THE GREAT COURSES®

Topic
Science
& Mathematics

Subtopic
Mathematics

The Mathematics of Games and Puzzles: From Cards to Sudoku

Course Guidebook

Professor Arthur T. Benjamin
Harvey Mudd College

PUBLISHED BY:

THE GREAT COURSES
Corporate Headquarters
4840 Westfields Boulevard, Suite 500
Chantilly, Virginia 20151-2299
Phone: 1-800-832-2412
Fax: 703-378-3819
www.thegreatcourses.com

Arthur T. Benjamin, Ph.D.

Professor of Mathematics
Harvey Mudd College

Professor Arthur T. Benjamin is a Professor of Mathematics at Harvey Mudd College. In 1983, he graduated from Carnegie Mellon University, earning his B.S. in Applied Mathematics with university honors. In 1989, he received his Ph.D. in Mathematical Sciences from Johns Hopkins University, where he was supported by a National Science Foundation graduate fellowship and a Rufus P. Isaacs fellowship. Since 1989, Professor Benjamin has been a faculty member of the Department of Mathematics at Harvey Mudd College, where he has served as Department Chair. He has spent sabbatical visits at the California Institute of Technology; Brandeis University; the University of New South Wales in Sydney, Australia; and the University of Oxford.

In 1999, Professor Benjamin received the Southern California-Nevada Section Award for Distinguished College or University Teaching of Mathematics from the Mathematical Association of America (MAA), and in 2000, he received the MAA's national Deborah and Franklin Tepper Haimo Award for Distinguished College or University Teaching of Mathematics. He was also named the 2006–2008 George Pólya Lecturer by the MAA. Professor Benjamin was chosen by The Princeton Review as one of its Best 300 Professors. In 2012, he was selected as an inaugural Fellow of the American Mathematical Society. Professor Benjamin has given wide-reaching TED talks, one of which has been viewed more than five million times. In 2005, *Reader's Digest* named him "America's Best Math Whiz." An avid game player, Professor Benjamin is a previous winner of the American Backgammon Tour.

Professor Benjamin's research interests include combinatorics, game theory, and number theory, with a special fondness for Fibonacci numbers. Many of these ideas appear in his book (coauthored with Jennifer Quinn) *Proofs*

That Really Count: The Art of Combinatorial Proof, published by the MAA. In 2006, *Proofs That Really Count* received the MAA's Beckenbach Book Prize. From 2004 to 2008, Professors Benjamin and Quinn served as the coeditors of *Math Horizons* magazine, which is published by the MAA and enjoyed by more than 20,000 readers, mostly undergraduate math students and their teachers. In 2009, the MAA published Professor Benjamin's latest book, *Biscuits of Number Theory*, coedited with Ezra Brown.

Professor Benjamin is also a professional magician. He has given thousands of "mathemagics" shows in venues around the world, from primary schools to scientific conferences, in which he demonstrates and explains his calculating talents. His techniques are explained in his book *Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks*. Prolific math and science writer Martin Gardner called the book "the clearest, simplest, most entertaining, and best book yet on the art of calculating in your head."

Professor Benjamin has appeared on dozens of television and radio programs, including the *TODAY* show and *The Colbert Report* as well as CNN and National Public Radio. He has been featured in *Scientific American*, *Omni*, *Discover*, *People*, *Esquire*, *The New York Times*, the *Los Angeles Times*, and *Reader's Digest*.

Professor Benjamin has taught three other Great Courses: *The Joy of Mathematics*, *Discrete Mathematics*, and *The Secrets of Mental Math*. ■

Table of Contents

INTRODUCTION

LECTURE GUIDES

Table of Contents

Acknowledgments

This course would not have been possible without the generous assistance of so many people, especially many of my former students. The idea for this course began with a class I taught at Harvey Mudd College called The Mathematics of Games and Gambling. My teaching assistant for that course was Louis Ryan (a master of many games and puzzles, and now a graduate student in Applied Mathematics at Harvard), and we met weekly to discuss course material. My friend Jay Cordes (who, among other accomplishments, has written a prize-winning poker-playing program) provided useful feedback on all of the lectures, especially in regard to poker. Joe Kisenwether (of Craftsman Gaming) and Fritz Juhnke (mathematics graduate student and past Arimaa World Champion) also looked at drafts of every lecture and provided me with extremely helpful advice. Finally, Elizabeth (Lizard) Reiland (soon to be a graduate student in mathematics) watched all of the video lectures with her ever-careful eye for detail.

I was given the opportunity to deliver many of these lectures to test audiences, who provided me with invaluable experience and useful feedback. For these opportunities, I am tremendously grateful to Lisa Loop (Director of Operations in the Teacher Education Program at the Claremont Graduate University), Dan Uhlman (head of the mathematics department at the Taipei American School) and George Gilbert (chair of the Department of Mathematics at Texas Christian University).

Although I have always been a lover of games and puzzles, I was fortunate to have the assistance of several game experts along the way. I learned the Rubik's Cube algorithm from John George and Tyson Mao and benefited from practice sessions with Sam Ettinger and Louis Ryan. Many thanks to "cube wranglers" David Chao, Theo Faust, and Peter Gunnarson for invaluable assistance in the studio. For my sudoku education, I am grateful to Linda Barrett, Jared Levine, Jason Linhart, and especially Palmer Mebane, Thomas Snyder (grandmasterpuzzles.com), and Laura Taalman (brainfreezepuzzles.com). My poker days would have been numbered (or at least not as numberful) without the assistance of Jon Jacobsen, Richard

Lederer, and especially Jay Cordes. Thanks to David Levy, Dana Mackenzie, and Scott Nollet for looking over my chess material. Frank Frigo and Perry Gartner are players that I "counted on" for backgammon advice. Special thanks to Harvey Gillis and the U.S. Backgammon Federation (USBGF.org) for permission to quote from Harvey's insightful article "Backgammon: Decision Analysis for Success," which appeared in the March 2011 issue of *PrimeTime Backgammon*.

I am also grateful to those people and organizations that let me use their products on camera. Specifically, I want to thank Jason Linhart for his sudoku app (Enjoy Sudoku), Xavier Dufaure de Citres for his backgammon app (XG Mobile), AppAnnex for their 15 Puzzle Challenge, Seven Towns for permission to use the Rubik's Cube, Robert Fuhrer and Nextoy for KenKen®, Ed Rosenblum (EssentialBG.com) for his magnificent massive magnetic backgammon board, Michael Gray for permission to display Hasbro products, and Jim Pressman for permission to display the products of Pressman Toy Corporation.

Most of the work for this course took place during my sabbatical, and I am ever grateful to Harvey Mudd College for supporting me during this time. My deepest thanks to Dr. Richard Hartzell of Taipei American School and Steven Biller of the University of Oxford Department of Physics for their hospitality during my sabbatical.

I couldn't have created this course without the patience and support of my family. My wonderful wife Deena offered suggestions about every aspect of the course and helped me rehearse every lecture. My brilliant daughters, Laurel and Ariel, were also great assistants.

Finally, it's always a pleasure to work with the very professional staff at The Great Courses, who helped me every step of the way. Special thanks to Jay Tate, Susan Dyer, and Zachary H. Rhoades for working so tirelessly behind the scenes to make this course the best it can be. ■

The Mathematics of Games and Puzzles: From Cards to Sudoku

Scope:

This course takes a mathematical approach to playing games and solving puzzles. In this course, you will be introduced to all kinds of games—from games of pure strategy (like chess) to games of pure luck (like many casino games) to games that mix strategy and luck (like blackjack, backgammon, and poker). You will analyze puzzles that have stumped people for centuries to modern favorites like sudoku and Rubik's Cube. The advice that you will receive ranges from the fundamentally practical to the mathematically interesting. You will improve your ability to play these games and solve these puzzles, but you will also learn some interesting mathematics along the way.

Lecture 1: Let the Games Begin!
How does a mathematician look at games and puzzles? Games and puzzles can be classified in a number of ways. Is there randomness involved or not? Do you have an intelligent adversary or not? Using examples from games and puzzles such as tic-tac-toe, ghost, 20 questions, the Tower of Hanoi, Mastermind, and bridge, you will learn some effective strategies, such as the art of working backward, exploiting symmetry, and careful counting.

Lecture 2: Games of Chance and Winning Wagers
What are the best games of chance to play (and avoid)? How can you quantify your advantage or disadvantage? Using popular games of chance that are encountered at casinos and carnivals—like roulette, chuck-a-luck, and craps—you will learn which games give you the most bang for your buck. Also, once you know the size of your advantage or disadvantage, how much should you bet?

Lecture 3: Optimal Blackjack and Simple Card Counting
When should you stand, hit, double down, or split? You will learn the optimal basic strategy and the underlying reasoning behind it. If you play this game

properly, it can be the fairest game in the casino. Play it wrong, and you're throwing your money away.

Lecture 4: Mixed Strategies and the Art of Bluffing
How should you play against an intelligent adversary whose interests are diametrically opposed to yours? In such games, you don't want your actions to be predictable. Whether you are playing rock-paper-scissors or poker, it definitely pays to vary your strategy.

Lecture 5: Practical Poker Probabilities
This lecture discusses the mathematics that is essential for successful poker playing. The lecture focuses on the game Texas Hold'em, but the mathematical ideas, like the counting of outs and pot odds, can be applied to all poker games. The lecture also presents optimal strategies for playing video poker.

Lecture 6: Expert Backgammon
The exciting dice game known as backgammon is loaded with mathematics. You will learn about the rules of the game, how to count shots, opening strategy, and the all-important doubling cube.

Lecture 7: Games You Can't Lose and Sneaky Puzzles
There are many games that seem very innocent and fair but actually offer you a big advantage. You will also explore challenging-sounding puzzles that have deviously simple solutions.

Lecture 8: Solving "Impossible" Puzzles
This lecture presents some puzzles that have been driving people crazy for decades, even centuries. These puzzles involve jumping pegs, sliding blocks, or blinking lights. Most of these puzzles have easy solutions, once you know the secret, but sometimes tweaking the problem a tiny bit can make it mathematically impossible.

Lecture 9: Mastering Rubik's Cube
You will learn a fast and easy strategy to solve the world's most popular puzzle, along with some of the mathematics behind it.

Lecture 10: Solving Sudoku
This lecture presents some simple strategies, and their underlying logic, that will enable you to solve more sudoku puzzles quickly and easily. The strategy given is the one recommended by a world sudoku champion.

Lecture 11: Mathematics and Chess
Mathematics and chess are both activities that train your brain to look for patterns and think logically. You will see where mathematical ideas appear in the opening, middle game, and endgame of chess and learn an easy solution to the world's most famous chess puzzle.

Lecture 12: Winning Ways—It's Your Move!
The course concludes with more games of strategy in which a player who is armed with the proper mathematical insights will have a decisive advantage.

Of course, the main reason we play games and solve puzzles is to have fun and to experience the intellectual satisfaction of rising to a challenge. Games train you to be a better decision maker in general. We take calculated risks every day of our lives. In this course, you will learn how to do some of these calculations while having fun at the same time.

Games are a great family activity that can be appreciated on multiple levels by children and adults. They are also a fun way to learn some really beautiful mathematics. And, of course, games and puzzles are a great way to keep your mind active and sharp at any age.

By the end of this course, you will have a fun and mentally stimulating set of skills that you can apply to countless games and puzzles. ■

Let the Games Begin!

Lecture 1

In this course, you will learn that with the simplest of mathematical tools, you can vastly improve your ability to play and understand a tremendous number of games and puzzles. In fact, mathematicians have been among the world's best games players, and games have also motivated some very interesting mathematics. Games and puzzles can be classified in a number of ways: Is there randomness involved? Do you have an intelligent adversary? If so, do both players have the same information? In this lecture, you will learn about the games of 21 and 15, tic-tac-toe, and the Tower of Hanoi.

The Game of 21

- To play the game of 21 (not the card game), the first person chooses a number between 1 and 3. Then, the second person adds a number from 1 to 3 to the first person's number to create a new total. This continues until the total reaches 21. Whichever person gets to 21 is the winner.

- After playing a few games, you'll notice that if you ever reach a total of 17, then you are guaranteed to win because you are 4 away from 21, so for any number that your opponent adds to 17—a 1, 2, or 3—you will be able to add the opposite number—3, 2, or 1—to reach 21. By working backward, the goal of this game is really to get to 17.

- If you continue that logic, then if you can get to 13, which is 4 below 17, then you can be assured of reaching 17 because if you can get to 17, then you can get to 21. Therefore, 13 is your new goal. Subtracting 4 from 13, you can force a win if you can get to 9—or, continuing to subtract 4, if you can get to 5 or 1.

- If you want to guarantee that you can win no matter how your opponent plays, you should start with a total of 1. Thereafter, you should jump to the totals of 5, 9, 13, 17, and finally 21. When playing this game against a new player, wait a few turns before jumping to one of your winning numbers because you don't want your opponent to see any pattern too soon.

- This game illustrates one of the strategies for successful game and puzzle solving: Work backward from your goal. This strategy is a useful tool for games of pure strategy, but it can also be applied to other games of chance.

The Game of 15

- In the game of 15, another game that is similar to 21, if you know the secret, you can never lose—although it is possible for the game to be a draw. In this game, two players take turns choosing numbers from 1 to 9, and they are not allowed to repeat numbers. The numbers 1 through 9 can be represented by cards from a deck of playing cards.

- Players take turns choosing numbers, and the first player to obtain 3 numbers that add to 15 is the winner. If nobody scores 15, then the game is a draw. Because the first player has the advantage, you should alternate who goes first each game.

- For example, suppose that you start with the number 2, and your opponent picks the number 9. Then, you pick the number 6. With 2 and 6, you are threatening to get 15, so your opponent is going to have to counter that by taking 7 so that you can't get to 15. Then, your opponent adds up to 16. You're not worried about your opponent scoring a 15 this time. You take the number 8. Now, you're in a guaranteed winning position because next turn, you are either going to make 15 by having 8, 6, and 1 or by having 8, 2, and 5. Your opponent won't be able to stop you.

- There are eight ways to create 15 numbers using 3 different numbers from 1 through 9. You can arrange these combinations in a tic-tac-toe board. All eight combinations appear as lines on a tic-tac-toe board—horizontally, vertically, and diagonally. Mathematicians call this a magic square.

- Playing the game of 15 is really the same as playing tic-tac-toe, and if your opponent doesn't notice this, then you'll win most of the time and never lose.

- The game of 15 illustrates another useful tip: Find a mathematical structure to represent your game or puzzle. You won't always be able to do this, but when you can, it will often provide you with insights that would not otherwise be apparent.

Tic-Tac-Toe

- In the game of tic-tac-toe, the first player, X, has the advantage because if X plays properly, he or she can never lose—and can even sometimes force a win. As a practical matter, X should start in one of the four corners because then O has only one safe response, which is to play in the middle. If O doesn't play in the middle, then X can force a win.

Tic-Tac-Toe game.

- Suppose we change the rules so that if someone gets three in a row, instead of becoming the winner, he or she loses. The game changes, becoming much more interesting. This time, because X goes first and has to make five of the nine moves, X is at a disadvantage. You would think that the worst place for X to start would be in the center because it's involved in the most three-in-a-row combinations, but if X plays anywhere other than the center, then O can actually force a win.

- This idea of exploiting symmetry is very useful in many games and puzzles. As for the strategy of mimicking your opponent, not only does it sometimes lead to a draw, but it can also sometimes lead to a win. For example, with Cram, a game that is a cross between checkers and dominoes, the strategy of copying your opponent's moves is sometimes known as strategy stealing, and it can be used to prove some interesting results about games.

The Tower of Hanoi

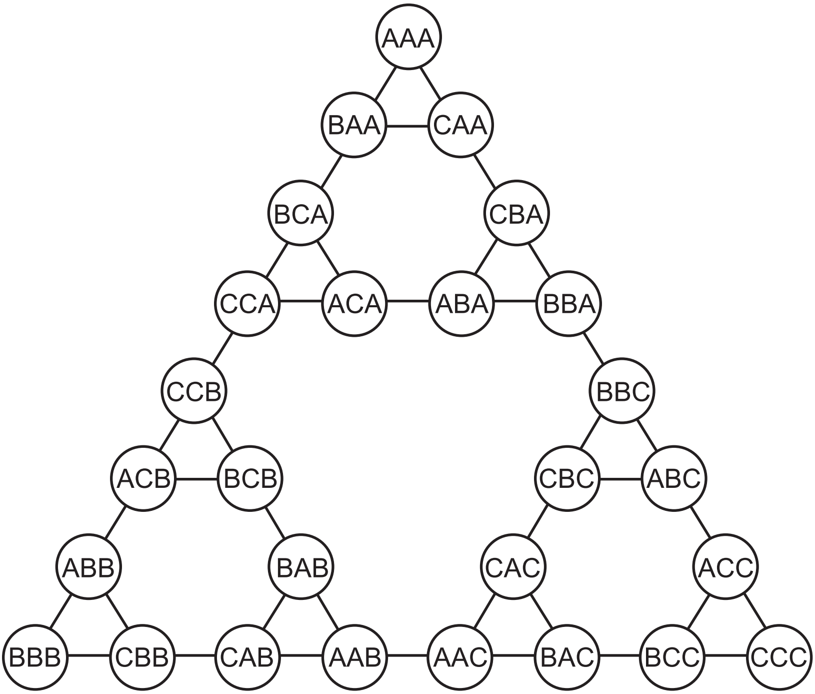
- Some of the world's best game players have been mathematicians. Mathematicians have been national and world champions at many games, including chess and checkers, as well as other games like Scrabble, backgammon, bridge, blackjack and poker. Mathematicians are leading authorities on famous puzzles like Rubik's Cube and sudoku.

- Mathematician Édouard Lucas was an expert on the Fibonacci numbers, and he invented a famous puzzle called the Tower of Hanoi, sometimes called the Lucas Tower in his honor.

- The challenge of this puzzle is to move the pieces one at a time from the first peg to one of the other pegs with the restriction that you're never allowed to put a bigger piece on top of a smaller piece. How do you begin to solve this puzzle, and can you do it in the fewest number of moves?

- For each piece, the number of moves doubles plus 1. In general, the problem with n pieces can be solved in $2^n - 1$ moves. The original puzzle, which comes with $n = 9$ pieces, can be solved in $2^9 - 1 = 511$ moves. There is a quick way to solve this puzzle in 31 moves.

- Another way to understand the Tower of Hanoi is by using an object that mathematicians call a graph. The graph shows you that the Tower of Hanoi puzzle is connected in that it's always possible to get from any position to any other position in a legal sequence of moves.



Games and Puzzles

- Mathematicians have been very good for improving our understanding of games. On the other hand, the reverse is also true: Games and puzzles have inspired many people to study mathematics and have even led to the creation of entirely new branches of mathematics.

- Games come in all different shapes and sizes, and mathematicians classify them in many different ways. For example, random games use objects like cards and dice. If a game has no randomness and is just pure strategy, like with chess or checkers, then it's called deterministic.

- In addition, games can be classified by the type of strategy your opponent uses. Is it predetermined, like in most casino games, or can your opponent use an intelligent strategy to maximize his chances of defeating you, like in poker or chess? Depending on the presence of randomness and how your opponent plays, we can put most games into one of four categories.

- Most games of pure strategy—like chess and checkers—are in the first category: deterministic versus an intelligent opponent. Deterministic games against neutral opponents are less common, but it's probably true of most video games where you're playing against a computer. Most puzzles could be classified as deterministic one-player games; typically, all of your information is present, and there's no randomness. Examples include the Tower of Hanoi, Rubik's Cube, and sudoku.

- Random games against intelligent opponents include games like poker, bridge, Scrabble, and backgammon. Random games against opponents with predetermined strategies include games like roulette, craps, blackjack, and slot machines. In fact, we could even classify random games according to where the randomness occurs. In bridge, it happens at the beginning of the game, when the cards are dealt. In backgammon, there's no hidden information. In games like Scrabble, blackjack, and poker, there is randomness on both ends.

Suggested Reading

Ball and Coxeter, *Mathematical Recreations and Essays.*

Beasley, *The Mathematics of Games.*

Bewersdorff, *Luck, Logic, and White Lies.*

Browne, *Connection Games.*

Browne, *Hex Strategy.*

Epstein, *The Theory of Gambling and Statistical Logic.*

Hess, *Mental Gymnastics.*

MacKinnon, *Bridge, Probability, and Information.*

Niederman, *The Puzzler's Dilemma.*

Packel, *The Mathematics of Games and Gambling.*

Rubens, *Expert Bridge Simplified.*

Wells, *Book of Curious and Interesting Puzzles.*

Problems

1. Suppose you and your friend play the game of 51, where you take turns adding numbers from 1 to 6. Whoever reaches 51 is the winner. Would you rather go first or second, and what is the winning strategy?

2. You play tic-tac-toe and start in the upper right corner. Your opponent plays in the square below you. What is the winning response?

3. Explain why the Tower of Hanoi with n pieces will have 3^n positions.

Games of Chance and Winning Wagers

Lecture 2

There's only one way to find out which games of chance—such as those found at carnivals or casinos—are the best and which are the worst: Do the math. Once you know the odds of winning a game, how much should you bet? The answer depends on whether you are playing a game where the odds are in your favor or against you and what your financial objectives are. In this lecture, you will be introduced to roulette, sic bo, and craps.

Roulette

- An American roulette wheel has 38 numbers, 18 of which are red and 18 of which are black. Two of the numbers, 0 and 00, are green. The simplest bet in roulette is to bet on one of the main colors—for example, red. Let's say that you bet $1 on red. It's an even money bet, which means that if you bet $1, then you'll either win or lose $1, depending on whether or not a red number appears.

- Now what are your chances of winning? Because there are 38 numbers, each of which has the same chance of occurring, and 18 of these numbers are red, then the probability that you win is 18/38, which is a little less than 50%. Clearly, you have a disadvantage at this game. We can quantify this disadvantage using the very important concept of expected value.

- The expected value of a bet is a weighted average of how much you can win or lose. When you bet on red in roulette, you'll either win $1 with probability 18/38, or you'll lose $1, which could also be interpreted as winning –$1 with probability 20/38 because there are 18 red numbers and 20 numbers that aren't red. Hence, your expected value is 1(18/38) + (–1)(20/38), which equals –2/38, or –0.0526. Therefore, on average, you'll lose about 5.3¢ for every dollar that you bet.

- In roulette, you can bet on other things besides color. For instance, you can bet that a number between 1 and 12 shows up. Let's say that you bet $1 that one of the first 12 numbers shows up. The casino pays two-to-one odds for this bet, which means that if you bet $1 and you win, then the casino pays you $2.

- When you bet $1, you're going to win $2 with a probability of 12 numbers out of 38, and you're going to lose $1 with probability 26/38 because if you win 12 times out of 38, you lose 26 times out of 38. Hence, the expected value of this bet is 2(12/38) + (–1)(26/38) = –2/38 = –0.0526, which is the same number as before.

- Suppose you bet on a single number, such as 17. In this case, the casino pays 35-to-1 odds. Thus, when you make this bet, then you either win $35 with probability 1/38, just one winning number out of all 38, or you lose $1 with probability 37/38. When you calculate the expected value, 35(1/38) + (–1)(37/38), once again you get –2/38, or –5.3¢.

- Interestingly, when you play roulette, practically every bet has the exact same expected value of –5.3¢ per $1 bet. When your expected value is negative like in roulette, then that bet is called unfavorable. If the expected value is positive, which is very rare in a casino, then the bet is called favorable. If a bet has an expected value of 0, then the bet is called fair.

- When you make a $1 bet on the number 17, the casino is paying you 35-to-1 odds, which is unfavorable. The true odds of this bet are 37 to 1 because there are 37 unfavorable numbers, or 37 numbers that make you lose, and just 1 good number. If the casino paid you 37-to-1 odds, then the game would be fair because 37(1/38) + (–1)(37/38) equals 0.

- If you decide to bet on red twice in a row, then you either win twice with a probability of about 22%, win once with a probability of about 50%, or lose twice with a probability of about 28%. Notice that the sum of these probabilities is exactly 100%, or 1. This is not a coincidence. Whenever you sum all distinct possibilities, the probabilities have to sum to 1.

Sic Bo

- A carnival game known as chuck-a-luck has been around for years, but more recently, a variation of the game has been brought to the casinos with the name "sic bo," which is Chinese for "dice pair." The game is played with 3 dice. You place a wager on one number from 1 to 6. Let's say you bet $1 on the number 4. Then, the dice are rolled, and if a 4 appears, you win $1.

- It almost sounds like a fair game, because each die has a 1-in-6 chance of being a 4. Furthermore, if your number shows up twice, then you'd win $2, and if it shows up three times, then you'd win $3.

- Your chance of winning $3 is extremely low. To win $3, each die must be 4, and the probability that the red die is 4 is 1 out of 6. The probability that the yellow die is 4 is 1 out of 6. The probability that the blue die is 4 is 1 out of 6, and because the dice rolls are very independent, the probability that all of them are 4s is (1/6)(1/6)(1/6) = $1/6^3$ = 1/216, which is less than 1%.

- On the other hand, the chance of losing $1 is actually pretty high. For that to happen, none of the dice can be 4s. What's the chance of that? Well, the chance that the red die is not a 4 is 5 out of 6. Likewise, the probability that the yellow die is not a 4 is 5 out of 6, and the probability that the blue die is not a 4 is 5 out of 6. Therefore, the probability that they are all not equal to 4 is $(5/6)^3$, which is 125/216, which is about 58%. So, you actually lose money in this game more than half the time.

- What are your chances of winning $1? To do this, you need exactly one of these 3 dice to be a 4. There are three ways that this can happen. You can either have a red 4 and yellow and blue not be 4, or you could have a yellow 4 only, or you could have a blue 4 only. The chance of having a red 4 and no yellow and blue 4 would be (1/6)(5/6)(5/6) = 25/216. Likewise, there's a 25/216 chance of seeing just a yellow 4 and a 25/216 chance of seeing just a blue 4. Therefore, the total chance of seeing exactly one 4 is 75/216, which is about 35%. Similarly, the probability of winning $2 is 15/216, about 7%.

- The probabilities sum to 1, or 216/216. If they don't, then you've made a mistake in at least one of your earlier calculations. Your expected profit is 75/216 (1) + 15/216(2) + 1/216 (3) + 125/216(–1) = –17/216 = –0.08. In other words, you can expect to lose about 8¢ for every dollar that you bet in this game. This game is very unfavorable and should be avoided.

Number of 4s	Probability
0 (Lose $1)	125/216
1 (Win $1)	75/216
2 (Win $2)	15/216
3 (Win $3)	1/216
Total	216/216 = 1

Craps

- The most popular dice game in the casino is the game of craps. The rules of the game are pretty simple. You place your bet and roll two dice. If the total is 7 or 11, you win. If the total is 2, 3, or 12, you lose. If the total is something else, then that number becomes your point, and you keep rolling the dice until your point appears or 7 appears. If your point appears first, you win. If 7 appears first, you lose.

- If you rolled a 4, for example, on your first roll, what are the chances that you win? There are a few ways to calculate this probability, all of which lead to the same answer: 1/3. If you look at all the possible totals when you roll two dice—for example, a red and a green die—you have 36 possible outcomes: 6 for the red and 6 for the green, each of which is equally likely. Of these, 3 have a total of 4, and 6 have a total of 7.

- Once you rolled that first 4, you just keep rolling the dice until a 4 appears or a 7 appears. So, there are nine outcomes that end the game: Three are winners; six are losers. Therefore, your chance of winning is 3 out of 9, or 1/3. Thus, your chance of losing is 2/3. This makes sense because you have twice as many ways to roll a 7 than to roll a 4.

- We can use these numbers to figure out your overall chance of winning using what's called the law of total probability, which says that your overall winning chance is a weighted average of your chances of winning from each opening roll. In other words, we weight each opening roll based on how likely it is to occur on the first roll. The sum of these weights is necessarily 1.

Opening Roll	Weight	Winning chances	Product
2	1/36	0	0
3	2/36	0	0
4	3/36	3/9	9/324 = .0278
5	4/36	4/10	16/360 = .0444
6	5/36	5/11	25/396 = .0631
7	6/36	1	6/36 = .1667
8	5/36	5/11	25/396 = .0631
9	4/36	4/10	16/360 = .0444
10	3/36	3/9	9/324 = .0278
11	2/36	1	2/36 = .0556
12	1/36	0	0
Total	1		244/495 = .493

- When we multiply each winning chance by its weight and take the sum, and then reduce the fraction, it turns out to be 244/495, which is 0.493. This equates to there being a 49.3% chance of winning the game of craps. When you bet $1, you're either going to win $1 with probability 0.493, or you're going to lose $1 with probability 0.507.

- Your expected value for craps is 1(0.493) + (–1)(0.507) = –0.014, which equals 1.4¢ for every dollar that you bet. This makes the game of craps much fairer than roulette and chuck-a-luck, but it's still an unfavorable game.

Suggested Reading

Beasley, *The Mathematics of Games.*

Bewersdorff, *Luck, Logic, and White Lies.*

Epstein, *The Theory of Gambling and Statistical Logic.*

Gillis, *Backgammon.*

Haigh, *Taking Chances.*

Packel, *The Mathematics of Games and Gambling.*

Poundstone, *Fortune's Formula.*

Rosenhouse, *The Monty Hall Problem.*

Problems

1. The casino game of sic bo allows other bets as well. For instance, you can bet that when you roll three dice, the total will be "small." If your total is 10 or smaller, but not three of a kind (all 1s, 2s, or 3s), then you win. Answer the following questions, keeping in mind that you are rolling three six-sided dice.

   a. What is the probability of rolling three of the same number?

   b. Use symmetry to explain why the probability that the total is 10 or less is the same as the probability that the total is 11 or greater.

   c. Use a.) and b.) to determine the probability of winning the sic bo bet described above.

   d. What is your expected value when placing a $1 bet?

2. In the game of craps, a “field bet” pays even money if your opening roll has a total of 3, 4, 9, 10, or 11, and it pays two to one if the opening roll is 2 or 12. What is the probability of winning, and what is the expected value of this bet?

3. Suppose you play a fair game where you win or lose \$5 on each bet.

a. Starting with \$15, what are the chances of reaching \$60 before going broke?

b. Suppose you boldly try to reach \$60 by betting everything twice in a row. What are your chances of success?

c. Answer questions a.) and b.) when your probability of winning each bet is 0.6.

Optimal Blackjack and Simple Card Counting

Lecture 3

Blackjack is the fairest game in the casino—if you play your cards right. In blackjack, there are many decisions for the player: Should you hit, stand, split, double down, or buy insurance? The answer will depend on your cards and the card that the dealer is showing. In this lecture, you will learn the optimal basic strategy for playing blackjack without counting cards. You will also learn techniques for counting cards, which can sometimes give you an advantage over the house.

Blackjack

- In blackjack, you and the dealer are initially dealt two cards, and you can see one of the dealer's cards, called the up card. Next, you add up your card values: Cards 2 through 9 have values 2 through 9; ten, jack, queen, and king have value 10; and aces are worth either 1 or 11.

- A hand with an ace in it can be 1 or 11. If it can be used as either a 1 or 11, it's called a soft hand. For example, a hand that consists of an ace and a 6 is a soft 17 because the hand can also be counted as a 7—in other words, it's a 17 that can be softened to a 7. However, a hand that consists of an ace, a 6, and a queen is a hard 17 because for these hands, 17 is the only legal total.

- You may continue to take cards—which is called hitting—as long as your total is under 21. If your total goes over 21, then you are busted, and you automatically lose your bet. If you stop taking cards—which is called standing—without going bust, then the dealer takes cards until his or her total is 17 or higher.

- If neither you nor the dealer busts, then whoever has a total that is closer to 21 wins the bet. If you and the dealer have the same total, then you neither win nor lose; it's a tie, known in blackjack as a push.

- If two cards that you are originally dealt add up to 21—for example, if you get an ace and a king as your first two cards—then your hand is called a blackjack, and the casino pays three to two. Thus, if you initially bet $2 and you get dealt a blackjack, then the casino pays you $3—unless the dealer also has a blackjack, in which case it is a push.

- Sometimes, the player has additional options. After you get your initial two cards, you can double your bet and get exactly one more card. This is called doubling down. For example, let's say your total is 11, which is excellent, and the dealer has a 9 showing. Then, if you were betting $2, you can double your bet to get $4. You tell the dealer that you want one more card. If your card ends up being a 3, for example, even though you'd like to take another card, you cannot because you doubled down. You just have to hope that the dealer busts, because if the dealer gets 17 to 21, the dealer wins.

When to Double Down

Your first two cards	Double if dealer has
Total 11	10 or below
Total 10	9 or below
Total 9	4, 5, or 6
A2 thru A7	4, 5, or 6

- If your first two cards have the same value, then you're allowed to split the cards and play the two hands for an additional bet. For example, let's say you bet $2 and you're dealt a pair of 8s, and the dealer has an up card of 5. Instead of playing one hand with a rather poor total of 16, you'd rather play two hands that start with a total of 8. You tell the dealer that you're splitting, and you can take your original bet and essentially double it, so now you're playing two hands each worth $2 against the dealer's 5.

When to Split

When do you split…	If dealer has…
A, 8	Any card
4, 5, 10	Never!
2, 3, 6, 7	2,3,4,5, or 6
9	2,3,4,5,6,8,9

- If you split aces, then the dealer will give you just one more card. If you get a blackjack, it only pays even money. Some casinos will even let you re-split any cards except for aces, and most casinos will not let you double down after splitting.

- The final option is called insurance. If the dealer's face-up card is an ace, then you are allowed to bet up to half of your initial bet that the dealer has a 10, jack, queen, or king underneath. If so, that bet pays two-to-one odds.

- Taking insurance is generally a bad bet. When you make an insurance bet, there are four cards that win for you. If the down card is a 10, jack, queen, or king, then you win your insurance bet on those four card values. The other nine card values—ace through 9—would lose for you. For the insurance bet to be fair with four winning cards and nine losing cards, the casino should pay you nine-to-four odds, but the casino is only offering you two-to-one odds.

- Who has the advantage in blackjack? On the one hand, the player has many more options than the dealer. After all, the player can stand before reaching 17, double down, split, and take insurance. The player also gets paid three to two for blackjacks. However, the dealer has one advantage that offsets all of the player's options: If the player and the dealer both bust, then the dealer still wins.

The Optimal Basic Strategy

- Most casinos play with more than one deck, meaning that they shuffle several decks together and you play with those cards. The optimal basic strategy can vary slightly, depending on the number of decks in use, but your overall advantage won't change much. Consider the following to be an extremely good strategy that's easy to remember and works for any number of decks.

- If your cards have a hard total of 17 or higher, then you should always stand. If your cards have a hard total of 11 or lower, then you should never stand—because there's nothing to lose by taking another card.

- The situation is more interesting when you have a hard total between 12 and 16. If the dealer shows a good card—a 7, 8, 9, 10, or ace—then you should hit. In your head, think of the dealer as having a total of at least 17. If the dealer has a bad card—a 4, 5, or 6—then take no chances and stand. These are bad cards because they have a tendency to become 14s, 15s, or 16s, which have a tendency to bust. If the dealer shows a 2 or a 3—a so-so card—then you should hit with a hard total of 12 but stand on 13, 14, 15, or 16.

- When you have a soft hand, the strategy changes. If you have a soft 16 or lower, such as an ace and a 5, then you should always hit because you have nothing to lose. Also, according to basic strategy, you should always hit if you have a soft 17. Some people are surprised by this rule, but it is supported by math.

- You should always stand on a soft 19 or higher. If you have a soft 18, such as an ace and a 7, then you should hit if the dealer shows a 9, 10, or ace. Otherwise, you should stand.

- The rules for doubling down are very simple: If you have a total of 11, then you should double down if the dealer's card is 10 or below. If you have a total of 10, you should double down if the dealer's card is 9 or below. In other words, if your total is 10 or 11, you should double down if your total is better than the dealer's up card. The only other times when you might double down is when your total is 9 or if your hand is soft. In these situations, you should only double down if the dealer shows a bad card—a 4, 5, or 6.

- The rules for splitting are easy to remember. Always split aces and 8s. Never split 4s, 5s, or 10s because you're going from a good total (8, 10, or 20) to a bad total (4 or 5) or, at least in the case of 10, to a worse total. For almost everything else, you should split your cards when the dealer has a low card showing—a 2, 3, 4, 5, or 6. There's only one exotic exception: You split your 9s when the dealer has 2 through 9, except 7.

Card Counting

- In 1966, mathematician Edward Thorp wrote a book called *Beat the Dealer* that changed the way people played blackjack, also known as the game of 21. Thorp observed that blackjack is different from most other casino games because your winning chances can actually change as you go from one deal to the next.

- Many people think that card counting involves memorizing all the cards that have been played, but in fact, card counting usually just consists of keeping track of a single number, and it's pretty easy to do. It's also not as profitable as people think.

- Specifically, if the deck contains a much higher proportion of 10s and aces, then that will benefit the player. The extra 10s and aces lead to more blackjacks, and that pays three to two to the player, but only even money to the dealer, so the player benefits more than the dealer the more blackjacks there are.

- The additional 10s help the player in a few other ways. Most of the time that you double down, you have a total of 10 or 11, and you're hoping to get a 10 card. Therefore, the more 10s that are left in the deck, the better it is for you. The 10s are bad for the dealer because they cause the dealer to bust more often. It is also true that the 10s cause the player to bust more often, but the player won't bust as often because the player is allowed to stop before reaching 17. On the other hand, the dealer has to keep taking cards until he or she gets to 17, so the dealer busts more often.

- When the count gets high—which is going to happen when there have been a lot of low cards that have been played and a lot of 10s and aces left in the deck—you can make some adjustments to basic strategy. For example, if there are enough 10s left in the deck, it can even be worthwhile to take insurance when it favors the player, but most of all, when there are enough 10s and aces, you typically want to bet more because you now have an advantage in the game.

- The essence of card counting is as follows: When the deck is against you, which is most of the time, bet low. When it's for you, bet high—but not too high.

- The simplest and most frequently used card-counting system is called the high-low system. You start with zero, and every time you see a low card—a 2, 3, 4, 5, or 6—you add one to your count. Every time you see a high card—a 10, jack, queen, king, or ace—you subtract one from your count. If you have a middle card—a 7, 8, or 9—you don't change your count at all (you add zero). There are five low cards (2, 3, 4, 5, 6) and five high cards (10, jack, queen, king, ace), so the count should usually be around zero.

Suggested Reading

Griffin, *The Theory of Blackjack.*

Schlesinger, *Blackjack Attack.*

Snyder, *Blackbelt in Blackjack.*

Thorp, *Beat the Dealer.*

Vancura and Fuchs, *Knock-Out Blackjack*

Problems

21 questions! For the following blackjack scenarios, what does basic strategy tell you to do: stand, hit, double down, or split?

1. 10 5 vs. 7 (that is, you have 10 and 5; the dealer's face-up card is 7).

2. 9 3 vs. 5.

3. 8 8 vs. 10.

4. 5 5 vs. 5.

5. 7 6 vs. 2.

6. A 7 vs. Q.

7. 8 3 vs. J.

8. A 6 vs. 7.

9. A 2 3 4 vs. 9.

10. Q 2 vs. 3.

11. J 7 vs. A.

12. 9 5 vs. 10.

13. A 6 vs. 4.

14. 9 9 vs. 7.

15. A 4 2 vs. 6.

16. A 7 vs. 5.

17. 7 2 vs. 4.

18. 4 4 vs. 6.

19. 7 7 vs. 3.

20. 7 4 vs. 8.

21. Basic strategy says to never take insurance, but if there are 26 unseen cards remaining, at least how many of them have to be 10, J, Q, or K for insurance to be the correct action?

Mixed Strategies and the Art of Bluffing

Lecture 4

In blackjack, your opponent is the dealer, who must always follow a fixed strategy, but there are other games in which your opponent can alter his or her strategy to actively try to defeat you. In this lecture, you will learn how to play against an intelligent adversary whose interests are diametrically opposed to yours. You will learn this fundamental skill, along with the underlying mathematics, with simple games like rock-paper-scissors, the penny-matching game, and Le Her—all of which show you how it pays to vary your strategy.

Rock-Paper-Scissors

- In many games like roulette, craps, and blackjack, your chance of winning can be calculated. In these games, your fate is determined by the spin of a wheel, the roll of the dice, or the dealer's cards. But what happens when you're faced with an intelligent adversary whose goals are diametrically opposed to yours?

- When both players are allowed to pick their own strategy, and when one person's gain is the other person's loss, these are called zero-sum games, and their analysis can be very interesting.

- Rock-paper-scissors is a typical zero-sum game because when one player wins, the other player loses, and the players are allowed to vary their strategy. When you're playing against an intelligent opponent, it can be dangerous to be predictable. This game is a fair game, because no player has a built-in strategic advantage over the other.

- In general, zero-sum games have the following features. They're played by two players, Rose and Colin, for example. The players each choose a strategy, and they reveal it simultaneously. In a zero-sum game, every dollar that Rose wins, Colin loses—and vice versa. A zero-sum game can be represented by a payoff matrix.

- For example, in the game rock-paper-scissors, Rose chooses one of the rows and Colin chooses one of the columns in the payoff matrix. A matrix is just a collection of numbers arranged in a rectangular box. In the matrix, if Rose plays rock and Colin plays scissors, then Rose wins $1. On the other hand, if Rose plays rock and Colin plays paper, then Rose loses $1. Colin's payoff matrix is just the opposite (the negatives) of Rose's.

Rock-Paper-Scissors: Rose's Payoff Matrix

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

The Penny-Matching Game

- In a game called the penny-matching game, Rose and Colin have just two strategies, but the payoffs are not symmetrical. In this game, Rose and Colin simultaneously show one side of a coin to each other. They don't actually flip the coins; rather, they just show each other one side of their coin.

- For example, maybe Rose announces that she is choosing heads and Colin announces that he is choosing tails. If both players show heads, then Colin wins $3. If both players show tails, then Colin wins $1. If one shows heads and the other shows tails, then Rose wins $2.

The Penny-Matching Game: Rose's Payoff Matrix

	H	T
H	-3	2
T	2	-1

- Suppose that Rose had to tell Colin her strategy in advance. Should she play heads or tails? Obviously, if Colin knows what she's going to do, she's going to pick tails because she's only going to lose \$1—whereas if she played heads, then Colin would play heads, and she'd lose \$3.

- Instead of simply choosing tails, Rose can actually do better by employing a randomized, or mixed, strategy. Suppose that she tells Colin that she's going to flip her coin and use whatever lands. If Colin knows that she's going to flip her coin, what should he do?

- Assuming that Rose is flipping a fair and balanced coin, then Colin knows that Rose will choose heads or tails with probability 0.5. Knowing that, what should Colin do? If Colin decides to play heads, then half the time Rose is going to lose \$3, and half the time she's going to win \$2, which turns out to be –1/2. So, on average, she'll lose 50¢ with that strategy.

- On the other hand, if Colin plays tails, then half the time Rose wins \$2, and half the time Rose loses \$1, which gives Rose an expected payoff of 1/2. Colin wants Rose to do poorly, so if he knew that Rose was flipping a fair coin, he would choose the heads strategy, giving Rose an expected value of –1/2.

- This coin-flipping strategy is better than Rose always picking tails because the worst Rose can lose on average is 50¢, whereas if she uses the strategy of always picking tails, then she's going to lose on average \$1, if Colin could exploit that.

- Suppose that instead of playing heads and tails with equal probability, she chooses heads with probability 3/8 and tails with probability 5/8. Knowing that, what should Colin do? Notice that when Colin plays heads, then Rose's expected value is 3/8(–3) + 5/8(2), which equals 1/8. Rose has a positive expected value.

- On the other hand, when Colin plays tails, then 3/8 of the time Rose wins \$2, and 5/8 of the time she loses \$1: 3/8(2) + 5/8(–1), which equals 1/8. No matter what Colin plays, Rose has an expected value of 1/8.

- In fact, even if Colin decides to get into the act and randomly mix between heads and tails, Rose will still get an expected value of 1/8, or 12.5¢. This game is good for Rose. In fact, there is nothing Rose can do to force this number any higher than 1/8.

- In this example, 1/8 is the value of the game ($V = 1/8$). Rose's (3/8, 5/8) strategy is called her optimal strategy, or equilibrium strategy. It guarantees the highest expected payoff, and it has the nice feature that Rose can reveal this strategy to Colin, and he won't be able to exploit this information. Another feature of the equilibrium strategy is that if your opponent is using his or her equilibrium strategy against you, then you can do no better than to use your equilibrium strategy against your opponent.

- Rose can execute the (3/8, 5/8) strategy by flipping her coin three times. When you flip a coin three times, there are eight equally likely outcomes, and three of those outcomes have exactly one head. In other words, when you flip your coin three times, the probability of getting one head is 3/8. Rose flips her coin three times: tails, tails, tails. If heads appears once, she can show heads. Otherwise, she shows tails.

- When a player has only two strategies, like showing heads or tails, then there's a simple algebraic solution. Suppose Rose chose heads with probability x and, therefore, tails with probability $1 - x$, then when Colin chooses heads, Rose gets an expected value of $-3x + 2(1 - x) = 2 - 5x$.

- On the other hand, when Colin chooses tails, then Rose gets an expected value of $2x + (-1)(1 - x) = 3x - 1$. When she plays heads with probability x, her expected value is guaranteed to be at least $2 - 5x$ or $3x - 1$, whichever is smaller. These lines intersect when $2 - 5x = 3x - 1$, which occurs when $x = 3/8$.

- For two-person games in which both players have more than two strategies, can we be sure that an equilibrium solution even exists? This question was proved by one of the 20th century's greatest thinkers, John Von Neumann. He proved that every two-person zero-sum game has a value and that both players have mixed strategies that achieve that value. This theorem was later extended by John Nash to games that were not required to be zero-sum games.

Le Her

- A poker variation called Le Her is a classic game that dates back to the 18th century and may have been the first game ever analyzed where the optimal solution employed a mixed strategy.

- The game begins with several players, but it eventually reduces to a two-person game: the dealer and the receiver. The game is played from a deck of 13 cards, ace through king, where ace is the lowest card and king is the highest card. Both players are dealt one card facedown.

- The players look at their own cards, and then the receiver must decide whether to keep the card he or she was dealt or switch it for the dealer's card, which is unknown to him or her. Once the receiver has switched the card or not, the dealer then decides whether to keep the card that is now in front of him or her or switch it for the top card of the deck. Whoever has the higher card wins.

- Most people expect that the dealer has the advantage in this game, but the receiver actually has the edge. The payoff matrix for the receiver is a 13-by-13 matrix. For example, when the receiver (Rose) plays strategy 8, that means that she will keep any card that is 8 or higher. When the dealer (Colin) plays strategy 9, that means that if Rose swaps her card, then he makes the obvious decision—because he knows what his card and what her card is—so he's going to keep his card if it's higher and swap it if it's lower. But when Rose keeps her card, then playing strategy 9 means that Colin will keep his card if it's 9 or above.

Le Her: Receiver's Payoff Matrix

	A	2	3	4	5	6	7	8	9	10	J	Q	K
A	.50	.46	.43	.40	.39	.37	.36	.36	.37	.39	.40	.43	.46
2	.54	.50	.47	.44	.42	.41	.40	.40	.41	.42	.44	.47	.50
3	.57	.54	.51	.48	.46	.45	.44	.44	.45	.46	.48	.50	.53
4	.60	.57	.54	.52	.50	.48	.47	.47	.48	.49	.51	.53	.56
5	.61	.59	.57	.55	.53	.51	.50	.50	.50	.51	.53	.55	.58
6	.62	.61	.59	.57	.56	.54	.53	.52	.52	.53	.54	.56	.59
7	.62	.61	.60	.59	.57	.56	.55	.54	.54	.54	.55	.57	.59
8	.61	.60	.60	.59	.58	.57	.56	.55	.55	.55	.55	.56	.57
9	.60	.59	.58	.58	.57	.57	.56	.56	.55	.55	.55	.56	.57
10	.57	.56	.56	56.	.55	.55	.55	.54	.54	.54	.53	.54	.55
J	.53	.52	.52	.52	.52	.52	.52	.51	.51	.51	.51	.51	.51
Q	.47	.47	.47	.47	.47	.47	.47	.47	.47	.47	.47	.47	.47
K	.41	.41	.41	.41	.41	.41	.41	41	.41	41	.41	.41	.41

- Even though the matrix has 13 rows and 13 columns, we can actually reduce this problem to a two-by-two game through something called dominated strategies. First, Rose should never use strategies 1 through 6 because no matter what column is played, all of those numbers are dominated by row 7. Because of that, we can remove rows 1 through 6 from the matrix.

- By similar logic, Colin prefers small numbers because they represent Rose's probabilities. If we examine the remaining rows, then we realize that column 9 dominates all of the columns to its left, so we can remove columns 1 through 8. Row 9 dominates rows 10 through king, so those can be removed as well. Then, column 10 dominates jack, queen, and king, so we can eliminate those columns. Finally, row 8 dominates row 7, so row 7 can be eliminated. We're left with just a two-by-two game.

The Reduced Two-by-Two Game

	9	10
8	.547	.548
9	.549	.545

- The receiver should play strategy 8 80% of the time and strategy 9 20% of the time. This means that the receiver should always swap 7s and below and should always keep 9s and above. The only questionable card is 8, and according to our mixed strategy, the receiver should keep 8s 80% of the time and swap 8s 20% of the time. Similarly, the dealer should always swap 8s and below, always keep 10s and above, and keep 9s 56% of the time.

- The value of this game is 0.547, so using this strategy, the receiver has a 54.7% chance to win. Most people are surprised by this result, because they expect the dealer to have the advantage. However, it is true that the dealer sometimes has more information than the receiver and can exploit this.

Suggested Reading

Beasley, *The Mathematics of Games.*

Binmore, *Fun and Games.*

Bewersdorff, *Luck, Logic, and White Lies.*

Epstein, *The Theory of Gambling and Statistical Logic*.

Packel, *The Mathematics of Games and Gambling.*

Von Neumann and Morgenstern, *Theory of Games and Economic Behavior.*

Problems

1. Determine the equilibrium strategies for the row and column players when the payoff matrix is given below, and determine the value of the game.

2	–3
–5	8

2. Consider a general two-by-two game, as follows.

A	B
C	D

Assuming that there are not dominated strategies and that $A + D \neq B + C$, define $E = A + D - B - C$.

a. Prove that the value of this game is $V = (AD - BC)/E$ by showing that the row player can assure a payout of V by playing row 1 with probability $(D - C)/E$ (and, therefore, row 2 with probability $(A - B)/E$) and that the column player can assure that same payout by playing column 1 with probability $(D - B)/E$ (and, therefore, column 2 with probability $(A - C)/E$).

b. Verify your answer in problem 1.) using this formula.

3. In the game of weighted rock-paper-scissors, the game is played in the usual way, but if you win with rock, you get \$10; if you win with paper, you get \$3; and if you win with scissors, you get \$1.

a. Construct the payoff matrix for this game.

b. Verify that the equilibrium strategy for both players is to play rock-paper-scissors with respective probabilities 1/14, 10/14, 3/14.

c. What would the equilibrium strategy be if the weights were positive numbers a, b, and c?

Practical Poker Probabilities

Lecture 5

This lecture is devoted to the game of poker, one of the most popular card games ever invented. Poker is an extremely complex game because players are faced with many types of decisions. In this lecture, you will learn the mathematics that is essential for successful poker playing. Although this lecture focuses on the game Texas Hold'em, the strategies that you learn can be applied to all variations of poker, including video poker.

Texas Hold'em

- The most popular poker variation by far is called Texas Hold'em. In this game, you can have as many as 10 people playing at the same table. Everyone is dealt two cards facedown, and then there's a round of betting. After that comes the flop, where three cards are turned over. These are common cards that every player is allowed to use in his or her hand. There's a round of betting, and then another common card is revealed. This is called the turn. There's another round of betting, and then a fifth common card is revealed called the river, followed by a final round of betting.

- The players who have not yet folded each have seven cards—their own two facedown cards and the five common cards—to create their best five-card poker hand. Whoever has the best hand wins.

- Before you've seen any cards in the deck, what's the probability that all three cards in the flop have the color red? Each card has a 50/50 chance, or 0.5 probability of being red. If someone offered to pay you seven to one odds that the flop is all red, would that be a fair bet?

- Let's first look at a related question. If someone offered you seven to one odds for flipping three heads in a row then that would be a fair bet because when you flip a coin, each flip would have heads probability 1/2, and the flips are independent, so the chance of getting three heads would be a 1/2 × 1/2 × 1/2 = 1/8. Therefore, the chance of not getting three heads would be 7/8. In this scenario, getting seven-to-one odds would be fair because your losing chances, 7/8, are exactly seven times as big as your winning chance of 1/8.

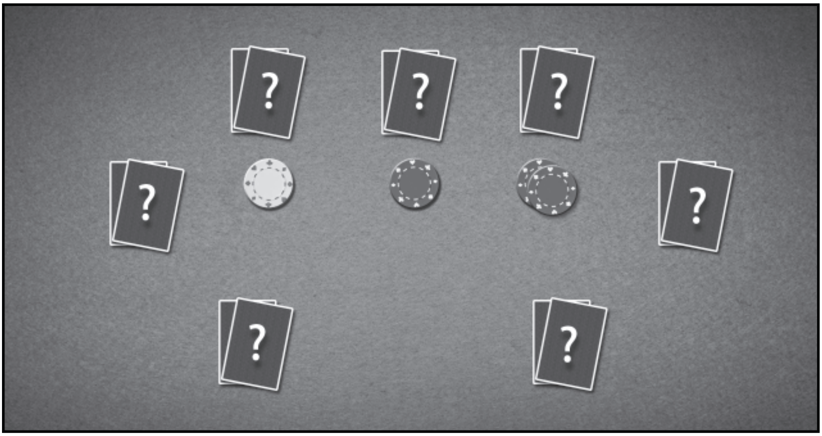
Poker Hand Rankings

Royal Flush	10♥	J♥	Q♥	K♥	A♥
Straight Flush	4♣	5♣	6♣	7♣	8♣
Four of a kind	K♠	K♥	K♣	K♦	3♠
Full House	10♥	10♠	10♦	A♠	A♣
Flush	10♠	K♠	2♠	6♠	7♠
Straight	7♣	8♠	9♦	10♠	J♥
Three of a Kind	5♠	5♥	5♣	J♦	A♦
Two Pair	A♠	A♥	3♣	3♠	J♣
One Pair	Q♦	Q♥	2♥	8♠	9♣

- However, coins and cards behave a little differently, so getting three heads in a row is not the same as getting three reds in a row. While coin flips are independent, the card outcomes are not. Because the cards are dealt without replacement, knowing the first card can affect the probability of the second card, and so on.

- The probability that the first card is red is indeed 1/2, but the probability that the second card is red, given that the first card is red, is 25/51 because there are 25 red cards among the remaining 51.

- If the first two cards are red, what's the chance that the third card is red? Because among the remaining 50 cards, 24 of them are red, the third card will be red with probability 24/50. When we multiply these numbers together, we get the probability that all three cards are red: 2/17, which is less than 1/8.



Texas Hold'em setup.

- Because you win with probability 2/17 and, therefore, lose with probability 15/17, then the fair odds of this bet would be 15 to 2, or 7.5 to 1, instead of 7 to 1.

- Suppose that after the flop, you determine that you don't have the best hand, but there are some cards that, if they appear in the next two cards, would give you the best hand. These winning cards are called your outs—as in, there are nine good cards *out* there for you.

- For example, suppose that your hand contains two hearts (a queen and a nine), and there are two hearts on the table, so there are nine hearts out there that will give you a flush. The queen-nine hand has nine outs.

- Suppose you started with the 10 and jack of spades, and the flop is 8 of spades, 9 of hearts, and 10 of clubs. Currently, you would have a pair of 10s, but your hand could get even better. How many cards are left in the deck that are especially good for you? To some extent, this depends on what your opponent has. With 10 or more outs, a more accurate probability is (3x + 9)%.

- Let's say that by your opponent's previous betting, you're pretty sure that he or she probably has you beat with a pair of aces or kings. Under these circumstances, your outs are two 10s (10 of diamonds and 10 of hearts), which give you three of a kind. There are three jacks, which would give you two pair. Receiving an 8 or a 9 would also give you two pair, but those cards would improve everyone else's hand, so they're not counted as outs. Because your hand has an 8, a 9, a 10, and a jack, there are four 7s and four queens that give you a straight. Altogether, there are 13 good cards for you, so you have 13 outs.

- The rule of 2 says that if you have x outs and there's just one more card to be shown, then your chance of winning is approximately $2x$%. For example, if you have 10 outs, then the chance that one of them shows up on the last card on the river is about 20%.

- If you already have seen six cards—namely, your two cards and four cards on the board—then the chance that you win on the last card is $x/46$ because there are 46 cards you don't know about, x of which are good for you. That's $0.022x$, or about $2x$%. If you can do the mental math and multiply by 2.2%, then you get an even better estimate. For instance, with 10 outs, you really have a 22% chance of success. We use the rule of 2—or, more accurately, 2.2—when there's one card left to be revealed.

- If there are two cards left to be revealed, the situation immediately after the flop, we use the rule of 4, which says that if you have x outs and two more cards to be shown, then the probability that one of your outs shows up is approximately $4x$%. For instance, with 10 outs and two cards to be revealed, your chance of success is about 40%.

- This rule makes sense because if you know five cards, two in your hand and three from the flop, then there are 47 cards left in the deck. The chance that the first card is one of your x outs is $x/47$. The chance that the second card is one of your outs, not knowing the first card, is also $x/47$. The chance that either card is good is slightly below $x/47 + x/47$, which is about $4x\%$. With 10 outs, your chance of success is about 40%. The actual probability is about 38.4%, so this estimate is pretty close. With 10 or more outs, a more accurate probability is $(3x + 9)\%$.

Number of Outs	Rule of 4	Exact	Modified
1	4%	4.4%	4%
2	8%	8.4%	8%
3	12%	12.5%	12%
4 (inside straight draw)	16%	16.5%	16%
5	20%	20.3%	20%
6	24%	24.1%	24%
7	28%	27.8%	28%
8 (open ended straight)	32%	31.5%	32%
9 (flush draw)	36%	35.0%	36%
10	40%	38.4%	39%
11	44%	41.7%	42%
12 (inside straight or flush)	48%	45.0%	45%
13	52%	48.1%	48%
14	56%	51.2%	51%
15 (straight flush draw)	60%	54.1%	54%
16	64%	57.0%	57%
17	68%	59.8%	60%

Video Poker

- Video poker is different from typical poker because you're not playing against opponents who are deliberately trying to beat you. There's no need to bluff against the computer. It's just you against the machine, so in many ways, it's like blackjack, where the dealer plays with fixed rules.

- The most popular version of video poker is known as Jacks or Better, which is played like traditional five-card draw poker. You are dealt five face-up cards, and you have to decide which cards to keep and which cards to let go. The machine replaces the cards you let go, and if your resulting hand is good enough, then you win.

- In video poker, you initially pay one unit—for example, $1. The payoff table is a bit misleading in that the payoff represents the amount of money that you get back after you've spent one unit. For example, if you bet $1 and you end up with a pair of jacks, then you get your original $1 back, so you really just broke even. Even though the payoff is pretty stingy, you can actually come close to breaking even in this game by following a few simple tips.

Jacks or Better Payoff Table

Pair of Js, Qs, Ks, or As	1
Two Pair	2
Three of a Kind	3
Straight	4
Flush	6
Full House	9
Four of a Kind	25
Straight Flush	50
Royal Flush	800

- The following is basic strategy for video poker. With two pair or higher—a profitable hand—do the obvious thing. For example, with three of a kind, draw two cards. With a straight, don't take any cards. With one pair, draw three cards. Never keep "kickers" like jacks, queens, kings, or aces. With no high cards—jacks, queens, kings, or aces—draw five cards. With one high card, draw four. With two high cards, draw three.

- Surprisingly, with three or four different high cards, you should only keep two of them. In other words, you should draw three. Which two high cards should you keep? If you have two high cards of the same suit, then keep them because you might get a flush. On the other hand, if you don't have two high cards of the same suit, then keep the two lowest high cards among them—because it increases your chance of getting a straight, and there are more ways to get a straight with a king and a jack, for example, than with a king and an ace.

- There are some exceptions to this basic strategy. First, with four cards in a straight flush, go for the straight flush: Take one card. The exception is if you were dealt a straight or a flush, don't break it up to go for a straight flush unless it could give you a royal flush. With a pair of jacks or higher in your hand, this first exception is the only exception that you need to know from basic strategy.

- For example, if you have three high cards (ace, king, and jack), but two of them are the same suit (the ace and jack of spades), hold onto those two cards. On the other hand, suppose that instead of having a jack of spades, you had a jack of diamonds, then because all three high cards have different suits, then you should hold the two lowest cards—the king and the jack—and discard the rest.

- Assuming that you don't have a pair of jacks or higher, the following are the other exceptions in order of priority. The second exception is if you have four cards in a flush, go for the flush. The third exception is with three cards in a royal flush, go for it: Take two cards.

- Finally, if your hand has no pairs in it—not even a pair of 2s—then take the following chances in order of priority. The fourth exception is if you have an open-ended straight draw, go for the straight. The fifth exception is with three straight flush cards, go for it: Draw two cards. The sixth exception is if you have a 10 and a high card of the same suit, keep the 10 since there's a potential royal flush.

- If you play with these tips, then you only expect to lose about 0.5¢ per dollar bet (the expected value). If you played the game perfectly, the expected value is –0.46¢ per dollar bet. Either way, you're down to about 0.5¢ per dollar bet, which makes video poker about as fair as basic strategy in blackjack, if played properly.

Suggested Reading

Chen and Ankenman, *The Mathematics of Poker.*

Duke and Vorhaus, *Decide to Play Great Poker.*

Guerrera, *Killer Poker by the Numbers.*

Harrington and Robertie, *Harrington on Hold'em.*

Moshman and Zare, *The Math of Hold'em.*

Paymar, *Video Poker.*

Sklansky, *The Theory of Poker.*

Problems

1. Suppose you are playing Texas Hold'em.

a. What is the probability that the first card of the flop is a low card (2 through 9)?

b. Use this to estimate the probability that all three cards in the flop are low.

c. Compute the exact probability that all three cards in the flop are low.

d. What would be the approximate fair odds for betting that all three cards in the flop are low?

2. You are dealt an 8 and 9 of diamonds in Texas Hold'em. The flop contains 6 of diamonds, 7 of diamonds, and queen of hearts. You suspect that one of your opponents has a pair that is higher than 9s.

a. How many outs do you have?

b. Estimate your probability of getting a straight or flush from the next two cards.

c. The next card is the two of clubs. Now estimate your probability of getting a straight or flush on the next card.

d. It's down to you and one other player. You have $10 in chips remaining, and your opponent bets $10, bringing the pot to $40. Should you call this bet?

3. In Jacks or Better video poker with the standard payouts, determine which cards to keep when dealt the following hands:

a. JS, JC, QD, 4D, 9D.

b. AS, QH, JH, 4H, 3D.

c. AS, QH, JH, 3H, 3D.

d. AS, QH, JH, 4H, 3H.

Expert Backgammon

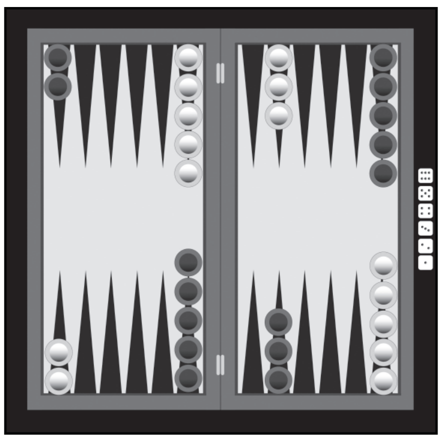
Lecture 6

Backgammon, one of the world's oldest games, trains you to make decisions under uncertainty, a skill that's useful in many of life's circumstances. In this lecture, you will learn the rules of the game, the basic strategy for winning, and the ideas behind the strategy. You will see how math enters the game—from figuring out the safest way to move your checkers to the all-important doubling cube, which makes the game even more interesting.

Backgammon: The Basics

- Backgammon is a dice game played between two players, represented by black and white pieces, or checkers. The player using the black checkers moves his or her checkers clockwise while the player using the white checkers moves them counterclockwise, with the goal of bringing all of the checkers into a quadrant called the inner board or home board. Once all of checkers are in a player's home board, he or she can start taking checkers from the board. Whoever removes all of their checkers off the board first is the winner.

- To start the game, both players roll one die, and whoever gets the higher number goes first using that roll. If both players roll the same number, then they keep rerolling until the dice are different.



Backgammon initial setup.

- When rolling both dice during the game, if you roll a 3 and a 1, you can move one of your checkers three spaces forward and another checker one space forward—you don't have to just move the total number of four spaces with one checker. Checkers are not allowed to move backward.

- When you have two or more checkers on one spot—which is called a point in backgammon—then you own that point, and your opponent is not allowed to land on it. There is no limit to the number of checkers that can go on a point.

- When you roll doubles in this game, it's like getting four of that number. For instance, if you roll double 6s, you get four 6s to play. Rolling doubles is usually a good thing. The probability of rolling doubles is 1/6. (If you throw two dice, one at a time, after the first die lands, the chances are 1/6 that the second die is going to match the first.)

- When a checker is on a point all by itself, it's called a blot, and if the opponent lands on it, it is hit and is sent all the way back to the starting point. The hit checker is placed on the bar, which is in the middle of the board. When a player has checkers on the bar, he or she must bring those checkers back into the game before moving any of the other checkers.

- When it is the player's turn to roll the dice, if his or her checkers that are on the bar are being blocked from entering the game by the other player's checkers, then his or her turn is over. (We say that the player has just "fanned" or "danced.")

- Although backgammon eventually becomes a race, each player's plan should be to try to build a blockade called a prime in front of the other player's entry point so that when the other player has checkers on the bar, there's nothing he or she can roll to get back on the board. In this case, we say that the other player is closed out and can't move until the player with the blockade opens up one of the entry points.

- The best opening rolls in backgammon are those that make a point as part of a blockade: 3 and 1, 4 and 2, 6 and 1, or 5 and 3. The roll 6 and 4 could be used to make the 2 point, but most good players don't do that because the 2 point and the 8 point can't be part of the same blockade.

Backgammon: Playing the Game

- Backgammon is a game of dice, and all dice probabilities are easy to calculate. When rolling two dice, there are 36 possible—equally likely—outcomes. A roll like 6-2 can happen two ways: with a 6 and 2 or a 2 and 6, but a roll like double 4s can only happen one way. In general, a specific roll of doubles, like double 4s, has a 1/36 chance of happening, and a specific non-doubles roll, like 6-2, has a 2/36 chance of happening.

Total Roll Values for Two Dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- If you need a total roll of 8 to hit your opponent's checker, then in backgammon terminology, we say that your checker is 8 pips away.

- In general, if you can hit your opponent in his or her home board or outer board on your first roll, then it's almost always right to do so.

- An indirect shot is a shot that requires a combination of both dice to hit your opponent's checker—so the checker is a distance of more than six pips away. On the other hand, when a checker is a distance of six away or closer, then it's called a direct shot because it can be hit with just a single die.

- How many dice rolls use the number 1? The answer is that 11 of the 36 rolls, about 30% of them, contain the number 1. Likewise, there are 11 rolls that contain a 2, and so on.

- If your opponent has a blot that is 4 pips away, how many rolls hit the blot directly with a 4 on one of the dice? There are 11 dice rolls that contain a 4 on one of them. Add to that the numbers with a total of 4—there are 3 of those. Don't forget to count double 1s, when you get to play 1 four times. That gives you a total of 11 + 3 + 1 = 15 shots on your opponent's blot. Hence, the probability that you hit the blot is 15/36. Being able to count shots can often help you determine the right play to make.

Rolls That Hit

	1	2	3	4	5	6
1	X		X	X		
2		X		X		
3	X			X		
4	X	X	X	X	X	X
5				X		
6				X		

- A double direct shot is a shot that can be hit by all 2s and all 5s, for example. As you know, direct shots get hit 11 ways plus combinations. With double direct shots, there are 20 shots plus combinations. For example, there are 20 dice rolls that contain a 2 or a 5 on one of the dice. There are 16 rolls that don't use a 2 or 5 anywhere; therefore, there are 36 – 16 = 20 numbers that do.

- Essentially, both players are trying to bring all of their checkers to their zero point. If a checker on the 6 point has to travel six points, then it has to travel a distance of six pips. A checker on the 4 point has to travel four pips, etc. We measure each player's race by doing what's called a pip count.

- Once players have all of their checkers in their home board, they can start removing them from the board, a process known as bearing off. When bearing off, you're essentially moving your checkers to the zero point.

- There's only one other rule for bearing off: Once there are no more checkers on the 6 point, then 6s can be used to take off checkers from the next largest point. Once the 6 and 5 points are cleared, then 6s and 5s can be used to clear checkers from the next largest point, and so on. Rolling double 6s is virtually always the best roll when you're trying to take off checkers quickly. As usual, the two players take turns rolling the dice, and whoever takes off all of their pieces first is the winner.

- If one player takes off all of his or her checkers before the other player removes any of his or hers, then that's called a gammon, and that player wins twice whatever the game was worth. If the game was worth 1 point—for example, in a tournament match—then winning a gammon earns you 2 points. If the first player earns a gammon while the other player still has a checker inside of the first player's home board, then that's called a backgammon, and the first player wins triple the stake.

The Doubling Cube

- A relatively recent development introduced to backgammon in America around the 1920s that makes the game even more interesting is called the doubling cube, which is used to raise the stakes in backgammon—much like raises do in poker.

- A doubling cube has the numbers 2, 4, 8, 16, 32, and 64 on it. It usually sits off to the side of the board. During a game of backgammon, if you feel that you have a much stronger position than your opponent, then before it's your turn to roll, you can offer to double the stakes by picking up the doubling cube, placing it in front of you with the number 2 face up, and saying, "I double."

- Your opponent has two options: drop (pass) or take the cube. If your opponent drops, which is like folding in poker, then the game is over, and you win. On the other hand, if your opponent takes the cube, then he or she puts the cube on his or her side of the board. Now, whoever wins the game gets 2 points. Then, it's your turn. You roll the dice and the game continues, but now the cube is on your opponent's side of the board, and you can't touch it.

- Later, if the tables turn and your opponent has a much stronger position than you do, then your opponent can redouble you to four, making the game worth 4 points. At this point, you can either drop or take. If you drop, then you lose 2 points, and if you take, then you will put the cube over in front of you on your side of the board so that you could potentially use it later. You could turn it to 8, for instance.

- The doubling cube takes some of the luck out of the game because the person being doubled has to decide if his or her chances are good enough to justify playing for higher stakes. How good do your chances have to be to take the double? Assume you're in a situation where neither side is likely to win a gammon—maybe the game has become a straight race, for instance—then in order to take the double, your chances need to be at least 25%.

- In practice, your doubling cube action can also depend on factors like your ability to use the cube if you turn the game around or your chance of getting gammoned. For instance, if the game is a pure race, you can often take with a 22% chance of winning because you probably have a better than 25% chance of being able to reach a position where you can double your opponent out. This allows you to take with a little less than 25% chance. On the other extreme, if you're in a position where you never win a gammon, but all of your losses are gammons, then your winning chances would have to be at least 50% in order to take the cube.

- Sometimes it can be tricky to estimate your winning chances, but the more you play, the better you become at doing this. A decent rule of thumb is you should double your opponent if you're comfortably ahead of the game, and you should take a double if you can see a clear way to turn the game around that doesn't require too much luck on your part.

Suggested Reading

Clay, *Backgammon in a Week.*

Magriel, *Backgammon.*

Trice, *Backgammon Boot Camp.*

Woolsey, *How to Play Tournament Backgammon.*

Woolsey and Beadles, *52 Great Backgammon Tips.*

Problems

1. The following questions apply to the game of backgammon.

   a. What are the chances of hitting a checker that is 3 pips in front of you?

   b. What are the chances of hitting a checker that is 5 pips in front of you?

   c. What are the chances of hitting a checker that is 12 pips in front of you?

   d. Suppose there are three blots in front of you that are 3 pips, 5 pips, and 12 pips away. What are the chances of hitting at least one of them?

2. When playing backgammon, you have two checkers on the bar, and your opponent has a two-point board (say, owning the five and six points). What are the chances that in your next roll:

a. both checkers come in?

b. both checkers stay on the bar?

c. one checker comes in and one checker stays on the bar?

3. The backgammon game is almost over. You have two checkers left on your two point. Your opponent has one checker left on her four point and one checker on her one point. You own the doubling cube.

a. What is the probability of taking both of your checkers off on this roll?

b. If you don't use the doubling cube, what is the probability that your opponent wins in this position?

c. If you double, should your opponent take?

d. Is it correct for you to double?

Games You Can't Lose and Sneaky Puzzles

Lecture 7

There are many games that seem very innocent and fair, but where you actually have a big advantage. The focus of this lecture is on scams and hustles that arise in games and puzzles. In this lecture, you will learn about games with nontransitive probabilities, involving cards, coins, dice, and even bingo cards. You will also learn how averages can be misleading by learning about Simpson's paradox. The strategies presented in this lecture may keep you from being exploited by a devious adversary.

Nontransitive Probabilities

- There is a dice game that uses four dice, but they are not ordinary dice. Die A has two 6s and four 2s. Die B has all 3s on it. Die C has four 4s and two 0s, and die D has three 5s and three 1s. Each of two players rolls a die, and whoever rolls the higher number is the winner. Player 1 chooses a die, and player 2 selects a different die.

- Die A is tempting because it has the largest number and the largest total, but if player 1 chooses die A, then player 2 should choose die B. Notice that because player 2's roll is guaranteed to be 3, he or she will win whenever player 1 rolls a 2, which happens 4 times out of 6. Thus, die B beats die A with probability 2/3. The notation is as follows: $P(B > A) = 2/3$.

- If player 1 picks die B instead, then player 2 should choose die C, which wins whenever player 2 rolls a 4—because player 1 has to roll a 3, and that happens 2/3 of the time. Therefore, die C is going to beat die B 2/3 of the time.

- Die D beats die C 2/3 of the time. Die D can beat die C in two ways: If die D is a 5, which happens half the time, then it's guaranteed to beat C, or if die D is a 1 and C is a 0, then D will beat C. The probability that die D is 1 and C is 0 is $(1/2)(1/3) = 1/6$, so the probability that die D beats die C is $1/2 + 1/6 = 2/3$.

- If player 1 chooses die D, then player 2 should choose A. Player 2 has two ways of winning: Either player 2 rolls a 6, which happens 1/3 of the time (because two of the six numbers are 6s), or player 2 can roll a 2 and player 1 can roll a 1. In that case, there's a 2/3 chance of player 2 rolling a 2, and there's a 1/2 chance that player 1 rolls a 1, so (2/3)(1/2) = 1/3. The probability that die A beats die D is 1/3 + 1/3 = 2/3.

- To summarize, die A loses to die B, die B loses to die C, die C loses to die D, and die D loses to die A—each with probability 2/3. This situation is what mathematicians call nontransitive probabilities.

- At first glance, this sounds impossible. Most competitive situations are transitive. If A is faster than B and B is faster than C, then A is faster than C. We encounter nontransitive situations in games like rock-paper-scissors, but when it shows up in other games, like dice games and poker, it's very unexpected.

Roulette

- In the game of roulette, every bet on average loses about 5.3¢ per dollar bet. Suppose that you like to bet $1 on your favorite number—for example, 17. If 17 shows up, then you win $35; otherwise, you lose $1. That's one of the ways that roulette is played.

- Let's say that you decide to bet on your favorite number (17) 35 times in a row. What are your chances of showing a profit after 35 bets? You may be surprised to learn that your chance of showing a profit is around 60%.

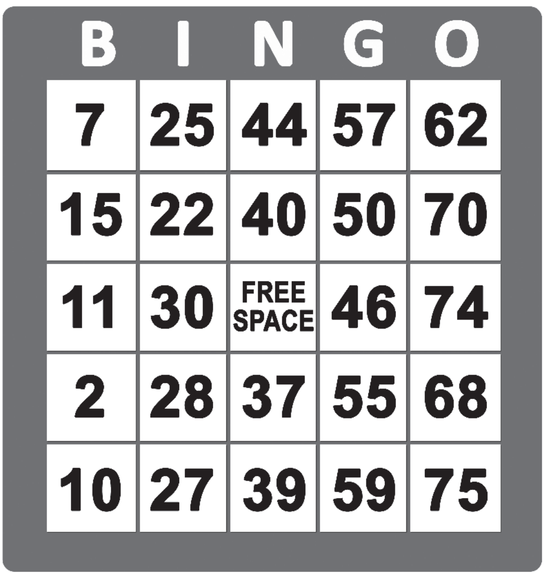
- If you win once, then you made a profit because you have one big win of $35 and 34 losses of $1. We need to figure out the chance of winning at least one of your 35 bets. To do that, answer the opposite question: What are the chances of losing all 35 bets—of not showing a profit?

- Each bet has a 1/38 chance of winning and, thus, a 37/38 chance of losing. Because the outcomes are independent, the probability that you lose all 35 bets is $(37/38)^5 = (0.9736\ldots)^5 \approx 0.39 = 39\%$. Hence, the probability that you don't lose all your bets—that is, the probability that you win at least one bet—is about 61%.

- The problem is that when you are profitable, you're usually only $1 profitable. Sometimes you win more, but when you lose, which is nearly 40% of the time, you always lose $35. Sometimes you win twice, and very occasionally you win three times, but on average, after 35 bets, you will be down about $2. You can't beat this game.

Bingo

- Bingo is not quite as fair as it seems. In a typical Bingo game, everyone gets a five-by-five card filled with numbers from 1 to 75. Numbers are drawn one at a time in random order, and the winner is the first person who gets a "bingo," consisting of five numbers in a row, column, or diagonal.



- If more than one person gets a bingo at the same time, then the prize money is shared between them. You would think that any bingo card is just as good as any other, but that's not always the case.

- To prove this point, let's simplify the game. If every card just had one number and all the cards were different, then indeed each card is just as good as any other. That would just be the same as a raffle.

- However, suppose that each card has two numbers. Let's say you have the following domino-shaped cards.

A	B	C	D	E
[1 2]	[3 4]	[1 5]	[3 1]	[4 1]

- The numbers 1 through 5 are drawn in random order, and whoever gets both of the numbers first is the winner. Does any one card seem to be better than the others?

- If the prize were $120 and the game were fair, then each card would have an expected value of $120/5 = $24. However, if we do the math, the expected values of these cards are 21, 40, 21, 19, and 19, respectively.

A	B	C	D	E
[1 2]	[3 4]	[1 5]	[3 1]	[4 1]
21	40	21	19	19

- Card B has an enormous advantage. So, if everyone paid $24 to play this game, then B has an expected profit of $16, while everyone else will lose, on average, $3 or $5.

- Why does B do so much better than the rest? Notice that all the other cards need the number 1 to win, and sometimes they have to share their prize, but B wins whenever the number 1 is drawn after the 3 and 4, and this happens 1/3 of the time. That makes sense because among the numbers 1, 3, and 4, 1 is just as likely to be the first, second, or third of these three numbers.

- Moreover, B never shares the prize. Hence, B's expected value is 1/3 of $120, which is $40. In fact, by the same logic, B wins 1/3 of the time—even if there were more cards that all had 1 in them.

- When the cards get larger, the game becomes more complicated, and a full five-by-five card game would be way too difficult to analyze. What happens when the cards have three numbers? Suppose that two players agree to play bingo with the following L-shaped cards.

A	B	C
3	4	1
1 2	3 2	4 2

- On these three cards—A, B, and C—the numbers 1 through 4 are chosen in random order, and A beats B if a 1 and 2 or a 1 and 3 appears before a 2 and 3 or a 3 and 4.

- For example, if A plays B and the numbers come out in order 2, 4, 1, 3, then A has the bingo, so A would win. On the other hand, if the numbers came out 1, 4, 3, 2, then both players get a bingo at the same time, so this situation would be a tie.

- This is another nontransitive game—where A is favored to beat B, B is favored to beat C, and C is favored to beat A. To see why A beats B, notice that the numbers 1, 2, 3, and 4 can be arranged 24 ways (4 choices for the first number, 3 choices for the next, 2 choices for the next, 1 choice for the next, or 4! = 24 ways).

- In 12 of those 24 ways, A beats B, and in 10 of those 24 ways, B beats A. In 2 of those 24 ways, it would be a tie. Therefore, A has a 12-to-10, or 6-to-5, advantage over B. By the same logic, if we exclude card A, then we see that B has a 6-to-5 advantage over C, and C has a 6-to-5 advantage over A.

- Suppose you walk into a Bingo hall and watch a game until it ends. Is the winning card more likely to be a horizontal bingo or a vertical bingo? Intuitively, you would think it would be just as likely for someone to have a horizontal bingo or a vertical bingo. However, it's actually more likely—in fact, about twice as likely—for the winning card to come from a horizontal than from a vertical bingo.

- This bingo result is closely related to a classic conundrum known as the birthday paradox. Suppose you walk into a room with 22 other people. What are the chances that two or more people in the room have the same birthday? Most people think it's pretty low, but there's actually more than a 50% chance that two people in that room have the same birthday.

Simpson's Paradox

- Is it possible for there to be two baseball players, A and B, for which A has a better batting average than B one year and the next year, but when we combine the two years, B does better than A? Not only is it possible for this to happen, it actually has happened.

- The batting records for Derek Jeter and David Justice in 1995 and 1996 are a great example. In both years, Justice had the better average, but when we combine the two years, Jeter has the better average.

	1995	**1996**	**Combined**
Jeter	12/48 = .250	183/582 = .314	195/630 = **.310**
Justice	104/411 = **.253**	45/140 = **.321**	149/551 = .270

- In fact, Justice even outperformed Jeter in the 1997 season, but when we combine all three years, Jeter has the better average.

Jeter batted .300, and Justice batted .298: 385/1284 = .300 versus 312/1046 = .298.

- Mathematicians call this Simpson's paradox, and it arises in many places even outside of games and puzzles. Simpson's paradox cannot happen if both players have the same number of at bats each season, but it can happen when the denominators are very different, like in this example.

	1995	**1996**	**1997**	**Combined**
Jeter	12/48 = .250	183/582 = .314	190/654 = .291	385/1284 = **.300**
Justice	104/411 = **.253**	45/140 = **.321**	163/495 = **.329**	312/1046 = .298

Suggested Reading

Bewersdorff, *Luck, Logic, and White Lies.*

Diaconis and Graham, *Magical Mathematics.*

Gardner, *The Colossal Book of Short Puzzles and Problems.*

Gardner, *Perplexing Puzzles and Tantalizing Teasers.*

Haigh, *Taking Chances.*

Problems

1. Look at the magic square below.

a. If you choose a random number from the first row and your opponent chooses a random number from the second row, who is more likely to have the higher number?

b. Find a nontransitive situation from the rows of the magic square. (A similar situation arises with the columns, too!)

4	9	2
3	5	7
8	1	6

2. You have three six-sided dice (A, B, and C) with the following numbers: A = (6,3,3,3,3,3), B = (5,5,5,2,2,2), and C = (4,4,4,4,4,1).

a. Find P(A > B).

b. Find P(B > C).

c. Find P(C > A).

d. From the above probabilities, if your opponent chooses die A, which die should you choose to maximize your probability of winning?

3. Choose your favorite baseball team, and your opponent will choose hers. Each team will play about 160 games in the season. Your opponent's score at the end of the season is the sum of her scores after each game. (So, if in the first five games, your opponent's team scores 1, 1, 2, 3, and 5 runs, then her total will be 12 at that point.) You get the product of the scores. (So, if your teams scored 1, 1, 2, 3, and 5 runs, then your product would be 30 at that point.) Who is favored to win this bet?

4. Your friend has two American coins in his hand totaling 30 cents, but one of them is not a nickel. What coins does he have?

Solving "Impossible" Puzzles

Lecture 8

In this lecture, you will learn how mathematics plays an interesting role in solving and understanding various puzzles. You will explore some classic puzzles, including the Fifteen Puzzle, an electronic game with lights, and peg solitaire, many of which have been driving people crazy for centuries. On the surface, they seem very different—involving sliding blocks, blinking lights, or jumping pegs—but they do have some common features that apply to many other puzzles as well.

The Fifteen Puzzle

- A classic puzzle called the Fifteen Puzzle is also sometimes referred to as the Rubik's Cube of the 19th century. In its classic form, the numbers 1 through 15 are displayed in a four-by-four square, and a blank square allows the tiles to move.

- You start with a randomly scrambled position, and your goal is to get it back to the original order. To solve the puzzle, you basically get the numbers in the right place one at a time—with just an occasional twist.

The Fifteen Puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

- Starting from a random position, you first want to move tiles 1 and 2 to their goal positions. Next, you want to bring 3 and 4 to their positions. To do this, first bring 3 to the upper right corner then bring 4 below it. After that, bring 3 and 4 to their goal positions. Once the numbers 1 through 4 are in their final positions, they won't move for the rest of the puzzle.

- The next row is pretty much the same. First, you bring the 5 and 6 to their proper positions. Then, bring the 7 and 8 to their proper positions. The first two rows are now finished, and you'll never have to touch them again.

- The next tiles you place are not 9 and 10, but 9 and 13. First, bring the 9 and 13 next to each other on the bottom row. Then, bring the 9 to its proper place and the 13 to the place below the 9. Next, you want to move the 10 and 14 next to each other and then to their final positions, in which the 14 ends up below the 10.

- Finally, you're left with the 11, 12, and 15 in some order. If the original puzzle was solvable, then they can simply be rotated to reach the goal position. World chess champion Bobby Fischer was able to do this puzzle consistently in under 25 seconds.

- The Fifteen Puzzle was invented by Noyes Chapman, a postmaster from New York, and was later popularized by the puzzle maker Sam Loyd, who offered $1,000 to anyone who could solve the original Fifteen Puzzle but with tiles 14 and 15 swapped. Loyd's money was safe because it turns out that this puzzle has no solution.

An Electronic Game with Lights

- In an electronic game, there is a three-by-three box of lights. At the start of the game, some of the lights are on and some are off. The lights are all buttons, and we can label the buttons as numbered: 1, 2, 3, 4, 5, 6, 7, 8, and 9. Buttons 2, 3, 5, and 8 are lit, and the rest are not. Your goal is to turn off all the lights by pressing the right combination of buttons.

Box of Lights

1	2	3
4	5	6
7	8	9

- There are three types of buttons: corner buttons (1, 3, 7, and 9), edge buttons (2, 4, 6, and 8), and the middle button (5). Pressing a corner button will toggle the corner and the three surrounding lights, turning each one from on to off and off to on. For example, pressing button 1 will toggle the lights in cells 1, 2, 4, and 5—turning 1 and 4 on and 2 and 5 off. Pressing button 1 again brings each button back to its original position.

- The following are more examples: Pressing button 3 toggles lights 2, 3, 5, and 6. Pressing button 7 toggles 4, 5, 7, and 8. Pressing button 9 toggles 5, 6, 8, and 9.

- Pressing an edge button toggles the edge and two of its surrounding neighbors. For example, pressing button 2 toggles the lights in cells 1, 2, and 3. The other rows don't change. Pressing button 2 again brings each back to its original position. Similar rules apply for pressing buttons 4, 6, and 8.

- Finally, pressing button 5, the middle button, toggles the middle and the four edge lights: 2, 4, 6, and 8. Pressing 5 again brings each back to its original position.

- The goal is to press a sequence of buttons so that you go from the initial position to the position where all lights are out—hopefully, using as few buttons as possible. The solution is far from obvious.

- When pressing buttons, does the order matter? For example, if you press the buttons 1, 2, 3, 4 in that order, will you get the same position if you press the buttons in a different order—for example, 2, 4, 1, 3? The answer is not immediately clear.

- To clarify this situation, it's time to introduce what mathematicians call mod 2 arithmetic. With mod 2 arithmetic, there are just two numbers—0 and 1. For the most part, addition works in the usual way: $0 + 0 = 0$, $0 + 1 = 1$, and $1 + 0 = 1$, but $1 + 1$ does not equal 2 because the number 2 is not in the original number system (which only allows 0 and 1). As a result, with mod 2 arithmetic, $1 + 1 = 0$.

- One way to make sense of mod 2 arithmetic is to think of 0 as representing any even number and 1 as representing any odd number. Then, the mod 2 addition table is really just the following: Even plus even equals even, even plus odd equals odd, and odd plus odd equals even.

- With the electronic game, mod 2 arithmetic comes in handy. First, every position can be represented by a vector of zeroes and ones, where 0 means that a light is off and 1 means that a light is on. For example, the beginning position would have vector representation $\mathbf{b} = (0, 1, 1, 0, 1, 0, 0, 1, 0)$, where we have a 1 in positions 2, 3, 5, and 8.

- Each button can also be represented by a vector. For example, pressing button 2, which toggles the lights in positions 1, 2, and 3, can be represented by the vector $\mathbf{v}_2 = (1, 1, 1, 0, 0, 0, 0, 0, 0)$. Pressing button 2 is just like adding the vector $\mathbf{v}_2$.

- For example, if we take the beginning position and press button 2, we get a new position where lights 1, 5, and 8 are on. If we take the vector representing the original position with 1s in positions 2, 3, 5, and 8—$\mathbf{b} = (0, 1, 1, 0, 1, 0, 0, 1, 0)$—and add the vector $\mathbf{v}_2$, which has 1s in positions 1, 2, and 3—$\mathbf{v}_2 = (1, 1, 1, 0, 0, 0, 0, 0, 0)$—and do that addition (mod 2), then the new vector $\mathbf{b} + \mathbf{v}_2$ will have ones in positions 1, 5, and 8—(1, 0, 0, 0, 1, 0, 0, 1, 0)—just like the lights.

- Similarly, pressing button 1, which toggles lights 1, 2, 4, and 5, is equivalent to adding the vector $\mathbf{v}_1$ with 1s in the first, second, fourth, and fifth positions (and similarly for the other buttons).

- By the way, pressing a button twice in a row leaves a position unchanged, and we can also see that from vectors. Because order does not matter, no solution would require you to press the same button twice in a row. In other words, if a puzzle has a solution, then it has a solution where every button is pressed at most once.

- Strictly speaking, finding a solution to this game is a problem from linear algebra, the mathematics of vector equations. Starting from the beginning position, represented by the vector **b**, you need to add some of the vectors $\mathbf{v}_1$ through $\mathbf{v}_9$ to reach the vector of all zeroes, which is the position where all lights are off.

- Instead of using linear algebra, there is a quick way to solve this problem in your head without the need to write down a single equation. Recall that when you solve this puzzle, the order that you press the buttons does not matter. Therefore, you can always solve the puzzle in the following order: First, decide on the edge buttons, then decide on the corner buttons, and then decide on the middle button.

Peg Solitaire

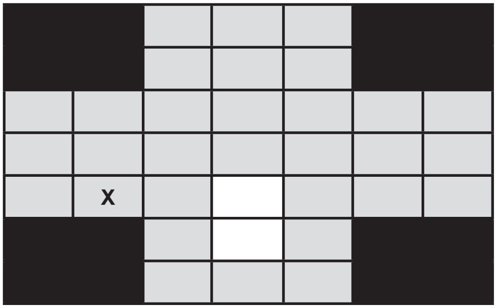
- Peg solitaire is typically played on a board that looks like a giant plus sign. Every spot, except for the central one, is occupied by a peg or sometimes a marble. Pieces can jump over other pieces horizontally or vertically, provided that there is an empty space on the other side, and the piece that's jumped over is removed from the board.

- The goal of the puzzle is to jump your pieces in such a way that all you are left with at the end is one piece, preferably in the middle of the board. Curiously, the middle square is practically the only place that a single peg can end up.

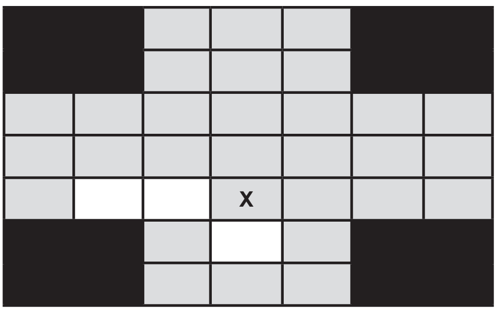
- To solve the puzzle, begin by making your initial move toward the center, or "bottoms up." To clear most of the pegs from the board, use the following mnemonic: right, up, left, then move it up. Repeat this four times, rotating the board 90° after each time you make this set of moves.

- After rotating the board the last time, you are left with what looks like a house, which is made from the remaining pegs. You are going to clear the rest of the pegs by going to the attic in the house, and one of the pegs is going to do a grand tour and annihilate most of the pegs, ending up with a T formation. At that point, the puzzle is pretty easy to solve.

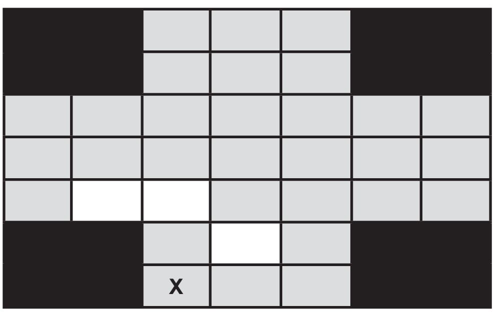
Peg Solitaire



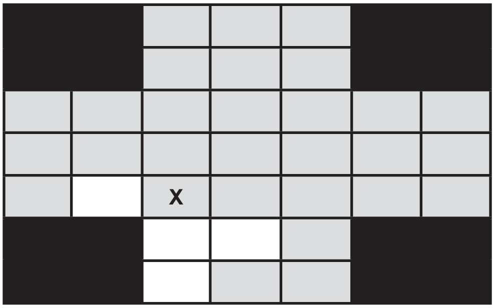
Right



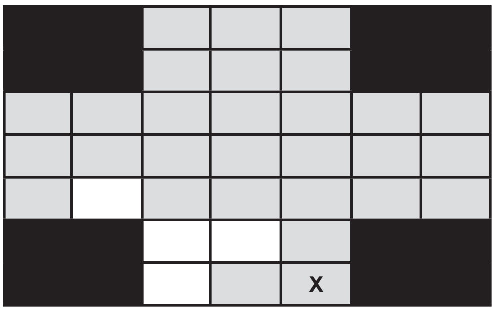
Right



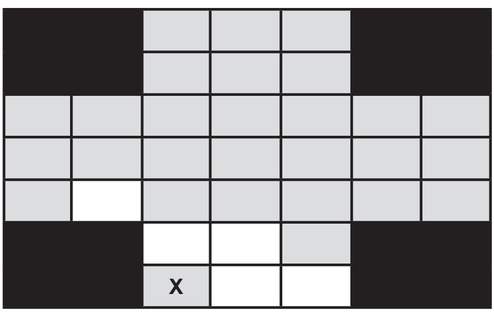
Up



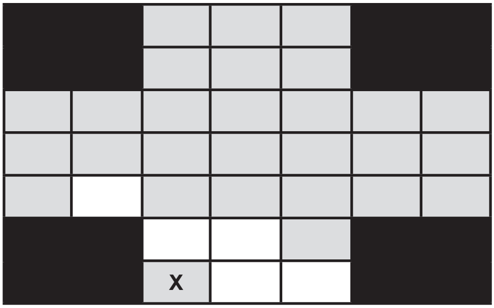
Up



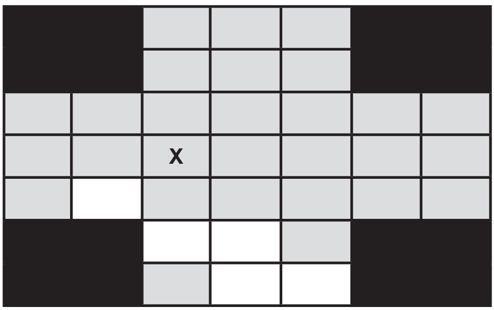
Left



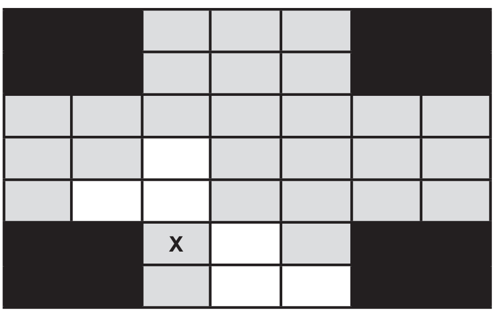
Left



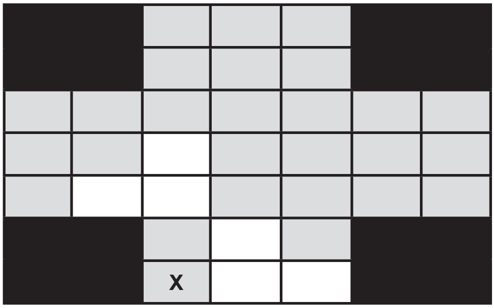
Then move it up



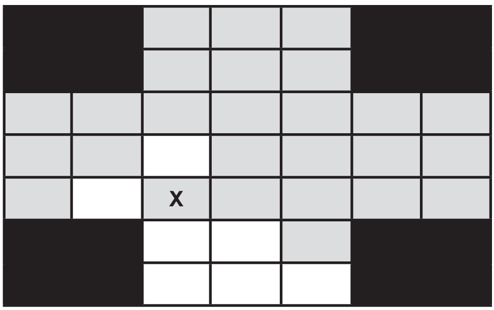
Then move it up



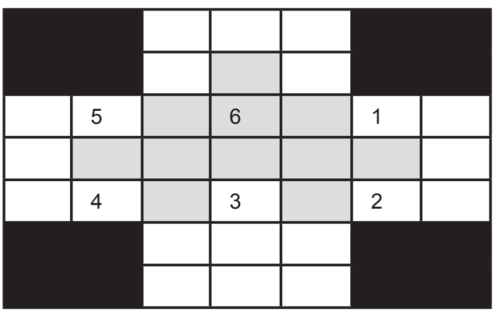
Then move it up



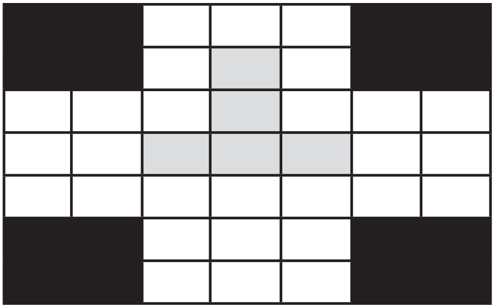
Then move it up



Now rotate the board counterclockwise and repeat 3 more times



The House Tour



T Formation

Suggested Reading

Ball and Coxeter, *Mathematical Recreations and Essays.*

Beasley, *The Ins and Outs of Peg Solitaire.*

Beasley, *The Mathematics of Games.*

Hess, *Mental Gymnastics.*

Gardner, *The Colossal Book of Short Puzzles and Problems.*

Hoffman, *Puzzles Old and New.*

Slocum and Sonneveld, *The 15 Puzzle.*

Wells, *Book of Curious and Interesting Puzzles.*

Problems

1. In the electronic game presented in the lecture, suppose the initial position looks like the following.

ON	ON	OFF
OFF	ON	ON
ON	OFF	OFF

Which of the edge buttons (2, 4, 6, or 8) should you press? What will you do after that?

2. Consider the following Fifteen Puzzle position.

3	14	15	9
2	6	5	13
8	7	12	4
10	1	11	

a. How many "oddballs" does it have?

b. Based on your answer to a.), determine if the Fifteen Puzzle position is solvable.

3. In the triangular peg solitaire with 15 pegs, where the first hole is unoccupied and all other holes are occupied, is it possible to reduce the puzzle to a single peg in the second row?

Mastering Rubik's Cube

Lecture 9

Rubik's Cube, perhaps the most famous puzzle of all time, combines mental dexterity with physical agility. In this lecture, you will learn some of the mathematics behind Rubik's Cube. In addition, you will learn a method that's very easy to learn that will allow you to solve the puzzle quickly and accurately every time. You probably won't be able to solve the cube immediately after learning the algorithm used in this lecture, but you can most likely learn it with a few hours of practice.

Rubik's Cube: The Math

- Rubik's Cube could well be the most famous puzzle ever invented. You start with a cube with six different-colored sides, then after a few twists, the cube is in a random mess. Most people find it a hopeless task to bring the cube back to its proper order.

- Traditionally, the sides of Rubik's Cube have stickers that come in six colors: red, orange, yellow, green, blue, and white. In fact, for all official Rubik's Cubes, the colors have the white face opposite the yellow face, the red face opposite the orange face, and the blue face opposite the green face. The opposite faces differ by the color yellow: White is opposite yellow; red is opposite orange; blue is opposite green.

- Because the cube has dimensions three by three by three, it should consist of 27 little pieces, sometimes called cubies. If you actually counted, you'd find 26 cubies because there's one cube, the one in the very center, that has no color—so there are just 26 pieces that you can see.

- There are three types of pieces. Every face has a center piece, which has just one sticker on it. For instance, the white center and the yellow center are on opposite sides of each other, and nothing you can do to the cube can change that. The same goes for the other centers: The red center is opposite the orange center, and the blue center is opposite the green center. When the cube is mixed up, it is impossible to have two centers on the same face.

- There are two other kinds of pieces: corners and edges. There are eight corner pieces, and there are 12 edge pieces, which have two stickers on them. No matter how you twist the cube, a corner piece can only move to another corner position and an edge piece can only move to another edge position.

- How many different Rubik's Cube positions are possible? The answer depends on what is meant by a "possible" position. For example, if we completely disassembled the cube and stripped it to its interiors and its six center cubes, then how many ways could we put it back together? Starting with the corners, there are eight choices for which piece goes in a corner. Once you pick a piece to go in the first chosen corner, then there are seven choices left that you can make for which piece goes in the next corner. Then, you have six choices for which corner piece goes in the next corner, and so on, down to the last corner.

- The number of ways to choose the locations of the corner pieces is 8!, or 40,320. Once you have placed them in the corners, then there are three ways of orienting the piece. You could perform those orientations 3^8—or 6,561—ways. Similarly, the 12 edge piece positions can be placed in their positions 12! ways, which is nearly half a billion ways, before you oriented them. Furthermore, the edge pieces can be oriented in 2^{12}—or 4,096—ways.

- When you multiply all of those numbers together, you find that there are $8! \times 3^8 \times 12! \times 2^{12} = 519{,}024{,}039{,}293{,}878{,}272{,}000$ (over 500 quintillion) ways that the cube can be reassembled. It turns out that exactly 1/12 of these positions can actually be achieved from the starting position, so the number of starting positions is really only about 40 quintillion: $(8! \times 3^8 \times 12! \times 2^{12})/12 = 43{,}252{,}003{,}274{,}489{,}856{,}000$.

- Another way to say this is that if you randomly assembled a cube, the chances that it would actually be solvable would only be 1 in 12. With the Fifteen Puzzle, every legal position had to be an even arrangement of the numbers 1 through 15. As it happens, the only achievable positions of a Rubik's Cube are even arrangements of the pieces.

- For instance, if you took a final position and swapped any two of the pieces, you would have an unsolvable position—no matter how you oriented the pieces—because you would be going from an even position to an odd position. Because 1/2 of all arrangements are even, this cuts the number of achievable arrangements in half. Furthermore, once you choose the orientation of 11 of the edge pieces, then orientation of the 12th edge piece is also forced.

- Suppose you were in the middle of solving the cube and you accidently dropped it on the floor, causing one of the edge pieces to come out. And suppose that you didn't know how it was placed before it came out, and you put it back in the cube. It matters which way you put the piece back in because if you put it in the wrong way, then the puzzle becomes unsolvable. This cuts down the number of achievable arrangements by 1/2 again.

- Finally, once you've oriented seven of the corner pieces, that forces the orientation of the eighth piece, and because each corner has three potential orientations, only 1/3 of its orientations are achievable. If you multiply (1/2) × (1/2) × (1/3), you get 1/12, which is the fraction of reassembled cubes that are legally achievable or solvable.

Rubik's Cube: The Algorithm

- General note: For the entire process, you will have the yellow side facing up and the white side facing down.

- Notation:
  - The move U means to turn the up face (always yellow) in the clockwise direction (one quarter turn).

- Likewise, R means to look at the right face and turn it clockwise (AWAY from you).

- L means to look at the left face and turn it clockwise (TOWARD you).

- F means to turn the front face clockwise.

- B means to turn the back face clockwise.

- D means to turn the downward face (always white) clockwise (you won't need this move).

- A move like U' means to turn the up face counterclockwise.

- A move like F^2 means to turn F twice.

- Rubik's Cube has 8 corner cubes (with three colors), 12 edge cubes (with two colors), and 6 center cubes (with one color that never moves). Saying "the yellow side" means the side with the yellow center.

Step 1 (Daisy)

- There are four edge cubes with the color white. One at a time, bring these four edge cubes to the yellow side (upward facing) with the white colors facing up. This forms a "daisy" with a yellow center and four white petals.

Step 2 (Easy)

- Rotate the up side until the edge cube white/red has the red part lined up with the red center. Then, rotate the red face 180° so that the edge cube is now on the bottom (white) face with the white facing down. Do this with all four colors. When you are done, you will have a white cross on the bottom. Those four edge cubes on the bottom are now in their final position. At the end of each subsequent step, they will stay where they are now.

Step 3 (The 1-2-3 Move)

- Look for a corner cube with white on it. The most convenient place to find one is on the top "rim" (but not on the up face). This corner cube will have another color on its rim—for example, green. Twist the top face (U) so that the green part of this cube is on the green side, and orient the cube so that the green side is facing you. The white part of the corner cube will now be either on the right side or on the left side.

- If it's on the right side, then with your right hand, perform the 1-2-3 move by doing RUR': 1) Twist the right side away from you, then 2) twist the top toward you, then 3) twist the right side toward you. If the corner cube is on the left side, then do the same move with your left hand (L'U'L): 1) Twist the left away from you, then 2) twist the top toward you, then 3) twist the left side toward you. This move puts the corner piece on the bottom where it belongs.

- If the white corner is on the up face, then twist the top face so that that white piece is above a corner on the bottom that has not yet been solved. Then, perform 1-2-2-3 (twisting the up face twice instead of once), and that will put the corner piece on the top rim, which you can then solve as above.

- If no white corner is on the top rim, but it's on the bottom rim, then do the 1-2-3 move, and that will move it to the top rim.

- Doing this for all four white corner cubes will complete the bottom layer.

Step 4

- Find an edge piece from the top face that has no yellow on it. Note that piece's "rim" color (not the top color on the yellow face) and twist the top so that the rim color matches its center. Note the "top" color of that piece. Its matching center will be either the right face or left face. If it's the right face, then perform U with the right hand, and then perform 1-2-3 with the right hand. If it's on the left face, then perform U' with the left hand. Then, perform 1-2-3 with the left hand. Either way, you have now dislodged a white corner cube from the bottom to the top. Restore that white corner cube to the bottom (using the process outlined in step 3). You now have placed the original edge piece in its final position.

- If you find yourself in the situation where all of the edge pieces on top have yellow on them, then orient the cube so that one of the non-yellow edge pieces on the second level is facing you. Then, do a 1-2-3 move, restore the white corner, and you will now have that non-yellow edge piece on top so that you can apply the previous paragraph.

- Do this for all four edge pieces that don't have yellow on them. After you finish this step, the bottom and middle level will be completely solved.

Step 5 (FURU'R'F')

- Look for two yellow edge cubes on the top face that are diagonally touching. Along with the yellow center, orient the cube so that they are in the "nine o'clock" position. Then, perform FURU'R'F'. This will produce a yellow cross on the top face (maybe with some yellow corners facing up, too).

- If you do not have two yellow edge cubes that are diagonally touching, then perform FURU'R'F' once or twice to create the nine o'clock position, and then do it one more time to create the yellow cross.

Step 6 (The Fun Move)

- You will now finish the top face so that everything on top has yellow facing up. Currently, the center and all four edge faces are yellow facing up. If all four corners have yellow face up (unlikely), then you can skip this step. If exactly one corner has yellow face up, then orient the cube so that the yellow corner is on the bottom-left corner of the top face. Next, perform the "fun move," which is all done with the right hand: RUR'URU^{2}R'. (In your mind, you can think of it as: out, twist, in, twist, out, twist, twist, in.) You may have to repeat this move to get all four corners yellow.

- If zero or two corners have yellow face up (three is impossible), then orient the cube so that a yellow sticker is in the top-left corner of the FRONT cube. Perform the fun move once or twice to get one yellow cube on the top face, and then proceed as detailed in the previous paragraph.

Step 7 (The R'F Move)

- The only remaining unsolved part of the cube is the top rim. If one side of the top rim has two matching corner colors, then orient the cube so that it's on the back side. (Otherwise, the orientation doesn't matter.) Next, perform $R'FR'B^2RF'R'B^2R^2$. If there were no matching colors the first time, then there will be now. Proceed as above, and you will be done; everything will be finished except for three or four edge cubes on the top rim.

Step 8 (The FURL Step)

- If there are only three edge cubes out of position, then one side of the cube will be completely solved, and this is your last step. Orient the cube so that the pristine face is on the back side. Next, perform the following move once or twice: FFUR'LFFRL'UFF. In your mind, you can think of it as: FFU; both sides down for R'L; dial around for FF; both sides up for RL'; and that's "en-uff" UFF.

- If there are four edge cubes on the top rim that are out of position, then perform the above move, and there will only be three cubes out of position. Performing the move (as above) one more time will solve the cube.

- If you practice this method, you can comfortably solve the cube in under five minutes, and if you use your fingers very efficiently, you can do it in about one minute. It can take about 180 twists to perform, so to do it in three minutes requires that you spend about one second per twist.

- What's the fewest number of moves needed to solve the cube? It was recently proved, in 2010, that every cube is never more than

20 twists away from being solved. That number can't be lowered because there are some positions that provably require at least 20 moves to complete, such as the superflip, which is known to require 20 moves.

- People who are speed cubers use algorithms that are different from what is used in this lecture. They move their fingers incredibly fast, and their methods combine several steps at once. The world's fastest solvers can do a randomly scrambled cube in under 10 seconds.

Suggested Reading

Beasley, *The Mathematics of Games.*

Frey and Singmaster, *Handbook of Cubik Math.*

Harris, *Speedsolving the Cube.*

Joyner, *Adventures in Group Theory.*

Slocum, Singmaster, Huang, Gebhardt, and Hellings, *The Cube.*

Questions to Consider

1. Have a friend scramble your Rubik's Cube and see if you can solve it. How long did it take?

2. From the original position, how many times do you need to perform the move R^2U^2 until you get back to the original position? (You need to have the cube in your hands to do this one. I don't think you can do this problem in your head.)

3. Perform these instructions and see what you get.

a. $F^2\,B^2\,R^2\,L^2\,U^2\,D^2$.

b. U D' R L' F B' U D'.

c. F U F R L^2 B D' R D^2 L D' B R^2 L F U F.

Solving Sudoku

Lecture 10

Puzzles like sudoku and KenKen® are fun because they rely entirely on logic. This lecture on sudoku is purely an intellectual challenge. In this lecture, you will develop a toolbox of simple strategies that will allow you to solve almost any sudoku puzzle you encounter. Strategies include tic-tac-toe and crosshatching processes, miniboxes, completed lines, right angles, and some advanced techniques for very challenging sudoku puzzles. In addition, you will be introduced to the rules of KenKen®.

Sudoku Strategies

- Without a doubt, Rubik's Cube has been the most successful physical puzzle ever invented, but in the last 10 years, the world's most popular puzzle, requiring less physical dexterity and more logical reasoning than Rubik's Cube, is sudoku.

- Part of sudoku's enduring popularity is that the challenge can be described in a single sentence: Enter the numbers one through nine in such a way that each number appears once in every row, column, and three-by-three box. Every sudoku puzzle will have a unique solution, and if the puzzle is well crafted, it should be solvable using just pure logic. No guessing should be required.

Sudoku Puzzle

		9			6			
		8			4			
					1	2		3
							4	
	5						7	
	2							
6		4	7					
			8			5		
			9			8		

- Sudoku is a mathematical game, and it's not because the grid is filled with numbers. It would be just as mathematical if each square had to be filled with a letter or color. What makes sudoku mathematical is that in order to solve it, you need to think like a mathematician by looking for patterns and using careful logic.

- A sudoku puzzle is played on a nine-by-nine grid consisting of 81 squares with some number of initial clues. It has nine rows, nine columns, and nine boxes. The numbers that can potentially go in a square are called the candidates for that square.

- The wrong way to start a sudoku puzzle is to enter the candidate numbers for each square in hopes of finding a square with a single candidate. You might want to do this if you get stuck near the end, but you definitely don't want to do this at the beginning of the puzzle. In fact, you want to take the opposite approach. Instead of determining which numbers can go in each square, you want to determine which squares can take each number.

- A sudoku puzzle known as the pi puzzle is an 18-clue puzzle. It appears in the book *Taking Sudoku Seriously* by mathematicians Jason Rosenhouse and Laura Taalman. Not only is the puzzle rotationally symmetric, but the digits of this puzzle also come from the first 18 digits of pi: 3.14159265358979323.

Pi Puzzle

7	2							
	5				9			
				3	8			
			4			5		
		3				9		
		1			3			
			2	5				
			6				3	
							1	9

- Every sudoku has nine rows and nine columns, but if you focus on the thick lines, it sort of looks like a big tic-tac-toe board. You can think of it as having three big rows and three big columns. Look for a number that appears twice in the same big row or big column. In the case of the pi puzzle, the number 3 appears twice in the middle big column (big column 2)—in little columns 5 and 6.

- Look at the other little column, which is little column 4, in big column 2. Where can the third 3 go in that little column? It can't go in the top box or the middle box because both of those already have 3s, so it must go in the bottom box. There's only one place it can go within the bottom box because columns 5 and 6 have 3s, and there is only one open spot in column 4.

7	2							
	5				9			
				3	8			
			4			5		
		3				9		
		1			3			
			2	5				
			6				3	
			3				1	9

Tic-Tac-Toe Process

- The number that was just entered into the puzzle is called a hidden single, which is a number that has only one legal square in a row, column, or box. The best way to find hidden singles is through the tic-tac-toe process.

- In this puzzle, you are able to fill in most of your numbers by judicious use of the tic-tac-toe and crosshatching processes. With tic-tac-toe you need two repeated numbers in the same big row or column. When two numbers are not in the same big row and column, then you can sometimes use crosshatching.

- For example, if you crosshatch the 3s in column 3 and in row 3, you find that no 3 can appear anywhere in the top left box of the puzzle

Crosshatching Process

7	2					3	9	
3	5				9	3		
				3	8		9	
			4			5		3
		3				9		
		1			3			
3	3		2	5				
			6				3	
			3				1	9

7	2					3	9	
3	5				9	3		
				3	8		9	
			4			5		3
		3				9		
		1			3			
3	3		2	5				
			6				3	
			3				1	9

7	2					3	9	
3	5				9			
				3	8		9	
			4			5		3
		3				9		
		1			3			
3	3		2	5				
			6				3	
			3				1	9

except for one position: column 1, row 2. Then, two more 3s are eliminated as potential candidates, leaving you with two more 3s being placed in the puzzle. Now, you've placed every 3 on the grid, and you won't ever have to deal with that number again.

- When a number can only find two possible squares in a box, lightly pencil in that number in both places. If both of those places were in the same row or column, that's called a pointing pair, and that can sometimes eliminate candidates from the other boxes. If you find two numbers that can only go in the same two spots in a box, then they form a hidden pair, and those spots must contain those numbers.

- Look for more places to crosshatch. When you scan the grid, look for certain structures. For example, the top left box has a two-by-two minibox. There is also a completed line inside the box that is located in the middle on the bottom of the puzzle. In addition, there are a few boxes containing right angles. All of these structures tend to be very helpful.

Miniboxes, Completed Lines, and Right Angles

7	2					3	9	
3	5				9			
				3	8		9	
			4			5		3
		3				9		
		1			3			
	3		2	5				
			6				3	
			3				1	9

- For example, the right angle located in the bottom right corner of the puzzle contains three unfilled squares in a row and three unfilled squares in a column, which is sometimes susceptible to crosshatching, which then sometimes causes a chain reaction of solutions.

- With tic-tac-toe and right angles, you need two numbers to make progress, but with miniboxes, you often need just one number to make progress.

- We started our puzzle by looking for numbers that appear a lot, because they tend to produce the most useful tic-tac-toes, right angles, and miniboxes, and because they tend to produce very useful crosshatches. In addition, there is a structure called a completed line, which consists of three numbers that completely fill a row or column of a box.

- Completed lines let you start a tic-tac-toe with one number instead of two. When starting a sudoku puzzle, the first thing you look for should be completed lines.

- At some point, the tic-tac-toe process stops working, and there may not be any crosshatches that jump out at you. What do you do when you get stuck? Walk away and come back to it later. After taking a break, you'll often see things that you missed before.

- Then, examine the numbers that are almost finished, but if you're still stuck, try the following techniques: Scrutinize your rows, columns, and boxes that are almost finished and look for naked singles, which are squares that can only take on one value. Naked singles become more common as the row, columns, and boxes start

Completed Pi Puzzle

7	2	9	1	4	6	3	5	8
3	5	8	7	2	9	1	4	6
4	1	6	5	3	8	7	9	2
8	9	7	4	1	2	5	6	3
2	4	3	8	6	5	9	7	1
5	6	1	9	7	3	8	2	4
9	3	4	2	5	1	6	8	7
1	8	2	6	9	7	4	3	5
6	7	5	3	8	4	2	1	9

to fill up. Don't forget chain reactions. Anytime you get a new piece of information, see what chain reaction that causes.

Solving Difficult Puzzles: Sudoku and KenKen®

- Using the techniques in this lecture, you can solve most if not all of the puzzles that you'll find in newspapers and magazines, but to solve the most difficult sudoku puzzles—typically labeled as tough, fiendish or diabolical—you might need a few more tools in your toolbox. To use these techniques, you sometimes need more information than you have written in your puzzle.

- The method that you've been learning focuses on numbers that can only go in a small number of squares. When you've reached a position where you're really stuck and you can't make any more progress through tic-tac-toe, crosshatching, hidden pairs, and naked singles, then it's time to reverse that approach and look for squares that can only take a few numbers. In other words, now and only now, it may be time to pencil in the candidates for each square. Be especially on the lookout for squares that can only take two numbers.

- The first thing to look for is a naked pair, which consists of two squares in the same row, column, or box that can only take the same two numbers. There's an extension of the naked pair concept called a naked triple, which consists of three squares in the same row, column, or box that can only take on three values.

- Another neat idea is known as X-wing. Suppose the number 5 only has two places it can go in column A, say in rows C and D, and suppose that in column B, 5 can only go in the same two rows. The X-wing rule says that there can't be any 5s in rows C and D. X-wings can be pretty tricky to spot, but when you find them, they tend to simplify the puzzle quite a bit.

- Another advanced solving technique is called unique rectangle, which is based on the fact that every sudoku puzzle is required to have a unique solution. This way, the newspaper or book with the puzzle can print the one solution to the puzzle instead of showing

X-Wing Rule

X	5	X	X	X	X	5	X	X
X	5	X	X	X	X	X	X	X

many possibilities. Another reason the solution has to be unique is because otherwise, a solution would require you to do some guessing, and all sudoku puzzles are supposed to be solvable without requiring any guessing. When solving sudoku, you can sometimes exploit uniqueness.

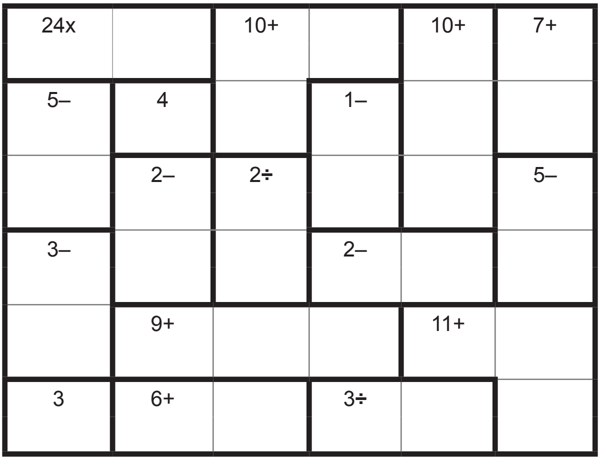
- Strictly speaking, all sudoku puzzles are supposed to be solvable by pure logic, and no guessing is required. On the other hand, if you filled up most of the puzzle and reach a point where you get stuck, then sometimes it is helpful to try a quick guess and see what happens.

- You should choose a square that has two possible values that will cause a big chain reaction. Circle your guess, and put everything else in colored pencil. Typically, one of two things will happen: Either the chain reaction will lead to a solution to the puzzle, or it will lead to some kind of logical contradiction, such as a row that must have two 5s in it, in which case you know that your initial assumption was wrong.

- There are many variations on the sudoku puzzle, and many of the techniques that you learned in this lecture will transfer to other puzzles. The sudoku variation known as KenKen® combines the logic of sudoku with some actual reasoning about numbers as well.

Similar to sudoku, every row and column must contain the numbers one through six, and each cage indicates what the numbers add, subtract, multiply, or divide to. Numbers can be repeated in the same cage, but not in the same row or column.

KenKen®



Suggested Reading

Gordon, *Solving Sudoku.*

Miyamoto, *Brain-Busting KenKen®.*

Riley and Taalman, *No-Frills Sudoku.*

Rosenhouse and Taalman, *Taking Sudoku Seriously.*

Snyder, *The Art of Sudoku.*

Stephens, *Mastering Sudoku Week by Week.*

Stuart, *The Logic of Sudoku.*

Problems

1. Solve the sudoku puzzle below. It only has 18 clues, but you can do it!

		9			6			
		8			4			
					1	2		3
							4	
	5						7	
	2							
6		4	7					
			8			5		
			9			8		

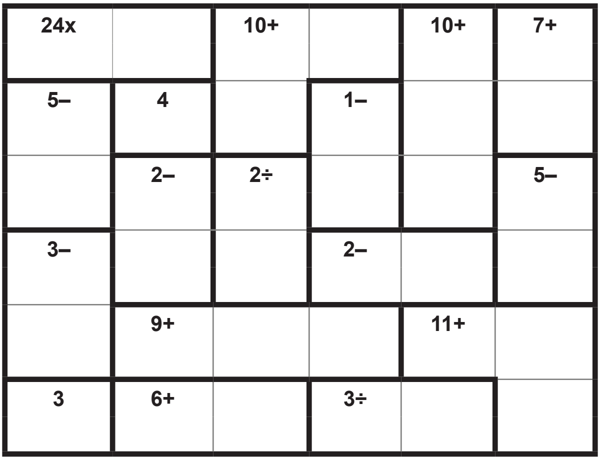
(Puzzle appears in the book *No Frills Sudoku,* by Brainfreeze Puzzles, Philip Riley and Laura Taalman, © 2011. Used with permission.)

2. Solve the "silicon sudoku" below.

2	8							
		1	6	7	9			
		5					4	
		6						
		7	2	4	5		3	
					1		9	
					6		5	
		3	8	1	4		2	7
1	4							

(Copyright © 2008 by Thomas Snyder. Used with permission. For more puzzles like this, see GrandmasterPuzzles.com.)

3. Solve the KenKen® puzzle below. Each row and column uses the numbers 1 through 6. The number in each cage (heavily outlined set of squares) indicates what the numbers must add, subtract, multiply, or divide to. Numbers may be repeated in a cage, but not in the same row or column.


24x
10+
10+
7+
5–
4
1–
2–
2÷
5–
3–
2–
9+
11+
3
6+
3÷

Mathematics and Chess

Lecture 11

In this lecture, you will learn about one of the world's oldest games: chess. When playing a game of chess, you don't perform many mathematical calculations, like you do in backgammon or poker, but the game is still very mathematical. When playing chess, you're constantly looking for patterns, experimenting with different strategies, and building a sophisticated toolbox from a simple set of rules. This lecture will draw connections between mathematics and chess and give you tips that are sure to improve your game.

The Game of Chess

- The game of chess is played on an eight-by-eight checkerboard. There are two players, white and black. Each player has 16 chessmen, consisting of eight pieces and eight pawns. Among the eight pieces, each player has a king, a queen, two rooks, two bishops, and two knights. The player with the white pieces always makes the first move.

Chessboard

R	**N**	**B**	**Q**	**K**	**B**	**N**	**R**
P	**P**	**P**	**P**	**P**	**P**	**P**	**P**
P	P	P	P	P	P	P	P
R	N	B	Q	K	B	N	R

- The squares of the checkerboard alternate as light and dark colors. Also, the bottom right-hand corner square is required to be light colored. Be sure to place the white queen on the light square and the black queen on the dark square. A way to remember this is to think that the queen's dress has to match her shoes.

- The king can move one space in any direction—forward, backward, right, left, or diagonally. The rook, the piece that's shaped like a castle, can move forward, backward, left, or right. It cannot move diagonally, and unlike the king, which can only move one square at a time, a rook can go as many squares as you'd like.

- While a rook can only move horizontally or vertically, a bishop can only move diagonally. A bishop in the center of the board attacks 13 other squares, but a bishop on the corner only attacks 7 other squares. That's one reason why a bishop is less valuable than a rook. On an empty board, a rook can attack 14 squares no matter where it's located. Also, a rook can get from anywhere to anywhere in just two moves, but a bishop is forever restricted to the color that it started on.

- The queen combines the powers of the rook and the bishop. A queen is like the king on steroids: It can move as many squares as it wants—vertically, horizontally, or diagonally. Thus, on an empty board, a queen located in the center can attack 27 other squares. Even a queen in the corner can attack 21 squares. The queen is the most powerful piece.

- All pieces can be blocked by their own pieces. On the other hand, if a piece encounters a piece of the opposite color, then it can capture that piece, and the captured piece is permanently removed from the board. When the pieces described so far—the king, queen, rook, and bishop—are in their initial positions, they are incapable of moving because they're all locked in by their own pieces.

- The only piece that's allowed to jump over other pieces is the knight, whose piece is usually represented by a horse. The knight makes an interesting L-shaped move. In the opening position, a knight has just two legal places it can go.

- Both players start with eight pawns. Pawns are only allowed to move forward; they can never go backward. From its original position, a pawn can move forward either one or two spaces. After its initial move, it can only move forward one space at a time. Although pawns move forward, they capture diagonally, so if an opponent's piece is directly in front of it, the pawn can't move forward, but if an opponent's piece is diagonally in front of it, the pawn can capture it.

- There's a capturing rule for pawns that doesn't occur very often. If your pawn is on the fifth row—or, as chess players call it, the fifth rank—and your opponent moves his or her pawn two spaces so that it's next to your pawn, then you can, for the next turn only, capture that pawn as if it only moved up one square. This is called capturing en passant.

- If a pawn reaches the last row of the board, the eighth rank, then that pawn is promoted to a queen. Technically, you could turn it into a rook, bishop, or knight, but it's extremely rare that you'd want to. It can even become a queen if you still have another queen on the board. When you promote your pawn to a queen, it's usually a decisive advantage because you've turned the weakest piece into the strongest piece.

- When a player's king is under attack, that's called check. When you're in check, your next move must take you out of check, and you can do this one of three ways: move the king to a safe spot, block the attack by placing a piece in between the attacker and the king, or capture the piece that is attacking the king. The king is never allowed to move into check, nor can you move a piece that would leave the king in check.

- Every move in chess uses just one piece—with one exception, called castling, which involves the king and the rook. If a king and rook have not yet moved during the game and there are no pieces in between, then the king can move two spaces toward the rook, and the rook goes on the other side of the king. This is either called castling on the king's side or castling on the queen's side, depending on which side you choose to perform this move. You are not allowed to castle if your king is in check, and the king can't land on or pass through a square that's under attack.

- The goal of chess is to put your opponent in check in such a way that he or she can't move out of check—that's called checkmate, and the game is over.

Chess Strategy and Tactics

- You can basically divide the game of chess into three parts: the opening, middle game, and endgame. In the opening, your goal is to move your pieces off of your back rank and onto more productive locations. That's called developing your pieces.

- A piece that's closer to the center of the board can attack more squares than a piece that's close to the boundary. The three key principles of the opening are to develop your pieces, control the center, and watch out for king's safety.

- In the opening, it's good to try to castle your king for two reasons: It allows one of your rooks to get into the action conveniently, and in the initial position, the king is somewhat vulnerable to attack.

- It is important to remember the following rule: Don't bring out your queen too early in the game because it allows your opponent to attack you while developing their pieces.

- Once both sides have developed most of their pieces, you move into the middle game phase of chess, during which you are trying to find and exploit weaknesses in your opponent's position while trying not to be exploited by our opponent. The secret to winning in the middle game is to be on the lookout for tactical opportunities.

- If each pawn is worth one point, then a knight or a bishop tends to be worth three points. The rook is worth five points, and the queen is worth nine points. The king, although not as powerful as the other pieces, has infinite value because if you lose your king, you lose the game. You can use these numbers to determine if it's worth making certain exchanges.

- What most chess tactics have in common is that they tend to do two things at once, such as attacking two pieces at the same time. When a single piece attacks two pieces at once, that's called a fork. When you put your opponent in check, he or she is required to defend the king in some way, so if you can attack your opponent's king and another piece, you may be able to capture the other piece. Queens and knights are especially well suited to this kind of attack.

- A tactic called pinning can happen when two valuable pieces are somehow in line with each other. For example, if the black knight is on the same file as the black king, then if the white rook attacks the black knight, then the knight is pinned because it can't move off the file without exposing the king to check. The player with the black pieces can try to defend the knight with his or her bishop, but then the white pawn attacks, and the black knight is lost. (See figures.)

- A great way to accomplish two things at once is through the use of discovered attack, especially a discovered check. This happens when moving one piece out of the way enables an attack from a different piece.

- Once you have a material advantage—for example, once you're up a piece or maybe a pawn or two—you should simplify by trading pieces of equal value for two reasons: By reducing the number of pieces on the board, you limit your opponent's ability to wage a counterattack, and as the amount of material decreases, your relative advantage increases. For example, if you have one more piece than your opponent, then instead of being up four pieces to three, it would be more decisive to be up three pieces to two or two pieces to one.

Pinning

R		**B**		**K**	**B**		**R**
	P	**P**			**P**	**P**	**P**
P		**P**					
				N			
					N		
P	P	P	P		P	P	P
R	N	B			R	K	

R		**B**		**K**	**B**		**R**
	P	**P**			**P**	**P**	**P**
P		**P**					
				N			
					N		
P	P	P	P		P	P	P
R	N	B		R		K	

- Be careful not to trade too many pawns away because you'll probably need them to win. For example, suppose you were able to capture all of your opponent's pieces, and you were left with just your king and one other chess piece. It would definitely be better for that other piece to be a pawn than a bishop because it turns out that it's impossible to checkmate your opponent's king with just a king and bishop or, for that matter, just a king and knight. Even if

Pinning

R				**K**	**B**		**R**
	P	**P**			**P**	**P**	**P**
P		**P**					
					B		
				N			
					N		
P	P	P	P		P	P	P
R	N	B		R		K	

R				**K**	**B**		**R**
	P	**P**			**P**	**P**	**P**
P		**P**					
					B		
				N			
			P		N		
P	P	P			P	P	P
R	N	B		R		K	

your opponent is trying to get checkmated, you can't do it. Such games must end in a draw.

- Likewise, it's impossible to force your opponent into checkmate with two knights against a king, and it's extremely difficult to do with one knight and one bishop against a king, assuming that there are no other pawns on the board. However, with a king and pawn, you have the potential to turn that pawn into a queen, and with a king and queen, it's easy to checkmate your opponent.

- At the end of the game, let's say that it's just you, your queen (perhaps a pawn that was turned into a queen), and the opposing king. Computer scientists have shown that if you have the queen, it's your move, and there are no other pieces on the board, then you can always force checkmate within 10 moves. The basic idea is that you have to drive your opponent's king to one of the edges of the board because it's impossible to checkmate him or her in the interior. After that, you walk your king to help the queen deliver the final blow.

- In recent years, our knowledge of the endgame has changed profoundly through the use of computers. Using a dynamic programming process called retrograde analysis, computers have worked backward from every possible checkmate position with up to six pieces on the board to determine the guaranteed best move from every chess position with six or fewer pieces. For each of these positions, the computer can tell you, if the position is not a draw, how many moves until the winning player can checkmate the other.

- Outside the endgame, computers don't yet play perfect chess. On the other hand, it's fair to say that computers have mastered the game. It was 1996 when a computer first beat the world's best player in a game of chess, and computers are now dominant in chess. No human has beaten the top computer chess program since 2005.

Suggested Reading

Emms, *Concise Chess.*

Fischer, *Bobby Fischer Teaches Chess.*

Nunn, *Learn Chess.*

Seirawan and Silman, *Play Winning Chess.*

Watkins, *Across the Board.*

Problems

1. What tends to be more valuable: a knight and a bishop or a rook and two pawns?

2. Find a winning move for White in the position below. (Note: The bold letters are black pieces; the non-bold letters are white pieces.)

	K				**R**		
P							**P**
	P		**R**		**P**		
		P		**P**			
	N						
	P				R	P	
P	B		P		P		P
	K						R

3. What's the right move for White in this position?

				K			
				K			
				P			

Winning Ways—It's Your Move!

Lecture 12

Hopefully, this course has given you a better appreciation of mathematics and games. Aside from being a lot of fun, games are also good for you and society as well. In this course, you've learned some of the calculations you do when playing games, but similar ones can be done in many aspects of your life. In this lecture, you will learn about impartial games, including NIM and chomp. You will also learn how computers and humans match up when it comes to games and puzzles.

NIM

- Very broadly speaking, most games seem to fall into one of three categories. The first one is games where whoever is the last player to move wins. This includes games like chess and checkers, but also games you may not have played before, such as NIM. Second, there are games where the goal is to be the first player to create some sort of structure, like in tic-tac-toe. Third, there are games where the winner is determined by whomever accumulates the most stuff—whether it is the most points, like in Scrabble, or the most territory, like in Go.

- NIM is a two-player game played with several piles of coins. The players take turns removing as many coins as they'd like from any pile. You can only take coins from one pile, and you must take at least one coin. Whoever takes the last coin is the winner. Once you know the math behind this game, you'll almost certainly be able to win against anyone who does not know the secret.

- The optimal strategy was given by the mathematician Charles Bouton in 1901. It was one of the first games ever to be given a complete mathematical strategy. Suppose the game reduces to a single pile of coins, then the optimal strategy is obvious: If it's your turn, just take all the coins and you win.

- Suppose the game has just two piles of coins: One pile has nine coins, and the other has five coins. If you take four coins away from the larger pile, then both piles now have five coins. Because both piles have the same number of coins, you can now win, by using a symmetry strategy. Thus, you are guaranteed to take the last coin.

- With two-pile NIM, the good positions, the ones that you want to create, are those that have two piles with an equal number of coins. The bad positions, which you don't want to create, are those that have two piles with a different number of coins. If you give your opponent two equal piles, then your opponent must make them unequal. Any move that they make, because they're only allowed to take from one pile, is going to cause those piles to be unequal.

- Conversely, when your opponent gives you two unequal piles, you can always make them equal again just by taking from the larger pile until the sizes are the same. That's a general feature of games like this. From a good position, all of your opponent's moves go to bad positions, and when your opponent moves to a bad position, there will always be at least one move that will take you to a good position.

- Suppose that you have a NIM position with three or more piles to choose from. In this case, the situation becomes much more interesting. In fact, you can still win this game by exploiting symmetry, but in a not-so-obvious way. It's based on the fact that all numbers can be uniquely expressed using powers of two.

- For example, the numbers one through seven can each be represented using the numbers one, two, and four, as follows.

1

2

3 = 2 + 1

4

$5 = 4 + 1$

$6 = 4 + 2$

$7 = 4 + 2 + 1$

- By inserting the number eight, we can get the numbers eight through 15, and so on.

8

$9 = 8 + 1$

$10 = 8 + 2$

$11 = 8 + 2 + 1$

$12 = 8 + 4$

$13 = 8 + 4 + 1$

$14 = 8 + 4 + 2$

$15 = 8 + 4 + 2 + 1$

- Suppose you were playing three-pile NIM, and the piles had sizes 13, 10, and 6. What should you do? If we write each number in terms of powers of 2, we get the following: 13 can be expressed as $8 + 4 + 1$, 10 as $8 + 2$, and 6 as $4 + 2$.

$$\begin{aligned} 13 &= 8 + 4 + + 1 \\ 10 &= 8 + 2 \\ 6 &= 4 + 2 \end{aligned}$$

- To reach a good position, you want to get an even number of 8s, 4s, 2s, and 1s. In this situation, you already have an even number of 8s, an even number of 4s, and an even number of 2s, but you have an odd number of 1s. By eliminating the 1 from the first number—that is, by turning 13 into 12—you arrive at a good position. This is a good position because any move that your opponent makes is guaranteed to cause an 8, 4, 2, or 1 to appear an odd number of times.

- For example, suppose your opponent takes three coins from the 10 pile, turning 10 into 7. Writing 7 in terms of powers of 2, you get that 7 is 4 + 2 + 1. You now have one 8, three 4s, two 2s, and one 1. You now have an odd number of 8s, an odd number of 4s, and an odd number of 1s.

$$12 = 8 + 4$$

$$7 = 4 + 2 + 1$$

$$6 = 4 + 2$$

- In order to get an even number of 8s, you're going to have to go from one 8 to no 8s. That can only happen if you take coins from the pile of 12. You also need to lose a 4 because you have three 4s, which is an odd number of 4s, and you need to gain a 1 because you have one 1 and need to get two of them (an even number of them). Hence, the 12 pile has to transform into the following.

$$1 = \quad + 1$$

$$7 = 4 + 2 + 1$$

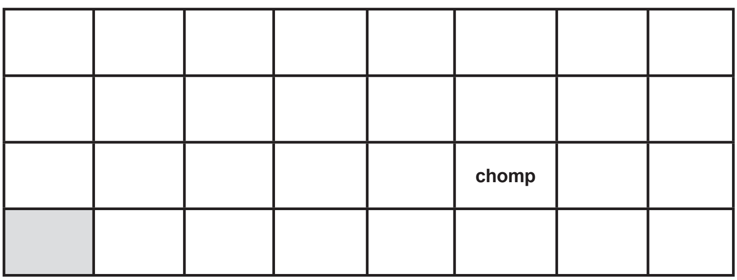
$$6 = 4 + 2$$

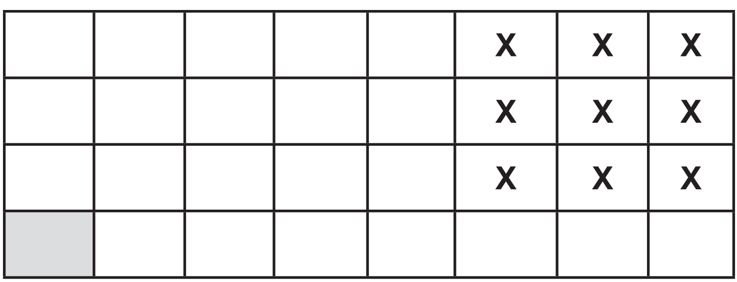
- Therefore, the winning move—in fact, the only winning move—is to reduce 12 coins down to one coin. You have two 4s, two 2s, and two 1s, so you have reached a good position, and if it's your opponent’s turn to move, any move he or she makes is going to disrupt the parity of the 1s, 2s, and 4s. At least one of those parities has to change, giving your opponent a bad position. Then, you can restore that to a good position.

- Almost all impartial games can be viewed as special cases of the game NIM. The major tools used in the analysis are two of the major problem-solving themes of this course: reducing to smaller problems and taking advantage of parity.

Chomp

- Chomp is an interesting impartial game that is not equivalent to NIM because the player who makes the last move is the loser, not the winner, in chomp. The game of chomp starts with a rectangle—for example, an eight-by-four checkerboard. Players take turns chomping off squares from the board. When a square is chomped off, it is removed, along with all squares above it and to the right of it.

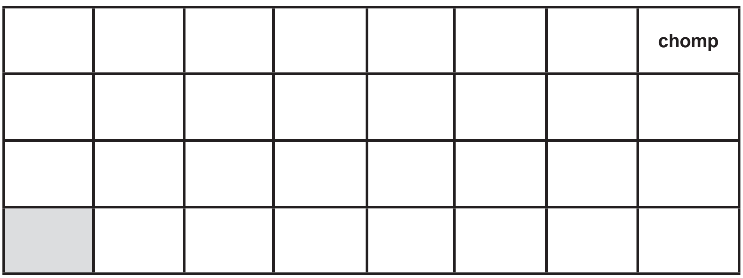




- Players take turns chomping off squares, and the goal is to not take the last square, which is located in the bottom left corner of the board and is known as the poison square. This rule is needed to keep the game interesting. Otherwise, the game would end on the first move because the first player would just chomp the square in the bottom left corner, and everything would be chomped off.

- Once an optimal strategy is found, a mathematical proof has shown that it has to be a win for the first player. This argument works for a rectangular board of any size with more than one square in it.

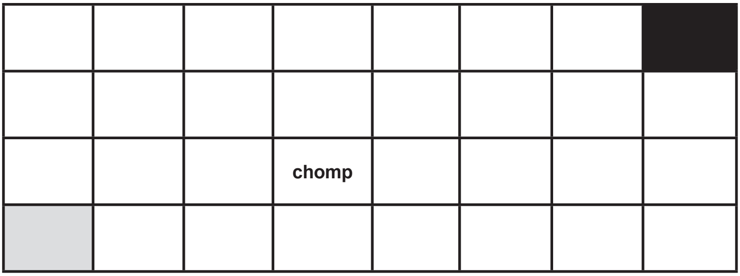
- Suppose that you are the first player. What should your first move be? Consider the position when you chomp off only the upper right corner. That creates either a good position or a bad position.



- If it’s a good position, then, assuming you play perfectly, your opponent can’t prevent you from winning from that position. If it’s a bad position, then that means that your opponent can somehow force a win by making a chomping move from that position that will give him or her a good position that you can’t beat.

- Suppose that your opponent’s winning response to your move is to chomp off the square in the following position.

- Then, that would make the following a winning position that you could not defeat.



- However, if that's a theoretically unbeatable position, then you could have made that move yourself on the first move, resulting in that same position, and then you would have a forced win from that position. In other words, playing the top corner is either a winning move or it's not. If it's a winning move, then you make it. If it's not a winning move, then you can legally play the response to that move and win from there. Mathematicians call this sort of argument an existence proof. You know who the winner is; you just don't know how the winner wins.

Games and Puzzles: Computers versus Humans

- The easiest games for computers are those where they can play perfectly from any position, including tic-tac-toe and NIM. Next in difficulty are games that computers know how to play perfectly from the beginning of the game, but not necessarily if thrown into a random middle position that it would never reach if it played well from the beginning. These include games like checkers.

- What's more interesting are the games where humans and computers are still pretty close in terms of ability. For example, it seems that computers are among the world's best players in chess, backgammon, Scrabble, Yahtzee, and the television game show *Jeopardy!*, but there are still some games that humans, at least for the top-performing people, still have an edge over computers—at least for now. Although there has been a lot of progress in the last decade, computers are not yet considered to be the top players of bridge, poker, or go.

- Mathematics, games, and puzzles all begin with a set of objects that have various rules defined for them. For instance, with chess or Rubik's Cube, you have pieces that move in certain ways. In mathematics, you have numbers that combine in different ways, and the better you are at combining these rules, the more often you'll be successful at defeating your opponent, solving the puzzle, or proving the theorem. Games, puzzles, and mathematics all provide great intellectual satisfaction, and they're all a lot of fun once you get the hang of them.

Suggested Reading

Albert, Nowakowski, and Wolfe, *Lessons in Play.*

Beasley, *The Mathematics of Games.*

Berlekamp, Conway, and Guy, *Winning Ways for Your Mathematical Plays.*

Epstein, *The Theory of Gambling and Statistical Logic.*

Guy, *Fair Game.*

Juhnke, *Beginning Arimaa.*

Nowakowski, *Games of No Chance.*

Stewart, *How to Cut a Cake and Other Mathematical Conundrums.*

Wells, *Book of Curious and Interesting Puzzles.*

Problems

1. Suppose a rook starts at the bottom left corner of a chessboard, and two players take turns moving the rook. On a player's turn, the rook can either be moved horizontally in the eastward direction or vertically in the northward direction. The player who brings the rook to the northeast corner is the winner.

    a. On an eight-by-eight board, what are the winning positions, and who has the advantage in this game?

    b. On a four-by-eight board, what are the winning positions, and who has the advantage in this game?

2. The following questions apply to the game of NIM.

    a. Find the only winning move when the piles have sizes 24, 14, and 19.

    b. If, for this turn only, you can add coins to a pile, then find two other winning moves.

3. On an eight-by-eight checkerboard, show that chomping the point (2,2) (diagonally next to the *poison* square (1,1)) is a winning move for the first player.

Solutions

LECTURE 1

1. Moving backward with multiples of 7, you see that 51 – (7 x 7) = 2. So, the first player can force a win by starting with 2 and then, on successive turns, jumping to the totals 9, 16, 23, 30, 37, 44, and finally 51.

2. Play in the upper left corner, forcing your opponent to play in the middle top. Then, play in the central square, and you will win on the following turn.

3. The largest piece can go on three pegs. Then, the next largest piece has three choices. Then, the next largest piece has three choices, and so on. Hence, a position can be created in 3 x 3 x … x 3 = 3n ways. By placing the pegs down from largest to smallest, you are assured of creating a legal position.

LECTURE 2

1.

a. There are 6 x 6 x 6 = 216 equally likely ways to roll three different-colored dice. Because six of these are three of a kind (namely, (1,1,1), (2,2,2), … (6,6,6)), then the probability of three of a kind is 6/216 = 1/36. Another way to do this problem is that regardless of what the first die is, the second die has a 1/6 chance of matching it, and the third die also has a 1/6 chance of matching it. Hence, the probability is 1/6 x 1/6 = 1/36.

b. Consider any roll (x, y, z) where the sum $x + y + z \leq 10$. By "flipping the dice over," you create the roll (7 – x, 7 – y, 7 – z) with a sum of 21 – ($x + y + z$), which will necessarily be greater than or equal to 11 (because $x + y + z$ is at most 10). Hence, every small roll "holds hands" with a large roll. So, by symmetry, half the dice rolls (108 of them) will be small, and half will be large.

c. There are 108 small rolls, but you lose if the three dice are (1,1,1), (2,2,2), or (3,3,3). Hence, your chance of winning a small bet is 105/216 = 35/72 (and your chance of losing is, therefore, 37/72).

d. Your expected value on a "small" \$1 bet is 35/72(1) + 37/72(–1) = –2/72 = –1/36 = –0.027777… . Hence, you expect to lose about 2.8 cents per dollar bet.

2. The number of ways to roll 2, 3, 4, 9, 10, 11, or 12 is (1 + 2 + 3 + 4 + 3 + 2 + 1)/36 = 16/36. Its expected value is 2(2/36) + 1(14/36) – 1(20/36) = –2/36 ≈ –0.0555. So, you lose, on average, about 5.55 cents per dollar bet.

3.

a. Treating \$5 as a single betting unit, this is the same problem as when you start with 3 units and your goal is to reach 12 units before going broke. For a fair game, the probability of success is 3/12 = 1/4 = 0.25.

b. The chance of winning two bets in a row is (0.5)(0.5) = 0.25.

c. Apply the gambler's ruin formula with $p = 0.6$ and $q = 0.4$, so $q/p = 2/3$. Thus, if you start with 3 units and bet 1 unit at a time, the probability of reaching 12 units is $(1 – (2/3)^3)/(1 – (2/3)^{12}) = (0.7037\ldots)/(0.9923\ldots) \approx 0.7092$. This is much better than placing two big bets in a row, where the chance of success is just (0.6)(0.6) = 0.36.

LECTURE 3

1. hit
2. stand
3. split
4. double down
5. stand
6. hit
7. double down
8. hit
9. stand
10. hit
11. stand
12. hit
13. double down
14. stand
15. hit
16. double down
17. double down
18. hit

19. split

20. double down

21. You would need at least 1/3 of the 26 cards to be 10, J, Q, or K, so at least 9 of the 26 must be 10, J, Q, or K.

LECTURE 4

1. Suppose that Rose plays row 1 with probability x and row 2 with probability $1 - x$. Then, when Colin plays column 1, her expected payoff is $2x - 5(1 - x) = 7x - 5$. When Colin plays column 2, her expected payoff is $-3x + 8(1 - x) = 8 - 11x$. These lines intersect when $7x - 5 = 8 - 11x$, which happens when $18x = 13$, or $x = 13/18$. Hence, Rose should play row 1 with probability $x = 13/18$ and play row 2 with probability $1 - x = 5/18$. Doing so, she achieves a value of $7x - 5 = 91/18 - 5 = 1/18$, regardless of the strategy that Colin uses. Likewise, Colin can ensure that Rose has no more than her expected payoff of 1/18 by choosing column 1 with probability y, where $2y - 3(1 - y) = 1/18$, so that $5y = 3 + 1/18 = 55/18$. Therefore, Colin chooses column 1 with probability $y = 11/18$ and column 2 with probability 7/18. (Verifying, $-5y + 8(1 - y) = -5(11/18) + 8(7/18) = 1/18$). The value of the game is 1/18, so the row player wins, on average, 1/18 of a unit per game under the equilibrium strategy.

2.

a. When Rose adopts the proposed strategy, then when Colin plays column 1, Rose's expected payoff is $A(D - C)/E + C(A - B)/E = (AD - BC)/E$. When Colin plays column 2, Rose's expected payoff is $B(D - C)/E + D(A - B)/E = (AD - BC)/E$ again. Either way, and even if Colin mixes strategies 1 and 2, Rose's expected payoff is $(AD - BC)/E$.

A similar calculation shows that when Colin adopts the proposed strategy, Rose has an expected value of $(AD - BC)/E$, regardless of which row she chooses, because $A(D - B)/E +$

Solutions

$B(A - C)/E = (AD - BC)/E$ and $C(D - B)/E + D(A - C)/E = (AD - BC)/E$.

b. Here, $A = 2$, $B = -3$, $C = -5$, and $D = 8$, so $E = A + D - B - C = 18$, and $V = (AD - BC)/E = (16 - 15)/18 = 1/18$, as desired.

3.

a. The payoff matrix for this game, where the rows and columns represent rock, paper, and scissors (in that order) is as follows.

0	–3	10
3	0	–1
–10	1	0

b. When Rose adopts the (1/14, 10/14, 3/14) strategy, then it's easy to verify that when Colin plays any column, Rose's expected payoff is zero. (For instance, against column 1, her expected payoff is $0(1/14) + 3(10/14) - 10(3/14) = 0$.) Likewise, if Colin adopts this strategy, Rose achieves an expected payoff of zero, regardless of which rows she plays. The value of this game is, of course, zero.

c. Here, the payoff matrix would look like the following.

0	$-b$	a
b	0	$-c$
$-a$	c	0

Naturally, because this is a symmetric game, the expected payoff should be zero. If Rose chooses her rows with proportions c:a:b (that is, with probabilities $c/(a + b + c)$, $a/(a + b + c)$, $b/(a + b + c)$), then she achieves an expected payoff of zero against any column strategy. The same result applies to Colin as well.

LECTURE 5

1.

a. The probability that the first card is low (2 through 9) is 8/13 ≈ 0.6

b. So, a reasonable estimate that all three cards are low would be about $(0.6)^3 = 0.216$.

c. The exact probability that all three cards are low is (32/52)(31/51)(30/50) ≈ 0.224.

d. Appropriate fair odds would be about 78 to 22, or roughly 3.5 to 1.

2.

a. Because you have the 6, 7, 8, and 9 of diamonds, your hand will improve with any diamond, any 5, or any 10. There are nine remaining diamonds, three 5s, and three 10s (because you are careful not to double count the 5 of diamonds or 10 of diamonds). Thus, your hand has 15 outs.

b. By the rule of 4, your chance of improving your hand is about 60%. But because you have more than 9 outs, the more accurate "3x + 9 rule" puts the probability closer to 54%.

c. With one more card remaining, the rule of 2 says that your chance of winning is about 30%. The more accurate rule of 2.2 says that your chance of winning is about 33%.

d. You are being asked to risk $10 to gain $40, so the pot is offering you four-to-one odds, which is very favorable because your chances are 33%. (It would have been a borderline decision if the pot were offering two-to-one odds.)

3.

a. Keep the pair of jacks.

b. Keep the honors with matching suits: QH and JH.

c. Keep the pair of 3s.

d. Keep the hearts. Discard the ace.

LECTURE 6

1.

a. There are 14 hitting numbers: All 11 direct 3s, plus the roll 21 (which is the same as the roll 12) and the roll 11. So, the probability of hitting is 14/36.

b. There are 15 hitting numbers: All 11 direct 5s, plus 41 and 32. So, the answer is 15/36.

c. A checker 12 away can only be hit by 66, 44, and 33, so the probability of hitting is 3/36.

d. There are 20 numbers that contain 3 or 5 directly, plus combination shots 11, 21, 41 (careful not to double count 32), and 44, and 66 for a total of 27/36. So, the probability of hitting is 27/36 = 3/4.

2.

a. (4/6)(4/6) = 16/36 = 4/9.

b. (2/6)(2/6) = 4/36 = 1/9.

c. Subtracting the two above answers from one: 1 – 4/9 – 1/9 = 4/9.

3.

a. You have 10 bad rolls (any roll that contains a 1, except for double ones). So, your chance of getting both checkers off this turn is 26/36 = 13/18 ≈ 0.72.

b. Your opponent has 7 bad rolls (32, 31, 21, 11) and, thus, 29 good rolls. In order for her to win, you must get a bad roll and she must get a good roll. Thus, her chance of winning is (5/18)(29/36) = 145/648 ≈ 0.22.

c. Even though her chance of winning is under 25%, she should actually take this double. The reason is that if you roll a bad number, which happens about 28% of the time, then she can win automatically, by redoubling you. Because your chance of winning from that position is only 7/36, you will have to drop the redouble.

d. Even though your opponent has a take, it is still correct to double. You currently own the cube at 2. If you don't double, then you win 2 points 78% of the time and lose 2 points 22% of the time for an expected value of 2(0.78) – 2(0.22) = 1.12. If you double, your winning chances go down to 72% (because your opponent will double you out if you fail to get two checkers off this turn), but your expected value is 4(0.72) – 4(0.28) = 1.76, which is much higher than 1.12 when you don't double.

LECTURE 7

1.

a. Because there are three choices for the row 1 number and three choices for the row 2 number, there are nine equally likely outcomes, five of which win for your opponent: namely, (4,5), (4,7), (2, 3), (2, 5), and (2,7). Hence, your opponent wins 5/9 of the time.

b. Row 2 loses to row 3 5/9 of the time—via (3,8), (3,6), (5,8), (5,6), (7,8)—yet row 3 loses to row 1 5/9 of the time—via (8,9), (1,4), (1,9), (1,2), (6,9).

2.

a. P(A > B) = 1/6 + (5/6)(1/2) = 7/12. (Explanation: A wins if A rolls a 6, or A wins if A rolls a 3 and B rolls a 2.)

b. P(B > C) = 1/2 + (1/2)(1/6) = 7/12.

c. P(C > A) = (5/6)(5/6) = 25/36.

d. Choose C because it has a better than 50% chance of winning against A.

3. Your opponent is an enormous favorite. (Why else would she offer you the bet?) Your score will be much higher than hers until your team has a game with zero runs. Then, your product will be zero for the rest of the season. It's almost never happened that a baseball team has gone the entire season scoring at least one run each game.

4. Your friend has a quarter and a nickel. *One* of the coins was not a nickel, and indeed the quarter is not a nickel.

LECTURE 8

1. The rectangles surrounding buttons 2 and 4 are even; the rectangles surrounding buttons 6 and 8 are odd, so you press buttons 6 and 8. After that, you'll press any lit corners and, if necessary, the center. After pressing buttons 6 and 8, corners 1 and 3 will be on (and corners 5 and 7 will be off). So, corners 1 and 3 should also be pressed. Light 5 will still be on, so it should be pressed as well.

2.

a. The following numbers are oddballs: 9, 2, 6, 13, 8, 4, 10.

b. Because there are seven oddballs, the puzzle cannot be solved.

3. Mark the holes of the puzzle with symbols x, y, and z, as follows.

$$
\begin{array}{ccccccccc}
 & & & & x & & & & \\
 & & & y & & z & & & \\
 & & z & & x & & y & & \\
 & x & & y & & z & & x & \\
y & & z & & x & & y & & z
\end{array}
$$

The symbols x, y, and z have the same meaning as before. Notice that every three consecutive squares contains an x, y, and z. If you add up all the occupied holes, you get a grand total of $y + z = x$. Thus, a single peg can only end up on a square labeled with x. Because the holes on the second row have labels y and z, you can't end up with a single peg there.

LECTURE 9

1. Answers may vary.

2. Perform this move six times, and you'll be back to where you started.

3.

a. This will create a checkerboard pattern.

b. This will create six "spots."

c. This will create a striped pattern.

LECTURE 10

1.

5	3	9	2	7	6	4	8	1
2	1	8	3	9	4	7	5	6
7	4	6	5	8	1	2	9	3
8	6	3	1	2	7	9	4	5
9	5	1	4	3	8	6	7	2
4	2	7	6	5	9	1	3	8
6	8	4	7	1	5	3	2	9
1	9	2	8	4	3	5	6	7
3	7	5	9	6	2	8	1	4

2.

2	8	9	4	5	3	7	1	6
4	3	1	6	7	9	2	8	5
7	6	5	1	8	2	3	4	9
5	1	6	9	3	8	4	7	2
8	9	7	2	4	5	6	3	1
3	2	4	7	6	1	5	9	8
9	7	8	3	2	6	1	5	4
6	5	3	8	1	4	9	2	7
1	4	2	5	9	7	8	6	3

3.

24x **4**	**6**	10+ **3**	**1**	10+ **2**	7+ **5**
5– **1**	4 **4**	**6**	1– **5**	**3**	**2**
6	2– **3**	2÷ **2**	**4**	**5**	5– **1**
3– **2**	**5**	**4**	2– **3**	**1**	**6**
5	9+ **2**	**1**	**6**	11+ **4**	**3**
3 **3**	6+ **1**	**5**	3÷ **2**	**6**	**4**

LECTURE 11

1. A knight and bishop tend to be worth about six points. A rook and two pawns tend to be worth seven points, so the rook and two pawns are generally better.

2. White's bishop should capture the pawn, pinning the black rook, which it can take next turn. If Black captures the bishop with its pawn, then the white rook can take the black rook.

3. White should move the pawn forward. This gives White the opposition, which will force the black king to move off of the file. (If the black king moves backward, then the white king moves forward, maintaining the opposition.) When the black king moves off of the file, White moves to the other file and eventually marches his or her pawn up the middle.

LECTURE 12

1.

a. The winning positions are when the rook is on the main diagonal (going from the bottom left corner to the upper right corner). This is a win for the second player because the first player is forced to move off the diagonal, from which the second player will be able to move back on the diagonal. Ultimately, the second player will reach the upper right hand corner, which is on the diagonal.

b. If the game were played on a four-by-eight board, then the first player can force a win by moving the rook to the right four squares (to the square (5,1)). The winning squares are (5,1), (6,2), (7,3), and (8,4).

2.

a. Writing the numbers in terms of powers of two, you have as follows.

$24 = 16 + 8$

$14 = \quad\quad 8 + 4 + 2$

$19 = 16 \quad\quad\quad + 2 + 1$

There are an even number of 16s, 8s, and 2s and an odd number of 4s and 1s. The only winning play must remove a 4 from the second number and gain a 1. In other words, you must remove 3 coins from the second pile, leaving 11 coins. In doing so, all of the 16s, 8s, 4s, 2s, and 1s appear an even number of times.

b. You can add 4 and 1 to the first pile, creating 29 coins, or you can add 4 and lose 1 from the third pile, creating 22 coins.

3. After the first player chomps the (2,2) square, the only squares remaining are in the first row and first column, which is a symmetrical position. From here, the first player can simply "mimic" the moves of the second player. For instance, if the second player chomps the fifth square in the first column, then the first player can respond by chomping the fifth square in the first row. Eventually, the second player will be forced to take the poison square.

Credits

Bibliography

Albert, Michael H., Richard J. Nowakowski, and David Wolfe. *Lessons in Play: An Introduction to Combinatorial Game Theory.* Wellesley, MA: A K Peters, 2007. This is the best introduction to the subject of combinatorial game theory, including impartial and non-impartial games. Very readable.

Ball, W. W. Rouse, and H. S. M. Coxeter. *Mathematical Recreations and Essays.* New York: Dover Publications, 1892. Reprinted in 2010. This is one of the original books pertaining to the mathematics of games, puzzles, and entertaining mathematical problems, including material on the Tower of Hanoi, the Fifteen Puzzle, and the knight's tour.

Beasley, John D. *The Ins and Outs of Peg Solitaire.* Oxford: Oxford University Press, 1989. Everything you ever wanted to know about peg-jumping games.

———. *The Mathematics of Games*. New York: Dover Publications, 1989. Reprinted in 2006. This is probably the book that you should buy for this course. It's inexpensive, well written, and covers most of the major topics: games of chance, zero-sum games, parity, puzzles, and games of pure strategy.

Berlekamp, Elwyn R., John H. Conway, and Richard K. Guy. *Winning Ways for Your Mathematical Plays*. 4 volumes, 2nd ed. Wellesley, MA: A K Peters, 2001–2004. These volumes provide the complete theory of combinatorial games, where two players take turns moving until one player can no longer move.

Binmore, Ken. *Fun and Games: A Text on Game Theory.* Lexington, MA: D. C. Heath, 1992. This textbook would be suitable for undergraduate math majors wishing to learn the basics of game theory.

Browne, Cameron. *Connection Games: Variations on a Theme.* Wellesley, MA: A K Peters, 2005. This book provides descriptions, strategies, and common plays in games like Hex, where the object of the game is to connect your pieces in some way. The book is comprehensive, but somewhat hard to read.

———. *Hex Strategy: Making the Right Connections.* Wellesley, MA: A K Peters, 2000. A comprehensive book on the game of Hex, including its history, mathematical theory, and especially strategies for playing the game.

Chen, Bill, and Jerrod Ankenman. *The Mathematics of Poker.* Pittsburgh, PA: ConJelCo LLC, 2006. This book goes beyond the basics and lays down a mathematical framework, including game theory, to explain the proper frequency of bluffing and how often it should be done.

Clay, Robin. *Backgammon in a Week.* London: Hodder & Stoughton, 1992. A quick introduction to the game that offers solid strategic advice.

Diaconis, Persi, and Ronald Graham. *Magical Mathematics: The Mathematical Ideas That Animate Great Magic Tricks.* Princeton, NJ: Princeton University Press, 2012. This book contains many new ideas that connect mathematics and magic. Written by world-famous mathematicians who are also experts in the performing arts.

Duke, Annie, and John Vorhaus. *Decide to Play Great Poker: A Strategy Guide to No-Limit Texas Hold'em.* Las Vegas, NV: Huntington Press, 2011. A very readable book offering good general strategic advice.

Emms, John. *Concise Chess: The Compact Guide for Beginners.* London: Everyman Chess Series, 2003. Short but effective introduction to chess and tactics.

Epstein, Richard A. *The Theory of Gambling and Statistical Logic.* 2nd ed. San Diego: Academic Press, 2009. A classic book on gambling and its mathematical analysis—from casino games to the stock market. Specific games include craps, blackjack, and bridge. Mathematics is at the undergraduate math major level.

Fischer, Robert. *Bobby Fischer Teaches Chess.* New York: Bantam, 1972. This is included as a sentimental favorite, because it was the book that Professor Benjamin read in middle school that turned him on to chess.

Frey, Alexander H., and David Singmaster. *Handbook of Cubik Math.* Cambridge, UK: Lutterworth Press, 1982. Singmaster is a recognized authority on the cube. The authors walk the reader through the concepts and notation for understanding Rubik's Cube so that they can both solve and understand the solution.

Gardner, Martin. *Aha! Gotcha* and *Aha! Insight*. 2-volume collection. Washington, DC: Mathematical Association of America, 2006. A collection of puzzles and paradoxes designed to develop critical-thinking skills. The *Gotcha* portion has more philosophy, and the *Insight* portion has more puzzles.

———. *The Colossal Book of Short Puzzles and Problems.* Edited by Dana Richards. New York: W. W. Norton, 2006. This book contains 340 puzzles that appeared in Martin Gardner's "Mathematical Games" column for *Scientific American*.

———. *Perplexing Puzzles* and *Tantalizing Teasers.* 2-volume collection. New York: Dover Publications, 1988. A fun collection of puzzles that should appeal to audiences of all ages.

Gillis, Harvey. *Backgammon: Decision Analysis for Success.* Appears in Vol. 2, Issue 2 of *PrimeTime Backgammon*, the official magazine of the U.S. Backgammon Federation, March/April 2011, located at www.usbgf.org.

Gordon, Peter. *Solving Sudoku: Hundreds of Puzzles Plus Techniques to Help You Crack Them All.* New York: Sterling Publishing, 2006. A solid introduction to sudoku-solving techniques.

Griffin, Peter A. *The Theory of Blackjack: The Compleat Card Counter's Guide to the Casino Game of 21.* 5th ed. Las Vegas: Huntington Press, 1996. This is the standard reference for the mathematical underpinnings of blackjack, from the derivation of basic strategy to the effectiveness of card-counting systems.

Guerrera, Tony. *Killer Poker by the Numbers: The Mathematical Edge for Winning Play.* New York: Kensington Publishing Corp, 2007. This book gives a reasonable explanation of expectation and pot odds and then applies it to many commonly occurring situations in no-limit hold'em poker.

Guy, Richard K. *Fair Game: How to Play Impartial Combinatorial Games.* Arlington, MA: COMAP Mathematical Exploration Series, 1989. A complete and lighthearted treatment of impartial combinatorial games, written by one of the pioneers of the field.

Haigh, John. *Taking Chances: Winning with Probability.* Oxford: Oxford University Press, 2003. Written for a general audience but with lots of surprises, even for the mathematically experienced. Highly recommended.

Harrington, Dan, and Bill Robertie. *Harrington on Cash Games: How to Win at No-Limit Hold'em Money Games.* Henderson, NV: Two Plus Two Publishing, 2008. One of the most popular books on hold'em strategy in cash games.

———. *Harrington on Hold'em: Expert Strategy for No-Limit Tournaments, Volume 1: Strategic Play.* Henderson, NV: Two Plus Two Publishing, 2004. One of the most popular books on tournament hold'em strategy. The first author has one of the best records in tournament poker. The second author is one of the best expositors on the game of backgammon, and his writing is equally clear with poker.

Harris, Dan. *Speedsolving the Cube: Easy-to-Follow, Step-by-Step Instructions for Many Popular 3-D Puzzles.* New York: Sterling Publishing, 2008. Learn to solve Rubik's Cube the way an expert does it. If you want to bring your time below 30 seconds, then this is the book for you.

Hess, Dick. *Mental Gymnastics: Recreational Mathematics Puzzles.* New York: Dover Publications, 2011. For the reader looking for new puzzles, with a strong mathematical flavor.

Hinebaugh, Jeffrey P. *A Board Game Education: Building Skills for Academic Success.* Lanham, MD: Rowman & Littlefield Education, 2009. This book promotes the idea that board games can provide a valuable educational experience for children of all ages.

Hoffman, Louis. *Puzzles Old and New.* London: Martin Breese, Ltd, 1893. Reprinted in 1988. One of the first books to describe mechanical puzzles, including the Fifteen Puzzle, with a vast assortment of arithmetical and word puzzles as well.

Joyner, David. *Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys.* Baltimore, MD: Johns Hopkins University Press, 2002. This book provides an interesting introduction to the advanced mathematical subject of group theory, using Rubik's Cube and its variations as motivation.

Juhnke, Fritz. *Beginning Arimaa: Chess Reborn beyond Computer Comprehension.* Flying Camel Publications, 2009. A strategy guide for this relatively new game where humans will likely outperform computers for a long time. The author is a two-time Arimaa world champion and did mathematics research with Professor Benjamin as an undergraduate.

MacKinnon, Robert F. *Bridge, Probability, and Information.* Toronto: Master Point Press, 2010. Mathematical concepts for experienced bridge players.

Magriel, Paul. *Backgammon.* 2nd ed. Clock & Rose Press, 2004. Updated from 1976 version. Widely considered to be the bible of backgammon, due to its clarity of exposition. This is probably the first book you should read on the subject.

Mao, Tyson. *You Can Solve the Cube.* John George Productions, 2008. This instructional DVD can teach anyone how to solve Rubik's Cube, using the method explained in this course. Strategies for speed cubing and blindfold cubing are also provided.

Miyamoto, Tetsuya. *Will Shortz Presents Brain-Busting KenKen®: 100 Challenging Logic Puzzles That Make You Smarter.* New York: Saint Martin's Griffin, 2011. The title says it all.

Moshman, Colin, and Douglas Zare. *The Math of Hold'em.* Suwanee, GA: Dimat Enterprises, 2011. Serious math for serious poker players.

Niederman, Derrick. *The Puzzler's Dilemma: From the Lighthouse of Alexandria to Monty Hall, a Fresh Look at Classic Conundrums of Logic, Mathematics, and Life.* London: Duckworth Overlook, 2012. Fresh insights on the design and analysis of puzzles, especially classical ones, and how they arise in real life. The author is a mathematician and an expert designer of word puzzles, logic puzzles, and crosswords.

Nowakowski, Richard J., ed. *Games of No Chance.* Cambridge, UK: Cambridge University Press, 1996. A collection of articles on games that involve no randomness. Includes a great article on how the game of checkers was solved and a biography of Marion Tinsley, a mathematician who played practically perfect checkers. Also contains articles on chess endgames, go, dots and boxes, and Nine Men's Morris.

Nunn, John. *Learn Chess.* London: Gambit Publications, 2010. This is Professor Benjamin's recommended introductory book. It covers all the basic tactics, and the writing is exceptionally clear. The author also happens to be a trained mathematician.

Packel, Edward. *The Mathematics of Games and Gambling.* 2nd ed. Washington, DC: Mathematical Association of America, 2006. A terrific textbook that covers gambling and betting strategies and the mathematics behind it. Only requires a high school mathematics background.

Paymar, Dan. *Video Poker: Optimum Play.* Pittsburgh: ConJelCo, 1998. This book covers strategies for playing various video poker games with the goal of making the right decisions to maximize your win rate.

Poundstone, William. *Fortune's Formula: The Untold Story of the Scientific Betting System That Beat the Casinos and Wall Street.* New York: Hill and Wang, 2005. A history of the Kelly criterion used for gambling and investing.

Riley, Philip, and Laura Taalman. *No-Frills Sudoku.* New York: Puzzlewright Press, 2011. All of these puzzles have just 18 clues to start, which is conjectured to be the bare minimum needed for a rotationally symmetric sudoku. Puzzles range from easy to very challenging.

Rosenhouse, Jason. *The Monty Hall Problem: The Remarkable Story of the World's Most Contentious Brain Teaser.* Oxford: Oxford University Press, 2009. Everything you ever wanted to know about this paradoxical problem, offering multiple explanations and variations.

Rosenhouse, Jason, and Laura Taalman. *Taking Sudoku Seriously: The Math behind the World's Most Popular Puzzle.* Oxford: Oxford University Press, 2011. In a clear and entertaining style, the authors show how mathematics can improve your understanding of sudoku. But more importantly, they demonstrate how sudoku can also improve your understanding of mathematics.

Ross, Ken. *A Mathematician at the Ballpark: Odds and Probabilities for Baseball Fans.* New York: Pearson Education, 2004. The game of baseball is full of numbers. This book, written by a past president of the Mathematical Association of America, will help you understand the game better.

Rubens, Jeff. *Expert Bridge Simplified: Arithmetic Shortcuts for Declarer.* New York: Bridge World Books, 2009. The author is a professor of mathematics, a bridge expert, and an editor. The book explains how to calculate the odds of making various plays in bridge. The book assumes a bit of bridge experience on the part of the reader.

Schlesinger, Don. *Blackjack Attack: Playing the Pros' Way.* 2nd ed. Oakland, CA: RGE Publishing, 2000. A valuable collection of articles for the practical card counter.

Seirawan, Yasser, and Jeremy Silman. *Play Winning Chess.* London: Everyman Chess Series, 2003. This book will take you from beginner to intermediate level with an emphasis on understanding the concepts of force, time, space, and pawn structure.

Sklansky, David. *The Theory of Poker*. 4th ed. Las Vegas: Two Plus Two Publishing, 2001. One of the first, and still one of the best, books for describing the mathematics that underlies many versions of poker, including Texas Hold'em. The author is considered one of the game's premiere authorities on the theory of poker.

Slocum, Jerry, and Dic Sonneveld. *The 15 Puzzle: How It Drove the World Crazy.* Beverly Hills, CA: Slocum Puzzle Foundation, 2006. Everything you ever wanted to know about the history and mathematics of this popular puzzle. Beautifully illustrated.

Slocum, Jerry, David Singmaster, Wei-Hwa Huang, Dieter Gebhardt, and Geert Hellings. *The Cube: The Ultimate Guide to the World's Bestselling Puzzle.* New York: Black Dog & Leventhal, 2009. Learn to solve and understand the standard Rubik's Cube and many of its variations. Very modern, with colorful illustrations.

Snyder, Arnold. *Blackbelt in Blackjack: Playing 21 as a Martial Art.* Oakland, CA: RGE Publishing, 1998. The editor of the respected journal *Blackjack Forum* gives the ins and outs of card counting (the hi-lo count and variations) along with shuffle tracking and team play.

Snyder, Thomas. *The Art of Sudoku.* San Francisco: Grandmaster Puzzles, 2012. Sudoku puzzles with an elegant artistic quality to them with increasing levels of difficulty. Created by a three-time world sudoku champion.

Stephens, Paul. *Mastering Sudoku Week by Week: 52 Steps to Becoming a Sudoku Wizard.* London: Duncan Baird Publishers, 2007. This is Professor Benjamin's recommended book for learning sudoku strategy, focusing on the thought process involved in solving challenging puzzles and practical tactics.

Stewart, Ian. *How to Cut a Cake and Other Mathematical Conundrums.* Oxford: Oxford University Press, 2006. A fun collection of recreational mathematics articles, including some on the topic of fair division. The author has a reputation for clear mathematical writing with considerable wit.

Stuart, Andrew C. *The Logic of Sudoku.* Somerset, UK: Michael Mepham, 2007. This book provides a rather comprehensive description of sudoku strategies, from beginner to very advanced. It is interesting from a mathematical perspective, but maybe not as practical as some of the other books for improving your puzzle-solving time.

Thorp, Edward O. *Beat the Dealer: A Winning Strategy for the Game of Twenty-One.* New York: Vintage Books, 1966. This classic book brought card counting to the masses and made Las Vegas change the rules.

Trice, Walter. *Backgammon Boot Camp.* San Francisco: The Fortuitous Press, 2004. After reading Magriel's book, this should be next on your list. Written by one of the most mathematically talented players of the game.

Vancura, Olaf. *Advantage Yahtzee.* Las Vegas, NV: Huntington Press, 2001. If you ever want to learn the optimal strategy for playing this dice game, then this short book is all you need. The results are based on dynamic programming.

Vancura, Olaf, and Ken Fuchs. *Knock-Out Blackjack: The Easiest Card-Counting System Ever Devised.* Las Vegas, NV: Huntington Press, 1998. This is an "unbalanced count" that tells the player exactly when it is right to take insurance. It's a very practical strategy and is highly recommended.

Von Neumann, John, and Oskar Morgenstern. *Theory of Games and Economic Behavior.* Princeton, NJ: Princeton University Press, 1944. This is the classic book from which modern game theory is derived.

Watkins, John J. *Across the Board: The Mathematics of Chessboard Problems.* Princeton, NJ: Princeton University Press, 2004. A beautifully written book with mathematical problems that come from the chessboard and its pieces. It's not about improving your game of chess, but it may increase your enjoyment of mathematics.

Wells, David. *Book of Curious and Interesting Puzzles.* New York: Dover Publications, 1992. Reprinted in 2006. A collection of classic puzzles from ancient times to the 20th century.

Winkler, Peter. *Bridge at the Enigma Club.* Toronto: Master Point Press, 2010. The author is a distinguished mathematician, computer scientist, cryptographer, and puzzle enthusiast who describes a "zero-knowledge" bidding system that communicates information to your partner without revealing any information to the opposition.

Winston, Wayne L. *Mathletics: How Gamblers, Managers, and Sports Enthusiasts Use Mathematics in Baseball, Basketball, and Football.* Princeton, NJ: Princeton University Press, 2009. The author describes methods that coaches and managers use (or should use) to evaluate players and improve performance. The articles on sports betting are interesting as well.

Woolsey, Kit. *How to Play Tournament Backgammon.* Arlington, MA: Gammon Press, 1993. A short but comprehensive treatment of how the score of your match affects how you play your checkers and use the doubling cube. The author is one of the top players of backgammon and bridge.

Woolsey, Kit, and Patti Beadles. *52 Great Backgammon Tips*. London: Batsford, 2007. A great book for improving your game, written by one of the best players in the world and one of his students.