Puzzles in Quantum Gravity

What can Black Hole Microstates teach us about Quantum Gravity?

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WHAT CAN BLACK HOLE MICROSTATES TEACH US ABOUT QUANTUM GRAVITY?

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor

aan de Universiteit van Amsterdam

op gezag van de Rector Magnificus

prof. dr. D.C. van den Boom

ten overstaan van een door het college voor promoties

ingestelde commissie,

in het openbaar te verdedigen in de Agnietenkapel

op woensdag 15 september 2009, te 12.00 uur

door

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geboren te Aramoun

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FACULTEIT DER NATUURWETENSCHAPPEN, WISKUNDE EN INFORMATICA

 This thesis was produced using gvim, Latex-Suite and Tex-Live.

This work is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie (FOM)", which is financially supported by the "Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)".

PUBLICATIONS

This thesis is based on the following publications:

① J. de Boer, P. de Medeiros, S. El-Showk, A. Sinkovics,
Open G_2 strings,
JHEP **02** (2008) 012 [hep-th/0611080].

② J. de Boer, P. de Medeiros, S. El-Showk, A. Sinkovics,
G_2 Hitchin functionals at one loop,
Class.Quant.Grav. **25** (2008) 075006 [arXiv:0706.3119 [hep-th]].

③ J. de Boer, F. Denef, S. El-Showk, I. Messamah, D. Van den Bleeken,
Black hole bound states in $AdS_3 \times S^2$,
JHEP **11** (2008) 050 [arXiv:0802.2257 [hep-th]].

④ J. de Boer, S. El-Showk, I. Messamah, D. Van den Bleeken,
Quantizing $\mathcal{N} = 2$ Multicenter Solutions,
JHEP **05** (2009) 002 [arXiv:0807.4556 [hep-th]].

⑤ V. Balasubramanian, J. de Boer, S. El-Showk, I. Messamah, D. Van den Bleeken,
Black Holes as Effective Geometries ,
Class.Quant.Grav **25** (2008) 214004 [arXiv:0811.0263 [hep-th]].

⑥ J. de Boer, S. El-Showk, I. Messamah, D. Van den Bleeken,
A bound on the entropy of supergravity?,
Unpublished [arXiv:0906.0011 [hep-th]].

Preface

This thesis is comprised of material developed by myself and collaborators over the four years of my doctoral training. During this period I participated in two essentially independent lines of research. The first, which comprises Part II of this thesis, resulted in two publications [1][2] and focused on toplogical string and field theories on seven dimensional manifolds with G_2 holonomy. In this exploratory work we found many interesting connections between open topological string theory on G_2 manifolds and various theories considered in the mathematics literature. We also explored the role of topological theories on G_2 manifolds in unifying the topological A and B models on Calabi-Yaus. It is my hope that this work will one day serve as the basis for a deeper understanding of so-called "Topological M-theory".

After completing this research my emphasis shifted and with several colleagues I began work on a program to better understand quantum aspects of supersymmetric black holes resulting in two publications and one preprint [3][4][5] and one published review [6]. This is the subject of Part I of this thesis. My motivation to embark on this new line of research was based on the desire to use string theory to learn general and relevant lessons about nature despite the lack of experimental evidence for the former. I believe the work in this thesis represents a degree of success in this regard as we have found some rather exotic phenomena, potentially relevant for resolving black hole information loss, and it is my belief (or hope) that the general mechanism for the resolution of this paradox should be somewhat universal to any consistent UV-complete theory of gravity.

Due to lack of time, as well as the desire to not reinvent the wheel, there is little introductory material in this thesis. The level of exposition through-out the bulk of this thesis is essentially geared towards an advanced PhD student already well-versed in string theory, supergravity, etc... Some expository material is provided but is primarily intended to address common misconceptions in the literature or provide some conceptual framework for the work here rather than to actually provide detailed background material. For the latter the reader is directed to any one of the several reviews or other theses cited in this work.

Much of the thesis is a recapitulation of results already present in [1][2][3][4][5] and our review [6]. The reader familiar with these may wish to focus instead on the first chapters where I have included my own conceptual overview of unifying threads running through these works. I have also attempted to present my perspective on some common issues or misconceptions plaguing the literature relevant to the research presented here.

It is my hope that this thesis will serve as a useful resource for students and researchers interested in building on the results presented here.

CONTENTS

CHAPTER 1

INTRODUCTION

In this thesis we will consider a diverse range of physical phenomena and theories explored in the works [1][2][3][4][5]. Our ultimate goal will be to enhance our understanding of the structure of quantum gravity by trying to learn general lessons from specific but exotic phenomena.

1.1 MOTIVATION

In the accumulated lore of string or M-theory we now have the makings of a theory of quantum gravity. Unfortunately, lacking either experimental underpinnings or even a full, non-perturbative formulation these theories may seem, at first instance, to be on rather unsound footing.

Despite these flaws string/M-theory has had profound successes on several fronts. Within this framework it has been possible to address several outstanding issues in black hole physics such as the black hole entropy and the essentially thermodynamic nature of black holes (and perhaps even gravity itself). In giving birth to and being an integral part of AdS/CFT these theories have also provided the first instance of a fully non-perturbative theory of quantized gravity that is manifestly holographic and that demonstrates that gravity emerges from very different underlying degrees of freedom. This correspondence has also been turned on its heels, allowing us to use string theory to study theories closely related to (and perhaps, in some sense, within the same universality class as) well-tested theories like QCD.

In and of themselves string and M-theory also exhibit a rich and beautiful structure, deeply interwoven with supersymmetry, which has been very fruitful for mathematics and also suggests, contingent upon the existence of this symmetry, that the theories are very natural.

Ideally we would like to find a first principles formulation of M-theory (with string theory as a derivative) that unifies both our perturbative and non-perturbative understandings of the theories. Doing so would also require elucidating various foundational questions that emerge in any theory of quantum gravity such as the role of boundary conditions, so-called "background independence", and the structure of the Hilbert space. Of course Matrix theory and AdS/CFT provide us with some insights or even examples of what such a theory would look like but these formulations are somewhat removed from our original notions of strings and spacetime and do not have the range of applicability we would like.

Given this state of affairs it is important to improve our understanding of these theories, in particular the ways they differ from non-gravitational quantum theories. In this thesis we hope to build upon the accumulated wisdom in the literature by contributing some new examples of somewhat exotic quantum gravitational phenomena that, moreover, yield potential resolutions to issues in black hole physics.

While our understanding of string theory still lacks the depth and completeness that might provide for a satisfactory and convincing resolution of all the puzzles associated with quantum gravity there is a growing sense that the general conceptual outline of this theory is beginning to reveal itself. Through a combination of diverse insights a general picture has emerged of gravity as an emergent or effective theory that has, as one manifestation of itself, a non-Abelian field theory in one dimension lower. Moreover string theory challenges our intuition, built on the foundation of effective field theory, as gravity seems essentially holographic and non-local and a new, still quite coarse, intuition is slowly emerging by incorporating these lessons.

It is our goal here to build on this emerging picture. We would like to understand, for instance, the structure of the gravitational spectrum, what super-selection sectors exist, how perturbative and non-perturbative states combine to generate e.g. a black hole ensemble, and what these states look like in spacetime. We will be able to address these questions to some degree in very specific contexts but, as our findings will be quite surprising from the point of view of standard field theory, we hope to tease from them some general features of the theory that will one day be string theory.

PART I

String theory is a quantum theory of gravity. Whether it is *the* theory of quantum gravity is not yet clear. As suggested above, here we take the attitude that any consistent theory of quantum gravity provides a framework to extract *qualitative* features of gravity associated with black hole paradoxes. Thus we would like to claim that the results in the first part of this thesis are of considerable interest to black hole physics somewhat independently of whether or not string theory proves to be the final theory of gravity.

The motivation for this perhaps surprising claim is the following. Black holes exhibit mysterious behaviour in a wide range of theories including the standard, well-tested, theory of four dimensional general relativity (Einstein gravity). The essential point is that black holes exhibit (almost) exactly the same paradoxes in more exotic theories and even higher dimensional theories like string theory and supergravity. Fortunately the latter are more symmetric and hence, to some degree, under better control than standard Einstein gravity. By studying black holes in such theories we can hope to use our enhanced technical control to resolve some of these issues, at least at a conceptual level. While it is possible, it seems highly implausible that the resolution to ubiquitous black hole quantum gravity puzzles, whose existence hinges on only the most basic features common to any gravitational theory, would somehow depend very sensitively on the details of the theory considered. In some sense then, the robustness of the information loss paradox is exactly what allows us to probe it, with some degree of confidence, using theories seemingly very different than the one where it was first discovered. That is to say, since the puzzle seems to emerge in any theory which has, as a consistent limit, low-energy Einstein gravity one would imagine that if this theory is a consistent theory of quantum gravity it has to capture, at least in broad strokes, the conceptual outlines of the "universal" resolution to this paradox. It is to the task of finding these universal principles that we turn in Part I of this thesis, working in the specific context of four and five dimensional black holes in supersymmetric string theory.

Another essential theme or lesson gleaned from string theory is that open degrees of freedom provide the non-perturbative completion of the theory. From black hole considerations it seems likely that gravity, in any UV completion, is an effective description of unknown (perhaps holographic) microscopic degrees of freedom and the lesson in string theory is that these degrees of freedom are non-perturbative in nature and enjoy some representation as non-Abelian degrees of freedom in a non-gravitational theory. It is of great interest to understand if and how these degrees of freedom emerge in a description where spacetime is manifest. In a weak-coupling limit we know they can be represented by branes but the interplay between non-Abelian degrees of freedom and spacetime is still quite obscure. We will not touch on this very directly in this thesis but it provides a backdrop for much of the work here and is somewhat evident in our considerations of the supergravity spectrum and its relation to black holes.

PART II

In a similar vein there we would like to use a vastly simplified version of string theory to extract lessons about the fundamental structure of the theory itself. Unlike Part I the lessons learnt here are unlikely to apply to general theories of quantum gravity as they are fundamentally intertwined with supersymmetry (so only apply if the latter is an essential part of quantum gravity).

Simplified, topological versions of string theory seem to exhibit many of the same structures as their full-blown counterparts with the former being far more amenable to detailed study and treatment. For instance, closed-open duality seems to be a feature of both physical and topological string theory but this duality is far easier to demonstrate in the topological version of the theory [7]. The relationship between the closed and open version of string theory seems to provide an underpinning for addressing non-perturbative aspects of the theory simply because open strings encode non-perturbative degrees of freedom from the closed perspective. Thus by better understanding this relationship, even in a simplified version of the theory (that is at least a derivative of the full physical theory) we may learn something about the way the full, non-perturbative version of string theory should be formulated.

Another non-perturbative duality is the relationship between strings and membranes in string/M-theory. A potential role for this duality has also recently emerged in topological string theory [8] as there are hints that topological A/B string theory may actually be united under the auspices of "topological M-theory". It is with the latter that we will concern ourselves in the second part of this thesis.

Although it is not clear what "topological M-theory" should be an immediate guess would be a topological membrane sigma model on G_2 manifolds that reduces, for manifolds of the form $\mathrm{CY}_3 \times \mathrm{S}^1$, to some combination of the A and B model. Such a direct analogy with the physical theory has proven somewhat difficult from a computational point of view as membrane theories seem hard to define and work with so we focus instead on a topological string theory on G_2 manifolds and hope to learn how such a theory may relate to the well studied topological A and B models and what lessons such a theory may yield for physical string theory. One particular hope, unfortunately unmet, is a clear understanding of how "topological M-theory" unifies, non-perturbatively, the A and B model and how the partition functions of the latter display a wavefunction like behaviour as a consequence of this.

1.2 Results

Here we would like to high-light, for the readers convenience, the major results of this thesis.

AdS/CFT for Multicentered Configurations

In Chapter 3 we find a decoupling limit for a large class of solutions to solutions of $\mathcal{N} = 1$ supergravity in five dimensions (which descend to all solutions to $\mathcal{N} = 2$ supergravity in four dimensions). This limit places the centers in asymptotically $\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{CY}_3$ space making them amenable to study via AdS/CFT, a task which we also embark upon.

Quantization of Multicentered Configurations

In Chapters 4 and 5 we determine a procedure to quantize large families of the aforementioned solutions, correctly reproducing the degeneracy found in [9] using split attractor arguments. Our quantization is made possible by restricting to the BPS locus of the phase space (and hence also the solution space) thereby avoiding the problems generally associated with quantizing gravity. Moreover, we exploit a non-renormalization theorem relating some gravitational (closed string) quantities to open string quantities in order to simplify our computation. This quantization also provides the groundwork for several of our other results.

Macroscopic Quantum Fluctuations

In Chapter 5 we exploit the quantum structure of the solution space described above to find the following fascinating result. Certain classical solutions, corresponding to points in the phase space of the theory (see Section 2.3), do not seem to support semi-classical quantum states localized on them. This is because the symplectic form is very sparse in a region of solution space so a large family of solutions that look macroscopically quite distinct nevertheless all sit in one "unit" of phase space once the latter is quantized. Because quantum states can be localized, at most, on and around a cell in phase space (but never within one) there are no semi-classical quantum states (with low variance) describing the very different solutions within the cell.

As this is a very important discovery let us go into somewhat more detail. The solutions described above have, as a defining characteristic, arbitrary long, deep throats that are nonetheless entirely smooth without any regions of high curvature. According to standard effective field theory intuition applied to general relativity these are good saddle-points to the path integral and can be trusted as classical solutions, accessible to macroscopic observers and having vanishingly small variance in the large charge limit. In our quantization a large family of such solutions sit in a region of solution space, which can be mapped to the phase space according to the arguments of Section 2.3, and are parameterized by a small number associated with the depth of the throat. The novel physics that emerges is that the symplectic form is somewhat insensitive to these many macroscopically different solutions; the symplectic volume associated with this region of the phase space is vanishingly small. Thus when quantum effects are included these geometries will fluctuate wildly into each other defying our intuition that classical solutions do not suffer from (significant) quantum fluctuations. Most of the differences between these geometries lie in the throat region so a general phase space density (the analog of a wavefunction) localized in this region of the phase space will have vanishingly small variance away from the throat but, at some depth down the throat, will start exhibiting wild quantum fluctuations over the macroscopic distances associated with the throat. This is exactly the kind of novel physical phenomena one might hope helps resolve puzzles such as information loss so we take this as an important qualitative lesson from string theory.

(Partial) Resolution of the Entropy Enigma

A somewhat new "enigma" that emerged after the discovery of the above mentioned multicentered black hole solutions was the realization that some two-centered black holes seem to have, in a particular region of charge space, parametrically more entropy than a single centered black hole with the same total charge. This is enigmatic from several perspectives. As the entropy of a black hole usually depends quadratically (or with some power greater than one) on its charges there is a general expectation that entropy can be maximized by having a single black hole. This is also consistent with the idea that a single thermodynamic system has more entropy than its two separated components. Finally, this seems to directly contradict the entropy scaling of the $\mathcal{N} = 2$ partition function conjectured by OSV [10].

In this thesis we find a partial resolution that essentially addresses all the points above except the last one. While a two centered configuration can have a parametrically larger entropy than a single centered one there is actually a large family of such two centered solutions and the dominant configuration in this family has all the entropy localized at just one of the two centers. In this configuration one center is entirely smooth and horizon free carrying no entropy while the other center is a black hole carrying (almost) all the entropy of the system. Thus in all cases it is a single black hole configuration that is most entropic but the kind of black hole that is dominant depends on the value of the charges. This dependence still seems to be in contradiction with OSV but various loopholes have been proposed for this in the literature [9][11].

Insufficient Entropy in Supergravity

A further application of our quantization procedure is the determination of the number of black hole states accessible via direct quantization of only supergravity fields. This is an important question as the latter are well understood and it turns out, perhaps surprisingly, sufficient in the case of so-called "small" $\mathcal{N} = 4$ black holes (preserving 8 supercharges) to account for all the entropy of the black hole. As a result there was some hope that something similar might occur for more realistic $\mathcal{N} = 2$ black holes, thereby providing access to the spacetime structure of a generic black hole microstate. The latter is of course interesting as it promises to shed light on the information loss paradox and indeed any quantum process associated with a black hole.

Our results, however, suggest that we are not so fortunate and that supergravity can only account for a parametrically small fraction of the black hole entropy. In particular, while the entropy of a large black hole grows as the square-root of the quartic invariant of the charges, the entropy of supergravity states grows only as a cube-root. That supergravity is capable of providing an exponential number of states justifies the initial hope at least but the difference in powers makes the latter very subleading at large charge.

DEFINITION AND SPECTRUM OF OPEN G_2 STRINGS

In Part II we extend the results of [12] by defining the open version of topological string theory on G_2 manifolds. The branes in this theory turn out to be the same as those in a physical theory on G_2 manifolds, namely associative three cycles and co-associative four cycles as well as zero branes and branes wrapping the whole manifold. We determine the spectrum of open string excitations for strings with any of these boundary conditions and find that they correspond to gauge fields on the branes and scalars encoding calibration-preserving fluctuations in the transverse space.

WORLDVOLUME THEORIES ON TOPOLOGICAL G_2 BRANES

Extending the above results we study the open string field theory for these topological theories and see that they reduce to Chern-Simons-like theories on associative and co-associative membranes in G_2 manifolds. In fact both these theories descend from a seven-dimensional theory, defined on the entire manifold, which is essentially the G_2 analog of holomorphic Chern-Simons theory.

For associative cycles this theory is nothing more than ordinary Chern-Simons theory coupled to essentially non-interacting matter (scalar degrees of freedom from the normal modes which couple minimally to the Chern-Simons gauge field). This is a powerful result as it provides a non-perturbative definition of the open theory which may, as in [7], be related to the closed version of the theory via a geometric transition. One could then hope to put the closed theory, presently defined only at genus zero, on more firm footing.

One-loop Comparison of Hitchin Functional and G_2 String

Although we do not include detailed results from [2] in this thesis, we do review it briefly so we include the results in this list. The primary result is that, unlike the case of the B-model [13], the one-loop partition functions of the (closed) topological G_2 string and the generalized Hitchin functional for G_2 manifolds do not match but, in fact, differ by power of the Ray-Singer torsion of the manifold.

In a somewhat different vein we do find agreement between the degrees of freedom of the dimensional reduction of the (original and the generalized) Hitchin functional on G_2 manifolds of the form $\mathrm{CY}_3 \times \mathrm{S}^1$ and the six-dimensional Hitchin functionals of [13]. This may seem to contradict the idea that topological M-theory, and indeed the G_2 Hitchin functional itself, encode both Kähler and complex structure deformations of the CY since the Hitchin functionals considered in [13] only incorporate complex structure deformations. A possible explanation for this discrepancy is the fact that the Kähler deformations are generally encoded in a purely topological gauge theory (such as that of [14]) and we neglect the topologically non-trivial sector of the G_2 theory in our analysis. A more careful analysis of the G_2 functional may therefore be needed to find a unification of the A and B models.

Part I

Black Hole Microstates

PRELUDE

In Part I of this thesis, based on [3][4][5], we report on our program to understand the quantum structure of black holes by studying and quantizing microstates originating from supergravity.

Our initial motivation when starting [3] was to understand which objects in string theory could provide corrections to OSV [10] following the developments of [15][16][9]. This work eventually took a different direction, however, and our focus shifted to extracting as much information as possible about $\mathcal{N} = 2$ black holes by studying only supergravity solutions with the same asymptotics. We first did this using AdS/CFT in [3] but soon found we could also directly quantize the space of such solutions providing much more fine grained control over the quantum mechanics of the BPS sector [4]. An important question throughout has been how far one could hope to get by restricting only to supergravity. While a considerable amount of technology exists to treat the latter, if a generic state in the black hole ensemble essentially requires knowledge of stringy or non-perturbative degrees of freedom to distinguish it then it is unlikely that we can carry out a complete program of study using only supergravity modes. This led to [5] where we established that it is very unlikely that supergravity will suffice to probe the generic black hole states.

Despite this setback several unexpected surprises proved that our efforts were hardly in vain as we established the existence of interesting quantum phenomena of exactly the kind one might imagine are necessary to resolve information loss and other associated paradoxes. In particular we have found that in these systems quantum effects can extend over macroscopic distances and our standard intuition from field theory seems to break down. Moreover, we have (further) opened the $\mathcal{N} = 2$ system to study via AdS/CFT and also via an unrelated 0+1 dimensional quiver quantum mechanics first pioneered in [17]. With these tools we may hope to go beyond supergravity and access the full quantum and stringy structure of a generic black hole microstate.

CHAPTER 2

BLACK HOLE PUZZLES

The first part of this thesis will focus on exploring various aspects of black hole physics. In particular we extend the technical tools available and apply them to learn qualitative lessons that may point towards resolutions of the well known paradoxes associated with black hole physics.

Before doing so we spend a chapter on some expository material. We will be relatively brief in our review of well-known material such as the information loss paradox as this thesis is not intended as a review of known results. Rather we will focus on a careful conceptual introduction to what has become known as the "fuzzball conjecture". Because the latter has been the source of somewhat ill-deserved controversy, we spend some time clarifying the main points and disavowing various points of view that are extensions of this conjecture rather than its core. We would like to emphasize, in particular, that the conjecture, in our view, is *not* about the role of semi-classical or even quantum *supergravity* states in making up the black hole ensemble. Rather the conjecture is a statement about the *spatial extent* of a generic state in the ensemble, be it stringy or quantum, and how this grows with the entropy of the ensemble. The main novelty in this proposal is the claim that stringy or quantum effects may be significant, in gravitational theories, even in regimes where curvatures are low (so we might naively think we can trust effective field theory). Put another way, the claim would be that a semi-classical analysis on a fixed background, such as Hawking's analysis in [18], may be invalid in a gravitational theory because quantum gravity effects can be relevant even in regimes with low curvature. We will, in fact, demonstrate a particular instance of this phenomena in Chapter 5.

We will also spend time here providing the conceptual background for some of our main tools such as AdS/CFT and phase space quantization. Again, the idea is not to be comprehensive or even detailed but rather to address conceptual issues relevant for this thesis that are perhaps unusual or novel and also to provide a modicum of background. A reader interested in more detailed background reading is referred to several other interesting theses in this field [19, 20] and to the literature survey of [6].

2.1 Information Loss

Perhaps the most persistent and troublesome puzzle associated with black holes is their ability to "lose" information in a quantum theory. This paradox is rather technical and there has been significant effort made to show that it does not in fact exist (see [21] for a nice summary and analysis of these arguments). It nonetheless seems to persist and a large part of this thesis will be motivated by attempting to qualitatively determine what sort of resolutions may be possible. Before doing so we should naturally introduce the puzzle itself.

Black hole information loss was first proposed by Hawking in the seminal paper [18]. Here we will present a very visual, heuristic and non-technical overview of this phenomena following [22]. Information loss will play a primarily motivational role in this thesis and will not directly be dealt with. We thus forego a detailed treatment here. Rather we hope to give the reader a sense for how the paradox emerges and to hint at what elements necessarily must play a role in any resolution.

If we consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (2.1)$$

we see that not only is there a coordinate singularity at $r = r_h = 2GM$ (with G Newton's constant and M the black hole mass) but also that this point corresponds to a "tipping" of the light-cone. Specifically the killing vector $\frac{\partial}{\partial t}$ which becomes the standard (Minkowski) time-like killing vector asymptotically, becomes null at this point, and is space-like within the horizon. This tipping is, as we shall see, an essential feature of information loss. Note that within the horizon it is $\frac{\partial}{\partial r}$ that is the timelike vector (though it is not a Killing vector).

To study the quantum mechanics of matter fields in this background, while neglecting quantum gravity effects, we would like to find a foliation of the spacetime corresponding to (2.1) on which the induced metric on each slice is everywhere spacelike and regular. Fixed t slices do not satisfy these requirements, as they might in a geometry without a horizon, since such slices become time-like inside the horizon. We can however, foliate spacetime as shown in figure 2.1. This foliation uses constant t slices outside the horizon, constant r slices inside the horizon and a "connector" region that crosses the horizon

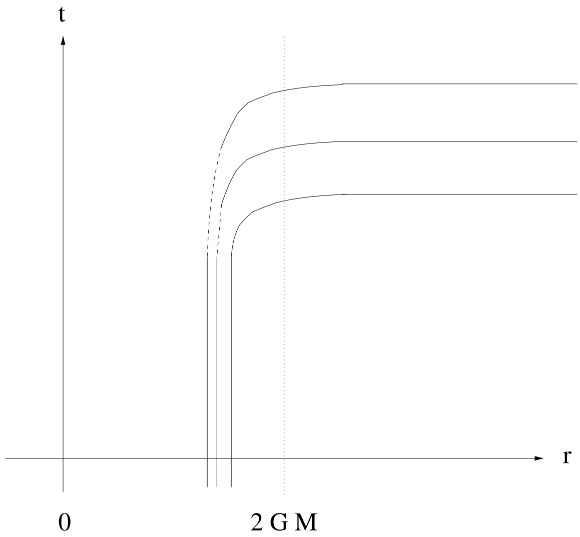

Figure 2.1: *A spatial foliation of the black hole geometry. The regions in side the horizon, $r < 2GM$, correspond to constant r slices whereas outside they are constant t slices. The "connector" region straddling the horizon has to stretch to accommodate the different rate of evolution of the slices. An attempt to foliate a general geometry in a similar manner would fail as this stretching would eventually make part of the connector region timelike. Based on figure 4 of [22].*

and is defined so that the foliation is always spatial. Because we want to keep our slices smooth and away from the singularity they only move inwards very slowly so while the external region of the slices are parameterized by values of t from t_0, when the black hole formed, to $t = \infty$ the internal slices run, within the same parameterization, from $r = r_h$ to $r = \epsilon$ where $\epsilon > 0$ is far enough away from the singularity to keep the curvature on each slice low, even at late times. The different rates of "time-evolution" of the slices implies a stretching of the slices in the connector region as indicated via the dotted lines in the figure.

Performing a standard field-theoretic analysis using this foliation has several interesting consequences. First, the curving of the spatial foliations reflects the fact that null geodesics move in different directions on either side of the horizon. Within the horizon they are directed inwards towards the singularity, whereas those outside the horizon move outwards towards asymptotic infinity (see figure 2.2). Moreover, the regions of the foliation away from the horizon (i.e. the complement of the connector region) have a very low curvature in the induced metric and do not vary much from slice to slice. Near the horizon region, however, there is a "stretching" effect as discussed before. The stretching of the background acts like a time-dependent potential for e.g. a scalar field in this background.

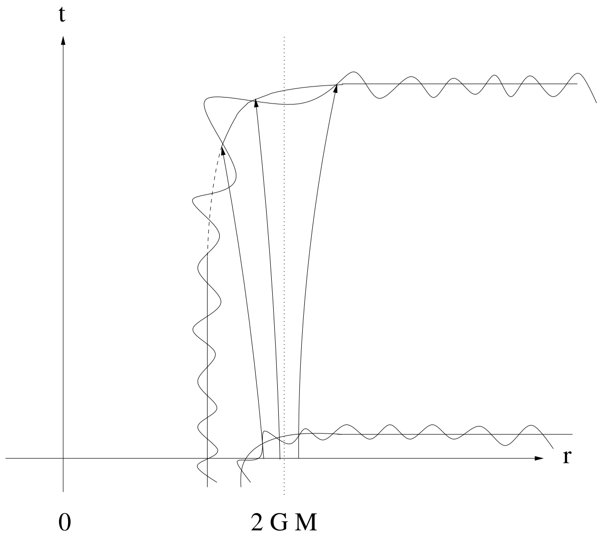

Figure 2.2: *Stretching of wavemodes as a result of changes in the spatial foliation. The upward pointing lines represent the paths of null geodesics in the black hole geometry. The wavemodes are time-evolved by keeping their phase constant along such null geodesic. Based on figure 9 of [22].*

Thus an empty region of spacetime near the horizon that is in a vacuum state at an early slice will no longer be in the vacuum state at later time slices. This can be seen by solving the wave-equation for a scalar on this spatial foliation at a particular time slice in terms of Fourier modes and comparing these modes on a different slice. Somewhat more concretely we can consider an outgoing mode of the form e^{iky^+} with $y^\pm$ light-cone coordinates near the horizon at an early time. We are interested in how these coordinates evolve with time and how they are related to light-cone coordinates at a future time-slice $X^\pm(y^\pm)$. This is because we will assume modes of the form given above time-evolve by keeping a constant phase along null-geodesics.

Recall that the horizon is a splicing point for null geodesics, with geodesics within the horizon heading inwards and those outside heading outwards. Because of this fact, and the way we foliate our geometry, the coordinate transformation between $X^\pm$ and $y^\pm$ will vary over the connector region with the largest variation being near the horizon (see figure 2.2). This coordinate transformation mixes creation and annihilation operators in the basis of Fourier modes and can transform the vacuum at one time slice to an excited state at another.

A simple analog for this is the vacuum state for a harmonic oscillator in a potential parameterized, in the standard way, by a frequency ω. If the potential changes fast enough to that of a different harmonic oscillator with frequency ω' the old ground state wavefunction will not adiabatically evolve into the new one but will keep its form and will now be given by a superposition of excited states of the new Hamiltonian.

In this case it can be shown that the resultant state is heuristically of the form

$$|\psi_e\rangle = e^{\sum_k \gamma b_k^\dagger c_k^\dagger}|0\rangle \tag{2.2}$$

where $b_k^\dagger$ and $c_k^\dagger$ are creation operators for localized quanta of the scalar field on the inside and outside of the horizon, respectively (of course k cannot label a momentum basis here since the quanta would then not be localized so we let k denote an index over some localized basis of creation operators). Thus the resultant state is an entangled state of excited quanta on either side of the horizon and, according to our previous arguments, these modes will either move inwards or outwards depending on which side of the horizon they were created.

This entangled state is at the heart of information loss. The quanta created by the $c_k^\dagger$ will move off to infinity while those created by $b_k^\dagger$ will slowly fall towards the singularity but they are in an entangled state. Generally this is not a problem as an entangled state is still a pure state. The problem arises when we realize that the black hole continues to radiate until it disappears. At this point the quanta at infinity are entangled with nothing!

As a larger and larger number of quanta $c_k^\dagger$ arrive at infinity the number of quanta $b_k^\dagger$ within the horizon must also grow at the same rate to keep the state pure (each $c_k^\dagger$ quanta is entangled with one $b_k^\dagger$ quanta). At the same time, the volume within the shrinking horizon is constantly decreasing so must support a higher and higher density of states. While one can allow this to happen leading to the notion of remnants it quickly becomes clear that this leads to a host of other problems (see e.g. [21]). On the other hand, if the number of quanta within the horizon is decreasing or vanishing then we are essentially tracing over the states $b_k^\dagger$ transforming $|\psi_e\rangle$ from an entangled state to a density matrix with an associated entropy. This entropy is not a result of our ignorance of the underlying degrees of freedom in this system but represents a genuine loss of information and violation of unitarity.

Note that small corrections near the horizon will *not* change the entangled nature of $|\psi_e\rangle$ as this would require a relatively large change to this state. For a more detailed discussion of why this cannot be avoided by simply appealing to small quantum gravity corrections see [22]. Our review here has been rather heuristic; for more detailed and technical arguments the reader is directed to consult the literature [18][23] and other references in [22].

2.2 FUZZBALLS

A spacetime geometry can carry an entropy in string theory via coarse graining over an underlying set of microstates. Since the initial success of string theory in accounting for the entropy of supersymmetric black holes by counting states in a field theory [24] there has been an ongoing effort to understand exactly what the structure of these microstates is in a setting where spacetime itself is manifest (i.e. in a closed string theory).

Recently, it was shown that, in examples with enough supersymmetry, including some extremal black holes, one can construct a basis of "coherent" microstates whose spacetime descriptions in the $\hbar \to 0$ limit approach non-singular, horizon-free geometries which resemble a topologically complicated "foam". Conversely, in these cases the quantum Hilbert space of states can be constructed by directly quantizing a moduli space of smooth classical solutions. Nevertheless, the *typical* states in these Hilbert spaces respond to semiclassical probes as if the underlying geometry was singular, or an extremal black hole. In this sense, these black holes are effective, coarse-grained descriptions of underlying non-singular, horizon-free states (see [6] for a review of examples with differing amounts of supersymmetry).

Such results suggest the idea, first put forward by Mathur and collaborators [25, 26], that *all* black hole geometries in string theory, even those with finite horizon area, can be seen as the effective coarse-grained descriptions of complex underlying horizon-free states[1] which have, essentially, an **extended spacetime structure**. Thus the main claim of the proposal is

Conjecture ***(form 1)*****:** *The generic state in the black hole ensemble, realized in a closed string theory, differs from the naive geometry up to the horizon scale.*

While this form captures the essence of the proposal it is perhaps somewhat imprecise as it is not immediately clear what the relation between a state and a geometry is so let us restate this somewhat differently (closely mirroring [22]).

Conjecture ***(form 2)*****:** *Quantum gravity effects are not confined to within a Plank scale and in particular may be relevant for describing the physics up to the horizon scale.*

The importance of this claim is that it would invalidate the implicit assumption made in

[1] The idea here is that a single microstate does not have an entropy, even if its coarse-grained description in gravity has a horizon. Thus the spacetime realization of the microstate, having no entropy, should be in some sense horizon-free, even though the idea of a horizon, or even a geometry, may be difficult to define precisely at a microscopic level.

Section 2.1 that in the region near the horizon quantum gravity effects are subleading (or highly suppressed) and effective field theory can be trusted. This idea seems initially unlikely because one might expect that the quantum effects that correct the classical black hole spacetime would be largely confined to regions of high curvature near the singularity, and would thus not modify the horizon structure.

While we are still quite far from a demonstration of this conjecture in the context of large black holes we do manage, in this thesis, to demonstrate the existence of large scale quantum or stringy structures that extend into a region of spacetime that standard reasoning would suggest is perfectly smooth and classical. More precisely we show that certain classical solutions to the equations of motion will, if embedded in a fully quantum string theory, receive quantum and stringy corrections over a large region of spacetime even though the associated solutions are smooth and of low curvature everywhere. This demonstrates that the precept underlying the conjecture may holds but it certainly does not imply the full conjecture.

Let us mention an important caveat at this point. While we claim (and demonstrate in some instances) that quantum or stringy effects can be "important" up to the horizon scale it is not clear exactly how an in-falling observer would perceive this. The idea that the black hole geometry is an averaged effective description of many underlying microstates, not unlike a thermal ensemble, would suggest that a macroscopic observer would have to make impractically precise measurements to distinguish the actual microstate from the ensemble average. There is some friction, however, between this philosophy and the apparent need for rather significant violations of the assumptions implicit in Section 2.1 (as promulgated in e.g. [22]) in order to resolve the information loss paradox. Thus, at this point it seems unclear, even within the context of the fuzzball proposal, whether the resolution to information loss will involve drastic violations of classical reasoning, immediately evident to an infalling observer, or if a more subtle resolution, wherein part of the classical picture survives, will emerge. A more complete understanding of this issue is essential if we are to have any hope of demonstrating the fuzzball proposal.

2.2.1 Background

In string theory, black holes can often be constructed by wrapping D-branes on cycles in a compact manifold X so they appear as point like objects in the spatial part of the non-compact spacetime, $\mathbb{R}^{1,d-1}$. As the string coupling is increased, these objects back-react on spacetime and can form supersymmetric spacetimes with macroscopic horizons. The entropy associated with these objects can be determined "microscopically" by counting BPS states in a field theory living on the branes and this has been shown in many cases to match the count expected from the horizon area (see [24, 27] for the prototypical calculations). Although the field theory description is only valid for very small values of the string coupling g_s the fact that the entropy counting in the two regimes coincides can be attributed to the protected nature of BPS states that persist in the spectrum at any value of the coupling unless a phase transition occurs or a wall of stability is crossed. The fact that the (leading) contribution to the entropy of the black hole could be reproduced from counting states in a sector of the field theory suggests that the black hole microstates dominate the entropy in this sector.

While it is already very impressive that these states can be counted at weak coupling, understanding the nature of these states in spacetime at finite coupling remains an open problem. As g_s is increased the branes couple to gravity and we expect them to start backreacting on the geometry. The main tools we have to understand the spacetime or closed string picture of the system are the AdS/CFT correspondence and the physics of D-branes.

Within the framework of the AdS/CFT correspondence black holes with near horizon geometries of the form $\mathrm{AdS}_m \times \mathcal{M}$ must correspond to objects in a dual conformal field theory that have an associated entropy[2]. A natural candidate is a thermal ensemble or density matrix, in the CFT, composed of individual pure states (see e.g. [28]). AdS/CFT then suggests that there must be corresponding pure states in the closed string picture and that these would comprise the microstates of the black hole. There is no reason, *a priori*, that such states will be accessible in the supergravity approximation. First, the dual objects should be closed string *states* and may not admit a classical description. Even if they do admit a classical description they may involve regions of high curvature and hence be inherently stringy. For BPS black holes[3], however, we may restrict to the BPS sector in the Hilbert space where the protected nature of the states suggests that they should persist as we tune continuous parameters (barring phase transitions or wall crossings). We may then hope to see a supergravity manifestation of these states, and indeed this turns out to be the case for systems with sufficient supersymmetry.

The large N limit[4], however, which must be taken for supergravity to be a valid description, bears many similarities with the $\hbar \to 0$ limit in quantum mechanics where we know that most states do not have a proper classical limit. Thus it is quite possible that only a vanishing fraction of the black hole states can be realized within supergravity and, indeed, we give evidence that this is the case for large (four supercharge) black holes in Chapter 6. We would like to emphasize once more that this is not a problem insofar as the conjecture stated above is concerned. The importance of quantum states supported on supergravity

[2]More generally objects in AdS with horizons, microscopic or macroscopic, are expected to have an associated entropy which should manifest itself in the dual CFT.

[3]Here "BPS" can mean either 1/2, 1/4 or 1/8 BPS states or black holes in the full string theory. The degree to which states are protected depends on the amount of supersymmetry that they preserve and our general remarks should always be taken with this caveat.

[4]N measures the size of the system. For black holes it is usually related to mass in the bulk and conformal weight in the CFT.

configurations or modes is purely technical as the latter are under some degree of control; conceptually there is no reason to believe or even hope that we can describe black holes by restricted our attention to supergravity modes. Nevertheless, as we will describe, this approach has been largely successful for very supersymmetric black holes (eight or more supercharges) so we will ultimately *begin* our study of less supersymmetric black holes this way.

Despite the potential problems and caveats mentioned above, recently, a very fruitful program has been undertaken to explore and classify the smooth supergravity duals of coherent CFT states in the black hole ensemble. Smoothness here is important because if these geometries exhibit singularities we expect these to either be resolved by string-scale effects, making them inaccessible in supergravity, or enclosed by a horizon implying that the geometry corresponds, not to a pure state, but rather an ensemble with some associated entropy.

Large classes of such smooth supergravity solutions, asymptotically indistinguishable from black hole[5] solutions, have indeed been found [29, 30, 31, 32, 33, 34, 35, 36] (and related [37, 38] to previously known black hole composites [39, 17, 40]). These are complete families of solutions preserving a certain amount of supersymmetry with fixed asymptotic charges[6] and with no (or very mild) singularities.

In constructing such solutions it has often been possible to start with a suitable probe brane solution with the correct asymptotic charges in a flat background and to generate a supergravity solution by backreacting the probe [29, 32, 41]. In a near-horizon limit these back-reacted probe solutions are asymptotically AdS, and by identifying the operator corresponding to the probe and the state it makes in the dual CFT, the backreacted solution can often be understood as the spacetime realization of a coherent state in the CFT. Lin, Lunin and Maldacena [29] showed that the back-reaction of such branes (as well their transition to flux) was identified with a complete set of asymptotically AdS_5 supergravity solutions (as described above) suggesting that the latter should be related to 1/2 BPS states of the original D3 probes generating the geometry. Indeed, in [42, 43] it was shown that quantizing the space of such supergravity solutions as a classical phase space reproduces the spectrum of BPS operators in the dual $\mathcal{N} = 4$ superconformal Yang-Mills (at $N \to \infty$).

In a different setting Lunin and Mathur [31] were able to construct supergravity solutions

[5]Throughout this thesis we will be discussing "microstates" of various objects in string theory but the objects will not necessarily be holes (i.e. spherical horizon topology) nor will they always have a macroscopic horizon. We will, none-the-less, somewhat carelessly continue to refer to these as "microstates" of a black hole for the sake of brevity.

[6]The question of which asymptotic charges of the microstates should match those of the black hole is somewhat subtle and depends on which ensemble the black hole is in. In principle some of the asymptotic charges might be traded for their conjugate potentials. Moreover, the solutions will, in general, only have the same isometries asymptotically.

related to configurations of a D1-D5 brane in six dimensions (i.e. compactified on a T^4) by utilizing dualities that relate this system to an F1-P system (see also [30]). The latter system is nothing more than a BPS excitation of a fundamental string quantized in a flat background. The back reaction of this system can be parametrized by a profile $F^i(z)$ in $\mathbb{R}^4$ (the transverse directions). T-duality relates configurations of this system to that of the D1-D5 system.

Recall that the naive back-reaction of a bound state of D1-D5 branes is a singular or "small" black hole in five dimensions. These geometries have naked singularities though horizons are believed to form when α' corrections are encorporated. The geometries arising from the F1-P system, on the other hand, are smooth after dualizing back to the D1-D5 frame, though they have the same asymptotics as the naive solution [30]. Each F1-P curve thus defines a unique supergravity solution with the same asymptotics as the naive D1-D5 black hole but with different subleading structure. Smoothness of these geometries led Lunin and Mathur to propose that these solutions should be mapped to individual states of the D1-D5 CFT. The logic of this idea was that *individual* microstates do not carry any entropy, and hence should be represented in spacetime by configurations without horizons. Lunin and Mathur also conjectured that the naive black hole geometry is somehow a coarse graining over all these smooth solutions, i.e. that the black hole itself is simply an effective, coarse-grained description. In this context a lot of evidence was put forth to demonstrate that this indeed likely the case and, due to the large amount of supersymmetry of the associated black holes, it seems that the black hole microstates can be realized directly within quantized supergravity. Although this initial success of the "fuzzball program" hinged upon using supergravity microstates it is not clear that extensions of the program to large black holes, with macroscopic horizon areas, will also be able to restrict purely to supergravity modes or if stringy modes will be essential (as argued in Chapter 6 current evidence seems to point to the latter).

2.2.2 Answers to some potential objections

The idea that black holes are simply effective descriptions of underlying horizon-free objects is confusing because it runs counter to well-established intuition in effective field theory; most importantly the idea that near the horizon of a large black hole the curvatures are small and hence so are the effects of quantum gravity. Indeed, it is not easy to formulate a precisely stated conjecture for black holes with finite horizon area, although for extremal black holes with enough supersymmetry a substantial amount of evidence has accumulated for the correctness of the picture, as reviewed in e.g. [6]. To clarify some potential misconceptions, we transcribe below a FAQ from [6], addressing some typical objections and representing our current point of view. See also [26, 22, 44].

1. How can a smooth geometry possibly correspond to a "microstate" of a black hole?

   Smooth geometries do *not* exactly correspond to states. Rather, as classical solutions they define points in the phase space of a theory (since a coordinate and a momenta define a history and hence a solution; see section 2.3 for more details) which is isomorphic to the solution space. In combination with a symplectic form, the phase space defines the Hilbert space of the theory upon quantization. While it is not clear that direct phase space quantization is the correct way to quantize gravity in its entirety this procedure, when applied to the BPS sector of the theory, seems to yield meaningful results that are consistent with AdS/CFT.

   As always in quantum mechanics, it is not possible to write down a state that corresponds to a point in phase space. The best we can do is to construct a state which is localized in one unit of phase space volume near a point. We will refer to such states as coherent states. Very often (but not always, as we will see later in these notes) the limit in which supergravity becomes a good approximation corresponds exactly to the classical limit of this quantum mechanical system, and in this limit coherent states localize at a point in phase space. It is in this sense, and only in this sense, that smooth geometries can correspond to microstates. Clearly, coherent states are very special states, and a generic state will *not* admit a description in terms of a smooth geometry.

   Let us note that, as the discussion above and throughout the thesis pertains to supergravity geometries, the relevant Hilbert space is an approximate factor, in the entire BPS Hilbert space of string theory, comprised of states that do not spread into the "stringy" directions of the full solution space of string theory.

2. How can a finite dimensional solution space provide an exponential number of states?

   The number of states obtained by quantizing a given phase space is roughly given by the volume of the phase space as measured by the symplectic form ω, $N \sim \int \omega^k/k!$ for a $2k$-dimensional phase space. Thus, all we need is an exponentially growing volume which is relatively easy to achieve.

3. Why do we expect to be able to account for the entropy of the black hole simply by studying smooth supergravity solutions?

   Well, actually, we do not really expect this to be true. In cases with enough supersymmetry, one does recover all BPS states of the field theory by quantizing the space of smooth solutions, but there is no guarantee that the same will remain true for large black holes, and the available evidence does not support this point of view. We do however expect that by including stringy degrees of freedom we should be able to accomplish this, in view of open/closed string duality.

4. If black hole "microstates" are stringy in nature then what is the content of the "fuzzball proposal"?

   The content of the fuzzball proposal is that the closed string description of a generic microstate of a black hole, while possibly highly stringy and quantum in nature, has interesting structure that extends all the way to the horizon of the naive black hole solution, and is well approximated by the black hole geometry outside the horizon.

   More precisely the naive black hole solution is argued to correspond to a thermodynamic ensemble of pure states. The *generic* constituent state will not have a good geometrical description in classical supergravity; it may be plagued by regions with string-scale curvature and may suffer large quantum fluctuations. These, however, are not restricted to the region near the singularity but extend all the way to the horizon of the naive geometry. This is important as it might shed light on information loss via Hawking radiation from the horizon as near horizon processes would now encode information about this state that, in principle, distinguish it from the ensemble average.

5. Why would we expect string-scale curvature or large quantum fluctuations away from the singularity of the naive black hole solution? Why would the classical equations of motion break down in this regime?

   As mentioned in the answer to question 1, it is not always true that a solution to the classical equations of motion is well described by a coherent state, even in the supergravity limit. In particular there may be some regions of phase space where the density of states is too low to localize a coherent state at a particular point. Such a point, which can be mapped to a particular solution of the equations of motion, is not a good classical solution because the variance of any quantum state whose expectation values match the solution will necessarily be large.

   Another way to understand this is to recall that the symplectic form effectively discretizes the phase space into $\hbar$-sized cells. In general all the points in a given cell correspond to classical solutions that are essentially indistinguishable from each other at large scales. It is possible, however, for a cell to contain solutions to the equation of motion that *do* differ from each other at very large scales. Since a quantum state can be localized at most to one such cell it is not possible to localize any state to a particular point within the cell. Only in the strict $\hbar \to 0$ limit will the cell size shrink to a single point suggesting there might be states corresponding to a given solution but this is just an artifact of the limit. A specific explicit example of such a scenario is discussed later in this thesis (in Chapter 5).

Thus, even though the black hole solution satisfies the classical equation of motion all the way to the singularity this does not necessarily imply that when quantum effects are taken into account that this solution will correspond to a good semi-classical state with very small (localized) quantum fluctuations.

6. So is a black hole a pure state or a thermal ensemble?

   In a fundamental theory we expect to be able to describe a quantum system in terms of pure states. This applies to a black hole as well. At first glance, since the black hole carries an entropy, it should be associated to a thermal ensemble of microstates. But, as we know from statistical physics, the thermal ensemble can be regarded as a technique for approximating the physics of the generic microstate in the microcanonical ensemble with the same macroscopic charges. Thus, one should be able to speak of the black hole as a coarse grained effective description of a generic underlying microstate. Recall that a typical or generic state in an ensemble is very hard to distinguish from the ensemble average without doing impossibly precise microscopic measurements. The entropy of the black hole is then, as usual in thermodynamics, a measure of the ignorance of macroscopic observers about the nature of the microstate underlying the black hole.

7. What does an observer falling into a black hole see?

   This is a difficult question which cannot be answered at present. The above picture of a black hole does suggest that the observer will gradually thermalize once the horizon has been passed, but the rate of thermalization remains to be computed. It would be interesting to do this and to compare it to recent suggestions that black holes are the most efficient scramblers in nature [45, 46, 47].

8. Does the fuzzball proposal follow from AdS/CFT?

   As we have defined it the fuzzball proposal does not follow from AdS/CFT. The latter is obviously useful for many purposes. For example, given a state or density matrix, we can try to find a bulk description by first computing all one-point functions in the state, and by subsequently integrating the equations of motion subject to the boundary conditions imposed by the one-point function. If this bulk solution is unique and has a low variance (so that it represents a good saddle-point of the bulk path integral) then it is the right geometric dual description. In particular, this allows us to attempt to find geometries dual to superpositions of smooth geometries. What it does not do is provide a useful criterion for which states have good geometric dual descriptions; it is not clear that there is a basis of coherent states that all have decent dual geometric descriptions, and it is difficult to determine the way in which bulk descriptions of generic states differ from each other. In particular, it is difficult to show that generic microstates have non-trivial structure all the way up to the location of the horizon of the corresponding black hole.

9. To what degree does it make sense to consider quantizing a (sub)space of supergravity solutions?

   In some instances a subspace of the solution space corresponds to a well defined symplectic manifold and is hence a phase space in its own right. Quantizing such a space defines a Hilbert space which sits (as a factor) in the larger Hilbert space of the full theory. Under some favourable circumstances the resulting Hilbert space may be physically relevant because a subspace of the total Hilbert space can be mapped to this smaller Hilbert space. That is, there is a one-to-one map between states in the Hilbert space generated by quantizing a submanifold of the phase space and states in the full Hilbert space whose support is localized on this submanifold.

   For instance, in determining BPS states we can imagine imposing BPS constraints on the Hilbert space of the full theory, generated by quantizing the full solution space, and expect that the resulting states will be supported primarily on the locus of points that corresponds to the BPS phase space; that is, the subset of the solution space corresponding to classical BPS solutions. It is therefore possible to first restrict the phase space to this subspace and then quantize it in order to determine the BPS sector of the Hilbert space.

2.3 Phase Space Quantization

An important technical tool in this thesis will be the quantization of large classes of smooth BPS solutions. Quantum mechanics on the moduli space of BPS solutions is a familiar topic in string theory but the solution spaces we consider here are somewhat special as they sit within the full phase space of the theory as a symplectic submanifold (hence defining a phase space of their own) rather than as a configuration space. The latter is more familiar from standard BPS systems and leads to such notion as supersymmetric sigma models on the BPS solution space with BPS wavefunctions corresponding to the cohomology of the BPS manifold. The solution spaces in our exampels our quite different, however, and require a different treatment. They are phase spaces and hence cannot be quantized via e.g. supersymmetric sigma models so we will resort to different techniques in order to understand the quantum structure of the theory. As this may be somewhat unfamiliar we devote this section to introducing the relevant notions.

The space of classical solutions of a theory is generally isomorphic to its classical phase space[7]. Heuristically, this is because a given point in the phase space, comprised of a configuration and associated momenta, can be translated into an entire history by integrating the equations of motion against this initial data; likewise, by fixing a spatial foliation, any solution can translated into a unique point in the phase space by extracting a configuration and momentum from the solution evaluated on a fixed spatial slice. This observation can be used to quantize the theory using a symplectic form, derived from the Lagrangian, on the space of solutions rather than on the phase space. This is an old idea [48] (see also [49] for an extensive list of references and [50] [51] [52] for more recent work) which was used in [42, 43] [53, 54] to quantize the LLM [29] and Lunin-Mathur [31] geometries. An important subtlety in these examples is that it is not the entire solution space which is being quantized but rather a subspace of the solutions with a certain amount of supersymmetry.

In general, quantizing a subspace of the phase space will not yield the correct physics as it is not clear that the resultant states do not couple to states coming from other sectors. It is not even clear that a given subspace will be a symplectic manifold with a non-degenerate symplectic pairing. As discussed in [43] we expect the latter to be the case only if the subspace contains dynamics; for gravitational solutions we thus expect stationary solutions, for which the canonical momenta are not trivial, to possibly yield a non-degenerate phase space. This still does not address the issue of consistency as states in the Hilbert space derived by quantizing fluctuations along a constrained submanifold of the phase space might mix with modes transverse to the submanifold. When the submanifold corresponds to the space of BPS solutions one can argue, however, that this should not matter. The number of BPS states is invariant under continuous deformations that do not cross a wall of marginal stability or induce a phase transition. Thus if we can quantize the solutions in a regime where the interaction with transverse fluctuations is very weak then the energy eigenstates will be given by perturbations around the states on the BPS phase space, and, although these will change character as parameters are varied the resultant space should be isomorphic to the Hilbert space obtained by quantizing the BPS sector alone. If a wall of marginal stability is crossed states will disappear from the spectrum but there are tools that allow us to analyze this as it occurs (see section 3.1.6).

Let us emphasize that the validity of this decoupling argument depends on what questions

[7]It is not entirely clear, however, which solutions should be included in defining the phase space. For instance, in treating gravity, it is not clear if trajectories which eventually develop singularities should also be included as points in the phase space or only solutions which are eternally smooth. As we will primarily be concerned with static or stationary solutions in these notes we will largely avoid this issue.

one is asking. If we were interested in studying dynamics then we would have to worry about how modes on the BPS phase space interact with transverse modes. For the purpose of enumerating or determining general (static) properties of states, however, as we have argued, it should be safe to ignore these modes. For an example of the relation between states obtained by considering the BPS sector of the full (i.e. non-BPS) Hilbert space and those obtained by quantizing just the BPS sector the phase space compare the two center states determined in [17] with those of section 4.3.2 (see [4] further discussion of this topic).

As mentioned, the LLM and Lunin-Mathur geometries have already been quantized and the resultant states were matched with states in the dual CFTs. Our focus here will be on the quantization of $\mathcal{N} = 2$ solutions in four (or $\mathcal{N} = 1$ in five) dimensions. For such solutions, although a decoupling limit has been defined [4], the dual $\mathcal{N} = (0, 4)$ CFT is rather poorly understood. Thus quantization of the supergravity solutions may yield important insights into the structure of the CFT and will be important in studying the microstates of the corresponding extremal black objects.

2.4 AdS/CFT

One of the most powerful tools to study properties of black holes in string theory is the AdS/CFT correspondence [55]. This conjecture relates string theory on backgrounds of the form AdS$_{p+1} \times \mathcal{M}$ to a CFT$_p$ that lives on the boundary of the AdS$_{p+1}$ space. Such backgrounds arise from taking a particular decoupling limit of geometries describing black objects such as black holes, black strings, black tubes, etc. The limit amounts to decoupling the physics in the near horizon region[8] of the black object from that of the asymptotically flat region by scaling the appropriate Planck length, l_p, which decouples the asymptotic gravitons from the bulk (i.e. the near horizon region). At the same time the appropriate spatial coordinates are rescaled with powers of l_p to keep the energies of some excitations finite. This procedure should be equivalent to the field theory limit of the brane bound states generating the geometry under consideration.

We are interested in black objects which describe normalizable deformations in the AdS$_{p+1}$ background. These correspond to a state/density matrix on the dual CFT according to the following dictionary

[8]In some of the cases treated in these notes the region will not be an actual near-horizon region as the original solutions may be horizon-free (or, in some cases, may have multiple horizons) but the decoupling limits are motivated by analogy with genuine black holes where the relevant region is the near horizon one.

BULK	BOUNDARY
$\exp\left(-S_{\text{bulk}}^{\text{on shell}}\right)$	$\text{Tr}[\rho\mathcal{O}_1\ldots\mathcal{O}_n] = \langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle_\rho$
classical geometries	semiclassical states ?
black hole	$\rho \sim \exp\{-\sum_i \beta_i\mathcal{O}_i\}$
entropy S	S = $-$ Tr$(\rho\log\rho)$
bulk isometry D	$\left[\rho, \hat{D}\right] = 0$
ADM quantum numbers of D	$\text{Tr}\left(\rho\hat{D}\right) = \left\langle\hat{D}\right\rangle = D_{\text{ADM}}$

In the first line $\mathcal{O}_i$ are operators dual to sources turned on in the boundary. They are included in the calculation of the on-shell bulk action, $\left(-S_{\text{bulk}}^{\text{on shell}}\right)$. The second line can be seen as the definition of the dual semiclassical state. More specifically, a semiclassical state is one that has an unambiguous dual bulk geometry (i.e. in the classical limit, $N \to \infty$ and $\hbar \to 0$, macroscopic observables take on a fixed expectation value with vanishing variance). In some ideal situations such semiclassical states turn out to be the analog of coherent states in the harmonic oscillator. In the third line, we describe a typical form of a density matrix that we expect to describe black holes. This form is motivated by the first law of thermodynamics: the entropy as defined in the fourth line obeys $dS = \sum_i \beta_i d\langle\mathcal{O}_i\rangle$, and by matching this to the first law as derived from the bulk description of the black hole we can identify the relevant set of operators $\mathcal{O}_i$ and potentials μ_i and guess the corresponding density matrix. The fourth line simply states that we expect a relation between the bulk and boundary entropies. In the fifth and the last line, $\hat{D}$ is the current/operator dual to the bulk isometry D.

Our use of AdS/CFT in this thesis will be quite standard and relatively basic. This is due, in part, due to the lack of control of the relevant $\mathcal{N} = (0, 4)$ CFT that we will encounter. Let us nonetheless be somewhat optimistic here and propose potentially interesting questions one might wish to explore in order to understand the spacetime structure of black hole microstates once greater control has been established over this CFT (something we hope to lay the groundwork for here).

One question relevant for understanding black holes via AdS/CFT is: "*Given a density matrix ρ on the CFT side, is there a dual geometry in the bulk?*". On general grounds one could have expected that a general density matrix ρ should be dual to a suitably weighted sum over geometries, each of which could be singular, have regions with high curvature, and perhaps not have good classical limits. As a result the dual gravitational description of a general density matrix will not generally be trustworthy. However, under suitable circumstances, it can happen that there is a dual "*effective*" geometry that describes the density matrix ρ very well. This procedure of finding the effective geometry is what we will call "*coarse graining*". In the gravity description, this amounts to neglecting the details that a classical observer cannot access anyway due to limitations associated to the resolution of their apparatus. So, one can phrase our question in the opposite direction, "*What are the characteristics of a density matrix on the CFT side, so that there is a good* *dual effective geometry that describes the physics accurately?*".

One can try to construct the dual effective geometry following the usual AdS/CFT prescription. To do so, one should first calculate all the non-vanishing expectation values of all operators dual to supergravity modes (assuming one knows the detailed map between the two). On the CFT side, these VEVs are simply given by

$$\langle \mathcal{O}_i \rangle = \mathrm{Tr}(\rho\, \mathcal{O}_i),$$

and they determine the boundary conditions for all the supergravity fields. The next step is to integrate the gravity equations of motion subject to these boundary conditions to get the dual geometry. This is in principle what has to be done according to AdS/CFT prescription. The problem with this straightforward approach is that it is not terribly practical, and so alternative approaches have been sought out [56, 57] [9].

[9]Though it would be interesting to study in some detail the connection between the two.

CHAPTER 3

MULTICENTERED SOLUTIONS

In chapters 3-5 we will turn our attention to a large class of classical supergravity solutions in four and five dimensions. These solutions furnish a laboratory within which we attempt to address several of the issues raised in chapter 2. The solutions look asymptotically like a black hole with the same total charge. We would like to understand their relation to the black hole microstates as this may yield insight into the spacetime structure of the latter.

To this end we first use AdS/CFT techniques to determine if these solutions can be related to black hole states via the duality between (string theory in) the near-horizon geometry and the dual CFT. Using this correspondence we can relate the geometries to semi-classical states in the dual CFT and might hope to study the CFT itself to learn about more quantum (i.e. less classical) states.

Another, more direct, approach to understanding the relation between these *classical* geometries and the quantum black hole microstates is to quantize this restricted class of solutions directly and study the resultant BPS states. While the original geometries are of course classical the BPS states are essentially phase space densities (the analog of wavefunctions) and need not be semi-classical objects.

The second approach will be the subject of Chapter 4. In this chapter we will focus on the classical solutions and their decoupling limits.

3.1 FOUR AND FIVE DIMENSIONAL SOLUTIONS

We begin with a brief review of multicentered solutions of $\mathcal{N} = 2$ supergravity in 4 dimensions and their lift to ($\mathcal{N} = 1$ solutions in) 5 dimensions. The four dimensional theory is obtained by compactifying IIA on a proper $SU(3)$ holonomy Calabi-Yau manifold X, the five dimensional theory from compactifying M-theory on the same Calabi-Yau manifold. In the regime of interest to us, we can restrict to the cubic part of the IIA prepotential.

The multicentered solutions are determined by specifying a number of charges, Γ_a, and their locations, $\vec{x}_a$, in the spatial $\mathbb{R}^3$. These charged centers correspond in the 10 dimensional picture to branes wrapping even cycles in the CY_3. There are $2b_2 + 2$ independent such cycles in homology, with b_2 the second Betti-number of X, each giving rise to a charge in 4d sourcing one of the $2b_2 + 2$ vector fields of the $\mathcal{N} = 2$ supergravity. We will often denote the charges by their coefficients in a basis of cohomology, i.e. $\Gamma = (p^0, p^A, q_A, q_0) = p^0 + p^A D_A + q_A \tilde{D}^A + q_0\, dV$, where the D_A form a basis of $\mathrm{H}^2(X, \mathbb{Z})$, the $\tilde{D}_A$ make up a dual basis and dV is the unit volume element of X; $\int_X dV \equiv 1$.

The moduli of the Calabi-Yau appear as scalar fields in the 4d/5d effective theories. In the solutions we will be considering the hypermultiplet moduli will be constant (and will mostly be irrelevant) while the moduli in the vector multiplets will vary dynamically in response to charged sources. An important boundary condition in these solutions is then the value of these vector multiplet moduli at infinity.

Our review of these solutions will be concise, as they are discussed in great detail in e.g. the references [58, 40, 9] (see also [34][32]). We will recall the *split attractor flow conjecture*, which relates the existence of solutions at particular values of the moduli at infinity to the existence of certain flow trees in moduli space. A short discussion of the concept of marginal stability, distinguishing between proper marginal stability and what we call threshold stability will also be included.

3.1.1 Four dimensional solutions

Our starting point is the set of multicentered solutions of [58, 59, 40]. The solutions are entirely determined in terms of a single function Σ, which is obtained from the charge (p^0, p^A, q_A, q_0) single centered BPS black hole entropy $S(p^0, p^A, q_A, q_0)$ by substituting

$$\Sigma := \frac{1}{\pi} S(H^0, H^A, H_A, H_0)\,, \tag{3.1}$$

where

$$H \equiv (H^0, H^A, H_A, H_0) := \sum_a \frac{\Gamma_a \sqrt{G_4}}{|x - x_a|} - 2\mathrm{Im}(e^{-i\alpha}\Omega)|_{r=\infty}\,, \tag{3.2}$$

and we will often denote the constant term in the harmonics by h

$$h := -2\mathrm{Im}(e^{-i\alpha}\Omega)|_{r=\infty}\,. \tag{3.3}$$

Here G_4 is the four dimensional Newton constant (i.e the Einstein-Hilbert action is of the form $S_4^{\rm EH} = \frac{1}{16\pi G_4}\int \sqrt{-g_4}\mathcal{R}_4$). We keep this dependence on G_4 explicit for now as it will be important when we take the decoupling limit. The Γ_a in the $2b_2+2$ harmonic functions take values in $H^{\rm ev}(X,\mathbb{Z})$, the integral even cohomology of the Calabi-Yau X, $e^{i\alpha}$ is the phase of the total central charge, $Z(\Gamma)$, and Ω is the normalized period vector defining the special geometry.

$$Z(\Gamma) = \langle \sum_a \Gamma_a, \Omega\rangle\,, \qquad e^{i\alpha} = \frac{Z}{|Z|} \tag{3.4}$$

Γ_a is the charge vector of the center at position $\vec{x}_a$. The constant term of the harmonic functions is such that $\Sigma|_{r=\infty} = 1$.

The solutions are now given by the following four dimensional metric, gauge fields and moduli [1]:

$$\begin{aligned} ds^2 &= -\frac{1}{\Sigma}(dt + \sqrt{G_4}\,\omega)^2 + \Sigma\, dx^i dx^i\,, \\ \mathcal{A}^0 &= \frac{\partial \log\Sigma}{\partial H_0}\left(\frac{dt}{\sqrt{G_4}} + \omega\right) + \omega_0\,, \\ \mathcal{A}^A &= \frac{\partial \log\Sigma}{\partial H_A}\left(\frac{dt}{\sqrt{G_4}} + \omega\right) + \mathcal{A}^A_d\,, \\ t^A &= B^A + i\,J^A = \frac{H^A - i\frac{\partial\Sigma}{\partial H_A}}{H^0 + i\frac{\partial\Sigma}{\partial H_0}}, \end{aligned} \tag{3.5}$$

The off diagonal metric components can be found explicitly too [40] by solving

$$\star\, d\omega = \frac{1}{\sqrt{G_4}}\langle dH, H\rangle\,, \tag{3.6}$$

where the Hodge $\star$ is on flat $\mathbb{R}^3$. The Dirac parts $\mathcal{A}^A_d$, $\omega_0 = \mathcal{A}^0_d$ of the vector potentials are obtained by solving

$$d\omega_0 = \frac{1}{\sqrt{G_4}}\star dH^0\,, \tag{3.7}$$

$$d\mathcal{A}^A_d = \frac{1}{\sqrt{G_4}}\star dH^A\,. \tag{3.8}$$

Again the Hodge star $\star$ is on flat $\mathbb{R}^3$. Asymptotically for $r\to\infty$ we have[2]

$$ds^2 = -dt^2 + d\vec{x}^2, \qquad \mathcal{A} = 2\,{\rm Re}\,(e^{-i\alpha}\Omega)|_\infty\,\frac{dt}{\sqrt{G_4}} + \mathcal{A}_d|_\infty \tag{3.9}$$

[1] We will work for the moment in conventions where we take $c = \hbar = 1$ but keep dimensions of length explicit. The formulae here can be compared with those of e.g. [40] by noting that there the convention $G_4 = 1$ was used. For more information concerning the conventions and different length scales used in this paper, see appendix A.

[2] Here $\mathcal{A}_d$ includes both ω_0 and $\mathcal{A}^A_d$.

The above form of the solution holds for any prepotential. However it still requires finding the entropy function $S(p,q)$ which in general cannot be obtained in closed form. If we take the prepotential to be cubic, which is tantamount to taking the large volume limit in IIA, we can be more explicit. First, the period vector becomes $\Omega = -\frac{e^{B+iJ}}{\sqrt{\frac{4J^3}{3}}}$, considered as an element of $H^{\mathrm{ev}}(X,\mathbb{R})$. Furthermore [60, 61],

$$\begin{aligned}
\mathcal{A}^0 &= \frac{-L}{\Sigma^2}\left(\frac{dt}{\sqrt{G_4}}+\omega\right)+\omega_0 \\
\mathcal{A}^A &= \frac{H^A L - Q^{3/2}y^A}{H^0\Sigma^2}\left(\frac{dt}{\sqrt{G_4}}+\omega\right)+\mathcal{A}^A_d\,, \\
t^A &= \frac{H^A}{H^0}+\frac{y^A}{Q^{\frac{3}{2}}}\left(i\Sigma-\frac{L}{H^0}\right), \\
\Sigma &= \sqrt{\frac{Q^3-L^2}{(H^0)^2}}\,, \\
L &= H_0(H^0)^2+\frac{1}{3}D_{ABC}H^AH^BH^C - H^AH_AH^0\,, \\
Q^3 &= (\frac{1}{3}D_{ABC}y^Ay^By^C)^2\,, \\
D_{ABC}y^Ay^B &= -2H_CH^0+D_{ABC}H^AH^B\,.
\end{aligned} \tag{3.10}$$

The entropy function Σ will play a central role in the discussion that follows. At the horizon of one of the bound black holes this function will be proportional to the entropy, i.e. $\Sigma(H)|_{(x\to x_a)} = \frac{G_4}{|x-x_a|^2}\Sigma(\Gamma_a)+\mathcal{O}(\frac{\sqrt{G_4}}{|x-x_a|})$ where $\pi\Sigma(\Gamma_a)=S(\Gamma_a)$ is the Bekenstein-Hawking entropy of the ath center[3].

Finally there are $N-1$ independent consistency conditions on the relative positions of the N centers, reflecting the fact that these configurations really are bound states and one can't move the centers around freely. These conditions arise from requiring integrability of (3.6). They take the simple form

$$\langle H,\Gamma_s\rangle|_{x=x_s}=0\,, \tag{3.11}$$

or written out more explicitly[4]

$$\sqrt{G_4}\sum_{b\neq \underline{a}}\frac{\langle\Gamma_a,\Gamma_b\rangle}{r_{ab}}=\langle h,\Gamma_a\rangle\,, \tag{3.12}$$

[3]Note that the entropy formula for black holes involving D6-charge is rather involved and might appear singular as H^0 (or p^0) goes to zero, see (3.10). This is however not the case and by analysing the formula in an expansion around small H^0 one finds that the leading term is the non-singular entropy function for a black hole without D6-charge, $\Sigma=\sqrt{\frac{D_{ABC}H^AH^BH^C}{3}(D^{AB}H_AH_B-2H_0)}$, as expected.

[4]For brevity we use unconventional notation here: by $\sum_{s\neq\underline{r}}$ we mean a sum over all s different from r whereas $\sum_{s\neq r}$ denotes a doubles sum over all s and r such that s and r are different.

where $r_{ab} = |x_{ab}| = |x_a - x_b|$ and the h are the constant terms in the harmonic functions given in (3.3). Note that, as these depend on the asymptotic values of the scalar fields, the equilibrium distances between the different centers do so as well.

Since there are $N-1$ independent position constraints, the dimension of the moduli space modulo the center of mass translations will generically be $2N-2$. Thus this space is always even dimensional which is good as we will eventually show that it is, in fact, a phase space.

3.1.2 Five dimensional solutions

In [61] (see also [37, 62, 38, 63, 64, 65, 34, 35]) these solutions were lifted to five dimensions via the connection between IIA and M-theory on a circle. These solutions were also independently discovered directly in five dimensions in [32][34]. The five dimensional solution can be expressed in terms of the four dimensional one as (see appendix A for more details about notations and conventions):

$$
\begin{aligned}
ds^2_{5d} &= \tilde{V}_{\mathrm{IIA}}^{2/3}\,\ell_5^2\left(d\psi + \mathcal{A}^0\right)^2 + \tilde{V}_{\mathrm{IIA}}^{-1/3}\,\frac{\hat{R}}{2}\,ds^2_{4\mathrm{d}}\,,\\
A^A_{5\mathrm{d}} &= \mathcal{A}^A + B^A\left(d\psi + \mathcal{A}^0\right)\,,\\
Y^A &= \tilde{V}_{\mathrm{IIA}}^{-1/3}\,J^A\,, \qquad \tilde{V}_{\mathrm{IIA}} = \frac{D_{ABC}}{6}J^AJ^BJ^C = \frac{1}{2}\left(\frac{\Sigma}{Q}\right)^3\,.
\end{aligned}
\tag{3.13}
$$

Here ψ parametrizes the M-theory circle with periodicity 4π and we define, in terms of the 11d Planck length l_{11} and the physical asymptotic M-theory circle radius R,

$$
\ell_5 := \frac{l_{11}}{4\pi\tilde{V}_M^{1/3}}\,, \qquad \hat{R} = \frac{R}{\ell_5}\,,
\tag{3.14}
$$

where $\tilde{V}_M = V_M/l_{11}^6$ is the M-theory volume of X in 11d Planck units. The reduced 5d Planck length ℓ_5 is related to the 4d Newton constant G_4 by

$$
\ell_5^3 = R\,G_4
\tag{3.15}
$$

and we have the relation $\hat{R} = 2\,\tilde{V}_{\mathrm{IIA}}^{1/3}|_\infty$. Note that unlike the M-theory volume in 11d Planck units, which is in a hypermultiplet and hence constant, the IIA volume in string units varies over space. Our normalizations are chosen such that asymptotically we have the metric

$$
\begin{aligned}
ds^2_{5d}|_\infty &= \frac{R^2}{4}\left(d\psi + \mathcal{A}^0\right)^2 + d\vec{x}^2 - dt^2\,,\\
\mathcal{A}^0 &= -2\cos\alpha_\infty\,\frac{dt}{R} + p^0\cos\theta\,d\phi\,,
\end{aligned}
\tag{3.16}
$$

where $\mathcal{A}^0$ was obtained from (3.9), and we recall that $e^{i\alpha}$ is the phase of the total central charge. Recall that p^0 is the total $D6$-charge of the solution which, for our decoupling analysis (and in fact through-out most of this thesis), we will take to be zero.

The five dimensional vector multiplet scalars Y^A are related to the M-theory Kähler moduli by $J^A_M = \tilde{V}_M^{1/3} Y^A$. Here $\tilde{V}_M = \frac{V_M}{l_{11}^6}$ is the volume of the internal Calabi-Yau as measured with the M-theory metric. This is constant throughout the solution as it is in a hypermultiplet and hence decoupled. For more details about all the different length scales and the relation between M-theory and IIA variables in our conventions see appendix A.

For practical computations it is often useful to express the metric (3.13) above more explicitly in terms of the functions (3.10):

$$
\begin{aligned}
ds^2_{5d} = 2^{-2/3} Q^{-2} \Bigg[&-\ell_5^2 \, (H^0)^2 \left(\sqrt{\frac{R}{\ell_5^3}} dt + \omega \right)^2 - 2\ell_5^2 \, L \left(\sqrt{\frac{R}{\ell_5^3}} dt + \omega \right) (d\psi + \omega_0) \\
&+ \ell_5^2 \, \Sigma^2 (d\psi + \omega_0)^2 \Bigg] + 2^{-2/3} \frac{R}{\ell_5} Q \, dx^i dx^i \, . \qquad (3.17)
\end{aligned}
$$

Finally, note that by construction, all these five dimensional solutions have a $U(1)$ isometry along the ψ direction. They are therefore not the complete set of five dimensional BPS solutions.

3.1.3 Angular Momentum

Let us briefly recall some relevant properties of these multicentered solutions.

The first new feature with respect to single black holes is that, as shown in [58], they carry an angular momentum equal to

$$
\vec{J} = \frac{1}{2} \sum_{a<b} \frac{\langle \Gamma_a, \Gamma_b \rangle \, \vec{x}_{ab}}{r_{ab}} \, . \qquad (3.18)
$$

Note that Dirac quantization of the charges is equivalent to half integral quantization of the angular momentum of a two centered solution. This angular momentum is associated to $SO(3)$ rotations in the three non-compact spacelike dimensions and should not be confused by the momentum around the M-theory circle (which, in the four dimensional picture, corresponds to the D0-charge q_0).

As angular momentum will play an important role in this part of the thesis, providing a natural coordinate on the solution space, we will derive some more useful ways of expressing it. Multiplying the condition (3.18) by x_a and then summing over the different centers shows that

$$J = \frac{1}{2}\sum_a \langle h, \Gamma_a\rangle x_a \,. \tag{3.19}$$

By using the fact that $\sum_a \langle h, \Gamma_a\rangle = 0$ one can rewrite the last expression as

$$J = \frac{1}{2}\sum_{a, a\neq b} \langle h, \Gamma_a\rangle x_{ab} \,. \tag{3.20}$$

Starting from this formula we can show that the size of the angular momentum can be compactly written in terms of the inter-center distances r_{ab}. Squaring (3.20) gives

$$J^2 = \frac{1}{4}\sum_{a, a\neq b}\sum_{c,\, c\neq b} \langle h, \Gamma_a\rangle \langle h, \Gamma_c\rangle\, x_{ab}\cdot x_{cb} \,. \tag{3.21}$$

Now we can use that for any three points labeled by a, b, c there is the relation $x_{ab}\cdot x_{cb} = \frac{1}{2}(r_{ab}^2 + r_{cb}^2 - r_{ac}^2)$ and a little more algebra reveals that

$$|J| = \frac{1}{2}\sqrt{-\sum_{a<b} \langle h, \Gamma_a\rangle \langle h, \Gamma_b\rangle\, r_{ab}^2}\,. \tag{3.22}$$

3.1.4 Solution Spaces

Another important property of a configuration with a sufficient number of centers is that although the centers bind to each other there is some freedom left to change their respective positions. These possible movements can be thought of as flat directions in the interaction potential. Equation (3.12) constrains the locations of the centers to the points where this potential is zero. As for a system with N centers there are $N-1$ such equations for $3N-3$ coordinate variables (neglecting the overall center of mass coordinate) there is, in general, a $2N-2$ dimensional moduli space of solutions for fixed charges and asymptotics. This space may or may not be connected and it may even have interesting topology. We will refer to this as the moduli space of solutions or solution space; the latter terminology will be preferred as it is less likely to be confused with the moduli space of the Calabi-Yau, in which the scalar fields t^A take value. The shape of this solution space does, in fact, depend quite sensitively on where the moduli at infinity, $t^A|_\infty$, lie in the Calabi-Yau moduli space (as the latter determine h on the RHS of eqn. (3.12)). We will return in more detail to the geometry of the solution space in section 3.4 and to its quantization in Section 4.3.

The space-time corresponding to a generic multicenter configuration can be rather complicated as there can be many centers of different kinds. Some properties of the 5 dimensional geometry have been discussed in the literature, e.g. [32, 38, 34, 61] and we won't repeat the details here. A basic understanding will be useful when considering the decoupling limit so we shortly summarize some points of interest. The four dimensional solutions are defined on a space that is topologically $\mathbb{R}^4$. When lifted to five dimensions, however, a Taub-NUT circle is fibred over this space, pinching at the location of any center with D6-charge. The resultant space typically has non-contractible two-spheres extending between centers with D6 charge and has been referred to as a "bubbling solution" [32]. Generically, a D4 charged center will lift to a black string unless it also caries D6-charge in which case it lifts to what locally looks like a BMPV black hole at the center of 5 dimensional Taub-NUT [37]. The topology of the horizon at a given center is that of an S^1-bundle over S^2 of degree p_a^0, i.e. $S^1 \times S^2$ for $p_a^0 = 0$ and $S^3/\mathbb{Z}_{|p_a^0|}$ otherwise.

Finally let us mention a symmetry of the solutions (which is closely related to the one observed in [64][38]) given by the following shift of the harmonic functions:

$$\begin{aligned} H^0 &\to H^0\,, \\ H^A &\to H^A + k^A H^0\,, \\ H_A &\to H_A + D_{ABC}H^B k^C + \frac{1}{2}D_{ABC}k^B k^C H^0\,, \\ H_0 &\to H_0 + k^A H_A + \frac{1}{2}D_{ABC}H^A k^B k^C + \frac{1}{6}D_{ABC}k^A k^B k^C (H^0)\,. \end{aligned} \tag{3.23}$$

Under which the metric and the constraint equations are invariant and the gauge field is transformed by a large gauge transformation

$$A^A \to A^A + k^A d\psi\,. \tag{3.24}$$

3.1.5 Smoothness and Zero-Entropy Bits

The multicenter solutions of sections 3.1.1 have played an important role in understanding the structure of the BPS Hilbert space of $\mathcal{N} = 2$ supergravity (in 4d) [58, 66, 9] and have also had related mathematical applications [67, 68]. More recently the five dimensional versions of these solutions came to prominence within the Mathur program of understanding black hole microstates as a result of [34, 32]. In this context we are interested in large families of smooth solutions because, as discussed in Chapter 2, such families provide various windows into the quantum structure of the theory, either as phase spaces which can be quantized yielding quantum states as described in section 2.3 or as duals to semi-classical states in a CFT as in section 2.4 (if the solutions in the family are asymptotically AdS). In either case it is important that the solutions be suitably smooth or entropy-less. Such solutions are often be referred to as "microstate geometries" though this unfortunate terminology is not intended to suggest that the classical solution necessarily corresponds to a black hole microstate. Rather, as we will see later, quantizing the space of such solutions as a phase space provides a set of quantum states that may or may not be semi-classical.

Although it is perhaps not entirely *a priori* clear what solutions we should include in defining the phase space of a our supergravity theory a natural criteria is that the solutions be non-singular in some sense which we will attempt to make precise. The reasoning for this is as follows. Singular supergravity solutions are only meaningful if stringy effects resolve or smooth out the singularity yielding a consistent solution to the full string theory. Although this is not something we generally have control over we will often use indirect arguments to determine which singularities are acceptable and which are not. For instance, we will occasionally have recourse to consider solutions with singularities coming from D0 branes and we allow these because these branes are believed to be good solutions to the full (closed) string theory. We have, moreover, an alternative open string formulation which suggests that such solutions are completely consistent. From this description we know that there is very little entropy associated with a single D0 so we argue that stringy effects will smooth out the geometry rather than generating a horizon. This is important as solutions with horizons have an associated entropy and are better thought of as an effective description of an ensemble of states (quantum or semi-classical) rather than a semi-classical state.

Let us now apply this philosophy to our multicentered solutions. Recall that the function Σ appearing in the metric in (3.5) is known as the entropy function. When evaluated at $\vec{x}_a$ it is proportional to the entropy of a black hole carrying the charge of the center lying at $\vec{x}_a$. This follows from the Bekenstein-Hawking relation and the fact that Σ determines the area of the horizon of a possible black hole at $\vec{x}_a$. If this area is zero then the center at $\vec{x}_a$ does not have any macroscopic entropy and, if the associated geometry does not suffer from large curvature in this region, then there is no reason to believe stringy corrections will change this.

The prototypical example of such geometries have centers with charges of the form $\Gamma = (1, p/2, p^2/8, p^3/48)$ corresponding to a single D6 branes wrapping the Calabi-Yau with all lower-dimensional charges induced by abelian flux. A configuration with a single such center can be spectral flowed (see e.g. [64, 3]) to a single D6 brane with no flux and hence no additional degrees of freedom in the Calabi-Yau; thus "integrating out" the Calabi-Yau degrees of freedom does not generate an entropy and the associated five dimensional solution is smooth. As discussed in [35, 69], "zero-entropy bits" can also be D4 and D2-branes with flux or D0 branes. Generically they can carry a "primitive" charge vector, i.e. some number of D6-branes with fractional fluxes such that the induced D2, D4 and D0 brane charges are all quantized, but have no common factor. Note that if the D6-brane charge is $N > 1$, the solution is not strictly smooth – there is relatively mild orbifold singularity (R^4/Z_N) in the five-dimensional theory. It was shown in [34, 35] that, in classical $\mathcal{N} = 8$ supergravity, such zero entropy smooth centers carry charges that are half-BPS. The half-BPS nature of the charge vector follows from smoothness.

Five-dimensional multicenter configurations with every center constrained to be of the above form have been studied in [32, 70, 34, 35] and numerous other works by the same authors. Note that the associated four dimensional solution can have singularities associated to Kaluza-Klein reduction on a non-trivial S^1 fibration. This high-lights an important distinction: while the entropy, determined by replacing H^A in the definition of $\Sigma(H)$ with Γ^A, is a duality invariant notion, the smoothness of the resulting supergravity solution is not (see e.g. [71]). Thus we will often generally assume that solutions with all centers that are "zero-entropy bits" (in the sense of vanishing "microscopic" entropy) are candidate "microstate geometries" even though they may have naked singularities in some duality frames. As suggested above a more fundamental criteria is provided by the open string formulation of the solutions. If this analysis suggests that a singular center does not carry additional (e.g. non-Abelian) degrees of freedom then we expect stringy effects to smooth it out. This allows us to distinguish between configurations such as small black holes which have vanishing horizon area and genuine zero-entropy bits such as D0 branes.

3.1.6 The Split Attractor Flow Conjecture

So far we have reviewed a class of 4 and 5 dimensional solutions. These solutions are relatively complicated and it is non-trivial to determine if they are well-behaved everywhere. In particular one should be concerned about the appearance of closed timelike curves or singularities. If the entropy function, Σ, which involves a square root, becomes zero or takes imaginary values in some regions the 4d solution is clearly ill behaved; this is equivalent to closed timelike curves in the 5d metric as discussed in [38] and [32]. One can on the other hand show that if $\Sigma^2 > \omega_i\omega^i$ everywhere then there can be no closed timelike curves [34]. This is a rather complicated condition to check for a generic multicenter solution however and furthermore it is sufficient but not necessary; the condition could be violated without closed timelike curves appearing. In [58] and [9] a simplified criteria was proposed for the existence of (well-behaved) solutions which we will now relate.

In [58] a conjecture is proposed whereby pathology-free solutions are those with a corresponding *attractor flow tree* in the moduli space. This conjecture was first posed for the multicentered four-dimensional solutions of Section 3.1.1.

Recall that the four dimensional moduli, $t^A(\vec{x}) = B^A(\vec{x}) + iJ^A(\vec{x})$, are the complexified Kähler moduli of the Calabi-Yau. The relation between these moduli and their five dimensional counterparts can be found in [37] [3]. To each charge vector, Γ_a, we can associate a complex number, the central charge, as

$$Z(\Gamma_a;t) := \langle \Gamma_a, \Omega(t) \rangle \qquad \Omega(t) := -\frac{e^t}{\sqrt{\frac{4}{3}J^3}} \tag{3.25}$$

Note that, since t^A is a two-form, Ω is a sum of even degree forms. The phase of the central charge, $\alpha(\Gamma_a) := \arg[Z(\Gamma_a;t)]$, encodes the supersymmetry preserved by that charge at the given value of the moduli. The even form Ω is related, asymptotically, to the constants in the harmonics (which define both the 4-d and 5-d solutions) as given in (3.3).

An *attractor flow tree* is a graph in the Calabi-Yau moduli space beginning at the moduli at infinity, $t^A|_\infty$, and ending at the attractor points for each center. The edges correspond to single center flows towards the attractor point for the sum of charges further down the tree. Vertices can occur where single center flows (for a charge $\Gamma = \Gamma_1 + \Gamma_2$) cross walls of marginal stability where the central charges are all aligned ($|Z(\Gamma)| = |Z(\Gamma_1)| + |Z(\Gamma_2)|$). The actual flow of the moduli $t^A(\vec{x})$ for a multi-centered solution will then be a thickening of this graph (see [58], [9] for more details). According to the conjecture a given attractor flow tree will correspond to a single connected set of solutions to the equations (3.11), all of which will be well-behaved. An example of such a flow is given in figure 3.1.

The intuition behind this proposal is based on studying the two center solution for charges Γ_1 and Γ_2. The constraint equations (3.11) imply that when the moduli at infinity are moved near a wall of marginal stability (where Z_1 and Z_2 are parallel) the centers are forced infinitely far apart

$$r_{12} = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{\langle h, \Gamma_1 \rangle} = \left. \frac{\langle \Gamma_1, \Gamma_2 \rangle \, |Z_1 + Z_2|}{2\,\mathrm{Im}(\bar{Z}_2 Z_1)} \right|_\infty \tag{3.26}$$

In this regime the actual flows in moduli space are well approximated by the split attractor trees since the centers are so far apart that the moduli will assume single-center behaviour in a large region of spacetime around each center. Thus in this regime the conjecture is well motivated. Varying the moduli at infinity continuously should not alter the BPS state count, which corresponds to the quantization of the two center moduli space, so unless the moduli cross a wall of marginal stability we expect solutions smoothly connected to these to also be well defined. Extending this logic to the general N center case requires an assumption that it is always possible to tune the moduli such that the N centers can be forced to decay into two clusters that effectively mimic the two center case. There is no general argument that this should be the case but one can run the logic in reverse, building certain large classes of solutions by bringing in charges pairwise from infinity and this can be understood in terms of attractor flow trees. What is not clear is that all solutions can be constructed in this way. For more discussion the reader should consult [72].

Although this conjecture was initially proposed for asymptotically flat solutions, when we consider the decoupling limit of the multicenter solutions in the next section we will see that the attractor flow conjecture and its utility in classifying solutions can be extended to AdS space.

For generic charges the attractor flow conjecture also provides a way to determine the entropy of a given solution space. The idea is that the entropy of a given total charge is the sum of the entropy of each possible attractor flow tree associated with it. Thus the partition function receives contributions from all possible trees associated with a given total charge and specific *moduli at infinity*. An immediate corollary of this is that, as emphasized in [9], the partition function depends on the asymptotic moduli. As the latter are varied certain attractor trees will cease to exist; specifically, a tree ceases to contribute when the moduli at infinity cross a wall of marginal stability (MS) for its first vertex, $\Gamma \to \Gamma_1 + \Gamma_2$, as is evident from (3.26).

For two center solutions one can determine the entropy most easily near marginal stability where the centers are infinitely far apart. In this regime locality suggests that the Hilbert state contains a product of three factors[5] [9]

$$\mathcal{H}(\Gamma_1 + \Gamma_2; t_{ms}) \supset \mathcal{H}_{\text{int}}(\Gamma_1, \Gamma_2; t_{ms}) \otimes \mathcal{H}(\Gamma_1; t_{ms}) \otimes \mathcal{H}(\Gamma_2; t_{ms}) \tag{3.27}$$

Since the centers move infinitely far apart as t_{ms} is approached we do not expect them to interact in general. There is, however, conserved angular momentum carried in the electromagnetic fields sourced by the centers and this also yields a non-trivial multiplet of quantum states. Thus the claim is that $\mathcal{H}_{\text{int}}$ is the Hilbert space of a single spin J multiplet where $J = \frac{1}{2}(|\langle\Gamma_1, \Gamma_2\rangle| - 1)$.[6]. $\mathcal{H}(\Gamma_1)$ and $\mathcal{H}(\Gamma_2)$ are the Hilbert spaces associated with BPS brane excitations in the Calabi-Yau and their dimensions are given in terms of a suitable entropy formula for the charges Γ_1 and Γ_2 valid at t_{ms}.

Thus, if the moduli at infinity were to cross a wall of marginal stability for the two center system above the associated Hilbert space would cease to contribute to the entropy (or the index). A similar analysis can be applied to a more general multicentered configuration like that in figure 3.1 by working iteratively down the tree and treating subtrees as though they correspond to single centers with the combined total charge of all their nodes. The idea is, once more, that we can cluster charges into two clusters by tuning the moduli and then treat the clusters effectively like individual charges. We can then iterate these arguments within each cluster. This counting argument mimics the constructive argument for building the solutions by bringing in charges from infinity and is hence subject to the same caveats, discussed above.

Altogether the above ideas allow us to determine the entropy associated with a particular attractor tree, which, by the split attractor flow conjecture corresponds to a single

[5]Since attractor flow trees do not *have* to split at walls of marginal stability, there will in general be other contributions to $\mathcal{H}(\Gamma_1 + \Gamma_2; t_{ms})$ as well.

[6]The unusual -1 in the definition of J comes from quantizing additional fermionic degrees of freedom [17] [4].

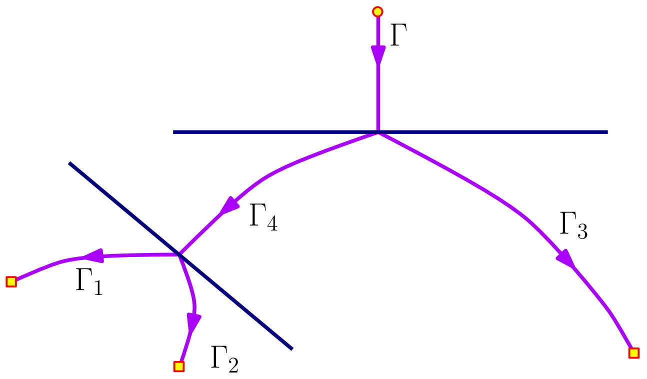

Figure 3.1: *Sketch of an attractor flow tree. The dark blue lines are lines of marginal stability, the purple lines are single center attractor flows. The tree starts at the yellow circle and flows towards the attractor points indicated by the yellow boxes.*

connected component of the solutions space. The entropy of a tree is the product of the angular momentum contribution from each vertex (i.e. $|\langle\Gamma_1, \Gamma_2\rangle|$, the dimension of $\mathcal{H}_{\text{int}}$) times the entropy associated to each node.

$$|\langle\Gamma_3, \Gamma_1 + \Gamma_2\rangle|\,|\langle\Gamma_1, \Gamma_2\rangle|\,\Omega(\Gamma_1)\,\Omega(\Gamma_2)\,\Omega(\Gamma_3) \tag{3.28}$$

In Chapter 4 we will show that it is possible, in the two and three center cases, to quantize the solution space directly and to match the entropy so derived with the entropy calculated using the split attractor tree. This provides a non-trivial check of both calculations.

Before proceeding we should mention an important subtlety in using the attractor flow conjecture to classify and validate solutions. Certain classes of charges will admit so called *scaling solutions* [69] [9] which are not amenable to study via attractor flows. These solutions are characterized by the fact that the constraint equations (3.11) have solutions that continue to exist at any value of the asymptotic moduli. We will discuss these solutions in greater detail in the section 3.4.2 but it is important to note here that the general arguments given in this section (such as counting of states via attractor flow trees) do not apply to scaling solutions.

3.2 Decoupling limit

As outlined in the introduction, we want to study the geometries dual to states of M5-branes wrapped on 4-cycles with total homology class $p^A D_A$, in the decoupling limit $R/l_{11} \to \infty$, V_M/l_{11}^6 fixed. A convenient way to take the limit is to adapt units such

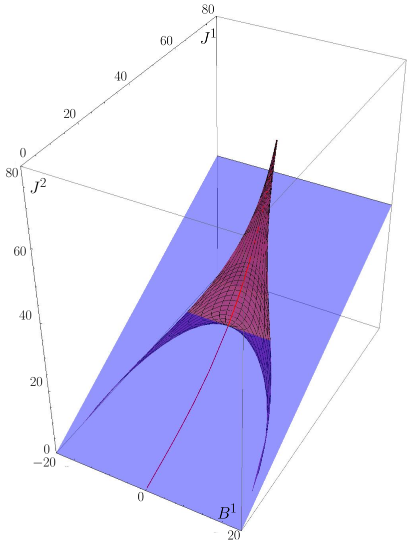

Figure 3.2: *In this figure the* $(50B^1, J^1, J^2)$ *subspace in moduli space is shown. The blue surface is a wall of marginal stability (which is always codimension 1). The central red line is the attractor flow for a single center solution with the same total charge as the two center. The pink surface shows the values the moduli take in the two center solution.*

that R remains finite — for example $R \equiv 1$ — while $\ell_5 \to 0$ (note that l_{11}/ℓ_5 is fixed because V_M/l_{11}^6 is fixed). Then the dynamics of finite energy excitations of the M5 are described by a (0,4) supersymmetric 1+1 dimensional nonlinear sigma model with target space naively[7] given by the classical M5 moduli space, the MSW string [27, 73], decoupled from bulk and KK modes. For example, Kaluza-Klein excitations along the 4-cycle decouple as their mass is of order $V_M^{-1/6}$, which scales to infinity.

We wish to find out how multicentered solutions with total charge $(0, p^A, q_A, q_0)$ behave when we take this limit. The IIA Kähler moduli J^A are related to the normalized scalars Y^A as $J^A \sim \frac{R}{\ell_5} Y^A$, hence $J \to \infty$. For two centered solutions involving D6-charges, the equilibrium separation following from the integrability condition (3.11) asymptotes to

$$|\vec{x}_1 - \vec{x}_2| = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2\,\mathrm{Im}(e^{-i\alpha} Z_1)|_\infty} \frac{\ell_5^{3/2}}{\sqrt{R}} \sim \frac{\langle \Gamma_1, \Gamma_2 \rangle}{R^2}\, \ell_5^3, \tag{3.29}$$

where we used that for total D6-charge zero, $\alpha \to 0$ when $J \to \infty$, while $Z_1 \sim i J^{3/2} \sim i(R/\ell_5)^{3/2}$.

To keep the coordinate separation finite in the limit $\ell_5 \to 0$, we should therefore rescale

[7] As discussed in the introduction and further in section 5.4.3, the precise M5-brane interpretation of the decoupling limit is rather mysterious and still poses various puzzles.

all coordinates as

$$\vec{x} = \ell_5^3\, \vec{\mathsf{x}}. \tag{3.30}$$

The finite $\vec{\mathsf{x}}$ region then has the expected properties for a decoupling limit. First, as we will see, at finite values of $\vec{\mathsf{x}}$, the metric converges to an expression of the form $ds^2 = \ell_5^2\, \mathsf{ds}^2$ with ds^2 finite. Finite fluctuations of ds^2 thus give rise to finite action fluctuations — the ℓ_5^2 metric prefactor cancels the ℓ_5^{-3} in front of the Einstein-Hilbert action [55]. Similarly, M2-branes wrapping the M-theory circle and stretched over finite $\vec{\mathsf{x}}$ intervals have finite energy. Finally, the geometry becomes asymptotically an S^2 bundle over AdS$_3$ at large $r = |\vec{\mathsf{x}}| \to \infty$:

$$\mathsf{ds}^2 \approx d\eta^2 + e^{\eta/U}(-d\tau^2 + d\sigma^2) + U^2\left(d\theta^2 + \sin^2\theta\,(d\phi + \tilde{A})^2\right), \tag{3.31}$$

$$\tilde{A} = \tfrac{J}{J_{\max}} d(\tau - \sigma) \tag{3.32}$$

$$A^A_{5\mathrm{d}} \approx -p^A \cos\theta\,(d\phi + \tilde{A}) + 2D^{AB} q_B\, d(\sigma + \tau), \tag{3.33}$$

$$Y^A \approx \frac{p^A}{U}. \tag{3.34}$$

where $U := (\frac{1}{6} D_{ABC} p^A p^B p^C)^{1/3}$ and we made the change of coordinates $(r, t, \psi) \to (\eta, \tau, \sigma)$ to leading order given by:

$$\eta := U\, \log \frac{R^2 r}{U}, \qquad \tau := \frac{t}{R}, \qquad \sigma := \frac{\psi}{2} - \frac{t}{R}. \tag{3.35}$$

Notice that the normalized Kähler moduli Y^A and the $U(1)$ vectors A^A are fixed at attractor values determined by the M5 and M2 charges. The flat connection $\tilde{A}$ determines the twisting of the S^2 over the AdS$_3$ base; J is the S^2-angular momentum of the solution and $J_{\max} := \frac{U^3}{2}$ is its maximal value for given p. Note that going around the M-theory circle in the new coordinates corresponds to

$$\sigma \to \sigma + 2\pi, \tag{3.36}$$

with all other coordinates fixed. Parallel transport of the S^2 along this circle produces a rotation $\Delta\phi = \frac{J}{J_{\max}} 2\pi$ around its z-axis (which is the axis determined by the direction of the four dimensional angular momentum). Because $\tilde{A} \sim d(\sigma - \tau)$, the sphere similarly gets rotated in time, resulting in angular momentum proportional to the amount of twisting around the S^1. Since the S^2 descends from the spatial sphere at infinity in four dimensions, this equals the 4d angular momentum of the 4d multicentered solution. In the dual CFT, it translates to $SU(2)_R$ charge.

The $\sigma \to \sigma + 2\pi$ circle smoothly connects to the asymptotic M-theory circle in the original asymptotically flat geometry. Fermions must be periodic around this circle, as antiperiodic fermions would produce a nonzero vacuum energy. Therefore we have periodic boundary conditions for the fermions on the AdS$_3$ boundary circle, so the supersymmetric black hole configurations we are describing must correspond to supersymmetric states in the Ramond sector of the boundary CFT.

It is not true, however, that *all* multicentered solutions with total charge $(0, p^A, q_A, q_0)$ give rise to such asymptotic $\mathrm{AdS}_3 \times S^2$ attractor geometries[8] in the decoupling limit. For example D4-D4 2-centered solutions (i.e. $p_1^0 = p_2^0 = 0$) turn out to have equilibrium separations in the original coordinates scaling as $|\vec{x}_1 - \vec{x}_2| \sim \langle \Gamma_1, \Gamma_2 \rangle \ell_5$. The different scaling compared to the case with nonzero D6-charges is due to the fact that now $\arg Z_1 \to 0$ in the decoupling limit. In the rescaled coordinates (3.30) the separation diverges, so these multicentered solutions therefore do not fit in the asymptotic $\mathrm{AdS}_3 \times S^2 \times X$ attractor geometry associated to the total M5 charge p^A. Rather, they give rise to two mutually decoupled $\mathrm{AdS}_3 \times S^2 \times X$ attractor geometries associated to the two individual centers. More elaborate configurations of this kind are possible too, for instance consisting of two clusters each with zero net D6-charge, but containing themselves more centers with nonzero D6-charge. The centers within each cluster will have rescaled coordinate separations of order 1, while the mutual separation between the clusters diverges like ℓ_5^{-2} in these coordinates.

These D4-D4 type BPS bound states exist in regions of Kähler moduli space separated from the overall M5 attractor point $Y^A = p^A/U$ by a wall of marginal stability. They correspond to ensembles of BPS states of the MSW string which exist at certain values of the Y^A but not at the attractor point. Their interpretation in the AdS-CFT context is therefore less clear — we will return to this in section 5.4.3.

In the following we wish to focus on solutions which do correspond to a single asymptotic $\mathrm{AdS}_3 \times S^2$ in the decoupling limit, and in particular find practical criteria to determine when this will be the case. We will proceed by rescaling coordinates as in (3.30) and carefully studying the behavior of the solutions when $\ell_5 \to 0$. As the explicit form of the multicenter solutions is rather complicated we will first make the dependence on ℓ_5 more clear by pulling it out through a rescaling of the variables in section 3.2.1. After this rescaling the dependence on ℓ_5 will simply be an overall factor in the metric as described above and a dependence left in the equilibrium distance between the centers and the constant terms of the harmonic functions. Once we have this simple form we will take the decoupling limit by sending $\ell_5 \to 0$. We calculate the asymptotics and some quantum numbers in sections 3.2.3 and 3.2.4 and finally we will discuss when the decoupling limit is well defined (in the sense that we do get a single asymptotically $\mathrm{AdS}_3 \times S^2$ geometry when $\ell_5 \to 0$) in section 3.2.5.

[8]Despite the nontrivial twist of the S^2, we will still loosely refer to the asymptotic geometry as $\mathrm{AdS}_3 \times S^2$.

3.2.1 Rescaling

As discussed above, to take the decoupling limit we want to work with the rescaled coordinates, x^i,

$$x^i = \ell_5^3\,\mathsf{x}^i\,. \tag{3.37}$$

Furthermore we want to extract a factor of ℓ_5 out of the 5d metric. As the multicenter solutions are rather complicated we will here first simplify the dependence on ℓ_5 by redefining various quantities. In the rescaled coordinates it is natural to define rescaled harmonic functions, H,

$$\mathsf{H} = \ell_5^{3/2} H = \sum_a \frac{\Gamma_a}{\sqrt{R}\,|\mathsf{x}-\mathsf{x}_a|} - 2\ell_5^{3/2}\mathrm{Im}(e^{-i\alpha}\Omega)|_\infty\,. \tag{3.38}$$

It is not difficult to verify that all functions appearing in (3.10) are actually homogenous under the rescaling of the coordinates and harmonic functions given above. For instance

$$\begin{aligned}
y^A(H) &= \ell_5^{-3/2} y^A(\mathsf{H})\,,\\
Q(H) &= \ell_5^{-3}\, Q(\mathsf{H})\,,\\
L(H) &= \ell_5^{-9/2} L(\mathsf{H})\,,\\
\Sigma(H) &= \ell_5^{-3}\,\Sigma(\mathsf{H})\,.
\end{aligned} \tag{3.39}$$

The scaling of ω is a little more subtle. Here one has to take into account that the $\star$ scales as well since the flat 3d metric scales as ℓ_5^{-6} under the coordinate rescaling. This implies

$$\star_x = \ell_5^{3(3-2p)}\star_{\mathsf{x}}\,, \tag{3.40}$$

for the $\star$ acting on a p-form . So from its equation of motion (3.10) we see that

$$\omega(H, dx, \sqrt{G_4}) = \ell_5^{-3/2}\omega(\mathsf{H}, d\mathsf{x}, R^{-1/2})\,, \tag{3.41}$$

where the factor $\ell_5^{3/2}$ out of $\sqrt{G_4} = \frac{\ell_5^{3/2}}{\sqrt{R}}$ is essential.

Note that the 4d metric from (3.5) scales as

$$ds^2_{4\mathrm{d}}(H, dx, \sqrt{G_4}) = \ell_5^{-3} ds^2_{4\mathrm{d}}(\mathsf{H}, d\mathsf{x}, R^{-1/2})\,. \tag{3.42}$$

Finally there are also some fields that remain invariant under the rescaling:

$$t^A(H) = t^A(\mathsf{H}) \tag{3.43}$$
$$\omega_0(H, dx, \sqrt{G_4}) = \omega_0(\mathsf{H}, d\mathsf{x}, R^{-1/2}) \tag{3.44}$$
$$\mathcal{A}(H, dx, \sqrt{G_4}) = \mathcal{A}(\mathsf{H}, d\mathsf{x}, R^{-1/2})\,. \tag{3.45}$$

It is clear from the discussion above that the whole solution transforms homogeneously under the rescaling of the coordinates and the redefinition of the harmonic functions.

In fact our solutions in rescaled coordinates take exactly the same form as the original solutions in Section 3.1, with the only changes being the replacement of $\sqrt{G_4}$ with $R^{-1/2}$ and H with H everywhere. For the readers convenience we provide the explicit rescaled form of the solutions in Appendix C.

The 5d metric in these coordinates now has a prefactor ℓ_5^2

$$\frac{1}{\ell_5^2}\, ds_{5d}^2 = 2^{-2/3}\, Q^{-2}\Big[-(\mathsf{H}^0)^2(\sqrt{R}dt+\omega)^2 - 2L(\sqrt{R}dt+\omega)(d\psi+\omega_0) + \Sigma^2(d\psi+\omega_0)^2\Big] + 2^{-2/3} R\, Q\, d\mathsf{x}^i d\mathsf{x}^i\,. \tag{3.46}$$

Otherwise, the only appearance of ℓ_5 is through the harmonic functions H in (3.38). It enters there in two ways. First, through the constant terms

$$-\,2\ell_5^{3/2}\mathrm{Im}(e^{-i\alpha}\Omega)|_\infty\,, \tag{3.47}$$

where it is important to recall that $\Omega|_\infty$ also depends on ℓ_5 as J_∞^A is related to ℓ_5 by $\frac{4J_\infty^3}{3} = \left(\frac{R}{l_5}\right)^3$. Secondly, the equilibrium positions x_i of the charged centers are determined by the consistency condition

$$\langle \Gamma_a, \mathsf{H}\rangle|_{\mathsf{x}_a} = 0\,. \tag{3.48}$$

By this equation they depend on the constant part of the harmonics and thus ℓ_5. We will elaborate in detail on this dependence in the next subsection when we consider the $\ell_5 \to 0$ limit.

From this point onwards we will always be working with rescaled coordinates (unless we explicitly state otherwise). Hence, for notational simplicity **we will revert to original notation** (e.g. Σ, $ds_{4\mathrm{d}}^2$, x, H) though we will be referring to the *rescaled* expressions (e.g. $\Sigma(\mathsf{H})$, $ds_{4\mathrm{d}}^2(\mathsf{H}, d\mathsf{t}, d\mathsf{x}, R^{-1/2})$, x, H). Hopefully this will not lead to excessive confusion.

3.2.2 Decoupling

Having rewritten our solutions in a rescaled form where the ℓ_5 dependence is transparent (see e.g. (C.5)) we can consistently take the decoupling limit, $\ell_5 \to 0$, while keeping $R, t, x^i, \psi, \tilde{V}_M$ and Γ_i fixed. As mentioned before, in the rescaled variables ℓ_5 only appears through the constants in the harmonic functions so taking the limit $\ell_5 \to 0$ will leave the whole structure of the solution invariant except for replacing the harmonic functions by their limiting form. Changing ℓ_5 also effects the equilibrium distances of the centers, x_a, in the solution due to the appearance of the constant terms in the constraint equation (3.12). In general the equilibrium distances will vary in a rather complicated (and not unique) way. Some interesting examples will be discussed explicitly in section 3.4.

Let us now examine the dependence on ℓ_5 in the small ℓ_5 regime. The constant terms of the rescaled harmonic functions are

$$h = -2\ell_5^{3/2}\mathrm{Im}(e^{-i\alpha}\Omega)|_\infty\,, \tag{3.49}$$

where $\Omega = -\frac{e^{B+iJ}}{\sqrt{\frac{4J^3}{3}}}$ and $J^A|_\infty = \frac{R}{2l_5}Y^A|_\infty$. We can write those constant terms in an expansion for small ℓ_5 as

$$\begin{aligned}
h^0 &= h^0_{(4)}\frac{\ell_5^4}{R^{5/2}} + \mathcal{O}(\ell_5^6)\,,\\
h^A &= h^A_{(2)}\frac{\ell_5^2}{\sqrt{R}} + h^A_{(4)}\frac{\ell_5^4}{R^{5/2}} + \mathcal{O}(\ell_5^6)\,,\\
h_A &= h_A^{(2)}\frac{\ell_5^2}{\sqrt{R}} + h_A^{(4)}\frac{\ell_5^4}{R^{5/2}} + \mathcal{O}(\ell_5^6)\,,\\
h_0 &= -\frac{R^{3/2}}{4} + h_0^{(2)}\frac{\ell_5^2}{\sqrt{R}} + h_0^{(4)}\frac{\ell_5^4}{R^{5/2}} + \mathcal{O}(\ell_5^6)\,,
\end{aligned} \tag{3.50}$$

where the leading terms are[9]

$$\begin{aligned}
h^0_{(4)} &= 8\frac{pYB - qY}{pYY}|_\infty\,,\\
h^A_{(2)} &= Y^A_\infty\,,\\
h_A^{(2)} &= (YB)_A|_\infty + \frac{Y_A^2}{pY^2}(qY - pYB)|_\infty\,,\\
h_0^{(2)} &= \frac{1}{2}YB^2|_\infty + \frac{BY^2}{pY^2}(qY - pYB)|_\infty + 2\frac{(qY-pYB)^2}{(pY^2)^2}|_\infty\,.
\end{aligned} \tag{3.51}$$

So in the limit $\ell_5 \to 0$ all the constants in harmonics are sent to zero except for the one in the D0 harmonic H_0 which reads

$$h_0 \to -\frac{R^{3/2}}{4}\,. \tag{3.52}$$

The equilibrium distances also depend on the asymptotic moduli through (3.11). These constraints can be written in the form

$$\sum_b \frac{\langle\Gamma_a, \Gamma_b\rangle}{\sqrt{R}\,|x_a - x_b|} = -\langle\Gamma_a, h\rangle\,. \tag{3.53}$$

So from the behavior (3.52) we see that in the decoupling limit $\ell_5 \to 0$ the consistency conditions (3.11) become

$$\sum_b \frac{\langle\Gamma_a, \Gamma_b\rangle}{|x_a - x_b|} = -\frac{p_a^0}{4}R^2\,. \tag{3.54}$$

[9] To keep the formulas in (3.50) readable we suppressed the various indices and contractions, these formulas should all be read as e.g. $XYZ = D_{ABC}X^AY^BZ^C\,,\quad (XY)_A = D_{ABC}X^BY^C\,,\quad XY = X_AY^A\,.$

Summarized, the decoupling limit corresponds to replacing the harmonic functions by

$$
\begin{aligned}
H^0 &= \sum_a \frac{p_a^0}{\sqrt{R}\,|x-x_a|}\,, \\
H^A &= \sum_a \frac{p_a^A}{\sqrt{R}\,|x-x_a|}\,, \\
H_A &= \sum_a \frac{q_A^a}{\sqrt{R}\,|x-x_a|}\,, \\
H_0 &= \sum_a \frac{q_0^a}{\sqrt{R}\,|x-x_a|} - \frac{R^{3/2}}{4}\,.
\end{aligned} \tag{3.55}
$$

Furthermore the equilibrium distances are now determined by the equations (3.54).

Note that this limit is similar to the usual near horizon limit, but not quite the same, since we are not simply dropping all constant terms from the harmonic functions. A similar situation was encountered for instance in [74], where a similar decoupling limit is defined for the three charge super tubes.

It is useful to note that although under the decoupling limit the D0 constant goes to a fixed non vanishing value, this constant can, however, be removed by the following formal transformations

$$
\begin{aligned}
H_0 &\quad \to H_0 + \frac{R^{3/2}}{4} \\
L &\quad \to L + \frac{R^{3/2}}{4}\,(H^0)^2 \\
t &\quad \to v = t - \frac{R}{4}\psi
\end{aligned} \tag{3.56}
$$

As this is the only effect of the constant term in the D0-brane harmonic function, we can set it to zero while replacing t by $v = t - R/4\ \psi$ and making a shift in L at the same time. This is sometimes technically convenient.

3.2.3 Asymptotics

Now that we have implemented the decoupling limit we want to study the new asymptotics of these solutions. This is completely determined by the asymptotics of the harmonic functions. For $r \to \infty$ the harmonic functions (3.55) can be expanded as

$$\begin{aligned} H^0 &\to R^{-1/2}\frac{e\cdot d^0}{r^2}\,, \\ H^A &\to R^{-1/2}\left(\frac{p^A}{r}+\frac{e\cdot d^A}{r^2}\right)\,, \\ H_A &\to R^{-1/2}\left(\frac{q_A}{r}+\frac{e\cdot d_A}{r^2}\right)\,, \\ H_0 &\to R^{-1/2}\left(\frac{q_0}{r}+\frac{e\cdot d_0}{r^2}\right)\,, \end{aligned} \tag{3.57}$$

where we have put the constant in H_0 to zero by the procedure explained at the end of the last subsection. In our notation

$$d := \sum_a \Gamma_a\, x_a \tag{3.58}$$

is the dipole moment and $\vec{e} = \frac{\vec{x}}{r}$, $r = |x|$, is the normalized position vector that gives the direction on the S^2 at infinity. Note that for H^0 the dipole term is leading as we take the overall D6 charge zero; the same is true for H_A if the total D2 charge is zero. As we will only consider cases of non-vanishing overall D4 charge here the dipole term is always subleading.

In studying the asymptotics of the physical fields it will be most straightforward to work in a coordinate system where d^0 lies along the z-axis. In this case

$$e\cdot d^0 = \cos\theta |d^0|\,, \tag{3.59}$$

with the standard spherical coordinates (θ, ϕ). To simplify the notation we will often write just d^0 for $|d^0|$; it should be clear from the context when the vectorial quantity is intended and when the scalar. Note that the different dipole moments don't have to align so in general there is no simple expression for e.g. $e\cdot d^A$ in this coordinate system.

In the decoupled geometry the d^0 plays a distinguished role as it is proportional to the total angular momentum of the system. To see this we start from the stability condition in the decoupled theory, (3.54), multiply by x_b and sum over b (note that this still is a vector identity):

$$J = \frac{1}{2}\sum_{a\neq b}\frac{\langle\Gamma_a,\Gamma_b\rangle x_b}{|x_a-x_b|} = \frac{R^2}{8}\sum_a p^0_a x_a = \frac{R^2}{8}d^0\,. \tag{3.60}$$

From the above asymptotic expansion of the harmonics (3.57), we can determine the asymptotic behavior of all the fields and functions appearing in our solution. First, let us determine the large r expansion of the functions y^A. These are given in the form of a quadratic equation which can be solved in a $1/r$ expansion as

$$y^A = H^A - H^0 D^{AB} H_B - \frac{1}{2}(H^0)^2 D^{FA} D_{FBC} D^{BD} H_D D^{CE} H_E + \mathcal{O}(\frac{1}{r^4})\,, \tag{3.61}$$

where we defined

$$D^{AB} = (D_{ABC}H^D)^{-1}\,. \tag{3.62}$$

Armed with this expression for y^A we compute

$$\begin{aligned} D_{ABC}y^Ay^By^C = D_{ABC}H^AH^BH^C - 3H^0H^AH_A \\ +\frac{3}{2}(H^0)^2H_AD^{AB}H_B + \mathcal{O}(\frac{1}{r^6})\,. \end{aligned} \tag{3.63}$$

We can now evaluate the $1/r$ expansion of the coefficient $\frac{\Sigma^2}{Q^2}$ appearing in front of $d\psi^2$ in the metric

$$\frac{\Sigma^2}{Q^2} = \left(H_AD^{AB}H_B - 2H_0\right)\left(\frac{D_{ABC}H^AH^BH^C}{3}\right)^{-1/3} + \mathcal{O}(\frac{1}{r})\,. \tag{3.64}$$

The expansion of L is straightforward, and the expansion for Q follows directly from (3.63). The last non-trivial expansions to be calculated are those of ω and ω_0. For those the following result is convenient: for any vector $n^i \in \mathbb{R}^3$ one has

$$d\left(\frac{\epsilon_{ijk}n^ir^jdr^k}{r^3}\right) = -\,*_3\,d\left(\frac{n^ir^i}{r^3}\right)\,. \tag{3.65}$$

In particular we find that

$$\omega_0 = -\epsilon_{ijk}\frac{(d^0)^ir^jdr^k}{r^3} + \mathcal{O}(\frac{1}{r^2}) = -\frac{\sin^2\theta d^0}{r}d\phi + \mathcal{O}(\frac{1}{r})\,, \tag{3.66}$$

where in the last equality we used our choice to take the z axis to be along the D6 dipole moment d^0. We will not need the explicit form of ω because its leading term goes like $\mathcal{O}(r^{-2})$. This follows from the asymptotic form of the equations of motion

$$d\omega = \sqrt{R}\star\left(-h_0dH^0 + \mathcal{O}(\frac{1}{r^4})\right)\,, \tag{3.67}$$

where we have once more shifted the D0 constant term to zero; see the end of section 3.2.2 for the details.

We are now ready to spell out the asymptotic expansion of the metric. We start from (3.46), use the expansions computed above and replace t by v to compensate for shifting the D0 constant h_0. The result one gets up to terms of order[10] $\mathcal{O}(\frac{1}{r})$ is

$$\begin{aligned} ds^2_{5\mathrm{d}} = &- r\frac{R}{U}dvd\psi + \frac{U^{-4}}{4}\Bigg[-R^2(d^0)^2dv^2 \\ &+ 2R\left(\frac{e\cdot d^AD_{ABC}p^Bp^C}{3} - \frac{p^Aq_Ad^0\cos\theta}{3}\right)dvd\psi + \mathcal{D}d\psi^2\Bigg] \\ &+ U^2\frac{dr^2}{r^2} + U^2\left(d\theta^2 + \sin^2\theta\,(d\phi + \tilde{A})^2\right) + \mathcal{O}(\frac{1}{r})\,. \end{aligned} \tag{3.68}$$

[10]In this power counting we consider $\mathcal{O}(dr) = \mathcal{O}(r)$.

Here we introduced the notation

$$\begin{aligned} &v = t - R/4\psi\,, \quad U^3 = \frac{p^3}{6}\,, \\ &\mathcal{D} = \frac{p^3}{3}\left(D^{AB}q_Aq_B - 2q_0\right), \quad \tilde{A} = \frac{J}{J_{\max}}\frac{2v}{R}\,. \end{aligned} \tag{3.69}$$

We used the relation between the D6-dipole moment d^0 and the angular momentum J given by (3.60) and the fact that there is a maximal angular momentum $J_{\max} = \frac{U^3}{2}$. Note that $\pi^2\mathcal{D} = S(\Gamma_t)^2$, so $\mathcal{D}$ is the discriminant of the total charge. With a coordinate transformation to a new radial variable ρ one can show that the angular dependent part in the second term of 3.68 is really further subleading. The coordinate ρ is given by

$$\frac{\rho^2}{4U^2} = -\frac{U^{-4}}{2}R\left(\frac{e\cdot d^A D_{ABC}p^Bp^C}{3} - \frac{p^Aq_Ad^0\cos\theta}{3}\right) + \frac{R}{U}r\,. \tag{3.70}$$

In this new radial coordinate the expansion in large ρ takes the following form

$$\begin{aligned} ds^2_{5\mathrm{d}} &= -\frac{\rho^2}{4U^2}dvd\psi + \frac{U^{-4}}{4}\left[-R^2(d^0)^2dv^2 + \mathcal{D}d\psi^2\right] + 4U^2\frac{d\rho^2}{\rho^2} \\ &\quad + U^2\left(d\theta^2 + \sin^2\theta\,(d\phi + \tilde{A})^2\right) + \mathcal{O}(\frac{1}{\rho^2})\,. \end{aligned} \tag{3.71}$$

Using the expansion formulas derived above it is straightforward to calculate the asymptotics of the gauge field and the scalars. Putting everything together we see that the solution asymptotes to

$$\begin{aligned} ds^2_{5\mathrm{d}} &= -\frac{\rho^2}{4U^2}dvd\psi + \frac{U^{-4}}{4}\left[-R^2(d^0)^2dv^2 + \mathcal{D}d\psi^2\right] \\ &\quad + 4U^2\frac{d\rho^2}{\rho^2} + U^2\left(d\theta^2 + \sin^2\theta\,(d\phi + \tilde{A})^2\right) + \mathcal{O}(\frac{1}{\rho^2})\,, \end{aligned} \tag{3.72}$$

$$A^A_{5\mathrm{d}} = -p^A\cos\theta d\alpha + D^{AB}q_Bd\psi + \mathcal{O}(\frac{1}{\rho^2})\,, \tag{3.73}$$

$$Y^A = \frac{p^A}{U} + \mathcal{O}(\frac{1}{r^2})\,. \tag{3.74}$$

It is clear that the metric is locally asymptotically $\mathrm{AdS}_3\times\mathrm{S}^2$ with $\mathrm{R}_{\mathrm{AdS}} = 2\mathrm{R}_{\mathrm{S}^2} = 2U$. We have kept track of some subleading terms as they will be important in reading off quantum numbers in the next section. Note that we have in fact a nontrivial S^2 fibration over AdS_3 described by the flat connection $\tilde{A} = \frac{J}{J_{\max}}(\frac{2dt}{R} - \frac{d\psi}{2})$. As $\tilde{A}$ depends on the time coordinate we see that as time progresses the sphere rotates, implying the solution has angular momentum as expected. In the same way, going once around the M-theory circle, i.e. $\psi \to \psi + 4\pi$, induces a rotation of $\frac{2\pi J}{J_{\max}}$ along the equator[11] of the S^2. The explicit coordinate transformation bringing the above metric in the form (3.31) after dropping the subleading terms will be given below.

[11] Remember we chose the canonical "z-axis" of our spherical coordinates along the total angular momentum of the solution.

3.2.4 CFT QUANTUM NUMBERS

In this subsection we will perform an analysis of the asymptotic conserved charges of the decoupled solutions. As we now have an asymptotic AdS geometry we can use the well developed technology for these spaces. In our case of AdS$_3$ a nice review can be found in [75]. The asymptotic charges as determined from the supergravity side can later be compared to various quantum numbers in the boundary CFT.

To proceed we first rewrite everything asymptotically in terms of a three dimensional theory on AdS$_3$ by reducing over the asymptotic sphere spanned by (θ, ϕ). Reducing five dimensional $\mathcal{N}$=1 supergravity over the S^2 will result in a three dimensional theory with an $SU(2)$ gauge group in addition to gravity (in an AdS$_3$ background) and the $U(1)$ vector multiplet fields that descend from five dimensions. The metric of the reduced theory is

$$ds^2_{\rm 3d} = -\frac{\rho^2}{4U^2}dv d\psi + \frac{U^{-4}}{4}\left[-R^2(d^0)^2dv^2 + \mathcal{D}d\psi^2\right] + 4U^2\frac{d\rho^2}{\rho^2}\,. \tag{3.75}$$

This can be put it into a standard form for the asymptotic expansion around AdS$_3$ by the coordinate transformations

$$\rho^2 = \frac{e^{\frac{\eta}{U}}4U^2}{R}\,, \qquad dv = -\frac{R}{2}d\bar{w}\,, \qquad d\psi = 2dw\,. \tag{3.76}$$

These are related to the coordinates τ, σ we used in (3.31)-(3.34) by $w = \sigma + \tau$, $\bar{w} = \sigma - \tau$. After Wick rotating $\tau \to i\tau$, these become the standard conjugate holomorphic coordinates on the boundary cylinder, with periodicity 2π. The metric reads

$$ds^2_{\rm 3d} = d\eta^2 + e^{\frac{\eta}{U}}dw d\bar{w} + \frac{1}{U^4}\left(\mathcal{D}dw^2 - \frac{R^4(d^0)^2}{16}d\bar{w}^2\right)\,, \tag{3.77}$$

which has the standard form $ds^2_{\rm 3d} = d\eta^2 + (e^{\frac{2\eta}{R_{\rm AdS}}}g^{(0)}_{ij} + g^{(2)}_{ij})du^i du^j$. We can now apply the formulas [75]:

$$\begin{aligned} T^{\rm grav}_{ww} &= \frac{1}{8\pi G_3 R_{\rm AdS}}g^{(2)}_{ww}\,, \\ T^{\rm grav}_{\bar{w}\bar{w}} &= \frac{1}{8\pi G_3 R_{\rm AdS}}g^{(2)}_{\bar{w}\bar{w}}\,. \end{aligned} \tag{3.78}$$

In our case this becomes[12]

$$\begin{aligned} T^{\rm grav}_{ww} &= \frac{\mathcal{D}}{8\pi U^3}\,, \\ T^{\rm grav}_{\bar{w}\bar{w}} &= \frac{-R^4(d^0)^2}{8\pi\,16U^3}\,. \end{aligned} \tag{3.79}$$

[12]We used $G_3 = \frac{\ell_5^3}{2R^2_{\rm S^2}}$. Note furthermore that the definitions (3.78) are given in unrescaled variables so that both $R_{\rm AdS}$ and $R_{\rm S^2}$ carry a factor ℓ_5. Thus when rescaling $g_{ij} \to \ell_5^2 g_{ij}$ all factors of ℓ_5 drop out of the energy momentum tensor. This is as expected since we defined our limit in such a way as to ensure that these energies stay finite as $\ell_5 \to 0$.

Apart from the metric, there are also gauge fields: the SU(2) gauge field coming from the reduction of the metric on S^2 and the U(1) vectors of the 5d supergravity. These gauge fields do contribute to the asymptotic energy momentum tensor because the 5-dimensional action contains a Chern-Simons term involving them. Here we will just present the results of the derivation that is detailed in appendix D. The contribution of all the different gauge fields to the energy momentum is given by

$$\begin{aligned} T^{\text{gauge}}_{ww} &= \frac{1}{4\pi}\left[\frac{(p^A\,q_A)^2}{p^3} - (q_A D^{AB} q_B)\right], \\ T^{\text{gauge}}_{\bar{w}\bar{w}} &= \frac{1}{4\pi}\,\frac{(p^A\,q_A)^2}{p^3} + \frac{R^4}{8\pi}\frac{(d^0)^2}{16U^3}\,. \end{aligned} \tag{3.80}$$

So by combining (3.79) and (3.80), we see that the total energy momentum tensor is:

$$T_{ww} = \frac{1}{4\pi}\left(\frac{(p^A\,q_A)^2}{p^3} - 2q_0\right), \qquad T_{w\bar{w}} = 0\,, \qquad T_{\bar{w}\bar{w}} = \frac{1}{4\pi}\,\frac{(p^A\,q_A)^2}{p^3}\,. \tag{3.81}$$

The Virasoro charges $(L_0)_{\text{cyl}}$ and $(\tilde{L}_0)_{\text{cyl}}$ on the cylinder are obtained from the energy-momentum tensor as

$$\begin{aligned} (L_0)_{\text{cyl}} &= \oint dw\, T_{ww} = \frac{(p^A\,q_A)^2}{2p^3} - q_0\,, \\ (\tilde{L}_0)_{\text{cyl}} &= \oint d\bar{w}\, T_{\bar{w}\bar{w}} = \frac{(p^A\,q_A)^2}{2p^3}\,, \end{aligned} \tag{3.82}$$

where the contour integral is taken along a contour wrapped once around the asymptotic cylinder, i.e. $w \to w + 2\pi$. These are related to the standard Virasoro charges on the $z = e^{iw}$-plane by the transformations

$$L_0 = (L_0)_{\text{cyl}} + \frac{c}{24}, \qquad \tilde{L}_0 = (\tilde{L}_0)_{\text{cyl}} + \frac{c}{24}, \tag{3.83}$$

with c the Brown-Henneaux central charge:

$$c = \frac{3R_{\text{AdS}}}{2G_3} = p^3. \tag{3.84}$$

These are exactly the quantum numbers of the BPS states of the dual CFT in the Ramond sector as determined in [16, 76], confirming our earlier assertion under (3.36). Naively one might have thought that the BPS condition would require $\tilde{L}_0 = c/24$. That this is not so follows from the particular structure of the $(0, 4)$ theory under consideration. It has, besides the usual $(0, 4)$ superconformal algebra, several additional $U(1)$ currents, as well as additional right-moving fermions — these are superpartners of the center of mass degrees of freedom of the original wrapped M5-brane description. As was analyzed in

[16, 76], the BPS conditions involve the right-moving fermions in a non-trivial way, and this modifies the BPS bound into $\tilde{L}_0 \geq \frac{(p^A q_A)^2}{2p^3} + \frac{p^3}{24}$, consistent with our result above.

Often, it is more convenient to work with different but closely related quantum numbers, L'_0 and $\tilde{L}'_0$, and similarly $(L'_0)_{\rm cyl}$ and $(\tilde{L}'_0)_{\rm cyl}$, which are obtained from the original ones by subtracting out the contributions of the zero modes of the additional currents, so only the oscillator contributions remain. In our case they are given by [16]:

$$\begin{aligned} L'_0 - \frac{c}{24} = (L'_0)_{\rm cyl} &= -\hat{q}_0 := -(q_0 - \frac{1}{2} D^{AB} q_A q_B)\,, \\ \tilde{L}'_0 - \frac{c}{24} = (\tilde{L}'_0)_{\rm cyl} &= 0\,. \end{aligned} \tag{3.85}$$

These reduced quantum numbers are in many cases more convenient. They are spectral flow invariant, and when we want to use Cardy's formula to compute the number of states with given $U(1)$ charges, we can simply use the standard Cardy formula with $L_0, \tilde{L}_0$ replaced by $L'_0, \tilde{L}'_0$. The reduced quantum numbers also have a simple interpretation in the AdS/CFT correspondence. They represent the contributions to $L_0, \tilde{L}_0$ from the gravitational sector, ignoring the additional contributions from the gauge fields.

The total energy and momentum, in units of $1/R$, are given by

$$H = (L_0)_{\rm cyl} + (\tilde{L}_0)_{\rm cyl} = \frac{(p^A q_A)^2}{p^3} - q_0, \qquad P = (L_0)_{\rm cyl} - (\tilde{L}_0)_{\rm cyl} = -q_0\,, \tag{3.86}$$

and the reduced energy and momentum by

$$H' = (L'_0)_{\rm cyl} + (\tilde{L}'_0)_{\rm cyl} = -\hat{q}_0, \qquad P' = (L'_0)_{\rm cyl} - (\tilde{L}'_0)_{\rm cyl} = -\hat{q}_0 = H'\,. \tag{3.87}$$

The energy H can be seen to equal the BPS energy $E = \frac{|Z|}{\sqrt{G_4}}$ of a D4-D2-D0 particle in a 4d asymptotically flat background with $J^A \to \infty p^A$, $B^A = 0$, with the diverging part subtracted off. The reduced energy is the same but now at $B^A = D^{AB} q_B$.

Finally, the $SU(2)_R$ charge can be read off from the sphere reduction connection appearing in the metric (3.31). In general it is given by

$$J_0^I = \oint \frac{d\bar{w}}{2\pi} J_{\bar{w}}^I = \frac{c}{12} \oint \frac{d\bar{w}}{2\pi} A_{\bar{w}}^I\,. \tag{3.88}$$

Details are given in appendix D. Thus the $SU(2)_R$ charge equals the four dimensional angular momentum:

$$J_0 = \frac{R^2 d^0}{8} = J\,, \tag{3.89}$$

where we used (3.60). This is as expected, since the S^2 descends from the spatial sphere at infinity in four dimensions.

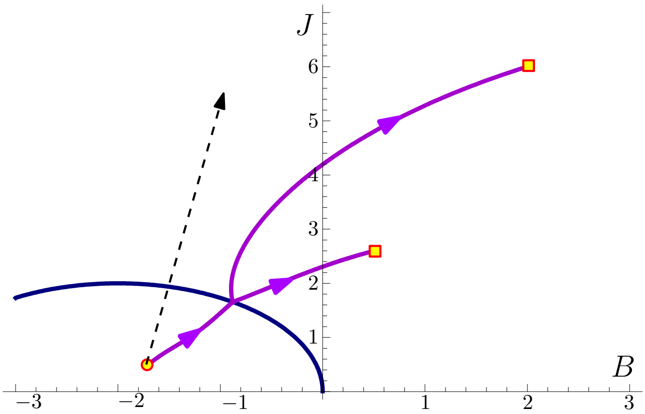

Figure 3.3: *This figure is an example of an attractor tree that exists in flat space but that will not survive in the decoupling limit, because the starting point of the split flow will move towards $J = \infty$ and hence cross a wall of marginal stability and decay.*

3.2.5 Existence and attractor flow trees

Not all choices of charges Γ_a give rise to multicentered solutions in asymptotically flat space at finite R/ℓ_5. Of those which do, not all survive the decoupling limit $R/\ell_5 \to \infty$. And of those which survive, not all give rise to a single $\mathrm{AdS}_3 \times S^2$ throat.

As reviewed in section 3.1.6, in four dimensional asymptotically flat space, the well supported split attractor flow conjecture states there is a one to one correspondence between attractor flow trees and components of the moduli space of multicentered solutions. In particular, the existence of flow trees implies the existence of corresponding multicentered configurations, which can be assembled or disassembled adiabatically by dialing the asymptotic moduli according to the flow tree diagram. By the uplift procedure we followed, the same correspondence holds for five dimensional solutions asymptotic to $\mathbb{R}^{1,3} \times S^1$ with a $U(1)$ isometry corresponding to the extra S^1.

The 4d Kähler moduli scalars J^A are related to the five dimensional normalized Kähler scalars Y^A and the radius R of the circle by

$$J^A = \frac{R}{2\ell_5} Y^A \tag{3.90}$$

and the four dimensional B-field moduli B^A equal the Wilson lines around the S^1 of the five dimensional gauge fields. Asymptotically $\mathbb{R}^{1,3} \times S^1$ solutions surviving the $R/\ell_5 \to \infty$ limit thus correspond to 4d flow trees surviving the $J^A \to \infty Y^A$ limit. Figure 3.3 gives an example of a class of flow trees not surviving in this limit.

Now, not all asymptotically $\mathbb{R}^{1,3} \times S^1$ configurations surviving in the limit fit into a single

$\mathrm{AdS}_3 \times S^2$ throat. For example D4-D4 bound states have center separations of order $p^3\,\ell_5$ in the original coordinates (see appendix D of [3]) , whereas multicentered configurations which do fit into an asymptotic $\mathrm{AdS}_3 \times S^2$ throat have separations of order $p^3 \ell_5^3/R^2$. The diverging hierarchy between these distance scales in the decoupling limit $R/\ell_5 \to \infty$ is manifest in the rescaled consistency condition (3.54) in the decoupling limit: for two D4 centers (or more generally clusters) with non-vanishing mutual intersection product, the (rescaled) equilibrium separation is infinite.

To understand this more systematically let us consider the fate of attractor trees in the decoupling limit. Looking at the asymptotics (3.31)-(3.34) of the decoupled solutions, we see that the value of Y^A at the boundary of AdS is proportional to p^A, and that the θ-averaged Wilson line $\frac{1}{4\pi}\oint A^A_{5d}$, equals $D^{AB}q_B$. This suggest asymptotic $\mathrm{AdS}_3 \times S^2$ solutions correspond to 4d attractor flow trees with starting point at the "asymptotic AdS_3 attractor point"

$$B^A + iJ^A = D^{AB}q_B + i\infty p^A. \tag{3.91}$$

As a test of this suggestion, note that, as pointed out in [9], this eliminates flow trees initially splitting into two flows carrying only D4-D2-D0 charges, and therefore configurations of two D4 clusters with nonvanishing intersection product, which as we just recalled indeed do not fit in a single $\mathrm{AdS}_3 \times S^2$ throat in the decoupling limit. To see this, it suffices to compute for $\Gamma_a = (0, p^A_a, q^a_A, q^a_0)$ at $B^A = D^{AB}q_B$, $J^A = \Lambda p^A$, $\Lambda \to \infty$:

$$\langle \Gamma_1, \Gamma_2 \rangle \mathrm{Im}(Z_1 \bar{Z}_2) = -\frac{3}{8}(p^A_1 q^2_A - p^A_2 q^1_A)^2 + \mathcal{O}(\Lambda^{-1}) < 0. \tag{3.92}$$

This inequality (valid when $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$) implies that the initial point can never be on the stable side of a wall of marginal stability, and hence a flow tree with this initial split cannot exist. Initial splits involving nonzero D6-charge on the other hand are not excluded in this way, consistent with expectations.

Thus, we arrive at the following

Conjecture: *There is a one to one correspondence between (i) components of the moduli space of multicentered asymptotically $AdS_3 \times S^2$ solutions with a $U(1)$ isometry and (ii) attractor flow trees starting at $J^A = p^A\infty$ and $B^A = D^{AB}q_B$.*

In what follows we will refer to this special point in moduli space as the *AdS point*. It is worth pointing out that the AdS point may lie on a wall of *threshold* stability,[13] as defined in appendix B, for which the inequality (3.92) may become an equality. As discussed

[13] Note that it cannot lie on a wall of *marginal* stability $\Gamma \to \Gamma_1 + \Gamma_2$: if the constituents have nonzero D6-charge, these D6-charges have to be opposite in sign, so in the $J \to \infty$ limit, the central charges cannot possibly align; if the constituents have zero D6-charge, (3.92) shows that their central charges cannot align either if their intersection product is non-vanishing.

there, the solution space becomes non-compact in this case, in the sense that constituents can be moved off to infinity — in this case to the boundary of AdS. An example is given by (B.3): since the overall D2-charge vanishes, the AdS point lies on the line $B = 0$, which is a line of threshold stability for splitting off the D0. The flow tree becomes degenerate as well, as it splits in a trivalent vertex. Keeping this in mind, the flow tree picture remains valid.

Finally, we should comment on our choice of B-field value for the AdS point. In general, the actual value of B^A at the boundary of AdS depends on the angle θ with the direction of the total angular momentum:

$$B^A|_{\partial AdS} = \frac{1}{4\pi}\oint A_{5d} = D^{AB}q_B - \cos\theta\,\frac{J}{2J_{\max}}\,p^A. \tag{3.93}$$

Hence there is a significant spread of the actual asymptotic value of the B-field, proportional to the total angular momentum, which moreover grows with p. Although natural, it is therefore not immediately obvious that picking the average value (or equivalently the value at $\theta = \frac{\pi}{2}$) as starting point is the right thing to do, and this is why our conjecture above is not an immediate consequence of the split attractor flow conjecture.

3.3 Some Decoupled Solutions

To explore what AdS/CFT can tell us about the states dual to the solutions introduced in Section 3.1.2 we will briefly describe the decoupling limit for some simple, but interesting multicentered configurations. The first example is rather straightforward as we show how the well known case of a single centered black hole/string fits in our more general story. Afterwards we discuss two 2-center systems of interest. First, we show that the decoupling limit of a purely fluxed $\mathrm{D6}-\overline{\mathrm{D6}}$ bound state is nothing but global $\mathrm{AdS}_3\times\mathrm{S}^2$ and we discuss the link of this interpretation with spectral flow in the CFT. Second we analyze configurations leading to the Entropy Enigma of [9] in asymptotic AdS space. In the Section 5.3 we will show how the Entropy Enigma translated to 5d coincides with a well know instability of small AdS black holes.

Note that from here on we put $R \equiv 1$.

3.3.1 One center: BTZ

In the case of a single black string we expect to reproduce the standard BTZ black hole (times S^2) as the decoupled geometry [28]. As a check on our results we show that this is indeed the case and that the entropy of the BTZ black hole corresponds to the one of the 4d black hole/5d black string we took the decoupling limit of. Given an M5M2P black string of charge $(0, p^A, q_A, q_0)$ one can easily calculate that the metric (3.46) in the decoupling limit is

$$ds^2 = \frac{r}{U}\left[-dtd\psi + \frac{1}{4}\left(1 + \frac{1}{rU^3}\left(\frac{S}{\pi}\right)^2\right)d\psi^2\right] + \frac{U^2 dr^2}{r^2} + U^2 d\Omega_2^2\,, \qquad (3.94)$$

where

$$S = 4\pi\sqrt{\frac{-\hat{q}_0 p^3}{24}}\,, \qquad \hat{q}_0 = q_0 - \frac{1}{2}D^{AB}q_A q_B\,, \qquad (3.95)$$

is the entropy of the 4d black hole. It is clear that this is indeed of the asymptotically local $\mathrm{AdS}_3\times\mathrm{S}^2$ form as found above. But in this case the full geometry, including the interior, is actually locally $\mathrm{AdS}_3\times\mathrm{S}^2$. To see this perform the coordinate transformation

$$\psi = 2(t+\alpha)\,, \qquad r = U(\rho^2 - \rho_*^2) \quad \text{and} \quad \rho_* = \frac{S}{\pi U^2}\,, \qquad (3.96)$$

to put this metric (3.94) into its well known BTZ form:

$$ds^2 = -\frac{(\rho^2-\rho_*^2)^2}{\rho^2}dt^2 + \frac{4\rho^2 U^2}{(\rho^2-\rho_*^2)^2}d\rho^2 + \rho^2(d\alpha + \frac{\rho_*^2}{\rho^2}dt)^2 + U^2 d\Omega_2^2\,. \qquad (3.97)$$

This is the geometry of a sphere times an extremal rotating BTZ black hole and as is well known [77], this can be viewed as a quotient of $\mathrm{AdS}_3\times\mathrm{S}^2$. Calculating the Bekenstein Hawking entropy of this BTZ black hole we find:

$$S_{\mathrm{BH}} = \frac{2\pi\rho_*}{4G_3} = S\,, \qquad (3.98)$$

in agreement with our expectations.

Note that the horizon topology is $S^1 \times S^2$, so from the 5d point of view we have a black ring.

3.3.2 Two centers: $\mathrm{D6} - \overline{\mathrm{D6}}$ and spinning $\mathrm{AdS}_3 \times S^2$

The first new configurations appear by taking the decoupling limit of 2-center bound states. As follows from the constraint (3.54), only 2-centered solutions where the centers carry (opposite) non-vanishing D6 charge exist in asymptotic $\mathrm{AdS}_3\times\mathrm{S}^2$ space. Such centers sit at a fixed distance completely determined by their charges:

$$r_{12} = \frac{-4\langle\Gamma_1, \Gamma_2\rangle}{p_1^0}\,. \qquad (3.99)$$

In general in the bulk the solution is now fully five-dimensional, mixing up the asymptotic sphere and AdS geometries in a complicated way.

The simplest two centered configuration is that of a bound state of a pure D6 and $\overline{\text{D6}}$ carrying only $U(1)$ flux, say $F = \pm\frac{p}{2}$. The two charges are then:

$$\begin{aligned}
\Gamma_1 &= e^{\frac{p}{2}} = [1, \frac{1}{2}, \frac{1}{8}, \frac{1}{48}]\,, \\
\Gamma_2 &= -e^{-\frac{p}{2}} = [-1, \frac{1}{2}, -\frac{1}{8}, \frac{1}{48}]\,,
\end{aligned} \tag{3.100}$$

where we introduced the following notation for (D6,D4,D2,D0)-charges:

$$[a, b, c, d] := (a, b\, p^A, c\, D_{ABC}p^B p^C, d\, D_{ABC}p^A p^B p^C)\,. \tag{3.101}$$

We now show that the lift of such a 2-centered configuration in the decoupling limit yields rotating global $\text{AdS}_3\times\text{S}^2$. In this limit the harmonic functions are:

$$\begin{aligned}
H^0 &= \frac{1}{|x-x_1|} - \frac{1}{|x-x_2|}\,, \\
H^A &= \frac{p^A}{2}\left(\frac{1}{|x-x_1|} + \frac{1}{|x-x_2|}\right)\,, \\
H_A &= \frac{D_{ABC}p^B p^C}{8}\left(\frac{1}{|x-x_1|} - \frac{1}{|x-x_2|}\right)\,, \\
H_0 &= \frac{p^3}{48}\left(\frac{1}{|x-x_1|} + \frac{1}{|x-x_2|}\right) - \frac{1}{4}\,.
\end{aligned} \tag{3.102}$$

The equilibrium distance, solution to (3.54), is given by:

$$|x_1 - x_2| = \frac{2\,p^3}{3} =: 4U^3\,. \tag{3.103}$$

After a change of coordinates (see also [78]):

$$\begin{aligned}
|x-x_1| &= 2U^3(\cosh 2\xi + \cos\tilde{\theta}) \\
|x-x_2| &= 2U^3(\cosh 2\xi - \cos\tilde{\theta}) && (3.104) \\
t &= \tau && (3.105) \\
\psi &= 2(\tau+\sigma)\,,
\end{aligned}$$

and letting ϕ be the angular coordinate around the axis through the centers (so the coordinates $(2\xi, \tilde{\theta}, \phi)$ are standard prolate spheroidal coordinates), the metric takes the simple form:

$$\begin{aligned}
ds^2 = (2U)^2(-\cosh^2\xi\, d\tau^2 + \sinh^2\xi\, d\sigma^2 + d\xi^2) \\
+U^2(\sin^2\tilde{\theta}\,(d\phi + \tilde{A})^2 + d\tilde{\theta}^2)\,,
\end{aligned} \tag{3.106}$$

where

$$\tilde{A} = d(\sigma - \tau). \tag{3.107}$$

The general asymptotic form (3.31) is obtained from this by the coordinate transformation $\xi = \frac{\eta}{2U} - \ln U$, $\tilde{\theta} = \theta$ and taking $\eta \to \infty$.

This metric describes an S^2 fibration over *global* AdS_3, with connection $\tilde{A}$. The connection is flat except at the origin, where it has a delta function curvature singularity. Hence this is essentially a particular case of the geometries considered in [79, 80].[14] The twist of the sphere around the AdS_3 boundary circle $\sigma \to \sigma + 2\pi$ is given by the Wilson line $\oint \tilde{A}$. In this case the twist equals a 2π rotation, in accordance with our general considerations under (3.36) and the fact that the angular momentum $J = p^3/12$ is maximal. Translated to the CFT, this means we have maximal $SU(2)_R$ charge. Moreover, as explained under (3.36), fermions are periodic around the AdS_3 boundary circle $\sigma \to \sigma + 2\pi$, so this geometry corresponds, in a semi-classical sense, to a maximally charged R-sector supersymmetric ground state.[15]

Since the twist amounts to a full 2π rotation of the sphere, the Wilson line can be removed by a large gauge transformation, that is, a coordinate transformation on the S^2,

$$\phi \to \phi' = \phi + \sigma - \tau\,, \tag{3.108}$$

which brings the metric to trivial $AdS_3 \times S^2$ direct product form, with $\tilde{A}' = 0$. In general, large gauge transformations of the bulk act as symmetries (or "spectral flows") of the boundary theory — in general they map states to physically different states. Here in particular this large gauge transformation will affect the periodicity of the fermions, since a 2π rotation of the sphere will flip the sign of the fermion fields. The fermions are then no longer periodic, but antiperiodic around $\sigma \to \sigma + 2\pi$ — we are now in the NS sector vacuum of the theory, consistent with the symmetries of global AdS_3 with $\tilde{A} = 0$.[16]

In the dual $(0, 4)$ CFT, this transformation acts as spectral flow generated by the $SU(2)_R$ charge J_0^3. The charges discussed in section 3.2.4 transform under this symmetry as [83]:

$$\begin{aligned} L_0 &\to L_0\,, \\ \tilde{L}_0 &\to \tilde{L}_0 + 2\epsilon J_0^3 + \frac{c}{6}\epsilon^2\,, \\ J_0^3 &\to J_0^3 + \frac{c}{6}\epsilon\,, \end{aligned} \tag{3.109}$$

[14] For the case of $AdS_3 \times S^3 \times Z$, i.e. the (4,4) D1-D5 CFT, these geometries were further studied in detail in [81, 82].

[15] There is of course a $2J + 1$ dimensional space of such ground states in the CFT. Correspondingly, on the gravity side, a spin J multiplet is obtained by quantizing the 2-particle $D6 - \overline{D6}$ system [17, 9], or equivalently the solution moduli space. This and related topics are studied in the companion paper [4].

[16] Spelled out in more detail, for a fermion field ψ, we have in the old coordinates $\psi(\sigma, \phi, \ldots) = \psi(\sigma + 2\pi, \phi, \ldots)$. Expressed in the new coordinates, this boundary condition is $\psi(\sigma, \phi, \ldots) = \psi(\sigma + 2\pi, \phi' + 2\pi, \ldots) = -\psi(\sigma + 2\pi, \phi', \ldots)$, where in the last equality we used the fact that ϕ' parametrizes rotations of the sphere.

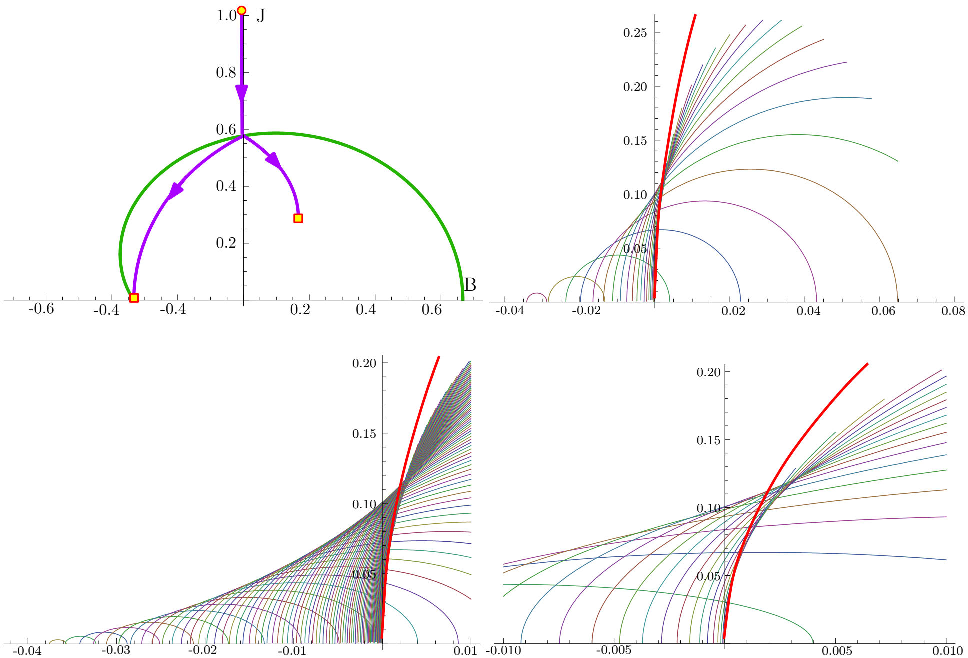

Figure 3.4: *In the upper left figure, the flow tree for the maximally entropic 2-centered configuration at $h = 0$ is shown (i.e. $u = 1/3$). The other three figures show the total entropy as a function of h for a number of uniformly spaced values of u between 0 and $1/2$, at three different zoom levels (and different u-spacings). The fat red line is the entropy of the BTZ black hole with the same total charge.*

with $\epsilon = 1/2$ and $c = p^3$. According to our general results (3.82) and (3.89), we get for the original geometry $L_0 = 0$, $\tilde{L}_0 = p^3/24$, $J_0^3 = -p^3/12$. Applying the above spectral flow, we obtain $L_0 = 0$, $\tilde{L}_0 = 0$, $J_0^3 = 0$, as expected for the NS vacuum.

More general geometries corresponding to states in the NS sector, at least in the case of axially symmetric solutions, can be obtained by applying the spectral flow coordinate transformation (3.108) to the R sector solutions we have constructed.

3.3.3 Enigmatic configurations

In [9] it was shown that there are some regions in charge space where the entropy corresponding to given total charges (with zero total D6 charge) is dominated not by single centered black holes, but by multicentered ones. This phenomenon was called the Entropy Enigma. For a short summary see [84].

Interestingly, these enigmatic configurations always survive the decoupling limit, because their walls of marginal stability are compact, with the stable side on the large type IIA volume side. This is to be contrasted with the 4d asymptotically flat case at fixed values of the asymptotic Kähler moduli; in this case, because the unstable region in Kähler moduli space grows with p, the enigmatic configurations always disappear when $p \to \infty$ as the asymptotic moduli will eventually become enclosed by the wall of marginal stability. In this sense, they are most naturally at home in the decoupled $\mathrm{AdS}_3 \times S^2$ setup under consideration, where they persist for all p.

In [9] section 3.4, a simple class of examples was given, consisting of 2-centered bound states with centers of equal entropy. However, this configuration is not the most entropic one for the given total charge: the total entropy can be increased by moving charge from one center to the other. The maximal entropic configuration is obtained when all entropy is carried by one center only; this can be traced back to the fact that the Hessian of the entropy function of a single black hole has some positive eigenvalues, making multi-black hole configurations generically thermodynamically unstable as soon as charges are allowed to be transported between the centers.

We have not been able to find other, more complicated configurations, involving more centers, with more entropy.

Thus we consider two charges $\Gamma_i = (p^0, p^A, q_A, q_0)_i$ of the form

$$\Gamma_1 \quad = \quad -e^{-up} = \left[-1, u, -\frac{u^2}{2}, \frac{u^3}{6}\right] \tag{3.110}$$

$$\Gamma_2 \quad = \quad p - h\,p^3 - \Gamma_1 = \left[1, (1-u), \frac{u^2}{2}, -h - \frac{u^3}{6}\right], \tag{3.111}$$

where we used the notation (3.101). The total charge of this system is

$$\Gamma = (0, p^A, 0, -h\,p^3) = [0, 1, 0, -h]\,. \tag{3.112}$$

If the bound state exists, the angular momentum (3.18) and rescaled equilibrium separation (3.54) between the centers are, respectively

$$J = \frac{1}{4}(u^2 - 2h)p^3\,, \qquad |\vec{x}_1 - \vec{x}_2| = 2(u^2 - 2h)p^3\,. \tag{3.113}$$

The entropy is given by

$$S_{2c} = S_1 + S_2,$$
$$S_1 = 0, \qquad S_2 = \frac{\pi}{3}\,p^3\,\sqrt{8(\tfrac{1}{2}-u)^3 - 9(\tfrac{1}{3} - h - u + \tfrac{u^2}{2})^2}\,. \tag{3.114}$$

To get a bound state in the decoupling limit, the equilibrium separation in (3.113) must of course be positive and the expression under the square root in (3.114) must be non-negative. A more detailed analysis using attractor flow trees shows that if we also require

$u \geq 0$, these conditions are necessary and sufficient. (The latter condition is necessary to prevent the wall of marginal stability to be enclosed by a wall of anti-marginal stability.)

The minimal possible value of h is $-\frac{1}{24}$, reached at $u = \frac{1}{2}$, where $\Gamma_2 = e^{up}$. This corresponds to the pure fluxed $\mathrm{D6} - \overline{\mathrm{D6}}$ of section 3.3.2. The maximal value of h attainable by the configurations under consideration is $9/128 \approx 0.07$.

The entropy for a single center of the same total charge (the BTZ black hole of section 3.3.1) is given by

$$S_{1\mathrm{c}} = 4\pi\sqrt{\frac{-q_0 p^3}{24}} = \frac{\pi\,\sqrt{2h}\,p^3}{\sqrt{3}}. \tag{3.115}$$

One way of phrasing the Entropy Enigma is that in the limit $p \to \infty$ keeping q_0 fixed, the 2-centered entropy is always parametrically larger than the 1-centered one,[17] as the former scales as p^3, while the latter scales as $p^{3/2}$. More generally this 2-centered parametric dominance will occur whenever $h = -q_0/p^3 \to 0$. A short computation starting from (3.114) shows that in this limit, the maximal 2-centered entropy is reached at $u = 1/3$, with entropy and angular momentum

$$S_{2\mathrm{c}} = \frac{\pi\,p^3}{18\sqrt{3}} \approx 0.100767\,p^3\,, \qquad J = \frac{p^3}{36} = \frac{J_{\max}}{3}\,. \tag{3.116}$$

Indeed this entropy is manifestly parametrically larger than $S_{1\mathrm{c}}$ when $h \to 0$. More precisely the crossover point between one and two-centered dominance is at $h_c \approx 0.00190622$. This is illustrated in fig. 3.4. The phase transition this crossover suggests will be discussed further in section 5.3.1.

We should note that we have only analyzed a particular family of 2-centered solutions here. A slight generalization would be to let both centers have nonzero entropy. However this turns out to give a lower total entropy for the same total charge — for example in the symmetric 2-centered case described in [9], the maximal attainable entropy is $S = \frac{\pi\,p^3}{48}$. Similarly for other generalizations such as sun-earth-moon systems (see [9]), we were unable to find configurations with higher entropy. We cannot exclude however that they exist. If so, this would affect the precise value of the crossover point h_c, but not its existence.

All of these 2-centered solutions have nonvanishing angular momentum, except in the degenerate limit of coalescing centers, when $u^2 = 2h$. In this case the entropy is always less than the single centered one, as it should be to not violate the holographic principle. One might therefore suspect that the Entropy Enigma disappears when restricting to configurations with zero angular momentum. This is not the case, however. A simple example of a multicentered solution with zero angular momentum but entropy $S \sim p^3$ is obtained

[17] Note that if $q_0 > 0$, there is no single centered black hole, so then this statement is trivially true.

as follows. Instead of one particle of charge $\Gamma_1 = -e^{-up}$ orbiting around a black hole of charge $\Gamma_2 = \Gamma - \Gamma_1$, consider $k > 1$ particles of charge $\Gamma_1(u) = -e^{-up}$ orbiting on a halo around a black hole of charge $\Gamma_2'(k, u, h) = \Gamma(h) - k\Gamma_1(u)$. Then by positioning the particles symmetrically on their equilibrium sphere around the black hole, we get configurations of zero angular momentum, but with entropy still of order p^3 at large p. This can be extended quantum mechanically: quantizing these configurations using the technology introduced in Chapter 4 (see also [17, 9, 4]), we get a number of spin zero singlets from tensoring k spin j single particle ground states.

Note that the entropy of the k-particle configuration at given u and h can be related to that of our original $k = 1$ solution by

$$S(k, u, h) = S(\Gamma_2'(k, u, h)) = \frac{1}{k} S(\Gamma_2'(1, ku, k^2 h)) = \frac{1}{k} S(\Gamma_2(ku, k^2 h)) \,. \tag{3.117}$$

The equilibrium separation between a $\Gamma_1(u)$ particle and the Γ_2' core, for given h and u, does not depend on k, so

$$x_{12}(k, u, h) = x_{12}(1, u, h) = \frac{1}{k^2} x_{12}(1, ku, k^2 h) \,. \tag{3.118}$$

From these relations, we can immediately deduce the existence conditions and maximal entropy configuration for $k > 1$ particles using the results for the $k = 1$ case derived above. In particular we see that the entropy is maximized at $u = 1/3k$, and for e.g. $h = 0$ equal to

$$S_{(1+k)\mathrm{c}} = \frac{1}{k} \cdot \frac{\pi\, p^3}{18\sqrt{3}} \,. \tag{3.119}$$

Note that due to the factor k in the denominator, the $k \geq 2$ (possible spin zero) configurations are thermodynamically disfavored compared to the $k = 1$ (necessarily spinning) configurations.

3.4 Some Solution Spaces

So far we have restricted our attention to one or two center configurations where the constraint equations (3.11) are essentially trivial. As mentioned in section 3.1.4 the solution space is $2N - 2$ dimensional so the two center case is two dimensional. For two centers the inter-center separation is fixed by the constraint equation (see e.g. eqn. (3.99)) but we still have the freedom to rotate the axis defined by the two centers giving an S^2 as the solution space (this is discussed further in section 4.3.2). For three centers the constraint equations do not completely fix the distance between the centers and there is generally a more complicated space of solutions. Some of these spaces can be characterized by particular limit points. Scaling solutions spaces are characterized by containing so called scaling solutions: solutions for which the inter-center coordinate space separation goes to zero. Another class of solutions enjoys the particular property of being (naively) non-compact in that certain centers are allowed to go to infinity. We will review these here for reference and then discuss their physics in more detail in the following chapters. We will also describe more intricate "Dipole Halo" solutions with an arbitrary number of centers that are nonetheless tractable. The latter have been considered in connection with black holes [78] and we will have reason to revisit them later.

3.4.1 Three Center Solution Space

The three center solution space is four dimensional. Placing one center at the origin (fixing the translational degrees of freedom) leaves six coordinate degrees of freedom but these are constrained by two equations. This leaves four degrees of freedom, of which three correspond to rotations in SO(3) and one of which is related to the separation of the centers.

The constraint equations take the form

$$\begin{aligned} \frac{a}{u} - \frac{b}{v} &= \frac{\Gamma_{12}}{r_{12}} - \frac{\Gamma_{31}}{r_{31}} = \langle h, \Gamma_1 \rangle =: \alpha \\ \frac{b}{v} - \frac{c}{w} &= \frac{\Gamma_{31}}{r_{31}} - \frac{\Gamma_{23}}{r_{23}} = \langle h, \Gamma_3 \rangle =: \beta \end{aligned} \tag{3.120}$$

in a self-evident notation. The nature of the solution space simplifies considerably if either α or β vanish so let us first consider this case (if both vanish there is an overall scaling degree of freedom and the centers are unbound). For definiteness we will take $\alpha = 0$; we also have $\sum_p \langle h, \Gamma_p \rangle = 0$ which implies $\langle h, \Gamma_2 \rangle = -\beta$. Thus from (3.19) we find

$$\vec{J} = \frac{\beta}{2} r_{23} \hat{z} \tag{3.121}$$

with $\hat{z}$ a unit vector in the $\vec{x}_3 - \vec{x}_2$ direction.

The solution has an angular momentum vector J^i directed between the centers 2 and 3 and the direction of this vector defines an S^2 in the phase space which we will coordinatize using θ and ϕ. The third center is free to rotate around the axis defined by this vector (since this does not change any of the inter-center distances) providing an additional $U(1)$, which we will coordinatize by an angle σ, fibred non-trivially over the S^2. Finally the angular momentum has a length which may be bounded from both below and above and this provides the final coordinate in the phase space, $j = |\vec{J}|$.

This construction is perhaps not the most obvious one from a spacetime perspective but, as we will see in the next chapters, these coordinates are the natural ones to use when quantizing the solution space as a phase space. When $\alpha = 0$ it is clear from (3.121) that j is a good coordinate on the solution space but this is not immediately obvious for the more complicated case of $\alpha \neq 0$. This is nonetheless true and, as shown in Appendix A of [4], this is always a good coordinatization of the three center solution space (though for $\alpha \neq 0$ the relation between $(j, \sigma, \theta, \phi)$ and the coordinates $\vec{x}_p$ is not as straightforward).

The quantization of these solutions is particularly interesting in certain cases where classical reasoning leads to pathologies. Before proceeding with quantization we will first briefly review these solutions.

3.4.2 Scaling Solutions

As noted in [69] and [9], for certain choices of charges it is possible to have points in the solution space where the coordinate distances between the centers goes to zero. Moreover, this occurs for any choice of moduli so it is, in fact, a property of the charges alone.

Three center examples of such solutions occur as follows. We take the inter-center distances to be given by $r_{ab} = \lambda\Gamma_{ab} + \mathcal{O}(\lambda^2)$ (fixing the order of the ab indices by requiring the leading term to always be positive). As $\lambda \to 0$ we can always solve (3.120) by tweaking the λ^2 and higher terms (we are no longer restricting to $\alpha = 0$). The leading behaviour will be $r_{ab} \sim \lambda\Gamma_{ab}$ but clearly this is only possible if the Γ_{ab} satisfy the triangle inequality. Thus any three centers with intersection products Γ_{ab} satisfying the triangle inequalities have this scaling property.

We will in general refer to such solutions as *scaling solutions* meaning, in particular, supergravity solutions corresponding to $\lambda \sim 0$. The space of supergravity solutions continuously connected (by varying the $\vec{x}_p$ continuously) to such solutions will be referred to as *scaling solution spaces*. We will, however, occasionally lapse and use the term scaling solution to refer to the entire solution space connected to a scaling solution. We hope the reader will be able to determine, from the context, whether a specific supergravity solution or an entire solution space is intended.

These scaling solutions are interesting because (a) they exist for all values of the moduli; (b) the coordinate distances between the centers go to zero; and (c) an infinite throat forms as the scale factor in the metric blows up as λ^{-2}. Combining (b) and (c) we see that, although the centers naively collapse on top of each other, the actual metric distance between them remains finite in the $\lambda \to 0$ limit. In this limit an infinite throat develops looking much like the throat of a single center black hole with the same charge as the total charge of all the centers. Moreover, as this configuration exists at any value of the moduli, it looks a lot more like a single center black hole (when the latter exists) than generic non-scaling solutions. As a consequence of the moduli independence of these solutions it is not clear how to understand them in the context of attractor flows; the techniques we develop in Chapter 4 provide an alternative method to quantize these solutions that applies even when the attractor tree does not.

Unlike the throat of a normal single center black hole the bottom of the scaling throat has non-trivial structure. If the charges, Γ_a, are zero entropy bits as described in section 3.1.5 (e.g. D6's with abelian flux) then the 5-dimensional uplifts of these solutions will yield smooth solutions in some duality frame and the throat will not end in a horizon but will be everywhere smooth, even at the bottom of the throat. Outside the throat, however, such solutions are essentially indistinguishable from single center black holes. Thus such solutions have been argued to be ideal candidate "microstate geometries" corresponding to single center black holes. In [9] it was noticed that some of these configurations, when studied in the Higgs branch of the associated quiver gauge theory, enjoy an exponential growth in the number of states unlike their non-scaling cousins which have only polynomial growth in the charges.

3.4.3 Barely Bound Centers

We will be brief here as such configurations are not discussed further in this thesis (though they are interesting and were treated in section 7 of [4]). For certain values of charges and moduli it is possible for some centers to move off to infinity. Although this would normal signal the decay of any associated states (as happens, for instance, for two centers at a wall of marginal stability [58]) it turns out that this is not always the case. In particular, it is important to distinguish between cases when centers are forced to infinity (marginal stability) versus those where there is simply an infinite (flat) direction in the solution space (threshold stability; see Appendix B). Although the first case clearly signals the decay of a state, in the second case, when centers move off to infinity along one direction of the solution space but may also stay within a finite distance in other regions of the solution space, it is still possible to have bound states, as was demonstrated in [4]. Quite essential to this argument is the fact that in some cases, although the solution space may seem naively non-compact (in the standard metric on $\mathbb{R}^{2N-2}$), its symplectic volume is actually finite and it admits normalizable wave-functions (whose expectation values are finite). There are also cases with unbound centers where the symplectic form on the solution space is degenerate and, in such cases, it is not clear if there is a bound state. Such cases are not amenable to treatment by the methods developed here.

3.4.4 Dipole Halos

So far we have restricted our analysis to some solutions with two or three centers. In Chapter 6 we will have recourse to study solutions with many more centers. Solution spaces corresponding to a large number of centers are only amenable to treatment using methods developed in this thesis if they have a particular, highly symmetric, structure.

In this section we will study an example of such a symmetric solution space. We consider the D6-$\overline{\text{D6}}$-D0 system, which seems closely related to the microstates of the D4-D0 black hole [78]. There exist such configurations with a purely fluxed D6-$\overline{\text{D6}}$ pair and an arbitrary number of anti[18]-$D0$'s. Depending on the sign of the B-field these D0's bind to the D6 or $\overline{\text{D6}}$ respectively. When we take the B-field to be zero at infinity the system is at threshold and the D0's are free to move in the equidistant plane between the D6 and anti-D6. This system and its behaviour under variations of the asymptotic moduli is studied in detail in appendix B.

In Chapter 6 we will be interested in counting the number of states arising from such configurations. As this number is independent of the asymptotic moduli (as long as we don't cross a wall of marginal stability) we are free to choose them such that the solution space has its simplest form. The solution space (and particularly the symplectic form on it) is easiest to analyse at threshold which, in our example, corresponds to $B|_\infty = 0$ so we work at this point. More specifically we will work at the *AdS point* as we will ultimately be interested in such configurations in asymptotically $\text{AdS}_3 \times \text{S}^2$ spaces. Thus the asymptotic moduli, h, assume their decoupled value

$$h = \left(0, 0, 0, -\frac{R^{3/2}}{4}\right) \tag{3.122}$$

with only the D0 constant, h_0, non-vanishing.

For a set of D0's with charges $\Gamma_a = \{0, 0, 0, -q_a\}$ with all the q_a positive and $\sum_a q_a = N$, bound to a D6, $\Gamma_6 = (1, \frac{p}{2}, \frac{p^2}{8}, \frac{p^3}{48})$, and $\overline{\text{D6}}$, $\Gamma_{\bar{6}} = (-1, \frac{p}{2}, -\frac{p^2}{8}, \frac{p^3}{48})$, the constraint equations take the following form at threshold:

$$-\frac{q_a}{x_{6a}} + \frac{q_a}{x_{\bar{6}a}} = 0 \tag{3.123}$$

$$-\frac{I}{r_{6\bar{6}}} + \sum_a \frac{q_a}{x_{6a}} = -\beta \tag{3.124}$$

Here $I = -\langle \Gamma_6, \Gamma_{\bar{6}} \rangle = \frac{p^3}{6}$ is given in terms of the total D4-charge p of the system and $\beta = \langle \Gamma_6, h \rangle$ with $I, \beta > 0$. From the first line we indeed see the D0's lie in the plane equidistant from the $D6$ and $\overline{D6}$, as we are at threshold, and so we can simply write $x_a := x_{6a} = x_{\bar{6}a}$.

We will find a set of coordinates on this space which are very natural from the perspective of quantization. In particular, the symplectic form (4.9) we introduce in Chapter 4 takes a

[18] In our conventions it is anti-D0's that bind to D4 branes. We will however often just refer to them as D0's.

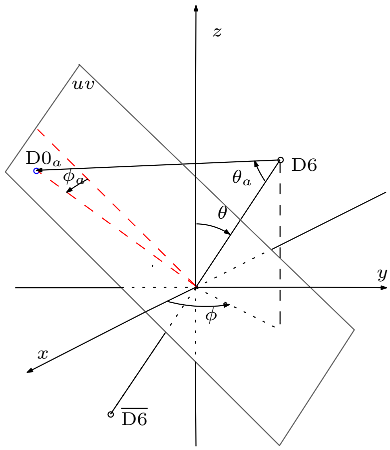

Figure 3.5: *The coordinate system used to parameterize the D6-$\overline{\text{D6}}$-N D0 solution space. The coordinates (θ, ϕ) define the orientation of the $(\hat{u}, \hat{v}, \hat{w})$ axis with respect to the fixed, reference, $(\hat{x}, \hat{y}, \hat{z})$ axis. The D6-$\overline{\text{D6}}$ lie along the $\hat{w}$ axis (with the origin between them) and the D0's lie on the $\hat{u}$-$\hat{v}$ plane at an angle ϕ_a from the $\hat{u}$-axis. The radial position of each D0 in the $\hat{u}$-$\hat{v}$ plane is encoded in the angle θ_a (between $\vec{x}_{6\bar{6}}$ and $\vec{x}_{6a}$).*

nice form in these coordinates. We define an orthonormal frame $(\hat{u}, \hat{v}, \hat{w})$ fixed to the D6-$\overline{\text{D6}}$ pair, such that the D6-$\overline{\text{D6}}$ lie along the w axis and with the D0's lying in the u-v plane. Rotations of the system can then be interpreted as rotations of the $(\hat{u}, \hat{v}, \hat{w})$ frame with respect to a fixed $(\hat{x}, \hat{y}, \hat{z})$ frame. We will parameterize these overall rotations in the standard fashion by two angles, (θ, ϕ). We can furthermore specify the location of the a'th D0 with respect to D6-$\overline{\text{D6}}$ pair by two additional angles, (θ_a, ϕ_a). The first angle, θ_a, is the one between $\vec{x}_{6\bar{6}}$ and $\vec{x}_{6a}$, while ϕ_a is a polar angle in the u-v plane (see figure 3.5). Our $2N + 2$ independent coordinates on solution space are thus $\{\theta, \phi, \theta_1, \phi_1, \ldots, \theta_N, \phi_N\}$.

The standard Euclidean coordinates of the centers are then given in terms of the angular coordinates by

$$\vec{x}_6 = \frac{j}{\beta}\hat{w} \qquad \hat{u} = \cos\phi\,\hat{x} - \sin\phi\,\hat{y}$$

$$\vec{x}_{\bar{6}} = -\frac{j}{\beta}\hat{w} \qquad \hat{v} = \sin\phi\cos\theta\,\hat{x} + \cos\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z}$$

$$\vec{x}_a = \frac{j\,\tan\theta_a}{\beta}(\cos\phi_a\hat{u} + \sin\phi_a\hat{v}) \qquad \hat{w} = \sin\phi\sin\theta\,\hat{x} + \cos\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z}$$

The angular momentum, $j(\theta_a, \phi_a)$, is a function of the other coordinates rather than an independent coordinate (when $N = 1$ it can be traded for θ_1), and is given by

$$j = \frac{I}{2} - \sum_a q_a \cos\theta_a \,. \tag{3.125}$$

In case $N \geq I/2$, which we will refer to as the *scaling regime*, it is possible to combine a sufficient number of centers and form a scaling throat. To see that in the scaling regime the centers can approach each other, let us place all the D0 charge at one center so $q_1 = N$ and consider solutions of the form

$$x_{6\bar{6}} = \lambda\, I + \mathcal{O}(\lambda^2) \qquad x_{61} = \lambda\, N + \mathcal{O}(\lambda^2) \tag{3.126}$$

For small λ solutions of this form can always be found so long as $N \geq I/2$; the latter requirement coming from the fact that $x_{6\bar{6}}$ and x_{61} are coordinate separations and must satisfy triangle inequalities. As $\lambda \to 0$ the *coordinate* distance between the centers goes to zero and the centers coincide in coordinate space. In physical space, however, warp factors in the metric blow up generating a deep throat that keeps the centers a fixed metric distance apart even as $\lambda \to 0$. Outside of this arbitrarily deep throat the solutions is almost indistinguishable from a D4D0 black hole. This regime is thus of great physical relevance and e.g. in [78] it was conjectured to correspond to the deconstruction of a D4D0 black hole.

This is discussed further in Chapter 6.

CHAPTER 4

QUANTIZATION

Before proceeding to our study of the black hole Hilbert space using AdS/CFT we would like to introduce another, complementary, picture and its associated technology. Namely we will be interested in directly quantizing the solutions introduced in sections 3.1.1 and 3.1.2, something we can do because the BPS phase space associated to these supergravity solutions is finite dimensional and is not subject to the subtleties generally associated with quantizing gravity. What's more the structure of the phase space is (in some fortuitous cases) particularly amenable to quantization via the well developed mathematical procedure of geometric quantization. In order to employ this procedure it will be very helpful to develop an open string picture of these solutions as was first done in [17]. Certain essential structures, such as the symplectic form on phase space, are much easier to determine in this open string picture.

4.1 OVERVIEW

As discussed in Section 2.3 there is, in general, a one-to-one map between the phase space and the solution space of a theory. In order to be able use this map to quantize a space of solutions, such as those described in Chapter 3, we need to determine its symplectic form which is induced from the supergravity Lagrangian. The idea is quite simple; one first determines the symplectic structure on the full space of solutions to the supergravity equations of motion, which takes the form

$$\Omega = \int d\Sigma_l \; \delta \left(\frac{\partial L}{\partial(\partial_l \phi^A)} \right) \wedge \delta\phi^A \,, \tag{4.1}$$

where L is the relevant supergravity Lagrangian. Here, the integral is over a Cauchy surface, ϕ^A represents a basis of the fields that appear in the Lagrangian and it is assumed that the Lagrangian does not contain second and higher order time derivatives. The induced symplectic form on the space of BPS solutions is then simply the restriction of (4.1) to that space.

In general, there is no guarantee that the symplectic form so obtained is non-degenerate. If it turns out to be degenerate then we should have included further degrees of freedom to arrive at a non-degenerate symplectic form. This happens for example when all centers are mutually BPS, i.e. if all inner products $\langle \Gamma_p, \Gamma_q \rangle$ vanishes. In this case the symplectic form, as we will see below, is identically zero, and the BPS solution space is therefore better thought of as being a configuration space. In order to obtain a non-degenerate symplectic form, we could try to include, for example, small velocities for the centers [85]. It is not clear, however, whether this can be done while saturating the BPS bound.

It may sound surprising that generically these BPS solutions spaces carry a non-degenerate symplectic form, since they are all time-independent. Crucially, the solutions we consider are stationary but in general not static. Stationary solutions do carry non-trivial momentum despite being time-independent and this is what gives rise to the non-trivial symplectic form. As a consequence of this we will see that the Hilbert space decomposes into angular momentum multiplets. The symplectic form (4.1), when evaluated on a family of static solutions, will simply vanish.

The idea to quantize spaces of BPS supergravity solutions using the restriction of (4.1) was successfully applied in [42][43][54] (see also [49] for an extensive list of older references and [50][51][52] for more recent, similar work). However, when we try to apply similar methods to the space of multicentered black hole solutions, the expressions become very lengthy and tedious and we run into serious technical difficulties due to the complicated nature of the multicentered solutions. We will therefore proceed differently and try to derive the required symplectic form from a dual open string description of the supergravity solutions.

From the open string point of view, which is appropriate for small values of the string coupling constant, black hole bound states correspond to supersymmetric vacua of a suitable quiver gauge theory. The connection between the supergravity solutions and gauge theory vacua becomes clear once we study the Coulomb branch of the gauge theory. For simplicity we will assume that all centers have primitive charges; the extension to non-primitive charges is straightforward, as the supergravity solutions are not specifically sensitive to whether charges are primitive or not.

In this section we determine the symplectic form of the theory explicitly for the three center case and show it to be non-degenerate (with some important exceptions where degenerations have a clear physical interpretation) and proceed to count the number of states in this moduli space via geometric quantization. The estimated number of states at large charge nicely matches the expected number of states from the wall-crossing formula given in [9].

Motivated by a non-renormalization theorem [17] and the exact agreement of our state counting with Denef and Moore's wall-crossing formula we propose that the same symplectic from should be derivable from the super-gravity action following the logic in [42] (see also [52] and references therein). Actually a structure similar to the quiver quantum mechanics symplectic form emerges from the gauge field contribution of the supergravity action. So we may propose the following conjecture: the other putative terms contributing to the symplectic form from supergravity cancel or only change the normalization [1].

Note that it is not *a priori* clear that our moduli space of solutions corresponds to a full phase space rather than a configuration space. The fact that the solution spaces are always even, $(2N-2)$, dimensional is a positive indication that they are indeed phase spaces as is the fact that the symplectic form is non-degenerate in the two and three center cases. Moreover, when degenerations do occur they have well-defined physical meaning further supporting this claim.

4.2 Open String Picture

Ultimately our goal will be to study the moduli space of our solutions using a symplectic form which is derived from the quiver quantum mechanics action of multiple intersecting branes in the coulomb branch [17]. To do so we first review an open string description of these multicenter solutions.

The material presented here is an incomplete review of relevant parts of the elegant paper [17] to which the reader is directed for more details.

4.2.1 Quiver Quantum Mechanics

When embedded within string theory the multicenter solutions of sections 3.1.1 are believed to incorporate the backreaction of a large number of D-branes wrapping cycles of a Calabi-Yau at large string coupling. Such objects can be alternatively understood in an open string picture as boundary conditions for the open string end-points (see e.g. [86]). The physics of D-branes is determined by the dynamics of the open strings ending on them. At low energies it is well-approximated by restricting to the zero modes of these strings yielding a weakly coupled gauge theory living on the D-brane world-volume with the gauge coupling related to the string coupling, g_s.

[1] The last possibility is seen for example in [42].

Let us consider the concrete case of several branes wrapping different cycles on the Calabi-Yau, which we denote by their associated charge vector Γ_a. As in [17] we will first consider D3-branes in IIB as the brane picture and string intersections are more straightforward in this context. As we will ultimately dimensionally reduce to a 0+1 dimensional quiver quantum mechanics in the non-compact spacetime, with the intersection product $\langle\Gamma_a, \Gamma_b\rangle$ as the only CY data we keep, the resulting theory also describes the mirror IIA setup.

The theory on each brane has several kinds of bosonic excitations: adjoint scalars encoding fluctuations of the brane inside the CY, adjoint scalars describing fluctuations in the non-compact directions and gauge fields on the brane world-volume. On the brane intersections there are bi-fundamental scalars from zero-modes of strings stretched between the branes. On the D3-brane lives an $\mathcal{N} = 1$ SYM theory in 4-d which can be dimensionally reduced along the Calabi-Yau cycle resulting in an $\mathcal{N} = 4$ theory in 0+1 dimensions. This theory has two kinds of multiplets: a vector multiplet $(a_a, \vec{x}_a, D_a, \lambda_a)$ for each brane Γ_a and $\kappa_{ab} = \langle\Gamma_a, \Gamma_b\rangle$ chiral[2] multiplets $(\phi_\alpha, F_\alpha, \psi_\alpha)$ for each pair ab with $\alpha = 1, \ldots, \kappa_{ab}$. The former arise from strings with both ends on the same brane while the latter arise from strings stretched between two branes.

The vector multiplet contains a 0+1 dimensional gauge field, a_a, three scalars, $\vec{x}_a$, describing motion of the brane in the transverse $\mathbb{R}^3$, an auxiliary field D_a, as well as fermions. If Γ_a wraps a non-rigid 3 cycle in the Calabi-Yau then there are additional vector multiplets encoding the fluctuations of the brane in the Calabi-Yau but, for the purpose of our analysis here, we will neglect these. In so doing we are assuming that the low-energy theory factorizes into a spacetime contribution and a contribution from the Calabi-Yau degrees of freedom. The fact that one-loop effects including only the non-compact degrees of freedom match perfectly onto the supergravity theory suggests that this factorization indeed holds.

Since we are interested in connecting to supergravity we will consider primitive branes described by abelian gauge theories. Whenever we treat non-primitive branes in subsequent chapters we essentially restrict our attention to the diagonal degrees of freedom. Since a direct connection to supergravity is only possible in the coulomb branch of the theory we can neglect the non-Abelian degrees of freedom.

The Lagrangian for this theory is then the sum of a chiral and a vector part which schematically take the following form [17]

[2]More precisely, κ_{ab} is an index measuring the difference in the number of chiral and anti-chiral modes but we assume that mass terms generically lift any non-chiral modes so we are left with only κ_{ab} chiral modes.

$$
\begin{aligned}
L_V =& \sum_a \frac{m_a}{2}(\dot{\vec{x}}_a^2 + D_a^2 + 2i\bar{\lambda}_a\dot{\lambda}_a) - \theta_a D_p \\
L_C =& \sum_{\alpha:a\to b} |D_t\phi_\alpha|^2 - (|\vec{x}_a - \vec{x}_b|^2 + D_b - D_a)|\phi_\alpha|^2 + |F_\alpha|^2 + i\bar{\psi}_\alpha D_t\psi_\alpha \\
& - \bar{\psi}_\alpha(\vec{x}_b - \vec{x}_a)\cdot\sigma\psi_\alpha - i\sqrt{2}(\bar{\phi}_\alpha\psi_\alpha\epsilon(\lambda_2 - \lambda_1) - (\bar{\lambda}_2 - \bar{\lambda}_1)\epsilon\bar{\psi}_\alpha\phi_\alpha)
\end{aligned}
$$

The covariant derivative $D_t = \partial_t + i(A_b - A_a)$ couples ϕ_α to the gauge fields on a and b branes and $m_a = |Z(\Gamma_a)|/l_P$ is the mass for the brane a in 4-d plank units. The θ_a are Fayet-Iliopoulos parameters whose value is related to the phase of the central charge, $Z(\Gamma_a)$ (i.e. a function of Γ_a and the CY moduli). An intuitive explanation for this is that they contribute to the mass ϕ_a when the auxiliary field D_a is integrated out and the former is proportional to the "angle" between intersecting branes. There are in principle also superpotential terms for the chiral multiplets but, as we are ultimately interested in the theory with these multiplets integrated out, we do not bother with them here (we are ultimately interested in the first order terms involving only the vector multiplets and, as we will see, these are essentially fixed by supersymmetry).

4.2.2 The Coulomb Branch

The theory described above has an very interesting vacuum structure which is explored in great depth in [17]. Although classically the theory only has a coulomb branch when $\theta_a = 0$, quantum effects renormalize the ground state energy generating zero-energy ground states in the coulomb branch even at non-zero values of θ_a. Put another way, integrating out the chiral matter (stretched strings), whose masses are controlled by $|\vec{x}_a - \vec{x}_b|$ and the θ_a, generates a potential for the vector multiplets which has supersymmetric minima. Moreover, as we will demonstrate below, this potential has minima corresponding precisely to the solution to (3.12) once the appropriate identifications between supergravity and gauge theory quantities are made. Although this is a beautiful story we will only need one small aspect of it for this thesis and so we direct the interested reader to [17] for more details.

Fortunately, as demonstrated in [17], the vacuum structure of the coulomb branch can essentially be deduced from supersymmetry alone. We will only need recourse to full theory, including chiral multiplets, to relate constants in the coulomb branch theory to the original D-brane theory. This is because supersymmetry alone constrains the low-energy effective action for N abelian vector multiplets to take the form

$$
L = \sum_a (-U_a(x)\, D_a + \vec{A}_a(x)\cdot\dot{\vec{x}}_a) + \text{ fermions } + \text{ higher order terms,} \tag{4.2}
$$

where U_a and $\vec{A}_a$ are functions of the $\vec{x}_a$ satisfying

$$\nabla_a U_b = \nabla_b U_a = \frac{1}{2}(\nabla_a \times \vec{A}_b + \nabla_b \times \vec{A}_a). \tag{4.3}$$

Note that the $\vec{A}_a(x)$ are vector-valued functions on the world-line and are not related to the gauge fields a_a in the vector multiplet.

This form of the action is determined by requiring that it is invariant under supersymmetry (see [17] for the supersymmetry transformation rules). The solutions to (4.3) are given in terms of harmonic functions. In particular we find that

$$U_a = \langle \Gamma_a, H(x_a) \rangle \tag{4.4}$$

where $H = \theta + \sum_a \frac{\Gamma_a}{|x - x_a|}$ and the pole at $x = x_a$ in H does not contribute to (4.4). We therefore see that except for the tree-level constant term θ, there are only one-loop contributions to U_a and A_a. Supersymmetry prohibits higher loop contributions to U_a and A_a.

From the supersymmetry transformation rules [17] it is clear that supersymmetric solutions must satisfy $\dot{\vec{x}}_a = 0$ and $D_a = 0$ so such solutions are governed completely by the first order part of the Lagrangian in (4.2) whose form is completely fixed by supersymmetry. Fortunately, this first order part also determines the symplectic form as follows from the general definition of the latter

$$\Omega = \left(\frac{\delta L}{\delta \dot{\phi}} \bigg|_{\dot{\phi}=0} \right) \wedge \delta\phi$$

when restricted to the supersymmetric, $\dot{\phi} = 0$, solution space. Thus we only need U_a and A_a to match the gauge theory to supergravity and to extract the symplectic form, and since they do not receive higher loop corrections we can safely extrapolate the results from the gauge theory regime at small g_s to the supergravity regime at large g_s.

Given (4.2), the space of supersymmetric vacua of the gauge theory is given by the solutions of the D-term equations $U_a = 0$. These are identical to the supergravity constraint equations (3.11), (3.12), as θ_a can be identified with the intersection of Γ_a and the constant term in the supergravity harmonic functions.

4.2.3 Symplectic Form from Quiver QM

The symplectic form then follows immediately by restricting $\sum_a \delta\vec{x}_a \wedge \delta\vec{A}_a$ obtained from (4.2) to the solution space $\bigcap_a \{U_a = 0\}$. This symplectic form is independent of g_s and must therefore agree with the symplectic form obtained from supergravity.

Phrased in terms of the supergravity solution, the symplectic form becomes

$$\tilde{\Omega} = \frac{1}{2} \sum_a \delta x_a^i \wedge \langle \Gamma_a, \delta \mathcal{A}_d^i(x_a) \rangle. \tag{4.5}$$

where $\mathcal{A}_d$ is the "spatial" part of the 4d gauge field defined in (3.5). Notice that

$$\delta \mathcal{A}_d^i(x_a) = (\delta \mathcal{A}_d^i(x))|_{x=x_a} + (\delta x_a^k \partial_k \mathcal{A}_d^i(x))|_{x=x_a}. \tag{4.6}$$

To proceed further we denote

$$f_a = \frac{1}{|\vec{x} - \vec{x}_a|}. \tag{4.7}$$

With some work, one can show that

$$\delta \mathcal{A}_d = \sum_a \Gamma_a \epsilon_{ijk} \delta x_a^i \, \partial_j f_a \, dx^k \tag{4.8}$$

Using the above form of $\delta\mathcal{A}_d$ and the fact that $\partial^k \mathcal{A}_d^i(x)$ can be replaced by $\frac{1}{2}(\partial^k \mathcal{A}_d^i(x) - \partial^i \mathcal{A}_d^k(x)) = \frac{1}{2}\mathcal{F}_d^{ki}(x)$ we finally obtain for the symplectic form

$$\tilde{\Omega} = \frac{1}{4} \sum_{a \neq b} \langle \Gamma_a, \Gamma_b \rangle \frac{\epsilon_{ijk} (\delta(x_a - x_b)^i \wedge \delta(x_a - x_b)^j) \, (x_a - x_b)^k}{|\vec{x}_a - \vec{x}_b|^3}. \tag{4.9}$$

Note that the overall translational mode does not contribute to this symplectic form.

The symplectic form (4.9) applies for any number of centers but it must still be restricted to the solution space defined by the constraint eqns. (3.12). Since these spaces are quite complicated we will only be able to analyse this restriction for the two and three center case and some simple examples with arbitrary number of centers.

4.3 Quantization

In Section 3.4.1 we described the solution space of three-centered solutions. Here, we will describe the quantization of these solution spaces using the symplectic form (4.9). It turns out that the structure of the solution space is determined to a large extent by the symmetries of the problem. Recall that we already removed the overall translational degree of freedom from the solution space, which we can do for example by fixing one center to be at the origin, e.g. $\vec{x}_1 = 0$, or by fixing the center of mass to be at the origin. The translational degrees of freedom and their dual momenta give give rise to a continuum of BPS states, but this continuum yields a fixed overall contribution to the BPS partition function. By factoring out this piece we are left with the reduced BPS partition function, and the quantization of the solution space we consider here yields contributions to this reduced BPS partition function.

4.3.1 The Solution Space as a Phase Space

We would now like to quantize the BPS solution space defined by eqns. (3.11) as a phase space. In order to do so we need to ensure that the symplectic form (4.9) is closed and non-degenerate on this space implying that the latter is indeed a (sub)phase space.

Ideally, we would demonstrate these properties for $\tilde{\Omega}$ in general but the second property (non-degeneracy) depends on the structure of the solution space (and in fact does not hold in some degenerate cases as we will see) so we will only be able to demonstrate it for the two and three center cases. As previously mentioned it is not *a priori* evident that the solution spaces are phase spaces so in principle $\tilde{\Omega}$ might have been degenerate on all these spaces.

A direct calculation suffices to show that $d\tilde{\Omega}$ is closed as a two-form on the solution space

$$d\tilde{\Omega} \sim \sum_{a\neq b} \frac{\langle\Gamma_a,\Gamma_b\rangle}{|x_{ab}|}\left(\epsilon^{ijk} - 3\frac{\epsilon_{lij}x^k_{ab}x^l_{ab}}{|x_{ab}|^2}\right)\delta x^i_{ab}\wedge\delta x^k_{ab}\wedge\delta x^j_{ab} = 0 \tag{4.10}$$

The last equality is most easily worked out in a coordinate basis. Non-degeneracy will follow in the two and three center case from the explicit form of the symplectic form as computed below. Before doing so it will be useful to highlight a general structure that emerges as part of any such solution space; namely, the overall rotational degrees of freedom.

Besides the overall translational symmetries, the constraint equations (3.12) are also invariant under global $SO(3)$ rotations of the $\vec{x}_a$. These rotations do appear in a non-trivial way in the symplectic form. Indeed, if we insert $\delta x^i_a = \epsilon^{ijk}n^j x^k_a$, which corresponds to an infinitesimal rotation around the $\vec{n}$-axis[3], in $\tilde{\Omega}$ we obtain

$$\begin{aligned}\tilde{\Omega} &= \frac{1}{4}\sum_{a\neq b}\langle\Gamma_a,\Gamma_b\rangle\frac{\epsilon_{ijk}\epsilon_{ilm}n^l(x_a-x_b)^m\delta(x_a-x_b)^j\,(x_a-x_b)^k}{|\vec{x}_a-\vec{x}_b|^3}\\ &= n^i\delta J^i\end{aligned} \tag{4.11}$$

where J^i are the components of the angular momentum vector (see Section 3.1.3)

$$J^i = \frac{1}{4}\sum_{a\neq b}\langle\Gamma_a,\Gamma_b\rangle\frac{x^i_a - x^i_b}{|\vec{x}_a-\vec{x}_b|} \tag{4.12}$$

The fact that the symplectic form takes the simple form in (4.11) is not surprising. This is merely a reflection of the fact that angular momentum is the generator of rotations. Fortunately, this simple form completely determines the symplectic form in the two and three-center case.

[3]In other words, we compute $\imath_X\tilde{\Omega}$ with X the vector field $\sum_a \epsilon^{ijk}n^j x^k_a\frac{\partial}{\partial x^i_a}$.

4.3.2 Two-Center Case

The two-center case is easy to describe. There is only a regular bound state for $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$ and $\langle h, \Gamma_p \rangle \neq 0$, and the constraint equations immediately tell us that x_{12} is fixed and given by

$$x_{12} = \frac{\langle h, \Gamma_1 \rangle}{\langle \Gamma_1, \Gamma_2 \rangle}. \tag{4.13}$$

In other words, $\vec{x}_1 - \vec{x}_2$ is a vector of fixed length but its direction is not constrained. Thus the solution space is consists of all possible orientations of this vector and can be parameterized by two angles (θ, ϕ) defining an S^2.

Since the solution space is simply the two-sphere the symplectic form must be proportional to the standard volume form on the two-sphere and this is indeed the case. In terms of standard spherical coordinates it is given by

$$\tilde{\Omega} = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \sin\theta \, d\theta \wedge d\phi = |J| \sin\theta \, d\theta \wedge d\phi. \tag{4.14}$$

We can now quantize the solution space using the standard rules of geometric quantization (see e.g. [87] [88]). We introduce a complex variable z by

$$z^2 = \frac{1 + \cos\theta}{1 - \cos\theta} e^{2i\phi} \tag{4.15}$$

and find that the Kähler potential corresponding to $\tilde{\Omega}$ is given by

$$\mathcal{K} = -2|J| \log(\sin\theta) = -|J| \log\left(\frac{z\bar{z}}{(1 + z\bar{z})^2} \right). \tag{4.16}$$

The holomorphic coordinate z represents a section of the line-bundle $\mathcal{L}$ (over S^2, the solution space) whose first Chern class equals $\tilde{\Omega}/(2\pi)$. The Hilbert space consists of global holomorphic sections of this line bundle and a basis of these is given by $\psi_m(z) = z^m$. However, not all of these functions are globally well-behaved. For example, regularity at $z = 0$ requires that $m \geq 0$, and to examine regularity at $z = \infty$ one could e.g. change coordinates $z \to 1/z$ and use the transition functions of $\mathcal{L}$ to find out that $m \leq 2|J|$. Equivalently, we can directly examine the norm of ψ_m by computing

$$|\psi_m|^2 \sim \int d\mathrm{vol}\, e^{-\mathcal{K}} |\psi_m(z)|^2 \tag{4.17}$$

where $d$vol is the volume form induced by the symplectic form. In our case we therefore find

$$|\psi_m|^2 \sim \int d\cos\theta \, d\phi \, (1 + \cos\theta)^{|J|+m} (1 - \cos\theta)^{|J|-m} \tag{4.18}$$

and clearly ψ_m only has a finite norm if $-|J| \leq m \leq |J|$. The total number of states equals $2|J| + 1$. This is in agreement with the wall-crossing formula up to a shift by 1. We will see later that the inclusion of fermions will get rid of this extra constant.

The integrand in (4.18) is a useful quantity as it is also the phase space density associated to the state ψ_m. According to the logic in [89, 90] the right spacetime description of one of the microstates ψ_m should be given by smearing the gravitational solution against the appropriate phase space density, which here is naturally given by the integrand in (4.18). We will come back to this point later, but observe, already, that since there are only $2|J|+1$ microstates, we cannot localize the angular momentum arbitrarily sharply on the S^2, but it will be spread out over an area of approximately $\pi/|J|$ on the unit two-sphere. It is therefore only in the limit of large angular momentum that we can trust the description of the two-centered solution (with two centers at fixed positions) in supergravity.

4.3.3 Three Center (Non-Scaling) Case

We now return to the symplectic form (4.11). In the three-center case, we expect four degrees of freedom. As discussed in section 3.4.1, three of those are related to the possibilities to rotate the system, whilst the fourth one can be taken to be the size of the angular momentum vector $|\vec{J}|$. The specific form of (4.11) strongly suggests that these are also the variables in which the symplectic form takes the nicest form.

We therefore take as our basic variables J^i and σ, where σ represents an angular coordinate for rotations around the $\vec{J}$-axis. Obviously, σ does not correspond to a globally well-defined coordinate, but rather should be viewed as a local coordinate on an S^1-bundle over the space of allowed angular momenta. Ignoring this fact for now, the rotation $\delta x^i_p = \epsilon^{iab} n^a x^b_p$ that we used in (4.11) corresponds to the vector field

$$X_n = \frac{n^i J^i}{|J|}\frac{\partial}{\partial\sigma} + \epsilon^{ijk} n^j J^k \frac{\partial}{\partial J^i}. \tag{4.19}$$

The second term is obvious, as $\vec{J}$ is rotated in the same way as the $\vec{x}_a$. The first term merely states that there is also a rotation around the $\vec{J}$-axis given by the component of n in the $\vec{J}$-direction. The final result in (4.11) therefore states that

$$\tilde{\Omega}(X_n, m^i\frac{\partial}{\partial J^i}) = n^i m^i \qquad \tilde{\Omega}(X_n, \frac{\partial}{\partial\sigma}) = 0. \tag{4.20}$$

It is now easy to determine that

$$\tilde{\Omega}(\frac{\partial}{\partial J^i}, \frac{\partial}{\partial J^j}) = \epsilon_{ijk}\frac{J^k}{|J|^2}, \qquad \tilde{\Omega}(\frac{\partial}{\partial J^i}, \frac{\partial}{\partial\sigma}) = -\frac{J^i}{|J|}. \tag{4.21}$$

Denoting $|\vec{J}|$ as j, and parameterize J^i in terms of j and standard spherical coordinates θ, ϕ, the symplectic form defined by (4.21) becomes

$$\tilde{\Omega} = j\sin\theta\, d\theta\wedge d\phi - dj\wedge d\sigma. \tag{4.22}$$

However, we clearly made a mistake since this two-form is not closed. The mistake was that σ was not a well-defined global coordinate but rather a coordinate on an S^1-bundle. We can take this into account by including a parallel transport in σ when we change J^i. The result at the end of the day is that the symplectic form is modified to

$$\tilde{\Omega} = j \sin\theta \, d\theta \wedge d\phi - dj \wedge D\sigma \tag{4.23}$$

with $D\sigma = d\sigma - A$, and $dA = \sin\theta \, d\theta \wedge d\phi$, so that A is a standard monopole one-form on S^2. A convenient choice for A is $A = -\cos\theta \, d\phi$ so that finally the symplectic form can be written as a manifestly closed two-form as[4]

$$\tilde{\Omega} = -d(j\cos\theta) \wedge d\phi - dj \wedge d\sigma. \tag{4.24}$$

This answer looks very simple but in order to quantize the solution space, we have to understand what the range of the variables is. Since θ, ϕ are standard spherical coordinates on S^2, ϕ is a good coordinate but degenerates at $\theta = 0, \pi$. The magnitude of the angular momentum vector j is bounded as can be seen from (4.12). By carefully examining the various possibilities in the three-center case (see appendix A of [4]), one finds that generically j takes values in an interval $j \in [j_-, j_+]$, where $j = j_-$ or $j = j_+$ only if the three-centers lie on a straight line. An exceptional case is if $j_- = 0$ implying that the three-centers can sit arbitrarily close to each other (see appendix A of [4]). Note that this latter case corresponds exactly to the scaling solutions described in section 3.4.2.

As we mentioned above, at $j = j_-$ and $j = j_+$ the centers align, and rotations around the $\vec{J}$-axis act trivially. In other words, at $j = j_\pm$ the circle parametrized by σ degenerates. Actually, we have to be quite careful in determining exactly which $U(1)$ degenerates where. Fortunately, what we have here is a toric Kähler manifold, with the two $U(1)$ actions given by translations in ϕ and σ, and we can use results in theory of toric Kähler manifolds from [91] (see also [92] and Appendix F) to describe the quantization of this space.

We start by defining $x = j$ and $y = j\cos\theta$ to be two coordinates on the plane. Then the ranges of the variables x and y are given by

$$x - j_- \geq 0, \quad j_+ - x \geq 0, \quad x - y \geq 0, \quad x + y \geq 0. \tag{4.25}$$

Together these four[5] inequalities define a Delzant polytope in $\mathbb{R}^2$ which completely specifies the toric manifold (see Appendix F). At the edges a $U(1)$ degenerates and at the vertices both $U(1)$'s degenerate. The geometry and quantization of the solution space can be done purely in terms of the combinatorial data of the polytope (see figure 4.1).

[4]This result can be re-derived in a more straightforward but tedious way as a special case of the Dipole Halo quantization in Chapter 6.

[5]In the case $j_- = 0$ the first equation is redundant and is not part of the characterization of the polytope.

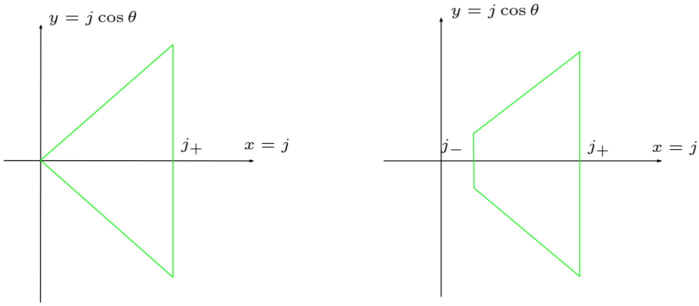

Figure 4.1: (Left) *The polytope for $j_- = 0$ corresponding to $\mathbb{CP}^2_{(1,1,2)}$.* **(Right)** *The polytope for $j_- > 0$. This corresponds to the second Hirzebruch surface, F_2, a blow-up of $\mathbb{CP}^2_{(1,1,2)}$.*

We have to distinguish two cases. First, when $j_- = 0$, which corresponds to a scaling point inside the solution space, our toric manifold is topologically $\mathbb{CP}^2_{(1,1,2)}$. In the case $j_- > 0$, where a scaling point is absent, the solution space becomes the blow-up of $\mathbb{CP}^2_{(1,1,2)}$, which can be identified as the second Hirzebruch surface F_2. These statements can all be verified by e.g. constructing the normal fan to the relevant polytopes.

For the purpose of quantizing the system we will assume that $j_- > 0$; we will return to the case of $j_- = 0$ in section 5.1.2. Thus the results of the rest of this section only apply for $j_- > 0$. Furthermore we assume that all three centers carry different charges; if two centers carry identical charge one needs to take into account their indistinguishability, quantum mechanically, and take a quotient of the corresponding solution space. We won't consider this possibility in this section but come back to it in detail in Chapter 6 when we consider the Dipole Halo system.

The construction of canonical complex coordinates is done by constructing a function g as

$$\begin{aligned} g = \frac{1}{2}\,&[(x - j_-)\log(x - j_-) + (j_+ - x)\log(j_+ - x) \\ &+(x - y)\log(x - y) + (x + y)\log(x + y)] \end{aligned} \tag{4.26}$$

which is related in an obvious way to the four inequalities in (4.25). Then the complex coordinates can be chosen to be $\exp(\partial_x g + i\sigma)$ and $\exp(\partial_y g + i\phi)$. Explicitly, and after removing some irrelevant numerical factors, the complex coordinates are

$$\begin{aligned} z^2 &= j^2 \sin^2\theta \left(\frac{j - j_-}{j_+ - j}\right) e^{2i\sigma} \\ w^2 &= \left(\frac{1 + \cos\theta}{1 - \cos\theta}\right) e^{2i\phi} \end{aligned} \tag{4.27}$$

and the Kähler potential ends up being equal to

$$\mathcal{K} = j_- \log(j - j_-) - j_+ \log(j_+ - j) + 2j. \tag{4.28}$$

Again, a basis for the Hilbert space is given by wave functions $\psi_{m,n} = z^m w^n$. Note that, as in the two center case, these wave functions are, by construction, sections of a line bundle, $\mathcal{L}$, whose curvature is given, once more, by $\tilde{\Omega}/(2\pi)$.

To find the range of n, m we look at the norm

$$|\psi_{m,n}|^2 \sim \int e^{-2r} \frac{(j_+ - r)^{j_+}}{(r - j_-)^{j_-}} \left(r^2 \sin^2\theta \left(\frac{r - j_-}{j_+ - r} \right) \right)^n \left(\frac{1 + \cos\theta}{1 - \cos\theta} \right)^m r \, dr \, d\cos\theta \, d\phi \, d\sigma. \tag{4.29}$$

This is finite if $j_- \le n \le j_+$ and $-n \le m \le n$. Not surprisingly, these equations are identical to the original inequalities that defined the polytope, and the number of states is equal to the number of lattice points in the polytope; notice that this is not quite identical to the area of the polytope. In our case that number of points is

$$\mathcal{N} = (j_+ - j_- + 1)(j_+ + j_- + 1). \tag{4.30}$$

This connection holds quite generally for toric Kähler manifolds. As in the two center case fermionic contributions will correct both the state count, (4.30), and the phase space measure, (4.29); this will be discussed in the next section.

The integrand in (4.29) can once more (as in the two center case) be viewed as a phase space density against which the supergravity solution has to be smeared in order to find the gravitational dual of each microstate [89] [90].

4.3.4 Fermionic Degrees of Freedom

From the open string point of view [17] we know that (4.30) is incorrect and that we must include fermionic degrees of freedom in order to account for all the BPS states (e.g. in the two center case). This is because, in the open string description, the centers are described by $\mathcal{N} = 4$, d=1 supersymmetric quiver quantum mechanics (QQM) with the position of each center encoded in the scalars of a vector multiplet and the latter also includes fermionic components (the λ_a of Section 4.2) which must be accounted for in any quantization procedure.

Since we expect to see the same number of BPS states in both the open and closed description and since the bosonic phase spaces in both cases match exactly (and the symplectic forms agree in view the non-renormalization theorem discussed above) we may ask what the closed string analog of the fermions in the QQM is?

Consider our phase space: the coordinates, x_a, subject to the constraint (3.12), parameterize the space of purely bosonic BPS solutions but, for each such solution, we may still be able to excite fermions if doing so is allowed by the equations of motion. If we consider only infinitesimal fermionic perturbations of the bosonic solutions then the former will always appear linearly in the equations of motion, acted on by a (twisted) Dirac operator. Thus fermions which are zero modes of this operator may be excited without altering the bosonic parts of the solution (to first order).

Determining the actual structure of these zero modes is quite non-trivial. A natural guess is that the bosonic coordinates of the centers must be augmented by fermionic partners (making the solution space a superspace) as is argued in [93] [94] where there is no potential. The fact that the bosonic coordinates are constrained by a potential complicates the problem in our case so we will simply posit the simplest and most natural guess and justify it, *a posteriori*, by reproducing the necessary correction to match the open string picture, the explicit two center and halo quantization of [17], as well as the split attractor conjecture [9].

Thus we will posit that the full solution space is actually the total space of the spin bundle over the Kähler phase space described in Section 4.3.3. The correct phase space densities are now harmonic spinors on the original phase space. This is natural from a mathematical point of view [95] and can be argued physically as follows.

The space of solutions in the open string picture is spanned by letting the bosonic coordinates take their allowed, constant, values and setting the fermionic coordinates to zero (we are neglecting the center of mass degrees of freedom). We could of course try to restrict the symplectic form to this space, and then quantize, but this would miss the possible non-trivial topology of the fermionic vacuum. Therefore we will proceed in a different way as follows.

We start with the full classical phase space of the quiver quantum mechanics including all the fermions. Next we are going to impose the constraints

$$\langle \Gamma_a, H(x_a) \rangle = 0 \tag{4.31}$$

which will restrict x_a to take values in the bosonic solution space. The constraint (4.31), however, is not invariant under all supersymmetries but only half of them. We can therefore impose (4.31) supersymmetrically as long as, at the same time, we remove half the fermions. The resulting system still has two supersymmetries left which one could, in principle, work out explicitly. If we assume that the solutions space is Kähler (which may in fact be a consequence of the two remaining supersymmetries), the resulting supersymmetry transformations will necessarily look like those of standard supersymmetric $\mathcal{N} = 2$ quantum mechanics on a Kähler manifold. Notice that so far we have not used the symplectic structure for the fermions at all.

Though it would be interesting to work this out in more detail, we finally expect that, after geometric quantization of the supersymmetric quantum mechanical system, the supersymmetric wave-functions will be $\mathcal{L}$-valued spinors which, at the same time, are zero-modes of the corresponding Dirac operator.

Recall (see e.g. [96]) that on a Kähler manifold $\mathcal{M}$ there is a canonical Spin^c structure where the spinors take values in $\Lambda^{0,*}(\mathcal{M})$. To define a spin structure we need to take a square root of the canonical bundle $K = \Lambda^{N,0}(\mathcal{M})$ and twist $\Lambda^{0,*}(\mathcal{M})$ by that. We also need to remember that the coordinate part of the wave functions were sections of a line bundle, $\mathcal{L}$. Thus altogether the spinors on the solution space are given by sections of

$$\mathcal{L} \otimes \Lambda^{0,*}(\mathcal{M}) \otimes K^{1/2}. \tag{4.32}$$

The Dirac operator is given by

$$D = \bar{\partial} + \bar{\partial}^* \tag{4.33}$$

and we have to look for zero modes of this Dirac operator. These are precisely the harmonic spinors on $\mathcal{M}$ and therefore the BPS states correspond to $H^{(0,*)}(\mathcal{M}, \mathcal{L} \otimes K^{1/2})$. By the Kodaira vanishing theorem, $H^{(0,n)}(\mathcal{M}, \mathcal{L} \otimes K^{1/2})$ vanishes unless $n = 0$ if we take $\mathcal{L}$ to be very ample, which it is for large enough charges. Thus, finally, the BPS states are given by the global holomorphic sections of $\mathcal{L} \otimes K^{1/2}$. The only difference with the previous purely bosonic analysis is that the line-bundle is twisted by $K^{1/2}$.

To find the number of BPS states we can therefore follow exactly the same analysis as in the bosonic case. We just have to make sure that in the inner product we use the norm appropriate for $\mathcal{L} \otimes K^{1/2}$. This can be accomplished by inserting an extra factor of $(\det \partial_i \partial_{\bar{j}} \mathcal{K})^{-1/2}$ in the inner product. For example, for the two-sphere, this introduces an extra factor of $(1 + \cos\theta)^{-1}$ in the integral, reducing the number of states by one compared to the purely bosonic analysis. This is in perfect agreement with [17].

For the three-center case (and more generally for toric Kähler manifolds) we find, after some manipulations, that

$$(\det \partial_i \partial_{\bar{j}} \mathcal{K})^{-1/2} \sim \exp\left(\sum_i \frac{\partial g}{\partial x^i}\right) \sqrt{\det \left[\frac{\partial^2 g}{\partial x_i \partial x_j}\right]} \tag{4.34}$$

in terms of the function g given for the three-center case in (4.26). To get this relation we used that $\mathcal{K}$ is the Legendre transform of g (see Appendix F). Evaluating this explicitly for the three-center case yields an extra factor

$$\frac{1}{j_+ - j} \sqrt{\frac{1 + \cos\theta}{1 - \cos\theta}}\, a(j) \tag{4.35}$$

with

$$a(j) = \sqrt{j(j_+ - j_-) + 2(j_+ - j)(j - j_-)} \tag{4.36}$$

This result indicates that, in the presence of spinors, we should take n integral and m half-integral with $-n \leq m + \frac{1}{2} \leq n$ and $j_- \leq n \leq j_+ - 1$. Then, the total number of normalizable wave-functions becomes

$$\mathcal{N} = (j_+ - j_-)(j_+ + j_-) \tag{4.37}$$

which does not have the unwanted shifts anymore!

For completeness let us provide the modified form of the norm for a wave function, including fermionic corrections,

$$|\psi_{m,n}|^2 \sim \int e^{-2j} \frac{(j_+ - j)^{j_+ - 1}}{(j - j_-)^{j_-}} \left(j^2 \sin^2 \theta \left(\frac{j - j_-}{j_+ - j} \right) \right)^n \\ \left(\frac{1 + \cos\theta}{1 - \cos\theta} \right)^{m + \frac{1}{2}} a(j)\, j\, dj\, d\cos\theta\, d\phi\, d\sigma. \tag{4.38}$$

Note that this is only the norm *for non-scaling solutions* with $j_- > 0$. The norm for wave-functions on solution spaces with a scaling point ($j_- = 0$) is given in section 5.1.2.

4.3.5 Comparison to the Split Attractor Flow Picture

In the previous subsections we computed the number of states corresponding to the position degrees of freedom of a given set of bound black hole centers. The approach we developed amounts essentially to calculating the appropriate symplectic volume of the solution space. To count the total number of BPS states of a given total charge one needs to take into account the fact that the different black hole centers may themselves carry internal degrees of freedom and that there may be many multicenter realizations of the same total charge. In the special case, however, when all the centers correspond to zero entropy bits without internal degrees of freedom the position degrees of freedom should account for all states. In this case it is interesting to compare the number of states obtained in our approach, using geometric quantization, with the number obtained by considering jumps at marginal stability as in [9] (see also section 3.1.6).

To make this comparison we use the attractor flow conjecture which states that to each component of solution space there corresponds a unique attractor flow tree. Given a component of solution space we can calculate its symplectic volume and hence the number of states. Given the corresponding attractor flow tree we can calculate the degeneracy using the wall crossing formula of [9]. To determine which attractor tree corresponds to which solution space (as needed to compare the two state counts) we will have to assume that part of the attractor flow conjecture holds. Although this might seem to weaken the comparison we should point out that the attractor flow tree has no inherent meaning outside the context of the attractor flow conjecture thus the need to assume the latter to relate the former to our solutions is not surprising. Moreover, the attractor flow conjecture (defined in [58][72]) is distinct from (and weaker than) the wall crossing formula (defined in [9]) which relies on it.

As mentioned before (around eqn. (4.18); see also the section about the addition of fermions), in the two center case we get a perfect agreement between the two calculations. This is not so surprising because both approaches are, in fact, counting the number of states in an angular momentum multiplet with $j = \frac{1}{2}\langle\Gamma_1,\, \Gamma_2\rangle - \frac{1}{2}$. Furthermore, there is no ambiguity in specifying the split attractor tree. Things become more interesting in the three centers case where there are now naively three attractor trees for a given set of centers. According to the attractor flow conjecture only one tree should correspond to any given solution space. It is possible to match solution spaces to attractor trees if we are willing to assume part of the attractor flow conjecture.

Let us consider the three center attractor flow tree depicted in figure 3.1. For the given charges, Γ_1, Γ_2, and Γ_3, there are, in fact, many different possible trees but, in terms of determining the relevant number of states, the only thing that matters is the branching order. In figure 3.1 the first branching is into charges Γ_3 and $\Gamma_4 = \Gamma_1 + \Gamma_2$ so the degeneracy associated with this split is $|\langle\Gamma_4, \Gamma_3\rangle|$ and the degeneracy of the second split is $|\langle\Gamma_1, \Gamma_2\rangle|$ giving a total number of states

$$\mathcal{N}_{\text{tree}} = |\Gamma_{12}|\, |(\Gamma_{13} + \Gamma_{23})| \tag{4.39}$$

where we have adopted an abbreviated notation, $\Gamma_{ij} = \langle\Gamma_i, \Gamma_j\rangle$ and have dropped the factors of $\Omega(\Gamma_a)$ in (3.28) (because we are only interested in the spacetime contribution to the state count so we consider centers with no internal states).

To compare this with the number of states arising from geometric quantization of the solution space, (4.37), we need to determine j_+ and j_-. As mentioned in Section 4.3.3 (see also Appendix A of [4]), j_+ and j_- correspond to two different collinear arrangements of the centers and, in a connected solution space, there can be only two such configurations. To relate this to a given attractor flow tree we will *assume part of the attractor flow conjecture*; namely, that we can tune the moduli to force the centers into two clusters as dictated by the tree. For the configuration in figure 3.1, for instance, this implies we can move the moduli at infinity close to the first wall of marginal stability (the horizon dark blue line) which will force Γ_3 very far apart from Γ_1 and Γ_2. In this regime it is clear that the only collinear configurations are Γ_1-Γ_2-Γ_3 and Γ_2-Γ_1-Γ_3; it is not possible to have Γ_3 in between the other two charges. Since j_+ and j_- always correspond to collinear configurations they must, up to signs, each be one of

$$j_1 = \frac{1}{2}(\Gamma_{12} + \Gamma_{13} + \Gamma_{23}) \tag{4.40}$$

$$j_2 = \frac{1}{2}(-\Gamma_{12} + \Gamma_{13} + \Gamma_{23}) \tag{4.41}$$

j_+ will correspond to the larger of j_1 and j_2 and j_- to the smaller but, from the form of (4.37), we see that this will only effect $\mathcal{N}$ by an overall sign (the state count depends only on the absolute value of $\mathcal{N}$). Thus

$$\mathcal{N} = \pm(j_1 - j_2)(j_1 + j_2) = \pm\Gamma_{12}\,(\Gamma_{13} + \Gamma_{23}) \tag{4.42}$$

which nicely matches (4.39).

Of course to obtain this matching we have had to assume the attractor flow conjecture itself (in part) so it does not serve as an entirely independent verification. Another drawback of this argument is that it is not applicable to the decoupling limit described in [3] where the asymptotic moduli, t^A, are fixed to the *AdS-point* [3]. However, it is possible to circumvent this limitation by gluing an asymptotic flat region to the interior geometry. This can always be done by choosing the moduli in the new added region to be equal to the asymptotic moduli, t_{AdS}, of the original solution. The physicality of such gluing relies on two important observations. The first one is that far away the centers behave like a one big black hole with charges the sum of all charges carried by the centers. The second important ingredient is that t_{AdS} is equal to the attractor value of the moduli associated to this big black hole. Doing so we are back to the asymptotic flat geometry where we have the freedom to play the asymptotic moduli once again. This is the same argument used in [3][68] to generalize the existence conjecture from asymptotic flat solutions to the decoupled limit.

Our result here provides another non-trivial consistency check for the conjecture that *there is a one-to-one map between split-attractor trees and BPS solutions to $\mathcal{N} = 2$ four dimensional supergravity* [9]. By explicitly evaluating (4.30), we find it is the product of two contributions exactly as predicted by the split attractor flow conjecture. Using geometric quantization it is clear that the three-center entropy always factorizes into a product of splits along walls of marginal stability matching an attractor flow tree (the only reason we need to assume part of the attractor flow conjecture for the matching is to determine *which* particular tree). This is strong evidence in favor of the fundamental underpinning of the attractor flow conjecture: namely, that by tuning moduli it is always possible to disassemble multicenter configurations pairwise.

Let us make some further remarks on the results derived here. The scaling solutions corresponding to $\lambda \to 0$ have $j_- = 0$ even if the centers don't align at this point. Therefore the connection to the wall crossing formula breaks down. The procedure of geometric quantization itself, however, does not seem to suffer any pathologies for these solutions. The curvature scales always stay small allowing us to trust the supergravity solutions. Thus one can see the resulting degeneracy as a good prediction. Although the symplectic form seems to degenerate at $j = j_-$ this is, in fact, nothing but a coordinate artifact as can be seen from studying the polytope associated with scaling solutions.

For fixed $j_\pm$ the Hilbert space is finite dimensional and it is not possible localize the centers arbitrarily accurately. Thus the supergravity solutions can only be well approximated in the large $j_\pm$ limit. In Section 5.1.2 we will study the nature of "classical" states defined in this limit. We will be interested in particular in the boundary of the solution spaces where classical pathologies such as infinitely deep throats or barely bound centers (see [3]) moving off to infinity may appear. We will show that quantum effects resolve these pathologies since there is less then one unit of phase space volume in the pathological regions (even for large finite charges) so the pathologies are purely classical artifacts.

CHAPTER 5

BLACK HOLE MICROSTATES

Having developed several powerful technical tools in Chapters 3 and 4 we proceed now to study the structure of these black hole microstates. We will be interested, in particular, in understanding how quantum effects help resolve some classical paradoxes. In particular we will show how the structure of the phase space implies that certain, nominally well-behaved classical geometries, do not correspond to well-defined semi-classical states in the quantum theory. The most interesting such examples are the *scaling solutions* of Section 3.4.2. These have arbitrarily deep throats resembling very much the naive black hole geometry but we will see that quantum effects stretching over macroscopic distances conspire to destroy these throats.

Another paradox we will investigate is the Entropy Enigma of [9]. Here the (partial) resolution of the enigma is not a consequence of quantum effects but merely of a more careful analysis of the partition function associated to the classical solutions. It is of interest as it suggests a phase transition in the dual CFT as a function of the total charge. Understanding whether such transitions occur and their exact nature may shed light on the correct structure of the thermal ensemble dual to a black hole and the super-selection structure of the dual CFT.

Although the issues discussed here do not directly relate the question of black hole information loss they nonetheless demonstrate the effectiveness of our techniques in resolving classical (or semi-classical) paradoxes, of which information loss is an example, by using these tools to study the quantum structure of the black hole. Moreover in the course of our analysis we will show, for the first time (to the author's knowledge), that some black hole microstates exhibit exotic characteristics such as quantum fluctuations across macroscopic spatial regions. Such behaviour has been suggested as a resolution to the information loss paradox and the emergence of such fluctuations here may be taken as a proof of concept.

5.1 Quantum Structure of Solutions

Having determined the structure of the quantum states associated with the two and three center solutions we can now investigate some potential problems with the classical solutions that we expect to be resolved by quantum effects. For instance, although the three center phase space appears to be non-compact for some choices of charges and moduli this turns out not to be a problem since the symplectic volume of the phase space is finite. We show, moreover, that centers cannot move off to infinity once quantum effects are taken into account. Likewise, as has already been observed in [69] and [9], scaling solutions can develop an infinitely deep throat classically but we expect the quantization of phase space to cap this throat off at some finite value and we find that this is indeed the case.

To investigate these issues we would like to consider the expectation value of the harmonic functions (3.2) defining the solutions. Of course this will depend on the particular state we are considering and there are, in general, many possible states one can construct so making any general statement is quite difficult. We do not, however, need detailed properties of $\langle H(r)\rangle$, only its behaviour in various asymptotic limits. We first discuss the non-compact case and then turn to scaling solutions in the next section. In both instances our main concern here will be the r dependence at the boundary of the solution space, which we will be able to extract for a general pure state.

5.1.1 Centers Near Infinity

Let us consider the two independent constraint equations (3.120) for the three center case once more

$$\frac{a}{u} - \frac{b}{v} = \frac{\Gamma_{12}}{r_{12}} - \frac{\Gamma_{31}}{r_{31}} = \langle h, \Gamma_1 \rangle =: \alpha \tag{5.1}$$

$$\frac{b}{v} - \frac{c}{w} = \frac{\Gamma_{31}}{r_{31}} - \frac{\Gamma_{23}}{r_{23}} = \langle h, \Gamma_3 \rangle =: \beta \tag{5.2}$$

where we have (re)introduced a hopefully obvious short-hand notation. We would like to see when one center can move off to infinity. We will assume that a, b and c are non-zero. In order to satisfy the triangle inequalities while taking at least one center to infinity we must have either two or three of the distances u, v, and w become infinite. Let us, for definiteness, try to set u and v to infinity which corresponds to centers 2 and 3 staying

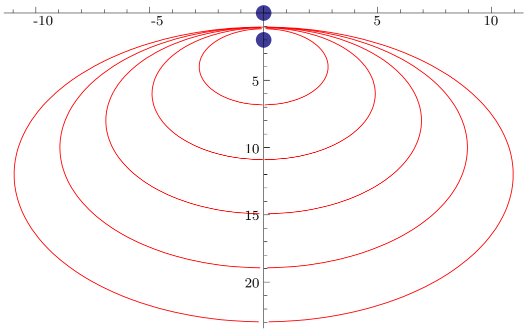

Figure 5.1: *Possible locations for $\vec{x}_1$ are given above. The centers $\vec{x}_3$ and $\vec{x}_2$ sit at $(0, 0)$ and $(0, -2)$ in the (x, z)-plane respectively and the different orbits represent different values of a/b. In the figure above $a/b < 1$ and as $a/b \to 1$ the orbit increases until it becomes the line $z = -1$ when $a = b$ (not depicted).*

a finite distance apart while center 1 moves off to infinity. From the constraint equation its clear that this can only be done if $\alpha = 0$. We can also consider the case when all three centers move infinitely far apart, which can only occur if $\beta = 0$ as well, but this is a somewhat trivial case as the angular momentum will then, by (3.19), vanish, as will the symplectic form, so the space cannot be quantized without adding additional degrees of freedom (momenta).

Thus we restrict to the case $\alpha = 0$, $\beta \neq 0$ which gives

$$\frac{r_{12}}{r_{31}} = \frac{u}{v} = \frac{a}{b} \tag{5.3}$$

When translated into coordinates this equation defines an ellipse for the possible locations of center 1 but it is easy to see from this that u and v must be bounded unless $a = b$, in which case the orbit is not an ellipse but rather center 1 must lie on the plane between centers 2 and 3. Hence all unbound three center solutions will be of this form (where center 1 is fixed on the plane normal to the axis defined by the other two centers). Orbits for various values of a/b are depicted in figure 5.1. Note that this proves our claim in Appendix B that for three centers "non-compactness" can only appear when the asymptotic moduli are at threshold stability, as $\alpha = 0,\ a = b$ is exactly the definition of threshold stability (as introduced in [3]) in different notation[1].

Recall that $\sum_p \langle h, \Gamma_p \rangle = 0$ so if we define $\gamma = \langle h, \Gamma_2 \rangle$ then $\alpha = 0$ implies $\beta = -\gamma$. Using (3.19) we find that

[1] More precisely: $a = b \Leftrightarrow \langle \Gamma_1, \Gamma_2 + \Gamma_3 \rangle = 0$ and $\alpha = 0 \Leftrightarrow \mathrm{Im}(Z_1 \bar{Z}_{2+3}) = 0$.

$$\vec{j} = \frac{\beta}{2} r_{23} \, \hat{z} = \frac{\beta w}{2} \, \hat{z} \tag{5.4}$$

with the positive z axis defined by $\vec{x}_3 - \vec{x}_2$.

Let us consider the other equation

$$\frac{b}{v} - \frac{c}{w} = \beta = 1 \tag{5.5}$$

where we rescale the coords so $\beta = 1$.[2] Note that this forces $c < 0$ if we want to allow $v \to \infty$. b can be positive or negative but we will assume $b < 0$ for concreteness (this will not alter the analysis).

Using the fact that $r_{23}/2 \leq r_{13} \leq \infty$ we find

$$j_+ = \frac{|c|}{2} \qquad j_- = \text{Min}\left(\frac{|c|}{2} - |b|, \, 0 \right) \tag{5.6}$$

with j_+ reached at $r_{13} \to \infty$ and j_- at $r_{13} = |c|/2 - |b|$ (unless this is less than zero[3]).

We can now consider the expectation value of

$$H(r) = \frac{1}{|\vec{r} - \vec{r}_1|} \tag{5.7}$$

The asymptotic behaviour of this function is particularly important in the decoupling limit considered in Section 3.2 since if $r_1 \to \infty$ the classical solutions will no longer be asymptotically AdS$_3$. Thus we would like to check if there are wave-functions for which $\langle H(r) \rangle$ does not decay as r^{-1}. If such states exist they would spoil the asymptotics of our solutions, particularly in a decoupled AdS$_3$ limit, and it would be hard to interpret them physically.

We are interested in studying wavefunctions localized near infinity so we could restrict our attention to the states with the largest angular momentum, $n = j_+ - 1$, but it turns out to be tractable to study more general pure states $\phi = \sum_{n,m} c_{n,m} \psi_{n,m}$ (though we will find contributions from $n < j_+ - 1$ are more strongly suppressed near infinity as suggested by figure 5.1, right). For this state the expectation value is given by

[2] We can permute the centers to force a positive sign since β appears in one of the three constraint equations and $-\beta$ on the other and we are free to use any two of the three.

[3] Since $a = b$, $b = c/2$ corresponds exactly to the beginning of the scaling regime and is, in fact, nothing more than $N = I/2$ in the specific example of D6$\overline{\text{D6}}$D0. Although we work here with $j_- > 0$ it should not matter much since the large n states we consider in this section have little support in the small j regime.

$$\langle\phi|H(r)|\phi\rangle = C\sum_{n',m',n,m}\int_{r_1^-}^{\infty} h(r,r_1)\, f_{n',m',n,m}(r_1)\, dr_1\, d\cos\theta\, d\phi\, d\sigma \tag{5.8}$$

$$h(r,r_1) = \frac{1}{(r^2+r_1^2-2rr_1\cos\alpha(\theta,\phi,\sigma))^{1/2}} \tag{5.9}$$

$$= \begin{cases} \sum_{l=0}^{\infty} C_{(l)}(\theta,\phi,\sigma)\frac{r_1^l}{r^{l+1}} & r > r_1 \\ \sum_{l=0}^{\infty} C'_{(l)}(\theta,\phi,\sigma)\frac{r^l}{r_1^{l+1}} & r < r_1 \end{cases} \tag{5.10}$$

$$f_{n',m',n,m}(r_1) = e^{-2\lambda j_+}(j_+-\lambda j_+)^{j_+-N/2-1}(\lambda j_+-j_-)^{N/2-j_-}g_{n',m',n,m}(\theta,\phi,\sigma)$$
$$\sqrt{(\lambda j_+)(j_+-j_-)+2j_+(1-\lambda)(\lambda j_+-j_-)}\frac{\lambda(r_1)^{2N+1}}{(r_1-b)^2} \tag{5.11}$$

$$\lambda(r_1) = \frac{r_1}{r_1-b} \tag{5.12}$$

Here $N = n+n'$ so $j_- \leq N/2 \leq j_+ - 1$ and $\vec{r}_1 = \vec{r}_{13}$ because $\vec{x}_3$ is at the origin. The function $f_{n',m',n,m} = c^*_{n',m'}\, c_{n,m}\, \psi^*_{n',m'}\, \psi_{n,m}\, dj/dr$ which we re-write in terms of r_1 using $j = \lambda(r_1)J_+$.[4] Since we are only interested in the large r behaviour of this function we can integrate out the angular dependence and also neglect constant factors (both of which have been absorbed into the function g).

The integral splits into two parts given by $r > r_1$ and $r < r_1$. Since $\lambda(r_1) \sim 1 + b/r_1 + \ldots$ for large r_1, in the second region we see that $f(r_1) \sim r_1^{-1-(J_+-N/2)}$ so, after performing the angular integrals, we are left with

$$\sum_{l,k=0}^{\infty} C_{(l,k)} r^l \int_r^{\infty} r_1^{-l-k-2-(J_+-N/2)} \tag{5.13}$$

The expansion in k comes from expanding $\lambda(r_1)$ in powers of r_1^{-1}. Clearly the $r_1 > r$ region only contributes negative powers of r to $\langle H(r)\rangle$.

In the region $r_1 < r$ we cannot expand $\lambda(r_1)$ since r_1 may not be larger than $|b|$ but we can split this integral once more into two regions: $r_- \leq r_1 \leq \tilde{r}$ and $\tilde{r} < r_1 \leq r$ for some $\tilde{r} \gg |b|$. In the first region the integration domain is r-independent so the r dependence is simply r^{-l-1}. In the second region we can repeat the analysis for the $r_1 > r$ integral, expanding λ in r_1^{-1}, and we find a similar greater than r_1^{-1} fall-off.

As was mentioned above, such configurations (with $\alpha = 0$ and $a/b = 1$) lie on walls of *Threshold Stability* discussed in Appendix B. In fact, our computation nicely agrees with the fact that, as follows from the wall crossing formula [9], no states actually decay when crossing such a wall of threshold stability. More loosely speaking, our calculation roughly excludes the possibility of “states running off to infinity”. This is important e.g.

[4]Note that we have absorbed a factor of $(r_1-b)^{-2}$ from the jacobian dj/dr_1 into our definition of $f(r_1)$.

in the consistency of the decoupling limit where the limit itself forces the moduli to a wall of threshold stability for many charge configurations and we do not want this to spoil the asymptotics of the solution. It is also more important in a more general context as unbounded centers do not admit an easy physical interpretation.

5.1.2 Infinitely Deep Throats

As described in Section 3.4.2, for certain choices of charge vectors there is a region in the solution space corresponding to solutions where an infinitely deep throat develops in spacetime [69, 9, 97]. Smooth solutions with arbitrarily deep throats are quite novel and are particularly enigmatic in the context of AdS/CFT. An infinite throat suggests that the bulk excitations localized deep in the throat will give rise to a continuum of states in the dual CFT. This is not consistent with the finite entropy expected from black hole physics and also what we know about the dual CFTs at weak coupling.

To make this connection with the dual $\mathcal{N} = (0, 4)$ CFT we uplift the 4-dimensional solutions to 5-dimensions and take the decoupling limit described in Chapter 3. The quantization procedure described in Chapter 4 will carry on *mutatis mutandis* to the uplifted solutions because they have exactly the same solution space. We will start with a general discussion but then specialize to a working example given by a three center D6$\overline{\text{D6}}$D0 scaling solution where the charge of the D0 center, N, satisfies $N > \langle\Gamma_6, \Gamma_{\bar{6}}\rangle/2$. We begin with some details on the structure of scaling three-center solution spaces since, as alluded to in section 4.3.3, there are some subtle differences in the geometry of scaling and non-scaling solution spaces. After constructing the appropriate wave functions from the Kähler geometry we will use them to estimate the depth of the throat. We will argue that these throats get capped at some scale ϵ and that this also sets the mass gap for the CFT. We will perform an estimate indicating that $\epsilon \sim N/\langle\Gamma_6, \Gamma_{\bar{6}}\rangle$ and argue that the corresponding mass gap in the CFT has the signature $1/c$ behaviour of a long-string sector when ϵ is of order 1.

Note that, although we will work mostly with decoupled $\text{AdS}_3 \times \text{S}^2$ solutions (since here the pathologies associated with these throats is most evident) the qualitative structure of the analysis, with quantum effects capping off the throat, are not particularly sensitive to the asymptotics of the solution so apply more generally.

Quantizing the Three Center Scaling Solutions

As was done in the non-scaling case we must first construct the appropriate polytope for these solutions (see figure 4.1). The only property that differentiates these solution spaces

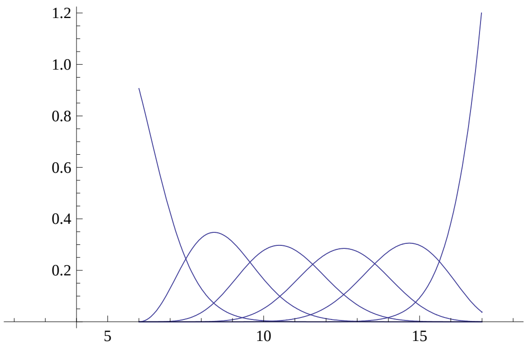

Figure 5.2: *Normalized probability densities $|\psi_{m,n}(j)|$ plotted as a function of j alone (neglecting angular dependence) for some values of n with $J_- \leq n \leq J_+ - 1$. In this example $J_+ = 18$ and $J_- = 6$. Note that as $J_- > 0$ this solution space does not admit scaling solutions.*

is that $j_- = 0$ (this is the *scaling point*). As a result, the associated polytope differs slightly from the non-scaling one; for instance, the first inequality in (4.25) is redundant. This may seem to be a small modification but it actually changes the topology of the solution space, taking the limit from non-scaling to scaling corresponds to a blow down with a non-contractible S^2 vanishing. Furthermore as we will discuss later, the probability densities at the boundary of solution space, $j = j_-$, will have a very different behavior in the $j_- = 0$ case (figure 5.3) than in the $j_- \neq 0$ case (fig 5.2). Note that in the scaling case the polytope doesn't satisfy the smoothness condition; this corresponds to the point $j = 0$ being the well known $\mathbb{Z}_2$ orbifold singularity of $\mathbb{CP}^2_{1,1,2}$. The presence of this orbifold fixed point doesn't seem too essential as the manifold can still be treated by toric orbifold techniques (see appendix F).

Using the coordinates $x = j$ and $y = j\cos\theta$ as in section 4.3.3, the scaling solution's polytope is defined by

$$j_+ - x \geq 0\,, \qquad x + y \geq 0\,, \qquad x - y \geq 0 \tag{5.14}$$

The construction of the complex coordinates is achieved through the function g (see appendix F). We will only need an expression for their norm squared, which is given by

$$|z_1|^2 = \frac{j^2\sin^2\theta}{j_+ - j}\,, \qquad |z_2|^2 = \frac{1+\cos\theta}{1-\cos\theta}\,. \tag{5.15}$$

The wave functions must have a finite norm using the measure $e^{-\mathcal{K}}$, modified by fermionic corrections as discussed in section 4.3.4. $\mathcal{K}$, as usual, stands for the Kähler potential. It is given by

$$\begin{aligned} \mathcal{K} &= j - j_+ \log(j_+ - j)\,, && (5.16)\\ \sqrt{\det\left(\partial_i \partial_{\bar{j}} \mathcal{K}\right)} &= \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\,\frac{\sqrt{2j_+ - j}}{j_+ - j}\,. && (5.17) \end{aligned}$$

Putting everything together gives the following form for $\psi_{n,m} = z_1^n\, z_2^m$

$$\begin{aligned} |\psi_{n,m}|^2 \sim \int e^{-j}\, \sqrt{2j_+ - j}\, (j_+ - x)^{j_+ - 1 - n}\, j^{2n+1} \\ (1+\cos\theta)^{n+(m+1/2)}\, (1-\cos\theta)^{n-(m+1/2)}\, dj\, d\cos\theta \qquad (5.18) \end{aligned}$$

Requiring that the norm is finite imposes the following restrictions

$$0 \le n \le j_+ - 1\,, \qquad -n \le m + 1/2 \le n \qquad (5.19)$$

So the number of states is given by

$$\mathcal{N} = j_+^2$$

Unfortunately, we cannot compare this prediction to wall-crossing because it is not clear how to treat scaling solutions within the framework of the attractor flow conjecture [9]. On the other hand this proves the usefulness of the tools developed here as they provide the only known way to compute the number of BPS states for scaling solutions.

Another important property that is worth mentioning is that the probability density, given by the integrand of (5.18), vanishes at $j = 0$. This suggests that, although classically the coordinate locations of the centers can be arbitrarily close together, quantum mechanically this is not true any more. The probability that the centers sit on top of each other is zero which implies that there is a minimum non-vanishing expectation value for the inter-center distance. Since the depth of the throat is related to the coordinate distance between the centers it follows that the throat will be capped off once quantum effects are taken into account. In the following section we will study this phenomena quantitatively and make some predictions for the depth of the throat and the corresponding mass gap in the CFT.

Macroscopic Quantum Effects

Before we analyse a specific type of scaling solution in some more quantitative detail let us point out some general features of any three center solution space with a scaling point. As long as we are interested in spherically symmetric quantities, e.g. sizes and distances, we can neglect the angular part of (5.18) as it will drop out after normalization. For such questions we can effectively use wave functions on j-space, which depend only on the quantum number n and the value of j_+. The corresponding probability densities are

$$\langle n, j_+; j | n, j_+; j \rangle = \frac{e^{-j} j^{2n+1} (j_+ - j)^{j_+ - n - 1} \sqrt{2j_+ - j}}{\int_0^{j_+} e^{-j} j^{2n+1} (j_+ - j)^{j_+ - n - 1} \sqrt{2j_+ - j}\, dj}\,, \qquad (5.20)$$

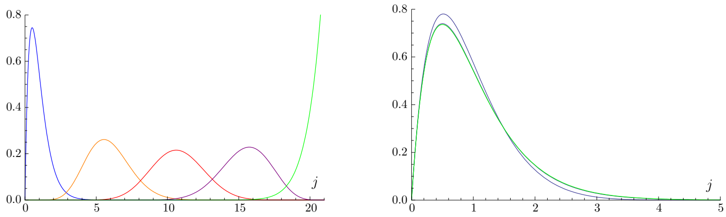

Figure 5.3: (Left) *Plot of the probability densities (5.20) for scaling solutions with* $j_+ = 21$. *The blue, orange, red, purple and green curves correspond to respectively* $n = 0, 5, 10, 15, 20$. **(Right)** *Plot of the probability density corresponding to the lowest state* $n = 0$ *for the values* $j_+ = 5, 50$ *(blue) and* $j_+ = \infty$ *(green) where the curve at* $j_+ = 50$ *is already barely distinguishable from the limiting curve* $j_+ = \infty$ *(see eqn. (5.22)). Note that the probability distribution vanishes at* $j = 0$.

These are plotted for $n = 0, 5, 10, 15, 20$ and $j_+ = 21$ in figure 5.3 (left).

The state that interests us the most is the $n = 0$ state as this has the greatest support near the classical scaling point, $j = 0$. Note that when there is no scaling point, i.e. $j_- \neq 0$, the lowest state, with $n = j_-$, peaks on j_- (see figure 5.2). In the scaling case the lowest state actually has zero support on $j_- = 0$ (see figure 5.3, right). This seems to indicate that the scaling solutions, supergravity solutions where all centers coincide in coordinate space and $j = 0$, are not well defined classical solutions as they correspond to a point in the phase space where no wave function has finite support. To get an idea of how well this classical point can be approximated by a quantum expectation value we calculate $\langle j \rangle$ in the lowest state, $|0, j_+; j\rangle$.

For a scaling solution space characterized by j_+ this expectation value is given by

$$\langle j \rangle_{j_+} = \int_0^{j_+} \langle 0, j_+; j | j | 0, j_+; j \rangle dj = \frac{\int_0^{j_+} e^{-j} j^2 (j_+ - j)^{j_+ - 1} \sqrt{2j_+ - j}\, dj}{\int_0^{j_+} e^{-j} j (j_+ - j)^{j_+ - 1} \sqrt{2j_+ - j}\, dj} \tag{5.21}$$

In general this expression is not analytically tractable. We are, however, particularly interested in the supergravity regime which coincides with $j_+ \to \infty$. In this limit the expression can be simplified considerably by using the well know expression for the exponential, $\lim_{j_+ \to \infty} (1 - \frac{j}{j_+})^{j_+} = e^{-j_+}$, giving

$$\lim_{j_+ \to \infty} \langle 0, j_+; j | 0, j_+; j \rangle = 4 e^{-2j} j\,, \tag{5.22}$$

this is the green curve plotted in figure 5.3 (right). Using this limiting behaviour it is straightforward to calculate that

$$\langle j \rangle_\infty = 1\,. \tag{5.23}$$

In other words, even in the lowest state the expected value of j is one quantum, i.e $\hbar$. Moreover, because the depth of the throat grows very rapidly in the region $j \sim 0$ many macroscopically different configurations sit within the range $0 < j < 1$ so, even though $j \sim 1$ is only one plank unit away from the scaling point the corresponding geometry (expectation value of the metric) is very different.

As further evidence for the formation of a cap we will also compute the behaviour of the harmonic functions appearing in the metric in this same state. To make this computation simpler we will work with the system introduced in section 5.1.1 ($\alpha = 0$ and $a = b$ but now we take $|b| > |c|/2$) and we will also use the notation used in there. We will first be interested in determining the r-dependence of the (expectation values of the) functions

$$h_1(r) = \frac{1}{|\vec{r} - \vec{r}_1|} \qquad h_2(r) = \frac{1}{|\vec{r} - \vec{r}_2|} \tag{5.24}$$

where r_1 and r_2 have the same meaning as in section 5.1.1 ($\vec{r}_3 = -\vec{r}_2$ so we need only work with one of them). Note that the functions $h_i(r)$ appear directly in the harmonics $\{H^0(r), H^A(r), H_A(r), H_0(r)\}$ but which harmonics they appear in depends on the specific form of Γ_i (which we do not fix at this point).

To compute the r-dependence of $h_i(r)$ we proceed very much as in section 5.1.1, and indeed the computation is mostly analogous,

$$h_i(r, r_i) = \begin{cases} \sum_{l=0}^{\infty} C_{(l)}(\theta, \phi, \sigma) \frac{r_i^l}{r^{l+1}} & r > r_i \\ \sum_{l=0}^{\infty} C'_{(l)}(\theta, \phi, \sigma) \frac{r^l}{r_i^{l+1}} & r < r_i \end{cases} \tag{5.25}$$

$$\begin{aligned} \int_0^{j_+} \langle 0, j_+; j | h_i(r) | 0, j_+; j \rangle dj = \frac{1}{L} \sum_l \Bigg[& r^{-l-1} \int_0^r C_{(l)}\, r_i^l\, f(r_i)\, dr_i \\ & + r^l \int_r^{\tilde{r}} C'_{(l)}\, r_i^{-l-1}\, f(r_i)\, dr_i \\ & + r^l \int_{\tilde{r}}^{r_+} C'_{(l)}\, r_i^{-l-1}\, f(r_i)\, dr_i \Bigg] \end{aligned} \tag{5.26}$$

$$f(r_i) = j(r_i)\, e^{-2j(r_i)} \sqrt{2J_+ - j(r_i)}\, \frac{dj(r_i)}{dr_i} \tag{5.27}$$

$$j(r_1) \sim \frac{r_1}{r_1 - b} j_+ \tag{5.28}$$

$$j(r_2) \sim r_2 \tag{5.29}$$

Here we are only interested in the regime $r \ll |b|$ and we define $\tilde{r}$ such that $r < \tilde{r} \ll |b|$. r_+ is defined by $j(r_+) = J_+$. We have also approximated $(1 - \frac{j}{J_+})^{J_+} = e^{-j}$. The factor of $1/L$ above is the normalization of the wavefunction which we will not need in this particular computation. The "$\sim$" in the last two equations reflects an ambiguity by a constant prefactor that depends on the moduli at infinity (which we set to one in section 5.1.1 by rescaling the coordinates).

The integrals in (5.26) can be solved (in terms of Γ-functions) by expanding in $r_1/|b|$ or r_2/J_+ yielding an answer in terms of a power-series in r. The lowest order term in the series is a constant so we find $\langle h_i(r)\rangle \sim a_i + b_i r^\alpha$ with $\alpha > 0$. This implies that the harmonics in the metric have the same small r behaviour so for $r \sim 0$, near the scaling point, we can evaluate the behaviour of the five dimensional metric (3.17) in the decoupling limit and find the metric does not develop a throat.

CUTTING THE THROAT

Finally we would like to translate our insight above, that quantum mechanically one cannot reach the scaling point, and hence no infinitely deep throat develops, into a rough quantitative estimate of a mass gap in the dual CFT.

We will do this in the particular example of the D6$\overline{\text{D6}}$D0 three center solution introduced in section 3.4.4, which is scaling when $N > I/2$. In section 3.4.4 we considered the non-scaling version of these solutions where $N < I/2$ and we considered multiple D0 centers; here we work with a single D0 center and scaling charges, but we will use the notation of that section. Note that, from eqns (3.123)-(3.124), this system is actually a particular instance of the general construction of section 5.1.1 since $\alpha = 0$ and $a = b$ (in the notation of section 5.1.1). Thus our computation of the harmonic functions in the previous subsection applies to this system.

The calculation will proceed in two steps. First, we want to estimate at which scale the quantum smearing cuts off the naive infinite throat. Second, we will translate this scale into a mass gap (in the dual CFT) by analysing a scalar field on a toy model geometry with a throat cut off at this scale. The scale at which we expect a deviation from the naive infinite throat is of the order of the minimum expected inter-center coordinate distance. In the case of the D6$\overline{\text{D6}}$D0 scaling solution $j_+ = \frac{I}{2}$ and, furthermore, the angular momentum is related to the inter-center distances by (see the constraint eqns. (3.123) with $\beta = \frac{1}{4}$ for asymptotic AdS space [3])

$$r_{6\bar{6}} = 8j\,, \qquad r_0 = \frac{8jN}{I-2j} \simeq 8j\,\frac{N}{I}\,, \tag{5.30}$$

where, in the second expression, we have made a large I approximation. As these expressions are linear in j we see that their expectation value in the $n = 0$ state is directly given in terms of $\langle j\rangle_\infty = 1$. So we find that quantum mechanically one expects the throat to be cut of at a scale of order $\epsilon \sim \frac{N}{I} > 1/2$.

A nice check on this estimate is to determine the charge dependence of the constant term in (the expansion of) $\langle h(r)\rangle$ in the $r \sim 0$ limit since, by a small r expansion,

$$h(r) = \frac{1}{|\vec{r}-\vec{r}_i|} = \frac{1}{r_i} + \mathcal{O}(r) \tag{5.31}$$

Note that this leading, r-independent, term feeds directly into the metric at small r so is a very relevant physical quantity to compute. A careful computation of this leading term, taking into account the normalization L in eqn. (5.26), yields

$$\langle h_1(r)\rangle \sim \frac{\gamma\, j_+}{|b|} + \mathcal{O}(r) \tag{5.32}$$

with γ a small number of order one. For the D6$\overline{\text{D6}}$D0 system this gives $\langle \epsilon^{-1}\rangle := \langle r_1^{-1}\rangle \sim \frac{I}{N}$ which confirms our previous computation. This shows that the estimate of ϵ is relatively robust and does not depend very strongly upon which particular quantum expectation value we use to compute it. Moreover, as the metric and solutions are defined via the harmonics this expectation value very directly relates to the quantum expectation value of the metric.

Now that we have understood the charge dependence of ϵ we wish to translate this into a mass gap in the dual CFT. The computation of the mass gap in terms of a scalar wave equation on a capped-throat geometry is somewhat technical and has thus been relegated to appendix E. Here we will quote the final result for the mass gap $\Delta(L_0+\bar{L}_0)$ in terms of ϵ:

$$\Delta(L_0+\bar{L}_0) \sim \frac{\epsilon}{c} > \frac{1}{2c}. \tag{5.33}$$

with $c = p^3 = 6I$, the central charge of the dual CFT. In the regime where $N \approx I/2$ (so the inequality above is saturated) this matches the expectation from the long string picture that the lowest energy excitation in the CFT is of order $1/c$ (see e.g. [26]). Whether the different scaling in the Cardy regime, $N \gg I$, reflects new physics of these solutions or is an artifact of our toy model geometry (see appendix E) would be interesting to explore.

It would also be interesting to find an interpretation of the dependence of the mass gap on the parameter $\epsilon \sim N/I \sim \frac{L_0}{c}$ which seems closely related to $h = (L_0 - c/24)/c$. We will see in Section 5.3.1 that the latter plays an interesting role as an order parameter in phase transitions in the dual CFT. Evidence is presented in Section 5.4.3 that this phase transition is due to a large number of winding modes being turned on. Although this is speculative, it might, itself, hint at a "long string" picture for the CFT at small h. It is interesting that the result (5.33) might also hint at such a picture. In any case these results and speculations only further highlight the importance of better understanding the dual $\mathcal{N} = (0,4)$ CFT (which is at least partially initiated in Section 5.4.3).

5.2 Macroscopic Quantum Fluctuations from AdS/CFT

That the quantum mechanics of scaling solutions would necessarily involve novel features was recognized shortly after their study in the AdS/CFT context as it is in this context that they present the most challenges. The existence of smooth, arbitrarily deep asymptotically AdS throats with low curvature everywhere seems to suggest the existence of a continuous spectrum in the dual conformal field theory. As already mentioned, this would not agree with the fact that large black holes are dual to thermal states and at the same time carry finite entropy. It would also disagree with our knowledge of the spectrum of the D1-D5-P CFT at weak coupling.

As presaged in [97, Section 6] resolving this would require quantum effects that extend across large portions of classical, smooth solutions and this is precisely what we was find in Section 5.1.2 by explicit computation. The emergence of such macroscopic quantum fluctuations can be traced to the fact (first observed in [4]) that the phase space volume of the system computed at weak coupling, for a system of weakly interacting D-branes, does not increase as the branes backreact and generate an infinitely deep throat. This follows naturally from supersymmetry but is nonetheless remarkable as it implies that the quantized "cells" of the BPS phase space stretch across macroscopic volume as an infinite throat forms. As such one might worry that this property is somehow an artifact of the BPS nature of the solutions. Here we would like to present, as supporting evidence for this phenomenon, a generic AdS/CFT based argument which uses some basic properties of the solution space, especially the fact that it is a phase space, but which does not explicitly rely on any supersymmetry.

To make an AdS/CFT based argument we of course require asymptotically AdS solutions (which was not essential for the arguments of the previous section) which can be obtained by taking an appropriate decoupling limit of the solutions. Thus we assume that the total charge $\Gamma = \sum_a \Gamma_a$ has vanishing D6 charge, which allows us to take the decoupling limit and which allows us to generate a family of asymptotically $\mathrm{AdS}_3\times\mathrm{S}^2$ solutions.

The essential observation is that generic harmonics in our solutions can be expanded asymptotically as

$$H = \sum_a \frac{\Gamma_a}{|\vec{x}-\vec{x}_a|} + h = \frac{\Gamma}{r} + h + \mathcal{O}\left(\frac{|\vec{x}_a|}{r}\right) \tag{5.34}$$

where the terms of order zero in x_a generate the base $\mathrm{AdS}_3\times\mathrm{S}^2$ geometry (rather they generate the geometry of an extremal BTZ black hole) and the subleading terms represent a modification of this base geometry. AdS/CFT arguments relate the expectation values of CFT operators in a particular state to subleading terms in a boundary expansion of the geometry dual to the state. In our solutions it is the terms proportional to $|\vec{x}_a|$ that generate these subleading terms in the expansion of the fields. For scaling solutions near the scaling point all the centers can be arbitrarily close to the origin so $|\vec{x}_a| \sim \lambda \ll 1$. As $\lambda \to 0$ the solution develops an infinitely deep scaling throat that closely resembles the naive black hole geometry and solutions in this region all have expectation values proportional to some positive power of λ.

Because the solution spaces we are studying map to a symplectic submanifold of the full phase space they contain both configuration and conjugate momenta variables. Hence we expect that these solutions can be parameterized, in the dual theory, by non-trivial expectation values of both an operator, O, and its conjugate momentum conjugate, π_O. Heisenberg's uncertainty principle, however, implies there will be an intrinsic variance in measuring these expectation values

$$\sigma_O \sigma_{\pi_O} \geq 1 \tag{5.35}$$

The crucial observation is that this bound on the variance is finite and independent of λ so as $\lambda \to 0$ (recall we are measuring expectation values in a state $|\lambda\rangle$ dual to a throat parameterized by λ) there will be some approximate value, λ_c, for which the variance is of the same order as either $\langle O \rangle$ or $\langle \pi_O \rangle$. For such states $|\lambda\rangle$ we can no longer think of the dual geometries as good classical solutions as an observer doing measurements would not be able to distinguish geometries corresponding to the different values of λ and these look macroscopically *very* different.

For instance, if we consider a dipole halo with only one D0 brane then from eqn. (3.125) (and the definition of the coordinates) we see $\lambda \sim j \sim \theta_a$ so the depth of the throat is controlled by the distance between the D6 and the D0. This can be measured in the CFT by measuring $\hat{J}_3$ (referred to as J_0 in Section 3.2.4). The conjugate variable in the bulk is ϕ_a which parameterizes the phase space and also appears asymptotically in the D0 dipole moment, $\vec{d}_0$ (see e.g. eqn. (3.58)). For scaling solutions both of these asymptotic coefficients will be first or higher order λ. Thus both $\hat{J}_3$ and its conjugate in the CFT will have expectation values and also variances of this order (for small enough λ). Even if we take the variance of $\langle \hat{J}_3 \rangle$ to be very small in a state $|\lambda\rangle$ implying a fixed throat depth the corresponding large variance in the dual operator implies that the location of the D0 is smeared in a circle around the origin at the bottom of the throat. Recall that because of the warp factor the centers remain at a fixed, macroscopic, distance apart so the throat ends in a large quantum foam rather than a classical cap.

It would clearly be interesting to explore this argument in more general cases, to make it more quantitative and to examine its validity.

5.3 Demystifying the Entropy Enigma

From the discussion in section 3.3.3, it transpires that the entropy "enigma" is in fact nothing but a supersymmetric version of a well known general instability phenomenon in the (nonsupersymmetric) microcanonical ensemble on $\mathrm{AdS}_p \times S^q$, first pointed out in [98]: Schwarzschild-AdS black holes become thermodynamically unstable once their horizon radius shrinks below a critical value of the order of the AdS radius — at this point it becomes entropically favorable at the given energy to form a Schwarzschild black hole localized on the S^q. Related thermodynamical as well as dynamical instabilities were studied in [99, 100, 101, 102, 103, 104, 105, 106, 107, 108] and other works.

We see something very similar here: when the BTZ black hole radius is lowered below a critical value of the order of the AdS radius, it becomes thermodynamically unstable — at this point it is entropically favorable at the given energy and total charge to form a BMPV-type BPS black hole [109] localized on the S^2, which is precisely the "enigmatic" configuration studied in the Section 3.3.3. This is illustrated in fig. 5.4.

In particular, we see now that the statement that multicentered black holes dominate the entropy in the small h regime is somewhat misleading. From the 4d point of view, the (presumably) dominant solution described in section 3.3.3 is two centered, with one zero entropy, pure fluxed D6 center; a naked timelike singularity. But from the 5d point of view, there is really only one black hole, since the 4d D6 singularity lifts to smooth geometry. Thus, the dominant configuration remains a single black hole — just one that is localized on the sphere.

To the best of our knowledge, this is the first instance of such an instability in a supersymmetric setting. The presence of supersymmetry makes it possible to write down completely explicit solutions, which is not possible in general nonsupersymmetric cases studied before. This might make explorations of this phenomenon as well as its dual CFT description more tractable.

5.3.1 PHASE TRANSITIONS

As suggested by figure 3.4 and the discussion in the previous subsection, the microcanonical ensemble exhibits a phase transition in the $p \to \infty$ limit. By microcanonical ensemble we mean more precisely here the statistical ensemble at fixed total charge $\Gamma = (0, p^A, q_A, q_0)$ and fixed total energy saturating the BPS bound, but variable S^2 angular momentum. Thus we introduce a potential μ dual to say the 3-component J^3. For concreteness we further specialize to the situation of section 3.3.3, putting $q_A = 0$ and $q_0 = -hp^3$.

Let us assume that, as our analysis suggest, the entropy below a critical value $h = h_c$ is indeed dominated by the black hole localized on the S^2, while for $h > h_c$ it is dominated by the BTZ black hole. Since the localized black holes have macroscopic angular momentum, we see that in the limit $p \to \infty$ keeping μ fixed, we get

$$\langle J^3 \rangle = \pm j_*(h) \quad (h < h_c), \qquad \langle J^3 \rangle = 0 \quad (h > h_c)\,, \tag{5.36}$$

where $j_*(h)$ is the angular momentum of the most entropic configuration and the sign is determined by the sign of μ. This is illustrated in fig. 5.5. If we assume either BTZ or single sphere localized black holes dominate, the critical value is $h_c \approx 0.00190622$, and in the large p limit, we have a sharp first order phase transition, with order parameter given by the angular momentum. However as we mentioned before, although we were unable to find any, we cannot exclude the existence of more complicated, more entropic multi-black hole / particle gas configurations which would push up h_c, and perhaps even smoothen the entropy and angular momentum as a function of h, changing the order of the phase transition

We can also consider the "canonical" ensemble, trading $-q_0$ for its dual potential $\beta = 1/T$ while still keeping the q_A fixed (say $q_A \equiv 0$, which for simplicity of exposition we assume for the rest of this section), and keeping the total energy at BPS saturation.[5] As we will see below, in the dual CFT, T has an interpretation as the "left-moving temperature", conjugate to $(L_0)_{\rm cyl} = H = hp^3 = -q_0$ (see section 5.4.1), while the constraint of BPS saturation can be enforced by taking the right-moving temperature $\tilde{T} \to 0$. Although T is strictly speaking not a real temperature, we will use terminology as if it were. The relation between h and T and the free energy are given by the Legendre transform

$$\frac{1}{T} = \frac{\partial S}{\partial H} = -\frac{\partial S}{\partial q_0}\,, \qquad F = H - TS\,.$$

For the BTZ black hole, (3.115) thus gives

$$h(T) = \frac{(2\pi T)^2}{24}\,, \qquad F(T) = -\frac{\pi^2 T^2}{6} p^3\,. \tag{5.37}$$

This means the BTZ black hole charge at thermal equilibrium is $\Gamma(T) = (0, p, 0, -h(T)\,p^3)$. For the localized black holes of section 3.3.3 we get more complicated expressions. The localized black hole charge and entropy in thermal equilibrium are, using the notation (3.101):

$$\begin{aligned} S_2 &= \pi^2 T \frac{(1-2u)^{3/2}\,p^3}{3\,(\pi^2T^2+1)^{1/2}}\,, \\ \Gamma_2 &= \left[1, (1-u), \frac{u^2}{2}, \frac{(1-2u)^{3/2}}{3\,(\pi^2T^2+1)^{1/2}} - \frac{u^3}{6} - \frac{u^2}{2} + u - \frac{1}{3}\right]. \end{aligned} \tag{5.38}$$

The resulting free energies as a function of T are shown in fig. 5.6. Again we see a phase transition in the large p limit: above a certain temperature T_c, the BTZ black hole minimizes the free energy due to its large entropy; below it the spinning global $AdS_3 \times S^2$ vacuum (3.106) (with $J^3 = \pm\frac{p^3}{12}$) takes over, as dumping energy into the reservoir becomes entropically favorable. (Both phases will also contain a thermal gas of particles, since we have coupled the system to a heat bath.) The free energy of the vacuum ($u =$

[5] As in the microcanonical ensemble we still allow the angular momentum J to vary and work at fixed μ, but we will suppress this in the explicit formulae below — its only effect in the end at $p \to \infty$ is to select a low temperature ground state.

$1/2$) is easy to compute as it has zero entropy: $F_{\rm vac} = H_{\rm vac} = -\frac{p^3}{24}$. By equating this with the BTZ free energy we get the critical temperature:

$$T_c = \frac{1}{2\pi} \tag{5.39}$$

(in units of $1/R$).

This phase transition is nothing but (a BPS version of) the Hawking-Page transition [23]. Its existence in a supersymmetric context was observed already in [110], by examining the elliptic genus of the Hilbert scheme of k points on $K3$ and its $\mathrm{AdS}_3 \times S^3 \times K3$ dual. Here we see its physical origin more directly.

Note that again the angular momentum jumps: from $\langle J^3 \rangle = 0$ at $T > T_c$ to $\langle J \rangle = \pm J_{\rm max} = \pm \frac{p^3}{12}$ at $T < T_c$. The AdS-CFT correspondence therefore implies a phase transition in the dual 1+1 dimensional CFT breaking the continuous $SU(2)_R$ symmetry. This is not in contradiction with the Coleman-Mermin-Wagner theorem [111, 112], since there is only a true phase transition in the strict limit $p \to \infty$. At any finite p, the combined free energy is smooth.

In any case, we are led to conclude that BTZ black holes much smaller than the AdS radius in fact do not provide stable classical ($p \to \infty$) backgrounds representing macroscopic (thermodynamic) states in the CFT. This is just as well, as the opposite situation would lead to various paradoxes. For example, according to the philosophy of the fuzzball proposal (see Section 2.2 and [6, 26, 89, 90, 113]), the BTZ black hole, when it exists as a proper classical geometry, should be obtained by coarse graining over all microstates of given energy or temperature, consistent with its interpretation as a purely thermal state [28]. However, when the BTZ black hole is small, it is hard to see how it could be the result of coarse graining over the ensembles of multicentered configurations, which typically extend far beyond the BTZ horizon size.

We end this subsection by giving an alternative way to arrive at the critical temperature (5.39). Let us start from the pure fluxed $\mathrm{D6} - \overline{\mathrm{D6}}$ system studied in section 3.3.2. Now add a number N of D0-branes (which according to (3.54) have to lie on the plane equidistant from the D6 and $\overline{\mathrm{D6}}$). This is essentially the setup of [78] and the Dipole Halos of Section 3.4.4. It was shown there that the D0-branes together with the D6 and anti-D6 can adiabatically[6] collapse into a scaling solution (or abyss) which approaches the single centered D4-D0 black hole arbitrarily closely, if and only if

$$N \geq \frac{p^3}{12} \qquad , \text{ i.e. } \; h = \frac{N - \frac{p^3}{24}}{p^3} \geq \frac{1}{24}\,.$$

This is in fact a direct consequence of the equilibrium constraints (3.54). In the $\mathrm{AdS}_3 \times S^2$ picture, what we have is a gas of gravitons and other massless modes orbiting at constant

[6] By adiabatic we mean here by a evolution process with energy arbitrarily close to the BPS bound.

radius in AdS_3 and at fixed ϕ on the equator of S^2, which can adiabatically collapse into a BTZ black hole if $h > \frac{1}{24}$. From the relation (5.37) between T and h, this is equivalent to $T > \frac{1}{2\pi}$, coinciding with the critical temperature (5.39).

Thus, below the critical temperature T_c, there is a potential barrier preventing adiabatic gravitational collapse of the system under consideration into a BTZ black hole, above T_c, this is not the case. We leave the clarification of the deeper meaning of this coincidence of critical temperatures, and its implications for the fuzzball proposal (for reviews see [26, 113]) to future work.

5.4 INTERPRETATION IN THE $(0,4)$ CFT

We will now discuss the interpretation in the dual CFT of the Entropy Enigma and other phenomena we observed.

5.4.1 TRANSLATION TO CFT

The quantum numbers of the decoupled solutions were given in section 3.2.4. In particular, L_0 and $\bar{L}_0$ were given in (3.82), and we also defined reduced quantum numbers L_0' and $\tilde{L}_0'$ in (3.85). In the regime $L_0' \gg \frac{c}{24}$, the Cardy formula gives the microcanonical entropy of the CFT:

$$S_{\text{Cardy}} = 4\pi\sqrt{\frac{c}{24}(L_0' - \frac{c}{24})} = 4\pi\sqrt{-\frac{\hat{q}_0 p^3}{24}} = S_{BTZ}\,, \tag{5.40}$$

where $c = p^3$, reproducing precisely the BTZ black hole entropy. Note that the regime where sphere localized black holes come to dominate is at $(L_0' - \frac{c}{24})/c \ll 1$; this is the opposite of the Cardy regime.

In both the microcanonical and the canonical ensembles we consider in the previous section, we kept the M2 charge q_A fixed and for simplicity we chose

$$q_A = 0\,. \tag{5.41}$$

We will do this here too. In this case the distinction between reduced and original Virasoro charges disappears, and we have the identifications

$$(L_0)_{\text{cyl}} = L_0 - \frac{c}{24} = -q_0 = hc, \qquad (\tilde{L}_0)_{\text{cyl}} = \tilde{L}_0 - \frac{c}{24} = 0. \tag{5.42}$$

This implies furthermore $H = hc$, explaining our notation $q_0 = -hc$ used in (3.112) and in the definition of the canonical ensemble in section 5.3.1.

The regime of particular interest to us is h small and positive, which is where the phase transitions are expected to occur based on the black hole picture.

5.4.2 Entropy for $L_0 \sim \frac{c}{24}$

There are not too many tools available to determine the number of states in a CFT for $h = (L_0 - \frac{c}{24})/c \to 0$. There is certainly no universal answer to this question, and in addition the answer may depend on moduli and other parameters — after all, it is not a protected quantity. In order for the $\mathcal{N} = (0,4)$ CFT, which is dual to the geometries we have been studying, to accommodate the sphere localized / multicenter solutions with entropy $S \sim p^3 = c$ near $h \to 0$, the number of states at small h in the CFT should grow accordingly. One can view this as a prediction of AdS/CFT for the (presumably strongly coupled) $\mathcal{N} = (0,4)$ CFT.

The simplest possible model where one could investigate this question is in the CFT of c free bosons, which has partition function $Z := \operatorname{Tr} q^{L_0 - \frac{c}{24}} = Z_1^c$ where

$$Z_1 = q^{-\frac{1}{24}} \prod_{i>0} \frac{1}{(1-q^i)} = \frac{1}{\eta(q)}\,. \tag{5.43}$$

Then the coefficient of q^0 can be estimated at large c by saddle point approximation. Parametrizing $q = e^{2\pi i \tau}$:

$$d(0) = \oint e^{c \log Z_1}\, d\tau \approx e^{c \log Z_1(\tau_*)}\,, \qquad \frac{\partial \log Z_1}{\partial \tau}|_{\tau_*} = 0\,. \tag{5.44}$$

The numerical solution to this is

$$\tau_* \approx 0.523524, \qquad \log d(0) \approx 0.176491\, c\,, \tag{5.45}$$

so this indeed gives an entropy of order $c = p^3$ at $h = 0$. Comparing to (3.116), we see that the coefficient is different; of course there was no reason to expect it to be the same, since the coefficient is model dependent. For example, replacing Z_1 with a more general weight w modular form

$$Z_1(q) = a_0 q^b + a_1 q^{b+1} + \cdots\,, \tag{5.46}$$

we can estimate (5.44) by writing $Z_1(\tau) = a_0(-i\tau)^{-w} e^{-\frac{2\pi i b}{\tau}} + \cdots$ which leads to

$$\tau_* \approx \frac{2\pi i b}{w}\,, \qquad \log d(0) \approx (\log a_0 - w(1 + \log(2\pi b/w)))\, c\,. \tag{5.47}$$

For this to be a good approximation we need $e^{\frac{-2\pi i}{\tau_*}} = e^{-w/b} \ll 1$. For the free boson, we have $w = -1/2$ and $b = -1/24$, so this is satisfied and indeed plugging in the numbers gives $\log d(0) \approx \frac{1}{2}(1 + \log\frac{\pi}{6})c$, reproducing (5.45) to very good accuracy.

In addition to similar saddle point approximations, a more refined analysis of the large c growth of $d(0)$ for various modular forms, using the Fareytail expansion, was done in [114], and was in agreement with the simple estimates given here.

Of course, since c is a measure for the number of degrees of freedom, it is hardly a surprise that the entropy for a fixed nonzero amount of energy per degree of freedom $L_0/c = 1/24$ grows linearly in the number of degrees of freedom c. More interesting would be to compute the actual proportionality constant. Despite the model dependence of this number, (3.116) nevertheless suggests a universal number for all CFTs dual to $\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{CY}_3$ in the large c limit:

$$\log d(0) = \frac{\pi}{18\sqrt{3}}\, c\,. \tag{5.48}$$

As mentioned earlier, this universality might however be an artifact of our lack of imagination in finding more entropic configurations.

In theories in which a “long string” picture exists, we can count the number of states in the long string CFT, which typically has reduced central charge $\hat{c} = c/k$ and increased excitation energy $\hat{L}_0 = kL_0$. For k sufficiently large, we can then use Cardy even if the original L_0 was of the order of $c/24$, and we find

$$\log d(0) = \frac{\pi}{6}\, c\,. \tag{5.49}$$

This does not agree with (5.48), but clearly our analyses on both sides are far from conclusive at this point.

To make further progress, it is necessary to delve into the intricacies of the actual dual CFTs. We will initiate this in the next subsection, improving the analysis of [27] by more carefully identifying entropic modes important at small h.

5.4.3 The MSW string

The MSW (0,4) 1+1 dimensional sigma model on $W = \mathbb{R} \times S^1$ arising from wrapping an M5 brane on $W \times P$ with P a very ample divisor has the following massless field content [27, 73, 115]:

- $h^{0,2}(P) \approx p^3/6$ complex non-chiral scalars z^i arising from holomorphic deformations of P.[7]

[7] Consistent with our practice throughout this thesis, we suppress (large p) subleading corrections to various Hodge numbers.

- 3 real scalars $\vec{x}$, the position in $\mathbb{R}^3$
- $b^2(P) \approx p^3$ real scalars from the reduction on P of the self-dual 2-form field b on the M5:
$$b = b^\alpha \Sigma_\alpha \,, \tag{5.50}$$
where $\{\Sigma_\alpha\}$ is an integral basis of the space of harmonic 2-forms $H^2(P)$. In such a basis the scalars are periodic: $b^\alpha \simeq b^\alpha + n^\alpha$, $n^\alpha \in \mathbb{Z}$. Furthermore they have to satisfy the self-duality constraint
$$db^\alpha \wedge \Sigma_\alpha = *_W db^\alpha \wedge *_P \Sigma_\alpha \,, \tag{5.51}$$
which implies there are $b^2_+(P) = 2h^{2,0}(P) + 1 \approx p^3/3$ right-moving ($*_W = +1$) degrees of freedom and $b^2_- = h^{1,1}(P) - 1 \approx 2p^3/3$ left-moving ($*_W = -1$). The left-right split depends on the deformation moduli z^i and the background complex and Kähler moduli.
- $4h^{2,0}(P) + 4 \approx 2p^3/3$ real right-moving fermions ψ^κ. These pair up with the in total $4h^{0,2}(P) + 4$ real right-moving scalars, as required by $(0,4)$ supersymmetry.

Motion of the string is supersymmetric if and only if it is (almost) purely left-moving[8] :
$$z^i(\tau,\sigma) = z^i(\tau+\sigma), \qquad b^\alpha(\tau,\sigma) = b^\alpha(\tau+\sigma) - 2(q\cdot\tilde{J})\tilde{J}^\alpha\,\tau\,. \tag{5.52}$$
Here $q\cdot\tilde{J} = q_A\tilde{J}^A$ with q_A the M2-charge and $\tilde{J} = \tilde{J}^A D_A$ proportional to the Kähler form of X, normalized such that $\int_P \tilde{J}^2 \equiv 1$. Furthermore the components $\tilde{J}^\alpha$ are defined by decomposing $\tilde{J}$ pulled back to P: $\tilde{J} = \tilde{J}^\alpha\Sigma_\alpha$. The reason for the presence of the τ-dependent term on the right hand side is the fact that supersymmetry is nonlinearly realized when $q\cdot\tilde{J}$ is nonvanishing [27], which is related to the fact that $q\cdot\tilde{J}$ is proportional to the imaginary part of the central charge Z, and therefore that a different subset of four supercharges out of the original eight is preserved for different $q\cdot\tilde{J}$. It is also closely related to the difference between L_0 and L_0' as discussed at the end of section 3.2.4.

In addition (5.52) is a solution to the equations of motion if and only if the selfduality constraint (5.51) is satisfied. On the profile $(z^i(s), b^\alpha(s))$, $s \in S^1$ introduced in (5.52) this constraint becomes the anti-selfduality condition
$$\dot{b}^\alpha(s)\,\Sigma_\alpha - (q\cdot\tilde{J})\tilde{J} = -*\left[(\dot{b}^\alpha(s)\,\Sigma_\alpha) - (q\cdot\tilde{J})\tilde{J}\right]. \tag{5.53}$$
The dot denotes derivation with respect to s, and we used the fact that the right-moving contribution in (5.52) automatically obeys the self-duality constraint (5.51). Harmonic

[8] As usual, the the extra winding term in b^α can be written, using $\tau = \frac{1}{2}(\tau+\sigma) + \frac{1}{2}(\tau-\sigma)$ as the sum of left-movers and right-movers, and the left-moving contribution can be absorbed in $b^\alpha(\tau+\sigma)$. We chose for convenience a convention in which the winding term depends on τ only.

2-forms on P are anti-selfdual if and only if they are of type $(1,1)$ and orthogonal to $\tilde{J}$. Following appendix G of [9], the first condition can be written as

$$\dot{b}^\alpha(s)\,\partial_i\Pi_\alpha(z(s)) = 0\,, \tag{5.54}$$

while the second one is

$$\dot{b}^\alpha(s)\,\tilde{J}_\alpha = q\cdot\tilde{J}\,. \tag{5.55}$$

Here $\Pi_\alpha(z)$ is the period of the holomorphic 3-form on a 3-chain with one boundary on the 2-cycle in $P(z)$ Poincaré dual to Σ_α, and J_α is the integral of the Kähler form J over the same 2-cycle.

SUPERSYMMETRIC SOLUTIONS

One could now try to get the BPS spectrum by quantizing this moduli space of supersymmetric configurations. In general however this is a complicated system of coupled equations.

Things simplify when we consider linearized oscillations around some arbitrary fixed point (z_*^i, b_*^α). Because there are about p^3 b^α and $p^3/3$ z^i real degrees of freedom, (5.54)-(5.55) will to lowest order just constrain the b^α to lie on a $2p^3/3$-dimensional plane, while δz^i can oscillate freely. Hence we can think of this as in total p^3 free bosonic modes. At large L_0, these oscillator modes will dominate the entropy, approximately reproducing the BTZ entropy.

In addition, since they are periodic, we can allow the scalars b^α to have nonzero winding number k^α in $H^2(P)$; this corresponds to turning on worldvolume flux on the M5 (and in particular these modes can therefore carry M2 charge). Still at fixed z_*, integrating (5.54) over the S^1 then gives the constraint

$$\partial_i W(z_*) = 0\,, \qquad W(z) := k^\alpha\Pi_\alpha(z)\,. \tag{5.56}$$

For generic z_* and generic integral k^α, this will not be satisfied. Only for k^α in the sublattice L_X of $H^2(P,\mathbb{Z})$ pulled back from the ambient Calabi-Yau X, this will be automatic (because these forms are always integral (1,1)).

Based on this and the fact that in the full M-theory, M2 instantons can interpolate between winding numbers except those in L_X, [27] rejected the possibility of turning on winding numbers except for those in L_X. However, at *special* points z_*, (5.56) *will* have solutions. Indeed these equations can be viewed as a superpotential critical point condition for z^i (formally identical to the one obtained for D4 flux vacua in appendix G of [9]), and as such it will have isolated critical points for sufficiently generic k^α; all z^i have become effectively massive. Integrating (5.55) over S^1 gives the constraint $k^\alpha\,J_\alpha = q\cdot J$. This is automatically satisfied, because the winding modes are exactly the origin of the M2 charge, as they correspond to M5 worldvolume flux; in general one can read off from the WZ terms in the M5-brane effective action that $q_A = \int_P D_A \wedge k^\alpha \Sigma_\alpha$.

So, once we specify a winding vector k^α, the string will still be supersymmetric when located at a critical point $z_*(k)$, and some or all of the z^i zeromodes will be lifted. At the semiclassical level, these are definitely valid supersymmetric ground states — and in fact there is a huge number of them, not quite unlike the landscape of string flux vacua. Instantons might tunnel between them and mix the states quantum mechanically, but this does not mean that they should not be considered; in particular when computing the Witten index, all these semiclassical vacua must be summed over (with signs).

The contribution of these winding modes to $-q_0 = P = (L_0)_{\rm cyl} - (\tilde{L}_0)_{\rm cyl}$ is half the topological intersection product:

$$\Delta P = -\frac{1}{2} Q_{\alpha\beta} k^\alpha k^\beta \,, \qquad Q_{\alpha\beta} := \int_P \Sigma_\alpha \wedge \Sigma_\beta \,. \tag{5.57}$$

If in addition to (5.56) we also set $q \cdot J = 0$ (for example by restricting to the $q_A = 0$ sector), then $k^\alpha \Sigma_\alpha$ is anti-selfdual, and therefore $\Delta L_0 = \Delta P \geq 0$. Moreover, in the notation of Section 3.2.4, we have $\Delta L_0' \geq 0$.

There are more complicated solutions to (5.54) possible, for example when we let the string loop around a nontrivial closed path $z(s)$ in the divisor moduli space and at the same time on some loop in the b^α-torus. This can give rise to complicated twisted sectors. As stressed in [73], there will in general be monodromies $b^\alpha \to M^\alpha{}_\beta b^\beta$ acting on the b-torus when circling around the discriminant locus in the divisor moduli space. Hence we should think of the target space of the string as a quotient of the total space of the b-torus fibration over Teichmüller space by the monodromy group. Closed strings can begin and end on different points identified by this group, leading to twisted sectors and possibly long strings.

Finally, we can form bound states of the localized winding strings described above. For example we can form a bound state of a closed string winding k_1 at some $z_*(k_1)$ and one winding k_2 at $z_*(k_2)$, by connecting them with two interpolating pieces of string. Note though that now the constraint (5.55) becomes important: indeed generically $k_1^\alpha J_\alpha \neq k_2^\alpha J_\alpha$, so the string we just described cannot have constant $\dot{b}^\alpha J_\alpha$ and we do not get a proper supersymmetric solution. It is conceivable however that in some cases at least the string will be able to relax down to a BPS configuration for which $\dot{b}^\alpha J_\alpha$ is constant everywhere.

This is reminiscent of brane recombination. Moreover, note that the condition of having $k_1^\alpha J_\alpha = k_2^\alpha J_\alpha$ corresponds to being on a wall of marginal stability for the two M5-branes represented by the two strings. Hence there is an obvious candidate for the gravitational interpretation of such configurations: they should correspond to the M5-M5 2-centered bound states. It would be interesting to make this more precise.

STATISTICAL MECHANICS

In this subsection we will give a rudimentary analysis of the statistical mechanics of the BPS sector of the MSW string, to see if we can reproduce some of the features we found on the black hole side.

We can roughly model the ensemble of winding and oscillator modes ignoring nonlinearities, say in the $q_A = 0$ sector, by the partition function

$$Z(q) = \operatorname{Tr} q^{L_0 - \frac{c}{24}} = \left(\frac{\vartheta_3(q)}{\eta(q)} \right)^c \tag{5.58}$$

with $c = p^3$. Here the theta function models the winding mode contributions and the eta function the oscillator contributions.[9] By numerical saddle point evaluation, the total entropy and the (entropy maximizing) distribution of it over the oscillator and winding modes at given $h = (L_0 - \frac{c}{24})/c$ can be straightforwardly computed. The result is shown in fig. 5.7. The inclusion of winding modes actually improves the match to the BTZ entropy compared to the most naive model with only free oscillators; it is almost perfect already slightly above the threshold. This can also be checked analytically: Because $Z(q)$ has weight 0, the total entropy computed by saddle point evaluation is exactly $S = 4\pi\sqrt{\frac{h}{24}}c = S_{\mathrm{BTZ}}$; for the free oscillator model, there are corrections.

We also see that at $h = 0$, there is still an entropy of order $c = p^3$, and almost all of it is in the winding modes. There are still no phase transitions in this model of course, since the system is noninteracting.

ANGULAR MOMENTUM AND $SU(2)_R$

Let us turn our attention now to the $SU(2)_R$ R-charge J^3; the S^2 angular momentum on the gravity side, which appeared as an order parameter J^3/p^3 for the phase transition we discussed. The only fields transforming nontrivially under $SU(2)_R$ are (i) the fermions, transforming in the **2**, but they are all rightmoving so cannot be excited except for their zeromodes, and (ii) the position $\vec{x}$ transforming in the **3**, but this represents only three oscillators out of order $c = p^3$, so one expects their contribution to the total R-charge to be negligible in the thermodynamic limit $p \to \infty$ (in the sense of their J^3 having an expectation value growing slower than p^3).

[9] Note that despite the fact that turning on winding modes is generically lifting *zeromodes* of z^i, it is not true that it also lifts the oscillator modes; in the presence of winding, it remains true that (5.54) reduces the number of *local* fluctuation (oscillator) degrees of freedom by $p^3/3$, so at our level of approximation the oscillator mode counting is essentially unaffected by winding: the number of oscillating degrees of freedom remains $p^3/3 + p^3 - p^3/3 = p^3 = c$.

So, where does the large angular momentum, $J = \frac{p^3}{12}$, of the $L_0 = 0$ gravity solution come from then? The answer is from the center of mass zero modes of the string. Since shifting the b^α by constants independent of the string coordinate s corresponds to a gauge transformation, the only physical zero mode space is the deformation moduli space $\mathcal{M}_P$ of P. These bosonic zero modes together with the fermionic ones (which we can have since they are independent of s) will give ground state wave functions in one to one correspondence with harmonic differential forms on $\mathcal{M}_P$. The form number corresponds to fermion number and therefore to R-charge — or in other words the $SU(2)_R$ is identified with Lefschetz $SU(2)_R$ on cohomology (see for example [17] for a pedagogical explanation). This is analogous to how angular momentum is produced in the D4-brane model [9]. Since the moduli space $\mathcal{M}_P = \mathbb{C}P^{p^3/6}$ (where as before we are dropping terms subleading to p^3), this means the $L_0 = 0$ ground states assemble into a spin $J = \frac{p^3}{12}$ multiplet, exactly as expected from the gravity side.

Now, when we turn on some small L_0, we expect from what we observed on the gravity side that J will go down somewhat (see fig. 5.5). We propose the following picture of how this happens on the CFT side. At very small L_0, a small number of winding modes will get turned on. This will typically freeze a small number of the moduli z^i, reducing the moduli space $\mathcal{M}_P$ to a lower dimensional space. The maximal Lefschetz spin always equals half the complex dimension n (this is the spin of the multiplet created by starting with 1 and subsequently wedging with the Kähler form on the moduli space till the volume is reached). Therefore the maximal J will go down. The higher L_0, the more winding modes get turned on, the smaller the dimension of the residual moduli spaces, and the smaller J. Eventually when L_0 becomes sufficiently large, so many winding modes will be turned on that all moduli will generically be frozen, and the expectation value of J becomes zero. This is in agreement with what we observe on the gravity side.

Again, this is only a rudimentary qualitative picture, and in particular too rough to be able to address how phase transitions could arise. Perhaps a variant of the toy models of [116] would be of help to make further progress. A more in depth analysis is left for future work.

THE FIELD THEORY DESCRIPTION OF THE MSW STRING

One puzzle we have encountered several times has to do with the nature of the MSW sigma model which describes the low-energy excitations of the wrapped M5-brane. This sigma model is obtained from a suitable KK reduction of the M5-brane theory over the four-cycle over which the M5-brane is wrapped. Classically, this sigma model is a $(0, 4)$ superconformal field theory, and the target space of the sigma model is the entire moduli space of supersymmetric four-cycles in the Calabi-Yau manifold.

The puzzle is that on the one hand, field theory arguments suggest that this sigma model also describes a quantum $(0,4)$ superconformal field theory which still probes the entire moduli space of supersymmetric four-cycles, whereas the bulk analysis shows that not all M5-brane bound states fit into a single asymptotically $\mathrm{AdS}_3 \times S^2$ geometry, which strongly suggests that a quantum SCFT which captures the entire moduli space does not exist.

The field theory arguments are based on claims in the literature that, unlike $(2,2)$ sigma models, $(0,4)$ sigma models are always finite [117, 118], in the sense that all renormalizations can be absorbed in finite field redefinitions, so that in particular the beta functions vanish and the theory is conformal also quantum mechanically. However there are potential caveats [119], to which in turn some counterarguments have been given in [120]; see also [121]. To the best of our knowledge, this issue remains not fully settled.

Perhaps our results shed some new light on this. As we observed in section 3.2.5 (see also appendix D of [3]) , M5-M5 bound states will not fit in a single asymptotically $\mathrm{AdS}_3 \times S^2$ geometry, but split in two (or more) separated $\mathrm{AdS}_3 \times S^2$ throats. At values of the normalized Kähler moduli Y^A sufficiently far away from the AdS attractor point $Y^A = p^A/U$, they do exist as supersymmetric states of the MSW string, and we suggested a possible explicit MSW string realization of them above. When moving the Y^A to the attractor point, all of these states decay. Hence they cannot be part of the CFT which is dual to a single $\mathrm{AdS}_3 \times S^2$ geometry.

There are therefore, in our view, two possibilities:

1. The MSW sigma model is a quantum SCFT for all values of the Kähler moduli Y^A. If so, it is not equivalent to quantum gravity in asymptotic $\mathrm{AdS}_3 \times S^2 \times X$, and therefore presents a situation very different from the usual AdS-CFT lore. It is not clear to us what the precise new prescription for a correspondence would be in this case.

2. The beta function in fact does not vanish for Y^A different from the attractor point and the Y^A undergo RG flow till they reach the attractor point, an IR fixed point. Along the flow, the constituents of M5-M5 bound states (whose gravity description is of the type studied in appendix D of [3]) decouple from each other; each of them has its own IR fixed point corresponding to an $\mathrm{AdS}_3 \times S^2$.

The second possibility seems much more attractive to us, but would imply that the MSW $(0,4)$ model does undergo RG flow. This need not be in contradiction with the finiteness of $(0,4)$ models, since the relevant non-renormalization theorems assume that the sigma model is weakly coupled and non-singular, and both assumptions are almost certainly violated for the MSW $(0,4)$ model. The latter can become strongly coupled whenever two-cycles in the moduli space shrink to zero volume (similar to what happens in the D1-D5 CFT), and is most likely singular when the four-cycle self-intersects: intersecting M5-branes support extra light degrees of freedom, coming from stretched M2-branes, and these need to taken into account in a proper low-energy description. The classical MSW CFT, however, does not take these additional light degrees of freedom into account, and usually this gives rise to singularities in the incomplete low-energy theory. Finally, the nontrivial interaction between the b^α and z^i modes leading to (5.54), will further complicate the RG flow.

It would be interesting to study this further.

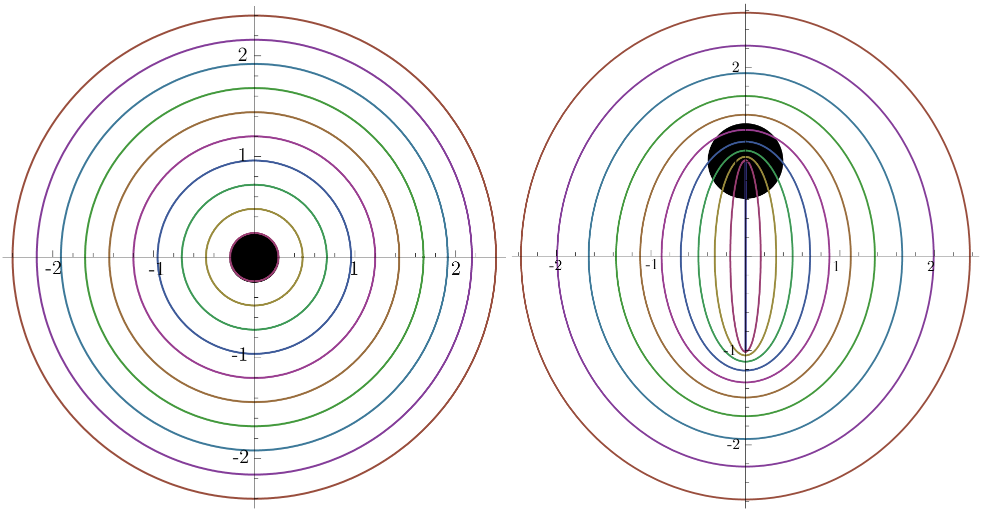

Figure 5.4: *On the left a representation is shown of the single centered 4d black hole; this lifts to the BTZ black hole (times S^2) at the center of AdS$_3$. Surfaces of constant spherical coordinate r in $\mathbb{R}^3$ are indicated — these become the S^2 fibers of AdS$_3 \times S^2$. On the right one of the 2-centered 4d configurations of section 3.3.3 is depicted; this lifts to a BMPV-like black hole roughly localized on the north pole of the S^2 and at the center of AdS$_3$. Surfaces of constant prolate spheroidal coordinate ξ are indicated. As is clear from (3.106), these are the S^2 fibers of AdS$_3 \times S^2$ in the zero size limit of the black hole at the north pole, i.e. the R vacuum. When the black hole has finite size, the metric near it will be deformed to that of a BMPV black hole in 5 dimensions.*

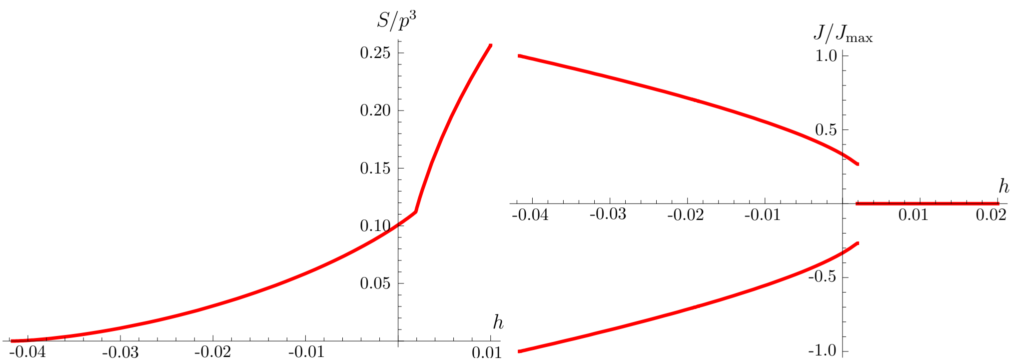

Figure 5.5: *Left: Entropy as a function of h in the limit $p \to \infty$. Right: $J^3/J_{\max}$ as a function of h (the branch depending on the sign of μ), for $p \to \infty$.*

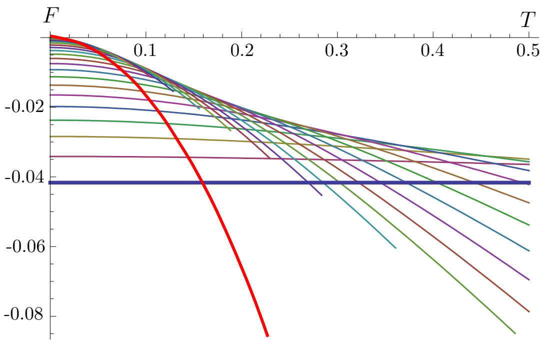

Figure 5.6: *Free energy F as a function of T in the limit $p \to \infty$ for BTZ (fat red line) and sphere localized black holes at different values of u ranging from 0 to $1/2$. The bottom fat blue line corresponds to $u = 1/2$, that is, $AdS_3 \times S^2$ without black holes. The end points of the black hole lines correspond to the 4d equilibrium separation and angular momentum becoming zero, i.e. becoming indistinguishable from BTZ in the asymptotic region.*

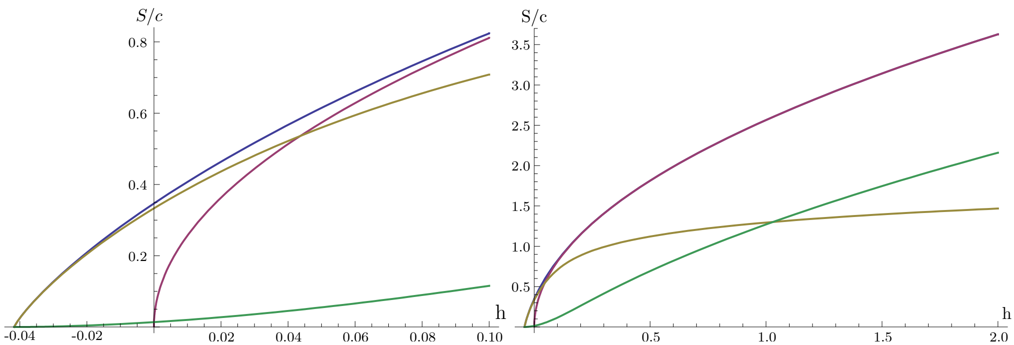

Figure 5.7: *Various entropies as a function of h, for h near 0 (left), and for a larger range of h (right). The blue line is the total entropy derived from (5.58), the yellow line is the entropy in the winding modes, the green line is the entropy in the oscillator modes, and the red line is the BTZ entropy.*

CHAPTER 6

SPECTRUM AND PHASE TRANSITIONS

Various arguments in the literature [69, 9, 97, 78] have suggested that scaling solutions carry vastly more entropy than their non-scaling cousins and may even account for a large fraction of the black hole entropy. This coincides nicely with the fact that scaling solutions are those that can mimic the black hole geometry to arbitrary accuracy. An immediate application of the technology developed in Chapters 3 and 4, relevant to the question of black hole entropy and information loss, is the determination of the entropy coming from scaling solutions.

In this chapter we compute the entropy of a large class of scaling solutions: the Dipole Halo configurations of Section 6.1 in both the scaling ($N \geq I/2$) and non-scaling regime ($N < I/2$). This is almost the most general class accessible using the tools we've developed so far. Unfortunately we will see that the resultant entropy is parametrically smaller than that of black hole with the same total charge.

One might imagine that there are much larger classes of scaling solutions, inaccessible using the technology developed here (or not yet even discovered), that would account for this discrepancy. However, as we will point out, the (leading) entropy coming from these solutions matches that of free gravitons in $\mathrm{AdS}_3 \times \mathrm{S}^2$. This suggests that the solutions we study here constitute the leading contribution to the black hole entropy from supergravity modes and, as a consequence, it is likely that generic black hole states will not be representable entirely in terms of supergravity modes.

While investigating this issue we will encounter some interesting surprises. The change in the leading degeneracy between the non-scaling and scaling regime seems to precisely take into account the stringy exclusion principle [28], which for a chiral primary in the NS sector states that $\tilde{L}_0 \leq c/12$. Moreover there seems to be a phase transition in the restricted ensemble of BPS supergravity states at $N \approx I$ that is somewhat reminiscent of the phase transition of Section 5.3.1 of the fully, unrestricted ensemble of BPS states. Whether deep meaning is to be ascribed to this new phase transition and its mimicry of the full theory is not clear.

6.1 COUNTING DIPOLE HALO STATES

In Section 3.4.4 the "Dipole Halo" system, consisting of a D6-$\overline{\text{D6}}$ pair orbited by a "Halo" of D0s, was introduced as well as a convenient coordinate system on the solution space of such configurations. In this section we will count states using techniques of geometric quantization of the supergravity solution spaces developed in Chapter 4; in the next section we will compare this to the calculation of free supergravity states on AdS$_3$ and see that the two results agree beautifully.

6.1.1 SYMPLECTIC FORM

Let us review the construction of the symplectic form on the solution space. Recall from Chapters 4 and 5 that once the symplectic form on the solution space (parameterized by the locations of centers satisfying (3.123)-(3.124)) has been found it can be used to quantize the system using methods of geometric quantization.

Using the explicit coordinatization in Section 3.4.4 and eqn. (4.9) the symplectic form turns out to be:

$$\Omega = -\frac{1}{4}d\left[2j\cos\theta\, d\phi + 2\sum_a q_a \cos\theta_a\, d\phi_a\right] \tag{6.1}$$

with d denoting the exterior derivative.

The symplectic form (6.1) is non-degenerate on the BPS solution space parameterized by the locations of the centers implying that the latter is in fact a phase space. By virtue of arguments in Section 2.3 and Chapter 4 this space can be quantized in its own right, ignoring the much larger non-BPS solution space in which it is embedded, and from this treatment one might hope to extract information about the BPS states of the full theory (including at least the number of such states).

Note that, as is manifest from our angular coordinatization, the phase space is actually toric with a U(1)$^{n+1}$ action coming from ϕ and the n ϕ_a's. This is a consequence of the fact that the D0's are mutually non-interacting; their sole interaction is via the D6$\overline{\text{D6}}$. As previously mentioned this toric structure is a technical (but not conceptual) requirement for quantization using the methods of Chapter 4.

6.1.2 PHYSICAL PICTURE

As much of the subsequent presentation will be a rather technical treatment of the phase space we would like to lend the reader some intuition. We begin by recalling [40] that the angular momentum carried by these solutions is

$$\vec{J} = \sum_{i<j} \vec{J}_{ij} \,, \qquad \vec{J}_{ij} := \frac{\langle \Gamma_i, \Gamma_j \rangle \vec{x}_{ij}}{2\, r_{ij}} \tag{6.2}$$

where now i, j run over all centers, including the D6s. Thus each pair of centers contributes angular momentum $\vec{J}_{ij}$ to the total. The length of these vectors is fixed to $\langle \Gamma_i, \Gamma_j \rangle / 2$ but their direction is not fixed. The dependence on the intersction product $\langle \Gamma_i, \Gamma_j \rangle$, pairing electric and magenetic sources, reflects the fact that this angular momentum is carried by the electromagnetic field and is due to crossed electric and magnetic fields. Since the D0's have vanishing intersection product with each other there are only $(2n + 1)$ momenta vectors: $\vec{J}_{6\bar{6}}$, $\vec{J}_{6a}$, and $\vec{J}_{\bar{6}a}$.

As we will see, our quantization can essentially be understood as quantizing the direction of these vectors, or more precisely the size of their projection on a given "z-axis", yielding familiar angular momentum multiplets. Naively the phase space of these angular momentum vectors is the direct product of $(2n+1)$ two-spheres and the number of states is just the product of the factors $(2|\vec{J}_{ij}| + 1)$ from each multiplet. The geometric origin of the momenta (i.e. endpoint of multiple vectors fixed to be the same center), however, as well as the constraint equations (3.123)-(3.124) fix the possible relative orientations of the different angular momentum vectors. As a result not all states of the full free angular momentum multiplets are allowed. Rather, the correct phase space is now a more complicated fibration of spheres of varying size and, although intuitively it is still insightful to think of the states as part of "angular momentum multiplets", they now only fill out a constrained subspace of the product of the full multiplets. For instance, since $\vec{J}_{6a}$ and $\vec{J}_{\bar{6}a}$ always end at the same point, their orientation relative to the w-axis is not independent so, rather than two angular momentum multiplets, these vectors yield only a single multiplet (the diagonal multiplet in their free product).

The best way to get some intuition for this is to consider the symplectic form on the phase space of our system. Using the coordinate system of Section 3.4.4 and introducing the notation

$$\vec{J} = J_z\hat{z} + J_y\hat{y} + J_x\hat{x} \tag{6.3}$$

$$\vec{J}_{6a} = \vec{J^a} = J^a_w\hat{w} + J^a_v\hat{v} + J^a_u\hat{u} \tag{6.4}$$

$$\vec{J}_{\overline{6}a} = \vec{J^{\overline{a}}} = J^{\overline{a}}_w\hat{w} + J^{\overline{a}}_v\hat{v} + J^{\overline{a}}_u\hat{u} \tag{6.5}$$

we can cast the dipole halo symplectic form in a more suggestive form

$$\Omega = -\frac{1}{2}\left[dJ_z \wedge d\phi + \sum_a dJ^a_w \wedge d\phi_a + \sum_a dJ^{\overline{a}}_w \wedge d\phi_{\overline{a}}\right]. \tag{6.6}$$

with $|\vec{J^a}| = |\vec{J^{\overline{a}}}| = q_a/2$. Because $\vec{J^a}$ and $\vec{J^{\overline{a}}}$ are related by the location of the D0 they end on the last two terms above can be combined yielding

$$\Omega = -\frac{1}{2}\left[dJ_z \wedge d\phi + 2\sum_a dJ^a_w \wedge d\phi_a\right]. \tag{6.7}$$

If there were no other constraints, the J^a_w would independently be able to take values between $\pm|\vec{J^a}|$. However, as we will discuss below, they have to satisfy the bounds $J^a_w > 0$ and $2\sum_a J^a_w \leq I/2$, leading to a more intricate phase space with a Hilbert space that is no longer a product of "free" angular momentum multiplets. There is also another angular momentum multiplet, coming from the total angular momentum $\vec{J}$, and this gives rise to a full multiplet with $-|J| < J_z < |J|$. The size of $\vec{J}$, however, depends on the J^a (even classically). Thus, each state in the Hilbert state labelled by J^a quantum numbers will be tensored with a J multiplet corresponding to the total $\vec{J}$ associated to its J^a quantum numbers (via $J = I/2 - 2\sum_a J^a_w$).

This is the intuitive physical picture for which we develop a precise mathematical treatment in the next subsection, using the observation above that the phase space is a toric manifold. The upshot is, however, that we are doing nothing more than quantizing angular momentum variables, but ones that are non-trivially connected and constrained.

There is a second physical phenomena that only appears when quantizing the system, which we would like to highlight here. As stressed above, we are in essence quantizing the classical angular momentum of the system. However, when we quantize we need to take into account the intrinsic spin of the particles involved, as was beautifully explained in [17]. As pointed out there, the centers are superparticles containing excitations in various spin states. Due to the presence of magnetic fields, however, the lowest energy BPS state is a spin half state, where energy is gained by aligning the intrinsic magnetic dipole moment with the magnetic field. The situation is sketched for our dipole halo system in figure 6.1. Including these quantum corrections the size of the total angular momentum is given by

$$J = \frac{I-1}{2} - \sum_a \left(q_a \cos\theta_a + \frac{1}{2}\right). \tag{6.8}$$

These spins are especially important when considering classical scaling solutions.

6.1.3 States and polytopes

In Section 3.4.4 we showed that for the D6$\overline{\text{D6}}$D0 system the solution space is a toric manifold. This allows us, in principle, to construct all the normalizable quantum states explicitly. Here, however, we will be less interested in the explicit form of the wave-functions than in their number. In appendix G we show how the number of states can be easily obtained from the combinatorics of the toric polytope. For more information on the technology of geometric quantization of toric manifolds we refer the reader to Appendices F and G.

From the symplectic form (6.1) we read off the coordinates on the polytope

$$y = j \ \cos\theta\,, \qquad\qquad y_a = q_a \ \cos\theta_a \geq 0 \tag{6.9}$$

So we see that the polytope is bounded by the inequalities

$$-\,j \leq y \leq j\,, \qquad 0 \leq y_a \leq q_a \tag{6.10}$$

and furthermore the requirement that the angular momentum is positive

$$j = \frac{I}{2} - \sum_a y_a \geq 0\,. \tag{6.11}$$

It is this last condition that differentiates the non-scaling regime $N = \sum_a q_a < I/2$ from the scaling regime $N \geq I/2$. In the former range the condition (6.11) is redundant in the definition of the polytope as it is automatically satisfied for all values of x_a allowed by the other constraints (6.10). In case $N > I/2$ the constraint (6.11) actually becomes essential and can make some of the constraints (6.10) redundant, although this depends on the specific values of the q_a. What is shared by all the solution spaces in the scaling case is that it is possible to approach the point where all centers coincide arbitrarily closely, which automatically implies that j has to approach zero. When this happens, an infinitely deep scaling throat forms in space-time [17, 69]. For more than a single D0 center there are however different types of solution spaces with a scaling point, depending on the specific values of the charges q_a. We show all the different possible polytope topologies for the case with two D0 centers in figure 6.2, clearly the number of topologies grows very fast with the number of D0-centers.

Given the defining inequalities (6.10) and (6.11) we can use eqn. (G.6) from the appendix (see also the example containing eqn. (G.2)) to see that there is a unique quantum state corresponding to each set of integers (m, m^a) satisfying

$$0 \leq m^a \leq q_a - 1\,, \qquad \sum_a (m^a + \frac{1}{2}) \leq \frac{I-1}{2}\,, \tag{6.12}$$

$$-\left[\frac{I-1}{2} - \sum_a \left(m^a + \frac{1}{2}\right)\right] \leq m + \frac{1}{2} \leq \left[\frac{I-1}{2} - \sum_a \left(m^a + \frac{1}{2}\right)\right] \tag{6.13}$$

The (m, m_a) above are simply quantized angular momenta corresponding to quantizing the angles (θ, θ_a) appearing in (6.10)-(6.11). The half-integral shifts are related to the fermionic nature of the centers as discussed in Section 4.3.4 and the coupling to the extrinsic spin, as explained at the end of Section 6.1.2.

To be precise the constraints above only hold under the assumption that all D0 centers carry different charges, q_a. To relax this assumption we introduce integer multiplicities, n_a, for each charge q_a so that $N = \sum_a n_a q_a$ and $n = \sum_a n_a$. We now have to take into account the quantum indistinguishability of these (fermionic) particles. This translates to taking the appropriate orbifold of the polytope (see [4] for a detailed explanation) or, in terms of (6.13), augmenting the m^a by an additional label i_a running from $1, \ldots, n_a$ and requiring them to satisfy

$$0 \leq m^a_1 < m^a_2 < .... < m^a_{n_a} < q_a\ , \qquad \sum_{a,i} \left(m^a_{i_a} + \frac{1}{2}\right) \leq \frac{I-1}{2} \tag{6.14}$$

$$-\left[\frac{I-1}{2} - \sum_{a,i} \left(m^a_{i_a} + \frac{1}{2}\right)\right] \leq m + \frac{1}{2} \leq \left[\frac{I-1}{2} - \sum_{a,i} \left(m^a_{i_a} + \frac{1}{2}\right)\right] \tag{6.15}$$

These constraints are fermionic, enforcing Pauli exclusion of indistinguishable centers. Note also that they reduce to (6.13) if all the $n_a = 1$.

6.1.4 The D6$\overline{\text{D6}}$D0 partition function

In this section we will count the combined number of supergravity states d_N of all D6$\overline{\text{D6}}$D0 systems with total charge $(0, p^A, 0, \frac{p^3}{24} - N)$. More precisely we will calculate the leading term of $S(N) = \log d_N$ in a large N expansion. We will notice there are two phases depending on the relative value of $I = p^3/6$ and N, separated by a transition at $N = I$. For larger N the appearance of scaling solutions slightly complicates the counting but can still be performed as shown below. What is interesting is that the existence of scaling solutions seems only to become dominant at $N = I$ where a phase transition occurs.

As we previously pointed out, in the scaling regime there is an additional constraint that complicates the polytope and makes the counting of integer points inside slightly more difficult. We will find it convenient not to calculate a fully explicit generating function Z

but rather, since we are only interested in the large N regime, it will be sufficient for us to find the leading term of $\log Z$ in a large N expansion.

The complication in the scaling regime arises because of the second constraint in equation (6.14). To proceed let us introduce the quantity

$$M = \sum_{a,i} \left(m^a_{i_a} + \frac{1}{2} \right) , \tag{6.16}$$

As the $m^a_{i_a}$ are the discrete analogues of the classical $q_a \cos\theta_a$ the interpretation of M is as the amount of angular momentum carried by the D0 centers (which, by the integrability constraints (3.123)-(3.124), is always opposite in direction to the angular momentum carried by the D6$\overline{\text{D6}}$ pair):

$$M = \frac{I}{2} - \frac{1}{2} - J\,. \tag{6.17}$$

Both the $\frac{1}{2}$ in the above formula and in (6.16) arise due to the spin contributions to the quantum mechanical angular momentum (see the end of section 6.1.2).

Now if we succeed in calculating the degeneracy $d_{N,M}$ as a function of M, then the full degeneracy will be

$$d_N = \sum_{M=1/2}^{(I-1)/2} d_{N,M} \tag{6.18}$$

The full degeneracy will clearly be less than $I/2$ times $d_{N,M'}$ where M' is the value of M which maximizes $d_{N,M}$. Thus instead of calculating the sum it will be sufficient for us to find the M' that maximizes $d_{N,M}$ because

$$S(N) = S(N, M') + \Delta S\,, \tag{6.19}$$

where we have defined

$$\Delta S = \log \sum_{M=1/2}^{(I-1)/2} e^{S(N,M)} - S(N, M') \leq \log I\,, \tag{6.20}$$

so, as long as the leading entropy is a power-law (rather than a logarithm) in the charges, we can find the leading term in $S(N)$ by calculating $S(N, M)$ and maximizing over M.

As we will now show it is not too hard to calculate a generating function for $d_{N,M}$

$$\mathcal{Z}(q, y) = \sum_{N,M}^{\infty} d_{N,M}\, q^N\, y^M\,. \tag{6.21}$$

Note that this does not reduce to a generating function for d_N by setting $y = 1$ as in this generating function we sum over $M = 1, \ldots, \infty$ while in the case of interest the range of M is restricted.

Let us derive an expression for (6.21) by approximating it in a few steps. A first key ingredient is that for a partition of $N = \sum_a n_a q_a$, one has

$$0 \leq m_1^a < ... < m_{n_a}^a < q_a \, , \qquad M = \sum_{a,i} \left(m_i^a + \frac{1}{2} \right) \tag{6.22}$$

Forgetting for the moment about N, the above relation is just a fermionic partition of M. This is given by

$$\mathcal{Z}_{ferm} = \prod_{l \geq 1} \left(1 + y^{l-1/2} \right) \tag{6.23}$$

We need to reintroduce the information about N. To do so remember that the sole role of the partition of N is to specify the number of m_i^a above. Keeping this key point in mind we proceed in two steps. First assume that we have n centers with the same charge k only ($N = nk$), then it is easy to see that the appropriate modification of $\mathcal{Z}_{ferm}$ (6.23) is

$$\mathcal{Z}_{int} = \prod_{1 \leq l \leq k} \left(1 + q^k y^{l-1/2} \right) \tag{6.24}$$

This comes about because in expanding the expression above the number of centers in each term is simply the number of q^k that appear in it. The product over possible l is then a reflection of the constraint (6.22). Now to generalize to an arbitrary partition of N we take a product of the above expression over all possible $k \geq 1$. This yields the *core* generating function

$$\mathcal{Z}_0 = \prod_{k \geq 1, 1 \leq l \leq k} \left(1 + q^k y^{l-1/2} \right) \tag{6.25}$$

To get the actual generating function we include the contribution from m in equation (6.15). The generating function is then

$$\mathcal{Z} = \left(I - 2y\partial_y \right) \mathcal{Z}_0 = \left(I - 2y\partial_y \right) \prod_{k \geq 1, 1 \leq l \leq k} \left(1 + q^k y^{l-1/2} \right) \tag{6.26}$$

In evaluating the leading contribution to the entropy we can neglect the overall multiplicative factor because it will be subleading. Thus we focus on $\mathcal{Z}_0$.

6.1.5 The entropy and a phase transition

As is familiar from thermodynamics we can study the large energy regime by evaluating the partition function at large temperature. We introduce the potentials β and μ through

$$q = e^{-\beta} \, , \qquad y = e^{-\mu}$$

and can then look for the behavior of the entropy for $\beta, \mu \ll 1$.

$$
\begin{aligned}
\log \mathcal{Z}_0 &= \sum_{k\geq 1, 1\leq l\leq k} \log\left(1 + q^k y^{l-1/2}\right) \\
&= \sum_{n\geq 1}\left(\frac{(-1)^{n+1}}{n}\left[\sum_{k\geq 1} q^{nk}\left(\sum_{l=1}^{k} y^{n(l-1/2)}\right)\right]\right) \\
&= \sum_{n\geq 1}\left(\frac{(-1)^{n+1}}{n}\,\frac{y^{n/2}}{1-y^n}\left[\sum_{k\geq 1} q^{nk}\left(1-y^{nk}\right)\right]\right) \\
&= \sum_{n\geq 1}\left(\frac{(-1)^{n+1}}{n}\,\frac{q^n\, y^{n/2}}{(1-q^n)\,(1-q^n\, y^n)}\right) \\
&\sim \left(\sum_{n>1}\frac{(-1)^{n+1}}{n^3}\right)\frac{1}{\beta\,(\mu+\beta)} =: \frac{\alpha}{\beta\,(\mu+\beta)} \qquad (6.27)
\end{aligned}
$$

with $\alpha = \frac{3}{4}\zeta(3)$. Using the above relation we find

$$
N = -\partial_\beta \log \mathcal{Z}_0 \sim \frac{\alpha(\mu+2\beta)}{\beta^2\,(\mu+\beta)^2} \qquad (6.28)
$$

$$
M = -\partial_\mu \log \mathcal{Z}_0 \sim \frac{\alpha}{\beta\,(\mu+\beta)^2} \qquad (6.29)
$$

From the equations above it follows that the approximation is valid for $N, M \gg 1$, which is exactly the regime we are interested in. Furthermore the relative size between M and N is determined by the ratio μ/β as

$$
N/M = 2 + \frac{\mu}{\beta}\,. \qquad (6.30)
$$

The entropy in the large M, N regime then reads

$$
S(N,M) = -\log \mathcal{Z}_0 + \beta N + \mu M \sim \frac{\alpha}{\beta\,(\mu+\beta)} \sim \left(\alpha\, M\, [N-M]\right)^{1/3}\,. \qquad (6.31)
$$

Maximizing $S(N,M)$ over M in the range[1] $1/2 < M < I/2$ we find that

$$
S(N) = \begin{cases} \left(\alpha \frac{N^2}{4}\right)^{1/3} & \text{if } N \leq I \\ \left(\alpha\, \frac{I}{2}(N - \frac{I}{2})\right)^{1/3} & \text{if } I \leq N \end{cases} \qquad (6.32)
$$

The most entropic configuration always has $M' = N/2$ until $N = I$ and then the bound (6.11) restricts $M' = I/2$. Thus most entropy is realized by low angular momentum

[1]Note that we are interested in the large charge limit $I \gg 1$, so throughout the paper we will often neglect quantum mechanical shifts of $1/2$ to I

states (remember $J \sim I/2 - M$) and, deep in the scaling regime where $N > I$, most of the entropy is given by the $j = 0$ states.

The saddle point approximation used to obtain eqn. (6.32) is only valid for charges $N \lesssim I^2$ because the discussion above shows we are interested in $M \approx \frac{I}{2}$ and in that regime $N \gtrsim I^2$ is not consistent with $\mu, \beta \ll 1$, as can be seen from (6.28)-(6.29). We will presently focus on the regime $N \gg I$ which is still consistent (in the large charge regime) so long as their ratio does not scale with I.

For $N \gg I$ Cardy's formula implies the leading entropy of the associated black hole grows as [27]

$$S_{BH}(N, I) \sim 4\pi\sqrt{\frac{N\,I}{4}} \tag{6.33}$$

Thus the D6$\overline{\text{D6}}$D0 configurations we are considering do not exhibit the correct growth of entropy as a function of the charges to dominate the black hole ensemble, especially for large charges they are parametrically subleading.

Associated with the change from the first to the second line of (6.32) appears to be a second order phase transition occurring at $N = I$. In this phase transition we seem to move from an asymmetric phase, $\langle j \rangle \neq 0$, to a symmetric phase $\langle j \rangle = 0$. It is not immediately clear that any physical meaning should be ascribed to this "phase transition" since these configurations are not the dominant constituents of this sector of the BPS Hilbert space. Curiously, however, this seems to mirror the phase transition of Section 5.3. Although the latter was analyzed for different constituents centers, if we simply equate the total charges of the two systems then the critical point of Section 5.3 would be at $N \approx I/4$ and would correspond to a (first-order) transition from a phase with $\langle j \rangle \neq 0$ to a $\langle j \rangle = 0$ phase as N increases (note that here there is a discontinuous jump in $\langle j \rangle$). It is both curious and interesting that the set of states we obtained, while relatively sparse in the overall Hilbert space, nonetheless exhibits a non-trivial phase structure that even seems to qualitatively share some of the structure of the full theory.

In the regime $N \gg I$ of [27], the number of states we obtained was substantially smaller than total number of BPS states of the conformal field theory. One may therefore wonder whether other solutions of supergravity exist with the same asymptotic charges and which could account for the missing states, or whether this is the best supergravity can do. Such additional solutions could look like complicated multi-centered solutions of the type we have been considering, or be of an entirely different form. To address this question we will now compute the spectrum and degeneracy of a gas of free supergravitons in $\text{AdS}_3 \times \text{S}^2$. As we will argue, this will provide an estimate for the maximal number of states we might expect to be obtainable from supergravity. It turns out that this computation yields a result whose asymptotic expansion agrees precisely with the number of D6$\overline{\text{D6}}$D0 states, which supports the claim that the supergravity does not give rise to significantly more states in addition to those that we described.

6.2 Free supergravity estimate

In the previous section we calculated the number of BPS states in a given D4D0-charge sector that can be associated to configurational degrees of freedom of a D6$\overline{\mathrm{D6}}$D0 system of that same total charge. As we pointed out, there is an exponential number of states leading to a macroscopic statistical entropy. However the entropy scales with a different power of the charges than the D4D0 black hole entropy, making it parametrically subleading in the large charge supergravity limit. In other words, although we found very many D6$\overline{\mathrm{D6}}$D0 states the corresponding single center black hole still has exponentially more of them, indicating that these are not generic states of the black hole.

One might still wonder, however, if this is due to our restriction to a specific set of smooth multicenter solutions and if perhaps a larger number of states can be found by quantizing more complicated multicentered configurations. In this section we will give some nontrivial evidence that this is **not** the case and that the black hole degrees of freedom have to be sought outside of supergravity. An example of such states could be those of the proposal [122, 123, 78, 124] or the possibly related setup of [125, 126]. Roughly speaking the degrees of freedom in these pictures seem to reside in non-abelian D-brane degrees of freedom; see also [35].

The approach we take to get a "bound" on the degrees of freedom coming from supergravity states is to exploit the fact that both the D4D0 black hole and the D6$\overline{\mathrm{D6}}$D0 system (and its generalizations) can be studied in asymptotically AdS space via the decoupling limit of Section 3.2. In this context, the counting of the previous section corresponds to counting backreacted supergravity solutions with the same asymptotics as the D4D0 BTZ black hole, whereas in this section we will simply count free supergravity modes in empty AdS. The advantage of working in this limit, where the supergravity fields become free excitations around a fixed $\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{CY}_3$ background, is that it becomes relatively easy to count them. Free supergravitons organize themselves in representations of the (0,4) superconformal isometry algebra, and we merely need to determine the quantum numbers of the highest weights of the representations. This can be done following e.g. [127, 128] by performing a KK-reduction of eleven dimensional supergravity fields on the compact $\mathrm{S}^2 \times \mathrm{CY}_3$ space[2] to fields living on AdS_3. The supergravity spectrum can then be determined using pure representation theoretic methods, in terms of the massless field content of the KK reduction of M-theory on the Calabi-Yau manifold.

[2]Note that we will assume the size of the CY_3 to be much smaller than that of the S^2 so that we will only consider the massless spectrum on the CY, while keeping track of the full tower of massive harmonic modes on the sphere.

6.2.1 SUPERCONFORMAL QUANTUM NUMBERS

We want to compare the number of states we found by counting the possible configurational degrees of freedom of a D6$\overline{\text{D6}}$D0 system to the number of chiral primaries given by KK reduction of 5d supergravity in the free field limit. To make this comparison as clear as possible let us first translate the conserved four dimensional charges of the solutions, as presented in the previous section, to quantum numbers under the (0,4) superconformal isometry algebra of the AdS$_3\times$S^2 background we consider here. Such a dictionary was derived in Section 3.2.4 and can be straightforwardly applied to the D6$\overline{\text{D6}}$D0 case. The map from supergravity to CFT quantum numbers is (recall that $c = 6I$)

$$L_0 = N\,, \quad \tilde{L}_0 = \frac{I}{4}\,, \quad J_3 = -J\,. \tag{6.34}$$

States with these quantum numbers are Ramond ground states, with minimum eigenvalues under $\tilde{L}_0$, as expected for BPS states. The calculation of the KK-spectrum on AdS$_3$, however, is most naturally phrased in the NS sector and thus we would like to work in this sector. Thus we relate the charges (6.34) by spectral flow [129] in the right moving sector to the charges of the corresponding states in the NS-sector. Performing the spectral flow explicitly (as in eqn. (3.109)) we find

$$L_0 = N\,, \quad \tilde{L}_0 = \frac{I}{2} - J\,, \quad J_3 = \frac{I}{2} - J\,. \tag{6.35}$$

As expected the BPS states manifest themselves in the NS sector as chiral primaries, satisfying the condition $\bar{L}_0 = J_3$. The well known unitarity bound [129] on the R-charge of chiral primaries implies a bound on the range of the 4d angular momentum:

$$0 \leq J \leq \frac{I}{2} \tag{6.36}$$

From the results of the previous section it is clear that the D6$\overline{\text{D6}}$D0 configurations satisfy this bound. This bound was first observed to have consequences for AdS$_3$/CFT$_2$ in [28], where it was called a stringy exclusion principle. As was argued there, it has to be imposed by hand on the free supergravity spectrum. What is perhaps surprising is that in the fully interacting supergravity theory the bound seems to emerge dynamically as it follows (at least for the D6$\overline{\text{D6}}$D0 system) from the integrability equations (3.123) which are essentially a consequence of the BPS equations of motion. We have no solid proof of this, but we were unable to find other multicentered supergravity configurations that violate the bound, even with flat space asymptotics where there is no direct connection to the exclusion principle in the CFT.

It is interesting to note that by (6.17) and (6.35) we see that for the D6$\overline{\text{D6}}$D0 system $\tilde{L}_0 = M + 1/2$ and that indeed also the bound on M, observed in the previous section, follows directly from the unitarity bound discussed above. Using the identification of M and $\tilde{L}_0$, we can write the following analogue of the generating function (6.21):

$$\mathcal{Z} = \mathrm{Tr}_{\mathrm{NS,BPS}}(-1)^F q^{L_0} y^{\tilde{L}_0 - 1/2}. \tag{6.37}$$

Some remarks are in order. First we would like to point out that, for computational simplicity, we will calculate, in this section, an index rather than an absolute number of states, the difference with (6.21) being an explicit insertion of $(-1)^F$. As one can see explicitly from the derivation below, the difference between the absolute number of states and the index will be only affect the numerical coefficient of the entropy, not its functional dependence on the charges. Second, note that at $y = 0$ the above index coincides with the standard elliptic genus for this theory.

6.2.2 The spectrum of BPS states

To calculate the degeneracies we are interested in, we need to enumerate the possible BPS states of linearized (free) supergravity on $\mathrm{AdS}_3 \times \mathrm{S}^2$. It is often easier to enumerate these states via their quantum numbers in the CFT so we will use this language.

As we only have supersymmetry in the right moving sector, there are no BPS constraints on the left moving fields and thus all descendants of highest weight states will appear. The right-moving sector has $N = 4$ supersymmetry and BPS states must be chiral primaries of a given weight. As a consequence, and as was shown in detail in e.g [127, 128, 130, 131], the full BPS spectrum can be written in the form[3]:

$$\{s, \tilde{h}\} = \oplus_{n \geq 0} \, (L_{-1})^n \, |\tilde{h} + s\rangle_L \otimes |\tilde{h}\rangle_R \tag{6.38}$$

where $|h\rangle_L$ are highest weight states of weight h of the left-moving Virasoro algebra and $|\tilde{h}\rangle_R$ are weight $\tilde{h}$ chiral primaries of the right-moving $\mathcal{N} = 4$ super-Virasoro algebra.

Each field of five dimensional supergravity gives rise to a set of BPS states and their descendants after KK-reduction, where $\tilde{h}$ essentially labels the different spherical harmonics, while n labels momentum excitations in AdS_3 and s the spin of the particle. It was shown in [127, 128, 130, 131] that, given the precise field content of 5d $\mathcal{N} = 1$ supergravity, the reduction on a 2-sphere gives the set of quantum numbers shown in table 6.1. Notice that the quantum numbers $\{s, \tilde{h}\}$ are of the form $\{s, \tilde{h}_{\min} + m\}$, and for each such set the partition function (6.37) has the following form

$$Z_{\{s, \tilde{h}_{\min}\}} = \prod_{n \geq 0} \prod_{m \geq 0} (1 - y^{m + \tilde{h}_{\min} - 1/2} q^{n + m + \tilde{h}_{\min} + s})^{(-1)^{2s+1}} \tag{6.39}$$

[3]Furthermore, to be fully precise we should point out that there remain so called singleton representations, but, for our purposes, we can ignore them as one can show they only contribute to subleading terms in the entropy in the regime studied in the last section: $N, I \gg 1$ and $N \ll I^2$.

5d origin	number	$\{s, \tilde{h}\}$-towers
hypermultiplets	$2h^{1,2}+2$	$\{\frac{1}{2}, \frac{1}{2}+m\}$
vectormultiplets	$h^{1,1}-1$	$\{0, 1+m\}$ and $\{1, m\}$
gravitymultiplet	1	$\{-1, 2+m\}$, $\{0, 2+m\}$, $\{1, 1+m\}$ and $\{2, 1+m\}$

Table 6.1: *Summary of the spectrum of chiral primaries on AdS$_3$. The states are organized in towers of the form (6.38), the number of such towers and their characteristics are determined by the properties of the original theory and the details of the reduction. In the above table, m is an arbitrary nonnegative integer.*

with the total partition function given by a product of such factors.

To extract the large N degeneracies we proceed as in (6.27) and calculate the free energy corresponding to this partition function. We then evaluate it in the $\beta, \mu \ll 1$ limit ($q = e^{-\beta}, y = e^{-\mu}$):

$$F_{\{s,h_{\min}\}} = (-1)^{2s} \sum_{n\geq 1} \frac{q^{n(\tilde{h}_{\min}+s)} y^{n\tilde{h}_{\min}}}{n(1-q^n)(1-y^n q^n)} \tag{6.40}$$

$$\approx \frac{(-1)^{2s}\zeta(3)}{\beta(\beta+\mu)} \tag{6.41}$$

Note that, as might have been expected, at high temperatures only the statistics of the particles matter, as $h_{\min}$ and s only change the lowest states of the towers. The total free energy is now the sum over all different towers. Using table 6.1 we find that

$$F \approx \left[-(2h^{1,2}+2) + 2(h^{1,1}-1) + 4\right] \frac{\zeta(3)}{\beta(\beta+\mu)} = \chi \frac{\zeta(3)}{\beta(\beta+\mu)} \tag{6.42}$$

where we used the definition of the Euler characteristic χ of the CY$_3$. Finally we can do a Legendre transform to obtain the entropy. This is completely analogous to (6.31) and the result is

$$S \approx (\chi\zeta(3) M(N-M))^{1/3} \tag{6.43}$$

This result is equivalent to (6.31) and maximization with respect to M proceeds analogously, again leading to the result

$$S(N) = \begin{cases} \left(\chi\zeta(3)\frac{N^2}{4}\right)^{1/3} & \text{if } N \leq I \\ \left(\chi\zeta(3)\,\frac{I}{2}(N-\frac{I}{2})\right)^{1/3} & \text{if } I \leq N \end{cases} \tag{6.44}$$

This might look somewhat unfamiliar when compared with other calculations of the elliptic genus, e.g [15, 75]. This is because those calculations were all performed in the regime $N \ll I$ where the unitarity bound on the spectrum can be ignored. It is exactly around $N \approx I$ that this bound starts to be relevant leading to a different, slower, growth of the number of states in the regime $I \ll N$. Such a behavior was also seen in the computation of the elliptic genus in [132].

Note that once more our computation above only applies for $N \lesssim I^2$ as the asymptotic form of the free energy is essentially the same as that of the dipole halo system.

6.2.3 COMPARISON TO BLACK HOLE ENTROPY

As we have seen, calculating the number of free supergravity states at fixed total charge in the large N, M limit proceeds rather analogously to the counting of section 6.1.5 and, more importantly, we found a precise match between the leading contributions, up to an overall prefactor.

It is not hard, however, to see that even this prefactor can be made to match. In the previous subsection we focussed on the 4 dimensional degrees of freedom of the D6$\overline{\text{D6}}$D0 -system ignoring the fact that the D0-branes bound to the D6$\overline{\text{D6}}$ still have degrees of freedom in the internal CY_3 manifold. These internal degrees of freedom can be quantized via a 0+1 dimensional sigma model[4] on the CY. The BPS states of this sigma model correspond to the cohomology of the Calabi-Yau with even degree mapping to bosonic states and odd degree to fermionic states. Thus there are exactly χ BPS states per D0 when counted with the correct sign, $(-1)^F$. Including this extra degeneracy in the calculation of section 6.1.5 will lead to a match with (6.44), including the prefactor.

That the two calculations provide the same amount of states is non-trivial, since earlier we restricted ourselves to counting only states realized as a D6$\overline{\text{D6}}$D0 system, while in the second calculation we count all free supergravity states in $\text{AdS}_3\times\text{S}^2$ with given momentum. This suggests that indeed the leading portion of supergravity entropy is realized as D6$\overline{\text{D6}}$D0 configurations once backreaction is included. This is a very strong result as clearly one can think of many, much more complicated, smooth multicenter configurations with the same total charge. Furthermore, we learn from these calculations that the number of such states is parametrically smaller than the number of black hole states. This seems to strongly indicate that the generic black hole state is associated to degrees of freedom beyond supergravity.

From another perspective, however, the match between the free regime and the D6$\overline{\text{D6}}$D0 entropy is not so surprising. If we consider first a D6$\overline{\text{D6}}$ bound state we can use a coordinate transformation from [78] (see also Section 3.3.2) to map this to global $\text{AdS}_3\times\text{S}^2$. Thus we can think of the D6$\overline{\text{D6}}$ as simply generating the empty AdS background. Recalling that D0 branes lift to gravitational shock waves in 5-dimensions one might already have anticipated that counting D0's in the D6$\overline{\text{D6}}$ background is closely related to counting

[4] In this simplistic model we neglect more complicated interactions coming from strings stretched between the D0's and the D6's in the CY.

free gravitons on an $AdS_3 \times S^2$ background. What makes the result non-trivial is that interactions are apparently not terribly relevant when counting BPS states, but then again the D0's only interact very indirectly with each other. We might, therefore, wonder if more exotic configurations, such as the supereggs of [78] or the wiggling rings of [125, 126], are perhaps not captured by the free theory and hence not subject to the bound we find above. The problem with this is that we can compute not only the entropy but also the index in both regimes and they exhibit the same leading growth. If additional supergravity configurations are to generate parametrically more states this would either require very precise cancellations (so that the index is very different from the number of states) or a phase transition at weak coupling (a phase transition in g_s, not the $N = I$ transition discussed above). Even if many states would cancel in the index, one would still need to explain why they become invisible in the limit in which interactions are turned off.

In the above, we have only counted multiparticle BPS supergravitons in 5d supergravity. It is conceivable that additional degrees of freedom could be obtained by allowing fluctuations in the Calabi-Yau as well. For example, as we discussed, D0-branes carry an extra degeneracy corresponding to the harmonic forms on the Calabi-Yau. Though this can contribute a finite multiplicative factor to the entropy, it does not change the functional form. In addition, 5d supergravity does include all massless degrees of freedom that one gets from the reduction on the Calabi-Yau, and the other massive degrees of freedom generically do not contain any BPS states.

One might also worry that multiparticle states, which in the free theory are not BPS, become BPS once interactions are included. Though this is a logical possibility, such degrees of freedom would not contribute to the index, and therefore the estimate of the index remains unaffected by this argument.

Finally, we notice that it is possible to do similar computations for $AdS_3 \times S^3$, which leads to the result that for $N \lesssim I$, $S \sim N^{3/4}$, while for $I \ll N \ll I^2$, $S \sim I^{1/2} N^{1/4}$. It would be interesting to reproduce these results by counting solutions of 6d supergravity as well.

6.2.4 The stringy exclusion principle

It is also somewhat intriguing to see that in the "free theory" we recover the phase transition noted in the previous section only after imposing (by hand) the CFT unitarity bound suggesting that the latter is taken into account by our scaling solutions. A priori this sounds somewhat mysterious as the stringy exclusion principle was argued in [28] to be inaccessible to perturbatively string theory. As noted earlier, however, the multicentered solutions seem to always satisfy this bound (though there is no general proof of this). In fact, the origin of the bound in this system can simply be traced back to the fact that the size of the angular momentum equals $j = I/2 - M$, which cannot be negative, and using

$M = L_0$ this then immediately implies that the unitarity bound will be satisfied by our solutions.

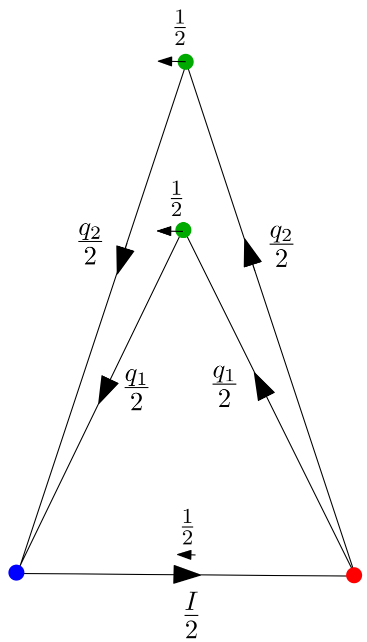

Figure 6.1: *In this figure all contributions to the total angular momentum are shown. The large arrows denote the classical angular momenta carried by the electromagnetic field. They are proportional to the intersection products of the charges, D6 (red), $\overline{\mathrm{D6}}$ (blue) and D0 (green). The small arrows extending from the D0's represent their spin aligning with the dipole magnetic field sourced by the D6$\overline{\mathrm{D6}}$-pair, the small arrow in the bottom is the spin of the center of mass multiplet of the D6$\overline{\mathrm{D6}}$-pair aligning with the magnetic fields.*

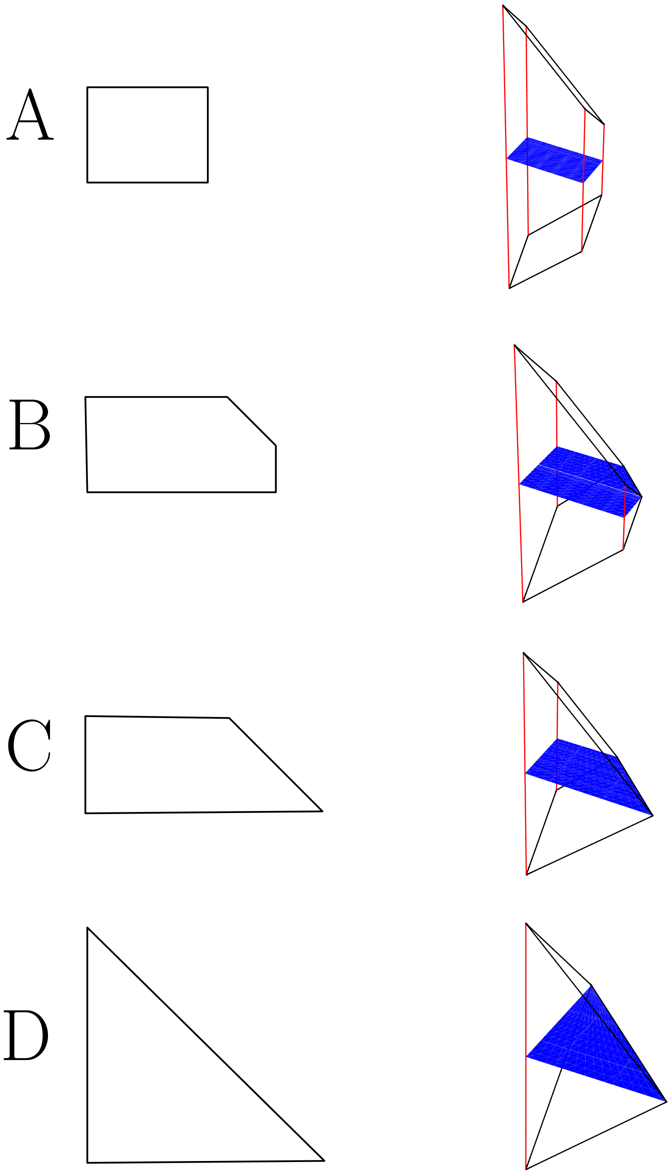

Figure 6.2: *These are the different types of polytopes corresponding to a D6$\overline{\mathrm{D6}}$D0 system with 4 centers. On the left the 'base' polytope determined by the coordinates y_1 and y_2 with $y = 0$ is shown, on the right also the fiber spanned by the coordinate y is included, the edges along this direction are drawn in red, the surface $y = 0$ is shown in blue. See (6.9) for a definition of the coordinates. The different cases correspond respectively to: Case A (non-scaling) $q_1 + q_2 < \frac{I}{2}$, Case B (scaling) $q_1 + q_2 \geq \frac{I}{2}$ and $q_1,\ q_2 \leq \frac{I}{2}$, Case C (scaling) $q_2 \leq \frac{I}{2} \leq q_1$, and Case D (scaling) $\frac{I}{2} \leq q_1,\ q_2$. So we see that from 4 centers onward there are different types of scaling polytopes, a feature that was absent for three scaling centers.*

Part II

BPS States from G_2 Manifolds

PRELUDE

In Part II of this thesis our focus shifts, somewhat dramatically, to topological string and field theory on manifolds of G_2 holonomy following [1][2].

The main part of this work, based on [1], is a follow up of [12] where closed topological string theory on manifolds with G_2 holonomy was defined and studied. The definition of the open theory proceeds rather naturally from the closed one but provides a potentially more computable framework since, mirroring [133], it turns out that topological open string field theory (OSFT) on G_2 manifolds reduces to various Chern-Simons-like field theories. These theories have, in fact, already occurred in the math literature suggesting the potential for further cross-fertilization between the mathematics and physics.

A driving theme behind both [1][2] is the relation between topological string theory (on CY_3) and topological M-theory (on e.g. $\mathrm{CY}_3 \times \mathrm{S}^1$). We explore this in the open theories by relating the various OSFTs on topological branes in G_2 manifolds to the CY_3 versions via dimensional reduction. This hints at a unification of the A and B model within the G_2 theory as expected on rather general grounds.

Although it is only very briefly reviewed here [2] was essentially motivated by this idea and pursued it by considering various Hitchin functionals on G_2 manifolds thought to be related to topological closed string field theory. Here again an attempt was made to connect the dimensional reduction of theories on $\mathrm{CY}_3 \times \mathrm{S}^1$ manifolds with the A and B model but this time at the one-loop level. It turns out that the G_2 theories match exactly onto the B-model at the one-loop level but this probably reflects the fact that the A-model topological string field theory is a Witten-type topological theory (i.e. locally trivial) and we have neglected, in our analysis, contributions of topologically non-trivial configurations.

It proved harder, on the other hand, to relate the G_2 Hitchin functional to closed topological string theory on G_2 manifolds meaning the former does not, unfortunately, provide a non-perturbative formulation of the latter. One might still hope to use the relation to Chern-Simons theories to put the topological G_2 string on better footing via geometric transitions; it may then be possible to explore the relationship between putative M-theory partition functions and the A and B model.

CHAPTER 7

G_2 MANIFOLDS AND NON-PERTURBATIVE STRING THEORY

The first part of this thesis focused quite concretely on particular kinds of states in quantum gravity related to black holes and also the relation of the latter to the full Hilbert space of string theory. Although great attention was given to the physics of black holes it should be recalled that the latter are mostly interesting because of their potentially important role in elucidating the fundamental principles of quantum gravity and it is along these lines that we made some interesting progress in the first half of this work.

In the second half of this thesis we will step back somewhat and consider seemingly unrelated questions about the structure of topological string theory. Although the works involved in the two parts of this thesis were conceived quite independently there has, in fact, been a strong interplay between black hole physics and topological string theory as the latter provides a very important window into various foundational questions in quantum gravity which are often deeply interwoven with the questions posed in Part I, at least for BPS black holes. An obvious instance of this was [10] which, in fact, was part of the initial motivation for [3] on which Part I of this thesis is partially based.

In Part II our focus will be topological theories on G_2 manifolds. This part of the thesis is based on [1][2]. Specifically, we will define and explore open topological strings on G_2 manifolds in order to extend the work done in [12]. We will also very briefly review our work in [2] on topological field theories on G_2 manifolds; we eschew a more detailed treatment, however, and refer the reader to the original paper for all details.

Although [1] and [2] may seem very far removed from black hole physics or even BPS states in physical string theory, a central aspect of this part will be how these theories potentially relates the A/B model topological string theories to each other. The latter have of course played an essential role in understanding the BPS partition function of $\mathcal{N} = 2$ string theory which plays an essential role in understanding BPS black holes. In [10] the BPS partition function of $\mathcal{N} = 2$ was argued to be given (to leading order) by the square of the topological string partition function, with the latter being thought of as some kind of a wavefunction. An essential goal of studying topological string/membrane theory on G_2 manifolds is to better understand the nature of the CY topological string partition function. For instance in [8] it is claimed that the wavefunction nature of the latter can be derived by uplifting to a G_2 manifold, namely $\mathrm{CY}_3 \times \mathrm{S}^1$.

7.1 G_2 Hitchin Functional at One Loop

Here we will very briefly review the content of [2]. We eschew a more detailed treatment of this material for reasons of length as this thesis is already somewhat extensive. The results of this work are nonetheless interesting and relate to those discussed below of [1]. We will this introduce the main ideas and results and refer the reader to the original paper for more details.

Topological string theory on Calabi-Yau manifolds has been the source of many recent insights in the structure of gauge theories and black holes. The traditional construction for topological strings is in terms of topologically twisted worldsheet A- and B-models, computing Kähler and complex structure deformations. The topological information these theories compute is encoded in Gromov-Witten invariants.

More recently a target space quantum foam reformulation of the A-model in terms of the Kähler structure has emerged [134, 14]. The topological information computed are the Donaldson-Thomas invariants, providing a powerful reformulation of Gromov-Witten invariants. For topological string theories on Calabi-Yau manifolds there are additional well-developed computational tools using open-closed duality such as the topological vertex or matrix models.

In comparison, topological theories on G_2 manifold target spaces are much less explored. One motivation to consider such theories is that G_2 structure couples Kähler and complex structure naturally so such a theory might couple topological A- and B-models non-perturbatively, a coupling which we expect to exist following recent work on topological string theory. A recent proposal for topological theories on G_2 manifolds that goes under the name of topological M-theory was given in [8].

The classical effective description of topological M-theory is in terms of a Hitchin functional [135]. Alternative topological theories on G_2 manifolds employing quantum worldsheet/worldvolume formulations have been proposed in terms of topological strings [12] and topological membranes [136, 137, 138, 139] [1]. The topological G_2 string and topological membrane theories [138] have the same structure of local observables associated to the de Rham cohomology of G_2 manifolds. The full quantum worldvolume formulation of these theories, especially the computation of the complete path integral is much more difficult though than for the usual topological string theories on Calabi-Yau target spaces [2] .

In [2] we attempted to understand the moduli space of topological M-theory in terms of a G_2 target space description. Our strategy was similar to the A-model quantum foam, where one considers fluctuations around a fixed background Kähler form. Here the quantum path integral is computed in terms of a topologically twisted six-dimensional abelian gauge theory.

Analogously, the stable closed 3-form encoding the G_2 structure in seven dimensions can be understood as a perturbation around a fixed background associative 3-form. Locally the fluctuation can be regarded as the field strength of an abelian 2-form gauge field. Unlike the A-model quantum foam, however, expanding the Hitchin functional to quadratic order around this fixed background gives a seven-dimensional gauge theory that is not quite topological but which is only invariant under diffeomorphisms of the G_2 manifold.

We analyzed the quantum structure of this theory by taking the 2-form gauge field to be topologically trivial. In practise this means we neglected certain 'total derivative' terms in the expansion of the Hitchin functional involving components of the bare 2-form gauge field [3] . This allowed us to generalize to seven dimensions the approach used by Pestun and Witten [13] to quantize the Hitchin functional for a stable 3-form in six dimensions to 1-loop order. This approach is based on the powerful techniques developed by Schwarz [141] for evaluating the partition function of a degenerate quadratic action functional. The structure of the partition function here is most naturally understood by fixing the gauge symmetry of the action using the antifield-BRST method of Batalin and Vilkovisky [142]. See also [8, 143] for possible alternatives to the perturbative quantization we considered in [2].

We first computed the 1-loop partition function of the ordinary G_2 Hitchin functional and

[1] A topological version of F-theory on $Spin(7)$ manifolds which are trivial torus fibrations over Calabi-Yau spaces was also considered in [140].

[2] The topological G_2 string partition function is only well-understood below genus two. At genus zero it computes the Hitchin functional while its genus one contribution was calculated in [2]. The topological membrane partition function is written only formally.

[3] For more conventional gauge theories such local 'total derivative' terms usually correspond to topological invariants computing certain characteristic classes for the gauge bundle from the patching conditions. Unlike in conventional abelian gauge theory where the gauge field corresponds to a connection on a line bundle over the base space, the 2-form gauge field we have here corresponds to a connection on a gerbe.

found agreement between the local degrees of freedom for the reduction of this theory on a circle and the corresponding theory of Pestun and Witten [13], obtained from the Hitchin functional for a stable 3-form in 6 dimensions. The calculation was repeated for the generalized G_2 Hitchin functional and a certain truncation of the circle reduction of this theory was related to the extended Hitchin functional in 6 dimensions, whose 1-loop partition function was equated with the topological B-model in [13].

The 1-loop partition function for the topological G_2 string [12] was also computed here and found to agree with the generalized G_2 theory only up to a power of the Ray-Singer torsion of the background G_2 manifold. It is not clear to us whether precise agreement could have been obtained by a more careful analysis incorporating the global topological structure of the local total derivative terms we dropped. Nonetheless, it seems that the topologically structure of such terms could potentially give rise to non-trivial 1-loop determinants which we ignored.

Our 1-loop quantization of the generalized G_2 Hitchin functional was in terms of linear variations of a closed stable odd-form in seven dimensions. However, the odd-form can be parameterized non-linearly in terms of other fields, that would be related to the dilaton, B-field, metric and RR flux moduli in compactifications of physical string theory on generalized G_2 manifolds. Hence an additional question is if we were using the appropriate degrees of freedom to describe the quantum theory. It would be interesting to see if our results could be checked by comparison with the couplings appearing in effective actions for generalized G_2 compactifications of physical string and M-theory.

Finally, since general background G_2 metric variations contain complex structure variations in 6 dimensions, it is natural to ask whether the wavefunction behaviour of B-model has a nice interpretation in 7 dimensions? Indeed this was one of the original motivations for the proposal of topological M-theory in [8]. It is possible that this could be understood from the structure of partition functions we calculated in [2] and this was one motivation for this work. Unfortunately the complexity of the 7-dimensional Hitchin functional makes such a relationship rather difficult to determine.

7.2 TOPOLOGICAL G_2 STRINGS

As mentioned above, topological strings have been studied quite intensively as a toy model of ordinary string theory. Besides displaying a rich mathematical structure, they partially or completely control certain BPS quantities in ordinary string theory, and as such have found applications e.g. in the study BPS black holes and non-perturbative contributions to superpotentials.

Unfortunately, a full non-perturbative definition of topological string theory is still lacking, but it is clear that it will involve ingredients from both the A- and B-model, and that both open and closed topological strings will play a role. Since M-theory is crucial in understanding the strong coupling limit and nonperturbative properties of string theory, one may wonder whether something similar is true in the topological case, i.e. does there exist a seven-dimensional topological theory which reduces to topological string theory in six dimensions when compactified on a circle? And could such a seven-dimensional theory shed light on the non-perturbative properties of topological string theory?

In order to find such a seven-dimensional theory one can use various strategies. One can try to directly guess the spacetime theory, as we described in Section 7.1 (see also [8, 144]), or one can try to construct a topological membrane theory as in [145, 137, 138, 139, 146] (after all, M-theory appears to be a theory of membranes, though the precise meaning of this sentence remains opaque). Here and in the next section we will describe a different approach involving studying a topological version of strings propagating on a manifold of G_2 holonomy, following [12] [1] (for an earlier work on G_2 sigma-models see [147]).

In [12] the topological twist was defined using the extended worldsheet algebra that sigma-models on manifolds with exceptional holonomy possess [148]. For manifolds of G_2 holonomy the extended worldsheet algebra contains the $c = 7/10$ superconformal algebra [147] that describes the tricritical Ising model, and the conformal block structure of this theory was crucial in defining the twist. In [12] it was furthermore shown that the BRST cohomology of the topological G_2 string is equivalent to the ordinary de Rham cohomology of the seven-manifold, and that the genus zero three-point functions are the third derivatives of a suitable prepotential, which turned out to be equal to the seven-dimensional Hitchin functional of [149]. The latter also features prominently in [8, 144], suggesting a close connection between the spacetime and worldsheet approaches.

In Chapter 8 we will study open topological strings on seven-manifolds of G_2 holonomy, using the same twist as in [12]. There are several motivations to do this. First of all, we hope that this formalism will eventually lead to a better understanding of the open topological string in six dimensions. Second, some of the results may be relevant for the study of realistic compactifications of M-theory on manifolds of G_2 holonomy[4], for a recent discussion of the latter see e.g. [150]. Third, by studying branes wrapping three-cycles we may establish a connection between topological strings and topological membranes in seven dimensions. And finally, for open topological strings one can completely determine the corresponding open string field theory [133], from which one can compute arbitrary higher genus partition functions and from which one can also extract highly non-trivial all-order results for the closed topological string using geometric transitions [151]. Repeating such an analysis in the G_2 case would allow us to use open G_2 string field theory

[4]This will require an extension of our results to singular manifolds which is an interesting direction for future research.

to perform computations at higher genus in both the open and closed topological G_2 string. This is of special importance since the definition and existence of the topological twist at higher genus has not yet been rigorously established in the G_2 case.

Along the way we will run into various interesting mathematical structures and topological field theories in various dimensions that may be of interest in their own right.

7.3 THE CLOSED TOPOLOGICAL G_2 STRING

Let us briefly review the definition of the *closed* topological G_2 string found in [12] as we will need this background when defining the open theory in Chapter 8. We will cover only essential points. For further details we refer the reader to [12].

7.3.1 SIGMA MODEL FOR THE G_2 STRING

The topological G_2 string constructs a topological string theory with target space a seven-dimensional G_2-holonomy manifold Y. This topological string theory is defined in terms of a topological twist of the relevant sigma-model. In order to have $\mathcal{N} = 1$ target space supersymmetry, one starts with an $\mathcal{N} = (1, 1)$ sigma model on a G_2 holonomy manifold. The special holonomy of the target space implies an extended supersymmetry algebra for the worldsheet sigma-model [148]. That is, additional conserved supercurrents are generated by pulling back the covariantly constant 3-form ϕ and its hodge dual $*\phi$ to the worldsheet as

$$\phi_{\mu\nu\rho}(\mathbf{X})D\mathbf{X}^{\mu}D\mathbf{X}^{\nu}D\mathbf{X}^{\rho} \ ,$$

where $\mathbf{X}$ is a worldsheet chiral superfield, whose bosonic component corresponds to the world-sheet embedding map. From the classical theory it is then postulated that the extended symmetry algebra survives quantization, and is present in the quantum theory. This postulate is also based on analyzing all possible quantum extensions of the symmetry algebra compatible with spacetime supersymmetry and G_2 holonomy.

A crucial property of the extended symmetry algebra is that it contains an $\mathcal{N} = 1$ SCFT sub-algebra, which has the correct central charge of $c = 7/10$ to correspond to the tri-critical Ising unitary minimal model. Unitary minimal models have central charges in the series $c = 1 - \frac{6}{p(p+1)}$ (for p an integer) so the tri-critical Ising model has $p = 4$.

The conformal primaries for such models are labelled by two integer Kac labels, n' and n, as $\phi_{(n',n)}$ where $1 \leq n' \leq p$ and $1 \leq n < p$. The Kac labels determine the conformal weight of the state as $h_{n',n} = \frac{[pn'-(p+1)n]^2-1}{4p(p+1)}$. The Kac table for this minimal model is reproduced in [12, Table 1]. Note that primaries with label (n', n) and

$(p+1-n', p-n)$ are equivalent. This model has six conformal primaries with weights $h_I = 0, 1/10, 6/10, 3/2$ (for the NS states) and $h_I = 7/16, 3/80$ (for the R states).

The conformal block structure of the weight 1/10, $\phi_{(2,1)}$, and of the weight 7/16 primary, $\phi_{(1,2)}$, is particularly simple,

$$\begin{aligned}\phi_{(2,1)} \times \phi_{(n',n)} &= \phi_{(n'-1,n)} + \phi_{(n'+1,n)} \ , \\ \phi_{(1,2)} \times \phi_{(n',n)} &= \phi_{(n',n-1)} + \phi_{(n',n+1)} \ ,\end{aligned}$$

where $\phi_{(n',n)}$ is any primary. This conformal block decomposition is schematically denoted as

$$\begin{aligned}\Phi_{(2,1)} &= \Phi^{\downarrow}_{(2,1)} \oplus \Phi^{\uparrow}_{(2,1)} \ , \\ \Phi_{(1,2)} &= \Phi^{-}_{(1,2)} \oplus \Phi^{+}_{(1,2)} \ .\end{aligned} \tag{7.1}$$

The conformal primaries of the full sigma-model are labelled by their tri-critical Ising model highest weight, h_I, and the highest weight corresponding to the rest of the algebra, h_r, as $|h_I, h_r\rangle$. This is possible because the stress tensors, T_I, of the tricritical subalgebra and of the 'rest' of the algebra, $T_r = T - T_I$ (where T is the stress tensor of the full algebra), satisfy $T_I \cdot T_r \sim 0$.

7.3.2 THE G_2 TWIST

The standard $\mathcal{N} = (2,2)$ sigma-models can be twisted by making use of the U(1) R-symmetry of their algebra. Using the U(1) symmetry, the twisting can be regarded as changing the worldsheet sigma-model with a Calabi-Yau target space by the addition of the following term:

$$\pm \frac{\omega}{2}\overline{\psi}\psi \ , \tag{7.2}$$

with ω the spin connection on the world-sheet. This effectively changes the charge of the fermions under worldsheet gravity to be integral, resulting in the topological A/B-model depending on the relative sign of the twist in the left and right sector of the theory (for fermions with holomorphic or anti-holomorphic target space indices). Here $\overline{\psi}$ and ψ can be either left- or right-moving worldsheet fermions and ω is the spin-connection on the worldsheet. In the topological theory, before coupling to gravity, there are no ghosts or anti-ghosts so these are the only spinors/fermions in the system.

This twist has been re-interpreted [152, 153] as follows. First think of the exponentiation of (7.2) as an insertion in the path integral rather than a modification of the action. By bosonising the world-sheet fermions we can write $\overline{\psi}\psi = \partial H$ for a free boson field so the above becomes

$$\int \frac{\omega}{2}\partial H = -\int H\frac{\partial\omega}{2} = \int HR \,, \tag{7.3}$$

where R is the curvature of the world-sheet. We can always choose a gauge for the metric such that R will only have support on a number of points given by the Euler number of the worldsheet.

For closed strings on a sphere the Euler class has support on two points which can be chosen to be at 0 and ∞ (in the CFT defined on the sphere) so the correlation functions in the topological theory can be calculated in terms of the original CFT using the following dictionary:

$$\langle \ldots \rangle_{twisted} = \left\langle e^{H(\infty)} \ldots e^{H(0)} \right\rangle_{untwisted} \,. \tag{7.4}$$

The 'untwisted' theory should not be confused with the physical theory, because it does not include integration over world-sheet metrics and hence has no ghost or superghost system and also it is still not at the critical dimension. The equation above simply relates the original untwisted $\mathcal{N} = 2$ sigma-model theory to the twisted one.

In [12] a related prescription is given to define the twisted 'topological' sigma-model on a 7-dimensional target space with G_2 holonomy. Here the role of the U(1) R-symmetry is played by the tri-critical Ising model sub-algebra. However, a difference is that the topological G_2 sigma-model is formulated in terms of conformal blocks rather than in terms of local operators. In particular the operator H in the above is replaced by the conformal block $\Phi^{+}_{(1,2)}$.

The main point of the topological twisting is to redefine the theory in such a way that it contains a scalar BRST operator. In the G_2 sigma model, the BRST operator is defined to be related to the conformal block of the weight 3/2 current $G(z)$ of the super stress-energy tensor[5],

$$Q = G^{\downarrow}_{-\frac{1}{2}} \,. \tag{7.5}$$

The states of the twisted G_2 theory are defined to be in the cohomology of this operator. See [12] for a more detailed definition of this operator.

[5]The super stress-energy tensor is given as $\mathbf{T}(z,\theta) = G(z) + \theta T(z)$. The current G(z) can be further decomposed as G(z) = $\Phi_{(2,1)} \otimes \Psi_{\frac{14}{10}}$, in terms of the tri-critical Ising-model part and the rest of the algebra, respectively. Since its tri-critical Ising model part contains only the primary $\Phi_{(2,1)}$, it can be decomposed into conformal blocks accordingly.

It should be pointed out that in [12] it was not possible to explicitly construct the twisted stress tensor, and although there is circumstantial evidence that the topological theory does exist beyond tree level this statement remains conjectural.

7.3.3 THE G_2 STRING HILBERT SPACE

In a general CFT the set of states can be generated by acting with primary operators and their decendants on the vacuum state, resulting in an infinite dimensional Fock space. In string sigma models this Fock space contains unphysical states, and so the physical Hilbert space is given by the cohomology of the BRST operator on this physical Hilbert space which is still generally infinite-dimensional.

In the topological A- and B-models a localization argument [153] implies that only BRST fixed-points contribute to the path integral and these correspond to holomorphic and constant maps, respectively. Thus the set of field configurations that when quantized, generate states in the Hilbert space is restricted to this subclass of all field configurations and so the Fock space is much smaller. Upon passing to BRST cohomology this space actually becomes finite-dimensional.

In the G_2 string the localization argument cannot be made rigorous, because the action of the BRST operator on the worldsheet fields is inherently quantum, and so is not well defined on the classical fields. Neglecting this issue and proceeding naively, however, one can construct a localization argument for G_2 strings that suggests that the path integral localizes on the space of constant maps [12]. Thus we will take our initial Hilbert space to consist of states generated by constant modes X_0^μ and ψ_0^μ on the world-sheet (in the NS-sector there is no constant fermionic mode but the lowest energy mode $\psi^\mu_{-\frac{1}{2}}$ is used instead). These correspond to solutions of worldsheet equations of motion with minimal action which dominate the path integral in the large volume limit.

In [153] the fact that the path integral can be evaluated by restricting to the space of BRST fixed points is related to another feature of the A/B-models: namely the coupling-invariance (modulo topological terms) of the worldsheet path integral. Variations of the path integral with respect to the inverse string coupling constant $t \propto (\alpha')^{-1}$ are Q-exact, so one may freely take the weak coupling limit $t \to \infty$ in which the classical configurations dominate. This limit is equivalent to rescaling the target space metric, and so we will refer to it as the large volume limit.

Accordingly, calculations in the A- and B- model can be performed in the limit where the Calabi-Yau space has a large volume relative to the string scale, and the worldsheet theory can be approximated by a free theory (this neglects, of course, the important instanton effects in the A-model). The G_2 string also has the characteristics of a topological theory, such as correlators being independent of the operator's positions, and the fact that the

BRST cohomology corresponds to chiral primaries. On the other hand since the theory is defined in terms of the conformal blocks, it is difficult to explicitly check the coupling constant independence. Based on the topological arguments, and on the postulate of the quantum symmetry algebra, in this thesis we will assume the coupling constant independence and the validity of localization arguments. Even if these arguments should fail for subtle reasons, the results derived here are still valid in the large volume limit.

7.3.4 The G_2 string and geometry

As in the topological A- and B-model, for the topological G_2 string there is a one-to-one correspondence between local operators of the form $O_{\omega_p} = \omega_{i_1 \ldots i_p} \psi^{i_1} \ldots \psi^{i_p}$ and target space p-forms $\omega_p = \omega_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$. In [12] it is found that the BRST cohomology of the left (right) sector alone maps to a certain refinement of the de Rham cohomology described by the 'G_2 Dolbeault' complex

$$0 \to \Lambda^0_{\mathbf{1}} \xrightarrow{\check{D}} \Lambda^1_{\mathbf{7}} \xrightarrow{\check{D}} \Lambda^2_{\mathbf{7}} \xrightarrow{\check{D}} \Lambda^3_{\mathbf{1}} \to 0 \; . \tag{7.6}$$

The notation is that $\Lambda^p_{\mathbf{n}}$ denotes differential forms of degree p, transforming in the irreducible representation $\mathbf{n}$ of G_2. The operator $\check{D}$ acts as the exterior derivative on 0-forms, and as

$$\begin{aligned} \check{D}(\alpha) &= \pi^2_{\mathbf{7}}(d\alpha) \quad \text{if} \quad \alpha \in \Lambda^1 \; , \\ \check{D}(\beta) &= \pi^3_{\mathbf{1}}(d\beta) \quad \text{if} \quad \beta \in \Lambda^2 \; , \end{aligned}$$

where $\pi^2_{\mathbf{7}}$ and $\pi^3_{\mathbf{1}}$ are projectors onto the relevant representations. The explicit expressions for the projectors and the standard decomposition of the de Rham cohomology are included in appendix H. Thus, the BRST operator $G^{\downarrow}_{-1/2}$ maps to the differential operator of the complex $\check{D}$. In the closed theory, combining the left- and right-movers, one obtains the full cohomology of the target manifold, accounting for all geometric moduli: metric deformations, the B-field moduli, and rescaling of the associative 3-form ϕ. The relevant cohomology for the open string states will be worked out in the next chapter.

CHAPTER 8

OPEN G_2 STRINGS

In the last chapter we reviewed the closed topological G_2 string and its Hilbert space as derived in [12]. Here we would like to study the open version of this theory. We start by considering open topological strings and their boundary conditions. Consistent boundary conditions are those which preserve one copy of the non-linear G_2 worldsheet algebra and were previously analyzed in [154, 155]. One finds that there are topological zero-, three-, four- and seven-branes in the theory[1]. The three- and four-branes wrap associative and coassociative cycles respectively and are calibrated by the covariantly constant three-form and its Hodge-dual which define the G_2 structure.

We would like to compute the topological open string spectrum in the presence of these branes which we do in Section 8.1. For a seven-brane, the spectrum has a simple geometric interpretation in terms of the Dolbeault cohomology of the G_2 manifold. To define the Dolbeault cohomology, we need to use the fact that $G_2 \subset SO(7)$ acts naturally on differential forms, and we can decompose them into G_2 representations. Recall that the notation $\pi^p_{\mathbf{n}}$ denotes the projection of the space of p-forms Λ^p onto the irreducible representation $\mathbf{n}$ of G_2 (see Appendix H). The Dolbeault complex is then

$$0 \longrightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{\pi^2_{\mathbf{7}} d} \pi^2_{\mathbf{7}}(\Lambda^2) \xrightarrow{\pi^3_{\mathbf{1}} d} \pi^3_{\mathbf{1}}(\Lambda^3) \longrightarrow 0 \, . \tag{8.1}$$

The topological open string BRST cohomology is the cohomology of this complex and yields states at ghost numbers $0, 1, 2, 3$. For zero-, three- and four-branes the cohomology is obtained by reducing the above complex to the brane in question.

In section 8.2 we will verify explicitly that the BRST cohomology in ghost number one contains not only the space of (generalized) flat connections on the brane but also the

[1] It is unclear to us how we could incorporate coisotropic six-branes in our theory, whose existence is suggested in [156].

infinitesimal moduli of the topological brane. In particular, we will see that the topological open string reproduces precisely the results in the mathematics literature [157] regarding deformations of calibrated cycles in manifolds of G_2 holonomy.

We briefly discuss scattering amplitudes in section 8.3 and use them to construct, in Section 8.4, the open topological string field theory following methods discussed in [133]. The final answer for the open topological string field theory turns out to be very simple. For seven-branes we obtain the associative Chern-Simons (CS) action

$$S = \int_Y *\phi \wedge CS_3(A) \;, \tag{8.2}$$

with $CS_3(A)$ the standard Chern-Simons three-form and $*\phi$ the harmonic four-form on the G_2 manifold Y. For the other branes we obtain the dimensional reduction of this action to the appropriate brane. The action (8.2) was first considered in [158, 159], and it is gratifying to have a direct derivation of this action from string theory. We will also discuss the dimensional reduction of this theory on $\mathrm{CY}_3 \times \mathrm{S}^1$, which leads to various real versions of the open A- and B-model, depending on the brane one is looking at. The situation is very similar to the closed topological G_2 string, which also reduced to a combination of real versions of the A- and B-models. It is presently unclear to us whether we should interpret this as meaning that the partition functions of the open and closed topological G_2 strings should not be interpreted as wave functions, as opposed to the partition functions of the open and closed A- and B-models, which are most naturally viewed as wave functions.

The last subject we discuss in section 8.4 is the emergence of worldsheet instanton contributions of the topological string theory on Calabi-Yau manifolds from the topological G_2 string on $\mathrm{CY}_3 \times \mathrm{S}^1$. Though our analysis is not yet conclusive, it appears that these worldsheet instanton effects arise from wrapped branes in the G_2 theory and not directly from worldsheet instantons.

We conclude with a list of open problems and briefly mention some additional results. The results of this chapter are based on [1].

We will adhere to the following conventions: M will refer to a calibrated submanifold of dimension 3 or 4 (i.e. calibrated by ϕ or $*\phi$, respectively); these are known, respectively, as associative and coassociative submanifolds. The ambient G_2 manifold will be denoted Y.

8.1 Open string cohomology

We will now consider the Q cohomology of the open string states. Later, we will interpret part of this cohomology in terms of geometric and non-geometric (gauge field) moduli on calibrated 3- and 4-cycles.

In [12] states in the G_2 CFT were shown to satisfy a certain non-linear bound in terms of h_I and h_r and states saturating this bound are argued to fall into shorter, BPS, representation of the non-linear G_2 operator algebra. Such states are referred to as chiral primaries. Analogous to the $\mathcal{N}=2$ case, it is the physics of these primaries that the twist is intended to capture and thus they are the states that occur in the BRST cohomology. The chiral primaries in the NS sector have $h=0,1/2,1,3/2$ and $h_I=0,1/10,6/10,3/2$ and they are the image of the RR ground states under spectral flow.

Recall that we are working in the zero mode approximation (corresponding to the large volume limit, $t\to\infty$, where oscillator modes can be neglected) and in this limit a general state is of the form $A_{\mu_1\ldots\mu_n}(X_0)\psi_0^{\mu_1}\ldots\psi_0^{\mu_n}$. On such states L_0 acts as $t\Box+\frac{n}{2}$ so states with $h=0,1/2,1,3/2$ correspond to 0, 1, 2, and 3 forms ($f(X_0)$, $A_\mu(X_0)\psi_0^\mu$, $\ldots$). As argued in [12] we can thus consider Q-cohomology on the space of 0, 1, 2, and 3 forms restricted to those that have $h_I=0,1/10,6/10,3/2$, respectively.

In general we are interested in harmonic representatives of the Q cohomology so we will look for operators (corresponding to states) that are both Q- and $Q^\dagger$-closed. The results we obtain are essentially the same as those for one side of the closed worldsheet theory [12].

8.1.1 Degree one

We will start by looking at the $h=1/2$ state, because it is the only one that will generate a marginal deformation of the theory. A general state with $h=1/2$ is of the form $A_\mu(X)\psi^\mu$ so long as [2]

$$[L_0,A_\mu(X)]=t\Box A_\mu(X)=0 \tag{8.3}$$

It also satisfies

$$[L_0^I,A_\mu(X)\psi^\mu]=\frac{1}{10}A_\mu(X)\psi^\mu\ ,$$

so it is a chiral primary (i.e. it saturates the chiral bound). Because it is a chiral primary, it has to be Q-closed [12]. Rather than proceed along these lines, however, we will consider the Q-cohomology directly from the definition of Q.

Let us determine the Q-cohomology of 1-forms $\mathcal{A}=A_\mu(X)\psi^\mu$. We first calculate $\{G_{-\frac{1}{2}},A_\mu(X)\psi^\mu\}$ in the CFT on the complex plane with z complex 'bulk' coordinates and y 'boundary' coordinates on the real line

[2] Although we will sometimes use the full fields X and ψ in the CFT and also consider OPE's which generate deriviatives of these fields the reader should recall that we are always working in the large volume limit where these reduce to X_0 and ψ_0.

$$
\begin{aligned}
\{G_{-1/2}, A_\mu(X)\psi^\mu\} &= \oint dz\, G(z) \cdot A_\mu(X)\psi^\mu(y)\ , \\
G(z) \cdot A_\mu(X)\psi^\mu(y) &= g_{\rho\sigma}(X)\psi^\rho \partial X^\sigma(z) \cdot A_\mu(X)\psi^\mu(y) \\
&\sim \partial(\ln|z-y|^2 + \ln|\overline{z}-y|^2)\nabla_\rho A_\mu \psi^\rho(z)\psi^\mu(y) \\
&+ \frac{1}{z-y}\partial X^\mu(z) A_\mu(X(y))\ .
\end{aligned}
\tag{8.4}
$$

This gives[3]

$$
\{G_{-1/2}, A_\mu(X)\psi^\mu\} = A_\mu \partial X^\mu(y) + \frac{1}{2}\partial_{[\mu}A_{\nu]}\psi^\mu\psi^\nu\ . \tag{8.5}
$$

To compute the action of Q we now project onto the $\downarrow$ part (recall the twisted BRST operator corresponds to $G^\downarrow_{-1/2}$), which includes only the part with tri-critical Ising weight $6/10$. The term $A_\mu \partial X^\mu$ vanishes in the zero mode limit so we only need to consider the second term. The condition that this term has $h_I = \frac{6}{10}$ is [12]

$$
(\pi^2_{\mathbf{14}})^{\rho\sigma}{}_{\mu\nu}\partial_{[\rho}A_{\sigma]} = 0\ , \tag{8.6}
$$

where $\pi^2_{\mathbf{14}}$ is the projector onto the 2-form subspace $\Lambda^2_{\mathbf{14}} \subset \Lambda^2$, in the **14** representation of G_2.

This result implies that the $\frac{6}{10}$ part of $\partial_{[\rho}A_{\sigma]}$ (or any 2-form) is in $\Lambda^2_{\mathbf{7}}$, so on a 1-form we can define Q as

$$
\{Q, A_\mu\psi^\mu\} = (\pi^2_{\mathbf{7}})\{G_{-\frac{1}{2}}, A_\mu\psi^\mu\} = 6\phi_{\mu\nu}{}^\gamma\phi_\gamma{}^{\rho\sigma}\partial_{[\rho}A_{\sigma]}dx^\mu \wedge dx^\nu = \check{D}A = 0\ , \tag{8.7}
$$

where we have used

$$
(\pi^2_{\mathbf{7}})^{\rho\sigma}{}_{\mu\nu} = 4(*\phi)^{\rho\sigma}{}_{\mu\nu} + \frac{1}{6}(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\rho_\nu) = 6\phi_{\mu\nu}{}^\gamma\phi_\gamma{}^{\rho\sigma}\ . \tag{8.8}
$$

Note that Q acting on 1-forms has reduced essentially to $\check{D}$; the same will occur for forms of other degrees.

[3]We have not been careful about the relative normalizations of the bosonic and fermionic bulk-boundary OPE's, but this is not relevant as in all computations of this type that occur below, we will only end up keeping one of the terms.

Let us now consider Q-coclosure. The inner product of states

$$\left\langle A_{[\mu\nu]}\psi^\mu\psi^\nu | B_{[\alpha\beta]}\psi^\alpha\psi^\beta \right\rangle \ ,$$

becomes the inner product of forms $\int_Y (*A \wedge B)$, so $Q^\dagger$ acting on A is given by $\langle Q \cdot f(X) | A_\mu(X)\psi^\mu \rangle = \left\langle f(X) | Q^\dagger \cdot A_\mu\psi^\mu \right\rangle$, which can be determined as

$$\langle Q \cdot f(X) | A_\mu(X)\psi^\mu \rangle = \int \sqrt{g}\partial_\mu f(X) A^\mu(X) = -\int \sqrt{g} f(X) \nabla_\mu A^\mu(X) \ . \quad (8.9)$$

So if A_μ is also required to satisfy

$$Q^\dagger \cdot A_\mu(X)\psi^\mu = -\nabla_\mu A^\mu(X) = 0 \ , \quad (8.10)$$

then it is Q- and $Q^\dagger$-closed and hence a harmonic represenative of Q-cohomology.

8.1.2 Degree zero

The cohomology in degree zero is rather trivial. Given a degree zero mode $f(X)$ we have $\{Q, f(X)\} = \partial_\mu f(x)\psi^\mu$. This follows from $Q = G^\downarrow_{-\frac{1}{2}} = G_{-\frac{1}{2}}$, because the projection onto the $\downarrow$ component is trivial since all operators of the form $A_\mu(X)\psi^\mu$ automatically have L_0^I weight $\frac{1}{10}$. So Q-closure implies

$$\partial_\mu f(X) = 0 \ . \quad (8.11)$$

The $Q^\dagger$-closure here is vacuous since there are no lower degree fields.

8.1.3 Degree two

In degree two we start with a two form $\omega_{\rho\sigma}\psi^\rho\psi^\sigma$ which should have L_0^I weight $\frac{6}{10}$, so it should satisfy $\pi^2_{\mathbf{7}}(\omega) = \omega$. The need to restrict $\omega \in \Lambda^2_{\mathbf{7}}$ comes from the way Q is defined in [12]. We must once more calculate the action of $G_{-\frac{1}{2}}$, and then project it onto the $\downarrow$ part

$$
\begin{aligned}
\{G_{-\frac{1}{2}}, \omega\} &= \oint dz\; g_{\mu\nu}\psi^\mu \partial X^\nu(z)\cdot \omega_{\rho\sigma}\psi^\rho\psi^\sigma \\
&= \oint dz\; \frac{1}{z} g_{\mu\nu}\partial^\nu \omega_{\rho\sigma}\psi^\mu\psi^\rho\psi^\sigma + \frac{1}{z} g_{\mu\nu}\partial X^\nu \omega_{\rho\sigma} g^{\mu\rho}\psi^\sigma - \frac{1}{z} g_{\mu\nu}\partial X^\nu \omega_{\rho\sigma} g^{\mu\sigma}\psi^\rho \\
&= \partial_\mu \omega_{\rho\sigma}\psi^\mu\psi^\rho\psi^\sigma + 2\omega_{\rho\sigma}\partial X^\rho \psi^\sigma \ .
\end{aligned} \tag{8.12}
$$

Once more we can drop the second term in the large volume limit in which we are working. We use the result in [12] that the projector onto the L_0^I weight $\frac{3}{2}$ corresponds to the projector onto $\Lambda^3_{\mathbf{1}}$, and is given by contracting with the associative 3-form ϕ. So for $\Omega \in \Lambda^3_{\mathbf{1}}$

$$
\phi^{\alpha\beta\gamma}\Omega_{\alpha\beta\gamma}\phi_{\mu\nu\rho} = 7\Omega_{\mu\nu\rho} \ . \tag{8.13}
$$

In particular, we can project onto the $\frac{3}{2}$ part of $\{G_{-\frac{1}{2}}, \omega\} = \partial_\mu \omega_{\rho\sigma}$ using $\phi^{\alpha\beta\gamma}$, so Q-closure implies

$$
\phi^{\alpha\beta\gamma}\partial_{[\alpha}\omega_{\beta\gamma]} = 0 \ . \tag{8.14}
$$

Note that this once again can be written as $\check{D}\omega = 0$.

We will now derive the $Q^\dagger$-closure condition. This is done in exactly the same way as was done for the degree one components

$$
\begin{aligned}
\langle \omega | Q \cdot A_\mu(X)\psi^\mu\rangle &= \int \sqrt{g}\,\omega^{\mu\nu}(\pi^2_{\mathbf{7}})^{\alpha\beta}{}_{\mu\nu}\partial_\alpha A_\beta \\
&= -\int \sqrt{g} A_\beta \big((\pi^2_{\mathbf{7}})^{\alpha\beta}{}_{\mu\nu}\nabla_\alpha \omega^{\mu\nu}\big) \ ,
\end{aligned} \tag{8.15}
$$

so

$$
\begin{aligned}
Q^\dagger \cdot \omega &= -(\pi^2_{\mathbf{7}})^{\mu\nu}{}_{\alpha\beta}\nabla^\alpha \omega_{\mu\nu} dx^\beta \\
&= -6\phi^{\mu\nu}{}_\gamma \phi^\gamma{}_{\alpha\beta}\nabla^\alpha\omega_{\mu\nu}dx^\beta = -\nabla^\alpha \omega_{\alpha\beta}dx^\beta = 0 \ .
\end{aligned} \tag{8.16}
$$

Here we have used $\pi^2_{\mathbf{7}}(\omega) = \omega$.

8.1.4 DEGREE THREE

A 3-form $\Omega_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho$ is first projected onto its $\Lambda^3_{\mathbf{1}}$ component by Q, so we take $\pi^3_{\mathbf{1}}(\Omega) = \Omega$, which means that Ω is a function times ϕ. From the definition of Q it is evident that it acts trivially on Ω since there is no higher L^I_0 eigenstate in the NS sector for Q to project onto. This implies $Q = 0$ on three forms which matches (7.6). Thus we see that the action of Q on states in the zero mode approximation maps into the complex (7.6) as anticipated in Section 7.3.4.

The Q-coclosure of Ω is derived similarly to the 1- and 2-form case and gives

$$Q^\dagger \cdot \Omega = \nabla^\mu \Omega_{\mu\nu\rho} dx^\nu \wedge dx^\rho = 0 \ . \tag{8.17}$$

8.1.5 HARMONIC CONSTRAINTS

In the previous subsections we considered the conditions for Q- and $Q^\dagger$-closure on the states in the G_2 CFT. These conditions are all linear in derivatives but they must be enforced simultaneously to generate unique representatives of Q-cohomology. As Q corresponds to the operator $\check{D}$ discussed in section 7.3.4, it generates the Dolbeault complex (7.6) which is known to be elliptic [160, 158] and so can be studied using Hodge theory. This implies that physical states in the theory correspond to the kernel of the Laplacian operator $\{Q, Q^\dagger\}$, so one can equivalently consider this single non-linear condition instead of the two seperate linear conditions imposed by Q and $Q^\dagger$.

These Q-harmonic conditions (derived from the actions of Q and $Q^\dagger$) are

$$\begin{aligned}
&\{Q, Q^\dagger\} \cdot f = \nabla_\mu \partial^\mu f = 0 \ , \\
&\{Q, Q^\dagger\} \cdot A_\nu \psi^\nu = \big(\nabla_\nu \nabla_\mu A^\mu + (\pi^2_{\mathbf{7}})_\nu{}^{\gamma\mu\sigma} \nabla_\gamma \nabla_\mu A_\sigma\big)\psi^\nu = 0 \ , \\
&\{Q, Q^\dagger\} \cdot \omega_{\mu\nu}\psi^\mu\psi^\nu = \big((\pi^2_{\mathbf{7}})_{\mu\nu}{}^{\alpha\beta} \nabla_\alpha \nabla^\gamma \omega_{\beta\gamma} + (\pi^3_{\mathbf{1}})_{\mu\nu\rho}{}^{\alpha\beta\gamma} \nabla^\rho \nabla_\alpha \omega_{\beta\gamma}\big)\psi^\mu\psi^\nu = 0 \ .
\end{aligned} \tag{8.18}$$

We have used $\pi^2_7(\omega) = \omega$ to simplify the last expression above.

8.2 OPEN STRING MODULI

In a general topological theory one can use elements of degree one cohomology to deform the theory using descendant operators. If $\mathcal{O}$ is a degree one operator, in the A/B-model this means that it has ghost number one, whereas in the G_2 string this means that it corresponds to one '+' conformal block. Then one can deform the action by adding a term $\int_{\partial\Sigma}\{G^{\uparrow}_{-\frac{1}{2}}, \mathcal{O}\}$, which is $Q = G^{\downarrow}_{-\frac{1}{2}}$ closed and of degree 0. Thus the elements of H^1_Q cohomology should correspond to possible deformations of the theory or tangent vectors to the moduli space of open topological G_2 strings.

Since open strings correspond to supersymmetric[4] branes, the full moduli space should include both the moduli space of the field theory on the brane as well as the geometric moduli of the branes. For G_2 manifolds the latter are simply the moduli of associative and coassociative 3- and 4-cycles, respectively, which have been studied in [157]. Below we will show that the operators $\mathcal{O}$ corresponding to normal modes do satisfy the correct constraints to be deformations of the relevant calibrated submanifolds. Since a priori it is not known what the field theory on these branes will be, in the topological case we will study the constraints on the tangential modes (which in physical strings would correspond to gauge fields on the brane), and attempt to interpret these as infinitesimal deformations in the moduli space of some gauge theory on the brane.

8.2.1 Calibrated geometry

In order to preserve the extended symmetry algebra (such as $\mathcal{N} = 2$ or G_2) of the worldsheet SCFT in the presence of a boundary, certain constraints must be imposed on the worldsheet currents. These have been studied in [161] [154], and more extensively in [162] [163] [155]. One imposes the boundary condition on the left- and right-moving components of the worldsheet fermions, $\psi^\mu_L = R^\mu_\nu(X)\psi^\nu_R$, and then conservation of the worldsheet currents in the presence of the boundary implies that, on the subspace M where open strings can end,

$$\begin{aligned}
\phi_{\mu\nu\sigma} &= \eta_\phi R^\alpha_\mu R^\beta_\nu R^\gamma_\sigma \phi_{\alpha\beta\gamma} \ , \\
(*\phi)_{\mu\nu\sigma\lambda} &= \eta_\phi R^\alpha_\mu R^\beta_\nu R^\gamma_\sigma R^\rho_\lambda (*\phi)_{\alpha\beta\gamma\lambda} \det(R) \\
&= R^\alpha_\mu R^\beta_\nu R^\gamma_\sigma R^\rho_\lambda (*\phi)_{\alpha\beta\gamma\lambda} \ .
\end{aligned} \tag{8.19}$$

Note that $R^\alpha_\mu(X)$ (for any $X \in M$) is generally a position-dependent invertible matrix, but locally it can be diagonalized with eigenvalues $+1$ in Neumann directions and -1 in Dirichlet directions. $\eta_\phi = \pm 1$ gives two different possible boundary conditions with the choice of $\eta_\phi = 1$ corresponding to open strings ending on a calibrated 3-cycle, while $\eta_\phi = -1$ corresponds to strings on a calibrated 4-cycle [154]. Calibrated submanifolds, first studied in [164], are characterized by the property that their volume form induced by the metric in the ambient space is the pull-back of particular global forms, in this case ϕ (for associative 3-cycles) or $*\phi$ (for coassociative 4-cycles). This implies the volume of the calibrated submanifold is minimal in its homology class.

[4] In the sense of preserving the extended worldsheet superalgebra.

Remark. There are several subtleties regarding boundary conditions in topological sigma-models that deserve to be mentioned. Below, we will advocate the perspective that any boundary condition preserving the extended algebra[5] should also be a boundary condition of the topological theory, because the presence of an extended algebra allows one to define a twisted theory. In the A- and B-model, however, although both the A- and B-brane boundary conditions preserve the $\mathcal{N} = 2$ algebra, each is compatible with only one of the twists, so a given topological twist is not necessarily compatible with an arbitrary algebra-preserving boundary condition. Moreover, a given topological twist might only depend on the existence of a subalgebra of the full extended algebra, so may be possible even with boundary conditions that do not preserve the full extended algebra. A concrete example of this is the Lagrangian boundary condition for the A-model branes proposed by Witten [133]. This condition is considerably less restrictive that the special Lagrangian condition required to preserve the full $\mathcal{N} = 2$ algebra in the physical string [161] and reflects the fact that the A-model is well-defined for any Kähler manifold and does not require a strict Calabi-Yau target space. While similar subtleties might, in principle, exist for the G_2 twist they are concealed by the fact that the twist does not have a classical realization that we know of. So we will tentatively assume the correct boundary conditions are those that preserve the full G_2 algebra on one half of the worldsheet theory.

8.2.2 Normal modes

Let us now consider the cohomology of open strings ending on a D-brane which wraps either an associative 3-cycle or a co-associative 4-cycle. We adopt the convention that $I, J, K, \ldots$ are indices normal to the brane while $a, b, c, \ldots$ are tangential, and Greek letters run over all indices. The state $A_\mu \psi^\mu$ decomposes into normal and tangential modes which will be denoted $\theta_I \psi^I$ and $A_a \psi^a$ respectively; all momenta is tangential, denoted by k_a. The normal modes will have the form $\mathcal{A} = \theta_I(X^a)\psi^I$ so $G_{-1/2} \cdot \mathcal{A} = \partial_a \theta_I(X^b)\psi^a \psi^I$. Here $\mathcal{A}$ will denote a *general* operator/state in the CFT and should not be confused with the gauge field (or operator) $A_\mu \psi^\mu$.

Associative 3-cycles. Let us now consider the Q-cohomology when restricted to an associative 3-cycle M. On the 3-cycle the form ϕ must satisfy [155]

$$\phi_{\mu\nu\sigma} = R^\alpha_\mu R^\beta_\nu R^\gamma_\sigma \phi_{\alpha\beta\gamma} \ . \tag{8.20}$$

[5] To be precise the boundary conditions preserve some linear combination of the extended algebra in the left/right sector of the worldsheet. So a brane may reduce an $\mathcal{N} = (2, 2)$ theory to an $\mathcal{N} = 2$ theory.

Since M is associative, ϕ acts as a volume form on this cycle and, from the above, it is only non-vanishing for an odd number of tangential indices[6]

$$\begin{aligned} \phi_{abc} &= \epsilon_{abc} \ , \\ \phi_{Ibc} &= 0 \ , \\ \phi_{IJK} &= 0 \ . \end{aligned} \tag{8.21}$$

The Q-closure of normal modes is given by (8.7)

$$\phi_{bK}{}^{J}\phi_{J}{}^{aI}\nabla_a\theta_I = 0 \ , \tag{8.22}$$

where the index structure is enforced by the requirement that ϕ has an even number of normal indices.

To understand the geometric significance of equation (8.22) in the abelian theory, recall that θ^I is just a section of the normal bundle NM of M in Y, which by the tubular neighborhood theorem can be identified with an infinitesimal deformation of M. This equation is the linear condition on θ^I such that the exponential map (defined by flowing along a geodesic in Y defined by θ^I) $\exp_\theta(M)$ takes M to a new associative submanifold M'. This is just a reformulation of the condition given in [157].

In [157] McLean defines a functional on the space of (integrable) normal bundle sections by

$$F_\gamma(\theta) = (*\phi(x))_{\mu\nu\rho\gamma}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}\frac{\partial x^\rho}{\partial\sigma^c}\epsilon^{abc} \propto (*\phi(x))_{\mu\nu\rho\gamma}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}\frac{\partial x^\rho}{\partial\sigma^c}\phi^{abc} \ . \tag{8.23}$$

Here $x(t,\theta,\sigma) = \exp_\theta(\sigma,t)$ is a geodesic curve parameterized by the variable $0 < t < t_1$, which starts at a point $\sigma \in M$ with $\dot{x}(\sigma) = \theta$ at $t = 0$, and flows after a fixed time to $x(t = t_1, \theta, \sigma) \in M'$, the new putative associative submanifold. The functional is just the pull-back[7] of $*\phi$ from M' to M and it should vanish if M' is associative.

For M' to be a associative it turns out to be sufficient to require that the time derivative of F at $t = 0$ vanishes, which gives

$$\dot{F}_\gamma(\theta)|_{t=0} = (*\phi(x))_{Ibc\gamma}\partial_a\theta^I\phi^{abc} = \phi_{I\gamma}{}^{a}\partial_a\theta^I. \tag{8.24}$$

[6]Here, and throughout the paper, we will take ϵ to be the volume form on the (sub)manifold not merely the antisymmetric tensor.

[7]More precisely we are pulling back $\chi \in \Omega^3(Y, TY)$, a tangent bundle valued 3-form, defined using the G_2 metric $\chi^\alpha{}_{\mu\nu\rho} = g^{\alpha\beta}(*\phi)_{\beta\mu\nu\rho}$.

This is equivalent to (8.22) since each choice of bK indices in that equation gives only one non-vanishing term. The space of such deformations is generally not a smooth manifold and currently the moduli space of associative submanifolds of a given G_2 manifold is not well understood (but see [165] for some recent work on this).

At first glance (8.22) looks like the linearized equation (4.7) in [138] but the fields in that action are actually embedding maps which are non-linear, whereas the θ^I above are more closely related to linearized fluctuations around fixed embedding maps [8] .

Remark. The harmonic condition as follows from eqn. (8.18) for normal modes is

$$(\pi^2_{\mathbf{7}})^a{}_I{}^{bJ}\nabla_a\nabla_b\theta_J = 0 \ . \tag{8.25}$$

This also has a nice geometrical interpretation as vector fields θ^I extremizing the action

$$\int_M \langle Q\cdot\theta, Q\cdot\theta\rangle \ , \tag{8.26}$$

on the associative 3-cycle. Theorem 5-3 in [157] shows that the zeros of this action (which are extrema since it is positive semi-definite) correspond to a family of deformations through minimal submanifolds.

Coassociative 4-cycles. The consideration of the 4-cycle M is similar to that of the 3-cycle, but now in the boundary condition we have $\eta_\phi = -1$, so the non-vanishing components of ϕ must have an odd number of normal indices and

$$\phi_{abc} = 0 \ . \tag{8.27}$$

Let us first consider the Q-closure of θ_I

$$\phi_{Ic}{}^b\phi_b{}^{aJ}\partial_{[a}\theta_{J]} = 0 \ . \tag{8.28}$$

[8] In [138], maps $x : \Sigma_3 \to Y$ from an arbitrary three-manifold to a G_2 manifold are considered and a functional which localizes on associative embeddings is defined. There a reference associative embedding x_0 is chosen and used to define a local coordinate splitting of x^μ into tangential x^a and normal y^I parts. This is different from the present situation where θ^I is an infinitesimal normal deformation of an associative cycle. θ^I can be identified with a section of the normal bundle (via the tubular neighborhood theorem) and is essentially a linear object, whereas the y^I above are a local coordinate representation of a non-linear map. Basically θ^I here are related to the linear variation $\delta y^I|_{x_0}$ (evaluated at $x = x_0$) in [138].

These are 24 equations depending on a choice of I and c. Examining the index structure, eqn. (8.28) reduces to 4 independent equations

$$\phi_b{}^{aJ}\nabla_a\theta_J = 0 \,, \tag{8.29}$$

where we replaced the commutator of a derivative with the covariant derivative on M in the induced metric.

Following [157], let us observe an isomorphism between the normal bundle NM of the 4-cycle M, and the space of self-dual 2-forms $\Lambda^2_+(M)$ on M, given by

$$\theta^I \to \theta^I\phi_{Iab} \equiv \Omega_{ab} \,, \tag{8.30}$$

$$\Omega_{ab} \to \phi^{Iab}\Omega_{ab} = \phi^{Iab}\phi_{Jab}\theta^J = \frac{1}{6}\theta^I \,, \tag{8.31}$$

where we have used the first identity in (H.3).

To see that Ω_{ab} is self-dual we use the second identity in (H.3) and the fact that $*\phi_{ab}{}^{cd} \propto \epsilon_{ab}{}^{cd}$ on M, so that

$$(*_4\Omega)_{ab} \propto *\phi_{ab}{}^{cd}\Omega_{cd} = \phi_{ab}{}^{cd}\theta^I\phi_{Icd} = \frac{1}{6}\phi_{Iab}\theta^I = \frac{1}{6}\Omega_{ab} \,. \tag{8.32}$$

Let us now use (8.31) to see what (8.29) implies for Ω_{ab};

$$0 = \phi_b{}^{aJ}\nabla_a\phi_J{}^{cd}\Omega_{cd} = \nabla_a\big(\phi_b{}^{aJ}\phi_J{}^{cd}\Omega_{cd}\big) = \nabla^a\big[\big(\frac{1}{9}\Omega_{ba} + \frac{1}{18}\Omega_{ba}\big)\big] = \frac{1}{6}\nabla^a\Omega_{ba} \,. \tag{8.33}$$

This equation is just $d^\dagger\Omega = 0$, and since Ω is self-dual, it also implies $d\Omega = 0$ so that Ω must be harmonic. Thus the Q-cohomology for the normal modes is given by θ^I which map to harmonic self-dual 2-forms on M.

Since the $Q^\dagger$-cohomology on the normal modes is trivial (eqn. (8.10) is trivially true for normal directions), such θ^I are Q-closed and co-closed, and hence Q-harmonic. Thus their Q-cohomology is isomorphic to the de Rham cohomology group $H^2_+(M)$ of harmonic self-dual 2-forms on M. This corresponds to the geometric moduli space of deformations of a coassociative 4-cycle, determined by McLean in [157].

8.2.3 Tangential modes

For the tangential modes the Q- and $Q^\dagger$-closure conditions are just (8.7) and (8.10) with all the indices replaced by worldvolume indices $a, b, c, \ldots$.

Associative 3-cycles. On the 3-cycle it is convenient to represent the Q-closure condition using the projector $\pi^2_{\mathbf{7}}$ in terms of ϕ which gives

$$\phi_{ab}{}^{c}\phi_{c}{}^{de}\partial_{[d}A_{e]} = 0\ . \tag{8.34}$$

When pulled back to the associative cycle, ϕ is proportional to the volume form and so this is

$$\epsilon_{ab}{}^{c}\epsilon_{c}{}^{de}\partial_{[d}A_{e]} = 0\ , \tag{8.35}$$

which is just multiple copies of the equation $\partial_{[d}A_{e]} = 0$. Therefore any tangential deformation corresponds to a flat connection on the 3-cycle.

Requiring the deformation $A_a\psi^a$ be also be $Q^\dagger$-closed, and hence a harmonic representative of Q-cohomology, implies (8.10), which can be viewed as enforcing a covariant gauge condition.

Combined together this means that the Q-cohomology for tangential modes on M is spanned by the space of gauge-inequivalent flat connections on M. This matches the result for Lagrangian submanifolds in the A-model and also the results derived using κ-symmetry for physical branes in [166].

Coassociative 4-cycles. On the 4-cycle it is easier to use the representation of Q-closure

$$\big((\delta^a_c\delta^b_d - \delta^b_c\delta^a_d) + 24(*\phi)^{ab}{}_{cd}\big)\partial_{[a}A_{b]}\psi^c\psi^d = 0\ , \tag{8.36}$$

in terms of the 4-form $*\phi$, which is now proportional to the volume form on M. Defining $F_{ab} = \partial_{[a}A_{b]}$ to be the field strength of the $U(1)$ gauge field, the equation above implies

$$(*_4F)_{ab} = 12(*\phi)^{cd}{}_{ab}F_{cd} = -F_{ab}\ . \tag{8.37}$$

Thus F_{ab} is constrained to be anti-self-dual (ASD) on M. Therefore any tangential deformation on the 4-cycle is given by a gauge field with ASD field strength. Note an important difference with the case of normal modes. In the latter case each θ^I is mapped uniquely to a harmonic self-dual 2-form Ω_{ab} on M, so there are exactly $b_+^2(M)$ such modes. In this case however the tangential mode A_a is the potential for a gauge field with ASD field strength (i.e. an (anti-)instanton configuration). Hence the tangential modes correspond to tangent vectors on the moduli space of instanton configurations on M.

Again the condition $\nabla_a A^a = 0$ for $Q^\dagger$-closure is simply a gauge choice, implying that each Q-harmonic representative is associated to a unique orbit of the gauge group (up to Gribov ambiguity in the path integral). In fact, these harmonic constraints $*_4 F = -F$, $d^\dagger A = 0$ are precisely (linearized versions of) the conditions cited in equation (5.22) of [167] as defining the deformations of an instanton configuration.

In physical string theory the anti-self-duality constraint on the field strength of a coassociative brane has been determined in [166] using κ-symmetry of the DBI action. In [168], a topological field theory is proposed on calibrated 4-cycles whose total moduli space is a product of the moduli space of geometric deformations with the moduli space of ASD connections on M. We will see shortly that this is indeed the worldvolume theory on coassociative 4-cycles for the open G_2 string.

8.3 SCATTERING AMPLITUDES

Before considering the nature of the worldvolume theory of the calibrated 3- and 4-cycles, it will be useful to consider some scattering amplitudes in the open G_2 theory, as these can be compared with field theoretic scattering amplitudes and will help constrain the interaction terms in the worldvolume action. In fact, as will be discussed in the next section, these interactions can actually be related to string field theory, not just to effective field theory, if one concedes that the G_2 string is independent of its coupling constant, as argued in [12].

8.3.1 3-POINT AMPLITUDE

The simplest amplitudes to calculate (and the only ones we will need) are the 3-point functions of degree one fields $A_\mu \psi^\mu$, which are essentially already calculated in [12]. Introducing Chan-Paton factors into the calculation performed there gives the 3-point function of three ghost number one fields as

$$\lambda^3 \frac{3}{2} f_{jik} \int_Y \phi^{\alpha\beta\gamma}(x) A^i_\alpha(x) A^j_\beta(x) A^k_\gamma(x) \ , \tag{8.38}$$

where f_{ijk} are the structure functions for the Lie algebra of the gauge group G and λ is the normalization of the bulk-boundary 2-point function in the G_2 CFT (these are generally not relevant and will not be treated with a great deal of care).

Tangential modes. For an associative 3-cycle embedding $i : M \to Y$, we have the relation $i^*(\phi) = \epsilon$, where ϵ is the volume form on M. If we now consider the previous calculation but where now the fields ψ^μ are restricted to be along the 3-brane (so they have indices $a, b, c \dots$), we find that

$$\langle AAA \rangle = \lambda^3 \frac{3}{2} f_{jik} \int_M \epsilon^{abc}(x) A_a^i(x) A_b^j(x) A_c^k(x) \ . \tag{8.39}$$

As will be discussed in the next section, this is an interaction vertex for Chern-Simons theory, which is the part of the effective worldvolume theory for the 3-cycle.

As mentioned in previous section, on a coassociative 4-cycle $\phi^{abc} = 0$ so the 3-point function of tangential modes vanishes.

Normal and mixed modes. We can now try to consider a mixture of normal or tangential modes in the 3-point function. The boundary conditions on the open G_2 string, preserving the extended algebra on a 3-cycle, imply [154] that only ϕ^{abc} and ϕ^{IJc} are non-vanishing. Thus ϕ is only non-vanishing for an even number of indices in Dirichlet directions, so we can only scatter two normal modes and one tangential mode. This gives

$$\langle \theta\theta A \rangle = \lambda^3 \frac{3}{2} f_{jik} \int_M \phi^{IJc}(x) \theta_I^i(x) \theta_J^j(x) A_c^k(x) \ . \tag{8.40}$$

On a 4-cycle the non-vanishing components of ϕ have an odd number of normal indices, and it is easy to see that the only non-vanishing 3-point functions of degree one modes are

$$\begin{aligned} \langle \theta A A \rangle &= \lambda^3 \frac{3}{2} f_{jik} \int_M \phi^{Iab}(x) \theta_I^i(x) A_a^j(x) A_b^k(x) \ , \\ \langle \theta\theta\theta \rangle &= \lambda^3 \frac{3}{2} f_{jik} \int_M \phi^{IJK}(x) \theta_I^i(x) \theta_J^j(x) \theta_K^k(x) \ . \end{aligned} \tag{8.41}$$

8.4 Worldvolume theories

We have already determined the BRST cohomology of normal and tangential modes on 3- and 4-cycles. These should be thought of as marginal deformations of the theory preserving the twisting on the worldsheet (by general arguments that map an element of BRST cohomology to a descendant that can generate a deformation). When considered from the spacetime perspective, the elements of BRST cohomology should translate into spacetime fields and we expect the BRST closure condition to correspond to the *linearized* spacetime equations of motion. This is true in physical string theory and can be derived more rigorously via open string field theory for topological strings, as will be reviewed below.

For the normal modes, the BRST cohomology condition can be translated into constraints on deformations of the calibrated submanifolds, such that these modes correspond to tangent vectors on the moduli space of (co)associative cycles in the G_2 manifold.

For tangential modes, the BRST cohomology condition looks different for the different cycles. On the 3-cycle, BRST closure and co-closure of the tangential mode A_a imply $dA = 0$ and $d^\dagger A = 0$, so that A is a flat connection in a fixed gauge, and we expect a gauge theory whose solutions correspond to gauge-inequivalent flat connections. On the 4-cycle, BRST closure and co-closure of A_a imply

$$*_4\, dA = -dA\ , \qquad\qquad d^\dagger A = 0\ . \tag{8.42}$$

These equations are the linearization of the condition for a variation of a gauge field to be a deformation of an instanton solution (c.f. equation (5.50) in [167]). This suggests, in analogy with the geometric moduli, that the theory on the worldvolume should be a gauge theory extremizing on instantons and that marginal tangential deformations of the worldsheet theory should correspond to tangent vectors on the moduli space of instantons.

In the case of both the 3- and 4-cycle, the worldvolume theory will include contributions from the normal and tangential modes, and so should result in a theory whose moduli space includes the normal and tangential deformations that we have determined in section 8.2. We also expect that the other physical states, which are massless in the twisted theory, may still play a role in the spacetime action even though they cannot be used to generate boundary deformations of the CFT[9], and hence are not moduli of the theory.

To determine the relevant spacetime actions and how the normal and tangential moduli, as well as the higher ghost number fields, come into play we will start by considering Witten's derivation of Chern-Simons theory from open string field theory (OSFT). We will find that by restricting our attention to tangential modes on a calibrated 3-cycle we can re-derive Witten's Chern-Simons theory simply by following the arguments of [133]. We will then attempt to generalize this derivation to include normal modes. Their contribution is expected to be related to the topological theories in [138, 139], whose actions also localize on the moduli space of associative 3-cycles (though, as we will see, this relation

[9]Only a ghost number one state has a 1-form descendant with ghost number 0; ghost number p states have p-form descendants with ghost number zero, so to preserve the ghost number in the worldsheet action we would have to integrate them over a p-cycle on the worldsheet.

is mostly at the level of equations of motion). Following a comment in [133], we expect the higher string modes to be related to additional fields generated by gauge-fixing the CS action. This is discussed in appendix B of [1].

Once we have transplanted Witten's arguments for special Lagrangian branes in a Calabi-Yau to associative branes in a G_2 manifold, we will apply them to branes wrapping coassociative cycles and branes wrapping all of Y.

8.4.1 Chern-Simons theory as a string theory

In [133] Witten argues that the open A-model on T^*M reduces exactly to Chern-Simons theory on M, for any 3-manifold M. There are several arguments supporting this claim and we will attempt to generalize them below to the G_2 case. Before doing so, we first review them briefly.

The first argument concerns Q-invariance of a boundary term in the string path integral. In general the open string path integral can be augmented by coupling to a 'classical' background gauge field. This is done by including an additional piece in the integrand of the path integral which is of the form

$$\mathrm{Tr} P \exp\left(\oint_{\partial\Sigma} X^*(A)\right) . \tag{8.43}$$

Here A is a (non-abelian) connection defined on the brane M and the term above is a Wilson loop for the pull-back of this connection along the boundary of the worldsheet Σ. Requiring that this new term preserve the Q-invariance of the action implies that the field strength $F = dA + A \wedge A$ must vanish. Hence open strings in the A-model can only couple to flat connections.

To more rigorously establish that the relevant spacetime theory is Chern-Simons theory, Witten considers the OSFT action

$$\int \mathcal{A} \star Q\mathcal{A} + \frac{2}{3}\mathcal{A} \star \mathcal{A} \star \mathcal{A} , \tag{8.44}$$

where $\mathcal{A}$ is a functional of the open string modes quantized on a fixed time slice, and Q is the appropriate BRST operator of the theory. The integration measure is defined by the path integral over the disc[10]. The linearized equations of motion (coming from

[10] There is a subtlety here. In OSFT for the bosonic string this measure involves gluing together several discs using conformal transformations, but in the setting of a topological theory all the states have conformal weight zero under the twisted stress-tensor so they do not transform under conformal transformations.

the quadratic part of the OSFT action) enforce the requirement that physical states are BRST-closed on-shell:

$$Q\mathcal{A} = 0 \; . \tag{8.45}$$

In the large coupling constant limit ($t \to \infty$) the Q-cohomology can be studied by restricting to functionals $\mathcal{A}$ that depend only on the string zero-modes, X_0^μ and ψ_0^μ. The BRST operator, Q, acting on such states, reduces to the exterior derivative d on T^*M (which we can write in terms of the zero modes)

$$d = dx^\mu \frac{\partial}{\partial x^\mu} = \psi_0^\mu \frac{\partial}{\partial X_0^\mu} \; . \tag{8.46}$$

Since the $t \to \infty$ limit is exact in the A-model (modulo world-sheet instantons which are not present when the target space is T^*M), these identifications are not approximations but rather exact statements. This allows one to identify the string field action with Chern-Simons theory.

To make this identification one must identify the string field $\mathcal{A}$ with the target space gauge field $A_\mu(x)dx^\mu$. The general form for $\mathcal{A}$ at large t is given by the expansion

$$\mathcal{A}(X^\mu, \psi^\mu) = f(X_0) + A_\mu(X_0)\psi_0^\mu + \beta_{\mu\nu}(X_0)\psi_0^\mu\psi_0^\nu + C_{\mu\nu\rho}(X_0)\psi_0^\mu\psi_0^\nu\psi_0^\rho \; , \tag{8.47}$$

in 3 dimensions. The reason that $\mathcal{A}$ reduces to $A_\mu(X_0)\psi_0^\mu$ is simply that only ghost number one string fields should be considered, and here ghost number coincides with fermion number. Witten comments that it is possible to relate the other terms in the expansion to ghost and anti-ghosts fields derived from gauge-fixing CS theory [169], or alternatively gauge-fixing OSFT. In appendix B of [1] it was shown that this is indeed the case when this derivation is repeated on an associative cycle in a G_2 manifold.

Witten provides a final argument for CS theory as the string field theory for the A-model, namely that the open string propagator on the strip reduces to the CS propagator in the large t limit. This is essentially the statement that $\frac{b_0}{L_0} = \frac{d^\dagger}{\Box}$. For the topological string, b_0 is replaced by the superpartner of the stress-energy tensor in the twisted theory (i.e. $Q^\dagger$ in $T = \{Q, Q^\dagger\}$). In the G_2 case this would be (tentatively) $G^\uparrow_{-\frac{1}{2}}$ [12].

We will now attempt to establish the validity of these arguments for the open G_2 string ending on a calibrated 3-cycle. Before doing so we should mention that what was missing in this treatment is a discussion of the normal modes on the brane. It is not immediately clear whether these modes modify the Chern-Simons action on the special Lagrangian cycle (though one would imagine they should in order to capture the dependence of the theory on the geometric moduli of the brane).

8.4.2 CHERN-SIMONS THEORY ON CALIBRATED SUBMANIFOLDS

If we consider only the tangential modes on a calibrated cycle then the Q-closure conditions become (in the free field approximation)

$$\partial_a f(X) = 0 \ , \tag{8.48}$$
$$\epsilon^{abc}\partial_a A_b = 0 \ ,$$
$$\epsilon^{abc}\partial_a \beta_{bc} = 0 \ , \tag{8.49}$$

for the degree 0, 1, and 2 components of the string field. Here we have already used that $\phi_{abc} \propto \epsilon_{abc}$ on the 3-cycle. This is consistent with the notion that $Q = G^{\downarrow}_{-\frac{1}{2}} = d$ in the large t limit. More generally, the complex (7.6), which encodes the BRST cohomology, reduces, when restricted to the tangential directions on an associative 3-cycle, to the de Rham complex so $Q = d$ and $Q^\dagger = d^\dagger$.

Recall, from the discussion in Section 7.3.3, that, in contrast to the situation in the A-model, we do not have an explicit worldsheet action to work with and hence do not have a Hamiltonian formulation which might directly establish the t invariance of the action. Assuming this invariance none-the-less, the equations above imply that the quadratic part of the string field action reduces to the quadratic part of Chern-Simons theory. That is, in the large t limit, the Q-closure constraint becomes the linearized CS equation of motion. Here we have also considered modes with fermion number different from one; these are discussed in appendix B of [1] in relation to gauge-fixing Chern-Simons theory.

Also in this limit (of free string theory approximation), the $Q^\dagger$-closure constraints become

$$\nabla_a A^a = 0 \ ,$$
$$\nabla_a \beta^{ab} = 0 \ . \tag{8.50}$$

The first term is just the gauge-choice $d^\dagger A = 0$. The spacetime interpretation of β_{ab} is discussed in appendix B of [1] but, as it is not needed here, we will not review it. Let us now translate the rest of Witten's arguments to the G_2 case.

The argument is essentially that open string field theory with the action (8.44) reduces to Chern-Simons theory in the large t limit, if one restricts the string field to have ghost number 1 (which, in the G_2 case, translates into fermion number 1 because the ghost number is the grading for the Q-cohomology, and that is given by the fermion number). That this holds for the kinetic term follows because we have shown that the linearized CS action is the same as the linearized Q-closure condition.

For the interaction term this just follows from the fact that the 3-pt function of the ghost number one parts of $\mathcal{A}$ reduces to the wedge products of the Lie algebra valued 1-forms, $A_a(x)dx^a$. This is because, at large t, $\mathcal{A}$ depends only on the zero modes so the ghost number one part has the form $A_a(X_0)\psi_0^a$ which can be mapped to one-forms in spacetime. We showed in section 8.3.1 that the 3-pt function of these modes is just the 3-pt correlator of CS theory.

Witten also shows that the propagator of the OSFT reduces, in the $t \to \infty$ limit, to the CS propagator. We will reproduce this argument briefly here for the G_2 case. A much more complete treatment (of the analogous A/B-model argument) can be found in section 4.2 of [133]. The open string propagator is simply given by the partition function of a finite strip, of length T and width 1 with the standard metric

$$ds^2 = d\sigma^2 + d\tau^2 \ . \tag{8.51}$$

In OSFT the moduli space of open Riemann surfaces is built by gluing such strips together. The strip has one modulus, namely its length, so in calculating the partition function one insertion of $G^{\uparrow}_{-\frac{1}{2}}$ folded against a Beltrami differential μ is required [12]

$$\int d\sigma d\tau \ \mu(\sigma,\tau) G^{\uparrow}(\sigma,\tau) \ . \tag{8.52}$$

The Beltrami differential here is just given by a change to the metric that changes the length of the strip and corresponds to a function $f(\tau) = \delta T \cdot \delta(\tau - \tau_0)$ for any τ_0 on the strip. Here δT is the infinitesimal change in the length of the strip generated by this differential. Thus the insertion becomes

$$\int d\sigma d\tau \ \delta T \cdot \delta(\tau - \tau_0) G^{\uparrow}(\sigma,\tau) = \delta T \int d\sigma \ G^{\uparrow}(\sigma,\tau_0) \ . \tag{8.53}$$

Because we have been working in the NS sector, the integral of the current $G^{\uparrow}(z)$ around a contour (given by fixed τ_0 which maps to a half-circle in the complex plane) will just give a $G^{\uparrow}_{-\frac{1}{2}}$ insertion in the world-sheet path integral, so its overall form is

$$\int_0^{\infty} DT (G^{\uparrow}_{-\frac{1}{2}}) e^{-TL_0} = \frac{G^{\uparrow}_{-\frac{1}{2}}}{L_0} \ . \tag{8.54}$$

By our previous identification of $G^{\uparrow}_{-\frac{1}{2}}$ with $d^{\dagger}$ (this becomes $d^{\dagger}$ on M for tangential modes) and L_0 with $\Box$ in the large t limit, this becomes $\frac{d^{\dagger}}{\Box}$ which is the CS propagator [133]. One should note that in the A-model this follows rather directly but in the G_2 string it depends on the fact that $\phi_{abc} \propto \epsilon_{abc}$ on the associative cycle (so, as previously mentioned, $Q = \check{D}$ reduces to d) and thus, in particular, might not hold on a coassociative cycle.

There is a final argument one can make in favour of CS theory, though it is more heuristic. We want to argue, as Witten has, that coupling the worldsheet to a classical background gauge field via a term such as (8.43) requires this background to satisfy $F = 0$ which is the equation of motion for Chern-Simons theory.

In [12], a heuristic version of the twisted G_2 action is derived using the decomposition of worldsheet fermions into $\uparrow$ and $\downarrow$ components, $\psi = \psi^{\uparrow} + \psi^{\downarrow}$. This is heuristic because this decomposition is essentially quantum and is not understood at the level of classical fields. Using this decomposition we can check Witten's argument for the BRST-invariance of a boundary coupling to a classical configuration of the gauge field

$$\mathrm{Tr}P \exp\left(\oint_{\partial\Sigma} A_\mu \partial_t X^\mu\right) . \tag{8.55}$$

The variation of this factor in the partition function under $[Q, X^\mu] = \delta X^\mu$ is given by

$$\mathrm{Tr}P \oint_{\partial\Sigma} \delta X^\mu \partial_t X^\nu F_{\mu\nu} d\tau \cdot \exp\left(\int_{\partial\Sigma;\tau} A_\mu \partial_t X^\mu\right) , \tag{8.56}$$

where the contour in the exponent must start and end at the point τ [133]. To make this variation vanish requires that the first term vanish and since [12]

$$\delta X^\mu = i\epsilon_L \psi_L^{\downarrow\mu} + i\epsilon_R \psi_R^{\uparrow\mu} , \tag{8.57}$$

this implies that $F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu] = 0$ for classical configurations of the background gauge field A. This is, of course, the Chern-Simons equation of motion.

In the physical theory one could also couple to a term of the form $C_{\mu\nu}\psi^\mu\psi^\nu$, but no such terms seem to effect the derivation of $F = 0$ above in the A-model, because any such coupling results in a variation which cannot cancel the gauge-field coupling.

The only boundary term in a topological theory should be generated by the descent procedure starting from a Q-closed ghost number one field whose descendent is a ghost number zero one-form that is given by

$$\{G^{\uparrow}_{-\frac{1}{2}}, A_\mu \psi^\mu\} = A_\mu \partial_t X^\mu + \pi^2_{\mathbf{14}}(\partial_\mu A_\nu \psi^\mu \psi^\nu) . \tag{8.58}$$

Both these terms have conformal weight 1 and, by virtue of a standard descent argument, are Q-closed up to a total derivative. To apply Witten's argument here it is necessary to understand why the second term cannot appear on the boundary. This follows because we are considering modes tangential to an associative cycle and one can check that on such a cycle $\Lambda^2 T^* M \subset \iota^*(\Lambda^2_7(Y))$ (here $\iota : M \to Y$ is the embedding of the three cycle into the ambient G_2).

To derive the Chern-Simons action we have considered only the ghost number one part of the string field $\mathcal{A}$ as this is the standard prescription in OSFT. In some cases, however, it is desirable to consider the full expansion of $\mathcal{A}$ and include fields of all ghost number in the action. This is because the higher modes just play the role of ghosts in gauge-fixing the OSFT action [170]. This is a special feature of Chern-Simons like theories [169] and so will apply for all the brane theories that we derive. Appendix B of [1] descibes the general form of the gauge-fixed actions for these theories but we omit such a discussion here as we will not consider the quantum versions of these theories and simply refer the reader to [1] instead.

8.4.3 Normal mode contributions

In the previous section we argued that the tangential modes of the G_2 worldsheet correspond to gauge fields in a CS theory on the 3-cycle and when higher string modes are included this becomes gauge-fixed CS theory.

We are also interested in terms in the effective action that include the normal modes. The most direct way to to get at a normal mode action is to simply expand the terms $\mathcal{A} \star Q\mathcal{A}$ and $\mathcal{A} \star \mathcal{A} \star \mathcal{A}$ in the OSFT action. Ignoring the higher string modes, we have

$$\begin{aligned} \mathcal{A} &= A_a \psi^a + \theta_J \psi^J \ , \\ Q\mathcal{A} &= \{Q, A_a \psi^a\} + \{Q, \theta_I \psi^I\} \\ &= \phi_{IJc} \phi^{cde} \nabla_d A_e \psi^I \psi^J + \phi_{abc} \phi^{cde} \nabla_d A_e \psi^a \psi^b + \phi_{aIJ} \phi^{JbK} \nabla_b \theta_K \psi^a \psi^I \ . \end{aligned} \tag{8.59}$$

Recall that the integration of expressions involving string fields, $\mathcal{A}$, in the OSFT action corresponds to evaluating the correlator of the integrand, decomposed in individual string modes on the disc. In the G_2 string only certain combinations of string modes will have a non-vanishing 3-pt function depending on the conformal blocks the modes correspond to (see [12]). In our calculation of the 3-pt functions above, this translates into non-vanishing 3-pt functions when we can contract the spacetime indices of the string modes with the 3-form ϕ. From our previous calculation of three point functions in sections 8.3.1 (see also Appendix B of [1]) we find the generic form of a 3-pt function on the disc

$$
\begin{aligned}
\langle \lambda\omega \rangle &= \int_M \phi^{\mu\nu\rho} \mathrm{Tr}\left(\lambda_\mu \omega_{\nu\rho}\right) \ , \\
\langle \alpha\beta\gamma \rangle &= \int_M \phi^{\mu\nu\rho} \mathrm{Tr}\left(\alpha_\mu \beta_\nu \gamma_\rho\right) \ ,
\end{aligned}
\tag{8.60}
$$

(where, e.g. $\omega = 1/2\, \omega_{\mu\nu}(x)\psi^\mu\psi^\nu$). Doing this gives the following action

$$
\begin{aligned}
S_{\text{deg 1}} = \int_M \phi^{abc}\, \mathrm{Tr}\left(A_a \nabla_b A_c + \frac{2}{3} A_a A_b A_c\right) \\
+ \phi^{IaJ}\, \mathrm{Tr}\big(\theta_I (\nabla_a \theta_J + [A_a, \theta_J])\big) \ ,
\end{aligned}
\tag{8.61}
$$

where the trace Tr is over Lie algebra indices. The interaction terms can be calculated directly in string perturbation theory by checking 3-pt disc amplitudes whereas the kinetic terms coming from $\mathcal{A} \star Q\mathcal{A}$ vanish in perturbation theory because on-shell string modes satisfy $Q\mathcal{A} = 0$. To determine them we either simply consider all the terms of the correct degree in the string mode decomposition of $\mathcal{A} \star Q\mathcal{A}$ or 'formally' calculate 3-pt functions assuming the field $\mathcal{A}$ is off-shell. Both result in the same action and as a consistency check, the linearized equations of motion for this action correspond to the BRST closure of the string modes. We have not been too careful with the coefficients in (8.61) but this is because most coefficients either follow from gauge invariance or can be absorbed into field redefinitions.

The equations of motion for this action are

$$
\epsilon^{abc} F_{bc} = \phi^{IaJ} [\theta_I, \theta_J] \ , \tag{8.62}
$$

$$
\phi^{aIJ} \big(\nabla_a \theta_J + [A_a, \theta_J]\big) = 0 \ . \tag{8.63}
$$

In the abelian case this just reduces to $F = 0$ and the geometric constraint (8.29) on the normal modes describing associative deformations. In the non-abelian case this is no longer true but of course in this setting we have lost the simple association of θ_I with normal deformations of the brane, as the string modes become matrix-valued.

At first glance the equations above look similar in form to the Seiberg-Witten type equations (32) and (40) in [165]. This reference is concerned with resolving the singular structure of the moduli space of deformations of associative submanifolds in a general G_2 manifold by considering a larger space of deformations where one is allowed to also deform the induced connection on the normal bundle to make the deformed submanifold associative. This amounts to a choice of complex structure on the normal bundle, for each deformation of the 3-submanifold, such that its reduced structure group $U(2) \subset SO(4)$ in the G_2 manifold is compatible with the induced metric connection. This additional topological restriction on the G_2 manifold is something we have not assumed and indeed, for general gauge group, there is no obvious relation between (8.62), (8.63) and the purely geometric equations in [165][11].

8.4.4 ANTI-SELF-DUAL CONNECTIONS ON COASSOCIATIVE SUBMANIFOLDS

We expect that the worldvolume theory on the 4-cycle should have equations of motion corresponding to the BRST closure of the associated string modes. Let us consider the following action

$$S[A, \theta] = \int_M \phi^{Iab} \mathrm{Tr}\big(\theta_I F_{ab}\big) + \frac{2}{3}\, \phi^{IJK} \mathrm{Tr}\big(\theta_I \theta_J \theta_K\big) \ . \tag{8.64}$$

As with the action on a 3-cycle we cannot directly check the quadratic terms by considering a string correlator because the relevant correlators vanish for on-shell states as dictated by the fact that the quadratic terms in the action determine the BRST closure condition. Rather, we can compare the linearized equations of motion (generated purely by the quadratic terms) and the string BRST closure condition and these should match.

The abelian θ_I equation of motion is now just $\phi^{Iab} F_{ab} = 0$, which implies anti-self-duality of F and so matches the BRST closure condition. The A_a equation of motion is

$$\phi^{abI} D_b \theta_I = 0 \ , \tag{8.65}$$

where $D_a = \nabla_a + [A_a,]$ on M. This equation is more conveniently expressed in terms of the self-dual 2-form $\omega_{ab} = \phi_{abI} \theta^I$ on M. At the linear level, the equation above implies ω is co-closed, and hence also closed since it is self-dual. Thus we have the correct linearized condition for coassociative deformations found by McLean.

We can also consider the formal structure of the term $\mathcal{A} \cdot Q\mathcal{A}$ in the OSFT action, letting $\mathcal{A}$ go 'off-shell', and indeed we find matching.

As a further check we should compare the interaction term to string scattering amplitudes. The 3-pt function for a general degree one vertex operator in the topological theory is given by

[11] It is possible that the $U(1)$ part of our gauge connection could be related to the $U(1) \subset U(2)$ part of the induced connection on the normal bundle with fixed complex structure in [165].

$$\lambda^3 \frac{3}{2} \int_Y \phi^{\alpha\beta\gamma}(x) \mathrm{Tr}\big(A_\alpha(x) A_\beta(x) A_\gamma(x)\big) \ . \tag{8.66}$$

On the 4-cycle the only non-vanishing components of ϕ must have an even number of tangential indices, which implies the following non-vanishing amplitudes

$$\begin{aligned} &\lambda^3 \frac{3}{2} \int_M \phi^{Iab} \mathrm{Tr}\big(\theta_I A_a A_b\big) \ , \\ &\lambda^3 \frac{3}{2} \int_M \phi^{IJK} \mathrm{Tr}\big(\theta_I \theta_J \theta_K\big) \ . \end{aligned} \tag{8.67}$$

The first line above corresponds to the cubic interaction θAA in the first term of (8.64) while second correlator in (8.67) implies the cubic vertex in the second term. This last term, of course, only corrects the non-abelian instanton equation of motion

$$\phi^{Iab} F_{ab} = -\phi^{IJK} [\theta_J, \theta_K] \ , \tag{8.68}$$

and so has no effect on the geometric interpretation in the abelian case.

In [168] Leung proposes a 1-form on the space $\mathcal{C} = \mathrm{Map}(M, Y) \times \mathcal{A}(M)$ where M is a 4-manifold, Y is a G_2 7-manifold and $\mathcal{A}(M)$ is the space of Hermitian connections on the gauge bundle $E \to M$ (with fibre G)

$$S(f, D_E)(v, B) = \int_M \mathrm{Tr}\left(f^*(\iota_v \phi) \wedge F_E + f^*(\phi) \wedge B\right) \ . \tag{8.69}$$

Here $(f, D_E) \in \mathcal{C}$ and $(v, B) \in T_{(f, D_E)}\mathcal{C}$ with v a section of TY, $B \in \Lambda^1(M, \mathrm{ad}\, G)$ and F_E the curvature of D_E (here f is an element of $\mathrm{Map}(M, Y)$ and should not be confused with the f used to denote the zero fermion component of the string field). The one-form S is invariant under diffeomorphisms of M and its zeros correspond to coassociative embeddings $f(M) \subset Y$ with anti-self-dual connections on them. This follows from the fact that S must vanish when evaluated on arbitrary vectors, B, implying $f^*(\phi) = 0$, and arbitrary v implying that $F_E = -{*}F_E$.

To compare with our theory we do not want to consider the space of all such maps, but only the local deformations of a given coassociative $f(M)$ in Y, so we only consider fluctuations around a fixed coassociative submanifold. Thus we will take f to be a coassociative embedding implying that the second term in the action above vanishes and $*\phi$

defines the volume form on the embedded coassociative 4-cycle $f(M)$. Thus, we rewrite Leung's functional to generate the following action functional[12]

$$S^0[A,\theta] = \int_M \mathrm{Tr}\left(f^*(\iota_\theta \phi) \wedge F\right) = \int_M \phi^{Iab}\mathrm{Tr}\big(\theta_I\left(\partial_a A_b + A_a A_b\right)\big) \ , \tag{8.70}$$

using the identity $\epsilon_{abcd}\,\phi^{cdI} = 2\,\phi_{ab}{}^I$ on the coassociative cycle.

Thus we see that the open G_2 string has reproduced the action Leung suggested in order to study SYZ in the G_2 setting and it has also introduced an additional term that is not present in Leung's action.

8.4.5 Seven-cycle worldvolume theory

As in physical string theory, it is natural to expect the 3- and 4-cycle theory to look like the dimensional reduction of a theory on the whole 7-manifold (which is trivially calibrated by its volume form $\phi \wedge *\phi$). Lee et al [171], who propose theories closely related to our 3- and 4-cycle theories, claim that this theory should be related to (deformed) Donaldson-Thomas theory [158].

The 7-cycle theory can be determined exactly the same way as the 3- and 4- cycle theory. For the interaction term we just calculate the 3-pt functions of the (ghost number one) terms in $\langle \mathcal{A} \star \mathcal{A} \star \mathcal{A} \rangle$ given by (8.60). The kinetic terms, defining the linearized equations of motion, should correspond to $Q\mathcal{A} = 0$ and they should match $\mathcal{A} \star Q\mathcal{A}$.

This gives the following action

$$S = \int_Y \phi^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho\right) = \int_Y *\phi \wedge CS_3(A) \ . \tag{8.71}$$

The equation of motion for this action is

$$*\,\phi \wedge F = 0 \ . \tag{8.72}$$

This is one of the equations in [158] where it is argued to be associated with the 7-dimensional generalization of Chern-Simons theory. In the abelian theory this equation of motion is simply $\check{D}A = 0$ which has no global solutions which are not exact (i.e. $A = df$) because $H^1_7(Y) = 0$ for G_2 manifolds. Of course, as a gauge field A need not be a global

[12]More precisely, Leung's one-form, Φ_0, descends to a closed one-form on the space '$\mathcal{C}$/Diffeo(M)' and this form is locally the derivative of a functional, $\mathcal{F}$, whose critical points are zeros of Φ_0. Our action is most closely related to this functional.

one form and then this result no longer applies. This is similar to the situation one finds for Chern-Simons theory on a simply connected manifold.

Note that for the action (8.71) to be gauge invariant under large gauge transformations $*\phi$ must actually be an integral cohomology class. A similar issue arises in holomorphic Chern-Simons theory as mentioned by Nekrasov in [144] but, as the three-form Ω is holomorphic, it is not clear that it can always be normalized to be integral. Nekrasov notes, however, that the integrality condition is precisely the condition on the complex moduli of the CY to be solutions of the attractor equations. It would be interesting to understand if the integrality of $*\phi$ has a similar interpretation.

In [171] the authors want to consider solutions to the deformed Donaldson-Thomas equation

$$* \phi \wedge F = \frac{1}{6} F \wedge F \wedge F \ , \tag{8.73}$$

which would involve adding a term $CS_7(A)$ to the Lagrangian above. It is not at all clear why such a term would appear in OSFT but in Section 7 of [1] we see that such a term does emerge in a rather interesting way when quantizing this theory.

8.4.6 Dimensional reduction, A- and B-branes

Reducing the open topological G_2 string on $\mathrm{CY}_3 \times \mathrm{S}^1$ gives rise to both special Lagrangian A-branes and holomorphic B-branes on CY_3. This follows from the decomposition of ϕ and $*\phi$ in terms of the holomorphic 3-form and Kähler form on CY_3 (see appendix H). The A-branes arise when reducing the associative 3-cycle action (8.61) in the normal direction. The resulting action

$$\int_M \epsilon^{abc} \mathrm{Tr} \left(A_a \nabla_b A_c + \frac{2}{3} A_a A_b A_c \right) + \rho^{aij} \mathrm{Tr}\big(\theta_i (\nabla_a \theta_j + [A_a, \theta_j])\big) \ , \tag{8.74}$$

is the real part of complex Chern-Simons theory, where the indices $a, b, c = 1, 2, 3$ are in the SLag while $i, j = 4, 5, 6$ are in the normal direction. The normal modes appear quadratically and can be integrated out (see Section 7 in [1] for a discussion of this issue on an associated cycle).

Similarly we can reduce the 4-cycle action (8.64) in the tangential direction. This is again a special Lagrangian brane in CY_3 but now calibrated by $\hat{\rho}$ instead of ρ, and the worldvolume action is given by the imaginary part of complex Chern-Simons theory

$$\int_M \rho^{iab} \mathrm{Tr}(\theta_i F_{ab}) + \frac{2}{3} \rho^{ijk} \mathrm{Tr}(\theta_i \theta_j \theta_k) \ , \tag{8.75}$$

with the additional constraint $D_a\theta_i = 0$ for the normal modes.

The B-branes are simplest to find starting from the 7-cycle worldvolume theory (8.71) and reducing on the CY_3. We find

$$\begin{aligned} S &= \int_{CY_3} \hat{\rho} \wedge CS(A) + k \wedge k \wedge \mathrm{Tr}(\lambda F) \\ &= \frac{1}{2i}\int_{CY_3} \Omega \wedge CS(A) - \frac{1}{2i}\int_{CY_3} \bar{\Omega} \wedge CS(\bar{A}) + \int_{CY_3} k \wedge k \wedge \mathrm{Tr}(\lambda F) \ , \end{aligned} \tag{8.76}$$

where $*\phi = \hat{\rho} \wedge dt + \frac{1}{2} k \wedge k$, t parametrizes the circle direction, $\Omega = \rho + i\hat{\rho}$ is the holomorphic 3-form of the Calabi-Yau, and $\lambda = A_t$ is the scalar component of the gauge field in the reduction. The action is the sum of B-model 6-brane and $\bar{\mathrm{B}}$-model 6-brane actions (the appearance of the imaginary part of the holomorphic 3-form rather than the real part is just a matter of convention). The extra term in the action comes with the Lagrange multiplier λ, and so it expresses the constraint

$$k \wedge k \wedge F = 0 \ .$$

This extra condition is related to stability of the brane (complexifies the $U(N)$ symmetry). Lower-dimensional 4-branes and 2-branes then follow by further dimensional reduction, where again we obtain B- and $\bar{\mathrm{B}}$-model actions together with a stability condition.

It is remarkable that like the closed topological M-theory, the open topological string also contains the A and B+$\bar{\mathrm{B}}$ models. Perturbatively the B+$\bar{\mathrm{B}}$-models are decoupled, and it would be interesting to understand if there is a non-perturbative coupling between them.

8.5 Remarks and open problems

So far we have determined the spectrum of the open G_2 string and related it to the world-volume field theories of branes in a G_2 manifold. In this section we would like to conclude by making some final remarks regarding issues that still need to be resolved as well as interesting directions for further research.

8.5.1 Quantization

Although we have omitted a discussion of quantum aspects of these theories in this thesis the latter were considered in detail in our original work [1]. There we made a preliminary investigation of the gauge-fixing and quantization of (8.2) and its reductions to four- and three-dimensional branes. As was the case in the open string field theory for A-branes, the gauge-fixed actions look very similar to the action (8.2) once we replace the ghost number one field A by a field of arbitrary ghost number. We also studied the one-loop partition functions of the various open string field theories, and found that they tend to have the effect of shifting the tree-level theories in a rather simple way. This is similar to the one-loop shift $k \to k + h(G)$ of the level k in ordinary Chern-Simons theory, with $h(G)$ the dual Coxeter number of the gauge group G. In particular, we found that $*\phi$ in (8.2) is shifted by a four-form proportional to the first Pontrjagin class of the manifold Y. We have not yet attempted to determine whether (8.2) is renormalizable and well-defined as a quantum theory (which, by naive power-counting, it is not) but we expect that it should be as it is equivalent to a string theory (a similar issue occurs for holomorphic Chern-Simons in the B-model). For a more detailed discussion of quantization of these theories the reader is referred to [1].

8.5.2 HOLOMORPHIC INSTANTONS ON SPECIAL LAGRANGIANS

In dimensionally reducing the G_2 branes on a Calabi-Yau Z times a circle, we have found that we almost reproduce the real versions of the gauge theories for the open A- and B-models. There is a discrepancy, however. If one considers a special Lagrangian $M \subset Z$, with holomorphic open curves $\Sigma \subset Z$ ending on M so that $\partial\Sigma \subset M$, then the A-model branes will receive worldsheet instanton corrections to the standard Chern-Simons action. A naive dimensional reduction of the associative theory on a G_2 manifold $Y = Z \times S^1$ gives a special Lagrangian in Z with the Chern-Simons action without instanton corrections.

This issue is already present in the closed topological G_2 string. When reducing on $\mathrm{CY}_3 \times \mathrm{S}^1$, the closed G_2 string gives a combination of A and B+$\bar{\mathrm{B}}$ models. But it is non-trivial to see where the worldsheet instanton corrections in the A-model would come from, given that the G_2 theory appears to localize on constant maps. A possible resolution suggested in [12] is that since, unlike a generic G_2 manifold, the manifold $\mathrm{CY}_3 \times \mathrm{S}^1$ has 2-cycles, worldsheet instantons may now wrap these 2-cycles. However, upon closer inspection, this possibility appears rather unlikely. A much more straightforward explanation is that the worldsheet instanton contribution is due to topological membranes (i.e. topological 3-branes of the type discussed in this paper) that wrap associative cycles of the form $\Sigma \times S^1$ in $\mathrm{CY}_3 \times \mathrm{S}^1$. Such 3-cycles are indeed associative as long as Σ is a holomorphic curve in the Calabi-Yau manifold.

Returning to the open worldsheet instanton contribution to branes in the A-model, there are two ways to obtain these from the topological G_2 string on $\mathrm{CY}_3 \times \mathrm{S}^1$. The first way is to lift the A-model brane together with the open worldsheet instanton to a single associative cycle in $\mathrm{CY}_3 \times \mathrm{S}^1$. This is similar to the M-theory lift in terms of a single M2-brane of a configuration of a fundamental string ending on a D2-brane in type IIA string theory. To describe it, we take a special Lagrangian 3-cycle C in a Calabi-Yau manifold X, plus an open holomorphic curve Σ. We denote the boundary of Σ by $\gamma \subset C$. We first lift C to $X \times S^1$, which we describe in terms of a map $C \to X \times S^1$ which takes $x \in C$ to $(x, \theta(x)) \in X \times S^1$. Here, $\theta(x)$ describes an S^1-valued function on C which we want to have the property that it winds once around the S^1 as we wind once around the curve $\gamma \subset C$. The lift is therefore one-to-many, as the image of a point in γ is an entire circle, and because of this the lift of C is an open submanifold of $X \times S^1$ with boundary $\gamma \times S^1$. We can now glue the naive lift of Σ, which is $\Sigma \times S^1$, to the lift of C to form a closed 3-manifold M, since the boundary of $\Sigma \times S^1$ is also $\gamma \times S^1$. In this way we have obtained a closed 3-manifold $M \subset X \times S^1$ which projects down to C and Σ upon reduction over the S^1. The 3-manifold M is not calibrated, but we can compute the integral of ϕ over M. The result is simply $\int_C \rho + \int_\Sigma k$ if we normalize the size of the S^1 appropriately. The fact that the lift of C winds around the circle does not yield any additional contribution to $\int_M \phi$ because the restriction of k to C vanishes identically.

We have thus constructed a closed 3-cycle M such that the integral of ϕ over it has the correct structure, geometrically, to yield the worldsheet instanton contribution. The final step is to minimize the volume of M while keeping its homology class fixed. This will not change $\int_M \phi$ but presumably lead to the sought-for associative 3-cycle with the right properties.

In order to push this program further and relate $\int_\Sigma k$ to the (exponentiated) weight of a holomorphic instanton we note that maps $\theta(x)$ which wind about γ n times will generate contributions such as $n \int_\Sigma k$. Carefully summing over all lifts of this form with the appropriate weight might properly reproduce the instanton contributions.

An entirely alternative approach is to lift both C and Σ to $C \times S^1$ and $\Sigma \times S^1$. In this way we obtain an open associative 3-cycle ending on a coassociative 4-cycle in $X \times S^1$. To analyze whether this makes sense, we consider the simple example of an open 3-brane in $\mathbb{R}^7$ stretched along the 123-direction, ending on a coassociative cycle stretching in the 2345-direction. If we vary the action (8.61) on the 3-brane we obtain a boundary term

$$S_{\rm bndry} = \int dx^2 dx^3 \mathrm{tr}(A_3 \delta A_2 - A_2 \delta A_3 + \theta_5 \delta\theta_4 - \theta_4 \delta\theta_5 + \theta_7 \delta\theta_6 - \theta_6 \delta\theta_7) \; . \quad (8.77)$$

We obviously want Dirichlet boundary conditions for θ_6 and θ_7 so that the endpoint of the open 3-brane is confined to lie in the 4-brane. We also want θ_4 and θ_5 to be unconstrained at the boundary. If we therefore choose the boundary condition

$$A_2 = \theta_5 \qquad A_3 = \theta_4 \; , \quad (8.78)$$

the variations all cancel. To preserve these boundary conditions under a gauge transformation, we need to restrict the gauge parameter in such a way that its derivatives in the $2, 3$ vanish at the boundary. In this way we indeed find a consistent open 3-brane ending on a 4-brane.

8.5.3 EXTENSIONS

The actions we have discovered on topological branes wrapping cycles in a G_2 manifold are variants of Chern-Simons theories derived from OSFT. OSFT itself, as a generator of perturbative string amplitudes, might need to be augmented by terms that are locally BRST trivial but none-the-less have global meaning deriving from the topological structure of the space of string fields. In the bosonic open string such questions are currently inaccessible but in the topological case we see some motivation for local total derivative terms to be added to the action. One such potential term is

$$\int_Y F \wedge F \wedge \phi \tag{8.79}$$

that might describe lower dimensional branes dissolved in the seven dimensional brane. Such terms might be motivated by analogy with the Wess-Zumino terms on physical branes. Note, also, that this reduces to $F \wedge F \wedge k$ in six-dimensions, a term which appears in the A-model Kähler quantum foam theory [14] which Nekrasov suggests should be related to holomorphic Chern-Simons theory [144] (the latter is, of course, related to our theory by dimensional reduction). It would be interesting to try and probe for the existence of such terms directly in the G_2 world-sheet or OSFT theory.

The appearance of the $CS_7(A)$ term in the one-loop partition function in [1] suggests that perhaps this term appears in quantizing the theory and so should have been included in the original classical action.

Understanding if such terms do actually appear in these effective actions is interesting as it may play a role in the conjectured S-duality of the A/B model topological strings. In the latter it seems that one may need to consider both the open and closed theory simultaneously and then terms such as (8.79) might play a role in coupling these theories.

8.5.4 RELATION TO TWISTS OF SUPER YANG-MILLS

The theories we have found on G_2 branes are all topological theories of the Schwarz type (see [167] for the terminology) which is no doubt linked to the fact that they are generated by OSFT. A similar statement holds for branes in the A- and B-model.

The worldvolume theory on a brane in a G_2 or Calabi-Yau manifold in a physical model is a twisted, dimensionally reduced super Yang-Mills (SYM) theory [172] whose ground states are topological in nature. These are related to the topological field theories that can be constructed by twisting SYM and considering only the supersymmetric states (by promoting the twisted supercharge to a BRST operator). Such theories include the topological action for Donaldson-Witten theory [173] as well as its generalizations to higher dimensions [174]. These are generally field theories of the Witten type meaning that the action is itself a BRST commutator plus a locally trivial term.

Aside from the obvious connection to Chern-Simons theory via OSFT it would be interesting to understand why the topological theories on branes in topological string theory are generally of the Schwarz type (which are locally non-trivial) while the supersymmetric states of the twisted theories on a physical brane can be studied in a theory that is of the Witten type.

8.5.5 Geometric invariants

One of the most interesting open directions is to investigate the geometric or topological invariants our open worldvolume gauge theories compute, and perhaps use them, via open-closed duality, to discover the connection to the closed topological G_2 theory. It would be interesting to explore the full quantum open string partition function on a few examples of G_2 manifolds. The theory on the 3-cycle is basically Chern-Simons theory, while on the 4-cycle the gauge theory of ASD connections will be related naturally to Donaldson theory. It would very interesting to find a role for the partition functions in terms of the full physical string theory, as well as deepen connections with the mathematics results in [168]. Another open problem is to analyze these invariants in the special case of $\mathrm{CY}_3 \times \mathrm{S}^1$, and find a physical understanding of related mathematical invariants such as the one proposed by Joyce [175] counting special Lagrangian cycles in a Calabi-Yau manifold.

8.5.6 Geometric transitions

Open-closed duality techniques have proven very useful for topological string theory on Calabi-Yau manifolds. In particular, geometric transitions provide nice examples where closed topological string amplitudes can be computed from the gauge theory on the branes, which in this case is just Chern-Simons theory with possible worldsheet instanton corrections. Geometric transitions on G_2 manifolds in general are less studied, but interesting examples from the full string theory point of view are exhibited in e.g. [176][177]. In the present paper we derived the relevant worldvolume gauge theory actions from open topological strings and so, one of the immediate applications of our results is to study geometric transitions from the topological G_2 string point of view.

8.5.7 Mirror symmetry for G_2

Mirror symmetry on a Calabi-Yau 3-fold can be described in terms of the Strominger-Yau-Zaslow (SYZ) conjecture. One starts with a special Lagrangian fibration, and then the mirror manifold is conjectured to be the dual torus fibration over the same base. In physics language, the action of mirror symmetry on the fibres is T-duality. In [171], a G_2 version of the SYZ conjecture was suggested, relating coassociative to associative geometry. Evidence for the G_2 mirror symmetry was also found in G_2 compactifications of the physical IIA/IIB string theory on G_2 holonomy manifolds [177] [178]. It would be interesting to explore the action of mirror symmetry in the case of the topological G_2 models. A good starting point for this is by examining automorphisms of the closed G_2 string algebra such as those discussed in [179].

8.5.8 Zero Branes

Although we have not attempted a treatment here it should be possible to reduce the action (8.71) to zero dimensions to determine the world-volume of $D0$-branes on the G_2 manifold. This will be a matrix model which may be related in an interesting way to the G_2 geometry.

Appendix A

Supergravity Conventions and Notation

For various common definitions we refer to appendix A of [9], whose notation we follow.

In this appendix we give some more details on the conventions we take for various physical quantities mostly relevant for Part I. We work in units in which $c = \hbar = k_B = 1$ but we will keep dimensions of length explicitly in much of Chapter 3 where we take the decoupling limit of multicentered solutions. The coordinates x, t we take to have dimension of length. Angular coordinates, most of the time denoted by Greek letters as $\alpha\,, \theta\,, \psi$ etc will be dimensionless however. Furthermore we will take forms to be dimensionless. As e.g. $\omega = \omega_i dx^i$ is dimensionless this implies the components of forms have dimensions of inverse length, i.e. $[dx^i] = L,\ [\omega_i] = L^{-1}$ and $[\omega] = 1$. This convention implies that the Hodge star is dimensionful: $[\star] = L^{d-2p}$ when acting on a p-form.

In each dimension we define a natural Planck length l_d ($[l_d] = L$ of course) by normalising the Einstein-Hilbert action as

$$S_d^{\mathrm{EH}} = \frac{2\pi}{(l_d)^{d-2}} \int \sqrt{-g_d}\,\mathcal{R}_d\,, \tag{A.1}$$

and a reduced planck length by

$$\ell_d = \frac{l_d}{4\pi}\,. \tag{A.2}$$

A.1 M-THEORY VS IIA CONVENTIONS

We start from the following $11d$ M-theory metric:

$$ds^2_{11d} = R^2\, e^{4\phi/3}\, \Theta^2 + e^{-2\phi/3}\, ds^2_{10d}\,, \tag{A.3}$$

where ds^2_{10d} is a ten-dimensional metric and R is a constant with dimensions of length. The one form $\Theta = d\theta + 2\pi\, A$, with $\theta = \theta + 2\pi$ and A is a one form on the ten dimensional space. Furthermore ϕ is normalized in such a way that $\phi(\infty) = 0$. The M2-branes of this theory have a tension

$$T_{M2} = \frac{2\pi}{l_M^3}\,, \tag{A.4}$$

with $l_M = l_{11}$ is the 11 dimensional Planck length.

We can relate these to IIA quantities by reduction on the θ circle. As an M2 wrapped around this circle is a fundamental string we find:

$$T_{F1} = \frac{2\pi}{l_s^2} = 2\pi R\, T_{M2} = \frac{4\pi^2}{l_M^3} R \quad \Rightarrow \quad l_M^3 = 2\pi R\, l_s^2\,. \tag{A.5}$$

From the relation between M2 and D2 one easily infers

$$l_M^3 = g_s\, l_s^3\,, \tag{A.6}$$

where in our conventions $T_{\mathrm{D}p} = \frac{2\pi}{g_s\, l_s^{p+1}}$. The constants g_s and l_s are respectively the string coupling constant (at infinity) and the string length. They are arbitrary constants related to the 10 dimensional Planck length l_{10} by

$$l_{10}^4 = g_s\, l_s^4\,. \tag{A.7}$$

We can now reduce both the 11d and 10d theory on the same Calabi-Yau manifold X. Since

$$ds^2_{11d} = R^2\, e^{4\phi/3}\, \Theta^2 + e^{-2\phi/3}\left(ds^2_{4d} + ds^2_{CY\,\mathrm{IIA}}\right)\,, \tag{A.8}$$

we can relate the effective 5d and 4d metrics:

$$ds^2_{5d} = R^2 \left(\frac{V_{\mathrm{IIA}}}{V_M}\right)^{2/3} \Theta^2 + \left(\frac{V_{\mathrm{IIA}}}{V_M}\right)^{-1/3} ds^2_{4d}\,, \tag{A.9}$$

where we used that

$$e^{2\phi} = \frac{V_{\mathrm{IIA}}}{V_M}\,. \tag{A.10}$$

In a slightly more transparent form this is

$$ds^2_{5d} = \tilde{V}_{\mathrm{IIA}}^{2/3}\, \ell_5^2\, (2\Theta)^2 + \tilde{V}_{\mathrm{IIA}}^{-1/3}\, \frac{\hat{R}}{2}\, ds^2_{4d}\,, \tag{A.11}$$

with ℓ_5 the reduced 5 dimensional Planck length, we will also use the notation $2\Theta = d\psi + A^0_{4\mathrm{d}}$. We use various dimensionless objects:

$$\tilde{V}_M = \frac{V_M}{l_M^6}\,, \qquad \tilde{V}_{\mathrm{IIA}} = \frac{V_{\mathrm{IIA}}}{l_s^6}\,, \qquad \hat{R} = \frac{R}{\ell_5}\,. \tag{A.12}$$

The Calabi-Yau reduction relates all the different parameters at infinity. We will give the relations that will be of most importance to us. The relation between the 4d Planck length l_4 and the string length is

$$l_4 = g_{4d}\, l_s\,, \tag{A.13}$$

where

$$g_{4d}^2 = \frac{g_s^2}{\tilde{V}_{\mathrm{IIA}}^{\infty}} = \frac{1}{\tilde{V}_M}\,. \tag{A.14}$$

The effective 4d string coupling g_{4d} is in a hypermultiplet and thus constant in the solutions we will consider. Note that the same is true for $\tilde{V}_M$. The 4d and 5d Planck lengths are related by the size of the M-theory circle:

$$\ell_5 = \sqrt{\frac{\hat{R}}{2}}\,\ell_4\,. \tag{A.15}$$

Furthermore this size of the circle is immediately related to the size of the Calabi-Yau at infinity and thus to the value of the Kähler moduli at infinity, i.e.

$$\frac{\hat{R}^3}{8} = \tilde{V}_{\mathrm{IIA}}^{\infty} = \frac{1}{6}(J_\infty)^3\,. \tag{A.16}$$

Finally let us relate the reduced 4d plank length ℓ_4 to Newton's constant that appears in front of the 4d Einstein-Hilbert action as

$$S_4^{\mathrm{EH}} = \frac{1}{16\pi\, G_4}\int \sqrt{-g_4}\mathcal{R}_4\,. \tag{A.17}$$

This gives the relation

$$\ell_4 = \sqrt{2\, G_4}\,, \tag{A.18}$$

and by (A.15) this implies

$$\sqrt{G_4} = \frac{\ell_5^{3/2}}{\sqrt{R}}\,. \tag{A.19}$$

Appendix B

Marginal vs threshold stability

In this appendix we refine the commonly used notion of *marginal stability*. This refinement is, in our view, useful as there are two different physical situations that both go under the name of marginal stability in the current literature. Distinguishing between them is useful in analyzing the decoupling limit of Chapter 3. A somewhat similar distinction was already proposed in [67].

The notion that we want to refine and that is commonly referred to as marginal stability is that of two BPS states having aligned central charges for certain values of the moduli. In our case of interest, multicentered black holes in $\mathcal{N}$=2 supergravity, the BPS states are characterised by their charge Γ and their central charge is determined in terms of this charge and the scalar moduli t by $Z(\Gamma, t) = \langle \Gamma, \Omega(t) \rangle$. The length of the central charge vector, $|Z|$, corresponds to the mass, as we are considering BPS states, and its phase, α, characterises the supersymmetries left unbroken by this state. In case the moduli are such that for two BPS states Γ_1 and Γ_2 the phase aligns, the two BPS particles preserve the same supersymmetries and the binding energy of a BPS bounds state of them vanishes (if it exists), as $|Z_{1+2}| = |Z_1| + |Z_2|$. This is equivalent to the condition

$$\mathrm{Im}(\bar{Z}_1 Z_2) = 0 \quad \text{and} \quad \mathrm{Re}(\bar{Z}_1 Z_2) > 0\,. \tag{B.1}$$

The second inequality is needed to ensure that the central charges not only align but also point in the same direction. As the condition (B.1) is a single real equation it will, in general, be satisfied on a codimension one surface in moduli space. Crossing such a surface or 'wall' may correspond to the decay of the bound state formed by the two charges, but it does not have to. Whether a bound state decays or not depends on the intersection product of these charges. In the case $\langle\Gamma_1,\Gamma_2\rangle = 0$, i.e. the charges are mutually local, there will be no decay whereas if charges are mutually non-local, $\langle\Gamma_1,\Gamma_2\rangle \neq 0$, there will be a decay. This follows because in the constraint equation, (3.12), the RHS depends on $\mathrm{Im}(\bar{Z}_1 Z_2)$ so the inter-center separation is given by

$$r_{12} = \frac{\langle\Gamma_1,\Gamma_2\rangle}{\langle h,\Gamma_1\rangle} = \frac{\langle\Gamma_1,\Gamma_2\rangle\,|Z_1+Z_2|}{2\,\mathrm{Im}(\bar{Z}_2 Z_1)}\bigg|_\infty . \tag{B.2}$$

This qualitative difference when approaching or crossing such a hypersurface in moduli space prompts us to name them differently so we can easily refer to the appropriate picture. Therefor we define

$$\begin{aligned} &\textbf{Marginal stability:} \qquad \mathrm{Im}(\bar{Z}_1 Z_2) = 0\,, \quad \mathrm{Re}(\bar{Z}_1 Z_2) > 0 \quad \text{and } \langle\Gamma_1,\Gamma_2\rangle \neq 0 \\ &\textbf{Threshold stability:} \qquad \mathrm{Im}(\bar{Z}_1 Z_2) = 0\,, \quad \mathrm{Re}(\bar{Z}_1 Z_2) > 0 \quad \text{and } \langle\Gamma_1,\Gamma_2\rangle = 0 \end{aligned}$$

Thus we will refer to the codimension one hypersurfaces on which this condition is satisfied as **walls of marginal/threshold stability**, respectively. As mentioned above the physics of bound states is rather different when crossing a wall of marginal stability or one of threshold stability. So let us shortly review this physics to make things clear. The discussion can be most easily understood when illustrated by an example although the story is general and holds for all multicenter black holes.

We take as our example a simple three center solution consisting of the charges

$$\Gamma_1 = (1, \frac{p}{2}, \frac{p^2}{8}, \frac{p^3}{48})\,, \quad \Gamma_2 = (-1, \frac{p}{2}, -\frac{p^2}{8}, \frac{p^3}{48}) \quad \text{and} \quad \Gamma_3 = (0,0,0,-n)\,. \tag{B.3}$$

This configuration is discussed in some detail in [9] and an attractor flow tree is given in figure B.1.

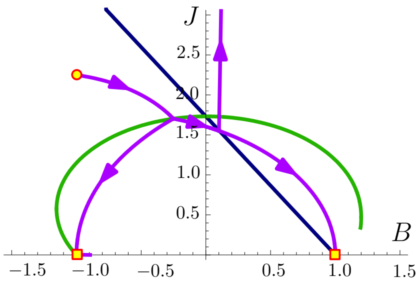

Figure B.1: *Attractor flow for the charges* $\Gamma_1 = (1,1,1/2,1/6)$, $\Gamma_2 = (-1,1,-1/2,1/6)$ *and* $\Gamma_3 = (0,0,0,-1/100)$. *The attractor point for* Γ_1 *is the box on the* B*-axis on the left, that for* Γ_2 *the one on the right. The attractor point for* Γ_3 *lies at infinite* J.

In this figure B.1 the green line is a wall of marginal stability for the charges Γ_2 and $\Gamma_1+\Gamma_3$. More precisely on this line $\mathrm{Im}(\bar{Z}_2 Z_{1+3}) = 0$. As the intersection product

$\langle \Gamma_2, \Gamma_1 + \Gamma_3 \rangle = \frac{p^3}{6} - n$ is non-vanishing this is thus a wall of marginal stability in our refined sense. In this same example the J-axis, i.e. $B = 0$, is a wall of threshold stability for the charges $\Gamma_1 + \Gamma_2$ and Γ_3, i.e. $\mathrm{Im}(\bar{Z}_{1+2} Z_3) = 0$ at $B = 0$ and $\langle \Gamma_1 + \Gamma_2, \Gamma_3 \rangle = 0$. We will now look at the behavior of the split flow and the solution space in approaching this wall of marginal or threshold stability respectively. In both cases we start from the attractor flow depicted in figure B.1, which has its starting point at $B = -1$ and $J = 9/4$. First we will discuss what happens while we keep B fixed and lower J thus approaching the wall of marginal stability discussed above. Secondly we will see what happens when one keeps J fixed but moves B towards positive values thus crossing the wall of threshold stability at $B = 0$ pointed to above.

Starting at a negative value for the B-field modulus and a large enough Kähler modulus a split flow $(\Gamma_2, (\Gamma_1, \Gamma_3))$ exists and in spacetime this corresponds to a supergravity solution corresponding to the $\overline{\text{D0}}$ "orbiting" the D6 which then together bind to the $\overline{\text{D6}}$, see figure B.2 (A). We can now see what happens in case we start moving the starting point of the attractor flow tree. We keep the value of the B-field fixed and lower the Kähler modulus towards zero. In this way we will approach the wall of marginal stability for the charges Γ_2 and $\Gamma_1 + \Gamma_3$, the green line in figure B.1. Approaching this wall corresponds to the (Γ_1, Γ_3) cluster being forced further and further away from the Γ_2 center. A plot of the solution space for values of the moduli closer and closer to marginal stability is given in figure B.2 (A) through (C). The centers are forced infinitely far apart and decay the moment the starting point coincides with the wall of marginal stability and the solution ceases to exist once the wall has been crossed. This is the familiar decay of multicenter bound states when crossing a wall of marginal stability. Also microscopically the bound state disappears out of the spectrum and the BPS index makes a jump. The way this is manifested in the split flow picture is by the fact that the split flow tree only exists on one side of the wall of marginal stability.

In case of crossing a wall of threshold stability the physics is different. We can start from the same initial configuration but now deform it in a different way. We now move the starting point in moduli space towards the J-axis along a trajectory of constant J. We have plotted the solution space along this trajectory in figure B.3 (A) through (E). Approaching the wall of threshold stability $B = 0$, the orbit of the D0 around the D6 becomes more and more deformed and it expands. Once we reach threshold stability the D0 is equally bound to the D6 as to the $\overline{\text{D6}}$ and can sit anywhere on the equidistant plane between D6 and $\overline{\text{D6}}$. Note that this plane is non-compact, i.e. the D0 can move arbitrarily far away along this plane, while the orbits before were large but always compact. Continuing further to positive values for B the orbit of the D0 becomes compact again but has now become an orbit around the $\overline{\text{D6}}$. This corresponds to the fact that the split flow has now changed topology from $(\Gamma_2, (\Gamma_1, \Gamma_3))$ to $(\Gamma_1, (\Gamma_2, \Gamma_3))$. In this process no states have decayed and no solutions have been lost.

This example illustrates the general behavior that we can summarize as follows:

- A wall of marginal stability (in the refined sense) corresponds to a boundary between a region in moduli space where a certain multicenter solution exists and a region where it no longer exists. In the supergravity picture the disappearance of the bound state happens as a number of centers separate further and further towards infinite separation at marginal stability. Crossing a wall of marginal stability corresponds to a decay of states and a jump in the index counting these states.
- A wall of threshold stability corresponds to a boundary between two regions of different 'topology'. This holds both on the level of the flow tree that changes topology, i.e. the type and order of splits changes, as on the level of the solution space that changes topology as a manifold. This change of topology of the solution manifold can happen as exactly at threshold stability the solution space becomes non compact. Note that when crossing a wall of threshold stability no states decay, they only change character.

So at threshold stability some centers are allowed to move of to infinity but it is also possible for them to sit at finite distance to the other centers; they are not forced to infinite separation as is the case for marginal stability. Although the solution space is non-compact, it turns out to have finite symplectic volume when considered as a phase space [4]. One can check explicitly that this number of states equals that on both sides of the wall of threshold stability and so crossing a wall of threshold stability does *not* correspond to a decay of states — rather, at the wall, the BPS states exist as bound states *at threshold* (hence the name), similar to D0-branes in type IIA string theory in flat space.

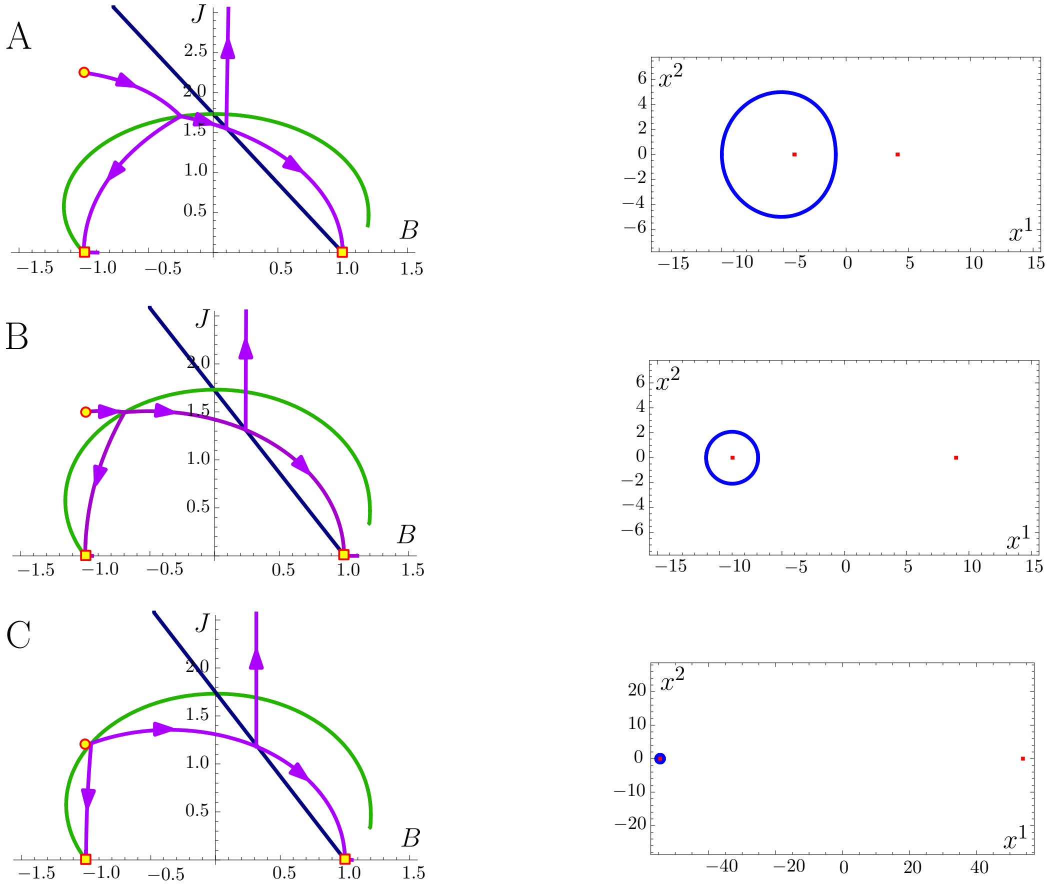

Figure B.2: *On the left the attractor flows for the charges of fig. B.1 are shown for different values of the starting moduli. On the right the corresponding solution moduli space is plotted. The red points are the positions of* Γ_1 *(right) and* Γ_2 *(left). In blue are the possible positions of* Γ_3*. Note the difference in the scale in the last plot, this as once we approach marginal stability the relative position of the centers diverges.*

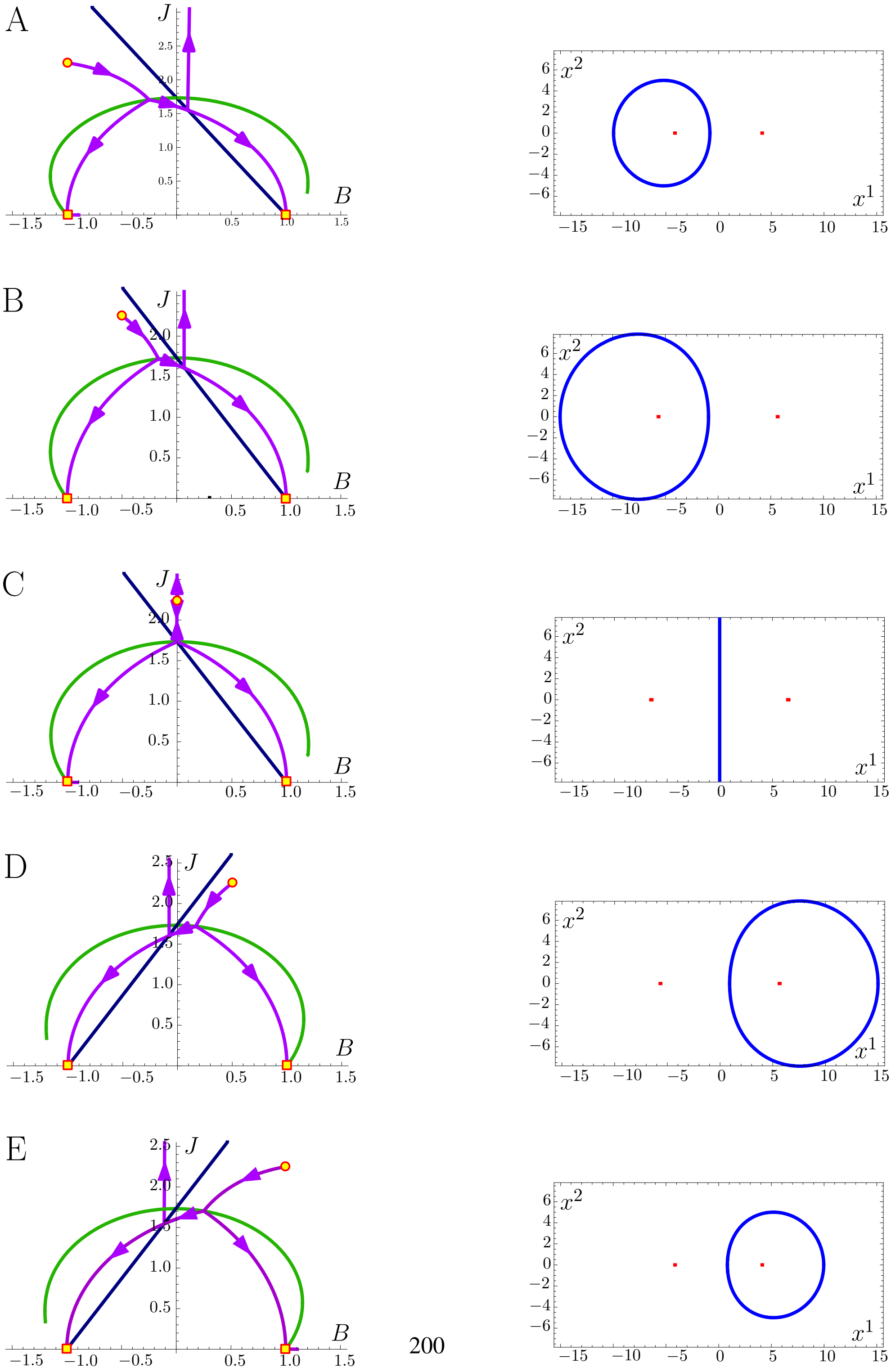

Figure B.3: *Here we show the same type of plots as in fig. B.2, but now taking the starting point through a wall of threshold stability, in this case the J-axis.*

Appendix C

Rescaled solutions

In this appendix we provide the explicit form of the multicentered solutions in rescaled coordinates x_i and in terms of the rescaled harmonics H (see Section 3.2.1),

$$\mathsf{H} = \sum_a \frac{\Gamma_a}{\sqrt{R}\,|\mathsf{x} - \mathsf{x}_a|} - 2\ell_5^{3/2}\mathrm{Im}(e^{-i\alpha}\Omega)|_\infty \,. \tag{C.1}$$

The rescaled solution is given by

$$\begin{aligned}
ds^2_{4\mathrm{d}} &= -\frac{1}{\Sigma}(dt + \frac{\omega}{\sqrt{R}})^2 + \Sigma\, d\mathsf{x}^i d\mathsf{x}^i \,, \\
\mathcal{A}^0 &= \frac{-L}{\Sigma^2}\left(\sqrt{R}dt + \omega\right) + \omega_0 \,, \\
\mathcal{A}^A &= \frac{\mathsf{H}^A L - Q^{3/2} y^A}{\mathsf{H}^0 \Sigma^2}\left(\sqrt{R}dt + \omega\right) + \mathcal{A}_d \,, \\
t^A &= B^A + i\, J^A = \frac{\mathsf{H}^A}{\mathsf{H}^0} + \frac{y^A}{Q^{\frac{3}{2}}}\left(i\Sigma - \frac{L}{\mathsf{H}^0}\right).
\end{aligned} \tag{C.2}$$

These relate to the other rescaled functions appearing in (C.2) through:

$$\begin{aligned} d\omega_0 &= \sqrt{R} \star d\mathsf{H}^0 \,, \\ d\mathcal{A}_d^A &= \sqrt{R} \star d\mathsf{H}^A \,, \\ \star d\omega &= \sqrt{R}\,\langle d\mathsf{H},, \mathsf{H}\rangle \\ \Sigma &= \sqrt{\frac{Q^3 - L^2}{(\mathsf{H}^0)^2}} \,, \\ L &= \mathsf{H}_0(\mathsf{H}^0)^2 + \frac{1}{3} D_{ABC}\mathsf{H}^A\mathsf{H}^B\mathsf{H}^C - \mathsf{H}^A\mathsf{H}_A\mathsf{H}^0 \,, \\ Q^3 &= (\frac{1}{3} D_{ABC} y^A y^B y^C)^2 \,, \\ D_{ABC} y^A y^B &= -2\mathsf{H}_C\mathsf{H}^0 + D_{ABC}\mathsf{H}^A\mathsf{H}^B \,. \end{aligned} \tag{C.3}$$

Of course the form of the rescaled consistency condition doesn't change:

$$\langle \mathsf{H}, \Gamma_s\rangle|_{\mathsf{x}=\mathsf{x}_s} = 0 \,. \tag{C.4}$$

The rescaled 5d lift is

$$\begin{aligned} \frac{1}{\ell_5^2}\, ds^2_{5d} &= \tilde{V}_{\rm IIA}^{2/3}\,\left(d\psi + \mathcal{A}^0\right)^2 + \frac{R}{2}\tilde{V}_{\rm IIA}^{-1/3}\, ds^2_{\rm 4D} \,, \\ A^A_{\rm 5d} &= \mathcal{A}^A + B^A\left(d\psi + \mathcal{A}^0\right) \,, \\ Y^A &= \tilde{V}_{\rm IIA}^{-1/3}\, J^A \,, \qquad \tilde{V}_{\rm IIA} = \frac{D_{ABC}}{6} J^A J^B J^C = \frac{1}{2}\left(\frac{\Sigma}{Q}\right)^3 \,. \end{aligned} \tag{C.5}$$

The more explicit form of the five dimensional metric becomes in terms of the rescaled variables

$$\begin{aligned} \frac{1}{\ell_5^2}\, ds^2_{5d} &= 2^{-2/3}\, Q^{-2}\Big[-(\mathsf{H}^0)^2(\sqrt{R}dt + \omega)^2 - 2L(\sqrt{R}dt + \omega)(d\psi + \omega_0) \\ &\quad + \Sigma^2(d\psi + \omega_0)^2\Big] + 2^{-2/3} R\, Q\, d\mathsf{x}^i d\mathsf{x}^i \,. \end{aligned} \tag{C.6}$$

APPENDIX D

GAUGE FIELD CONTRIBUTION TO CONSERVED CHARGES

In this appendix we give some more detail concerning the contribution of the various gauge fields to the conserved boundary charges in the decoupled solutions of Chapter 3. The five dimensional $\mathcal{N}$=1 supergravity theory of which our asymptotic $\mathrm{AdS}_3\times\mathrm{S}^2$ configurations are solutions has b_2 U(1) vectorfields, where b_2 is the second Betti number of the Calabi-Yau we compactified on. After reduction over the asymptotic two-sphere we end up with an additional SU(2) gauge field as will be explained in some detail below. Analyzing how the action on the boundary of AdS varies with respect to these gauge fields and the metric gives the conserved currents of the boundary theory that can be identified with a 2d CFT.

Before doing this analysis for our solutions we can avoid some work by considering the behavior of a general theory near an asymptotic AdS_3 boundary. As in (3.77) we can, in general, write an asymptotic AdS_3 metric as

$$ds^2_{3\mathrm{d}} = d\eta^2 + (e^{\frac{2\eta}{R_{\mathrm{AdS}}}} g^{(0)}_{ij} + g^{(2)}_{ij}) du^i du^j \,, \tag{D.1}$$

where the boundary is at $\eta = \infty$ and $g^{(0)}_{ij}$ is the metric on the boundary. A generic action for a gauge field in 3 dimensions has the following form

$$S = a \int \mathrm{Tr}(F \wedge \star F) + b \int \mathrm{Tr}(A \wedge F + \frac{2}{3} A \wedge A \wedge A) \,, \tag{D.2}$$

with a and b some coupling constants. Now one should remark that due to the appearance of the metric in the first term it decreases as $e^{-\eta}$ near the boundary while the second term is purely topological and will thus dominate near the boundary. This implies that to calculate the boundary charges we only need to keep track of the topological Chern-Simons part of the gauge field action. In the following section D.1 we calculate these 3d Chern-Simons terms explicitly for the case we are concerned with. In section D.2 we quickly review the general idea behind the calculation of the boundary charges from Chern-Simons theory and in D.3 and D.4 we calculate these for our solutions.

D.1 Reduction of the Chern-Simons term

The three dimensional Chern-Simons term is a reduction over the sphere of the Chern-Simons term of five dimensional $\mathcal{N}$=1 supergravity, which itself has its origin in such a topological term in the M-theory action. Starting from the Chern-Simons part of the 11-dimensional supergravity action and reducing over a CY_3, one gets the following action in 5-dimensions (in Euclidean signature)

$$I_{CS} = \frac{i}{192\pi^2} \int D_{ABC} A^A \wedge F^B \wedge F^C \,. \tag{D.3}$$

The ansatz for the gauge field A^A to further reduce to 3 dimensions is given by the asymptotic form found in (3.73). So we propose as our general reduction ansatz a field strength of the form

$$F^A = \frac{1}{2} p^A e_2 + dC^A \,, \tag{D.4}$$

where C^A is a one-form on AdS_3. The two-from e_2 is known in the literature as the global angular 2-form [180], [181], [182]; it is the generalisation of the standard volume of the sphere to an S^2 fibration and is defined as follows:

$$\begin{aligned} e_2 &= \epsilon_{ijk}(Dy^i \wedge Dy^j - \tilde{F}^{ij}) y^k \,, \\ ds^2 &= ds^2_{\mathrm{AdS}_3} + \frac{1}{l^2}(dy^i - \tilde{A}^{ij} y^j)(dy^i - \tilde{A}^{ik} y^k) \,, \\ Dy^i &= dy^i - \tilde{A}^{ij} y^j \,, \\ \tilde{F}^{ij} &= d\tilde{A}^{ij} - \tilde{A}^{ik} \wedge \tilde{A}^{kj} \,. \end{aligned} \tag{D.5}$$

Summation over repeated indices is assumed and the y^i are the embedding coordinates of S^2 in flat $\mathbb{R}^3$, i.e. $y^i y^i = 1$. The $\tilde{A}$ are the SU(2) gauge fields coming from the reduction of the metric over the S^2. Keep in mind that $\tilde{A}$ depends only on the AdS_3 coordinates.

To make the reduction a bit more tractable we introduce the following quantities

$$\tilde{A}^{ij} = \epsilon_{ijk} A^k, \qquad F^{ij} = \epsilon_{ijk} F^k \,. \tag{D.6}$$

To get compact expressions, we will associate to every quantity with an SU(2) index i, j, ... the following notation

$$\mathcal{O} = \frac{i}{2} O^j \sigma_j \,, \tag{D.7}$$

where σ_j are the Pauli matrices satisfying

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \qquad \text{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}, \qquad \text{Tr}(\sigma_i\sigma_j\sigma_k) = 2i\epsilon_{ijk}\,. \tag{D.8}$$

$$\tag{D.9}$$

For example, one has

$$\begin{aligned} D\mathcal{Y} &= d\mathcal{Y} + [\mathcal{Y}, \mathcal{A}], \quad \mathcal{F} = d\mathcal{A} - \mathcal{A}\wedge\mathcal{A}\,, \\ e_2 &= 4\text{Tr}\left[\mathcal{Y}d\mathcal{Y}\wedge d\mathcal{Y} + d(\mathcal{Y}\mathcal{A})\right] = 2[\sin\theta d\theta \wedge d\phi - d(y^i A^i)]\,, \end{aligned} \tag{D.10}$$

where in the last equation we used spherical coordinates.

Plugging in eqn (D.3) and reducing over S^2, bearing in mind that the only dependence on S^2 resides in e_2, one ends up with the following Chern-Simons term on AdS^3:

$$I_{gauge} = -\frac{i}{4\pi}\frac{p^3}{6}\int \text{Tr}\left(\mathcal{A}\wedge d\mathcal{A} - \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) + \frac{i}{16\pi}D_{AB}\int C^A \wedge dC^B\,. \tag{D.11}$$

As one sees the $\mathcal{A}$ and C fields don't interact with each other, this is as expected from SU(2) gauge invariance. The SU(2) gauge field $\mathcal{A}$ does change under such a gauge transformation but C does not. So one needs two $\mathcal{A}$ and one C for a consistent interaction term. But $\text{Tr}\mathcal{A}\wedge\mathcal{A} = 0$. So there is no interaction term between $\mathcal{A}$ and C.

D.2 Boundary charges: lightning review

How in general the presence of Chern-Simons terms leads to contributions to both the boundary SU(2) and U(1) currents and the boundary energy momentum tensor is very nicely reviewed in [75] and so we will restrict ourselves to a short recapitulation here. Essential in the derivation is the addition of extra boundary terms to the bulk Chern-Simons action. Let's take the simple example of single U(1) field:

$$S = ik\int_{AdS} A\wedge dA\,. \tag{D.12}$$

We can make the gauge choice $A_\eta = 0$ and furthermore the equations of motion imply that A is a flat connection. As argued in [75] there are two reasons to include an additional boundary term to this bulk action. The first is that imposing Dirichlet conditions for both components of the gauge fields, i.e. $\delta A|_{\partial AdS} = 0$, is too strict. Second is that one wants the current associated to the gauge field to be purely left or rightmoving. This last argument is natural from the canonical quantization of Chern-Simons theory [183]. Without the boundary term one has $\delta S \sim \int_{\partial AdS} p\delta q + q\delta p$, where p and q are both a component of the boundary gauge field. Adding the correct boundary term cancels the second term and gives the natural interpretation to p as the momentum conjugate to q. The boundary term that does this is

$$S_{bd} = -\frac{|k|}{2}\int_{\partial AdS} A \wedge \star A\,. \tag{D.13}$$

The absolute value of k is needed to have positive energy as we will see shortly. Introducing the standard complex coordinates $w, \bar{w}$ on the boundary cylinder and noting that $\star dw = idw$, $\star d\bar{w} = -id\bar{w}$ one can verify that that once one adds the boundary term indeed

$$\delta S = \begin{cases} 2i\int_{\partial AdS}(\delta A_w)A_{\bar{w}} & \text{if } k > 0\,, \\ 2i\int_{\partial AdS}(\delta A_{\bar{w}})A_w & \text{if } k < 0\,, \end{cases} \tag{D.14}$$

where we have assumed the bulk fields to be on shell. Now we can impose the Dirichlet boundary conditions $\delta A_w = 0$ and leave $\delta A_{\bar{w}}$ arbitrary in case $k > 0$ and vice versa if $k < 0$. The addition of this boundary term influences the boundary currents; these are defined as

$$\delta S = \int_{\partial AdS} \sqrt{g^{(0)}}\left(\frac{i}{2\pi}J^i\delta A_i + \frac{1}{2}T_{ij}\delta g^{ij}_{(0)}\right), \tag{D.15}$$

so we sees that e.g. the contribution to the energy momentum tensor comes completely from the boundary term as the bulk term is purely topological. It is now easy to calculate these currents:

$$\begin{aligned} T_{ww} &= \frac{|k|}{2}A_w A_w\,, \quad T_{w\bar{w}} = 0\,, \quad T_{\bar{w}\bar{w}} = \frac{|k|}{2}A_{\bar{w}}A_{\bar{w}}\,, \\ J_w &= \begin{cases} 0 & \text{if } k > 0\,, \\ 2\pi A_w & \text{if } k < 0\,, \end{cases} \\ J_{\bar{w}} &= \begin{cases} 2\pi A_{\bar{w}} & \text{if } k > 0\,, \\ 0 & \text{if } k < 0\,. \end{cases} \end{aligned} \tag{D.16}$$

Having reviewed the general philosophy we can now calculate the charges in our case of interest. Note that the story generalizes straightforward to the non-abelian case [75].

D.3 THE U(1) PART

The U(1) part of the Chern-Simons term (D.11) is given by

$$\frac{i}{16\pi}D_{AB}\int C^A \wedge dC^B\,. \tag{D.17}$$

Due to the fact that D_{AB} as a metric on $\mathrm{H}^2(X)$ has a single positive eigenvector and $b_2 - 1$ negative ones (see e.g. [27]) we have to treat the two cases slightly differently (see the discussion above). The projectors to the positive and negative eigenspaces are

$$(P^+)^A_B = \frac{1}{p^3}\, p^A\, D_{BC}\, p^C\,, \qquad (P^-)^A_B = \delta^A_B - (P^+)^A_B\,, \tag{D.18}$$

which gives

$$\begin{aligned} C^{A+} &= \frac{1}{p^3}\, p^A\, D_{BC}\, p^B\, C^C = \frac{1}{p^3}\,(p^B q_B)\, p^A\, d\psi = \frac{2}{p^3}\,(p^B q_B)\, p^A\, dw, \\ C^{A-} &= \left(D^{AB} - \frac{1}{p^3}\, p^B\, p^A\right)\, q_B\, d\psi = 2\left(D^{AB} - \frac{1}{p^3}\, p^B\, p^A\right)\, q_B\, dw\,, \end{aligned} \tag{D.19}$$

where we used the asymptotic from of our gauge field, eqn. (3.73).

As explained in the previous section, once we add the correct boundary term we have the following boundary conditions left $\delta C_w^{+A} = 0$ and $\delta C_{\bar w}^{-A} = 0$. So we have to choose a fixed value for those gauge fields at the boundary. It turns out that the correct choice is $C_w^{+A} = 0$ and $C_{\bar w}^{-A} = 0$.

However our asymptotic gauge field (3.73) doesn't satisfy this boundary condition as one can see from (D.19). This is however easily corrected by the following gauge transformation

$$C^A \longrightarrow C^A - \frac{4}{R}\frac{p^B q_B}{p^3} p^A\, dt\,,$$

which gives

$$C^{A+} = 2\frac{1}{p^3}\, p^B\, q_B\, p^A\, d\bar w\,. \tag{D.20}$$

Given this split into positive and negative modes one can now apply the general procedure as reviewed in the previous section to find

$$\begin{aligned} T_{\bar w \bar w} &= \frac{1}{4\pi}\,\frac{1}{p^3}\,(p^A\, q_A)^2\,, \\ T_{ww} &= \frac{1}{4\pi}\,\frac{1}{p^3}\,\left[(p^A\, q_A)^2 - p^3(q_A D^{AB} q_B)\right]\,, \\ J^+_{\bar w} &= \frac{1}{4}\frac{1}{p^3}\, p^B\, q_B p^A\,, \\ J^-_w &= \frac{1}{4}\left(D^{AB} - \frac{1}{p^3}\, p^B\, p^A\right)\, q_B\,. \end{aligned} \tag{D.21}$$

D.4 The SU(2) part

The SU(2) part of (D.11) is

$$-\frac{i}{4\pi}\frac{p^3}{6}\int \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A}-\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) \tag{D.22}$$

with $k=\frac{1}{4\pi}U^3>0$. Let's look at the value the SU(2) gauge field $\mathcal{A}$ takes in our solution. The general sphere reduction Ansatz has the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{\alpha\beta}\left(dx^\alpha + A^I_\mu X^\alpha_I dx^\mu\right)\left(dx^\beta + A^I_\nu X^\beta_I dx^\nu\right)\,, \tag{D.23}$$

where the x^μ are in our case the coordinates on AdS_3 and the x^α coordinates on the S^2, the $X^\alpha_I\partial_\alpha$ are the killing vectors of the sphere. So using the form of the asymptotic metric (3.72), we have

$$d\theta^2+\sin^2\theta(d\phi+\frac{Rd^0}{2U^3}dv)^2=(d\theta+A^I_\mu X^\theta_I dx^\mu)^2+\sin^2\theta(d\phi+A^I_\mu X^\phi_I dx^\mu)^2 \tag{D.24}$$

which then implies that the only non vanishing component of the gauge field is

$$A^3_v = \frac{Rd^0}{2U^3}\,, \tag{D.25}$$

or in the complex coordinates we introduced

$$A^3_{\bar{w}} = -\frac{R^2 d^0}{4U^3}\,. \tag{D.26}$$

And following the by now standard procedure its contribution to the energy momentum tensor is

$$T_{\bar{w}\bar{w}} = \frac{R^4}{8\pi}\frac{(d^0)^2}{16U^3}\,. \tag{D.27}$$

Appendix E

Gravitational Throats and CFT Mass Gaps

Having an infinitely deep throat in AdS spaces seems paradoxal as it suggests a continuous spectrum on the CFT side, as one can make arbitrarily small energy excitations by localizing them deep enough in the throat. In this appendix we would like to calculate the correspondence between the size of the mass gap and a smoothly capped off throat in the bulk (as we find in Section 5.1.2). We are going to approximate the finite throat by a toy model metric with a throat that is crudely cut off at a scale ϵ and try to solve the scalar wave equation in this background. Due to the technical difficulty of the full problem we resort to matching the far ($r \gg \epsilon$) and near ($r \ll \epsilon$) region solutions. By doing so we are able to relate the inverse of the depth of the throat to the mass gap on the CFT. The result we find agrees with what one expects for CFT's with a long string picture. We use this result in section 5.1.2 to relate the cutoff scale ϵ we calculate there to the size of a mass gap.

To get a reasonable guess to what the capped off geometry would be, we use that far away from the tip of the throat the geometry looks like the one of a D4D2D0 BTZ black hole. So our starting point is the following metric (see e.g. (3.94))

$$ds^2 = -\frac{r}{U}dtd\psi + \frac{1}{4}\frac{r+C}{U}d\psi^2 + U^2\frac{dr^2}{r^2} \tag{E.1}$$

with $C = \frac{S^2}{\pi^2 C^3}$ some constant determined by the charges. Let us assume this metric to be a good approximation for $r \geq \epsilon$, whereas we take

$$ds^2 = -\frac{\epsilon}{U}dtd\psi + \frac{r^2U^2}{\epsilon^2}d\psi^2 + \frac{U^2}{\epsilon^2}dr^2 \tag{E.2}$$

valid for $r \leq \epsilon$. We find the following radial equations for a free scalar field in these two "sub-geometries". In the outer region

$$\left(-\partial_r \frac{r^2}{U^2}\partial_r + m^2 - \frac{\omega^2(C+r)U}{r^2} - \frac{4U\omega k}{r}\right)\phi = 0 \tag{E.3}$$

and in the inner region

$$\left(-\partial_r \frac{\epsilon^2}{U^2}\partial_r + m^2 - \frac{\omega^2 r^2 U^4}{\epsilon^4} - \frac{4U\omega k}{\epsilon}\right)\phi = 0. \tag{E.4}$$

Let us solve these equations. The first one has two solutions but only one is normalizable at infinity. Define λ via

$$\frac{1}{4} - \lambda^2 = -m^2U^2 \tag{E.5}$$

and also

$$\xi \equiv -i\frac{\omega}{2}\sqrt{\frac{U}{C}}(\omega + 4k) \tag{E.6}$$

and the variable

$$z = 2\omega i\sqrt{UC}/r \tag{E.7}$$

then the field equation for ϕ becomes

$$\left(\partial_z^2 + \left(-\frac{1}{4} + \frac{\xi}{z} + \frac{\frac{1}{4} - \lambda^2}{z^2}\right)\right)\phi = 0 \tag{E.8}$$

and the solution in the outer region in terms of Whittaker functions is (see e.g. eq (84) in [184])

$$\phi(z) = M_{\xi,\lambda}(z) + M_{-\xi,\lambda}(-z). \tag{E.9}$$

These two terms are proportional to each other but this answer is the linear combination which yields a real answer. The large z behavior is

$$\phi(z) \sim \frac{e^{z/2}z^{-\xi}}{\Gamma(\frac{1}{2} + \lambda - \xi)} + c.c. \tag{E.10}$$

Next we turn to the cap region. Here, the field equation is usually solved in terms of parabolic cylinder functions. We need the right linear combination which vanishes as $r \to 0$ to be smooth there. Define

$$a = \frac{\epsilon m^2}{2U} - 2k, \qquad r = x\sqrt{\frac{\epsilon^3}{2\omega U^3}} \tag{E.11}$$

then the field equation is $(\partial_y^2 + \frac{y^2}{4} - a)\phi = 0$ and then we find the following large x behavior of the solution (notice that $r = \epsilon$ is at large x)

$$\phi \sim \exp(-ix^2/4)x^{ia-1/2}\frac{(-1)^{3/8}2^{-1/4-ia/2}e^{a\pi/4}\sqrt{\pi}}{\Gamma(\frac{3}{4} + \frac{ia}{2})} + c.c. \tag{E.12}$$

Notice that the coefficients are important, only for this particular coefficient (and including the c.c.) this function has the asymptotics of the regular solution.

In order to match the solutions (E.9) and (E.12) at $r = \epsilon$, both the fields and their first derivatives should match. A priori this need not be possible since we already have unique solutions up to overall normalization. Since the overall normalization is not fixed, the condition that we need to impose is

$$\frac{\partial z}{\partial r}\frac{\phi'(z)}{\phi(z)}\bigg|_{r=\epsilon} = \frac{\partial x}{\partial r}\frac{\phi'(x)}{\phi(x)}\bigg|_{r=\epsilon}. \tag{E.13}$$

In the small ϵ-limit the derivatives of the exponentials give the largest contributions to ϕ' thus we only need to differentiate those.

To proceed, we define $e^{i\rho}$ to be the phase of $1/\Gamma(1/2+\lambda-\xi)$, and $e^{i\sigma}$ to be the phase of

$$\frac{(-1)^{3/8}2^{-1/4-ia/2}e^{a\pi/4}\sqrt{\pi}}{\Gamma(\frac{3}{4}+\frac{ia}{2})}. \tag{E.14}$$

Then we find

$$\begin{aligned}\phi_{\text{outer}} &\sim \cos(\frac{z}{2i} - (\xi/i)\log(z/i) + \rho),\\ \phi_{\text{inner}} &\sim \frac{1}{\sqrt{x}}\cos(-\frac{x^2}{4} + a\log(x) + \sigma)\end{aligned} \tag{E.15}$$

up to irrelevant overall normalizations. We now evaluate the matching condition keeping only the leading terms for small ϵ, i.e. we only differentiate $z/2i$ and $-x^2/4$. After doing this, various factors of ϵ and ω happily cancel and we are left with the matching condition (evaluated at $r = \epsilon$)

$$-\frac{U^{5/2}}{C^{1/2}}\tan(-\frac{x^2}{4} + a\log(x) + \sigma) = \tan(\frac{z}{2i} - (\xi/i)\log(z/i) + \rho). \tag{E.16}$$

As we increase ω, but keeping ω small enough so the matching strategy remains sensible, the first tangent seems to oscillate most rapidly (as long as $U^3 > 2\sqrt{UC}$, otherwise the other tangent seems to win). Because of this oscillatory nature we get a gapped spectrum. A crude estimate for the gap can therefore be made by looking at the values of ω for which the first tangent makes a π period. Since $x^2/4$, evaluated at $r = \epsilon$, is $\omega U^3/(2\epsilon)$, we find the following rough estimate:

$$\Delta\omega \sim \frac{2\epsilon\pi}{U^3}. \tag{E.17}$$

Since the eigenvalue of ∂_t is like that of $L_0 + \bar{L}_0$, without any further factors, we finally conclude that

$$\Delta(L_0 + \bar{L}_0) \sim \frac{2\epsilon\pi}{U^3}. \tag{E.18}$$

In case ϵ is of order 1 the gap in the conformal weights scales like $1/U^3$ which is precisely $1/c$, with c the central charge of the dual CFT, as one would get from long string fractionation. This suggests that long strings might play an important role in the physics of four dimensional black holes and the dual $N = (0, 4)$ SCFT.

Appendix F

Aspects of Toric Geometry

In this appendix we review some techniques in toric geometry that we use in Part I. By construction we start in our description of solution space in the main text from a symplectic point of view. It is however more convenient for geometrical quantization to have a Kähler description, which can always be made in the case of a symplectic toric manifold. The main formulas in this appendix are thus the expressions (F.5)-(F.6) for the complex coordinates and Kähler potential in terms of the symplectic coordinates on a symplectic toric manifold. Before giving these formulas we review some of the basics of symplectic toric manifolds and symplectic toric orbifolds.

Polytopes

As is customary we will refer to the convex hull of a finite number of points in $\mathbb{R}^n$ as a *polytope*. The boundary of such a polytope is itself the union of various lower dimensional polytopes that are called *faces*. In particular a zero-dimensional face is called a $vertex$, a one-dimensional face an *edge* and a $n-1$-dimensional face a *facet*. Note that we can view any polytope as the intersection of a number of affine half spaces in $\mathbb{R}^n$. A polytope P can thus be uniquely characterized by a set of inequalities, namely $\vec{x} \in P$ iff $\forall i = 1, \ldots, \#(\text{facets})$

$$\langle \vec{c}_i, \vec{x} \rangle \geq \lambda_i \quad \Leftrightarrow \quad \sum_j c_{ij} x_j \geq \lambda_i \, . \tag{F.1}$$

Given a polytope we will call the set $\vec{c}_i \in \mathbb{Z}^n$, given by the inward pointing normals to the various facets, the *normal fan*.

An n-dimensional polytope is called a *Delzant polytope* if it satisfies the following three conditions

- **simplicity:** in each vertex exactly n edges meet,
- **rationality:** each of the n edges that meet at the vertex p is of the form $p + tu_i$ with $t \in \mathbb{R}^+$ and $u_i \in \mathbb{Z}^n$,
- **smoothness:** for each vertex the u_i form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

The polytope is called *rational* instead of Delzant if we replace in the third condition the requirement of a $\mathbb{Z}$-basis by that of a $\mathbb{Q}$-basis.

SYMPLECTIC TORIC MANIFOLDS

Before giving the precise technical definition of a symplectic toric manifold, let us first sketch the idea. Roughly speaking a toric manifold is a $\mathbb{T}^n$ fibration over a given n-dimensional polytope, such that at each facet a single $U(1)$ inside the $\mathbb{T}^n$ shrinks to zero size. On the intersections of the different facets multiple $U(1)$'s collapse, e.g. at the vertices all circles have shrunk. On the interior of the polytope the toric manifold is simply of the form $P^\circ \times \mathbb{T}^n$ and the full toric manifold is the compactification of this space. On the interior there is thus a standard set of coordinates of the form (x_i, θ_i) with $x_i \in P^\circ$ and $\theta_i \in \mathbb{T}$ and the manifold comes with a standard symplectic form $\Omega = \sum_i dx_i \wedge d\theta_i$. It is of course rather non-trivial that this manifold can be smoothly compactified, but when the polytope is Delzant this is the case. Let us now state the above ideas more precisely.

A *symplectic toric manifold* is a compact connected $2n$-dimensional symplectic manifold (M, Ω) that allows an effective Hamiltonian action of an n-dimensional torus $\mathbb{T}^n$. Remember that the action of a Lie group on a symplectic manifold is called Hamiltonian if there exists a *moment map* μ from the manifold to the dual Lie algebra that satisfies

$$d\langle \mu(p), X \rangle = \Omega(\cdot, \tilde{X}) \,, \tag{F.2}$$

with $p \in M$, X a generator of the Lie algebra and $\tilde{X}$ the corresponding vector field. Furthermore the moment map should be equivariant with respect to the group action, i.e. $\mu(g(p)) = \mathrm{Ad}^*_g \circ \mu(p)$, with Ad^* the coadjoint representation.

By a theorem of Delzant [185] every symplectic toric manifold is uniquely characterized by a Delzant polytope. Given a symplectic toric manifold the corresponding polytope is given by the image of the moment map. To conversely reconstruct the manifold from the polytope is slightly more involved and relies on the technique of symplectic reduction; we refer readers interested in further details to e.g. [186]. Note that the normal fan to the polytope can be interpreted as a *fan*, which is used to characterize toric varieties in algebraic geometry, see e.g. [187] for a nice introduction. This can be useful to identify a symplectic manifold given by a polytope and furthermore provides an embedding in projective space.

KÄHLER TORIC MANIFOLDS

What will be of use in this thesis is that Delzant's construction also associates a set of canonical complex coordinates to every symplectic toric manifold, effectively implying that every closed symplectic toric manifold is actually a Kähler manifold. As throughout Part I we will make use of the explicit construction of these complex coordinates in terms of the symplectic coordinates (x_i, θ_i), we will detail the general procedure here, be it without proofs or motivation. Those can be found in references [91, 92].

As mentioned above (F.1) any polytope P is characterized by a set of inequalities. Given this combinatorial data of the polytope one can define associated functions

$$l_i(x) = \sum_j c_{ij} x_j - \lambda_i \,, \qquad\qquad l_\infty = \sum_{i,j} c_{ij} x_j \,, \tag{F.3}$$

which are everywhere positive on P. Using these functions one can define a 'potential' as follows

$$g(x) = \frac{1}{2} \sum_i l_i(x) \log l_i(x) \,. \tag{F.4}$$

In case the polytope is Delzant, it is shown in [91] that this potential defines good complex coordinates on the toric manifold as follows

$$z_i = \exp\left(\frac{\partial}{\partial x_i} g(x) + i\theta_i \right) \,. \tag{F.5}$$

Furthermore a Kähler potential for the corresponding Kähler metric $\Omega(\cdot, J\cdot)$ is given by

$$\mathcal{K} = \sum_i \lambda_i \log l_i(x) + l_\infty \,. \tag{F.6}$$

It follows from the construction [91, 92] that $\mathcal{K}$ is the Legendre transform of g, i.e. $\mathcal{K}(z) = \frac{\partial g}{\partial x} x - g(x)$. This can be used to derive that

$$\det \partial_i \partial_{\bar{j}} \mathcal{K} = \exp\left(\sum_i \frac{\partial g}{\partial x_i} \right) \det \frac{\partial^2 g}{\partial x_i \partial x_j} \,, \tag{F.7}$$

which will be a useful formula in the bulk of the thesis.

TORIC ORBIFOLDS

As we also consider quotients of symplectic toric manifolds by a permutation group in this thesis, it will be necessary to introduce the generalization of the above construction of complex coordinates to that of symplectic toric orbifolds. As in the manifold case, a *symplectic toric orbifold* is a $2n$-dimensional symplectic orbifold that allows a Hamiltonian $\mathbb{T}^n$ action. As was shown in [188] such symplectic toric orbifolds are in one to one correspondence with labeled rational polytopes. Such a labeled rational polytope is nothing but a rational polytope with a natural number attached to each facet. The label m_i denotes that the i'th facet is a $\mathbb{Z}_{m_i}$ singularity. Again the explicit construction of the toric orbifold from the labeled polytope is rather involved and we refer those who are interested to [188]. The labeled polytope corresponding to the quotient of a symplectic toric manifold by a group respecting the torus action is, however, easy to find. It is given by the quotient of the original polytope and attaching a label m to each facet that is a $\mathbb{Z}_m$ fixed point under the group action.

Given a labeled rational polytope one can construct complex coordinates on the toric orbifold in a way similar to the manifold case. The functions l_i from (F.3) are generalized to [188, 189]

$$l_i(x) = m_i \left(\sum_j c_{ij} x_j - \lambda_i \right) , \qquad l_\infty = \sum_{i,j} m_i c_{ij} x_j \, , \tag{F.8}$$

where m_i is the label attached to the facet orthogonal to the vector $\vec{c}_i$. The construction of the complex coordinates and the kähler potential from these functions then carries on analogously to (F.5)-(F.6).

APPENDIX G

NUMBER OF STATES AS DISCRETE POINTS INSIDE THE POLYTOPE

In this appendix we show that for *rational* polytopes the number of normalizable modes can simply be computed from the 'discretized volume' of the polytope. This can be useful as it saves time in cases where we are only interested in the number of states and not in the explicit wave functions.

Let us very shortly review the description we use for polytopes. For the full definition and the algorithms and formulae to calculate the associated complex coordinates and wave-functions we refer the reader to appendix F.

Recall that we can think of a toric polytope as a region in $\mathbb{R}^n$, parameterized by coordinates x^i, on which a certain set of first order polynomials are positive. That is, given such a set of m first order functions:

$$l_j(x) = \sum_{i=1}^{n} c_{ij} x^i + \lambda_j \tag{G.1}$$

the polytope is defined as $P_l = \{x \in \mathbb{R}^n | l_j(x) \geq 0\}$. A few remarks are in order:

- It follows from this definition that the polytope is an intersection of m half spaces.
- Note that from that interpretation it follows that $m \geq 2 + n$ to have a compact polytope, so c_{ij} is never a square matrix! (this has some consequences later)

- Note that c_{ij} here is actually the transpose of the one defined in appendix F. This because we now use the more natural definition $c_{ij} = \frac{\partial l_j}{\partial x^i}$.
- Finally note that c and λ cannot be completely arbitrary (i.e. not every intersection of half planes gives a sensible polytope).

Example For the readers convenience we will give the defining functions corresponding to the dipole halos of equations (6.10)-(6.11). Let us first define a coordinate system using the $n+1$ coordinates $(y_0, y_1, \ldots, y_n)$ corresponding to (6.9) with $y_0 = y$.. To encode these constraints in a polytope we require an $(n+1) \times (2n+3)$ c_{ij} matrix which which we will think of instead as $2n+3$ vectors of length $n+1$ given below. In addition we will also need a $2n+3$-component vector λ with components given below as well.

$$\begin{aligned}
\vec{c}_0 &= (-1, \ldots, -1) & \lambda_0 &= I/2 \\
\vec{c}_1 &= (1, -1, \ldots, 1) & \lambda_1 &= I/2 \\
\vec{c}_{2a} &= (0, \ldots, -1, \ldots, 0) & \lambda_{2a} &= q_a \\
\vec{c}_{2a+1} &= (0, \ldots, 1, \ldots, 0) & \lambda_{2a+1} &= 0 \\
\vec{c}_{2n+2} &= (0, -1, \ldots, -1) & \lambda_{2n+2} &= I/2
\end{aligned} \tag{G.2}$$

The $\vec{c}_{2a}$ and $\vec{c}_{2a+1}$ are non-zero only on the $(a+1)$'th entry (recall that $a = 1, \ldots, n$ and our coordinates are labelled from $0, \ldots, n$). Note that the indices on $\vec{c}$ correspond to the labels j in (G.1). With this in mind the reader can check that the $2n+3$ equations defined by substituting the $\vec{c}$ and λ above into (G.1) reproduce (6.10)-(6.11). The corresponding polytopes are shown in figure 6.2.

As discussed in the Appendix F and references therein, all relevant functions (i.e. complex coordinates, Kähler potential, etc.) are defined in terms of the c and λ. Hence we can write the norm square of the wavefunction $\psi_a = \prod_i (z^i)^{(a^i)}$, with $a \in \mathbb{Z}^n$ and the z_i appropriate complex coordinates on the toric manifold, in terms of these objects:

$$\begin{aligned}
|\psi_a|^2 &\sim e^{\sum_i \partial_i g} \sqrt{\det \partial_i \partial_j g}\, e^{-\mathcal{K}} \prod_i |z^i|^{(2a^i)} \\
&\sim e^{\sum_i \partial_i g} \sqrt{\det \partial_i \partial_j g} \prod_{j=1}^m l_j^{(\sum_{i=1}^n c_{ij} a^i + \lambda_j)} && \text{(G.3)} \\
&\sim \prod_{j=1}^m l_j^{\left(\sum_{i=1}^n c_{ij}(a_i + 1/2) + \lambda_j - 1/2\right)} && \text{(G.4)}
\end{aligned}$$

where $\sim$ indicates proportionality up to constants and functions that have no poles and also contain no overall l_j factors. The first step is rather straightforward while the last step is more subtle to prove so that proof is relegated to a separate subsection below.

Let us first focus on the interpretation of the above result. We see that, without taking into account the fermionic contribution $e^{\sum_i \partial_i g}\sqrt{\det \partial_i \partial_j g}$, normalizability of the wavefunctions requires the $a \in \mathbb{Z}^n$ to satisfy

$$\sum_{i=1}^{n} c_{ij} a^i + \lambda_j > -1 \tag{G.5}$$

while, when also including those necessary fermionic corrections, we find the final precise condition is

$$\sum_{i=1}^{n} c_{ij}(a_i + 1/2) + \lambda_j - 1/2 > -1 \tag{G.6}$$

Up to some shifts these equations essentially tell us that a has to lie "inside" the polytope, making the number of states essentially the discretized volume of the polytope, i.e. the volume dived in Planck size cells. Furthermore, as we discuss in detail in the specific case studied in the main text, the shifts of 1/2 introduced by taking into account the fermionic nature of the wavefunctions has a very natural physical interpretation. As was discussed in Part I, the quantization of the polytopes roughly corresponds to quantizing the angular momentum of the system. That the lowest energy state corresponds to a specific alignment of the spins of the constituents then leads to various half integer shifts of the quantum angular momentum, giving rise to the 1/2's in (G.6).

Plugging the $\vec{c}$'s and λ's defined in (G.2) into eqn. (G.6) should allow the reader to reproduce the constraints found in (6.13).

G.1 Evaluation of $\det \partial_i \partial_j g$

In this section we give a detailed description of the steps which lead from (G.3) to (G.4). These steps are based on an intermediate result, which states that

$$\det \partial_i \partial_j g = \left(\prod_{j=1}^{m} \frac{1}{l_j} \right) A(l)\,, \tag{G.7}$$

where, as we will show, $A(l)$ is a homogeneous polynomial of order $m - n$ in the l_j with such coefficients that for no rational polytopes will it contain an overall l_j factor.

We will prove this in two steps. First we will evaluate the relevant determinant to show the form (G.7) explicitly. In the second step we then use this explicit form of $A(l)$ to argue its relevant properties, namely that it has no poles nor contains an overall l_j factor.

Calculating the determinant It is straightforward to check that

$$\partial_i \partial_j g = \frac{1}{2} \sum_{k=1}^{m} \frac{c_{ik} c_{jk}}{l_k} = \frac{1}{2} \left(C \cdot L^{-1} \cdot C^T \right)_{ij} \tag{G.8}$$

with $C_{ij} = c_{ij}$ (recall that $\dim C = n \times m$) and $L_{ij} = l_j \delta_{ij}$ an $m \times m$ matrix. So indeed $CL^{-1}C^T$ is a square $n \times n$ matrix and the determinant makes sense. Sadly the factors inside are not square matrices, making the evaluation a bit less straightforward (we are not interested in constant factors so we will forget about the 1/2 in the following).

Using the basic definition of the determinant and using some symmetry properties it is not too difficult, though somewhat tedious, to show that

$$\det \left(CL^{-1}C^T \right) = \left(\prod_{j=1}^{m} l_j^{-1} \right) \left(\sum_S l^S (\det C_S)^2 \right) . \tag{G.9}$$

The second factor might need some explanation as it uses some unconventional notation. The sum is over all different subsets $S \subset \{1, \ldots, m\}$ with $m - n$ elements, i.e. $\#S = m - n$. Furthermore we use the shorthand $l^S := \prod_{j \in S} l_j$. Finally there is the definition of the $n \times n$ matrix C_S. Note that C was an $n \times m$ matrix, C_S is now defined as the matrix C but with the $i_1, \ldots, i_{m-n}$'th columns removed where $S = \{i_1, \ldots, i_{m-n}\}$.

Properties of $A(l)$ We found the result (G.7) with the explicit form $A(l) = \sum_S l^S (\det C_S)^2$. We now want to show that $A(l)$ has no poles nor contains an overall l_j factor. As is clear from its definition $A(l)$ is a homogeneous polynomial of order $m - n$ in the l_j. As the l_j themselves are simply first order in the x^i, the polynomial A has no poles in the x^i. The second point, that there is no overall l_j factor, is more subtle to see. To show it, pick a particular element $j \in 1, \ldots, m$. By relabeling we can just take $j = 1$. Now from its definition it is clear that $A(l)$ only has an overall l_1 factor if the coefficients of all the terms $l^{\tilde{S}}$, with $\tilde{S}$ such that $1 \notin \tilde{S}$, vanish.

We can now easily show that this never happens using some basic properties of C and $C_{\tilde{S}}$. We will argue that there is always at least one $\tilde{S}_\star$ among the $\tilde{S}$ for which $\det C_{\tilde{S}_\star}$ doesn't vanish. By the definition of the C_S, all the $C_{\tilde{S}}$ include the first column of C, given by c_{i1}. Furthermore let us go go back to the definition of C and the c_{ij}. Note that the original definition of c_{ij} was that it consisted of the n components of the $\vec{c}_j$, which were the normals to the m facets of the polytope. The statement $\exists \tilde{S}_\star \mid \det C_{\tilde{S}_\star} \neq 0$ thus translates to: "there exists a set of $(n-1)$ vectors among the m different normals $\vec{c}_j$ that together with $\vec{c}_1$ form a basis of $\mathbb{R}^n$". We will use the notation $\vec{c}_a$ for these n vectors and now show their existence.

Pick one of the vertices that is a corner of the facet orthogonal to $\vec{c}_1$ and let's call it v_1. As the polytopes of our interest are rational there are exactly n edges $\vec{e}_i$ meeting in the vertex

v_1, that furthermore form a basis of $\mathbb{R}^n$. Now the different facets meeting in v_1 each lie in a subspace generated by a set of $(n-1)$ of the n edges e_i [1]. So we find n facets that all meet in the vertex v_1. Let us label the n normals to these facets as $\vec{c}_a$, by their definition they can be labelled such that they satisfy $\vec{e}_i \cdot \vec{c}_a \sim \delta_{i,j}$. So we see that the $\vec{c}_a$ form a basis of $\mathbb{R}^n$ that includes $\vec{c}_1$, which concludes the proof, i.e. we now know that $\det C_{\tilde{S}_\star} \neq 0$ for $(C_{\tilde{S}_\star})_{ia} = c_{ia}$.

[1] Note that all facets are of this form by the definition of the edges. That furthermore each of the n combinations of $n-1$ linearly independent edges generates a facet is maybe less straightforward and actually not true for a generic non-rational polytope. However here the fact that for each subspace generated by (n-1) of the n-edges there is only one remaining edge not contained in that subset, implies that the subspace must be on the boundary of the polytope and hence generate a facet.

APPENDIX H

G_2 CONVENTIONS

In this section we will detail the conventions used in dealing with the associative 3-form and coassociative 4-form on a 7-manifold with G_2 holonomy that we will use in Part II of this thesis. We adopt the conventions of [12] since we use many results from that paper. More details and original references for G_2 holonomy manifolds can be found in that paper.

Although we will generally not have need for the explicit form of ϕ or $*\phi$ we provide a definition in terms of local coordinates, using the conventions of [12]

$$\phi = \omega^{123} + \omega^1 \wedge (\omega^{45} + \omega^{67}) + \omega^2 \wedge (\omega^{46} - \omega^{57}) - \omega^3 \wedge (\omega^{47} + \omega^{56}) \; , \tag{H.1}$$
$$*\phi = \omega^{4567} + \omega^{23} \wedge (\omega^{67} + \omega^{45}) + \omega^{13} \wedge (\omega^{57} - \omega^{46}) - \omega^{12} \wedge (\omega^{56} + \omega^{47}) \; , \tag{H.2}$$

where ω^i are vielbeins and $\omega^{ij} = \omega^i \wedge \omega^j$ etc.

We also reproduce some identities for ϕ and $*\phi$ from [12] that we will have need of. The precise factors in these identities depends on a choice of conventions and normalizations (e.g. their normalizations are related to those used in [138] by $\phi^{\text{here}}_{\mu\nu\rho} = \frac{1}{3!}\phi^{\text{there}}_{MNP}$ and $*\phi^{\text{here}}_{\mu\nu\rho\sigma} = \frac{1}{4!}*\phi^{\text{there}}_{MNPQ}$).

$$
\begin{aligned}
\phi^{\mu\alpha\beta}\phi_{\alpha\beta\nu} &= \frac{1}{6}\delta^{\mu}_{\nu} \ , \\
(*\phi)_{\mu\nu\alpha\beta}\phi^{\alpha\beta\gamma} &= \frac{1}{6}\phi_{\mu\nu}{}^{\gamma} \ , \\
\phi_{\mu\nu\gamma}\phi^{\gamma\alpha\beta} &= \frac{2}{3}(*\phi)_{\mu\nu}{}^{\alpha\beta} + \frac{1}{18}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]} \ , \\
(*\phi)_{\mu\nu\gamma\rho}(*\phi)^{\gamma\rho\alpha\beta} &= \frac{1}{12}(*\phi)_{\mu\nu}{}^{\alpha\beta} + \frac{1}{72}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]} \ .
\end{aligned}
\tag{H.3}
$$

The exterior algebra on a G_2 manifold can be decomposed into irreducible representations of G_2. The decomposition is given as follows

$$
\begin{aligned}
\Lambda^0 &= \Lambda^0_{\mathbf{1}} \ , \qquad & \Lambda^1 &= \Lambda^1_{\mathbf{7}} \ , \\
\Lambda^2 &= \Lambda^2_{\mathbf{7}} \oplus \Lambda^2_{\mathbf{14}} \ , \qquad & \Lambda^3 &= \Lambda^3_{\mathbf{1}} \oplus \Lambda^3_{\mathbf{7}} \oplus \Lambda^3_{\mathbf{27}} \ .
\end{aligned}
\tag{H.4}
$$

Subscripts here indicate the dimension of the irreducible representation of G_2. The decomposition of higher degree forms follows by Hodge duality $*\Lambda^i_{\mathbf{n}} = \Lambda^{7-i}_{\mathbf{n}}$.

We will frequently have use for the explicit form of the projectors onto these representations

$$
\begin{aligned}
(\pi^2_{\mathbf{7}})_{\mu\nu}{}^{\alpha\beta} &= 6\phi_{\mu\nu\gamma}\phi^{\gamma\alpha\beta} \ , \\
(\pi^2_{\mathbf{14}})_{\mu\nu}{}^{\alpha\beta} &= -4(*\phi)_{\mu\nu}{}^{\alpha\beta} + \frac{2}{3}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]} \ , \\
(\pi^3_{\mathbf{1}})_{\mu\nu\rho}{}^{\alpha\beta\gamma} &= \frac{1}{7}\phi_{\mu\nu\rho}\phi^{\alpha\beta\gamma} \ .
\end{aligned}
\tag{H.5}
$$

When a G_2 manifold has the structure $\mathrm{CY}_3\times\mathrm{S}^1$, there is a decomposition of ϕ and $*\phi$ in terms of $\rho = \mathrm{Re}(e^{i\alpha}\Omega)$, $\hat{\rho} = \mathrm{Im}(e^{i\alpha}\Omega)$ and k (where Ω is the holomorphic 3-form and k is the Kähler form on CY_3). Let η be the volume form on S^1 such that $\int_{S^1}\eta = 2\pi R$, then one has the decompositions

$$
\begin{aligned}
\phi &= \rho + k\wedge\eta \ , \\
*\phi &= \hat{\rho}\wedge\eta + \frac{1}{2}k\wedge k \ .
\end{aligned}
\tag{H.6}
$$

Note that the arbitrary phase α implies that the real/imaginary part of Ω is not canonically related to ϕ or $*\phi$. In Chapter 8 we frequently take $\alpha = 0$ but it is possible to have a CY_3 sitting in a G_2 with a different alignment of its complex structure.

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SUMMARY

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SAMENVATTING

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ACKNOWLEDGEMENTS

I acknowledge ...