$ term and $S_{\mu\nu}$ to the $F^2(1/\square)C$ term.

2.4 Conformal and scaling properties of the effective action

In the one loop effective action, we have found two terms that are proportional to the beta function coefficient, b_i . These can be referred to as the $\ln \square$ term and the $1/\square$

term. We will see that both of them are required, but by somewhat different scale symmetry transformations. As we will describe below, the $\ln \square$ responds directly to dilations while the $1/\square$ responds to conformal transformations. The existence of both allows us to relate the two symmetries in this context. The $\ln \square$ behavior and the $1/\square$ behavior are much discussed in the literature. For example, Deser and Schwimmer [13] refer to the $\ln \square$ terms as Type B anomalies and $1/\square$ as Type A. It is interesting that both types emerge in this calculation. The $1/\square$ terms are also associated with the Riegert anomaly action [24], which will be commented on in section 6.

Let us now discuss the dichotomy between scaling and conformal symmetry breaking in the effective action constructed in the previous section. The scaling behavior of $\Gamma^{(0)}[A]$ was discussed in the introduction. Before we repeat the same exercise for $\Gamma^{(1)}[A, h]$, we note that since $h_{\mu\nu}$ has a mass dimension zero, it has a vanishing scaling dimension. Accordingly, under a scale transformation the 1-loop EA transforms as follows

$$\Gamma^{(1)}[A, h] \rightarrow \Gamma^{(1)}[A, h] + \frac{b_i}{2} \int d^4x h^{\mu\nu} [\log \lambda^2 T_{\mu\nu}^{cl}] \quad . \quad (2.71)$$

Using eq. (2.3) and taking $\Gamma^{(0)}[A]$ into account as well, we find

$$T_\mu^\mu = \frac{b_i}{2} (\eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + 2h^{\mu\nu} T_{\mu\nu}^{cl}) \quad (2.72)$$

which is indeed the desired anomalous operator expanded around flat space.

Notice in particular the feature that when performing this rescaling, the $1/\square$ portion of the answer is scale invariant. However, when forming the energy momentum tensor, it is precisely the $1/\square$ part that yields the *tracefull* contribution to the energy-momentum tensor. To explain this, we need to understand the violation of conformal symmetry present in the effective action. Once again, we need to determine the transformation properties of the metric perturbation $h_{\mu\nu}$. This is best

achieved by linearizing the classical action and performing an infinitesimal conformal transformation, namely

$$g_{\mu\nu} \rightarrow (1 + 2\sigma)g_{\mu\nu} \quad . \quad (2.73)$$

This allows to read off the transformation of the metric perturbation

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\sigma\eta_{\mu\nu} \quad (2.74)$$

One can readily check that the linearized action of eq. (5.1) is indeed invariant under the above transformation provided $\phi \rightarrow (1 - \sigma)\phi$. Both $S_{EM}[A, h]$ and $\Gamma^{(0)}[A]$ are invariant. Moreover,

$$\Gamma^{(1)}[A, h] \rightarrow \Gamma^{(1)}[A, h] - b_i \int d^4x \, \sigma \frac{1}{\square} (\partial_\lambda F_{\mu\nu} \partial^\lambda F^{\mu\nu}) \quad . \quad (2.75)$$

By using eqs. (3.13) and (2.33), one reproduces the flat space limit of the anomalous operator

$$T_\mu^\mu = \frac{b_i}{2} \eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \quad (2.76)$$

We have seen that when expanding to first order around flat space, two terms arise which are both related to the anomaly. When forming the energy momentum tensor, the log term multiplies the classical energy momentum tensor and hence is itself traceless. However under scale transformations the log produces an anomaly which combines with the lowest order piece in the proper way. On the other hand, conformal transformations directly produce the trace of the energy-momentum tensor, and this is manifest in the $1/\square$ term of the one-loop result.

2.5 The on-shell energy-momentum matrix element at one loop

For completeness, let us display the matrix element of the energy momentum tensor found in the previous section. The energy momentum tensor for on-shell photons has the general form

$$\begin{aligned} \langle \gamma(p') | T_{\mu\nu} | \gamma(p) \rangle = & \epsilon_\beta^*(p') \epsilon_\alpha(p) \left[\mathcal{M}_{\mu\nu, \alpha\beta}^0 G_1(q^2) \right. \\ & + Q_\mu Q_\nu (p'_\alpha p_\beta - p \cdot p' \eta_{\alpha\beta}) G_2(q^2) \\ & \left. + (q_\mu q_\nu - q^2 \eta_{\mu\nu}) (p'_\alpha p_\beta - p \cdot p' \eta_{\alpha\beta}) G_3(q^2) \right] \end{aligned} \quad (2.77)$$

where

$$\begin{aligned} \mathcal{M}_{\mu\nu, \alpha\beta}^0 = & p'_\mu p_\nu \eta_{\alpha\beta} + p_\mu p'_\nu \eta_{\alpha\beta} + \eta_{\mu\nu} p'_\alpha p_\beta - p_\mu p'_\alpha \eta_{\nu\beta} - p'_\mu p_\beta \eta_{\alpha\nu} - p_\nu p'_\alpha \eta_{\mu\beta} \\ & - p'_\nu p_\beta \eta_{\alpha\mu} + p \cdot p' (\eta_{\mu\alpha} \eta_{\beta\nu} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}) \end{aligned} \quad (2.78)$$

is the tree level matrix element and $G_{1,2,3}$ are form-factors.

We can extract this result from the energy momentum tensor found in the previous section. Unlike the effective action, the photons are dynamical in the matrix element computation and thus we include the field-strength renormalization graphs shown in figure 2.3. These remove the dependence on the unphysical parameter μ and bring in mass singularities, and thus we use the off-shellness condition $p^2 = p'^2 = \lambda^2$ to regulate these. The net effect is to replace the μ^2 dependence within the logarithm with λ^2 . The results for the massless conformally coupled scalar are

$$G_1 = 1 + e^2 b_s \ln(q^2/\lambda^2), \quad G_2 = \frac{e^2}{96\pi^2 q^2}, \quad G_3 = -\frac{e^2}{96\pi^2 q^2} \quad . \quad (2.79)$$

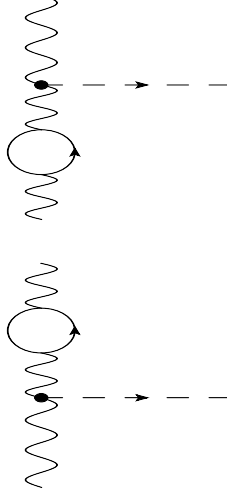


Figure 2.3. External leg corrections relevant for the matrix element.

Note also the pole, $1/q^2$, in G_2 , G_3 , which we also saw in the effective action. The equivalent result for a massless fermion [18] corresponds to

$$G_1 = 1 + e^2 b_f \ln(q^2/\lambda^2), \quad G_2 = -\frac{e^2}{48\pi^2 q^2}, \quad G_3 = -\frac{e^2}{48\pi^2 q^2} . \quad (2.80)$$

We note that the trace anomaly relation emerges correctly in both cases, in that

$$\langle \gamma(p') | T_\mu^\mu | \gamma(p) \rangle = \epsilon_\beta^*(p') \epsilon_\alpha(p) \left[(p'_\alpha p_\beta - p \cdot p' \eta_{\alpha\beta}) q^2 (-G_2(q^2) - 3G_3(q^2)) \right] \quad (2.81)$$

with

$$q^2 (-G_2(q^2) - 3G_3(q^2)) = \frac{\beta^{(s,f)}}{e} . \quad (2.82)$$

In each case, the result is consistent with the relation

$$T_\mu^\mu = \frac{\beta^{(s,f)}}{2e} F_{\mu\nu} F^{\mu\nu} \quad (2.83)$$

with the appropriate β function. Although the matrix element has a $1/q^2$ pole, the trace is a constant.

2.6 Gravity and a non-linear completion of the action

The connection between the non-local effective action and the trace anomaly is more obvious if we construct a non-linear form of the action using gravitational curvatures. There has been a lot of controversy in the literature about the correct form of the non-local action that gives rise to the anomaly. Some authors, see for example [9, 22, 23], argue for the Riegert action first obtained in [24, 25] while others dismissed it based on several arguments [15, 13, 17] and proposed alternative forms. Moreover, another group of authors has used a renormalization group approach to argue that both forms exist in the effective action [26]. One might try developing a non-linear completion based on the perturbative result, however this opens up extra questions about general covariance and uniqueness of the result. The answer to these questions will be addressed in the next chapter.

When dealing with massive charged fields, the covariant form involving the curvatures could readily be found by one of two ways; non-linear completion or heat kernel methods. For massive fields, all Lagrangians are local and the expansion in the curvatures coincides with the energy or derivative expansion - higher powers of the curvature involve higher derivatives. To shed light on the difficulties of the construction when dealing with non-locality, we review a local action given by Drummond and Hathrell [27] corresponding to the one-loop effect of a massive charged fermion

$$\Gamma_{local}[g, A] = \frac{e^2}{m^2} \int d^4x \sqrt{g} \left[l_1 F_{\mu\nu} F^{\mu\nu} R + l_2 F_{\mu\sigma} F_{\nu}{}^{\sigma} R^{\mu\nu} + l_3 F^{\mu\nu} F_{\beta}^{\alpha} R_{\mu\nu\alpha}{}^{\beta} + l_4 \nabla_{\mu} F^{\mu\nu} \nabla_{\alpha} F_{\nu}^{\alpha} \right] . \quad (2.84)$$

These operators comprise a complete basis up to third order in the generalized curvature expansion. In [27] they were determined using the two methods mentioned above; matching the above operators onto the perturbative calculation of [18] in the

low-energy limit and using the Schwinger-DeWitt technique to compute the heat kernel. Indeed the outcome of the two methods agreed, with the result

$$l_1 = -\frac{1}{576\pi^2}, \quad l_2 = \frac{13}{1440\pi^2}, \quad l_3 = -\frac{1}{1440\pi^2}, \quad l_4 = -\frac{1}{120\pi^2} \quad . \quad (2.85)$$

With non-local actions the curvature expansion is not equivalent to the derivative or energy expansion because the calculations require factors of $1/q^2$ or $1/\square$. Higher powers of $(1/\square)R$ are not suppressed in the energy expansion. Since there is no mass scale in the problem, derivatives acting on curvatures can not be deemed small and thus all powers of derivatives must be taken into account. One can think of the non-local form as a non-analytic expansion summarizing the results of a one-loop calculation. Nevertheless, the curvature expansion as in eq. (2.84) is useful because it accommodates the general covariance of the theory in a more explicit fashion.

In the local expansion the term involving the constant l_4 in eq. (2.84) is the only term which survives in flat space. It comes from the vacuum polarization and is the analogue of the $\ln \square$ of our non-local form. However, this coefficient has no relation to the beta function. For the other terms, the factors of $1/m^2$ have to be replaced by a different factor with the same dimensionality. This can be done schematically by replacing $1/m^2$ by $1/\square$ in eq. (2.84). The $1/m^2$ is the leading term in the low-energy expansion of a massive propagator, and thus for massless particles $1/\square$ is the obvious generalization. Of course, the replacement is not exact, and we need to adjust the coefficients to match the perturbative result.

We find the following form to be the most informative

$$\Gamma_{anom.}[g, A] = \int d^4x \sqrt{g} \left[n_R F_{\rho\sigma} F^{\rho\sigma} \frac{1}{\square} R + n_C F^{\rho\sigma} F^\gamma_\lambda \frac{1}{\square} C_{\rho\sigma\gamma}{}^\lambda \right] \quad . \quad (2.86)$$

In this basis, $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d' Alembertian and $C_{\rho\sigma\gamma}{}^\lambda$ is the Weyl tensor which in $4D$ reads

$$\begin{aligned}
C_{\mu\nu\alpha\beta} &= R_{\mu\nu\alpha\beta} - \frac{1}{2}(g_{\mu\alpha}R_{\nu\beta} - g_{\mu\beta}R_{\nu\alpha} - g_{\nu\alpha}R_{\mu\beta} + g_{\nu\beta}R_{\mu\alpha}) \\
&\quad + \frac{R}{6}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})
\end{aligned} \tag{2.87}$$

and

$$n_R^{(s,f)} = -\frac{\beta^{(s,f)}}{12e}, \quad n_C^s = -\frac{e^2}{96\pi^2}, \quad n_C^f = \frac{e^2}{48\pi^2} \quad . \tag{2.88}$$

The term with the Weyl tensor is unrelated to the beta function and the trace anomaly. The term involving the scalar curvature in the form $(1/\square)R$ is the nonlinear completion of the $1/\square$ effects which leads to the conformal anomaly above. The latter is consistent with the leading part of the Riegert action whose non-local piece reads

$$\Gamma_{Riegert} = \frac{b}{4} \int d^4x \sqrt{g} F^2 \frac{1}{\Delta_4} \left(E - \frac{2}{3} \square R \right) \tag{2.89}$$

where E is the $4D$ Gauss-Bonnet topological invariant and Δ_4 is the fourth order operator [24]

$$\Delta_4 = \square^2 - 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{2}{3}R\square^2 - \frac{1}{3}(\nabla^\mu R)\nabla_\mu \quad . \tag{2.90}$$

The Riegert action has additional contributions which are purely gravitational that we do not display. One immediately sees that the piece relevant for a linear expansion around flat space has the required form $F^2(1/\square)R$ with $b = \beta/2e$. This aspect of the effective action was noticed before in [9] as well.

2.7 Quantum equivalence principle violation

Quantum loops will upset the predictions of classical general relativity. In this section, we display the quantum corrected formula for the bending angle of light and

show the violation of the equivalence principle. The classical prediction of general relativity can be found in almost every textbook on general relativity. There is no reliable fully quantum treatment that can be applied to the bending of light. We follow the semiclassical approach presented in [20]. The inverse Fourier transform of the amplitude is first obtained, from which one can define a semiclassical potential describing the interaction between a photon and a massive object like a star. This allows the bending angle to be computed via

$$\theta \approx \frac{b}{E} \int_{-\infty}^{\infty} du \frac{V'(b\sqrt{1+u^2})}{\sqrt{1+u^2}} \quad (2.91)$$

where b is the classical impact parameter and E is the photon energy. Although this formula might look naive, it was shown in [20] that it indeed yields the correct result for the post-Newtonian correction to the bending angle when graviton loops are considered.

Because there are no completely massless charged particles⁸, our result would only apply in the real world at energies far above the particle mass. However, it is interesting as a theoretical laboratory. What aspects of the equivalence principle can be violated by quantum effects? As a technical aspect, we allow the mass to provide an infrared cutoff to the infrared singularity of the energy-momentum matrix element. The coupling of photons to gravity is given by the one-loop energy-momentum tensor given in the previous section with λ replaced by m .

Since we work in the static limit, the scalar particle mass is large compared to the momentum transfer $M_{\odot} \gg |\mathbf{q}|$ and so we ignore the recoil. We also remind that the polarization vectors for physical photons are purely spatial and thus the amplitude takes the simple form

⁸However, note that in the early universe above the electroweak phase transition, the elementary particles are massless.

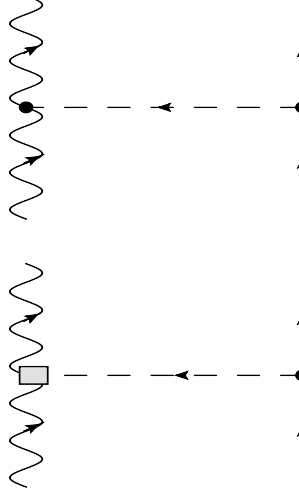


Figure 2.4. Gravitational scattering of a photon off a static massive target. The diagram on the top is the tree level process, while the square in the bottom diagram represents the non-local effects.

$$\mathcal{M} = \frac{(\kappa M_\odot)^2}{2\mathbf{q}^2} \left[1 - \frac{\beta^{(s,f)}}{e} \ln \left(\frac{\mathbf{q}^2}{\mu^2} \right) \right] (E^2 \epsilon^\star \cdot \epsilon (1 + \cos \theta) - \mathbf{k} \cdot \epsilon^\star \mathbf{k}' \cdot \epsilon) \quad (2.92)$$

where E is the photon energy, \mathbf{k} is the incoming 3-momentum, \mathbf{k}' is the outgoing 3-momentum and the polarization vectors are purely spatial.

It is convenient to work with circularly polarized photons, and we find that the helicity conserving amplitude includes the contribution of the logarithm, yielding

$$\mathcal{M}(++) = \mathcal{M}(--) = \frac{(\kappa M_\odot E)^2}{2\mathbf{q}^2} \left[1 - \frac{\beta^{(s,f)}}{e} \ln \left(\frac{\mathbf{q}^2}{m^2} \right) \right] (1 + \cos \theta) \quad . \quad (2.93)$$

In the non-relativistic limit, the semiclassical potential is simply

$$V(r) = -\frac{1}{4M_\odot E} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \mathcal{M}(q) \quad (2.94)$$

where the pre-factor accounts for non-relativistic normalization. Employing the following relations,

$$\begin{aligned}
\int \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2} \ln\left(\frac{\mathbf{q}^2}{m^2}\right) &= -\frac{\ln(mr) + \gamma_E}{2\pi r} \\
\int \frac{d^3\vec{q}}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \ln\left(\frac{\mathbf{q}^2}{m^2}\right) &= -\frac{1}{2\pi r^3}, \quad \cos\theta = 1 - \frac{\mathbf{q}^2}{2E^2}
\end{aligned} \tag{2.95}$$

we simply find

$$\begin{aligned}
V_{++}(r) = V_{--}(r) &= -\frac{2GM_\odot E}{r} + \frac{16\pi GM_\odot}{E} \delta^{(3)}(\mathbf{x}) \\
&+ \frac{4\beta GM_\odot E}{er} \left(\frac{1}{4E^2 r^2} - \ln mr - \gamma_E \right)
\end{aligned} \tag{2.96}$$

Notice in particular that the corrections to the Newtonian piece are not necessarily attractive. The short-range delta function does not lead to any modifications to the bending angle. Using eq. (2.91), we find

$$\theta_{non-flip} \approx \frac{4GM_\odot}{b} + \frac{8\beta GM_\odot}{eb} (\ln mb + \gamma_E - \ln 2) - \frac{4\beta GM_\odot}{eE^2 b^3} \tag{2.97}$$

In contrast to this, the $1/q^2$ portion of the energy momentum tensor leads to helicity flip amplitudes. Here, one finds the result

$$\mathcal{M}(+-) = \mathcal{M}(-+) = -\frac{(\kappa e M_\odot E)^2}{\mathbf{q}^2} b_s + \frac{(\kappa e M_\odot)^2}{4} b_s \tag{2.98}$$

for bosons and

$$\mathcal{M}(+-) = \mathcal{M}(-+) = \frac{(\kappa e M_\odot E)^2}{\mathbf{q}^2} b_f + \frac{(\kappa e M_\odot)^2}{4} b_f \tag{2.99}$$

for fermions. This result has interesting features; first of all the sign in front of the Coulomb-like piece is different for both species. Moreover, the $1/q^2$ terms do not

modify the helicity non-flip part of the amplitude. Thus the non-relativistic potential is spin-dependent. If we proceed with the calculation of the bending angle, we find

$$\theta_{flip} \approx \begin{cases} -4e^2 b_s G M_\odot / b, & \text{bosons} \\ 4e^2 b_f G M_\odot / b, & \text{fermions} \end{cases} \quad (2.100)$$

The interpretation of this result is less clear. However, the overall picture is clear: quantum physics has modified the classical prediction for light bending. In particular, photons of different energies will follow different trajectories.

2.8 Conclusion

We have been discussing low energy aspects of the conformal (trace) anomaly of QED using the one-loop effective action obtained by integrating out the massless charged particles. This is non-local because of the long distance propagation of the massless particles. However, after renormalization it is this non-local object that encodes the information on the anomaly. We also constructed the non-local energy-momentum tensor quadratic in the gauge field. This has the correct non-vanishing trace arising from a $1/q^2$ pole, which nevertheless yields a local trace. In the effective action, both the $\log \square$ and $1/\square$ terms were required, with the log piece being related to scale symmetry and the $1/\square$ piece being related to conformal symmetry. These non-local terms are interesting in their own right. For example, we showed that such corrections lead to an energy dependence of the bending of light, signaling a violation of some classical versions of the Equivalence Principle.

Another aspect of our exploration is an initial construction of the non-local action for a curved background, the correct form of which has been an ongoing controversy since the seminal work on gravitational anomalies by Deser, Isham and Duff [10]. This construction constitutes a fundamental ingredient if one wants to consider the effects of the anomaly on various gravitational phenomena beyond the linear approximation.

Over the years, multiple authors have investigated the effects of anomalies on different phenomena ranging from cosmology and astrophysics [32, 28, 29, 30, 31, 33] to black holes [34, 35]. We will continue the discussion of the covariant form of the effective action in the next chapter.

CHAPTER 3

COVARIANT NON-LOCAL ACTION FOR MASSLESS QED AND THE CURVATURE EXPANSION

3.1 Introduction

While the fundamental Lagrangians describing our known physical theories are all local, quantum loops of massless or nearly massless particles yield non-local effects. It is often useful to arrange those loop effects into a non-local effective action which enables a systematic investigation of the quantum effects on the classical background fields. For theories where the symmetries relate the couplings of different types of particles, such as chiral theories or general relativity, the evaluation of a single loop using the background field method allows the loop corrections to a large number of processes to be calculated at once. For example in chiral perturbation theory, the renormalized non-local effective action [4] is useful for many different reactions.

In general relativity, Barvinsky, Vilkovisky and collaborators (hereafter referred to collectively as BV) have developed techniques for calculating and displaying the non-local gravitational effective actions that arise due to graviton loops or those of other massless fields [36, 37, 38, 39, 40, 41, 42, 44, 45, 43]. The results are presented using an expansion in the curvature. In effective field theory we are used to an expansion in the curvature for local Lagrangians. This corresponds to an energy or derivative expansion in which operators are suppressed by a mass scale which is typically the mass of the 'integrated out' field. If the light fields present in the effective action are slowly varying, each term in the expansion is correspondingly smaller. Quantum mechanically, this corresponds to low energies. However, with

non-local actions the curvature expansion has a different nature. Because non-local operators such as the inverse d' Alembertian $1/\nabla^2$ can appear, higher powers of the curvature such as $[(1/\nabla^2)R]^n$ are not automatically suppressed at low energy and the curvature expansion is not the same as the energy expansion. Instead, it is a way to describe the (calculable) infrared physics from quantum loops. The effects of these infrared non-local effects from loops are just starting to be explored [46, 47, 48, 49, 50, 51, 52, 53, 54]. We are going to present few applications of the formalism in the coming chapters.

In this chapter, however, we explore the non-local curvature expansion in a relatively simple setting - that of photons coupled to a massless charged scalar and to gravity. The analysis is based on the results of the previous chapter. We also display the results relevant for massless fermions to highlight interesting features of the non-local action. Both the spacetime metric and the gauge field are treated as classical background fields. In the previous chapter, we focused on obtaining the flat-space non-local effective action and the associated energy-momentum tensor that gives rise to the trace anomaly. Here we are concerned with generalizing the flat-space results to curved backgrounds. This is achieved via a technique that we refer to as the *non-linear completion* of the action where, similar to CPT, the action is displayed as an expansion in the curvatures. The non-local effective actions are a relatively unexplored topic and there remain interpretive issues that we explore in the current chapter.

Most notable is the issue of the covariant nature of the non-local *form factors* such as $\ln \nabla^2$. In particular, we pay special attention to the generalization of the flat d' Alembertian to curved space which turns out to be a non-trivial aspect of the effective action. Moreover, direct use of the Feynman graph expansion of the effective action allows us to identify the terms which is related to the beta function of the theory and those which are not related to the latter. Our exploration leads to a

better understanding of the non-local action that generates the QED trace (conformal) anomaly. To the best of our knowledge, this is an unsettled issue in the literature and the procedure of non-linear completion yields interesting insight into the correct form.

The plan this chapter is the following. In section 3.2 we provide an overview of the main problem discussed in this chapter and also present our results. In section 3.3 we discuss some of the methodological issues with this program, pointing out the main difficulties of constructing non-local actions in curved spaces and in section 3.4 we describe the non-linear completion matching technique. Section 3.5 is devoted for the non-linear completion of the quadratic action while the cubic action is displayed in section 3.6. We then move in section 3.7 to show how the terms in the effective action generates the trace anomaly. Finally, we summarize and conclude in section 3.8.

3.2 The problem of $\ln \square$

In flat-space the one loop effective action for a photon, obtained by integrating out a massless charged scalar or fermion, has the form

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} \left[\frac{1}{e^2(\mu)} - b_i \ln(\square/\mu^2) \right] F^{\mu\nu} \right] \quad (3.1)$$

where b_i is the leading coefficient of the beta function, $b_s = 1/(48\pi^2)$ for a charged scalar and $b_f = 1/(12\pi^2)$ for a charged fermion, and $\square = \partial^2$. Here the action is expressed in quasi-local form and the $\ln \square/\mu^2$ operator is a shorthand for the fully non-local realization

$$\langle x | \ln \left(\frac{\square}{\mu^2} \right) | y \rangle \equiv L(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq \cdot (x-y)} \ln \left(\frac{-q^2}{\mu^2} \right) . \quad (3.2)$$

When one desires a formulation in curved spacetime, one requires that the logarithm generalizes to the covariant form, with tensor indices raised and lowered with the metric, and the \square operator also being covariant. We will reserve the notation \square for the flat-space d' Alembertian and use ∇^2 for the covariant version. That is, one requires

$$\frac{b_i}{4} \int d^4x \, \eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} \ln(\square/\mu^2) F_{\alpha\beta} \rightarrow \frac{b_i}{4} \int d^4x \sqrt{-g} \, g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} \ln(\nabla^2/\mu^2) F_{\alpha\beta} \quad (3.3)$$

This can be made more usable through the definition of the log as

$$\ln(\nabla^2/\mu^2) = - \int_0^\infty dm^2 \left[\frac{1}{\nabla^2 + m^2} - \frac{1}{\mu^2 + m^2} \right] \quad (3.4)$$

which then involves propagators that can be co-variantly defined. Even here the result is not simple as the inverse operator is acting on the tensor indices of $F_{\alpha\beta}$ and itself becomes a bi-tensor [55]. Later, we expand the covariant form in eq. (3.3) to first order in the expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, for photons satisfying $p^2 = p'^2$, resulting in

$$\int d^4x \sqrt{g} F^{\alpha\beta} \ln\left(\frac{\nabla^2}{\mu^2}\right) F_{\alpha\beta} = \int d^4x \left[F^{\alpha\beta} \ln(\square/\mu^2) F_{\alpha\beta} + h_{\mu\nu} (\mathcal{O}_1^{\mu\nu} + \mathcal{O}_2^{\mu\nu}) \right] \quad (3.5)$$

where

$$\begin{aligned} \mathcal{O}_1^{\mu\nu} &= \frac{1}{2} \eta^{\mu\nu} F^{\alpha\beta} \ln(\square/\mu^2) F_{\alpha\beta} - 2 F^\mu{}_\alpha \ln(\square/\mu^2) F^{\nu\alpha} \\ \mathcal{O}_2^{\mu\nu} &= \partial^\mu \partial^\nu F_{\alpha\beta} \frac{1}{\square} F^{\alpha\beta} + \partial^\mu \partial^\nu F^{\alpha\beta} \frac{1}{\square} F_{\alpha\beta} - \eta^{\mu\nu} \partial_\lambda F_{\alpha\beta} \frac{1}{\square} \partial^\lambda F^{\alpha\beta} \end{aligned} \quad (3.6)$$

and indices are raised and lowered with the flat metric. We note that near the mass shell, $p^2 = p'^2 = \lambda^2 \approx 0$, the $F(1/\square)F$ terms are particularly dangerous as they involve the inverse photon “mass”. Notice also that the logarithms in eq. (3.6) are infrared singular.

On the other hand, in the previous chapter, we have explicitly calculated the $h_{\mu\nu}$ corrections to the effective action for a conformally coupled scalar field and on-shell photons, and have extracted the fermionic analogy from the work of [18]. Interestingly, *none* of the above $h_{\mu\nu}$ terms in eq. (3.5) are found in the result. Instead we get a relatively simple answer, in that the terms that are proportional to the beta function coefficient¹ are

$$\frac{b_i}{4} \left\{ F^{\alpha\beta} \ln(\Box/\mu^2) F_{\alpha\beta} + h_{\mu\nu} \left[2 \ln(\Box) T_{cl}^{\mu\nu} - \frac{2}{3} \frac{1}{\Box} A^{\mu\nu} \right] \right\} \quad (3.7)$$

with

$$T_{\mu\nu}^{cl} = -F_{\mu\sigma} F_{\nu}{}^{\sigma} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \quad (3.8)$$

and

$$A_{\mu\nu} = \partial_{\mu} F_{\alpha\beta} \partial_{\nu} F^{\alpha\beta} + F_{\alpha\beta} \partial_{\mu} \partial_{\nu} F^{\alpha\beta} - \eta_{\mu\nu} \partial_{\lambda} F_{\alpha\beta} \partial^{\lambda} F^{\alpha\beta} \quad (3.9)$$

This result is itself generally covariant to this order in $h_{\mu\nu}$, although different in structure from eq. (3.5). One can easily check that the *full* result is invariant under local coordinate transformations $h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_{(\mu} \xi_{\nu)}$. In contrast to eq. (3.5) we see that eq. (3.7) does not contain any of the dangerous $F(1/\Box)F$ terms - the inverse photon mass does not arise in perturbation theory.

Both of the $\mathcal{O}(h_{\mu\nu})$ terms in eq. (3.7) are required by trace anomaly considerations and hence must be proportional to the beta function coefficient b_i . The terms with the logarithm yield the correct trace anomaly for a pure scale transformation

$$x' = \lambda x, \quad A'_{\mu}(x') = \lambda^{-1} A_{\mu}(x), \quad h'_{\mu\nu}(x') = h_{\mu\nu}(x), \quad \ln \Box' = \ln \Box - \ln \lambda^2, \quad (3.10)$$

¹There is also a term independent of the beta function which we include below.

where the non-invariance of $\ln \square$ leads to

$$T_\mu^\mu = \frac{b_i}{2} (\eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + 2h^{\mu\nu} T_{\mu\nu}^{cl}) \quad (3.11)$$

which is the correct expansion of the covariant density $\sqrt{g}F^2$. Under this rescaling the last term in eq. (3.7) is invariant. However, under a conformal transformation ($g_{\mu\nu} \rightarrow \exp(2\sigma(x))g_{\mu\nu}$) restricted to flat-space

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\sigma\eta_{\mu\nu} \quad (3.12)$$

the first two terms in eq. (3.7) are invariant while the last term is not. Using the on-shell condition $\square A_\mu = 0$ we have that²

$$\partial_\lambda F_{\alpha\beta} \partial^\lambda F^{\alpha\beta} = \frac{1}{2} \square (F_{\mu\nu} F^{\mu\nu}), \quad \eta^{\mu\nu} \frac{1}{\square} A_{\mu\nu} = -\frac{3}{2} F_{\mu\nu} F^{\mu\nu} \quad (3.13)$$

and we see that last term yields the correct trace anomaly. The two related transformations, rescaling the coordinates and rescaling the metric, act differently in the effective action yet both yielding the same anomaly relation. We see that both types of non-locality, i.e. the logarithm and the massless pole in eq. (3.7) are required by direct calculation as well as by anomaly considerations.

We seek the covariant curvature expansion which reproduces the perturbative results. For nomenclature, the term of order F^2 is referred to as second order in the curvature, while that with an extra gravitational curvature, e.g. $F^2 R$, is called third order in the curvature. The details of the matching will be given below, while here we summarize the results.

The mismatch of the two expressions eqs. (3.5) and (3.7) makes the expansion in the curvature relatively complicated. Because one is starting out with the $F \ln \nabla^2 F$

²The details of these steps were given in the previous chapter.

expression as the covariant form which is second order in the curvature, one needs to add and subtract correction terms in order to reproduce the actual calculated result. These *counter-terms* are third order in the curvature as we show below. This does not modify the covariance of the result - both expressions are covariant. Nevertheless it does make the resulting expression at third order quite complicated. This matching procedure, which we refer to as *non-linear completion*, occupies most of the work described below. We find that the result to this order in the curvature is³

$$\begin{aligned}
\Gamma_{log} = & \frac{b_i}{4} \int d^4x \sqrt{g} \left\{ F_{\alpha\beta} \ln(\nabla^2/\mu^2) F^{\alpha\beta} - \frac{1}{3} F_{\alpha\beta} F^{\alpha\beta} \frac{1}{\nabla^2} R \right. \\
& + 4R^{\mu\nu} \frac{1}{\nabla^2} \left[\log(\nabla^2) \left(-F_{\mu\sigma} F_\nu{}^\sigma + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \right. \\
& + F_{\mu\sigma} \log(\nabla^2) F_\nu{}^\sigma - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} \log(\nabla^2) F^{\alpha\beta} \left. \right] \\
& \left. + \frac{1}{3} R F_{\alpha\beta} \frac{1}{\nabla^2} F^{\alpha\beta} - C_{\beta\mu\nu}^\alpha F_\alpha{}^\beta \frac{1}{\nabla^2} F^{\mu\nu} \right\} \quad (3.14)
\end{aligned}$$

where $C_{\beta\mu\nu}^\alpha$ is the Weyl tensor. Note that the logarithms within the square brackets [...] do not need a factor of μ^2 as the $\log \mu^2$ would cancel between the two terms. In particular, these terms are scale invariant as we will discuss later on.

We will show that eq. (3.14) has the correct anomaly properties. The way that this is accomplished is interesting. For a scale transformation as in eq. (3.10), it is the first term - the logarithm - which yields the anomaly. However for a local Weyl (conformal) transformation it is the second term - $F^2 \frac{1}{\nabla^2} R$ - which is the active ingredient. This latter term appears as one of the portions of the Riegert anomaly action [24] when appropriately displayed in a curvature expansion. Finally for a global rescaling of the metric $g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$, with σ being a spacetime constant, there is a

³The placement of the differential operators appears somewhat different than the expressions in the body of the chapter. This is allowed indeed under integration by parts as we are assuming asymptotically flat spacetimes.

simpler path that again involves the logarithm. The latter is equivalent to a scale transformation as in eq. (3.10). We conclude that both the logarithm and the Riegert term (massless pole) are required by anomaly considerations. We comment on this dichotomy in regard to the geometric program to classify anomalies set forth by Deser and Schwimmer [13].

Finally in order to match the full result found in the direct one-loop calculation found in the previous chapter, one must add a non-anomalous term that has no relation to the beta function

$$\Gamma_{Weyl} = n_C^i \int d^4x \sqrt{g} F^{\mu\nu} F_\alpha{}^\beta \frac{1}{\nabla^2} C^\alpha{}_{\beta\mu\nu} \quad (3.15)$$

where

$$n_C^s = -\frac{1}{96\pi^2}, \quad n_C^f = \frac{1}{48\pi^2} \quad . \quad (3.16)$$

This is different for fermions and scalars and is invariant under both scaling and conformal transformations.

Our final result for the covariant one-loop effective action is

$$\Gamma_{tot} = S_{cl} + \Gamma_{log} + \Gamma_{Weyl} \quad (3.17)$$

where

$$S_{cl} = \int d^4x \sqrt{g} - \frac{1}{4e^2(\mu)} F_{\mu\nu} F^{\mu\nu} \quad (3.18)$$

is the classical action.

3.3 Covariant non-local actions: General remarks

General relativistic actions are readily described when local. Using the metric, covariant derivatives and curvature tensors one can construct generally covariant local

functions of the field variables. The ultraviolet divergences of quantum loops are therefore simple to treat because they are also local [56, 57]. However non-local objects are difficult to describe in a generally covariant form because they sample the metric at a continuum of points in spacetime. For a general metric, explicit expressions for such actions are not possible.

For massless scalar QED and after integrating out the charged scalars at one loop the effective action must be gauge invariant and thus involves only the field strength tensor. Up to quadratic order in the gauge field and using dimensional regularization, a general form in curved spacetime is

$$\Gamma[g, A] = \frac{1}{e_0^2} S_{EM} + \int d^4x \int d^4y F_{\mu\nu}(x) M_{\alpha\beta}^{\mu\nu}(x, y; \mu) F^{\alpha\beta}(y) \quad (3.19)$$

where S_{EM} is the classical Maxwell action, e_0 is the bare electric charge and $M_{\alpha\beta}^{\mu\nu}(x, y; \mu)$ is an antisymmetric second-rank bi-tensor *density* of unit weight which explicitly depends on the renormalization scale. As we show below, this bi-tensor samples the full space-time and not just the pair of points (x, y) since it involves the effects of massless propagators. The practical question is what the form of this bi-tensor is and how we can best describe it.

The divergence contained in $M_{\alpha\beta}^{\mu\nu}(x, y)$ is local and calculable. It has the form

$$M_{\alpha\beta}^{\mu\nu}(x, y; \mu) = \frac{1}{192\pi^2\epsilon} \sqrt{g(x)}^{1/4} \delta^4(x - y) \sqrt{g(y)}^{1/4} I_{\alpha\beta}^{\mu\nu} + L_{\alpha\beta}^{\mu\nu}(x, y; \mu) \quad (3.20)$$

where

$$I_{\alpha\beta}^{\mu\nu} = \frac{1}{2} (\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu) \quad (3.21)$$

This divergence is absorbed into the renormalization of the electric charge. After removing this divergence, the residual bi-tensor $L_{\alpha\beta}^{\mu\nu}(x, y; \mu)$ is finite.

One might expect that there are also local terms proportional to the geometric curvatures, such as FFR which would correspond to $M^{\mu\nu,\alpha\beta} \sim \frac{g^{\mu\alpha}g^{\nu\beta}}{\sqrt{g}}\delta^4(x-y) R$. Such terms are found when one integrates out a *massive* charged particle [27]. However, they are absent in our problem, that of integrating out a massless field, simply on dimensional grounds. The curvatures involve two derivatives of the metric, and hence the coefficient of any local term of the form FFR must have dimensions of $1/\text{mass}^2$. Because all fields are massless, there is no way to obtain such a coefficient. Any factors of the curvature in the action must be balanced by non-local factors such as $1/\nabla^2$. This tells us that once we have dealt with charge renormalization, which is of course a local operator, the remainder of the effective action will be purely non-local.

In flat space, the non-local function was obtained in the previous chapter

$$L_{\alpha\beta}^{(0)\mu\nu}(x, y; \mu) = \frac{b_s e^2}{4} I_{\alpha\beta}^{\mu\nu} L(x - y; \mu) \quad (3.22)$$

where $b_s = 1/(48\pi^2)$ is the leading coefficient of the QED beta function for a charged scalar, e is the physical charge and $L(x - y; \mu)$ is displayed in eq. (3.2). As a warm-up for later usage, let us pause at this stage to show how one can convert from a non-local form to a quasi-local one employing non-local *form factors*. The latter are the building blocks of the curvature expansion. Through the position-space representation

$$\langle x | \ln \left(\frac{\partial^2}{\mu^2} \right) | y \rangle \equiv L(x - y) \quad (3.23)$$

one can re-write eq. (3.19) in quasi-local form as

$$\Gamma^{(0)}[A] = S_{EM} + \frac{b_s e^2}{4} \int d^4x F_{\mu\nu} \left[\ln \left(\frac{\partial^2}{\mu^2} \right) \right] F^{\mu\nu} . \quad (3.24)$$

To appreciate the subtleties in the construction of the bi-tensor, let us quote the effective action linear in metric perturbation around flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In

non-local form, it reads

$$\Gamma^{(1)}[A, h] = -\frac{1}{2} \int d^4x \int d^4y h^{\mu\nu}(x) \left[b_s L(x-y) T_{\mu\nu}^{cl}(y) - i \frac{b_s}{2} \Delta_F(x-y) \tilde{T}_{\mu\nu}^s(y) \right] \quad (3.25)$$

where photons are taken to be on-shell, i.e. dropping factors of $\square F_{\mu\nu}$. Here we have defined

$$\tilde{T}_{\mu\nu}^s = 2\partial_\mu F_{\alpha\beta} \partial_\nu F^{\alpha\beta} - \eta_{\mu\nu} \partial_\lambda F_{\alpha\beta} \partial^\lambda F^{\alpha\beta} \quad (3.26)$$

and

$$T_{\mu\nu}^{cl} = -F_{\mu\sigma} F_\nu{}^\sigma + \frac{1}{4} \eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \quad (3.27)$$

is the classical energy-momentum tensor. We also have the massless propagator⁴

$$\Delta_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 + i0} e^{-iq \cdot (x-y)} \quad (3.28)$$

In this case, the result is local in the relative position of the gauge fields, but contains both logarithmic and massless-pole non-localities with respect to the gravitational field. Allowing the gauge fields to be off-shell would lead to a non-locality in all three field variables due to the appearance of the triangle diagram⁵. Let us arrange the bi-tensor density at this order in metric perturbation, it reads

⁴The boundary condition imposed on the propagator depends on the application one is considering. For instance, for time-dependent systems one should choose the re-tarted propagator.

⁵See the discussion in the previous chapter.

$$L_{\alpha\beta}^{(1)\mu\nu}(x, y; \mu) = \int d^4z h^{\sigma\lambda}(z) \left[L(z - x; \mu) J_{\alpha\beta\sigma\lambda}^{\mu\nu} + \Delta_F(z - x) H_{\alpha\beta\sigma\lambda}^{\mu\nu} \right] \delta^{(4)}(x - y) \quad (3.29)$$

where

$$\begin{aligned} J_{\alpha\beta\sigma\lambda}^{\mu\nu} &= \frac{b_s}{8} (\delta_\alpha^\nu \delta_\sigma^\mu \eta_{\beta\lambda} + \delta_\alpha^\nu \delta_\lambda^\mu \eta_{\beta\sigma} - \delta_\alpha^\mu \delta_\sigma^\nu \eta_{\beta\lambda} - \delta_\alpha^\mu \delta_\lambda^\nu \eta_{\beta\sigma} - I_{\alpha\beta}^{\mu\nu} \eta_{\sigma\lambda}) \\ H_{\alpha\beta\sigma\lambda}^{\mu\nu} &= i \frac{b_s}{4} I_{\alpha\beta}^{\mu\nu} (2\partial_\sigma \partial_\lambda - \eta_{\sigma\lambda} \square) \quad . \end{aligned} \quad (3.30)$$

One immediately notices that the bi-tensor density samples the gravitational field over the whole spacetime manifold. This is the main reason that the explicit construction of such non-local objects is not possible in arbitrary geometries. Instead, one can use the quasi-local form factors to express the loop correction as follows

$$\Gamma^{(1)}[A, h] = -\frac{1}{2} \int d^4x h^{\mu\nu} \left[b_s \log \left(\frac{\square}{\mu^2} \right) T_{\mu\nu}^{cl} + \frac{b_s}{2} \frac{1}{\square} \tilde{T}_{\mu\nu}^s \right] \quad (3.31)$$

where the position-space representation of the inverse d' Alembertian is given above. In this chapter, we seek a generally covariant non-linear completion of the above results that is accomplished by employing the non-local form factors.

3.4 Non-linear completion: Expansion in the curvature

The curvature expansion is a covariant method to display the effective action with arbitrary background fields. For local actions, the heat kernel expansion is the most elegant technique to resolve the functional determinant of any operator [58, ?, 60, 61, 62]. Its usage encompasses many applications in physics and mathematics, but unfortunately it becomes somewhat complicated when we deal with a massless operator. Moreover, the correspondence with the more familiar perturbative expansion of the effective action in terms of Feynman graphs is not very obvious [63, 64]. In

this chapter, we propose a new technique to obtain the effective action which we call non-linear completion. The logic is very similar to the matching procedure well known in effective field theory (EFT). This procedure proceeds by perturbative matching of the full theory onto the effective theory. What makes the construction of the EFT Lagrangian possible is the fact that it must inherit all the exact symmetries of the full theory. This is the pathway we are going to employ in our case as well.

In our example, the symmetries of the full theory are diffeomorphism and gauge invariances and hence the non-local action must be constructed from the generalized curvatures. As we have shown in the previous section, the form factors are an important tool as they enable the action to be written in quasi-local form where the action is manifestly covariant. One starts by listing the relevant curvature basis and organize it in terms of a power series. For the example at hand, we have

$$\begin{aligned} \mathcal{R}^2 : & F_{\mu\nu} F^{\mu\nu} \\ \mathcal{R}^3 : & F_{\mu\nu} F^{\mu\nu} R, \quad F_{\mu\alpha} F_\nu{}^\alpha R^{\mu\nu}, \quad F^{\mu\nu} F^{\alpha\beta} R_{\mu\nu\alpha\beta}, \quad \nabla_\mu F^{\mu\nu} \nabla_\alpha F_\nu{}^\alpha \quad . \end{aligned} \quad (3.32)$$

The field strength is the curvature of the gauge-connection and thus counts as one power of the curvature. The effective action will be displayed as an expansion in these *generalized* curvatures. The last operator in eq. (3.32) does not contribute when the photons are on-shell and thus we are not going to discuss it further. Then one proposes all possible non-local functionals of the d' Alembertian which could possibly act on the different terms in the curvature basis

$$\begin{aligned} \mathcal{F}_2 : & \ln \left(\frac{\nabla^2}{\mu^2} \right) \\ \mathcal{F}_3 : & \frac{1}{\nabla^2}, \quad \frac{\ln(\nabla_i^2/\mu^2)}{\nabla_i^2} \quad . \end{aligned} \quad (3.33)$$

where the subscripts denote the curvature upon which the operator acts. As far as \mathcal{F}_3 is concerned, one can arrange more operators such as

$$\frac{\ln(\nabla_i^2/\nabla_j^2)}{f(\nabla^2)} \quad (3.34)$$

where $f(\nabla^2)$ is some function to be determined. However, we will see that no from factor of this kind arises in our example due to the on-shell condition. Although the above form factors look very complicated, these are all well defined via their Laplace transform

$$\mathcal{F}(\nabla^2) = \int_0^\infty ds \mathcal{F}(s) e^{-s\nabla^2} \quad (3.35)$$

The last step is perturbatively matching the full theory diagrams onto the non-local action. The 'Wilson' coefficients in this case only depends on the coupling constants of the full theory and are to be adjusted via the matching procedure. Since a massless field is being integrated out, these coefficients can not depend on any mass or renormalization scale, i.e. the non-local action is completely insensitive to the UV.

3.5 The \mathcal{R}^2 action: The elusive logarithm

In this section, we discuss the non-linear completion of the flat-space action in eq. (3.24). It reads

$$\overset{(2)}{\Gamma}[g, A] = \frac{b_s}{4} \int d^4x \sqrt{g} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \log\left(\frac{\nabla^2}{\mu^2}\right) F_{\mu\nu} \quad (3.36)$$

where $\nabla^2 = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d' Alembertian. The matching onto eq. (3.24) is immediate. Now one must raise the question: what is the expansion of the above action around flat space? In partiucular, the piece linear in the metric perturbation and its connection to the perturbative computation. The answer to these questions is very important in understanding the covariant nature of the quasi-local expansion. In the remainder of this section, we show how to consistently expand the logarithm

and prove that the $\mathcal{O}(h)$ term in the action is entirely absent from the perturbative computation. We start by showing the steps for a scalar field as a toy example and then discuss the more interesting example of a 2-form.

3.5.1 Toy example: A scalar field

Let us consider the following action

$$\Gamma[g, \phi] = \int d^4x \sqrt{g} \phi \ln \left(\frac{\nabla^2}{\mu^2} \right) \phi . \quad (3.37)$$

The goal is to expand the action around flat space to linear order in the metric perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The most convenient way to accomplish this is to first vary the action with respect to the metric and then restrict the result to flat space. Using eq. (3.4), we find

$$\delta_g \Gamma[\eta, \phi] = \int d^4x \int_0^\infty dm^2 \left\{ \phi (\square + m^2)^{-1} [\delta_g \nabla^2]_{g=\eta} (\square + m^2)^{-1} \phi \right\} + \dots \quad (3.38)$$

where the ellipses denote terms resulting from the variation of \sqrt{g} which do not matter to our discussion. To arrive at the above expression, we have used the formal variation of an inverse operator

$$\delta_g \frac{1}{\nabla^2 + m^2} = - \frac{1}{\nabla^2 + m^2} (\delta_g \nabla^2) \frac{1}{\nabla^2 + m^2} . \quad (3.39)$$

The variation of the d' Alembertian depends on the tensor field in the action. For a scalar field, we have

$$(\delta_g \nabla^2) \Psi = (\delta g^{\mu\nu} \partial_\mu \partial_\nu - \delta g^{\mu\nu} \Gamma_{\mu\nu}^\alpha \partial_\alpha - g^{\mu\nu} \delta \Gamma_{\mu\nu}^\alpha \partial_\alpha) \Psi \quad (3.40)$$

where

$$\delta \Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} (\partial_\mu \delta g_{\beta\nu} + \partial_\nu \delta g_{\beta\mu} - \partial_\beta \delta g_{\mu\nu}) . \quad (3.41)$$

It is advisable at this stage to express eq. (3.38) in a non-local form which is accomplished via the identity

$$\frac{1}{\square + m^2} \Psi = \int d^4 y \Delta(x - y) \Psi(y) \quad (3.42)$$

where

$$(\square + m^2) \Delta(x - y) = \delta^{(4)}(x - y) \quad . \quad (3.43)$$

If we recall that $\delta g^{\mu\nu} = -h^{\mu\nu}$ around flat space, we find

$$\begin{aligned} \delta_g \Gamma[\eta, \phi] &= \int d^4 x d^4 y d^4 z \int_0^\infty dm^2 \phi(x) \Delta(x - y) \\ &\quad \left(-h^{\mu\nu} \partial_\mu \partial_\nu - \partial^\mu h_{\mu\nu} \partial^\nu + \frac{1}{2} \partial^\alpha h \partial_\alpha \right) \Delta(y - z) \phi(z) \end{aligned} \quad (3.44)$$

where

$$\Delta(x - y) = - \int \frac{d^4 l}{(2\pi)^4} \frac{e^{-il \cdot (x - y)}}{l^2 - m^2} \quad . \quad (3.45)$$

Although the above must be defined with some boundary condition, this is not going to affect our discussion. Notice that one could obtain the same result using the more explicit variation of the propagator

$$\frac{\delta G(x, x')}{\delta g^{\mu\nu}(z)} = - \int d^4 y G(x, y) \left[\frac{\delta \nabla^2}{\delta g^{\mu\nu}(z)} \right] G(y, x') \quad . \quad (3.46)$$

To facilitate comparison with the perturbative calculation, we can Fourier transform the above expression and find

$$\int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \phi(p) \phi(p') h^{\mu\nu}(q) \mathcal{P}_{\mu\nu} \frac{\ln p'^2 - \ln p^2}{p^2 - p'^2}, \quad q = -p - p' \quad (3.47)$$

where

$$\begin{aligned}\mathcal{P}_{\mu\nu} = & \frac{1}{2}p'_\mu p'_\nu + \frac{1}{2}p_\mu p_\nu + \frac{1}{4}q_\mu p'_\nu + \frac{1}{4}q_\nu p'_\mu + \frac{1}{4}q_\mu p_\nu + \frac{1}{4}q_\nu p_\mu \\ & - \frac{1}{4}q \cdot p' \eta_{\mu\nu} - \frac{1}{4}q \cdot p \eta_{\mu\nu}\end{aligned}\quad (3.48)$$

3.5.2 2-forms

We now turn to the treatment of 2-forms which is our main interest. There are two distinct pieces that arise from the variation procedure. The first comes from varying the explicit factors of the metric tensor in eq. (3.36) while the second comes from varying the logarithm and the procedure is almost identical to the scalar example aside from some differences related to the tensor rank that we now discuss. First, we generalize eq. (3.40) to the variation of the d' Alembertian when it acts on a 2-form

$$\begin{aligned}(\delta_g \nabla^2 A_{\mu\nu})|_{g=\eta} = & (-h^{\alpha\beta} \partial_\alpha \partial_\beta - \eta^{\alpha\beta} \delta\Gamma_{\alpha\beta}^\sigma \partial_\sigma) A_{\mu\nu} - \partial^\beta (\delta\Gamma_{\beta\mu}^\sigma A_{\sigma\nu} + \delta\Gamma_{\beta\nu}^\sigma A_{\sigma\mu}) \\ & - \delta\Gamma_{\beta\mu}^\sigma \partial^\beta A_{\sigma\nu} - \delta\Gamma_{\beta\nu}^\sigma \partial^\beta A_{\sigma\mu} \quad .\end{aligned}\quad (3.49)$$

Second, we need to generalize eq. (3.42)

$$\frac{1}{\square + m^2} A_{\mu\nu} = \int d^4 y \Delta_{\mu\nu}^{\alpha\beta}(x-y) A_{\alpha\beta}(y), \quad \Delta_{\mu\nu}^{\alpha\beta} = I_{\mu\nu}^{\alpha\beta} \Delta(x-y) \quad . \quad (3.50)$$

We recognize in eq. (3.49) a structure identical to the scalar field and the result is the same as before but with the difference that both transversality and on-shellness are taken into account in the previous chapter. We now show how to treat the new structures in eq. (3.49). In position-space, we have the following piece

$$\begin{aligned}\int d^4 x d^4 y d^4 z \int_0^\infty dm^2 A^{\mu\nu}(x) \Delta(x-y) \Big[& -2\delta\Gamma_{\lambda\mu}^\sigma(y) (\partial^\lambda \Delta(y-x)) A_{\sigma\nu}(z) \\ & - (\partial^\lambda \delta\Gamma_{\lambda\mu}^\sigma(y)) \Delta(y-z) A_{\sigma\nu}(z) \Big]\end{aligned}\quad (3.51)$$

where we used eq. (3.50). We now have all ingredients and after a laborious computation in momentum-space one finds

$$\begin{aligned} \Gamma^{(2)}[g, A] &= \Gamma^{(0)}[A] \\ &+ \frac{b_s}{4} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} A^\alpha(p) A^\beta(-p') h^{\mu\nu}(-q) (\mathcal{D}_{\alpha\beta\mu\nu} - \mathcal{N}_{\alpha\beta\mu\nu}) + \mathcal{O}(h^2) \end{aligned} \quad (3.52)$$

where

$$\mathcal{D}_{\mu\nu\alpha\beta} = \frac{1}{2} (q^2 \eta_{\mu\nu} - q_\mu q_\nu - Q_\mu Q_\nu) (p \cdot p' \eta_{\alpha\beta} - p'_\alpha p_\beta) \frac{\ln p'^2 - \ln p^2}{p^2 - p'^2} \quad (3.53)$$

$$\mathcal{N}_{\mu\nu\alpha\beta} = \mathcal{M}_{\mu\nu\alpha\beta}^0 \log \left(\frac{-p^2}{\mu^2} \right) \quad (3.54)$$

and $\mathcal{M}_{\mu\nu,\alpha\beta}^0$ is the tensor is the lowest-order matrix element describing the local coupling of photons to gravity. Explicitly, it reads

$$\begin{aligned} \mathcal{M}_{\mu\nu,\alpha\beta}^0 &= p'_\mu p_\nu \eta_{\alpha\beta} + p_\mu p'_\nu \eta_{\alpha\beta} + \eta_{\mu\nu} p'_\alpha p_\beta - p_\mu p'_\alpha \eta_{\nu\beta} - p'_\mu p_\beta \eta_{\alpha\nu} - p_\nu p'_\alpha \eta_{\mu\beta} \\ &- p'_\nu p_\beta \eta_{\alpha\mu} + p \cdot p' (\eta_{\mu\alpha} \eta_{\beta\nu} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}) \quad . \end{aligned} \quad (3.55)$$

The first tensor is the result of varying the metric tensor inside the logarithm, while the second comes from the metric tensors in the rest of the action. Notice that we enforce both transversality and on-shellness except in non-analytic expressions that are infrared singular. Apart from being gauge-invariant, the above tensors respects local energy-momentum conservation

$$\begin{aligned} q^\mu \mathcal{D}_{\mu\nu\alpha\beta} &= q^\nu \mathcal{D}_{\mu\nu\alpha\beta} = 0 \\ q^\mu \mathcal{N}_{\mu\nu\alpha\beta} &= q^\nu \mathcal{N}_{\mu\nu\alpha\beta} = 0 \quad . \end{aligned} \quad (3.56)$$

Indeed this property is guaranteed for the tensor $\mathcal{N}_{\mu\nu\alpha\beta}$ since it is the variation of a local operator, but it is gratifying to see that the same applies for $\mathcal{D}_{\mu\nu\alpha\beta}$ which is the variation of a purely non-local object.

3.6 The \mathcal{R}^3 action

In this section, we perform the matching procedure outlined in section 3.4. It is more convenient to work in momentum space, and so we list the momentum-space expansions of the different curvature invariants in an appendix.

3.6.1 Terms including $1/\nabla^2$

Here we display the non-linear completion of the anomalous contribution to the effective action. At the linear level, we had from the previous chapter

$$\Gamma_{pole}[A, h] = \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} A^\alpha(p) A^\beta(-p') h^{\mu\nu}(-q) \frac{1}{q^2} \mathcal{M}_{\mu\nu\alpha\beta}^s \quad (3.57)$$

where

$$\mathcal{M}_{\mu\nu\alpha\beta}^s = \frac{1}{192\pi^2} (p'_\alpha p_\beta - p \cdot p' \eta_{\alpha\beta}) (Q_\mu Q_\nu - q_\mu q_\nu + q^2 \eta_{\mu\nu}) \quad (3.58)$$

The non-linear completion commences by proposing the ansatz

$$\begin{aligned} \Gamma_{pole}^{(3)}[g, A] = \int d^4 x \sqrt{g} \left(P^S F_{\mu\nu} F^{\mu\nu} \frac{1}{\nabla^2} R + P^{Ric} F_\mu^\beta F^{\alpha\mu} \frac{1}{\nabla^2} R_{\alpha\beta} \right. \\ \left. + P^{Riem} F_\alpha^\beta F^{\mu\nu} \frac{1}{\nabla^2} R_{\beta\mu\nu}^\alpha \right) \end{aligned} \quad (3.59)$$

where the choice of the form factor is easily motivated by the presence of the massless pole

$$\frac{1}{-q^2} \rightarrow \frac{1}{\square} \quad (3.60)$$

Using the expansions provided in the appendix, one can form a linear system to solve for the three coefficients. It naively appears that the system is overdetermined since the expansion of the curvature invariants contain tensor structures that do not appear

in eq. (3.58). Nevertheless, one only finds exactly three *independent* equations which uniquely yields

$$P^S = -\frac{1}{192\pi^2}, \quad P^{Ric} = \frac{1}{48\pi^2}, \quad P^{Riem} = -\frac{1}{96\pi^2} . \quad (3.61)$$

We can use the Weyl tensor to change the curvature basis which is very useful to discuss the conformal (non)-invariance of the effective action. In $4D$, the Weyl tensor reads

$$\begin{aligned} C_{\mu\nu\alpha\beta} &= R_{\mu\nu\alpha\beta} - \frac{1}{2}(g_{\mu\alpha}R_{\nu\beta} - g_{\mu\beta}R_{\nu\alpha} - g_{\nu\alpha}R_{\mu\beta} + g_{\nu\beta}R_{\mu\alpha}) \\ &\quad + \frac{R}{6}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) . \end{aligned} \quad (3.62)$$

Hence, eq. (3.59) becomes

$$\Gamma_{pole}^{(3)}[g, A] = \int d^4x \sqrt{g} (\bar{P}^S F_{\mu\nu} F^{\mu\nu} \frac{1}{\nabla^2} R + P^C F_\alpha{}^\beta F^{\mu\nu} \frac{1}{\nabla^2} C_{\beta\mu\nu}^\alpha) \quad (3.63)$$

where

$$\bar{P}^S = -\frac{1}{576\pi^2}, \quad P^C = -\frac{1}{96\pi^2} . \quad (3.64)$$

In fact, the coefficient of the Ricci scalar piece is indeed related to the beta function of the theory as could easily be checked by consulting the effective action in fermionic QED given in the previous chapter. One finds

$$\bar{P}^S = -\frac{b_i}{12}, \quad b_{boson} = \frac{1}{48\pi^2}, \quad b_{fermion} = \frac{1}{12\pi^2} . \quad (3.65)$$

3.6.2 Terms including $(\log \nabla^2)/\nabla^2$

In the linear action, we also found a logarithmic non-locality which reads

$$\Gamma[A, h] = -\frac{b_i}{4} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} A^\alpha(p) A^\beta(-p') h^{\mu\nu}(-q) \log\left(\frac{-q^2}{\mu^2}\right) \mathcal{M}_{\mu\nu\alpha\beta}^0 \quad (3.66)$$

where $\mathcal{M}_{\mu\nu,\alpha\beta}^0$ has been given in the previous chapter. Although the appearance of \mathcal{M}^0 might suggest that the above action could be matched onto the quadratic basis, this is in fact impossible. We show next that the action can only be matched onto the cubic basis with the following form factor

$$\begin{aligned} \Gamma_{\log}^{(3)}[g, A] = \int d^4 x \sqrt{g} \left(L^S F_{\mu\nu} F^{\mu\nu} \frac{\log(\nabla^2/\mu^2)}{\nabla^2} R \right. \\ \left. + L^{Ric} F_{\mu}^{\beta} F^{\alpha\mu} \frac{\log(\nabla^2/\mu^2)}{\nabla^2} R_{\alpha\beta} \right. \\ \left. + L^{Riem} F_{\alpha}^{\beta} F^{\mu\nu} \frac{\log(\nabla^2/\mu^2)}{\nabla^2} R^{\alpha}_{\beta\mu\nu} \right) . \end{aligned} \quad (3.67)$$

The $1/\square$ is inserted for dimensional consistency at this stage as it comprises the only possible non-local object one can employ. The matching procedure is the only way to decide on the consistency of the ansatz. Once again, using the curvature expansions in the appendix one ends up with three *independent* equations which uniquely fixes the coefficients

$$L^S = \frac{b_s}{4}, \quad L^{Ric} = -b_s, \quad L^{Riem} = 0 \quad . \quad (3.68)$$

The $1/q^2$ factor which results from inserting the inverse d' Alembertian cancels out against factors of q^2 in the curvature invariants. Using eq. (3.27), one can rewrite the above action in a more transparent form which will prove useful in discussing the conformal (non)-invariance of the action

$$\Gamma_{\log}^{(3)}[g, A] = b_s \int d^4 x \sqrt{g} T_{\mu\nu}^{cl} \frac{\log(\nabla^2/\mu^2)}{\nabla^2} R^{\mu\nu} \quad (3.69)$$

3.6.3 Counter-terms for the logarithm

Here we display the counterterms that we need to cancel out the $\mathcal{O}(h)$ piece that appears in the expansion of the quadratic action eq. (3.52). As we show next, these are third order in the curvature. There are two independent tensors in eq. (3.52) which should be matched onto two different ansatz. For the tensor $\mathcal{N}_{\mu\nu\alpha\beta}$, the ansatz is the following

$$\begin{aligned} \Gamma_{ct.1}^{(3)}[g, A] = \int d^4x \sqrt{g} \left[C^S F_{\mu\nu} \log(\nabla^2/\mu^2) F^{\mu\nu} \frac{1}{\nabla^2} R \right. \\ \left. + C^{Ric} F_{\mu}^{\beta} \log(\nabla^2/\mu^2) F^{\alpha\mu} \frac{1}{\nabla^2} R_{\alpha\beta} \right. \\ \left. + C^{Riem} F_{\alpha}^{\beta} \log(\nabla^2/\mu^2) F^{\mu\nu} \frac{1}{\nabla^2} R^{\alpha}_{\beta\mu\nu} \right] . \end{aligned} \quad (3.70)$$

A straightforward matching as before yields

$$C^S = -\frac{b_s}{4}, \quad C^{Ric} = b_s, \quad C^{Riem} = 0 . \quad (3.71)$$

Moving to the tensor $\mathcal{D}_{\mu\nu\alpha\beta}$, we first notice that in the limit $p^2 = p'^2$ the non-analytic structure becomes

$$\lim_{p'^2 \rightarrow p^2} \frac{\ln p'^2 - \ln p^2}{p^2 - p'^2} = -\frac{1}{p^2} \quad (3.72)$$

which enables us to propose the following ansatz

$$\begin{aligned} \Gamma_{ct.2}^{(3)}[g, A] = \int d^4x \sqrt{g} \left[T^S F_{\mu\nu} \frac{1}{\nabla^2} F^{\mu\nu} R + T^{Ric} F_{\mu}^{\beta} \frac{1}{\nabla^2} F^{\alpha\mu} R_{\alpha\beta} \right. \\ \left. + T^C F_{\alpha}^{\beta} \frac{1}{\nabla^2} F^{\mu\nu} C^{\alpha}_{\beta\mu\nu} \right] . \end{aligned} \quad (3.73)$$

We choose to work directly in the conformal basis, since it is more convenient. The matching yields

$$T^S = \frac{b_s}{12}, \quad T^{Ric} = 0, \quad T^C = -\frac{b_s}{4} \quad . \quad (3.74)$$

The same result holds for fermions, substituting b_f for b_s .

3.7 Remarks on the trace anomaly

In this section we explore the conformal transformation properties of the different terms in the action⁶. We find an interesting dichotomy regarding the terms that give rise to the anomaly in response to conformal transformations. This requires a separate treatment of scale (global) and Weyl (local) transformations. Since the seminal work of Deser, Duff and Isham [10], there has been a consistent effort to understand the precise form of the non-local effective action that gives rise to gravitational anomalies. In [13], anomalies were geometrically classified to fall into two types. Type A anomalies arise from scale-invariant actions, i.e. invariant under a global Weyl rescaling. These are unique and strictly proportional to the Euler density of the dimension. On the other hand, type B anomalies arise from scale-dependent actions⁷ but the local anomaly itself when *densitized* is invariant under local Weyl transformations. For example, for a massless minimally coupled scalar in 2D the anomaly reads

$$T_\mu{}^\mu = \frac{1}{24\pi} R \quad (3.75)$$

whose density $\sqrt{g}R$ is indeed the Euler density in 2D. So this is a type A anomaly, and one can check easily that the non-local Polyakov action [65] giving rise to the anomaly is scale-invariant. Reigert, following Polyakov, constructed a non-local action in 4D by integrating the anomaly [24]. However, the Riegert action was criticized in [13, 66, 17] based on several reasons while others [9, 22] argued for its validity.

⁶See a parallel discussion in [11].

⁷This also means that the action carries an explicit dependence on the renormalization scale μ .

The QED trace anomaly falls into type B since its densitized version is indeed (locally) conformally invariant, and according to the above classification the generating non-local action should be scale-dependent. We show below that the two non-local structures present in the action are required to generate the correct trace relation whether one performs a global or local conformal transformation. Remarkably, the different terms have completely different behavior under both types of transformations. In particular, the trace relation is generated from the logarithmic non-locality under a scale transformation while the massless pole non-locality is responsible for the latter under local ones.

3.7.1 Weyl transformations

Let us commence by considering local transformations. Under an infinitesimal transformation, we have

$$\delta_\sigma g_{\mu\nu} = 2\sigma(x)g_{\mu\nu} \quad (3.76)$$

which leads to the following transformation of the Christoffel symbol

$$\delta_\sigma \Gamma_{\mu\nu}^\lambda = \delta_\mu^\lambda \nabla_\nu \sigma + \delta_\nu^\lambda \nabla_\mu \sigma - g_{\mu\nu} \nabla^\lambda \sigma \quad (3.77)$$

From these one readily determines the transformation of the different curvature tensors. The ones we need are

$$\delta_\sigma R_{\mu\nu} = 2\nabla_\mu \nabla_\nu \sigma + g_{\mu\nu} \nabla^2 \sigma, \quad \delta_\sigma R = 6\nabla^2 \sigma - 2\sigma R \quad (3.78)$$

Another object we will need its transformation is the d' Alembertian operator acting on different tensors, in particular, 2-forms

$$\delta_\sigma (\nabla^2 A_{\mu\nu}) = -2\sigma \nabla^2 A_{\mu\nu} - 2(\nabla^2 \sigma) A_{\mu\nu} - 2(\nabla_\mu \sigma) \nabla^\lambda A_{\lambda\nu} + 2(\nabla_\nu \sigma) \nabla^\lambda A_{\mu\lambda} \quad (3.79)$$

where it is understood that $A_{\mu\nu}$ is invariant. Once again, let us apply the transformation to the quadratic action

$$\delta_\sigma^{(2)}\Gamma[e^{2\sigma}g, A] = \frac{b_s}{4} \int d^4x \sqrt{g} \int_0^\infty dm^2 F^{\mu\nu} (\nabla^2 + m^2)^{-1} (\delta_\sigma \nabla^2) (\nabla^2 + m^2)^{-1} F_{\mu\nu} \quad . \quad (3.80)$$

Counting the powers of curvature is very important at this stage. The function $\sigma(x)$ counts as a power of the curvature which means that we can freely commute covariant derivatives. For example,

$$[\nabla_\mu, (\nabla^2 + m^2)^{-1}] \sim \mathcal{O}(R) \quad . \quad (3.81)$$

Using eq. (3.79) and integrating by parts, we find

$$\begin{aligned} \delta_\sigma^{(2)}\Gamma[e^{2\sigma}g, A] &= \frac{b_s}{4} \int d^4x \sqrt{g} \int_0^\infty dm^2 F^{\mu\nu} (\nabla^2 + m^2)^{-1} (\nabla^2 + m^2)^{-1} \\ &\quad (-2\sigma \nabla^2 F_{\mu\nu} - 2(\nabla^2 \sigma) F_{\mu\nu} + 2\sigma \nabla^\lambda \nabla_\mu F_{\lambda\nu} - 2\sigma \nabla^\lambda \nabla_\nu F_{\mu\lambda}) \quad . \end{aligned} \quad (3.82)$$

Now we employ the Bianchi identity

$$\nabla_\mu F_{\lambda\nu} + \nabla_\nu F_{\mu\lambda} + \nabla_\lambda F_{\nu\mu} = 0 \quad (3.83)$$

to find

$$\delta_\sigma^{(2)}\Gamma[e^{2\sigma}g, A] = -\frac{b_s}{2} \int d^4x \sqrt{g} (\nabla^2 \sigma) F^{\mu\nu} \frac{1}{\nabla^2} F_{\mu\nu} \quad . \quad (3.84)$$

Although a prescription to integrate over (dm^2) might not seem obvious with the inverse operators present in eq. (3.82), one could easily check the above equation

by linearilizing eq. (3.82) around flat space. It is very important to notice that the above computation clearly shows that under the local transformation the log piece does not give rise to the anomaly as it does not possess the correct pole structure. Moreover, we show next that eq. (3.84) cancels identocally against the contribution coming from the transformation of the counter-term.

Indeed we need not worry about terms containing the Weyl tensor. Moreover, from the transformation listed in eq. (3.78) one easily finds

$$\delta_\sigma^{(3)} \Gamma_{log}[e^{2\sigma} g, A] = \delta_\sigma^{(3)} \Gamma_{ct.1}[e^{2\sigma} g, A] = 0 \quad (3.85)$$

given that the field strength is on-shell

$$\nabla_\mu F^{\mu\nu} = 0 \quad . \quad (3.86)$$

The other counter-term transforms as

$$\delta_\sigma^{(3)} \Gamma_{ct.2}[e^{2\sigma} g, A] = \frac{b_s}{2} \int d^4x \sqrt{g} (\nabla^2 \sigma) F^{\mu\nu} \frac{1}{\nabla^2} F_{\mu\nu} \quad (3.87)$$

exactly cancelling eq. (3.84) as promised.

Lastly, the massless pole non-locality of eq. (3.63) is the piece that yields the correct trace. To this order in the curvature we only need to keep the $\delta_\sigma R = 6\nabla^2 \sigma + \dots$ term in the transformation of eq. (3.78), and neglect the variation of $1/\nabla^2$. Doing this yields

$$\delta_\sigma^{(3)} \Gamma_{pole}[e^{2\sigma} g, A] = -\frac{b_s}{2} \int d^4x \sqrt{g} \sigma F^{\mu\nu} F_{\mu\nu} \quad . \quad (3.88)$$

which yields the desired trace. In order to see this more simply, and make contact with the literature, we can show that all corrections to this result are higher order in

the curvature by employing the Riegert action [24]. By defining the Paneitz operator [67]

$$\Delta_4 = \nabla^2 \nabla^2 + 2\nabla_\mu (R^{\mu\nu} - \frac{1}{3}g^{\mu\nu}R)\nabla_\nu \quad (3.89)$$

and

$$\mathcal{R} = \sqrt{-g} \left(\nabla^2 R - \frac{3}{2}G \right) \quad (3.90)$$

where G is the Gauss-Bonnet term

$$G = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta} R_{\alpha\beta} + R^2 \quad (3.91)$$

we can see that the Riegert form of this action

$$\Gamma_R[g, A] = \int d^4x \sqrt{g} (\bar{P}^S F_{\mu\nu} F^{\mu\nu} \frac{1}{\Delta_4} \mathcal{R}) \quad (3.92)$$

is equivalent to the first term of eq. (3.63) up to terms which are higher order in the curvature, $\Gamma_R[g, A] = \overset{(3)}{\Gamma}_{pole}[g, A] + \mathcal{O}(F^2 R^2)$. With this form, one can show without approximation [67] that

$$\delta_\sigma \frac{1}{\Delta_4} = 0 \quad , \quad \delta_\sigma \mathcal{R} = 6\Delta_4 \sigma \quad (3.93)$$

yielding

$$\delta_\sigma \Gamma_R[g, A] = -\frac{b_s}{2} \int d^4x \sqrt{g} \sigma F^{\mu\nu} F_{\mu\nu} \quad . \quad (3.94)$$

The expansion in the curvature has yielded a term which, to this order in the curvature, is equivalent to the Reigert action.

Now we know that a conformal variation of a generic action reads

$$\delta_\sigma S = - \int d^4x \sqrt{g} \sigma T_\mu{}^\mu \quad (3.95)$$

and thus indeed eq. (3.94) (likewise eq. (3.88)) yields the correct trace relation.

3.7.2 Scale transformations

A global scale transformation can take a couple of forms. One involves the scaling relations shown in eq. (3.10). It is simple to see that this transformation leaves all terms invariant, except the covariant logarithm. The logarithmic terms inside the square brackets [...] of eq. (3.14) are both shifted by $\ln \nabla^2 \rightarrow \ln \nabla^2 - \ln \lambda^2$, but $\ln \lambda^2$ cancels out leaving the whole expression invariant. So in contrast to the above Weyl transformation, this form of rescaling yields an anomaly that comes from the covariant logarithm.

Interestingly in the presence of the metric, there is another way to achieve a global scale transformation. In this case the transformation on the metric acts as follows

$$g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu} \quad (3.96)$$

where σ is a constant, *not* necessarily infinitesimal. This may seem like a sub-case of the Weyl transformation, but in fact it is distinct [5, 6]. Computationally, a distinction arises in that derivatives of σ vanish, so that many of the integration-by-parts steps from the previous section are not available.

In this case, the transformation properties of the different curvature tensors proceeds easily

$$R_{\mu\nu} \rightarrow R_{\mu\nu}, \quad R \rightarrow e^{-2\sigma} R, \quad C^\mu{}_{\nu\alpha\beta} \rightarrow C^\mu{}_{\nu\alpha\beta} \quad . \quad (3.97)$$

With these relations in hand, we can apply a scale transformation to the covariant action to recover the trace relation. We start with the quadratic action

$$\delta_\sigma \Gamma^{(2)}[e^{2\sigma}g, A] = -\sigma \frac{b_i}{2} \int d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu} . \quad (3.98)$$

All terms with the form factor $1/\nabla^2$ are scale invariant, hence

$$\delta_\sigma \Gamma_{pole}^{(3)}[e^{2\sigma}g, A] = 0, \quad \delta_\sigma \Gamma_{ct.2}^{(3)}[e^{2\sigma}g, A] = 0 \quad (3.99)$$

while terms with the form factor $\ln \nabla^2/\nabla^2$ cancel each other identically as described above

$$\delta_\sigma \Gamma_{log}^{(3)}[e^{2\sigma}g, A] = -\delta_\sigma \Gamma_{ct.1}^{(3)}[e^{2\sigma}g, A] . \quad (3.100)$$

The anomalous trace of the energy-momentum tensor is easily determined from eq. (3.95) and hence eq. (3.98) correctly reproduces the trace relation

$$T_\mu{}^\mu = \frac{b_i}{2} F_{\mu\nu} F^{\mu\nu} . \quad (3.101)$$

Again it is the logarithm which is the determining factor for the anomaly.

3.8 Summary

We have used a method which we refer to as *non-linear completion* in order to match the one-loop perturbative expansion of the QED effective action to a covariant expansion in the generalized curvatures. Within this procedure, the matching has been unique and relatively simple to implement. The results are given in eqs. (3.17), (3.14) and (3.15). These summarize the one-loop perturbative calculation involving one gravitational vertex.

The effective action also encodes the anomaly structure of the theory. For the anomaly, the important aspect is to generalize the feature that appeared as $\ln \square$ in flat space. Our generalized result eq. (3.14) contains many terms when expressed in terms of covariant derivatives and curvatures. All of these are required in order to both match the one loop perturbative calculation and to respect general covariance. There is also an interplay between these terms and various forms of scale and/or conformal invariance. There is a dispute in the literature about whether the anomaly comes from logarithmic terms or from the Riegert action, e.g. see [13, 66, 17] and [9, 22]. In our explicit computation, we showed that both forms are required in order for the action to respond properly to different types of transformation.

Given the simplicity of the perturbative result eq. (3.7), and the complexity of the expansion in the curvature eq. (3.14), one suspects that there is a better covariant representation for this result. However, the expansion in the curvature is one of the few covariant approximation schemes available and therefore needs to be well explored. We are not prepared to propose an improved representation at this stage, and are only trying to match the perturbative result to the standard form found when performing an expansion in the curvature. We (hopefully) reserve this improved representation to a future publication.

In addition, we note that some of the higher order terms in the curvature expansion have the potential to be singular in the infrared, and these higher order terms have only been lightly explored. It is precisely the anomalous portion of the action that is going to be used in chapter 5 to realize inflationary magneto-genesis.

CHAPTER 4

NON-LOCAL QUANTUM EFFECTS IN COSMOLOGY: QUANTUM MEMORY, NON-LOCAL FLRW EQUATIONS AND SINGULARITY AVOIDANCE

4.1 Introduction

Massless particles can propagate over long distances, and loops involving massless particles generate nonlocal effects. In cosmology, where the evolution of the scale factor depends only on time, this means that loops can generate temporal non-localities. There will be modifications to the FLRW (Friedmann, Lemaître, Robertson, Walker) equations governing the scale factor $a(t)$, which in the classical theory are local differential equations. The effects of loops will generate new contributions where the equation for the scale factor depends on what the scale factor was doing in the past. We refer to this effect as the quantum memory of the scale factor and it is the subject of the present paper. Such non-local effects are calculable, even if we do not know the full theory of quantum gravity, because they come from the low energy portion of the effective field theory [1] where the interactions are those of general relativity.

Quantum non-local effects produce modifications to standard cosmological behavior at scales below, but approaching, the Planck scale. In an expanding universe, we explore how classical behavior emerges from the quantum regime. In a contracting universe, singularities are inevitable in the classical theory, as shown by the Hawking-Penrose singularity theorems [3]. We study whether quantum effects could lead to the avoidance of singularities. Our work contains some approximations, described below, but within the context of those approximations it does seem that quantum effects do lead to non-singular bounce solutions in at least some situations.

We will provide results for all forms of matter. However, two cases are of particular importance. One is obviously pure gravity, studying the effects of graviton loops. The other is the case of a large number of matter fields. Conceptually this situation is distinctive because when the number (N) of matter fields is large, the non-local quantum effects become important at a scale M_P/\sqrt{N} , at which point general relativity itself can be treated classically. For example, in such a theory the effect of the graviton vacuum polarization from N scalar particles can be summed to produce a modification to the graviton propagator

$$\frac{1}{q^2} \rightarrow \frac{1}{q^2 - \frac{GNq^2}{120\pi} \log(-q^2/\mu^2)} . \quad (4.1)$$

The logarithmic term is crucial for restoring unitarity to scattering amplitudes in this theory [68, 69]. It is the momentum space equivalent of the non-local terms that we will be studying in this paper. We are interested in the effect of this loop, not in scattering amplitudes but in cosmology. The large N limit is also relevant for the physical Universe, as the Standard Model has roughly a hundred effective degrees of freedom (fermions, vectors and scalars, as defined in section 4) producing quantum effects that are larger than graviton loops¹. We also display results for the Standard Model set of particles.

The study of quantum field theory and gravity is a vast subject - many fundamental developments can be traced in the references of books such as [58, ?, 60]. In connection with non-localities, we should mention some previous work in particular. As described in the previous chapters, Barvinsky, Vilkovisky and collaborators have developed powerful heat kernel techniques to uncover non-local effects. We compare our results with theirs in section 4. Espriu and collaborators [46, 47] have made important preliminary investigations of possible non-local effects during inflation. We

¹At the energy scales being probed, the Standard Model particles can be treated as massless.

are building on these earlier results. In addition there are a wide variety of works in non-local models which were cited in the previous chapter. These however are of a quite different character than the quantum effects that we study.

The plan of the paper is as follows. In sections 4.2 and 4.3, we first treat simple perturbation theory around flat space. This is useful to show the nature of the non-locality in time, and to show how one obtains causal behavior in the equations of motion. We then provide a non-linear form of this result, matching to the heat kernel methods in section 4.4, with the corresponding non-linear FLRW equations of motion being displayed in section 4.5. The expanding universe emerging from the quantum regime is studied in section 4.6, while section 4.7 is devoted to the exploration of singularity avoidance in a collapsing phase. In this paper, we discuss matter and radiation dominated FLRW cosmologies, reserving de Sitter cosmology for future work because of the extra complication of the de Sitter case [70, 71, 72, 73, 74, 75, 76, 77, 78]. Comments, caveats and further work are discussed in the summary.

4.2 Perturbative analysis

We first start with a perturbative treatment of the graviton vacuum polarization. This provides us with a basis for later treatment of the non-linear equations, separating the non-local effect from the renormalization of the local terms in the action. It also allows us to explore the impact of using the appropriate field theoretic formalism to generate causal behavior for cosmology in the next section.

We compute perturbatively the effective action for a massless free scalar field minimally coupled to gravity with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (4.2)$$

After performing the functional integral integrating out the scalar field, the operator of interest reads

$$\begin{aligned}
\mathcal{D} &= \sqrt{g}(\square) \\
&= \sqrt{g}g^{\mu\nu} (\partial_\mu \partial_\nu - \Gamma_{\mu\nu}^\alpha \partial_\alpha) .
\end{aligned} \tag{4.3}$$

The last equality holds because the covariant d' Alembertian acts on a scalar field. The metric is expanded around flat space (we use the mostly minus signature)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} . \tag{4.4}$$

Likewise, the differential operator can be expanded in powers of $h_{\mu\nu}$ to yield

$$\mathcal{D} = \partial^2 + \delta^{(1)} + \delta^{(2)} + \mathcal{O}(h^3) \tag{4.5}$$

where,

$$\partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu, \quad \delta^{(1)} = -h^{\mu\nu} \partial_\mu \partial_\nu + \frac{1}{2} h \partial^2 - \eta^{\mu\nu} \underline{\Gamma}_{\mu\nu}^\alpha \partial_\alpha \tag{4.6}$$

$$\begin{aligned}
\delta^{(2)} &= h^{\mu\nu} h_\nu^\alpha \partial_\mu \partial_\alpha - \frac{1}{2} h h^{\mu\nu} \partial_\mu \partial_\nu + \left(\frac{1}{4} h_{\mu\nu} h^{\mu\nu} + \frac{1}{8} h^2 \right) \partial^2 \\
&\quad + \left(h^{\mu\nu} + \frac{1}{2} h \eta^{\mu\nu} \right) \underline{\Gamma}_{\mu\nu}^\alpha \partial_\alpha - \eta^{\mu\nu} \underline{\Gamma}_{\mu\nu}^\alpha \partial_\alpha .
\end{aligned} \tag{4.7}$$

The indices are raised and lowered using the flat metric, and we have defined

$$\underline{\Gamma}_{\mu\nu}^\alpha = \frac{1}{2} (\partial_\mu h_\nu^\alpha + \partial_\nu h_\mu^\alpha - \partial^\alpha h_{\mu\nu}) \tag{4.8}$$

$$\underline{\Gamma}_{\mu\nu}^\alpha = -\frac{1}{2} h^{\alpha\beta} (\partial_\mu h_{\nu\beta} + \partial_\nu h_{\mu\beta} - \partial_\beta h_{\mu\nu}) . \tag{4.9}$$

To find the effective action, we take the logarithm of the differential operator and expand in powers of $h_{\mu\nu}$ to find

$$Tr(\log \mathcal{D}) = Tr(\log \partial^2) + Tr \left(G\delta^{(1)} + G\delta^{(2)} - \frac{1}{2} G\delta^{(1)} G\delta^{(1)} \right) + \mathcal{O}(h^3) \tag{4.10}$$

In the above, G is the Feynman propagator of a massless scalar. Terms with one propagator vanish when regularized dimensionally. The first non vanishing contribution is at second order in $h_{\mu\nu}$. We find at this order

$$Tr(\log \mathcal{D}) = -\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} h^{\mu\nu}(k) h^{\alpha\beta}(-k) \int \frac{d^4 p}{(2\pi)^4} \frac{V_{\mu\nu}(k, p) V_{\alpha\beta}(k, p)}{(p^2 + i0)((p+k)^2 + i0)} \quad (4.11)$$

where

$$V_{\mu\nu}(k, p) = p_\mu p_\nu - \frac{1}{2} \eta_{\mu\nu} p^2 + \frac{1}{2} k_\mu p_\nu + \frac{1}{2} k_\nu p_\mu - \frac{1}{2} \eta_{\mu\nu} k \cdot p. \quad (4.12)$$

This can be calculated straightforwardly, with the final result

$$Tr(\log \mathcal{D}) = -\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} h^{\mu\nu}(k) h^{\alpha\beta}(-k) T_{\mu\nu\alpha\beta}(k) \quad (4.13)$$

where

$$\begin{aligned} T_{\mu\nu\alpha\beta}(k) = & \frac{i}{3840\pi^2} \left(\frac{1}{\bar{\epsilon}} - \log \left(\frac{-k^2}{\mu^2} \right) \right) \\ & [k^4 (6\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}) + 8k_\mu k_\nu k_\alpha k_\beta \\ & - k^2 (6k_\mu k_\nu \eta_{\alpha\beta} + 6k_\alpha k_\beta \eta_{\mu\nu} + k_\mu k_\alpha \eta_{\nu\beta} + k_\mu k_\beta \eta_{\nu\alpha} \\ & + k_\nu k_\alpha \eta_{\mu\beta} + k_\nu k_\beta \eta_{\mu\alpha})] \end{aligned} \quad (4.14)$$

and

$$\frac{1}{\bar{\epsilon}} \equiv \frac{1}{\epsilon} - \gamma + \log 4\pi \quad (4.15)$$

with $2\epsilon = 4 - d$.

In order to write the effective action, we transition back to position space. The momentum factors turn into derivatives acting on the external field. For example, the divergent term can be written as

$$S_{div} = \frac{1}{3840\pi^2} \frac{1}{\epsilon} \int d^4x \left[2\partial_\mu \partial_\nu h^{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \frac{3}{2} \partial^2 h \partial^2 h + \frac{1}{2} \partial^2 h_{\mu\nu} \partial^2 h^{\mu\nu} - 3\partial_\mu \partial_\nu h^{\mu\nu} \partial^2 h - \partial_\mu \partial^\nu h_{\alpha\nu} \partial^\mu \partial_\beta h^{\beta\alpha} \right]. \quad (4.16)$$

The divergent contribution to the effective action goes into the renormalization of local operators in the gravitational action. Counting the number of derivatives in the above expression shows that the local operator we seek is composed of terms proportional to R^2 . Hence, we seek the expansions of the different invariants up to second order in h .

$$\begin{aligned} R &= -\partial_\mu \partial_\nu h^{\mu\nu} + \partial^2 h \\ R^2 &= \partial_\mu \partial_\nu h^{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} - 2\partial^2 h \partial_\mu \partial_\nu h^{\mu\nu} + \partial^2 h \partial^2 h \end{aligned} \quad (4.17)$$

and

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2} (-\partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} + \partial^2 h_{\mu\nu} + \partial_\mu \partial_\nu h) \\ 2R_{\mu\nu} R^{\mu\nu} &= \frac{1}{2} \partial^2 h \partial^2 h + \frac{1}{2} \partial^2 h_{\mu\nu} \partial^2 h^{\mu\nu} + \partial_\mu \partial_\nu h^{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} \\ &\quad - \partial^2 h \partial_\mu \partial_\nu h^{\mu\nu} - \partial_\mu \partial_\nu h^{\mu\alpha} \partial^\beta \partial^\nu h_{\beta\alpha}. \end{aligned} \quad (4.18)$$

Note that we have freely integrated by parts in these expressions. The gravitational effective Lagrangian is

$$S = \int d^4x \sqrt{g} \left(\frac{1}{16\pi G} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} \right). \quad (4.19)$$

Matching with the perturbative calculation allows us to identify the renormalized coupling constants as

$$c_1 = c_1^r(\mu) - \frac{1}{3840\pi^2} \left(\frac{1}{\epsilon} - \gamma + \log 4\pi \right) \quad (4.20)$$

$$c_2 = c_2^r(\mu) - \frac{1}{1920\pi^2} \left(\frac{1}{\epsilon} - \gamma + \log 4\pi \right). \quad (4.21)$$

Notice the explicit scale-dependence of the renormalized parameters which ensures the scale-independence of the effective action.

The non-local part of the effective action follows closely from the divergent part because the coefficient of $\log(-k^2)$ is uniquely tied to the divergent $1/\bar{\epsilon}$ term. Following the logarithm in the transition to coordinate space, we find

$$\begin{aligned} S_{non-local} = \frac{1}{3840\pi^2} \int d^4x \int d^4y [& \partial^2 h(x) \bar{\mathfrak{L}}(x-y) \partial^2 h(y) + \partial_\mu \partial_\nu h^{\mu\nu}(x) \bar{\mathfrak{L}}(x-y) \partial_\alpha \partial_\beta h^{\alpha\beta}(y) \\ & - \partial^2 h(x) \bar{\mathfrak{L}}(x-y) \partial_\mu \partial_\nu h^{\mu\nu}(y) - \partial_\mu \partial_\nu h^{\mu\nu}(x) \bar{\mathfrak{L}}(x-y) \partial^2 h(y) \\ & + \partial_\mu \partial^\nu h_{\nu\alpha}(x) \bar{\mathfrak{L}}(x-y) \partial^\mu \partial_\beta h^{\alpha\beta}(y) + \partial_\mu \partial^\nu h_{\nu\alpha}(x) \bar{\mathfrak{L}}(x-y) \partial^\alpha \partial_\beta h^{\mu\beta}(y) \\ & - \partial_\mu \partial^\nu h_{\nu\alpha}(x) \bar{\mathfrak{L}}(x-y) \partial^2 h^{\mu\alpha}(y) - \partial_\mu \partial^\nu h_{\nu\alpha}(x) \bar{\mathfrak{L}}(x-y) \partial^\mu \partial^\alpha h(y) \\ & - \partial^2 h_{\mu\nu}(x) \bar{\mathfrak{L}}(x-y) \partial^\mu \partial_\beta h^{\nu\beta}(y) - \partial_\mu \partial_\nu h(x) \bar{\mathfrak{L}}(x-y) \partial^\mu \partial_\beta h^{\nu\beta}(y) \\ & + \frac{1}{2} \partial^2 h_{\mu\nu}(x) \bar{\mathfrak{L}}(x-y) \partial^2 h^{\mu\nu}(y) + \frac{1}{2} \partial^2 h_{\mu\nu}(x) \bar{\mathfrak{L}}(x-y) \partial^\mu \partial^\nu h(y) \\ & + \frac{1}{2} \partial_\mu \partial_\nu h(x) \bar{\mathfrak{L}}(x-y) \partial^2 h^{\mu\nu}(y) + \frac{1}{2} \partial_\mu \partial_\nu h(x) \bar{\mathfrak{L}}(x-y) \partial^\mu \partial^\nu h(y)] \end{aligned} \quad (4.22)$$

where

$$\bar{\mathfrak{L}}(x-y) = - \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \log \left(\frac{-k^2}{\mu^2} \right). \quad (4.23)$$

We note that each term in the momentum-space expression contributes to more than one term in the above position-space expression, so it needs some work to pass to

eq. (4.22). Using the curvature expansions listed above, we easily realize a possible non-linear form of the non-local action

$$S_{non-local} = \frac{1}{3840\pi^2} \int d^4x \int d^4y (\sqrt{g(x)}\sqrt{g(y)})^{\frac{1}{2}} \quad (4.24)$$

$$\left[R(x)\bar{\mathfrak{L}}(x-y)R(y) + 2R_\nu^\mu(x)\bar{\mathfrak{L}}(x-y)R_\mu^\nu(y) \right] .$$

We note that the perturbative calculation alone does not enable us to differentiate between alternate forms of the non-linear completion which differ by application of the Gauss-Bonnet identity. The Gauss-Bonnet identity relates local terms involving the curvatures squared, but cannot be used for non-local terms. Indeed in section 4.4, we will see that the form of eq. (4.24) is not fully correct and we will display the appropriate non-linear completion. Note that the $\log \mu^2$ portion of $\bar{\mathfrak{L}}(x-y)$ corresponds to a delta function and hence is a finite local addition to c_1 and c_2 . For N scalar fields, the actions S_{div} and $S_{nonl-local}$ are multiplied by a factor of N .

4.3 Causal behavior

The effective action of the previous section is *not* appropriate for generating causal effects in the equations of motion. The reason is that the Feynman propagators involve both advanced and retarded solutions, and any variation of the effective action with respect to a field at time t will involve the non-local effects both before and after t . This is appropriate for scattering amplitudes but not for the equations of motion. Rather one needs to calculate the effects of the loops on the equations of motion using the in-in (or Schwinger-Keldysh or closed-time-path) formalism [79, 80, 81, 82, 83, 84, 85, 86], which is designed to produce causal behavior. This is relatively more complicated and unfamiliar than usual perturbation theory. However, Bavinsky and Velkovsky [81, 82] suggest the simple prescription - that one merely varies the effective action (which they calculate in Euclidean space) and then afterwards imposes causal

behavior or scattering behavior on the final result when one writes the answer in Lorentzian space. We perform the calculation below and confirm the validity of their prescription. The reader who is not interested in the details can skip to the results of Eqs. (4.34), (4.36) and (4.39), which are reasonably intuitive.

The in-in formalism deals not with the effective action but with expectation values. It is well known that the variation of the effective action yields the energy-momentum tensor of the quantum fields, and hence our strategy is to use the in-in formalism to calculate the *causal* energy-momentum tensor. The set-up of the formalism is laid out in the appendix, and our starting point is the expectation value of a Heisenberg operator

$$\langle \mathcal{O}(t) \rangle = {}_I \langle \Phi(-\infty) | S^\dagger(t, -\infty) \mathcal{O}_I(t) S(t, -\infty) | \Phi(-\infty) \rangle_I. \quad (4.25)$$

It is very useful to insert the identity operator in the form $S^\dagger(\infty, t) S(\infty, t) = 1$ to the left of the operator

$$\langle \mathcal{O}(t) \rangle = {}_I \langle \Phi(-\infty) | S^\dagger(\infty, -\infty) T [\mathcal{O}_I(t) S(\infty, -\infty)] | \Phi(-\infty) \rangle_I. \quad (4.26)$$

One then obtains various propagators - the normal Feynman propagators associated with purely time-ordered contractions, and others associated with mixed contractions as will be explicitly shown below.

For our case, we are calculating the expectation value of the energy momentum tensor to lowest order in the external field $h_{\mu\nu}$. In order to derive the result, one uses eq. (4.26) with the operator \mathcal{O}_I being $T_{\mu\nu}$ and considers contractions with the interactions contained in S or S^\dagger . In our case, we only have two bubble diagrams each with two propagators where one space-time point is the observation time. The

first diagram arises from the $\mathcal{O}(h_{\mu\nu})$ term in $S(\infty, -\infty)$ and therefore contains the usual Feynman propagators. One obtains the non-local part of the expectation value

$$\begin{aligned} \langle T_{\mu\nu}^{NL}(x) \rangle = & \frac{1}{3840\pi^2} \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \log\left(\frac{-k^2}{\mu^2}\right) h^{\alpha\beta}(-k) [8k_\mu k_\nu k_\alpha k_\beta \\ & - k^2(6k_\alpha k_\beta \eta_{\mu\nu} + 6k_\mu k_\nu \eta_{\alpha\beta} + k_\nu k_\beta \eta_{\mu\alpha} \\ & + k_\alpha k_\nu \eta_{\mu\beta} + k_\mu k_\beta \eta_{\alpha\nu} + k_\alpha k_\mu \eta_{\beta\nu}) \\ & + k^4(\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\alpha\nu} + 6\eta_{\mu\nu} \eta_{\alpha\beta})] \end{aligned} \quad (4.27)$$

where

$$h^{\alpha\beta}(-k) = \int d^4y e^{ik \cdot y} h^{\alpha\beta}(y). \quad (4.28)$$

This can be obtained either by direct calculation or by varying the effective action of the previous section. If we specialize to gravitational fields $h_{\mu\nu}(x)$ which are independent of spatial coordinates, we have

$$\begin{aligned} \langle T_{\mu\nu}^{NL}(t) \rangle = & \frac{1}{3840\pi^2} \int \frac{d\omega}{2\pi} e^{-i\omega t} \left[\log\left(\frac{-\omega^2}{\mu^2}\right) \right] h^{\alpha\beta}(-\omega) \\ & [8k_\mu k_\nu k_\alpha k_\beta - k^2(6k_\alpha k_\beta \eta_{\mu\nu} + 6k_\mu k_\nu \eta_{\alpha\beta} \\ & + k_\nu k_\beta \eta_{\mu\alpha} + k_\alpha k_\nu \eta_{\mu\beta} + k_\mu k_\beta \eta_{\alpha\nu} + k_\alpha k_\mu \eta_{\beta\nu}) \\ & + k^4(\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\alpha\nu} + 6\eta_{\mu\nu} \eta_{\alpha\beta})] \end{aligned} \quad (4.29)$$

where now the momentum is purely temporal $k^\mu = (\omega, \vec{0})$ and

$$h^{\alpha\beta}(-\omega) = \int dt' e^{i\omega t'} h^{\alpha\beta}(t'). \quad (4.30)$$

Note that this result displays non-causal behavior because it is sensitive to times both before and after t .

The second diagram arises from the $\mathcal{O}(h_{\mu\nu})$ term in $S^\dagger(\infty, -\infty)$. To calculate such diagram, the algebra of contractions needs a modification to Wick's theorem to incorporate anti-time-ordered product of operators. The details of the construction is included in the appendix. Only the last two terms in eq. (D.5) involving products of positive-frequency Wightman functions contribute to the calculation. We denote this particular Wightman function by an underline

$$\underline{\phi(x)\phi(y)} \equiv [\phi^+(x), \phi^-(y)] = \langle 0 | [\phi^+(x), \phi^-(y)] | 0 \rangle = \langle 0 | \phi^+(x) \phi^-(y) | 0 \rangle = \langle 0 | \phi(x) \phi(y) | 0 \rangle \quad (4.31)$$

and it explicitly reads

$$\underline{\phi(x)\phi(y)} = 2\pi \int \frac{d^4 p}{(2\pi)^4} \theta(p^0) \delta(p^2) e^{-ip \cdot (x-y)}. \quad (4.32)$$

The result is a simple addition to the expectation value, with a total result that reads

$$\begin{aligned} \langle T_{\mu\nu}^{NL}(t) \rangle = & \frac{1}{3840\pi^2} \int \frac{d\omega}{2\pi} e^{-i\omega t} \left[\log \left(\frac{-\omega^2}{\mu^2} \right) + 2i\pi\theta(-\omega) \right] h^{\alpha\beta}(-\omega) \\ & [8k_\mu k_\nu k_\alpha k_\beta - k^2(6k_\alpha k_\beta \eta_{\mu\nu} + 6k_\mu k_\nu \eta_{\alpha\beta} + k_\nu k_\beta \eta_{\mu\alpha} + k_\alpha k_\nu \eta_{\mu\beta} \\ & + k_\mu k_\beta \eta_{\alpha\nu} + k_\alpha k_\mu \eta_{\beta\nu}) + k^4(\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\alpha\nu} + 6\eta_{\mu\nu} \eta_{\alpha\beta})]. \quad (4.33) \end{aligned}$$

Again we transform the above expression to real space, with momentum factors turning into derivatives. This yields

$$\langle T_{\mu\nu}^{NL}(t) \rangle = \int dt' \mathcal{L}(t-t') \mathcal{D}_{\mu\nu\alpha\beta} h^{\alpha\beta}(t') \quad (4.34)$$

where

$$\begin{aligned}
\mathcal{D}_{\mu\nu\alpha\beta} = & \frac{1}{3840\pi^2} [8\partial_\mu\partial_\nu\partial_\alpha\partial_\beta + \partial^4(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\alpha\nu} + 6\eta_{\mu\nu}\eta_{\alpha\beta}) \\
& - \partial^2(6\partial_\alpha\partial_\beta\eta_{\mu\nu} + 6\partial_\mu\partial_\nu\eta_{\alpha\beta} + \partial_\nu\partial_\beta\eta_{\mu\alpha} + \partial_\alpha\partial_\nu\eta_{\mu\beta} \\
& + \partial_\mu\partial_\beta\eta_{\alpha\nu} + \partial_\alpha\partial_\mu\eta_{\beta\nu})]
\end{aligned} \tag{4.35}$$

and where we have identified our key non-local function

$$\mathfrak{L}(t - t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \left[\log\left(\frac{-\omega^2}{\mu^2}\right) + 2i\pi\theta(-\omega) \right]. \tag{4.36}$$

In order to evaluate this integral, we first note that the usual $i\epsilon$ prescription for the Feynman propagator implies

$$\log\left(\frac{-\omega^2}{\mu^2}\right) = \log\left(\frac{\omega^2}{\mu^2}\right) - i\pi, \quad -i\pi + 2i\pi\theta(-\omega) = -i\pi \operatorname{sgn}(\omega) \tag{4.37}$$

and hence

$$\begin{aligned}
\mathfrak{L}(t - t') &= -2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \left[\log\left(\frac{\mu}{|\omega|}\right) + \frac{i\pi}{2} \operatorname{sgn}(\omega) \right] \\
&= -2 \mathcal{P} \frac{\theta(t - t')}{t - t'}.
\end{aligned} \tag{4.38}$$

Here \mathcal{P} denotes the principal value distribution [87] defined by

$$\mathcal{P} \frac{\theta(t - t')}{t - t'} = \lim_{\epsilon \rightarrow 0} \left[\frac{\theta(t - t' - \epsilon)}{t - t'} + \delta(t - t') (\log(\mu\epsilon) + \gamma) \right]. \tag{4.39}$$

Unlike eq. (4.23), this function is clearly causal and real. It also provides a precise definition of how the non-local integration is to be performed as the term with the delta function yields the desired feature that the non-local effect is finite. This result verifies the Bavinsky-Velkovisky procedure of varying the effective action and then simply imposing causal behavior.

4.4 Non-linear completion and quasi-local form

The perturbative analysis gives us a reference for the form of the non-local quantum effects and the precise causal prescription. In order to have a more complete description appropriate for application to FLRW cosmology, we can match to the work by Barvinsky, Vilkovisky and collaborators mentioned in the previous chapter. These authors have explored non-local aspects of the heat kernel expansion and expressed the results in quasi-local form. Normally the heat kernel methodology is used to capture local quantum effects. For example, the second coefficient in the expansion of the one-loop effective action, commonly called $a_2(x)$, gives the divergent terms that go into the renormalization of the effective Lagrangian quadratic in curvature invariants. For massless fields, this is the only one-loop divergence. However, the asymptotic form of the heat kernel expansion also reveals non-analytic terms. These are expanded in powers of the curvature. The results that we are studying are second order in the curvature. As described in the previous chapter one is able to obtain the non-analytic terms and display the results using quasi-local actions of the form

$$S_{QL} = \int d^4x \sqrt{g} \left[R \log \left(\frac{\square}{\mu^2} \right) R \right]. \quad (4.40)$$

Despite the fact that this appears to be expressed in local form, we show below that matching to the perturbative calculation of the preceding sections confirms that it corresponds to a non-local effect. The quasi-local forms provide a non-linear covariant completion of the perturbative calculation.

If we resolve the operator $\log \left(\frac{\square}{\mu^2} \right)$ by introducing position space eigenstates we find

$$S_{QL} = \int d^4x \sqrt{g(x)} R(x) \int d^4y \sqrt{g(y)} \langle x | \log \left(\frac{\square}{\mu^2} \right) | y \rangle R(y). \quad (4.41)$$

Here the states are normalized covariantly

$$\langle x|y\rangle = \frac{\delta^{(4)}(x-y)}{\left(\sqrt{g(y)}\sqrt{g(x)}\right)^{1/2}}. \quad (4.42)$$

If we also define

$$\langle x|\log\left(\frac{\square}{\mu^2}\right)|y\rangle = \left(\sqrt{g(y)}\sqrt{g(x)}\right)^{-1/2} L(x,y;\mu) \quad (4.43)$$

we can write the action in explicitly non-local form

$$S_{NL} = \int d^4x \int d^4y \sqrt{g(x)}^{1/2} R(x) L(x,y;\mu) \sqrt{g(y)}^{1/2} R(y). \quad (4.44)$$

Again, we note that the $\log \mu$ dependence in these equations corresponds to a local effect. Here, we see that replacing the covariant d' Alembertian in eq. (4.41) by its Minkowski counterpart yields the first term in eq. (4.24).

There are three terms in the general non-local Lagrangian. Reverting temporarily to quasi-local form, these can be written as

$$S_{QL} = \int d^4x \sqrt{g} \left(\alpha R \log\left(\frac{\square}{\mu_\alpha^2}\right) R + \beta R_{\mu\nu} \log\left(\frac{\square}{\mu_\beta^2}\right) R^{\mu\nu} + \gamma R_{\mu\nu\alpha\beta} \log\left(\frac{\square}{\mu_\gamma^2}\right) R^{\mu\nu\alpha\beta} \right) \quad (4.45)$$

where α, β, γ are numerical coefficients which we will display below. We allow for the possibility that the renormalization scales are different for the three terms as the coupling constants of the local Lagrangian could be measured at different scales. For local terms, there are only two quadratic invariants to be considered due to the Gauss-Bonnet identity which holds strictly in four dimensions

$$\int d^4x \sqrt{g} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = \int d^4x \sqrt{g} [4R_{\mu\nu} R^{\mu\nu} - R^2] + \text{total derivative}. \quad (4.46)$$

While eq. (4.45) is simple and easy to apply, an alternate form reveals some interesting physics. For this form we employ the Weyl tensor in four dimensions

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - \frac{1}{2}(g_{\mu\alpha}R_{\nu\beta} - g_{\mu\beta}R_{\nu\alpha} + g_{\nu\alpha}R_{\mu\beta} - g_{\nu\beta}R_{\mu\alpha}) + \frac{1}{6}R(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \quad (4.47)$$

to rewrite

$$S_{QL} = \int d^4x \sqrt{g} \left[\bar{\alpha} R \log\left(\frac{\square}{\mu_1^2}\right) R + \bar{\beta} C_{\mu\nu\alpha\beta} \log\left(\frac{\square}{\mu_2^2}\right) C^{\mu\nu\alpha\beta} + \bar{\gamma} (R_{\mu\nu\alpha\beta} \log(\square) R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} \log(\square) R^{\mu\nu} + R \log(\square) R) \right]. \quad (4.48)$$

This form has several theoretical advantages. Here the last term, similar in structure to the Gauss-Bonnet term, does not have any μ dependence because its local form does not contribute to the equations of motion. The FLRW metric that we use below is conformally flat and thus its Weyl tensor vanishes. Thus the second term will not contribute to our cosmological application. In turn this tells us that the cosmology study dependence on local short distance physics comes through the first term only, and there is only one parameter $\mu_1 \equiv \mu$ which describes this local term. In addition this first term is not generated by conformally invariant field theories (fermions, photons and conformally coupled scalars) and their quantum effects will be purely non-local. The coefficients in these two different bases are related by

$$\alpha = \bar{\alpha} + \frac{\bar{\beta}}{3} + \bar{\gamma}, \quad \beta = -2\bar{\beta} - 4\bar{\gamma}, \quad \gamma = \bar{\beta} + \bar{\gamma}. \quad (4.49)$$

We can identify the coefficients in the non-local Lagrangian because the logarithms are tied to the divergences in the one-loop effective action, as shown by the perturbative calculation. The latter have been calculated in the background field method, and

results are known before the Gauss-Bonnet identity has been applied². For example, the divergent effective Lagrangian for a massless field reads

$$\mathcal{L}_{div} = \sqrt{|g|} \frac{a_2(x)}{16\pi^2 \epsilon} . \quad (4.50)$$

The coefficient $a_2(x)$ is known for scalars, fermions and photons [58]

$$a_2^S(x) = \frac{1}{180} \left(\frac{5}{2} R^2 - R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \right) \quad (4.51)$$

$$a_2^F(x) = \frac{1}{360} (-5R^2 + 8R_{\mu\nu} R^{\mu\nu} + 7R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}) \quad (4.52)$$

$$a_2^V(x) = \frac{-1}{180} (20R^2 - 86R_{\mu\nu} R^{\mu\nu} + 11R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}) . \quad (4.53)$$

Here, the result for fermions assumes a four-component spinor field. The result for the massless vector field also includes the ghost contribution, which is twice the scalar field result with an appropriate minus sign. Finally, the classic paper by 't Hooft and Veltman [88] gave the result for gravitons only after using the Gauss-Bonnet relation, but the general result has since been calculated, see e.g. [89]. This enables us to read off the result for gravitons which also includes the ghost contribution

$$a_2^G(x) = \frac{215}{180} R^2 - \frac{361}{90} R_{\mu\nu} R^{\mu\nu} + \frac{53}{45} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} . \quad (4.54)$$

In table (4.1), we collect the coefficients of different fields.

The results are shown for a scalar with a coupling $\xi R\phi^2$ and the parameter ξ enters the α couplings

$$\alpha = \bar{\alpha} = \frac{(6\xi - 1)^2}{2304\pi^2} \quad (4.55)$$

²This background field method resolves the problem of identifying the complete form of the non-linear completion that we had in discussing eq. (4.24).

	α	β	γ	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\gamma}$
Scalar	$5(6\xi - 1)^2$	-2	2	$5(6\xi - 1)^2$	3	-1
Fermion	-5	8	7	0	18	-11
Vector	-50	176	-26	0	36	-62
Graviton	430	-1444	424	90	126	298

Table 4.1. Coefficients of different fields. All numbers should be divided by $11520\pi^2$.

with β , γ , $\bar{\beta}$, $\bar{\gamma}$ independent of ξ . Unless stated otherwise, our results are presented for a minimally coupled scalar ($\xi = 0$), while a conformally coupled scalar has $\xi = 1/6$. For conformally invariant fields the coefficient $\bar{\alpha}$ will vanish. Because the FLRW metric is conformally flat, the coupling $\bar{\beta}$ does not contribute to our analysis as mentioned previously. This leaves only the coefficient $\bar{\gamma}$ as the active parameter. For N_S scalars, N_f fermions and N_V gauge bosons, this coupling has the value

$$\bar{\gamma} = -\frac{1}{11520\pi^2}[N_S + 11N_f + 62N_V]. \quad (4.56)$$

Note that all conformally invariant matter fields carry the same sign of $\bar{\gamma}$ and will have similar effects, differing just in magnitude. Moreover, this case is independent of the parameter μ because the Gauss-Bonnet non-local term (the one proportional to $\bar{\gamma}$) has no local contribution to the equations of motion.

Finally, we can also add up the contributions of all the SM particles (plus the graviton) to find effective SM coefficients which are calculated as follows

$$\alpha_{SM} = N_S\alpha_S + N_l\alpha_F + N_cN_q\alpha_F + N_V\alpha_V + \alpha_G \quad (4.57)$$

and likewise for β and γ . Here, we have broken the fermion contribution up into quark and lepton terms $N_f = N_l + N_cN_q$ where N_l is the number of leptons, N_q and N_c are the numbers of quarks and colors respectively. For the standard model with a minimally coupled Higgs, these numbers read

$$N_S = 4, \quad N_l = 6, \quad N_c = 3, \quad N_q = 6, \quad N_V = 12. \quad (4.58)$$

Hence, for this case we find

$$\alpha_{SM} = \frac{-7}{1152\pi^2}, \quad \beta_{SM} = \frac{287}{1440\pi^2}, \quad \gamma_{SM} = \frac{-17}{1440\pi^2} \quad (4.59)$$

for the Standard Model particles alone, or also including gravitons

$$\alpha_{SMG} = \frac{-3}{128\pi^2}, \quad \beta_{SMG} = \frac{71}{960\pi^2}, \quad \gamma_{SMG} = \frac{1}{40\pi^2}. \quad (4.60)$$

For a conformally coupled Higgs field we find the conformally invariant result (without gravitons) $\bar{\alpha}_c = 0$ and

$$\bar{\gamma}_c = -\frac{253}{2880\pi^2}. \quad (4.61)$$

Of course, we recognize that we expect to find new particles between the weak scale and the Planck scale, and so these numbers would likely be modified when the formalism is applied near the Planck scale.

4.5 Non-local FLRW equations

The equations of motion can be obtained by varying the effective action, specializing to the FLRW metric and then imposing causal prescription. We do that in this section, displaying the corresponding non-local effects in the FLRW equations.

We are working to second order in the curvature. As we uncovered in the previous section, graphs with triangular topology generate terms at third order in the curvature expansion. Some of these effects could be incorporated through a modification of the non-local function $L(x, y; \mu)$ to depend on the background curvature. However,

since the quasi-local action is already quadratic in the curvature, we will proceed by dropping such higher curvature terms and employing the approximation

$$L(x, y; \mu) \approx \bar{\mathfrak{L}}(x - y) \quad (4.62)$$

when we pass to the non-local form of the action. This approximation confines our study to quadratic corrections to the gravitational action. Because the non-local function $\bar{\mathfrak{L}}(x - y)$ falls as $1/(t - t')$ our approximation captures the behavior where the integrand is the largest, but will differ past the Hubble time where the integrated curvature becomes large. With this approximation, the non-local function depends only on $|x - y|$ so that

$$\frac{\partial}{\partial x} \bar{\mathfrak{L}}(x - y) = -\frac{\partial}{\partial y} \bar{\mathfrak{L}}(x - y) \quad (4.63)$$

allowing derivatives acting on $\bar{\mathfrak{L}}$ to be transferred to derivatives acting on the scale factor $a(t')$.

The non-linear FLRW equations can be derived in one of two ways. One can vary $g_{\mu\nu}$ in general and then specialize to the FLRW metric. Equivalently one may use the general metric $ds^2 = f^2(t)dt^2 - a^2(t)d^2\mathbf{x}$, varying with respect to both f and a and then setting $f = 1$ at the end. Either way we obtain the 0 - 0 component of the modified equations of motion

$$\begin{aligned} \frac{3a\dot{a}^2}{8\pi} + N \left[6(\sqrt{a}\ddot{a})_t \int dt' \mathfrak{L}(t - t') \mathcal{R}_1 + 6 \left(\frac{\dot{a}^2}{\sqrt{a}} \right)_t \int dt' \mathfrak{L}(t - t') \mathcal{R}_2 \right. \\ \left. + 12(\sqrt{a}\dot{a})_t \int dt' \mathfrak{L}(t - t') \frac{d\mathcal{R}_3}{dt'} \right] = a^3 \rho. \end{aligned} \quad (4.64)$$

Here, N represents the number of particles and the different functions read

$$\mathcal{R}_1 = -\sqrt{a}\ddot{a}(6\alpha + 2\beta + 2\gamma) - \frac{\dot{a}^2}{\sqrt{a}}(6\alpha + \beta) \quad (4.65)$$

$$\mathcal{R}_2 = -\sqrt{a}\ddot{a}(12\alpha + \beta - 2\gamma) - \frac{\dot{a}^2}{\sqrt{a}}(12\alpha + 5\beta + 6\gamma) \quad (4.66)$$

$$\mathcal{R}_3 = \sqrt{a}\ddot{a}(6\alpha + 2\beta + 2\gamma) + \frac{\dot{a}^2}{\sqrt{a}}(6\alpha + \beta). \quad (4.67)$$

For mixed combinations of particles, N can be absorbed in the definitions of α_{tot} , β_{tot} , γ_{tot} as described in the previous section. As described in section 4.3, the equations of motion must use the causal non-local function

$$\mathfrak{L}(t - t') = \lim_{\epsilon \rightarrow 0} \left[\frac{\theta(t - t' - \epsilon)}{t - t'} + \delta(t - t') \log(\mu_R \epsilon) \right] \quad (4.68)$$

obtained therein and we absorbed Euler's constant into the renormalization scale μ_R . We finally remind that in a covariant theory the space-space equation of motion is *not* an independent equation. This is not true in our case since we employed an approximation for the function $L(x, y; \mu)$ that manifestly breaks general covariance but the treatment is consistent at second order in the curvatures.

4.6 Emergence of classical behavior

In assessing the effects of the non-local behavior, we treat the new terms as a perturbation in the equation of motion. They have certainly been calculated as perturbations to the leading behavior, so this is a conservative approach. We will address the limits of such perturbative treatment in the final section.

In an expanding universe, the quantum effects are expected to be felt most in the early phases of expansion when the curvature is largest. In principle, these effects could change the character of the expansion, perhaps by an instability. In addition, the memory effect which is sensitive to past values of the curvature with the weight $1/(t - t')$ could have an effect which builds up with time. Within our approximation,

neither of these happens. We will explore the situation by 'switching on' the non-local effect at the Planck time. The evolution of the scale factor is influenced by the non-local effect very close to the Planck time. However, subsequent evolution turns essentially classical and the effect of non-local terms fades away.

We will treat both a dust-filled universe and a radiation-filled universe. We set $G = 1$ in the numerical evaluation. The lower limit of the integrals is then taken to be $t_0 = 1$ which corresponds to the Planck time as mentioned earlier. In treating the new terms as a perturbation, we use the known classical solutions as input to the integrands, integrating up to the observation time t . This allows the integrals over time to be done by hand and converts the integro-differential equation into a simpler differential equation, albeit one with a reference back to a starting time t_0 .

For a scalar field, we use the coefficients listed in the previous section to find the functions

$$\begin{aligned}\mathcal{R}_1 &= \frac{-1}{\pi^2} \left(\frac{\sqrt{a}\ddot{a}}{384} + \frac{7\dot{a}^2}{2880\sqrt{a}} \right), & \mathcal{R}_2 &= \frac{-1}{\pi^2} \left(\frac{3\sqrt{a}\ddot{a}}{640} + \frac{31\dot{a}^2}{5760\sqrt{a}} \right) \\ \mathcal{R}_3 &= \frac{1}{\pi^2} \left(\frac{\sqrt{a}\ddot{a}}{384} + \frac{7\dot{a}^2}{2880\sqrt{a}} \right).\end{aligned}\tag{4.69}$$

If we treat the matter input as dust, the classical solution is $a(t) = (t/t_0)^{2/3}$ and thus the $0 - 0$ equation of motion reads

$$a\dot{a}^2 - \frac{N_S}{2430\pi} \left(\frac{19E_1(t; t_0)}{t_0^2 t} + \frac{26E_2(t; t_0)}{t_0^2} \right) = \frac{8\pi\rho_0}{3}.\tag{4.70}$$

We note that the normalization time is chosen to coincide with the initial time t_0 , and hence the energy density is $\rho_0 = 1/(6\pi t_0^2)$. We also defined the functions

$$\begin{aligned}E_1(t; t_0) &= \frac{\log(\mu_R t) + \log(t/t_0 - 1)}{t} \\ E_2(t; t_0) &= \frac{\log(\mu_R t) + \log(t/t_0 - 1) + (t/t_0 - 1)}{t^2}.\end{aligned}\tag{4.71}$$

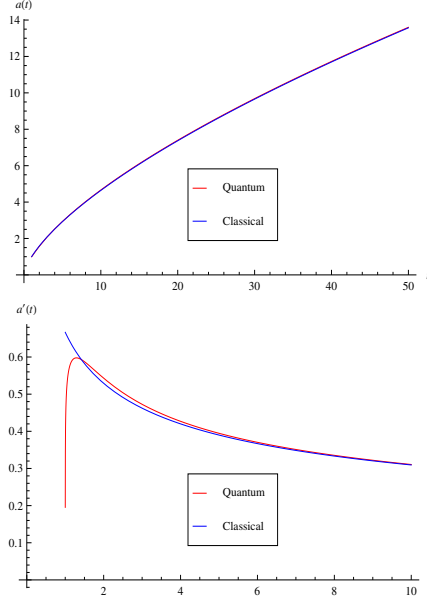


Figure 4.1. The evolution of the scale factor and its time derivative in an expanding dust-filled universe for $N=10$.

Results are shown in figures 4.1-4.3 for different numbers of scalar fields. In each case, the quantum correction provides an initial deviation from the straight classical behavior. However, as the scale factor evolves, the curvature decreases and the evolution is driven by the lowest order FLRW equation with the usual classical form. This is perhaps expected but indicates, at least within our approximations, that the quantum terms do not destabilize the evolution of the scale factor. One can see that increasing the number of scalars increases the magnitude of the quantum effect, but does not change the character of the effect. For these plots we have used $\mu_R = 1$, but a reasonable range of other values of μ_R leads to qualitatively similar results.

We also show the case of pure graviton loops in figure 4.4. This is qualitatively similar to that of scalars, with the graviton making a somewhat larger effect than would an individual scalar.

For a radiation dominated universe the situation is also interesting,

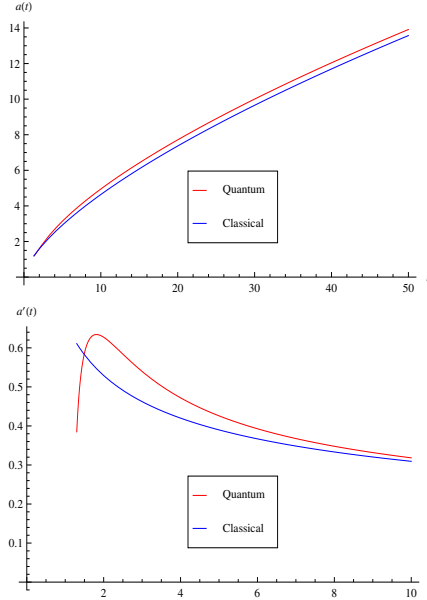


Figure 4.2. The evolution of the scale factor and its time derivative in an expanding dust-filled universe for $N=100$.

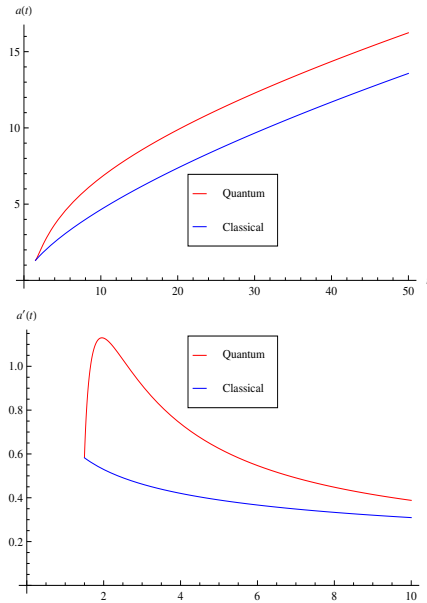


Figure 4.3. The evolution of the scale factor and its time derivative in an expanding dust-filled universe for $N=1000$.

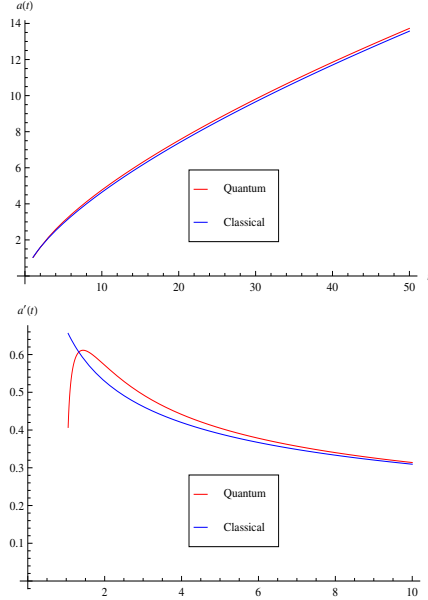


Figure 4.4. The evolution of the scale factor and its time derivative in an expanding dust-filled universe with quantum graviton loops.

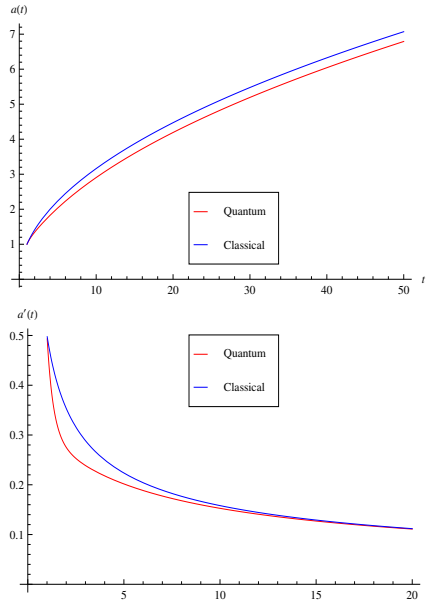


Figure 4.5. The evolution of the scale factor and its time derivative in an expanding dust-filled universe with quantum graviton loops.

$$a\dot{a}^2 - \frac{N_S}{1152\pi} \left(\frac{E_{5/4}(t; t_0)}{t_0^{3/2}t^{5/4}} - \frac{E_{9/4}(t; t_0)}{t_0^{3/2}t^{1/4}} \right) = \frac{8\pi\rho_0}{3a}. \quad (4.72)$$

In this case the energy density is $\rho_0 = 3/(32\pi t_0^2)$. The expansion functions read

$$E_{5/4}(t, t_0) = \frac{1}{t^{5/4}} \left[\log(\mu_R t) + \log \left(\frac{t^{1/4} - t_0^{1/4}}{t^{1/4} + t_0^{1/4}} \right) + 4 \left(\frac{t}{t_0} \right)^{1/4} + 2 \arctan \left(\frac{t_0}{t} \right)^{1/4} \right. \\ \left. + \log(8) - 4 - \frac{\pi}{2} \right] \quad (4.73)$$

$$E_{9/4}(t, t_0) = \frac{1}{t^{9/4}} \left[\log(\mu_R t) + \log \left(\frac{t^{1/4} - t_0^{1/4}}{t^{1/4} + t_0^{1/4}} \right) + 4 \left(\frac{t}{t_0} \right)^{1/4} + \frac{5}{4} \left(\frac{t}{t_0} \right)^{5/4} \right. \\ \left. + 2 \arctan \left(\frac{t_0}{t} \right)^{1/4} + \log(8) - \frac{21}{4} - \frac{\pi}{2} \right]. \quad (4.74)$$

The equation of motion shows the interesting feature that the dependence on $\log \mu_R$ cancels out, which means that the effect is purely non-local. The reason is that the classical solution in the case of radiation $a(t) = (t/t_0)^{1/2}$ furnishes an exact solution to *local* quadratic gravity. We show results for the expanding radiation universe with a thousand scalar fields in figure 4.5. The quantum effects are somewhat smaller in the radiation case, but have the same qualitative behavior as the dust-filled universe. Situations involving fermions, photons and gravitons are also quite similar and we do not display figures for each case.

Overall these results are satisfying in that the quantum corrections are well behaved and turn off as we enter the period of classical evolution.

4.7 Contracting universe and the possibility of a bounce

Of perhaps greater interest is the physics of a collapsing phase. Here the initial conditions are purely classical and the natural evolution bring the universe into the quantum regime. The classical evolution is headed towards a singularity - the big crunch. We will explore this case and see that within our approximation the quantum effects can lead to an avoidance of the singularity.

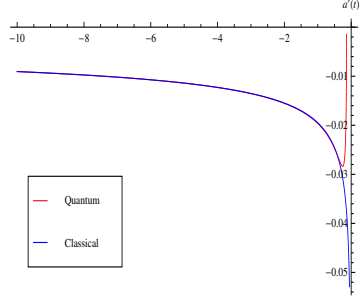


Figure 4.6. Collapsing dust-filled universe with $\mu_R = 1$ and a single scalar field. The time derivative of the scale factor quickly stops diverging when the quantum correction becomes active.

Our procedures are similar to those of the previous section. We input the classical solution into the non-local functions. For scalar fields in the case of collapsing dust, this results in

$$a\dot{a}^2 - \frac{N_S}{2430\pi} \left(\frac{19C_1(t)}{t_0^2 t} + \frac{26C_2(t)}{t_0^2} \right) = \frac{8\pi\rho_0}{3}. \quad (4.75)$$

The collapse functions are defined as

$$C_1(t) = \frac{\log(-\mu_R t)}{t}, \quad C_2(t) = \frac{\log(-\mu_R t) + 1}{t^2}. \quad (4.76)$$

We note that the initial time in this case is taken to be $-\infty$ as there is no need to cut off the non-local integrals. The normalization time t_0 can be chosen arbitrarily but in a regime where the classical behavior remains dominant.

As an example of what happens in a collapsing phase, consider the case $N_S = 1$, $\mu_R = 1$, shown in figure 4.6. Here we see that $\dot{a}(t)$, which is diverging classically, slows down and in fact turns around. This appears as a bouncing solution rather than a singular one. Because of the choice $\mu_R = 1$, $\log \mu_R = 0$ and there is no local effect in these units.

If we change the number of scalars, we can lower the energy that this behavior occurs at, in accord with the expected N scaling. This is shown in figure 4.7 by

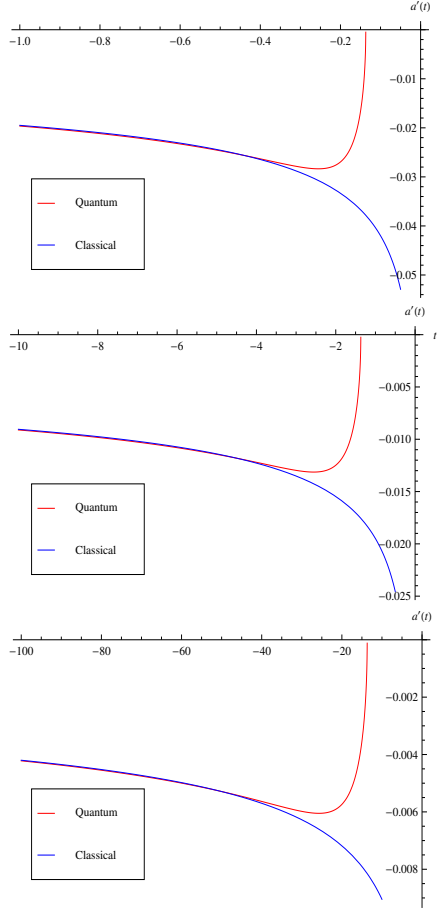


Figure 4.7. Varying both the scale μ and the number of scalar particles N_S in a collapsing dust-filled universe. The plots from top to bottom involve $(N_S = 1, \mu_R = 1)$, $(N_S = 10^2, \mu_R = 0.1)$ and $(N_S = 10^4, \mu_R = 0.01)$. Note the change of scale along the time axis in the figures. The results illustrate the similarity of the quantum corrections with an energy scale that scales as $E \sim M_P/\sqrt{N}$.

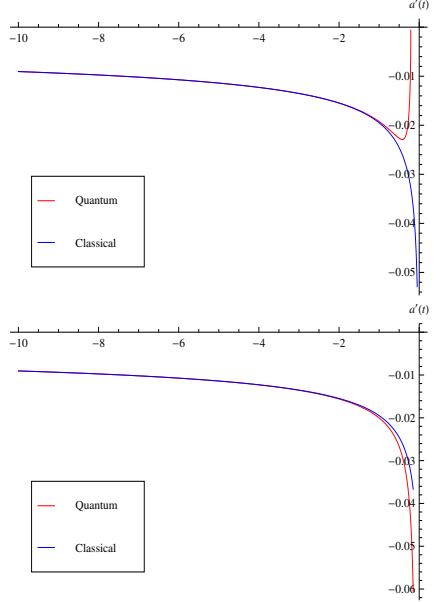


Figure 4.8. Varying the scale μ_R in a collapsing dust-filled universe, with $\mu_R = 0.1$ on top and $\mu_R = 10$ on the bottom.

adjusting N_S and μ_R together such that the number of scalars changes by a factor of 100 between figures, while μ_R changes by a factor of 10. This modifies the location of the bounce in a predictable way. The figures look similar even though the horizontal scale changes by a factor of 10 between pictures. The physics does scale as $1/\sqrt{N_S}$ as long as we rescale μ_R by this factor, and we can have this effect occur well below the Planck scale if the number of scalars is large enough.

However, not all cases lead to singularity avoidance. There is a dependence on the scale μ_R and for some choices the local terms overwhelm the effect of the non-local terms. This can be seen in figure 4.8. Here the local terms drive the scale factor in a more singular direction and the singularity happens more rapidly. It is possible that yet higher orders in the curvature tensors could eventually solve this and perhaps also remove singularities in these cases. However, we do not explore this possibility further in this paper.

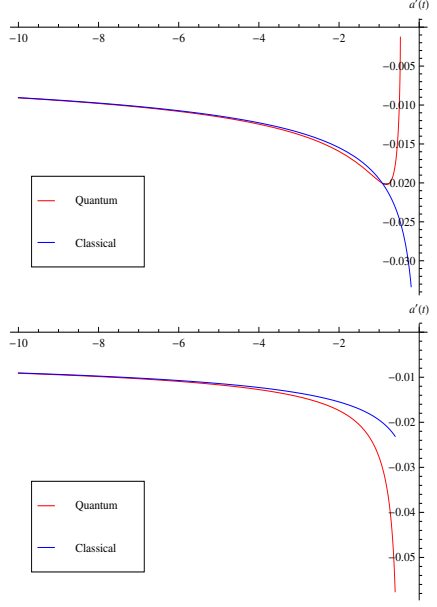


Figure 4.9. The effect of graviton loops on a dust-filled universe. These have $\mu_R = 0.1$ on top and $\mu_R = 1$ on the bottom.

The bounce is also seen in the case of pure gravity, figure 4.9. The nonlocal coefficients for the graviton are larger than those for a single scalar and the change in the scale factor happens at a slightly earlier time than the single scalar case.

A very interesting case is the Standard Model with a conformally coupled Higgs. As explained in section 5, this situation is purely non-local and completely independent of the parameter μ_R because in the basis of eq. (4.48), only the Gauss-Bonnet non-local term contributes and this has no local effect. So this prediction is particu-

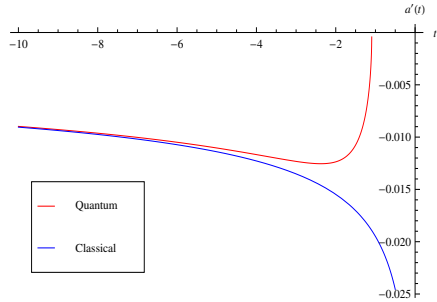


Figure 4.10. Collapsing dust-filled universe with the Standard Model particles and a conformally coupled Higgs. The result is purely non-local and hence independent of any scale μ_R .

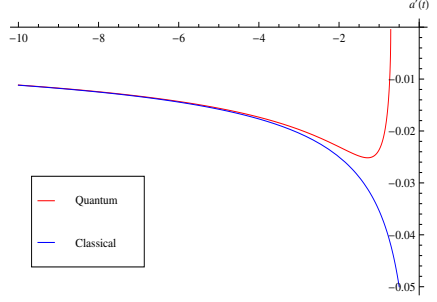


Figure 4.11. Collapsing radiation-filled universe with gravitons only considered.

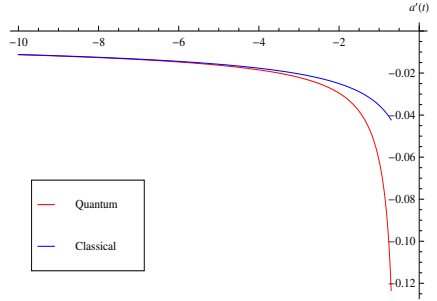


Figure 4.12. Collapsing radiation-filled universe all the Standard Model particles included, as well as graviton loops.

larly simple and beautiful. The result with all the Standard Model particles is shown in figure 4.10 and demonstrates the non-local bounce effect in a parameter independent fashion. Note that all conformally coupled fields contribute with the same sign, so that increasing the number of matter fields will always enhance this effect³.

For a radiation-filled universe, the effect is always independent of the scale μ_R . With just graviton loops, we see a very similar bounce, see figure 4.11. Unfortunately, matter fields have an effect in the opposite direction, and overwhelm the effects of gravity. So with the full set of Standard Model particles plus gravitons, the net effect does not lead to singularity avoidance, as shown in figure 4.12.

³Gravitons are not conformally coupled, but we have checked that their quantum effect (with μ near unity) is smaller than the effect of the Standard Model particles, and do not change the character of figure 4.10.

4.8 Summary

Quantum loops bring a unique feature to cosmology, i.e. non-locality. The local classical theory is supplemented by effects which depend on the past behavior of the scale factor. Because of the power-counting theorems of general relativity, these effects are small except at times of large curvature. However, with enough light fields they can become important below the Planck scale.

Within the context of matter and radiation dominated FRLW cosmologies, we have explored the non-local effects that correspond most closely to the graviton vacuum polarization. Our work has been perturbative, in that we treat the new non-local effects to first order only. This is appropriate for a correction that has been calculated at one-loop order only. Actually the large N case can be used to argue that the one-loop result is the most important in the limit of large N . The one-loop integral is proportional to GN . For matter fields that have only gravitational interactions, higher loops would either involve extra gravitons in loops (which do not bring extra factors of N) or would be the iteration of the simple vacuum polarization. Counting the powers of G and N reveals that the iteration of the one-loop diagram is the only effect of order $(GN)^n$, with other diagrams suppressed by at least a power of N .

In addition to the unavoidable use of perturbation theory, we have also approximated the non-local function by its free field behavior. The use of the full propagators is not realistically tractable in a general FLRW space time. The approximation amounts to neglecting higher powers of the curvature which appear in the propagators. This is reasonable when paired with the general use of perturbation theory. The approximation should be good in the region where the non-local integrand, $1/(t - t')$, is the largest. We have not seen any problematic effects from the long-time tail of this integrand.

The most interesting effect uncovered is the tendency towards singularity avoidance in some collapsing FLRW universes. The classical theory, with only the Einstein

action, collapses towards an inevitable singularity. The quantum effects can oppose this collapse and can turn around converging geodesics. Because of the perturbative treatment, we cannot be certain of the ultimate fate of such effect, but within the limits of our approximations it appears to have the characteristics of a bounce.

There is clearly much more work needed to fully understand the effects of quantum non-locality in general relativity. Some of these aspects, especially those related to general covariance, will be explored further in chapter 6.

CHAPTER 5

INFLATIONARY MAGNETOGENESIS AND NON-LOCAL ACTIONS: THE CONFORMAL ANOMALY

5.1 Introduction

Magnetic fields of various strengths have been observed at several length scales in our Universe. For instance, in galaxies they are of the order of few μG . While the origin of galactic fields remains mysterious, it is widely accepted that a seed field which predates structure formation is required to produce the observed fields today. There exist several dynamo mechanisms able to amplify a relatively weak field to the currently observed field strengths [90]. A conservative estimate demands a field of strength 10^{-23} G at the 1 Mpc scale to appropriately seed the galactic dynamo.

On the other hand, magnetic fields also exist in the intergalactic medium (IGM) where recent bounds have been inferred in [91, 92, 93] from the lack of observation of GeV electromagnetic cascades initiated by TeV gamma rays in the IGM. These fields are especially interesting from a cosmological standpoint since it is unlikely that they are due to some astrophysical mechanism [94]. These measurements have thus reopened the door to further investigate a primordial origin of cosmic magnetic fields. Although astrophysical processes could generate the required seed for the galactic dynamo, a primordial origin remains an attractive possibility as well in this case. Having the ability to amplify quantum fluctuations, the inflationary epoch offers the perfect setting to establish magneto-genesis in the early Universe. Moreover, understanding inflationary magneto-genesis could help better constrain the landscape of inflationary models if future experiments confirmed the primordial origin of cosmic magnetic fields.

One obstacle to achieving inflationary magneto-genesis is the conformal invariance of classical electromagnetism [95]. In a spatially flat Universe, this implies that the conformal vacuum is preserved and magnetic fields can not be amplified [96]. Thus the starting point in model building is the breakage of conformal invariance. Ratra [97] studied a model with the gauge-invariant Lagrangian $-\frac{1}{4}I^2F_{\mu\nu}F^{\mu\nu}$ where I is a function of conformal time. For instance, a coupling of the gauge field to the inflaton during slow-roll would give rise to such scenario [98]. Another proposed mechanism is the axion model [99] where a psuedoscalar inflaton is coupled to $F_{\mu\nu}\tilde{F}^{\mu\nu}$. Parity violation in particular has the advantage of producing *maximally helical* fields [100] whose coherence scale grows much faster than non-helical fields during cosmic evolution. A recently proposed model [101] is a hybrid of the previous ones where a time dependent function appears in front of both the parity-preserving and parity-violating invariants. A UV realization of the latter model was proposed in the context of $\mathcal{N} = 1$ four-dimensional supergravity.

On the other hand, the breakdown of conformal symmetry takes place naturally due to vacuum fluctuations. Although less appreciated, one important aspect of anomalies is their *infrared* origin. It is precisely the low energy portion of quantum loops of massless particles that breaks the classical symmetry. In particular, this implies that any new physics that might appear in the UV would not alter the anomaly structure. By using dispersive techniques, this piece of physics was originally emphasized in [102, 103] in the context of the axial anomaly and later by [8] for the conformal anomaly.

In part I of this thesis, the infrared physics of the conformal anomaly was developed further by constructing the effective action of massless QED. The anomaly could be elegantly reproduced from the effective action that results from integrating out the massless charged particle. Being produced by long-distance fluctuations, the renormalized effective action is non-local in position space. Non-local field theories

have just started to be explored with various applications especially in cosmology [46, 47, 48, 51, 52, 54], which is surely an incomplete list. However, the construction of non-local actions over curved spaces is far from being trivial. For the QED case, this has been systematically carried out in Part I of this thesis.

To our knowledge, Dolgov [104] made the first attempt to employ the conformal anomaly to derive inflationary magnetogenesis¹. Only knowledge of the *local* anomalous operator was used in [104] with the strength of the field required to seed the galactic dynamo being highly dependent on the sign of the one loop beta function of an $SU(N)$ gauge theory². It is also worth mentioning that mechanisms including *local* gravitational-electromagnetic couplings were explored by many authors, see for example [106, 107, 108, 109]. These couplings would naturally arise if massive fields heavier than the inflationary scale are present. Another mechanism was discussed in [110] where a radiatively induced photon mass during inflation is used to generate magnetic fields.

In this chapter, we are concerned with the non-local action that generates the QED trace anomaly. If the Standard Model (SM) electroweak symmetry is unbroken during inflation, then all charged fermions are massless. In particular, integrating out the latter yields a non-local action which encodes the conformal anomaly³. We present a thorough analysis to investigate the viability of magnetogenesis during inflation using the QED trace anomaly as the driving mechanism. Although the action is very complicated, we will see that the anomalous portion is rather simple to handle during inflation assuming an exact de-Sitter phase. In particular, the constancy of

¹See [105] for a similar treatment using purely gravitational anomalies.

²Note that the author in [104] uses the beta function of the full $SU(N)$ gauge theory as the coefficient of the anomalous operator. We argue that this is inaccurate since one must formally use the electric charge beta function.

³To be precise, a photon is not an active degree of freedom in the unbroken phase. Our presentation is exploratory in this regard and more comments appear in the concluding remarks.

the scalar curvature enables the action to be written in a form similar to the models previously described which simplifies the analysis greatly. Despite this similarity, it is important to realize that the mechanism discussed here does not require any physics beyond the SM. The action is parameter-free and thus we need not worry about any possible constraints usually discussed in the model building literature. We find a rather blue spectrum at the end of inflation given that the QED beta function is positive. The evolution of the initial conditions till the present day is carried out via two pathways. We first evolve the magnetic field based on the simple requirement of flux conservation. The reheating temperature has to be relatively low to satisfy the lower bound on the IGM field reported in [91, 92, 93]. Second, we summarize the main features of the magneto-hydrodynamic (MHD) evolution [94, 111] and argue that the simple evolution is largely accurate with our initial conditions.

The plan of this chapter is as follows. We describe in some detail the non-local effective action in section 5.2. Then in section 5.3 we describe how to cast the non-local action in a simple form. The theory is canonically quantized and approximate solutions for the mode functions of the gauge field are found in section 5.4. In section 5.5 the properties of the magnetic field at the end of inflation are determined. In section 5.6 the evolution of the initial conditions are carefully carried out. In section 5.7, we test whether the present day properties of the magnetic field are consistent with the lower bound in [91, 92, 93]. We conclude and discuss future directions in section 5.8.

5.2 The non-local action

The effective action is an extremely useful object in field theory, in particular, it embodies all the effects of quantum fluctuations. By construction it is the generating functional of one-particle irreducible (1PI) correlation functions. Its prominent use is when the problem involves classical background fields and one aims to study the

effect of quantum loops in a semiclassical context. In particular, its importance in gravitational physics can not be overestimated [89, 58, 59]. Formally, one computes the effective action by integrating out a field from the path integral of the theory. If the field is heavy, the result is a local effective Lagrangian built from the light degrees of freedom organized in a derivative (energy) expansion and the cut-off of the effective theory is the mass of the heavy field. On the other hand, loops of massless fields leads to non-analyticity in momentum space or equivalently non-locality in position space as described in the previous chapters. These effects strictly arise from the infrared fluctuations of massless particles and the resulting effective Lagrangian is non-local. There has been a consistent effort to understand the construction, properties and phenomenology of non-local Lagrangians.

Anomalies in field theory remains to date an active area of research due to their wide array of applications. The common lore in the literature is that anomalies are understood through the UV properties of Feynman diagrams. Using different approaches, several authors pointed out that it is the low-energy portion of quantum loops that give rise to anomalies [102, 8]. In the gravity sector, the seminal work of Deser, Isham and Duff [10] was the first attempt to reproduce gravitational anomalies from a non-local action. On the gauge theory side and in the context of massless QED, both the non-local action and the associated energy-momentum tensor (e.m.t) were constructed in chapter 1 with the initial results displayed for flat space. Subsequently, these results were carried over to curved space employing a technique referred to as *non-linear completion* in chapter 2. The latter shares similar features with the Covariant Perturbation Theory formalism developed by Barvinsky, Vilkovisky and collaborators [36, 37, 38].

Here we only quote the main results and refer the interested reader to the previous chapters for more details. The classical theory under consideration is QED coupled to either charged scalars or fermions. For instance, in case of a charged scalar the

classical action reads

$$S = S_{EM} + \int d^4x \sqrt{g} [g^{\mu\nu} (D_\mu \phi)^\star (D_\nu \phi) - \xi \phi^\star \phi R] \quad (5.1)$$

where S_{EM} is the standard maxwell action, $D_\mu = \partial_\mu + ie_0 A_\mu$ is the gauge-covariant derivative and e_0 is the bare electric charge. For $\xi = 1/6$, the action is indeed invariant under local Weyl transformations

$$g_{\mu\nu} \rightarrow e^{2\sigma(x)} g_{\mu\nu}, \quad \phi \rightarrow e^{-\sigma(x)} \phi, \quad A_\mu \rightarrow A_\mu \quad . \quad (5.2)$$

Conformal invariance is manifest in the traceless-ness of the classical e.m.t. After integrating out the massless charged field, one ends up with a variety of terms that exhibits different behavior under conformal and scale transformations. It was shown in chapter 2 that the piece that ultimately generates the anomaly is given by

$$\Gamma_{anom.}[g, A] = S_{EM} - \frac{b_i e^2}{12} \int d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu} \frac{1}{\nabla^2} R \quad . \quad (5.3)$$

Here b_i is the leading coefficient of the electric charge beta function

$$b_s = \frac{1}{48\pi^2}, \quad b_f = \frac{1}{12\pi^2} \quad (5.4)$$

and $\nabla^2 = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d' Alembertian. We also include the Maxwell action for consistency and later usage. To see how the anomaly arises, we employ an infinitesimal conformal transformation given by

$$\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}, \quad \delta_\sigma R = 6\nabla^2 \sigma - 2\sigma R \quad . \quad (5.5)$$

A generic action transforms as follows

$$\delta_\sigma S = - \int d^4x \sqrt{g} \sigma T_\mu{}^\mu \quad . \quad (5.6)$$

and thus transforming the action in eq. (5.3) immediately yields the correct trace relation⁴

$$T_\mu{}^\mu = \frac{b_i}{2} F_{\mu\nu} F^{\mu\nu} \quad (5.7)$$

Written in this form, we say that the action in eq. (5.3) is *quasi-local*. In purely non-local form, we have

$$\Gamma_{anom.}[A] = S_{EM} - \frac{b_i e^2}{12} \int d^4x d^4y \sqrt{g(x)} \sqrt{g(y)} (F_{\mu\nu} F^{\mu\nu})_x G(x, y) R_y \quad (5.8)$$

where the propagator satisfies

$$\nabla_x^2 G(x, y) = \frac{\delta^{(4)}(x - y)}{(\sqrt{g(x)} \sqrt{g(y)})^{1/2}} \quad (5.9)$$

5.3 The set-up

Many models of magneto-genesis start with the following Lagrangian [97, 96, 99, 101]

$$\mathcal{L} = I^2(\tau) \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \quad (5.10)$$

where $I(\tau)$ is some specified function that contains the parameters of the model and indices are raised and lowered using the flat metric. Inspection of eq. (5.8) shows that we can cast the action in the form of eq. (5.10) since the scalar curvature is constant during an exact de-Sitter phase. This however requires knowledge of the propagator on a de-Sitter background which fortunately could be obtained in closed form. We show in this section how to manipulate eq. (5.8) to identify the function $I^2(\tau)$.

⁴The details of these steps are explained clearly in the previous chapters.

We work in the cosmological slice of de-Sitter and write the metric in conformal coordinates

$$ds^2 = a^2(\tau) (d\tau^2 - d\vec{x} \cdot d\vec{x}), \quad a(\tau) = (-H\tau)^{-1}, \quad -\infty < \tau < 0 \quad . \quad (5.11)$$

We start by solving for the propagator in eq. (5.9) where we impose the usual retarded boundary conditions⁵. In the above metric, eq. (5.9) becomes

$$\frac{1}{a^2(\tau)} \left(\partial^2 + \frac{2a'}{a} \partial_\tau \right) G(x, y) = \frac{\delta^{(4)}(x - y)}{a^2(\tau) a^2(\tau')} \quad (5.12)$$

where $\tau = x^0$ while $\tau' = y^0$. It suffices to determine the inverse of the operator appearing in brackets on the lhs. With flat spatial slices, the propagator can be expanded as a Fourier integral

$$G(x, y) = \int \frac{d^3 \vec{k}}{(2\pi)^3} G(\tau, \tau'; k) e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \quad . \quad (5.13)$$

Hence, the function $G(\tau, \tau'; k)$ satisfies the equation

$$\left(\frac{d^2}{d\tau^2} + k^2 - \frac{2}{\tau} \frac{d}{d\tau} \right) G(\tau, \tau'; k) = \frac{\delta(\tau - \tau')}{a^2(\tau')} \quad . \quad (5.14)$$

The retarded propagator of the operator in brackets is well known [101]. Hence, our function reads

$$\begin{aligned} G(\tau, \tau'; k) = \frac{H^2}{k^3} & \left((1 + k^2 \tau \tau') \sin k(\tau - \tau') \right. \\ & \left. + k(\tau' - \tau) \cos k(\tau - \tau') \right) \Theta(\tau - \tau') \quad . \end{aligned} \quad (5.15)$$

⁵It has been shown in the previous chapter that using the in-in formalism yields a causal prescription for the non-local functions.

We finally plug everything back in the action eq. (5.8) and notice that the $d^3\vec{y}$ -integral is trivial since the scalar curvature is constant. The integral yields a delta function $\delta^{(3)}(k)$ and so we must first expand the propagator around $\vec{k} = 0$ to find

$$\Gamma_{anom.}[A] = S_{EM} - \frac{Rb_ie^2}{36H^2} \int d^4x \sqrt{g} F^2 \int \frac{d\tau'}{\tau'^4} (\tau^3 - \tau'^3) \Theta(\tau - \tau') \quad . \quad (5.16)$$

It is gratifying to see that the answer is completely well-behaved. Now $R = 12H^2$ and thus we can now identify the time-dependent function

$$I^2(\tau) = 1 + \frac{4b_ie^2}{3} \int \frac{d\tau'}{\tau'^4} (\tau^3 - \tau'^3) \Theta(\tau - \tau') \quad . \quad (5.17)$$

The second piece in the bracket leads to a logarithmic divergence⁶. However, this causes no trouble since it is always plausible to cut off the integral at an early time τ_0 corresponding to the beginning of inflation. The effect of this arbitrary parameter on the physical observables we consider is thoroughly discussed in subsequent sections. Finally we obtain

$$I^2(\tau) = 1 + \frac{4b_ie^2}{3} \left[\frac{1}{3} \left(\frac{\tau}{\tau_0} \right)^3 - \frac{1}{3} + \ln \left(\frac{\tau_0}{\tau} \right) \right] \quad . \quad (5.18)$$

It is desirable to pause at this stage and comment on some issues regularly discussed in the model building literature. The first aspect concerns whether the theory is strongly coupled⁷ [96]. We easily see that $I^2(\tau) \geq 1$ during inflation and hence we are definitely in a weak coupling regime. This is guaranteed with a positive definite beta function. But let us now imagine that $I^2(\tau)$ was in fact less than unity due to

⁶This is not surprising since the long-time behavior of the non-local functions corresponds to the far infrared. When integrated against a source, the long-time behavior of the result might become singular.

⁷See [112] for a choice of the time-dependent function that avoids strong coupling.

a negative beta function. Even in this hypothetical situation, no problem arises in our case. The effective action is the result of integrating out the massless charged particles and thus, formally, the latter can not appear as external states in the theory. We argue that unlike magneto-genesis models the issue of strong coupling does not posit a concern all together. Along the same lines, it was shown in [113, 114] that a serious challenge to magneto-genesis models emerges if the time-dependent function is the result of coupling the gauge field to the rolling inflaton. In this case the amplified gauge field couples to the inflaton perturbations leading to observable non-gaussianities which provides an extra constraint on the parameters of any such model. Such constraints do not apply in our case as the action is parameter-free and relies only on the existence of massless charged particles during inflation. Nevertheless, the non-local coupling in eq. (5.3) inevitably contribute to the curvature perturbation⁸. This is one exciting direction that we leave for the future.

Let us now include the effect of multiple particles in the loop and define the following constant

$$\beta \equiv \frac{4}{3} \left(\sum_f g_f b_f Q_f^2 + \sum_s g_s b_s Q_s^2 \right) . \quad (5.19)$$

where $g_f(g_s)$ is the number of fermionic (scalar) internal degrees of freedom and $Q_f(Q_s)$ is the electric charge of each species. We can now rewrite eq. (5.18) as

$$I^2(\tau) = 1 + \beta \left[\frac{1}{3} \left(\frac{\tau}{\tau_0} \right)^3 - \frac{1}{3} + \ln \left(\frac{\tau_0}{\tau} \right) \right] . \quad (5.20)$$

To get an idea about the range of values β can take, let us restrict to the charged fermions in the Standard Model and find

⁸For example, see the construction in [115] and [116].

$$\beta_{SM} = \frac{4}{3} \sum_l \frac{Q_l^2}{12\pi^2} + 4 \sum_q \frac{Q_q^2}{12\pi^2} \quad (5.21)$$

where l and q refer to leptons and quarks respectively. We now use the one-loop beta function to run the electric charge from the weak scale up to the energy scale of inflation. Hence

$$\frac{1}{e^2(E_{\text{inf}})} = \frac{1}{e^2(M_Z)} - \frac{4}{3\pi^2} \ln \left(\frac{E_{\text{inf}}}{M_Z} \right) \quad (5.22)$$

where M_Z is the Z-boson mass. Using input from [117] and taking the energy scale of inflation to be above the electroweak scale yields

$$\beta_{SM} \simeq 10^{-2} \div 10^{-3} \quad . \quad (5.23)$$

5.4 Canonical Quantization

In this section we perform the quantization procedure and find approximate solutions to the mode functions. It is straightforward to derive the equations of motion from the action in eq. (5.10)

$$\partial^\mu (\eta^{\beta\nu} I^2 F_{\mu\nu}) = 0 \quad (5.24)$$

where $\partial^\mu = \eta^{\mu\alpha} \partial_\alpha$ and we kept a flat metric inside the brackets manifest so that no confusion arises. In the absence of currents, we employ Coulomb gauge $\partial^i A_i = 0$ that forces $A_0 = 0$ and hence eq. (5.24) becomes

$$\left(\partial^2 + \frac{2I'}{I} \partial_\tau \right) A_i = 0, \quad \partial^i A_i = 0 \quad . \quad (5.25)$$

The quantization of the gauge field proceeds as usual via the canonical formalism

$$\hat{A}_i(x) = \sum_{\sigma=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \epsilon_i(\mathbf{k}, \sigma) a(\mathbf{k}, \sigma) A(k, \eta) e^{i\mathbf{k}\cdot\mathbf{x}} + h.c. \quad (5.26)$$

where $a(k, \eta)$ and $a^\dagger(k, \eta)$ are creation and annihilation operators satisfying

$$[a(\mathbf{k}, \sigma), a^\dagger(\mathbf{k}', \sigma')] = \delta^{(3)}(\mathbf{k} - \mathbf{k}') \delta_{\sigma\sigma'} \quad . \quad (5.27)$$

Indeed the polarization tensors are transverse but notice here that they are covariantly normalized, in particular, they carry explicit time dependence [98]

$$\epsilon(\mathbf{k}, \sigma) \cdot \epsilon(\mathbf{k}, \sigma') = -\delta_{\sigma\sigma'} \quad . \quad (5.28)$$

Now we can define a canonically normalized mode function by

$$\tilde{A}(\mathbf{k}, \eta) = a I A(\mathbf{k}, \eta) \quad . \quad (5.29)$$

The reason the scale factor is inserted is to cancel the time dependence explicit in the polarization tensors. Now applying eq. (5.25), we find

$$\left(\partial_\tau^2 + k^2 - \frac{I''}{I} \right) \tilde{A}(k, \tau) = 0 \quad . \quad (5.30)$$

The power spectrum is readily found from the two-point function which reads

$$\langle 0 | \hat{A}^\mu(\tau, \vec{x}) \hat{A}_\mu(\tau, \vec{y}) | 0 \rangle = -\frac{2}{a^2 I^2} \int \frac{d^3 k}{(2\pi)^{3/2}} \tilde{A}(k, \tau) \tilde{A}^*(k, \tau) e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \quad . \quad (5.31)$$

From the coincidence limit, we determine the power spectrum

$$\mathcal{P}_A(k, \tau) = \sqrt{\frac{k^3 |\tilde{A}(k, \tau)|^2}{2\pi^2 a^2 I^2}} \quad . \quad (5.32)$$

5.4.1 Solving for the mode functions

Solving eq. (5.30) exactly is not possible due to the non-trivial nature of $I(\tau)$. Nevertheless, all what we really need is an approximate solution at the end of inflation

which is sufficient to determine the power spectrum as well as the amplitude of the magnetic field and its correlation length, i.e. the initial conditions. First of all, we easily find

$$\frac{I''}{I} = \frac{\beta}{2I^2} \frac{1}{\tau^2} \left[\left(1 + \frac{2\tau^3}{\tau_0^3} \right) - \frac{\beta}{2I^2} \left(1 - \frac{\tau^3}{\tau_0^3} \right)^2 \right] . \quad (5.33)$$

At the onset of inflation ($\tau \sim \tau_0$) all modes of cosmological interest are inside the horizon, i.e. $k|\tau| \gg 1$, and hence the modes reside in the Bunch-Davies vacuum. The positive energy solution to eq. (5.30) reads

$$\tilde{A}(\tau, k) \simeq \frac{1}{\sqrt{2k}} e^{-ik(\tau - \tau_i)}, \quad \tau \rightarrow \tau_0 \quad (5.34)$$

where τ_i is arbitrary and will later be chosen for convenience. As the size of the horizon decreases, the modes start to leave their vacuum state and get amplified. When a mode approaches horizon exit, we can approximate

$$\frac{I''}{I} \simeq \frac{\beta}{2I^2} \frac{1}{\tau^2} \left[1 - \frac{\beta}{2I^2} \right] \quad (5.35)$$

valid because $(\tau/\tau_0)^3 \ll 1$ at this stage. We can further process the above expression if we notice that

$$I^2 \simeq 1 + \beta N(\tau) \quad (5.36)$$

where $N(\tau)$ is the number of e-folds since the beginning of inflation. Thus the second term in the brackets in eq. (5.35) is much smaller than unity and could be dropped. This turns eq. (5.30) into a rather simple form

$$\left(\partial_\tau^2 + k^2 - \frac{\alpha(\tau)}{\tau^2} \right) \tilde{A}(k, \tau) = 0, \quad |\tau| \lesssim |\tau_k| = 1/k \quad (5.37)$$

where we defined

$$\alpha(\tau) \equiv \frac{\beta}{2[1 + \beta N(\tau)]} . \quad (5.38)$$

Eq. (5.37) is readily solved with Bessel functions if α was constant. Can we treat α as a constant? It is reasonable to adopt this approximation as the rate of change of the last term in eq. (5.37) is controlled by⁹ $1/\tau^2$. Hence

$$\begin{aligned} \tilde{A}(k, \tau) &\simeq \frac{1}{\sqrt{k}} [c_1 (-k\tau)^{1/2} J_\nu(-k\tau) + c_2 (-k\tau)^{1/2} J_{-\nu}(-k\tau)] \\ \nu &= \frac{1}{2} \sqrt{1 + 4\alpha} \end{aligned} \quad (5.39)$$

where c_1 and c_2 are constants to be determined. Notice here that the order of the Bessel functions is treated as *time-dependent*. We match the solutions and their first derivative at the time of horizon crossing, i.e. $\tau_k = -1/k$, onto the free solutions in eq. (5.34). Fixing $\tau_i = \tau_k$ we find

$$\vec{c} = \hat{\gamma}^{-1} \vec{r} \quad (5.40)$$

where $\vec{c}^T = (c_1, c_2)$ and $\vec{r}^T = (1/\sqrt{2}, i/\sqrt{2})$ while the matrix $\hat{\gamma}$ reads

$$\begin{aligned} \gamma_{11} &= J_{\nu_k}(1) & \gamma_{21} &= \frac{1}{2} J_{\nu_k}(1) + J'_{\nu_k}(1) \\ \gamma_{12} &= J_{-\nu_k}(1) & \gamma_{22} &= \frac{1}{2} J_{-\nu_k}(1) + J'_{-\nu_k}(1) . \end{aligned} \quad (5.41)$$

Here, $\nu_k = (1 + 4\alpha(\tau_k))/2$ is determined by the number of e-folds until a certain mode crosses the horizon. One can easily check that the coefficients c_1 and c_2 are

⁹One can easily check this statement by taking a time derivative of the aforementioned term and using the fact that after a few e-folds α becomes negligible *compared to unity*.

$\mathcal{O}(1)$ complex numbers and thus will not change our final results in any significant manner. Now at the end of inflation ($\tau \rightarrow \tau_e$), all modes of cosmological interest are way outside the horizon implying $-k\tau \ll 1$ and thus we can approximate the Bessel functions and turn the result into a power law. The solution multiplying c_1 contribute negligibly for the considered modes and hence the mode functions take the rather simple form

$$\tilde{A}(k, \tau_e) \simeq \frac{\bar{c}_2}{\sqrt{k}} (-k\tau_e)^{(1-\sqrt{1+4\alpha_e})/2}, \quad \bar{c}_2 = \frac{2^{1/2\sqrt{1+4\alpha_e}}}{\Gamma(1-1/2\sqrt{1+4\alpha_e})} c_2 \quad (5.42)$$

where $\alpha_e = \alpha(\tau_e)$ is given in terms of the total number of e-folds during inflation. Indeed \bar{c}_2 is an $\mathcal{O}(1)$ number as well.

5.5 The magnetic field at the end of inflation

Our task in this section is to determine the properties of the magnetic field at the end of inflation: the amplitude of the field, its coherence scale and the spectral index. These initial conditions will be subsequently evolved to the present time. We start from the covariant definition of the magnetic field in curved space [98]

$$B_\mu = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} u^\beta F^{\nu\alpha} \quad (5.43)$$

where u^μ is the 4-velocity vector field tangent to an observer's worldline and $\epsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric tensor, i.e. $\epsilon_{0123} = \sqrt{g}$. For a comoving observer $u^\mu = (1/a, \mathbf{0})$ and

$$B_i = \frac{1}{a} \epsilon_{ijk} \partial_j A_k \quad . \quad (5.44)$$

It is now straightforward to find the square of the magnetic field power spectrum from the two-point function

$$\mathcal{P}_B(k, \tau) = \sqrt{\frac{k^5 |\tilde{A}(k, \tau)|^2}{\pi^2 a^4 I^2}} . \quad (5.45)$$

Notice the extra power of the scale factor in the denominator relative to the gauge field power spectrum. Plugging in the solution in eq. (5.42), we can easily read off the spectral index

$$\begin{aligned} n_B &= 2 + \frac{1}{2} (1 - \sqrt{1 + 4\alpha_e}) \\ &\simeq 2 - \alpha_e \end{aligned} \quad (5.46)$$

valid since $\alpha_e < 1$. A precise knowledge of the spectral index is crucial to determine the strength of the magnetic field at the present epoch and thus one should investigate at this stage the exact size of α_e . The total number of e-folds strongly depends on the dynamics of inflation so we are going to fix $N = 60$ since, as evident from eq. (5.38), lowering the number of e-folds yields a larger α_e . Moreover, to obtain a *best value* we will imagine dialing up the number of particles in the loop such that the spectral index asymptotes to¹⁰

$$n_B \simeq 1.991 . \quad (5.47)$$

It is rather important to pause at this stage and notice that *reversing* the sign of the beta function would change the whole picture. If β is small but negative one would be able to achieve a noticeably larger α_e and in turn a spectrum which is less blue. In fact, one could even obtain a (nearly) scale-invariant spectrum by adjusting the number of particles, a result that might be enough to generate the present day IGM field as well as to ignite the galactic dynamo [94]. This somewhat echoes the observation made in [104] and we reserve considering this possibility to a future work.

¹⁰Working instead with β_{SM} does not alter the spectral index significantly.

The second quantity of interest is the average strength of the magnetic field which reads

$$B^2(\tau_e) = (\pi I_e a_e^2)^{-2} \int_{k_{\min}}^{k_{\max}} dk k^4 |\tilde{A}(k, \tau_e)|^2 \quad (5.48)$$

where $k_{\min}(k_{\max})$ is an IR(UV) cut-off. The value of k_{\max} is naturally dictated by the size of the horizon at the end of inflation, namely $k_{\max} = Ha_e$ corresponding to the last mode that crossed the horizon and felt the amplification. On the other hand, strictly speaking k_{\min} should be determined by the size of the horizon *today* but for simplicity we are going instead to take $k_{\min} = (|\tau_0|)^{-1}$. This choice does not alter the result as we show next. The above integral could readily be performed and yields

$$B^2(\tau_e) = \frac{\mathcal{O}(1)}{(4 - 2\alpha_e)\pi^2 I_e^2} H^4 \quad . \quad (5.49)$$

Indeed, the coefficient \bar{c}_2 depends implicitly on the wavenumber and should have been included in the integral but this complicates the analysis without gaining any insight. The lower limit of the integral contributes negligibly to the amplitude and thus the precise choice of the IR cut-off does not affect the result, which is a manifestation of the blue spectrum.

Finally we need the *comoving* coherence scale of the magnetic field at the end of inflation. As we describe in the next section, the value of the present day magnetic field is determined by the evolution of the coherence scale. It is defined as [94, 111]

$$\lambda_B(\tau_e) = 2\pi \frac{\int dk k^{-1} B^2(k, \tau_e)}{\int dk B^2(k, \tau_e)} \quad (5.50)$$

where $\mathbf{B}(k, \tau_e)$ is the Fourier decomposition of the magnetic field. Performing the integrals, we find

$$\lambda_B(\tau_e) = \mathcal{O}(1) \frac{(4 - 2\alpha_e)}{(3 - 2\alpha_e)} \frac{2\pi}{Ha_e} \quad . \quad (5.51)$$

5.6 The current magnetic field

The results of the previous section provides the initial conditions for the subsequent evolution of the magnetic field. As is well known [94, 111], to trace the exact evolution of the magnetic field is quite complicated. The conventional treatment is to assume that the magnetic field *freezes* in the cosmic plasma quickly after inflation ends. This is because the electric conductivity of the plasma becomes effectively infinite leading the gauge field to become almost static after inflation [98]. Inspection of the power spectrum eq. (5.45) shows that the magnetic field is simply diluted by the scale factor squared which is nothing but the requirement of magnetic flux conservation.

In this simple picture it suffices to know the ratio (a_0/a_{end}) where a_0 is the scale factor today while a_{end} is that at the end of inflation. This ratio precisely depends on three *independent* parameters: the energy scale of inflation, the reheating temperature and the equation of state parameter during reheating [118]. It reads

$$\frac{a_{\text{end}}}{a_0} = \mathcal{R} \left(\Omega_{\text{rad}}^0 \frac{3H_0^2}{M_P^2} \right)^{1/4} \left(\frac{\rho_{\text{end}}}{M_P^4} \right)^{-1/2}, \quad \rho_{\text{end}} = 3H^2 M_P^2. \quad (5.52)$$

The parameter \mathcal{R} is a function of the three variables $(w_{\text{reh}}, T_{\text{reh}}, \rho_{\text{end}})$ and it determines the amplitude and coherence scale of the present day magnetic field. Notwithstanding, its precise form is not important for our analysis but rather the range of values it could take. A *model-independent* estimate for the latter was carried out in [98]

$$\frac{1}{4} \ln (\rho_{\text{nuc}}/M_P^4) < \ln \mathcal{R} < -\frac{1}{12} \ln (\rho_{\text{nuc}}/M_P^4) + \frac{1}{3} \ln (\rho_{\text{end}}/M_P^4) \quad (5.53)$$

where ρ_{nuc} is the radiation energy density at nucleosynthesis. Both the upper and lower bounds assume the lowest possible reheating temperature. The lower bound

assumes $w_{\text{reh}} = -1/3$ while the upper bound assumes¹¹ $w_{\text{reh}} = 1$. It is clear from eq. (5.52) that the larger \mathcal{R} becomes the stronger the present day magnetic field would be. Yet, inspection of eq. (5.51) shows that a larger \mathcal{R} leads to a shorter coherence scale. We shall see in the next section how to obtain a lower bound on \mathcal{R} .

However, this simple picture of the evolution is inaccurate as was first pointed out by Banerjee and Jedamzik in [111]. The coupling of the magnetic field to the cosmic plasma results in non-linear energy *cascades* in Fourier space. In particular, the above estimate does not describe the physics at the coherence scale. To obtain a more robust prediction, one ideally has to evolve the non-linear magneto-hydrodynamical equations from the moment of genesis to the present day. It is needless to say that this is impossible to perform analytically. Fortunately, numerical simulations show that the gross features of the evolution is rather simple to understand [111, 94].

The magnetic field evolves in three main stages depending on the initial conditions and the properties of the plasma. We briefly state the main features of each phase.

- Free turbulent decay: This phase is characterized by a large Reynolds number.

The latter is given by [94]

$$R_k(T) = \frac{v_k \lambda_k}{\lambda_{\text{mfp}}(T)} \quad (5.54)$$

where v_k is the velocity of the fluid at some scale λ_k and λ_{mfp} is the *comoving* mean-free-path of the particles in the plasma. During this phase, the power spectrum at scales larger than λ_B retains its original shape while at smaller scales the spectrum develops a universal slope [94] *irrespective* of the initial conditions. Overall, λ_B grows while the amplitude decays.

¹¹This equation of state is realizable in models based on the *quintessential inflation* scenario [119] where, after inflation, the kinetic energy of the inflaton dominates the energy density.

- Viscous phase: The system enters this phase once the mean-free-path of the least coupled particle becomes large enough that the Reynolds number becomes of order unity. The high viscosity suppresses plasma motions on scales up to the coherence scale. This leads the magnetic field to decouple from the plasma. Overall, λ_B stays constant and the magnetic field gets diluted only by expansion [111].
- Free streaming: Close to decoupling (e.g. neutrino decoupling), the mean-free-path grows beyond λ_B . Neutrinos, being too weakly coupled, do not provide *true* viscosity at this stage but rather contribute a friction term in the Euler equation [111]. The coefficient of the latter is inversely proportional to the mean-free-path and thus the turbulent phase is restored shortly before decoupling [94, 111]. Afterwards, the whole cycle is repeated but now with photons instead.

The magnetic field and the coherence scale evolve according to a power-law during turbulence and free-streaming [111]. The commencement/termination of each phase depends on the initial conditions and the properties of the plasma. Let us estimate the Reynolds number in eq. (5.54) right after inflation and for simplicity instantaneous reheating will be assumed. The *proper* mean-free-path in the plasma above the electroweak scale reads [94]

$$l_{\text{mfp}} = \frac{22}{T} \quad . \quad (5.55)$$

At the coherence scale λ_B , the velocity of the fluid is taken to be the Alfvén speed [94] and thus

$$R_{\lambda_B} \simeq \sqrt{H/M_P} \ll 1 \quad . \quad (5.56)$$

Hence, the flow at the coherence scale is not turbulent with our initial conditions. As discussed in [94], this condition is typical in inflationary magnetogenesis scenarios

unless there exist a mechanism able to set the magnetic field in *equipartition* with the flow¹². In particular, with our initial conditions the system starts in the viscous phase which means the magnetic field stays comovingly constant. For this reason it suffices to predict the present day amplitude and coherence scale based on flux conservation as we described above¹³.

5.7 The lower bound on the IGM field

In this section, we employ the previous analysis to determine the properties of the present day magnetic field. In particular, we are concerned with satisfying the lower bound inferred on the IGM field which was given in [91, 92, 93]

$$B_{\text{meas.}} \geq 6 \times 10^{-18} \sqrt{\frac{1 \text{Mpc}}{\lambda_B}} \text{ G} \quad (5.57)$$

where account is taken of coherence scales shorter than 1 Mpc. Notice that this is a combined bound on both the magnetic field and coherence scale, in particular, it does not constrain the spectral index. Using eqs. (5.49) and (5.52) yields a present day magnetic field

$$B_0 \simeq \frac{2 \times 10^{18}}{(1 + \beta N)^{1/2}} \Delta^2 \text{ G} \quad (5.58)$$

and we defined the dimensionless quantity

$$\Delta \equiv \frac{H}{1 \text{GeV}} \frac{a_{\text{end}}}{a_0} . \quad (5.59)$$

¹²The occurrence of parity violation is able to amplify the field to equipartition as shown in [101].

¹³It is possible that turbulence develops at a later stage in the evolution, e.g. at neutrino decoupling. Yet, we do not consider such a possibility since it is unlikely that it affects our conclusion in any substantial manner.

To obtain the best value, we obviously need to minimize the denominator in eq. (5.58) and thus we choose $N = 60$ and $\beta = \beta_{SM}$. Using eq. (5.51) the bound in eq. (5.57) could be written as follows

$$\Delta \gtrsim 10^{-34/3} . \quad (5.60)$$

Inspection of eq. (5.52) reveals that the explicit dependence on the Hubble scale disappears from Δ all together. In fact, the above bound is readily turned into a lower bound on \mathcal{R}

$$\ln \mathcal{R} \gtrsim 4 . \quad (5.61)$$

This is the main result of our analysis. Now one must inquire if this value for \mathcal{R} is realizable. Assuming the highest possible scale of inflation, eq. (5.53) leads to [98]

$$-47 \lesssim \ln \mathcal{R} \lesssim 10 . \quad (5.62)$$

We conclude that the QED trace anomaly is in principle capable of producing the IGM field although the reheating temperature must be very low¹⁴.

5.8 Summary and conclusions

Quantum loops of massless particles bring a unique feature to gravitational phenomena, i.e. non-locality. These effects have received recent interest in the literature especially in regard to cosmology. One open question of present day cosmology and astrophysics is the large-scale magnetic fields observed across our Universe. Such

¹⁴Assuming the highest scale of inflation and $w_{\text{reh}} = 1$, the bound on \mathcal{R} is equivalent to an upper bound on the reheating temperature of about 100 GeV.

fields can not be produced by standard electromagnetism because conformal symmetry preserves the vacuum of the theory. As is well known, conformality is anomalously broken by loops of massless particles and precisely by the low energy portion of loops. It is important then to try achieving magneto-genesis using this basic field theoretic mechanism. The first attempt in this direction was carried out by Dolgov in [104].

In this chapter, we exploited the effective action of massless QED to discuss this scenario. Although non-local actions defined over curved space are quite cumbersome, we showed how to cast the anomalous portion of the action into a usable form that resembles the starting Lagrangian for plenty of models that exist in the literature. In particular, we found the spectral index to depend on both the number of e-folds, the number of charged particles that run in the loop and most importantly on the sign of the beta function. With a positive beta function and dialing up the number of fermions we obtained a rather blue spectrum at the end of inflation. Demanding magnetic flux conservation, we found that a very low reheating temperature is required to produce a present day magnetic field consistent with the lower bound inferred on the IGM field [91, 92, 93].

There is an important caveat about our presentation: the photon is not an active degree of freedom before spontaneous symmetry breaking. Thus one should ideally perform the analysis for the gauge bosons of the whole electroweak sector and evolve the system down to $T_{EW} = 100 \text{ GeV}$ before projecting onto the photon field. In this regard our analysis is exploratory. One important lesson is the effect of altering the sign of the beta function on the final result. In particular, it is possible to obtain a (nearly) scale-invariant spectrum with a negative beta function and an appropriate number of particles in the loop. An exciting future direction is to include gravitational loops in the presence of a positive cosmological constant. As emphasized by Toms in [120], the latter can render QED asymptotically free. We will hopefully pursue various directions and report on our findings in a future work.

CHAPTER 6

QUANTUM GRAVITY OF KERR-SCHILD SPACETIMES AND THE LOGARITHMIC CORRECTION TO SCHWARZSCHILD BLACK HOLE ENTROPY

6.1 Introduction

General relativity is a well-behaved quantum theory at low energies [2, 1]. Treated as an effective field theory (EFT), quantum predictions can systematically be quantified. The clear separation of scales provided by the EFT framework enables the extraction of the leading quantum effects. The latter are precisely due to the low-energy portion of the theory which is dictated by the symmetries of general relativity. On the other hand, the unknown high-energy physics is manifested only in the Wilson coefficients of the most general Lagrangian. All observables are then expressed in terms of the low energy constants, which are experimentally measured. As an EFT, the theory is renormalizable order by order in the counting parameter, i.e. E/M_P , which makes it fully predictive.

Massless particles can propagate over long distances. The quantum fluctuations of massless excitations offer a unique feature in field theory; non-locality. For example, the non-analytic portion of scattering amplitudes is due to the low energy propagation of massless particles. Using EFT techniques, Donoghue and collaborators determined the leading long-distance modification to the Newtonian potential [2, 1, 121]. More generally, this class of quantum corrections establish a set of low-energy theorems of quantum gravity [122]. Apart from scattering amplitudes, previous investigations focused primarily on the regime of weak gravity where gravitons propagate through

flat space. For instance, quantum corrections to various black hole geometries in the asymptotic region were computed in [123].

It is very natural then to pose the following question: What is the full structure of the loop-induced modifications to general relativity? In order to treat the non-linear regime of gravity, we clearly need to quantify these infrared corrections in curved spacetimes. Here, the technical aspect concerns the construction and properties of non-local effective actions. These are somewhat easy to understand in Minkowski space but become quite complicated when considered in curved space [125, 124, 126, 127, 128, 129]. The non-local corrections provide a *quantum memory* and could become appreciable even below the Planck scale. For example, the analysis presented in chapter 3 hints at the possible avoidance of cosmological singularities¹.

On a different front, the startling discovery that a black hole is a thermodynamic system endowed with entropy stands out as a remarkable achievement of twentieth century physics. A complete understanding of the Bekenstein-Hawking (BH) area law [133, 134] is believed by many to be our window to learn profound lessons about quantum gravity. There exist plenty of *macroscopic* derivations of the BH entropy using different approaches that we briefly discuss below. Nevertheless, the conundrum we face concerns the statistical or *microscopic* description of black hole entropy. There has been partial success to address this question in string theory [135], holography [136] and quantum geometry [137] but we are still far from a definitive answer².

It is well known that the BH area law does not hold in more general theories of gravity [138, 139]. In this light it is crucial to study quantum corrections to Einstein gravity and their corresponding effect on the area law. Thus, even on the macroscopic side it is quite possible to gain new insights about quantum gravity. One

¹See also a host of papers [46, 47, 48, 49, 50, 51, 52, 54, 131, 132] that explore the phenomenology of non-local models.

²We only include a restricted list of references since microscopic derivations lie beyond the scope of this work.

might nevertheless be tempted to think that an exact knowledge of these deviations requires a UV completion of gravity. This is certainly not the case if the corrections emerge from the infrared limit of quantum loops of massless particles. As described above, these *parameter-free* corrections are genuine predictions of quantum gravity. Once known, they furnish a test laboratory for any proposed UV completion.

In this chapter, we adopt the EFT framework to study quantum gravity³ with free massless minimally coupled (MMC) matter fields in Kerr-Schild (KS) spacetimes. For KS spacetimes, there exist coordinates such that the spacetime metric reads⁴

$$g_{\mu\nu}^{\text{KS}} = \eta_{\mu\nu} - k_{\mu}k_{\nu} \quad (6.1)$$

with k^{μ} - the KS vector - being a null vector field. It is a remarkable fact that black holes in vacuum Einstein gravity are of the KS type. In particular, the Kerr solution was originally found using the KS ansatz, thanks to the extreme reduction in complexity provided by the formalism [142, 143, 144]. Although we do not intend to review the formalism at length⁵, a short version is provided in appendix E, where we show how one can obtain both the Schwarzschild and Kerr solutions starting from the KS ansatz.

We have two goals in mind for the present chapter:

- To address some of the subtleties associated with the construction of non-local actions in curved spacetimes. Previous studies [125, 124, 126, 127, 128, 129] have focused on obtaining results appropriate for a generic metric. Albeit robust, the results are complicated for an arbitrary geometry and some questions remain unanswered in regard to the nature of the so called *form factors*. It is not

³See [140, 141] for detailed reviews.

⁴Throughout this chapter, we assume a vanishing cosmological constant.

⁵The interested reader could consult [145, 146] for thorough accounts.

clear whether the available results provide the best pathway to explore the phenomenology.

The special form of the KS metric enables us to *exactly* resolve the heat kernel for various operators. Hence, we can probe the structure of non-local actions in a non-trivial context. In spite of being special, the KS class contains black holes which are phenomenologically the most relevant. Our results pave the way to interesting further progress in the quantum physics of black holes.

- To compute the logarithmic correction to the Schwarzschild black hole entropy. The non-analytic dependence on the horizon area hints that the underlying action is non-local. The effective action can readily be used to identify the logarithmic correction by constructing the Euclidean partition function. Moreover, knowledge of the partition function is a precursor to explore quantum aspects of black hole thermodynamics. We posit a few interesting questions in section 6.6.

A quick review of the literature regarding the mentioned goals is in place. First, a significant amount of work has been undertaken to uncover the structure of non-localities in gravitational effective actions, see [125, 124, 126, 127, 128, 129] and references therein. Results are customarily displayed as an expansion in gravitational curvatures. Nevertheless, this expansion is quite different from *local* Lagrangians familiar in (non)-renormalizable quantum field theories. For instance, the effective Lagrangian of quantum gravity is arranged according to the energy or derivative expansion and only local polynomials of curvature invariants appear. This is the typical story when one integrates out a heavy field from the path integral of the theory. On the other hand, quantum loops of massless fields yield a non-local effective theory. The so called form factors are fundamental objects in the non-local expansion and the covariance properties of the latter were scrutinized in previous chapters.

One great advantage of fixing the background geometry to have the KS form is an unambiguous definition of the form factors. In this case, the results turn out to be much simpler than those which exist in the literature [125, 124, 126, 127, 128, 129]. In addition, the KS form of the metric allows for a transparent analysis of the curvature expansion, which we shall review in section 6.4. The nature of the non-local expansion becomes manifest, which provides invaluable clues for future endeavors

Moving to the second goal where a decent amount of work has been done as well. Fursaev, to the best we know, provided the first hint about the logarithmic correction in [34] using the conical singularity method. Recently, Sen and collaborators used Euclidean methods to uncover the logarithmic correction for both extremal [147, 148, 149] as well as non-extremal [150] black holes. When available, the results remarkably agree with microscopic results in the extremal case. Carlip employed Cardy's formula, which counts states in 2d conformal field theory, to find the logarithmic correction to the BH entropy [151]. The authors of [152] computed the exact partition function of the BTZ black hole to uncover the logarithmic correction. Banerjee and collaborators used the tunneling approach to identify corrections to Hawking temperature which then yield a logarithmic correction in the entropy of various black holes [153, 154]. Other authors used the anomaly-induced action, i.e. Riegert action, to compute the same correction this time via Wald's Noether charge formalism [155]. The authors of [156] obtained exact black hole solutions to the semi-classical Einstein equations including the conformal anomaly. A direct computation revealed a logarithmic correction to the BH entropy. Finally, the logarithmic correction was also found based on the quantum geometry program [157].

Now we summarize our results. Our starting point is the EFT action

$$\mathcal{S} = \mathcal{S}_{\text{GEFT}} + \mathcal{S}_{\text{matter}} \quad . \quad (6.2)$$

The gravitational effective action is

$$\mathcal{S}_{\text{GEFT}} = \int d^d x \sqrt{g} \left(\frac{M_{\text{P}}^2}{2} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + c_4 \nabla^2 R \right) \quad (6.3)$$

where only operators containing up to four derivatives are included. Notice here that the above is not usually how the action is displayed [2, 1]. The last term is customarily omitted because it is a total derivative and does not contribute to the Feynman rules while the Riemann piece is omitted via an implicit use of the Gauss-Bonnet identity. We shall see below that we need to keep all the terms in order to carry out the renormalization program. The second portion $\mathcal{S}_{\text{matter}}$ describes free MMC matter fields of spin 0, 1/2, 1. The constants (c_1, c_2, c_3, c_4) are the *bare* Wilson coefficients⁶ and the dimensionality of spacetime is extended in order to employ dimensional regularization, i.e. $d = 4 - 2\epsilon$. The one-loop effective action is evaluated fixing the background geometry to be a KS spacetime. Upon integrating out the matter degrees of freedom and graviton fluctuations at the one-loop level⁷, we obtain

$$\Gamma[\bar{g}] = \Gamma_{\text{local}} + \Gamma_{\text{ln}} \quad (6.4)$$

where the renormalized action now reads

$$\Gamma_{\text{local}}[\bar{g}] = \int d^d x \left(\frac{M_{\text{P}}^2}{2} R + c_1^r(\mu) R^2 + c_2^r(\mu) R_{\mu\nu} R^{\mu\nu} + c_3^r(\mu) R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + c_4^r(\mu) \nabla^2 R \right) . \quad (6.5)$$

Here, \bar{g} is the background metric that takes the form in eq. (6.1) and μ is the scale of dimensional regularization. Notice in particular that Newton's constant does *not*

⁶As per usual, the bare constants remain dimensionless in d dimensions.

⁷The one-loop graviton fluctuations arise solely from the Einstein-Hilbert action. There is indeed a contribution from the $\mathcal{O}(\partial^4)$ pieces but these are suppressed by the Planck mass. To be consistent with the power counting of the EFT, these are included only when one considers the two-loop effective action.

get renormalized because the divergences arising from massless loops are proportional to the quadratic invariants. Of utmost importance is the finite pieces that exhibit a logarithmic non-locality

$$\Gamma_{\ln}[\bar{g}] = - \int d^4x \left(\alpha R \ln \left(\frac{\square}{\mu^2} \right) R + \beta R_{\mu\nu} \ln \left(\frac{\square}{\mu^2} \right) R^{\mu\nu} + \gamma R_{\mu\nu\alpha\beta} \ln \left(\frac{\square}{\mu^2} \right) R^{\mu\nu\alpha\beta} + \Theta \ln \left(\frac{\square}{\mu^2} \right) \square R \right) \quad (6.6)$$

where $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$. The different coefficients depend on the particle species and are listed in table 6.1.

Focusing on the Schwarzschild solution, we use the effective action to construct the partition function. From the latter, the entropy is determined and our main result reads

$$S_{\text{bh}} = S_{\text{BH}} + 64\pi^2 \left(c_3^r(\mu) + \Xi \ln(\mu^2 \mathcal{A}) \right) \quad (6.7)$$

Here $S_{\text{BH}} = \mathcal{A}/4G$ is the BH entropy and $\mathcal{A} = 16\pi(GM)^2$ is the horizon area. The constant Ξ sums up the contributions from all the massless particles in the theory and reads

$$\Xi = \frac{1}{11520\pi^2} (2N_s + 7N_f - 26N_V + 424) \quad (6.8)$$

where we allowed for variable number of particles. The logarithmic dependence on the horizon area and the associated coefficient is in exact agreement with [34, 150, 153, 155] albeit using different approaches than ours. Furthermore, eq. (6.7) contains a subtle feature: the entropy is manifestly renormalization-group (RG) invariant. The demonstration of this property is made clear in section 6.5. In fact, this feature is

mandatory if black hole entropy is to be identified as a physical quantity. We can further employ dimensional transmutation to rewrite eq. (6.7) as

$$S_{\text{bh}} = S_{\text{BH}} + 64\pi^2 \Xi \ln \left(\frac{\mathcal{A}}{\mathcal{A}_{\text{QG}}} \right) \quad (6.9)$$

where \mathcal{A}_{QG} corresponds to a length (energy) scale *uniquely* set by the full theory, i.e. the UV completion of quantum gravity. As we shall discuss further below, the result uncovers an intricate connection between the UV and IR properties of quantum gravity. More comments about the content of the result are reserved to section 6.5.2.

The plan of this chapter is as follows. We commence in section 6.2 by developing a set of Feynman-like rules to resolve the heat kernel for the d' Alembertian operator in KS spacetimes. The Einstein equations are solved with the KS ansatz in appendix E while the non-local expansion of the heat kernel is described in appendix F. In section 6.3 the curvature expansion is introduced and the technique of non-linear completion is used to express the heat kernel trace in the desired form. We then move in section 6.4 to find the effective action by integrating over proper time. There, we uncover what we would like to call a *UV-IR correspondence*. Among other things, this correspondence allows us to extend the results to matter fields of various spins and gravitons. This is achieved knowing only the divergences of the theory. In section 6.5 the partition function is determined using the effective action. The behavior of the partition function under a global scale transformation provides an elegant pathway to extract the logarithmic correction to the BH entropy. We discuss possible future directions in section 6.6. In appendix G we collect useful formulas used throughout.

6.2 The heat kernel for the covariant d' Alembertian

In this section, we commence by resolving the heat kernel of the d' Alembertian operator. Knowing the latter enables a straightforward determination of the effective

action which results from integrating out a massless free scalar. One can otherwise directly compute the effective action via Feynman graphs as was done in the previous chapters but we choose to work with the heat kernel for reasons that we shall spell out below. The basic definitions and properties of the heat kernel are given in appendix F. Now we restrict our consideration to KS spacetimes of the form displayed in eq. (6.1). An immediate consequence of the null property of the KS vector is the set of relations

$$\sqrt{g} = 1, \quad g^{\mu\nu} = \eta^{\mu\nu} + \lambda k^\mu k^\nu, \quad g^{\mu\nu} k_\mu k_\nu = \eta^{\mu\nu} k_\mu k_\nu = 0 \quad (6.10)$$

where the Minkowski metric is expressed in standard coordinates. Here λ is a trivial counting parameter which is set to unity at the end of the computation. In order to treat operators with no associated mass scale, we use the non-local expansion of the heat kernel developed by Barvinsky, Vilkovisky and collaborators [124, 125, 126, 127]. For the convenience of the reader, we provide an essential review of the formalism in appendix F.

We seek an expansion of the heat kernel in powers of λ . Let us quote the d'Alembertian operator as it acts on a scalar density of weight 1/2

$$\nabla^2 \Psi = \frac{1}{\sqrt[4]{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu) \frac{1}{\sqrt[4]{g}} \Psi \quad (6.11)$$

The KS form of the metric drastically simplifies the structure of the operator

$$\nabla^2 \Psi = \left(\partial^2 + \lambda k^\mu k^\nu \partial_\mu \partial_\nu + \frac{\lambda}{2} \partial_\mu (k^\mu k^\nu) \partial_\nu + \frac{\lambda}{2} \partial_\nu (k^\mu k^\nu) \partial_\mu \right) \Psi \quad (6.12)$$

It is important to pause at this stage and comment on the above result. Let us imagine that we aim to study the same operator on a generic background spacetime. The conventional treatment is to expand the metric around flat space as $g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$

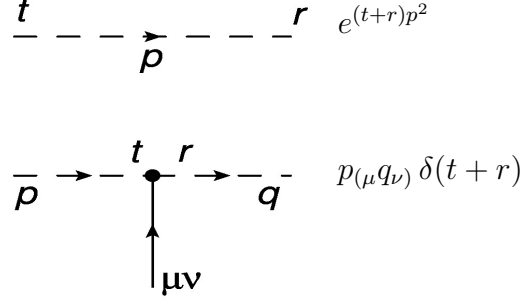


Figure 6.1. Feynman-like rules for the heat kernel trace. The solid line corresponds to an insertion of the external tensor field $K_{\mu\nu}$ which carries a power of λ .

and proceed to evaluate the heat kernel in powers of the *external* classical field $H_{\mu\nu}$. Both the inverse metric and metric determinant are expanded accordingly and the result is an infinite series in $H_{\mu\nu}$. Consequently the d' Alembertian operator contains arbitrarily high powers of the external field. On the contrary, there is an immediate truncation for KS spacetimes as evident from eq. (6.12). More comments about similar simplifications are made as we go along.

In the notation of appendix F, we identify the interaction term

$$V = \lambda \left(k^\mu k^\nu \partial_\mu \partial_\nu + \frac{1}{2} \partial_\mu (k^\mu k^\nu) \partial_\nu + \frac{1}{2} \partial_\nu (k^\mu k^\nu) \partial_\mu \right) . \quad (6.13)$$

For later convenience, we define the following tensor

$$K_{\mu\nu} \equiv k_\mu k_\nu . \quad (6.14)$$

We seek an expansion of the heat kernel *trace* in powers of λ . Using eqs. (F.17) and (F.19) one can easily introduce Fourier transforms to derive a set of Feynman-like rules which read:

- The rule for the vertex and propagator are given in the figure 6.2.

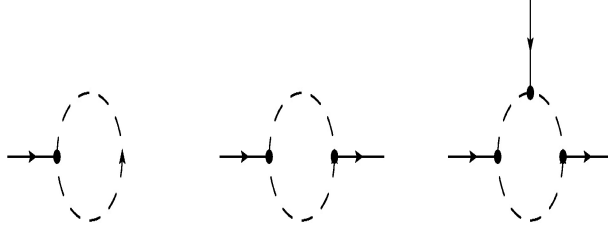


Figure 6.2. The diagrammatic expansion of the heat kernel trace.

- The internal propagator that carries the loop momentum gets an extra factor of 1 in the exponent⁸.
- Add a factor of s in the exponent of all propagators.
- Impose momentum conservation at each vertex.
- Integrate over the loop momentum and proper-time⁹.

From the Feynman-like rules, we easily develop a diagrammatic expansion as shown in figure 6.2. Here, a great simplification emerges thanks to the KS form of the metric: there is a single diagram in the expansion at each order in λ . On the contrary, for a generic background the number of diagrams proliferate as we go to higher orders in the expansion.

6.2.1 Lowest order

Let us compute the first diagram in figure 6.2. Applying the rules given above, we find

$$\mathcal{H}^{(1)}(s) = s K_0^{\mu\nu} \int \frac{d^d l}{(2\pi)^d} l_\mu l_\nu e^{s l^2} \quad (6.15)$$

⁸This is due to the flat space kernel that appears convoluted in eq. (F.17).

⁹Here, we mean the integration variable in the exponent of eq. (F.19).

where the subscript on the background field denotes its momentum, i.e. $K_0^{\mu\nu} \equiv K_{\mu\nu}(0)$. Using the tensor integrals given in appendix G we find

$$\mathcal{H}^{(1)}(s) = -\frac{i}{2(4\pi s)^{d/2}} K_0^{\mu\nu} \eta_{\mu\nu} \quad . \quad (6.16)$$

By construction the KS vector is null with respect to the Minkowski metric, and thus

$$\mathcal{H}^{(1)}(s) = 0 \quad . \quad (6.17)$$

This is the trace of the heat kernel to lowest order in λ and the result is exact. Nevertheless, we shall see in the next section that we need to compute the heat kernel in the coincidence limit rather than the trace. This is necessary in order to carry out the non-linear completion procedure that we explain in the next section.

6.2.2 Next-to-leading order

At $\mathcal{O}(\lambda^2)$ we encounter the second diagram in figure 6.2. We display the steps in some detail to elucidate the construction. Straightforward application of the rules yields

$$\begin{aligned} \mathcal{H}^{(2)}(s) = s^2 \int \frac{d^d p}{(2\pi)^d} K_p^{\mu\nu} K_{-p}^{\alpha\beta} \int_0^1 dt_1 \int_0^{t_1} dt_2 \int \frac{d^d p}{(2\pi)^d} V_{\mu\nu}(l, p) V_{\alpha\beta}(l, p) \\ e^{s((1-t_1+t_2)l^2 + (t_1-t_2)(l+p)^2)} \end{aligned} \quad (6.18)$$

where

$$V_{\mu\nu}(l, p) = l_\mu l_\nu + l_{(\mu} p_{\nu)} \quad . \quad (6.19)$$

We first need to put the exponent in eq. (6.18) in quadratic form. In particular, this enables dropping odd powers of the loop momentum. This is accomplished via

shifting the loop momentum by sending $l \rightarrow l + (t_1 - t_2)p$. If we moreover perform the tensor integrals using appendix G we find

$$\begin{aligned} \mathcal{H}^{(2)}(s) = s^2 \int \frac{d^d p}{(2\pi)^d} K_p^{\mu\nu} K_{-p}^{\alpha\beta} \int_0^1 dt_1 \int_0^{t_1} dt_2 e^{\sigma(1-\sigma)sp^2} \\ \left[J_{\mu\nu\alpha\beta} - \sigma(1-\sigma) (J_{\mu\nu} p_\alpha p_\beta + J_{\alpha\beta} p_\mu p_\nu) + \frac{1}{4}(1-2\sigma)^2 (J_{\mu\alpha} p_\nu p_\beta + J_{\mu\beta} p_\alpha p_\nu \right. \\ \left. + J_{\nu\alpha} p_\mu p_\beta + J_{\nu\beta} p_\mu p_\alpha) + \sigma^2(1-\sigma)^2 p_\mu p_\nu p_\alpha p_\beta \right] \end{aligned} \quad (6.20)$$

where $\sigma \equiv t_1 - t_2$. The above expression can be simplified greatly if one notices that any function $f(\sigma)$ that is invariant under $\sigma \rightarrow (1-\sigma)$ has the property

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 f(\sigma) = \frac{1}{2} \int_0^1 d\sigma f(\sigma) \quad . \quad (6.21)$$

The final result then becomes

$$\mathcal{H}^{(2)}(s) = \frac{is^2}{2(2\pi s)^{d/2}} \int \frac{d^d p}{(2\pi)^d} K_p^{\mu\nu} K_{-p}^{\alpha\beta} \int_0^1 d\sigma e^{\sigma(1-\sigma)sp^2} \mathcal{M}_{\mu\nu\alpha\beta} \quad (6.22)$$

where

$$\begin{aligned} \mathcal{M}_{\mu\nu\alpha\beta} = \left[\frac{1}{4s^2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}) - \frac{1}{8s} (1-2\sigma)^2 (\eta_{\mu\alpha} p_\nu p_\beta + \eta_{\mu\beta} p_\alpha p_\nu \right. \\ \left. + \eta_{\nu\alpha} p_\mu p_\beta + \eta_{\nu\beta} p_\mu p_\alpha) + \sigma^2(1-\sigma)^2 p_\mu p_\nu p_\alpha p_\beta \right] \quad . \end{aligned} \quad (6.23)$$

All tensor structures that vanish because of the null property of the KS vector have been dropped, which comprises an extra simplification special to the KS geometry.

6.2.3 Next-to-next-to-leading order

The third diagram in figure 6.2 could easily be carried out similar to the previous diagrams. Nevertheless, it has an extra subtle feature: the triangular topology

of the graph with massless internal lines inevitably leads to an *infrared* singularity when we pass to the effective action¹⁰. This is due to the *long-time* behavior of the heat kernel being singular. The existence of infrared singularities in gauge theory scattering amplitudes is conventionally dealt with by adding real emission graphs which guarantees all observables are IR finite [158, 159]. Similar story takes place in gravitational scattering, see for example [160, 161]. On the contrary, the treatment of infrared singularities present in the effective action is a widely unexplored topic. It is not clear how to obtain finite predictions in this case. Although this issue is crucial for understanding non-local effective actions, its discussion lies beyond the scope of this chapter. We show below that the leading non-locality is captured by the results already obtained, which suffices for the applications to be considered in this work.

6.2.4 A brief comment on the result

It is important to pause at this stage to stress that the heat kernel trace given in eqs. (6.17) and (6.22) is exact for any KS spacetime. This is indeed true regardless of the underlying gravity theory. For example, for a Schwarzschild or Kerr black hole one can use the results to obtain the *on-shell* one-loop effective action or the Euclidean partition function. One merely has to determine the Fourier decomposition of the KS vector and plug back in eqs. (6.17) and (6.22) in order to perform the last momentum integral. Nevertheless, we choose not to follow this pathway and present an alternative procedure which is very useful in achieving the goals of our study.

6.3 The curvature expansion

In this section we describe in detail how to express the heat kernel trace in an expansion utilizing the geometric curvatures. Albeit being elegant, this is not the

¹⁰As long as the external legs are *off-shell* the singularity is soft. Yet, these singularities could disappear for specific external kinematics. For example, see the discussion in chapter 1.

main reason why we take this direction. First, the non-local expansion is controlled by the form factors. The non-analytic logarithm in eq. (6.6) is one example of a form factor. As we alluded to in the introduction, it is of utmost importance to study the covariance properties of the form factors. The first step in this direction was presented in the previous chapters. An exact solution of the heat kernel over a non-trivial background spacetime supplies us with important clues about the form factors. After we display the computation, we return back to this point in section 6.3.4. Second, having the action expanded in geometric objects facilitates the determination of the leading correction to the BH entropy. Finally, if one hopes to track the back-reaction of quantum fluctuations on the spacetime, it is desirable to express the effective action using geometric objects.

There exist two techniques to construct the curvature expansion. The first is the *covariant perturbation theory* extensively developed in [124, 125, 126, 127]. The second is *non-linear completion* described and utilized in the previous chapters. We employ the latter which is relatively simple. The procedure here is quite similar to matching computations in effective field theories whereby the Wilson coefficients are determined. One starts by proposing a *local* operator basis using the classical fields and their derivatives. This basis is typically arranged as a power series expansion in *generalized* curvatures. At each order in the curvature expansion, one supplements the operators with various non-local form factors. The latter are uniquely fixed via matching onto the results obtained in the last section. We now move to apply this procedure.

6.3.1 The heat kernel at zeroth order

To zeroth order in the curvature, the only invariant available is

$$\mathcal{H}(s) = \frac{i}{(4\pi s)^{d/2}} \int d^d x [\mathcal{E}_0 + \mathcal{O}(R)] \quad (6.24)$$

where we stripped off some factors for convenience. Here \mathcal{E}_0 is the form factor which will turn out to be trivial in this case. It is also important to notice that for KS spacetimes, eq. (6.10) holds so no factor of \sqrt{g} appears. One immediately finds¹¹

$$\mathcal{E}_0 = 1 \quad . \quad (6.25)$$

6.3.2 The heat kernel at linear order

To lowest order in the curvature, the Ricci scalar is the only invariant that can show up in the heat kernel trace

$$\mathcal{H}(s) = \frac{i}{(4\pi s)^{d/2}} \int d^d x \left[\mathcal{E}_0 + s \mathcal{G}_R(s\Box) R + \mathcal{O}(R^2) \right] \quad (6.26)$$

where \Box is the flat space d ' Alembertian and the form factor $\mathcal{G}_R(s\Box)$ can only depend on the dimensionless combination $s\Box$. The common lore in the literature is to *covariantize* the derivative operators but we do not adopt this approach here. More comments appear in section 6.3.4. The matching step is most easily done in momentum space and at $\mathcal{O}(\lambda)$ the Ricci scalar reads

$$R^{(1)} = \partial_\mu \partial_\nu K^{\mu\nu} \quad . \quad (6.27)$$

Here the situation is subtle because the spacetime integral in eq. (6.26) forces the momentum variable to vanish. Hence the derivatives in the above equation forces a null result which matches the result in eq. (6.17). Nevertheless, we still can not determine the form factor. An alternative route is to compute the heat kernel in the *coincidence* limit, i.e. without invoking the spacetime integral, and then perform the matching. This way one finds a non-trivial result that enables the determination of

¹¹This precisely comes from the first term in the expansion of eq. (F.17) where the proper-time evolution operator is approximated by unity.

the form factor. Let us go back to section 6.2.1 and compute the coincidence limit of the heat kernel. One finds

$$H^{(1)}(x, x; s) = -\frac{is}{(4\pi s)^{d/2}} \int \frac{d^d p}{(2\pi)^d} K_p^{\mu\nu} p_\mu p_\nu \int_0^1 d\sigma \sigma(1-\sigma) e^{\sigma(1-\sigma)sp^2} e^{-ipx} \quad . \quad (6.28)$$

The matching is immediate and the form factor reads

$$\mathcal{G}_R(s\Box) = \int_0^1 d\sigma \sigma(1-\sigma) e^{-\sigma(1-\sigma)s\Box} \quad . \quad (6.29)$$

In appendix G, we derive a nice identity that enables us to reexpress the above result in a simpler form

$$\mathcal{G}_R(s\Box) = \frac{1}{4}f(s\Box) + \frac{1}{2s\Box}[f(s\Box) - 1] \quad (6.30)$$

where the *fundamental* form factor is¹²

$$f(s\Box) = \int_0^1 d\sigma e^{-\sigma(1-\sigma)s\Box} \quad . \quad (6.31)$$

Later on we shall see that only the value of the form factor at zero momentum is important. In particular we find

$$\mathcal{G}_R(0) = \frac{1}{6} \quad . \quad (6.32)$$

6.3.3 The heat kernel at quadratic order

Along the same lines of the last section, we match the heat kernel trace given in eq. (6.22) onto a curvature basis. Counting the number of derivatives this must be

¹²We stick to the name given to this special form factor in [130], which we find very illustrative.

second order in curvatures and hence

$$\begin{aligned} \mathcal{H}(s) = \frac{i}{(4\pi s)^{d/2}} \int d^d x \left[\mathcal{E}_0 + s \mathcal{G}_R(s\Box) R + s^2 R \mathcal{F}_R(s\Box) R + s^2 R_{\mu\nu} \mathcal{F}_{Ric}(s\Box) R^{\mu\nu} \right. \\ \left. + s^2 R_{\mu\nu\alpha\beta} \mathcal{F}_{Riem}(s\Box) R^{\mu\nu\alpha\beta} + \mathcal{O}(R^3) \right] . \end{aligned} \quad (6.33)$$

We need to expand the curvature invariants to $\mathcal{O}(\lambda^2)$ which are given in appendix G. Here comes an important part of the construction: the form factor $\mathcal{G}_R(0)$ plays role in the matching procedure. Although the form factors are defined with the flat d' Alembertian the curvature tensors must be expanded appropriately. Notice as well that only $\mathcal{G}_R(0) = 1/6$ is needed as the rest of this form factor contains total derivatives and thus vanishes by momentum conservation.

Inspection of the expressions given in appendix G we see that there are three tensor structures available which appears sufficient to determine the three form factors. But in fact only two equations turn out to be independent and they read

$$\frac{s}{48} + \frac{s^2 p^2}{8} \mathcal{F}_{Ric}(sp^2) + \frac{s^2 p^2}{2} \mathcal{F}_{Riem}(sp^2) = \frac{s}{16} \int_0^1 d\sigma (1 - 2\sigma)^2 e^{\sigma(1-\sigma)sp^2} \quad (6.34)$$

$$s^2 \mathcal{F}_R(sp^2) + \frac{s^2}{2} \mathcal{F}_{Ric}(sp^2) + s^2 \mathcal{F}_{Riem}(sp^2) = \frac{s^2}{2} \int_0^1 d\sigma \sigma^2 (1 - \sigma)^2 e^{\sigma(1-\sigma)sp^2} . \quad (6.35)$$

Note the first term on the LHS of eq. (6.34) which comes from $\mathcal{G}_R(0)$. We show next in detail how to uniquely fix the form factors. Once again, with the help of identities that are proven in appendix G we can express the RHS in terms of the fundamental form factor. Hence

$$\frac{s}{48} - \frac{s^2 p^2}{8} \mathcal{F}_{Ric}(sp^2) - \frac{s^2 p^2}{2} \mathcal{F}_{Riem}(sp^2) = \frac{1}{8p^2} (f(sp^2) - 1) \quad (6.36)$$

$$\begin{aligned} \mathcal{F}_R(sp^2) + \frac{1}{2} \mathcal{F}_{Ric}(sp^2) + \mathcal{F}_{Riem}(sp^2) &= \frac{1}{32} f(sp^2) - \frac{1}{8sp^2} f(sp^2) \\ &+ \frac{1}{16sp^2} + \frac{3}{8s^2 p^4} (f(sp^2) - 1) . \end{aligned} \quad (6.37)$$

One might suspect that the issue we face here is special to KS spacetimes since some tensor structures vanish due to the null property of the KS vector. In fact, this is a generic feature that takes place at second order in the curvature expansion. One could easily check that the same issue arises even for an arbitrary metric, see for example [130].

6.3.3.1 Fixing the form factors

We saw above that there are only two available equations for three form factors that appear at second order. Usually this is circumvented by making use of the following identity [127]

$$\int d^4x \sqrt{g} \left(R_{\mu\nu\alpha\beta} (\nabla^2)^n R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} (\nabla^2)^n R^{\mu\nu} + R (\nabla^2)^n R \right) = \int d^4x \sqrt{g} \mathcal{R}^3 . \quad (6.38)$$

Here the rhs refers to cubic curvature terms. The proof of the above takes a few lines and relies on using the Bianchi identities. Hence, to second order in the curvature one can set one of the form factors in eq. (6.33) to zero since the *error* would be higher order in the curvature expansion. The canonical choice made in the literature is [124, 125, 126, 127]

$$\mathcal{F}_{Riem} = 0 . \quad (6.39)$$

Indeed there is nothing special about this choice: it is nothing but one possible solution to the undetermined system of equations. Here we proceed differently because of two central reasons. First, the above choice essentially hides some of the physics contained in the computation. As we shall see below, the choice in eq. (6.39) becomes dangerous when applications are considered¹³. Second, the form factors in eq. (6.33)

¹³This point has been noted before in chapter 3.

strictly contain the flat space d' Alembertian and thus, formally, eq. (6.38) does not hold anymore.

The question remains: how can we make progress given that we have an undetermined system? This is achieved via an indirect approach, namely we consult the local UV divergences. The one-loop divergences are exactly known and expressed in a covariant manner from the coincidence limit of the Seeley-DeWitt-Gilkey series¹⁴, see for example [58, 59, 61, 62]. Our procedure is discussed in the next section when we consider the effective action. For now we impose a seemingly *ad hoc* extra relation between the form factors

$$\mathcal{F}_{Riem}(sp^2) + \mathcal{F}_{Ric}(sp^2) = 0 \quad (6.40)$$

and the consistency of this choice shall become clear in the next section. We can now solve for the form factors and find

$$\mathcal{F}_{Ric}(sp^2) = -\mathcal{F}_{Riem}(sp^2) = \frac{1}{18sp^2} + \frac{1}{3s^2p^4} - \frac{1}{3s^2p^4}f(sp^2) \quad (6.41)$$

$$\mathcal{F}_R(sp^2) = \frac{13}{144sp^2} - \frac{5}{24s^2p^4} + \frac{5}{24s^2p^4}f(sp^2) + \frac{1}{32}f(sp^2) - \frac{1}{8sp^2}f(sp^2) \quad . \quad (6.42)$$

This completes the matching procedure up to this accuracy in the curvature expansion. The practice is identical if one aims to consider the $\mathcal{O}(R^3)$ basis. Nevertheless, the last diagram in figure 6.2 must be computed for the matching procedure to work properly. From the vertex rules given in section 6.2, it is clear this diagram is $\mathcal{O}(\partial^6)$. Hence, the latter must be included for the non-linear completion procedure to work.

¹⁴The Seeley-DeWitt-Gilkey expansion is local and assumes a massive operator. Nevertheless, the divergences that arise at second order in the curvature are valid in the massless limit.

6.3.4 Comments on the form factors

So far we have shown how to re-express the *exact* results of the previous section employing the curvature expansion. One of the main concerns of the present chapter is to better understand the properties of the form factors. In particular, should we enforce the following replacement?

$$\mathcal{G}(s\Box) \rightarrow \mathcal{G}(s\nabla^2), \quad \mathcal{F}(s\Box) \rightarrow \mathcal{F}(s\nabla^2) \quad . \quad (6.43)$$

This is the conventional approach in the literature. Let us point out some features of the form factors that were described in chapter 2. There - in the context of massless QED with gravitational couplings - it has been shown that the expansion of the covariant form factor $\ln(\nabla^2)$ contributes terms in the action that does *not* match the diagrammatic expansion from perturbation theory. A proposed cure for this problem was developed in chapter 2 and referred to as the *counterterm method*. One has to introduce terms at higher order in the curvature expansion which are then fixed by requiring that the result matches that from perturbation theory. Albeit very complicated, it was shown that the procedure is robust and yields a unique answer for the action.

What does the current computation tell us about this issue? The results we presented are exact for KS spacetimes which shows that the replacement in eq. (6.43) is clearly superfluous. This is the main advantage of fixing the background geometry: it enables the heat kernel to be fully determined with an unambiguous definition of the form factors. Further comments appear in section 6.6 regarding the fate of the form factors.

6.4 The effective action

In this section, we compute the effective action up to second order in the curvature expansion. This is easily accomplished by integrating over proper time as in eq. (F.2).

Hence,

$$\Gamma[g] = -\frac{i\hbar}{2} \int d^d x \int_0^\infty \frac{ds}{s} H(x, x; s) \quad . \quad (6.44)$$

The integral over the proper time has two interesting regimes which are known as the *early* and *late* times. The former corresponds to the small s behavior and encodes the *short distance* behavior of the theory. The late time on the other hand corresponds to the large s asymptotics of the heat kernel and controls the *long distance* behavior of the theory. Let us describe a simple method to uncover the UV divergences. First, recall that the heat kernel is expressed solely in terms of the fundamental form factor. We can expand the exponential in eq. (6.31) and retain the first few terms. One then integrates over a small neighborhood, say $0 \leq s \leq 1$. The divergences then appear as a simple pole in ϵ as per usual in dimensional regularization.

Instead of studying limits of the proper-time integral, we proceed to perform the integral all at once using a simple trick. This procedure is very useful as it reveals a close link between the UV divergences and the IR logarithmic non-locality that emerges at second order in the curvature expansion. Without any further computation, we will be able to display the answer for matter fields of various spins as well as gravitons.

6.4.1 The action at zeroth order

If one plugs eq. (6.24) back in eq. (6.44) the integral is seen to be *scaleless*. What should we do in this case? Let us try regulating the integral as follows

$$\begin{aligned} \int_0^\infty ds s^{-d/2} &= \lim_{\delta \rightarrow 0^+} \int_0^\infty ds s^{-d/2} e^{-\delta s} \\ &= \lim_{\delta \rightarrow 0^+} \delta^{d/2-1} \Gamma(1 - d/2) \end{aligned} \quad (6.45)$$

which vanishes for $d > 2$ upon taking the limit. We conclude that scaleless integrals similar to the above can be set conveniently to zero. If a mass scale was present in the operator, the above integral would yield a divergent result proportional to m^4 , which in turn renormalizes the cosmological constant.

6.4.2 The action at linear order

We now move to the piece in eq. (6.26) with the form factor displayed in eq. (6.30). The trick to evaluate the effective action is to interchange the order of integration, namely to perform the proper time integral before the σ integral. Once again, all scaleless integrals are dropped. We present the details of the calculation for the convenience of the reader. Let us focus on the first piece in eq. (6.30)

$$\begin{aligned}\Gamma[g] &\propto \int d^d x \int_0^1 d\sigma \int_0^\infty ds s^{\epsilon-2} f(s\Box) R \\ &= \int d^d x \int_0^1 d\sigma \int_0^\infty ds s^{\epsilon-2} e^{-\sigma(1-\sigma)s\Box} R\end{aligned}\tag{6.46}$$

where we used $d = 4 - 2\epsilon$. The integral over proper time is easily written in terms of the Euler gamma function

$$\Gamma[g] \propto \int d^d x \int_0^1 d\sigma [\sigma(1-\sigma)]^{1-\epsilon} \Gamma(\epsilon-1) \Box^{1-\epsilon} R \quad .\tag{6.47}$$

We recognize immediately the UV divergence in the gamma function. The above expression is then expanded in ϵ and the σ integral is readily evaluated

$$\Gamma[g] \propto \int d^4 x - \frac{1}{6} \left(\frac{1}{\bar{\epsilon}} - \ln \Box \right) \Box R, \quad \frac{1}{\bar{\epsilon}} = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi\tag{6.48}$$

where we dropped a numerical constant which amounts to a finite renormalization. The rest of the form factor is treated the same way and we end up with the divergent part

$$\Gamma_\epsilon[g] = -\frac{\hbar}{60\epsilon} \int d^4x \square R \quad (6.49)$$

which is indeed the correct divergence found in the Seeley-DeWitt-Gilkey expansion [58, 59, 61, 62]. In particular, dropping the scaleless integrals is fully consistent as promised. It is worth mentioning that for a massive operator, the corresponding integrals would yield divergences proportional to m^2 which would then renormalize Newton's constant.

More importantly is the finite IR contribution to the action which reads

$$\Gamma[g] = \frac{\hbar}{60} \int d^4x \ln\left(\frac{\square}{\mu^2}\right) \square R \quad (6.50)$$

where μ^2 is the scale associated with dimensional regularization. Of utmost important is that the logarithmic non-locality comes tied to the UV divergence. Thus, it suffices to know the latter in order to determine the finite part of the action. It is then immediate to read off the result for any particle species other than minimally coupled scalars¹⁵. As we show below, this *UV-IR correspondence* continues to hold for the quadratic action. It is also crucial to point out that this correspondence is only true for the pieces in the action with four derivatives, i.e. $(\square R, R^2)$ terms. This is easily seen by dimensional analysis: the only non-local structure that can show up is logarithmic which dictates $\log \mu^2$ to appear as well. The latter is a UV scale whose coefficient must be tied to the divergences. At $\mathcal{O}(R^3)$ and beyond, the one-loop effective action is finite.

¹⁵We are not going to pursue this further simply because eq. (6.50) is not going to contribute in the applications we wish to consider. The last column in table 6.1 is left empty except from the scalar result that we already obtained.

6.4.3 The action at quadratic order

We now transition to the quadratic action which is the main concern of our work. The form factors are given in eq. (6.41) and the computation proceeds similar to the previous subsection albeit one subtlty. The scaleless integrals can *not* be set to zero using the steps given in eq. (6.45): divergences are logarithmic and can not be regulated as in eq. (6.45). Nevertheless, let us press on by discarding those integrals as before and examine what the outcome is. Following the same steps one find the divergent piece

$$\Gamma_\epsilon[g] = \frac{\hbar}{32\pi^2\epsilon} \int d^4x \left(\frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \right) \quad (6.51)$$

which is the correct set of divergences found in the Seeley-DeWitt-Glikey expansion [58, 59, 61, 62]. As advertised, dropping scaleless integrals is consistent. Here we pause to comment on the relation imposed in eq. (6.40). This choice was enforced based on knowledge that the divergent coefficients associated with the Riemann and Ricci pieces in eq. (6.51) are identical but carry an opposite sign. In other words, eq. (6.40) is an educated guess that ensured we obtain the correct result for the effective action.

Moving on, the finite non-local portion follows immediately

$$\begin{aligned} \Gamma_{\ln}[g] = -\frac{\hbar}{32\pi^2} \int d^4x & \left(\frac{1}{72} R \ln \left(\frac{\square}{\mu^2} \right) R - \frac{1}{180} R_{\mu\nu} \ln \left(\frac{\square}{\mu^2} \right) R^{\mu\nu} \right. \\ & \left. + \frac{1}{180} R_{\mu\nu\alpha\beta} \ln \left(\frac{\square}{\mu^2} \right) R^{\mu\nu\alpha\beta} \right) \end{aligned} \quad (6.52)$$

and once again we see that indeed the logarithmic non-locality is intimately tied to the divergences. This correspondence allows us to display the $\mathcal{O}(R^2)$ action given any matter field as well as gravitons from the knowledge of Γ_ϵ which is carried out below.

6.4.4 The total action and renormalization

We now carry out the renormalization program. The total action is composed of three parts

$$\Gamma[\bar{g}] = \mathcal{S}_{\text{GEFT}} + \Gamma_\epsilon + \Gamma_{\text{ln}} \quad (6.53)$$

where \bar{g} denotes the background KS metric. Here the first piece is the gravitational effective action up to $\mathcal{O}(\partial^4)$

$$\mathcal{S}_{\text{GEFT}} = \int d^4x \left(\frac{M_P^2}{2} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + c_4 \nabla^2 R \right) . \quad (6.54)$$

Notice here that we included the Riemann tensor explicitly in the curvature basis which is not how the action is usually displayed. The last piece is usually dropped since it is a total derivative. Inspection of eq. (6.49) shows that we must retain this operator¹⁶. Moreover, it is conventional to invoke the Gauss-Bonnet identity in order to get rid of the Riemann piece. This choice has no effect on the equations of motion. As we show in the next section, it is mandatory *not* to use Gauss-Bonnet in order to correctly compute the entropy. This is one crucial advantage of not adopting the naive approach - setting $\mathcal{F}_{\text{Riem}} = 0$ - as we explained in the last section. The second piece in eq. (6.53) is the equivalent of eq. (6.51) but generalized to any matter field as well as gravitons. It reads

$$\Gamma_\epsilon[\bar{g}] = \frac{\hbar}{\epsilon} \int d^4x \left(\alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \Theta \square R \right) \quad (6.55)$$

where the coefficients are listed in table 6.1. Now from the UV – IR correspondence uncovered before, we know how to construct the non-local portion of the action for

¹⁶Notice that for a KS spacetime we have $\nabla^2 R \rightarrow \square R$ via a simple integration by parts. This is indeed consistent with the divergence in eq. (6.49).

	α	β	γ	Θ
Scalar	5	-2	2	-6
Fermion	-5	8	7	-
U(1)boson	-50	176	-26	-
Graviton	430	-1444	424	-

Table 6.1. The coefficients appearing in the effective action due to massless fields of various spins. All numbers are divided by $11520\pi^2$.

any particle species

$$\begin{aligned} \Gamma_{\ln}[\bar{g}] = & -\hbar \int d^4x \left(\alpha R \ln \left(\frac{\square}{\mu^2} \right) R + \beta R_{\mu\nu} \ln \left(\frac{\square}{\mu^2} \right) R^{\mu\nu} \right. \\ & \left. + \gamma R_{\mu\nu\alpha\beta} \ln \left(\frac{\square}{\mu^2} \right) R^{\mu\nu\alpha\beta} + \Theta \ln \left(\frac{\square}{\mu^2} \right) \square R \right) . \end{aligned} \quad (6.56)$$

The renormalization program is now straightforward to perform by replacing the bare constants with their renormalized values¹⁷

$$c_1 = c_1^r(\mu) - \frac{\alpha}{\bar{\epsilon}}, \quad c_2 = c_2^r(\mu) - \frac{\beta}{\bar{\epsilon}}, \quad c_3 = c_3^r(\mu) - \frac{\gamma}{\bar{\epsilon}}, \quad c_4 = c_4^r(\mu) - \frac{\Theta}{\bar{\epsilon}} . \quad (6.57)$$

The renormalized constants carry an explicit scale dependence such that the renormalized action is μ independent. A standard RG analysis dictates

$$\begin{aligned} c_1^r(\mu) &= c_1^r(\mu_\star) - \alpha \ln \left(\frac{\mu^2}{\mu_\star^2} \right) & c_2^r(\mu) &= c_2^r(\mu_\star) - \beta \ln \left(\frac{\mu^2}{\mu_\star^2} \right) \\ c_3^r(\mu) &= c_3^r(\mu_\star) - \gamma \ln \left(\frac{\mu^2}{\mu_\star^2} \right) & c_4^r(\mu) &= c_4^r(\mu_\star) - \Theta \ln \left(\frac{\mu^2}{\mu_\star^2} \right) \end{aligned} \quad (6.58)$$

where μ_\star is some fixed (matching) scale where the effective theory is matched onto the full theory. Clearly, the previous statement is academic since we have no knowledge

¹⁷We are using the $\overline{\text{MS}}$ scheme.

of the full theory. The EFT treatment of quantum gravity is built in a bottom-up approach much like chiral perturbation theory. In such theories, the renormalized couplings must be measured experimentally [1]. When we discuss the correction to the BH entropy, we shall discover an interesting sensitivity to UV physics.

6.5 The partition function and entropy

We now turn to the second goal mentioned in the introduction which is to identify the logarithmic correction to the Schwarzschild black hole entropy. On the *macroscopic* side, there exist a handful of methods to compute the entropy associated to a black hole. On the one hand, Gibbons and Hawking pioneered the Euclidean gravity approach [162]. Subsequently, a host of Euclidean-based methods appeared in the literature as well [163, 164, 165, 166, 167]. On the other hand, Wald’s Noether charge approach [168, 169, 170] expresses the entropy of a stationary black hole as an integral of a local geometric quantity - the Noether charge - over the bifurcation surface of the horizon.

One immediate advantage of knowing the effective action is to enable the use of Wald’s technique. Nevertheless, the formalism as it is originally presented assumes the action to be *local* and a direct application of the results is not possible in our case. One general trick is to render the action local by introducing auxiliary fields and then move to apply Wald’s formula. This trick was used by Myers [171] to discuss the contribution of the Polyakov action to the entropy of 2d black holes. Likewise, the authors of [155] employed the same method to discuss the logarithmic correction to the BH entropy starting from the Riegert action [24]. Yet, it remains quite interesting to adapt Wald’s approach to non-local field theories. We hopefully reserve this endeavor to a future publication.

Here we choose to employ the Euclidean partition function to directly compute the entropy. Let us recall the definition of the partition function in the canonical

ensemble

$$Z(\beta) = \int \mathrm{D}\Psi \mathrm{D}g e^{-\mathcal{S}_E} \quad (6.59)$$

where \mathcal{S}_E is the Euclidean action, Ψ denotes any matter field and g is the spacetime metric. The functional integral runs over periodic field configurations, i.e. $\Psi(0, \vec{x}) = \Psi(\beta, \vec{x})$, for bosons and anti-periodic for fermions. The metrics that appear in the path-integral are those with *asymptotically flat* (AF) boundary conditions [172], i.e. approaching the flat metric on $\mathbb{R}^3 \times \mathbb{S}^1$.

For the theory we are considering the Euclidean action reads

$$\mathcal{S}_E = -\mathcal{S}_{\text{GEFT}} - \mathcal{S}_{\text{boundary}} + \mathcal{S}_{\text{matter}}^E \quad (6.60)$$

where $\mathcal{S}_{\text{GEFT}}$ is given in eq. (6.54), $\mathcal{S}_{\text{boundary}}$ is the Gibbons-Hawking-York boundary term [162, 173] and \mathcal{S}_m^E is the matter action evaluated on the class of Euclidean metrics described above. Indeed one can not compute the functional integral unless some approximation is made. Note that the matter sector we consider is one-loop exact since self interactions are ignored, i.e. the path-integral is Gaussian. For metric fluctuations, we need to expand around a gravitational instanton which leads to a well-defined loop expansion for the partition function¹⁸. At the one-loop level, the partition function now appears

$$\ln Z(\beta) = \Gamma[\bar{g}_E] + \mathcal{S}_{\text{boundary}} \quad (6.61)$$

Here, \bar{g}_E is the Euclidean instanton which obeys the KS form and $\Gamma[\bar{g}_E]$ denotes the effective action evaluated *on-shell*. The only subtlety here is that we have to affect

¹⁸Stationary, but non-static, black hole solutions do not have a *Euclidean section* [168]. For example, the analytic continuation of the Kerr solution yields an imaginary metric. Nevertheless, the Euclidean procedure is well-defined [162].

the following replacement in eqs. (6.55) and (6.56)

$$\square \rightarrow -\Delta \quad (6.62)$$

where Δ is the 4d Laplacian on $\mathbb{R}^3 \times \mathbb{S}^1$.

6.5.1 Schwarzschild black hole

In this section we use the partition function to directly compute the entropy of Schwarzschild black hole. We have the fundamental relation

$$S = (1 - \beta \partial_\beta) \ln Z(\beta) \quad (6.63)$$

The Euclidean section of the Schwarzschild solution reads

$$ds^2 = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (6.64)$$

with $0 \leq \tau \leq \beta$. Customarily, a conical singularity at $r = 2GM$ is avoided by fixing $\beta = \beta_H \equiv 8\pi GM$ which defines the Hawking temperature. In order for us to use the effective action in eq. (6.53) to evaluate the partition function, we need to affect a coordinate transformation similar to eq. (E.22) in order to cast the above metric in its KS form. One then proceeds to carry out the spacetime integrals in eq. (6.53). Although this could readily be done, the evaluation of the non-local portion in eq. (6.56) is quite cumbersome¹⁹. As we show next, the logarithmic correction can be extracted in a much simpler fashion by studying the scaling properties of Γ_{ln} .

¹⁹The interested reader can consult the previous chapters for the position-space representation of $\ln \square$.

Consider two background metrics \bar{g} and \bar{g}_Λ related as follows²⁰

$$\bar{g}_\Lambda = \Lambda^2 \bar{g} \quad (6.65)$$

where Λ is a spacetime constant. In other words, they are related by a global scale transformation. If the original metric \bar{g} solves Einstein equations, so would the scaled metric. In particular, the scaled metric is an instanton. One then inquires about the corresponding change in the entropy. As evident from eq. (6.61), this requires knowledge of the transformation properties of the effective action. The various curvature tensors transform as follows

$$\begin{aligned} \sqrt{\bar{g}_\Lambda} &= \Lambda^4 \sqrt{\bar{g}}, \quad R^\mu_{\nu\alpha\beta}(\bar{g}_\Lambda) = R^\mu_{\nu\alpha\beta}(\bar{g}) \\ R_{\mu\nu}(\bar{g}_\Lambda) &= R_{\mu\nu}(\bar{g}), \quad R(\bar{g}_\Lambda) = \Lambda^{-2} R(\bar{g}) \quad . \end{aligned} \quad (6.66)$$

On the other hand, the logarithm in eq. (6.56) transforms as

$$\ln \left(\frac{-\Delta}{\mu^2} \right) \rightarrow \ln \left(\frac{-\Delta}{\mu^2} \right) - \ln \Lambda^2 \quad . \quad (6.67)$$

Finally, we have

$$S_\Lambda - S \propto \ln \Lambda^2 (1 - \beta \partial_\beta) \Upsilon[\bar{g}_E] \quad (6.68)$$

where

$$\Upsilon[\bar{g}_E] = \int d^4x \left(\alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - \Theta \Delta R \right) \quad . \quad (6.69)$$

²⁰One could achieve this scaling by transforming the coordinates as $x^\mu \rightarrow \Lambda x^\mu$ and simultaneously rescaling $M \rightarrow \Lambda M$.

It is easily verified that under the scale transformation in eq. (6.65) the ADM mass of Schwarzschild black hole becomes

$$M \rightarrow \Lambda M \quad . \quad (6.70)$$

Since the mass of the black hole is the only dimensionful parameter in the solution, it is evident from eq. (6.68) that the correction to the entropy is proportional to the logarithm of the horizon area. The coefficient is easily computed from eq. (6.69) where only the Riemann piece contributes non-trivially. This point makes it obvious why we should keep all independent invariants present in the action²¹.

Finally, taking the local portion of the action into account we arrive at²²

$$S_{\text{bh}} = S_{\text{BH}} + 64\pi^2 (c_3^r(\mu) + \Xi \ln(\mu^2 \mathcal{A})) \quad (6.71)$$

where Ξ is given in eq. (6.8). This is the second result of this chapter²³. We observe a rather important feature in the result: the entropy is invariant under RG evolution

$$\frac{d}{d \ln \mu} S_{\text{bh}} = 0 \quad (6.72)$$

where use has been made of eq. (6.58). Conversely, we could have deduced the logarithmic correction by enforcing RG invariance. Notice that $\ln \mu^2$ in eq. (6.56) contributes a local piece in the partition function. By dimensional consistency, there

²¹Another way to see the same physics is to realize that the Euler number of the Schwarzschild instanton is non-vanishing. Hence, a naive implementation of the Gauss-Bonnet identity is incorrect.

²²Notice that \hbar has been set to unity.

²³It is shown in [150] that the possible contribution of zero modes to the partition function should be handled carefully. The scalar operator treated in section 6.2 is positive definite on the Euclidean black hole. Nevertheless, a thorough analysis of zero modes should be performed when considering higher-spin fields.

must exist a geometric quantity with the correct mass-dimension to render the logarithm dimensionless as it must be. For the Schwarzschild instanton, the only quantity available is the area of the event horizon.

6.5.2 Dimensional transmutation and final remarks

The physical character of the entropy is elegantly emphasized if we use dimensional transmutation. The constant in eq. (6.71) is dimensionless and could be traded for a dimensionful scale by writing

$$c_3^r(\mu) = -\Xi \ln(\mu^2 \mathcal{A}_{\text{QG}}) \quad . \quad (6.73)$$

Every UV completion of quantum gravity *must* predict a unique value for the above constant at same matching scale. This in turn fixes the value of \mathcal{A}_{QG} which has dimensions of area. In other words, the latter scale defines the theory of quantum theory. We can now rewrite eq. (6.71) with no reference to the unphysical scale μ

$$S_{\text{bh}} = S_{\text{BH}} + 64\pi^2 \Xi \ln\left(\frac{\mathcal{A}}{\mathcal{A}_{\text{QG}}}\right) \quad . \quad (6.74)$$

The result exhibits a manifest correspondence between the UV and IR. This elegant dichotomy is brought about by the structure of the logarithmic non-locality in the partition function. Here, one clearly sees the power of the EFT framework. Induced by the non-analytic portion of the action, the logarithmic dependance on the horizon area and the associated coefficient furnish a test laboratory for any proposed theory of quantum gravity. Yet, a short-distance scale, characteristic of the UV completion, shows up hand-in-hand with the infrared effect.

Some remarks are due in place. It is quite intriguing that the coefficient of the logarithm in eq. (6.8) is not positive definite. The gauge fields in the theory yields a negative contribution. In fact, dialing up the number of particles could render the

quantum correction large even in a regime where the effective field theory is valid. In other words, the logarithm might compete with the BH term in the large- N limit. The inevitable existence of massless gauge fields makes it possible to attain a state of vanishing entropy. Nevertheless, it is not clear to us if this observation hides any deep physics. One might also inquire if higher curvature (loop) corrections would alter the result. The uncovered UV/IR properties of the correction lead us to believe that the logarithmic correction does not receive any modification.

6.6 Future outlook

There exist a handful of open questions which we reserve for future work. Let us outline them in some detail:

- The fate of the form factors and their covariance properties remains unclear in an arbitrary spacetime. In our case, we lost *general* coordinate invariance by fixing the background geometry to be a KS spacetime. Yet, we gained the ability to obtain the exact effective action up to second order in the curvature. In particular, we uncovered the non-analytic structure of the form factors which turned out to be rather simple. Only the flat space derivative operators appear in the form factors. The counter-term method initiated in chapter 2 was *unnecessary* in our construction. More work is needed to clarify if there exists a better way to display the answer in a generic spacetime.
- It is also possible to extend the analysis beyond minimally coupled fields. For example, introducing the non-minimal coupling (ξR) into the *scalar* kinetic operator will change the coefficient α in table 6.1 by a multiplicative factor $(6\xi - 1)^2$. This modification does not affect the correction to the BH entropy. One the other hand, the effect of including self interactions in the matter sector

is worth investigating. In particular, one would like to check if the logarithmic correction receives any modification in this case.

- To realize a successful program of infrared quantum gravity, it is crucial to understand how to handle infrared singularities in effective actions. Although the result at second order in the curvature is free of the latter, they become omnipresent at higher orders. It was found in chapter 2 that the effective action of massless QED - with gravitational coupling - could be made infrared safe if one chooses the background fields to satisfy their lowest order equations of motion²⁴. Nevertheless, this procedure is neither justified nor is it guaranteed to work. Clearly, we need further insight.
- Wald's Noether charge approach stands out as the most elegant technique to define and compute the entropy. In particular, it endows black hole entropy with a geometric meaning. It is rather important to obtain the logarithmic correction via Wald's approach. In 2d, Myers [171] has made a pioneering step to adapt Wald's technique to study the non-local Polyakov action. Nevertheless, the non-local structure in the latter comprises a massless pole, i.e. $1/\nabla^2$, and so it is not clear how to generalize the treatment in our case. A geometric derivation is highly desirable in order to go beyond specific black holes and generalize our results.
- It is always interesting to derive Hawking radiation using various approaches. As the effective action encodes the vacuum fluctuations, an elegant pathway to Hawking radiation should start from the effective action. Progress has been made for 2d black holes, see for example [174]. In 4d, Mukhanov et. al. [178] made an initial step in this direction by considering the contribution of *s-modes*

²⁴Here, we mean both the gauge and metric fields.

to the effective action. In this case, the computation is very similar to the 2d case. Nevertheless, more work needs to be done in 4d.

- Perhaps the most important future step is to study the back-reaction on the spacetime. This is mandatory in order to track the process of black hole evaporation. Much work has been devoted to study the physics in 2d, see for example [175, 176, 177] which is surely an incomplete list. There exist little work, if any, regarding realistic 4d black holes. It is quite unlikely that one would be able to find analytic solutions to the equations of motion given the non-local structures present. Nevertheless, numerical solutions will indeed provide invaluable insight.

APPENDICES

APPENDIX A

SCALE CURRENTS

Let us give a quick review of scale and conformal symmetries in a bit more detail than described in the introduction. In general the consequence of dilatation symmetry is to generate a current

$$J_{Noether}^\mu = \Theta_\nu^\mu x^\nu - j^\mu \quad (\text{A.1})$$

where j^μ is called the virial current and $\Theta_{\mu\nu}$ is the *canonical* energy-momentum tensor. Scale symmetry then implies that

$$\partial_\mu J_{Noether}^\mu = \Theta_\mu^\mu - \partial_\mu j^\mu \quad (\text{A.2})$$

For example, if we apply Noether's theorem to S_{EM} we find

$$J_{Noether}^\mu = \Theta_\nu^\mu x^\nu - F^{\mu\alpha} A_\alpha \quad (\text{A.3})$$

where $\Theta_{\mu\nu}$ is

$$\Theta_{\mu\nu} = \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F_{\mu\alpha} \partial_\nu A^\alpha \quad . \quad (\text{A.4})$$

The current is easily seen to be conserved upon using the classical equation of motion, but notice that it looks quite different from the dilatation current in Eq. (2.3).

Moreover, the *asymmetric* canonical energy-momentum tensor is not the same as $T_{\mu\nu}$ quoted in the same equation. The trick is to use scale invariance to construct an *improved* traceless tensor much like using the Belinfante procedure for finding a symmetric energy-momentum tensor exploiting the Lorentz invariance of the theory. These aspects are well explained in [?, 160]. The procedure is to judiciously add a conserved symmetric second-rank tensor to form the Belinfante tensor such that its trace reads

$$T_{\mu}^{\mu} = \partial_{\mu} J_{Noether}^{\mu} \Big|_{off-shell} \quad (\text{A.5})$$

and hence the *improved* tensor $T_{\mu\nu}$ will be traceless on-shell. For electromagnetism, the Belinfante procedure yields the desired tensor without any further modifications¹

$$T_{\mu\nu} = -F_{\mu\sigma}F_{\nu}^{\sigma} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \quad . \quad (\text{A.6})$$

With this object in hand, Eq. (2.3) defines the dilatation current. When coupled to gravity, the photon action is conformally invariant.

A similar story holds for the scalar field, starting from the Lagrangian of Eq. (5.1). For the minimally coupled field, the energy momentum tensor is not traceless and the dilatation current is

$$J_{Noether}^{\mu} = T_{\nu}^{(\xi=0)\mu} x^{\nu} - [\phi^{\star} \partial^{\mu} \phi + (\partial^{\mu} \phi^{\star}) \phi] \quad (\text{A.7})$$

However, if we use the improved energy momentum tensor with conformal coupling, the energy momentum tensor is now traceless

¹Note that the energy-momentum tensor is traceless even off-shell.

$$T_{\mu}^{(\xi=1/6)\ \mu} = 0 \tag{A.8}$$

and we do not need the virial current. The scalar field is only conformally invariant for $\xi = 1/6$.

APPENDIX B

REDUCTION OF THE TRIANGLE AND BUBBLE INTEGRALS

Bubbles

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 + i0)((k+l)^2 + i0)} = -\frac{1}{2} l^\mu I_2(l) \quad (\text{B.1})$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 + i0)((k+l)^2 + i0)} = \frac{1}{4(D-1)} [Dl^\mu l^\nu - l^2 \eta^{\mu\nu}] I_2(l) \quad (\text{B.2})$$

$$\begin{aligned} \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\alpha}{(k^2 + i0)((k+l)^2 + i0)} &= \frac{1}{8(D-1)} [l^2 (\eta^{\mu\nu} l^\alpha + \eta^{\mu\alpha} l^\nu + \eta^{\alpha\nu} l^\mu) \\ &\quad - (D+2) l^\mu l^\nu l^\alpha] I_2(l) \end{aligned} \quad (\text{B.3})$$

where l is an arbitrary four-momentum and I_2 is the scalar bubble function

$$I_2(p) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + i0)((k+p)^2 + i0)} \quad (\text{B.4})$$

Triangles

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{(k^2 + i0)((k+l)^2 + i0)((k+l')^2 + i0)} = A Q^\mu \quad (\text{B.5})$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 + i0)((k+l)^2 + i0)((k+l')^2 + i0)} = B \eta^{\mu\nu} + C Q^\mu Q^\nu + D q^\mu q^\nu \quad (\text{B.6})$$

$$\begin{aligned} \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\alpha}{(k^2 + i0)((k+l)^2 + i0)((k+l')^2 + i0)} &= E(Q^\mu \eta^{\nu\alpha} + \text{perm}) \\ &+ F Q^\mu Q^\nu Q^\alpha \\ &+ G(Q^\mu q^\nu q^\alpha + \text{perm}) \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\alpha k^\beta}{(k^2 + i0)((k+l)^2 + i0)((k+l')^2 + i0)} &= H(\eta^{\mu\nu} \eta^{\alpha\beta} + \text{perm}) \\ &+ I(\eta^{\mu\nu} Q^\alpha Q^\beta + \text{perm}) \\ &+ J(\eta^{\mu\nu} q^\alpha q^\beta + \text{perm}) \\ &+ K Q^\mu Q^\nu Q^\alpha Q^\beta + L q^\mu q^\nu q^\alpha q^\beta \\ &+ M(Q^\mu Q^\nu q^\alpha q^\beta + \text{perm}) \end{aligned} \quad (\text{B.8})$$

where

$$l^2 = l'^2 = \lambda^2 \rightarrow 0, \quad Q = l + l' \quad q = l - l' \quad (\text{B.9})$$

We ignored any analytic dependence on λ^2 , and only retained it inside logarithms.

The different coefficients read

$$\begin{aligned} A &= \frac{1}{q^2} (I_2(q) - I_2(l)), \quad B = \frac{1}{2(D-2)} I_2(q), \quad C = \frac{1}{q^2} \left(\frac{1}{4} I_2(l) - \frac{D-3}{2(D-2)} I_2(q) \right) \\ D &= \frac{1}{q^2} \left(\frac{1}{4} I_2(l) - \frac{1}{2(D-2)} I_2(q) \right), \quad E = -\frac{1}{4(D-1)} I_2(q) \\ F &= \frac{1}{4q^2(D-1)} \left((D-3) I_2(q) - \frac{D}{4} I_2(l) \right), \quad G = \frac{1}{4q^2(D-1)} \left(I_2(q) - \frac{D}{4} I_2(l) \right) \\ H &= -\frac{q^2}{8D(D-1)} I_2(q), \quad I = \frac{1}{8D} I_2(q), \quad J = \frac{1}{8D(D-1)} I_2(q) \\ K &= \frac{1}{8q^2} \left(\frac{D+2}{8(D-1)} I_2(l) - \frac{D-3}{D} I_2(q) \right), \quad L = \frac{1}{8q^2(D-1)} \left(\frac{D+2}{8} I_2(l) - \frac{3}{D} I_2(q) \right) \\ M &= \frac{1}{8q^2} \left(\frac{D+2}{8(D-1)} I_2(l) - \frac{1}{D} I_2(q) \right) \end{aligned} \quad (\text{B.10})$$

APPENDIX C

MOMENTUM SPACE REPRESENTATION

In this appendix, we collect all the momentum space representations of the different curvature invariants. The quadratic density $\sqrt{g}F^2$ is very simple and we do not list here. Moving to the cubic invariants, we find

$$\int d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu} R = \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} A^\alpha(p) A^\beta(-p') h^{\mu\nu}(-q) \mathcal{M}_{\alpha\beta\mu\nu}^S + \mathcal{O}(h^2) \quad (\text{C.1})$$

where

$$\mathcal{M}_{\alpha\beta\mu\nu}^S = 2(p \cdot p' \eta_{\alpha\beta} - p_\beta p'_\alpha)(q_\mu q_\nu - q^2 \eta_{\mu\nu}) \quad . \quad (\text{C.2})$$

The invariant including the Ricci tensor reads

$$\int d^4x \sqrt{g} F_\mu^\beta F^{\alpha\mu} R_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} A^\alpha(p) A^\beta(-p') h^{\mu\nu}(-q) \mathcal{M}_{\alpha\beta\mu\nu}^{Ric} + \mathcal{O}(h^2) \quad (\text{C.3})$$

where

$$\begin{aligned} \mathcal{M}_{\alpha\beta\mu\nu}^{Ric} = & \frac{1}{4} p \cdot p' [(Q_\mu Q_\nu + q_\mu q_\nu) \eta_{\alpha\beta} - 2(p'_\mu p_\beta \eta_{\alpha\nu} + p'_\nu p_\beta \eta_{\alpha\mu} + p'_\alpha p_\mu \eta_{\beta\nu} + p'_\alpha p_\nu \eta_{\beta\mu}) \\ & - q^2 (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\nu\alpha} \eta_{\mu\beta} + \eta_{\beta\alpha} \eta_{\mu\nu}) - 2p_\beta p'_\alpha \eta_{\mu\nu}] - \frac{1}{2} q_\mu q_\nu p_\beta p'_\alpha \quad . \end{aligned} \quad (\text{C.4})$$

Lastly, the invariant including the Riemann tensor reads

$$\int d^4x \sqrt{g} F_\alpha^\beta F^{\mu\nu} R^\alpha_{\beta\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} A^\alpha(p) A^\beta(-p') h^{\mu\nu}(-q) \mathcal{M}_{\alpha\beta\mu\nu}^{Riem} + \mathcal{O}(h^2) \quad (\text{C.5})$$

where

$$\begin{aligned}\mathcal{M}_{\alpha\beta\mu\nu}^{Riem} = & \frac{1}{4} [2p'_\alpha p_\beta (Q_\mu Q_\nu - q_\mu q_\nu) + q^4 (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}) \\ & + 2q^2 (p_\mu p'_\alpha \eta_{\nu\beta} + p_\nu p'_\alpha \eta_{\mu\beta} + p_\beta p'_\nu \eta_{\mu\alpha} + p_\beta p'_\mu \eta_{\nu\alpha})] \quad .\end{aligned}\quad (\text{C.6})$$

In the above, $Q = p + p'$ and $q = p - p'$. Moreover, transversality and on-shellness are assumed

$$p^2 = p'^2 = 0, \quad p \cdot A(p) = p' \cdot A(p') = 0 \quad . \quad (\text{C.7})$$

One can easily check that the matrix elements are both gauge-invariant and satisfy energy-momentum conservation. For example,

$$p^\alpha \mathcal{M}_{\alpha\beta\mu\nu}^S = p'^\beta \mathcal{M}_{\alpha\beta\mu\nu}^S = q^\mu \mathcal{M}_{\alpha\beta\mu\nu}^S = 0 \quad . \quad (\text{C.8})$$

Moreover, one can use the Weyl tensor given in eq. (3.62) to get

$$\int d^4x \sqrt{g} F_\alpha{}^\beta F^{\mu\nu} C_{\beta\mu\nu}^\alpha = \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} A^\alpha(p) A^\beta(-p') h^{\mu\nu}(-q) \mathcal{M}_{\alpha\beta\mu\nu}^C + \mathcal{O}(h^2) \quad (\text{C.9})$$

where

$$\mathcal{M}_{\alpha\beta\mu\nu}^C = \frac{1}{6} (p \cdot p' \eta_{\alpha\beta} - p_\beta p'_\alpha) (q_\mu q_\nu - 3Q_\mu Q_\nu - q^2 \eta_{\mu\nu}) \quad . \quad (\text{C.10})$$

The above tensor is clearly traceless as required since it stems from a conformally invariant Lagrangian.

APPENDIX D

ASPECTS OF THE IN-IN FORMALISM

The aim of the in-in formalism is to derive an expression for the time-dependent expectation value of a Heisenberg operator $\mathcal{O}_H(t)$. For systems out of equilibrium, the Hamiltonian has explicit time dependence. For systems under equilibrium, the common practice in perturbation theory is to switch to the interaction picture by splitting the Hamiltonian into free and interaction pieces. For our case, we switch to the interaction picture by splitting the full Hamiltonian to a time-independent piece, which might itself contain interactions, and a time-dependent interaction; $H(t) = H_0 + H_{int}(t)$. Hence,

$$\mathcal{O}_H(t) = \mathcal{U}^\dagger(t, 0) e^{-iH_0 t} \mathcal{O}_I(t) e^{iH_0 t} \mathcal{U}(t, 0) \equiv S^\dagger(t, 0) \mathcal{O}_I(t) S(t, 0)$$

where $\mathcal{U}(t, t')$ is the *fundamental* time-evolution operator and we choose all pictures to coincide at $t = 0$. The operator $S(t, t')$ is readily seen to satisfy a Schrodinger-like equation whose solution reads

$$S(t, t') = T \exp \left(-i \int_{t'}^t dt_1 H_I(t_1) \right), \quad H_I(t) \equiv e^{iH_0 t} H_{int}(t) e^{-iH_0 t}. \quad (\text{D.1})$$

It remains to relate the states in different pictures where it is convenient for our problem to change the reference time such that all pictures coincide at $t = -\infty$. Hence,

$$|\Phi\rangle_H = |\Phi(-\infty)\rangle_I. \quad (\text{D.2})$$

Using the fundamental unitarity property of the time evolution operator, we find the time-dependent expectation value of an arbitrary operator

$$\langle \mathcal{O}_H(t) \rangle = {}_I \langle \Phi(-\infty) | S^\dagger(t, -\infty) \mathcal{O}_I(t) S(t, -\infty) | \Phi(-\infty) \rangle_I. \quad (\text{D.3})$$

As mentioned in the text, it is very useful to insert the identity operator in the form $S^\dagger(\infty, t) S(\infty, t) = 1$ to the left of the operator

$$\langle \mathcal{O}(t) \rangle = {}_I \langle \Phi(-\infty) | S^\dagger(\infty, -\infty) T [\mathcal{O}_I(t) S(\infty, -\infty)] | \Phi(-\infty) \rangle_I. \quad (\text{D.4})$$

One then obtains various propagators - the normal Feynman propagators associated with purely time-ordered contractions, and others associated with mixed contractions. Wick's theorem must then be generalized to include the anti-time-ordered products of fields, which we now describe.

The goal is to modify Wick's theorem to incorporate an anti-time-ordered product of operators. Here, we do not prove the modified theorem but rather only derive the needed expression for our calculation which is

$$\begin{aligned} \hat{T}[AB]T[CD] = & N[ABCD + \overline{AB}\overline{CD} + \overline{CD}\overline{AB} + \overline{CD}\overline{AB} + \underline{BC}\underline{AD} + \underline{BD}\underline{AC} \\ & + \underline{AD}\underline{BC} + \underline{AC}\underline{BD} + \underline{BC}\underline{AD} + \underline{BD}\underline{AC}]. \end{aligned} \quad (\text{D.5})$$

Here, the operators A, B, C, D may represent different fields or the same field evaluated at different spacetime points and the hat denotes the anti-time-ordering symbol. The underline symbol denotes the positive-frequency Wightman function defined in section 3. We also have the usual Feynmann and Dyson propagators

$$\overline{AB} \equiv \langle 0 | T[AB] | 0 \rangle, \quad \underline{AB} \equiv \langle 0 | \hat{T}[AB] | 0 \rangle. \quad (\text{D.6})$$

To derive eq. (D.5), we start with the simpler product

$$\widehat{T}[AB]C = N[ABC + C \underline{AB} + B \underline{AC} + A \underline{BC}] \quad (\text{D.7})$$

which is proved by employing

$$\widehat{T}[AB] = N[AB] + \underline{AB}, \quad N[AB]C = N[ABC + A \underline{BC} + B \underline{AC}]. \quad (\text{D.8})$$

Left-multiplying eq. (D.7) by an operator, one finds

$$\begin{aligned} \widehat{T}[AB]CD = & N[ABCD + AB \underline{CD} + AC \underline{BD} + BC \underline{AD} + CD \underline{AB} + BD \underline{AC} + AD \underline{BC} \\ & + \underline{CDAB} + \underline{BD AC} + \underline{AD BC}]. \end{aligned} \quad (\text{D.9})$$

The above expression is obtained by deriving the analog of the second equation in (D.8), albeit with an extra operator to the left. Using the basic definition of time-ordered products along with eq. (D.9) readily yields eq. (D.5).

APPENDIX E

KERR-SCHILD SPACETIMES

for the convenience of the reader we review the derivation of the Schwarzschild solution starting from the KS ansatz for the metric. The approach presented here is due to Adler et. al. [179]. This approach is purely algebraic which is quite different from the geometric approach originally employed by Kerr et. al. in [142, 143, 144].

If we substitute the metric in eq. (6.10) into the Ricci tensor, the vacuum Einstein equations appear as a power series in λ

$$\sum_{i=1}^4 R_{\mu\nu}^{(i)} = 0 \quad . \quad (\text{E.1})$$

The expansion goes to fourth order since the Christoffel symbols truncate at second order. The Ricci tensor must vanish at each order in λ . Moreover, since $\sqrt{g} = 1$, we have that $\Gamma_{\mu\nu}^{\mu} = 0$ and thus

$$R_{\mu\nu} = -\partial_{\alpha}\Gamma_{\mu\nu}^{\alpha} + \Gamma_{\beta\mu}^{\alpha}\Gamma_{\nu\alpha}^{\beta} \quad . \quad (\text{E.2})$$

The null property of the KS vector leads to important identities

$$k^{\mu} = g^{\mu\nu}k_{\nu} = \eta^{\mu\nu}k_{\nu}, \quad k^{\mu}\partial_{\nu}k_{\mu} = 0 \quad . \quad (\text{E.3})$$

It is easy to verify that $R_{\mu\nu}^{(4)} = 0$ is satisfied. Setting $R_{\mu\nu}^{(3)} = 0$, we have another important equation

$$\eta^{\alpha\beta}x_{\alpha}x_{\beta} = 0, \quad x_{\alpha} \equiv k^{\beta}\partial_{\beta}k_{\alpha} = k^{\beta}\nabla_{\beta}k_{\alpha} \quad . \quad (\text{E.4})$$

Hence, x_α is null and moreover it is orthogonal to k_α as can easily be checked. Indeed two null vectors which are orthogonal at each point on the manifold must be proportional to each other

$$k^\beta \nabla_\beta k_\alpha = -A k_\alpha \quad (\text{E.5})$$

where A is a scalar function¹. We conclude that k^α must be a null geodesic with non-affine parameterization. It is shown in [179] that the $\mathcal{O}(\lambda^2)$ equation is automatically satisfied once the $\mathcal{O}(\lambda)$ equation is solved. The linear equation is elegantly expressed if we define an extra scalar function $L \equiv -\partial_\mu k^\mu$

$$\square(k_\mu k_\nu) = -2\partial_{(\mu}[(L + A)k_{\nu)}] \quad . \quad (\text{E.6})$$

To simplify the equations, we write the KS vector as $k_\alpha = (\kappa, \kappa \mathbf{w}) = (\kappa, \kappa w_1, \kappa w_2, \kappa w_3)$. For stationary spacetimes, eq. (E.6) leads to three² coupled second-order equations for κ and \mathbf{w} . The latter could be manipulated into an equation involving only first derivatives of \mathbf{w} which reads

$$(\partial_m w_i)(\partial_m w_j) = P(\partial_i w_j + \partial_j w_i), \quad P \equiv \frac{L + A}{2\kappa} \quad . \quad (\text{E.7})$$

This equation is solved analytically by a linear algebraic approach [179]. If we define a real matrix $M_{ij} \equiv \partial_i w_j$ then the above equation becomes

$$M + M^T = P^{-1} M M^T \quad . \quad (\text{E.8})$$

¹We stick to the notation of [179] as much as possible.

²The fourth equation comes from the null constraint which forces $w_i w_i = 1$.

Using eq. (E.5), one finds that \mathbf{w} lies in the null space of both M and M^T . We shall see next that the analysis is greatly simplified. Let R be an orthogonal matrix defined such that

$$\mathbf{w}' = R\mathbf{w}, \quad \mathbf{w}'^T = (1, 0, 0) \quad . \quad (\text{E.9})$$

Indeed the matrix $M' = R^T M R$ satisfies an identical relation as eq. (E.8). Moreover, the rotated vector \mathbf{w}' lies in the null space of M' . In particular, we must have

$$M' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & N_{21} & N_{22} \end{bmatrix} \quad (\text{E.10})$$

which yields

$$N + N'^T = P^{-1} N N'^T \quad . \quad (\text{E.11})$$

The above equation is easily solved in terms of an $U \in SO(2)$ matrix such that $N' = P(1 - U)$. The $SO(2)$ group is parameterized in terms of a single continuous variable, say θ . Plugging everything back, we find

$$M_{ij} = P(1 - \cos \theta)(R_{2i}R_{2j} + R_{3i}R_{3j}) + P \sin \theta(R_{2i}R_{3j} - R_{3i}R_{2j}) \quad . \quad (\text{E.12})$$

Notice that R is orthogonal and has unit determinant which enables us to write

$$M_{ij} = P(1 - \cos \theta)(\delta_{ij} - R_{1i}R_{1j}) + P \sin \theta \epsilon_{ijk} R_{1k} \quad . \quad (\text{E.13})$$

In particular, the elements of the first row fully determine the matrix M . Recall that \mathbf{w} is in the null space of M which forces $w_i = R_{1i}$. Finally we end up with³

$$\partial_i w_j = \alpha(\delta_{ij} - w_i w_j) + \beta \epsilon_{ijk} w_k, \quad \alpha \equiv P(1 - \cos \theta), \quad \beta \equiv P \sin \theta \quad . \quad (\text{E.14})$$

The above equation is both linear and first order in derivatives. Yet, we still need to decouple the rhs which turns out to be an exercise in vector calculus. From the above expression we can form all possible vector and scalar quantities, i.e. $\nabla^2 \mathbf{w}$, $\nabla \cdot \mathbf{w}$ and $\nabla \times \mathbf{w}$. Taking the triple cross product of \mathbf{w} and comparing the resulting expression with $\nabla^2 \mathbf{w}$ yields an equation for the gradient of α

$$\nabla \alpha = \nabla \beta \times \mathbf{w} + (\beta^2 - \alpha^2) \mathbf{w} \quad . \quad (\text{E.15})$$

From the above equation and using $\nabla \times \mathbf{w}$ we obtain a similar expression for β

$$\nabla \beta = -\nabla \alpha \times \mathbf{w} - 2\alpha\beta \mathbf{w} \quad . \quad (\text{E.16})$$

It is rather remarkable that we can remove \mathbf{w} entirely from the above relations. In terms of the complex function $\gamma = \alpha + i\beta$, we compute

$$\nabla^2 \gamma = 0, \quad (\nabla \gamma)^2 = \gamma^4 \quad . \quad (\text{E.17})$$

The KS vector, and hence the spacetime metric, is determined in terms of κ and \mathbf{w} . This is easily achieved in terms of $\xi \equiv \gamma^{-1}$. A straightforward manipulation of $\nabla \xi \times \nabla \xi^*$ and $\nabla \xi \cdot \nabla \xi^*$ yields the desired result

$$\mathbf{w} = \frac{i \nabla \xi \times \nabla \xi^* + \nabla \xi + \nabla \xi^*}{(1 + \nabla \xi \cdot \nabla \xi^*)} \quad . \quad (\text{E.18})$$

³Notice that θ is a function of \mathbf{w} .

It remains to find κ . We note that eq. (E.6) yields

$$\nabla^2(\kappa^2 \mathbf{w}) = \nabla[(L + A)\kappa], \quad \nabla^2 \kappa^2 = 0 \quad . \quad (\text{E.19})$$

Remarkably, these two equations are simultaneously satisfied with the choice $\kappa^2 = c \alpha$, where c is an arbitrary constant.

Let us apply the formalism to find the Schwarzschild solution. A real function solving eq. (E.17) is transparent

$$\gamma = \frac{c}{r} = \frac{c}{(x^2 + y^2 + z^2)^{1/2}} \rightarrow k_\mu = \frac{c}{\sqrt{r}} \left(1, \frac{\mathbf{x}}{r}\right) \quad (\text{E.20})$$

which yields

$$ds^2 = dt_\star^2 - (d\mathbf{x} \cdot d\mathbf{x}) - \frac{c^2}{r} (dt_\star + dr)^2 \quad . \quad (\text{E.21})$$

This is the Schwarzschild solution in Eddington coordinates. A simple coordinate transformation

$$t_\star = t + c^2 \ln(r/c^2 - 1) \quad (\text{E.22})$$

yields the usual form of the Schwarzschild metric. The free constant is determined as per usual from the Newtonian limit of the solution, $c^2 = 2GM$.

APPENDIX F

HEAT KERNEL

Definition

At the one-loop level, one is interested in computing a functional trace of the logarithm of some operator. That is

$$\Gamma[g, \Phi] \propto \text{Tr} \ln \left(\frac{\mathcal{D}}{\mathcal{D}_0} \right) \quad (\text{F.1})$$

where Φ comprises extra background fields present in the system and Tr denotes a trace operation over spacetime as well as internal degrees of freedom. Using the identity

$$\ln \left(\frac{\mathcal{D}}{\mathcal{D}_0} \right) = \int_0^\infty \frac{ds}{s} (e^{-s\mathcal{D}_0} - e^{-s\mathcal{D}}), \quad (\text{F.2})$$

the heat kernel is defined as follows

$$H(x, y; s) = e^{-s\mathcal{D}} \delta^{(d)}(x - y) \quad . \quad (\text{F.3})$$

The parameter s is conventionally called *proper time*. Notice that the Dirac-delta distribution is not covariant in the above expression¹. This choice of normalization

¹The delta distribution contains an implicit identity tensor acting in field space.

appeared in [130] and is convenient for our purposes. The eigenmodes of the operator \mathcal{D} are tensor *densities* of weight 1/2 normalized as follows

$$\mathcal{D}\varphi_n = \lambda_n \varphi_n, \quad \int d^d x \varphi_n \varphi_m = \delta_{nm}, \quad \delta^{(d)}(x - y) = \sum_n \varphi_n(x) \varphi_n(y) \quad . \quad (\text{F.4})$$

Hence eq. (F.3) becomes

$$H(x, y; s) = \sum_n e^{-s\lambda_n} \varphi_n(x) \varphi_n(y) \quad (\text{F.5})$$

which shows that the heat kernel defined as such is a *bi-tensor density* of weight 1/2.

The trace of the heat kernel is defined as

$$\mathcal{H}(s) = \text{tr}_I \int d^d x H(x, x; s) \quad (\text{F.6})$$

where tr_I denotes a trace over internal degrees of freedom, i.e. spacetime indices, spin and so on. Now from eq. (F.3), we see that the heat kernel satisfies the following first order differential equation

$$(\partial_s + \mathcal{D}_x)H(x, y; s) = 0, \quad H(x, y; 0) = \delta^{(d)}(x - y) \quad . \quad (\text{F.7})$$

This last equation allows the perturbative expansion of the heat kernel to be developed.

Perturbative expansion

The heat kernel could be determined exactly if one knows the eigenvalues of the operator under consideration. This might be possible to obtain in few simple cases, for instance, Schwinger pair creation in constant electromagnetic field [180]. In general one has to content with some sort of perturbative expansion which enables a

systematic study of a certain problem. Here we describe in some detail the formalism first presented in [124, 125, 126, 127] and reviewed in [130]. Such formalism offers a non-local expansion of the heat kernel and is highly suitable for operators without a given mass scale and thus naturally lends itself to our computation. Recall the KS metric reads $g_{\mu\nu} = \eta_{\mu\nu} - \lambda K_{\mu\nu}$. Consequently, the operator reads²

$$\mathcal{D} = \partial^2 + V \quad (\text{F.8})$$

where V is a function of $K_{\mu\nu}$ and any extra background fields present. Let us take $V = 0$ and solve for the flat space heat kernel. Now eq. (F.7) becomes

$$(\partial_s + \partial_x^2)H_0(x, y; s) = 0 \quad (\text{F.9})$$

This is easily solved by going to Fourier space

$$H_0(p, p'; s) = (2\pi)^d \delta^{(d)}(p + p') e^{sp^2} \quad (\text{F.10})$$

which then yields

$$H_0(x, y; s) = \frac{i}{(4\pi s)^{d/2}} \exp \left[\frac{(x - y)^2}{4s} \right] . \quad (\text{F.11})$$

It is convenient to introduce a matrix notation at this stage if we recognize the heat kernel as a matrix in position space. For instance, the flat-space heat kernel satisfies the following property

$$H_0(x, y; s + t) = \int d^d z H_0(x, z; s) H_0(z, y; t) \quad (\text{F.12})$$

²Any operator must start with the full spacetime d' Alembertian that results from the kinetic term in the action.

which could be written as

$$H_0(s+t) = H_0(s) \times H_0(t) \quad . \quad (\text{F.13})$$

Note in particular the following identity

$$\mathbb{K} = H_0(s) \times H_0(-s) \quad . \quad (\text{F.14})$$

To set up the perturbative expansion, we define a proper-time evolution operator as follows [130]

$$U(s) = H_0(-s) \times H(s) \quad (\text{F.15})$$

which, using eqs. (F.7) and (F.9), is easily seen to satisfy the following differential equation

$$\partial_s U(s) = -H_0(-s) \times V \times H(s), \quad U(0) = \mathbb{K} \quad . \quad (\text{F.16})$$

Now the interaction V is also a matrix in position space. The above equation is not yet in the desired form, but we can use eq. (F.12) to rewrite eq. (F.15) as follows

$$H(s) = H_0(s) \times U(s) \quad . \quad (\text{F.17})$$

Hence, eq. (F.16) becomes

$$\partial_s U(s) = -H_0(-s) \times V \times H_0(s) \times U(s) \quad (\text{F.18})$$

and has the familiar solution

$$U(s) = \text{T exp} \left(- \int_0^s dt H_0(-t) \times V \times H_0(t) \right) \quad . \quad (\text{F.19})$$

Here, T is the proper-time ordering operator. We observe here that the proper time plays the role of it in real-time perturbation theory. It proves easier to turn the integration variables into dimensionless quantities by rescaling $t \rightarrow t/s$ [130]

$$U(s) = T \exp \left(- \int_0^1 dt H_0(-st) \times V \times H_0(st) \right) . \quad (\text{F.20})$$

This equation is the basis of the non-local expansion of the heat kernel [124, 125, 126, 127]. We finally plug the above formula in eq. (F.15) to obtain the heat kernel.

APPENDIX G

USEFUL IDENTITIES

The form factors

There are various ways to relate the form factors to the fundamental one in eq. (6.31). Let us process the following integral

$$I(n) = \frac{1}{2} \int_0^1 d\sigma (1 - 2\sigma)^2 (\sigma(1 - \sigma))^n \quad (\text{G.1})$$

which is easily expressed in terms of the Euler gamma function

$$I(n) = \frac{1}{n+1} \frac{\Gamma(2+n)\Gamma(2+n)}{\Gamma(4+2n)} .$$

This can be put back into an integral representation

$$I(n) = \frac{n!}{(n+1)!} \int_0^1 d\sigma \sigma^{n+1} (1 - \sigma)^{n+1} \quad (\text{G.2})$$

which enables us to derive the following identity

$$\frac{x}{2} \int_0^1 d\sigma (1 - 2\sigma)^2 e^{\sigma(1-\sigma)x} = f(x) - 1 \quad (\text{G.3})$$

where $f(x)$ is the form factor in eq. (6.31). Using the above, we can derive the following identities as well

$$\int_0^1 d\sigma \sigma(1 - \sigma) e^{\sigma(1-\sigma)x} = \frac{1}{4} f(x) - \frac{1}{2x} [f(x) - 1] \quad (\text{G.4})$$

$$\int_0^1 d\sigma \sigma^2(1 - \sigma)^2 e^{\sigma(1-\sigma)x} = \frac{1}{32} f(x) - \frac{1}{8x} f(x) + \frac{1}{16x} + \frac{3}{8x^2} [f(x) - 1] . \quad (\text{G.5})$$

Tensor integrals

We here list the tensor integrals needed for the computation of the heat kernel.

$$\int \frac{d^d p}{(2\pi)^d} e^{sp^2} = \frac{i}{(4\pi s)^{d/2}} \quad (\text{G.6})$$

$$\int \frac{d^d p}{(2\pi)^d} p_\mu p_\nu e^{sp^2} = \frac{i}{(4\pi s)^{d/2}} \frac{-1}{2s} \eta_{\mu\nu} \quad (\text{G.7})$$

$$\int \frac{d^d p}{(2\pi)^d} p_\mu p_\nu p_\alpha p_\beta e^{sp^2} = \frac{i}{(4\pi s)^{d/2}} \frac{1}{4s^2} (\eta_{\mu\nu} \eta_{\alpha\beta} + \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}) \quad (\text{G.8})$$

Curvature invariants in momentum space

Here we provide the momentum space representation of the different curvature invariants which are needed to determine the heat kernel at second order in the curvature. For KS spacetimes with a flat background metric in Cartesian coordinates, the quadratic invariants read at lowest order

$$\int d^d x \text{Riem}^2 = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} K_p^{\mu\nu} K_{-p}^{\alpha\beta} (p^4 T_{\mu\nu\alpha\beta} - p^2 \mathcal{P}_{\mu\nu\alpha\beta} + 2p_\mu p_\nu p_\alpha p_\beta) \quad (\text{G.9})$$

$$\int d^d x \text{Ric}^2 = \frac{1}{8} \int \frac{d^d p}{(2\pi)^d} K_p^{\mu\nu} K_{-p}^{\alpha\beta} (p^4 T_{\mu\nu\alpha\beta} - p^2 \mathcal{P}_{\mu\nu\alpha\beta} + 4p_\mu p_\nu p_\alpha p_\beta) \quad (\text{G.10})$$

$$\int d^d x R^2 = \int \frac{d^d p}{(2\pi)^d} K_p^{\mu\nu} K_{-p}^{\alpha\beta} p_\mu p_\nu p_\alpha p_\beta \quad (\text{G.11})$$

where we defined

$$T_{\mu\nu\alpha\beta} = \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}, \quad \mathcal{P}_{\mu\nu\alpha\beta} = p_\mu p_\alpha \eta_{\nu\beta} + p_\mu p_\beta \eta_{\nu\alpha} + p_\nu p_\alpha \eta_{\mu\beta} + p_\nu p_\beta \eta_{\mu\alpha} \quad . \quad (\text{G.12})$$

We also need the expansion of the Ricci scalar to order λ^2 which reads

$$\int d^d x R^{(2)} = \frac{1}{8} \int \frac{d^d p}{(2\pi)^d} K_p^{\mu\nu} K_{-p}^{\alpha\beta} (p^2 T_{\mu\nu\alpha\beta} - \mathcal{P}_{\mu\nu\alpha\beta}) \quad . \quad (\text{G.13})$$

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