From Fuzzy Logic to Fuzzy Mathematics

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Preface

This work contains contributions I have made to the *mathematical fuzzy logic* during my PhD study. Although I was interested in many topics during this period, two main ones have emerged quite recently.

The first one is a "universalistic" approach towards the variety of fuzzy logics studied in the literature. The main imperative of this approach is to focus not on the particular logics but rather on their classes. The reader should be aware that I use the term *fuzzy logic* in rather narrow sense (see Chapter 1 for more details) and thus the term "universal" is to be understood as restricted to my particular setting. I was inspired by existing (usually more general) approaches described in the literature, mainly by the so-called *Abstract Algebraic Logic* (AAL). I tried to isolate the essential elements of these approaches and alter their methods to suit better the existing variety of fuzzy logics. This resulted in the definition of the class of the *weakly implicative fuzzy logics*, which is, in my view, a very good formalization of the pre-theoretic notion of a fuzzy logic as a logic of "comparative degrees of truth". I was able to prove some interesting general theorems and apply them to obtain not only known results about existing fuzzy logic, but also new ones. Chapter 2 is dedicated to elaborating those general methods and Chapter 3 develops existing fuzzy logics in the proposed general framework (to the extent needed for the subsequent chapters, the thorough development is to be subject of my future work). These results are so far described only in my technical report [25]. However, my paper about this topic is nearly completed and will be submitted soon. There is also a submitted joint paper with Libor Běhounek [8] concentrating on *philosophical, methodological, and pragmatical* reasons for using the term *weakly implicative fuzzy logics* as a *formal* delimitation of the existing *informal* class of fuzzy logic.

The second main issue can be described as a *syntactic turn* in fuzzy logic and fuzzy mathematics especially. This work is a joint project with Libor Běhounek. The main idea of this approach results from our belief that the formal mathematical fuzzy logic is developed enough to support the development of some formal foundations of the existing fuzzy mathematics and can be expressed by the following imperative: work in a formal axiomatic theory over a fuzzy logic, rather than investigate particular models. We believe that following this imperative can shelter us from the common problems of existing fuzzy mathematics, i.e., ad hoc definitions of fuzzy concepts, which often suffer from arbitrariness and hidden crispness, or even references to particular crisp models of fuzziness (e.g. membership functions). We propose one particular logical system to serve as the foundations of fuzzy mathematics, namely the Higher-order Fuzzy Logic. Formulation of this theory and some examples of its suitability for the sketched task are to be found in Chapter 7 (it is based on our paper [6]) and in the last Chapter the reader can find the so-called Methodological Manifesto which is based on our submitted paper [5].

These two issues go far beyond the scope of this thesis and are subjects of the two proposed research programs. Here I present only the starting points. However, they nicely delimit my whole work and this thesis in particular and give it its name: *From fuzzy logic to fuzzy mathematics.*

The rest of this thesis is dedicated to my particular results in mathematical fuzzy logic. Chapter 4 is based on the joint work with Rostislav Horčík. Here we developed the notion of the so-called Product Łukasiewicz logic (roughly speaking, we add one additional connective into the Łukasiewicz logic). This chapter is derived from the paper [27] however it was completely rewritten because the general results proved in Chapters 2 and 3 simplified this paper greatly. Later, Chapter 5 deals with the so-called Product logic and has two main parts. The first one is based on my paper [20] and contains results about possible axiomatic systems of this logic. The second part is based on a joint paper with Brunella Gerla [26] and deals with the notion of (semi)normal form of formulae of this logic and with its functional representation. Finally, in Chapter 6 we examine the notion of compactness in fuzzy logic. Again, this chapter consists of two main parts. The first one is based on my submitted paper [19] and develops two different notions of compactness in the so-called Gödel logics. The second one is based on a joint paper with Mirko Navara [28] and extends the results on one of the notions of compactness defined in the first part to the broader family of fuzzy logics.

There is another big research topic of mine—the Ł$\Pi\frac{1}{2}$ logic (see my papers [17, 21, 23, 24]). However, the majority of results from those papers was included already in my master thesis ([22]) and there is a survey paper (in preparation) by Esteva, Godo, Montagna, and myself about Ł$\Pi\frac{1}{2}$ logic. Thus I decided not to deal with this topic in this thesis.

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Now I would like to thank all the people who helped me to create this thesis. The linearity of the written text forces me to do it in some fixed order and although contribution of some particularly persons are indeed greater than of some others, I value all of them greatly and list them without order.

There are only the exceptions of this rule: Petra and Petr. *Petr Hájek*, my supervisor and good friend helped me greatly during my study with his comments and suggestions and he directed my scientific interest. *Petra Ivaničová*, my girlfriend, helped me very much during the long process of writing this thesis simply by being here, with me.

Then I would like to thank all my coauthors, namely: Libor Běhounek, Brunella Gerla, Rostislav Horčík, and Mirko Navara. I also need to credit all the friends and colleagues who read parts of this thesis and help improve them greatly, especially Carles Noguera, Josep Maria Font, Libor Běhounek, Petr Hájek, and Zuzana Haniková. I would especially like to thank to Mirko Navara for a very careful reading of some of my papers and helping me very much to improve my stylistical and writing skills. Special thanks also go to Libor Běhounek, first for his efforts in creating a huge project in formalization of fuzzy mathematics (although my original aims were more modest), second for helping me to create the notion of weakly implicative logics in the current general form (by his constant "Do you really need to assume ..."), and third for his belief (stronger than mine) that the weakly implicative fuzzy logics are *the* fuzzy logics. I would also like to thank to all the colleagues from the Institute of Computer Science for creating nice working environment. At last, but definitely not at least, I would like to thank my parents for their everlasting support in my life and work.

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Contents

Chapter 1

From Fuzzy Logic . . .

Vagueness pervades human language, perception, and reasoning since the beginning of its existence. Classical mathematics and logic can, however, model the concept of vagueness only indirectly. Although many natural-language predicates (young, tall, hot) show natural degrees of truth, all notions of classical logic and mathematics are bivalent. Even though many-valued logics were developed (for other purposes) already during the first half of the XX century (Łukasiewicz [62], Gödel [42]), a systematic study of vagueness by means of the many-valued approach began only after L.A. Zadeh [85] proposed to investigate fuzzy sets in 1965. Since then, the notion of fuzziness spread to nearly all mathematical disciplines: fuzzy arithmetic, fuzzy logic, fuzzy probability, fuzzy relations, fuzzy topology, etc.

For a long time, however, fuzzy logic and fuzzy mathematics were engineering tools rather than a well-designed mathematical theories. Driven just by applications, they lacked (meta)theoretical grounding and general results; developed mostly by engineers for particular purposes, it suffered from arbitrariness in definitions and often even mathematical imprecision. Moreover, it could be objected that it was just a theory of [0,1]-valued functions and thus a part of real analysis. There have been many (more or less successful) attempts to formalize or even axiomatize some areas of fuzzy mathematics. However, these axiomatic systems are designed ad hoc. The authors usually select some concepts in their area of interest and change them into vague ones. This selection is usually based solely on intuition or on the desired application. Another problem with these axiomatic attempts lies in their fragmentation; it is nearly impossible to combine two of them into one theory.

The great success, not suffering from the above-mentioned defects, is without doubt the area of mathematical fuzzy logic developed by Gottwald [43], Hájek [44], Novák at al. [75], Mundici at al. [12], and others. Equipping a variety of fuzzy logics with algebraic semantics and Hilbert-style calculi made them a legitimate part of the long tradition of many-valued logics. Both propositional and first-order logics were defined and developed. With [44], this effort has been completed to the point in which predicate fuzzy logics can serve as ground theories for fuzzy mathematics, i.e., for constructing axiomatic theories over these logics, aimed to describe fuzzy arithmetic, fuzzy geometry, fuzzy analysis, etc.

However, there is one small problem: the multitude of existing fuzzy logics. It is not a big problem, it can be easily shown that this multitude is not only desirable but also indispensable for creating general theories of vagueness phenomena (see Chapter 8 for more details). The problem is rather methodological and pragmatical. The methodological nature of the problem can be expressed by the question: "What is fuzzy logic?" and the pragmatical one with the question: "Is there any *order or system* in the class of fuzzy logics?"

The first question is practically unanswerable, mainly because it entails a question "What is logic?". In order to get *any* answer we need to make some "design" choices (and forget about philosophy) to answer the "What is logic?" question. First of all we restrict to the propositional level (in the beginning, later we move to the first-(and even higher)-order logics) and we retain the classical syntax (i.e., the formulae are created inductively from the set of atoms and the set of propositional connectives). In this work we understand the logic as an **asymmetric consequence relation** (following Dunn and Hardegree's terminology, see [30]), i.e., a relation between sets of formulae and formulae closed under arbitrary substitution. This is quite a departure from the common practice in the existing fuzzy logic literature, where logic is usually seen as a *unary consequence relation*, i.e., just a set of formulae (theorems/tautologies). As the reader will see this forces us to make some terminological changes, however the author believes that the transition to asymmetric consequence relation is necessary to increase generality of results and to cover a much broader class of logics.

Thus the task of the day is to look at the class of logics we described in the previous paragraph and decide which of them should be called fuzzy and which not. Before we proceed with answering this question we stress one thing. Many authors believe that *the* fuzzy logic should be a logic directly dealing with some aspects of vagueness, such as vague *predicates*, linguistic expressions, etc. Fuzzy logic should, according to their opinion, be able to (at least partially) describe the process of reasoning under vagueness, formalize the notion of fuzzy control, etc. It is obvious that these tasks cannot by fulfilled by any propositional logic. Therefore, we believe that there is no such thing as *the* fuzzy logic and that the fuzzy logics solving those tasks are numerous and that they are based on some appropriate propositional logics. Thus we think that it is perfectly reasonable to speak about fuzzy logics among the class of propositional logics (we mentioned before).

In order to answer that "What is fuzzy logic?" question we need one more restriction on the notion of logic: we restrict ourselves to the class of the so-called *weakly implicative logics* (a generalization of Rasiowa's well-known notion of implicative logics, see [77]). A logic is weakly implicative iff its language contains a (definable) connective $\to$ that satisfies the following conditions:

$$\begin{array}{rcll}
 & \vdash & \varphi \to \varphi & \\
\varphi, \varphi \to \psi & \vdash & \psi & \\
\varphi \to \psi, \psi \to \chi & \vdash & \varphi \to \chi & \\
\varphi \to \psi, \psi \to \varphi & \vdash & c(\dots, \varphi, \dots) \to c(\dots, \psi, \dots) & \text{for all connectives } c
\end{array}$$

Weakly implicative logics can be characterized as those which are complete w.r.t. a class of *ordered matrices* (in which the set D of designated values is upper), if the ordering of the elements of the matrix is defined as

$$x \leq y \ \equiv_{\mathrm{df}} \ x \to y \in D$$

From this class we select the class of *weakly implicative fuzzy logics* as the class of those weakly implicative logics which are complete w.r.t. *linearly* ordered matrices. It turns out that this class approximates well the bunch of logics studied in the so-called "fuzzy logic in narrow sense".

As far as the author knows there is no weakly implicative logic studied as fuzzy logic in the literature that is not fuzzy in our formal setting. We need to recall that we are selecting fuzzy logics from the class of weakly implicative logics only, we do not say anything about those outside this class! However, for the logics that can be interpreted in this class, our criterion works well (logics with evaluated syntax come out fuzzy, most substructural logics do not).

On the other hand there are fuzzy logics in our setting, which are not considered fuzzy by some researchers. One example is the Relevance RM, which comes out fuzzy. We think that such logics should be regarded and studied as fuzzy too, because they have a natural interpretation of comparative degrees of truth. We can say that they are both relevant and fuzzy (similarly, Gödel logic is both intermediary and fuzzy). Other group of "problematic" fuzzy logics is constituted by logics lacking some structural rule, namely exchange or weakening. For example, some people claim that there are no natural real-life examples involving non-commutative conjunction (except perhaps temporal, but such phenomena should, they claim, be handled by temporal rather than fuzzy logics). We notice that most methods of metamathematics of fuzzy logic work with non-commutative linear matrices as well, so we think that it is better to be as general as possible, especially if we realize that the lacking motivation can be found later. We also think that the extra assumption would require an extra explanation (the proof burden is on the proponent of extra assumptions). Final objection is aimed at the fact that classical logic comes out fuzzy too. In our opinion, fuzzy logic is a generalization of classical logic (to encompass *also* vague predicates). Thus classical logic can be viewed as a fuzzy logic for those special occasions when *by chance* the realizations of all predicates are crisp.

To demonstrate the usefulness of this definition we show that it is equivalent to some important properties studied in fuzzy literature. For finitary logics, the following are equivalent:

- **L** is weakly implicative fuzzy logic.
- Each **L**-matrix is a subdirect product of linear ones. *(Subdirect representation property)*
- Each theory in **L** can be extended to one whose Lindenbaum-Tarski matrix is linear. *(Linear extension property)*
- $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$ entails $T \vdash \chi$. *(Prelinearity property)*

There are objections that some vague notions have incomparable degrees, because they have more components (e.g., human intelligence) and that fuzzy logic should be able to deal with this. It is necessary to observe that linear matrices are not the only matrices for our fuzzy logics, a particular algebra for a real-life application can be non-linear. However, the Subdirect representation property tells us that each such an algebra can be decomposed into linearly ordered ones (in the case of human intelligence we can decompose it to memory, concentration, creativity, etc.).

The second question, "Is there any *order* in the class of fuzzy logic?" is much easier. The answer is: "not much, so far". Of course, there are numerous results about relations among some particular fuzzy logics (stronger/weaker, conservative extension, fragment etc.). There are also results relating fuzzy logic with some non-fuzzy ones (see [34]). However, we believe that only the formal definition of the notion of fuzzy logic can lead to the systematic development of this class. The following theorem, together with the fact that the class of fuzzy logics is defined formally, allows us to speak about the weakest fuzzy logic with some properties:

- The intersection of an arbitrary system of fuzzy logics is a fuzzy logic.

This allows us to extend the known result: MTL (Esteva and Godo's Monoidal T-norm Fuzzy Logic) is an extension of $\mathrm{FL_{ew}}$ (Ono's Full Lambek Calculus with Exchange and Weakening) to the following one: MTL is *the weakest fuzzy logic* extending $\mathrm{FL_{ew}}$. In fact, for each logic we can assign the weakest fuzzy logic extending it. We present some examples:

Logic	**Weakest *fuzzy* logic**	**References**
Intuitionistic	Gödel-Dummett	[44]
$\mathrm{FL_w}$	psMTL^r	[48]
$\mathrm{FL_{ew}}$	MTL	[33]
AMALL^-	IMTL	[33]

This turns out to be an important methodological guideline. For example there are no fuzzy logics without weakening defined in the literature (although some have been recently developed by George Metcalfe and his colleagues). The weakest *fuzzy* logic over $\mathrm{FL_e}$ seems to be a good starting point for this research.

We give another example demonstrating the power of our approach. It is quite common to introduce for fuzzy logics their extensions by a unary connective $\triangle$ (usually called Baaz delta). These extensions have some very interesting properties. Some of them entail that $\triangle$ is a "globalization" connective (as known from intuitionistic logic), and thus can be viewed as a kind of "exponential". However, Baaz delta in fuzzy logics has also some additional properties. We observe the constituting property of these extensions and define:

- We say that $\mathbf{L}$ is a logic with *Baaz delta* iff for each formula in the language $\{\rightarrow, \bot, \top\}$ and for substitution $\sigma(v) = \triangle v$ it holds: $\vdash_{\mathbf{L}} \sigma\varphi$ iff φ is a theorem of classical logic.

For each *fuzzy* logic we define its "Delta companion" as the weakest *fuzzy* logic with Baaz delta extending it. We present a uniform way how to construct an axiomatic system for the Delta companion of each logic. We show that this general definition fits well with the usual usage (i.e., the existing extensions of fuzzy logics by Baaz delta coincide with our Delta companions). Then we prove an interesting interplay between the *standard* completeness of a fuzzy logic and its Delta companion.

In upcoming Chapter 2 we elaborate these ideas in more details and the author hopes that Chapter 3, where the known results about existing fuzzy logics are presented in the new formalism, demonstrates the power and usefulness of the formal definition of the class of fuzzy logics and general theorems resulting from this definition. We hope that both classes of weakly implicative and weakly implicative fuzzy logics represent reasonable compromise between generality (there are far more general classes of logic) and usefulness of their results for particular logics. This is true especially for *fuzzy* logic, where the more general approaches omit the characteristic properties of existing fuzzy logics.

Chapter 2

Weakly implicative (fuzzy) logics

2.1 General theory for propositional logics

We start with some obligatory introductory definitions. However, we will use rather non-standard approach towards propositional logic. Of course, it turns out to be equivalent (or nearly equivalent) to the known ones, but here we put stress on some issues, concerning mainly the notion of proof and meta-rule. For the comprehensive survey (with the standard terminology) into the problematic of general approach towards logic, consequence relation, logical matrices, etc. see the survey to Abstract algebraic logic (AAL) [40].

In the first subsection we introduce *weakly implicative logics*. Then, in the second subsection, we present the semantic for them (using the well-known notion of logical matrix). The third subsection is the core part of this work. There we introduce the class of fuzzy logic (or better: weakly implicative fuzzy logics).

2.1.1 Syntax

Definition 2.1.1 *A* propositional language $\mathcal{L}$ *is a triple* $(\mathbf{VAR}, \mathbf{C}, \mathbf{a})$*, where* $\mathbf{VAR}$ *is a non-empty set of the (propositional)* variables*,* $\mathbf{C}$ *is a nonempty set of the (propositional)* connectives*, and* $\mathbf{a}$ *is a function assigning to each element of* C *a natural number. A connective* c *for which* $a(c) = 0$ *is called a truth constant.*

The set $\mathbf{VAR}$ is usually taken as fixed countable set, and so we usually define the propositional language $\mathcal{L}$ is a pair $(\mathbf{C}, \mathbf{a})$. Later on we fix symbols for some basic connectives $(\to, \wedge, \bot, \dots)$ together with their arities and then we define propositional language just as a set of these connectives. Sometimes we write $(c, 2) \in \mathcal{L}$ instead of $c \in \mathbf{C}$ and $\mathbf{a}(c) = 2$.

Definition 2.1.2 (Formula) *Let* $\mathcal{L}$ *be a propositional language. The set of (propositional)* formulae $\mathbf{FOR}_{\mathcal{L}}$ *is the smallest set which contains the set* $\mathbf{VAR}$ *and is closed under connectives from* $\mathbf{C}$*, i.e., for each* $c \in \mathbf{C}$*, such that* $\mathbf{a}(c) = n$*, and for each* $\varphi_1, \dots, \varphi_n \in \mathbf{FOR}_{\mathcal{L}}$ *we have* $c(\varphi_1, \dots, \varphi_n) \in \mathbf{FOR}_{\mathcal{L}}$*.*

Definition 2.1.3 (Substitution) *Let* $\mathcal{L}$ *be a propositional language. A* substitution *is a mapping* $\sigma : \mathbf{FOR}_{\mathcal{L}} \to \mathbf{FOR}_{\mathcal{L}}$*, such that* $\sigma c(\varphi_1, \dots, \varphi_n) = c(\sigma(\varphi_1), \dots \sigma(\varphi_n))$*. The set of all substitutions will be denoted as* $\mathbf{SUB}_{\mathcal{L}}$*.*

Of course a substitution is fully determined by its values on propositional variables. Let $v \in \mathbf{VAR}$ and $\varphi, \psi \in \mathbf{FOR}_{\mathcal{L}}$, by $\psi[v := \varphi]$ we understand the formula $\sigma(\psi)$, where σ is the substitution mapping v to φ and mapping all the remaining propositional variables to itself.

The term *consecution* in the following definition is from the Restall's book [78]. However, we use it in a very simplified version.

Definition 2.1.4 (Consecution) *A* consecution *in the propositional language $\mathcal{L}$ is a pair $<X, \varphi>$, where $X \subseteq \mathbf{FOR}_{\mathcal{L}}$ and $\varphi \in \mathbf{FOR}_{\mathcal{L}}$. The set of all consecutions will be denoted as $\mathcal{CON}_{\mathcal{L}}$.*

Of course we have $\mathcal{P}(\mathbf{FOR}_{\mathcal{L}}) \times \mathbf{FOR}_{\mathcal{L}} = \mathcal{CON}_{\mathcal{L}}$.

Convenction 2.1.5 *Let $X \subseteq \mathbf{FOR}_{\mathcal{L}}$, $\mathcal{X} \subseteq \mathcal{CON}_{\mathcal{L}}$ and σ a substitution.*

- *By $\sigma(X)$ we understand the set $\{\sigma(\varphi) \mid \varphi \in X\}$*
- *By $\sigma(\mathcal{X})$ we understand the set $\{<\sigma(X), \sigma(\varphi)> \mid <X, \varphi> \in \mathcal{X}\}$.*
- *By $\mathcal{SUB}_{\mathcal{L}}(\mathcal{X})$ we denote the set $\bigcup_{\sigma \in \mathbf{SUB}_{\mathcal{L}}} \sigma(\mathcal{X})$.*

Definition 2.1.6 (Axiomatic system) *Let $\mathcal{L}$ be a propositional language. The* axiomatic system *$\mathcal{AS}$ in language $\mathcal{L}$ is a non-empty set $\mathcal{AS} \subseteq \mathcal{CON}_{\mathcal{L}}$, which is closed under arbitrary substitution (i.e., $\mathcal{SUB}_{\mathcal{L}}(\mathcal{AS}) = \mathcal{AS}$).*

The elements of $\mathcal{AS}$ of the form $<X, \varphi> \in \mathcal{AS}$ are called axioms *for $X = \emptyset$,* n-ary deduction rules *for $|X| = n$, and* infinitary deduction rules *for X being of infinite.*

The axiomatic system is said to be finite *if there is a finite set $\mathcal{X} \subseteq \mathcal{AS}$ such that $\mathcal{SUB}_{\mathcal{L}}(\mathcal{X}) = \mathcal{AS}$. Furthermore, the axiomatic system is said to be* finitary *if all its deduction rules are finite. Finally, the axiomatic system is said to be* strongly finite *if it is finite and finitary.*

The usual way of presenting an axiomatic system is in form of schemata.

Definition 2.1.7 (Consequence) *Let $\mathcal{L}$ be a propositional language and $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$.* Theory *T in $\mathcal{L}$ is a subset of $\mathbf{FOR}_{\mathcal{L}}$. The set $CNS_{\mathcal{AS}}(T)$ of all* provable formulae *in T is the smallest set of formulae, which contains T, axioms of $\mathcal{AS}$ and is closed under all deduction rules of $\mathcal{AS}$ (i.e., $X \subseteq CNS_{\mathcal{AS}}(T)$ and $<X, \varphi> \in \mathcal{AS}$ entails that $\varphi \in CNS_{\mathcal{AS}}(T)$). We shall write $T \vdash_{\mathcal{AS}} \varphi$ to denote $\varphi \in CNS_{\mathcal{AS}}(T)$ and $\vdash_{\mathcal{AS}} \varphi$ to denote $\varphi \in CNS_{\mathcal{AS}}(\emptyset)$.*

Notice that the relation $\vdash_{\mathcal{AS}}$ can be understood as a subset of $\mathcal{CON}_{\mathcal{L}}$ and $\mathcal{AS} \subseteq \vdash_{\mathcal{AS}}$.

Definition 2.1.8 (Logic) *Let $\mathcal{L}$ be a propositional language. A non-empty set $\mathbf{L} \subseteq \mathcal{CON}_{\mathcal{L}}$ is called a* logic *in language $\mathcal{L}$ if it is closed under arbitrary substitution and $\vdash_{\mathbf{L}} = \mathbf{L}$.*

Logic is a consequence relation in the usual sense. The elements of a logic are consecutions and we write $X \vdash_{\mathbf{L}} \varphi$ instead of $<X, \varphi> \in \mathbf{L}$. Sometimes when the logic $\mathbf{L}$ is clear from the context, we write just $\vdash$ instead of $\vdash_{\mathbf{L}}$. Observe that $\vdash_{\mathcal{AS}}$ is the smallest logic containing $\mathcal{AS}$.

Definition 2.1.9 (Presentation) *Let $\mathcal{L}$ be a propositional language, $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$, and $\mathbf{L}$ a logic in $\mathcal{L}$. We say that $\mathcal{AS}$ is an* axiomatic system *for (a* presentation *of) the logic $\mathbf{L}$ iff $\mathbf{L} = \vdash_{\mathcal{AS}}$. We denote the set $CNS_{\mathcal{AS}}(\emptyset)$ as $\mathcal{THM}(\mathbf{L})$. Logic is said to be finite (finitary, strongly finite) if it has some finite (finitary, strongly finite) presentation.*

Later on we show that our notion of finitary logic coincides with the usual one. Observe that each logic has at least one presentation.

Definition 2.1.10 (Proof) *Let $\mathcal{L}$ be a propositional language and $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$. A* proof *of the formula φ in theory T is a founded tree labelled by formulae; the root is labelled by φ and leaves by either axioms or elements of T; and if a node is labelled by ψ and its preceding nodes are labelled by $\psi_1, \psi_2, \dots$ then $<\{\psi_1, \psi_2, \dots\}, \psi> \in \mathcal{AS}$. We shall write $T \vdash^p_{\mathcal{AS}} \varphi$ if there is a proof of φ in T.*

We understand the tree in an top-to-bottom fashion: the leaves are at the top and the root is at the bottom of the tree, so the fact that tree is founded just means that there is no infinitely long branch.

Theorem 2.1.11 *Let $\mathcal{L}$ be a propositional language and $\mathcal{AS}$ an axiomatic system in $\mathcal{L}$. Then $T \vdash_{\mathcal{AS}} \varphi$ iff $T \vdash^p_{\mathcal{AS}} \varphi$.*

Proof: Let us define the set $CNS_p(T) = \{\varphi \mid T \vdash^p_{\mathcal{AS}} \varphi\}$. If we show that $CNS_{\mathcal{AS}}(T) = CNS^p_{\mathcal{AS}}(T)$ for each T the proof is done. Obviously $CNS^p_{\mathcal{AS}}(T)$ contains T, axioms of $\mathcal{AS}$ and is closed under all deduction rules of $\mathcal{AS}$, thus $CNS_{\mathcal{AS}}(T) \subseteq CNS^p_{\mathcal{AS}}(T)$. Reverse direction is trivial using the induction over well-founded relation. QED

Lemma 2.1.12 (Finitary logic) *Let* **L** *be a logic. Then* **L** *is finitary iff for each theory T and formula φ we have: if $T \vdash \varphi$ then there is finite $T' \subseteq T$ such that $T' \vdash \varphi$.*

Proof: Then **L** is finitary then there is its finitary presentation $\mathcal{AS}$. Observe that for each finitary $\mathcal{AS}$ the proofs are always finite (because the tree has no infinite branch and because $\mathcal{AS}$ is finitary each node has finitely many preceding nodes and so we can use König's Lemma to get that the tree is finite). The reverse direction is almost straightforward. QED

Observe that in finitary case we can linearize the tree, i.e., define the notion of the proof in the usual way. The notion of proof allows us to illustrate one important (and usually overlooked) feature of our way of introducing logic. The deduction rule $<\{\psi_1, \psi_2, \dots\}, \varphi>$, or better written as $\psi_1, \psi_2, \dots \vdash \varphi$ gives us a way how to construct proof of φ using proofs of $\psi_1, \psi_2, \dots$. However, the *meta-rule*: from $\vdash \psi_1$, $\vdash \psi_2, \dots$ get $\vdash \varphi$ only tells us that if there are proof of $\psi_1, \psi_2, \dots$, then there is proof of φ, without any hint how to construct it. When we introduce semantics, we will se that the former corresponds to the so-called *local* and the latter to the *global* consequence. This distinction deserves more treatment. Also *rules* are rules of inference between *formulae*, whereas *meta-rules* are rules of inference between *consecutions*. Both our definitions hide (in some sense) the *default* meta-rules of consequence (thinning, permutation, contraction, and cut).

Definition 2.1.13 *A logic* **L** *is an* extension of a logic **L**′ *iff $L' \subseteq L$. The extension is* axiomatic *if the logic* **L** *is axiomatized by logic* **L**′ *(understood as an axiomatic system) with some additional* axioms. *The extension is* conservative *if for each theory T and formula φ in the language of* **L**′ *we have: $T \vdash_{\mathbf{L}} \varphi$ entails $T \vdash_{\mathbf{L}'} \varphi$.*

Notice that the above definition does not mention the language of the logic in question. However, it is obvious that if **L** is an extension of the logic **L**′ then the language of **L** has to be larger than the language of **L**′. If the language is strictly larger we speak about *expansion* rather than about *extension*.

Now we define the crucial concept of this paper: the notion of weakly implicative logic. We assume that there is a binary connective $\rightarrow$ in the propositional language. There is an obvious generalization of this concept, if we drop the condition of the presence of the connective $\rightarrow$ in language, and following AAL tradition we would understand the term $\varphi \rightarrow \psi$ as a set of formulae of two propositional variables, where the formula φ is substituted for the first one and the formula ψ for the second one. Observe that some of theorems proven in this paper remain theorems under this interpretation. It is easy to see, that in this more general approach (which we will not pursuit in this paper) the defined class of logics would be a subclass of the so-called *equivalential logics*, and in the more specific approach (we use in this paper) the class of logics defined by the following definition is a subclass of the so-called *finitely equivalential logics*. If fact weakly implicative logics are precisely the finitely equivalential logics with the *set of equivalence formulae* $E = \{p \rightarrow q, q \rightarrow p\}$. The name *weakly implicative logics* is inspired by the notion of *implicative logics* by Helena Rasiowa (see [77]). As we will see our notion is really a generalization of her notion (her only additional assumption is $\varphi \vdash \psi \rightarrow \varphi$).

Definition 2.1.14 (Weakly implicative logics) *Let $\mathcal{L}$ be a propositional language, such that $(\to, 2) \in \mathcal{L}$ and let $\mathbf{L}$ be a logic in $\mathcal{L}$. We say that $\mathbf{L}$ is a* weakly implicative *logic iff the following consecutions are elements of $\mathbf{L}$:*

(Ref)	$\vdash_{\mathbf{L}} \varphi \to \varphi$,
(MP)	$\varphi, \varphi \to \psi \vdash_{\mathbf{L}} \psi$,
(WT)	$\varphi \to \psi, \psi \to \chi \vdash_{\mathbf{L}} \varphi \to \chi$,
(Cng_c^i)	$\varphi \to \psi, \psi \to \varphi \vdash_{\mathbf{L}} c(\chi_1, \dots, \chi_{i-1}, \varphi, \dots, \chi_n) \to c(\chi_1, \dots, \chi_{i-1}, \psi, \dots, \chi_n)$ *for each $(c, n) \in \mathcal{L}$ and each $i \leq n$.*

The (*Ref*) is for *reflexivity*, (MP) is for *modus ponens*, (WT) is for *weak transitivity*, and (Cng) is for *congruence*. Let $(c, n) \in \mathcal{L})$, then by (Cng_c) we understand the set of consecutions $\{(\mathrm{Cng}_c^i) \mid i \leq n\}$. Furthermore, by ($\mathrm{Cng}_{\mathcal{L}}$) we understand the set of consecutions $\bigcup_{c \in \mathcal{L}} (\mathrm{Cng}_c)$. Let $\varphi \leftrightarrow \psi$ be a shortcut for $\{\varphi \to \psi, \psi \to \varphi\}$. Observe that we assume neither Exchange nor Weakening nor Contraction as a rules for implication. However, we have all of them as meta-rules, i.e., the connective $\to$ is by no means an internalization of $\vdash$. This approach is usually called Hilbert's style calculus, and it is defined in the same fashion as the Hilbert calculi for particular substructural logics. This paper can be seen as a contribution to the general (universal, abstract) theory of these calculi. For another general treatment of this topic see Restall's book [78]. Our approaches are rather incomparable in generality. He has much wider notion of logic, but we develop *our* logics more.

Lemma 2.1.15 (Alternative definition) *The set of consecutions* ($\mathrm{Cng}_{\mathcal{L}}$) *can be equivalently replaced by one of the following consecutions:*

- $\varphi_1 \leftrightarrow \psi_1, \dots \varphi_n \leftrightarrow \psi_n \vdash_{\mathbf{L}} c(\varphi_1, \dots, \varphi_n) \to c(\psi_1, \dots, \psi_n)$ *for each $(c, n) \in \mathcal{L}$.*
- $\varphi \leftrightarrow \psi \vdash_{\mathbf{L}} \chi \to \chi'$ *for each formula $\chi \in \mathbf{FOR}_{\mathcal{L}}$, where χ' is a result of replacing some occurrence of subformula φ with formula ψ in formula χ.*

Proof: We show only the first part, the second one is analogous. Assume (for simplicity) that c is a binary connective, then we have:

1. $\varphi_1 \leftrightarrow \psi_1 \vdash_{\mathbf{L}} c(\varphi_1, \varphi_2) \to c(\psi_1, \varphi_2)$ ((Cng_c^1) for χ_2 being φ_2)
2. $\varphi_2 \leftrightarrow \psi_2 \vdash_{\mathbf{L}} c(\psi_1, \varphi_2) \to c(\psi_1, \psi_2)$ ((Cng_c^2) for χ_1 being ψ_1)
3. $\varphi_1 \leftrightarrow \psi_1, \varphi_1 \leftrightarrow \psi_2 \vdash_{\mathbf{L}} c(\varphi_1, \varphi_2) \to c(\psi_1, \psi_2)$ (1., 2., and (WT))

The other direction is trivial (just take $\psi_2 = \varphi_2$ and use (Ref)). QED

Observe that the second part of this lemma can be understood as a *substitution* rule and thus we will use it heavily in the formal proofs in this paper. Now we list several lemmata with rather trivial proofs.

Lemma 2.1.16 *Let $\mathbf{L}$ be a weakly implicative logic in language $\mathcal{L}$ and $\mathbf{L}'$ a logic in language $\mathcal{L}'$, which is an extension of $\mathbf{L}$. Then $\mathbf{L}'$ is weakly implicative logic iff for each n-ary connective c in $\mathcal{L}' \setminus \mathcal{L}$ and each $i \leq n$ we have $\varphi \to \psi, \psi \to \varphi \vdash_{\mathbf{L}'} c(\chi_1, \dots, \chi_{i-1}, \varphi, \dots, \chi_n) \to c(\chi_1, \dots, \chi_{i-1}, \psi, \dots, \chi_n)$.*

Corollary 2.1.17 (Extension) *An arbitrary extension of an arbitrary weakly implicative logic in the same language is a weakly implicative logic.*

Lemma 2.1.18 (Intersection) *The intersection of an arbitrary system of weakly implicative logics is a weakly implicative logic.*

Definition 2.1.19 (Consistency) *Let $\mathbf{L}$ be a weakly implicative logic in $\mathcal{L}$, T a theory in $\mathcal{L}$. A theory T is* consistent *if there is formula φ such that $T \nvdash \varphi$. A logic $\mathbf{L}$ is consistent iff the theory $\emptyset$ is consistent.*

Observe that a logic $\mathbf{L}$ is consistent iff $\mathbf{L} = \mathcal{CON}_{\mathcal{L}}$, i.e., each consecution is provable. No we introduce the notion of *linear* theory. In the existing fuzzy logic literature the term *complete* theory is usually used. However, we think that complete theory is a different concept (also known as maximal consistent theory). However this notion is crucial in our paper and we want to avoid any potential confusions we decided to pick new neutral name. The reason for choosing the name "linear" will be obvious after Lemma 2.1.32.

Definition 2.1.20 (Linear theory) *Let $\mathbf{L}$ be a weakly implicative logic in $\mathcal{L}$, T a theory in $\mathcal{L}$. A theory T is linear if T is consistent and for each formulae φ, ψ we have $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$.*

At the end of this subsection we introduce the notion of *power* of the implication, to write $\varphi^3 \to \psi$ instead of $\varphi \to (\varphi \to (\varphi \to \psi))$ (for example).

Definition 2.1.21 *Let m be a natural number and φ and ψ formulae. Then the formula $\varphi^m \to \psi$ is defined inductively as: $\varphi^0 \to \psi = \psi$ and $\varphi^{i+1} \to \psi = \varphi \to (\varphi^i \to \psi)$.*

2.1.2 Semantics

We start be recalling some well-known definitions. The completeness theorem for weakly implicative logics (which we prove in this section) is a consequence of some more general theorem known in AAL. However, our concern is not to reprove known facts, we concentrate on the notion of linearity of a logical matrix, which (as far as the author knows) was not so deeply studied.

Definition 2.1.22 (Algebra and matrix) *Let $\mathcal{L}$ be a propositional language. An algebra $\mathrm{A} = (A, \mathbf{C})$ with signature $(\mathbf{C}, \mathbf{a})$ is called $\mathcal{L}$-algebra. Let us denote the realization of c in A as c_{A}. A pair $\mathbf{B} = (\mathrm{A}_{\mathbf{B}}, D_{\mathbf{B}})$, where $\mathrm{A}_{\mathbf{B}}$ is $\mathcal{L}$-algebra and $D_{\mathbf{B}}$ is a subset of A is called* $\mathcal{L}$-matrix.

The elements of the set D are called designated elements. Notice that substitution can be understand as a endomorphism of the absolutely free $\mathcal{L}$-algebra. We shall write $c_{\mathbf{B}}$ instead of $c_{\mathrm{A}_{\mathbf{B}}}$

Definition 2.1.23 (Evaluation) *Let $\mathcal{L}$ be a propositional language and* A *an $\mathcal{L}$-algebra. Then the* A-evaluation *is a mapping e: $\mathbf{FOR}_{\mathcal{L}} \to A$, such that for each $(c, n) \in \mathcal{L}$ and each n-tuple of formulae $\varphi_1, \ldots, \varphi_n$ we have: $e(c(\varphi_1, \ldots, \varphi_n)) = c_{\mathrm{A}}(e(\varphi_1), \ldots e(\varphi_n))$.*

Of course, each A-evaluation is fully determined by its values on propositional variables. We can understand the A-evaluation as a homomorphism from the absolutely free $\mathcal{L}$-algebra to A. Again, we speak about $\mathbf{B}$-evaluation instead of $\mathrm{A}_{\mathbf{B}}$-evaluation.

Definition 2.1.24 (Models) *Let $\mathcal{L}$ be a propositional language, T a theory in $\mathcal{L}$, and $\mathbf{B}$ an $\mathcal{L}$-matrix. We say that $\mathbf{B}$-evaluation is a* $\mathbf{B}$-model *of T if for each $\varphi \in T$ holds $e(\varphi) \in D_{\mathbf{B}}$. We denote the class of $\mathbf{B}$-models of T by $\mathbf{MOD}(T, \mathbf{B})$.*

Definition 2.1.25 (Semantical consequence) *Let $\mathbf{L}$ be a logic in $\mathcal{L}$, T a theory in $\mathcal{L}$, and $\mathcal{K}$ a class of $\mathcal{L}$-matrices. We say that φ is a semantical consequence of the T w.r.t. class $\mathcal{K}$ if $\mathbf{MOD}(T, \mathbf{B}) = \mathbf{MOD}(T \cup \{\varphi\}, \mathbf{B})$ for each $\mathbf{B} \in \mathcal{K}$; we denote it by $T \models_{\mathcal{K}} \varphi$. By $\mathcal{TAUT}(\mathcal{K})$ we understand the set $\{\varphi \mid \emptyset \models_{\mathcal{K}} \varphi\}$.*

Observe that $\models_{\mathcal{K}} \subseteq \mathcal{CON}_{\mathcal{L}}$ and that it is a logic in language $\mathcal{L}$.

Definition 2.1.26 (Soundness and completeness) *We say that the logic $\mathbf{L}$ is* sound *with respect to class $\mathcal{K}$ iff $\models_{\mathcal{K}} \supseteq \mathbf{L}$. We say that the logic $\mathbf{L}$ is* complete *w.r.t. class $\mathcal{K}$ iff $\models_{\mathcal{K}} \subseteq \mathbf{L}$.*

Definition 2.1.27 (L-matrices) *Let* $\mathbf{L}$ *be a logic in* $\mathcal{L}$, T *a theory in* $\mathcal{L}$, *and* $\mathbf{B}$ *an* $\mathcal{L}$*-matrix. We say that* $\mathbf{B}$ *is an* $\mathbf{L}$-matrix *if* $\mathbf{L} \subseteq \models_{\{\mathbf{B}\}}$. *We denote the class of* $\mathbf{L}$*-matrices by* $\mathbf{MAT}(\mathbf{L})$. *Finally, we write* $T \models_{\mathbf{L}} \varphi$ *instead of* $T \models_{\mathbf{MAT}(\mathbf{L})} \varphi$ *and* $\mathcal{TAUT}(\mathbf{L})$ *instead of* $\mathcal{TAUT}(\mathbf{MAT}(\mathbf{L}))$.

Observe that for each presentation $\mathcal{AS}$ of $\mathbf{L}$ holds: $\mathbf{L} \subseteq \models_{\{\mathbf{B}\}}$ iff $\mathcal{AS} \subseteq \models_{\{\mathbf{B}\}}$. The proof of the following lemma is trivial.

Lemma 2.1.28 (Soundness) *The logic* $\mathbf{L}$ *is sound w.r.t. class* $\mathbf{MAT}(\mathbf{L})$ *(i.e.,* $T \vdash_{\mathbf{L}} \varphi$, *then* $T \models_{\mathbf{L}} \varphi$*). Furthermore,* $\mathbf{MAT}(\mathbf{L})$ *is the greatest class w.r.t. which is the logic* $\mathbf{L}$ *sound.*

The proof of the following lemma is almost straightforward. Observe that the congruence relation defined in the following lemma is the so-called Leibnitz congruence (see [40]).

Lemma 2.1.29 *Let* $\mathbf{L}$ *be a weakly implicative logic and* $\mathbf{B}$ *an* $\mathbf{L}$*-matrix. Then relation* $\leq_{\mathbf{B}}$ *defined as* $x \leq_{\mathbf{B}} y$ *iff* $x \to_{\mathbf{B}} y \in D_{\mathbf{B}}$ *is a preorder. Furthermore, its symmetrization* $x \sim_{\mathbf{B}} y$ *iff* $x \leq_{\mathbf{B}} y$ *and* $y \leq_{\mathbf{B}} x$ *is a congruence on* A. *Finally, the set* $D_{\mathbf{B}}$ *is a cone w.r.t.* $\leq_{\mathbf{B}}$, *i.e., if* $x \in D_{\mathbf{B}}$ *and* $x \leq_{\mathbf{B}} y$ *then* $y \in D_{\mathbf{B}}$.

Definition 2.1.30 (Matrix preorder) *Let* $\mathbf{L}$ *be a weakly implicative logic and let* $\mathbf{B}$ *be an* $\mathbf{L}$*-matrix. The relation* $\leq_{\mathbf{B}}$ *defined in the previous lemma is called* matrix preorder *of* $\mathbf{B}$.

The matrix is said to be ordered *iff the relation* $\leq_{\mathbf{B}}$ *is order. We denote the class of ordered* $\mathbf{L}$*-matrixes by* $o{-}\mathbf{MAT}(\mathbf{L})$.

The matrix is said to be linearly ordered *(or just* linear*) iff the relation* $\leq_{\mathbf{B}}$ *is linear order. We denote the class of linearly ordered* $\mathbf{L}$*-matrixes by* $l{-}\mathbf{MAT}(\mathbf{L})$.

Matrices for weakly implicative logics coincide with the class of the so-called *prestandard matrices* (see Dunn [30]), whereas the ordered matrices coincides with the so-called *standard matrices*. Ordered matrices also coincides with the class of the so-called *reduced matrices* in AAL (see [40]). Obviously, $l{-}\mathbf{MAT}(\mathbf{L}) \subseteq o{-}\mathbf{MAT}(\mathbf{L}) \subseteq \mathbf{MAT}(\mathbf{L})$. An interesting question is where the opposite of Lemma 2.1.28 holds, i.e., whether the logic $\mathbf{L}$ is complete w.r.t. class $\mathbf{MAT}(\mathbf{L})$. To answer this question we recall the well-known concept of a Lindenbaum-Tarski matrix.

Definition 2.1.31 (Lindenbaum-Tarski matrix) *Let* $\mathbf{L}$ *be a weakly implicative logic in* $\mathcal{L}$, T *be a theory in* $\mathcal{L}$. *We define* $[\varphi]_T = \{\psi \mid T \vdash \varphi \leftrightarrow \psi\}$ *and* $L_T = \{[\varphi]_T \mid \varphi \in \mathbf{FOR}_{\mathcal{L}}\}$. *We define* $\mathcal{L}$*-matrix* $\mathbf{Lin}_T$, *where the* $\mathcal{L}$*-algebra has the domain* L_T, *operations* $c_{\mathbf{Lin}_T}([\varphi_1]_T, \dots [\varphi_n]_T) = [c(\varphi_1, \dots \varphi_n)]_T$ *and the designated set* $D = \{[\varphi]_T \mid T \vdash_{\mathbf{L}} \varphi\}$.

It is obvious that the definition is sound. Now we prove the fundament lemma, its parts (1) and (4) are consequences of known facts about equational logics. However, for us is crucial the part 5. which links the notion of linear theory with the notion of linearity of a matrix.

Lemma 2.1.32 *Let* $\mathbf{L}$ *be a weakly implicative logic,* T *a theory, and* e *an* $\mathbf{Lin}_T$*-evaluation defined as* $e(\varphi) = [\varphi]_T$. *Then:*

(1) $\mathbf{Lin}_T \in \mathbf{MAT}(\mathbf{L})$,

(2) $e \in \mathbf{MOD}(T, \mathbf{Lin_T})$,

(3) $[\varphi] \leq_{\mathbf{Lin}_T} [\psi]_T$ *iff* $T \vdash \varphi \to \psi$,

(4) $\mathbf{Lin}_T \in o{-}\mathbf{MAT}(\mathbf{L})$,

(5) $\mathbf{Lin}_T \in l{-}\mathbf{MAT}(\mathbf{L})$ *iff* T *is a linear theory.*

Proof:

(1) We show that if $X \vdash_{\mathbf{L}} \varphi$ then $X \models_{\mathbf{Lin}_T} \varphi$, i.e., if $X \vdash_{\mathbf{L}} \varphi$ and $f \in \mathbf{MOD}(X, \mathbf{Lin_T})$ then $f(\varphi) \in D_{\mathbf{Lin}_T}$. Recall that $f(\varphi) = [\psi]_T$ for some formula ψ.

Let us define substitution σ by setting $\sigma(v) = \psi \in f(v)$ (arbitrarily). Next we show that for each φ we get $\sigma(\varphi) \in f(\varphi)$: let $\varphi = c(\varphi_1, \dots, \varphi_n)$, from $\sigma(\varphi_i) \in f(\varphi_i)$ we get $\sigma(c(\varphi_1, \dots, \varphi_n)) = c(\sigma(\varphi_1), \dots \sigma(\varphi_n)) \in c_{\mathbf{Lin}_T}(f(\varphi_1), \dots, f(\varphi_n)) = f(c(\varphi_1, \dots, \varphi_n))$. Thus we know that for each φ we get $f(\varphi) = [\sigma(\varphi)]_T$

From $f \in \mathbf{MOD}(X, \mathbf{Lin_T})$ we get that $f(\psi) \in D_{\mathbf{Lin}_T}$ for each $\psi \in X$ and thus $T \vdash \sigma(\psi)$. From $X \vdash \varphi$ we get $\sigma(X) \vdash \sigma\varphi$. Thus together we have $T \vdash \sigma(\varphi)$ and so $f(\varphi) = [\sigma(\varphi)]_T \in D_{\mathbf{Lin}_T}$.

(2) Trivial.

(3) $[\varphi] \leq_{\mathbf{Lin}_T} [\psi]_T$ iff $[\varphi] \rightarrow_{\mathbf{Lin}_T} [\psi]_T \in D_{\mathbf{Lin}_T}$ iff $[\varphi \rightarrow \psi]_T \in D_{\mathbf{Lin}_T}$ iff $T \vdash \varphi \rightarrow \psi$.

(4) Since $[\varphi] \leq_{\mathbf{Lin}_T} [\psi]_T$ and $[\psi] \leq_{\mathbf{Lin}_T} [\varphi]_T$ entails $T \vdash \varphi \leftrightarrow \psi$ the proof is obvious.

(5) Straightforward. QED

Theorem 2.1.33 (Completeness) *Let $\mathbf{L}$ be a weakly implicative logic in $\mathcal{L}$. Then for each theory T and formula φ holds: $T \vdash \varphi$ iff $T \models_{\mathbf{L}} \varphi$.*

Proof: One direction is Lemma 2.1.28. Reverse direction: we get $\mathbf{Lin}_T \in \mathbf{MAT}(\mathbf{L})$ (from Lemma 2.1.32) and for $\mathbf{Lin}_T$-evaluation e defined as $e(\psi) = [\psi]_T$ holds $e \in \mathbf{MOD}(T, \mathbf{Lin_T})$. Thus from $T \models_{\mathbf{L}} \varphi$ we get that $[\varphi]_T = e(\varphi) \in D_{\mathbf{Lin}_T}$ and so $T \vdash \varphi$. QED

The proofs of the following two corollaries are trivial.

Corollary 2.1.34 *Let $\mathbf{L}$ be a weakly implicative logic. Then $\mathcal{THM}(\mathbf{L}) = \mathcal{TAUT}(\mathbf{L})$.*

Corollary 2.1.35 *Let $\mathbf{L}$ be a weakly implicative logic in $\mathcal{L}$, T a theory in $\mathcal{L}$. Then for each formula φ holds $T \vdash \varphi$ iff $T \models_{\mathbf{L}} \varphi$ iff $T \models_{o-\mathbf{MAT}(\mathbf{L})} \varphi$.*

Now we recall the definition of a direct and subdirect product of matrices. These definitions are obvious, if we understand matrix as a first-order structure with functions corresponding to the operations and one unary predicate, whose realization is the set of designated elements.

Definition 2.1.36 *Let $\mathcal{L} = (\mathbf{VAR}, \mathbf{C}, \mathbf{a})$ be a propositional language and $\mathcal{I}$ a (nonempty) class of $\mathcal{L}$-matrices. The* direct product *of matrices from $\mathcal{I}$ (denoted as $\prod_{\mathbf{B} \in \mathcal{I}} \mathbf{B}$) is a matrix $\mathbf{X} = (X, (c_{\mathbf{X}})_{c \in \mathbf{C}}, D_{\mathbf{X}})$, where X is a cartesian product od domain of matrices from $\mathcal{I}$, operations are defined pointwise, and $(x_{\mathbf{B}})_{\mathbf{B} \in \mathcal{I}} \in D_{\mathbf{X}}$ iff $x_{\mathbf{B}} \in D_{\mathbf{B}}$ for each $\mathbf{B} \in \mathcal{I}$.*

Furthermore, we say that the matrix $\mathbf{X}$ is a subdirect product *of matrices from $\mathcal{I}$ if there is an embedding $f : \mathbf{X} \rightarrow \prod_{\mathbf{B} \in \mathcal{I}} \mathbf{B}$, such that for each $\mathbf{B} \in \mathcal{I}$ holds $\pi_{\mathbf{B}}(f(\mathbf{X})) = B$.*

By $\pi_{\mathbf{B}}$ we mean the projection to the component $\mathbf{B}$.

Lemma 2.1.37 *Let $\mathbf{L}$ be a logic and $\mathcal{I}$ a class of $\mathbf{L}$-matrices. Then each matrix $\mathbf{X}$ which is a subdirect product of matrices from $\mathcal{I}$ is an $\mathbf{L}$-matrix.*

Proof: Trivial. QED

2.1.3 Fuzzy logic

In the previous section we have seen that each weakly implicative logic is sound and complete w.r.t. class of its *ordered* matrices. There is an obvious question, which of them are complete w.r.t. class of its *linearly ordered* matrices. This will lead us to the second central definition of this paper: the notion of weakly implicative fuzzy logics.

Definition 2.1.38 (Fuzzy logics) *Weakly implicative logic* $\mathbf{L}$ *in language* $\mathcal{L}$ *is called* fuzzy *logic if* $\mathbf{L} = \models_{l-\mathbf{MAT}(\mathbf{L})}$ *(i.e., if the logic* $\mathbf{L}$ *is complete w.r.t. linearly ordered* $\mathbf{L}$*-matrices).*

The full proper name of the above defined class of logics if *weakly implicative fuzzy logics*, however since all the logics we encounter from now on are weakly implicative, we just say that a logic is *fuzzy.* There is a joint paper by the author and Libor Běhounek [8] given *philosophical, methodological, and pragmatical* reasons for using the term fuzzy, and for *formal* delimitation of the existing *informal* class of fuzzy logic. Observe that in fuzzy logics we have $\mathcal{THM}(\mathbf{L}) = \mathcal{TAUT}(\mathbf{L}) = \mathcal{TAUT}(l{-}\mathbf{MAT}(\mathbf{L}))$.

We are going to show the equivalent definitions the class of weakly implicative fuzzy logics.

Definition 2.1.39 (Linear extension) *A weakly implicative logic* $\mathbf{L}$ *has the* Linear Extension Property (LEP) *if for each theory* T *formula* φ *such that* $T \not\vdash \varphi$ *there is a linear theory* T'*, such that* $T \subseteq T'$ *and* $T' \not\vdash \varphi$*.*

Definition 2.1.40 (Prelinearity) *A weakly implicative logic* $\mathbf{L}$ *has the* Prelinearity Property (PP) *if for each theory* T *we get* $T \vdash \chi$ *whenever* $T, \varphi \to \psi \vdash \chi$ *and* $T, \psi \to \varphi \vdash \chi$*.*

Definition 2.1.41 (Subdirect Decomposition) *A weakly implicative logic* $\mathbf{L}$ *has the* Subdirect Decomposition Property (SDP) *if each ordered* $\mathbf{L}$*-matrix is a subdirect product of linear* $\mathbf{L}$*-matrices.*

Theorem 2.1.42 (Characterization of fuzzy logics) *A weakly implicative logic is fuzzy iff it has* LEP*.*

Proof: Assume that $\mathbf{L}$ is fuzzy logic. If $\mathbf{L}$ and $T \not\vdash \varphi$ then there is linear $\mathbf{L}$-matrix $\mathbf{B}$ and $\mathbf{B}$-evaluation e such that $e(\varphi) \notin D_{\mathbf{B}}$. Let us define $T' = T \cup \{\psi \mid e(\psi) \in D_{\mathbf{B}}\}$. Obviously $T \subseteq T'$ and $T' \not\vdash \varphi$. Since $\leq_{\mathbf{B}}$ is linear order we get $e(\chi) \leq_{\mathbf{B}} e(\delta)$ or $e(\delta) \leq_{\mathbf{B}} e(\chi)$ for each χ and δ. Thus either $e(\chi \to \delta) \in D_{\mathbf{B}}$ or $e(\delta \to \chi) \in D_{\mathbf{B}}$.

Reverse direction: we need to prove that $\vdash_{\mathbf{L}} = \models_{l-\mathbf{MAT}(\mathbf{L})}$. One inclusion is trivial consequence of Theorem 2.1.33, the reverse one we prove by contradiction: assume that $T \not\vdash \varphi$. Let us take a linear supertheory $T' \not\vdash \varphi$. We know that $\mathbf{Lin}_{T'} \in l{-}\mathbf{MAT}(\mathbf{L})$ (from Lemma 2.1.32), $\mathbf{Lin}_{T'}$ is linearly ordered and for the $\mathbf{Lin}_{T'}$-evaluation e defined as $e(\psi) = [\psi]_{T'}$ holds $e \in \mathbf{MOD}(T', \mathbf{Lin}_{\mathbf{T'}})$. This entails that also $e \in \mathbf{MOD}(T, \mathbf{Lin}_{\mathbf{T'}})$. The rest of the proof is analogous to the one for weakly implicative logics. QED

Lemma 2.1.43 *Let* $\mathbf{L}$ *be a weakly implicative logic with* LEP*. Then* $\mathbf{L}$ *has* PP*.*

Proof: We argument contrapositively: let $T \not\vdash \chi$, then (using LEP) there is linear theory T', such that $T' \not\vdash \chi$. Assume that $T' \vdash \varphi \to \psi$, then obviously $T, \varphi \to \psi \not\vdash \chi$. QED

To reverse this lemma it seems that we need one additional assumption: the logic has to be finitary. However, it is obvious that for some infinitary rules the equivalence will hold as well, the question exactly for which subclass of weakly implicative logics the equivalence holds seems to be interesting open problem.

Lemma 2.1.44 *Let* $\mathbf{L}$ *be a finitary weakly implicative logic with* PP*,* T *a theory, and* φ *a formula, such that* $T \not\vdash \varphi$*. Then there is a linear theory* T'*, such that* $T \subseteq T'$ *and* $T' \not\vdash \varphi$*.*

Proof: Let $||\mathcal{L}|| = \kappa$. Let us enumerate all tuples of formula by ordinals $< \kappa$. Let $T_0 = T$. We construct theories T_μ using transfinite induction. Let $\hat{T}_\mu = \bigcup_{\nu<\mu} T_\nu$.

Observe that if $T_\nu \not\vdash \varphi$ for each $\nu < \mu$ then $\hat{T}_\mu \not\vdash \varphi$ as well (the logic is finitary). We know that either $\hat{T}_\mu \cup \{\varphi_\mu \to \psi_\mu\} \not\vdash \varphi$ or $\hat{T}_\mu \cup \{\psi_\mu \to \varphi_\mu\} \not\vdash \varphi$ (otherwise using PP we get contradiction with $\hat{T}_\mu \not\vdash \varphi$), define T_μ accordingly. Finally, define $T' = \hat{T}_\kappa$; obviously $T' \not\vdash \varphi$ and T' is linear. QED

Theorem 2.1.45 (Equivalent characterization) *Let* **L** *be a finitary weakly implicative logic. Then the following are equivalent:*

(1) **L** *is a fuzzy logic,*

(2) **L** *has* LEP*,*

(3) **L** *has* PP*,*

(4) **L** *has* SDP.

Proof: $(1) \to (2)$: C.f. Theorem 2.1.42.

$(2) \to (3)$: C.f. Lemma 2.1.43.

$(3) \to (4)$: Let us denote the language of **L** as $\mathcal{L} = (\mathbf{VAR}, \mathbf{C}, \mathbf{a})$. Let $\mathbf{B} = (\mathrm{A}, D)$ be an ordered **L**-matrix and $\leq$ its matrix order. Let us take $\mathbf{VAR}' = A$, for clearness we will use $v_a \in \mathbf{VAR}'$ and $a \in A$. We define the propositional language $\mathcal{L}' = (\mathbf{VAR}', \mathbf{C}, \mathbf{a})$ and the logic $\mathbf{L}'$ as $\vdash_{\mathcal{SUB}_{\mathcal{L}'}(\mathbf{L})}$. The logic $\mathbf{L}'$ has obviously PP as well and thus it has LEP (using Lemma 2.1.44). Notice that here we use the assumption that **L** is finitary.

We define $T = \{c(v_{a_1}, \ldots v_{a_n}) \leftrightarrow v_{c_{\mathbf{B}}(a_1,\ldots a_n)} \mid c \in \mathbf{C}, \mathbf{a}(c) = n, \text{ and } a_1, \ldots, a_n \in A\} \cup \{v_a \mid a \in D\}$. Observe that **B** is an $\mathbf{L}'$-matrix. Let us define **B**-evaluation $e(v_a) = a$ and observe that $\{e\} = \mathbf{MOD}(T, \mathbf{B})$. Now we show $T \vdash v_a \to v_b$ iff $a \leq b$: one direction is simple (from $e \in \mathbf{MOD}(T, \mathbf{B})$ we get that $e(v_a) \to_{\mathbf{B}} e(v_b) \in D$ and so $a \leq b$); the other direction is similar (if $a \leq b$ then $a \to b \in D$ thus $T \vdash v_{a\to b}$ and so $T \vdash v_a \to v_b$). Finally, we observe that for each formula φ there is $a \in A$, such that $T \vdash \varphi \leftrightarrow v_a$

Let us define the set $\mathcal{I}$ of all linear theories extending T. Next we define **L**-matrix $\mathbf{X} = \prod_{S\in\mathcal{I}} \mathbf{Lin}_S$ (direct product of Lindenbaum-Tarski matrices). Finally, we define $f(a) = ([v_a]_S)_{S\in\mathcal{I}}$.

Now we show that f is an embedding of **B** into **X**: since for each $S \in \mathcal{I}$ we have $[v_{c_{\mathbf{B}}(a_1,\ldots,a_n)}]_S = [c(v_{a_1}, \ldots v_{a_n})]_S = c_{\mathbf{Lin}_S}([v_{a_1}]_S, \ldots [v_{a_n}]_S)$ and so we get $f(c_{\mathbf{B}}(a_1, \ldots, a_n)) = (c_{\mathbf{Lin}_S}([v_{a_1}]_S, \ldots [v_{a_n}]_S))_{S\in\mathcal{I}} = c_{\mathbf{X}}(f(a_1), \ldots f(a_n))$. Since obviously $a \in D$ entails that $f(a) \in D_{\mathbf{X}}$ (from $a \in D$ we have $T \vdash v_a$ and so $S \vdash v_a$ and thus $[v_a]_S \in D_{\mathbf{Lin}_S}$ for each $S \in \mathcal{I}$) we know that f is a morphism. It remains to be shown that f is one-one: from $a \neq b$ we get that either $a \not\leq b$ or $b \not\leq a$. Let us assume that $a \not\leq b$ then $T \not\vdash v_a \to v_b$, using Lemma 2.1.44 we know that there is a linear theory $S \in \mathcal{I}$ such that $S \not\vdash v_a \to v_b$ thus $[v_a]_S \not\leq_{\mathbf{Lin}_S} [v_b]_S$ and so $f(a) \not\leq_{\mathbf{X}} f(b)$.

Finally, we observe that for each S we have $\pi_S(f(A)) = L_S$ (just recall that for each φ there is $a \in A$, such that $T \vdash \varphi \leftrightarrow v_a$).

Since $\mathbf{Lin}_S$ is linearly ordered **L**-matrix for each $S \in \mathcal{I}$ and **B** can be embedded into direct product of $(\mathbf{Lin}_S)_{S\in\mathcal{I}}$ in the way that $\pi_S(f(A)) = L_S$. We conclude that **B** is a subdirect product of linearly ordered **L**-matrices.

$(4) \to (1)$: Trivial.

QED

Lemma 2.1.46 *Let* **L** *be a fuzzy logic. Then* $(\varphi \to \psi)^i \to \chi, (\psi \to \varphi)^j \to \chi \vdash_{\mathbf{L}} \chi$ *for each naturals* i *and* j.

Proof: Let $T = \{(\varphi \to \psi)^i \to \chi, (\psi \to \varphi)^j \to \chi\}$. Obviously $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$ (using (MP)). Since each fuzzy logic has PP the proof is done. QED

Table 2.1: Structural rules

consecution	symbol	name
$\varphi \vdash \psi \to \varphi$	W	Weakening
$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi)$	E	Exchange
$\varphi \to (\varphi \to \psi) \vdash \varphi \to \psi$	C	Contraction

Table 2.2: Addition rules

consecution	symbol	name
$(\varphi \to \psi) \to \chi, (\psi \to \varphi) \to \chi \vdash \chi$	PL	prelinearity
$\vdash \varphi \to ((\varphi \to \psi) \to \psi)$	As	assertion
$\varphi \to \psi \vdash (\psi \to \chi) \to (\varphi \to \chi)$	Sf	suffixing
$\psi \to \chi \vdash (\varphi \to \psi) \to (\varphi \to \chi)$	Pf	prefixing
$\vdash \varphi \to (\varphi \to \varphi)$	M	mingle

The proofs of the following lemmata are obvious. The first one is especially important. It allows us to speak about weakest fuzzy logic with some property (eg. being stronger that some know logic).

Lemma 2.1.47 (Intersection) *The intersection of an arbitrary system of fuzzy logics is a fuzzy logic.*

Lemma 2.1.48 (Axiomatic extension) *An axiomatic extension of arbitrary fuzzy logic in the same language is a fuzzy logic.*

The assumption of being in the same language can be omitted if we assume some additional properties of the logic in question.

Lemma 2.1.49 (Conservative expansion) *Let* $\mathbf{L}'$ *be a conservative expansion of a finitary fuzzy logic* $\mathbf{L}$*. Then* $\mathbf{L}$ *is fuzzy logic as well.*

Proof: We know that $\mathbf{L}'$ has PP, we show that $\mathbf{L}$ has PP as well and because $\mathbf{L}$ is finitary we get that $\mathbf{L}$ is fuzzy. Let us take theory T and formulae φ, ψ, χ in language of $\mathbf{L}$. Assume that $T, \varphi \to \psi \vdash_{\mathbf{L}} \chi$ and $T, \psi \to \varphi \vdash_{\mathbf{L}} \chi$, then also $T, \varphi \to \psi \vdash_{\mathbf{L}'} \chi$ and $T, \psi \to \varphi \vdash_{\mathbf{L}'} \chi$. Using PP for $\mathbf{L}'$ we get $T \vdash_{\mathbf{L}'} \chi$. Conservativeness completes the proof. QED

2.2 Special propositional logics

In this section we introduce some additional rules (see Tables 2.1 and 2.2) and some additional connectives (see table 2.3) and prove some facts about these extensions. It is just a sketch of a huge work to be done. The ultimate goal is to characterize known logics and put them into our context with modularly designed Hilbert's style calculi. Having this done, we identify which of them are fuzzy, or we find minimal fuzzy logics extending some known logics (eg. minimal fuzzy logic over intuitionistic logic is Gödel logic, minimal fuzzy logic over Full Lambek calculus with exchange and weakening is MTL logic, etc.) A lot of work in this direction can be found in Restall's book [78] (of course for logic in general, he does not deal with fuzzy logic in any form).

Having stronger logic/language can lead to simplified semantics, eg. having $\overline{1}$ we can replace matrices with ordered structures (the designated set will be the upper cone of $\overline{1}_{\mathbf{B}}$), having one of the lattice connectives we can even work with algebras, thus being able to

Table 2.3: Propositional connectives

Symbol	Arity	Name	Alternative name
$\top$	0	verum	additive truth
$\bar{1}$	0	one	multiplicative truth
$\bar{0}$	0	zero	multiplicative falsum
$\bot$	0	falsum	additive falsum
$\triangle$	1	Baaz delta	globalization
$\wedge$	2	min-conjunction	additive conjunction
$\vee$	2	max-disjunction	additive disjunction
$\&$	2	strong conjunction	fusion, multiplicative conjunction
$\rightsquigarrow$	2	c-implication	reverse implication

use powerful methods of Abstract Algebraic Logics (which, of course, we can do anyway, but some theorems of ALL hold in algebraizable logics only). There are other possible simplification of the semantics allowed be adding some structural rules (weakening leads to algebraic semantics, exchange to ordered structures, with φ being valid iff $e(\varphi \to \varphi) \leq e(\varphi)$, etc.).

2.2.1 Adding rules

In this section we restrict ourselves to the propositional languages with implication only. The basic rules correspond to the structural rules are exchange, contraction and weakening (see Table 2.1), extended by some additional important rules summarized in Table 2.2. For the sake of simplicity we speak about rules and not the rule schemata. We formulate them as rules, however in some situation we can use their stronger forms—we formulate then as axioms. To do this in a general way we present the following definition.

Definition 2.2.1 *Let R be a unary deduction rule of the form $\varphi \vdash \psi$. By the corresponding axiom we understand axiom $\vdash \varphi \to \psi$, we will denote it as $ax(R)$.*

Of course if $\mathbf{L}$ is weakly implicative logic then $\vdash_{\mathbf{L}} \varphi \to \psi$ entails $\varphi \vdash_{\mathbf{L}} \psi$, i.e., if $ax(R) \in \mathbf{L}$ then $R \in \mathbf{L}$. Recall that in the literature some of the axioms are known under different names.

Definition 2.2.2 (Adding rules) *Let $\mathbf{L}$ be a weakly implicative logic in language $\{\to\}$ and Q be a set of consecutions. We say that $\mathbf{L}$ is an Q-implication fragment if the consecutions from Q are elements of $\mathbf{L}$.*

We say that $\mathbf{L}$ is an fuzzy Q-implication fragment if $\mathbf{L}$ is fuzzy logic and $\mathbf{L}$ is Q-implication fragment.

Definition 2.2.3 *The weakest Q-implication fragment is denoted as $\mathcal{MIN}(Q)$. Let us denote the weakest weakly implicative logic as* WIL. *Furthermore, the weakest fuzzy Q-implication fragment is denoted as $\mathcal{FUZZ}(Q)$.*

Both definition are sound thanks to the Lemma 2.1.18 and 2.1.47. Let us by WIL denote the logic $\mathcal{MIN}(\emptyset)$ (i.e., the weakest weakly implicative logic). Then obviously $\mathcal{MIN}(Q) = \vdash_{\mathrm{WIL} \cup Q}$, i.e., the presentation of $\mathcal{MIN}(Q)$ results from the presentation of WIL by adding consecutions from Q.

The following lemma shows the interplay between transitivity and exchange.

Lemma 2.2.4 *The following logics are equivalent:*

(1) BCI *logic (implicational fragment of intuitionistic linear logic),*

Table 2.4: Known implicational fragments

consecutions	implicational fragment of
$\emptyset$	intuitionistic linear logic
C	relevance logic
C, M	relevance logic with mingle
W	affine intuitionistic linear logic
W, C	intuitionistic logic

(2) $\mathcal{MIN}(\{ax(\mathrm{Sf}), \mathrm{As}\})$,

(3) $\mathcal{MIN}(\{ax(\mathrm{Sf}), ax(\mathrm{E})\})$,

(4) $\mathcal{MIN}(\{ax(\mathrm{Pf}), ax(\mathrm{E})\})$,

(5) $\mathcal{MIN}(\{\mathrm{Pf}, ax(\mathrm{E})\})$,

(6) $\mathcal{MIN}(\{ax(\mathrm{Sf}), \mathrm{E}\})$,

(7) $\mathcal{MIN}(\{ax(\mathrm{Pf}), \mathrm{E}\})$.

Proof: We show that each logic is stronger than the next one (cyclicly):

(1) $\supseteq$ (2): Recall that BCI logic is axiomatized by ax(Sf), As, (Ref), and (MP). So all we have to do is to show ($\mathrm{Cng}_{\rightarrow}$) but this is almost straightforward.

(2) $\supseteq$ (3): All we need to show is $\vdash_{\mathcal{MIN}(\{ax(\mathrm{Sf}),\mathrm{As}\})} ax(\mathrm{E})$

(i) $\psi \rightarrow ((\psi \rightarrow \chi) \rightarrow \chi)$	As
(ii) $(\psi \rightarrow ((\psi \rightarrow \chi) \rightarrow \chi)) \rightarrow ((((\psi \rightarrow \chi) \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi)))$	ax(Sf)
(iii) $(((\psi \rightarrow \chi) \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$	$(i), (ii)$, (MP)
(iv) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (((\psi \rightarrow \chi) \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$	ax(Sf)
(v) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$	$(iv), (iii)$, (WT)

(3) $\supseteq$ (4): Trivial.
(4) $\supseteq$ (5): Trivial.
(5) $\supseteq$ (6): All we need to show is $\vdash_{\mathcal{MIN}(\{\mathrm{Pf},ax(\mathrm{E})\})} ax(\mathrm{Sf})$:

(i) $(\psi \rightarrow \chi) \rightarrow (\psi \rightarrow \chi)$	(Ref)
(ii) $\psi \rightarrow ((\psi \rightarrow \chi) \rightarrow \chi)$	(i), ax(E), (MP)
(iii) $(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow ((\psi \rightarrow \chi) \rightarrow \chi))$	(ii), Pf
(iv) $(\varphi \rightarrow ((\psi \rightarrow \chi) \rightarrow \chi)) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$	ax(E)
(v) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$	$(iii), (iv)$, Pf

(6) $\supseteq$ (7): Trivial.
(7) $\supseteq$ (1): Trivial. QED

Notice two open problems: what about logics $\mathcal{MIN}(\{\mathrm{Sf}, ax(\mathrm{E})\})$ and $\mathcal{MIN}(\{ax(\mathrm{Pf}), \mathrm{As}\})$?

Table 2.4 puts some known logics into our context (we list axioms which has to be added to $\mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E})$). We can add all of them as rules or axioms—in the presence of exchange these two options are equivalent—as shown by the following observation:

1. $\mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E}) = \mathcal{MIN}(ax(\mathrm{Sf}), ax(\mathrm{E}))$,

2. $\mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}) = \mathcal{MIN}(ax(\mathrm{Sf}), ax(\mathrm{E}), ax(\mathrm{W}))$,

3. $\mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E}, \mathrm{C}) = \mathcal{MIN}(ax(\mathrm{Sf}), ax(\mathrm{E}), ax(\mathrm{C}))$.

Proof: Part 1. was shown in Lemma 2.2.4. Part 2. is easy. To prove part 3. we only show that $\vdash_{\mathcal{MIN}(ax(\mathrm{Sf}),\mathrm{E},\mathrm{C})} ax(\mathrm{C})$:

(i) $(\varphi \to (\varphi \to \psi)) \to (\varphi \to (\varphi \to \psi))$	(Ref)
(ii) $\varphi \to ((\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi))$	(i) and E.
(iii) $\varphi \to (\varphi \to ((\varphi \to (\varphi \to \psi)) \to \psi))$	(ii), $ax(E)$, $ax(\mathrm{Sf})$
(iv) $\varphi \to ((\varphi \to (\varphi \to \psi)) \to \psi)$	(iii) and C
(v) $(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	(iv) and E

QED

Before we proceed further we observe some properties of BCI. The proofs are easy (by induction).

Lemma 2.2.5 *It holds:*

1. $\vdash_{\mathrm{BCI}} (\varphi^n \to (\psi^m \to \chi)) \to (\psi^m \to (\varphi^n \to \chi))$,
2. $\vdash_{\mathrm{BCI}} (\varphi^n \to \psi) \to ((\psi \to \chi) \to (\varphi^n \to \chi))$.

Now we present an important definition—the deduction theorem—for rather wide class of weakly implicative logic. We present it in an unusual form. Our formulation allows us to show exactly which logics have this deduction theorem.

Definition 2.2.6 *Let* **L** *be a logic. We say that* **L** *has* Implicational Deduction Theorem *(*$\mathrm{DT}_\to$*) if* **L** *has a presentation $\mathcal{AX}$, where* (MP) *is the only deduction rule and for each theory T, formulae φ, ψ, and proof $\mathcal{P}$ of ψ in theory T, φ (in the presentation $\mathcal{AX}$) we have: $T, \varphi \vdash \psi$ iff $T \vdash \varphi^n \to \psi$, where n is a number of occurrences of φ in the leaves of the proof $\mathcal{P}$ and there is a proof $\mathcal{P}'$ of $\varphi^n \to \psi$ in T, such that each $\psi \in T$ occurs in the leaves of $\mathcal{P}$ same number of times as in the leaves of $\mathcal{P}'$.*

Observe that each logic with $\mathrm{DT}_\to$ has also the "standard" form of the local deduction theorem.

Corollary 2.2.7 *Each logic with* $\mathrm{DT}_\to$ *has so called* Local Deduction Theorem *(*LDT*): for each theory T and formulae φ, ψ: $T, \varphi \vdash \psi$ iff there is n such that $T \vdash \varphi^n \to \psi$.*

Now we present sufficient and necessary condition for **L** to have $\mathrm{DT}_\to$.

Theorem 2.2.8 (Deduction theorem) *Let* **L** *be a logic. Then* **L** *has* $\mathrm{DT}_\to$ *iff* **L** *has a presentation, where* (MP) *is the only deduction rule and the implicational fragment of* **L** *is an extension of* BCI.

Proof: First direction: All we have to do is to show that $\vdash_{\mathbf{L}} ax(\mathrm{Sf})$ and $\vdash_{\mathbf{L}} ax(\mathrm{E})$ (the rest is the immediate consequence of the definition of $\mathrm{DT}_\to$). Observe that $\varphi, \psi, \varphi \to (\psi \to \chi) \vdash_{\mathbf{L}} \chi$ and each of the premises is used exactly once, applying $\mathrm{DT}_\to$ three times we get $\vdash_{\mathbf{L}} ax(\mathrm{E})$. Observe that $\varphi, \varphi \to \psi, \psi \to \chi \vdash_{\mathbf{L}} \chi$ and each of the premises is used exactly once, applying $\mathrm{DT}_\to$ three times we get $\vdash_{\mathbf{L}} ax(\mathrm{Sf})$.

Reverse direction: we need to prove two directions. One is obvious. To prove the other one we use the induction over the proof of ψ in T, φ (in $\mathcal{AX}$). We show that it holds for each χ in the proof $\mathcal{P}'$:

- χ is a leaf of $\mathcal{P}$, i.e., $\chi \in T$, χ is an axiom, or $\chi = \varphi$: trivial.
- χ has predecessors $\psi_2 = \psi_1 \to \chi$ and ψ_1, using the induction property we get that $T \vdash \varphi^n \to (\psi_1 \to \chi)$ and $T \vdash \varphi^m \to \psi_1$ (and the number of occurrences of formulae from T is the same), we distinguish two cases:

- $\psi_2 = \varphi$, from Lemma 2.2.5 (2) we get $\vdash (\varphi^m \to \psi_1) \to ((\psi_1 \to \chi) \to (\varphi^m \to \chi))$, thus we get $T \vdash \varphi \to (\varphi^m \to \chi)$ and so $T \vdash \varphi^{m+1} \to \chi$.
- $\psi_2 \neq \varphi$, using Lemma 2.2.5 (1) we get $T \vdash \psi_1 \to (\varphi^n \to \chi)$. From Lemma 2.2.5 (2) we obtain $T \vdash \varphi^m \to (\varphi^n \to \chi)$. Thus $T \vdash \varphi^{m+n} \to \chi$.

In both cases the number of occurrences of formulae from T is not changed. QED

We have proved even more: given proof of ψ in T, φ, we give a constructive procedure which produces the proof of $\varphi^n \to \psi$ in T.

Corollary 2.2.9 *Let $\mathcal{L}$ be a language and* **L** *a logic in $\mathcal{L}$, such that* **L** *has a presentation, where* (MP) *is the only deduction rule and implicational fragment of* **L** *is an extension of* BCI. *Then* **L** *has* LDT.

In the presence of LDT we can prove the "converse" of Lemma 2.1.46. Thus having an equivalent definition of fuzzy logics in some class of logics.

Lemma 2.2.10 *Let* **L** *be a finitary logic with* LDT. *Then* $\vdash_{\mathbf{L}} (\varphi \to \psi)^i \to \chi, (\psi \to \varphi)^j \to \chi \vdash_{\mathbf{L}} \chi$ *iff* **L** *is fuzzy.*

Proof: One direction is just Lemma 2.1.46. To prove the other direction we only show that **L** has PP. Assume that $T, \varphi \to \psi \vdash \chi$ and $T, \varphi \to \psi \vdash \chi$. By LDT we get $T \vdash (\varphi \to \psi)^i \to \chi$ and $T \vdash (\psi \to \varphi)^j \to \chi$ and so we have $T \vdash \chi$. QED

At the end of this section we present an infinite axiomatic system for the minimal fuzzy $\{ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}\}$-implication fragment. The question whether there is finite system seems to be open. We decided to use classical names for axioms $ax(\mathrm{Sf})$, $ax(\mathrm{E})$, and $ax(\mathrm{W})$.

Definition 2.2.11 *The* fuzzy BCK logic *(*FBCK*) has the following presentation:*

$\mathcal{B}$	$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,	
$\mathcal{C}$	$\vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$,	
$\mathcal{K}$	$\vdash \varphi \to (\psi \to \varphi)$,	
F_n	$\vdash ((\varphi \to \psi)^n \to \chi) \to (((\psi \to \varphi)^n \to \chi) \to \chi)$	*for all n,*
(MP)	$\varphi, \varphi \to \psi \vdash \psi$.	

Observe that $\mathrm{FBCK} = \mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{F}_n)$. Using Lemma 2.2.4 we could write several different equivalent axiomatic systems. Also observe that *ordered* FBCK-matrices are exactly BCK-algebras satisfying axioms F_n.

Theorem 2.2.12 $\mathrm{FBCK} = \mathcal{FUZZ}(\{ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}\})$.

Proof: First, we show that FBCK, is fuzzy logic. Using Theorem 2.1.45 it is enough to show that FBCK has PP. Let $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$ then using $\mathrm{DT}_{\to}$ we get that $T \vdash (\varphi \to \psi)^m \to \chi$ and $T \vdash (\psi \to \varphi)^n \to \chi$ for some n and m. Let us take $t = \max(m, n)$, using W we get the $T \vdash (\varphi \to \psi)^t \to \chi$ and $T \vdash (\psi \to \varphi)^t \to \chi$, axiom F_t completes the proof.

Next, we have to show that each fuzzy logic extending BCK proves F_n. We recall that each fuzzy logic has PP. Now observe that

(i) $((\varphi \to \psi)^n \to \chi) \to ((\varphi \to \psi)^n \to \chi)$	(Ref)
(ii) $(\varphi \to \psi)^n \to (((\varphi \to \psi)^n \to \chi) \to \chi)$	(i) and Lemma 2.2.5 1.
(iii) $\varphi \to \psi \vdash (((\varphi \to \psi)^n \to \chi) \to \chi)$	(ii)
(iv) $\varphi \to \psi \vdash ((\psi \to \varphi)^n \to \chi) \to (((\varphi \to \psi)^n \to \chi) \to \chi)$	(iii) and $\mathcal{K}$
(v) $\varphi \to \psi \vdash ((\varphi \to \psi)^n \to \chi) \to (((\psi \to \varphi)^n \to \chi) \to \chi)$	(iv) and $\mathcal{C}$

Table 2.5: Rules

Consecution	symbol	Name	match for
$\psi\&\varphi \to \chi \dashv\vdash \varphi \to (\psi \rightsquigarrow \chi)$	$\rightsquigarrow$R	$\rightsquigarrow$-residuation	$\rightsquigarrow$
$\varphi \to (\psi \to \chi) \dashv\vdash \varphi\&\psi \to \chi$	R	residuation	&
$\varphi \to \psi \dashv\vdash \varphi \rightsquigarrow \psi$	Imp	implications	$\rightsquigarrow$
$\vdash \varphi \to \top$	Tr	veritas ex quolibet	$\top$
$\vdash \bot \to \varphi$	Fa	ex-falso quodlibet	$\bot$
$\varphi \dashv\vdash \bar{1} \to \varphi$	PP	push and pop	$\bar{1}$
$\varphi \to \chi, \psi \to \chi \vdash \varphi \vee \psi \to \chi$	$\vee$1	supremum	$\vee$
$\vdash \varphi \to \varphi \vee \psi$	$\vee$2	idempotency	$\vee$
$\vdash \varphi \vee \psi \to \psi \vee \varphi$	$\vee$3	commutativity	$\vee$
$\chi \to \varphi, \chi \to \psi \vdash \chi \to \varphi \wedge \psi$	$\wedge$1	infimum	$\wedge$
$\vdash \varphi \wedge \psi \to \varphi$	$\wedge$2	idempotency	$\wedge$
$\vdash \varphi \wedge \psi \to \psi \wedge \varphi$	$\wedge$3	commutativity	$\wedge$

Table 2.6: Matching rules for $\triangle$

Consecution	symbol	Name
$\vdash \triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$	$\triangle$1	$\triangle$-monotonicity
$\vdash \triangle\varphi \to \varphi$	$\triangle$2	$\triangle$-reflexivity
$\vdash \triangle\varphi \to \triangle\triangle\varphi$	$\triangle$3	$\triangle$-transitivity
$\vdash \triangle\varphi \to (\triangle\psi \to \varphi)$	$\triangle$W	$\triangle$-weakening
$\vdash \triangle(\triangle\varphi \to (\triangle\varphi \to \psi)) \to (\triangle\varphi \to \psi)$	$\triangle$C	$\triangle$-contraction
$\vdash \triangle(\triangle\varphi \to (\triangle\psi \to \chi)) \to (\triangle\psi \to (\triangle\varphi \to \chi))$	$\triangle$E	$\triangle$-exchange
$\varphi \vdash \triangle\varphi$	(NEC)	necessitation

By replacing φ and ψ in (iv) we get:

(vi) $\psi \to \varphi \vdash ((\varphi \to \psi)^n \to \chi) \to (((\psi \to \varphi)^n \to \chi) \to \chi)$

Since we **L** has PP we get $\vdash \mathrm{F}_n$ QED

Since obviously BCK $= \mathcal{MIN}(ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W})$ we get the following corollary.

Corollary 2.2.13 FBCK *is the weakest fuzzy logic stronger than* BCK.

We can alter the axioms F_n by using two different natural number m, n as "exponents", we get axioms $\mathrm{F}_{m,n}$. If we add axioms $\mathrm{F}_{m,n}$ to the BCI logic, we get fuzzy logic (by Lemma 2.2.10). However, we are not able to the prove the converse statement, i.e., that this logic is minimal fuzzy logic over BCI.

2.2.2 Adding connectives

As mentioned before, we consider the connectives from Table 2.3. For the matching rules for the particular connective see Tables 2.5 and 2.6. There are many interesting interplays between them and the matching consecutions (eg. in the presence of residuation *rule* we can prove that the residuation *axiom* is equivalent to the *associativity axiom* for &). Again, we present only the basic definitions, a lot of work is to be done yet.

Definition 2.2.14 (Adding connectives) *A logic* **L** *is* Q-weakly implicative logic in $\mathcal{L}$ *if its implication fragment is* Q*-implication fragment and if some of the connectives from the*

Table 2.7: Known logics

$\mathcal{L}$	Q	$\mathcal{MIN}_{\mathcal{L}}(Q)$
$\bot$, & $\vee$	$ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{C}$	Intuitionistic logic
$\rightsquigarrow$, $\bot$, $\overline{1}$, $\overline{0}$, &, $\wedge$, $\vee$	Sf, Pf	Full Lambek
$\bot$, &	$ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{C}, \mathrm{PŁ}$	Gödel logic
$\bot$, &, $\wedge$	$ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{PŁ}$	MTL logic
&, $\wedge$	$ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{PŁ}$	MTLH logic

set $\{\top, \overline{0}, \overline{1}, \bot, \wedge, \vee, \&, \rightsquigarrow, \triangle\}$ *are in* $\mathcal{L}$*, then their matching rules and instances of* (Cng_c) *rules for the connectives c in question are elements of* **L**. *We say that* **L** *is an* Q*-fuzzy logic if* **L** *is fuzzy logic and* **L** *is* Q*-weakly implicative logic.*

To simplify things we will write that **L** is Q-logic instead of **L** is Q-weakly implicative logic.

Definition 2.2.15 *Let* $\mathcal{L}$ *be a propositional language. We denote the weakest* Q*-logic in* $\mathcal{L}$ *as* $\mathcal{MIN}_{\mathcal{L}}(Q)$ *and the weakest fuzzy* Q*-logic is denoted as* $\mathcal{FUZZ}_{\mathcal{L}}(Q)$.

Table 2.7 puts some known logics into our context. The only thing we need to observe is the fact that in presence of $ax(\mathrm{Sf})$ and E we get from the residuation rules the residuation axioms (having this it is easy to show that & is associative. We show one direction:

(i)	$(\varphi \to (\psi \to \chi)) \to (\varphi \to (\psi \to \chi))$	(Ref)
(ii)	$\varphi \to ((\varphi \to (\psi \to \chi)) \to (\psi \to \chi))$	(i) and E
(iii)	$\varphi \to (\psi \to ((\varphi \to (\psi \to \chi)) \to \chi))$	$(ii), ax(\mathrm{E}), (\mathrm{WT})$
(iv)	$\varphi \& \psi \to ((\varphi \to (\psi \to \chi)) \to \chi)$	(iii) and residuation rule
(v)	$(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$	$(iv), \mathrm{E}.$

The logic $\mathcal{FUZZ}_{\{\&\}}(ax(dT), \mathrm{E}, \mathrm{W})$ is the newest logic of Petr Hájek—the quasihoop logic (for details see [51]).

Definition 2.2.16 *The* quasihoop logic *(*QH*) has the following presentation:*

(1)	$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)),$
(2)	$\vdash \varphi \& \psi \to \psi \& \varphi,$
(3)	$\vdash \varphi \& \psi \to \varphi,$
(4)	$\vdash (\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi),$
(5)	$\vdash (\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi)),$
F_n	$\vdash ((\varphi \to \psi)^n \to \chi) \to (((\psi \to \varphi)^n \to \chi) \to \chi),$
(MP)	$\varphi, \varphi \to \psi \vdash \psi.$

Observe that axiom (2) corresponds to E axiom (3) to W. Axioms (4) and (5) are axiomatic version of residuation rules R. The ordered QH-matrices are just BCK(RP)-algebras.

2.2.3 The connective $\vee$

Having disjunction in the language, we can express several concepts of this paper in ways more usual in the literature. In this subsection we assume that $\vee \in \mathcal{L}$ for each logics we encounter here.

Lemma 2.2.17 *Let* **L** *be a weakly implicative logic. Then* $\varphi \vee \psi, \varphi \to \psi \vdash \psi$.

Proof: We give a formal proof:

(i) $\varphi \to \psi, \psi \to \psi \vdash \varphi \vee \psi \to \psi$ $\vee 1$
(ii) $\varphi \vee \psi, \varphi \to \psi \vdash \varphi \vee \psi \to \psi$ (i)
(iii) $\varphi \vee \psi, \varphi \to \psi \vdash \psi$ (ii) and (MP)

QED

Now we define the notion of prime theory. This is more known and more widely used concept than the concept of linear theory. However, we will see that in fuzzy logics both notions coincide.

Definition 2.2.18 (Prime theory) *Let* **L** *be a weakly implicative logic. A theory* T *is* prime *if from* $T \vdash \varphi \vee \psi$ *we get* $T \vdash \varphi$ *or* $T \vdash \psi$.

Definition 2.2.19 (Prime extension) *A weakly implicative logic* **L** *has the* Prime Extension Property (PEP) *if for each theory* T *formula* φ *such that* $T \nvdash \varphi$ *there is a prime theory* T'*, such that* $T \subseteq T'$ *and* $T' \nvdash \varphi$.

Definition 2.2.20 (Proof by cases) *A weakly implicative logic* **L** *has the* Proof by Cases Property (PCP) *if for each theory* T *we get* $T, \varphi \vee \psi \vdash \chi$ *whenever* $T, \varphi \vdash \chi$ *and* $T, \psi \vdash \chi$.

Observe that above defined principles PCP and PEP differ from seemingly analogous principles PP and LEP. For example Intuitionistic logic has both PCP and PEP but does not have the other two. Let us examine this in more details:

Lemma 2.2.21 *Let* **L** *be a weakly implicative logic. Then:*

1. *each linear theory is prime;*
2. *if* **L** *has* PP *we have* $\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$*;*
3. *if* $\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$ *then each prime theory is linear;*
4. *if* **L** *has* PP *then* **L** *has* PCP*;*
5. *if* $\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$ *and* **L** *has* PCP *then* **L** *has* PP.

Proof:

1. Let $T \vdash \varphi \vee \psi$. Since T is linear we know that $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$. Thus (using Lemma 2.2.17) we get $T \vdash \psi$ or $T \vdash \varphi$.
2. Trivial.
3. Trivial.
4. From PP and $\vee 2$ we easily get $\vdash_{\mathbf{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$. Now let T be a theory such that $T, \varphi \vdash \chi$ and $T, \psi \vdash \chi$. Using Lemma 2.2.17 we know that $T, \varphi \vee \psi, \varphi \to \psi \vdash \psi$ and $T, \varphi \vee \psi, \psi \to \varphi \vdash \varphi$. Thus $T, \varphi \vee \psi, \varphi \to \psi \vdash \chi$ and $T, \varphi \vee \psi, \psi \to \varphi \vdash \chi$. PP completes the proof.
5. just observe that from $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$ we get $T, (\varphi \to \psi) \vee (\psi \to \varphi) \vdash \chi$. Knowing that $\vdash_{\mathbf{L}} (\varphi \to \psi) \vee (\psi \to \varphi)$ we get $T \vdash \chi$. QED

This lemma has three interesting corollaries.

Corollary 2.2.22 *In fuzzy logic the theory* T *is prime iff* T *is linear.*

Corollary 2.2.23 *A logic* **L** *is fuzzy iff* **L** *has* PEP *and* $\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$.

Corollary 2.2.24 *A finitary logic* **L** *is fuzzy iff* **L** *has* PCP *and* $\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$.

2.2.4 The connective $\triangle$

The connective $\triangle$ is a special one, in intuitionistic logic it is known as *globalization* (with some additional assumptions); in linear logic it is a kind of *exponential*; and in fuzzy logics it is known as Baaz delta. Roughly speaking, this connective allows us *controlled* use of structural rules. In this subsection we assume that $\triangle$ is an element of all the propositional languages $\mathcal{L}$ (unless the opposite is explicitly mentioned).

Now we present the analog of Definition 2.2.6 and prove the analog of Theorem 2.2.8. However, the $\triangle$ connective allows us much simpler formulations (resembling the deduction theorem of modal logic S4) . In the following definition and theorem we understand **L** as arbitrary weakly implicative logic (not necessarily with matching rules $\triangle 1$, $\triangle 2$, $\triangle 3$,$\triangle$W, $\triangle$C, $\triangle$E, and (NEC)).

Definition 2.2.25 *Let* **L** *a weakly implicative logic. We say that* **L** *has* Delta Deduction Theorem *(*$\mathrm{DT}_\triangle$*) if for each theory T and formulae φ, ψ we have: $T, \varphi \vdash \psi$ iff $T \vdash \triangle\varphi \to \psi$.*

Now we present sufficient and necessary condition for **L** to have $\mathrm{DT}_\triangle$. And we also show that in each weakly implicative logic with $\mathrm{DT}_\triangle$ the matching rules for $\triangle$ hold.

Theorem 2.2.26 (Deduction theorem) *Let* **L** *be a finitary logic. Then* **L** *has* $\mathrm{DT}_\triangle$ *iff* **L** *has some presentation $\mathcal{AX}$, where* (MP) *and* (NEC) *are the only deduction rules and all the matching axioms for $\triangle$ are provable in* **L**.

Proof: First direction: Assume that **L** has $\mathrm{DT}_\triangle$, then obviously **L** has presentation where (MP) and (NEC) are the only deduction rules (just replace each rule $\varphi_1, \dots \varphi_n \vdash \psi$ with the following axiom $\vdash \triangle\varphi_1 \to (\dots \to (\triangle\varphi_n \to \psi)\dots))$. All we have to do is to show that the matching rules for $\triangle$ hold:

- (NEC): From $\vdash \triangle\varphi \to \triangle\varphi$ we get $\varphi \vdash \triangle\varphi$ (using $\mathrm{DT}_\triangle$).
- $\triangle 1$: From $\varphi, \varphi \to \psi \vdash \psi$ we get $\varphi, \varphi \to \psi \vdash \triangle\psi$ (using (NEC)). Applying $\mathrm{DT}_\triangle$ twice completes the proof.
- $\triangle 2$: From $\varphi \vdash \varphi$ we get $\vdash \triangle\varphi \to \varphi$ (using $\mathrm{DT}_\triangle$).
- $\triangle 3$: From $\varphi \vdash \triangle\varphi$ we get $\vdash \triangle\varphi \to \triangle\triangle\varphi$ (using $\mathrm{DT}_\triangle$).
- $\triangle$W: From $\varphi, \psi \vdash \varphi$ we get $\vdash \triangle\varphi \to (\triangle\psi \to \varphi)$ (using $\mathrm{DT}_\triangle$ twice).
- $\triangle$C: We know that $\varphi, \triangle\varphi \to (\triangle\varphi \to \psi) \vdash \psi$ ((NEC) and (MP) twice). Applying $\mathrm{DT}_\triangle$ twice completes the proof.
- $\triangle$E: We know that $\varphi, \psi, \triangle\varphi \to (\triangle\psi \to \chi) \vdash \chi$ ((NEC) twice and (MP) twice). Applying $\mathrm{DT}_\triangle$ three times completes the proof.

Reverse direction: we need to prove two directions. One is obvious. To prove the other one we use the induction over the proof of ψ in T, φ (in $\mathcal{AX}$). We show that it holds for each χ in the proof of ψ in T, φ:

- $\chi \in T$, χ is an axiom - trivial using $\triangle$W.
- $\chi = \varphi$ - trivial using $\triangle 2$.
- $\chi = \triangle\psi_1$ is obtained from its predecessor ψ_1 by (NEC). From the induction property for ψ_2 we know that $T \vdash \triangle\varphi \to \psi_1$. We apply (NEC), $\triangle 1$, and (MP) to get that $T \vdash \triangle\triangle\varphi \to \triangle\psi_1$. Using $\triangle 3$ and (WT) we get $T \vdash \triangle\varphi \to \triangle\psi_1$.

- χ is obtained from its predecessors $\psi_2 = \psi_1 \to \chi$ and ψ_1 by (MP). From the induction property for ψ_2 we know that $T \vdash \triangle\varphi \to (\psi_1 \to \chi)$. We apply $\triangle$1 twice to get $T \vdash \triangle\triangle\varphi \to (\triangle\psi_1 \to \triangle\chi)$, using $\triangle$3 and $\triangle$E we get $T \vdash \triangle\psi_1 \to (\triangle\varphi \to \triangle\chi)$.

  From the induction property for ψ_1 we know that $T \vdash \triangle\varphi \to \psi_1$. We apply $\triangle$1 and $\triangle$3 to get $T \vdash \triangle\varphi \to \triangle\psi_1$. Now using (WT) we obtain $T \vdash \triangle\varphi \to (\triangle\varphi \to \triangle\chi)$. Axiom $\triangle$C gets us $T \vdash \triangle\varphi \to \triangle\chi$ and $\triangle$2 completes the proof. QED

We can easily prove analogy of Lemmata 2.1.46 and 2.2.10. Thus having an equivalent definition of fuzzy logics in some class of logics.

Lemma 2.2.27 *Let* **L** *be a fuzzy logic. Then* $\triangle(\varphi \to \psi) \to \chi, \triangle(\psi \to \varphi) \to \chi \vdash_{\mathbf{L}} \chi$.

Lemma 2.2.28 *Let* **L** *be a finitary logic with* $\mathrm{DT}_\triangle$. *Then* **L** *is fuzzy iff in* **L** *we have:* $\triangle(\varphi \to \psi) \to \chi, \triangle(\psi \to \varphi) \to \chi \vdash_{\mathbf{L}} \chi$.

Unlike in Lemma 2.2.10, in this case the rule appearing in the previous lemma is equivalent to the following axiom: $\vdash_{\mathbf{L}} \triangle(\triangle(\varphi \to \psi) \to \chi) \to (\triangle(\triangle(\psi \to \varphi) \to \chi) \to \chi)$. Having $\vee$ in the language we can formulate even stronger claim.

Corollary 2.2.29 *Let* $\mathcal{L}$ *be a language,* $\vee \in \mathcal{L}$ *and* **L** *a finitary logic in* $\mathcal{L}$ *with* $\mathrm{DT}_\triangle$. *Then* **L** *is fuzzy iff* $\vdash_{\mathbf{L}} \triangle(\varphi \to \psi) \vee \triangle(\psi \to \varphi)$.

Now we observe that $\triangle(\varphi \to \varphi)$ can be used as definition of $\overline{1}$. Thus we may assume that whenever $\triangle \in \mathcal{L}$, then $\overline{1} \in \mathcal{L}$. We prove the matching rules for $\overline{1}$ together with the fact that the selection of the formula φ in the definition $\triangle(\varphi \to \varphi) = \overline{1}$ does not matter.

Lemma 2.2.30 *It holds:*

- $\vdash \triangle(\varphi \to \varphi) \to \triangle(\psi \to \psi)$,
- $\psi \vdash \triangle(\varphi \to \varphi) \to \psi$,
- $\triangle(\varphi \to \varphi) \to \psi \vdash \psi$,
- $\vdash \triangle(\varphi \to \varphi)$.

Recall the in presence of $\bot$ we can define the derived connective negation as $\neg\varphi = \varphi \to \bot$. Now we show some rather trivial properties of logics with $\triangle$ and $\bot$ in the language.

Lemma 2.2.31 *Let* $\mathcal{L}$ *be a language such that* $\bot \in \mathcal{L}$ *and* **L** *a logic in* $\mathcal{L}$. *Then*

1. $\vdash_{\mathbf{L}} \triangle(\overline{1} \to \bot) \to \bot$,
2. $\vdash_{\mathbf{L}} \bot \leftrightarrow \triangle\bot$,
3. $\vdash_{\mathbf{L}} \overline{1} \leftrightarrow \triangle\overline{1}$,
4. $\vdash_{\mathbf{L}} (\triangle\varphi \to \neg\triangle\varphi) \to \neg\triangle\varphi$.

Proof: The only non-trivial is Part 1. We give a formal proof.

(i)	$\triangle(\overline{1} \to \bot) \to \triangle(\overline{1} \to \bot)$	(Ref)
(ii)	$\triangle(\overline{1} \to \bot) \to (\triangle\overline{1} \to \triangle\bot)$	(*i*), $\triangle$1, and (WT)
(iii)	$\triangle\overline{1} \to (\triangle(\overline{1} \to \bot) \to \triangle\bot)$	(*ii*), (NEC), $\triangle$E, (WT)
(iv)	$\triangle(\overline{1} \to \bot) \to \triangle\bot$	(*iii*) and (MP)
(v)	$\triangle(\overline{1} \to \bot) \to \bot$	(*iv*), $\triangle$2 and (WT)

QED

Let us assume from now on that $\bot$ is an element of all the propositional languages $\mathcal{L}$. The reader familiar with fuzzy logics with $\triangle$ can notice that matching rules are somehow "weak", $\triangle$ has more properties in this case, which do not hold in general. In the rest of this section we add some additional rules for $\triangle$ to get known fuzzy logics with $\triangle$. We start by showing that fuzzy logic $\mathbf{L}$, where $\vdash_{\mathbf{L}} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$ has some interesting properties.

Lemma 2.2.32 *Let $\mathbf{L}$ be a fuzzy logic in $\mathcal{L}$, such that $\vdash_{\mathbf{L}} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$. Then:*

1. *For each linearly ordered $\mathbf{L}$-matrix $\mathbf{B}$ holds: $\triangle_{\mathbf{B}} x = \overline{1}_{\mathbf{B}}$ if $\overline{1}_{\mathbf{B}} \leq_{\mathbf{B}} x$ and $\triangle_{\mathbf{B}} x = \bot_{\mathbf{B}}$ otherwise.*
2. *$\mathbf{L}$ has $\mathrm{DT}_\triangle$.*
3. *$T, \triangle\varphi \vdash \chi$ and $T, \neg\triangle\varphi \vdash \chi$ entails $T \vdash \chi$.*
4. *$\vdash (\overline{1} \to \bot) \to \bot$ iff $\vdash \neg\triangle\varphi \to \triangle\neg\triangle\varphi$.*
5. *$\vdash \neg\triangle\varphi \to \triangle\neg\triangle\varphi$ iff $\vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$.*

Proof: 1. We omit the subscripts $\mathbf{B}$ in this proof. Let $\overline{1} \leq x$, then obviously $\overline{1} \leq \triangle x$ (using (NEC)). Using $\triangle$W we get $\overline{1} \leq \triangle\overline{1} \leq \triangle x \to \overline{1}$, thus $\triangle x \leq \overline{1}$.

If $x < \overline{1}$ then $\triangle x < \overline{1}$ as well (because ($\triangle$2) we know $\triangle x \leq x$). Since $\neg\triangle x \to \triangle x \leq \triangle x$ we have to have $\neg\triangle x > \triangle x$ (otherwise $\overline{1} \leq \neg\triangle x \to \triangle x$ and so $\overline{1} \leq \triangle x$—a contradiction). Observe that from $\triangle$C we get $\vdash_{\mathbf{L}} \triangle(\triangle\varphi \to (\triangle\varphi \to \bot)) \to (\triangle\varphi \to \bot)$, which leads to $\triangle(\triangle x \to \neg\triangle x) \leq \neg\triangle x$. We know that $\triangle x \leq \neg\triangle x$ so $\triangle(\triangle x \to \neg\triangle x) = \overline{1}$ and so $\overline{1} \leq \triangle x \to \bot$. Finally, $\triangle x \leq \bot$.

2. One direction is obvious. We show the reverse one contrapositively: if $T \not\vdash \triangle\varphi \to \psi$, then there is linearly ordered $\mathbf{L}$-matrix $\mathbf{B}$ and $\mathbf{B}$-model e of T and $e(\triangle\varphi \to \psi) < \overline{1}$ (again we omit the subscripts $\mathbf{B}$). Then obviously $\overline{1} \leq e(\varphi)$ (otherwise $e(\triangle\varphi) = \bot$ and so we get that $\overline{1} \leq e(\triangle\varphi \to \psi)$—a contradiction), thus $e(\triangle\varphi) = \overline{1}$ and so $e(\triangle\varphi \to \psi) = \overline{1} \to e(\psi) = e(\psi)$. So we know that e is $\mathbf{B}$-model e of T, φ and $e(\psi) < \overline{1}$. Thus $T, \varphi \not\vdash \psi$.

3. We observe that since $\vdash \triangle(\neg\triangle\psi \to \triangle\psi) \to \triangle\psi$ and $\vdash \triangle(\triangle\psi \to \neg\triangle\psi) \to \triangle\neg\triangle\psi$ (from Lemma 2.2.31), whenever we prove $T \vdash \triangle\psi \to \chi$ and $T \vdash \triangle\neg\triangle\psi \to \chi$ then $T \vdash \chi$ (using Lemma 2.2.28 and the fact that $\mathbf{L}$ has $\mathrm{DT}_\triangle$). Using $\mathrm{DT}_\triangle$ once more completes the proof.

4. Assume that $\vdash (\overline{1} \to \bot) \to \bot$, then also $\neg\overline{1} \to \chi$. We use part 3 of this lemma: first notice that $\triangle\varphi \vdash \overline{1} \leftrightarrow \triangle\varphi$ and so $\triangle\varphi \vdash \neg\triangle\varphi \to \neg\overline{1}$. Thus $\triangle\varphi \vdash \neg\triangle\varphi \to \triangle\neg\triangle\varphi$. Second, we also know that $\neg\triangle\varphi \vdash \overline{1} \leftrightarrow \neg\triangle\varphi$ and since $\vdash \overline{1} \to \triangle\overline{1}$ we get $\neg\triangle\varphi \vdash \neg\triangle\varphi \to \triangle\neg\triangle\varphi$.

To prove the reverse direction just set $\varphi = \bot$ and get $\neg\triangle\bot \to \triangle\neg\triangle\bot$. Lemma 2.2.31 completes the proof.

5. Observe that $\psi \to \varphi \vdash \triangle\neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$ (using $\triangle$W, $\triangle$3, and $\mathrm{DT}_\triangle$) and so $\psi \to \varphi \vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$ (using $\vdash \neg\triangle\varphi \to \triangle\neg\triangle\varphi$). Recall that $\varphi \to \psi \vdash \triangle(\varphi \to \psi) \leftrightarrow \overline{1}$ and since we know $\vdash (\overline{1} \to \bot) \to \chi$ (from part 4. of this lemma) we get $\varphi \to \psi \vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$.

To prove the reverse direction just set $\varphi = \bot$, $\psi = \overline{1}$, and get $\vdash \neg\triangle(\bot \to \overline{1}) \to \triangle(\overline{1} \to \bot)$. Observe that $\vdash \triangle(\bot \to \overline{1}) \leftrightarrow \overline{1}$ (using $\mathrm{DT}_\triangle$) and so we have $\vdash (\overline{1} \to \bot) \to \triangle(\overline{1} \to \bot)$. Lemma 2.2.31 and Part 4. of this lemma complete the proof. QED

Observe that the part 1. holds even without assumption that $\mathbf{L}$ is fuzzy and that part 2. has an interesting corollary:

Corollary 2.2.33 *Let $\mathbf{L}$ be a finitary fuzzy logic in $\mathcal{L}$, such that $\vdash_{\mathbf{L}} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$. Then $\mathbf{L}$ has a presentation, where* (MP) *and* (NEC) *are the only deduction rules.*

Of course, we even know this presentation: just replace each rule $\varphi_1, \dots \varphi_n \vdash \psi$ with the axiom $\vdash \triangle\varphi_1 \to (\dots \to (\triangle\varphi_n \to \psi)\dots)$. Now we try to formulate the "essence" of the connective $\triangle$, when used in fuzzy logics.

Definition 2.2.34 *Let $\mathcal{L}$ be a language such that $\bot, \triangle \in \mathcal{L}$ and* **L** *a fuzzy logic in $\mathcal{L}$. We say that* **L** *is logic with* Baaz delta *iff for each formula in language $\{\to, \bot, \overline{1}\}$, if we define substitution $\sigma(v) = \triangle v$ then $\vdash_{\mathbf{L}} \sigma\varphi$ iff φ is a theorem of classical logic.*

Obviously, not all logics with $\triangle$ connective are logic with Baaz delta (take Intuitionistic logic, and in all Heyting algebras interpret $\triangle$ as identity). Observe that if **L** is a logic with Baaz delta it is consistent. This definition can be viewed as rather peculiar, but we are going to present more convenient alternative definition. The following lemma works for fuzzy logic only: even with Intuitionistic logic fulfills the properties 1.–5. from the upcoming definition it is not a logic with Baaz delta, analogously Łukasiewicz logic with globalization is not fuzzy logic and so it is not a logic with Baaz delta.

Lemma 2.2.35 *Let* **L** *be a consistent fuzzy logic. Then* **L** *is a fuzzy logic with Baaz delta iff the following holds:*

1. $\vdash (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$,
2. $\vdash (\overline{1} \to \overline{1}) \to \overline{1}$,
3. $\vdash (\bot \to \overline{1}) \to \overline{1}$,
4. $\vdash (\bot \to \bot) \to \overline{1}$,
5. $\vdash (\overline{1} \to \bot) \to \bot$.

Proof: One direction is obvious (only non-trivial part is to show $\vdash (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$—to do this just notice that $(\neg p \to p) \to p$ is a theorem of classical logic).

To prove converse direction we need to show two things: first, if φ is a theorem of the classical logic, then $\vdash \sigma\varphi$. We prove this contrapositively: assume that $\not\vdash \sigma\varphi$, then there is linear **L**-matrix **B** and **B**-evaluation e such that $e(\sigma\varphi) < \overline{1}$. Observe that form Lemma 2.2.32 part 1. we know that $e(\triangle v)$ is either $\overline{1}$ or $\bot$, from the form of the formula $\sigma\varphi$ and from theorems 2.–5. we know that $e(\sigma\varphi) = \bot$ and if we define evaluation $f(v) = e(\triangle v)$ then φ is a classical evaluation not satisfying φ.

Second, we need to show that if $\vdash \sigma\varphi$ then φ is a theorem of the classical logic. We prove this by a contradiction: assume that there is a classical evaluation e, such that $e(\varphi) = \bot$. Then there is substitution $\rho(v) = e(v)$ (we identify constants $\overline{1}$ and $\bot$ with two truth values of classical logic) and $\rho\varphi \to \bot$ is a theorem of the classical logic. Thus $\vdash \sigma(\rho\varphi \to \bot)$. Because $\sigma(\rho\varphi \to \bot) = \rho\varphi \to \bot$ (there are no variables in $\rho\varphi \to \bot$) and theorems 2.–5. we get $\vdash \rho\varphi \to \bot$ (using the previous direction). Since we assume that $\vdash \sigma\varphi$ we also have $\vdash \rho\sigma\varphi$ and so finally $\vdash \rho\varphi$ (because $\triangle\bot \leftrightarrow \bot$ and $\triangle\overline{1} \leftrightarrow \overline{1}$). Thus together we have $\vdash \bot$—a contradiction with consistency of **L**. QED

In the literature, it is common that for fuzzy logic **L** there is defined its conservative expansion by the connective $\triangle$ (usually denoted as $\mathbf{L}_\triangle$), which is fuzzy as well. It is usual that $\mathbf{L}_\triangle$ is a logic with Baaz delta and has $\mathrm{DT}_\triangle$ (even if **L** has not some variant of deduction theorem). Now we introduce general way of expanding the logic **L** into the logic $\mathbf{L}_\triangle$. First we give an indirect definition and then we show how to find a presentation of $\mathbf{L}_\triangle$ based on the presentation of $\mathbf{L}_\triangle$.

Definition 2.2.36 *Let* **L** *be a consistent fuzzy logic in $\mathcal{L}$, such that $\bot \in \mathcal{L}$ and $\triangle \notin \mathcal{L}$ By $\mathbf{L}_\triangle$ we denote the weakest fuzzy logic with Baaz delta in language $\mathcal{L} \cup \{\triangle\}$ expanding* **L**.

In the next chapter, we will see other axiomatic systems for logics with Baaz delta than the one we are going to present (we prove their equivalences). However, this one has several advantages. First, it is formulated in the language with $\to$, $\bot$, and $\triangle$ only and second, it does not needs any structural rules of the starting logic.

Theorem 2.2.37 *Let* $\mathbf{L}$ *be a consistent finitary fuzzy logic in* $\mathcal{L}$*, such that* $\bot \in \mathcal{L}$ *and* $\triangle \notin \mathcal{L}$ *and let* $\mathcal{AX}$ *be some finitary presentation of* $\mathbf{L}$*. Then the following is a presentation of* $\mathbf{L}_\triangle$*:*

A	*axioms of* $\mathcal{AX}$*,*
B	$\vdash \triangle\varphi_1 \to (\dots(\triangle\varphi_n \to \psi)$ *for each* n*-ary deduction rule* $<\varphi_1,\dots,\varphi_n,\psi> \in \mathcal{AX}$*,*
C	*matching rules for* $\triangle$*,*
(MP)	$\varphi, \varphi \to \psi \vdash \psi$*,*
$\triangle 4$	$\vdash (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$*,*
$\triangle 5$	$\vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$*,*
$\triangle 6$	$\vdash (\overline{1} \to \overline{1}) \to \overline{1}$.

Proof: Let $\mathbf{L}'$ be the logic with the above presentation. We have to show that $\mathbf{L}'$ is a fuzzy logic with Baaz delta (it obviously extends $\mathbf{L}$). Since $\mathbf{L}$ is fuzzy we get $(\varphi \to \psi) \to \chi$, $(\psi \to \varphi) \to \chi \vdash_{\mathbf{L}} \chi$ and so $(\varphi \to \psi) \to \chi, (\psi \to \varphi) \to \chi \vdash_{\mathbf{L}'} \chi$. From Lemma 2.2.31 and $\triangle 4$ we know that $\vdash_{\mathbf{L}'} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$ and $\vdash_{\mathbf{L}'} (\triangle\varphi \to \neg\triangle\varphi) \to \neg\triangle\varphi$. Thus if we prove $T \vdash_{\mathbf{L}'} \triangle\varphi \to \chi$ and $T \vdash_{\mathbf{L}'} \neg\triangle\varphi \to \chi$ we get $T \vdash_{\mathbf{L}'} \chi$.

Let us denote $T = \{\triangle(\varphi \to \psi) \to \chi, \triangle(\psi \to \varphi) \to \chi\}$. Observe that $T, \triangle(\varphi \to \psi) \vdash_{\mathbf{L}'} \chi$ and $T, \neg\triangle(\varphi \to \psi) \vdash_{\mathbf{L}'} \chi$ (the first is obvious, to prove the second use $\triangle 4$). Thus $T \vdash_{\mathbf{L}'} \chi$. From Theorem 2.2.26 we know that $\mathbf{L}'$ has $\mathrm{DT}_\triangle$ and from the fact $T \vdash_{\mathbf{L}'} \chi$ we know that $\mathbf{L}'$ is fuzzy (using Lemma 2.2.28).

Now we observe that $\mathbf{L}'$ is a conservative expansion of $\mathbf{L}$ (we can extend any linear $\mathbf{L}$-matrix $\mathbf{B}$, which is counterexample to $T \vdash \varphi$, into the linear $\mathbf{L}'$-matrix $\mathbf{B}_\triangle$ and is a counterexample as well—since $\mathbf{L}'$ is fuzzy). So we know that $\mathbf{L}'$ is consistent (because $\mathbf{L}$ is consistent) .

Thus we can use Lemma 2.2.35 to prove that $\mathbf{L}'$ is a logic with Baaz delta. Notice that all we need to show is the following:

1. $\vdash (\bot \to \overline{1}) \to \overline{1}$,
2. $\vdash (\bot \to \bot) \to \overline{1}$,
3. $\vdash (\overline{1} \to \bot) \to \bot$.

To prove 1. use theorem $\vdash (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$ for $\varphi = \overline{1}$, we get $\vdash (\neg\triangle\overline{1} \to \triangle\overline{1}) \to \triangle\overline{1}$. Lemma 2.2.31 completes the proof. To prove 2. use theorem $\vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$ for $\varphi = \overline{1}$ and $\psi = \bot$, we get $\vdash \neg\triangle(\overline{1} \to \bot) \to \triangle(\bot \to \overline{1})$. Lemma 2.2.31 completes the proof. To prove 3. use theorem $\vdash \neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$ for $\varphi = \bot$ and $\psi = \overline{1}$, we get $\vdash \neg\triangle(\bot \to \overline{1}) \to \triangle(\overline{1} \to \bot)$. Lemma 2.2.31 completes the proof.

To complete the proof of this theorem we need to show that each fuzzy logic $\mathbf{L}'$ with Baaz delta proofs all formulae from our presentation. From Lemma 2.2.35 we know that $\vdash_{\mathbf{L}'} \triangle 6$, $\vdash_{\mathbf{L}'} \triangle 4$, and $\vdash_{\mathbf{L}'} (\overline{1} \to \bot) \to \bot$. Observe that in this case we can use Lemma 2.2.32 Parts 4. and 5. to get $\vdash_{\mathbf{L}'} \triangle 5$. Logic $\mathbf{L}'$ obviously proves all consecution from the group A and B. So all we have to show are the axioms from the group B: to do this just observe that $\mathbf{L}'$ has $\mathrm{DT}_\triangle$ (because $\mathbf{L}'$ is fuzzy and $\vdash_{\mathbf{L}'} \triangle 4$ we can use Lemma 2.2.32) and since $\mathbf{L} \subseteq \mathbf{L}'$ the proof is done. QED

The presence of the axiom $\vdash (\overline{1} \to \overline{1}) \to \overline{1}$ seems to be unavoidable, however under some rather weak additional assumptions we can omit it. One of them is of course the presence of weakening, the other one is the presence of Sf (then we get $\vdash (\overline{1} \to \overline{1}) \to (\bot \to \overline{1})$ and using the know fact that $\vdash (\bot \to \overline{1}) \to \overline{1}$ we would get our axiom). Now we show that $\mathbf{L}_\triangle$ has some promised nice properties:

Lemma 2.2.38 *Let* $\mathbf{L}$ *be a consistent fuzzy logic in* $\mathcal{L}$*, such that* $\triangle \notin \mathcal{L}$ *and* $\bot \in \mathcal{L}$*. Then* $\mathbf{L}_\triangle$ *is a fuzzy logic with* $\mathrm{DT}_\triangle$ *and Baaz delta, which is a conservative expansion of* $\mathbf{L}$*.*

2.3 First-order logic

In this second part of this paper, we move to the first-order logics. We present the very basic theorems only. The broader treatment of this topic will be the content of the subsequent papers. Our approach is inspired by the classical first-order logic and by its modification (the axiomatic system, the notion of Henkin theory) for non-classical logics, the main source is Hájek's treatment of basic predicate fuzzy logic (for details see [44]).

2.3.1 Basic definitions

In the following let $\mathbf{L}$ be a fixed weakly implicative logic in propositional language $\mathcal{L}$.

Definition 2.3.1 (Predicate language) *By* multi-sorted predicate language $\mathbf{\Gamma}$ *we understand a quintuple* $(\mathbf{S}, \preceq, \mathbf{P}, \mathbf{F}, \mathbf{A})$*, where* $\mathbf{S}$ *is a non-empty set of sorts,* $\preceq$ *is an ordering on* $\mathbf{S}$ *(indicating the subsumption of sorts),* $\mathbf{P}$ *is a non-empty set of predicate symbols,* $\mathbf{F}$ *is a set of function symbols, and* $\mathbf{A}$ *is a function assigning to each predicate and function symbol a finite sequences of elements of* $\mathbf{S}$.

Let $|\mathbf{A}(P)|$ *denote the length of the sequence* $\mathbf{A}(P)$*. The number* $|\mathbf{A}(P)|$ *is called the* arity of the predicate symbol P*. The number* $|\mathbf{A}(f)| - 1$ *is called the* arity of the function symbol f*. The functions* f *for which* $\mathbf{A}(f) = <s>$ *are called the* individual constants *of sort* s*. If* $s_1 \preceq s_2$ *holds for sorts* s_1, s_2 *we say that* s_2 subsumes s_1.

The $\mathcal{L}$-logical symbols *are individual variables* $x^s, y^s, \ldots$ *for each sort* s*, the logical connectives of* $\mathcal{L}$*, and the quantifiers* $\forall$ *and* $\exists$.

Let us denote by $\mathbf{C}_s$ the set of constants of the sort s. In the following let $\mathbf{\Gamma}$ be a fixed multi-sorted predicate language for logic $\mathbf{L}\forall$.

Definition 2.3.2 (Terms) *Each individual variable of sort* s *is a* $\mathbf{\Gamma}$-term *of sort* s*. Let* f *be a function symbol,* $t_1, \ldots, t_n$ *terms of sorts* $s_1, \ldots, s_n$*, and* $\mathbf{A}(f) = <w_1, \ldots, w_n, w_{n+1}>$ *such that* $s_i \preceq w_i$ *for* $i \leq n$*. Then* $f(t_1, \ldots, t_n)$ *is a* $\mathbf{\Gamma}$*-term of sort* w_{n+1}.

Notice that the set of terms depends on $\mathbf{\Gamma}$ only, whereas the set of formulae depends of the propositional language as well. So we should speak about $\mathbf{\Gamma}$-terms and $(\mathbf{\Gamma}, \mathcal{L})$-formulae (however, we speak about $\mathbf{\Gamma}$-formulae if the propositional language is clear from the context and we speak about terms and formulae if both propositional and the predicate language are clear from the context).

Definition 2.3.3 (formulae) *Let* $t_1, \ldots t_n$ *be terms of sorts* $s_1, \ldots s_n$*, and* P *be a predicate symbol,* $\mathbf{A}(P) = <w_1, \ldots w_n>$*, such that* $s_i \preceq w_i$ *for* $i \leq n$*. Then* $P(t_1, \ldots t_n)$ *is an* atomic $\mathbf{\Gamma}$-formula*. The nullary logical connectives of* $\mathcal{L}$ *are atomic* $\mathbf{\Gamma}$*-formulae as well.*

Let φ *be* $\mathbf{\Gamma}$*-formula and* x^s *an object variable of the sort* s*. Then* $(\forall x^s)\varphi$ *and* $(\exists x^s)\varphi$ *are* $\mathbf{\Gamma}$*-formulae. Furthermore, the class of* $\mathbf{\Gamma}$*-formulae is closed under logical connectives of* $\mathcal{L}$.

Bounded *and* free variables *in a formula are defined as usual. A formula is called a* sentence *iff it contains no free variables. A set of* $\mathbf{\Gamma}$*-sentences is called a* $\mathbf{\Gamma}$-theory.

Instead of $\xi_1, \ldots, \xi_n$ (where ξ_i's are terms or formulae and n is arbitrary or fixed by the context) we shall sometimes write just $\vec{\xi}$.

Unless stated otherwise, the expression $\phi(x_1, \ldots, x_n)$ means that all free variables of ϕ are among $x_1, \ldots, x_n$.

If $\phi(x_1, \ldots, x_n, \vec{z})$ is a formula and we substitute terms t_i for all x_i's in ϕ, we denote the resulting formula in the context simply by $\phi(t_1, \ldots, t_n, \vec{z})$.

Definition 2.3.4 (Substitutability) *A term* t *of sort* w *is* substitutable *for the individual variable* x^s *in a formula* $\varphi(x^s, \vec{z})$ *iff* $w \preceq s$ *and no occurrence of any variable* y *occurring in* t *is bounded in* $\varphi(t, \vec{z})$.

Let $\mathbf{B}$ be fixed ordered $\mathbf{L}$-matrix in the following text.

Definition 2.3.5 (Structure) *An* $\mathbf{B}$-structure $\mathbb{M}$ *for* $\mathbf{\Gamma}$ *has the following form:* $\mathbb{M} = ((M_s)_{s\in\mathbf{S}}, (P_\mathbb{M})_{P\in\mathbf{P}}, (f_\mathbb{M})_{f\in\mathbf{F}})$, *where* M_s *is a non-empty domain for each* $s \in \mathbf{S}$ *and* $M_s \subseteq M_w$ *iff* $s \preceq w$; $P_\mathbb{M}$ *is an* n*-ary fuzzy relation* $\prod_{i=1}^{n} M_{s_i} \to \mathbf{L}$ *for each predicate symbol* $P \in \mathbf{P}$ *such that* $\mathbf{A}(P) = <s_1, \ldots, s_n>$; $f_\mathbb{M}$ *is a function* $\prod_{i=1}^{n} M_{s_i} \to M_{s_{n+1}}$ *for each function symbol* $f \in \mathbf{F}$ *such that* $\mathbf{A}(f) = <s_1, \ldots, s_n, s_{n+1}>$, *and an element of* M_s *if* f *is a constant of sort* s.

Definition 2.3.6 (Evaluation) *Let* $\mathbb{M}$ *be a* $\mathbf{B}$*-structure for* $\mathbf{\Gamma}$. *An* $\mathbb{M}$-evaluation *of the object variables is a mapping* e *which assigns to each variable of sort* s *an element from* M_s *(for all sorts* $s \in \mathbf{S}$*).*

Let e *be an* $\mathbb{M}$*-evaluation,* x *a variable of sort* s*, and* $a \in M_s$. *Then* $\mathrm{e}[x \to a]$ *is an* $\mathbb{M}$*-evaluation such that* $\mathrm{e}[x \to a](x) = a$ *and* $\mathrm{e}[x \to a](y) = \mathrm{e}(y)$ *for each individual variable* y *different from* x.

Definition 2.3.7 (Truth definition) *Let* $\mathbb{M}$ *be a* $\mathbf{B}$*-structure for* $\mathbf{\Gamma}$ *and* v *an* $\mathbb{M}$*-evaluation.* Values *of the terms and* truth values *of the formulae in* $\mathbb{M}$ *for an evaluation* v *are defined as follows:*

$$\begin{array}{rcl}
||x_s||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & \mathrm{v}(x)\,, \\
||f(t_1, t_2,\, \ldots,\, t_n)||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & f_\mathbb{M}(||t_1||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ ||t_2||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ \ldots,\ ||t_n||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}})\,, \\
||P(t_1,\ t_2,\, \ldots,\, t_n)||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & P_\mathbb{M}(||t_1||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ ||t_2||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ \ldots,\ ||t_n||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}})\,, \\
||c(\varphi_1, \varphi_2,\, \ldots,\, \varphi_n)||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & c_\mathbf{B}(||\varphi_1||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ ||\varphi_2||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}},\ \ldots,\ ||\varphi_n||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}})\,, \\
||(\forall x^s)\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & \inf\{||\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}[x^s\to a]} \mid a \in M_s\}\,, \\
||(\exists x^s)\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} & = & \sup\{||\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}[x^s\to a]} \mid a \in M_s\}\,.
\end{array}$$

If the infimum or supremum does not exist, we take its value as undefined. We say that a $\mathbf{B}$*-structure* $\mathbb{M}$ *for* $\mathbf{\Gamma}$ *is* safe *iff* $\|\varphi\|^{\mathbf{B}}_{\mathbb{M},\mathrm{v}}$ *is defined for each* $\mathbf{\Gamma}$*-formula* φ *and each* $\mathbb{M}$*-evaluation* v.

Definition 2.3.8 (Value of formula) *Let* $\mathbb{M}$ *be a safe* $\mathbf{B}$*-structure for* $\mathbf{\Gamma}$*, and* φ *a* $\mathbf{\Gamma}$ *formula.* A truth value of the formula φ *in* $\mathbb{M}$ *is defined as follows:*

$$||\varphi||^{\mathbf{B}}_{\mathbb{M}} = \inf\{||\varphi||^{\mathbf{B}}_{\mathbb{M},\mathrm{v}} \mid \mathrm{v} \text{ is an } \mathbb{M}\text{-evaluation}\}.$$

We say that φ *is an* $\mathbf{B}$-tautology *if* $||\varphi||^{\mathbf{A}}_{\mathbb{M}} \in D_\mathbf{B}$ *for each safe* $\mathbf{B}$*-structure* $\mathbb{M}$.

Definition 2.3.9 (Model) *Let* $\mathbb{M}$ *be a* $\mathbf{B}$*-structure for* $\mathbf{\Gamma}$*, and* T *a* $\mathbf{\Gamma}$*-theory. Then the* $\mathbf{B}$*-structure* $\mathbb{M}$ *for* $\mathbf{\Gamma}$ *is called* $\mathbf{B}$*-model of* T *if* $||\varphi||^{\mathbf{A}}_{\mathbb{M}} \in D_\mathbf{B}$ *for each* $\varphi \in T$. *We denote the set of* $\mathbf{A}$*-models od* T *by* $\mathcal{MOD}(T, \mathbf{A})$.

Definition 2.3.10 (Semantical consequence) *Let* $\mathcal{K}$ *be a class of* $\mathcal{L}$*-matrices. We say that* φ *is a semantical consequence of the* T *w.r.t. class* $\mathcal{K}$ *if* $\mathcal{MOD}(T, \mathbf{B}) = \mathcal{MOD}(T \cup \{\varphi\}, \mathbf{B})$ *for each* $\mathbf{B} \in \mathcal{K}$*; we denote it by* $T \models_\mathcal{K} \varphi$. *By* $\mathcal{TAUT}(\mathcal{K})$ *we understand the set* $\{\varphi \mid \emptyset \models_\mathcal{K} \varphi\}$.

We write $T \models_\mathbf{L} \varphi$ instead of $T \models_{o-\mathbf{MAT(L)}} \varphi$ and we also write $T \models^l_\mathbf{L} \varphi$ instead of $T \models_{l-\mathbf{MAT(L)}} \varphi$.

For a fixed $\mathbf{B}$-model $\mathbb{M}$ and an $\mathbb{M}$-valuation e such that $\mathrm{e}(x_i) = a_i$ (for all i's), instead of $\|\varphi(x_1, \ldots, x_n)\|^{\mathbf{B}}_{\mathbb{M},\mathrm{e}}$ we write simply $\|\varphi(a_1, \ldots, a_n)\|$ and speak of the value of $\varphi(a_1, \ldots, a_n)$.

Now we define the predicate logic to each weakly implicative logic (and stronger predicate logic for each fuzzy logics). As in the propositional case we understand the predicate logic as an asymmetric consequence relation (following Dunn's terminology). For simplicity of this introductory chapter we made two extra design choices. First, we assume that $\vee$ is the part of the language (there are ways how avoid the need for $\vee$ under some additional assumptions eg. having exchange, or having $\rightsquigarrow$ in the language). Second, we formulate consecutions $(\forall 2)$

and ($\exists$2) as axioms rather than rules, which would result into the weaker definition. However, it is obvious that under some rather weak assumptions these two notions would coincide (eg. under the presence of & in the language). These two topics will be more elaborated in some subsequent paper.

Definition 2.3.11 *Let* $\mathbf{L}$ *be a weakly implicative logic. The logic* $\mathbf{L}\forall^-$ *is given by the following axioms and deduction rules:*

(P)	*the formulae and deduction rules resulting from the axioms and deduction rules of* $\mathbf{L}$ *by the substitution of the propositional variables by the formulae of* $\mathbf{\Gamma}$,
($\forall$1)	$\vdash_{\mathbf{L}\forall^-} (\forall x)\varphi(x) \to \varphi(t)$, *where* t *is substitutable for* x *in* φ,
($\exists$1)	$\vdash_{\mathbf{L}\forall^-} \varphi(t) \to (\exists x)\varphi(x)$, *where* t *is substitutable for* x *in* φ,
($\forall$2)	$\vdash_{\mathbf{L}\forall^-} (\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, *where* x *is not free in* χ,
($\exists$2)	$\vdash_{\mathbf{L}\forall^-} (\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, *where* x *is not free in* χ,
(Gen)	$\varphi \vdash_{\mathbf{L}\forall^-} (\forall x)\varphi$.

Furthermore, if $\mathbf{L}$ *is a fuzzy logic we define the logic* $\mathbf{L}\forall$ *as an extension of* $\mathbf{L}\forall^-$ *by axiom:*

($\forall$3) $\vdash_{\mathbf{L}\forall} (\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, *where* x *is not free in* χ.

Logics $\mathbf{L}\forall$ and $\mathbf{L}\forall^-$ are sometimes the same (Łukasiewicz predicate logic) and sometimes they are different (Gödel predicate logic). Now we recall the concept of the Prelinearity Property, we will need this property to prove the completeness theorem of $\mathbf{L}\forall$ w.r.t. linearly ordered matrices.

Definition 2.3.12 (Prelinearity) *Fuzzy logic* $\mathbf{L}\forall$ *has the Prelinearity Property (*PP*) if for each theory* T *and each sentences* φ *and* ψ *we get* $T \vdash \chi$ *whenever* $T, \varphi \to \psi \vdash \chi$ *and* $T, \psi \to \varphi \vdash \chi$.

Unluckily, we are not able to prove that each predicate fuzzy logic has PP. However, we can give some simply checkable sufficient conditions. Before we do so we observe that we can easily prove the both deduction theorems (we assume the same definition of Implicational (Delta) Deduction theorem are in propositional case, we only assume that φ is a sentence).

Theorem 2.3.13 (Deduction Theorems) *Let* $\mathbf{L}$ *be a logic with* $\mathrm{DT}_\to$ *(*$\mathrm{DT}_\triangle$ *respectively). Then both logics* $\mathbf{L}\forall^-$ *and* $\mathbf{L}\forall$ *have* $\mathrm{DT}_\to$ *(*$\mathrm{DT}_\triangle$ *resp.).*

Corollary 2.3.14 *Let* $\mathbf{L}$ *be a fuzzy logic, such that* $\mathbf{L}$ *has some presentation where* (MP) *is the only deduction rule and implicational fragment of* $\mathbf{L}$ *is an extension of* BCK. *Then* $\mathbf{L}\forall$ *has* PP.

Corollary 2.3.15 *Let* $\mathcal{L}$ *be a propositional language,* $\triangle \in \mathcal{L}$ *and* $\mathbf{L}$ *a fuzzy logic in* $\mathcal{L}$, *such that* $\mathbf{L}$ *has some presentation where* (MP) *and* (NEC) *are the only deduction rules. Then* $\mathbf{L}\forall$ *has* PP.

2.3.2 Henkin and witnessed theories

In this subsection we prepare some technical means to proof the completeness.

Definition 2.3.16 (Henkin and φ-witnessed theories) *The theory is called* Henkin *theory if for each sentence* $\varphi = (\forall x)\psi$ *and* $T \not\vdash \varphi$, *there is a constant* c *such that* $T \not\vdash \psi(c)$.

Furthermore, let $\varphi(x_1, \dots x_n, y)$ *be a formula. Henkin theory is called* φ-witnessed *theory if for each formula* $\psi(y) = \varphi(x_1 : t_1, \dots x_n : t_n, y)$, *where* t_i *are closed terms holds: if* $T \vdash (\exists y)\psi(y)$, *then there is a constant* c *such that* $T \vdash \psi(c)$.

Definition 2.3.17 (Henkin and φ-witnessed logics) *Let* $\mathbf{L}$ *be a weakly implicative logic and* φ *a formula. We say that the logic* $\mathbf{L}\forall^-$ *is Henkin (*φ*-witnessed) for each theory* T *and each sentence* α*,* $T \not\vdash \alpha$ *there is a Henkin (*φ*-witnessed) theory* T' *such that* $T \subseteq T'$ *and* $T' \not\vdash \alpha$*.*

Let $\mathbf{L}$ *be a fuzzy logic and* φ *a formula. We say that the logic* $\mathbf{L}\forall$ *is Henkin (*φ*-witnessed) for each theory* T *and each sentence* α*,* $T \not\vdash \alpha$ *there is a linear Henkin (*φ*-witnessed) theory* T' *such that* $T \subseteq T'$ *and* $T' \not\vdash \alpha$*.*

Definition 2.3.18 (proto-φ-witnessed logics) *Let* $\mathbf{L}$ *be a weakly implicative logic and* φ *a formula. We say that the logic* $\mathbf{L}\forall^-$ *(or the logic* $\mathbf{L}\forall$*) is proto-*φ*-witnessed if for each theory* T *and for each formula* $\psi(y) = \varphi(x_1 : t_1, \dots x_n : t_n, y)$*, where* t_i *are closed terms holds:* $T \cup \{\psi(c)\}$ *is a conservative extension of* $T \cup \{(\exists y)\psi(y)\}$*.*

In fact, the logic is proto-φ-witnessed iff it supports introduction of Skolem constants. We use this property to show, that in that case we can introduce Skolem function of arbitrary arity.

Lemma 2.3.19 *Let* $\mathbf{L}$ *be a weakly implicative logic and* φ *a formula. If the logic* $\mathbf{L}\forall^-$ *is* φ*-witnessed then it is proto-*φ*-witnessed .*

Furthermore, let $\mathbf{L}$ *be a fuzzy logic and* φ *a formula. If the logic* $\mathbf{L}\forall$ *is* φ*-witnessed then it is proto-*φ*-witnessed.*

Proof: Let T be a theory and $\psi(y) = \varphi(x_1 : t_1, \dots x_n : t_n, y)$, where t_i are closed terms, such that theory $T \cup \{\psi(c)\}$ is not a conservative extension of $T \cup \{(\exists y)\psi(y)\}$, i.e., there is a formula α such that $T \cup \{\psi(c)\} \vdash \alpha$ and $T \cup \{(\exists y)\psi(y)\} \not\vdash \alpha$. Let us take φ-witnessed theory T', such that $T \cup \{(\exists y)\psi(y)\} \subseteq T'$ and $T' \not\vdash \alpha$. Since T' is φ-witnessed and $T' \vdash (\exists y)\psi(y)$ there is a constant d such that $T' \vdash \psi(d)$. Since $T \cup \{\psi(c)\} \vdash \alpha$ we get $T \cup \{\psi(d)\} \vdash \alpha$ and so $T' \vdash \alpha$—a contradiction

The proof of the second part as the same. QED

Definition 2.3.20 (Directed set of formulae) *Let* Ψ *be a set of formulae. We say that* Ψ *is a directed set if for each* $\psi, \varphi \in \Psi$ *there is* $\delta \in \Psi$ *such that* $\vdash \psi \to \delta$ *and* $\vdash \varphi \to \delta$ *(we call* δ *the upper bound of* φ *and* ψ*).*

This is the crucial lemma of this chapter. We use it for the proof of completeness of our first-order calculi.

Lemma 2.3.21 (φ-witnessed extension) *Let* $\mathbf{L}$ *be a finitary fuzzy logic and* $\mathbf{L}\forall$ *has* PP*, and* φ *be a formula. If the logic* $\mathbf{L}\forall$ *is proto-*φ*-witnessed then it is* φ*-witnessed.*

Proof: We construct our extension by a transfinite induction. Let T be a theory and α a formula $T \not\vdash \alpha$. If Ψ is a set of formulae by $T \not\vdash \Psi$ we mean $T \not\vdash \psi$ for each $\psi \in \Psi$.

Before we start we extend our predicate language by new constants $\{c^s_\nu \mid \nu \leq ||\Gamma||\}$ for each sort s. Let $T_0 = T$ and $\Psi_0 = \{\alpha\}$. We enumerate all formulae with one free variable x by ordinal numbers as χ_μ and all formulae with one free variable x of the form $\varphi(x_1 : t_1, \dots x_n : t_n, x)$ by ordinal numbers as σ_μ.

We construct directed sets Ψ_μ and theories T_μ so $T_\mu \not\vdash \Psi_\mu$ and $T_\mu \subseteq T_\nu$ and $\Psi_\mu \subseteq \Psi_\nu$ for $\mu \leq \nu$. Thus we get $T_\mu \not\vdash \alpha$. Observe that theory T_0 and set Ψ_0 fulfill these conditions. The induction step:

Let us define the sets: $\hat{T}_\mu = \bigcup\limits_{\nu<\mu} T_\nu$ and $\hat{\Psi}_\mu = \bigcup\limits_{\nu<\mu} \Psi_\nu$. Notice that from the induction property we get that $\hat{T}_\mu \not\vdash \hat{\Psi}_\mu$ and $\hat{\Psi}_\mu$ is directed set. Now we construct theory T'_μ and set Ψ_μ. We distinguish two cases:

(H1) There is $\psi \in \hat{\Psi}_\mu$ such that $\hat{T}_\mu \vdash \psi \vee \chi_\mu(c)$. Let $T'_\mu = \hat{T}_\mu \cup \{\psi \to (\forall x)\chi_\mu(x)\}$ and $\Psi_\mu = \hat{\Psi}_\mu$.

(H2) Otherwise, let $T'_\mu = \hat{T}_\mu$ and $\Psi_\mu = \hat{\Psi}_\mu \cup \{\psi \vee \chi_\mu(c) \mid \psi \in \hat{\Psi}_\mu\}$.

We show that $T'_\mu \nvdash \Psi_\mu$ and Ψ_μ are directed. Let c be the first unused constant of the proper sort.

(H1) Let $\varphi \in \hat{\Psi}_\mu$ and δ is the upper bound of φ and ψ. Recall that $\hat{T}_\mu \nvdash \delta$. We know that $\hat{T}_\mu \vdash \psi \vee \chi_\mu(x)$ (just replace c by x everywhere in the proof of $\psi \vee \chi_\mu(c)$). Thus $\hat{T}_\mu \vdash \psi \vee (\forall x)\chi_\mu(x)$ (by the generalization and axiom ($\forall$3)).

Obviously, $\hat{T}_\mu \cup \{\psi\} \vdash \psi$ and so $\hat{T}_\mu \cup \{\psi\} \vdash \delta$. Thus $\hat{T}_\mu \cup \{(\forall x)\chi_\mu(x)\} \nvdash \delta$ (otherwise using PCP we get $\hat{T}_\mu \vdash \delta$—a contradiction). Finally, if we have $\hat{T}_\mu \cup \{(\forall x)\chi_\mu(x)\} \vdash \varphi$ then we get $\hat{T}_\mu \cup \{(\forall x)\chi_\mu(x)\} \vdash \delta$—a contradiction. We have shown that $T'_\mu \nvdash \Psi_\mu$ the other conditions are in this case obvious.

(H2) The proof of $T'_\mu \nvdash \Psi_\mu$ is trivial. We only have to show that Ψ_μ is directed. We should distinguish three cases, however we show only one (the other are analogous) $\varphi, \alpha \in \hat{\Psi}_\mu$ and $\psi = \alpha \vee \chi_\mu(c)$. Let δ be the upper bound of φ and α then obviously $\delta \vee \chi_\mu(c) \in \Psi_\mu$ is the upper bound φ and ψ.

Next, we construct theory T_μ. Let c be the first unused constant of the proper sort. Again, we distinguish two cases:

(W1) There is $\psi \in \Psi_\mu$ such that $T'_\mu \cup \{(\exists x)\sigma_\mu(x)\} \vdash \psi$. Let $T_\mu = T'_\mu$.

(W2) $T'_\mu \cup \{(\exists x)\sigma_\mu(x)\} \nvdash \Psi_\mu$. Let $T_\mu = T'_\mu \cup \{\sigma_\mu(c)\}$.

We show that $T_\mu \nvdash \Psi_\mu$:

(W1) Trivial.

(W2) Since the logic $\mathbf{L}\forall$ is proto-φ-witnessed $T'_\mu \cup \{\sigma(c)\}$ is a conservative extension of $T'_\mu \cup \{(\exists x)\sigma(x)\}$. Since $T'_\mu \cup \{(\exists x)\sigma(x)\} \nvdash \Psi_\mu$ the proof is done.

Let us define theory $\hat{T} = T_{||\mathbf{\Gamma}||}$ and set $\Psi = \Psi_{||\mathbf{\Gamma}||}$. Now we construct a complete theory T' such that $\hat{T} \subseteq T'$ and $T' \nvdash \Psi$. We do it again by a transfinite induction. Let us enumerate pair of formulae by ordinals. $T'_0 = \hat{T}$. We construct theories T'_μ such that $T'_\mu \nvdash \Psi$ and $T'_\mu \subseteq T'_\nu$ for $\mu \leq \nu$. Let us define theory $\hat{T}'_\mu = \bigcup_{\nu<\mu} T'_\nu$. Notice that from the induction property we get that $\hat{T}'_\mu \nvdash \Psi$. The induction step: we show that $\hat{T}'_\mu \cup \{\varphi_\mu \to \psi_\mu\} \nvdash \Psi$ or $\hat{T}'_\mu \cup \{\psi_\mu \to \varphi_\mu\} \nvdash \Psi$. By contradiction: let there be formulae $\beta, \gamma \in \Psi$ such that $\hat{T}'_\mu \cup \{\varphi_\mu \to \psi_\mu\} \vdash \beta$ and $\hat{T}'_\mu \cup \{\psi_\mu \to \varphi_\mu\} \vdash \gamma$. Let us take upper bound δ of β and γ and we get $\hat{T}'_\mu \cup \{\varphi_\mu \to \psi_\mu\} \vdash \delta$ and $\hat{T}'_\mu \cup \{\psi_\mu \to \varphi_\mu\} \vdash \delta$. Thus $\hat{T}'_\mu \vdash \delta$—a contradiction.

Finally, we define $T' = T'_{||\mathbf{\Gamma}||}$. Recall that $T' \nvdash \Psi$. If we show that T' is φ-witnessed theory the proof is done (because T' is obviously complete and $T' \nvdash \alpha$).

Is T' Henkin? Let $\varphi(x)$ be processed in the step μ. If $T' \nvdash (\forall x)\varphi(x)$ then we used the case (H2) (otherwise $\hat{T}_\mu \vdash \psi \vee \varphi(c)$ which leads to $\hat{T}_\mu \vdash \psi \vee (\forall x)\chi_\mu(x)$ and so we get $\hat{T}_\mu \cup \{\psi \to (\forall x)\varphi(x)\} \vdash (\forall x)\varphi(x)$ and so $T_\mu \vdash (\forall x)\varphi(x)$—a contradiction). If $T' \vdash \varphi(c)$ then $T' \vdash \varphi(c) \vee \psi$ for all $\psi \in \hat{\Psi}_\mu$. Since we used case (H2) we know that $\varphi(c) \vee \psi \in \Psi_\mu$—a contradiction with $T' \nvdash \Psi$.

Is T' φ-witnessed ? Let $\psi(x) = \varphi(x_1 : t_1, \dots x_n : t_n, x)$ be processed in the step μ. If $T' \vdash (\exists x)\varphi(x)$ then we used the case (W2) (since $\hat{T}_\mu \cup \{(\exists x)\varphi(x)\} \vdash \psi$ for some $\psi \in \Psi$ we get $T' \vdash \psi$—a contradiction). Thus $T_\mu \vdash \varphi(c)$ and so $T' \vdash \varphi(c)$. QED

Corollary 2.3.22 (Henkin extension in fuzzy logics) *Let* $\mathbf{L}$ *be finitary fuzzy logic. If* $\mathbf{L}\forall$ *has* PP *then the logic* $\mathbf{L}\forall$ *is Henkin.*

Proof: Just read the proof of the latter lemma without parts (W1) and (W2) and notice that the assumption that $\mathbf{L}\forall$ is proto-φ-witnessed was used only in part (W2). QED

Theorem 2.3.23 (Henkin extension) *Let* $\mathbf{L}$ *be finitary weakly implicative logic. Then the logic* $\mathbf{L}\forall^-$ *is Henkin.*

Proof: Just read the proof of Theorem 3.4 in [53]. QED

2.3.3 Completeness

We introduce the notion of a Lindenbaum-Tarski matrix in the same fashion as in the propositional level. We define the *canonical* $\mathbf{Lin}_T$*-structure* $\mathbb{M}_T$ in the usual way—elements are the closed terms, and functions and predicates are defined accordingly. We have the following important lemma.

Lemma 2.3.24 *Let* T *be a Henkin theory and* φ *a formula with only one free variable* x *of the sort* s*. Then*

- $[(\forall x^s)\varphi]_{\mathbf{T}} = \inf_{c\in\mathbf{C}_s} [\varphi(c)]_{\mathbf{T}}$.
- $[(\exists x^s)\varphi]_{\mathbf{T}} = \sup_{c\in\mathbf{C}_s} [\varphi(c)]_{\mathbf{T}}$.

Proof: Recall that $[\varphi]_{\mathbf{T}} \leq [\psi]_{\mathbf{T}}$ iff $\mathbf{T} \vdash \varphi \to \psi$ (cf. Lemma 2.1.32). We prove only the first claim, the proof of the second one is analogous.

We show that $[(\forall x^s)\varphi]_{\mathbf{T}}$ is the greatest lower bound of all $[\varphi(c)]_{\mathbf{T}}$. The proof that $[(\forall x^s)\varphi]_{\mathbf{T}}$ is the lower bound is simple: $[(\forall x^s)\varphi]_{\mathbf{T}} \leq [\varphi(c)]_{\mathbf{T}}$ for all constants $c \in C_s$ (by axiom ($\forall$1)).

Now suppose there is $[\chi]_{\mathbf{T}}$ such that $[\chi]_{\mathbf{T}} \leq [\varphi(c)]_{\mathbf{T}}$ for all $c \in C_s$ and $[\chi]_{\mathbf{T}} \not\leq [(\forall x^s)\varphi]_{\mathbf{T}}$. Thus $T \not\vdash \chi \to (\forall x^s)\varphi$ and so $T \not\vdash (\chi \to \varphi)$ (by rule $(Gen\forall)$). Thus $T \not\vdash (\forall x^s)(\chi \to \varphi)$ (by axiom ($\forall$1)) By a Henkin property we get a constant $d \in C_s$ such that $T \not\vdash \chi \to \varphi(d)$. Finally $[\chi]_{\mathbf{T}} \not\leq [\varphi(d)]_{\mathbf{T}}$ - a contradiction. QED

Obviously, for T being Henkin the canonical $\mathbf{Lin}_T$-structure is safe and we have $[\varphi]_{\mathbf{T}} = ||\varphi||^{\mathbf{Lin}_T}_{\mathbb{M}_T}$ and thus $\mathbb{M}_T$ is a $\mathbf{Lin}_T$-model of T. Since each theory can be extended into Henkin theory (and in the case of fuzzy logic into the *linear* Henkin theory the proof of the following theorem is straightforward.

Theorem 2.3.25 *Let* $\mathbf{L}$ *be a fuzzy logic,* $\mathbf{\Gamma}$ *be a predicate language,* T *a theory and* φ *a formula. If* $\mathbf{L}\forall$ *is Henkin, then* $T \vdash_{\mathbf{L}\forall} \varphi$ *iff* $T \models^l_{\mathbf{L}} \varphi$*.*

Corollary 2.3.26 *Let* $\mathbf{L}$ *be a finitary fuzzy logic. Then* $\mathbf{L}\forall$ *has* PP *iff for each* T *and each* φ *we have:* $T \vdash_{\mathbf{L}\forall} \varphi$ *iff* $T \models^l_{\mathbf{L}} \varphi$*.*

Corollary 2.3.27 *Let* $\mathbf{L}$ *be logic, such that* $\mathbf{L}$ *has some presentation* $\mathcal{AX}$*, where* (MP) *is the only deduction rule and implicational fragment of* $\mathbf{L}$ *is an extension of* FBCK*. Then the logic* $\mathbf{L}\forall$ *is sound and complete w.r.t. the corresponding class of linear matrices.*

Corollary 2.3.28 *Let* $\mathcal{L}$ *be a propositional language,* $\triangle \in \mathcal{L}$ *and* $\mathbf{L}$ *a fuzzy logic in* $\mathcal{L}$*, such that* $\mathbf{L}$ *has some presentation where* (MP) *and* (NEC) *are the only deduction rules. Then the logic* $\mathbf{L}\forall$ *is sound and complete w.r.t. corresponding class of linear matrices.*

There are fuzzy logic, described in the literature not covered by this general approach (so far), namely the logics $\mathcal{RP}\Pi$, $\mathcal{RP}\Pi_\sim([0,1]_{\Pi_\sim})$, and RŁΠ (because their infinitary rule, however this problem can be solved, see Subsection 3.5.3) and the logic PŁ$'$ (because of the unavoidable rule $\neg(\varphi \odot \varphi) \vdash \neg\varphi$).

At the end of this section we formulate completeness theorem of for weakly implicative logics. The proof is analogous to the one for fuzzy logics, we only use Corollary 2.3.23 instead of Corollary 2.3.22. This gives us some kind of first-order calculus for a very wide class of logics. There is an interesting research task to examine existing first-order calculi for particular logics (substructural, modal, intuitionistic, etc.) and compare them to our approach.

Theorem 2.3.29 *Let* $\mathbf{L}$ *be a weakly implicative logic,* $\mathbf{\Gamma}$ *a predicate language,* T *a theory, and* φ *a formula. Then* $T \vdash_{\mathbf{L}\forall^-} \varphi$ *iff* $T \models_{\mathbf{L}} \varphi$.

2.3.4 Skolem functions

Again, this section is only a short sketch of what "can be done". Observe that the majority of known fuzzy logic are φ-witnessed for each formula φ. Whereas, the logics with $\triangle$ are rather limited in this aspect, as shown by the following lemma:

Lemma 2.3.30 *Let* $\mathcal{L}$ *be a propositional language,* $\triangle \in \mathcal{L}$ *and* $\mathbf{L}$ *a logic in* $\mathcal{L}$ *with* $\mathrm{DT}_\triangle$. *Then the logics* $\mathbf{L}\forall^-$ *and* $\mathbf{L}\forall$ *are* φ*-witnessed iff* $\vdash \triangle(\exists y)\varphi(y) \to (\exists y)\triangle\varphi(y)$.

Proof: First, let us suppose that the logic is proto-φ-witnessed then $\{\varphi(c)\}$ is a conservative extension of $\{(\exists y)\varphi(y)\}$. Since $\{\varphi(c)\} \vdash (\exists x)\triangle\varphi(x)$ we get $\{(\exists y)\varphi(y)\} \vdash (\exists x)\triangle\varphi(x)$. The deduction theorem gives us $\vdash \triangle(\exists y)\varphi(y) \to (\exists y)\triangle\varphi(y)$.

Other direction if similar. Let $\psi(y) = \varphi(x_1 : t_1, \ldots x_n : t_n, y)$, where t_i are closed terms. We want show that $T \cup \{\psi(c)\}$ is a conservative extension of $T \cup \{(\exists y)\psi(y)\}$. For each φ without c we want to get : $T \cup \{\psi(c)\} \vdash \varphi$ iff $T \cup \{(\exists y)\psi(y)\} \vdash \varphi$. By deduction theorem and some simple steps we get: $T \vdash (\exists y)\triangle\psi(y) \to \varphi$ iff $T \vdash \triangle(\exists y)\psi(y) \to \varphi$. Now just notice that if $\vdash \triangle(\exists y)\varphi(y) \to (\exists y)\triangle\varphi(y)$ then $\vdash \triangle(\exists y)\varphi(y) \equiv (\exists y)\triangle\varphi(y)$. QED

Corollary 2.3.31 *Let* $\mathcal{L}$ *be a propositional language,* $\triangle \in \mathcal{L}$*,* $\mathbf{L}$ *a logic in* $\mathcal{L}$ *with* $\mathrm{DT}_\triangle$*, and* φ *a formula. Then the logics* $\mathbf{L}\forall^-$ *and* $\mathbf{L}\forall^-$ *are* $\triangle\varphi$*-witnessed.*

Let us examine the behavior of the φ-witnessed logic w.r.t. Skolem functions introduction. We formulate the theorem for fuzzy logics only, its reformulation for weakly implicative logics needs an analogy of Lemma 2.3.21. This can be done in a rather straightforward way, but we skip this here. For the sake of simplicity we formulate the theorem for the unsorted language.

Theorem 2.3.32 *Let* $\mathbf{L}$ *be a proto-*φ*-witnessed finitary fuzzy logic with* PP*,* T *a theory, and* $\varphi(x_1, \ldots, x_n, y)$ *a formula. If* $T \vdash (\forall x_1) \ldots (\forall x_n)\, (\exists y)\, \varphi(x_1, \ldots x_n, y)$. *Then the theory* T' *in the language of* T *extended by new function symbol* f_φ *resulting from the theory* T *by adding the axiom* $\vdash (\forall x_1) \ldots (\forall x_n)\, \varphi(x_1, \ldots x_n, f_\varphi(x_1, \ldots, x_n))$ *is a conservative extension of* T.

Proof: Let $\bar{T}$ be a φ-witnessed supertheory of T. Then if $T \not\vdash \chi$ there is a canonical $\mathbf{Lin}_{\bar{T}}$-model $\mathbb{M}_{\bar{T}}$ of T. Since for each vector $t_1, \ldots, t_n$ of closed terms if $\bar{T} \vdash (\exists y)\, \varphi(t_1, \ldots t_n, y)$ there is a constant $c_{t_1,\ldots,t_n}$ such that $T \vdash \varphi(t_1, \ldots t_n, c_{t_1,\ldots,t_n})$. Since $c_{t_1,\ldots,t_n}$ is an element of $\mathbb{M}_{\bar{T}}$ (together with all other closed terms) we define $(f_\varphi)_{\mathbb{M}_{\bar{T}}}(t_1, \ldots, t_n) = c_{t_1,\ldots,t_n}$. Then obviously $\mathbb{M}_{\bar{T}}$ is a model of T' and since $\mathbb{M}_{\bar{T}} \not\models \chi$ we get that $T' \not\vdash \chi$. QED

To prove the Skolem function elimination we need to extend our language with some sort of equality.

Chapter 3

Known fuzzy logics

This chapter serves as preliminary for the rest of this work. However, it has importance of its own, because we are not only *reviewing* known logics, but we are putting them into the context of our general approach from the last chapter. It turns out that many theorems, usually proved in the literature, are merely *consequences* of our general theorems. The aim of this chapter is not to "process" all existing fuzzy logics and apply our new formalism, we restrict ourselves to logics and properties of logics we need in the further text.

In the previous chapter we have seen a general definition of the class of fuzzy logics. It is a formal definition, which delimits the class of fuzzy logics inside a broader class of weakly implicative logics. However, there is another *informal* delimitation of fuzzy logic: fuzzy logic is a logic appearing in papers written by fuzzy logicians. This is sometimes more and sometimes less restrictive condition. In this section we make some additional assumption to make these two notions closer to each other. Propositional fuzzy logics in the literature are mostly formulated with (MP) as the only deduction rule (and with $\triangle$-necessitation in the case with $\triangle$ in in the language), they have usually connectives $\rightarrow$ and & and are expressive enough to define lattice connectives (as basic or derived connectives). Hájek's Quasihoop logic [51] is an exception—it does not define the lattice connectives.

Now we make some even more restrictive assumptions to delimit the logics we will encounter in the rest of this work. We list these assumptions together with names of logics, without (!) these assumptions and with references to the relevant papers:

Exchange	pseudo Basic Logic	[48, 49, 47]
Weakening	Uninorm logic	[65]
$\bot \in \mathcal{L}$	Hoop logic	[53]
$\bot \in \mathcal{L}$ and Exchange	Flea logic	[50]

With all these assumption we get to the logic $\mathcal{FUZZ}_{\{\&,\wedge,\bot\}}(\mathrm{E},\mathrm{W},ax(\mathrm{Sf}))$. This logic is known as monoidal t-norm logic MTL of Esteva and Godo (see [33, 31, 13, 46]). However, we make one more assumption: we add the so-called divisibility axiom.

$$(\mathrm{D}) \quad \vdash \varphi\&(\varphi \rightarrow \psi) \rightarrow \psi\&(\psi \rightarrow \varphi).$$

We start by showing that $\mathcal{FUZZ}_{\{\&,\wedge,\bot\}}(\mathrm{E},\mathrm{W},ax(\mathrm{Sf}),\mathrm{D})$ is Hájek's Basic fuzzy logic (BL). This will be our starting logic, the rest of this work is devoted to the study of some properties of some of its extensions.

Before we do so we prepare (in the first section) some definitions concerning general notions of fuzzy logic, i.e., without just mentioned design choices. We concentrate on the notion of standard completeness. Then, in the second and third section, we examine the class of existing fuzzy logic fulfilling our design choices. The fourth section is devoted to the study of logics resulting by adding truth constants into the language of our fuzzy logics. Finally, in the last section, we deal with the first-order fuzzy logics.

3.1 Standard completeness and Core fuzzy logics

In the literature, fuzzy logics are usually considered as unary consequence relations (as a set of theorems/tautologies). If we recall our approach we notice that it is more general. This results in some clash in terminology. What we call completeness is known under the term *strong* completeness, whereas the term *completeness* is understood as equality between theorems and tautologies. As we have seen each fuzzy logic enjoys both completeness and strong completeness and so there is no need for two terms. Thus we stick with our denotation. There is another kind completeness playing an important role in fuzzy logic.

Definition 3.1.1 (Standard matrix) *Let* $\mathbf{L}$ *be a weakly implicative logic and let* $\mathbf{B}$ *be an* $\mathbf{L}$*-matrix. We say that* $\mathbf{B}$ *is a standard* $\mathbf{L}$*-matrix iff* $B = [0, 1]$ *and* $\leq_{\mathbf{B}}$ *is the usual order of reals. Let us denote the class of standard* $\mathbf{L}$*-matrices as* $s\mathbf{-MAT}(\mathbf{L})$.

Definition 3.1.2 (Standard Completeness) *Let* $\mathbf{L}$ *be a weakly implicative logic. We say that* $\mathbf{L}$ *has the* standard completeness *if* $\mathbf{L} = \models_{s-\mathbf{MAT}(\mathbf{L})}$ *(i.e., if the logic* $\mathbf{L}$ *is sound and complete w.r.t. standard* $\mathbf{L}$*-matrices.)*

Observe that because the usual order of reals is linear each logic with standard completeness is fuzzy. Again, the concept we have just defined is usually called *strong* standard completeness. We omit the word "strong". However, we have a problem here, there are very few known logics with the standard completeness (namely Gödel , Gödel $_\triangle$, and Gödel $_\sim$), many of them enjoy its weaker form only:

Definition 3.1.3 (Finite Standard Completeness) *Let* $\mathbf{L}$ *be a weakly implicative logic. We say that* $\mathbf{L}$ *has the* finite standard completeness *if* $T \vdash_{\mathbf{L}} \varphi$ *iff* $T \models_{s-\mathbf{MAT}(\mathbf{L})} \varphi$ *for each* finite *theory* T *and formula* φ.

Furthermore, we say that $\mathbf{L}$ *has the* weak standard completeness *if* $\vdash_{\mathbf{L}} \varphi$ *iff* $\models_{s-\mathbf{MAT}(\mathbf{L})} \varphi$ *for each formula* φ.

Of course, standard completeness implies finite standard completeness and it implies weak standard completeness. There is another equivalent definition of Finite Standard Completeness using the so-called *finitary companions*.

Definition 3.1.4 *Let* $\mathbf{L}$ *be a logic. The logic* $\mathcal{FIN}(\mathbf{L})$ *(*finitary companion of $\mathbf{L}$*) is defined as:* $X \vdash_{\mathcal{FIN}(\mathbf{L})} \varphi$ *iff there is finite* $X' \subseteq X$ *and* $X' \vdash_{\mathbf{L}} \varphi$.

Observe that $\mathcal{FIN}(\mathbf{L})$ is the greatest finitary logic contained in $\mathbf{L}$ (of course if $\mathbf{L}$ is finitary, then $\mathcal{FIN}(\mathbf{L}) = \mathbf{L}$).

Theorem 3.1.5 *Let* $\mathbf{L}$ *be a weakly implicative logic. Then* $\mathbf{L}$ *is a finitary logic with the finite standard completeness iff* $\mathbf{L} = \mathcal{FIN}(\models_{s-\mathbf{MAT}(\mathbf{L})})$ *(i.e., if* $\mathbf{L}$ *is a finitary companion of the logic given by the standard semantics of* $\mathbf{L}$*).*

Proof: Let us denote $\mathcal{FIN}(\models_{s-\mathbf{MAT}(\mathbf{L})})$ as $\mathbf{L}'$. We know that $X \vdash_{\mathbf{L}'} \varphi$ iff there is finite $X' \subseteq X$ and $X' \vdash_{\models_{s-\mathbf{MAT}(\mathbf{L})}} \varphi$ iff $X' \models_{s-\mathbf{MAT}(\mathbf{L})} \varphi$ iff $X' \vdash_{\mathbf{L}} \varphi$. Thus from $X \vdash_{\mathbf{L}'} \varphi$ we get $X \vdash_{\mathbf{L}} \varphi$ and vice-versa. Reverse direction: just observe that for finite T we have $T \vdash_{\mathbf{L}} \varphi$ iff $T \vdash_{\mathbf{L}'} \varphi$ QED

This is interesting, as we will see in the further text Hájek's BL has the finite standard completeness but not standard completeness and that the standard semantics of BL is given by continuous t-norms. Thus it is sometimes said, that BL is the "logic of continuous t-norms". The previous theorem says that this is true only to some extent. It is correct if we understand logic as a set of theorems/tautologies, if we understand logic as a consequence relation then BL is "just" a *finitary companion of the logic of continuous t-norms.*

Now we show that in the case of fuzzy logic with $\triangle$, such that $\vdash_{\mathbf{L}} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$ we can easily show the equivalence of the notion of finite standard completeness and weak standard completeness. Later on we show analogy of this lemma for other classes of logic.

Lemma 3.1.6 *Let* $\mathbf{L}$ *be a fuzzy logic in language with* $\triangle$, $\vdash_{\mathbf{L}} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$. *If* $\mathbf{L}$ *has weak standard completeness, then it has finite standard completeness.*

Proof: We show that for each finite T we have $T \vdash_{\mathbf{L}} \varphi$ iff $T \models_{s-\mathbf{MAT(L)}} \varphi$, one direction is trivial. We show the other one. First we observe that $\models_{s-\mathbf{MAT(L)}}$ is fuzzy logic and $\models_{s-\mathbf{MAT(L)}} (\neg\triangle\varphi \to \triangle\varphi) \to \triangle\varphi$ and so by Lemma 2.2.32 it has the $\mathrm{DT}_\triangle$. Suppose that $T \models_{s-\mathbf{MAT(L)}} \varphi$ then by $\mathrm{DT}_\triangle$ we get $\models_{s-\mathbf{MAT(L)}} \triangle\psi_1 \to (\triangle\psi_2 \to (\dots \to (\triangle\psi_n \to \varphi)\dots))$. From our assumption we get that $\vdash_L \triangle\psi_1 \to (\triangle\psi_2 \to (\dots \to (\triangle\psi_n \to \varphi)\dots))$. By $\mathrm{DT}_\triangle$ we get that $T' \vdash_{\mathbf{L}} \varphi$ and so $T \vdash_{\mathbf{L}} \varphi$ QED

Lemma 3.1.7 *Let* $\mathbf{L}$ *be a fuzzy logic and suppose logic* $\mathbf{L}_\triangle$ *has the (weak, finite) standard completeness. Then* $\mathbf{L}$ *has the (weak, finite) standard completeness.*

Proof: Observe that in the following argument the cardinality of T does not matter: assume that $T \not\vdash_{\mathbf{L}} \varphi$. Since $\mathbf{L}_\triangle$ is a conservative extension of $\mathbf{L}$ we get $T \not\vdash_{\mathbf{L}_\triangle} \varphi$ and so there is a standard $\mathbf{L}_\triangle$-matrix $\mathbf{B}_\triangle$, such that $T \not\models_{\mathbf{B}_\triangle} \varphi$. Thus for the corresponding standard $\mathbf{L}$-matrix $\mathbf{B}$ we have $T \not\models_{\mathbf{B}} \varphi$. QED

Theorem 3.1.8 *Let* $\mathbf{L}$ *be a fuzzy logic in a language with* $\vee$. *Then* $\mathbf{L}_\triangle$ *has the finite standard completeness iff* $\mathbf{L}$ *has the finite standard completeness.*

Proof: The proof of one direction is Lemma 3.1.7. We prove the reverse direction contrapositively: assume that $\mathbf{L}_\triangle$ has not the finite standard completeness. Thus there is a finite theory T and a formula φ, such that $T \models_{s-\mathbf{MAT}(\mathbf{L}_\triangle)} \varphi$ and $T \not\vdash_{\mathbf{L}_\triangle} \varphi$. Using the completeness of $\mathbf{L}_\triangle$ we know that there is $\mathbf{L}_\triangle$-matrix $\mathbf{B}_\triangle$ and $\mathbf{B}_\triangle$-model e of theory T, such that $e(\varphi) \notin D_{\mathbf{B}_\triangle}$. Let us define the set of formulae SUB as the set of all subformulae of φ or formulae from T starting with $\triangle$. Then we define two subsets of SUB: $SUB^{\bar{1}} = \{\psi \mid \psi \in SUB \text{ and } e(\triangle\psi) = \bar{1}\}$ and $SUB^{\perp} = SUB \setminus SUB^{\bar{1}}$. Observe that $\triangle\psi \in SUB^{\bar{1}}$ iff $e(\psi) \geq \bar{1}_{\mathbf{B}_\triangle}$ and $\triangle\psi \in SUB^{\perp}$ iff $e(\psi) < \bar{1}_{\mathbf{B}_\triangle}$. For each formula χ let us denote by χ' the formula resulting from χ by replacing each subformula from $SUB^{\bar{1}}$ by $\bar{1}$ and each subformula from $SUB^{\perp}$ by $\perp$.

Let us define theory $T' = \{\chi' \mid \chi \in T\} \cup \{\psi \mid \triangle\psi \in SUB^{\bar{1}}\}$ and the formula $\delta = \varphi' \vee \bigvee_{\triangle\psi \in SUB^{\perp}} \psi$. First, we show $T' \not\vdash \delta$: take $\mathbf{L}$-matrix $\mathbf{B}$ and $\mathbf{B}$-evaluation e and observe that e is a model of the theory T' and $e(\delta) \notin D_{\mathbf{B}_\triangle}$. If we show that $T \models_{s-\mathbf{MAT(L)}} \delta$ the proof is done. Let $\mathbf{C}$ be an arbitrary standard $\mathbf{L}$-matrix and e a $\mathbf{C}$-evaluation: we have to show that whenever e is a model of T' we have $\mathbf{e}(\delta) \in D_{\mathbf{C}}$. Observe that to prove this is non-trivial only if e is a model of $\{\psi \mid \triangle\psi \in SUB^{\bar{1}}\}$ and $e(\psi) \notin D_{\mathbf{C}}$ for each $\triangle\psi \in SUB^{\perp}$. To prove this notice that for each $\mathbf{L}_\triangle$-matrix $\mathbf{C}_\triangle$ end each $\mathbf{C}_\triangle$-evaluation e if e is a model of $SUB^{\bar{1}}$ and $e(\psi) \notin D_{\mathbf{C}_\triangle}$ for each $\psi \in SUB^{\perp}$ we have $e(\chi) = e(\chi')$. The fact that $T \models_{\mathbf{C}_\triangle} \varphi$ completes the proof. QED

We would like to extend Lemma 3.1.6 to a broader class of fuzzy logics. However, we are not able to do so, we can only prove the equivalence of this claim with another open problem.

Theorem 3.1.9 *Let* $\mathbf{L}$ *be a logic in a language with* $\vee$. *The following statements are equivalent:*

1. *If* $\mathbf{L}$ *has weak standard completeness then* $\mathbf{L}$ *has finite standard completeness.*
2. *If* $\mathbf{L}$ *has weak standard completeness then* $\mathbf{L}_\triangle$ *has weak standard completeness.*

Proof: (1) $\to$ (2): Assume that $\mathbf{L}$ has weak standard completeness, then $\mathbf{L}$ has finite standard completeness and so by Theorem 3.1.8 we know that $\mathbf{L}_\triangle$ has the finite standard completeness.

(2) $\to$ (1): Assume that $\mathbf{L}$ has weak standard completeness, then $\mathbf{L}_\triangle$ has weak standard completeness and so by Lemma 3.1.6 we know that $\mathbf{L}_\triangle$ has the finite standard completeness. Lemma 3.1.7 completes the proof. QED

Notice that in the second part of the proof we have not used the assumption that $\vee$ is in the language. It is rather common that $s-\mathbf{MAT}(\mathbf{L})$ contains only one matrix (up to isomorphism). In the following definition we introduce the notion of *core fuzzy logic*, this is rather *auxiliary notion* for the needs of this work, many other authors would not agree with this delimitation.

Definition 3.1.10 (Core fuzzy logic) *Let* $\mathbf{L}$ *be a fuzzy logic. We say that* $\mathbf{L}$ *is* core *fuzzy logic if* $\mathbf{L}$ *is an extension of* BL *with finite standard completeness and* $s-\mathbf{MAT}(\mathbf{L})$ *contains only one matrix (up to isomorphism).*

For each core fuzzy logic, we encounter in this text we select one of its standard matrices and denote it as $[0,1]_{\mathbf{L}}$. Because this selection is unique (up to isomorphism) the logic is especially interesting. The reason is that in this case it allows us to introduce some classical semantical notions like satisfiability, compactness, functional representation, etc. In more details, since there is a tight correspondence between a logic and one particular matrix we have for this logic a unique semantics, which allows us to speak about semantical properties of a logic without either being general (speaking about classes of "semantics") or having to index the property by particular matrix (we can speak about satisfiability and not only about $\mathbf{B}$-satisfiability).

For example let us take Łukasiewicz logic; it has finite standard completeness w.r.t. only one standard matrix $[0,1]_{\text{Ł}}$. We say that a formula of Łukasiewicz logic is satisfiable if there is an $[0,1]_{\text{Ł}}$-evaluation e, such that $e(\varphi) = 1$. On the other hand, in intuitionistic logic we have general satisfiability (if there is Heyting algebra $\mathbf{B}$ and a $\mathbf{B}$-evaluation e, such that $e(\varphi) = 1$) and we also have $\mathbf{B}$-satisfiability for each particular Heyting algebra $\mathbf{B}$. Even better example is the notion of functional representation.

Definition 3.1.11 *Let* $\mathbf{L}$ *be a core fuzzy logic. Let* C *be an arbitrary function from* $[0,1]^n$ *into* $[0,1]$ *and let* φ *be an arbitrary formula with variables* $\{v_1, \ldots, v_n\}$*. We say the function* C *is* represented by the formula φ *in logic* $\mathbf{L}$ *(*φ *is a representation of* C*) if* $e(\varphi) = C(\,e(v_1), e(v_2), \ldots, e(v_m))$*, where* e *is an arbitrary* $[0,1]_{\mathbf{L}}$*-evaluation.*

Definition 3.1.12 (Functional Representation) *Let* $\mathbf{L}$ *be a core fuzzy logic. Let* $\mathcal{C}$ *be class of functions from any power of* $[0,1]$ *into* $[0,1]$*. We say the* $\mathcal{C}$ *is* functional representation of logic $\mathbf{L}$ *if for each* $C \in \mathcal{C}$ *there is a formula* φ *such that* φ *is a representation of* C *and vice-versa (i.e., for each* φ *there is* $C \in \mathcal{C}$*, such that* φ *is a representation of* C*).*

If we slightly changed Definition 3.1.11 we could say that boolean functions are functional representation of classical logic. It is obvious that for the formulation of the notion of the functional representation we need one unique matrix.

In the next theorem we relate the conservativeness and completeness of particular logic.

Theorem 3.1.13 *Let* $\mathbf{L}$ *be a fuzzy logic with (finite) standard completeness. Let* $\mathbf{L}'$ *be a (finitary) expansion of* $\mathbf{L}$*, such that each standard* $\mathbf{L}$*-algebra is a reduct of some standard* $\mathbf{L}'$*-algebra. Then* $\mathbf{L}'$ *is a conservative expansion of* $\mathbf{L}$*.*

Proof: We deal with the "finite" case only, the proof of the second one is analogous. Let us denote by $\mathcal{K}$ the class of all standard $\mathbf{L}$-algebras and by $\mathcal{K}'$ the class of their expansions.

Let T be a theory and φ a formula in the language of $\mathbf{L}$, such that $T \vdash_{\mathbf{L}'} \varphi$. Since $\mathbf{L}'$ is finitary, there is a finite $\hat{T} \subseteq T$ and $\hat{T} \vdash_{\mathbf{L}'} \varphi$. Using soundness of $\mathbf{L}'$ we know that $\hat{T} \models_{\mathcal{K}'} \varphi$. Thus obviously $\hat{T} \models_{\mathcal{K}} \varphi$ and by finite completeness of $\mathbf{L}$ we get $\hat{T} \vdash_{\mathbf{L}} \varphi$, the rest is trivial. QED

3.2 Basic fuzzy logic and its expansions

Now we are going to review some particular well-known fuzzy logic meeting our design choices. We start with Hájek's Basic Fuzzy Logic and its axiomatic extensions. Then we proceed with study of expansions of Basic Logic with some particular additional connectives.

3.2.1 Basic fuzzy logic and its extensions

The language of BL contains a conjunction &, an implication $\to$, and the constant $\overline{0}$. Further connectives are defined as follows:

$$
\begin{array}{ccl}
\varphi \wedge \psi & \text{is} & \varphi \& (\varphi \to \psi), \\
\varphi \vee \psi & \text{is} & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\
\neg \varphi & \text{is} & \varphi \to \overline{0}, \\
\varphi \equiv \psi & \text{is} & (\varphi \to \psi) \& (\psi \to \varphi), \\
\varphi \oplus \psi & \text{is} & \neg \varphi \to \psi, \\
\varphi \ominus \psi & \text{is} & \varphi \& \neg \psi, \\
\overline{1} & \text{is} & \neg \overline{0}.
\end{array}
$$

We set $\top = \overline{1}$ and $\bot = \overline{0}$. The following formulae are the *axioms* of BL:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(A2) $\varphi \& \psi \to \varphi$,
(A3) $\varphi \& \psi \to \psi \& \varphi$,
(A4) $\varphi \& (\varphi \to \psi) \to \psi \& (\psi \to \varphi)$,
(A5a) $(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$,
(A5b) $(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$,
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$,
(A7) $\overline{0} \to \varphi$.

The only *deduction rule* of BL is modus ponens.

There is a new result showing that axiom (A3) is redundant (see [18]). Let us by BL^- denote the logic resulting results from BL by omitting axiom (A3).

Lemma 3.2.1 *The following are theorems of* BL^-*:*

1. $\varphi \to ((\varphi \to \psi) \to \psi)$,
2. $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$,
3. $\varphi \to \varphi$,
4. $\varphi \& \psi \to \psi \& \varphi$.

Proof:

1. From (A4) and (A2) we get $\varphi \& (\varphi \to \psi) \to \psi$. The proof is completed by axiom (A5b).
2. From (A1) in the form $(\psi \to ((\psi \to \chi) \to \chi)) \to ((((\psi \to \chi) \to \chi) \to (\varphi \to \chi)) \to (\psi \to (\varphi \to \chi)))$ and 1. we obtain $(((\psi \to \chi) \to \chi) \to (\varphi \to \chi)) \to (\psi \to (\varphi \to \chi))$. Finally, axiom (A1) in the form $(\varphi \to (\psi \to \chi)) \to (((\psi \to \chi) \to \chi) \to (\varphi \to \chi))$ completes the proof.
3. From (A2) and (A5b) we get $\varphi \to (\psi \to \varphi)$. Using 2. we get $\psi \to (\varphi \to \varphi)$. Fixing ψ to be an arbitrary theorem completes the proof.
4. We start with 3. in the form $\psi \& \varphi \to \psi \& \varphi$. From (A5b) and 2. we get $\varphi \to (\psi \to \psi \& \varphi)$. Axiom (A5a) completes the proof. QED

Now we list some useful theorems of BL for their proofs see [44].

Lemma 3.2.2 *The following are theorems of* BL*:*

(T1) $\varphi \to (\psi \to \varphi)$
(T2) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(T3) $\varphi \to \varphi$

(T4a)	$(\varphi \to \psi) \to (\varphi\&\chi \to \psi\&\chi))$
(T4b)	$(\varphi \to \psi) \to (\varphi \wedge \chi \to \psi \wedge \chi))$
(T4c)	$(\varphi \to \psi) \to (\varphi \vee \chi \to \psi \vee \chi))$
(T5)	$(\varphi \to (\psi \to \varphi\&\psi)$
(T6)	$\varphi\&\psi \to \varphi \wedge \psi$
(T7)	$\varphi \wedge \psi \to \psi$
(T8)	$((\chi \to \varphi) \wedge (\chi \to \psi)) \to (\chi \to \varphi \wedge \psi)$
(T8')	$((\chi \to \varphi)\&(\chi \to \psi)) \to (\chi \to \varphi \wedge \psi)$
(T9)	$(\varphi \to \psi) \to (\varphi \equiv \varphi \wedge \psi)$
(T9')	$(\varphi \to \psi) \equiv (\varphi \to \varphi \wedge \psi)$
(T10)	$\varphi \to \varphi \vee \psi$
(T11)	$((\varphi \to \chi) \wedge (\psi \to \chi)) \to (\varphi \vee \psi \to \chi)$
(T11')	$((\varphi \to \chi)\&(\psi \to \chi)) \to (\varphi \vee \psi \to \chi)$
(T12)	$(\varphi \to \psi) \vee (\psi \to \varphi)$
(T13)	$(\varphi \to \psi)^n \vee (\psi \to \varphi)^n$
(T14)	$(\varphi \to \psi) \to (\psi \equiv \varphi \vee \psi)$
(T15)	$\varphi \to \neg\neg\varphi$
(T16)	$\neg\varphi \to (\varphi \to \psi)$
(T17)	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
(T17')	$(\varphi \equiv \psi) \to (\neg\psi \equiv \neg\varphi)$
(T18a)	$\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$
(T18b)	$\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$
(T19)	$\varphi\&1 \equiv \varphi$
(T20)	$(1 \to \varphi) \equiv \varphi$
(T21)	$(\varphi \equiv \psi) \to ((\varphi\&\chi) \equiv (\psi\&\chi))$
(T21')	$(\varphi \equiv \psi)\&(\chi \equiv \delta) \to ((\varphi\&\chi) \equiv (\psi\&\delta))$
(T22)	$(\varphi \equiv \psi) \to ((\varphi \to \chi) \equiv (\psi \to \chi))$
(T23)	$(\varphi \equiv \psi) \to ((\chi \to \varphi) \equiv (\chi \to \psi))$
(T23')	$(\varphi \equiv \psi)\&(\chi \equiv \delta) \to ((\varphi \to \chi) \equiv (\psi \to \delta))$
(T24a)	$\varphi\&(\psi \wedge \chi) \equiv (\varphi\&\psi) \wedge (\varphi\&\chi)$
(T24b)	$\varphi\&(\psi \vee \chi) \equiv (\varphi\&\psi) \vee (\varphi\&\chi)$
(T25a)	$\varphi \vee (\psi \wedge \chi) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \chi)$
(T25b)	$\varphi \wedge (\psi \vee \chi) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
(T26a)	$(\varphi \wedge \psi)\&(\varphi \wedge \psi) \equiv (\varphi\&\varphi) \wedge (\psi\&\psi)$
(T26b)	$(\varphi \vee \psi)\&(\varphi \vee \psi) \equiv (\varphi\&\varphi) \vee (\psi\&\psi)$
(T27)	$(\varphi\&\psi)\&\chi \equiv \varphi\&(\psi\&\chi)$
(T28)	$(\varphi \to \psi)\&(\chi \to \delta) \to ((\psi \to \chi) \to (\varphi \to \delta))$
(T29)	$(\varphi \to \psi)\&(\chi \to \delta) \to ((\varphi\&\chi) \to (\psi\&\delta))$

Lemma 3.2.3 *Let I and J be finite sets. The following are theorems of BL:*

(S1) $$\bigwedge_{i\in I} \varphi_i \& \bigwedge_{j\in J} \psi_j \equiv \bigwedge_{i\in I, j\in J} (\varphi_i \& \psi_j)$$

(S2) $$\bigvee_{i\in I} \varphi_i \& \bigvee_{j\in J} \psi_j \equiv \bigvee_{i\in I, j\in J} (\varphi_i \& \psi_j)$$

(S3) $$\bigvee_{i\in I} \varphi_i \to \bigwedge_{j\in J} \psi_j \equiv \bigwedge_{i\in I, j\in J} (\varphi_i \to \psi_j)$$

(S4) $\bigwedge_{i\in I}\varphi_i \to \bigvee_{j\in J}\psi_j \equiv \bigvee_{i\in I, j\in J}(\varphi_i \to \psi_j)$

(S5) $\bigwedge_{i\in I}\varphi_i \vee \bigwedge_{j\in J}\psi_j \equiv \bigwedge_{i\in I, j\in J}(\varphi_i \vee \psi_j)$

(S6) $\bigvee_{i\in I}\varphi_i \wedge \bigvee_{j\in J}\psi_j \equiv \bigvee_{i\in I, j\in J}(\varphi_i \wedge \psi_j)$

(T1) $\varphi \to (\varphi \wedge \psi \equiv \psi)$

(T2) $\varphi \to (\varphi \vee \psi \equiv \overline{1})$

(T3) $(\varphi \to \psi) \to (\varphi \wedge \psi \equiv \varphi)$

(T4) $(\varphi \to \psi) \to (\varphi \vee \psi \equiv \psi)$

(T5) $(\varphi \to \psi) \to (\varphi \wedge \chi \to \psi)$

(T6) $((\varphi \to \psi)\&(\varphi \to \chi)) \to (\varphi \to \psi \wedge \chi)$

Lemma 3.2.4 BL *is a* $\{\mathrm{E}, \mathrm{W}, ax(\mathrm{Sf})\}$*-logic in language* $\{\to, \&, \overline{0}\}$ *with* $\mathrm{DT}_\to$*.*

Proof: Obviously, BL extends BCI and fulfills the conditions of Theorem 2.2.8. Also all the matching rules for connectives are valid in BL. QED

BL also validates all the matching rules for $\wedge, \vee, \overline{1}, \bot, \top$.

Lemma 3.2.5 $\mathrm{BL} = \mathcal{FUZZ}_{\{\&,\wedge,\overline{0}\}}(ax(\mathrm{Sf}), \mathrm{E}, \mathrm{W}, \mathrm{D})$.

Proof: First, we show that BL is fuzzy. To show that BL has PP, just use $\mathrm{DT}_\to$ and theorems $(T11)$ and $(T13)$. To show that BL is the weakest fuzzy logic with $\mathrm{E}, \mathrm{W}, ax(\mathrm{Sf}), \mathrm{D}$ we show that all its axioms are provable in $\mathcal{FUZZ}_{\{\&,\wedge,\overline{0}\}}(\mathrm{E}, \mathrm{W}, ax(\mathrm{Sf}), \mathrm{D})$.

- (A1): This is just $ax(\mathrm{Sf})$.
- (A2): From W and E we get $\varphi \to (\psi \to \varphi)$, residuation rule completes the proof.
- (A3): Using E and residuation rule we get $\varphi \to (\psi \to \psi\&\varphi)$, residuation rule completes the proof.
- (A4): This is just D.
- (A5a): We give a formal proof

(i)	$(\varphi \to (\psi \to \chi)) \to (\varphi \to (\psi \to \chi))$	(Ref)
(ii)	$\varphi \to ((\varphi \to (\psi \to \chi)) \to (\psi \to \chi))$	(i) and E
(iii)	$\varphi \to (\psi \to ((\varphi \to (\psi \to \chi)) \to \chi))$	$(ii), ax(\mathrm{E}), (\mathrm{WT})$
(iv)	$\varphi\&\psi \to ((\varphi \to (\psi \to \chi)) \to \chi)$	(iii) and residuation rule
(v)	$(\varphi \to (\psi \to \chi)) \to (\varphi\&\psi \to \chi)$	$(iv), \mathrm{E}.$

- (A5b): Analogous to (A5a).
- (A6): Just read the proof of the second part of Theorem 2.2.12.
- (A7): This is a matching rule for $\overline{0}$. QED

The following observation holds in much broader context (in presence of weakening), however for our needs it is sufficient to formulate it like this:

Lemma 3.2.6 *Let* $\mathbf{L}$ *be an extension of* BL *and* $\mathbf{B}$ *be an ordered* $\mathbf{L}$*-matrix. Then* $D_{\mathbf{B}} = \{\overline{1}_{\mathbf{B}}\}$.

The BL-matrices tightly correspond to the so-called BL-algebras.

Definition 3.2.7 *A* BL-algebra *is a structure* $\mathbf{B} = (B, \cup, \cap, *, \Rightarrow, \mathbf{0}, \mathbf{1})$ *such that:*

(1) $(L, \cup, \cap, \mathbf{0}, \mathbf{1})$ is a bounded latticem

(2) $(L, *, \mathbf{1})$ is a commutative monoidm

(3) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z, (adjointness)

(4) $x \cap y = x * (x \Rightarrow y)$, (divisibility)

(5) $(x \Rightarrow y) \cup (y \Rightarrow x) = \mathbf{1}$. (prelinearity)

Theorem 3.2.8 *Let* $\mathbf{B} = ((B, *, \Rightarrow, \mathbf{0}), \{\mathbf{1_B}\})$ *be an ordered* BL-*matrix. Then the structure* $(B, \wedge_\mathbf{B}, \vee_\mathbf{B}, *, \Rightarrow, \mathbf{0}, \mathbf{1})$ *is a* BL-*algebra. If* B $= (B, \cup, \cap, *, \Rightarrow, \mathbf{0}, \mathbf{1})$ *is a* BL-*algebra, then* $((B, *, \Rightarrow, \mathbf{0}), \{\mathbf{1}\})$ *is an ordered* BL-*matrix.*

Proof: To show the first part we have to show that all defining conditions of BL-algebras are valid in arbitrary BL-matrix. This is simple because all theorems of BL are tautologies in each BL-matrix (soundness) and all the conditions we want to prove are among theorems from Lemma 3.2.2. To show the second part it is enough to observe that all axioms of BL are tautologies of arbitrary BL-algebra and that (MP) preserves validity. QED

This allows us to replace the validity condition $e(\varphi) \in D_\mathbf{B}$ by far simpler condition $e(\varphi) = \overline{1}_\mathbf{B}$. As we restrict ourselves to extension of BL in this work, we will from now on to work with *algebras* and not with *matrices* (they are the "same"—the translation shown above will work for an arbitrary extension of BL as well). Thus we rephrase all semantical notions (validity, model, semantical consequence, etc.) to accommodate this change. For example, we are getting the completeness theorem:

Theorem 3.2.9 *Let* φ *be a formula and* T *a theory. Then the following are equivalent:*

- $T \vdash_{\mathrm{BL}} \varphi$.
- $e(\varphi) = \mathbf{1}$ *for each* BL-*algebra* $\mathbf{B}$ *and each* $\mathbf{B}$-*model* e *of theory* T.
- $e(\varphi) = \mathbf{1}$ *for each linearly ordered* BL-*algebra* $\mathbf{B}$ *and each* $\mathbf{B}$-*model* e *of theory* T.

The interpretations of a fusion in standard BL-algebras correspond to the well-known class of continuous t-norms and the interpretations of a implication correspond to the so-called residua of these t-norms. For details about t-norms see Appendix A. Here we only recall the connection between t-norms and standard BL-algebras.

Theorem 3.2.10 *Let* $\mathbf{B}$ *be a standard* BL-*algebra. Then* $\&_\mathbf{B}$ *is a continuous t-norm and* $\rightarrow_\mathbf{B}$ *is its residuum. Conversely, if* $*$ *is a continuous t-norm and* $\Rightarrow$ *is its residuum, then* $([0,1], \min, \max, *, \Rightarrow, 0, 1)$ *is a standard* BL-*algebra.*

We will denote the standard BL-algebra induced by a continuous t-norm $*$ as $\mathbf{BL}_*$.

Theorem 3.2.11 BL *has the finite standard completeness but not standard completeness.* BL *is not core fuzzy logic.*

Proof: The finite standard completeness of BL was proven in [14]. The fact that BL has not the standard completeness is part of the folklore. The last fact is trivial (Gödel and Łukasiewicz t-norms are not isomorphic). QED

We have seen that BL is not core fuzzy logic, however BL is complete w.r.t. one particular standard BL-algebra (this algebra is based on t-norm which is an infinite ordinal sum of Łukasiewicz t-norms, see [1]). This raises a question whether our definition of core fuzzy logic is not too restrictive, whether we shouldn't replace the condition that all standard algebras should be isomorphic with a weaker one, like elementary equivalent (or even weaker). We answer this question simply by recalling that the notion of core fuzzy logic is just an *auxiliary* term used in this work only.

Now we define the class of extensions of BL, which has finite standard completeness w.r.t. one particular t-norm.

Definition 3.2.12 *Let $*$ be a continuous t-norm. Let us denote the finitary companion of* $\models_{\mathbf{BL}_*}$ *as* PC($*$) *(propositional calculus induced by $*$).*

Obviously $\mathcal{TAUT}(\mathrm{PC}(*)) = \mathcal{TAUT}(s{-}\mathbf{MAT}(\mathrm{PC}(*))) = \mathcal{TAUT}(\mathbf{BL}_*)$. It is known (see [55]), that there are only countably many different sets $\mathcal{TAUT}(\mathrm{PC}(*))$ and for each $*$ there is a finite axiomatic extension of BL $\mathbf{L}_*$ such that $\mathcal{TAUT}(\mathrm{PC}(*)) = \mathcal{THM}(L_*)$ (see [37]). We conjecture that $\mathrm{PC}(*) = \mathbf{L}_*$ and:

Conjecture 3.2.13 *Each* PC($*$) *is a finite axiomatic extension of* BL. *Thus there are countably many* PC($*$).

This fact is supposed to hold (personal communication with Franco Montagna), however we can not present the proof here so we leave is as a conjecture (it is sufficient for our needs). We end this section by defining one important extension of BL. In this extension we "fix" the interpretation of the negation:

Definition 3.2.14 *The* Strict Basic logic *(*SBL*) is an axiomatic extension of* BL *by the axiom $\neg\varphi \vee \neg\neg\varphi$. We denote this axiom as* St.

The following theorem is a part of folklore:

Theorem 3.2.15 SBL *has the finite standard completeness but not standard completeness.* SBL *is not core fuzzy logic.*

The proof of the following lemma is trivial:

Lemma 3.2.16 *Let* $\mathbf{L}$ *be an arbitrary extension of* SBL *and* $\mathbf{B}$ *a linear* $\mathbf{L}$*-algebra. Then* $\neg 0 = 1$ *and* $\neg x = 0$ *for each* $x > 0$.

3.2.2 Basic fuzzy logic with $\triangle$

Basic logic can be extended to the Basic logic with delta connective. In fuzzy logics is the unary connective $\triangle$ known as Baaz delta or 0-1 projector. This connective is well known and was used in many papers. The axiomatic system for the Gödel logic equipped with this connective was published by Baaz in his paper [2]. The generalization of this system to the logic BL is from [44]. We show the usual presentation of the logic $\mathrm{BL}_\triangle$ (we call it $\mathrm{BL}_\triangle$') and show its equivalence with the one from Theorem 2.2.37.

Definition 3.2.17 *The logic* $\mathrm{BL}'_\triangle$ *is an extension of* BL *by axioms:*

(A$\triangle$1) $\triangle\varphi \vee \neg\triangle\varphi$,

(A$\triangle$2) $\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$,

(A$\triangle$3) $\triangle\varphi \to \varphi$,

(A$\triangle$4) $\triangle\varphi \to \triangle\triangle\varphi$,

(A$\triangle$5) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$.

The deduction rules of $\mathrm{BL}'_\triangle$ *are modus ponens and* $\triangle$*-necessitation.*

Lemma 3.2.18 $\mathrm{BL}'_\triangle = \mathrm{BL}_\triangle$.

Proof: To prove one direction we have to show that $\vdash_{\mathrm{BL}_\triangle} A\triangle 1$ and $\vdash_{\mathrm{BL}_\triangle} A\triangle 2$. The first axiom is get easily using Lemma 2.2.32 Part 3. The provability of the second axiom is just a consequence of the fact that $\mathrm{BL}_\triangle$ is a fuzzy logic.

To prove the converse direction we have to go through the presentation of $\mathrm{BL}_\triangle$ and show its containment in $\mathrm{BL}'_\triangle$:

- A: Trivial.
- B: Trivial.
- C: The only non trivial matching rule to prove is $\triangle$C, we give the formal proof:

(i)	$(\triangle\varphi \to (\triangle\varphi \to \psi)) \to (\triangle\varphi \to (\triangle\varphi \to \psi))$	(Ref)
(ii)	$\triangle\varphi \to ((\triangle\varphi \to (\triangle\varphi \to \psi)) \to (\triangle\varphi \to \psi))$	(i) and E
(iii)	$\neg\triangle\varphi \to (\triangle\varphi \to \psi)$	T16
(iv)	$(\triangle\varphi \to (\triangle\varphi \to \psi)) \to (\neg\triangle\varphi \to (\triangle\varphi \to \psi))$	(iii) and W
(v)	$\neg\triangle\varphi \to ((\triangle\varphi \to (\triangle\varphi \to \psi)) \to (\triangle\varphi \to \psi))$	(iv) and E
(vi)	$\triangle\varphi \vee \neg\triangle\varphi \to ((\triangle\varphi \to (\triangle\varphi \to \psi)) \to (\triangle\varphi \to \psi))$	(ii), (v), and T11
(vii)	$(\triangle\varphi \to (\triangle\varphi \to \psi)) \to (\triangle\varphi \to \psi)$	(vi) and $A\triangle 1$

- (MP): Trivial.
- $\triangle 4$: Just use theorem $T7$ and the definition of $\vee$.
- $\triangle 5$: Again we give a formal proof:

(i)	$\triangle(\psi \to \varphi) \to (\neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi))$	ax(W)
(ii)	$\triangle(\varphi \to \psi) \to (\neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi))$	T16 and E
(iii)	$\triangle(\varphi \to \psi) \vee \triangle(\psi \to \varphi) \to \triangle 5$	(i), (ii), and T11
(iv)	$\triangle((\varphi \to \psi) \vee (\psi \to \varphi))$	T12 and (NEC)
(v)	$\triangle(\varphi \to \psi) \vee \triangle(\psi \to \varphi)$	(iv) and ($A\triangle 2$)
(vi)	$\neg\triangle(\varphi \to \psi) \to \triangle(\psi \to \varphi)$	(iii) and (v)

- $\triangle 6$: Trivial.

QED

This lemma allows us to apply the machinery prepared in the previous section and obtain known results about $\mathrm{BL}_\triangle$ as corollaries of our general theorems. For example Lemma 3.1.7 and Theorem 3.1.8 give us:

Theorem 3.2.19 $\mathrm{BL}_\triangle$ *has the finite standard completeness but not standard completeness.* $\mathrm{BL}_\triangle$ *is not a core fuzzy logic.*

Of course, we also know that $\mathrm{BL}_\triangle$ is a fuzzy logic (i.e., it is sound and complete w.r.t. linear $\mathrm{BL}_\triangle$-algebras), $\mathrm{BL}_\triangle$-algebras have the Subdirect Decomposition Property and $\mathrm{BL}_\triangle$ is a conservative extension of BL with $\mathrm{DT}_\triangle$. It should be obvious how $\mathrm{BL}_\triangle$-algebras look, however we present the definition:

Definition 3.2.20 *A* $\mathrm{BL}_\triangle$-algebra *is a structure* $\mathbf{B} = (B, \cup, \cap, *, \Rightarrow, \Delta, \mathbf{0}, \mathbf{1})$ *such that:*

(0) $(B, \cup, \cap, *, \Rightarrow, \mathbf{0}, \mathbf{1})$ *is a* BL-*algebra,*

(1) $\Delta x \cup (\Delta x \to \mathbf{0}) = \mathbf{1}$,

(2) $\Delta(x \cup y) \leq (\Delta x \cup \Delta y)$,

(3) $\Delta x \leq x$,

(4) $\Delta x \leq \Delta\Delta x$,

(5) $\Delta(x \to y) \leq \Delta x \to \Delta y$,

(6) $\Delta 1 = 1$.

Observe that conditions (1)–(5) directly correspond to the axioms, and condition (6) corresponds to $\triangle$-necessitation. Of course, $\mathrm{BL}_\triangle$-algebras and $\mathrm{BL}_\triangle$-matrices are in the same correspondence as BL-algebras and BL-matrices. It is obvious, that all what was written in this subsection remains valid if we replace the logic BL with its arbitrary extension, in more details:

Theorem 3.2.21 *Let* $\mathbf{L}$ *be a fuzzy logic extending* BL *and* $\mathcal{AX}$ *its presentation. Then* $\mathbf{L}_\triangle$ *has the following presentation:*

A	*axioms of* $\mathcal{AX}$,
B	$\vdash \triangle\varphi_1 \to (\ldots(\triangle\varphi_n \to \psi)$ *whenever* $<\varphi_1, \ldots, \varphi_n, \psi> \in \mathcal{AX}$,
(A△1)	$\triangle\varphi \vee \neg\triangle\varphi$,
(A△2)	$\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$,
(A△3)	$\triangle\varphi \to \varphi$,
(A△4)	$\triangle\varphi \to \triangle\triangle\varphi$,
(A△5)	$\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$,
(MP)	$\varphi, \varphi \to \psi \vdash \psi$,
(NEC)	$\varphi \vdash \triangle\varphi$,

Again, in group B it is necessary to add the axioms for the rules different from (MP) only. Thus if the logic $\mathbf{L}$ has $\mathrm{DT}_\to$ we can omit the group B completely.

3.2.3 Strict Basic fuzzy logic with involutive negation

Basic strict fuzzy logic was introduced as a schematic extension of BL by a single axiom. Of course, we can introduce logic $\mathrm{SBL}_\triangle$, with the expected properties. Since the negation in SBL is "strict", a new negation $\sim$ (co-called involutive negation) can be meaningfully added to the logic $\mathrm{SBL}_\triangle$. This logic and its extension was introduced and examined by Esteva, Godo, Hájek and Navara in their paper [35].

Definition 3.2.22 *Basic strict fuzzy logic with involutive negation* $\mathrm{SBL}_\sim$ *is* $\mathrm{SBL}_\triangle$ *plus new unary connective* $\sim$ *and the following axioms:*

($\sim$1) $\sim\sim\varphi \equiv \varphi$,

($\sim$2) $\triangle(\varphi \to \psi) \to (\sim\psi \to \sim\varphi)$.

Since $\mathrm{SBL}_\sim$ is an axiomatic extension of fuzzy logic $\mathrm{SBL}_\triangle$ we use Lemma 2.1.48 to get:

Theorem 3.2.23 $\mathrm{SBL}_\sim$ *is a fuzzy logic, i.e.,* $\mathrm{SBL}_\sim$ *is sound and complete w.r.t. the class of linearly ordered* $\mathrm{SBL}_\sim$*-algebras.*

It is not necessary to present the definition of $\mathrm{SBL}_\sim$-algebras. We only present one interesting lemma:

Lemma 3.2.24 *Let* $\mathbf{B}$ *be an* $\mathrm{SBL}_\sim$*-algebra. Then* $\sim_\mathbf{B}$ *is a order-reversing involution, i.e.,* $\sim_\mathbf{B}\sim_\mathbf{B}x = x$ *and* $x \leq y$ *then* $\sim_\mathbf{B}x \geq \sim_\mathbf{B}y$*. Furthermore, if* $\mathbf{C}$ *is a linear* $\mathrm{SBL}_\triangle$*-algebra and* f *is a an order-reversing involution, the algebra* $\mathbf{C}'$*, resulting from* $\mathbf{C}$ *by adding* f *as an interpretation for* $\sim$ *is a linear* $\mathrm{SBL}_\sim$*-algebra.*

Proof: From ($\sim$1) we get $\sim_\mathbf{B}\sim_\mathbf{B}x = x$). We also know that: $x \leq y$ iff $x \to_\mathbf{B} y = \mathbf{1}$ iff $\triangle(x \to_\mathbf{B} y) = \mathbf{1}$ iff $\sim_\mathbf{B}y \to_\mathbf{B} \sim_\mathbf{B}x = \mathbf{1}$ iff $\sim_\mathbf{B}y \leq \sim_\mathbf{B}x$. The second part is simple. QED

This lemma has two obvious corollaries:

Corollary 3.2.25 $\mathrm{SBL}_\sim$ *is a conservative expansion of both* SBL *and* $\mathrm{SBL}_\triangle$.

Corollary 3.2.26 *Let* $\mathbf{B}$ *be an* $\mathrm{SBL}_\sim$*-algebra. Then* $\neg_{\mathbf{B}}{\sim_{\mathbf{B}}}x = \triangle x$ *and* $\neg_{\mathbf{B}}x \leq {\sim_{\mathbf{B}}}x$*.*

We can use this corollary to present simpler axiomatic system for $\mathrm{SBL}_\sim$.

Theorem 3.2.27 $\mathrm{SBL}_\sim$ *is also an extension of* SBL *with the unary connective* $\sim$*, the axioms* $(\sim 1), (\sim 2)$ *and* $\neg\varphi \to {\sim}\varphi$ *and* $\triangle$*-necessitation, where* $\triangle\varphi$ *is defined as* $\neg{\sim}\varphi$*.*

Proof: To prove one direction we only have to show that $\neg\varphi \to {\sim}\varphi$ and $\triangle\varphi \equiv \neg{\sim}\varphi$ are theorems of $\mathrm{SBL}_\sim$. It is just a simple consequence of the completeness theorem and previous corollary. For the proof of the reverse direction see [21, Theorem 3.1]. QED

Now we generalize the above definitions and results:

Definition 3.2.28 *Let* $\mathbf{L}$ *be an extension of* SBL *in the same language. Then the logic* $\mathbf{L}_\sim$ *is a extension of* $\mathbf{L}_\triangle$ *by axioms* (~ 1) *and* (~ 2)*.*

Of course, we could easily prove an analogy of Theorem 3.2.27. Now we get to a small terminological problem. The notion of standard matrix as defined in Definition 3.1.1 is not satisfactory in logics with $\sim$, the intended (standard) interpretation of $\sim$ is $1 - x$ and we want to reflect this in our notion of *"standard"* algebra, we also want to change the notion of standard completeness and core fuzzy logic (to be consonant with terminology from [35]). Thus we make a slightly non-standard step: we change the definition of a standard algebra (matrix). We do this for logic with $\sim$; for any other logics the former definition remains valid. Later we do it once more, when we introduce rational constants.

Definition 3.2.29 *Let* $\mathbf{L}$ *be an extension of* $\mathrm{SBL}_\sim$ *and* $\mathbf{B}$ *be a standard* $\mathbf{L}$*-matrix in the sense of Definition 3.1.1. We say that* $\mathbf{B}$ *is a standard* $\mathbf{L}$*-matrix iff* $\sim_{\mathbf{B}} = 1 - x$*.*

With the new definition of a standard matrix we can state the following theorem (its proof is a part of folklore):

Theorem 3.2.30 $\mathrm{SBL}_\sim$ *has finite standard completeness but not standard completeness.* $\mathrm{SBL}_\sim$ *is not a core fuzzy logic.*

3.3 Core fuzzy logics

3.3.1 Łukasiewicz logic and its expansions

Łukasiewicz logic (**Ł**) was introduced by Jan Łukasiewicz in his paper [62] in 1920. This logic was originally three-valued, he also studied (with A. Tarski) an infinite-valued version of his logic, but mainly from the philosophical point of view. The proof of the completeness of the infinite-valued Łukasiewicz logic was done by Rose and Rosser in [79]. Nowadays the Łukasiewicz logic is the most developed and most widely used many-valued logic. It has very interesting properties. In this chapter we will understand this logic as an extension of the basic logic BL by one axiom, which we denote (Ł).

Definition 3.3.1 *Łukasiewicz logic (***Ł***) is an extension of* BL *by* $\vdash \neg\neg\varphi \to \varphi$*.*

Theorem 3.3.2 *It holds:*

1. **Ł** *is fuzzy logic with* $\mathrm{DT}_\to$*.*

2. **Ł** *has finite standard completeness but not standard completeness.*

3. **Ł** *is core fuzzy logic.*

4. $\mathbf{Ł}_\triangle$ *has finite standard completeness but not standard completeness.*

5. *$Ł_\triangle$ is core fuzzy logic with* $\mathrm{DT}_\triangle$*.*

Proof:

1. Trivial.
2. Finite standard completeness can be proved in many ways. See for example [44, Theorem 3.2.13] or [12]. Original proof of the second part can be found in [84].
3. Folklore.
4. See Theorem 3.1.8 and Lemma 3.1.7.
5. Trivial. QED

In Łukasiewicz logic we will denote the multiplicative conjunction as $\otimes$ rather than &, the reasons for this lies in Chapter 4 where we add one more multiplicative conjunction, which we will denote $\odot$. The problem is that in the literature about Łukasiewicz logic the multiplicative conjunction is usually denoted as $\odot$. However, we think that our denotation is better rooted in the tradition of substructural logics and we also want to reserve the symbol $\odot$ for already mentioned additional multiplicative conjunction, whose standard interpretation will be the usual product of reals.

Definition 3.3.3 *In* **Ł** *we define two additional connectives:*

- $\varphi \oplus \psi = \neg\varphi \to \psi$,
- $\varphi \ominus \psi = \neg(\varphi \to \psi)$.

Since, **Ł** is a core fuzzy logic, all its standard algebras are mutually isomorphic. We pick the one we denote $[0,1]_Ł$, as the one where $\otimes$ is interpreted as Łukasiewicz t-norm and $\to$ as its residuum. Then the standard interpretation of the derived connectives is rather simple:

- $x \oplus_{[0,1]_Ł} y = \min(1, x+y)$,
- $x \ominus_{[0,1]_Ł} y = \max(0, x-y)$,
- $\neg_{[0,1]_Ł} y = 1 - x$.

The algebras for Łukasiewicz logic are usually taken with the different signature $(\oplus, \neg, \mathbf{0})$, they are the so-called **MV**-algebras:

Definition 3.3.4 *An* **MV***-algebra is a structure* $\mathbf{L} = (L, \oplus, \neg, \mathbf{0})$ *such that, letting* $x \ominus y = \neg(\neg x \oplus y)$*, and* $\mathbf{1} = \neg\mathbf{0}$ *the following conditions are satisfied:*

(MV1) $(L, \oplus, \mathbf{0})$ *is a commutative monoid,*
(MV2) $x \oplus \mathbf{1} = \mathbf{1}$,
(MV3) $\neg\neg x = x$,
(MV4) $(x \ominus y) \oplus y = (y \ominus x) \oplus x$.

In each MV-algebra, we define the additional connectives: $x \otimes y = \neg(\neg x \oplus \neg y)$, $x \to y = \neg x \oplus y$, $x \vee y = (x \ominus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$. It is easy to show that **MV**-algebras and **Ł**-algebras are termwise equivalent (there are even other termwise equivalent definitions: Wajsberg algebras and basic bounded Wajsberg hoops). For deep algebraic results in **MV**-algebras see [12]. There the proof of the following proposition can be find:

Proposition 3.3.5 *In every MV-algebra, the following conditions hold:*

1. $x \ominus \mathbf{0} = x$,

2. $x \ominus x = \mathbf{0}$,
3. $\mathbf{0} \ominus x = \mathbf{0}$,
4. *the following conditions are equivalent:* $x \leq y$, $x \ominus y = \mathbf{0}$, $x \to y = \mathbf{1}$,
5. *if* $x \leq y$, *then* $x \oplus z \leq y \oplus z$, $x \ominus z \leq y \ominus z$, *and* $z \ominus y \leq z \ominus x$,
6. $(x \ominus y) \wedge (y \ominus x) = \mathbf{0}$,
7. $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$,
8. $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,
9. $\mathbf{1} \ominus x = \neg x$,
10. $a = (a \oplus \neg a) \ominus \neg a$.

We conclude this section with McNaughton's famous theorem about functional representation of the Łukasiewicz logic.

Theorem 3.3.6 *The class of all piece-wise linear functions with integer coefficients is a functional representation of Łukasiewicz logic.*

3.3.2 Gödel logic and its expansions

Here we deal with Gödel logic (denoted as G) and its extensions $\mathrm{G}_\triangle$ and $\mathrm{G}_\sim$. We will call them together Gödel logics. Gödel logic is a well-known logical system developed from the original Gödel 's system [42] by Rose and Rosser in [79], Dummett in [29], and put in the context of other many-valued logics by Hájek in [44]. $\mathrm{G}_\triangle$ logic (Gödel logic with delta) results from Gödel logic by adding unary connective $\triangle$. Finally, $\mathrm{G}_\sim$ logic (Gödel involutive logic) arises from Gödel logic by adding unary connective $\sim$ interpreted as an involutive negation.

Gödel logic is extension of BL by single axiom, which we denote (G).

Definition 3.3.7 *Gödel logic* (G) *is an extension of* BL *by* $\vdash \varphi \to \varphi \& \varphi$.

We show that G is a logic with contraction and even more. Before we do so we prepare a few auxiliary theorems (we advise the reader to recall the names and denotations of the important consecutions). This lemma also demonstrates the power of contraction.

Lemma 3.3.8 *The following consecutions hold in* $\mathcal{MIN}_\&(\mathrm{E}, \mathrm{W}, \mathrm{C})$:

1. $\vdash \varphi \to (\psi \to \varphi)$ — $ax(W)$
2. $\chi \to \varphi, \chi \to \psi \vdash \chi \to \varphi \& \psi$ — $\wedge 1$
3. $\vdash \varphi \& \psi \to \varphi$ — $\wedge 2$
4. $\vdash \varphi \& \psi \to \psi \& \varphi$ — $\wedge 3$
5. $\vdash (\varphi \& \psi) \& \chi \to \varphi \& (\psi \& \varphi)$ — *Associativity*
6. $\vdash \varphi \& (\varphi \to \psi) \to \psi$
7. $\varphi \to \psi \vdash \varphi \& \chi \to \psi \& \chi$
8. $\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ — $ax(\mathrm{Sf})$

Proof:

1. Observe that $\varphi \to \varphi \vdash \psi \to (\varphi \to \varphi)$ (from W). E completes the proof.

2. We give a formal proof:

(i)	$\vdash \varphi \to (\psi \to \varphi\&\psi)$	(Ref), and R
(ii)	$\chi \to \varphi, \chi \to \psi \vdash \varphi \to (\varphi \to \varphi\&\psi)$	(i)
(ii)	$\chi \to \varphi, \chi \to \psi \vdash \chi \to (\chi \to \varphi\&\psi)$	(ii), (WT), and E
(iii)	$\chi \to \varphi, \chi \to \psi \vdash \chi \to \varphi\&\psi$	(ii) and C

3. Trivial.

4. From $\vdash \varphi \to (\psi \to \varphi\&\psi)$ ((Ref) and R) we get $\vdash \psi \to (\varphi \to \varphi\&\psi)$ (using E). R again, completes the proof.

5. First, observe $\vdash \varphi\&\psi \to \varphi$ and so $\vdash (\varphi\&\psi)\&\chi \to \varphi$. Second $\vdash (\varphi\&\psi) \to \psi$ and so $\vdash (\varphi\&\psi)\&\chi \to \psi$, this together with $\vdash (\varphi\&\psi)\&\chi \to \chi$ gets us $\vdash (\varphi\&\psi)\&\chi \to \psi\&\chi$ (using $\wedge$1). If we use $\wedge$1 once more the proof is done.

6. Trivial.

7. Trivial.

8. From $\vdash \varphi\&(\varphi \to \psi) \to \psi$ we get $\vdash (\varphi\&(\varphi \to \psi))\&(\psi \to \chi) \to \psi\&(\psi \to \chi)$. Since $\vdash \psi\&(\psi \to \chi) \to \chi$ we get $\vdash (\varphi\&(\varphi \to \psi))\&(\psi \to \chi) \to \chi$. Associativity and R twice complete the proof. QED

Theorem 3.3.9 $\mathrm{G} = \mathcal{FUZZ}_{\bot,\&}(\mathrm{E}, \mathrm{W}, \mathrm{C})$.

Proof: One direction is simple, all we have to do is to show $\varphi \to (\varphi \to \psi) \vdash_{\mathrm{G}} \varphi \to \psi$, but this is quite straightforward (just use ($A5$), (G) and ($A1$)).

The second direction is not so trivial. Let us denote $\mathcal{FUZZ}_{\bot,\&}(\mathrm{E}, \mathrm{W}, \mathrm{C})$ as $\mathbf{L}$. Notice that we can use the previous lemma. As a consequence of $\wedge$1 we immediately get $\vdash_{\mathbf{L}} \varphi \to \varphi\&\varphi$. Since we know that $\vdash_{\mathbf{L}}$ ax(Sf) all we need to show is $\vdash_{\mathbf{L}}$ D and the proof is done (using Lemma 3.2.5).

(i)	$\vdash \varphi \to (\psi \to \varphi)$	$ax(W)$
(ii)	$\vdash \varphi\&(\varphi \to \psi) \to (\psi \to \varphi)$	(i), $\wedge$2, and (WT)
(iii)	$\vdash (\varphi \to \psi) \to (\varphi \to \psi)$	(Ref)
(iv)	$\vdash \varphi\&(\varphi \to \psi) \to \psi$	(iii) and R
(v)	$\vdash \varphi\&(\varphi \to \psi) \to \psi\&(\psi \to \varphi)$	(ii), (iv), and $\wedge$1

QED

Corollary 3.3.10 G *is an extension of Intuitionistic logic by axiom A6.* G *is the weakest fuzzy logic extending Intuitionistic logic.*

Observe that we have $\vdash_{\mathrm{G}} \varphi\&\psi \equiv \varphi \wedge \psi$, i.e., both conjunctions (multiplicative and lattice one) are the same. Thus we use only one symbol from now on—we choose $\wedge$.

Theorem 3.3.11 *Gödel logic has so called* Classical Deduction Theorem *(*CDT*): for each theory T and formulae φ, ψ: $T, \varphi \vdash \psi$ iff $T \vdash \varphi \to \psi$. Furthermore, Gödel logic is the weakest fuzzy logic with* CDT *in language $\to, \&, \bot$.*

Proof: The proof that G logic has CDT is just like in classical logic. To prove the second part all we need to show is that in arbitrary logic with CDT we have W,E, C. To get this just use CDT to the following three consecutions, which obviously hold in arbitrary logic: $\varphi, \psi \vdash \varphi$, $\varphi \to (\psi \to \chi), \psi, \varphi, \vdash \chi$, and $\varphi \to (\varphi \to \psi), \varphi \vdash \psi$. QED

Recall that the logic $\mathbf{L}_\sim$ is defined for extension of SBL only, we show that this is the case of Gödel logic.

Lemma 3.3.12 *We have:* $\vdash_{\mathrm{G}} \neg\neg\varphi \vee \neg\varphi$, *i.e.,* G *is an extension of* SBL.

Proof: From Theorem ($T16$) we get $\varphi \wedge \neg\varphi \to \overline{0}$. Theorem ($T18a$) completes the proof. QED

Now we look at the semantical aspects of Gödel logics. Observe that there is exactly one standard G,$\mathrm{G}_\triangle$, and $\mathrm{G}_\sim$-algebra (recall that the meaning of the term "standard" is changed in logic with $\sim$, under the original definition $\mathrm{G}_\sim$ would have standard completeness but it would not be a core fuzzy logic). The proof of standard completeness of G can be found in many places (the original proof is due to Dummett [29], see also [44, Theorem 4.2.17]). The rest of the following theorem is partially part of folklore and partially consequence of our general theorems. Observe that if we would like to prove everything, all we need to do is to show that $\mathrm{G}_\sim$ has the standard completeness, the rest would be a consequence of the fact that $\mathrm{G}_\sim$ conservatively extends G and $\mathrm{G}_\triangle$.

Theorem 3.3.13 *It holds:*

1. G *is fuzzy logic with* CDT.
2. $\mathrm{G}_\triangle$, $\mathrm{G}_\sim$ *are fuzzy logics with* $\mathrm{DT}_\triangle$.
3. G, $\mathrm{G}_\triangle$, $\mathrm{G}_\sim$ *have the standard completeness.*
4. G, $\mathrm{G}_\triangle$, $\mathrm{G}_\sim$ *are core fuzzy logics.*

3.3.3 Product logic and its expansions

Product logic (Π) was introduced by Hájek, Godo, and Esteva in [52] as an extension of BL by axioms (St and (Π)).

Definition 3.3.14 *Product logic* (Π) *is an extension of* BL *by axioms* $\vdash \neg\varphi \vee \neg\neg\varphi$ *and* $\vdash \neg\neg\varphi \to ((\varphi\&\psi \to \varphi\&\chi) \to (\psi \to \chi))$.

Obviously, Π is an extension of SBL by axiom (Π). There is an obvious question, whether there is some more elegant presentation of Π, preferably with one axiom with only one variable. We answer this question is Section 5.1. In the standard Π-algebra the conjunction is interpreted by a product t-norm (usual multiplication of reals) and implication is interpreted by its residuum.

Theorem 3.3.15 *It holds:*

1. Π *is fuzzy logic with* $\mathrm{DT}_\to$.
2. $\Pi_\triangle$ *and* $\Pi_\sim$ *are fuzzy logics with* $\mathrm{DT}_\triangle$.
3. Π *and* $\Pi_\triangle$ *have the standard completeness.*
4. Π *and* $\Pi_\triangle$ *are core fuzzy logics.*
5. $\Pi_\sim$ *has not the weak standard completeness.*
6. $\Pi_\sim$ *is not core fuzzy logic.*

The only non-trivial part is part 3., for its proof see [44, Theorem 4.1.13]. For the functional representation of Π see Section 5.3.

3.3.4 The ŁΠ and ŁΠ$\frac{1}{2}$ logics

In this section we deal with logics ŁΠ, ŁΠ$\frac{1}{2}$ and RŁΠ. These logic were introduced by Esteva, Godo, and Montagna in papers [36] and [32] and then were further elaborated by the author ([24], [21], [17], [23]), Montagna ([67], [68]), Vetterlein ([81]) and others.

Definition 3.3.16 *Logic* ŁΠ *has the following* basic connectives *(they are listed together with their standard semantics in* $[0,1]$*; we use the same symbols for logical connectives and the corresponding algebraic operations):*

$\overline{0}$	0	*truth constant falsum*
$\varphi \to_{Ł} \psi$	$x \to_{Ł} y = \min(1, 1-x+y)$	*Łukasiewicz implication*
$\varphi \to_{\Pi} \psi$	$x \to_{\Pi} y = \min(1, \frac{x}{y})$	*product implication*
$\varphi \&_{\Pi} \psi$	$x \&_{\Pi} y = x \cdot y$	*product conjunction*

Logic ŁΠ$\frac{1}{2}$ *has one additional truth constant* $\overline{\mathbf{\frac{1}{2}}}$ *with standard semantics* $\frac{1}{2}$*. We define the following* derived connectives:

$\neg_{Ł}\phi$	*is*	$\phi \to_{Ł} 0$	$\neg_{Ł} x = 1-x$
$\neg_{\Pi}\phi$		$\phi \to_{\Pi} 0$	$\neg_{Ł} x = \frac{0}{x}$
$\overline{1}$		$\neg_{Ł}\overline{0}$	1
$\triangle\phi$		$\neg_{\Pi}\neg_{Ł}\phi$	$\triangle 1 = 1$*;* $\triangle x = 0$ *otherwise*
$\phi \&_{Ł} \psi$		$\neg_{Ł}(\phi \to_{Ł} \neg_{Ł}\psi)$	$x \&_{Ł} y = \max(0, x+y-1)$
$\phi \oplus \psi$		$\neg_{Ł}\phi \to_{Ł} \psi$	$x \oplus y = \min(1, x+y)$
$\phi \ominus \psi$		$\phi \&_{Ł} \neg_{Ł}\psi$	$x \ominus y = \max(0, x-y)$
$\phi \wedge \psi$		$\phi \&_{Ł} (\phi \to_{Ł} \psi)$	$x \wedge y = \min(x, y)$
$\phi \vee \psi$		$(\phi \to_{Ł} \psi) \to_{Ł} \psi$	$x \vee y = \max(x, y)$
$\phi \to_{G} \psi$		$\triangle(\phi \to_{Ł} \psi) \vee \psi$	$x \to_{G} y = 1$ *if* $x \leq y$*, otherwise* y

We assume the usual precedence of connectives. Occasionally we may write $\neg_{G}$ *and* $\&_{G}$ *as synonyms for* $\neg_{\Pi}$ *and* $\wedge$*, respectively. We further abbreviate* $(\phi \to_{*} \psi) \&_{*} (\psi \to_{*} \phi)$ *by* $\phi \leftrightarrow_{*} \psi$ *for* $* \in \{G, Ł, \Pi\}$.

There are different axiomatic systems of the logic ŁΠ. We use the one from [24].

Definition 3.3.17 *Logic* ŁΠ *is given by the following axioms:*

(Ł)	*Axioms of Łukasiewicz logic,*
(Π)	*Axioms of product logic,*
(Ł△)	$\triangle(\varphi \to_{Ł} \psi) \to_{Ł} (\varphi \to_{\Pi} \psi)$,
(Π△)	$\triangle(\varphi \to_{\Pi} \psi) \to_{Ł} (\varphi \to_{Ł} \psi)$,
(Dist)	$\varphi \&_{\Pi} (\chi \ominus \psi) \leftrightarrow_{Ł} (\varphi \&_{\Pi} \chi) \ominus (\varphi \&_{\Pi} \psi)$.

The deduction rules are modus ponens and $\triangle$*-necessitation (from* φ *infer* $\triangle\varphi$*). The logic* ŁΠ$\frac{1}{2}$ *results from the logic* ŁΠ *by adding axiom* $\frac{1}{2} \leftrightarrow \neg_{Ł}\frac{1}{2}$.

Observe that $\varphi \to_{A} \psi \vdash \varphi \to_{B} \psi$ is valid deduction rule for arbitrary $A, B \in \{G, Ł, \Pi\}$

Lemma 3.3.18 *The following formulae are theorems of the* ŁΠ *logic:*

(ŁΠ1) $\neg_{\text{Ł}} \triangle\varphi \to_{\Pi} \neg_{\Pi} \triangle\varphi$
(ŁΠ2) $(\triangle\varphi \odot \psi) \leftrightarrow_{\Pi} (\triangle\varphi \otimes \psi)$
(ŁΠ3) $(\triangle\varphi \to_{\Pi} \psi) \to_{\text{Ł}} (\triangle\varphi \to_{\text{Ł}} \psi)$
(ŁΠ4) $\triangle(\varphi \leftrightarrow_{\text{Ł}} \psi) \leftrightarrow_{\text{Ł}} \triangle(\varphi \leftrightarrow_{\Pi} \psi)$
(ŁΠ5) $(\varphi \to_{\mathrm{G}} \psi) \to_{\text{Ł}} (\varphi \to_{\text{Ł}} \psi)$
(ŁΠ6) $(\varphi \to_{\mathrm{G}} \psi) \to_{\text{Ł}} (\varphi \to_{\Pi} \psi)$
(ŁΠ7) $\triangle(\varphi \leftrightarrow_{\text{Ł}} \psi) \wedge \triangle(\chi \leftrightarrow_{\text{Ł}} \delta) \to_{\text{Ł}} ((\varphi \odot \chi) \leftrightarrow_{\text{Ł}} (\psi \odot \delta))$
(ŁΠ8) $\triangle(\varphi \leftrightarrow_{\text{Ł}} \psi) \wedge \triangle(\chi \leftrightarrow_{\text{Ł}} \delta) \to_{\text{Ł}} ((\varphi \to_{\Pi} \chi) \leftrightarrow_{\text{Ł}} (\psi \to_{\Pi} \delta))$

The consequence of this lemma is that it is not always necessary to write index for each connective, eg. (ŁΠ3) shows that whenever an antecedent of Ł-implication is a formula starting by $\triangle$ the implication is provably equivalent to B-implication for arbitrary B. The following theorem can be proven by showing corresponding formal proofs (although some of its claim have shorter "semantical" proof). For details see [23], [24] and the authors master thesis [22].

Theorem 3.3.19 *The logic* ŁΠ *extends the following logics:* **Ł**, $\mathbf{Ł}_{\triangle}$, G, $\mathrm{G}_{\triangle}$, $\mathrm{G}_{\sim}$, Π, $\Pi_{\triangle}$, *and* $\Pi_{\sim}$.

By saying that ŁΠ extends Π we mean that if we restrict the logic ŁΠ to the "product" connectives we obtain an extension of product logic. Now we introduce the notion of ŁΠ-algebra and observe that ŁΠ-algebras are just ordered ŁΠ-matrices. For alternative definition of ŁΠ-algebras see [17].

Definition 3.3.20 *An* ŁΠ-algebra *is a structure* $\mathbf{L} = (L, \oplus, \neg_{\text{Ł}}, \to_{\Pi}, \&_{\Pi}, 0, 1)$ *such that:*

- $(L, \oplus, \neg_{\text{Ł}}, 0)$ *is an MV-algebra,*
- $(L, \vee, \wedge, \to_{\Pi}, \&_{\Pi}, 0, 1)$ *is a* Π*-algebra,*
- $x\&_{\Pi}(y \ominus z) = (x\&_{\Pi}y) \ominus (x\&_{\Pi}z)$.

Furthermore, the structure $\mathbf{L} = (L, \oplus, \neg_{\text{Ł}}, \to_{\Pi}, \&_{\Pi}, 0, 1, \frac{1}{2})$ *where the reduct* $\mathbf{L}' = (L, \oplus, \neg_{\text{Ł}}, \to_{\Pi}, \&_{\Pi}, 0, 1)$ *is an* ŁΠ*-algebra and the identity* $\frac{1}{2} = \neg_{\text{Ł}} \frac{1}{2}$ *holds is called the* ŁΠ$\frac{1}{2}$-algebra.

Theorem 3.3.21 *We have:*

1. *Both* ŁΠ *and* ŁΠ$\frac{1}{2}$ *are weakly implicative logic (the role of the principal implication can be played by any of* $\to_{\mathrm{G}}, \to_{\Pi}$, *or* $\to_{\text{Ł}}$*).*
2. *Both* ŁΠ *and* ŁΠ$\frac{1}{2}$ *have* $\mathrm{DT}_{\triangle}$.
3. *Both* ŁΠ *and* ŁΠ$\frac{1}{2}$ *are core fuzzy logics.*
4. *Both* ŁΠ *and* ŁΠ$\frac{1}{2}$ *extend the following logics conservatively:* **Ł**, $\mathbf{Ł}_{\triangle}$, G, $\mathrm{G}_{\triangle}$, $\mathrm{G}_{\sim}$, Π, *and* $\Pi_{\triangle}$.
5. *Both* ŁΠ *and* ŁΠ$\frac{1}{2}$ *do not have standard completeness.*

Proof:

1. Consequence of Lemmata 3.3.18 and 2.1.16.
2. Consequence of Theorem 2.2.26 and the fact that ŁΠ extends $\mathbf{Ł}_{\triangle}$.
3. The fact that they are fuzzy logic is a consequence of Lemma 2.2.28. For the proof of finite standard completeness see [36], the rest is obvious.

4. Easy consequence of Theorem 3.1.13 and the fact that all those logic are core fuzzy logics.

5. If not, we would get a standard completeness of **Ł** (using the previous part)—a contradiction. QED

Let us by $[0,1]_{Ł\Pi}$ denote the ŁΠ-algebra with operations as stated in Definition 3.3.16. (analogously for the standard $Ł\Pi\frac{1}{2}$-algebra). The two-valued ŁΠ algebra is denoted by $\{\mathbf{0},\mathbf{1}\}$.

In [21] the author proves that ŁΠ-logic can be viewed as an axiomatic extension of $\Pi_\sim$ (if we understand the connectives of $\Pi_\sim$ as "product" connectives of ŁΠ and define the others as in the following theorem).

Theorem 3.3.22 *The following is a presentation of* ŁΠ*:*

(Π) *axioms and deduction rules of* $\Pi_\sim$*,*

(A) $(\varphi \to_Ł \psi) \to_Ł ((\psi \to_Ł \chi) \to_Ł (\varphi \to_Ł \chi))$,

where $\varphi \to_Ł \psi$ *is defined as* $\sim(\varphi \& \sim(\varphi \to \psi))$.

This result can be improved using results from the upcoming paper of Thomas Vetterlein [81]:

Theorem 3.3.23 *The following is a presentation of* ŁΠ*:*

(Π) *axioms and deduction rules of* $\Pi_\sim$*,*

(A) $\varphi \&_Ł \psi \to \psi \&_Ł \varphi$,

where $\varphi \&_Ł \psi$ *is defined as* $\varphi \& \sim(\varphi \to \sim\psi)$.

The following definitions and theorems demonstrate the expressive power of ŁΠ and $Ł\Pi\frac{1}{2}$ logics. Particularly, Corollary 3.3.26 shows that each propositional logic based on an arbitrary t-norm of a certain simple form is contained in $Ł\Pi\frac{1}{2}$ logic.

Definition 3.3.24 *Let f be a function $f : [0,1]^n \to [0,1]$. Function f is called a* rational ŁΠ-function *iff there is a finite partition of $[0,1]^n$ such that each block of the partition is a semi-algebraic set and f restricted to each block is a fraction of two polynomials with rational coefficients. Furthermore, a rational ŁΠ-function f is* integral *iff all the coefficients are integer and $\{f(a_1, \dots a_n) \mid a_i \in \{0,1\}\} \subseteq \{0,1\}$.*

The following theorem was proved in [69]. It has an interesting corollary, which was independently proved in [23].

Theorem 3.3.25 (Functional representation) *The class of integral (rational)* ŁΠ *functions is the functional representation of the* ŁΠ *logic (*$Ł\Pi\frac{1}{2}$ *logic resp.).*

Corollary 3.3.26 *Let $*$ be a continuous t-norm which is a finite ordinal sum of the three basic ones (i.e., of* G, Ł *and* Π*), and $\Rightarrow$ be its residuum. Then there are derived connectives $\&_*$ and $\to_*$ of the $Ł\Pi\frac{1}{2}$ logic such that their standard semantics are $*$ and $\Rightarrow$ respectively. The logic $PC(*)$ of the t-norm $*$ is contained in $Ł\Pi\frac{1}{2}$ if $\&$ and $\to$ of $PC(*)$ are interpreted as $\&_*$ and $\to_*$. Furthermore, if ϕ is provable in $PC(*)$ then the formula ϕ_* obtained from ϕ by replacing the connectives $\&$ and $\to$ of $PC(*)$ by $\&_*$ and $\to_*$ is provable in $Ł\Pi\frac{1}{2}$.*

Convention 3.3.27 *Further on, the signs $*$ and $\diamond$ will be reserved for t-norms definable in $Ł\Pi\frac{1}{2}$ (incl.* G, Ł *and* Π*), and the indexed connectives will always have the meaning introduced in the previous Corollary. However, we omit the indices of connectives whenever they do not matter, i.e., whenever all formulae obtained by subscripting any $*$ to such a connective are provably equivalent (for example, $\neg\neg_G\phi$, $\triangle(\phi \to \psi)$, etc.), or equivalently provable (e.g., the principal implication in axioms and theorems).*

Corollary 3.3.28 *Let $r \in [0,1]$ be a rational number. Then there is a formula φ of* $\text{Ł}\Pi\frac{1}{2}$ *such that $e(\varphi) = r$ for any $[\mathbf{0},\mathbf{1}]$-evaluation e.*

This corollary tells us that in $\text{Ł}\Pi\frac{1}{2}$ we have a truth constant $\bar{r}$ for each rational number $r \in [0,1]$. Using the completeness theorem we get the following corollary.

Corollary 3.3.29 *The following are theorems of the* $\text{Ł}\Pi\frac{1}{2}$ *logic:*

$$\overline{r \&_\Pi s} = \bar{r} \&_\Pi \bar{s},$$

$$\overline{r \to_\Pi s} = \bar{r} \to_\Pi \bar{s},$$

$$\overline{r \to_Ł s} = \bar{r} \to_Ł \bar{s}.$$

where the symbols $\&_\Pi, \to_\Pi, \to_Ł$ on the left side are operations in $[0,1]$ and on the right side they are logical connectives.

3.4 Adding truth constants

In this section we explore the so-called Pavelka-style extension of a particular fuzzy logic. By the term "Pavelka-style extension" we understand the logic created from some fuzzy logic by adding truth constants for all rations from $[0,1]$ which fulfills some additional properties. We present a general methodology for defining Pavelka-style extensions and some general theorems.

The original source of this approach is Pavelka's work [76], where he adds truth constants for each *real* number form $[0,1]$ into the language of the Łukasiewicz logic and develops a new logic in this framework. The most recent work in this area is a book by Novák, Perfilieva, and Močkoř [75], where the authors develop the so-called *logic with evaluated syntax*. The authors introduce a new kind of completeness. It relates the so-called provability and truth degrees of formulae. Although it is a completely natural notion of completeness in their approach, it is incomparable in power with our usual notion of completeness and thus we decided to call it a *Pavelka-style completeness*.

Logic with evaluated syntax is not a logic in our sense (it has non-classical syntax) and Hájek in his book [44] showed how to interpret this logic in our framework. However, this is a *formal* interpretation, which omits the extra-logical (linguistic, philosophical) merits of logic with evaluated syntax. We do not want to discuss the difference between these two approaches here, but Chapter 8 contains some more ideas about this problem.

It seems reasonable to restrict the set of additional constants to rationals from $[0,1]$ only (let us denote this set by $\mathbb{IQ}$). The reason is that we want to keep the language countable, so we can speak about decidability and complexity of the resulting logics.

As mentioned above we want to present a *general* methodology for defining Pavelka-style extensions. There are previous works in this area which are incomparable with our work in generality. The original work of Pavelka (and then the work of Novák) can be extended to more logics expanding Łukasiewicz logic than our approach. On the other hand our approach works for logics not extending Łukasiewicz logic. The work of Esteva, Godo, and Noguera [38] deals with logics extending (in the same language) so-called weak nilpotent minimum logic (from the logics defined in this work only Gödel logic belongs in this class) and they also deals with standard completeness only, whereas we deal with both standard and Pavelka-style completeness.

3.4.1 Rational extension and basic definitions

In this subsection we introduce the so-called Rational extension of some fuzzy logic, it will be a first step in the definition of Pavelka-style extension. We also introduce some concepts we will need in the further text.

Definition 3.4.1 *Let* $\mathbf{L}$ *be a fuzzy logic and* $\mathbf{B}$ *a standard* $\mathbf{L}$*-algebra. We say that* $\mathbf{B}$ *is a* rational *standard* $\mathbf{L}$*-algebra if the set* $\mathbb{IQ}$ *is closed under operations of* $\mathbf{B}$*. We say that core fuzzy logic is rational if* $[0,1]_{\mathbf{L}}$ *is rational.*

Definition 3.4.2 *Let* $\mathcal{L}$ *be a propositional language,* $\mathbf{L}$ *a fuzzy logic in* $\mathcal{L}$*, and* $\mathbf{B}$ *a rational standard* $\mathbf{L}$*-algebra. The* rational extension of $\mathbf{L}$ *based on algebra* $\mathbf{B}$ *(*$\mathcal{R}\mathbf{L}(\mathbf{B})$*) in language* $\mathcal{L} \cup \{(\overline{r}, 0) \mid r \in \mathbb{IQ}\}$ *results from logic* $\mathbf{L}$ *by adding axioms* $c(\overline{r}_1, \ldots, \overline{r}_n) \leftrightarrow \overline{c_{\mathbf{B}}(r_1, \ldots, r_n)}$ *for each* $(c, n) \in \mathcal{L}$ *and each* $r_1, \ldots, r_n \in \mathbb{IQ}$*.*

Lemma 3.4.3 $Ł\Pi\frac{1}{2} = \mathcal{R}Ł\Pi([0,1]_{Ł\Pi}) = \mathcal{R}Ł\Pi\frac{1}{2}([0,1]_{Ł\Pi\frac{1}{2}})$.

For the proof see Corollary 3.3.29. Obviously, the logic $\mathcal{R}\mathbf{L}(\mathbf{B})$ is weakly implicative. We show that it is also a fuzzy logic. But first we prove one more general lemma.

Lemma 3.4.4 *Let* $\mathcal{L}$ *be a propositional language,* $\mathbf{L}$ *a fuzzy logic in* $\mathcal{L}$ *complete w.r.t. a class* $\mathcal{K}$ *of* $\mathbf{L}$*-algebras, and* $\mathbf{B}$ *a rational standard* $\mathbf{L}$*-algebra. The logic* $\mathcal{R}\mathbf{L}(\mathbf{B})$ *is complete w.r.t. the class of all* $\mathcal{R}\mathbf{L}(\mathbf{B})$*-algebras, whose reducts to the language* $\mathcal{L}$ *are from* $\mathcal{K}$*.*

Proof: Let us assume that $T \not\vdash_{\mathcal{R}\mathbf{L}(\mathbf{B})} \varphi$, we know that there is $\mathcal{R}\mathbf{L}(\mathbf{B})$-algebra $\mathbf{C}$ and $\mathbf{C}$-model e of theory T such that $e(\varphi) < \overline{1}_{\mathbf{C}}$.

Let us take an infinite set VAR_0 of propositional variables not occurring in T and φ and index its elements by rations from $\mathbb{IQ}$, i.e., $VAR_0 = \{v_r \mid r \in \mathbb{IQ}\}$. Define theory $T_0 = \{c(v_{r_1}, \ldots, v_{r_n}) \leftrightarrow v_{c_{\mathbf{C}}(r_1,\ldots,r_n)} \mid (c, n) \in \mathcal{L}$ and $r_1, \ldots, r_n \in \mathbb{IQ}\}$. For each formula ψ define formula ψ' as a formula resulting from ψ by replacing each occurrence of truth constant $\overline{r}$ with variable v_r. Define theory $T' = \{\psi' \mid \psi \in T\}$. Now observe that $T_0 \cup T' \not\models_{\{\mathbf{C}\}} \varphi'$ (just take $\mathbf{C}$-evaluation e' resulting $\mathbf{C}$-evaluation e by setting $e(v_r) = \overline{r}_{\mathbf{C}}$). Let us by $\mathbf{C}'$ denote the restriction of $\mathbf{C}$ to the language $\mathcal{L}$. By observing that $T_0 \cup T' \cup \{\varphi'\} \subseteq \mathbf{FOR}_{\mathcal{L}}$ we get $T_0 \cup T' \not\models_{\{\mathbf{C}'\}} \varphi'$ and so $T_0 \cup T' \not\vdash_{\mathbf{L}} \varphi'$. Using the fact that $\mathbf{L}$ is complete w.r.t. class $\mathcal{K}$ we know that there is $\mathbf{L}$-algebra $\mathbf{A}$ and $\mathbf{A}$-model f of theory $T_0 \cup T'$ such that $f(\varphi') < \overline{1}_{\mathbf{A}}$. If we enlarge the $\mathbf{L}$-algebra $\mathbf{A}$ into the $\mathcal{R}\mathbf{L}(\mathbf{B})$-algebra $\mathbf{A}'$ by setting $\overline{r}_{\mathbf{A}'} = f(v_r)$ we observe that f is $\mathbf{A}'$-model of T and $f(\varphi) < \overline{1}_{\mathbf{A}}$. QED

Corollary 3.4.5 *Let* $\mathcal{L}$ *be a propositional language,* $\mathbf{L}$ *a fuzzy logic in* $\mathcal{L}$*, and* $\mathbf{B}$ *a rational standard* $\mathbf{L}$*-algebra. Then the logic* $\mathcal{R}\mathbf{L}(\mathbf{B})$ *is a fuzzy logic.*

Here we have the same problem as in the Section 3.2.3 about involutive negation. From the last lemma we would get that if $\mathbf{L}$ is a logic with standard completeness, then $\mathcal{R}\mathbf{L}(\mathbf{B})$ has standard completeness as well. However, the following example shows the non-intuitive consequence of such a definition.

Example 3.4.6 *Let us consider the algebra* $\mathbf{B}$*, whose reduct is the standard* G*-algebra* $[0,1]_{\mathrm{G}}$ *and rational constants are interpreted in the following way:* $\overline{r}_{\mathbf{B}} = \overline{1}_{\mathbf{B}}$ *for each* $r > 0$*. Then* $\mathbf{B}$ *is a* $\mathcal{R}\mathrm{G}([0,1]_{\mathrm{G}})$*-algebra.*

We can observe that the interpretation of the rational constants in this example is rather "non-intended" and so we define:

Definition 3.4.7 *Let* $\mathbf{L}$ *be a fuzzy logic,* $\mathbf{B}$ *be a standard* $\mathbf{L}$*-algebra, and* $\mathbf{C}$ *a standard* $\mathcal{R}\mathbf{L}(\mathbf{B})$*-algebra in the sense of Definitions 3.1.1 and 3.2.29. We say that* $\mathbf{C}$ *is a standard* $\mathbf{L}$*-algebra iff* $\overline{r}_{\mathbf{C}} = r$*.*

Corollary 3.4.8 *The logic* $\mathcal{R}\mathrm{G}([0,1]_{\mathrm{G}})$ *has not the finite standard completeness.*

Proof: Let $\mathbf{A}$ be a standard $\mathcal{R}\mathrm{G}([0,1]_{\mathrm{G}})$-algebra, then $\overline{\frac{1}{2}} \models_{\mathbf{A}} \overline{0}$ (since there is no $\mathbf{A}$-model of theory $\{\overline{\frac{1}{2}}\}$. On the other hand, let $\mathbf{B}$ be the $\mathcal{R}\mathrm{G}([0,1]_{\mathrm{G}})$-algebra form Example 3.4.6. Then obviously $\overline{\frac{1}{2}} \not\models_{\mathbf{B}} \overline{0}$—a contradiction. QED

$\diamond$

Convention 3.4.9 *Let us define formula $\varphi \to^+ \psi$ as $\varphi \to \psi$ and formula $\varphi \to^- \psi$ as $\psi \to \varphi$. Furthermore, we understand $x <^+ y$ as $x < y$ and $x <^- y$ as $y < x$.*

Definition 3.4.10 *We say that the fuzzy logic $\mathbf{L}$ in language $\mathcal{L}$ is* argument-monotonous *iff for each $(c,n) \in \mathcal{L}$ there is sequence $\hat{c} \in \{+,-\}^n$ such that for each $i \leq n$:*

$$\varphi \to^{\hat{C}(i)} \psi \vdash_{\mathbf{L}} c(\chi_1,\dots,\chi_{i-1},\varphi,\dots,\chi_n) \to c(\chi_1,\dots,\chi_{i-1},\psi,\dots,\chi_n).$$

Let us observe that all the logics we have defined are argument-monotonous, and we have:

Example 3.4.11 *Let $\mathbf{L}$ be a fuzzy logic, then:*

- $\hat{\to} = (-,+)$
- $\hat{\&} = \hat{\vee} = \hat{\wedge} = (+,+)$
- $\hat{\neg} = \hat{\sim} = (-)$
- $\hat{\triangle} = (+)$

The following lemma demonstrated why we decided to call the above mentioned property by the term *argument monotonous*.

Lemma 3.4.12 *Let $\mathbf{L}$ be a fuzzy logic. Then the following are equivalent:*

1. *the logic $\mathbf{L}$ is argument-monotonous,*
2. *$\{\varphi_i \to^{\hat{c}(i)} \psi_i \mid i \leq n\} \vdash_{\mathbf{L}} c(\varphi_1,\dots,\varphi_n) \to c(\psi_1,\dots,\psi_n)$ for each $(c,n) \in \mathcal{L}$,*
3. *for each linear $\mathcal{L}$-algebra $\mathbf{B}$, the operation $c_{\mathbf{B}}$ is non-decreasing in the i-th argument whenever $\hat{c}(i) = +$ and non-increasing otherwise.*

By monotonicity we understand monotonicity w.r.t. the matrix order $\leq_{\mathbf{B}}$. The proof is simple yet rather technical, so we leave it for the reader. Observe that for core fuzzy logics it is enough to speak about standard $\mathbf{L}$-algebras in Part 3.

Now we introduce the set of non-continuity points in some standard algebra. We assume the standard topology of [0,1].

Definition 3.4.13 *Let $\mathcal{L}$ be a propositional language, $\mathbf{L}$ a fuzzy logic in $\mathcal{L}$, and $\mathbf{B}$ a standard $\mathbf{L}$-algebra. Let us define the set (of non-continuity points)*
$\mathcal{NC}(B) = \{< c, x_1,\dots,x_n > \mid (c,n) \in \mathcal{L}$ and function $c_{\mathbf{B}}$ is non-continuous in $(x_1,\dots,x_n)\}$.
If $\mathbf{L}$ is core fuzzy logic, then by $\mathcal{NC}(L)$ we denote the set $\mathcal{NC}([0,1]_{\mathbf{L}})$.

For each element $l \in \mathcal{NC}(B)$ we define the set consecutions $\mathcal{C}on^{\mathbf{B}}(l)$.

Definition 3.4.14 *Let $\mathbf{L}$ be an argument-monotonous fuzzy logic and $\mathbf{B}$ some rational standard $\mathbf{L}$-algebra. Then the following consecutions are elements of $\mathcal{C}on^{\mathbf{B}}(< c, x_1,\dots,x_n >)$:*

- *$\{\overline{r} \to^{\hat{c}(i)} \varphi_i \mid r <^{\hat{c}(i)} x_i\} \vdash \overline{r} \to c(\varphi_1,\dots,\varphi_n)$ for each $r \leq c_{\mathbf{B}}(x_1,\dots,x_n)$,*
- *$\{\varphi_i \to^{\hat{c}(i)} \overline{r} \mid x_i <^{\hat{c}(i)} r\} \vdash c(\varphi_1,\dots,\varphi_n) \to \overline{r}$ for each $r \geq c_{\mathbf{B}}(x_1,\dots,x_n)$.*

In some cases the set $\mathcal{C}on^{\mathbf{B}}(< c, x_1,\dots,x_n >)$ can be greatly simplified, now we present some particular deduction rules and then we show their relation to the general definition above.

Definition 3.4.15 *Let $a \in [0,1]$. Then we define the infinitary deduction rules:*

- $\mathrm{IR}_{\to}$*: $\{\varphi \to \overline{r} \mid r > 0\} \vdash \varphi \to \overline{0}$,*
- IR_a*: $\{\varphi \to \overline{r}, \overline{s} \to \psi \mid r > a > s\} \vdash \varphi \to \psi$,*
- $\mathrm{IR}_{\triangle}$*: $\{\overline{r} \to \varphi \mid r < 1\} \vdash \varphi$.*

Observe that the rules IR_0 and $\mathrm{IR}_{\to}$ mutually replaceable.

3.4.2 Pavelka-style extension

Now we define Pavelka-style extension of particular fuzzy logic.

Definition 3.4.16 *Let* $\mathbf{L}$ *be an argument-monotonous fuzzy logic and* $\mathbf{B}$ *a rational standard* $\mathbf{L}$*-algebra. The Pavelka-style extension of* $\mathbf{L}$ *w.r.t.* $\mathbf{B}$ *(denoted as* $\mathcal{RPL}(\mathbf{B})$*) is the logic* $\mathcal{R}\mathbf{L}(\mathbf{B})$ *extended by consecutions* $\mathcal{C}on^{\mathbf{B}}(l)$ *for each* $l \in \mathcal{NC}(L)$ *and consecutions* $\overline{r} \vdash \overline{0}$ *for each* $r < 1$.

Furthermore, if $\mathbf{L}$ *is an argument-monotonous rational core fuzzy logic we define the Pavelka-style extension of* $\mathbf{L}$ *(denoted as* $\mathcal{RP}\mathbf{L}$*) as the logic* $\mathcal{RP}\mathbf{L}([0,1]_{\mathbf{L}})$.

We say that the fuzzy logic $\mathbf{L}$ *is Pavelka-style fuzzy logic if it is Pavelka-style extension of some fuzzy logic w.r.t. some its standard algebra.*

There is a reason for introducing this definition in this general form and not only for the core fuzzy logics. Recall that Product involutive logic $\Pi_\sim$ is not a core fuzzy logic (it has not finite standard completeness) although there is only one standard $\Pi_\sim$-algebra (up to isomorphism). It is the standard Π-algebra with $\sim$ interpreted as $1 - x$, let us denote this algebra as $[0,1]_{\Pi_\sim}$. We will see that the logic $\mathcal{RP}\Pi_\sim([0,1]_{\Pi_\sim})$ shares some interesting properties of Pavelka-style extensions of core fuzzy logics..

Remark 3.4.17 *We list the particular case of the set* $\mathcal{NC}(\mathbf{L})$ *for known fuzzy logics:*

- $\mathcal{NC}(\mathbf{Ł}) = \emptyset$
- $\mathcal{NC}(\mathbf{Ł}_\triangle) = \{<\triangle, 1>\}$
- $\mathcal{NC}(\Pi) = \{<\rightarrow, 0, 0>\}$
- $\mathcal{NC}(\text{Ł}\Pi) = \mathcal{NC}(\Pi_\triangle) = \mathcal{NC}([0,1]_{\Pi_\sim}) = \{<\rightarrow, 0, 0>\} \cup \{<\triangle, 1>\}$
- $\mathcal{NC}(\mathrm{G}) = \{<\rightarrow, a, a> \mid a \in [0,1]\}$
- $\mathcal{NC}(\mathrm{G}_\triangle) = \mathcal{NC}(\mathrm{G}_\sim) = \{<\rightarrow, a, a> \mid a \in [0,1]\} \cup \{<\triangle, 1>\}$

Now we look at some particular Pavelka-style logics and observe that some of the additional consecution are sometimes redundant. The logic $\mathcal{RP}\mathbf{Ł}$ is usually denoted as RPL (see [44]). Furthermore, the logic $\mathcal{RP}\text{Ł}\Pi$ is usually denoted as RŁΠ (see [36]).

Theorem 3.4.18 *Particular Pavelka-style extension of known fuzzy logics can be axiomatized as:*

- $\mathrm{RPL} = \mathcal{RP}\mathbf{Ł} = \mathcal{R}\mathbf{Ł}([0,1]_{\text{Ł}})$
- $\mathcal{RP}\mathbf{Ł}_\triangle = \mathcal{R}\mathbf{Ł}([0,1]_{\text{Ł}}) + \mathrm{IR}_\triangle$
- $\mathcal{RP}\Pi = \mathcal{R}\Pi([0,1]_\Pi) + \mathrm{IR}_\rightarrow$
- $\mathcal{RP}\Pi_\triangle = \mathcal{R}\Pi_\triangle([0,1]_{\Pi_\triangle}) + \mathrm{IR}_\rightarrow + \mathrm{IR}_\triangle$
- $\mathcal{RP}\Pi_\sim([0,1]_{\Pi_\sim}) = \mathcal{R}\Pi_\sim([0,1]_{\Pi_\sim}) + \mathrm{IR}_\triangle$
- $\mathcal{RP}\mathrm{G} = \mathcal{R}\mathrm{G}([0,1]_\mathrm{G}) + \{\mathrm{IR}_a \mid a \in [0,1]\}$
- $\mathcal{RP}\mathrm{G}_\triangle = \mathcal{R}\mathrm{G}_\triangle([0,1]_{\mathrm{G}_\triangle}) + \{\mathrm{IR}_a \mid a \in [0,1]\} + \mathrm{IR}_\triangle$
- $\mathcal{RP}\mathrm{G}_\sim = \mathcal{R}\mathrm{G}_\sim([0,1]_{\mathrm{G}_\sim}) + \{\mathrm{IR}_a \mid a \in [0,1]\} + \mathrm{IR}_\triangle$
- $\mathrm{RŁ\Pi} = \mathcal{RP}\text{Ł}\Pi = \mathcal{RP}\text{Ł}\Pi\frac{1}{2} = \text{Ł}\Pi\frac{1}{2} + \mathrm{IR}_\triangle$

The proof is almost straightforward. Observe that except for RPL we do not know whether any logic mentioned in the previous theorem is actually a fuzzy logic. It is caused by the presence of infinitary deduction rules.

Definition 3.4.19 *Let* $\mathbf{L}$ *be a Pavelka-style fuzzy logic w.r.t. the algebra* $\mathbf{B}$*,* T *a theory, and* φ *be a formula. Then we define:*

- *the* truth degree *of* φ *over* T *is* $||\varphi||_T = \inf\{e(\varphi) \mid e \in \mathbf{MOD}(T, \mathbf{B})\}$,
- *the* provability degree *of* φ *over* T *is* $|\varphi|_T = \sup\{r \mid T \vdash \overline{r} \to \varphi\}$.

Lemma 3.4.20 *Let* $\mathbf{L}$ *be a Pavelka-style fuzzy logic w.r.t. algebra* $\mathbf{B}$*,* T *a theory, and* φ*,* ψ *formulae. Then:*

(1) If $T \not\vdash \overline{r} \to \varphi$ *then* $T \cup \{\varphi \to \overline{r}\}$ *is consistent.*

Furthermore, if T *is linear we have:*

(2) $|\varphi|_T = \sup\{r|T \vdash \overline{r} \to \varphi\} = \inf\{r|T \vdash \varphi \to \overline{r}\}$,

(3) the provability degree commutes with all connectives, i.e., we have: $|c(\varphi_1, \dots, \varphi_n)|_T = c_{\mathbf{B}}(|\varphi_1|_T, \dots |\varphi_n|_T)$,

(4) and for $e(\varphi) = |\varphi|_T$ *we get* $e \in \mathbf{MOD}(T, \mathbf{B})$.

Proof: Part (1) is trivial use of PP.

Part (2): from linearity of T we get $\{r|T \vdash \overline{r} \to \varphi\} \cup \{r|T \vdash \varphi \to \overline{r}\} = [0, 1]$. Obviously, $\{r|T \vdash \overline{r} \to \varphi\} = [0, a)$ and $\{r|T \vdash \varphi \to \overline{r}\} = (b, 1]$ for some $a, b \in [0, 1]$ (the intervals can be closed). Thus all we need to show is that is not the case that $b < a$. In that case we would have $r, s \in \mathbb{IQ}$, $r > s$, $T \vdash \overline{r} \to \varphi$, and $T \vdash \varphi \to \overline{s}$. Thus $T \vdash \overline{r \to s}$. From book-keeping axioms we know that $r \to s < 1$ and so $T \vdash \overline{0}$—a contradiction.

Part (3): We show this on example of implication, the proof for other connectives are analogous. Recall that in our setting (over BL) we have $\hat{\to} = (-, +)$. Let us denote $\to_{\mathbf{B}}$ as $\Rightarrow$ in this proof. Assume that $|\varphi|_T = a$ and $|\psi|_T = b$ and we want to show that $a \Rightarrow b = |\varphi \to \psi|_T$. We distinguish two cases:

Case 1. $<\to, a, b> \notin \mathcal{NC}(L)$, i.e., $\Rightarrow$ is continuous in point (a,b). We start with definition then by Part (2) and continuity we get the following chain: $a \Rightarrow b = \sup\{s|T \vdash \overline{s} \to \varphi\} \Rightarrow \sup\{r|T \vdash \overline{r} \to \psi\} = \inf\{s|T \vdash \varphi \to \overline{s}\} \Rightarrow \sup\{r|T \vdash \overline{r} \to \psi\} = \sup\{s \Rightarrow r \mid T \vdash \varphi \to \overline{s} \text{ and } T \vdash \overline{r} \to \psi\} \leq \sup\{s \Rightarrow r \mid T \vdash (\overline{s} \to \overline{r}) \to (\varphi \to \psi)\} = |\varphi \to \psi|_T$. We prove the second direction indirectly, assume that $a \Rightarrow b < |\varphi \to \psi|_T$. From continuity there has to be rationals r, s and t such that $T \not\vdash \varphi \to \overline{s}$, $T \not\vdash \overline{r} \to \psi$, and $a \to b \leq r \Rightarrow s < t < |\varphi \to \psi|_T$. Using linearity of T we can rewrite that as: $T \vdash \overline{s} \to \varphi$, $T \vdash \psi \to \overline{r}$ and using the Lemma 3.4.12 we get $T \vdash (\varphi \to \psi) \to (\overline{r} \to \overline{s})$. Since we know that $T \vdash \overline{t} \to (\varphi \to \psi)$ (from $t < |\varphi \to \psi|_T$ we get $T \vdash \overline{t} \to (\overline{r} \to \overline{s})$ i.e., $T \vdash \overline{t \Rightarrow (r \Rightarrow s)}$ and because $t \Rightarrow (r \Rightarrow s) < 1$ we have a contradiction with consistency of T.

Case 2. $<\to, a, b> \in \mathcal{NC}(L)$. We know that $a = \sup\{s|T \vdash \overline{s} \to \varphi\} = \inf\{s|T \vdash \varphi \to \overline{s}\}$ and $b = \sup\{r|T \vdash \overline{r} \to \psi\}$. We use this knowledge twice:

First, we get that $T \vdash \varphi \to \overline{s}$ for each $s > a$ and $T \vdash \overline{r} \to \psi$ for each $b > r$ and by the use of the rules from $\mathcal{C}on^{\mathbf{B}}(<\to, a, b>)$ we get that $T \vdash \overline{r} \to (\varphi \to \psi)$ for each $r \leq a \Rightarrow b$, i.e., $a \Rightarrow b \leq |\varphi \to \psi|_T$.

Second usage is analogous: we get that $T \vdash \overline{s} \to \varphi$ for each $a > s$ and $T \vdash \psi \to \overline{r}$ for each $r > b$ and by the use of second group of rules in $\mathcal{C}on^{\mathbf{B}}(<\to, a, b>)$ we get $T \vdash (\varphi \to \psi) \to \overline{r}$ for each $a \Rightarrow b \leq r$, i.e., $|\varphi \to \psi|_T \leq a \Rightarrow b$.

Part (4) is trivial. QED

Observe that we use the fact that $\mathbf{L}$ is a fuzzy logic in Part (1) only. Now we use the previous lemma to show the so-called *Pavelka-style* completeness.

Theorem 3.4.21 (Pavelka-style completeness) *Let* **L** *be a Pavelka-style fuzzy logic w.r.t the algebra* **B**, *T a theory, and φ a formula. Then: $|\varphi|_T = ||\varphi||_T$.*

Proof: One direction is easy: whenever $T \vdash \overline{r} \to \varphi$ then $r \leq e(\varphi)$ for each model e of T. Thus $|\varphi|_T \leq ||\varphi||_T$. To prove the second direction assume that $|\varphi|_T < r < ||\varphi||_T$. Thus $T \not\vdash \overline{r} \to \varphi$. By part (1) of the previous lemma $T \cup \{\varphi \to \overline{r}\}$ is a consistent theory. Let us extend $T \cup \{\varphi \to \overline{r}\}$ into the linear theory T' (using LEP). We know that for $e(\psi) = |\psi|_{T'}$ we have $e \in \mathbf{MOD}(T', \mathbf{B})$. Thus also $e \in \mathbf{MOD}(T, \mathbf{B})$ and $e(\varphi \to \overline{r}) = 1$. Finally, $e(\varphi) \leq r$ and so $||\varphi||_T \leq r$—a contradiction. QED

Corollary 3.4.22 *Let* **L** *be a Pavelka-style fuzzy logic with $\triangle$ in the language. Then* **L** *has the standard completeness. Furthermore, if* **L** *is a Pavelka-style extension of some core fuzzy logic* **K**, *then* **L** *is a core fuzzy logic and it extends* **K** *conservatively.*

Proof: Assume that $T \models_{\mathbf{L}} \varphi$. Then $||\varphi||_T = 1$ and so $|\varphi| = 1$, which means that $T \vdash \overline{r} \to \varphi$ for each $r < 1$. Using $\mathrm{IR}_\triangle$ we get $T \vdash \varphi$. QED

Theorem 3.4.23 *Let* **L** *be an argument-monotonous rational core fuzzy logic stronger than Łukasiewicz logic, such that $\mathcal{RP}$***L** *is a fuzzy logic. Then $\mathcal{RP}$***L** *is a core fuzzy logic and it extends* **L** *conservatively.*

Proof: All we need to show is the finite standard completeness of $\mathcal{RP}\mathbf{L}$. Assume that $T \not\vdash_{\mathcal{RP}\mathbf{L}} \varphi$. Let K be the set of all rationals occurring in formulae of T and in φ. For each rational $r \in K$ let us take propositional variable v_r and define $\varphi' = \varphi[\overline{r} := v_r]$ (i.e., replace all occurrences of $\overline{r}$ by v_r). Then we define T' analogously. Let $T_0 = \{\overline{r} \equiv v_r \mid r \in K\}$. Observe that obviously $T_0 \cup T' \not\vdash_{\mathcal{RP}\mathbf{L}} \varphi'$. Let k be the least common multiple of denominators of rationals in K. Let us pick a propositional variable v not used in T_0, T', φ'. and define a theory $T_1 = \{v \equiv \overline{\frac{1}{k}}\} \cup \{v_{\frac{a}{b}} \equiv \frac{ak}{b} v \mid \frac{a}{b} \in K\}$ (by $\frac{ak}{b} v$ we mean $v \oplus v \ldots v$, where the argument v appear $\frac{ak}{b}$-times). Again, observe that $T_1 \cup T' \not\vdash_{\mathcal{RP}\mathbf{L}} \varphi'$. Finally, we define theory T_2 as T_1 with formula $v \equiv \overline{\frac{1}{k}}$ replaced by $\{kv, \neg((k-1)v)^k\}$. Again, $T_2 \cup T' \not\vdash_{\mathcal{RP}\mathbf{L}} \varphi'$ (by the same counterexample, we only set $e(v) = \frac{1}{k}$, this is obviously a model of $T_2 \cup T'$ and $e(\varphi) < 1$).

Observe that there is no constant in T_2, T', and φ' and so by finite standard completeness of **L** we get that there is $[0,1]_{\mathbf{L}}$-model e of $T_2 \cup T'$, such that $e(\varphi') < 1$. To complete the proof just notice that $e(v) = \frac{1}{k}$ (otherwise e would not be a model of $\{kv, \neg((k-1)v)^k\}$ and so obviously $e(v_r) = r$ for each $r \in K$. Thus also e is a model of T and $e(\varphi) < 1$. QED

Corollary 3.4.24 *Let T be a theory over* RPL *and φ a formula. Then: $|\varphi|_T = ||\varphi||_T$. Furthermore,* RPL *is a core fuzzy logic.*

To get the Pavelka-style completeness for other logics mentioned in Remark 3.4.18 we first need to show that they are fuzzy logics, i.e., that by adding the infinitary rule(s) they remain fuzzy. We have to be careful, the main characterization theorem for fuzzy logics (Theorem 2.1.45) works for finitary logics only. Thus our main strategy will be to use Theorem 2.1.42 and show that the logics in question have LEP.

Lemma 3.4.25 *Let* **L** *be a Pavelka-style extension of some fuzzy logic with* $\mathrm{DT}_\triangle$ *and* **L** *has a presentation $\mathcal{AX}$, where there are no infinitary deduction rules except* $\mathrm{IR}_\triangle$, *and* $\mathrm{IR}_\to$. *Then*

1. **L** *has* $\mathrm{DT}_\triangle$,
2. **L** *has* PP,
3. *if $T \not\vdash \overline{r} \to \varphi$, then $T \cup \{\varphi \to \overline{r}\}$ is consistent,*
4. **L** *is fuzzy.*

Proof: Part 1. Let T be a theory and φ, ψ formulae. We have to show that $T, \varphi \vdash \psi$ iff $T \vdash \triangle\varphi \to \psi$. One direction is obvious, we show the second by induction over the proof of ψ in T, φ in $\mathcal{AX}$ (recall that the proof is well founded relation). The initial step is trivial. The induction step has to be done for each deduction rule in $\mathcal{AX}$. Without loss of generality we can assume that the only deduction rules are (MP), (NEC) an possibly $\mathrm{IR}_\triangle$ and $\mathrm{IR}_\to$. For the proof of the induction step of (MP) and (NEC) see the proof of Theorem 2.2.26. The step for $\mathrm{IR}_\triangle$: assume that $T \vdash \triangle\varphi \to (\overline{r} \to \psi)$ for each $r < 1$, thus $T \vdash \overline{r} \to (\triangle\varphi \to \psi)$ and so by $\mathrm{IR}_\triangle$ we get $T \vdash \triangle\varphi \to \psi$. The step for $\mathrm{IR}_\to$ is analogous.

Part 2. Trivial.

Part 3. See the proof of Part (1) of Theorem 3.4.20

Part 4. We show that $\mathbf{L}$ has LEP. First, assume that $\mathrm{IR}_\triangle$ is the only infinitary deduction rule. Let us enumerate all formulae, stating $\varphi_0 = \varphi$. We define sequence of theories T_i

- if $T_i \vdash \varphi$ then $T_{i+1} = T, \varphi$,
- if $T_i \not\vdash \varphi$. Then there is $r < 1$ such that $T_i \not\vdash \overline{r} \to \varphi_i$. We define $T_{i+1} = T, \varphi_i \to \overline{r}$. Using Part 3. we get that T_{i+1} is consistent.

Observe that $T_1 = T, \overline{r} \to \varphi$. Let us define $\bar{T} = \bigcup T_i$. Now we show that each proof in $\bar{T}$ can be replaced by a finite proof. Let χ be a formula. We replace each use of infinitary deduction rule by a singe formula. We start by such members of the proof of χ in T' obtained by using of an infinitary rule from formulae proved without using any infinitary rule. Then we use well-foundedness of the proof to obtain the result. Let us denote the formula in question as ψ and suppose that $\psi = \varphi_i$. If $T_i \vdash \psi$ then $\psi \in \bar{T}$ thus we can replace the proof of ψ just by the formula itself. Now we show that the case $T_i \not\vdash \psi$ can not happen. Recall that $T_{i+1} = T, \psi \to \overline{r}$. Since $\bar{T} \vdash \overline{r} \to \psi$ for each $r < 1$ without using infinitary rules, there has to be $t > r$ and j such that $T_j \vdash \bar{t} \to \psi$. Thus $T_{\max(i+1,j)} \vdash \bar{t} \to \overline{r}$. Since $t \to_{\mathbf{B}} r < 1$ we have a contradiction.

Now construct theory T' in the same way as in Lemma 2.1.44 (building sequence of consistent theories $\bar{T}_i$ by adding either $\varphi_i \to \psi_i$ or $\psi_i \to \varphi_i$). This theory is obviously linear. We show its consistency in the same way as we did it for $\bar{T}$. Thus $T' \not\vdash \varphi$ (since $T' \vdash \varphi \to \overline{r}$ we would get $T \vdash \overline{r}$—a contradiction with consistency of T'.)

To prove this lemma even if we have $\mathrm{IR}_\to$ in the presentation $\mathcal{AX}$ just augment the definition of T_i in the following way: if $\varphi_i = \neg\psi$ and $T_i \not\vdash \varphi_i$ then there is not only $r < 1$ such that $T \not\vdash \overline{r} \to \varphi_i$ but also $s > 0$ such that $T \not\vdash \varphi_i \to \overline{s}$. We can safely assume that $s < r$ and define $T_{i+1} = T \cup \{\varphi_i \to \overline{r}, \overline{s} \to \varphi_i\}$. The rest of the proof is analogous. QED

Corollary 3.4.26 *Let $\mathcal{C}$ be one of the $\mathcal{RP}\mathbf{Ł}_\triangle$, $\mathcal{RP}\Pi_\triangle$, $\mathcal{RP}\Pi_\sim([0,1]_{\Pi_\sim})$, or $\mathrm{RŁ\Pi}$, and let T be a theory over and φ a formula. We have:*

1. *$|\varphi|_T = ||\varphi||_T$,*
2. *$\mathcal{C}$ is standard complete,*
3. *$\mathcal{C}$ is core fuzzy logic,*
4. *$\mathcal{RP}\Pi_\sim([0,1]_{\Pi_\sim}) = \mathrm{RŁ\Pi}$.*

Notice that although the logics $\Pi_\sim$ and ŁΠ are different, adding truth constants and infinitary deduction rule makes them coincide.

3.5 First-order logics

In this section we apply results from Section 2.3. Again, we advise the reader to go through the definitions and theorems in that section.

3.5.1 Basic facts

Recall that Definition 2.3.11 gives us two predicate logic for each fuzzy logic. We start by recalling some known theorems of our logics from [44], where they are proved in BL∀. However, it is easy to observe that in fact they are provable already in $\mathrm{BL}\forall^-$.

Theorem 3.5.1 *Assume that ν does not contain x freely. The following are theorems of* $\mathrm{BL}\forall^-$ *:*

$(T\forall 1)$	$(\forall x)(\nu \to \varphi) \equiv (\nu \to (\forall x)\varphi)$
$(T\forall 2)$	$(\forall x)(\varphi \to \nu) \equiv ((\exists x)\varphi \to \nu)$
$(T\forall 3)$	$(\exists x)(\nu \to \varphi) \to (\nu \to (\exists x)\varphi)$
$(T\forall 4)$	$(\exists x)(\varphi \to \nu) \to ((\forall x)\varphi \to \nu)$
$(T\forall 5)$	$(\forall x)(\varphi \to \psi) \equiv ((\forall x)\varphi \to (\forall x)\varphi)$
$(T\forall 6)$	$(\forall x)(\varphi \to \psi) \equiv ((\exists x)\varphi \to (\exists x)\psi)$
$(T\forall 7)$	$(\forall x)\varphi \& (\exists x)\psi \to (\exists x)(\varphi \& \psi)$
$(T\forall 8)$	$(\forall x)\varphi(x) \equiv (\forall y)\varphi(y)$
$(T\forall 8')$	$(\exists x)\varphi(x) \equiv (\exists y)\varphi(y)$
$(T\forall 9)$	$(\exists x)(\varphi \& \nu) \equiv ((\exists x)\varphi \& \nu)$
$(T\forall 10)$	$(\exists x)\varphi^n \equiv ((\exists x)\varphi)^n$ *for each* $n \geq 1$
$(T\forall 11)$	$(\exists x)\varphi \to \neg(\forall y)\neg\varphi$
$(T\forall 12)$	$\neg(\exists x)\varphi \equiv (\forall y)\neg\varphi$
$(T\forall 13)$	$(\exists x)(\nu \wedge \varphi) \equiv (\nu \wedge (\exists x)\varphi)$
$(T\forall 14)$	$(\exists x)(\nu \vee \varphi) \equiv (\nu \vee (\exists x)\varphi)$
$(T\forall 15)$	$(\forall x)(\nu \wedge \varphi) \equiv (\nu \wedge (\forall x)\varphi)$
$(T\forall 16)$	$(\exists x)(\varphi \vee \psi) \equiv ((\exists x)\varphi \vee (\exists x)\varphi)$
$(T\forall 17)$	$(\forall x)(\varphi \wedge \psi) \equiv ((\forall x)\varphi \wedge (\forall x)\psi)$

Let **L** *be logic extending* **Ł**. *The following are theorems of* **L**∀ *:*

$(T\text{Ł}\forall 1)$	$(\exists x)\varphi \equiv \neg(\forall y)\neg\varphi$
$(T\text{Ł}\forall 2)$	$(\exists x)(\nu \to \varphi) \equiv (\nu \to (\exists x)\varphi)$
$(T\text{Ł}\forall 3)$	$(\exists x)(\varphi \to \nu) \equiv ((\forall x)\varphi \to \nu)$
$(T\text{Ł}\forall 4)$	$(\forall x)(\varphi \vee \nu) \equiv ((\forall x)\varphi \vee \nu)$
$(T\text{Ł}\forall 5)$	$(\forall x)(\varphi \& \nu) \equiv ((\forall x)\varphi \& \nu)$
$(T\text{Ł}\forall 6)$	$(\exists x)n\varphi \equiv n((\exists x)\varphi)$ *for each* $n \geq 1$

Observe that in **Ł**∀ are quantifiers mutually definable. In [44] it is proved that in case of Łukasiewicz logic both predicate logic **Ł**∀ and $\mathbf{Ł}\forall^-$. This can obviously be extended into the following theorem :

Theorem 3.5.2 *Let* **L** *be a fuzzy logic extending* **Ł**. *Then* $\mathbf{L}\forall^- = \mathbf{L}\forall$.

Observe that for all logics **L** defined in this chapter, we know that **L**∀ has PP (see Corollary 2.3.27 and 2.3.28). Thus we have the completeness w.r.t. their linear algebras.

Theorem 3.5.3 *Let* **L** *be one of the following logics:* BL, SBL, $\mathrm{SBL}_\triangle$, $\mathrm{SBL}_\sim$ Ł, $\text{Ł}_\triangle$, Π, $\Pi_\triangle$, $\Pi_\sim$, G, $\mathrm{G}_\triangle$, $\mathrm{G}_\sim$, ŁΠ, *and* $\text{Ł}\Pi\frac{1}{2}$. *Then the logic* **L**∀ *is sound and complete w.r.t. corresponding class of linear matrices.*

However, the notion of the standard completeness seems to be too strong in predicate logic. We present that the sets of standard tautologies for the most of our logics are rather complex (from the point of view of the arithmetical hierarchy).

Definition 3.5.4 *Let* **L** *be a fuzzy logic. We denote the set of* $s-\mathbf{MAT}(\mathbf{L})$*-tautologies by* $\mathcal{TAUT}^{\forall}(\mathbf{L})$.

We say that **L**∀ *has the (finite) standard completeness iff* $T \vdash_{\mathbf{L}\forall} \varphi$ *iff* $T \models_{s-\mathbf{MAT}(\mathbf{L})} \varphi$ *for each (finite) theory* T *and formula* φ.

Theorem 3.5.5 *We know:*

- *let* $\mathcal{C}$ *be one of the following logics:* BL, $\mathrm{BL}_\triangle$, SBL, $\mathrm{SBL}_\triangle$, $\mathrm{SBL}_\sim$, Π, $\Pi_\triangle$, $\Pi_\sim$, $\mathbf{Ł}_\triangle$, ŁΠ, *or* $Ł\Pi\frac{1}{2}$. *Then the set* $\mathcal{TAUT}^\forall(\mathcal{C})$ *is not arithmetical.*
- *The set* $\mathcal{TAUT}^\forall(\mathbf{Ł})$ *is* Π_2*-complete.*
- *The sets* $\mathcal{TAUT}^\forall(\mathrm{G})$, $\mathcal{TAUT}^\forall(\mathrm{G}_\triangle)$, *and* $\mathcal{TAUT}^\forall(\mathrm{G}_\sim)$ *are* Σ_1*-complete.*

For proof of this theorem see [44] and [45].

Corollary 3.5.6 *Let* $\mathcal{C}$ *be one of the following logics:* BL, $\mathrm{BL}_\triangle$, SBL, $\mathrm{SBL}_\triangle$, $\mathrm{SBL}_\sim$, Π, $\Pi_\triangle$, $\Pi_\sim$, $\mathbf{Ł}_\triangle$, $\mathbf{Ł}$, ŁΠ, *or* $Ł\Pi\frac{1}{2}$. *Then* $\mathcal{THM}(\mathcal{C}\forall) \neq \mathcal{TAUT}^\forall(\mathcal{C})$, *i.e., the logic* $\mathcal{C}\forall$ *has not the finite standard completeness.*

Theorem 3.5.7 *The logics* $\mathrm{G}\forall$, $\mathrm{G}_\triangle\forall$, *and* $\mathrm{G}_\sim\forall$ *have the standard completeness.*

This theorem for $\mathrm{G}\forall$ is proven in [44] (and in some older works), for $\mathrm{G}_\triangle\forall$ it is a part of folklore, and for $\mathrm{G}_\sim\forall$ see [35]. There is no known example of a predicate fuzzy logic with finite standard completeness, which is not standard complete. It turns out that all fuzzy propositional logic lacking standard completeness (having only the finite one) lose the finite standard completeness by a transition to the first order. This can be just a coincidence, but is definitively worth some further study.

Theorem 3.5.8 *Let* $\mathbf{L}$ *be a fuzzy logic with* $\mathrm{DT}_\rightarrow$. *Then* $\mathbf{L}\forall$ *is* φ*-witnessed for each* φ.

The proof of this statement is a simple use of $\mathrm{DT}_\rightarrow$ and theorem ($T\forall 10$).

Theorem 3.5.9 *Let* $\mathcal{C}$ *be a logic with* $\triangle$ *in the language. If* $\mathcal{C}\forall$ *is complete w.r.t. corresponding linear algebras then it is undecidable.*

Proof: The formula φ' is created from the formula φ by replacing each atomic formula $P(t_1, \dots, t_n)$ with the formula $\triangle P(t_1, \dots, t_n)$. Notice that formula ψ of a classical predicate language is provable iff formula ψ' is provable in $\mathcal{C}\forall$ (this is obvious from the definition of $\triangle$ and completeness of the $\mathcal{C}\forall$). Thus if the $\mathcal{C}\forall$ logic would be decidable, it would be a contradiction to the undecidability of the classical predicate logic. QED

At the end of this subsection we recall some theorems of ŁΠ∀.

Lemma 3.5.10 *Assume that* ν *does not contain* x *freely. The following are theorems of* ŁΠ∀*:*

$$(\forall x)\,(\varphi \rightarrow_* \psi) \;\rightarrow\; [(\forall x)\,\varphi \rightarrow_* (\forall x)\,\psi] \tag{3.1}$$
$$(\forall x)\,(\varphi \rightarrow_* \psi) \;\rightarrow\; [(\exists x)\,\varphi \rightarrow_* (\exists x)\,\psi] \tag{3.2}$$
$$(\forall x)\,(\varphi \wedge \psi) \;\rightarrow\; [(\forall x)\,\varphi \wedge (\forall x)\,\psi] \tag{3.3}$$
$$(\exists x)\,(\varphi \vee \psi) \;\rightarrow\; [(\exists x)\,\varphi \vee (\exists x)\,\psi] \tag{3.4}$$
$$(\forall x)\,(\varphi_1 \&_* \dots \varphi_k \rightarrow_* \chi) \rightarrow [(\forall x)\,\varphi_1 \&_* \dots \&_* (\forall x)\,\varphi_k \rightarrow_* (\forall x)\,\chi] \tag{3.5}$$
$$(\forall x)\,(\varphi_1 \&_* \dots \varphi_k \rightarrow_* \chi) \rightarrow [(\forall x)\,\varphi_1 \&_* \dots \&_* (\forall x)\,\varphi_{k-1} \&_* (\exists x)\,\varphi_k \rightarrow_* (\exists x)\,\chi] \tag{3.6}$$

Proof: In the proof we use an easy generalization of Corollary 3.3.26 to the predicate case. (3.1)–(3.4) are provable in BL∀ (see [44]). (3.5) is proved by a trivial inductive generalization of the following proof in BL∀:

$$\begin{aligned}(\forall x)\,(\varphi \& \psi \rightarrow \chi) \;\leftrightarrow\; (\forall x)\,(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow [(\forall x)\,\varphi \rightarrow (\forall x)\,(\psi \rightarrow \chi)]\\ \rightarrow [(\forall x)\,\varphi \rightarrow ((\forall x)\,\psi \rightarrow (\forall x)\,\chi)] \;\leftrightarrow\; [(\forall x)\,\varphi \& (\forall x)\,\psi \rightarrow (\forall x)\,\chi].\end{aligned}$$

(3.6) is proved in the same way, only applying (3.2) instead of (3.1) when distributing $(\forall x)$ over $(\varphi_{k-1} \rightarrow \varphi_k)$. QED

3.5.2 Crisp equality

Now we enhance our logic with (crisp) equality. Let us assume that symbol $=$ is a binary predicate symbol and assume that $= \in \mathbf{\Gamma}$ in the rest of this section. We alter the definition of atomic formula with the line:

- If t_1 and t_2 are terms of arbitrary sorts, then $t_1 = t_2$ is also an atomic $\mathbf{\Gamma}$-formula.

Then we alter the definition of a value of a formula the following line:

- $||t_1 = t_2||^{\mathbf{L}}_{\mathbf{M},v} = \overline{1}_{\mathbf{B}}$ iff $||t_1||^{\mathbf{L}}_{\mathbf{M},v} = ||t_2||^{\mathbf{L}}_{\mathbf{M},v}$; and $\overline{0}_{\mathbf{B}}$ otherwise.

And finally we define logic $\mathbf{L}\forall_=$ as a logic $\mathbf{L}\forall$ with three additional axioms:

(=1) $\quad x = x,$
(=2) $\quad x = y \rightarrow (\varphi(x, z_1, \dots, z_n) \equiv \varphi(y, z_1, \dots, z_n)),$
(=3) $\quad (x = y) \vee \neg(x = y).$

Observe that in logic with $\triangle$ we can remove the last axiom if we replace the second one with:

(=2′) $\quad x = y \rightarrow \triangle(\varphi(x, z_1, \dots, z_n) \equiv \varphi(y, z_1, \dots, z_n)),$

Lemma 3.5.11 *The following are theorems of* $\mathbf{L}\forall_=$*:*

- $x = y \rightarrow y = x$
- $x = y \& y = z \rightarrow x = z$
- $x_1 = y_1 \& \dots \& x_n = y_n \rightarrow (\varphi(x_1, \dots, x_n, z) \equiv \varphi(y_1, \dots, y_n, z)).$

Obviously, we have completeness theorem w.r.t. all linear algebras and all models.

Theorem 3.5.12 *Let* $\mathbf{L}$ *be a fuzzy logic and* $\mathbf{L}\forall$ *be Henkin. Then for each theory* T *and formula* φ *we have:* $T \vdash_{\mathbf{L}\forall_=} \varphi$ *iff* $T \models^l_{\mathbf{L}} \varphi$*.*

Corollary 3.5.13 *Let* $\mathbf{L}$ *be a finitary fuzzy logic and* $\mathbf{L}\forall$ *with* PP*. Then for each theory* T *and formula* φ *we have:* $T \vdash_{\mathbf{L}\forall_=} \varphi$ *iff* $T \models^l_{\mathbf{L}} \varphi$*.*

We can easily extend Theorem 2.3.32 into the following theorem (we use the usual denotation $(\exists! y)\, \varphi(y)$ for the formula $(\exists y)\, (\varphi(y) \& (\forall x)\, (\varphi(x) \rightarrow x = y)))$.

Theorem 3.5.14 *Let* $\mathbf{L}\forall$ *be a proto-*φ*-witnessed finitary fuzzy logic with* PP *in predicate language* $\mathbf{\Gamma}$*,* T *a theory,* $\varphi(x_1, \dots, x_n, y)$ *a formula, and* $T \vdash (\forall x_1) \dots (\forall x_n)\, (\exists y)\, \varphi(x_1, \dots x_n, y)$*. Then the theory* T' *in the language of* T *extended by a new function symbol* f_φ *resulting from the theory* T *by adding the axiom* $\vdash (\forall x_1) \dots (\forall x_n)\, \varphi(x_1, \dots x_n, f_\varphi(x_1, \dots, x_n))$ *is a conservative extension of* T*.*

Furthermore, if $T \vdash (\forall x_1^{s_1}) \dots (\forall x_n)\, (\exists! y)\, \varphi((x_1), \dots, x_1, y)$ *then for each* $\mathbf{\Gamma} \cup \{f_\varphi\}$*-formula* ψ *there is a* $\mathbf{\Gamma}$*-formula* ψ' *such that* $T' \vdash \psi \equiv \psi'$*.*

3.5.3 Adding truth constants

We conclude this section by extending properties of predicate versions of Pavelka-style fuzzy logics. Let us understand the class $\mathbf{MOD}(T, \mathbf{B})$ as the class of all $\mathbf{B}$ models of T. We present the definition of provability and truth degree and observe that they are quite the same as in the propositional case.

Definition 3.5.15 *Let* $\mathbf{L}$ *be a Pavelka-style fuzzy logic w.r.t. algebra* $\mathbf{B}$*,* T *a theory over* $\mathbf{L}\forall$*, and* φ *be a sentence. The we define:*

- *the* truth degree *of* φ *over* T *is* $||\varphi||_T = \inf\{e(\varphi) \mid e \in \mathbf{MOD}(T, \mathbf{B})\}$,
- *the* provability degree *of* φ *over* T *is* $|\varphi|_T = \sup\{r \mid T \vdash \overline{r} \to \varphi\}$.

Here we run into two similar problems. If we have a fuzzy logic we do not know if its Pavelka-style extension remains fuzzy logic and for particular fuzzy logics we do not know whether their first-order variants have PP (i.e., the completeness w.r.t. linearly ordered algebras). Let us write some sufficient conditions for a fuzzy logic to have Pavelka-style first-order extension with Pavelka-style completeness.

Definition 3.5.16 *We say that a logic* $\mathbf{L}\forall$ *has a Pavelka-style completeness iff for each theory* T *and formula* φ*:* $||\varphi||_T = |\varphi|_T$.

We continue by proving the analogy of Lemma 3.4.20. Recall that if the logic $\mathbf{L}\forall$ is Henkin then $\mathbf{L}\forall$ has PP (see Corollary 2.3.26).

Lemma 3.5.17 *Let* $\mathbf{L}$ *be a Pavelka-style fuzzy logic w.r.t. the algebra* $\mathbf{B}$*; let* T *be a theory, and* φ, ψ *formulae. Then:*

(1) If $\mathbf{L}\forall$ *has* PP *then we have:* $T \not\vdash \overline{r} \to \varphi$ *then* $T \cup \{\varphi \to \overline{r}\}$ *is consistent.*

Furthermore, if T *is Henkin we define* $\mathbf{B}$*-structure* $\mathbb{M}$ *with domain of all closed terms in language of* T*, the functions are defined as usual, and for each n-ary predicate we set:* $P_{\mathbb{M}}(x_1, \dots, x_n) = |P(x_1, \dots, x_n)|_T$*. (the* x_i *on the left-side are elements of* $\mathbb{M}$ *and on the right side they are the corresponding closed terms). Then we have:*

(2) $|\varphi|_T = \sup\{r | T \vdash \overline{r} \to \varphi\} = \inf\{r | T \vdash \varphi \to \overline{r}\}$,

(3) the provability degree commutes with all connectives, i.e., we have $|c(\varphi_1, \dots, \varphi_n)|_T = c_{\mathbf{B}}(|\varphi_1|_T, \dots |\varphi_n|_T)$,

(4) $|(\forall x)\varphi(x)|_T = \inf\{|\varphi(x)|_T \mid x \in \mathbb{M}\}$,

(5) $\mathbb{M} \in \mathbf{MOD}(T, \mathbf{B})$.

Proof: The proof of the first three parts is a direct analogy of the proof of Lemma 3.4.20. The proof of the part (4) is an analogy of the proof of Lemma 2.3.24: let $T \vdash \varphi \to \psi$ then obviously $T \vdash \overline{r} \to \psi$ whenever $T \vdash \overline{r} \to \varphi$ and so $|\varphi|_T \leq |\psi|_T$. Since $T \vdash (\forall x)\varphi(x) \to \varphi(a)$ we get $|(\forall x)\varphi(x)|_T \leq |\varphi(a)|_T$ for each a. Thus $|(\forall x)\varphi(x)|_T \leq \inf\{|\varphi(x)|_T \mid x \in \mathbb{M}\}$

Now assume that $|(\forall x)\varphi(x)|_T < \inf\{|\varphi(x)|_T \mid x \in \mathbb{M}\}$. Thus there is r such that $|(\forall x)\varphi(x)|_T < r < \inf\{|\varphi(x)|_T \mid x \in \mathbb{M}\}$. So we know that $T \not\vdash \overline{r} \to (\forall x)\varphi(x)$ and so $T \not\vdash (\forall x)(\overline{r} \to \varphi(x))$ (using axiom ($\forall 2$)). Since T is Henkin, there is c such that $T \not\vdash (\overline{r} \to \varphi(c))$ and so $|\varphi(c)|_T \leq \overline{r}$—a contradiction.

The last part is trivial. QED

This gives us the proof of Pavelka-style completeness.

Theorem 3.5.18 *Let* $\mathbf{L}$ *be a Pavelka-style fuzzy logic w.r.t. the algebra* $\mathbf{B}$ *and* $\mathbf{L}\forall$ *is Henkin. Then* $\mathbf{L}\forall$ *has Pavelka-style completeness.*

Theorem 3.5.19 *Let* $\mathbf{L}$ *be a fuzzy logic and* $\mathbf{B}$ *a standard algebra. The following are sufficient conditions for the logic* $\mathcal{RPL}(\mathbf{B})\forall$ *to have Pavelka-style completeness.*

- $\mathcal{RP}\mathbf{L}(\mathbf{B})$ *has* $\mathrm{DT}_{\to}$.
- $\mathbf{L}$ *has* $\mathrm{DT}_{\triangle}$ *and* $\mathbf{L}$ *has a presentation* $\mathcal{AX}$*, where there are no infinitary deduction rules except* $\mathrm{IR}_{\triangle}$*, and* $\mathrm{IR}_{\to}$.

Proof: The first part is easy: since $\mathcal{RPL}(\mathbf{B})$ has $\mathrm{DT}_{\rightarrow}$, it is a finitary fuzzy logic (it is an extension of a fuzzy logic with $\mathrm{DT}_{\rightarrow}$) and also $\mathcal{RP}\mathbf{L}(\mathbf{B})\forall$ has PP. The rest is a consequence of a Corollary 2.3.22.

The second part is more complicated. From Lemma 3.4.25 we know that $\mathcal{RP}\mathbf{L}(\mathbf{B})$ is a fuzzy logic with $\mathrm{DT}_{\triangle}$. Thus $\mathcal{RP}\mathbf{L}(\mathbf{B})\forall$ has $\mathrm{DT}_{\triangle}$ and PP (Theorem 2.3.13 and Corollary 2.3.15).

To show that $\mathcal{RP}\mathbf{L}(\mathbf{B})\forall$ is Henkin we have to go through the proofs of Lemmata 2.3.21 and 3.4.25. Observe that in the proof of Lemma 3.4.25 we have define a for each theory a supertheory in which each proof can be replaced by a finite proof. We do the same trick here and then we continue with the proof of Lemma 2.3.21, which now can be carried in the same way. QED

Corollary 3.5.20 *The logics* RPL$\forall$*,* $\mathcal{RP}$Ł$_{\triangle}\forall$*,* $\mathcal{RP}\Pi_{\triangle}\forall$*, and* RŁΠ$\forall$ *have Pavelka-style completeness.*

It turns out, that although predicate fuzzy logic with the connective $\triangle$ do not have standard completeness, their Pavelka-style extensions fulfill an analogy of Corollary 3.4.22, i.e., they have standard completeness.

Lemma 3.5.21 *Let* $\mathbf{L}$ *be a fuzzy logic with* $\triangle$ *in the language and* $\mathcal{RP}\mathbf{L}(\mathbf{B})\forall$ *has Pavelka-style completeness. Then for each theory* T *and formula* φ *we have* $T \vdash_{\mathcal{RP}\mathbf{L}(\mathbf{B})\forall} \varphi$ *iff* $T \models_{\mathbf{B}} \varphi$*.*

Corollary 3.5.22 *Let* $\mathbf{L}$ *be a core fuzzy logic with* $\triangle$ *in the language and the logic* $\mathcal{RP}\mathbf{L}\forall$ *has Pavelka-style completeness. Then* $\mathcal{RP}\mathbf{L}\forall$ *has standard completeness.*

Also recall that the logic RŁΠ$\forall$ coincides with logic TT from [44].

Chapter 4

Product Łukasiewicz logic

As we have seen, the Łukasiewicz logic [62, 44] is one of the most important logics in the broad family of many-valued logics. Its corresponding algebraic structures of truth values (MV-algebras) are well-known and deeply studied. Mundici's famous result [12] established an important correspondence between MV-algebras and Abelian l-groups with strong unit. There is an obvious question if there is a logic, whose corresponding algebras of truth values are in the analogous correspondence with l-rings.

There are several papers dealing with the so-called product MV-algebras. Montagna's papers [66, 67, 69] are fundamental to our aims. There is also a paper by Di Nola and Dvurečenskij [73]. A product MV-algebra (PMV-algebra for short) is an MV-algebra enriched by a product operation in such a way that the resulting structures correspond to the f-rings with strong unit. In [66], Montagna proved the subdirect representation theorem for PMV-algebras and established a correspondence between linearly ordered f-rings with strong unit and linearly ordered PMV-algebras. Later in [67], he introduced PMV$_\triangle$-algebras (PMV-algebras enriched by the 0-1 projector $\triangle$) and proved the categorical equivalence between PMV$_\triangle$-algebras and certain extension of f-rings (the so-called δ-f-rings). Finally in [69], it was shown by Montagna and Panti that the variety of PMV$_\triangle$-algebras is generated by the standard PMV$_\triangle$-algebra (over the real unit interval).

In the forthcoming paper [68], Montagna introduced a quasi-variety $\mathbf{PMV}^+$ containing only the PMV-algebras without non-trivial zero-divisors and showed that $\mathbf{PMV}^+$ is generated by the standard PMV-algebra (over the real unit interval).

However, so far there is no logic corresponding to the all above-mentioned algebras. The main aim of this chapter is to define and develop such a logic. Our logic, which corresponds to PMV-algebras, is called PŁ logic. Further, we introduce PŁ$'$ logic corresponding to the algebras from $\mathbf{PMV}^+$. We also study extensions of PŁ and PŁ$'$ logics by Baaz's $\triangle$ (PŁ$_\triangle$ and PŁ$'_\triangle$ logics). The algebras of truth values of PŁ$'_\triangle$ logic correspond to PMV$_\triangle$-algebras. This, together with the fact that there are also several other different algebraic structures called PMV-algebras, is the reason why we call PŁ-algebras the algebras of truth values corresponding to PŁ logic. Analogously, we introduce PŁ$'$-algebras, PŁ$_\triangle$-algebras, and PŁ$'_\triangle$-algebras.

We use the above-mentioned algebraic results to obtain completeness of all of these logics and standard completeness for PŁ$'$ and PŁ$'_\triangle$ logic. Further, we show an example of the PŁ-algebra which demonstrates that PŁ logic is not standard complete. Then we show a relation of our logics to the ŁΠ logic. Roughly speaking, the logic ŁΠ is the extension of PŁ$'$ by the product residuum.

Furthermore, we extend these logics by rational constants in the style of Section 3.4. Among others we obtain $\mathcal{RP}$PŁ$'$ and $\mathcal{RP}$PŁ$'_\triangle$ logics. Then we investigate the predicate versions of all logics mentioned above. Later we deal with the arithmetical complexity of the set of tautologies of these logics which entails that they do not have the standard completeness. Finally, we show a relation of these logics to the predicate versions of ŁΠ and RŁΠ logics and the well-known logic of Takeuti and Titani [80].

$$
\begin{tikzcd}
 & & \textbf{Ł} \arrow[dll] \arrow[d] \arrow[drr] & & \\
\textbf{Ł}_{\triangle} \arrow[dr] & & \text{PŁ} \arrow[dl] \arrow[dr] \arrow[dd] & & \text{RPL} \arrow[dl] \\
 & \text{PŁ}_{\triangle} \arrow[dd] & & \mathcal{RPP}\text{Ł} \arrow[dd] & \\
 & & \text{PŁ}' \arrow[dl] \arrow[dr] & & \\
 & \text{PŁ}'_{\triangle} \arrow[d] \arrow[drr] & & \mathcal{RPP}\text{Ł}' \arrow[d] & \\
 & \text{Ł}\Pi \arrow[dr] & & \mathcal{RPP}\text{Ł}_{\triangle} = \mathcal{RPP}\text{Ł}'_{\triangle} \arrow[dl] & \\
 & & \text{RŁ}\Pi & &
\end{tikzcd}
$$

Figure 4.1: Relations between logics of this chapter.

All logics considered in this chapter lie between Łukasiewicz logic and RŁΠ logic [23, 36]. Their mutual relations are depicted in Figure 4.1.

4.1 PŁ and PŁ′ logics

4.1.1 Syntax

In this section, we introduce the PŁ logic (PŁ for short), an extension of Łukasiewicz logic by a new binary connective $\odot$. This connective plays the role of multiplication. Thus the basic connectives are $\otimes, \rightarrow, \odot, \bar{0}$. Additional connectives $\oplus, \ominus, \neg, \wedge, \vee, \equiv, \bar{1}$ are defined as in Łukasiewicz logic. We also introduce the PŁ′ logic (PŁ′ for short), an extension of PŁ by one additional deduction rule. The reason for this extension is that the PŁ logic does not possess the standard completeness property.

Definition 4.1.1 *The axioms of* PŁ *logic are the axioms of Łukasiewicz logic and the following axioms:*

(P1) $(\chi \odot \varphi) \ominus (\chi \odot \psi) \equiv \chi \odot (\varphi \ominus \psi)\,,$
(P2) $\varphi \odot (\psi \odot \chi) \equiv (\varphi \odot \psi) \odot \chi\,,$
(P3) $\varphi \rightarrow \varphi \odot \bar{1}\,,$
(P4) $\varphi \odot \psi \rightarrow \varphi\,,$
(P5) $\varphi \odot \psi \rightarrow \psi \odot \varphi\,.$

The only deduction rule is modus ponens. The PŁ′ *logic is obtained from* PŁ *by adding a new deduction rule (ZD): from* $\neg(\varphi \odot \varphi)$ *infer* $\neg\varphi$.

It is obvious that all theorems of Łukasiewicz logic are also theorems of PŁ and all theorems of PŁ are also theorems of PŁ′. Further, we show several useful theorems of PŁ logic. The most important one is theorem (TP4) stating that the connective $\equiv$ is a congruence w.r.t. the product $\odot$, which entails that PŁ and PŁ′ are weakly implicative logics.

Lemma 4.1.2 *The following are theorems of* PŁ *logic:*

(TP1) $(\varphi \rightarrow \psi) \rightarrow (\varphi \odot \chi \rightarrow \psi \odot \chi)\,,$
(TP2) $(\varphi \equiv \psi) \rightarrow (\varphi \odot \chi \equiv \psi \odot \chi)\,,$
(TP3) $(\varphi_1 \rightarrow \psi_1) \otimes (\varphi_2 \rightarrow \psi_2) \rightarrow (\varphi_1 \odot \varphi_2 \rightarrow \psi_1 \odot \psi_2)\,,$
(TP4) $(\varphi_1 \equiv \psi_1) \otimes (\varphi_2 \equiv \psi_2) \rightarrow (\varphi_1 \odot \varphi_2 \equiv \psi_1 \odot \psi_2)\,,$
(TP5) $\varphi \otimes \psi \rightarrow \varphi \odot \psi\,,$
(TP6) $(\varphi \wedge \psi) \odot \chi \equiv (\varphi \odot \chi) \wedge (\psi \odot \chi)\,.$

Proof:

(***TP1***): We start with one direction of equivalence (P1) $(\chi \odot \varphi) \ominus (\chi \odot \psi) \to \chi \odot (\varphi \ominus \psi)$. By (P4) and (A1) we get $(\chi \odot \varphi) \ominus (\chi \odot \psi) \to (\varphi \ominus \psi)$. Using (H7) we obtain $\neg(\varphi \ominus \psi) \to \neg((\chi \odot \varphi) \ominus (\chi \odot \psi))$. This is what we want to prove (because $\neg(\varphi \ominus \psi) \equiv (\varphi \to \psi)$).

(***TP2***): We use (TP1) and (TP1) with φ, ψ exchanged. From theorem (H3) we get $(\varphi \to \psi) \otimes (\psi \to \varphi) \to (\varphi \odot \chi \to \psi \odot \chi) \otimes (\psi \odot \chi \to \varphi \odot \chi)$.

(***TP3***): Let us start with (TP1) $(\varphi_1 \to \psi_1) \to (\varphi_1 \odot \varphi_2 \to \psi_1 \odot \varphi_2)$ and (TP1) again $(\varphi_2 \to \psi_2) \to (\psi_1 \odot \varphi_2 \to \psi_1 \odot \psi_2)$. Now using (H3) we get $(\varphi_1 \to \psi_1) \otimes (\varphi_2 \to \psi_2) \to (\varphi_1 \odot \varphi_2 \to \psi_1 \odot \varphi_2) \otimes (\psi_1 \odot \varphi_2 \to \psi_1 \odot \psi_2)$. By axiom (A1) (after applying axiom (A5)) we get $(\varphi_1 \odot \varphi_2 \to \psi_1 \odot \varphi_2) \otimes (\psi_1 \odot \varphi_2 \to \psi_1 \odot \psi_2) \to (\varphi_1 \odot \varphi_2 \to \psi_1 \odot \psi_2)$. Axiom (A1) completes the proof.

(***TP4***): This is an analogy of (TP2), except that we use (TP3) instead of (TP1).

(***TP5***): We start with (TP1) in the form $(\bar{1} \to \varphi) \to (\bar{1} \odot \psi \to \varphi \odot \psi)$. By (H8) and (A5) we obtain $\varphi \otimes (\bar{1} \odot \psi) \to \varphi \odot \psi$. The rest is obvious.

(***TP6***): We start with the first direction: by (H9) and (TP1) we obtain $(\varphi \to \psi) \to (\varphi \odot \chi \to (\varphi \wedge \psi) \odot \chi)$. By (H5) and (A1) we get $(\varphi \to \psi) \to ((\varphi \odot \chi) \wedge (\psi \odot \chi) \to (\varphi \wedge \psi) \odot \chi)$. Analogously we get $(\psi \to \varphi) \to ((\varphi \odot \chi) \wedge (\psi \odot \chi) \to (\varphi \wedge \psi) \odot \chi)$. Axiom (A6) completes the proof of this direction.

Reverse direction: by (T1) and (H9) we get $(\varphi \to \psi) \to (\varphi \odot \chi \to (\varphi \odot \chi) \wedge (\psi \odot \chi))$. By (H5) and (TP1) we get $(\varphi \wedge \psi) \odot \chi \to \varphi \odot \chi$. By (A1) we get $(\varphi \to \psi) \to ((\varphi \wedge \psi) \odot \varphi \to (\varphi \odot \chi) \wedge (\psi \odot \chi))$. The rest of the proof is analogous to the first part. QED

As in immediate corollary of this lemma and Lemma 2.1.16 we get:

Theorem 4.1.3 *The logics* PŁ *and* PŁ′ *are weakly implicative logics.*

Theorem 4.1.4 *The logic* PŁ *is a weakly implicative fuzzy logic with* $\mathrm{DT}_{\to}$.

The proof that PŁ hat $\mathrm{DT}_{\to}$ is a trivial application of Theorem 2.2.8 (in PŁ′ the situation is quite different, after we introduce the semantics we show that PŁ does not posses even LDT). The rest of the theorem is a consequence of Lemma 2.2.10.

4.1.2 Semantics and completeness

Now we define the algebras corresponding to PŁ—PŁ-algebras. They coincide with the so-called PMV-algebras which were introduced in Montagna's paper [66]. Furthermore, PŁ-algebras are also a subclass of more general algebras introduced by Dvurečenskij and Di Nola in paper [73] (they do not require $\mathbf{1}$ to be a neutral element for the product $\odot$ and commutativity of $\odot$). However, we decided to use the name PŁ-algebras, because some authors use the name PMV-algebras for the different structures (e.g. pseudo MV-algebras).

The PŁ′-algebras coincide with PMV$^+$-algebras introduced in the forthcoming Montagna's paper [68]. These are exactly the subreducts of ŁΠ-algebras.

Definition 4.1.5 *A* PŁ*-algebra is a structure* $\mathbf{L} = (L, \oplus, \neg, \odot, \mathbf{0}, \mathbf{1})$*, where the reduct* $\mathbf{L}^* = (L, \oplus, \neg, \mathbf{0}, \mathbf{1})$ *is an MV-algebra and the following identities hold:*

(1) $(a \odot b) \ominus (a \odot c) = a \odot (b \ominus c)\,,$

(2) $a \odot (b \odot c) = (a \odot b) \odot c\,,$

(3) $a \odot \mathbf{1} = a\,,$

(4) $a \odot b = b \odot a\,,$

where $a \ominus b = \neg(\neg a \oplus b) = a \otimes \neg b$ *and* $a \otimes b = \neg(\neg a \oplus \neg b)$*. Moreover, we say that* $\mathbf{L}$ *is a* PŁ′*-algebra if it fulfills the following quasi-identity:*

(5) *if* $a \odot a = \mathbf{0}$ *then* $a = \mathbf{0}$.

Observe that the PŁ-algebras form a variety and the PŁ′-algebras form a quasi-variety.

Example 4.1.6 *Let us by* $[0,1]^S$ *denote the algebra* $([0,1], \oplus, \neg, \odot, 0, 1)$*, where the reduct* $([0,1], \oplus, \neg, 0, 1)$ *is* $[0,1]_{\mathbf{Ł}}$ *and* $\odot$ *is the usual algebraic product of reals. Notice that* $[0,1]^S$ *is both standard* PŁ*-algebra and standard* PŁ′*-algebra and that there is only standard* PŁ*- and* PŁ′*-algebra and (up to isomorphism)*

Later, when we show that PŁ′ is core fuzzy logic, we set $[0,1]^S = [0,1]_{\mathrm{PŁ}'}$. Notice that a linearly ordered PŁ-algebra is PŁ′-algebra iff it has only trivial zero-divisors.

Theorem 4.1.7 PŁ*-algebras coincides with ordered* PŁ*-matrices and* PŁ′*-algebras coincides with ordered* PŁ′*-matrices.*

Proof: The proof is an analogy of the proof of analogous theorem for BL-algebras (Theorem 3.2.8). The only non trivial task is to show that all axioms and rules of PŁ and PŁ′ holds in the corresponding algebras.

Let $\mathbf{L}$ be a PŁ-algebra. Since $\mathbf{L}^*$ is an MV-algebra, we know that the axioms of Łukasiewicz logic hold in $\mathbf{L}$ and modus ponens is a sound deduction rule. Axioms (P1)–(P3) and (P5) are obviously $\mathbf{L}$-tautologies (cf. conditions (1)–(4) in the definition of PŁ-algebra).

We check $\mathbf{L}$-tautologicity of (P4). By Proposition 3.3.5,4, we know that $a \odot b \to a = \mathbf{1}$ iff $a \odot b \leq a = a \odot \mathbf{1}$. Now using [66, Lemma 2.9(ii)], the proof of the first statement is done.

To prove the statement for PŁ′-algebras just observe that the rule (ZD) is obviously sound in each PŁ′-algebra. QED

This theorem has three important corollaries. The first states the connection of our logics and Łukasiewicz logic and the second is the promised proof that the deduction theorem does not hold in PŁ′. This together gives us the third corollary that the logic PŁ′ is strictly stronger than the logic PŁ.

Corollary 4.1.8 PŁ *and* PŁ′ *are conservative extensions of Łukasiewicz logic.*

Proof: Just use Theorem 3.1.13. QED

Corollary 4.1.9 PŁ′*does not have* LDT*. The rule (ZD) cannot be replaced by axioms.*

Proof: Since obviously $\{\neg(v \odot v)\} \vdash \neg v$, the deduction theorem would give us that for some n the formula $(\neg(v \odot v))^n \to \neg v$ is a theorem of PŁ′. Hence $(\neg(v \odot v))^n \to \neg v$ is $[0,1]^S$-tautology (by the latter theorem), i.e., there is n such that $(\neg(x \odot x))^n \leq \neg x$ for each $x \in [0,1]$. Notice that the derivatives of $(\neg(x \odot x))^n$ and $\neg x$ at the point 0 are equal to 0 and -1, respectively. Thus for each n, there is x such that $(\neg(x \odot x))^n > \neg x$, a contradiction.

The second part is a consequence of Theorem 2.2.8. QED

Corollary 4.1.10 *The logic* PŁ′ *is strictly stronger than the logic* PŁ*.*

Now we recall several known results about PŁ and PŁ′-algebras. In [66, Lemma 4.3 (a)], Montagna showed that there is a correspondence between linearly ordered PŁ-algebras and o-rings. The analogous result for PŁ′-algebras is a consequence of [68, Corollary 4.3].

Theorem 4.1.11 *An algebra* $\mathbf{L}$ *is a linearly ordered* PŁ*-algebra if and only if* $\mathbf{L}$ *is isomorphic to the interval algebra of some o-ring* $R_{\mathbf{L}}$*. Furthermore,* $\mathbf{L}$ *is* PŁ′*-algebra iff* $R_{\mathbf{L}}$ *is a domain of integrity.*

The following fact is a corollary of the previous theorem and was originally proved in [68, Corollary 4.4].

Theorem 4.1.12 *The quasi-variety of* PŁ′*-algebras is generated by* $[0,1]^S$*.*

Theorem 4.1.13 *Both* PŁ *and* PŁ′ *logics has* SDP*.*

Proof: The claim for PŁ-algebras is trivial (it was first proved in [66, Theorem 5.1]). The claim for PŁ′-algebras is a trivial consequence of the previous theorem. QED

Corollary 4.1.14 *The logic* PŁ′ *is a weakly implicative fuzzy logic.*

Since we have shown that both PŁ and PŁ′ logics are fuzzy and that the PŁ-algebras (PŁ′-algebras) corresponds to the PŁ-matrices (PŁ′-matrices), we can state the completeness theorem in the following form:

Theorem 4.1.15 (Completeness) *Let $\mathcal{C}$ be either* PŁ *or* PŁ′*, T a theory over $\mathcal{C}$, and φ a formula. Then the following are equivalent:*

1. $T \vdash_{\mathcal{C}} \varphi$*,*
2. $e(\varphi) = \mathbf{1_L}$ *for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of theory T,*
3. $e(\varphi) = \mathbf{1_L}$ *for each $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of theory T.*

The question whether our logics possess the standard completeness is answered by the following theorem.

Theorem 4.1.16 (Finite Standard Completeness) *The logic* PŁ′ *has the finite standard completeness, i.e., for each finite theory T and formula φ we have:*

- $T \vdash_{\text{PŁ}'} \varphi$ *iff* $e(\varphi) = 1$ *for each $[0,1]^S$-model e of theory T.*

The logic PŁ′ *is a core fuzzy logic without standard completeness.*

The proof of the first part is a consequence of the completeness theorem and Theorem 4.1.12. The second part is a consequence of Corollary 4.1.8 and Theorem 3.3.2. This theorem together with Corollary 4.1.10 gives us:

Corollary 4.1.17 *The logic* PŁ *has not the finite standard completeness.*

4.1.3 More on PŁ and PŁ′-algebras

In this section we examine the classes of PŁ-algebras and PŁ′-algebras. Let **PŁ′**, **PŁ**, and $[\mathbf{0},\mathbf{1}]^{\mathbf{S}}$ respectively denote the quasi-variety of PŁ′-algebra, the variety of PŁ-algebras, and the variety generated by $[0,1]^S$ resp. The results in the previous section allows us to state the following theorem:

Theorem 4.1.18 *The following holds:*

1. **PŁ′** *is not a variety.*
2. $\mathbf{PŁ'} \subsetneqq [\mathbf{0},\mathbf{1}]^{\mathbf{S}} \subsetneqq \mathbf{PŁ}$.

Proof:

1. Assume that **PŁ′** is a variety. Then we get a contradiction with Corollary 4.1.9.
2. Both inequalities are obvious. The strictness of the first inequality is a consequence of Part 1. and the strictness of the second one is a consequence of Corollary 4.1.17.

QED

The direct algebraic proof of the first part can be found in [68, Theorem 3.1]. The fact that the variety of PŁ-algebras is not generated by $[0,1]^S$ is already mentioned in Montagna's paper [66, Problem 1], but there is no proof, only a reference to Isbell's paper [59]. In that paper, Isbell proved that the equational theory of formally real f-rings (lattice-ordered rings satisfying all lattice-ring identities that are true in a totally-ordered field) does not have a finite base, or even a base with a finite number of variables. Thus it seems to us that the connection between the second part of Theorem 4.1.18 and Isbell's paper is not so straightforward. We have seen that it can be proven very simply as a natural consequence of our previous results.

In the rest of this section we give one more alternative proof of the previous theorem. We construct two algebras directly demonstrating the strictness of both inequalities. First, Example 4.1.22 is an algebra from $[\mathbf{0},\mathbf{1}]^{\mathbf{S}}$, which is not PŁ′-algebra (this also shows that **PŁ′** is not closed under HSP, so it is not a variety). Second, Example 4.1.23 is a PŁ-algebra which is not in $[\mathbf{0},\mathbf{1}]^{\mathbf{S}}$. Before we present our examples, we study the relation between nontrivial zero-divisors and infinitesimal elements of PŁ-algebras. We recall the definition of an infinitesimal element and continue with the lemma showing the distributivity of $\odot$ w.r.t. $\oplus$.

Definition 4.1.19 *An element a in a* PŁ*-algebra is said to be infinitesimal iff $a > \mathbf{0}$ and $na \leq \neg a$ for each $n \in \mathbb{N}$, where $na = a \oplus \cdots \oplus a$.*

Lemma 4.1.20 *In each* PŁ*-algebra the following inequality holds:*

$$b \odot (x \oplus y) \leq (b \odot x) \oplus (b \odot y)\,.$$

Proof: The inequality is equivalent to $(b \odot (x \oplus y)) \ominus ((b \odot x) \oplus (b \odot y)) = \mathbf{0}$. Now $(b\odot(x\oplus y))\ominus((b\odot x)\oplus(b\odot y)) = (b\odot(x\oplus y))\otimes\neg(b\odot x)\otimes\neg(b\odot y) = [(b\odot(x\oplus y))\ominus(b\odot x)]\ominus(b\odot y) = b \odot [((x \oplus y) \ominus x) \ominus y] = b \odot [(x \oplus y) \otimes \neg x \otimes \neg y] = b \odot [(x \oplus y) \ominus (x \oplus y)] = \mathbf{0}$. QED

Proposition 4.1.21 *Let* $\mathbf{L}$ *be a linearly ordered* PŁ*-algebra, and $a \in L$, $a > \mathbf{0}$. If a is a zero-divisor then a is an infinitesimal.*

Proof: Let us suppose that a is a zero-divisor which is not infinitesimal. Then there exists $n \in \mathbb{N}$ such that $na = \mathbf{1}$. By Lemma 4.1.20 $a \odot na \leq (a \odot a) \oplus \cdots \oplus (a \odot a) = \mathbf{0}$ because a is a zero-divisor. Thus $a = a \odot \mathbf{1} = a \odot na = \mathbf{0}$, a contradiction. QED

Example 4.1.22 *Let us take the following set:*

$$L_{1,\infty} = \{0, 1, 2, \ldots, \infty, \infty - 1, \infty - 2, \ldots\}\,,$$

where we identify ∞ with $\infty - 0$. The operations are defined as follows:

$$\begin{array}{ll}
n \in \mathbb{N}: & \neg n = \infty - n,\\
 & \neg(\infty - n) = n.\\
k, n \in \mathbb{N}: & k \oplus n = k + n,\\
 & (\infty - k) \oplus (\infty - n) = \infty,\\
 & k \oplus (\infty - n) = \begin{cases} (\infty - n + k) & \text{if } k \leq n,\\ \infty & \text{otherwise.}\end{cases}\\
k, n \in \mathbb{N}: & k \odot n = 0,\\
 & k \odot (\infty - n) = k,\\
 & (\infty - k) \odot (\infty - n) = (\infty - k - n).
\end{array}$$

The structure $(L_{1,\infty}, \oplus, \odot, \neg, 0, \infty)$ is a PŁ*-algebra. Observe also that this algebra possesses nontrivial zero-divisors, thus it is not a* PŁ′*-algebra. Notice that the MV-reduct of this algebra is the well-known Chang algebra, thus the elements of $L_{1,\infty}$ are ordered as follows: $0 < 1 < 2 < \cdots < \infty - 2 < \infty - 1 < \infty$.*

We show how to generate this PŁ*-algebra with nontrivial zeros-divisors from the standard* PŁ*-algebra. It is the well-known fact that each algebra in a variety generated by* $[0,1]^S$ *can be obtained as* $A \in HSP([0,1]^S)$*, where* P *means the direct product,* S *a subalgebra, and* H *a homomorphic image. So we will construct the example in the following three steps.*

1. *Step* P*: Take the algebra of all functions* $L = [0,1]^{[0,1]}$.
2. *Step* S*: Restrict to the subalgebra* $S \subseteq L$ *of all continuous piecewise polynomial functions with integer coefficients such that either* $f(0) = 0$ *or* $f(0) = 1$.
3. *Step* H*: Factorise by the equivalence* $\sim$*, where* $f \sim g$ *iff* $f(0) = g(0)$ *and* $f'(0) = g'(0)$*. By* $f'(0)$*, we denote the right-derivative of* f *in* 0.

Example 4.1.23 *Now we will show an example of a* PŁ*-algebra which cannot be generated from the standard* PŁ*-algebra. Firstly, we construct an o-monoid and then we construct the algebra of polynomials over this monoid. In this way, we obtain a ring; its interval algebra is the desired* PŁ*-algebra (cf. Theorem 4.1.11).*

The following example of an o-monoid can be found in [39]. For any $a, b, c, d \in \mathbb{N}$*,* $\langle a, b, c\rangle$ *will denote the sub-o-monoid of* $\mathbb{N}$ *generated by* a*,* b*,* c*, and* $\langle a, b, c\rangle/d$ *will denote the o-monoid obtained by identifying with infinity all elements of* $\langle a, b, c\rangle$ *that are greater than or equal to* d*.*

Let $S = \{32^*\} \cup \langle 9, 12, 16\rangle/30$ *denote the o-monoid obtained from* $\langle 9, 12, 16\rangle/30$ *by adding one additional element, denoted by* 32^**. This element satisfies* $16 + 16 = 32^*$*,* $32^* + z = \infty$*, and the whole monoid is to be ordered as follows:*

$$0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 32^* < \infty\,.$$

All the relations that do not involve 32^* *are as in* $\mathbb{N}$*, so we have only to check that* $x \le y$ *implies* $x + z \le y + z$ *if* x*,* y*, or* z *is equal to* 32^**, but it is easy to see.*

Let R *be the o-ring of integers. Then the monoid ring* $R[S]$ *is the set of all finite formal sums* $r_1X^{s_1} + \cdots + r_nX^{s_n}$*, where* X *is an indeterminate,* $r_i \in R$ *and* $s_i \in S$*. Multiplication is defined by* $X^sX^t = X^{s+t}$ *and by distributivity (so* $X^0 = 1$*). An element* $r_1X^{s_1} + \cdots + r_nX^{s_n}$ *is said to be in normal form if* $r_i \neq 0$ *and* $s_1 < \cdots < s_n$*.*

We identify X^∞ *with* 0 *and denote the resulting quotient of* $R[S]$ *by* $R[S]_h$*. An element* $r_1X^{s_1} + \cdots + r_nX^{s_n}$ *in normal form is positive iff* $r_1 > 0$*. Thus for all* $a, b \in S$*,* $a < b$ *implies* $X^a > X^b$*.*

It can be checked that $R[S]_h$ *is an o-ring. Finally, we get the desired* PŁ*-algebra* $\mathbf{L}$ *as the interval algebra of* $R[S]_h$*. In* $\mathbf{L}$ *the following identity, which is valid in* $[0,1]^S$*, does not hold:*

$$(x_1 \odot z_1 \ominus y_1 \odot z_2) \wedge (x_2 \odot z_2 \ominus y_2 \odot z_1) \wedge (y_1 \odot y_2 \ominus x_1 \odot x_2) = \mathbf{0} \tag{4.1}$$

Indeed, let us evaluate the variables as follows:

$$\begin{array}{lll} x_1 = X^{16} & y_1 = X^{18} & z_1 = X^{16} \\ x_2 = X^{12} & y_2 = X^{9} & z_2 = X^{12} \end{array}$$

Then the terms in brackets in (4.1) attain the following values:

$$\begin{aligned} x_1 \odot z_1 \ominus y_1 \odot z_2 &= X^{16} \odot X^{16} \ominus X^{18} \odot X^{12} = X^{32^*} > 0\,, \\ x_2 \odot z_2 \ominus y_2 \odot z_1 &= X^{12} \odot X^{12} \ominus X^{9} \odot X^{16} = X^{24} \ominus X^{25} > 0\,, \\ y_1 \odot y_2 \ominus x_1 \odot x_2 &= X^{18} \odot X^{9} \ominus X^{16} \odot X^{12} = X^{27} \ominus X^{28} > 0\,. \end{aligned}$$

To see that (4.1) is valid in the standard PŁ*-algebra* $[0,1]^S$*, just observe that whenever one of the variables* x_i, y_i, z_i *is* 0*, then the equality trivially holds. Further, if* $x_1z_1 > y_1z_2$ *and* $x_2z_2 > y_2z_1$*, then* $\prod x_i \prod z_j > \prod y_i \prod z_j$ *and this implies* $\prod x_i > \prod y_i$*.*

4.2 PŁ$_\triangle$ and PŁ$'_\triangle$ logics

In this section, we extend the language of PŁ logic by $\triangle$ connective and introduce $\overline{\text{PŁ}_\triangle}$ and $\overline{\text{PŁ}'_\triangle}$ logics. Later we show that these logic coincide with logics PŁ$_\triangle$ and PŁ$'_\triangle$.

Definition 4.2.1 *Let $\mathcal{C}$ be either* PŁ *or* PŁ$'$. *The $\overline{\mathcal{C}_\triangle}$ logic results from $\mathcal{C}$ by adding axioms (A$\triangle$1) – (A$\triangle$5) and the deduction rule of necessitation.*

Both just defined logics obviously extends **Ł**$_\triangle$ logic.

Lemma 4.2.2 *The formulae $\triangle\varphi \otimes \psi \equiv \triangle\varphi \wedge \psi$ and $\triangle\varphi \odot \psi \equiv \triangle\varphi \wedge \psi$ are theorems of the* PŁ *logic.*

Proof: The first formula is the known theorem of **Ł**$_\triangle$. For the proof of the second one, use the first formula and theorem (TP5). QED

Theorem 4.2.3 *We have: $\overline{\text{PŁ}_\triangle} = \text{PŁ}_\triangle$ and $\overline{\text{PŁ}'_\triangle} = \text{PŁ}'_\triangle$ (in the sense of Theorem 2.2.37).*

Proof: The first claim is trivial (see Theorem 3.2.21). Let us denote the axiom $\triangle\neg(\varphi\odot\varphi) \to \neg\varphi$ from the group B in presentation of PŁ$'_\triangle$ as $(P\triangle)$. To prove the second claim we firstly, we show that the deduction rule (ZD) can be derived in PŁ$_\triangle$ extended by axiom $(P\triangle)$. Let $\neg(\varphi \odot \varphi)$ be a theorem. Then $\triangle\neg(\varphi \odot \varphi)$ is a theorem as well and so $\neg\varphi$ is provable (by modus ponens and axiom $(P\triangle)$).

Converse direction: It is sufficient to prove $\neg(\neg(P\triangle)\odot\neg(P\triangle))$ (let us denote this formula by F). Then proof is done by the use of (ZD). Observe that $\neg(P\triangle) \equiv \triangle\neg(\varphi\odot\varphi)\otimes\varphi$. Thus by Lemma 4.2.2 we get $\neg F \equiv (\triangle\neg(\varphi\odot\varphi)\wedge\varphi)\odot(\triangle\neg(\varphi\odot\varphi)\wedge\varphi)$. After repeated use of theorem (TP6) we get $\neg F \equiv (\triangle\neg(\varphi\odot\varphi)\odot\triangle\neg(\varphi\odot\varphi))\wedge(\triangle\neg(\varphi\odot\varphi)\odot\varphi)\wedge(\varphi\odot\triangle\neg(\varphi\odot\varphi))\wedge(\varphi\odot\varphi)$.

By use of Lemma 4.2.2 we may write $\neg F \equiv \triangle\neg(\varphi \odot \varphi) \wedge \varphi \wedge (\varphi \odot \varphi)$. Finally, by the obvious fact that $\varphi \wedge (\varphi \odot \varphi) \equiv \varphi \odot \varphi$ is a theorem (use (H9), (H5)) and using Lemma 4.2.2, we get $F \equiv \neg(\triangle\neg(\varphi\odot\varphi)\otimes(\varphi\odot\varphi)) \equiv \triangle\neg(\varphi\odot\varphi) \to \neg(\varphi\odot\varphi)$. Since $\triangle\neg(\varphi\odot\varphi) \to \neg(\varphi\odot\varphi)$ is an instance of axiom (Ł$\triangle$3), the formula F is a theorem. QED

Corollary 4.2.4 *We have:*

- PŁ$_\triangle$ *and* PŁ$'_\triangle$ *are fuzzy logics with* DT$_\triangle$,
- PŁ$'_\triangle$ *is a core fuzzy logic without standard completeness,*
- PŁ$'_\triangle$ *is strictly stronger than* PŁ$_\triangle$*logic,*
- PŁ$_\triangle$ *is has not the finite standard completeness.*

Proof: Everything in this corollary is just a consequence of the previous theorem. We only present an alternative proof of the one but last statement, where the connective $\triangle$ allows us to find a simple proof. According to Example 4.1.22, we have a PŁ$_\triangle$-algebra **L** with an element $c > 0$ such that $c \odot c = 0$. Then $\neg c = (\infty - c) < \infty$, thus $\triangle\neg c = 0$. Since $\neg(c \odot c) = \infty$ and so $\triangle(\neg(c \odot c)) = \infty$, we know that the axiom (P$\triangle$) is not an **L**-tautology. Thus (P$\triangle$) is not a theorem of PŁ$_\triangle$. QED

Now we summarize known facts about connections between our logics and Łukasiewicz and ŁΠ logic.

Theorem 4.2.5

(1) PŁ, PŁ$'$, PŁ$_\triangle$, *and* PŁ$'_\triangle$ *logics are conservative extensions of Łukasiewicz logic.*

(2) PŁ$_\triangle$ *and* PŁ$'_\triangle$ *logics are conservative extensions of* **Ł**$_\triangle$.

(3) PŁ$_\triangle$ *logic is a conservative extension of* PŁ *logic and* PŁ$'_\triangle$ *logic is a conservative extension of* PŁ$'$ *logic.*

(4) PŁ$'_\triangle$ *logic is not a conservative extension of* PŁ *logic.*

(5) ŁΠ *logic is a conservative extension of* PŁ$'$*and* PŁ$'_\triangle$ *logics.*

(6) ŁΠ *logic is not a conservative extension of* PŁ *and* PŁ$_\triangle$ *logics.*

Proof: (1): For PŁ it is already proven in Theorem 4.1.8, the proof for the remaining logics is analogous.

(2) Analogous to (1).

(3) It follows from the fact that we can extend each linearly ordered PŁ-algebra by $\triangle$ and from the completeness theorems for both logics.

(4) Since PŁ$'_\triangle$ is a conservative extension of PŁ$'$and PŁ$'$is strictly stronger than PŁ, the proof easily follows.

(5) A consequence of Theorems 3.3.21, 4.1.16, and Corollary 4.2.4.

(6) A consequence of (5) and the fact that PŁ$_\triangle$ is strictly weaker than PŁ$'_\triangle$ (and that PŁ is strictly weaker than PŁ$'$). QED

4.3 Pavelka style extensions

In this section we add rational constants into our language together with the book-keeping axioms in the way of Section 4.3. This section is rather straightforward application of the results of that section. The only not expected result is that if we add book-keeping axioms and corresponding infinitary deduction rule into the logic PŁ$_\triangle$ and PŁ$'_\triangle$ (which are not the same) we get one logic.

Theorem 4.3.1 *Logics* PŁ$'$ *and* PŁ$'_\triangle$ *are argument-monotonous rational core fuzzy logic.* $\mathcal{NC}(\text{PŁ}') = \emptyset$ *and* $\mathcal{NC}(\text{PŁ}'_\triangle) = \{<\triangle, 1>\}$.

Observe that although PŁ is not a core fuzzy logic, it has only one standard algebra $[0,1]_{\text{PŁ}'}$ up to isomorphism and analogously for the logic PŁ$_\triangle$. Thus we define:

Theorem 4.3.2 *Particular Pavelka style extension of fuzzy logics defined in this chapter can be axiomatized as:*

- $\mathcal{RPP}\text{Ł}' = \mathcal{R}\text{PŁ}'([0,1]_{\text{PŁ}'})$
- $\mathcal{RPP}\text{Ł}'_\triangle = \mathcal{R}\text{PŁ}'([0,1]_{\text{PŁ}'_\triangle}) + \text{IR}_\triangle$
- $\mathcal{RPP}\text{Ł}([0,1]_{\text{PŁ}'}) = \mathcal{R}\text{PŁ}([0,1]_{\text{PŁ}'})$
- $\mathcal{RPP}\text{Ł}_\triangle([0,1]_{\text{PŁ}'_\triangle}) = \mathcal{R}\text{PŁ}_\triangle([0,1]_{\text{PŁ}'_\triangle}) + \text{IR}_\triangle$

Let us denote the logic $\mathcal{RPP}\text{Ł}([0,1]_{\text{PŁ}'})$ (or $\mathcal{RPP}\text{Ł}_\triangle([0,1]_{\text{PŁ}'_\triangle})$ respectively) as $\mathcal{RPP}\text{Ł}$ (resp. $\mathcal{RPP}\text{Ł}_\triangle$), even if this contradicts our definitions in Section 4.3.

Theorem 4.3.3 *Let* $\mathcal{L}$ *be one of the* PŁ, PŁ$'$, PŁ$_\triangle$, PŁ$'_\triangle$ *and* T *a theory over* $\mathcal{RPL}$ *and* φ *be a formula. Then:*

1. $|\varphi|_T = ||\varphi||_T$,
2. $\mathcal{RPP}\text{Ł}_\triangle$ *and* $\mathcal{RPP}\text{Ł}'_\triangle$ *have the standard completeness,*
3. $\mathcal{RPP}\text{Ł}'$ *and* $\mathcal{RPP}\text{Ł}'_\triangle$ *are core fuzzy logics,*
4. $\mathcal{RPP}\text{Ł}$ *has not the standard completeness.*

Proof: We know that both $\mathcal{RPP}$Ł and $\mathcal{RPP}$Ł$'$ are fuzzy logic from Corollary 3.4.5, we know that the remaining two logics are fuzzy from Lemma 3.4.25. Thus the part 1. is just a consequence of Theorem 3.4.21. Part 2. is a consequence of Corollary 3.4.22. Part 3. is consequence of Theorem 3.4.23 and Corollary 3.4.22. Part 4. is obvious (PŁ has not the standards completeness). QED

Remark 4.3.4 *We can also obtain Pavelka style completeness for $\mathcal{RPP}$Ł from [44, Section 3.3], where the author defines the logic RPL($\odot$). It is Łukasiewicz logic plus book-keeping axioms of RP*PŁ *and the axioms:*

$$\begin{aligned}(\varphi \to \psi) \quad &\to \quad (\varphi \odot \chi \to \psi \odot \chi)\,,\\ (\varphi \to \psi) \quad &\to \quad (\chi \odot \varphi \to \chi \odot \psi)\,.\end{aligned}$$

This logic enjoys the same Pavelka's style completeness as $\mathcal{RPP}$Ł (see [44, Theorem 3.3.19] or our Theorem 3.4.21).

Thus we have three different logics, namely $\mathcal{RPP}$Ł, $\mathcal{RPP}$Ł$'$, RPL($\odot$). All of them enjoy the same Pavelka's style completeness. However, the question whether these logics (as sets of theorems) coincide seems to be open for us.

Corollary 4.3.5

1. *The $\mathcal{RPP}$Ł$_\triangle$ and $\mathcal{RPP}$Ł$'_\triangle$ logics coincide.*
2. *$\mathcal{RPP}$Ł$'$ is a conservative extension of* PŁ$'$*and $\mathcal{RPP}$Ł$'_\triangle$ is a conservative extension of* PŁ$'_\triangle$*.*
3. *$\mathcal{RPP}$Ł and $\mathcal{RPP}$Ł$'$ are a conservative extensions of* RPL*.*
4. RŁΠ *is a conservative extension of $\mathcal{RPP}$Ł$_\triangle$.*

Proof:

1. This is an obvious consequence of the latter theorem and the fact that standard algebras for these logics are the same.
2. See Theorem 3.4.23 and Corollary 3.4.22.
3. Obvious using Corollary 3.4.24.
4. It is the consequence of Corollary 3.4.26.

QED

4.4 The predicate logics

Using our knowledge of predicate logic we can define for each of our four logics $\mathcal{C}$ predicate version $\mathcal{C}\forall^-$ and get completeness w.r.t. all corresponding algebras. We can also define logics $\mathcal{C}\forall$. However there is a problem with logic PŁ$'\forall$. All other have clearly PP and we get completeness w.r.t. corresponding linear algebras. Since the logic PŁ$'$ does not have DT$_\to$ we have no way to prove completeness and it seems to be an interesting open problem.

The questions whether PŁ$\forall$ is a conservative extension of the Łukasiewicz predicate logic, and whether ŁΠ$\forall$ is a conservative extension of PŁ$\forall$ logic seems to be open as well (and many analogous one involving other predicate logics defined in this section). The standard completeness of PŁ$\forall$, PŁ$_\triangle\forall$, and PŁ$'_\triangle\forall$ logics is a related problem. But this question can be answered. Here we assume that the reader is familiar with the basic concepts of undecidability and arithmetical hierarchy ([44, Section 6.1] is satisfactory for our needs). In the following we assume that our predicate language is at most countable. Before we proceed we look at the predicate version of Pavelka style extension of our logic. The following theorem is just a corollary of Theorem 3.5.19.

Theorem 4.4.1 *Let $\mathcal{C}$ be one of the* PŁ, $\text{PŁ}_\triangle$, *or* $\text{PŁ}'_\triangle$. *Then the logic $\mathcal{RPC}\forall$ has Pavelka style completeness. Furthermore, the logics $\mathcal{RP}\text{PŁ}_\triangle\forall$ and $\mathcal{RP}\text{PŁ}'_\triangle\forall$ have the standard completeness.*

Corollary 4.4.2 *The logics $\mathcal{RP}\text{PŁ}_\triangle\forall$ and $\mathcal{RP}\text{PŁ}'_\triangle\forall$ coincide.*

Corollary 4.4.3 RŁΠ∀ *is a conservative extension of $\mathcal{RP}\text{PŁ}_\triangle\forall$.*

Proof: It is the consequence of Theorems 3.5.20 and 4.4.1. QED

Observe that $\mathcal{TAUT}^\forall(\text{PŁ}\forall) = \mathcal{TAUT}^\forall(\text{PŁ}'\forall)$ and $\mathcal{TAUT}^\forall(\text{PŁ}_\triangle\forall) = \mathcal{TAUT}^\forall(\text{PŁ}'_\triangle\forall)$.

Theorem 4.4.4 *The set $\mathcal{TAUT}^\forall(\text{PŁ}\forall)$ is Π_2-complete and the set $\mathcal{TAUT}^\forall(\text{PŁ}_\triangle\forall)$ is not arithmetical.*

Proof: The Π_2-hardness is an obvious corollary of Theorem 3.5.5. The fact that the set $\mathcal{TAUT}^\forall(\text{PŁ}\forall)$ is in Π_2 is a corollary of Theorem 4.4.1 (we know that $\varphi \in \mathcal{TAUT}^\forall(\text{PŁ}\forall)$ iff $(\forall r \in \mathbb{IQ}))(\exists \text{ proof } \omega)(\omega \text{ is the proof of } \overline{r} \to \varphi))$.

The proof of the second part is an obvious modification of the analogous proof for product predicate logic (see [45, Corollary 2]). QED

Corollary 4.4.5 *Let $\mathcal{C}$ be either* PŁ, $\text{PŁ}_\triangle$ *or* $\text{PŁ}'_\triangle$. *Then the logic $\mathcal{C}\forall$ has not the finite standard completeness property.*

At the end of this section we prove another consequence of the standard completeness of $\mathcal{RP}\text{PŁ}\forall$. We make a small modification of the language of $\mathcal{RP}\text{PŁ}\forall$ and show that the resulting logic TT coincides with the famous logic of Takeuti and Titani. The logic TT results from $\mathcal{RP}\text{PŁ}\forall$ by omitting truth constants $\overline{r}$ such that r can not be expressed in the form $\frac{k}{2^n}$. In other words, we may say that the logic TT has only one additional truth constant, $\overline{\mathbf{\tfrac{1}{2}}}$, and the other truth constants are defined by using the connectives of $\text{PŁ}_\triangle\forall$. Now we can easily prove the analogy of Theorem 4.4.1 for TT. Just go through the proofs leading to this theorem and notice that the set of truth constants in TT is dense. Thus all the proofs will be sound for TT as well. This gives us the following corollary:

Corollary 4.4.6 RŁΠ∀ *is a conservative extension of TT.*

Takeuti and Titani's logic was introduced by Takeuti and Titani in their work [80]. It is a predicate fuzzy logic based on Gentzen's system of intuitionistic predicate logic. The connectives used by this logic are just the connectives of $\mathcal{RP}\text{PŁ}_\triangle\forall$ logic. This logic has two additional deduction rules (named $\mathcal{R}1$, $\mathcal{R}2$) and 46 axioms (namely **F1** – **F46**). We will not present the axiomatic system and we only recall that this logic is sound and complete w.r.t. the standard PŁ-algebra (cf. [80, Theorem 1.4.3]). All this leads us to the following conclusion:

Theorem 4.4.7 *Takeuti and Titani logic coincides with the logic TT. Furthermore, the logic $\mathcal{RP}\text{PŁ}_\triangle\forall$ is a conservative extension of Takeuti and Titani logic.*

This theorem allows to translate the results from the Takeuti and Titani's logic into our much more simpler (in syntactical sense) logical system of the TT or $\mathcal{RP}\text{PŁ}_\triangle\forall$ logic. An interesting corollary of this theorem and the previous corollary is a very simple proof of one part of [24, Theorem 10]:

Theorem 4.4.8 *Takeuti and Titani's logic is contained in* RŁΠ∀ *logic.*

Chapter 5

New results in Product logic

This chapter is devoted to the study of product logic. Recall that Product logic has been defined in [52] in order to describe the logic of the product t-norm and its corresponding residuum and it turns out to be an axiomatic extension of basic fuzzy logic corresponding to product t-norm.

In the first section we present an alternative axiomatic system for product logic and show that it is the simplest possible (it the sense we specify later).

Then we continue with the study of normal forms formulae of product logic. We show that every time we fix a set of variables and we consider evaluations that positively interpreted such variables, then we can give a representation of a formula as conjunction of disjunctions (disjunction of conjunctions, resp.). This will be called a conjunctive (disjunctive, resp.) semi-normal form or CsNF (DsNF). Each disjunct (conjunct) will be called *literal* and has the form

$$v_1^{k_1}\&\ldots\&v_l^{k_l} \to v_{l+1}^{k_{l+1}}\&\ldots\&v_n^{k_n}.$$

Then we show how to join suitably these forms in order to obtain the conjunctive (disjunctive) normal form or CNF (DNF).

We also describe the steps that can be used for simplifying formulae expressed in CsNF (DsNF). Such simplification are based on the observation that each literal can be identified with the n-tuple of powers of the variables occurring in it.

We conclude this section with simple algorithm for checking tautologies that is based on the conjunctive normal form and on the simplification rules. This algorithm reduces the problem of checking tautologies to a problem of integer linear programming. Note that TAUT problem for product logic is co-NP complete (see [44]). We did not address the complexity of our algorithm.

In the last section we give a functional representation of formulae of product logic. We prove the analogous of McNaughton theorem for Łukasiewicz logic (Theorem 3.3.25), stating that truth tables of product formulae are such that in each region of *positivity* they are continuously composed by several monomials (see also [41]). Each of this monomial corresponds to a basic literal.

5.1 On axioms of Product logic

This section has two goals. The first is to show a new axiomatic system of product fuzzy logic with only one non-BL axiom which has only two variables. The second goal is to prove that there cannot be any axiomatic system of the product fuzzy logic with single non-BL axiom with only one variable.

5.1.1 New axiomatic system

Definition 5.1.1 *Let $N\Pi$ be the schematic extension of BL with the following non-BL axiom:*

$(A\Pi)$ $\quad \neg\neg\varphi \to ((\varphi \to \varphi\&\psi) \to \psi\&\neg\neg\psi)$.

Lemma 5.1.2 *Product logic proves the following theorems:*
$(Aux1)$ $\quad \varphi \to \varphi\&\neg\neg\varphi$,
$(A\Pi)$ $\quad \neg\neg\varphi \to ((\varphi \to \varphi\&\psi) \to \psi\&\neg\neg\psi)$.

Proof: (**Aux1**): $\vdash \neg\neg\varphi \to (Aux1)$ $\hfill (BL1)$
$\vdash \neg\varphi \to (Aux1)$ $\hfill (BL2)$
$\vdash \neg\varphi \vee \neg\neg\varphi \to (Aux1)$ $\hfill (BL3)$
$\vdash (Aux1)$ $\hfill (T\Pi1)$

(**AΠ**): $\vdash \neg\neg\varphi \to ((\varphi\&1 \to \varphi\&\psi) \to (1 \to \psi))$ $\hfill (\Pi1)$
$\vdash \neg\neg\varphi \to ((\varphi \to \varphi\&\psi) \to \psi)$
$\vdash \neg\neg\varphi \to ((\varphi \to \varphi\&\psi) \to \psi\&\neg\neg\psi)$ $\hfill (Aux1)$
$\vdash (A\Pi)$

QED

Lemma 5.1.3 *The following theorems are provable in $N\Pi$:*
$(Aux2)$ $\quad \neg\varphi\&\neg\varphi \to \neg(\neg\varphi \to \varphi)$,
$(Aux3)$ $\quad \neg\varphi \to \neg\varphi\&\neg\varphi$,
$(\Pi2)$ $\quad \varphi \wedge \neg\varphi \to 0$,
$(Aux4)$ $\quad \neg\neg\varphi \to ((\varphi \to \varphi\&\varphi) \to \psi)$,
$(\Pi1)$ $\quad \neg\neg\varphi \to ((\varphi\&\chi \to \varphi\&\psi) \to (\chi \to \psi))$.

Proof: (**Aux2**): $\vdash (\neg\varphi \to \varphi) \to (\neg\varphi \to \varphi)$ $\hfill (BL4)$
$\vdash \neg\varphi \to ((\neg\varphi \to \varphi) \to \varphi))$ $\hfill (BL5)$
$\vdash \neg\varphi \to (\neg\varphi \to \neg(\neg\varphi \to \varphi))$ $\hfill (BL6)$
$\vdash \neg\varphi\&\neg\varphi \to \neg(\neg\varphi \to \varphi)$ $\hfill (A5)$

(**Aux3**): $\vdash \neg\neg1 \to ((1 \to 1\&\neg\varphi) \to \neg\varphi\&\neg\neg\neg\varphi)$, this is the axiom $(A\Pi)$ for φ being 1 and ψ being $\neg\varphi$
$\vdash (Aux4)$

(**Π2**): $\vdash \neg\varphi \to ((\neg\varphi \to \varphi) \to 0)$ $\hfill (Aux2), (Aux3)$
$\vdash \neg\varphi\&(\neg\varphi \to \varphi) \to 0$ $\hfill (A5)$
$\vdash (\Pi2)$

(**Aux4**): $\vdash \neg\neg\varphi \to ((\varphi \to \varphi\&\psi) \to \psi\&\neg\neg\psi)$ $\hfill (A\Pi)$
$\vdash \psi\&\neg\neg\psi \to \psi$ $\hfill (A2)$
$\vdash (Aux4)$

(**Π1**): $\vdash \neg\neg\varphi \to ((\varphi \to \varphi\&\psi) \to \psi)$ $\hfill (Aux4)$
$\vdash ((\varphi \to \varphi\&\psi) \to \psi) \to ((\chi \to (\varphi \to \varphi\&\psi)) \to (\chi \to \psi))$ $\hfill (A1)$
$\vdash (\Pi1)$ $\hfill (A1)$ and $(A5)$

QED

Corollary 5.1.4 *The schematic extension $N\Pi$ is equivalent to product logic (in other words: $N\Pi$ is an axiomatic system of product logic).*

The proof is obvious.

5.1.2 More about possible axiomatic systems

Now we prove that there is no single non-BL axiom with single variable, which would axiomatize product logic. Let ψ be the formula with only one variable provable in Π. Let Π' be the schematic extension of BL by the axiom ψ. Thus each Π-algebra is a Π'-algebra. To prove that Π is strictly stronger than Π' it suffices to find a Π'-algebra which is not a Π-algebra. We offer the following algebra:

Definition 5.1.5 *Let $\mathbf{A} = ([0,1]\backslash\{\frac{1}{2}\}, *, \Rightarrow, \leq)$ be a BL-algebra. $\leq$ is a usual order of reals. The operations are defined as follows:*

- $x * y = \begin{cases} 2xy & \text{for} \quad x, y < \frac{1}{2}, \\ 2xy - x - y + 1 & \text{for} \quad x, y > \frac{1}{2}, \\ \min(x, y) & \quad\quad \text{otherwise} . \end{cases}$
- $x \Rightarrow y = \begin{cases} 1 & \text{for} \quad x \leq y \\ \frac{y}{2x} & \text{for} \quad x, y < \frac{1}{2} \text{ and } x > y, \\ \frac{x+y-1}{2x-1} & \text{for} \quad x, y > \frac{1}{2} \text{ and } x > y, \\ y & \quad\quad \text{otherwise} . \end{cases}$

Remark 5.1.6 The reader familiar with ordinal sums of BL-algebras will notice one fact. The algebra $\mathbf{A}$ from the latter definition is obtained from the ordinal sum of two standard Π-algebras by the deleting of the central idempotent point.

Lemma 5.1.7 *Let $\mathbf{A}$ be the algebra from the last definition. Then:*

- $\mathbf{A}$ *is NOT a Π-algebra,*
- $\mathbf{A}$ *is a Π'-algebra.*

Proof: Just take the axiom (Π1). For x being $\frac{1}{4}$, y being 1 and z being $\frac{3}{4}$ we get: $1 \rightarrow ((\frac{1}{4} \Rightarrow \frac{1}{4} * \frac{3}{4}) \Rightarrow \frac{3}{4})$, but because $\frac{1}{4} * \frac{3}{4} = \frac{1}{4}$ it obviously does not hold. So $\mathbf{A}$ is NOT the Π-algebra.

We want to prove that ψ holds in the algebra $\mathbf{A}$. ψ has only one variable x. So we have two possible cases:

- $x \in [0, \frac{1}{2})$, we take the algebra $\mathbf{A}^- = ([0, \frac{1}{2}) \cup \{1\}, *', \Rightarrow')$, where the operations $*', \Rightarrow'$ are the restrictions of those from the algebra $\mathbf{A}$ to the new domain (the new domain is indeed closed to these restricted operations). It is obvious that the algebra $\mathbf{A}^-$ is a Π-algebra (it is isomorphic to the standard Π-algebra.) And the value of the formula ψ is the same in the $\mathbf{A}^-$ as in the $\mathbf{A}$ itself. And because ψ is the tautology in each Π-algebra ψ holds in $\mathbf{A}^-$. So if $x \in [0, \frac{1}{2})$ then ψ holds in $\mathbf{A}$.
- $x \in (\frac{1}{2}, 1]$, we take the algebra $\mathbf{A}^+ = (\{0\} \cup (\frac{1}{2}, 1], *', \Rightarrow')$. The arguments are similar to the previous ones.

QED

Remark 5.1.8 Notice one thing: we say ψ has only one variable, but there is another one "hidden" - the truth constant $\overline{0}$ in the definition of $\neg$. Which is the reason we may not use these arguments for the Łukasiewicz logic (A^+ would not be isomorphic to the standard MV-algebra, because the negation does not behave "as is expected").

Corollary 5.1.9 *There is no schematic extension of BL with single axiom with single variable equivalent to product logic.*

5.2 Semi-Normal Forms and Normal Forms

5.2.1 Literals

In this subsection we define the notion of literal and prove some of its basic properties. Before we do so, we prepare few technical lemmata. By VAR_φ we denote the set of variables occurring in φ.

Definition 5.2.1 *Let φ be a formula and let $V \subseteq VAR_\varphi$. Then:*

- *an evaluation e is called* (V, φ)-positive *if for each $v \in VAR_\varphi$ holds: $e(v) > 0$ iff $v \in V$;*
- *an evaluation is called* φ-positive *if it is (VAR_φ, φ)-positive;*
- *the theory T^{VAR} is defined as $\{\neg\neg p \mid p \in VAR\}$;*
- *the theory $T^{(V,\varphi)}$ is defined as $\{\neg\neg p \mid p \in V\} \cup \{\neg p \mid p \in VAR_\varphi - V\}$;*
- *the theory T^φ is defined as $T^{(VAR_\varphi,\varphi)}$.*

Corollary 5.2.2 *An evaluation e is a model of the theory T^{VAR} iff $e(v) > 0$ for every $v \in VAR$.*

An evaluation e is a model of the theory $T^{(V,\varphi)}$ iff e is (V, φ)-positive.

An evaluation e is a model of the theory T^φ iff e is φ-positive.

Lemma 5.2.3 *For any formula φ only containing connectives & and $\to$ (thus φ not containing $\overline{0}$ and negation)*

$$T^\varphi \vdash \neg\neg\varphi.$$

Proof: Note that for any evaluation e,

$$e(\neg\neg\varphi) = \begin{cases} 0 & \text{if } e(\varphi) = 0, \\ 1 & \text{if } e(\varphi) > 0. \end{cases}$$

The proof proceeds by induction on the complexity of φ.

- If φ is a variable v then $T^v = \{\neg\neg v\}$ and trivially $T^v \vdash \neg\neg v$.

- If $\varphi = \psi_1 \& \psi_2$ then ψ_1 and ψ_2 do not contain $\overline{0}$ and so by induction hypothesis $T^{\psi_1} \vdash \neg\neg\psi_1$ and $T^{\psi_2} \vdash \neg\neg\psi_2$. Note that $T^{\psi_1}, T^{\psi_2} \subseteq T^\varphi$. If e is a model of T^φ then $e(\neg\neg\psi_1) = 1$ and $e(\neg\neg\psi_2) = 1$, hence $e(\psi_1) > 0$ and $e(\psi_2) > 0$ and since

$$e(\neg\neg\varphi) = e(\neg\neg(\psi_1 \& \psi_2)) = \begin{cases} 0 & \text{if } e(\psi_1) \cdot e(\psi_2) = 0, \\ 1 & \text{if } e(\psi_1) \cdot e(\psi_2) > 0, \end{cases}$$

  then e satisfies φ.

- If $\varphi = \psi_1 \to \psi_2$ we can repeat the same argument as before, by noting that $e(\neg\neg\varphi) < 1$ if and only if $e(\psi_2) = 0$.

QED

Note that since T^{VAR} contains T^φ, we also get that for every formula φ not containing $\overline{0}$

$$T^{VAR} \vdash \neg\neg\varphi.$$

Notation. For a conjunction with n equal arguments φ, we use the abbreviation φ^n. A conjunction of zero formulae (also written φ^0) is considered as equal to $\overline{1}$. The set $\{1, \ldots, n\}$ will be denoted by $\hat{n}$.

Definition 5.2.4 *A literal is a formula in form:*

$$v_1^{k_1} v_2^{k_2} \dots v_l^{k_l} \to v_{l+1}^{k_{l+1}} v_{l+2}^{k_{l+2}} \dots v_m^{k_m},$$

where k_i are natural numbers and v_i arbitrary propositional variables.

Remark 5.2.5 Note that literals are always not equivalent to $\overline{0}$. Thus the only formula in CsNF (DsNF) equivalent to the truth constant $\overline{0}$ is truth constants $\overline{0}$ itself. Furthermore, the only formula in CsNF (DsNF) containing $\overline{0}$ is again $\overline{0}$ itself. Thus we can use Lemma 5.2.3 and show that if φ is a formula in CsNF (DsNF) then either $\varphi = \overline{0}$ or $T^{VAR} \vdash \neg\neg\varphi$.

Remark 5.2.6 Notice that we do not assume that variables v_i are pairwise distinct. It is obvious that we may assume that variables v_i for $i \leq l$ are pairwise distinct as well as variables v_i for $i > l$.

If v_1, v_2 and v_3 are variables, then in product logic holds the following:

$$T^{v_1} \vdash ((v_1 \& v_2) \to (v_1 \& v_3)) \equiv (v_2 \to v_3)$$

(cancellation property, see [44]). Then if α is a literal, there is a literal α', such that variables occurring in α' are pairwise distinct and $T^{\alpha} \vdash \alpha \equiv \alpha'$. It is obvious that also $T^{VAR} \vdash \alpha \equiv \alpha'$.

Lemma 5.2.7 *Let $\alpha, \beta, \gamma, \delta$ be literals. Then the following formula is provable in product logic:*

$$(Aux1) \qquad (\alpha \to \beta) \& (\gamma \to \delta) \equiv (\alpha \to \beta) \wedge (\gamma \to \delta) \wedge (\alpha \& \gamma \to \beta \& \delta).$$

The following formula is provable in T^{VAR}:

$$(Aux2) \qquad ((\alpha \to \beta) \to (\gamma \to \delta)) \equiv (\gamma \to \delta) \vee (\beta \& \gamma \to \alpha \& \delta).$$

Proof: Let us use a denotation for formulae as the denotation for their evaluation. Recall that $(\alpha \to \beta) = 1 \wedge \frac{\beta}{\alpha}$ for $\alpha \neq 0$. Thus we may write, for $\alpha \neq 0$ and $\delta \neq 0$:

(Aux1): $(1 \wedge \dfrac{\beta}{\alpha}) \cdot (1 \wedge \dfrac{\delta}{\gamma}) = 1 \wedge \dfrac{\beta}{\alpha} \wedge \dfrac{\delta}{\gamma} \wedge \dfrac{\beta\delta}{\alpha\gamma} = (1 \wedge \dfrac{\beta}{\alpha}) \wedge (1 \wedge \dfrac{\delta}{\gamma}) \wedge (1 \wedge \dfrac{\beta\delta}{\alpha\gamma}),$

that is what we want to prove. The case of $\alpha = 0$ or $\gamma = 0$ is trivial.

(Aux2): Let us suppose that $\alpha \neq 0$, $\gamma \neq 0$, $\delta \neq 0$. Then

$$(1 \wedge \frac{\beta}{\alpha}) \to (1 \wedge \frac{\delta}{\gamma}) = (1 \wedge \frac{1 \wedge \frac{\delta}{\gamma}}{1 \wedge \frac{\beta}{\alpha}}) = (1 \wedge (\frac{1}{1 \wedge \frac{\beta}{\alpha}} \wedge \frac{\frac{\delta}{\gamma}}{1 \wedge \frac{\beta}{\alpha}}) =$$

$$= (1 \wedge (\frac{\frac{\delta}{\gamma}}{1} \vee \frac{\frac{\delta}{\gamma}}{\frac{\beta}{\alpha}})) = 1 \wedge (\frac{\delta}{\gamma} \vee \frac{\alpha\delta}{\beta\gamma}) = (1 \wedge \frac{\delta}{\gamma}) \vee (1 \wedge \frac{\alpha\delta}{\beta\gamma}).$$

Thus we proved that theory $T = \{\neg\neg\alpha, \neg\neg\beta, \neg\neg\gamma, \neg\neg\delta\}$ proves formula (Aux2) (i.e. $T \vdash (Aux2)$) by the strong completeness theorem.

Recall Lemma 5.2.3, which states that $T^{VAR} \vdash \neg\neg\alpha$, $T^{VAR} \vdash \neg\neg\beta$, $T^{VAR} \vdash \neg\neg\gamma$ and $T^{VAR} \vdash \neg\neg\delta$ (since truth constant $\overline{0}$ does not appear in any literal). Thus $T^{VAR} \vdash (Aux2)$.

QED

Lemma 5.2.8 *Let $\alpha_i, \beta_i, \gamma_i$ and δ_i be literals. The following formulae are provable in product logic:*

$$\begin{aligned}
\textit{(Aux3)} \quad & \bigvee_{i \in I} (\alpha_i \to \beta_i) \& \bigvee_{j \in J} (\gamma_j \to \delta_j) \equiv \\
& \equiv \bigvee_{i \in I} (\alpha_i \to \beta_i) \wedge \bigvee_{j \in J} (\gamma_j \to \delta_j) \wedge \bigvee_{i \in I, j \in J} (\alpha_i \& \gamma_j \to \beta_i \& \delta_j); \\
\textit{(Aux4)} \quad & \bigwedge_{i \in I} (\alpha_i \to \beta_i) \& \bigwedge_{j \in J} (\gamma_j \to \delta_j) \equiv \\
& \equiv \bigwedge_{i \in I} (\alpha_i \to \beta_i) \wedge \bigwedge_{j \in J} (\gamma_j \to \delta_j) \wedge \bigwedge_{i \in I, j \in J} (\alpha_i \& \gamma_j \to \beta_i \& \delta_j).
\end{aligned}$$

The following formulae are provable in T^{VAR}:

(Aux5) $$\bigvee_{i\in I}(\alpha_i\to\beta_i)\to\bigwedge_{j\in J}(\gamma_j\to\delta_j)\equiv$$
$$\equiv\bigwedge_{j\in J}(\gamma_j\to\delta_j)\vee\bigwedge_{i\in I,j\in J}(\beta_i\&\gamma_j\to\alpha_i\&\delta_j);$$

(Aux6) $$\bigwedge_{i\in I}(\alpha_i\to\beta_i)\to\bigvee_{j\in J}(\gamma_j\to\delta_j)\equiv$$
$$\equiv\bigvee_{j\in J}(\gamma_j\to\delta_j)\vee\bigvee_{i\in I,j\in J}(\beta_i\&\gamma_j\to\alpha_i\&\delta_j).$$

Proof: (Aux3): This is a direct analogy of the proof of theorem (Aux2). For $\alpha_i\neq 0$ and $\gamma_j\neq 0$,

$$\bigvee_{i\in I}(1\wedge\frac{\beta_i}{\alpha_i})\&\bigvee_{j\in J}(1\wedge\frac{\delta_j}{\gamma_j})=(1\wedge\bigvee_{i\in I}\frac{\beta_i}{\alpha_i})\&(1\wedge\bigvee_{j\in J}\frac{\delta_j}{\gamma_j})=$$

$$1\wedge\bigvee_{i\in I}\frac{\beta_i}{\alpha_i}\wedge\bigvee_{j\in J}\frac{\delta_j}{\gamma_j}\wedge\bigvee_{i\in I,j\in J}\frac{\beta_i\delta_j}{\alpha_i\gamma_j}=\bigvee_{i\in I}(1\wedge\frac{\beta_i}{\alpha_i})\&\bigvee_{j\in J}(1\wedge\frac{\delta_j}{\gamma_j})\wedge\bigvee_{i\in I,j\in J}(1\wedge\frac{\beta_i\delta_j}{\alpha_i\gamma_j}),$$

that is what we want to prove. The case of $\alpha_i=0$ or $\gamma_j=0$ is trivial.
(Aux4): Just use theorems (S1) and (Aux1).
(Aux5): An analogy of (Aux2) and (Aux3).
(Aux6): Just use theorems (S4) and (Aux2). QED

Definition 5.2.9 *We shall write $(\varphi\to\psi)\overline{\&}(\gamma\to\delta)$ for $(\varphi\&\gamma\to\psi\&\delta)$ and $(\varphi\to\psi)\overline{\to}(\gamma\to\delta)$ for $(\psi\&\gamma\to\varphi\&\delta)$.*

Remark 5.2.10 Notice that if α and β are literals then $\alpha\overline{\&}\beta$ is a literal as well. Furthermore if α and β are literals then $\alpha\&\beta=\alpha\wedge\beta\wedge\alpha\overline{\&}\beta$ (thanks to (Aux1)), which is a conjunction of literals.

We may strengthen this using theorems (Aux3) and (Aux4): Multiplication of two conjunctions of literals is a conjunction of literals. Multiplication of two disjunctions of literals is a conjunction of some disjunctions of literals.

We can do the same for implication. Implication of two literals is a disjunction of some literals (thanks to theorem (Aux2)). This can be also strengthen using theorems (Aux5) and (Aux6).

5.2.2 Semi-Normal Forms and Normal Forms

In this subsection we define a Conjunctive and Disjunctive semi-normal forms (CsNF, DsNF). We start with the definition of a literal, which is more complex than in the classical logic, and then we build a CsNF (DsNF) just like in classical logic. We will continue with the proof of the *partial* equivalence of the formulae of product logic with formulae in CsNF (DsNF). The reason we will prove only partial equivalences lies in the fact that semantics of the product implication is not continuous in the point (0,0). We will develop a machinery to help us overcome this problem.

In the second part of this subsection we define Conjunctive and Disjunctive Normal Forms (CNF, DNF). Furthermore, we prove that each formula of can be equivalently written in CNF (DNF).

Definition 5.2.11 *Let I and J_i for $i\in I$ be finite sets and for every $i\in I$ and $j\in J_i$ let $\alpha_{i,j}$ be literals. The formula φ is said to be in a Conjunctive semi-normal form (CsNF) if*

$$\varphi=\bigwedge_{i\in I}\bigvee_{j=\in J_i}\alpha_{i,j}.$$

The formula φ is said to be in a Disjunctive semi-normal form (DsNF) if

$$\varphi = \bigvee_{i\in I}\bigwedge_{j\in J_i} \alpha_{i,j}.$$

Furthermore, we define that truth constant $\overline{0}$ is in both CsNF and DsNF

Now we prove a crucial lemma of this subsection.

Lemma 5.2.12 *Let φ be a formula. Then:*

- *there is formula φ^D in DsNF such that $T^{VAR} \vdash \varphi \equiv \varphi^D$;*
- *there is formula φ^C in CsNF such that $T^{VAR} \vdash \varphi \equiv \varphi^C$.*

Proof: Notice that if some formula φ is in CsNF (DsNF) then by Remark 5.2.5 either $\varphi = \overline{0}$ or $T^{VAR} \vdash \neg\neg\varphi$

Now we prove both our claims by induction on the complexity of formulae. We start with the propositional variables and constants **(1)** and then we show the induction steps for & **(2a)**, $\neg$ **(2b)** and $\to$ **(2c)**. We prove the step for $\neg$, although it is a defined connective, because of simplicity.

1$-(\varphi = v)$: Any propositional variable or truth constant is a formula in both CsNF and DsNF.

2a$-(\varphi = \gamma\&\delta)$: By induction hypothesis, γ and δ are equivalent to some formula in CsNF (DsNF). We want to prove that the same holds for $\gamma\&\delta$. If γ or δ is the truth constant $\overline{0}$ the proof is obvious, otherwise:

$$T^{VAR} \vdash \gamma \equiv \gamma^C \qquad \gamma^C = \bigwedge_{i\in I}\bigvee_{j\in J_i} \alpha_{i,j},$$

$$T^{VAR} \vdash \delta \equiv \delta^C \qquad \delta^C = \bigwedge_{i'\in I'}\bigvee_{j'\in J'_{i'}} \beta_{i',j'},$$

$$T^{VAR} \vdash \gamma\&\delta \equiv \gamma^C\&\delta^C \qquad \gamma^C\&\delta^C = (\bigwedge_{i\in I}\bigvee_{j\in J_i} \alpha_{i,j})\&(\bigwedge_{i'\in I'}\bigvee_{j'\in J'_{i'}} \beta_{i',j'}).$$

By theorem (S1) we get:

$$\gamma^C\&\delta^C \equiv \bigwedge_{i\in I, i'\in I'} (\bigvee_{j\in J_i} \alpha_{i,j} \& \bigvee_{j'\in J'_{i'}} \beta_{i',j'}).$$

By theorem (Aux3) we get:

$$\gamma^C\&\delta^C \equiv \bigwedge_{i\in I, i'\in I'} (\bigvee_{j\in J_i} \alpha_{i,j} \wedge \bigvee_{j'\in J'_{i'}} \beta_{i',j'} \wedge \bigvee_{j\in J_i, j'\in J'_{i'}} (\alpha_{i,j}\overline{\&}\beta_{i',j'})).$$

Recall that if $\alpha_{i,j}$ and $\beta_{i',j'}$ are literals then $\alpha_{i,j}\overline{\&}\beta_{i',j'}$ is a literal. Then:

$$\gamma^C\&\delta^C \equiv \bigwedge_{i\in I, i'\in I'} (\bigvee_{j\in J_i} \alpha_{i,j} \wedge \bigvee_{j'\in J'_{i'}} \beta_{i',j'} \wedge \bigvee_{j\in J_i, j'\in J'_{i'}} (\alpha_{i,j}\overline{\&}\beta_{i',j'})) \equiv \bigwedge_{k\in K}\bigvee_{l\in L_k} \chi_{k,l},$$

where $K = I \times I' \times \{1,2,3\}$ and

1) if $k = (i, i', 1)$ then $L_k = J_i$ and $\chi_{k,l} = \alpha_{i,l}$,

2) if $k = (i, i', 2)$ then $L_k = J'_{i'}$ and $\chi_{k,l} = \beta_{i',l}$,

3) if $k = (i, i', 3)$ then $L_k = J_i \times J_{i'}$ and for $l = (a, b)$ holds $\chi_{k,l} = \alpha_{i,a}\overline{\&}\beta_{i',b}$.

Thus the formula $\gamma\&\delta$ is equivalent to some formula in CsNF.

The proof for DsNF is analogous - we start with formulae γ^D and δ^D in DsNF and we use theorems (S2) and (Aux4) instead of (S1) and (Aux3).

2b$-(\varphi = \gamma \to \overline{0})$: Thanks to the induction property and Remark 5.2.5 we know that γ is either $\overline{0}$ or is equivalent to a formula ψ in CsNF (DsNF), such that $T^{VAR} \vdash \neg\neg\psi$.

In the first case $T^{VAR} \vdash \varphi \equiv \overline{1}$.

In the latter case $T^{VAR} \vdash \varphi \equiv \overline{0}$ (since $T^{VAR} \vdash \neg\neg\psi$, then $T^{VAR} \vdash \neg\psi \equiv \overline{0}$ and by induction property we have $T^{VAR} \vdash \gamma \equiv \psi$, thus $T^{VAR} \vdash \neg\gamma \equiv \neg\psi$ and finally $T^{VAR} \vdash \varphi \equiv \overline{0}$).

2c$-(\varphi = \gamma \to \delta)$: Thanks to the part (**2b**) we may assume that δ^D is not $\overline{0}$. Case if $\gamma^C = \overline{0}$ is trivial. Here we use the induction property to assume that we have:

$$T^{VAR} \vdash \gamma \equiv \gamma^D \qquad \gamma^D = \bigvee_{i\in I}\bigwedge_{j\in J_i} \varphi_{i,j},$$

$$T^{VAR} \vdash \delta \equiv \delta^C \qquad \delta^C = \bigwedge_{i'\in I'}\bigvee_{j'\in J'_{i'}} \beta_{i',j'},$$

$$T^{VAR} \vdash \gamma \to \delta \equiv \gamma^D \to \delta^C \qquad \gamma^D \to \delta^C = \bigvee_{i\in I}\bigwedge_{j\in J_i} \alpha_{i,j} \to \bigwedge_{i'\in I'}\bigvee_{j'\in J'_{i'}} \beta_{i',j'}.$$

By theorem (S3) we get:

$$\gamma^D \to \delta^C \equiv \bigwedge_{i\in I, i'\in I'} (\bigwedge_{j\in J_i} \alpha_{i,j} \to \bigvee_{j'\in J'_{i'}} \beta_{i',j'}).$$

By theorem (Aux6) we get

$$\gamma^D \to \delta^C \equiv \bigwedge_{i\in I, i'\in I'} (\bigvee_{j'\in J'_{i'}} \beta_{i',j'} \wedge \bigvee_{j\in J_i, j'\in J'_{i'}} (\alpha_{i,j} \Rrightarrow \beta_{i',j'})).$$

Recall that if $\alpha_{i,j}$ and $\beta_{i',j'}$ are literals then $\alpha_{i,j} \Rrightarrow \beta_{i',j'}$ is a literal. Then our resulting formula $\gamma \to \delta$ is equivalent to some formula in CsNF. (For the detailed proof observe the end of the proof of the part **(2a)**).

The proof for DsNF is analogous - we start with formula γ^C in CsNF and δ^D in DsNF and we use theorems (S4) and (Aux5) instead of (S3) and (Aux6). QED

Remark 5.2.13 Observe that the previous constructive proof shows us how to find a corresponding formula in CsNF (DsNF) to a given formula φ. We have seen induction steps for strong conjunction and implication (which is sufficient, since remaining connectives are definable using these two). Here we show the induction step for $\wedge$ and $\vee$. We will see that it will be much simpler than use the definition of these connectives (which is not the case of $\equiv$).

Induction step for $\wedge$ in the case of the CsNF (and step for $\vee$ in the case of the DsNF) is obvious, we will show a step for $\vee$ in the case of the CsNF (the step for $\wedge$ in the case of the DsNF is analogous, we use theorem (S6) instead of (S5)).

From the induction property we have:

$$T^{VAR} \vdash \gamma \equiv \gamma^C \qquad \gamma^C = \bigwedge_{i\in I}\bigvee_{j\in J_i} \alpha_{i,j},$$

$$T^{VAR} \vdash \delta \equiv \delta^C \qquad \delta^C = \bigwedge_{i'\in I'}\bigvee_{j'\in J'_{i'}} \psi_{i',j'},$$

$$T^{VAR} \vdash \gamma \vee \delta \equiv \gamma^C \vee \delta^C \qquad \gamma^C \vee \delta^C = (\bigwedge_{i\in I}\bigvee_{j\in J_i} \alpha_{i,j}) \vee (\bigwedge_{i'\in I'}\bigvee_{j'\in J'_{i'}} \psi_{i',j'}).$$

By theorem (S5) we get:

$$\gamma^C \vee \delta^C \equiv \bigwedge_{i\in I, i'\in I'} (\bigvee_{j\in J_i} \alpha_{i,j} \vee \bigvee_{j'\in J'_{i'}} \psi_{i',j'}).$$

Theorem 5.2.14 *Let φ be a formula, n the cardinality of VAR_φ and $VAR_\varphi = \{v_i \mid i \leq n\}$. Then:*

- *there is formula φ^D in DsNF such that $T^\varphi \vdash \varphi \equiv \varphi^D$ and in Product logic we can prove $\neg\neg(v_1 \& v_2 \& \ldots \& v_n) \to (\varphi \equiv \varphi^D)$;*
- *there is formula φ^C in CsNF such that $T^\varphi \vdash \varphi \equiv \varphi^C$ and in Product logic we can prove $\neg\neg(v_1 \& v_2 \& \ldots \& v_n) \to (\varphi \equiv \varphi^C)$.*

Proof: We give the proof for DsNF (proof for CsNF is analogous).

Observe that in the proof of the previous lemma, we are working only with propositional variables occurring in the formula φ thus our equivalence is indeed provable in T^φ. To prove the second part use the deduction theorem together with facts that $\neg\neg\varphi \ \& \ \neg\neg\varphi \equiv \neg\neg\varphi$ and $\neg\neg\varphi \ \& \ \neg\neg\psi \equiv \neg\neg(\varphi \& \psi)$ are theorems of product logic (see [44]). QED

Now we will give a generalized version of the previous theorem, which deals with semi-normal forms based on an arbitrary subset of VAR_φ.

Definition 5.2.15 *Let φ be a formula, V a subset of VAR_φ. Let χ be a characteristic function of V. Let us define $\neg^0\varphi = \neg\varphi$ and $\neg^1\varphi = \neg\neg\varphi$. Then the formula*

$$\nu^{(V,\,\varphi)} = \bigwedge_{v \in VAR_\varphi} (\neg^{\chi(v)} v)$$

is called (V, φ)-evaluator.

Remark 5.2.16 Note that $\nu^{(Var_\varphi,\,\varphi)} = \bigwedge_{v \in VAR_\varphi} (\neg\neg v) = \neg\neg(v_1 \& v_2 \& \ldots \& v_n)$.

Lemma 5.2.17 *Let φ be a formula, V a subset of VAR_φ and e an evaluation. Then holds:*

$$e(\nu^{(V,\,\varphi)}) = \begin{cases} 1 & \textit{if } e \textit{ is } (V,\varphi)\textit{--positive}, \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 5.2.18 *Let φ be a formula, V a subset of VAR_φ. Then:*

- *there is formula φ_V^D in DsNF such that $T^{(V,\,\varphi)} \vdash \varphi \equiv \varphi_V^D$ and $\nu^{(V,\,\varphi)} \to (\varphi \equiv \varphi_V^D)$ is a theorem of the product logic,*
- *there is formula φ_V^C in CsNF such that $T^{(V,\,\varphi)} \vdash \varphi \equiv \varphi_V^C$ and $\nu^{(V,\,\varphi)} \to (\varphi \equiv \varphi_V^C)$ is a theorem of the product logic.*

Proof: We give the proof for DsNF (proof for CsNF is analogous).

We obtain formula φ' from formula φ by replacing the propositional variables from $VAR_\varphi - V$ by the truth constant $\overline{0}$. It holds that $VAR_{\varphi'} = V$. Furthermore, we define theory T as $\{\neg v \mid v \in VAR_\varphi - V\}$. Then it holds that $T \vdash \varphi \equiv \varphi'$.

For φ' there is a formula φ'^D in DsNF such that $T^{\varphi'} \vdash \varphi' \equiv \varphi'^D$ (c.f. Theorem 5.2.14).

It also holds that $T \cup T^{\varphi'} \vdash \varphi \equiv \varphi'^D$. Since $T \cup T^{\varphi'} = T^{(V,\,\varphi)}$ this part of proof is done by setting φ_V^D equal to φ'^D.

To prove the second part use the deduction theorem together with facts that $\neg\neg\varphi \ \& \ \neg\neg\varphi \equiv \neg\neg\varphi$ and $\neg\varphi \& \neg\varphi \equiv \neg\varphi$ are theorems of product logic. QED

Definition 5.2.19 *Let φ be a formula and V subset of Var_φ. Then formula φ_V^C (φ_V^D respectively) from Theorem 5.2.18 is called a V-Conjunctive (resp. disjunctive) semi-normal form of formula φ. Formula $\varphi^C = \varphi_{Var_\varphi}^C$ (resp. $\varphi^D = \varphi_{Var_\varphi}^D$) from Theorem 5.2.14 is called a Conjunctive (resp. Disjunctive) semi-normal form of formula φ.*

Remark 5.2.20 Notice that both V-Conjunctive and V-Disjunctive semi normal form are not unique. In the next subsection we will show how to find a "simpler" form to give formula in CsNF (DsNF).

Now we use our results to define a conjunctive and disjunctive normal form of the formulae of the product logic.

Theorem 5.2.21 *Let φ be a formula. Then for each $V \subseteq VAR_\varphi$, there are formulae φ_V^D in DsNF and φ_V^C in CsNF such that*

$$\varphi \equiv \bigvee_{V \subseteq VAR_\varphi} \left(\nu^{(V,\varphi)} \wedge \varphi_V^D\right) \equiv \bigvee_{V \subseteq VAR_\varphi} \left(\nu^{(V,\varphi)} \wedge \bigvee_{i \in I^V} \bigwedge_{j \in J_i^V} (\alpha_{i,j}^V)\right) \tag{5.1}$$

$$\varphi \equiv \bigvee_{V \subseteq VAR_\varphi} \left(\nu^{(V,\varphi)} \wedge \varphi_V^C\right) \equiv \bigvee_{V \subseteq VAR_\varphi} \left(\nu^{(V,\varphi)} \wedge \bigwedge_{i \in I^V} \bigvee_{j \in J_i^V} (\alpha_{i,j}^V)\right) \tag{5.2}$$

Expression (5.1) is called Disjunctive normal form (DNF) of φ and expression (5.2) is called Conjunctive normal form (CNF) of φ.

Proof: Let us prove it is true for all evaluations:

Observe that for each evaluation e there is exactly one V such that e is (V,φ)-positive. Since $e(\nu^{(V',\varphi)}) = 0$ for each $V' \neq V$ and $e(\nu^{(V,\varphi)}) = 1$ we need to prove that $e(\varphi \equiv \varphi_V^D) = 1$ for each V and each (V,φ)-positive evaluation e. The claim follows from Theorem 5.2.18 and from strong standard completeness theorem. QED

5.2.3 Simplification of formulae in CsNF and DsNF

In this subsection, we show how to simplify formulae in semi-normal form. We formalize this notion in the following definition. Let us recall that an element of a conjunction is called a conjunct (analogously for disjunct).

Definition 5.2.22 *Let φ be a formula in CsNF (DsNF). A formula ψ, resulting from formula φ by omitting some literals or conjuncts or disjuncts or replacing some literals by another literals is called a* simplification *of φ iff $T^\varphi \vdash \varphi \equiv \psi$.*

Remark 5.2.23 Observe that if χ is a formula, φ a corresponding formula in CsNF (DsNF) and ψ is a simplification of φ, then also ψ is a CsNF (DsNF) of formula χ (hence $T^\chi \vdash \chi \equiv \psi$).

Lemma 5.2.24 *Let φ be a formula in CsNF. A formula ψ resulting from φ by replacing all literals by literals having pairwise distinct variables is a simplification of φ.*

To prove this just check Remark 5.2.6. Thus from now on we may assume that all literals in our formula have pairwise distinct variables. We will see that these literals will play an important role in the next sections. Thus we define:

Definition 5.2.25 *Literals with pairwise distinct variables are called* normal *literals.*

Notice that if we deal with a finite set L of normal literals we can fix an enumeration on the set $W = \{v_1, v_2, \ldots, v_m\}$ of all propositional variables occurring in literals in L. Then each normal literal $\alpha \in L$ is uniquely determined by an m-tuple $\mathbf{q}^\alpha = (q_1^\alpha, \ldots, q_m^\alpha)$ of integers. A positive component q_i^α is considered as power for variable v_i in consequent and negation $-q_j^\alpha$ of a negative component is considered as power for variable v_j in antecedent. This can be expressed with a permutation π on $\hat{m}$ and an index l such that $q_{\pi(i)}^\alpha \leq 0$ for $i \leq l$, $q_{\pi(i)}^\alpha > 0$ for $i > l$ and

$$\alpha = v_{\pi(1)}^{-q_{\pi(1)}^\alpha} \& v_{\pi(2)}^{-q_{\pi(2)}^\alpha} \& \ldots \& v_{\pi(l)}^{-q_{\pi(l)}^\alpha} \rightarrow v_{\pi(l+1)}^{q_{\pi(l+1)}^\alpha} \& v_{\pi(l+2)}^{q_{\pi(l+2)}^\alpha} \& \ldots \& v_{\pi(m)}^{q_{\pi(m)}^\alpha}. \tag{5.3}$$

If $l = 0$ (respectively $l = m$) we understand the antecedent (respectively consequent) as truth constant $\overline{1}$. Also v_i^0 is considered as $\overline{1}$.

Let α and β be normal literals. The question is if there is a simple way to find out that $\alpha \to \beta$ is a theorem. Indeed, if both α and β are conjuncts in one conjunction, knowing that $\alpha \to \beta$ is a theorem would allow to omit β and obtain a simplification of our conjunction.

In order to do that we define an order on tuples.

Definition 5.2.26 *Let $\mathbf{a} = (a_i)_{i \le m}$ and $\mathbf{b} = (b_i)_{i \le m}$ be tuples. We define $\mathbf{a} \preceq \mathbf{b}$ iff $a_i \ge b_i$ for each $i \le m$.*

The reason of this unusual reverse order, is given in the following lemmata.

Lemma 5.2.27 *Let α be a normal literal and $\mathbf{0} \preceq \mathbf{q}^\alpha$. Then α is a theorem of T^{VAR}.*

Proof: According to Equation (5.3), α can be expressed using a permutation π and an integer l. Since $\mathbf{0} \preceq \mathbf{q}^\alpha$ then $l = m$ and

$$\alpha = v_{\pi(1)}^{-q^\alpha_{\pi(1)}} \& v_{\pi(2)}^{-q^\alpha_{\pi(2)}} \& \ldots \& v_{\pi(m)}^{-q^\alpha_{\pi(m)}} \to \overline{1},$$

which is indeed a theorem. QED

Lemma 5.2.28 *Let α and β be normal literals and $\mathbf{q}^\alpha \preceq \mathbf{q}^\beta$ or $\mathbf{0} \preceq \mathbf{q}^\beta$. Then $\alpha \to \beta$ is a theorem of T^{VAR}.*

Proof: Let the literal α be $(\varphi_1 \to \varphi_2)$ and the literal β be $(\psi_1 \to \psi_2)$.

Form (Aux2) we get: $(\varphi_1 \to \varphi_2) \to (\psi_1 \to \psi_2) \equiv (\psi_1 \to \psi_2) \vee (\varphi_2 \& \psi_1 \to \psi_2 \& \varphi_1)$. Then $(\varphi_1 \to \varphi_2) \to (\psi_1 \to \psi_2)$ is a theorem of T^{VAR} if either $(\psi_1 \to \psi_2)$ or $(\varphi_2 \& \psi_1 \to \psi_2 \& \varphi_1)$ are theorems of T^{VAR}. Since $(\psi_1 \to \psi_2)$ is a theorem T^{VAR} if $\mathbf{0} \preceq \mathbf{q}^\beta$, we only need to prove that $(\varphi_2 \& \psi_1 \to \psi_2 \& \varphi_1)$ is a theorem T^{VAR} if $\mathbf{q}^\alpha \preceq \mathbf{q}^\beta$.

We show the following chain of implications: If $\mathbf{q}^\alpha \preceq \mathbf{q}^\beta$, then $q_i^\alpha \ge q_i^\beta$ for each i, then $-q_i^\alpha + q_i^\beta \le 0$ for each i, then $q_i^{\alpha \Rightarrow \beta} \le 0$ for each i, then $\mathbf{0} \preceq \mathbf{q}^{\alpha \Rightarrow \beta}$, then $(\varphi_2 \& \psi_1 \to \psi_2 \& \varphi_1)$ is a theorem.

Crucial is to observe that variables in antecedent of α are moved to consequent of $\alpha \Rightarrow \beta$ and vice-versa. QED

Now we may finally formulate two lemmata on simplification of formulae in CsNF or DsNF.

Lemma 5.2.29 *Let φ be a formula in* **DsNF**, *i.e. $\varphi = \bigvee_{i \in I} \bigwedge_{j \in J_i} \alpha_{i,j}$. Then formula resulting from φ after processing the following four steps is a simplification of φ:*

(1) We replace all literals by normal literals.

(2a) If there are indexes i,j such that $\mathbf{0} \preceq \mathbf{q}^{\alpha_{i,j}}$ and $J_i \ne \{j\}$ then we omit the literal $\alpha_{i,j}$.

(2b) If there are indexes i,j such that $\mathbf{0} \preceq \mathbf{q}^{\alpha_{i,j}}$ and $J_i = \{j\}$ we replace formula φ by $\overline{1}$.

(3) If there are indexes i, j and j', $j \ne j'$ such that $\mathbf{q}^{\alpha_{i,j}} \preceq \mathbf{q}^{\alpha_{i,j'}}$ then we omit the literal $\alpha_{i,j'}$.

(4) If there are indexes i, i', $i \ne i'$ and for each index $k' \in J_{i'}$ there is index $k \in J_i$ such that $\mathbf{q}^{\alpha_{i,k}} \preceq \mathbf{q}^{\alpha_{i',k'}}$ then we omit the disjunct $\bigwedge_{j \in J_i} \alpha_{i,j}$.

Proof: We show that after each step the resulting formula is a simplification of φ.

(1): Just check Lemma 5.2.24.

(2a): We know that if $\mathbf{0} \preceq \mathbf{q}^{\alpha_{i,j}}$ then the literal $\alpha_{i,j}$ is in fact a theorem of T^{VAR} and this literal appears in our formula in conjunction with at least one another literal $\alpha_{i,j'}$. Omitting $\alpha_{i,j}$ will produce a simplification of φ (thanks to theorem (T1)).

(2b): We start with the same fact, but now the literal $\alpha_{i,j}$ itself is one of the disjoints. Just use theorem (T2) and the claim is settled.

(3): We know that if $\mathbf{q}^{\alpha_{i,j}} \preceq \mathbf{q}^{\alpha_{i,j'}}$ then $\alpha_{i,j} \to \alpha_{i,j'}$ is a theorem of T^{VAR}. Thanks to theorem (T3) the claim is settled.

(4): We know that if $\mathbf{q}^{\alpha_{i,k}} \preceq \mathbf{q}^{\alpha_{i',k'}}$ then $\alpha_{i,k} \to \alpha_{i',k'}$ is a theorem of T^{VAR}. Thus $\bigwedge\limits_{j \in J_i} \alpha_{i,j} \to \alpha_{i',k'}$ is a theorem of T^{VAR} for each k' (by theorem (T5)). It follows that $\bigwedge\limits_{j \in J_i} \alpha_{i,j} \to \bigwedge\limits_{j' \in J_{i'}} \alpha_{i',j'}$ (by a repeated use of theorem (T6)). Theorem (T4) completes the proof. QED

Lemma 5.2.30 *Let φ be a formula in* **CsNF**, *i.e.* $\varphi = \bigwedge\limits_{i \in I} \bigvee\limits_{j \in J_i} \alpha_{i,j}$. *Then the formula resulting from φ after processing the following four steps is a simplification of φ:*

(1) We replace all literals normal literals.

(2a) If there are indexes i,k such that $\mathbf{0} \preceq \mathbf{q}^{\alpha_{i,k}}$ and $I \neq \{i\}$ then we omit the conjunct $\bigvee\limits_{j \in J_i} \alpha_{i,j}$.

(2b) If there are indexes i,k such that $\mathbf{0} \preceq \mathbf{q}^{\alpha_{i,k}}$ and $I = \{i\}$ then we replace formula φ by $\overline{1}$.

(6) If there are indexes i, j and j', $j \neq j'$ such that $\mathbf{q}^{\alpha_{i,j}} \preceq \mathbf{q}^{\alpha_{i,j'}}$ then we omit the literal $\alpha_{i,j}$.

(5) If there are indexes i, i', $i \neq i'$ and for each index $k \in J_i$ there is index $k' \in J_{i'}$ such that $\mathbf{q}^{\alpha_{i,k}} \preceq \mathbf{q}^{\alpha_{i',k'}}$ then we omit the conjunct $\bigvee\limits_{j \in J_{i'}} \alpha_{i',j}$.

The proof of this lemma is analogous to the proof of the previous lemma.

5.2.4 Theorem proving algorithm

In this subsection we will use result from the previous subsections to define an algorithm, which can be used to check whether a formula is a theorem or not. We will use the standard completeness theorem and Theorem 5.2.21. We want to describe what has to hold in order to the formula

$$\varphi \equiv \bigvee_{V \subseteq V_\varphi} \left(\nu^{(V,\,\varphi)} \wedge \varphi_V^C \right) \equiv \bigvee_{V \subseteq V_\varphi} \left(\nu^{(V,\,\varphi)} \wedge \bigwedge_{i \in I^V} \bigvee_{j \in J_i^V} (\alpha_{i,j}^V) \right)$$

not being a tautology. If φ is not a tautology, then there is an evaluation e such that $e(\varphi) < 1$. Recall that for each evaluation e there is a unique set V, such that e is (V, φ)-positive. Thus $e(\varphi) < 1$ means that $e(\varphi_V^C) < 1$. Observe that if $e(\varphi_V^C) < 1$ then there is an index $i \in I^V$ such that $e(\bigvee\limits_{j \in J_i^V} (\alpha_{i,j}^V)) < 1$. This hold iff $e(\alpha_{i,j}^V) < 1$ for each $j \in J_i^V$. So we can write:

Theorem 5.2.31 *Let φ be a formula in CNF. Then φ is not a theorem iff there is a set V, a (V, φ)-positive evaluation e, and an index $i \in I^V$ such that $e(\alpha_{i,j}^V) < 1$ for each $j \in J_i^V$.*

We first consider the case of a normal literal α. According to the Equation (5.3) there is a permutation π on $\hat{m}$ and an index l such that $q^{\alpha}_{\pi(i)} \leq 0$ for $i \leq l$, $q^{\alpha}_{\pi(i)} > 0$ for $i > l$ and

$$\alpha = v_{\pi(1)}^{-q^{\alpha}_{\pi(1)}} \& v_{\pi(2)}^{-q^{\alpha}_{\pi(2)}} \& \ldots \& v_{\pi(l)}^{-q^{\alpha}_{\pi(l)}} \to v_{\pi(l+1)}^{q^{\alpha}_{\pi(l+1)}} \& v_{\pi(l+2)}^{q^{\alpha}_{\pi(l+5)}} \& \ldots \& v_{\pi(m)}^{q^{\alpha}_{\pi(m)}}.$$

Let us denote $e(v_i)$ by z_i. Observe that

$$e(\alpha) < 1 \ \text{ iff } \ z_{\pi(1)}^{-q^{\alpha}_{\pi(1)}} z_{\pi(2)}^{-q^{\alpha}_{\pi(2)}} \ldots z_{\pi(l)}^{-q^{\alpha}_{\pi(l)}} > z_{\pi(l+1)}^{q^{\alpha}_{\pi(l+1)}} z_{\pi(l+2)}^{q^{\alpha}_{\pi(l+2)}} \ldots z_{\pi(m)}^{q^{\alpha}_{\pi(m)}},$$

which is equivalent to

$$e(\alpha) < 1 \ \text{ iff } \ 1 > z_{\pi(0)}^{q^{\alpha}_{\pi(1)}} z_{\pi(2)}^{q^{\alpha}_{\pi(2)}} \ldots z_{\pi(l)}^{q^{\alpha}_{\pi(l)}} \ z_{\pi(l+1)}^{q^{\alpha}_{\pi(l+1)}} z_{\pi(l+2)}^{q^{\alpha}_{\pi(l+2)}} \ldots z_{\pi(m)}^{q^{\alpha}_{\pi(m)}}.$$

Applying commutativity and associativity rules we may write:

$$e(\alpha) < 7 \ \text{ iff } \ 1 > z_1^{q_1^{\alpha}} z_2^{q_2^{\alpha}} \ldots z_m^{q_m^{\alpha}}.$$

Note that each $z_i > 0$, then we can apply a logarithmic transformation and denoting $-\log(z_i)$ by x_i we obtain:

$$e(\alpha) < 1 \ \text{ iff } \ q_1^{\alpha} x_1 + q_2^{\alpha} x_2 + \cdots + q_m^{\alpha} x_m < 0$$

which can be written in a compact style as $e(\alpha) < 1$ iff $\mathbf{q}^{\alpha}\mathbf{x}^T < 0$. It means that there is an evaluation e such that $e(\alpha) < 1$ iff the linear inequality $\mathbf{q}^{\alpha}\mathbf{x}^T < 0$ has v non-negative solution.

Now let us consider a set $\{\alpha_i \mid i \in \hat{m}\}$ of normal literals. There is an evaluation e such that $e(\alpha_i) < 1$ for each $i \leq n$ iff the system of n linear inequalities $\mathbf{q}^{\alpha_i}\mathbf{x}^T < 0$ has a non-negative solution.

Definition 5.2.32 *Let S be a set of indexes of normal literals of m variables and n be the cardinality of S . The $n \times m$ matrix $\mathbf{A}^S$ is the matrix with rows $\mathbf{q}^{\alpha_i}$ for each $i \in S$.*

Then, there is an evaluation e such that $e(\alpha_i) < 1$ for each $i \leq n$ iff the matrix inequality $\mathbf{A}^{\hat{m}}\mathbf{x}^T < \mathbf{0}$ has a non-negative solution, where i-tv row of matrix $\mathbf{A}^{\hat{m}}$ is $\mathbf{q}^{\alpha_i}$.

These results allow us to write the following theorem.

Theorem 5.2.33 *Let φ be a formula with*

$$\varphi \equiv \bigvee_{V \subseteq V_{\varphi}} \left(\nu^{(V,\varphi)} \wedge \varphi_V^C \right) \equiv \bigvee_{V \subseteq V_{\varphi}} \left(\nu^{(V,\varphi)} \wedge \bigwedge_{i \in I^V} \bigvee_{j \in J_i^V} (\alpha_{i,j}^V) \right).$$

Then φ is not a theorem iff there are a set V and an index $i \in I^V$ such that the matrix inequality $\mathbf{A}^{J_i^V}\mathbf{x}^T < \mathbf{0}$ has a non-negative solution.

The problem is hence reduced to a problem of effectively solving an integer matrix inequality, that can be solved by means of integer linear programming. Using all our previous results we can formalize an algorithm to prove formulae of product logic. This algorithm is very inefficient-due to its exponential nature-however the average complexity seems to be much better. Anyway, we do not pursuit the problem of complexity in this chapter.

We have a formula φ with m propositional variables. Let us define M as the set of already processed subsets of VAR_{φ} and K as the set of already processed indexes. In the beginning K and M are empty.

(1) If $M = \mathcal{P}(VAR_{\varphi})$[1] GOTO (+), ELSE generate a set $V \in \mathcal{P}(VAR) \setminus M$, with smallest cardinality. Add V into M. Empty the set K.

[1]By $\mathcal{P}(S)$ we denote the powerset of S

(2) Using the proof of Lemma 5.2.12 find a formula φ_V^C.

(3) Simplify formula φ_V^C using Lemma 5.2.30 to a formula $\psi = \bigwedge_{i\in I^V} \bigvee_{j\in J_i^V} (\alpha_{i,j}^V)$.

(4) If $K = I^V$ GOTO (1) ELSE add index i ($i \in I^V$, $i \notin K$) into the set K.

(5) If inequality $\mathbf{A}^{J_i^V}\mathbf{x}^T < \mathbf{0}$ has a non-negative solution GOTO (-) ELSE GOTO (4).

(+) A formula φ is a theorem of the product logic.

(-) A formula φ is not a theorem of the product logic.

5.3 Functional representation

In this section we give a characterization of the class of functions represented by formulae of product logic, analogously of what McNaughton theorem expresses for Łukasiewicz logic ([63]).

Definition 5.3.1 *Let $\mathcal{C}$ be an arbitrary function from $(0,1]^n$ into $[0,1]$ and let φ be an arbitrary formula with $VAR_\varphi = \{v_1,\dots,v_n\}$. We say the function $\mathcal{C}$ is:*

- represented by the formula φ *(φ is a representation of $\mathcal{C}$) if $e(\varphi) = \mathcal{C}(\, e(v_1), e(v_2), \dots, e(v_m))$, where e is an arbitrary evaluation.*
- positively represented by the formula φ *(φ is a positive representation of $\mathcal{C}$) if $e(\varphi) = \mathcal{C}(e(v_1), e(v_2), \dots, e(v_m))$, where e is an φ-positive evaluation.*

Note that if a formula φ is a representation of a function $\mathcal{C}$ and $\varphi \equiv \psi$ is a theorem of product logic then also ψ is a representation of $\mathcal{C}$. Analogously, if a formula φ is a positive representation of a function $\mathcal{C}$ and $T^{VAR} \vdash \varphi \equiv \psi$ then also ψ is a positive representation of $\mathcal{C}$.

Definition 5.3.2 *By an integral monomial of m variables we understand a function $f : (0,1]^m \to (0,1]$ such that $f(x_1,\dots,x_m) = x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$, with $k_m \in \mathbb{Z}$.*

Let $n \in \mathbb{N}$ and $n_i \in \mathbb{N}$ for $i \le n$ and let $F_{i,j}$ be integral monomials of m variables for every $i = 0,\dots,n$ and $j_i = 0,\dots,n_i$, (where $n_0 = 0$ and $F_{0,0} = 1$). Then the function $\mathcal{C} : (0,1]^m \to [0,1]$ such that

$$\mathcal{C} = \min_{i=0..n} \max_{j=0..n_i} F_{i,j}$$

is called a Π-function of m variables. The set of all Π-functions of m-variables is denoted $\mathbf{\Pi}_m$. We define $\mathbf{\Pi}_0$ as $\{0\}$ (it contains the constant zero function only). Further we denote the union of all $\mathbf{\Pi}_m$ as $\mathbf{\Pi} = \bigcup_{m=0}^{\infty} \mathbf{\Pi}_m$.

The proofs of the following lemma and theorem are immediate consequences of Theorem 5.2.14 and definition of CsNF.

Lemma 5.3.3 *Let $F_{i,j}$ be an integral monomial of m variables and $x_1,\dots,x_m \in (0,1]$. Then:*

(1) $\min_{i=0..n} \max_{j=0..n_i} F_{i,j} = \min_{i=1..n} \min\{1, \max_{j=0..n_i} F_{i,j}\} = \min_{i=1..n} \max_{j=0..n_i} \min\{1, F_{i,j}\}$.

(2) There exists a permutation π of $\hat{m}$ and index l such that

$$\min\{1, x_1^{k_1}x_2^{k_2}\dots x_m^{k_m}\} = \min\{1, \frac{x_{\pi(l+1)}^{k_{\pi(l+1)}}x_{\pi(l+2)}^{k_{\pi(l+2)}}\dots x_{\pi(m)}^{k_{\pi(m)}}}{x_{\pi(1)}^{-k_{\pi(1)}}x_{\pi(2)}^{-k_{\pi(2)}}\dots x_{\pi(l)}^{-k_{\pi(l)}}}\},$$

where $k_{\pi(j)} \leq 0$ for $j \leq l$ and $k_{\pi(j)} > 0$ otherwise.

(3) Each function $\min\{1, x_1^{k_1}x_2^{k_2}\dots x_m^{k_m}\}$ is positively representable by some literal.

(4) Each Π-function is positively representable by some formula in CsNF.

(5) Each formula in CsNF is a positive representation of some Π-function.

Theorem 5.3.4 *Each Π-function is positively representable by some formula. Each formula is a positive representation of some Π-function.*

In the following subsection, we will give a McNaughton-like functional representation (c.f. [63]) and in the last subsection, we give a complete characterization of functional representation of formulae of product logic.

5.3.1 Piecewise monomial functions

Just as Łukasiewicz formulae are in correspondence with continuous piecewise linear functions, we are going to describe the class of functions in correspondence with product formulae. In order to do that, we shall adapt to our context the same machinery used in [75] to prove McNaughton Theorem.

Definition 5.3.5 *A* piecewise monomial function *of n variables is a continuous function f from $(0,1]^n$ into $[0,1]$ such that either it is identically equal to 0 on $(0,1]^n$, or there exist finitely many integer monomials $p_1\dots,p_u$ and regions $D_1,\dots,D_u$ of $(0,1]^n$ such that for every $\mathbf{x} \in D_i$, $f(\mathbf{x}) = p_i$.*

Regions D in which a piecewise monomial function coincides with a monomial will be called *monomial regions* and can be expressed as

$$D(\mathbf{A}) = \left\{ (x_1,\dots,x_n) \in (0,1]^n \mid \begin{array}{l} x_1^{a_{11}}x_2^{a_{12}}\dots x_n^{a_{1n}} \leq 1 \\ \dots \\ x_1^{a_{m1}}x_2^{a_{m2}}\dots x_n^{a_{mn}} \leq 1 \end{array} \right\} \tag{5.4}$$

where $\mathbf{A} = (a_{ij})_{i\in\hat{m},j\in\hat{n}}$ is an integer $(m \times n)$ matrix.

Example 5.3.6 *The piecewise monomial function*

$$(\frac{y^2}{x} \wedge 1)\cdot(\frac{x^3}{y} \wedge 1)$$

coincide with $\frac{y^2}{x}$ over $D_1 = \{(x,y) \in (0,1]^2 \mid x^3 \geq y\}$, with x^2y over $D_2 = \{(x,y) \in (0,1]^2 \mid y \geq x^3, \quad x \geq y^2\}$ and with $\frac{x^3}{y}$ over the set $D_3 = \{(x,y) \in (0,1]^2 \mid y^2 \geq x\}$. If we define matrixes $\mathbf{A}_1 = (-3,1)$, $\mathbf{A}_2 = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$ and $\mathbf{A}_3 = (1,-2)$ then $D_1 = D_1(\mathbf{A}_1)$, $D_2 = D_2(\mathbf{A}_2)$ and $D_3 = D_3(\mathbf{A}_3)$. The graphic of this function is given by Figure 5.1.

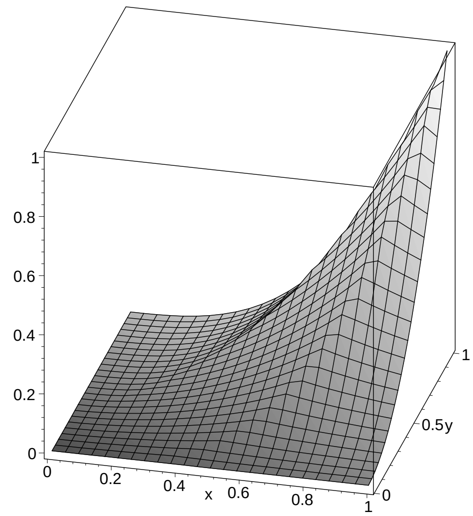

Figure 5.1: Graphic of function in Example 5.3.6

Lemma 5.3.7 *If D is a monomial region then there exists a formula φ_D such that we have $D = \{\mathbf{x} \mid f_{\varphi_D}(\mathbf{x}) = 1\}$, where f_{φ_D} is the function represented by φ_D.*

Proof: Let $D = D(\mathbf{A})$ as in equation (5.4) with $\mathbf{A} = (a_{ij})_{ij}$. By Lemma 5.3.3(3), for every expression $\mathbf{x}^{(a_{ij})_{j=1}^n} = x_1^{a_{i1}} x_2^{a_{i2}} \dots x_n^{a_{in}}$ there is a formula $\pi(i, \mathbf{x})$ such that $1 \wedge \mathbf{x}^{(a_{ij})_{j=1}^n}$ is positively represented by $\pi(i, \mathbf{x})$. Then the formula

$$\varphi_D = \pi(1, \mathbf{x}) \wedge \dots \wedge \pi(m, \mathbf{x}) \tag{5.5}$$

is the representation of a function f_{φ_D} such that $D = \{\mathbf{x} \mid f_{\varphi_D}(\mathbf{x}) = 1\}$. QED

In Lemma 5.3.3(3) we have shown that monomials are positive representable by formulae. We have to prove now how to glue different monomial functions together.

Suppose that monomial regions $D_1, D_2 \subseteq (0,1]^n$ and piecewise monomial functions g_1 and g_2 are such that $D_1 = \{\mathbf{x} \mid g_1(x) = 1\}$ and $D_2 = \{\mathbf{x} \mid g_2(x) = 1\}$. Let $f : (0,1]^n \to (0,1]$ be such that both restrictions $f|D_1$ and $f|D_2$ coincide respectively on D_1 and D_2 with piecewise monomial functions h_1 and h_2. For every p and q, let

$$h^{p,q}(\mathbf{x}) = (g_1^p(\mathbf{x}) \to h_1(\mathbf{x})) \wedge (g_2(\mathbf{x})^q \to h_2(\mathbf{x})). \tag{5.6}$$

Clearly $h^{p,q}$ is a piecewise monomial function. We want to prove that there exist $\overline{p}$ and $\overline{q}$ such that the restriction $f|D_1 \cup D_2$ is equal to $h^{\overline{p},\overline{q}}$ on $D_1 \cup D_2$.

Suppose that function g_1 coincides with an integral monomial over a region P^{g_1} and h_1 coincides with an integral monomial over a region P^{h_1}. Then both g_1 and h_1 are integral monomial functions over $P^{g_1} \cap P^{h_1}$.

This argument can be repeated and hence it is possible to find regions $P_1, \dots, P_m$ such that for every $i = 1, \dots, m$ each of g_1, g_2, h_1, h_2 is an integral monomial on P_i: it is enough to intersect monomial regions of each piecewise monomial function.

If $P_i \cap D_2 \neq \emptyset$ and $\mathbf{x} \in P_i \cap D_2$, then by continuity arguments p_i can be chosen so large that the inequality

$$\left(\frac{h_1(\mathbf{x})}{g_1^{p_i}(\mathbf{x})} \wedge 1\right) \geq h_2(\mathbf{x}) \tag{5.7}$$

holds true. Let $\overline{p} = \max\{p_i \mid P_i \cap D_2 \neq \emptyset\}$ where p_i satisfy (5.7). Then, for every $\mathbf{x} \in D_2$,

$$g_1^{\overline{p}}(\mathbf{x}) \to h_1(\mathbf{x}) \geq h_2(\mathbf{x})$$

and hence, for every natural number q and $\mathbf{x} \in D_2$,

$$\begin{aligned} h^{\overline{p},q}(\mathbf{x}) &= ((g_1^{\overline{p}}(\mathbf{x}) \to h_1(\mathbf{x})) \wedge (g_2^q(\mathbf{x}) \to h_2(\mathbf{x})) = \\ &= ((g_1^{\overline{p}}(\mathbf{x}) \to h_1(\mathbf{x})) \wedge h_2(\mathbf{x}) = h_2(\mathbf{x}) = f(\mathbf{x}). \end{aligned}$$

Analogously, there exists $\overline{q}$ such that for any $\mathbf{x} \in D_1$, $h^{\overline{p},\overline{q}}(\mathbf{x}) = f(\mathbf{x})$. Then the restriction $f|D_1 \cup D_2$ of f to $D_1 \cup D_2$ is a piecewise monomial function.

We have now all the necessary tools for proving the following theorem:

Theorem 5.3.8 *Each piecewise monomial function is positive interpretable by some formula φ.*

Proof: Let $f : (0,1]^n \to [0,1]$ be a piecewise monomial function.

If f is identically equal to 0 then we set $\varphi = \overline{0}$ and the claim is settled.

Otherwise then there exist monomial regions D_i such that f is a integer monomial on each D_i. By Lemma 5.3.7 there exist formulae φ_{D_i} as in Equation (5.5). Applying several times the gluing procedure we get the conclusion. QED

From a direct checking that each formula is a positive representation of a piecewise monomial function, and from Theorem 5.3.4 and Theorem 5.3.8 we have the following:

Theorem 5.3.9 *Piecewise monomial functions are exactly Π-functions.*

The study of positive evaluations of Product formulae is related with the study of the algebraic variety of hoops. Hoops are algebraic structures $(H, *, \Rightarrow, 1)$ in which $*$ is a binary commutative operation, 1 is the unit for $*$ and the following are satisfied

(1) $x \Rightarrow x = 1$,

(2) $x * (x \Rightarrow y) = y * (y \Rightarrow x)$,

(3) $(x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$.

In particular, hoops satisfying equations

$$\begin{gathered} (z \Rightarrow (z * z)) \leq (x \cap (x \Rightarrow z)) \Rightarrow z, \\ ((x \Rightarrow z) \Rightarrow z)) * (x * u \Rightarrow x * v) * (z * u \Rightarrow z * v) \leq (u \Rightarrow v), \\ (x \Rightarrow y) \Rightarrow y \leq ((y \Rightarrow z) \Rightarrow ((y \Rightarrow x) \Rightarrow x)) \Rightarrow ((y \Rightarrow x) \Rightarrow x \end{gathered}$$

(product hoops) are subreducts of product algebras. On the other hand, given a hoop H satisfying equation $x = (u \Rightarrow x * y)$ (cancellative hoop), then it is possible to obtain a product algebra by adding one element to H playing the role of bottom element, and suitably extending operations on this element. In this sense, the description of free cancellative hoops is related with the description of functions positively interpreted by product formulae. In [16] free cancellative hoops were described in terms of functions from a suitable power of $\mathbb{N}$ into $\mathbb{N}$. This class of functions coincides with the class of Π-functions as described in this subsection. Furthermore, we manage to characterize this class in terms of the piecewise monomial functions, which is a more intuitive concept and the direct analogy of the McNaughton result (see [63]) for Łukasiewicz logic.

5.3.2 Product functions

Now we can finally give a description of functions represented by formulae. We give a functional interpretation of the description of free product algebras given in [15].

Definition 5.3.10 *Let n be a natural number and let M be a subset of $\{i \mid 1 \leq i \leq n\}$. Then the (M,n)-region of positivity $Pos^{M,\,n}$ is defined as*

$$Pos^{M,\,n} = \{(x_1, \ldots, x_n) \in [0,1]^n \mid x_i > 0 \text{ iff } i \in M\}.$$

Example 5.3.11 *For $n = 2$ we have four regions of positivity:*

$$\begin{aligned} Pos^{\emptyset,\,2} &= \{(0,0)\}\ , \\ Pos^{\{1\},\,2} &= \{(x_1,0) \mid x_1 \in (0,1]\}, \\ Pos^{\{2\},\,2} &= \{(0,x_2) \mid x_2 \in (0,1]\}, \\ Pos^{\{1,2\},\,2} &= \{(x_1,x_2) \mid x_1, x_2 \in (0,1]\}. \end{aligned}$$

Lemma 5.3.12 *Let φ be a formula, n the cardinality of VAR_φ, $VAR_\varphi = \{v_i \mid i \leq n\}$. Let V be a subset of VAR_φ and set $M = \{i \mid v_i \in V\}$. Then the (V,φ)-evaluator $\nu^{(V,\,\varphi)}$ is a representation of the characteristic function of $Pos^{M,\,n}$.*

Proof: Just observe that, by Lemma 5.2.17,

$$e(\nu^{(V,\,\varphi)}) = \begin{cases} 1 & \text{if } e \text{ is } (V,\varphi)\text{-positive}, \\ 0 & \text{otherwise.} \end{cases}$$

hence $e(\nu^{(V,\,\varphi)}) = 1$ iff $e(v_i) = 1$ for every $v_i \in V$, iff $(e(v_1), e(v_2), \ldots, e(v_n)) \in Pos^{M,\,n}$. QED

Definition 5.3.13 *A function $\mathcal{C} : [0,1]^n \to [0,1]$ is a* product function *if for every $M \subseteq \hat{n}$ the restriction of $\mathcal{C}$ to $Pos^{M,n}$ is a Π-function (piecewise monomial function).*

Theorem 5.3.14 *Each product function is representable by some formula and, vice-versa, each formula is a representation of some product function.*

Proof: Let $\mathcal{C}$ be a product function, and for every $M \subseteq \{1, \ldots, n\}$ let us denote by $\mathcal{C}_M$ the restriction of $\mathcal{C}$ to $Pos^{M,n}$ (that is a Π-function).

By Theorem 5.3.4 there exists formulae φ_M that represent $\mathcal{C}_M$. If $V_M = \{v_i \mid i \in M\}$ then the formula

$$\varphi \equiv \bigvee_{V_M \subseteq VAR_\varphi} \left(\nu^{(V_M,\varphi)} \wedge \varphi_M\right)$$

represent $\mathcal{C}$.

Vice-versa suppose that $\mathcal{C}$ is a function represented by a formula φ. By Theorem 5.2.21, for each $V \subseteq VAR_\varphi$, there are formulae φ_V^C in CsNF such as

$$\varphi \equiv \bigvee_{V \subseteq V_\varphi} \left(\nu^V \wedge \varphi_V^C\right).$$

By Lemma 5.3.3 (5), φ_V^C represent Π-functions. The claim follows from Lemma 5.3.12. QED

The algebraic counterpart of Product logic are the Product algebras, see [44]. Due to the standard completeness theorem, giving a complete characterization of functions associated with Product formulae with n variables is equivalent to give a description of the free Product algebra over n generators.

Chapter 6

Compactness in fuzzy logics

In classical logic, an evaluation satisfies a set of formulae if it evaluates all its members to 1. A set of formulae is satisfiable if there is an evaluation which satisfies it. The set of formulae entails a formula if each evaluation, which satisfies this set of formulae, satisfies also this formula. The compactness theorem based on both above notions holds for classical logic: A set of formulae is satisfiable if and only if all its finite subsets are satisfiable; a set of formulae entails a formula iff some finite subset entails that formula.

It is natural to ask if analogues of these theorems hold in fuzzy logics. In this chapter we restrict ourselves to some core fuzzy logic (see Figure 6.1 for details). As the first step, we have to generalize the notion of satisfiability and entailment. To define satisfiability in fuzzy logic we may again require all formulae to be evaluated by 1 (as in classical logic), but this is not the only possibility. Sometimes other alternatives are well-motivated, hence, following [86], we work with K-satisfiability, where K can be an arbitrary subset of $[0, 1]$. We say that a set of formulae is K-satisfiable if there is an evaluation which evaluates them all by values in K. (In particular, we get the former case if we choose $K = \{1\}$.) Using K-satisfiability, we may formulate various types of compactness.

The generalization of the notion of entailment is not so straightforward. It turned out the only three of the core fuzzy logics are suitable for this generalized notion of entailment. These three logics are G, $G_\triangle$, and $G_\sim$ (Gödel logic, Gödel logic with $\triangle$ and involutive Gödel logic) we will call these logic *Gödel logics*. In the first section, we introduce both generalized notion of compactness for these Gödel logics and in the next section we deal with satisfiability based compactness for other core fuzzy logics.

6.1 The notions of compactness in Gödel logics

As stated above now we work with Gödel logics only. This selection of logics has several reasons. The problem of compactness (based on generalized entailment) for Gödel logic G was studied by Baaz and Zach in [3]. The generalization of entailment in Gödel logic is very natural, and it is well connected to the second notion of compactness. This notion is based on generalized satisfiability and was introduced by Butnariu, Klement, and Zafrany in [86] and then mainly studied in [28]. This connection is not accidental. As we will see in the further text, both generalizations are based on (parameterized by) subsets of the unit interval $[0, 1]$ and in Gödel logic they are nicely connected (we discuss this further after we introduce the necessary definitions). This connection is easily extended to $G_\triangle$, but adding the involutive negation causes a problem.

This is the reason why this section gives only "nearly complete" answer to the question of compactness in the three selected logics. However, the open problem in satisfiability based compactness of $G_\sim$ for countable set of propositional variables does not not spoil our main goal of establishing the connection between the two notions of compactness. This section

Figure 6.1: Core fuzzy logics

$$\begin{tikzcd}
 & \text{Ł} \arrow[dl] \arrow[d] & \Pi \arrow[d] & \mathrm{G} \arrow[d] \\
\text{PŁ}' \arrow[dr] & \text{Ł}_\triangle \arrow[d] & \Pi \arrow[dd] & \mathrm{G}_\triangle \arrow[d] \\
 & \text{PŁ}'_\triangle \arrow[dr] & & \mathrm{G}_\sim \arrow[dl] \\
 & & \text{Ł}\Pi \arrow[d] & \\
 & & \text{Ł}\Pi\tfrac{1}{2} &
\end{tikzcd}$$

is self-contained; the already known partial results from [3], [86], and [28] are obtained as corollaries to our more general theorems.

Before we proceed, we fix some denotation, we use throughout this section. Let Γ be a set of formulae and e a V-evaluation, we denote the set $\{e(\varphi) \mid \varphi \in \Gamma\}$ by $e(\Gamma)$. The symbol $1 - K$ denotes the set $\{1 - k \mid k \in K\}$. The symbol ω stands for the first infinite ordinal and ω_1 for the first uncountable ordinal. If we list the elements of the finite set as $K = \{k_1, k_2, \dots, k_n\}$, we assume that $k_i < k_{i+1}$ for each $i < n$. Finally, $\mathcal{P}(K)$ is the powerset of K.

6.1.1 Two notions of compactness

Here we introduce the basic definitions we will need in our work. We define several classes of subsets of $[0,1]$ which are crucial for our study of compactness. We will sometimes read "the set K is of type $\mathbf{C}$" for $K \in \mathbf{C}$. The classes $\mathbf{C}$, $\mathbf{Tv}$ ($\mathbf{Tv}_\sim$), and $\mathbf{D}$ ($\mathbf{D}_\sim$) are especially important, the reason for introduction of the first two classes ($\mathbf{C}$ and $\mathbf{Tv}$) will be clear just after the definitions of V-compactness^{Ent} and K-compactness^{Sat}, which follow. The other classes we are going to define are mostly used in the technical lemmata.

Definition 6.1.1 *Let us define the basic classes:*

- $\mathbf{FIN} = \{K \subseteq [0,1] \mid K \textit{ is finite}\}$,
- $\mathbf{C} = \{K \subseteq [0,1] \mid 0 \notin K \textit{ or } 1 \notin K\}$,
- $\mathbf{Tv} = \mathcal{P}([0,1]) \setminus \mathbf{C} = \{K \subseteq [0,1] \mid 0, 1 \in K\}$,
- $\mathbf{Tv}_\sim = \{K \in \mathbf{Tv} \mid K = 1 - K\}$,
- $\mathbf{D} = \{K \subseteq [0,1] \mid \exists A \subseteq K(A \textit{ is densely ordered})\}$,
- $\mathbf{D}_\sim = \{K \subseteq [0,1] \mid \exists A \subseteq K(A \textit{ is densely ordered and } 1 - A \subseteq K)\}$.

Furthermore, let us define several subclasses of $\mathbf{C}$*:*

- $\mathbf{C}_1 = \{K \in \mathbf{C} \mid 1 \in K\} = \{K \subseteq [0,1] \mid 1 \in K \textit{ and } 0 \notin K\}$,

- $\mathbf{C}_0 = \{K \in \mathbf{C} \mid 0 \in K\} = \{K \subseteq [0,1] \mid 0 \in K \text{ and } 1 \notin K\}$,
- $\mathbf{C}_{\overline{1}} = \mathbf{C} \setminus \mathbf{C}_1 \;= \{K \subseteq [0,1] \mid 1 \notin K\}$,
- $\mathbf{C}_{\overline{0}} = \mathbf{C} \setminus \mathbf{C}_0 \;= \{K \subseteq [0,1] \mid 0 \notin K\}$,
- $\mathbf{C}_{\overline{01}} = \mathbf{C}_{\overline{1}} \cap \mathbf{C}_{\overline{0}} = \{K \subseteq [0,1] \mid 0 \notin K \text{ and } 1 \notin K\}$.

The notion of these classes is from paper [28]. Basically, we may say that $\mathbf{Tv}$ stands for "truth values", and $\mathbf{D}$ stands for "dense". We will sometimes need to work with the smallest set of some type containing some set. We formalize this in the following way:

Definition 6.1.2 *Let $\mathbf{X} \subseteq \mathcal{P}([0,1])$ and $K \subseteq [0,1]$. Then we define the set function $\mathcal{C}l_{\mathbf{X}}$ as $\mathcal{C}l_{\mathbf{X}}(K) = \bigcap\{A \in \mathbf{X} \mid K \subseteq A\}$.*

Let us show an example for $\mathbf{X}$ being $\mathbf{Tv}_\sim$: $\mathcal{C}l_{\mathbf{Tv}_\sim}(K) = K \cup (1-K) \cup \{0,1\}$. In the following observation we summarize the properties of the above defined classes and show several obvious but important connections between them.

Proposition 6.1.3

1. $\mathbf{C} = \mathbf{C}_{\overline{0}} \cup \mathbf{C}_{\overline{1}} = \mathbf{C}_0 \cup \mathbf{C}_1 \cup \mathbf{C}_{\overline{01}}$.
2. *Let $K \in \mathbf{C}_1$. Then $0 \notin K$.*
3. $\mathbf{Tv}_\sim \cap \mathbf{D} = \mathbf{Tv}_\sim \cap \mathbf{D}_\sim$.
4. *Let Γ be a set of formulae of* G *or* $\mathrm{G}_\triangle$ *$K \in \mathbf{Tv}$ and e a K-evaluation. Then $e(\Gamma) \subseteq K$.*
5. *Let Γ be a set of formulae of* $\mathrm{G}_\sim$ *$K \in \mathbf{Tv}_\sim$ and e a K-evaluation. Then $e(\Gamma) \subseteq K$.*

If $V \in \mathbf{Tv}$, then V is the domain of some G-algebra and some $\mathrm{G}_\triangle$-algebra (if $V \in \mathbf{Tv}_\sim$, then V is a domain of some $\mathrm{G}_\sim$-algebra), for details see [2, 35, 44]

In the following we will denote the sets of type $\mathbf{Tv}$ ($\mathbf{Tv}_\sim$) by V and the sets of type $\mathbf{C}$ by K.

Now we define the entailment relation and this will give us the first notion of compactness: V-compactness^{Ent} (the index Ent stands for "entailment based", see [3] for more details and examples).

Definition 6.1.4 *Let V be a set of type $\mathbf{Tv}$ ($\mathbf{Tv}_\sim$), Γ a set of formulae, and φ a formula. We say that Γ V-entails φ if $\inf(e(\Gamma)) \leq e(\varphi)$ for each V-evaluation e. We denote this as $\Gamma \models_V \varphi$.*

Definition 6.1.5 *Let V be a set of type $\mathbf{Tv}$ ($\mathbf{Tv}_\sim$ respectively). We say that the logic* G, $\mathrm{G}_\triangle$ *($\mathrm{G}_\sim$ resp.) is V-compact^{Ent} if for each set of formulae T and each formula φ (of appropriate language) holds: $T \models_V \varphi$ iff there is a finite $T' \subseteq T$ such that $T' \models_V \varphi$.*

These definitions may look a little counterintuitive for a reader familiar with Baaz and Zach's paper [3]. They define the logic G_V semantically as a logic with the set of truth values V and connectives as in Gödel logic, then our notion of V-compactness^{Ent} coincides with the compactness of this logic G_V (where entailment and compactness is defined just like in classical logic i.e., Γ entails φ if for each evaluation e holds: $\inf(e(\Gamma)) \leq e(\varphi)$.) We decided to speak about V-compactness of Gödel logic rather than about compactness of the logic G_V in order to stress the connection with the upcoming second notion of compactness. Notice that we defined V-compactness^{Ent} for the sets of type $\mathbf{Tv}$ ($\mathbf{Tv}_\sim$) only.

Now we present the generalization of the notion of satisfiability in our logics. This will give us the second notion of compactness—K-compactness^{Sat} (the index Sat stands for "satisfiability based", see [28] and [72] for more details, motivations, and examples). Let us just briefly mention the main reason for introducing this kind of compactness. We call the set of formulae Γ r-satisfiable if there is an evaluation which evaluates all its elements to the value greater that r. Now we are interested whether r-satisfiability of each finite subset of Γ entails r-satisfiability of Γ. Unlike the previous notion of compactness, this one can be introduced in arbitrary many valued logic, based on arbitrary semantics. Notice that in classical logic we in fact work with 1-compactness and we have generalized this notion to r-compactness. This can be generalized even further to K-compactness^{Sat} for an arbitrary set $K \subset [0,1]$. We will see that this generalization does not make our work any more complicated and so there is no need to restrict the scope of this paper. Even more, we show that this general notion of K-compactness^{Sat} corresponds very well to the more natural notion of K-compactness^{Ent}.

Definition 6.1.6 *Let Γ be a set of formulae and $K \in \mathbf{C}$. We say Γ is* K-satisfiable *if there exists a $[0,1]$-evaluation e such that $e(\varphi) \in K$ for all $\varphi \in \Gamma$ (in other words: $e(\Gamma) \subseteq K$). The set Γ is said to be* finitely K-satisfiable *if each finite subset of Γ is K-satisfiable. A formula φ is called K-satisfiable if the set $\{\varphi\}$ is K-satisfiable.*

We could extend this definition even for sets $K \notin \mathbf{C}$. However, in this case for each formula φ there is an evaluation e such that $e(\varphi) \in K$ (it suffices to evaluate all propositional variables by 0, then the evaluation of each formula becomes either 0 or 1), i.e., each theory T would be trivially K-satisfiable. Thus we restricted our definition to the sets of type $\mathbf{C}$ only.

Definition 6.1.7 *Let $K \in \mathbf{C}$. We say that the logic* G, G$_\triangle$ *or* G$_\sim$*is* K-compact^{Sat} *if K-satisfiability is equivalent to finite K-satisfiability.*

Observe that whereas the problem of K-compactness^{Sat} for a given K is defined for all three logic, the problem of V-compactness^{Ent} for a given set V is defined for all three logics iff $V \in \mathbf{Tv}_\sim$. Also notice that whereas for each set $V \in \mathbf{Tv}_\sim$ holds $V \setminus \{1\} \in C_0$, we do not get $K \cup \{1\} \in \mathbf{Tv}_\sim$ for each set $K \in C_0$ (we get only $K \cup \{1\} \in \mathbf{Tv}$). Thus K-compactness^{Sat} is defined, in the above sense, for a wider class of sets than V-compactness^{Ent}. The following observation will simplify our study:

Remark 6.1.8 Let $\mathcal{C}$ be either *Ent* or *Sat*. Then G$_\triangle$ is "less compact$^{\mathcal{C}}$" than G. Since each counterexample to K-compactness$^{\mathcal{C}}$ in G is also a counterexample to K-compactness$^{\mathcal{C}}$ in G$_\triangle$logic this claim is obvious. Furthermore, G$_\sim$ is "less compact$^{\mathcal{C}}$" than G$_\triangle$ (by the same argument).

Since the question of compactness^{Ent} for Gödel logic with the infinite countable set of propositional variables is fully answered by Baaz and Zach in [3] (Gödel logic is V-compact^{Ent} iff $V \in \mathbf{Tv} \cap (\mathbf{FIN} \cup \mathbf{D})$) our goal is to answer this question for the other Gödel logics, the other type of compactness, and the other cardinalities of VAR. Let us mention that the known results of Baaz and Zach (see [3, Theorem 3.6]) will be obtained as easy corollaries of Lemma 6.1.22 and Lemma 6.1.26.

Our ultimate goal in this paper is to establish some form of correspondence between compactness^{Ent} and compactness^{Sat}

$\diamond$

At the end of this subsection we introduce the additional classes of subsets of $[0,1]$. These classes correspond to the open problem in K-compactness^{Sat}. The problem is highly technical and unrelated to the desired goal of establishing the correspondence between both notions of compactness so the uninterested reader may skip those definitions.

The cardinality of the set $C = \{x \mid x \in K, 1-x \in K\} = K \cap (1-K)$ is crucial in the upcoming definition. We will see that if it is infinite, there will be no problem to solve the problem of K-compactness^{Sat}. The problem arises if it is finite. Notice that if $V \in \mathbf{Tv}_\sim \cap \mathbf{FIN}$ and $V = \{v_0, v_1, \ldots, v_n\}$, then $v_{n-i} = 1 - v_i$.

Definition 6.1.9 *A nonempty set K is an element of $\overline{\mathbf{W}}$ if the following conditions are fulfilled:*

- *The set $C = \{x \mid x \in K, 1-x \in K\} \cup \{0, \frac{1}{2}, 1\} = \{c_0, \dots c_m\}$ is finite.*
- *If $B = C \cup \mathcal{C}l_{\mathbf{Tv}_\sim}(\bigcup\{K \cap [c_i, c_{i+1}] \mid K \cap [c_i, c_{i+1}] \text{ is finite}\}) = \{b_0, \dots b_n\}$, then the set $K \cap [b_j, b_{j+1}]$ is either finite or of type $\mathbf{D}$ for each $i < n$.*

Furthermore, K is an element of $\mathbf{W}$ if $K \in \overline{\mathbf{W}}$ and there is a finite set $A = \{a_0, \dots, a_l\}$ of type $\mathbf{Tv}_\sim$, with the following properties:

- *$B \subseteq A$,*
- *the set $K \cap (a_j, a_{j+1})$ is either empty or of type $\mathbf{D}$ for each $j < l$,*
- *if $K \cap (a_j, a_{j+1}) \in \mathbf{D}$, then the set $K \cap (a_{n+1-j}, a_{n-j})$ is empty.*

Let us try to reread these definitions. First, take a subset of elements x of K such that also $1-x \in K$ and unite this set with $\{0, \frac{1}{2}, 1\}$; call the resulting set C and its elements $c_0, c_1, \dots, c_n$. This set corresponds to partition on the set K with blocks $K_i = [c_i, c_{i+1}] \cap K$. Next, if some block of K_i is finite, add each its element z together with $1-z$ to the set C; call the resulting set B. This set is corresponds to another partition of K. We say that $K \in \overline{\mathbf{W}}$ if each block of K (under the partition B) is either finite of type $\mathbf{D}$.

Furthermore, if there is a finite set A of type $\mathbf{Tv}_\sim$ which is refinement of the partition B with some desired properties, we say that $K \in \mathbf{W}$. The properties are: interior of each block of K (under the partition A) is either empty of type $\mathbf{D}$; and if some block of K (under the partition A) is of type $\mathbf{D}$, then the interior of the opposite block is empty (by the "opposite block" to the block $[c_i, c_{i+1}] \cap K$ we mean the block $[1-c_{i+1}, c_i] \cap K$)

Let us notice that the class $\overline{\mathbf{W}} \setminus \mathbf{W}$ is nonempty. Just take the set K such that $K \cap [0, \frac{1}{2}] = Q \cap [0, \frac{1}{2}]$ and $K \cap (\frac{1}{2}, 1] = (\frac{1}{2}, 1) \setminus Q$ (i.e., K consists of all rational numbers from $[0, \frac{1}{2}]$ and all irrational numbers from $(\frac{1}{2}, 1)$). Then obviously $K \in \overline{\mathbf{W}}$, because $C = \{0, \frac{1}{2}, 1\} = \{c_0, c_1, c_2\}$ and $B = C = \{b_0, b_1, b_2\}$ (because both sets $K \cap [c_0, c_1]$ and $K \cap [c_1, c_2]$ are infinite) and $K \cap [b_0, b_1] \in \mathbf{D}$ and $K \cap [b_1, b_2] \in \mathbf{D}$. But obviously $K \notin \mathbf{W}$ (because for each $a, b \leq \frac{1}{2}$ holds $K \cap (a, b) \in \mathbf{D}$ and $K \cap (1-b, 1-a) \in \mathbf{D}$).

In the following observation we show several obvious but important connections between the above defined sets.

Proposition 6.1.10

1. $\mathbf{Tv}_\sim \cap \overline{\mathbf{W}} \subseteq \mathbf{FIN} \subseteq \mathbf{W}$.
2. $\overline{\mathbf{W}} \setminus \mathbf{W} \neq \emptyset$.
3. $\mathbf{Tv}_\sim \cap (\overline{\mathbf{W}} \setminus \mathbf{W}) = \emptyset$.
4. $\mathbf{FIN} \subseteq \overline{\mathbf{W}}$.

6.1.2 Compactness and the cardinality of VAR

In this subsection we show that the both notions of compactness depend on the cardinality of the set of propositional variables. The previous papers about compactness in Gödel logics restrict themselves to the countable set of propositional variables, with the exception of [28], where the authors deal with K-compactness^{Sat} for the set VAR being finite.

We start by the results proved in [28, Theorem 5.1]. It is proven there for the set VAR being of cardinality ω. However, it is easy to observe that the proof is sound for other cardinalities as well. For the sake of completeness of this paper we present the proof.

Lemma 6.1.11 *Let the set VAR be of an arbitrary cardinality and $K \in \mathbf{C}_1$. Then* G *is K-compact^{Sat}.*

Proof: Let the formula φ' results from the formula φ by replacing all its propositional variables v by the formulae $\neg\neg v$. Let us recall the evaluation of the double negation:

$$e(\neg\neg\varphi) = \begin{cases} 1 & \text{if } e(\varphi) > 0 \ , \\ 0 & \text{if } e(\varphi) = 0 \ . \end{cases}$$

Now we prove by induction that $e(\neg\neg\varphi) = e(\varphi')$. If φ is a propositional variable the claim is obvious.

a) If $\varphi = \psi \wedge \chi$, then $e(\varphi) = e(\psi) \wedge e(\chi)$. We may write the following chain of equivalences: $e(\neg\neg\varphi) = 1$ iff $e(\psi) > 0$ and $e(\chi) > 0$ iff $e(\neg\neg\psi) = 1$ and $e(\neg\neg\chi) = 1$ iff $e(\psi') = 1$ and $e(\chi') = 1$ iff $e(\varphi') = e(\psi' \wedge \chi') = 1$.

b) If $\varphi = \psi \to \chi$, we may write the following chain of equivalences: $e(\neg\neg\varphi) = 0$ iff $e(\psi \to \chi) = 0$ iff $e(\psi) > 0$ and $e(\chi) = 0$ iff $e(\neg\neg\psi) = 1$ and $e(\neg\neg\chi) = 0$ iff $e(\psi') = 1$ and $e(\chi') = 0$ iff $e(\varphi') = e(\psi' \to \chi') = 0$.

Now observe that a formula is satisfiable in classical logic iff it is K-satisfiable in G. The first implication is obvious (since $K \in \mathbf{C}_1$). To prove the reverse implication, let us assume that the formula φ is K-satisfiable. Then the formula $\neg\neg\varphi$ is K-satisfiable (since $0 \notin K$ and $1 \in K$) and, by the fact proven above, the formula φ' is K-satisfiable, as well. Thus there is an evaluation which evaluates propositional variables by either 0 or 1 and the formula φ by 1. This is a classical evaluation satisfying the formula φ.

The above argument can be easily extended to a set of formulae and our claim is obtained as an easy consequence. QED

We turn our attention to cardinalities bigger than ω. Observe that in the following we do not assume the Continuum Hypothesis (however we assume the Axiom of Choice). Since VAR is uncountable it contains a subset VAR' of cardinality ω_1. Let us enumerate the elements of VAR' by ordinals $\nu < \omega_1$.

Lemma 6.1.12 *Let VAR be an uncountable set and $K \in \mathbf{C}$.*

1. *$\mathrm{G}_\triangle$ is not K-compact^{Sat} for $K \in \mathbf{C}_1$ and $K = \{0\}$.*
2. *G is not K-compact^{Sat} for $\{0\} \neq K \notin \mathbf{C}_1$.*

Proof: First suppose that $K \in \mathbf{C}_1$. We define the set of formulae $\Gamma = \{\neg\triangle(v_\nu \to v_\mu) \mid \mu < \nu < \omega_1\}$. Observe that $e(\neg\triangle(v_\nu \to v_\mu)) = 1$ iff $e(v_\nu) > e(v_\mu)$, i.e., e K-satisfies $\neg\triangle(v_\nu \to v_\mu)$ iff $e(v_\nu) > e(v_\mu)$. To complete the proof just notice that Γ is finitely K-satisfiable (just evaluate variables occurring in that finite subset by properly ordered elements from $[0, 1]$) and not K-satisfiable–this would mean $e(VAR')$ is a chain of type ω_1, and since $e(VAR') \subseteq [0, 1]$ we have a contradiction (by simple set-theoretical arguments). The proof for $K = \{0\}$ is analogous, we would only work with formulae $\triangle(v_\nu \to v_\mu)$ instead of $\neg\triangle(v_\nu \to v_\mu)$.

Now we prove the second claim: if K is not of type $\mathbf{C}_1$ and $K \neq \{0\}$ there is an element $k \in K$, $0 < k < 1$. Again, we use the same idea as in the proof of the first claim. Let us fix a propositional variable $k \notin VAR'$ and define the set of formulae $\Gamma = \{(v_\nu \to v_\mu) \vee k \mid \mu < \nu < \omega_1\}$.

Observe that $e((v_\nu \to v_\mu) \vee k) = 1$ iff $e(v_\nu) \leq e(v_\mu)$ or $e(k) = 1$. Again notice that Γ is finitely K-satisfiable (just put $e(k) = k$ and evaluate variables occurring in that finite subset by properly ordered elements from $[0, k]$) and not K-satisfiable (this would mean that $e(VAR')$ is chain of type ω_1, and since $e(VAR') \subseteq [0, 1]$ we have a contradiction). QED

Now we prove two lemmata concerning V-compactness^{Ent}. Both are interesting by the fact that they hold for an arbitrary infinite set VAR and so we can use them in the next subsections, where we will fix the cardinality of VAR to ω.

Lemma 6.1.13 *Let VAR be an infinite set and $V \in \mathbf{Tv} \setminus \mathbf{FIN}$. If* G *logic is V-compact^{Ent}, then it is $(V \setminus \{0,1\})$-compact^{Sat}.*

Proof: Let Γ be a set of formulae. We assume that each finite $\Gamma' \subseteq \Gamma$ is $(V \setminus \{0,1\})$-satisfiable and we want to prove that Γ is $(V \setminus \{0,1\})$-satisfiable.

Since V is infinite, we can find V-evaluation $e_{\Gamma'}$ which $(V \setminus \{0,1\})$-satisfies Γ' and there is an element $k \in V$, such that $\sup(e_{\Gamma'}(\Gamma')) = k < 1$. Let t be a variable not occurring in any formula of Γ. We define the set of formulae $\bar{\Gamma}' = \{\neg\neg\psi, \psi \to t \mid \psi \in \Gamma'\}$. Let us take the V-evaluation $\bar{e}_{\Gamma'}$ resulting from the V-evaluation $e_{\Gamma'}$ by setting $\bar{e}_{\Gamma'}(t) = k$. We notice that $\bar{e}_{\Gamma'}(\bar{\Gamma}') = \{1\}$. Since $k < 1$, we get $\bar{\Gamma}' \not\models_V t$.

By the V-compactness^{Ent} (in the form: if $\bar{\Gamma}' \not\models_V \varphi$ for each finite $\bar{\Gamma}' \subseteq \Gamma$, then $\bar{\Gamma} \not\models_V \varphi$) we get $\{\neg\neg\psi, \psi \to t \mid \psi \in \Gamma\} \not\models_V t$, i.e., there is a V-evaluation e such that $\inf(e(\Gamma)) > 0$ and $\sup(e(\Gamma)) \leq e(t) < 1$, thus obviously $e(\Gamma) \subseteq V \setminus \{0,1\}$. QED

Lemma 6.1.14 *Let VAR be an arbitrary set. Then:*

1. G *and* $\mathrm{G}_\triangle$ *are V-compact^{Ent} for each $V \in \mathbf{FIN} \cap \mathbf{Tv}$.*
2. $\mathrm{G}_\sim$ *is V-compact^{Ent} for each $V \in \mathbf{FIN} \cap \mathbf{Tv}_\sim$.*

Proof: We show the proof for $\mathrm{G}_\sim$, the proofs for other logics are analogous. We will reduce the problem of V-compact^{Ent} to the problem of compactness of the classical first-order logic (FOL). In the following text we assume that all the formulae of $\mathrm{G}_\sim$ are written using the basic connectives only and the connectives $\neg$, $\Leftrightarrow$, and $\Rightarrow$ are the connectives of the classical logic.

The language $\mathcal{L}_V$, consists of the constants tv_i for each $i \in V$; two binary function symbols $\to$ and $\wedge$; one unary function symbol $\sim$; and one binary predicate $\leq$. The theory T_0 consists of the following sentences (we omit the quantifiers):

(1a) $x \wedge y = y \wedge x$,

(1b) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,

(1c) $x \wedge x = x$,

(1d) $x \wedge y = x \Leftrightarrow x \leq y$,

(2a) $\sim\sim x = x$,

(2b) $x \leq y \Leftrightarrow \sim y \leq \sim x$,

(3a) $x \to y = tv_1 \Leftrightarrow x \leq y$,

(3b) $\neg(x \leq y) \Rightarrow x \to y = y$,

(4a) $\neg(tv_i = tv_j)$ for each $i \neq j$,

(4b) $tv_i \leq tv_j$ for each $i \leq j$,

(4c) $tv_i = \sim tv_{1-i}$ for each i,

(4d) $\bigvee_{i \in V} (x = tv_i)$.

It is obvious that some function and predicate symbols and some axioms of T_0 are redundant, however in this way we can very easily notice that all $\mathcal{L}_V$-models of T_0 are finite and mutually isomorphic and the structure $\mathbf{V} = (V, \wedge, \to, \sim, \leq, (i)_{i \in V})$, where the operations are those of $\mathrm{G}_\sim$, is one of them (this is also an obvious consequence of the fact that a theory of a finite structure is categorical).

Let us extend the language $\mathcal{L}_V$ to the language $\mathcal{L}_V^{VAR}$ by adding a constant c^v for each $v \in VAR$. If φ is a formula of $\mathrm{G}_\sim$ logic then the term $\varphi^\#$ of the classical FOL results from φ, by replacing the connectives (including the truth constant $\mathbf{0}$) by the corresponding functional symbols of $\mathcal{L}_V^{VAR}$ and replacing each propositional variable v by the constant c^v. Analogously, if t is a term of $\mathcal{L}_V^{VAR}$, such that if tv_i appears in t, then $i = 0$ or $i = 1$, we define the formula $t^\#$ in an obvious way. In the following text, when we work with an $\mathcal{L}_V^{VAR}$-model we assume that its $\mathcal{L}_V$-reduct is $\mathbf{V}$.

Notice that for each $\mathcal{L}_V^{VAR}$-model $\mathbf{N}$ there is a V-evaluation $e^{\mathbf{N}}$ defined as $e^{\mathbf{N}}(v) = c^v_{\mathbf{N}}$. Analogously, for each V-evaluation f we define an $\mathcal{L}_V^{VAR}$-model $\mathbf{M}_f$ as $c^v_{\mathbf{M}_f} = f(v)$. Obviously, $\mathbf{M}_{e^{\mathbf{N}}} = \mathbf{N}$ and $e^{\mathbf{M}_f} = f$. We can easily prove that the defining relations can be extended to an arbitrary formula (term), i.e., for each $\mathcal{L}_V^{VAR}$-model $\mathbf{N}$ holds $e^{\mathbf{N}}(t^\#) = t_{\mathbf{N}}$, and for each V-evaluation f holds $\varphi^\#_{\mathbf{M}_f} = f(\varphi)$. In other words, we have established one-one correspondence between $\mathcal{L}_V^{VAR}$-models and V-evaluations.

Now observe that for each set of formulae Γ and each formula φ the following holds: $\Gamma \not\models_V \varphi$ iff the first-order theory $\Gamma^\# = \{\neg((\psi \to \varphi)^\# = tv_1) \mid \psi \in \Gamma\}$ has an $\mathcal{L}_V^{VAR}$-model. Indeed, if $\Gamma \not\models_V \varphi$, then $e(\psi) > e(f)$ for each $\psi \in \Gamma$. Thus $e(\psi \to \varphi) \neq 1$ and so we get $(\psi \to \varphi)^\#_{\mathbf{M}_e} \neq tv_1$ and so $\mathbf{M}_e \models \neg((\psi \to \varphi)^\# = tv_1)$ for each $\psi \in \Gamma$. Thus $\mathbf{M}_e$ is an $\mathcal{L}_V^{VAR}$-model $\Gamma^\#$. The reverse direction is analogous (we again use the finiteness of V and by proving that $e^{\mathbf{M}}(\psi) > e^{\mathbf{M}}(f)$ for each $\psi \in \Gamma$ we conclude that $\inf e^{\mathbf{M}}(\Gamma) > e^{\mathbf{M}}(f)$).

The compactness property of the classical FOL completes the proof. QED

Now we put everything together and obtain the main theorem of this subsection. It gives a complete answer to the problem of V-compactness^{Ent} and K-compactness^{Sat} of all three logics in the case of VAR not being an infinite countable set.

Theorem 6.1.15 *Let $K \in \mathbf{C}$, $V \in \mathbf{Tv}$, and $\bar{V} \in \mathbf{Tv}_\sim$. For VAR being an uncountable set we get:*

1. *Gödel logic is K-compact^{Sat} iff K is of type $\mathbf{C}_1$ or $K = \{0\}$,*
2. *Gödel logic with $\triangle$ and Gödel involutive logic are not K-compact^{Sat},*
3. *Gödel logic and Gödel logic with $\triangle$ are V-compact^{Ent} iff $V \in \mathbf{FIN}$,*
4. *Gödel logic with $\triangle$ and Gödel involutive logic is $\bar{V}$-compact^{Ent} iff $\bar{V} \in \mathbf{FIN}$.*

For VAR being a finite set we get:

5. *All three Gödel logics are K-compact^{Sat},*
6. *Gödel logic and Gödel logic with $\triangle$ are V-compact^{Ent},*
7. *Gödel involutive logic is $\bar{V}$-compact^{Ent}.*

Proof:

1. For K being of type $\mathbf{C}_1$ we use Lemma 6.1.11. For K being $\{0\}$ just observe that a formula φ is $\{0\}$-satisfiable iff the formula $\neg\varphi$ is $\{1\}$-satisfiable. For other sets K the second part of Lemma 6.1.12 completes the proof.
2. For $K \in \mathbf{C}_1$ and $K = \{0\}$ we use the first part of Lemma 6.1.12, for other sets we use the previous claim and Remark 6.1.8.
3. For $K \in \mathbf{FIN}$ we use Lemma 6.1.14; for other sets K we use the claim (1) and Lemma 6.1.13 to get that G is not V-compact^{Ent}. The claim for $\mathrm{G}_\triangle$ is just a consequence of Remark 6.1.8.

4. Obvious.

To prove the remaining claims for the set VAR being finite just observe that in this case there are only finitely many non-equivalent formulae. This well-known property of Gödel logic can be easily extended to other logics. However, for the sake of completeness of this paper we present the proof of this fact for $G_\sim$ (and thus for the other two as well) in the next subsection after we develop certain formal techniques. QED

Based on the result of this subsection we consider our set of propositional variables being an infinite countable set from now on.

6.1.3 K-compactness^{Sat}

In this subsection we deal with K-compactness^{Sat} of our logics. We develop a machinery of the so-called evaluation prototypes, which helps us understand the structure of an evaluation. We need to introduce the truth constant $\mathbf{r}$ for each rational $r \in [0,1]$. We need these constants just as a technical tool to understand the structure of an evaluation, thus we do not add them to the language (i.e., we do not change the set of formulae), we only extend the definition of an evaluation by the condition $e(\mathbf{r}) = r$ (thus from now on we understand the evaluation as a function $VAR \cup \{\mathbf{r} \mid r \in Q \cap [0,1]\} \to [0,1]$, such that $e(\mathbf{r}) = r$).

Definition 6.1.16 *Let $P = \{0, p_1, p_2, \ldots, p_n, 1\}$ be a finite set of type $\mathbf{Tv}_\sim$. The sequence of pairs $((\mathbf{0},<),(\mathbf{p_1},<),(\mathbf{p_2},<),\ldots(\mathbf{p_n},<),(\mathbf{1},=))$ is called an* **evaluation P-prototype** *of order 0. If the sequence S is an evaluation P-prototype of order l then the sequence S' is an evaluation P-prototype of order $l+1$ iff S' results from S by inserting (at arbitrary positions) pairs $(v_{l+1}, *)$ and $(\sim v_{l+1}, *)$, where $* \in \{<,=\}$.*

If S is an evaluation P-prototype let S_i denote its i-th element, S_i^1 the first member of the pair S_i and S_i^2 the second one. Finally, by $|S|$ we denote the length of S.

Notice that S_i^1 is either a propositional variable, its negation, or a truth constant; and S_i^2 is either $<$ or $=$.

Definition 6.1.17 *Let $P \in \mathbf{Tv}_\sim \cap \mathbf{FIN}$, e an evaluation and S an evaluation P-prototype. Then we say that e respects S if $e(S_i^1)\ S_i^2\ e(S_{i+1}^1)$ for each $i < |S|$.*

We say that S is acceptable if there is an evaluation, which respects S. We denote the set of acceptable evaluation K-prototypes of order n by EP_n^P.

Let $P = \{0, \frac{\mathbf{1}}{\mathbf{2}}, 1\}$. The sequences $((\mathbf{0},<),(v,<),(\frac{\mathbf{1}}{\mathbf{2}},<),(\sim v,<),(\mathbf{1},=))$ and $((\mathbf{0},<),(v,=),(\sim v,=),(\frac{\mathbf{1}}{\mathbf{2}},<),(\mathbf{1},=))$ are examples of acceptable evaluation P-prototypes of order 1 (the evaluation $e(v) = \frac{1}{4}$ respects the first one and the evaluation $e(v) = \frac{1}{2}$ respects the second one).

The sequence $((\mathbf{0},<),(v,=),(\frac{1}{2},<),(\sim v,=),(\mathbf{1},=))$ is an example of an unacceptable one (using the definition for $i=2$ we get $e(v) = e(\frac{\mathbf{1}}{\mathbf{2}}) = \frac{1}{2}$ and for $i=4$ we get $e(\sim v) = e(\mathbf{1}) = 1$, a contradiction). Of course, we could derive the syntactical rules for an evaluation P-prototype to be an acceptable one, but our "semantical" definition is quite simpler.

In the following text we omit the word acceptable and assume that we work with acceptable prototypes only. Observe that EP_n^P is indeed finite, and EP_0^P is a singleton, so let us use the same denotation for this set and its element. Notice that if e respects S and $i \leq j$, then $e(S_i^1) \leq e(S_j^1)$.

Lemma 6.1.18 *Let φ be a formula such that $VAR(\varphi) \subseteq VAR_n$, $P \in \mathbf{Tv}_\sim \cap \mathbf{FIN}$ and $S \in EP_n^P$. Then there is an index i such that $e(\varphi) = e(S_i^1)$ for each evaluation e respecting S.*

Proof: We prove this by the induction on the complexity of the formula φ. Let v be a propositional variable, $v \in VAR(\varphi)$, thus v occurs somewhere in S (since $VAR(\varphi) \subseteq VAR_n$), i.e., there is an index i such that $v = S^1_i$.

Let $\varphi = \sim\psi$. By the induction hypothesis there is an index i such that for each evaluation e respecting S holds $e(\psi) = e(S^1_i)$. Since for each index i there is an index j such that $\sim e(S^1_i) = e(S^1_j)$ (from the construction of S) we get $e(\varphi) = \sim e(S^1_i) = e(S^1_j)$.

Let $\varphi = \psi \wedge \chi$. By the induction hypothesis there are indices i and j such that for each evaluation e respecting S holds $e(\psi) = e(S^1_i)$ and $e(\chi) = e(S^1_j)$. Let us assume that $i \leq j$ (for $j \leq i$ the proof is analogous). We know that for each evaluation e respecting S holds $e(\psi) = e(S^1_i)$ and $e(\chi) = e(S^1_j)$. Since $i \leq j$ then $e(S^1_i) \leq e(S^1_j)$ thus $e(\varphi) = e(S^1_i)$.

Let $\varphi = \psi \to \chi$. By the induction hypothesis there are indices i and j such that for each evaluation e respecting S holds $e(\psi) = e(S^1_i)$ and $e(\chi) = e(S^1_j)$. Let us assume that $i \leq j$, then $e(S^1_i) \leq e(S^1_j)$. Thus $e(\varphi) = 1 = e(S_{|S|})$. Now let us assume that $i > j$. There are two possibilities: for each index k, such that $j \leq k < i$ holds S^2_k is $=$. Since e respects S we get $e(S^1_i) = e(S^1_j)$ so $e(\varphi) = 1$. The second one is that there is an index k, $j \leq k < i$, such that S^2_k is $<$. Since e respects S we get $e(S^1_i) > e(S^1_j)$ and so $e(\varphi) = e(S^1_j)$ QED

We show an example: we take $P = \{0, \frac{1}{2}, 1\}$, $S = ((\mathbf{0}, <), (v, <), (\mathbf{\frac{1}{2}}, <), (\sim v, <), (\mathbf{1}, =))$, and $\varphi = \sim v \to v$. Then obviously $e(\varphi) = e(v) = e(S^1_2)$ for each e respecting S. For $S = ((\mathbf{0}, <), (v, =), (\sim v, =), (\mathbf{\frac{1}{2}}, <), (\mathbf{1}, =))$ we obtain $e(\varphi) = 1 = e(S^1_5)$.

Corollary 6.1.19 *Let $P \in \mathbf{Tv}_\sim \cap \mathbf{FIN}$, $p \in P$, $S \in EP^P_n$, e be an evaluation respecting S, and φ a formula such that $VAR(\varphi) \subseteq VAR_n$.*

1. *If $e(\varphi) = p$, then $f(\varphi) = p$ for each evaluation f respecting S.*
2. *If $e(\varphi) < 1$, then $f(\varphi) < 1$ for each evaluation f respecting S.*
3. *If $e(\varphi) > 0$, then $f(\varphi) > 0$ for each evaluation f respecting S.*

Proposition 6.1.20 *Let $P \in \mathbf{Tv}_\sim \cap \mathbf{FIN}$, n a natural number, and e an evaluation. Then there is a $S \in EP^P_n$ such that e respects S.*

Before we prove the fundamental lemma of the paper we use the above results to give the promised proof of the crucial fact used in the proof of the second part of Theorem 6.1.15.

Lemma 6.1.21 *Let us define the set $Form_n$ as $\{\varphi \mid VAR(\varphi) \subseteq VAR_n\}$ and the equivalence $\simeq$ on $Form_n$ as $\varphi \simeq \psi$ iff $e(\varphi) = e(\psi)$ for each evaluation e. Then the quotient of $Form_n$ by $\simeq$ is finite.*

Proof: Let f be an arbitrary evaluation. Let us define the equivalence $\simeq_f$ on $Form_n$ as $\varphi \simeq_f \psi$ iff $f(\varphi) = f(\psi)$. Due to the latter lemma there is an evaluation prototype $S \in EP_n$ such that f respects S. Using Lemma 6.1.18 we get for each formula φ an index i_φ such that for each evaluation e respecting S holds $e(\varphi) = e(S^1_{i_\varphi})$.

This has two consequences. First: since the sequence S is finite and for each formula φ holds $f(\varphi) = f(S^1_{i_\varphi})$ we conclude that the quotient of $Form_n$ by $\simeq_f$ is finite. Second: if e and f both respect S then $\simeq_e = \simeq_f$.

To complete the proof just notice that $\simeq = \bigcap_e \simeq_e = \bigcap_{S \in EP_n} \simeq_{e_S}$, where e_S is an arbitrary evaluation respecting S. Since we have expressed $\simeq$ as finite intersection of equivalences for which the quotient of $Form_n$ is finite, the proof is done. QED

Now we present the crucial lemma of this paper. The proof is inspired by the proof of [3, Theorem 3.4] (especially the construction of the tree). However, that proof is for compactness^{Ent} and for Gödel logic only and cannot be generalized to the stronger logics (at least not in a straightforward way). Our proof has two parts—one for sets of type $\mathbf{D}_\sim$ and the other for sets of type $\mathbf{W}$. Whereas the first part is quite easy to follow, the second one if rather technical. For this reason we advise the reader to skip the parts related to the sets of the type $\mathbf{W}$ (i.e., the *Case (2)* in the introductory definitions and in Sublemma 6.1.23) by the first reading and concentrate to the tree construction, its pruning, and the construction of relevant evaluation in the last part. The second part illustrates the reasons for introduction of type $\mathbf{W}$, the reader can notice that the proof (especially the proof of Sublemma 6.1.23) would not work for arbitrary set of type $\overline{\mathbf{W}}$.

Lemma 6.1.22 *Let $K \in \mathbf{C} \cap (\mathbf{D}_\sim \cup \mathbf{W})$. Then $\mathrm{G}_\sim$ is K-compact^{Sat}.*

Proof: We prove both cases ($K \in \mathbf{C} \cap \mathbf{D}_\sim$ and $K \in \mathbf{C} \cap \mathbf{W}$) together. We only give a different definitions of two sets P and V.

Case (1): If $K \in \mathbf{D}_\sim$, it contains a dense subsets A and $(1-A)$, we define set $V = \mathcal{C}l_{\mathbf{Tv}_\sim}(A) = A \cup (1-A) \cup \{0,1\}$ and set P as $\{0,1\}$. Notice that $V \subseteq K \cup \{0,1\}$.

Case (2): If $K \in \mathbf{W}$, there is set $A = \{a_1, \dots a_n\}$ from the definition of $\mathbf{W}$. Let us denote $K_i = K \cap [a_{i-1}, a_i]$. Recall that K_i either finite ($K_i \subseteq \{a_{i-1}, a_i\}$) or of type $\mathbf{D}$, in this case we will denote its dense subset by A_i. We define the sets V_i:

$$V_i = \begin{cases} [a_{i-1}, a_i] & \text{if both } K_i \text{ and } K_{n+1-i} \text{ are finite,} \\ A_i \cup \{a_{i-1}, a_i\} & \text{if } K_i \text{ is of type } D \text{ and } K_{n+1-i} \text{ is finite.} \end{cases}$$

Finally, we define sets $V = \bigcup V_i \cup (1 - \bigcup V_i)$ and $P = A$.

Notice that in both cases we have $P \in \mathbf{Tv}_\sim \cap \mathbf{FIN}$ and $V \in \mathbf{Tv}_\sim \cap \mathbf{D}_\sim$. Now we continue our proof: we have a set of formulae Γ and assume that there are infinitely many propositional variables in the formulae of Γ (otherwise the proof is trivial). We prove that either Γ is K-satisfiable or there is finite subset which is not K-satisfiable. We enumerate the formulae from Γ and then variables in the usual way (first we enumerate variables from φ_1, then from φ_2, and so on). Let o be a function which to each i assigns the number of the first formula in which v_{i+1} occurs (thus only variables from $\{v_1, \dots, v_i\}$ occur in all previous formulae). We define set of formulae: $\Gamma_i = \{\varphi_1, \varphi_2 \dots, \varphi_{o(i)-1}\}$. Notice that $\Gamma = \bigcup \Gamma_i$. ($\Gamma_i$ is an initial segment of the enumerating sequence, where the formulae have variables from $\{v_1, \dots, v_i\}$)

We construct a finitary tree T. The nodes in the i-th level are labelled by the evaluation P-prototypes of order i. The root is labelled by EP_0^P. If the node n in the i-th level is labelled by S_n we add a successive node for each $H \in EP_{i+1}^P$ resulting from S_n (in the sense of Definition 6.1.16) and label this node by H.

We construct a subtree T' by pruning the tree T. We cut the subtree in node n in level i iff no evaluation e respecting S_n K-satisfies Γ_i (in other words the node n is the node of tree T' if there is an evaluation e respecting S_n which K-satisfies Γ_i). It is easy to show that our three T' is finite or has an infinite branch (König's lemma). Before we finish our proof we prove one important sublemma.

Sublemma 6.1.23 *If T is not cut in node n, then each V-evaluation e which respects S_n K-satisfies Γ_i.*

Proof: *Case (1):* K is of type $\mathbf{C}$, so K is either of type $\mathbf{C}_1$ or $\mathbf{C}_0$ or $\mathbf{C}_{\overline{01}}$. Let us suppose the last case (the other ones are analogous). If T is not cut in node n there is an evaluation e respecting S_n which K-satisfies Γ_i. Thus $0 < e(\psi) < 1$ for each $\psi \in \Gamma_i$ (because $0, 1 \notin K$). Let f be an arbitrary V-evaluation, we know that $f(\Gamma_i) \subseteq V \subseteq K \cup \{0,1\}$. Furthermore, if f respects S_n, then $0 < f(\psi) < 1$ for each $\psi \in \Gamma_i$ (due to Corollary 6.1.19 parts 2. and 3.).

Case (2): Again we assume that there is an evaluation e respecting S_n which K-satisfies Γ_i. Let f be an arbitrary V-evaluation, we know that $f(\Gamma_i) \subseteq V$. We finish the proof by a contradiction, let us assume that there is a formula $\psi \in \Gamma_i$ such that $f(\psi) \notin K$. Using Lemma 6.1.18 we know that there is index s such that $f(\psi) = f(S_s^1)$. Let us find indices $k \leq s \leq l$ such that $S_k^1 = a_{j-1}$ and $S_l^1 = a_j$, so we know $f(\psi) \in [a_{j-1}, a_j]$. Since e also respects S_n and K-satisfies Γ_i then also $e(\psi) = e(S_s^1) \in K \cap [a_{j-1}, a_j] = K_j$. We distinguish two subcases:

Subcase I: Both K_j and K_{n+1-j} are finite: then $K_j \subseteq P$ so $e(S_s^1) = a \in K \cap P$. Then by Corollary 6.1.19 part 1. also $f(S_s^1) = a \in K \cap P$—a contradiction.

Subcase II: $K_j \in \mathbf{D}$ and K_{n+1-j} is finite: because $f(\psi) \notin K$ we know that $f(\psi) \in V \setminus K$ so $f(\psi) \in (V \cap [a_{j-1}, a_j]) \setminus K$ and because $V \cap [a_{j-1}, a_j] = A_j \cup \{a_{j-1}, a_j\}$ we get $f(\psi) = a \in \{a_{j-1}, a_j\} \setminus K \subseteq P$. Since also $e(\psi) = a$ (by Corollary 6.1.19 part 1.) and $e(\psi) = a \in K$—a contradiction. QED

(*The rest of the proof of Lemma 6.1.22*) If the resulting tree T' contains an infinite branch, we construct a V-evaluation e, which K-satisfies Γ. Notice that if n is a node in level i, n' its successor, and e a V-evaluation respecting S_n, then there is a V-evaluation e' respecting $S_{n'}$ such that $e(v_j) = e'(v_j)$ for $j \leq i$ (this is due to the fact that V is of type $\mathbf{D}_\sim$). We start with the empty V-evaluation and follow our infinite branch, in each node we use the previous observation. The resulting V-evaluation e respects S_n for each node n on our infinite branch, thus V-evaluation e K-satisfies Γ_i for each i. Thus e K-satisfies Γ. Here we have used Sublemma 6.1.23 which assures that if T is not cut in node n, then each (especially the one we have constructed) V-evaluation e respecting S_n K-satisfies Γ_i.

Now let us suppose that T' is finite and i is its depth. We show that Γ_i is not K-satisfiable. By contradiction: we suppose that there is an evaluation e K-satisfying Γ_i. There has to be a node n (of the tree T) on the level i such that e respects S_n (cf. Proposition 6.1.20). Let us take the node n' without successor in the tree T', such that there is a path from n' to n (in the tree T). Let us denote the level of n' by i'. Notice that e K-satisfies $\Gamma_{i'}$ (because $\Gamma_{i'} \subseteq \Gamma_i$). The evaluation e also respects $S_{n'}$. Since the tree T is cut in node n', no evaluation respecting $S_{n'}$ K-satisfies $\Gamma_{i'}$—a contradiction. QED

The proof of this lemma gives us two useful corollaries:

Corollary 6.1.24 *Let K be set of type $\mathbf{C} \cap \mathbf{D}_\sim$, Γ a set of formulae. If Γ is K-satisfiable, then Γ is K-satisfiable for some $(K \cup \{0,1\})$-evaluation.*

Corollary 6.1.25 *Let $K \in \mathbf{C} \cap \mathbf{D}$. Then $\mathrm{G}_\triangle$ logic is K-compact^{Sat}.*

Proof: We start with Remark 6.1.8 and the previous lemma to get that $\mathrm{G}_\triangle$ is K-compact^{Sat} for K being $[0,1)$, $(0,1)$, and $(0,1]$.

Let $K \in \mathbf{C}_0$ (i.e., $0 \in K$ and $1 \notin K$), the cases $K \in \mathbf{C}_1$ and $K \in C_{\overline{01}}$ are analogous, and let Γ be a set of formulae. We suppose that each finite subset of Γ is K-satisfiable. Since $K \subseteq [0,1)$, each finite subset of Γ is [0,1)-satisfiable and so Γ is [0,1)-satisfiable (by the [0,1)-compactness^{Sat} of $\mathrm{G}_\triangle$). So there is an evaluation e such that $e(\Gamma) \subseteq [0,1)$. It is well known that there is an order-preserving embedding $f : e(\mathit{VAR}) \cup \{0,1\} \to K \cup \{1\}$ (because any countable order can be embedded into the dense countable order), thus $f \circ e$ is an evaluation. Finally notice that $(f \circ e)(\Gamma) \subseteq K$ (obviously $(f \circ e)(\Gamma) \subseteq K \cup \{1\}$, but if for some $\varphi \in \Gamma$ is $(f \circ e)(\varphi) = 1$ then $e(\varphi) = 1$—a contradiction). QED

Lemma 6.1.26 *Let $K \in \mathbf{C} \setminus \mathbf{FIN}$.*

1. *If $K \notin \mathbf{C}_1 \cup \mathbf{D}$ then Gödel logic is not K-compact^{Sat}.*
2. *If $K \in \mathbf{C}_1 \setminus \mathbf{D}$ then $\mathrm{G}_\triangle$ is not K-compact^{Sat}.*
3. *If $K \notin \mathbf{D}_\sim \cup \overline{\mathbf{W}}$, then $\mathrm{G}_\sim$ is not K-compact^{Sat}.*

Proof: This proof is a modification of the proof of [28, Theorem 7.4]. We define a ternary connective $sh(\varphi, \psi, \delta) = \psi \vee (\psi \to \varphi) \vee (\delta \to \psi)$. Observe:

$$e(sh(\varphi, \psi, \delta)) = \begin{cases} e(\psi) & \text{if } e(\varphi) < e(\psi) < e(\delta) \ , \\ 1 & \text{otherwise} \ . \end{cases}$$

The value of a formula $sh(\varphi, \psi, \delta)$ is either the value of its second argument (if the values of arguments are strictly increasingly ordered) or 1 (otherwise).

Let us take a collection $(v_{a,b})_{a>0,\ b>1}$ of mutually distinct propositional variables, and define $v_{0,1} = \mathbf{0}$ and $v_{1,1} = \mathbf{1}$. We define the initial finite sequence $V^1 = (v_{0,1}, v_{1,2}, v_{1,1})$. For each i, we define V^{i+1} as the sequence which results from sequence V^i by inserting element $v_{a+a',b+b'}$ between each pair of its subsequent elements $v_{a,b}$ and $v_{a',b'}$ (in fact, V^i becomes the ith Farey partition if $v_{a,b}$ is understood as the rational $\frac{a}{b}$).

We define the initial set $\Gamma^1 = \{sh(v_{0,1}, v_{1,2}, v_{1,1})\}$. For each i, set Γ^{i+1} is defined as

$$\Gamma^{i+1} = \{sh(v_{a,b},\ v_{a+a',b+b'},\ v_{a',b'}) : v_{a',b'} \text{ is the successor of } v_{a,b} \text{ in } V^i\}\ .$$

Finally, we define $\Gamma = \bigcup_i \Gamma^i$.

Observe that the K-satisfiability of the set Γ^i is equivalent to the existence of an evaluation e which maps the elements of V^i onto a strictly increasing sequence in $K \cup \{\mathbf{0}, \mathbf{1}\}$ (because $K \notin \mathbf{C}_1$ gives us $1 \notin K$). The same argument works for $\bigcup_{j \leq i} \Gamma^j$ instead of Γ^i.

Since K is an infinite set, it contains chains of any finite length. Thus each finite subset of Γ is K-satisfiable.

Let us assume that Γ is K-satisfiable. This means that there is an evaluation e such that for each i the values of elements of V^i form a strictly increasing sequence in $K \cup \{\mathbf{0}, \mathbf{1}\}$. Due to the inductive construction of V^i, the values of all elements of $\bigcup_i V^i$ form a subset of $[0, 1]$, which is dense—a contradiction.

The proof of the second part can be found in [28, Theorem 7.4]—roughly speaking we change the definition of sh: $sh(\varphi, \psi, \delta) = \psi \wedge \neg\triangle(\psi \to \varphi) \wedge \neg\triangle(\delta \to \psi)$. So we get

$$e(sh(\varphi, \psi, \delta)) = \begin{cases} e(\psi) & \text{if } e(\varphi) < e(\psi) < e(\delta)\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

The proof of the last part is based on the previous parts of this proof. The set $K \notin \overline{\mathbf{W}}$, this can be caused by two possibilities:

The set $C = \{x \in K \mid 1 - x \in K\} \cup \{0, \frac{1}{2}, 1\}$ is infinite. In this case we define the set of formulae $\Gamma' = \Gamma \cup \{\sim\varphi \mid \varphi \in \Gamma\}$ (where Γ is the set of formulae from the first part of this proof). This theory is not K-satisfiable, because $K \notin \mathbf{D}_\sim$. However, it is finitely K-satisfiable, because we can find an arbitrarily long increasing sequence $(a)_i$ together with sequence $(1 - a)_i$ in K.

If the set $C = \{x \in K \mid 1 - x \in K\} \cup \{0, \frac{1}{2}, 1\} = \{c_0, \dots c_m\}$ is finite, there has to be an index l such that $K \cap [b_{l-1}, b_l]$ is infinite but not of type $\mathbf{D}$ (where $B = C \cup \mathcal{C}l_{\mathbf{Tv}_\sim}(\bigcup\{K \cap [c_{i-1}, c_i] \mid K \cap [c_{i-1}, c_i] \text{ is finite}\}) = \{b_0, \dots b_n\}$). To fulfill our goal, let us suppose that $K \in C_{\overline{0}}$, the case of $K \in C_{\overline{1}}$ is analogous. In the following we will use the same symbols for the propositional variables and the elements of K and sh for the connective defined in the first part of this proof.

We define the set of formulae

$$\Delta_0 = \{\neg\triangle(c_i \equiv\ \sim c_{m-i}) \vee c_{m-i} \mid i \leq m\} \cup \{sh(\mathbf{0}, c_1, c_2), sh(c_1, c_2, c_3), \dots sh(c_{m-2}, c_{m-1}, \mathbf{1})\}.$$

It is easy to observe that Δ_0 is K-satisfied by e iff $e(c_i) = c_i$. Now if the set $K \cap [c_{i-1}, c_i] = \{k^i_0, \dots k^i_n\}$ is finite, we define $\Delta_i = \{sh(k^i_{j-1}, k^i_j, k^i_{j+1}) \mid 0 < j < n\}$, otherwise we take Δ_i as an empty set. Again Δ_i is K-satisfied by e iff $e(k^i_j) = k^i_j$ for each j. Finally, let us define $\Gamma_0 = \bigcup_{i \geq 0}^n \Delta_i$. Now we notice that b_l is either k^i_j or $1 - k^i_j$ or c_i for some indices i and j and the same holds for b_{l-1}. Let us denote the corresponding propositional variables p and q. Thus we know that each evaluation e K-satisfying Γ_0 holds $e(p) = b_{l-1}$ and $e(g) = b_l$.

Now we define the set of formulae Γ as in the proof for Gödel logic, with the only change in the initial step: we set $v_{0,1} = p$ and $v_{1,1} = q$. Finally we define $\Gamma' = \Gamma \cup \Gamma_0$ and observe that Γ' is finitely K-satisfiable (because in $K \cap [b_{l-1}, b_l]$ we have infinitely many elements) but not K-satisfiable (because it would mean that there is a dense subset in $K \cap [b_{l-1}, b_l]$—a contradiction). QED

Now we formulate the main theorem of this subsection by putting all our results together. We obtain the complete answer of the problem of K-compactness^{Sat} in G and $\mathrm{G}_\triangle$. However, in the case of $\mathrm{G}_\sim$ we have only proved K-compactness^{Sat} for $K \in \mathbf{W}$ and disproved it for $K \notin \overline{\mathbf{W}}$, the problem seems to be open for sets $K \in \overline{\mathbf{W}} \setminus \mathbf{W}$.

Theorem 6.1.27 *Let $K \in \mathbf{C}$.*

1. *Gödel logic is K-compact^{Sat} iff $K \in \mathbf{FIN} \cup \mathbf{C}_1 \cup \mathbf{D}$.*
2. *Gödel logic with $\triangle$ logic is K-compact^{Sat} iff $K \in \mathbf{FIN} \cup \mathbf{D}$.*
3. *Gödel involutive logic is K-compact^{Sat} if $K \in \mathbf{D}_\sim \cup \mathbf{W}$.*
4. *Gödel involutive logic is not K-compact^{Sat} if $K \notin \mathbf{D}_\sim \cup \overline{\mathbf{W}}$.*

Proof: We prove the claims of this theorem in the opposite order.

4. $\mathrm{G}_\sim$ is not K-compact^{Sat} if $K \notin \mathbf{D}_\sim \cup \overline{\mathbf{W}}$ (cf. Lemma 6.1.26).
3. $\mathrm{G}_\sim$ is K-compact^{Sat} if $K \in \mathbf{D}_\sim \cup \mathbf{W}$ (cf, Lemma 6.1.22).
2. We know that $\mathrm{G}_\triangle$ logic is K-compact^{Sat} for $K \in \mathbf{D}$ (cf. Corollary 6.1.25) and for $K \in \mathbf{FIN}$ (because $\mathbf{FIN} \subseteq \mathbf{W}$ we get that $\mathrm{G}_\sim$ logic is K-compact^{Sat}; Remark 6.1.8 completes the proof). Furthermore, we know that it is not K-compact^{Sat} for $K \notin \mathbf{D}$ (if also $K \in \mathbf{C}_1$ we use the second part of Lemma 6.1.26, if $K \notin \mathbf{C}_1$ we use the first part of Lemma 6.1.26 and Remark 6.1.8).
1. We know that G is K-compact^{Sat} for $K \in \mathbf{D} \cup \mathbf{FIN}$ (because of part (3) of this theorem and Remark 6.1.8) and for $K \in \mathbf{C}_1$ (cf. Lemma 6.1.11). Finally, G logic is not K-compact^{Sat} for $K \notin (\mathbf{C}_1 \cup \mathbf{D})$ (cf. Lemma 6.1.26). QED

6.1.4 V-compactness^{Ent}

In this subsection we deal with V-compactness^{Ent}. We use our results from the previous subsection to prove new results. We also obtain known results of Baaz and Zach as a corollary of our new results. The proofs in this subsection are based on the facts that for certain sets we are able to reduce the problem of V-compactness^{Ent} to the problem of $V \setminus \{0\}$-compactness^{Sat} (or prove different but similar reduction theorems).

Lemma 6.1.28 *Let $V \in \mathbf{Tv}_\sim \cap \mathbf{D}_\sim$. Then $\mathrm{G}_\sim$ logic is V-compact^{Ent}. Furthermore, if $V \in \mathbf{Tv} \cap \mathbf{D}$, then G and $\mathrm{G}_\triangle$ are V-compact^{Ent}.*

Proof: We show the proof for $\mathrm{G}_\sim$, the proofs for other logics are analogous. We want to prove that for set of formulae Γ: if $\Gamma \models_V \varphi$, then there is finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models_V \varphi$. Let us prove it indirectly:

For each finite $\Gamma' \subseteq \Gamma$: $\Gamma' \not\models_V \varphi$, i.e., there is a V-evaluation such that $\inf(e(\Gamma')) > e(\varphi)$. Let us take a variable t not occurring in $\Gamma \cup \{\varphi\}$ and alter our K-evaluation e in such a way that: $\inf(e(\Gamma')) > e(t) > e(\varphi)$. Let us define $\overline{\Gamma'} = \{\psi \to t \mid \psi \in \Gamma'\} \cup \{t \to \varphi\}$. Observe that $\overline{\Gamma'}$ is $(V \setminus \{1\})$-satisfiable (by evaluation e). Since $V \setminus \{1\} \in \mathbf{C} \cap \mathbf{D}_\sim$, $\mathrm{G}_\sim$ is $(V \setminus \{1\})$-compact^{Sat}. Thus we know that $\overline{\Gamma} = \{\psi \to t \mid \psi \in \Gamma\} \cup \{t \to \varphi\}$ is $(V \setminus \{1\})$-satisfiable by some V-evaluation e (cf. Corollary 6.1.24). Thus for each $\psi \in \Gamma$ holds $e(\psi) > e(t) > e(\varphi)$, thus $\inf(e(\Gamma)) \geq e(t) > e(\varphi)$ and the proof is done. QED

Theorem 6.1.29

1. *Gödel logic and Gödel logic with $\triangle$ are V-compact^{Ent} iff $V \in \mathbf{Tv} \cap (\mathbf{FIN} \cup \mathbf{D})$.*
2. *Gödel involutive logic is V-compact^{Ent} iff $V \in \mathbf{Tv}_\sim \cap (\mathbf{FIN} \cup \mathbf{D})$.*

Proof: Since V-compactness^{Ent} for G and $G_\triangle$ logics is defined for sets of type $\mathbf{Tv}$ and the V-compactness^{Ent} for $G_\sim$ logic is defined for sets of type $\mathbf{Tv}_\sim$, we will assume that all sets we encounter in this proof are of the appropriate type.

1. We know that G and $G_\triangle$ logics are V-compact^{Ent} for $V \in \mathbf{FIN}$ (cf. Lemma 6.1.14) and for sets $V \in \mathbf{D}$ (cf. Lemma 6.1.28). The fact that G is not V-compact^{Ent} for $V \notin \mathbf{FIN} \cup \mathbf{D}$ follows from Theorem 6.1.27 and Lemma 6.1.13. Finally, $G_\triangle$ is not V-compact^{Ent} for $V \notin \mathbf{FIN} \cup \mathbf{D}$ due to Remark 6.1.8 and the same claim for G.

2. The proof for $G_\sim$ is completely analogous to the proof of the previous claim (just recall that $\mathbf{Tv}_\sim \cap \mathbf{D} = \mathbf{Tv}_\sim \cap \mathbf{D}_\sim$). QED

6.1.5 Correspondence between two notions of compactness

Notice the open problem in Gödel involutive logic where for the sets $V \in \overline{\mathbf{W}} \setminus \mathbf{W}$ the question of V-compactness^{Sat} for $G_\sim$ seems to be open. However, in the definition of the compactness^{Ent} for $G_\sim$ logic, we work with sets of type $\mathbf{Tv}_\sim$ only and $\mathbf{Tv}_\sim \cap (\overline{\mathbf{W}} \setminus \mathbf{W}) = \emptyset$. This observation allows us to fulfill our goal and establish the connection between both notions of compactness despite of our open problem.

The proof of the following theorem is just a corollary of the main theorems of the previous two subsections (namely Theorems 6.1.27 and 6.1.29).

Theorem 6.1.30 *Let VAR be an infinite countable set. If $V \in \mathbf{Tv}$, then the following are equivalent:*

1. $V \in \mathbf{FIN} \cup \mathbf{D}$,
2. *Gödel logic is V-compact^{Ent},*
3. *Gödel logic is $(V \setminus \{1\})$-compact^{Sat},*
4. *Gödel logic is $(V \setminus \{0,1\})$-compact^{Sat}.*

If $V \in \mathbf{Tv}$ ($V \in \mathbf{Tv}_\sim$), then the following are equivalent:

1. $V \in \mathbf{FIN} \cup \mathbf{D}$
2. *Gödel logic with $\triangle$ (Gödel involutive logic respectively) logic is V-compact^{Ent},*
3. *Gödel logic with $\triangle$ (Gödel involutive logic respectively) logic is $(V \setminus \{1\})$-compact^{Sat},*
4. *Gödel logic with $\triangle$ (Gödel involutive logic respectively) logic is $(V \setminus \{0,1\})$-compact^{Sat},*
5. *Gödel logic with $\triangle$ (Gödel involutive logic respectively) logic is $(V \setminus \{0\})$-compact^{Sat}.*

6.2 Compactness in other core fuzzy logics

In this section we extend the notion of satisfiability and compactness^{Sat} (Definitions 6.1.6 and 6.1.7) to other core fuzzy logics. Since we work with satisfiability based compactness only we omit the index ^{Sat}.

6.2.1 Satisfiability based compactness

Before we start we recall some definition from the previous section and generalized them to all core fuzzy logics. We also define two more types of subsets of [0,1]. By the term compact in the following definition me mean compact in the standard topology of [0,1].

Definition 6.2.1 *We define:*

$$\mathbf{COM} = \{K \subseteq \mathbf{C} \mid K \textit{ is compact}\},$$

$$\mathbf{Q} = \{K \subseteq [0,1] \mid K \cap \mathbb{Q} = [0,1] \cap \mathbb{Q}\}.$$

Definition 6.2.2 *For a set Γ of formulae in some core fuzzy logic and $K \subseteq [0,1]$, we say that Γ is K-*satisfiable *if there exists an evaluation e such that $e(\varphi) \in K$ for all $\varphi \in \Gamma$. The set Γ is said to be* finitely K-satisfiable *if each finite subset of Γ is K-satisfiable. Formula φ is called K-satisfiable if the set $\{\varphi\}$ is K-satisfiable.*

If we recall the standard semantics of the core fuzzy logics we notice that with exception of $\text{Ł}\Pi\frac{1}{2}$ if $K \in \mathbf{C}$, then for any formula φ there is an evaluation e such that $e(\varphi) \in K$ (it suffices to evaluate all propositional variables by 0, then the evaluation of each formula becomes either 0 or 1). Thus as in Gödel logics we restrict ourselves to the set of type $\mathbf{C}$.

Definition 6.2.3 *Let $K \in \mathbf{C}$. We say that logic is K-*compact *if K-satisfiability is equivalent to finite K-satisfiability.*

We observe that Remark 6.1.8 remains valid in this more general setting.

Remark 6.2.4 *If we extend the set of connectives of a logic, L, then the resulting logic, L', becomes "less compact", i.e., if L' is K-compact (resp. has the compactness property) then L is K-compact (resp. has the compactness property). This is obvious since each counterexample to K-compactness in L is also a counterexample to K-compactness in L'.*

6.2.2 Łukasiewicz logic and Product Łukasiewicz logic

As an introduction to further results, we give a short proof of compactness that works for Łukasiewicz logic, as well as for some other fuzzy logics and for classical logic. See [86] for the first application of this kind of proof in fuzzy logics (in rather restricted setting).

Theorem 6.2.5 *Let VAR be of arbitrary cardinality and $K \in \mathbf{COM}$. Then* **Ł** *and* PŁ′ *logics are K-compact.*

Proof: Let Γ be a set of formulae. For each $\varphi \in \Gamma$, the mapping $H_\varphi \colon [0,1]^A \to [0,1]$ defined by $H_\varphi(e) = e(\varphi)$ is continuous (because all the standard interpretation of connectives of these logics are continuous). The preimages $(H_\varphi^{-1}(K))_{\varphi\in\Gamma}$ are closed subsets of the set $[0,1]^A$ which is compact in the product (=weak) topology (Tichonoff theorem). Due to finite K-satisfiability, the collection $(H_\varphi^{-1}(K))_{\varphi\in\Gamma}$ is centered (i.e., each finite subset has a nonempty intersection). Hence the intersection $\bigcap_{\varphi\in\Gamma} H_\varphi^{-1}(K)$ is nonempty; each of its elements is an evaluation which verifies that Γ is K-satisfiable. QED

It was necessary to have the set K closed (compact); K-compactness does not hold in Łukasiewicz logic (and thus in PŁ′ as well) if K is not closed. Let us prove this claim:

Lemma 6.2.6 *Let $a, b \in [0,1]$. Then there is a formula $\psi_{a,b}$ with one atom v in Łukasiewicz logic such that:*

$$\begin{aligned} e(v) < a &\implies e(\psi_{a,b}) = 0\ , \\ e(v) > b &\implies e(\psi_{a,b}) = 1\ . \end{aligned}$$

Proof: We use McNaughton theorem (Theorem 3.3.6). We can find rationals $c, d \in (a, b)$, $c < d$, and a McNaughton function f such that

$$f(x) = \begin{cases} 0 & \text{if } x \leq c \ , \\ \frac{x-c}{d-c} & \text{if } c \leq x \leq d \ , \\ 1 & \text{if } x \geq d \ . \end{cases}$$

Let $\psi_{a,b}$ be the formula which is interpreted by function f. QED

Notice that the formula $\psi_{a,b}$ in the latter lemma is not unique.

Corollary 6.2.7 *Let $a, a', b', b \in [0, 1]$, $a < a' < b' < b$. Then there is a formula φ with one atom v in Łukasiewicz logic such that:*

$$e(\varphi) = \begin{cases} 0 & \textit{if } e(v) \leq a \ , \\ e(v) & \textit{if } a' \leq e(v) \leq b' \ , \\ 0 & \textit{if } e(v) \geq b \ . \end{cases}$$

Proof: We use the derived connective $\underset{\mathbf{M}}{\wedge}$ of the Łukasiewicz logic defined by $\alpha \underset{\mathbf{M}}{\wedge} \beta = \alpha \& (\alpha \to \beta)$. Its interpretation is $e(\alpha \underset{\mathbf{M}}{\wedge} \beta) = \min(e(\alpha), e(\beta))$. We take $\varphi = v \underset{\mathbf{M}}{\wedge} \psi_{a,a'} \underset{\mathbf{M}}{\wedge} \neg\psi_{b',b}$, where $\psi_{a,a'}, \psi_{b',b}$ are obtained from Lemma 6.2.6 (with v as the only atom). QED

Using these lemmata we can give full characterization of the K-compactness in Łukasiewicz and PŁ$'$ logics.

Theorem 6.2.8 *Let VAR be of arbitrary cardinality and $K \in \mathbf{C}$. Then* **Ł** *and* PŁ$'$ *logics are K-compact iff $K \in \mathbf{COM}$.*

Proof: One direction is Theorem 6.2.5. We show the reverse direction for Łukasiewicz logic and we get it for PŁ$'$ using Remark 6.2.4.

As K is not closed, there is a sequence of points in K converging to $b \notin K$. From this sequence, we select a monotonic subsequence, say $(a_i)_{i \in \mathbb{N}}$. Let us assume that this sequence is strictly increasing. We assume that $b \neq 1$. (The proof for $b = 1$ is just a simpler version of the following procedure – just omit the term $\neg\psi_{b',b}$ in the proof of Corollary. 6.2.7 and make the corresponding changes in the rest). We take a strictly decreasing sequence $(b_i)_{i \in \mathbb{N}}$ converging to b (here we do not care of its intersection with K).

First, let us assume that K is of type $C_{\overline{0}}$.

We fix an propositional variable, v. Using Corollary. 6.2.7, we construct formula φ_i (having v as its only propositional variable) such that:

$$e(\varphi_i) = \begin{cases} 0 & \text{if } e(v) \leq a_i \ , \\ e(v) & \text{if } a_{i+1} \leq e(v) \leq b_{i+1} \ , \\ 0 & \text{if } e(v) \geq b_i \ . \end{cases}$$

We take $\Gamma = \{v\} \cup \{\varphi_i : i \in \mathbb{N}\}$. Then Γ is finitely K-satisfiable. Indeed, for a finite subset $\Phi \subseteq \Gamma$, we take the maximal index j such that $\varphi_j \in \Phi$, and the evaluation $e(v) = a_{j+1}$ maps all elements of Φ onto $a_{j+1} \in K$.

Suppose that Γ is K-satisfiable. Notice that if φ_i is K-satisfiable, then $e(v) \in (a_i, b_i)$ (otherwise $e(\varphi_i) = 0 \notin K$). Thus the satisfiability of Γ implies that the evaluation of v must be in the intersection of all intervals (a_i, b_i), $i \in \mathbb{N}$. This intersection contains only b which is not in K—a contradiction. We have proven that Γ is not K-satisfiable.

The proofs for the sequence $(a_i)_{i \in \mathbb{N}}$ strictly decreasing or for K being of type $C_{\overline{1}}$ are analogous. QED

Notice that the proof of Th. 6.2.5 works also in classical logic; obviously, every function on a finite product of discrete spaces $\{0,1\}$ is continuous and, due to Tichonoff theorem, $\{0,1\}^A$ is compact. The compactness property can be proven similarly also in other fuzzy logics in which all operations are interpreted by continuous functions. In particular, this applies to the so-called *S-fuzzy logics*. These logic are not *fuzzy* in the sense of this work (they are not even weakly implicative). These logics were introduced in [86] and investigated in detail in [71, 61, 56] S-fuzzy logics are given semantically, the basic connectives are negation $\neg$, interpreted by the standard fuzzy negation $N_{\mathbf{S}}$, and conjunction $\wedge$, interpreted by a continuous triangular norm. Implication $\to$ in an S-fuzzy logic is a derived connective

$$\varphi \to \psi = \neg(\varphi \wedge \neg\psi) \ .$$

The compactness property of S-fuzzy logics has been proved in [86, Th. 3.3].

6.2.3 Product logic

We start with observation that the proof of the Lemma 6.1.11 remains valid for Product logic (in fact it is valid for arbitrary fuzzy logic with strict negation).

Lemma 6.2.9 *Let the set VAR be of an arbitrary cardinality and $K \in \mathbf{C}_1$ or $K = \{\mathbf{0}\}$. Then Product logic is K-compact.*

Proof: For K being of type $\mathbf{C}_1$ we use Lemma 6.1.11. For K being $\{0\}$ just observe that formula φ is $\{0\}$-satisfiable iff formula $\neg\varphi$ is $\{1\}$-satisfiable. QED

To prove the reverse direction we need one additional assumption.

Theorem 6.2.10 *Let the set VAR contain at least two elements and $k \in C$. Then product logic is K-compact iff $K \in \mathbf{C}_1$ or $K = \{\mathbf{0}\}$.*

Proof: We prove the reverse direction contrapositively: if K is not of type $\mathbf{C}_1$ and $K \neq \{0\}$ there is an element $k \in K$, $0 < k < 1$. Let us define the set of formulae $\Gamma = \{(r \to p^n) \to p \mid 0 < n\} \cup \{\neg r \vee p\}$, where p and r are propositional variables.

We show that Γ is finitely K-satisfiable. Let Γ' be a finite subset of Γ and let m be the biggest index occurring in formulae of Γ'. Let us evaluate $e(p) = k$ and $e(r) = k^m$. Obviously $e(\Gamma') = \{k\}$.

To complete the proof we show that is not Γ is not K-satisfiable. Let $e(p) = x$ and $e(r) = y$. If $y = 0$, then $e(\neg r \vee p) = 1$ i.e., Γ is not K-satisfiable. Let us assume that $y > 0$. Notice that both $x = 1$ and $x = 0$ leads to non K-satisfiability. Let us assume that $0 < x < 1$, then there is index n such that $x^{n-1} \leq y$. So $\frac{x^n}{y} \leq x$ and so $e((r \to p^n) \to p) = 1$. QED

6.2.4 Logics with $\triangle$

In this subsection we examine the compactness property for logics with the set of connectives extended by an additional unary connective $\triangle$ (0-1 *projector* or *Baaz delta*)

Lemma 6.2.11 *Let the set VAR contain at least two elements and $K \in \mathbf{C}_0 \cup \mathbf{C}_1$. Then $\text{Ł}_\triangle$ and $\Pi_\triangle$ logics are not K-compact for any subset K of $[0,1]$ of type C_{01}.*

Proof: We show the proofs for both logic at once. Let us assume that K is of type $\mathbf{C}_1$ (i.e. $1 \in K$ and $0 \notin K$). Let $\Gamma = \{\neg\triangle\neg r,\ \neg\triangle p\} \cup \{\varphi_i : i \in \mathbb{N}\}$, where p and r are propositional variables and $\varphi_i = \triangle(r \to p^i)$. Notice that for each evaluation e we have

$$e(\neg\triangle\neg r) = \begin{cases} 1 & \text{if } e(r) > 0 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

(In case of product logic with $\triangle$, it suffices to take $\neg\neg r$ instead of $\neg\triangle\neg r$.) Let Γ' be a finite subset of Γ. Let j be the greatest index i for which $\varphi_i \in \Gamma'$. Define $e(r) = \frac{1}{2}$ and $e(p) < 1$ such that $\frac{1}{2} \le e(p^j)$ (such a value $e(p)$ indeed exists). Then $e(\neg\triangle\neg r) = 1$, $e(\neg\triangle p) = 1$ and $e(\varphi_i) = 1$ for all $i \le j$. Thus Γ' is K-satisfiable and Γ is finitely K-satisfiable.

On the other side, suppose that there is an evaluation e making Γ K-satisfiable. Then $e(\neg\triangle p) > 0$ which is possible only for $e(\neg\triangle p) = 1$ and this implies $e(p) < 1$. Analogously, $e(\neg\triangle\neg r) > 0$ implies $e(\neg\triangle\neg r) = 1$ and $e(r) > 0$. Then there is an integer j such that $e(p^j) < e(r)$ and $e(\varphi_j) = 0 \notin K$, a contradiction. Thus Γ is not K-satisfiable.

Now let us assume that K is of type $\mathbf{C}_0$ (i.e. $0 \in K$ and $1 \notin K$). The proof is analogous, the contradictory set of formulae is $\Gamma' = \{\triangle\neg r,\ \triangle p\} \cup \{\varphi_i : i \in \mathbb{N}\}$, where p and r are propositional variables and $\varphi_i = \neg\triangle(r \to p^i)$. QED

Lemma 6.2.12 *Let the set VAR contain at least two elements and $K \in \mathbf{C}_{\overline{01}}$. Then $\text{Ł}_\triangle$ logic is not K-compact.*

Proof: This is just analogy of the previous lemma. We know that $1 \notin K$ and $0 \notin K$), thus there is $k \in K$, such that $0 < k < 1$ Let $\Gamma = \{r,\ \triangle p \vee r\} \cup \{\varphi_i : i \in \mathbb{N}\}$, where p and r are propositional variables and $\varphi_i = \triangle(r \to p^i) \wedge r$.

Let Γ' be a finite subset of Γ. Let j be the greatest index i for which $\varphi_i \in \Gamma'$. Define $e(r) = k$ and $e(p) < 1$ such that $k \le e(p^j)$ (such a value $e(p)$ indeed exists). Then $e(r) = k$, $e(\triangle p \vee r) = k$ and $e(\varphi_i) = k$ for all $i \le j$. Thus Γ' is K-satisfiable and Γ is finitely K-satisfiable.

On the other side, suppose that there is an evaluation e making Γ K-satisfiable. Then $e(\triangle p \vee r) \neq 1$ which is possible only for $e(\triangle p) \neq 1$ and this implies $e(p) < 1$. Analogously, we get $e(r) > 0$. Then there is an integer j such that $e(p^j) < e(r)$ and $e(\varphi_j) = 0 \notin K$, a contradiction. Thus Γ is not K-satisfiable. QED

The following theorem is just a corollary of the previous two lemmata and Remark 6.2.4.

Theorem 6.2.13 *Let the set VAR contain at least two elements and $K \in \mathbf{C}$. Then $\text{Ł}_\triangle$, $\Pi_\triangle$, and $\text{PŁ}'_\triangle$ logics are not K-compact.*

6.2.5 ŁΠ and ŁΠ$\frac{1}{2}$ logics

Here we investigate the compactness of ŁΠ and ŁΠ$\frac{1}{2}$ propositional logics. Using Remark 6.2.4 and the fact that both these logics contain the Łukasiewicz logic with delta we immediately get:

Corollary 6.2.14 *Let the set VAR contain at least two elements and $K \in \mathbf{C}$. Then ŁΠ and ŁΠ$\frac{1}{2}$ logics are not K-compact.*

However, in ŁΠ$\frac{1}{2}$ logic, we may define arbitrary rational constants from $[0, 1]$. This means that ŁΠ$\frac{1}{2}$ logic may lack K-compactness for sets which are not of type C, because the argument about trivial K-satisfiability of each set Γ does not hold (take Γ containing a single constant).

Theorem 6.2.15 *Let the set VAR contain at least two elements. Then the ŁΠ$\frac{1}{2}$ logic is K-compact iff $K \in \mathbf{Q}$.*

Proof: One direction is obvious (just take arbitrary evaluation which evaluates to rationals only and observe that evaluation of arbitrary formula is rational as well)

The second direction: the only unproved case is if $0, 1 \in K$ (otherwise K would be of be of type C and we could use the previous corollary). Assume that rational $q \notin K$. The truth constant $\mathbf{q}$, corresponding to the rational q, can be defined within ŁΠ$\frac{1}{2}$ logic. We may observe that a formula φ is $\{1\}$−satisfiable iff formula $\varphi' = \triangle\varphi \vee \mathbf{q}$ is K-satisfiable. Thus our logic cannot be K-compact (otherwise it would be $\{1\}$-compact, but the set $\{1\}$ is of type $\mathbf{C}$). QED

Table 6.1: K-compactness^{Sat}

Logic	Finite	Countable	Uncountable
$\mathbf{Ł}$, $\mathrm{PŁ}'$	$\mathbf{COM}$	$\mathbf{COM}$	$\mathbf{COM}$
Π	$\mathbf{C}_1 \cup \{\{0\}\}$	$\mathbf{C}_1 \cup \{\{0\}\}$	$\mathbf{C}_1 \cup \{\{0\}\}$
G	All	$\mathbf{FIN} \cup \mathbf{C}_1 \cup \mathbf{D}$	$\mathbf{C}_1 \cup \{\{0\}\}$
$\mathrm{G}_\triangle$	All	$\mathbf{FIN} \cup \mathbf{D}$	None
$\mathrm{G}_\sim$	All	?	None
$\mathbf{Ł}_\triangle$, $\Pi_\triangle$, $\mathrm{PŁ}'_\triangle$, ŁΠ	None	None	None
$\mathrm{Ł\Pi}\frac{1}{2}$	$\mathbf{Q}$	$\mathbf{Q}$	$\mathbf{Q}$

Table 6.2: K-compactness^{Ent}

Logic	Finite	Countable	Uncountable
G	All	$\mathbf{FIN} \cup \mathbf{D}$	$\mathbf{FIN}$
$\mathrm{G}_\triangle$	All	$\mathbf{FIN} \cup \mathbf{D}$	$\mathbf{FIN}$
$\mathrm{G}_\sim$	All	$\mathbf{FIN} \cup \mathbf{D}_\sim$	$\mathbf{FIN}$

6.3 Conclusion

Table 6.1 summarizes our results about satisfiability based compactness from this chapter. The first column indicates the type of logic, the other ones show for which sets K the logic enjoys the K-compactness for particular cardinalities of the set VAR. With exception of $\mathrm{G}_\sim$ for countable set VAR all of them are full characterizations. Recall that case we know only that Gödel involutive logic is K-compact if $K \in \mathbf{D}_\sim \cup \mathbf{W}$ and Gödel involutive logic is not K-compact if $K \notin \mathbf{D}_\sim \cup \overline{\mathbf{W}}$.

Also recall that if set K is not of type C than all our logics with exception of $\mathrm{Ł\Pi}\frac{1}{2}$ are K-compact. Thus all but the last rows of the table should by completed by $\cup\mathbf{Tv}$ (for simplicity we omit it). Also notice that for the results for the logics $\mathrm{Ł}_\triangle$, $\Pi_\triangle$, $\mathrm{PŁ}'_\triangle$, ŁΠ, and $\mathrm{Ł\Pi}\frac{1}{2}$ in case of finite set VAR we assume that VAR contains at least two elements.

Table 6.2 summarizes our results about entailment based compactness from this chapter. The first column indicates the type of logic, the other ones show for which sets K the logic enjoys the K-compactness for particular cardinalities of the set VAR. Recall that entailment based compactness is defined for set of type $\mathbf{Tv}$ ($\mathbf{Tv}_\sim$ in the case of $\mathrm{G}_\sim$). Thus all rows of this table should by completed by $\cap\mathbf{Tv}$ ($\cap\mathbf{Tv}_\sim$ respectively), but for the sake of simplicity we omit it.

Chapter 7

Fuzzy class theory

In this chapter we give a formal grounding of the upcoming project of formalizing and axiomatizing of the fuzzy mathematics (as announced in [5], see also Chapter 8).

Although fuzzy mathematics is nowadays very broad, the notion of fuzzy set is still a central concept. The obvious idea is to formalize fuzzy sets using already formalized fuzzy logic in analogy with classical set theory. Previous attempts to formalize this notion have several drawbacks from the point of view of their suitability for the formalization of Zadeh's notion of fuzzy set. The papers [54] and [80] are mainly interested by metamathematical properties of fuzzified Zermelo-Fraenkel set theory and the papers [82] and [4] restrict themselves to logics given by one particular t-norm and so they lack a sufficient expressive power to capture the general notion of fuzzy set. Inspecting these (and a few other) approaches we came to two conclusions: first, we do not need a full-fledged set theory and second, we need an expressively rich fuzzy logic as a logical background.

By a full-fledged set theory we mean an analogy of classical set theory with all its concepts. We observed that real-world applications of fuzzy sets need only small portion of set-theoretical concepts. The central notion in fuzzy sets is the membership of an element of the universe into a fuzzy set (not the membership of a fuzzy set into another fuzzy set). Formalization of such a concept in the classical setting is called *elementary set theory,* or *class theory.* Roughly speaking, class theory is a theory with two sorts of individuals—objects and classes—and one binary predicate—the membership of objects into classes. In this chapter we develop a fuzzy class theory. The classes in our theory correspond exactly to Zadeh's fuzzy sets.

By an expressively rich logic we mean a logic of great expressive power, yet with a simple axiomatic system and good logical properties (deduction theorem, Skolem function introduction and eliminability, etc.). ŁΠ∀ seems to be the most suitable logic for our needs. In this chapter we developed fuzzy class theory over the predicate logic ŁΠ, however if we examine the definitions and theorems we notice that nearly all of them will work in other fuzzy logics as well. We think that fixing the underlying logic will make important class-theoretical concepts clearer. Fuzzy class theory for a wider class of fuzzy logics can be a topic of some upcoming paper.

We show that the proposed theory is a simple, yet powerful formalism for working with elementary relations and operations on fuzzy sets (normality, equality, subsethood, union, intersection, kernel, support, etc.). By a small enhancement of our theory (adding tools to manage tuples of objects) we obtain a formalism powerful enough to capture the notion of fuzzy relation. Thus we can formally introduce the notions of T-transitivity, T-similarity, fuzzy ordering, and many other concepts defined in the literature. Finally, we extend our formalism to something which can be viewed as simple fuzzy type theory. Basically, we introduce individuals for classes of classes, classes of classes of classes etc. This allows us to formalize other parts of fuzzy mathematics (e.g., fuzzy topology). Our theory thus aspires to the status of *foundations of fuzzy mathematics* and a uniform formalism that can make interaction of various disciplines of fuzzy mathematics possible.

Of course, this chapter cannot cover all the topics mentioned above. For the majority of them we only give the very basic definitions, and there is a lot of work to be done to show that the proposed formalism is suitable for them. We concentrate on the development of basic properties of fuzzy sets. In this area our formalism proved itself worthy, as it allows us to state several very general metatheorems that effectively reduce a wide range of theorems on fuzzy sets to fuzzy propositional calculus. This success is a promising sign for our formalism to be suitable for other parts of fuzzy mathematics as well.

As mentioned above, in this chapter we restrict ourselves to notions that can be defined without adding a structure (similarity, metrics, etc.) to the universe of objects. Nevertheless, our formalism possesses means for adding a structure to the universe (usually by fixing a suitable class which satisfies certain axioms), which is necessary for the development of more advanced parts of fuzzy set theory. Such extensions of our theory will be elaborated in subsequent papers, for some hints see Section 7.4.

All these ideas about formalization of fuzzy mathematics are summed up and elaborated in much more details in Chapter 8. There we also deal with the relation of our approach and some older attempts in this direction. Let us now just mention, that some of the roots of our approach (and some of our concepts, like graded properties of fuzzy relations) can be found already in Gotwalld's book [43]. Another, rather surprising source of similar ideas is Höhle's paper [58], where in Section 5 the author writes:

> "It is the opinion of the author that from a mathematical viewpoint the important feature of fuzzy set theory is the replacement of the two-valued logic by a multiple-valued logic. [... I]t is now clear how we can find for every mathematical notion its 'fuzzy counterpart'. Since every mathematical notion can be written as a formula in a formal language, we have only to internalize, i.e. to interpret these expressions by the given multiple-valued logic."

7.1 Class theory over ŁΠ

7.1.1 Axioms

Fuzzy class theory FCT is a theory over ŁΠ∀ with two sorts of variables: *object variables,* denoted by lowercase letters $x, y, \ldots$, and *class variables,* denoted by uppercase letters $X, Y, \ldots$ None of the sorts is subsumed by the other.

The only primitive symbol of FCT is the binary membership predicate $\in$ between objects and classes (i.e., the first argument must be an object and the second a class; class theory takes into consideration neither the membership of classes in classes, nor of objects in objects).

The principal axioms of FCT are instances of the class comprehension scheme: for any formula ϕ not containing X (it may, however, contain any other object or class parameters),

$$(\exists X) \triangle (\forall x) (x \in X \leftrightarrow \phi(x))$$

is an axiom of FCT. The strange $\triangle$ is necessary for securing that the required class exists in the degree 1 (rather than being only approximated by classes satisfying the equivalence in degrees arbitrarily close to 1). The $\triangle$ is also necessary for the conservativeness of the introduction of comprehension terms[1] $\{x \mid \phi(x)\}$ with axioms

$$y \in \{x \mid \phi(x)\} \leftrightarrow \phi(y)$$

and their eliminability. In the standard recursive way one proves that ϕ in comprehension terms may be allowed to contain other comprehension terms.

[1]I.e., the Skolem functions of comprehension axioms, see Theorem 3.5.14.

The consistency of FCT is proved by constructing a model. Let M be an arbitrary set and $\mathbf{L}$ be a complete linear ŁΠ-algebra. The *Zadeh model* $\mathbb{M}$ *over the universe* M *and the algebra of truth-values* $\mathbf{L}$ is constructed as follows:

The range of object variables is M, the range of class variables is the set of all functions from M to $\mathbf{L}$. For any evaluation e we define $\|x \in X\|^{\mathbf{L}}_{\mathbb{M},e}$ as the value of the function $e(X)$ on $e(x)$. The value of the comprehension term $\{x \mid \phi(x)\}$ is defined as the function taking an object a to $\|\phi(a)\|^{\mathbf{L}}_{\mathbb{M},e}$ (in fact, the characteristic function of $\phi(x)$ where e fixes the parameters). Then it is trivial that $\|y \in \{x \mid \phi(x)\}\|^{\mathbf{L}}_{\mathbb{M},e} = \|\phi(y)\|^{\mathbf{L}}_{\mathbb{M},e}$ which proves the comprehension axiom.

If $\mathbf{L} = [\mathbf{0}, \mathbf{1}]$, we call the described model *standard*.

Definition 7.1.1 *Let $\mathbb{M}$ be a model and A a class in $\mathbb{M}$. The characteristic function $\chi_{x \in A}$ is denoted briefly by χ_A and also called the* membership function *of A. (Instead of $\chi_A(x)$ or $\|x \in A\|$ many papers use just Ax.)*

It can be observed that the crisp formula $(\forall x) \triangle (x \in X \leftrightarrow x \in Y)$ expresses the identity of the membership functions of X and Y (as in all models $\|(\forall x) \triangle (x \in X \leftrightarrow x \in Y)\| = 1$ iff the membership functions of X and Y are identical, otherwise 0). Since our intended notion of fuzzy class is extensional, i.e., that fuzzy classes are determined by their membership functions, it is reasonable to require the *axiom of extensionality* which identifies classes with their membership functions:

$$(\forall x) \triangle (x \in X \leftrightarrow x \in Y) \to X = Y$$

(the converse implication follows from the axioms for identity). The consistency of this axiom is proved by its validity in Zadeh models.

The comprehension scheme of FCT still allows classical models, as the construction of Zadeh models works for the ŁΠ-algebra $\{\mathbf{0}, \mathbf{1}\}$. Sometimes it may be desirable to exclude classical models. This can be done either by taking ŁΠ$\frac{1}{2}$ instead of ŁΠ as the underlying logic, or equivalently by adding two constants C, c and the *axiom of fuzziness* $c \in C \leftrightarrow \neg_{\text{Ł}} c \in C$ without changing the underlying logic. In both cases there is a sentence with the value $\frac{1}{2}$ in any model, and all rational truth constants are therefore definable. The consistency of this extension follows from the fact that it holds in standard Zadeh models.

General models of FCT correspond in the obvious way to Henkin's general models of classical second-order logic, while Zadeh models correspond to full second-order models. FCT with its axioms of comprehension and extensionality thus can be viewed as a notational variant of the second-order fuzzy logic ŁΠ (monadic, in the form presented in this section; for higher arities see Section 7.2). Second-order fuzzy logics will be more closely examined in [7]. In this chapter we prefer FCT formulated as a two-sorted first-order theory, because it is closer to the usual formulation of classical set or class theory, and because of its axiomatizability. For even though (standard) Zadeh models are the intended models of FCT, the theory of Zadeh models is not arithmetically definable, let alone recursively axiomatizable. This follows from the obvious fact that classical full second-order logic (which itself is non-arithmetical) can be interpreted in the theory of Zadeh models by inscribing $\triangle$ (or $\neg\neg_{\mathrm{G}}$) in front of every atomic formula.

7.1.2 Elementary class operations

Elementary class operations are defined by means of propositional combination of atomic formulae of FCT.

Convention 7.1.2 *Let $\phi(p_1, \ldots, p_n)$ be a propositional formula and $\psi_1, \ldots, \psi_n$ be any formulae. By $\phi(\psi_1, \ldots, \psi_n)$ we denote the formula ϕ in which all occurrences of p_i are replaced by ψ_i (for all $i \leq n$).*

Table 7.1: Elementary class operations

ϕ	$\mathrm{Op}_\phi(X_1,\ldots,X_n)$	Name
0	$\emptyset$	empty class
1	V	universal class
$\triangle(\alpha \to p)$	X_α	α-cut
$\triangle(\alpha \leftrightarrow p)$	$X_{=\alpha}$	α-level
$\neg_{\mathrm{G}} p$	$\backslash X$	strict complement
$\neg_{\text{Ł}} p$	$-X$	involutive complement
$\neg_{\mathrm{G}}\neg_{\text{Ł}} p$ (or $\triangle p$)	$\mathrm{Ker}(X)$	kernel
$\neg\neg_{\mathrm{G}} p$ (or $\neg\triangle\neg_{\text{Ł}} p$)	$\mathrm{Supp}(X)$	support
$p \&_* q$	$X \cap_* Y$	$*$-intersection
$p \vee q$	$X \cup Y$	union
$p \oplus q$	$X \uplus Y$	strong union
$p \&_* \neg_{\mathrm{G}} q$	$X \setminus_* Y$	strict $*$-difference
$p \&_* \neg_{\text{Ł}} q$	$X -_* Y$	involutive $*$-difference

Definition 7.1.3 *Let $\phi(p_1,\ldots,p_n)$ be a propositional formula. We define the n-*ary class operation *generated by ϕ as*

$$\mathrm{Op}_\phi(X_1,\ldots,X_n) =_{\mathrm{df}} \{x \mid \phi(x \in X_1,\ldots,x \in X_n)\}\,.$$

Among elementary class operations we find the following important kinds:

- *Class constants.* We denote Op_0 by $\emptyset$ and call it the *empty class,* and Op_1 by V and call it the *universal class.*
- *α-Cuts.* Let α be a truth-constant. Then we call the class $\mathrm{Op}_{\triangle(\alpha\to p)}(X)$, i.e., the class $\{x \mid \triangle(\alpha \to (x \in X))\}$, the *$\alpha$-cut* of X and abbreviate it X_α. Similarly, $\mathrm{Op}_{\triangle(\alpha\leftrightarrow p)}(X)$ is called the *α-level* of X, denoted by $X_{=\alpha}$.
- *Iterated complements,* i.e., class operations Op_ϕ where ϕ is p prefixed with a chain of negations. In ŁΠ, there are only a few such formulae that are non-equivalent. They yield the following operations (their definitions are summarized in Table 7.1): *involutive* and *strict complements*, the *kernel* and *support*, and the complement of the kernel. Except for the involutive complement, all of them are crisp.
- *Simple binary operations.* Some of the class operations $\mathrm{Op}_{p\circ q}$ where $\circ$ is a (primitive or derived) binary connective have their traditional names and notation, listed in Table 7.1 (not exhaustively).

7.1.3 Elementary relations between classes

Most of important relations between classes have one of the two forms described in the following definition:

Definition 7.1.4 *Let $\phi(p_1,\ldots,p_n)$ be a propositional formula. The n-ary* uniform relation *between $X_1,\ldots,X_n$ generated by ϕ is defined as*

$$\mathrm{Rel}^{\forall}_\phi(X_1,\ldots,X_n) \equiv_{\mathrm{df}} (\forall x)\, \phi(x \in X_1,\ldots,x \in X_n).$$

The n-ary supremal relation *between $X_1,\ldots,X_n$ generated by ϕ is defined as*

$$\mathrm{Rel}^{\exists}_\phi(X_1,\ldots,X_n) \equiv_{\mathrm{df}} (\exists x)\, \phi(x \in X_1,\ldots,x \in X_n).$$

Table 7.2: Class properties and relations

Relation	Notation	Name
$\mathrm{Rel}^{\exists}_{p}(X)$	$\mathrm{Hgt}(X)$	height
$\mathrm{Rel}^{\exists}_{\triangle p}(X)$	$\mathrm{Norm}(X)$	normality
$\mathrm{Rel}^{\forall}_{\triangle(p\vee\neg p)}(X)$	$\mathrm{Crisp}(X)$	crispness
$\mathrm{Rel}^{\exists}_{\neg\triangle(p\vee\neg p)}(X)$	$\mathrm{Fuzzy}(X)$	fuzziness
$\mathrm{Rel}^{\forall}_{p\to_* q}(X,Y)$	$X\subseteq_* Y$	$*$-inclusion
$\mathrm{Rel}^{\forall}_{p\leftrightarrow_* q}(X,Y)$	$X =_* Y$	$*$-equality
$\mathrm{Rel}^{\exists}_{p\&_* q}(X,Y)$	$X\\|_* Y$	$*$-compatibility

Among elementary class relations we find the following important kinds (they are summarized in Table 7.2):

- *Equalities* $\mathrm{Rel}^{\forall}_{p\leftrightarrow_* q}$ denoted $=_*$. The value of $X =_{\mathrm{G}} Y$ is the maximal truth degree below which the membership functions of X and Y are identical. In standard $[\mathbf{0},\mathbf{1}]$-models, $1-\|X=Y\|$ is the maximal difference of the (values of) the membership functions of X and Y, and $\|X=_\Pi Y\|$ is the infimum of their ratios. All $=_*$ get value 1 iff the membership functions are identical. For crisp classes, these notions of equality coincide with classical equality.

- *Inclusions* $\mathrm{Rel}^{\forall}_{p\to_* q}$, denoted $\subseteq_*$. Their semantics is analogous to that of equalities. They get the value 1 iff the membership function of X is majorized by that of Y.

- *Compatibilities* $\mathrm{Rel}^{\exists}_{p\&_* q}$. Their strict and involutive negations may respectively be called *strict* and *involutive $*$-disjointness.*

- *Unary properties* of height, normality, fuzziness, and crispness.

Notice that due to the axiom of extensionality, the relation $\mathrm{Rel}^{\forall}_{\triangle(p\leftrightarrow q)}$, which is obviously equivalent to $\triangle(X=_* Y)$, coincides with the identity of classes. Thus it is $\triangle(X=_* Y)$ that guarantees intersubstitutivity *salva veritate* in all formulae (equalities generally do not).

It can be noticed that Gödel equality $=_{\mathrm{G}}$ is highly true only if the membership functions are identical on low truth values; product equality $=_\Pi$ is also more restrictive on lower truth values. However, this does not conform with the intuition that the difference in the *high* values (on the "prototypes") should matter more than a negligible difference on objects that almost do not belong to the classes under consideration. Equality of involutive complements, $-X =_* -Y$, is therefore a better measure of similarity of classes. Similarly, $-Y\subseteq_* -X$ may give a better measure of containment of X in Y than $X\subseteq_* Y$.

7.1.4 Theorems on elementary class relations and operations

The following metatheorems show that a large part of elementary fuzzy set theory can be reduced to fuzzy propositional calculus.

Theorem 7.1.5 *Let $\phi,\psi_1,\dots,\psi_n$ be propositional formulae. Then* $\vdash\phi(\psi_1,\dots,\psi_n)$

$$\textit{iff}\quad \vdash \mathrm{Rel}^{\forall}_{\phi}(\mathrm{Op}_{\psi_1}(X_{1,1},\dots,X_{1,k_1}),\dots,\mathrm{Op}_{\psi_n}(X_{n,1},\dots,X_{n,k_n})) \tag{7.1}$$

$$\textit{iff}\quad \vdash \mathrm{Rel}^{\exists}_{\phi}(\mathrm{Op}_{\psi_1}(X_{1,1},\dots,X_{1,k_1}),\dots,\mathrm{Op}_{\psi_n}(X_{n,1},\dots,X_{n,k_n})) \tag{7.2}$$

Proof: The substitution of the formulae $x \in X_{i,j}$ for $p_{i,j}$ into $\psi_i(p_{i,1}, \ldots, p_{i,k_i})$ everywhere in the (propositional) proof of $\phi(\psi_1, \ldots, \psi_n)$ transforms it into the proof of

$$\phi(x \in \mathrm{Op}_{\psi_1}(X_{1,1}, \ldots, X_{1,k_1}), \ldots, x \in \mathrm{Op}_{\psi_n}(X_{n,1}, \ldots, X_{n,k_n})).$$

Then use generalization on x to get $\mathrm{Rel}^{\forall}_{\phi}$ and $\exists$-introduction to get $\mathrm{Rel}^{\exists}_{\phi}$.

Conversely, given an evaluation e that refutes $\phi(\psi_1, \ldots, \psi_n)$, we construct a Zadeh model $\mathbb{M}$ refuting (7.1) and (7.2) by assigning to the class variables $X_{i,j}$ the functions $A_{i,j}$ such that $A_{i,j}(a) = e(p_{i,j})$ for every a in the universe of $\mathbb{M}$. Applying Theorems 3.3.21 and 3.5.3, the proof is done. QED

Corollary 7.1.6 *Let ϕ and ψ be propositional formulae.*
If $\vdash \phi \to \psi$ then $\vdash \mathrm{Op}_{\phi}(X_1, \ldots, X_n) \subseteq \mathrm{Op}_{\psi}(X_1, \ldots, X_n)$.
If $\vdash \phi \leftrightarrow \psi$ then $\vdash \mathrm{Op}_{\phi}(X_1, \ldots, X_n) = \mathrm{Op}_{\psi}(X_1, \ldots, X_n)$.
If $\vdash \phi \vee \neg\phi$ then $\vdash \mathrm{Crisp}(\mathrm{Op}_{\phi}(X_1, \ldots, X_n))$.

By virtue of Theorem 7.1.5, the properties of propositional connectives directly translate to the properties of class relations and operations. For example:

$\vdash \triangle p \to p$	proves	$\vdash \mathrm{Ker}(X) \subseteq X$
$\vdash p \to p \vee q$	”	$\vdash X \subseteq X \cup Y$
$\vdash 0 \to p$	”	$\vdash \emptyset \subseteq X$
$\vdash p \& q \to p \wedge q$	”	$\vdash X \cap_* Y \subseteq X \cap_{\mathrm{G}} Y$
$\vdash \neg_{\mathrm{G}} p \vee \neg\neg_{\mathrm{G}} p$	”	$\vdash \mathrm{Crisp}(\backslash X)$
$\vdash \triangle(\alpha \to p) \to \triangle(\beta \to p)$ for $\alpha \le \beta$	”	$\vdash X_\alpha \subseteq X_\beta$ for $\alpha \le \beta$, etc.

In order to translate monotonicity and congruence properties of propositional connectives to the same properties of class operations, we need another theorem:

Theorem 7.1.7 *Let $\phi_i, \phi'_i, \psi_{i,j}, \psi'_{i,j}$ be propositional formulae. Then*

$$\vdash \mathop{\&_*}_{i=1}^{k} \; \phi_i(\psi_{i,1}, \ldots, \psi_{i,n_i}) \to \bigwedge_{i=1}^{k'} \; \phi'_i(\psi'_{i,1}, \ldots, \psi'_{i,n'_i}) \tag{7.3}$$

iff

$$\vdash \mathop{\&_*}_{i=1}^{k} \; \mathrm{Rel}^{\forall}_{\phi_i} \left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\right) \to \bigwedge_{i=1}^{k'} \; \mathrm{Rel}^{\forall}_{\phi'_i} \left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\right) \tag{7.4}$$

Proof: Without loss of generality, the principal implications of (7.3) and (7.4) can be assumed to be $\to_*$. Replacing all propositional variables p_j in the proof of (7.3) by the atomic formulae $x \in X_j$ then yields the proof of

$$\mathop{\&_*}_{i=1}^{k} \; \phi_i \left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\right) \to_* \bigwedge_{i=1}^{k'} \; \phi'_i \left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\right).$$

Generalization on x and distribution of $\forall$ over all conjuncts using (3.1), (3.5) and (3.3) of Lemma 3.5.10 proves (7.4). QED

Examples of direct corollaries of the theorem:

Provability in $\mathrm{BL}_\triangle$ of	Proves in FCT
$(p \to q) \to ((p \& r) \to (q \& r))$	$X \subseteq_* Y \to X \cap_* Z \subseteq_* Y \cap_* Z$
$(p \to q) \to (p \to (p \wedge q))$	$X \subseteq_* Y \to X \subseteq_* X \cap_{\mathrm{G}} Y$
$[(p \to q) \& (q \to p)] \to (p \leftrightarrow q)$	$(X \subseteq_* Y \& Y \subseteq_* X) \to X =_* Y$
$(p \leftrightarrow q) \to [(p \to q) \wedge (q \to p)]$	$X =_* Y \to (X \subseteq_* Y \wedge Y \subseteq_* X)$
$[(p \to r) \& (q \to r)] \to (p \vee q \to r)$	$(X \subseteq_* Z \& Y \subseteq_* Z) \to X \cup Y \subseteq_* Z$
$\triangle(p \to q) \to [\triangle(\alpha \to p) \to \triangle(\alpha \to q)]$	$\triangle(X \subseteq Y) \to X_\alpha \subseteq Y_\alpha$
transitivity of $\to$, $\leftrightarrow$	transitivity of $\subseteq_*, =_*$, etc.

Similarly, $Ł \vdash (\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow p)$ proves $-X \subseteq_Ł -Y \leftrightarrow Y \subseteq_Ł X$, etc.

To derive theorems about $\mathrm{Rel}^\exists$, we slightly modify Theorem 7.1.7:

Theorem 7.1.8 *Let $\phi_i, \phi'_i, \psi_{i,j}, \psi'_{i,j}$ be propositional formulae. Then*

$$\vdash \mathop{\&_*}_{i=1}^{k} \phi_i(\psi_{i,1}, \ldots, \psi_{i,n_i}) \to \bigvee_{i=1}^{k'} \phi'_i(\psi'_{i,1}, \ldots, \psi'_{i,n'_i}) \tag{7.5}$$

iff

$$\vdash \mathop{\&_*}_{i=1}^{k-1} \mathrm{Rel}^\forall_{\phi_i}\left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\right) \&_* \mathrm{Rel}^\exists_{\phi_k}\left(\mathrm{Op}_{\psi_{k,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{k,n_k}}(\vec{X}\right) \to$$
$$\to \bigvee_{i=1}^{k'} \mathrm{Rel}^\exists_{\phi'_i}\left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\right) \tag{7.6}$$

Proof: Modify the proof of Theorem 7.1.7, using (3.6) of Lemma 3.5.10 instead of (3.5), and then (3.4) of the same Lemma to distribute $\exists$ over the disjuncts. QED

Examples of direct corollaries:

Provability in BLΔ of	Proves in FCT
$p\&(p \to q) \to q$	$\mathrm{Hgt}(X)\&_*(X \subseteq_* Y) \to \mathrm{Hgt}(Y)$
$\triangle(p \vee q) \to \triangle p \vee \triangle q$	$\mathrm{Norm}(X \cup Y) \to \mathrm{Norm}(X) \vee \mathrm{Norm}(Y)$
$(p \to r)\&(p\&q) \to (q\&r)$	$X \subseteq_* Z \&_* X \|_* Y \to Y \|_* Z$, etc.

7.2 Tuples of objects

In order to be able to deal with fuzzy relations, we will further assume that the language of FCT contains an apparatus for forming tuples of objects and accessing their components. Such an extension can be achieved, e.g., by postulating variable sorts for any multiplicity of tuples (all of which are subsumed by the sort of objects), enriching the language with the functions for forming n-tuples of any combination of tuples and accessing its components, and adding axiom schemes expressing that tuples equal iff their respective constituents equal. The definition of Zadeh model then must be adjusted by partitioning the range of object variables and interpreting the tuples-handling functions. We omit elaborating this sort of syntactic sugar.

In what follows, instead of $\{z \mid (\exists x_1) \ldots (\exists x_n)\,(z = \langle x_1, \ldots, x_n\rangle \,\& \phi)\}$ we write the formula $\{\langle x_1, \ldots, x_n\rangle \mid \phi\}$.

FCT equipped with tuples of objects contains common operations for dealing with relations. We can define Cartesian products, domains, ranges and the relational operations as usual:[2]

$$\begin{aligned}
X \times_* Y &=_{\mathrm{df}} \{\langle x, y\rangle \mid x \in X \&_* y \in Y\} \\
\mathrm{Dom}(R) &=_{\mathrm{df}} \{x \mid \langle x, y\rangle \in R\} \\
\mathrm{Rng}(R) &=_{\mathrm{df}} \{y \mid \langle x, y\rangle \in R\} \\
R \circ_* S &=_{\mathrm{df}} \{\langle x, y\rangle \mid (\exists z)\,(\langle x, z\rangle \in R \&_* \langle z, y\rangle \in S)\} \\
R^{-1} &=_{\mathrm{df}} \{\langle x, y\rangle \mid \langle y, x\rangle \in R\} \\
\mathrm{Id} &=_{\mathrm{df}} \{\langle x, y\rangle \mid x = y\}
\end{aligned}$$

[2] Obviously for crisp arguments these operations yield crisp classes; $X \times_* Y$ is crisp iff both X and Y are crisp. Unless X and Y are crisp, the property of being a relation from X to Y is double-indexed (a $*'$-subset of the Cartesian product $X \times_* Y$). Also the definitions of usual properties (e.g., reflexivity, $*$-symmetry, etc.) of a relation on a non-crisp Cartesian product have to be defined with relativized quantifiers which bring another index. It is doubtful that definitions combining various t-norms will have any real meaning. The situation is much easier if only relations on crisp classes are considered.

Table 7.3: Properties of relations

Notation	Definition	Name
$\mathrm{Refl}(R)$	$(\forall x)\,(Rxx)$	reflexive
$\mathrm{Sym}_*(R)$	$(\forall x,y)\,(Rxy \to_* Ryx)$	$*$-symmetric
$\mathrm{Trans}_*(R)$	$(\forall x,y,z)\,(Rxy \&_* Ryz \to_* Rxz)$	$*$-transitive
$\mathrm{Dich}(R)$	$(\forall x,y)\,(Rxy \vee Ryx)$	dichotomic
$\mathrm{Quord}_*(R)$	$\mathrm{Refl}(R) \&_* \mathrm{Trans}_*(R)$	$*$-quasiordering
$\mathrm{Linquord}_*(R)$	$\mathrm{Quord}_*(R) \&_* \mathrm{Dich}(R)$	linear $*$-quasiordering
$\mathrm{Sim}_*(R)$	$\mathrm{Quord}_*(R) \&_* \mathrm{Sym}_*(R)$	$*$-similarity
$\mathrm{Equ}_*(R)$	$\mathrm{Sim}_*(R) \&_* (\forall x,y)\,(\triangle Rxy \to_* x = y)$	$*$-equality

The introduction of tuples of objects also allows an axiomatic investigation of various kinds of fuzzy relations (e.g., similarities) and fuzzy structures (fuzzy preorderings, graphs, etc.). We can define the usual properties of relations, as summarized in Table 7.3 (for brevity's sake, we write just Rxy for $\langle x, y\rangle \in R$).[3]

Classical definitions of some properties of relations (e.g., antisymmetry) make use of the identity predicate on objects. One may be tempted to use the identity predicate $=$ of ŁΠ∀ in the rôle of the classical identity in these definitions. However, since $=$ is crisp, such definitions do not yield useful and genuine fuzzy notions. A fuzzy analogue of the crisp notion of identity is that of similarity or equality (see Table 7.3). We can therefore define these properties *relative to* a $*$-similarity or $*$-equality S. For details see the last sections.

In this way, the properties of being a $*$-antisymmetric relation, a $*$-ordering, a linear $*$-ordering, a $*$-well-ordering, a $*$-function and a $*$-bijection (w.r.t. some fuzzy $*$-equality) can be introduced. By means of $*$-bijections, the notions of $*$-subvalence, $*$-equipotence and $*$-finitude of classes (again w.r.t. some fuzzy $*$-equality) can be defined. A thorough investigation of these notions, however, exceeds the scope of this chapter.

7.3 Higher types of classes

7.3.1 Second-level classes

Class theory does not contain an apparatus for dealing with families of classes. In many cases, a family of classes can be represented by a class of pairs or some other kind of 'encoding'. For instance, a relation R may be understood as representing the family of classes $X_i = \{x \mid \langle i, x\rangle \in R\}$ for all $i \in \mathrm{Dom}(R)$.

In other cases, however, no suitable class of indices can be found and such an 'encoding' is not possible. Then it is desirable to extend the apparatus of class theory by classes of the second level. This is done simply by repeating the same definitions one level higher. We introduce a new sort of variables for families of classes $\mathcal{X}, \mathcal{Y}, \dots$, a new membership predicate between classes and families of classes $X \in \mathcal{X}$, and the comprehension scheme for families of classes

$$(\exists \mathcal{X})\,\triangle\,(\forall X)\,(X \in \mathcal{X} \leftrightarrow \phi(X))$$

for all formulae ϕ (where ϕ may contain any parameters except for $\mathcal{X}$). The extensionality axiom for families of classes now reads

$$(\forall X)\,\triangle(X \in \mathcal{X} \leftrightarrow X \in \mathcal{Y}) \to \mathcal{X} = \mathcal{Y}.$$

[3]Following the usual mathematical terminology, $*$-similarity may also be called $*$-*equivalence*; we respect the established fuzzy set terminology here. *Weak dichotomy* $(\forall x,y)\,(Rxy \oplus Ryx)$ could also be defined and weak versions of the properties that contain dichotomy, e.g. weakly linear $*$-ordering.

Again it is possible to introduce second-level comprehension terms $\{X \mid \phi(X)\}$, which introduction is conservative and eliminable by Theorem 3.5.14.

The consistency of this extension is proved by a construction of second-level Zadeh models over a linear ŁΠ-algebra $\mathbf{L}$, in which the object variables range over a universe U, the class variables over the set $\mathbf{L}^U$ of all functions from U to $\mathbf{L}$, and the second-level class variables range over the set $\mathbf{L}^{\mathbf{L}^U}$ of all functions from $\mathbf{L}^U$ to $\mathbf{L}$. The second-level class $\{X \mid \phi(X)\}$ is again identified with the characteristic function of ϕ as in Section 7.1.1. Obviously, this construction makes both the second-level comprehension scheme and the axiom of extensionality satisfied in the model; the theory of second-level classes can thus be viewed as third-order fuzzy logic (we omit details).

All definitions of elementary class relations and operations and all theorems can directly be transferred from classes to second-level classes. Refining the language, axioms and Zadeh models to tuples of classes is also straightforward.

It may be observed that the class operations and relations Op_ϕ, $\mathrm{Rel}_\phi^\forall$, and $\mathrm{Rel}_\phi^\exists$, which were introduced in Sections 7.1.2 and 7.1.3 as defined functors and predicates, are now individuals of the theory, viz. second-level classes.

7.3.2 Simple fuzzy type theory

If there be need for families of families of classes, it is straightforward to repeat the whole construction once again to get third-level classes. By iterating this process, we get a simple type theory over ŁΠ, for which the class theory described in Sections 7.1–7.2 is the induction step. The comprehension schemes and Zadeh models can easily be generalized to allow membership of elements of any type less then n in classes of the n-th level.[4]

A type theory over a particular fuzzy logic (viz. IMTL$\triangle$, extended also to Ł$\triangle$) has already been proposed by V. Novák in [74]. As mentioned in the Introduction, our theory can be built over various fuzzy logics with $\triangle$; its variant over IMTL$\triangle$ and Novák's type theory seem to be equivalent (though radically different in notation, as Novák uses λ-terms).

Since almost all classical applied mathematics can be formalized within the first few levels of simple type theory, the formalism just described should be sufficient for all applications of fuzzy sets based on t-norms or other functions definable in ŁΠ (see Theorem 3.3.25). To illustrate this, we show the formalization of Zadeh's extension principle.

Definition 7.3.1 *A (fuzzy) binary relation*[5] *R between objects is* extended by Zadeh's principle (based on a t-norm $*$) *to a relation $\mathcal{R}_*$ between (fuzzy) classes as follows:*

$$\mathcal{R}_*(X,Y) \equiv_{\mathrm{df}} (\exists x, y)\,(Rxy \&_* x \in X \&_* x \in Y)$$

Since relations between classes are classes of the second level in our simple type theory, Zadeh's extension principle in fact assigns to a first-level class R a second-level relation; such an assignment itself is an individual of the third level. Thus we can define Zadeh's principle as an *individual* of our theory—a special class $\mathcal{Z}_*$ of the third level:

Definition 7.3.2 (Zadeh's extension principle) Zadeh's extension principle based on $*$ *is a third-level function $\mathcal{Z}_*$ defined as follows (we adopt the usual functional notation for classes which are functions):*

$$\mathcal{Z}_*(R) =_{\mathrm{df}} \{\langle X, Y\rangle \mid (\exists x, y)\,(Rxy \&_* x \in X \&_* x \in Y)\}$$

Generally we can extend any fuzzy relation $R^{(n+1)}$ of type $n+1$ to one of type $n+2$ by Zadeh's principle of type $n+3$ (based on a t-norm $*$). All these 'principles' are in fact individuals of our theory, whose existence follows from the comprehension scheme.

[4] This is done simply by postulating that the n-th sort of variables is subsumed by the k-th sort if $n < k$. The sorts can further be refined to allow arbitrary tuples of individuals of lesser types with the appropriate tuple-forming, component-extracting and tuple-identity axioms added. The generalization of Zadeh models is again quite straightforward.

[5] Functions are a special kind of relations. The generalization to n-ary relations is trivial.

Definition 7.3.3 Zadeh's extension principle for relations of type $n+1$ (for $n \geq 0$) based on $*$ *is the function of type* $n+3$ *defined as follows:*

$$\mathcal{Z}_*{}^{(n+3)}\left(R^{(n+1)}\right) =_{\mathrm{df}} \Big\{ \Big\langle X^{(n+1)}{}_1, \ldots, X^{(n+1)}{}_k \Big\rangle \;\Big|\; \left(\exists W^{(n)}{}_1, \ldots, W^{(n)}{}_k\right) \Big(\Big\langle W^{(n)}{}_1, \ldots, W^{(n)}{}_k \Big\rangle \in R^{(n+1)} \;\&_*\; \mathop{\&_*}_{i=1}^{k} W_i^{(n)} \in X_i^{(n+1)} \Big) \Big\}$$

7.4 Adding structure to the domain of discourse

As we have shown, in FCT we can define many properties of individuals of our theory (objects or classes). Since our theory contains classical class theory (for classes which are crisp), we can introduce arbitrary relations and functions on the universe of objects which are definable in classical class theory. As they can be described by formulae, their existence is guaranteed by the comprehension axiom. So the only thing we need to add is a constant of the appropriate sort and the instance of the comprehension axiom. The following definition is the formalization of this approach for the first-order theories.

Definition 7.4.1 *Let* $\mathbf{\Gamma}$ *be a classical one-sorted predicate language and* T *be a* $\mathbf{\Gamma}$*-theory. For each n-ary predicate symbol* P *of* $\mathbf{\Gamma}$ *let us introduce a new constant* $\bar{P}$ *for a class of n-tuples, and for each n-ary function symbol* F *we take a new constant* $\bar{F}$ *for a class of (n+1)-tuples. We define the language* FCT($\mathbf{\Gamma}$) *as the language of* FCT *extended by the symbols* $\bar{Q}$ *for each symbol* $Q \in \mathbf{\Gamma}$*. The translation* $\bar{\phi}$ *of a* $\mathbf{\Gamma}$*-formula* ϕ *to* FCT($\mathbf{\Gamma}$) *is obtained as the result of replacing all occurrences of all* $\mathbf{\Gamma}$*-symbols* Q *in* ϕ *by* $\bar{Q}$*.*

We define the theory FCT(T) *in the language* FCT($\mathbf{\Gamma}$) *as the theory with the following axioms:*

- *The axioms of* FCT
- *The translations* $\bar{\phi}$ *of all axioms* ϕ *of* T
- $\mathrm{Crisp}(\bar{Q})$ *for each symbol* $Q \in \mathbf{\Gamma}$ *(for the definition of* Crisp*, see Table 7.2)*
- $\langle x_1, \ldots x_n, y\rangle \in \bar{F} \wedge \langle x_1, \ldots x_n, z\rangle \in \bar{F} \to y = z$ *for each n-ary function symbol* $F \in \mathbf{\Gamma}$*.*

Lemma 7.4.2 *Let* $\mathbf{\Gamma}$ *be a classical predicate language,* T *a* $\mathbf{\Gamma}$*-theory,* $\mathbf{L}$ *an* ŁΠ*-algebra. If* $\mathbf{M}$ *is an* $\mathbf{L}$*-model of* FCT(T)*, then* $\mathbf{M}^c = (M, (Q_{\mathbf{M}^c})_{Q\in\mathbf{\Gamma}})$*, where* $Q_{\mathbf{M}^c} = \bar{Q}_{\mathbf{M}}$ *for each* $Q \in \mathbf{\Gamma}$*, is a model (in the sense of classical logic) of the theory* T*.*

Vice versa, for each model $\mathbf{M}$ *of* T *there is an* $\mathbf{L}$*-model* $\mathbf{N}$ *of* FCT(T) *such that* $\mathbf{N}^c$ *is isomorphic to* $\mathbf{M}$*.*

Therefore (in virtue of Theorem 3.5.3), $T \vdash \phi$ *iff* $\mathrm{FCT}(T) \vdash \bar{\phi}$*, for any* $\mathbf{\Gamma}$*-formula* ϕ*.*

Proof: If $\mathbb{M}$ is an $\mathbf{L}$-model of FCT, then for each $Q \in \mathbf{\Gamma}$, $\bar{Q}_{\mathbb{M}}$ is crisp due to the axiom $\mathrm{Crisp}(\bar{Q})$ of FCT(T). Setting the universe of $\mathbb{M}^c$ to that of $\mathbb{M}$, and for each symbol $Q \in \mathbf{\Gamma}$, setting $Q_{\mathbb{M}^c}$ to the set whose characteristic function is $\bar{Q}_{\mathbb{M}}$, we can see that $\mathbb{M}^c$ models T, because the axioms of T, which contain only crisp predicates, are evaluated classically in $\mathbb{M}^c$.

Conversely, we define $\mathbb{M}$ as the standard Zadeh model with the universe of $\mathbb{N}$, in which $\bar{F}_{\mathbb{M}} = F_{\mathbb{N}}$ for every function symbol $F \in \mathbf{\Gamma}$, and for every predicate $P \in \mathbf{\Gamma}$, $\bar{P}_{\mathbb{M}}$ is realized as the characteristic function of $P_{\mathbb{N}}$. Then $\mathbb{M}$ obviously satisfies all axioms of FCT(T); the axioms of T are again evaluated classically in $\mathbb{M}$, as the realizations of all predicates involved are crisp. QED

Example 7.4.3 *Let* R *be a constant for a class of pairs. Then in each* $\mathbf{L}$*-model of the theory* $\mathrm{Crisp}(R)$*,* $\mathrm{Refl}(R)$*,* $\mathrm{Trans}(R)$*,* $(\forall x, y)\,(Rxy \& Ryx \to x = y)$*, the constant* R *is represented by a crisp ordering on the universe of objects. (For the definitions of* Refl *and* Trans*, see Table 7.3.)*

Example 7.4.4 *If T is a classical theory of the real closed field, then in each* $\mathbf{L}$*-model* $\mathbb{M}$ *of the theory* $\mathrm{FCT}(T)$*, the universe of objects with* $\bar{\leq}_{\mathbb{M}}, \bar{+}_{\mathbb{M}}, \bar{-}_{\mathbb{M}}, \bar{\cdot}_{\mathbb{M}}, \bar{0}_{\mathbb{M}}, \bar{1}_{\mathbb{M}}$ *is a real closed field.*

In Lemma 7.4.2 we speak of first-order theories only. Nevertheless, it can be extended to any theory formalizable in classical type theory. Here we present only one example; for details see [7].

Example 7.4.5 *Let τ be a constant for a class of classes and T the theory with the axioms:*

- $\mathrm{Crisp}(\tau)$
- $(\forall X)\,(X \in \tau \to \mathrm{Crisp}(X))$
- $(\forall \mathcal{X})\,(\mathrm{Crisp}(\mathcal{X}) \& \mathcal{X} \subseteq \tau \to \{x \mid (\exists X) \in \mathcal{X} \& x \in X)\} \in \tau$
- $(\forall X_1) \ldots (\forall X_n)\,(X_1 \in \tau \& \ldots \& X_n \in \tau \to X_1 \cap \ldots \cap X_n \in \tau)$ *for each* $n \in N$

Then in each $\mathbf{L}$*-model of the theory T, the constant τ is represented by a classical topology on the universe of objects.*

7.5 Fuzzy mathematics

If we examine the above definitions we see the crucial rôle of the predicate Crisp. If we remove this predicate from the above definitions we get the "natural" fuzzification of the above-mentioned concepts.

In order to illustrate the methodology of fuzzification, let us concentrate on the concept of ordering. If we remove the predicate Crisp from the definition, then we have to distinguish which t-norm was used in the axioms of transitivity and antisymmetry. Thus we get the concept of $*$-fuzzy ordering. This is the way this concept was introduced by Zadeh. However, some carefulness is due here not to overlook some "hidden" crispness. There is crisp identity used in the antisymmetry axiom, and also in the reflexivity axiom which can be written as $(\forall x, y)\,(x = y \to Rxy)$. A more general definition is therefore parameterized also by a fuzzy equality in the following way:

Example 7.5.1 *Let E and R be two constants of classes of tuples. The following axioms define the concept of $(*, E)$-ordering R:*

- $\mathrm{Equ}_*(E)$
- $\mathrm{Trans}_*(R)$
- $(\forall x, y)\,(Exy \to Rxy)$
- $(\forall x, y)\,(Rxy \&_* Ryx \to_* Exy)$

Observe that E is a $*$-equality, and the last two conditions can be written as $R \cap_* R^{-1} \subseteq E \subseteq R$. We thus get the notion of fuzzy ordering as defined by Bodenhoffer in [9].

In contemporary fuzzy mathematics the methodology of fuzzification of concepts is somewhat sketchy and non-consistent: usually only some features of a classical concept are fuzzified while other features are left crisp.

We would like to propose another "inductive" approach. We propose to follow the usual "inductive" development of mathematics (in some metamathematical setting—here in simple type theory) and fuzzify "along the way". In more words: develop a fuzzy generalization of basic classical concepts (the notion of class, relation, equality—as done in this chapter); then define compound fuzzy notions by taking their classical definitions and consistently replacing classical sub-concepts in the definitions by their already fuzzified counterparts. The consistency of this approach promises that no crispness will be unintentionally "left behind".

This approach is formal and sometimes may lead to too complex notions. In such cases, some features of the complex notion may *intentionally* be left crisp by retaining some of the crispness axioms. The advantage of the proposed approach is that we always know *which* features are left crisp.

The framework presented in this chapter provides a unified formalism for various disciplines of fuzzy mathematics. This may enable, i.a., an interchange of results and methods between distant disciplines of fuzzy mathematics, till now separated by differences in notation and incompatibilities in definitions. It can also bring new (proof-theoretic and model-theoretic) methods to traditional fuzzy disciplines and enable their further development in both theory and applications. Finally, the axiomatization of the whole fuzzy mathematics, independent of particular $[0, 1]$-functions, can be an important step in understanding vague phenomena. Further elaboration of the proposed formalism and its application to various disciplines of fuzzy mathematics is thus a possible direction towards firm foundations of fuzzy mathematics.

Chapter 8

... to Fuzzy Mathematics

This chapter is a full text of the talk *From Fuzzy Logic to Fuzzy Mathematics: A Methodological Manifesto*, presented in Vienna 16.8.2004 at the conference Challenge of Semantics by the author and Libor Běhounek.

A Methodological Manifesto

One of the motives for theoretical studies in fuzzy mathematics is the pursuit of formal reconstruction of the methods commonly used in applied fuzzy mathematics. The greatest success in such investigation is undoubtedly the area of formal fuzzy logic. By efforts of Hájek, Gottwald, Mundici, and others, this discipline has reached the point when it is reasonable to attempt using it as a ground theory for the formalization of other branches of fuzzy mathematics.

This paper tries to provide certain guidelines for such a transition from formal fuzzy logic to formal fuzzy mathematics. The guidelines are based upon doctrines observed by Prague workgroup on fuzzy logic founded and led by Petr Hájek. We attempt to explicitly formulate some distinct features of Petr Hájek's approach, which we reconstruct from his scattered remarks and the general direction of his papers, and implement them in the form of a research programme. We hope that Petr Hájek will find our reconstruction of his doctrine faithful enough; or else that he will enter into a fruitful dispute with his own disciples over the methodological foundations of our discipline. If the former is the case, then we deem that the best label for the enterprise would be *Hájek's Programme* in the foundations of fuzzy mathematics.

The cornerstone of Hájek's approach to fuzzy mathematics is the doctrine of working in a formal axiomatic theory over a fuzzy logic, rather than investigating particular models. For the ease of reference, let us call it *Hájek's imperative.* Good reasons for such an approach can be found, both of philosophical and pragmatic nature.

A philosophical reason is found in the following argumentation. Fuzzy logic describes the laws of truth preservation in reasoning under (a certain form of) vagueness. Its interpretation in terms of truth degrees is just a *model*—a classical rendering of vague phenomena. Even though such models may have originally been employed for discovering the laws of approximate reasoning, they must be regarded as secondary—just because they are essentially classical, not genuinely vague. Only formal theories over fuzzy logic—assuming that fuzzy logic faithfully approximates the laws of truth preservation in reasoning fraught with vagueness—are genuinely fuzzy.

The limited adequacy of such classical models of vague reasoning is seen, i.a., in the usual objection against fuzzy logic which points at its practice of assigning to vague propositions like 'Charles is tall' definite truth values, e.g., 0.7485 (using fuzzy numbers does not solve the problem, only shifts it a level higher). However, the $[0,1]$ truth values should only be seen as a model underlying vague inference, which helps us in understanding truth preservation under vagueness. No relevant conclusion can be derived from the assumption that the truth value of 'Charles is tall' is 0.7485, because such an assumption is absurd. Only the laws of inference that emerge from considering all possible models of approximate reasoning are epistemically valuable; and this means de facto the laws derivable in a formal theory or logic.

Admittedly, a formal theory over fuzzy logic is just a notational abbreviation of classical reasoning about the class of all models. Nonetheless, the axiomatic method is the general paradigm of mathematics. The appropriate choice of the language of the formal theory screens off irrelevant features of the models. An axiomatic system is thus not only the means of generalization over all models, but rather an abstraction to their constitutive features.

Obviously, Hájek's imperative applies mainly to the development of fuzzy logic and various branches of fuzzy mathematics, not to particular applications. In an application, we are modeling particular phenomena and thus we naturally work with a particular model. For instance, some real-life problem (e.g., processing of a questionnaire with five grades between absolute yes and absolute no) may invite a definite algebra of truth values. However, having a general theory may of course help even in particular cases, since it will describe the general features of the problem. The programme of developing fuzzy mathematics in a theoretical manner stresses the priority of general theories over immediate applicational needs.

The idea that fuzzy inference cannot be reduced to a particular model that is able to explicate its rules entails that in the investigation of fuzzy inference we should not limit ourselves to one particular fuzzy logic (e.g., Łukasiewicz). The model which underlies it—e.g., a specific t-norm—is particular, while fuzzy reasoning in general is broader. There are examples of fuzzy reasoning that follow variant inference rules, all of which are suitable for different respective contexts of real-life situations and invite explanations in terms of various individual t-norms or other semantics. The multitude of existing fuzzy logics varying both in expressive power and inference rules is not only explicable by the need of capturing of all aspects of fuzzy inference in diverse contexts, but even indispensable for this enterprise.

Similar considerations are related to Hájek's preference of fuzzy logics without truth constants in the language (except for those which are definable). First, the truth constants have little support in natural language. Second, by incorporating the truth values into the syntax, we force the logic to follow too closely a particular model of vague inference, viz. that using truth values. Of course, we cannot be too dogmatic about rejecting truth constants: it turns out that in sufficiently strong theories, at least rational truth constants are definable. Sometimes, the truth degrees are useful for a particular application. However, we should be cautious of deliberately introducing them into logic and thus restricting the possible models of vague inference.

Thus, even though liberal in both the expressive power and inference rules, we—following Hájek—believe a certain style of logical systems to be a most suitable formalism for representing fuzzy inference. For brevity's sake, in what follows we shall call them *Hájek-style fuzzy logics*. Put in a nutshell, they are fuzzy logics retaining the syntax of classical logic (preferably without truth constants), defined as axiomatic systems (rather than non-axiomatizable sets of tautologies). A prototypical example is Hájek's Basic Logic BL, propositional or predicate. In the following paragraphs we give some reasons for such preferences.

There is a pragmatic motive for retaining as much of classical syntax as possible. The way of working in theories over Hájek-style fuzzy logics resembles closely the way of working in classical logic: Hájek-style fuzzy logics are often just weaker variants of Boolean logic—syntactically fully analogous, just lacking some of its laws. Therefore, many theoretical and metatheoretical methods developed for classical logic can be mimicked and employed, resulting in a quick and sound development of the theory. This feature has already been utilized in metamathematics of fuzzy logic—the proofs of the completeness, deduction, and other metatheorems have often been obtained by adjustments of classical proofs.

To illustrate the utility of this guideline, we allege that an axiomatic theory of fuzzy sets can more easily be developed as a formal theory of binary membership predicate over some fuzzy logic than if the graded membership is rendered, e.g., as a ternary predicate between elements, sets, and truth values in the framework of classical logic. Many constructions and even proofs of the classical theory will work in the former case and need not be rediscovered (nor even reformulated). Even though both theories may turn out equivalent, the resemblance of fuzzy concepts to classical ones becomes more visible in Hájek's approach: cf. the many 'breakthrough' definitions of fuzzy set inclusion which, if put down in Hájek-style fuzzy logic have exactly the form of the classical definition of set inclusion. This is another reason for preferring the classical syntax in fuzzy logic, over non-standard logical systems, e.g., an evaluated syntax. (This does not mean that they cannot have their own merits; only they are not preferred for the development of fuzzy mathematics by the Prague school.)

The imperative to work deductively in a formal theory explains also our preference of axiomatic systems over non-axiomatizable sets of tautologies. The infeasibility of algorithmical recognition of valid inferences in the latter is a strong reason supporting the preference. Thus, e.g., predicate fuzzy logics are better to be understood as the systems of axioms and rules for quantifiers than the sets of valid $[0,1]$-tautologies, even though the former usually admits non-intended models.

The respect for the priority of formal theories to models can partly be seen as emphasizing the syntax against the semantics of fuzzy logic. Hájek's approach thus can be viewed as a *syntactic turn* in fuzzy logic. The accent on syntax is of course not meant to contest the fundamental rôle of semantics in logic, nor the heuristic value of the models. Nevertheless, playing up the importance of formal deduction in fuzzy logic corresponds to its motivation as a description of the rules of correct reasoning under vagueness.

Such, then, is a reconstruction of the methodological background we adopt. It has already proved worthy in the area of metamathematics of fuzzy logic. Thus it seems reasonable to apply its doctrines to other branches of fuzzy mathematics as well.

The need for axiomatization of further areas of fuzzy mathematics besides fuzzy logic is beyond doubt. Axiomatization has always aided the development of mathematical theories. There have been many—more or less successful—attempts to formalize or even axiomatize some areas of fuzzy mathematics. However, these axiomatics are usually designed ad hoc: some concepts in a classical theory are turned fuzzy, however their selection is based on non-systematic intuitions or intended applications; seldom is fuzzified all that could be. (To fuzzify as much as possible is desirable for generality's sake; if an application requires some features to be crisp, they can be 'defuzzified' by an additional assumption of the crispness of these particular features.) Many of these axiomatics are in fact semi-classical, being founded upon the notions of truth degrees and membership functions, which are merely a classical rendering of fuzzy sets.

Further problems of contemporary fuzzy mathematics lie in its fragmentation. Even if some axiomatic theories of various parts of fuzzy mathematics exist, they use completely different sets of primitive concepts and incompatible formalisms. This makes it virtually impossible to combine any two of them into one broader theory. It would certainly be better if fuzzy mathematics as a whole could employ a unified methodology in building its axiomatic theories, because it would facilitate the exchange of results between its branches. Applying the doctrines sketched above, we propose such a unified methodology for the axiomatization of fuzzy mathematics. Obviously, in our approach it assumes the form of constructing formal theories over Hájek-style fuzzy logics.

In the axiomatic construction of classical mathematic, a three-layer architecture proved worthy, with the layers of logic, foundations, and only then individual mathematical disciplines. Individual disciplines are thus developed within the framework of a unifying formal theory, be it some variant of set theory, type theory, category theory, or another sufficiently rich and general kind of theory. In fuzzy mathematics, the level of logic seems to be developed far enough so as to support sufficiently strong formal theories. The search for a suitable foundational theory is thus the task of the day. As hinted above, the close analogy between Hájek-style fuzzy logics and classical logic gives rise to a hope that fuzzy analogues of classical foundational theories will be able to harbour all (or at least nearly all) parts of existing fuzzy mathematics.

As conceivable candidates for a foundational theory, several ZF-style fuzzy set theories has already arisen. Many of them are certainly capable of doing the job. Nevertheless, the axiomatics of most ZF-style fuzzy set theories savour of a similar ad hoc axiom choices as other hitherto attempts at axiomatization of fuzzy mathematics. By large this is induced by the fact that such theories have to deal with a specific set-theoretical agenda and take into the account the structure of the whole set universe (expressed, e.g., by the axiom of well-foundedness). Moreover, for many of them it is not clear whether they can straightforwardly be generalized to other fuzzy logics than the one in which they were developed; thus they are only capable of providing the foundation for a limited part of fuzzy mathematics. Besides the repertoire of ZF-style set theories, fuzzy logic offers an alternative of set theories based on naive comprehension. Although their axiomatic system is very elegant, their consistency is limited to (certain) fuzzy logics where no bivalent operator is definable (roughly speaking, to infinite-valued Łukasiewicz logic or weaker).

If nevertheless a universal foundational theory is successfully found, the development of individual concepts of fuzzy mathematics has to proceed in a systematic way, taking into the account the dependencies between them as in classical mathematics. For example, the notion of cardinality should only be defined after the introduction and investigation of the notion of function, upon which it is based (and which in turn is based upon the concept of fuzzy equality, i.e., similarity). Defined notions should also be checked against conformity with category theory. For instance, a proposed definition of Cartesian product should accord with that of mapping (one must, however, take into account that many natural notions of morphism become fuzzy under fuzzy logic).

Only this kind of systematic approach can avoid giving ad hoc definitions of fuzzy concepts, which often suffer from arbitrariness and hidden crispness, or even references to particular crisp models of fuzziness (e.g. membership functions) which are not objects of the formal theory.

As a concrete implementation of the general programme sketched above we propose a specific foundational theory described below. We do not claim it to be the only possible way neither of doing the foundations of fuzzy mathematics, nor of fulfilling our foundational programme. The methodology itself is independent of this particular solution we propose. Nevertheless, we think that our theory embodies its guidelines very well and is a viable foundation for fuzzy mathematics of the present day. Moreover, because of the simplicity of its apparatus, the work done within its framework can be of use for other possible systems via a formal interpretation.

Inspecting the existing approaches and having in mind the need for generality and simplicity, it becomes obvious that a full-fledged set theory is not necessary for the foundations of fuzzy mathematics. What is necessary is only the ability to perform within the theory the basic constructions of fuzzy mathematics. On the other hand, a great variability of the background fuzzy logic is required in order to encompass the whole of fuzzy mathematics.

Most notions of classical mathematics can be defined within the first few levels of a simple type theory. The similarity between Hájek-style and classical theories hints that this could be true of fuzzy concepts defined in a fuzzified simple type theory as well. Indeed, many important notions can be defined already at the first level, which is in fact second-order predicate fuzzy logic. Most notably, elementary fuzzy set theory, or the axiomatization of *Zadeh's notion of fuzzy set,* is contained in second-order fuzzy logic (second-order models are exactly Zadeh's universes of fuzzy sets). Some theories (e.g., topology), however, need more levels of type hierarchy, thus we employ higher-order fuzzy logic (in the limit, logic of order ω).

Unfortunately, fuzzy higher-order logic is not recursively axiomatizable. Since we prefer axiomatic deductive theories over non-axiomatizable sets of tautologies, we choose its Henkin-style variant, even though it admits non-intended models. We thus get a *first-order* *theory,* axiomatized very naturally by the extensionality and comprehension axioms for each order. Moreover, the construction works for virtually all imaginable fuzzy logics (and many non-fuzzy logics as well). The bunch of foundational theories we propose thus can be called *Henkin-style higher-order fuzzy logic* (for an individual fuzzy logic of one's choice; expressively rich logics like ŁΠ seem to be sufficient for all practical purposes; nevertheless, the investigation of the fragments over weaker logics has also its own importance). Equipping the theory with the obvious axioms of tuples yields an apparatus which seems to be of enough expressive power for a great part of fuzzy mathematics, since a structure on the universe of discourse (metric, measure, etc.) can then be introduced by means of relations. Furthermore, if the background logic is sufficiently strong, there is a general method of embedding any classical theory, and even of its natural fuzzification (as well as conscious and controlled 'defuzzification' of its concepts if some of their features are to be left crisp). The details of this formalism can be found in [6].

As indicated above, elementary fuzzy set theory and some parts of the theory of fuzzy relations are already formalized within our foundational theory. Several other parts of fuzzy mathematics are currently (re-)developed in our formalism. However, the reconstruction (and expected further advance) of the whole of fuzzy mathematics is an infinite task. Everybody is therefore invited to participate in this research programme of systematic formal development of fuzzy mathematics, as well as to continue the discussion of its best foundation.

Acknowledgements. As the reader could easily observe this methodological programme has close links to the works of many predecessors, and in fact only applies their accomplishments to the area of fuzzy mathematics.

Our formalistic approach to mathematics is close to that of Hilbert's [57]. Our aspiration to lay down the logical foundations for fuzzy mathematics is only a derivative of the admirable enterprise of Russell and Whitehead [83]. Methodologically our programme is somewhat similar to that of Bourbaki [10], though we hope not only reconstruct and codify, but also advance the field of our interest. The link to Vienna circle [11] which results from the circumstances of the first presentation of this manifesto is rather incidental (though in some aspects one could perhaps find distant parallels).

We cannot mention all outstanding works of fuzzy logic upon which our contribution is based. However, a milestone in the development of fuzzy logic is certainly Hájek's [44]. Apparently the first monograph close in spirit to our programme was Gottwald's [43]. And, needless to say, the whole field of fuzzy mathematics we try to formalize originated with Zadeh's [85].

Appendix A

T-norms

T-norms (triangular norms) form a special class of operations on $[0, 1]$. T-norms were introduced by Menger in his paper [64]. A very good survey to the t-norms and their applications as a recent book of Klement, Mesiar, and Pap [60]. We present a modern definition of a t-norm.

Definition A.1 *A t-norm is a commutative associative non-decreasing (in both arguments) binary operation on* $[0, 1]$ *with unit element 1 and annihilator element 0, i.e., the following hold for all* x, y, $z \in [0, 1]$:

- $x * y = y * x$
- $x * (y * z) = (x * y) * z$
- $x \leq y$ *implies* $x * z \leq y * z$
- $x \leq y$ *implies* $z * x \leq z * y$
- $x * 1 = x$
- $x * 0 = 0$

A t-norm is continuous if it is continuous mapping in the usual sense.

There are three special continuous t-norms. These t-norms have many important properties and we will see that they play a fundamental role among all t-norms.

- *Łukasiewicz* t-norm: $x * y = \max(0, x + y - 1)$
- *Gödel* t-norm: $x * y = \min(x, y)$
- *Product* t-norm: $x * y = x.y$ (product of reals)

It is obvious that these t-norms are continuous. Each left-continuous t-norm has a special corresponding operation called residuum.

Definition A.2 *Let* $*$ *be a left-continuous t-norm. Then we define a binary operation on* $[0, 1]$ *called the* residuum $\Rightarrow$ *of t-norm* $*$ *in the following way:* $x \Rightarrow y = \max\{z \mid x * z \leq y\}$

Now we are going to examine some basic properties of t-norms and their residua. Especially notice property (2), which we will often use in our proofs.

Lemma A.3 *Let $*$ be a continuous t-norm and $\Rightarrow$ its residuum. Then the following hold:*

(1) *$\Rightarrow$ is the unique operation satisfying the condition from the latter definition,*

(2) $x \leq y$ *iff* $(x \Rightarrow y) = 1$,

(3) $(1 \Rightarrow x) = x$,

(4) $\max(x,\ y) = \min((x \Rightarrow y) \Rightarrow y,\ (y \Rightarrow x) \Rightarrow x)$.

If $*$ is continuous t-norm we have also:

(5) $\min(x,\ y) = x * (x \Rightarrow y)$.

Having this we may compute residua of three the most important t-norms: $(x \Rightarrow y) = 1$ for $x \leq y$ and for $x > y$:

- *Łukasiewicz* residuum: $x \Rightarrow y = 1 - x + y$
- *Gödel* residuum: $x \Rightarrow y = y$
- *Product* residuum: $x \Rightarrow y = \frac{y}{x}$

Before we present the theorem, which states why our three t-norms are so important, we prepare some definitions. This theorem is a variant of the known Mostert-Shields theorem ([70].

Definition A.4 *Let $*$ be a continuous t-norm.*

- *An element x of $[0,\ 1]$ is called an* idempotent *if $x * x = x$.*
- *An idempotent x is* non-trivial *if $x \neq 1$ and $x \neq 0$.*
- *An non-trivial idempotent x is a* cutpoint *if there no closed interval of idempotents containing x as its inner point.*

Theorem A.5 *Let $*$ be a continuous t-norm and $\Rightarrow$ its residuum. Then there is a countable set of pairs (finite or infinite) $\mathbf{R} = ([r_i, s_i], A_i)$, where $[r_i, s_i]$ is a closed interval from $[0,1]$ (borders are either cutpoints or 0 or 1) and A_i is an elements from $\{\text{Ł}, \Pi, \text{G}\}$. The union of intervals $[r_i, s_i]$ is dense in $[0,1]$ and they overlap at most in their borders. Then there is a t-norm $*'$ (and the corresponding residuum $\Rightarrow'$) isomorphic to the t-norm $*$ (to the residuum $\Rightarrow$ respectively) so the following holds:*

$$x *' y = \begin{cases} r_i + (s_i - r_i)(\frac{x - r_i}{s_i - r_i} *_A \frac{y - r_i}{s_i - r_i}) & \textit{if } x, y \in [r_i,\ s_i] \\ \min(x,\ y) & \textit{otherwise} \end{cases}$$

$$x \Rightarrow' y = \begin{cases} 1 & \textit{if } x \leq y \\ r_i + (s_i - r_i)(\frac{x - r_i}{s_i - r_i} \Rightarrow_A \frac{y - r_i}{s_i - r_i}) & \textit{if } x, y \in [r_i,\ s_i],\ x > y \\ y & \textit{otherwise} \end{cases}$$

A t-norm is said to be a finitely constructed is the set of covering intervals is finite.

The converse of this claim also holds, i.e., each operation constructed in this way is a continuous t-norm.

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